Some alternative pricing and optimization techniques in financial mathematics

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PhD THESIS

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THESIS ADVISOR: EMMANUEL LÉPINETTE

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Abstract

This thesis presents four problems in pricing and optimization in financial mathematics. The first three problems are completely solved and the fourth one is in progress.

In the first part, we consider the hedging problem in presence of dynamic risk measures defined on the space L^0 of random variables. In particular, we provide a no arbitrage (NA) condition under which the risk-hedging price is attained. Moreover, we show that under NA, the set of all risk-hedging prices is closed. We then prove a version of Fundamental Theorem of Asset Pricing and a dual characterization of the risk-hedging prices of a European option. At last, we provide an example where the dual representation of the risk-measure on L^0 is possible.

In the second part, we solve a classical problem of characterizing the prices of European options in financial market models with transaction costs. In the Kabanov model, it is well known that the infimum super-hedging price is presented via a dual characterization through Consistent Price Systems (**CPS**) under some appropriate NA condition, see the book [55]. However, it is difficult to characterize **CPS** given the only attempt proposed in [68] for finite probability spaces. In this work, we shall tackle directly the primal problem of super-hedging. To do so, we first prove a general version of Dynamic Programming Principle (DPP) for conditional essential infimum. We then introduce a weak NA condition under which the DPP is implementable. The interesting feature of this approach is that it works also for non-convex financial market models.

In the third part, we apply the theoretical result established in the second part by providing an algorithm to compute the super-hedging prices in practice. In particular, we prove the efficiency of the algorithm using the idea of (random) epiconvergence. Moreover, the *exact* prices will be deduced for the case of proportional transaction cost and the case of fixed cost.

In the last part, we present our current progress on the problem of portfolio optimization under credit risk constraints. Our problem fits into the framework of optimal control under stochastic target *pathwise* constraints. We then follow the idea in [10] to characterize the value function as a viscosity solution to a PDE. Our next step is to provide a condition for the uniqueness of our PDE and a numerical scheme to compute the value function.

Contents

Chapter 1

Introduction générale

Donner un *juste prix* pour un actif financier ξ a été l'une des principales tâches des mathématiques financières. Il existe de nombreuses notions de prix, y compris, mais sans s'y limiter, *le prix viable*, *le prix de sur-réplication*, *le prix d'indifférence*, voir le livre [9] pour une brève introduction. Dans cette thèse, nous considérons principalement les prix de sur-réplication des options européennes, c'est-à-dire des produit financiers qui paient un montant aléatoire ξ à maturité $T > 0$.

Avant de donner le prix d'un actif, nous devons d'abord définir le marché financier sous-jacent. Nous appelons un marché financier *marché sans friction* s'il permet aux participants d'acheter et de vendre librement les différents actifs, sans coûts de transaction ni les taxes. En temps discret, nous travaillons généralement sur un espace de probabilité complet $(\Omega, \mathcal{F}, P)$ muni d'une filtration $\mathbb{F} := (\mathcal{F}_t)_{0\leq t\leq T}$ telle que $\mathcal{F}_T = \mathcal{F}$. En finance, la filtration $\mathcal{F}$ peut être interprétée comme le flux d'informations. On note $L^0(\mathbf{R}^d, \mathcal{F}_t)$ l'espace des variables aléatoires $\mathcal{F}_t$-mesurables ayant des valeurs dans $\mathbf{R}^d$. Nous considérons un processus de prix $S := (S_t)_{0\leq t\leq T}$ tel que $S_t \in L^0(\mathbf{R}^d, \mathcal{F}_t)$. Un processus $(\theta_t)_{0\leq t\leq T}$ est une *stratégies financières* si $\theta_t \in L^0(\mathbf{R}^d, \mathcal{F}_t)$. θ^i_{t-1} représente le nombre de stocks investissant dans l'actif S^i_t pendant la période $[t, t+1]$. Nous en déduisons ensuite au temps t la valeur $V^{x,\theta}_t$ d'un portefeuille en utilisant la stratégie θ et en partant du capital initial $x \in \mathbf{R}$:

$$V^{x,\theta}_t := x + \sum_{u=1}^{t} \theta_{u-1}(S_u - S_{u-1}) = x + \sum_{u=1}^{t} \theta_{u-1}\Delta S_u.$$

où l'on note ab le produit scalaire de vecteurs a at b. Considérons maintenant un actif contingent ξ qui est une variable aléatoire dans $L^0(\mathbf{R}, \mathcal{F}_T)$

et dénotons l'ensemble des stratégies par Θ. Nous définissons le prix de. sur-réplication de ξ comme suit:

$$p(\xi) := \inf \left\{ p \in \mathbf{R} : \ \exists\ \theta \in \Theta \text{ s.t. } V_T^{p,\theta} \geq \xi, \text{ p.s.} \right\}.$$

En théorie des mathématiques financières, le prix de sur-réplication peut être caractérisé en supposant une condition d'absence d'arbitrage (**AOA**). En particulier, nous nous restreignons au cas où aucun profit ne peut être réalisé avec une probabilité positive à partir de rien. En langage mathématique, un arbitrage opportunité est une stratégie financière θ satisfaisant:

$$V_T^{0,\theta} \geq 0, \ P - \text{p.s. et } P(V_T^{0,\theta} > 0) > 0.$$

AOA est vérifiée s'il n'y a pas d'opportunité d'arbitrage. Un résultat classique en mathématiques financières appelé Fundamental Theorem of Asset Pricing (FTAP) a été formalisé pour la première fois dans [43]. FTAP donne une caractérisation équivalente de **AOA** et l'existence de mesures de martingales. En particulier, on note $\mathcal{M}(S)$ l'ensemble des mesures de martingale, c'est-à-dire une collection de mesures de probabilité $Q \sim P$ tel que S est une Q-martingale.

Theorem 1.0.1 (Le premier FTAP)**.** *Les suivants sont équivalentes:*

1) **AOA** *est vérifiée,*

2) $\mathcal{M}(S) \neq \emptyset$*.*

Étant donné l'ensemble des mesures de martingale équivalentes, nous sommes maintenant en mesure de en déduire la caractérisation duale du prix de sur-réplication.

Theorem 1.0.2 (Formulation duale)**.** *Supposons que* **NA** *est vérifiée pour que par FTAP,* $\mathcal{M}(S) \neq \emptyset$*, on a*

$$p(\xi) = \sup_{Q \in \mathcal{M}(S)} E_Q[\xi].$$

où E_Q *dénote l'espérance sous* Q*.*

De plus, si nous supposons que tous les actifs contingents sont replicables ou que le marché est *complet*, c'est-à-dire pour tout ξ, il existe $(p, \theta) \in \mathbf{R} \times \Theta$ tel que, $V_T^{p,\theta} = \xi$ P-p.s., on obtient le deuxième FTAP.

Theorem 1.0.3 (Le deuxième FTAP)**.** *Supposons que $\mathcal{M}(S) \neq \emptyset$. Puis le marché est complet si et seulement si $\mathcal{M}(S)$ est un singleton.*

Le marché complet est une propriété souhaitable non seulement en théorie mais aussi en pratique. Les praticiens qui étudient le prix de sur-réplication supposent souvent que le marché est complet de sorte que le prix de sur-réplication pour ξ peut être calculé par $E_Q(\xi)$, où Q est l'unique mesure de martingale équivalente dans $\mathcal{M}(S)$. $E_Q(\xi)$ peut être évalué en utilisant des méthodes de Monte Carlo ou par des méthodes pour les EDPs paraboliques, voir par exemple les livres [9] ou [37].

Dans de nombreux cas intéressants, marché complet n'est plus satisfait et le prix de sur-réplication devient désormais impossible à calculer lors de l'utilisation de la caractérisation duale. Cela motive le besoin d'un cadre alternatif pour calculer les prix de sur-réplication. Dans les travaux récents [17], au lieu de supposer **AOA** depuis le début, les auteurs ont abordé directement le problème de la sur-réplication dans un marché sans friction. Ils ont proposé une condition faible de non-arbitrage appelée absence de profit instantané (**API**) qui est l'exigence minimale pour que le prix de sur-réplication soit fini. Un marché satisfait à la condition **API** si le prix de la sur-réplication pour le payoff nul est identique à zéro, c'est-à-dire $p(0) = 0$. **API** est strictement plus faible que **AOA**, donc l'ensemble des mesures de martingale équivalentes peut être vide par le premier FTAP. Par conséquent, nous appellerons ce cadre alternatif *valorisation sans mesure de martingale*.

Dans cette thèse, nous adoptons **API** comme point de départ et nous le développerons dans deux directions. D'abord, dans la définition du prix de sur-réplication, la contrainte presque sûre est maintenant remplacée par $\rho(V_T^{p,\theta} - \xi) \leq 0$ pour une certaine mesure de risque ρ. Deuxièmement, nous travaillons avec des marchés financiers où les coûts de transaction sont encourus chaque fois que nous achetons ou vendons des actifs risqués. Les matériaux du chapitre 3, du chapitre 4 et du chapitre 5 sont basés sur la publication et les prépublications suivantes.

i) Coherent risk measure on L^0: NA condition, Pricing and Dual representation, with Lépinette E. IJTAF. 2021.

ii) Dynamic programming principle and computable prices in financial market models with transaction costs, with Lépinette E. Preprint. 2022.

iii) Limits theorems for super-hedging prices in general models with transaction costs, with Lépinette E. Preprint. 2022.

Chapter 2

General introduction

2.1 Motivation

How to determine a *fair price* for a financial asset ξ has been one of the main task in financial mathematics. There are many notions of price including but not limited to *viable price, super-hedging price, indifference price*, see the book [9] for a brief introduction. In this thesis, we consider mainly the *super-hedging prices* of European options, i.e. financial derivatives that pay a random amount ξ at *maturity* $T > 0$.

Before giving price for an asset, we first need to define the underlying financial market. We call a financial market *frictionless* if it allows participants to buy and sell different assets freely, without trading restriction nor price impact or taxes. In discrete-time setting, we usually work on a complete probability space $(\Omega, \mathcal{F}, P)$ and a filtration $\mathbb{F} := (\mathcal{F}_t)_{0\leq t\leq T}$ such that $\mathcal{F}_T = \mathcal{F}$. In finance, the filtration $\mathbb{F}$ can be interpreted as the flow of information. We denote by $L^0(\mathbf{R}^d, \mathcal{F}_t)$ the set of $\mathbf{R}^d$-valued $\mathcal{F}_t$-measurable random variables. We consider a price process $S := (S_t)_{0\leq t\leq T}$ such that $S_t \in L^0(\mathbf{R}^d, \mathcal{F}_t)$. A process $(\theta_t)_{0\leq t\leq T-1}$ is trading strategy if $\theta_t \in L^0(\mathbf{R}^d, \mathcal{F}_t)$. θ^i_{t-1} represents the number of shares investing in the asset S^i_t during the period $[t, t+1]$. We then deduce the value $V^{x,\theta}_t$ at time t of a portfolio using trading strategy θ and starting from initial capital $x \in \mathbf{R}$:

$$V^{x,\theta}_t := x + \sum_{u=1}^{t} \theta_{u-1}(S_u - S_{u-1}) = x + \sum_{u=1}^{t} \theta_{u-1}\Delta S_u.$$

where we use the notation ab as the scalar product of two vectors a and b.

Now, consider a contingent claim ξ which is a random variable belongs to $L^0(\mathbf{R}, \mathcal{F}_T)$ and denote the set of trading strategies as Θ, we now define the super-hedging price of ξ as:

$$p(\xi) := \inf \left\{ p \in \mathbf{R} : \ \exists\ \theta \in \Theta \text{ s.t. } V_T^{p,\theta} \geq \xi, \text{ a.s.} \right\}.$$

In theory of financial mathematics, the super-hedging price can be characterized assuming some no arbitrage (**NA**) condition holds true. In particular, we restrict ourselves to the case where no profit can be made with positive probability starting from nothing. In mathematical language, an arbitrage opportunity is a financial strategy θ satisfying:

$$V_T^{0,\theta} \geq 0, \ P - \text{a.s. and } P(V_T^{0,\theta} > 0) > 0.$$

NA holds if there is no arbitrage opportunity. A classical result in financial mathematics called Fundamental Theorem of Asset Pricing (FTAP) was first formalised in [43]. FTAP gives an equivalent characterization of **NA** and the existence of martingale measures. In particular, we denote the set of martingale measures as $\mathcal{M}(S)$ which is the collection of measure $Q \sim P$ such that under Q, S is a martingale.

Theorem 2.1.1 (The first FTAP)**.** *The following are equivalent:*

1) **NA** *holds.*

2) $\mathcal{M}(S) \neq \emptyset$.

Given the set of equivalent martingale measures, we are now able to deduce the dual characterization of super-hedging price:

Theorem 2.1.2 (Dual formulation)**.** *Suppose that* **NA** *holds so that by FTAP,* $\mathcal{M}(S) \neq \emptyset$*, we then have*

$$p(\xi) = \sup_{Q \in \mathcal{M}(S)} E_Q[\xi].$$

where E_Q *denotes the expectation under probability measure* Q*.*

Moreover, if we suppose that all contingent claims are replicable or the market is *complete*, i.e. for any ξ, there exists $(p, \theta) \in \mathbf{R} \times \Theta$ such that $V_T^{p,\theta} = \xi$ P-a.s., we get the second FTAP.

Theorem 2.1.3 (The second FTAP)**.** *Assume that $\mathcal{M}(S) \neq \emptyset$. Then, the market is complete if and only if $\mathcal{M}(S)$ is a singleton.*

Market completeness is a desirable property not only in theory but also in practice. Practitioners who study super-hedging price often assume the market is complete so that the super-hedging price for ξ can be computed as $E_Q(\xi)$, where Q is the unique equivalent martingale measure in $\mathcal{M}(S)$. $E_Q(\xi)$ can be evaluated using Monte Carlo methods or by methods for parabolic PDEs, see for examples the books [9] or [37].

In many interesting cases, market completeness fails to hold and super-hedging price now becomes infeasible to compute when using dual characterization. This motivates the need for an alternative framework in order to compute super-hedging prices. In the recent work [17], instead of assuming **NA** from the beginning, the authors tackled directly the super-hedging problem in frictionless market. They proposed a weak no arbitrage condition called Absence of Instantaneous Profit (**AIP**) which is the minimal requirement for the super-hedging price to be finite. A market satisfies **AIP** condition if the super-hedging price for the zero payoff is identical to zero, i.e. $p(0) = 0$. **AIP** is strictly weaker than **NA**, hence the set of equivalent martingale measures can be empty by the first FTAP. As a result, we shall call this alternative framework *pricing without martingale measure*.

In this thesis, we adopt **AIP** as our starting point and we will develop it in two directions. Firstly, we consider the case where a possibility of mishedge is allowed. In particular, in the definition of super-hedging price, the almost sure constraint is now replaced by $\rho(V_T^{p,\theta} - \xi) \leq 0$ for some risk measure ρ. Secondly, we work with financial markets where transaction costs are triggered whenever we buy or sell risky assets. In the remaining part of this chapter, we shall discuss briefly our main contributions. The materials of Chapter 3, Chapter 4 and Chapter 5 are based on the following publication and pre-publications.

i) Coherent risk measure on L^0: NA condition, Pricing and Dual representation, with Lépinette E. IJTAF. 2021.

ii) Dynamic programming principle and computable prices in financial market models with transaction costs, with Lépinette E. Preprint. 2022.

iii) Limits theorems for super-hedging prices in general models with transaction costs, with Lépinette E. Preprint. 2022.

2.2 Coherent risk measure on L^0: NA condition, Pricing and Dual representation

We consider a dynamic coherent risk-measure $X \mapsto (\rho_t(X))_{t\leq T}$ defined on the space $L^0(\overline{\mathbf{R}}, \mathcal{F}_T)$, $\overline{\mathbf{R}} = [-\infty, \infty]$. In this paper, the risk-measure is constructed from its closed acceptance sets $(\mathcal{A}_t)_{t\leq T}$ of acceptable financial positions $\mathcal{A}_t$ at time $t \leq T$. We suppose that $\mathcal{A}_t$ is a closed convex cone. We then define

$$\begin{aligned} \operatorname{Dom} \mathcal{A}_t &:= \{X \in L^0(\mathbf{R}, \mathcal{F}_T) : \ \mathcal{A}_t^X \neq \emptyset\}, \\ \mathcal{A}_t^X &:= \{C_t \in L^0(\mathbf{R}, \mathcal{F}_t) |\, X + C_t \in \mathcal{A}_t\}. \end{aligned}$$

We recall the definition the conditional essential infimum of a random variable, see [17] for a short proof of the existence.

Proposition 2.2.1 (Conditional essential infimum)**.** *Let $\mathcal{H}$ and $\mathcal{F}$ be complete σ-algebras such that $\mathcal{H} \subseteq \mathcal{F}$ and let $\Gamma = (\gamma_i)_{i\in I}$ be a family of real-valued $\mathcal{F}$-measurable random variables. There exists a unique (up to a P-negligible set) random variable $\gamma_{\mathcal{H}} \in L^0(\overline{\mathbf{R}}, \mathcal{H})$, denoted by $\operatorname{ess\,inf}_{\mathcal{H}} \Gamma$, which satisfies the following properties*

1) For every $i \in I$, $\gamma_{\mathcal{H}} \leq \gamma_i$ a.s.

2) If $\zeta \in L^0(\overline{\mathbf{R}}, \mathcal{H})$ satisfies $\zeta \leq \gamma_i$ a.s. for all $i \in I$, then $\zeta \leq \gamma_{\mathcal{H}}$ a.s.

For $X \in L^0(\overline{\mathbf{R}}, \mathcal{F}_T)$, we define $\rho_t(X) = \operatorname{ess\,inf}_{\mathcal{F}_t} \mathcal{A}_t^X$ if $X \in \operatorname{Dom} \mathcal{A}_t$ and we consider its extension to the whole space $L^0(\overline{\mathbf{R}}, \mathcal{F}_T)$ by [67]. For $X \in L^0(\overline{\mathbf{R}}, \mathcal{F}_T)$, $\rho_t(X)$ may be infinite and $\rho_t(X) \in \mathbf{R}$ a.s. if and only if $X \in \operatorname{Dom} \mathcal{A}_t$. We are interested in the super-hedging problem in the presence of the (random) risk measure ρ_t. Precisely, we consider the one-period risk-hedging problem

Definition 2.2.2. *A payoff $h_{t+1} \in L^0(\mathbf{R}, \mathcal{F}_{t+1})$ is said to be risk-hedged at time t if there exists $P_t \in L^0(\mathbf{R}, \mathcal{F}_t)$ and a strategy θ_t in $L^0(\mathbf{R}^d, \mathcal{F}_t)$ such that $P_t + \theta_t \Delta S_{t+1} - h_{t+1}$ is acceptable at time t.*

In that case, we say that P_t is a risk-hedging price. Let $\mathcal{P}_t(h_{t+1})$ be the set of all risk-hedging prices $P_t \in L^0(\mathbf{R}, \mathcal{F}_t)$ at time t. The minimal risk-hedging price of the contingent claim h_{t+1} at time t is defined as

$$P_t^* := \operatorname*{ess\,inf}_{\theta_t \in L^0(\mathbf{R}^d, \mathcal{F}_t)} \mathcal{P}_t(h_{t+1}). \tag{2.2.1}$$

Some contributions:

1. We introduce a no-arbitrage condition (NA) under which the minimal risk-hedging price of the contingent claim is attained. The result is extended to multi-period framework using the idea of *Consistent risk measure*.

2. We prove a version of the Fundamental Theorem of Asset Pricing in presence of a risk-measure ρ_t. In particular, we prove that the set of all risk-hedging prices is closed under NA, it then suffices to apply Hahn-Banach theorem. Moreover, a dual characterization of the risk-hedging prices of a European option follows.

3. We provide a dual representation of the risk-measure on L^0 under some conditions. This result is an extension to the dual representation of risk measure on L^∞ in the literature.

2.3 Dynamic programming principle and computable prices in financial market models with transaction costs

We consider a financial market where transaction costs are charged when the agents buy or sell risky assets. The typical case is a model defined by a bond whose discounted price is $S^1 = 1$ and $d-1$ risky assets that may be traded at some bid and ask discounted prices S^b and S^a, respectively, when selling or buying. Our general model is defined via a set-valued process $(\mathbf{G}_t)_{t=0}^T$ adapted to the filtration $(\mathcal{F}_t)_{t=0}^T$. Precisely, we suppose that for all $t \leq T$, $\mathbf{G}_t$ is $\mathcal{F}_t$-measurable in the sense of the graph $\text{Graph}(\mathbf{G}_t) = \{(\omega, x) : \ x \in G_t(\omega)\}$ that belongs to $\mathcal{F}_t \otimes \mathcal{B}(\mathbf{R}^d)$ (see [70]), where $\mathcal{B}(\mathbf{R}^d)$ is the Borel σ-algebra on $\mathbf{R}^d$ and $d \geq 1$ is the number of assets. $\mathbf{G}_t$ is usually called *solvency set* in the literature. In the classical approach of models with transaction costs, the analogs of FTAP and hedging theorem are proposed, see the book [55] for a comprehensive treatment. In this thesis, we use an alternative approach called *cost value process*.

We suppose that $\mathbf{G}_t(\omega)$ is closed for every $\omega \in \Omega$ and $\mathbf{G}_t(\omega) + \mathbf{R}^d_+ \subseteq \mathbf{G}_t(\omega)$, for all $t \leq T$. The cost value process $\mathrm{C} = (\mathrm{C}_t)_{t=0}^T$ associated to $\mathbf{G}$ is

defined as:

$$\mathrm{C}_t(z) = \inf\{\alpha \in \mathbf{R} : \ \alpha e_1 - z \in \mathbf{G}_t\} = \min\{\alpha \in \mathbf{R} : \ \alpha e_1 - z \in \mathbf{G}_t\}, \quad z \in \mathbf{R}^d.$$

A portfolio process is a stochastic process $(V_t)_{t=-1}^T$ where $V_{-1} \in \mathbf{R}e_1$ is the initial endowment expressed in cash that we may convert immediately into $V_0 \in \mathbf{R}^d$ at time $t = 0$. By definition, we suppose that

$$\Delta V_t = V_t - V_{t-1} \in -\mathbf{G}_t, \ a.s., \quad t = 0, \cdots, T.$$

This means that any position $V_{t-1} = V_t + (-\Delta V_t)$ may be changed into the new position V_t, letting aside the residual part $(-\Delta V_t)$ that can be liquidated without any debt. Let $\xi \in L^0(\mathbf{R}^d, \mathcal{F}_T)$ be a contingent claim. Our goal is to characterize the set of all portfolio processes $(V_t)_{t=-1}^T$ such that $V_T = \xi$. We are mainly interested by the infimum cost one needs to hedge ξ, i.e. the infimum value of the initial capitals $V_{-1}e_1 \in \mathbf{R}$ among the portfolios $(V_t)_{t=-1}^T$ replicating ξ.

Some contributions:

1. We first provide a dynamic programming principle in a very general setting in discrete time. In particular, the dynamic programming is stated using the notion of conditional essential supremum.

2. Secondly, we propose some weak no-arbitrage conditions under which it is possible to implement the dynamic programming principle. In particular, we show that under this NA condition, the infimum hedging cost defined as conditional essential infimum coincides with ω-wise infimum. This result is interesting given the only attempt in [68] which is proposed for finite probability space.

2.4 Limit theorems for the super-hedging prices in general models with transaction costs

This project developes the numerical methods for the theoretical results in the second project.

In particular, we consider the market where for each time t there is a family of $\mathcal{F}_t$-measurable random variables the following holds: $(\alpha_t^m)_{m\geq 1}$ such

that $S_{t+1} \in \{\alpha_t^m : \ m \geq 1\}$ a.s. and that $P(S_{t+1} = \alpha_t^m | \mathcal{F}_t) > 0$ a.s. for all $m \geq 1$. We aim to estimate $\text{ess sup}_{\mathcal{F}_t} f(S_{t+1})$ via a sequence of random variables $\{b_{t+1}^i, i \geq 1\}$, $b_{t+1}^i \in L^0(\mathbf{R}^d, \mathcal{F}_{t+1})$ which are independent and identically distributed conditional on $\mathcal{F}_t$ (notation $\mathcal{F}_t$-i.i.d.) in the following sense:

$$P\left[b_{t+1}^i \in B | \mathcal{F}_t\right] = P\left[b_{t+1}^j \in B | \mathcal{F}_t\right], \text{ a.s., } i, j \geq 1,$$
$$P\left[\bigcap_{j \in J} \left\{b_{t+1}^j \in B_j\right\} \Big| \mathcal{F}_t\right] = \prod_{j \in J} P\left[b_{t+1}^j \in B_j \big| \mathcal{F}_t\right], \text{ a.s.}$$

for all finite set $J \subset \mathbb{N}$, and Borel sets $B, B_j, j \in J$. Moreover, we also require $b_{t+1}^i \in \{\alpha_t^n, n \geq 1\}$ a.s. and $P(b_{t+1}^i = \alpha_t^n | \mathcal{F}_t) > 0$ a.s. for all $n, i \geq 1$. This is a first attempt to approximate the conditional essential supremum and is related to our super-hedging problem.

Some contributions

Using the idea of convergence of epigraph (epiconvergence), we establish some results:

1. For the first goal, we prove the validity of the approximation of $\mathcal{F}_t$ conditional essential supremum via a sequence of $\mathcal{F}_t$-i.i.d. random variables. In particular, we prove that:

$$\max_{i \leq m} f(b_{t+1}^i) =: \theta_t^m \to \theta_t := \text{ess sup}_{\mathcal{F}_t}(S_{t+1}), \text{ a.s.}$$

   as $m \to \infty$. Subsequently, using this convergence result, we then prove the convergence almost surely of the sequence of *randomized* super-hedging costs to the desired one. Finally, the result is extended to multi-period period by the help of Dynamic Programming Principle.

2. We give the prices for models with proportional and fixed costs. We consider here the Multinomial price process.

2.5 Portfolio optimization under credit risk constraints

Consider a financial market model defined on a stochastic basis $(\Omega, (\mathcal{F}_t)_{t \in [0,T]}, P)$ satisfying the usual assumptions. We denote by S^0 the risk-free asset of the

market and we suppose without loss of generality that $S^0 = 1$ so that the risk-free interest rate is $r^0 = 0$. In the following, we consider at any time $t \in [0, T]$ a firm characterized by it debts $(D_r)_{r\in[t,T]}$ and its asset $(A_r)_{r\in[t,T]}$ so that the equity is given by $(E_r)_{r\in[t,T]}$ such that $E = (A - D)^+$. We suppose that D satisfies the SDE:

$$dD_u = r_u D_u du - K_u du, \quad u \in [t, T], \tag{2.5.2}$$

where $r \geq 0$ is the debt interest rate (interpreted as a risk premium since $r^0 = 0$) and K_u is the amount of the firm reimbursement per unit of time. We suppose that $K_u := k_u D_u$ for some process k. Asset A of the firm satisfies by assumption $A_r = \theta_r^0 S_r^0 + \theta_r S_r$ where θ^0 and θ are quantities invested in asset S^0 and some risky assets $S = (S_1, \cdots, S_d)$, $d \geq 1$, held by the firm. We suppose the following self-financing condition:

$$dA_r = \theta_r dS_r - c_r dr, \quad r \in [t, T], \tag{2.5.3}$$

where c is a process such that $c \geq K$. We interpret $c_t - K_t$ as the amount of dividends distributed at time t. We only consider admissible strategies θ such that $A_r \geq \kappa^\theta$ for all $r \in [t, T]$, for some $\kappa^\theta \in \mathbf{R}$. Liquidation value of the asset firm at time $u \geq t$ is defined as $L_u := A_u - D_u$ so that we have $E = L^+$. Note that the dynamics of L is :

$$dL_u = \theta_u dS_u - d_u du - r_u D_u du, \quad u \in [t, T], \tag{2.5.4}$$

where $d_t = c_t - k_t D_t$ is the amount of dividends. The dynamics above shows that the liquidation value of the firm's financial position is naturally controlled by the investment strategy θ but it is adversely impacted by the dividends $d \geq 0$ paid to the share holders and by the credit risk premium r as well. Taking into account a possible default, the payoff delivered to the credit holders is as the Merton model:

$$h_T^D(r) := \int_t^T k_u D_u du + [A_T]^+ \wedge D_T$$

The goals of this project is to first find a *fair* value for the risk premium r. And then, given this *fair* credit risk premium and a utility function U we solve the following utility maximization problem:

$$V(t, x, y) := \sup_{\theta,c} J^0(t, x, y, \theta, c) := E\left(U\left(\int_t^T d_u du + L_T^{x,y,\theta,c}(r)\right)\right) \tag{2.5.5}$$

where θ and $c = (d_t, k_t)$ are progressively measurable process and satisfy the pathwise constraints:

i) Self-financing constraint: $A_s \geq 0$ a.s. for all $t \leq s \leq T$.

ii) *Fair* price condition (Market pricing or MP condition):

$$E\left(h_T^D(r)|(D_t, L_t) = (x, y)\right) = x.$$

Some contributions and future works

1. We prove that in a complete market such that S is a martingale, there exists a unique fair price satisfying MP condition: $E(h_T^D(r)|\mathcal{F}_t) = D_t$. This result guarantees that the problem 2.5.5 is well-defined.

2. Translate the problem 2.5.5 into a regular form of optimal control problem with stochastic pathwise constraints. Then, we establish a Dynamic Programming Principle for our problem.

3. We show that the value function should be interpreted as a viscosity solution of a HJB equation with state constraints, see the work in [10]. Our next goal is to provide a condition for uniqueness and a numerical scheme to compute the value function V.

2.6 Measurable selection theorems

In the following chapters, whenever we use the phrase *measurable selection argument*, we refer to the Theorem below. For a (lengthy) proof, see the book [27].

Theorem 2.6.1 (Measurable selection)**.** *Let $(\Omega, \mathcal{F}, P)$ be a complete probability space E be a Polish space equipped with a Borel sigma-algebra $\mathcal{E}$ and let $\Gamma \in \mathcal{F} \otimes \mathcal{E}$. Then, the projection $\pi_\Omega \Gamma$ of Γ onto Ω belongs to $\mathcal{F}$, and there exists an E-valued random variables ξ such that $\xi(\omega) \in \Gamma_\omega$ for all non-empty ω-sections Γ_ω of Γ.*

We also include here a *universal* measurable selection called Jankov-von Neumann theorem which will appear in Chapter 6 and Chapter 7. To do so, we first recall the definition of analytic sets and upper semianalytic function, see the books [8] or [27] for a detailed analysis.

Definition 2.6.2. *A subset of a Polish space X is analytic, if either it is empty or a continuous image of a Polish space. A function $g : X \to \overline{\mathbf{R}}$ is upper semianalytic (usa) if the set $\{x \in X : g(x) > c\}$ is analytic for every $c \in \mathbf{R}$. A function $g : X \to \overline{\mathbf{R}}$ is lower semianalytic (lsa) if the set $\{x \in X : g(x) < c\}$ is analytic for every $c \in \mathbf{R}$.*

Definition 2.6.3. *Let $(\Omega, \mathcal{F})$ be a measurable space. We denote $\mathcal{F}^P$ be the completion of $\mathcal{F}$ with respect to probability measure P. The universal completion of $\mathcal{F}$ is the σ-algebra defined as the intersection of $\mathcal{F}^P$ for all probability measures P on $(\Omega, \mathcal{F})$. A function $g : X \to \overline{\mathbf{R}}$ is universally measurable if the set $\{x \in X : g(x) > c\}$ belongs to the universal completion of $\mathcal{F}$ for every $c \in \mathbf{R}$.*

Theorem 2.6.4 (Jankov-von Neumann theorem)**.** *Let X and Y be Polish spaces, and A an analytic subset of $X \times Y$. There exists a universally measurable function $\varphi : \pi_X A \to Y$ such that* $\mathrm{Graph}(\varphi) \subset A$.

Theorem 2.6.5. *Let X and Y be Polish spaces, A an analytic subset of $X \times Y$ and $f : X \times Y \to \overline{\mathbf{R}}$ an upper semianalytic function. We define $f^* : \pi_X(A) \to \overline{\mathbf{R}}$ by*

$$f^*(x) := \sup_{y \in A_x} f(x, y).$$

Then, for every $\epsilon > 0$, there exists a universal measurable function $\varphi^\epsilon : \pi_X(A) \to Y$ such that $\mathrm{Graph}(\varphi^\epsilon) \subset A$ *and for all $x \in \pi_X(A)$:*

$$\begin{aligned} f(x, \varphi^\epsilon(x)) &\geq f^*(x) - \epsilon, \text{ if } f^*(x) < \infty, \\ f(x, \varphi^\epsilon(x)) &\geq \epsilon^{-1}, \text{ if } f^*(x) = \infty. \end{aligned}$$

Chapter 3

Coherent Risk Measure on L^0: NA Condition, Pricing and Dual Representation

Abstract

The NA condition is one of the pillars supporting the classical theory of financial mathematics. We revisit this condition for financial market models where a dynamic risk-measure defined on L^0 is fixed to characterize the family of acceptable wealths that play the role of non negative financial positions. We provide in this setting a new version of the fundamental theorem of asset pricing and we deduce a dual characterization of the super-hedging prices (called risk-hedging prices) of a European option. Moreover, we show that the set of all risk-hedging prices is closed under NA. At last, we provide a dual representation of the risk-measure on L^0 under some conditions.

3.1 Introduction

The NA condition originates from the work of Black and Scholes [7] and Merton [69]. In these articles, the risky asset is modeled by a geometric Brownian motion. The NA condition means the absence of arbitrage opportunities, i.e. a nonzero terminal portfolio value can not be acceptable if it starts from the zero initial endowment. A financial position in the classical arbitrage theory is acceptable if it is non negative almost surely. In our work, the new con-

tribution is that we consider a larger class of acceptable positions which are defined from a risk-measure.

The NA condition is characterized through the famous Fundamental Theorem of Asset Pricing (FTAP) for a variety of financial models. Essentially, NA is equivalent to the existence of a so-called risk-neutral probability measure, under which the price process is a martingale. In discrete-time, the well known FTAP theorem has been proved by Dalang, Morton and Willinger [20]. We may also mention the papers [49], [52], [54], [77], [78]. In continuous time, the formulation of the FTAP theorem is only possible once continuous-time self-financing portfolios are defined, see the seminal work of Black and Scholes [7]. This gave rise to an extensive development of the stochastic calculus, e.g. for semi-martingales [44], making possible formulation of several versions of the FTAP theorem as given in [21], [22], [23], [24] and [40].

The main contribution of the FTAP theorems is the link between the concept of arbitrage and the pricing technique which is deduced. It is now very well known that the super-hedging prices of a European claim are dually identified through the risk-neutral probability measures characterizing the NA condition. We may notice that the NA condition has been suitably chosen in the models of consideration in such a way that the set of all attainable claims is closed, see [55, Theorem 2.1.1]. This allows one to apply the Hahn-Banach separation theorem, see [81], and obtain dual elements that characterize the super-hedging prices. This is also the case for financial models with proportional transaction costs, see [55, Section 3] and the references mentioned therein.

The growing use of risk-measures in the context of the Basel banking supervision naturally calls into question the definition of the super-hedging condition which is commonly accepted in the usual literature. Recall that a portfolio process $(V_t)_{t\in[0,T]}$ super-replicates a contingent claim h_T at the horizon date $T > 0$ means that $V_T \geq h_T$ a.s.. In practice, this inequality remains difficult to achieve and practitioners accept to take a moderate risk, choosing for example $\alpha \in (0,1)$ small enough so that $P(V_T - h_T \geq 0) \geq 1-\alpha$ is close to 1. This is the case when considering the Value At Risk measure, see [50], and we say that $V_T - h_T$ is acceptable. More generally, $V_T - h_T$ is said acceptable for a risk-measure ρ if $\rho(V_T - h_T) \leq 0$, see [2], [25], [26], [29],[33] and [30] for frictionless markets and [3], [6], [32], [51] for conic models. The acceptable positions play the role of the almost surely non negative random

variables and allow one to take risk controlled by the risk measure we choose. Moreover, by considering a larger family of acceptable positions, the hedging prices may be lowered as shown in [71] for the Black and Scholes models with proportional transaction costs, see also the discussion in [63].

Pricing with a coherent risk-measure has been explored and developed by Cherny in two major papers [14] and [15] for coherent risk-measures defined on the space of bounded random variables. Cherny supposes that the risk-measure ρ (or equivalently the utility measure $u = -\rho$) is defined by a weakly compact determining set $\mathcal{D}$ of equivalent probability measures, i.e. such that $\rho(X) = \sup_{Q\in\mathcal{D}} E_Q(-X)$ for any $X \in L^\infty$. This representation automatically holds for coherent risk-measures defined on L^∞. This motivates the choice of Cherny to suppose such a representation for the risk-measures he considers on L^0 as he claims that *it is hopeless to axiomatize the notion of a risk measure on L^0 and then to obtain the corresponding representation theorem*, see [15].

Actually, the recent paper [67] proposes an axiomatic construction of a dynamic coherent risk-measure on L^0 from the set of all acceptable positions. We consider such a dynamic risk-measure and we define the discrete-time portfolio processes as the processes $(V_t)_{t\leq T}$ adapted to a filtration $(\mathcal{F}_t)_{t\leq T}$ such that $V_t + \theta_t\Delta S_{t+1} - V_{t+1}$ is acceptable at time t for some $\mathcal{F}_t$-measurable strategy $\theta_t \in L^0(\mathbf{R}^d, \mathcal{F}_t)$. This is a generalization of the classical definition where, usually, acceptable means non negative so that $V_t + \theta_t\Delta S_{t+1} \geq V_{t+1}$ almost surely. We then introduce a no-arbitrage condition we call NA as in the classical literature and we show that it coincides with the usual NA condition if the acceptable positions are the non negative random variables. This NA condition allows one to dually characterize the super-hedging prices, at least when ρ is time-consistent. One of our main contribution is a version of the Fundamental Theorem of Asset Pricing in presence of a risk-measure.

Similarly, Cherny proposes in his papers [14] and [15] a no-arbitrage condition *No Good Deal* (NGD) which is the key point to define the super-hedging prices. The approach is a priori slightly different: The NGD condition holds if there is no bounded claim X attainable from the zero initial capital such that $\rho(X) < 0$. In our setting, the NA condition is formulated from the minimal price super-hedging the zero claim, which is supposed to be non negative under NA. Clearly, there is a link between the NA and the NGD condition as $\rho(X)$ appears to be a possible super-hedging price for the zero claim. Actually, the NGD and the NA conditions are equivalent in the setting of Cherny, see Corollary 3.4.16. Although, in our paper we do not need

to suppose the existence of a priori given probability measure representing the risk-measure. This is why the proof of the FTAP theorem we formulate is more challenging as we cannot directly use an immediate compactness argument as done in [15] to obtain a risk-neutral probability measure. We then deduce a dual representation of the super-hedging prices in the case where the risk-measure is time-consistent. Under NA, we show that the set of all risk-hedging prices is closed. At last, we formulate a dual representation for a risk-measure defined on the whole set L^0, which is also a new contribution.

3.2 Framework

In discrete-time, we consider a stochastic basis $(\Omega, \mathcal{F} := (\mathcal{F}_t)_{t=0}^T, \mathrm{P})$ where the complete [1] σ-algebra $\mathcal{F}_t$ represents the information of the market available at time t. For any $t \leq T$, $L^0(\mathbf{R}^d, \mathcal{F}_t)$, $d \geq 1$, is the space of all $\mathbf{R}^d$-valued random variables which are $\mathcal{F}_t$-measurable, and endowed with the topology of convergence in probability. Similarly, $L^p(\mathbf{R}^d, \mathcal{F}_t)$, $p \in [1, \infty)$ (resp. $p = \infty$), is the normed space of all $\mathbf{R}^d$-valued random variables which are $\mathcal{F}_t$-measurable and admit a moment of order p under the probability measure P (resp. bounded). In particular, $L^p(\mathbf{R}_+, \mathcal{F}_t) = \{X \in L^p(\mathbf{R}, \mathcal{F}_t) | X \geq 0\}$ and $L^p(\mathbf{R}_-, \mathcal{F}_t) = -L^p(\mathbf{R}_+, \mathcal{F}_t)$ when $p = 0$ or $p \in [1, \infty]$. All equalities and inequalities between random variables are understood to hold everywhere on Ω up to a negligible set. If A_t is a set-valued mapping (i.e. a random set of $\mathbf{R}^d$), we denote by $L^0(A_t, \mathcal{F}_t)$ the set of all $\mathcal{F}_t$-measurable random variables X_t such that $X_t \in A_t$ a.s.. We say that $X_t \in L^0(A_t, \mathcal{F}_t)$ is a measurable selection of A_t. In our paper, a random set A_t is said $\mathcal{F}_t$-measurable if it is graph-measurable, see [70], i.e.

$$\mathrm{Graph}\, A_t = \{(\omega, x) \in \Omega \times \mathbf{R}^d : \ x \in A_t(\omega)\} \in \mathcal{F}_t \otimes \mathcal{B}(\mathbf{R}^d).$$

We consider a dynamic coherent risk-measure $X \mapsto (\rho_t(X))_{t \leq T}$ defined on the space $L^0(\overline{\mathbf{R}}, \mathcal{F}_T)$, $\overline{\mathbf{R}} = [-\infty, \infty]$. Precisely, we consider the risk-measure of [67], where an extension to the whole space $L^0(\overline{\mathbf{R}}, \mathcal{F}_T)$ is proposed. Recall that, in this paper, the risk-measure is constructed from its closed acceptance sets $(\mathcal{A}_t)_{t \leq T}$ of acceptable financial positions $\mathcal{A}_t$ at time $t \leq T$. We suppose that $\mathcal{A}_t$ is a closed convex cone. In the following, we use the conventions:

[1] This means that the σ-algebra contains the negligible sets so that an equality between two random variables is understood up to a negligible set.

$$0 \times (\pm\infty) = 0, \quad (0,\infty) \times (\pm\infty) = \{\pm\infty\},$$
$$\mathbf{R} + (\pm\infty) = \pm\infty, \quad \infty - \infty = -\infty + \infty = +\infty.$$

For $X \in L^0(\overline{\mathbf{R}}, \mathcal{F}_T)$, $\rho_t(X)$ may be infinite and $\rho_t(X) \in \mathbf{R}$ a.s. if and only if $X \in \operatorname{Dom} \mathcal{A}_t$ where

$$\begin{aligned} \operatorname{Dom} \mathcal{A}_t &:= \{X \in L^0(\mathbf{R}, \mathcal{F}_T) : \ \mathcal{A}_t^X \neq \emptyset\}, \\ \mathcal{A}_t^X &:= \{C_t \in L^0(\mathbf{R}, \mathcal{F}_t) |\, X + C_t \in \mathcal{A}_t\}. \end{aligned}$$

Actually, we have $\rho_t(X) = \operatorname{ess\,inf}_{\mathcal{F}_t} \mathcal{A}_t^X$ if $X \in \operatorname{Dom} \mathcal{A}_t$. Recall that the following properties hold (see [67]):

Proposition 3.2.1. *The risk-measure ρ_t satisfies the following properties:*

Normalization: $\rho_t(0) = 0$;

Monotonicity: $\rho_t(X) \geq \rho_t(X')$ whatever $X, X' \in L^0(\overline{\mathbf{R}}, \mathcal{F}_T)$ s.t. $X \leq X'$;

Cash invariance: $\rho_t(X + m_t) = \rho_t(X) - m_t$ if $m_t \in L^0(\mathbf{R}, \mathcal{F}_t)$, and $X \in L^0(\overline{\mathbf{R}}, \mathcal{F}_T)$;

Subadditivity: $\rho_t(X + X') \leq \rho_t(X) + \rho_t(X')$ if $X, X' \in L^0(\overline{\mathbf{R}}, \mathcal{F}_T)$;

Positive homogeneity: $\rho_t(k_t X) = k_t \rho_t(X)$ if $k_t \in L^0(\mathbf{R}_+, \mathcal{F}_t)$, $X \in L^0(\overline{\mathbf{R}}, \mathcal{F}_T)$.

Moreover, ρ_t is lower semi-continuous i.e., if $X_n \to X$ a.s., then $\rho_t(X) \leq \liminf_n \rho_t(X_n)$ a.s., and we have

$$\mathcal{A}_t = \{X \in \operatorname{Dom} \mathcal{A}_t \,|\, \rho_t(X) \leq 0\}. \tag{3.2.1}$$

In the following, we define $\mathcal{A}_{t,u} := \mathcal{A}_t \cap L^0(\overline{\mathbf{R}}, \mathcal{F}_u)$ for $u \in [t, T]$. Let $(S_t)_{t \leq T}$ be a process describing the discounted prices of d risky assets such that $S_t \in L^0(\mathbf{R}_+^d, \mathcal{F}_t)$ for any $t \geq 0$. A contingent claim with maturity date $t+1$ is defined by a real-valued $\mathcal{F}_{t+1}$-measurable random variable h_{t+1}. In the paper [67], the super-hedging problem for the payoff h_{t+1} is solved with respect to the dynamic risk-measure $(\rho_t)_{t \leq T}$. Precisely:

Definition 3.2.2. *A payoff $h_{t+1} \in L^0(\mathbf{R}, \mathcal{F}_{t+1})$ is said to be risk-hedged at time t if there exists $P_t \in L^0(\mathbf{R}, \mathcal{F}_t)$ and a strategy θ_t in $L^0(\mathbf{R}^d, \mathcal{F}_t)$ such that $P_t + \theta_t \Delta S_{t+1} - h_{t+1}$ is acceptable at time t. In that case, we say that P_t is a risk-hedging price.*

Let $\mathcal{P}_t(h_{t+1})$ be the set of all risk-hedging prices $P_t \in L^0(\mathbf{R}, \mathcal{F}_t)$ at time t as in Definition 3.2.2. In the following, we suppose that $\mathcal{P}_t(h_{t+1}) \neq \emptyset$. This is the case if there exist $a_t, b_t \in L^0(\mathbf{R}, \mathcal{F}_t)$ such that $h_{t+1} \leq a_t S_{t+1} + b_t$. This inequality trivially holds for European call and put options.

Definition 3.2.3. *The minimal risk-hedging price of the contingent claim h_{t+1} at time t is defined as*

$$P_t^* := \operatorname*{ess\,inf}_{\theta_t \in L^0(\mathbf{R}^d, \mathcal{F}_t)} \mathcal{P}_t(h_{t+1}). \tag{3.2.2}$$

Note that the minimal risk-hedging price P_t^* of h_{t+1} is not necessarily a price, i.e. it is not necessarily an element of $\mathcal{P}_t(h_{t+1})$ if this set is not closed. One contribution of our paper is to study a no-arbitrage condition under which $P_t^* \in \mathcal{P}_t(h_{t+1})$.

Starting from the contingent claim h_T at time T, we recursively define

$$P_T^* := h_T \ , \ P_t^* := \operatorname*{ess\,inf}_{\theta_t \in L^0(\mathbf{R}^d, \mathcal{F}_t)} \mathcal{P}_t(P_{t+1}^*),$$

where P_{t+1}^* may be interpreted as a contingent claim h_{t+1}. The interesting question is whether P_t^* is actually a price, i.e. an element of $\mathcal{P}_t(P_{t+1}^*)$, or equivalently whether $\mathcal{P}_t(P_{t+1}^*)$ is closed. In the classical setting, recall that closedness is obtained under the NA condition.

Definition 3.2.4. *A stochastic process $(V_t)_{t \leq T}$ adapted to $(\mathcal{F}_t)_{t \leq T}$, starting from an initial endowment V_0 is a portfolio process if, for all $t \leq T-1$, there exists $\theta_t \in L^0(\mathbf{R}^d, \mathcal{F}_t)$ such that $V_t + \theta_t \Delta S_{t+1} - V_{t+1}$ is acceptable at time t. Moreover, we say that it super-hedges the payoff $h_T \in L^0([-\infty, \infty], \mathcal{F}_T)$ if $V_T \geq h_T$ a.s..*

Note that $V_{T-1} + \theta_{T-1} \Delta S_T - V_T$ is supposed to be acceptable at time $T-1$. Therefore, $V_T \geq h_T$ implies that $V_{T-1} + \theta_{T-1} \Delta S_T - h_T$ is acceptable at time $T-1$. In the following, we actually set $V_T = h_T$ where $h_T \in L^0(\mathbf{R}, \mathcal{F}_T)$ is a European claim. Notice that, if $P_{T-1}^* = -\infty$ on some non null set, then, the one step pricing procedure of [67] may be applied as the risk-measure is defined on $L^0([-\infty, \infty], \mathcal{F}_T)$. Actually, this is trivial to super hedges $P_{T-1}^* = -\infty$ by $P_{T-2}^* = -\infty$. This means that the backward procedure of [67] may be applied without any no-arbitrage condition. Let us now recall this procedure.

We define $P_T^* = h_T =: h$ and let us consider the set $\mathcal{P}_t(P_{t+1}^*)$ of all prices p_t at time t allowing one to start a portfolio strategy $\theta_t \in L^0(\mathbf{R}^d, \mathcal{F}_t)$ such that $p_t + \theta_t \Delta S_{t+1} = P_{t+1}^* + a_{t,t+1}$ where $a_{t,t+1} \in L^0(\mathbf{R}, \mathcal{F}_t)$ is an acceptable position at time t. This is a generalization of the classical super-hedging inequality $p_t + \theta_t \Delta S_{t+1} \geq P_{t+1}^*$. We have

$$\mathcal{P}_t(P_{t+1}^*) = \{\theta_t S_t + \rho_t(\theta_t S_{t+1} - P_{t+1}^*) : \theta_t \in L^0(\mathbf{R}^d, \mathcal{F}_t)\} + L^0(\mathbf{R}_+^d, \mathcal{F}_t),$$

and, recursively, we define:

$$P_t^* = \operatorname*{ess\,inf}_{\theta_t \in L^0(\mathbf{R}^d, \mathcal{F}_t)} \mathcal{P}_t(P_{t+1}^*).$$

In [67], a jointly measurable version of the random function g_t that appears above in the characterization of $\mathcal{P}_t(P_{t+1}^*)$, i.e.

$$g_t^h(\omega, x) := xS_t + \rho_t(xS_{t+1} - P_{t+1}^*), \tag{3.2.3}$$

is constructed in the one-dimensional case. With the same arguments, we may obtain a jointly measurable version of $g_t^h(\omega, x) := xS_t + \rho_t(xS_{t+1} - P_{t+1}^*)$ if $x \in \mathbf{R}^d$. Moreover, by similar arguments, we also show that $P_t^* = \inf_{x \in \mathbf{R}^d} g_t^h(x)$.

Let V be a portfolio process with $V_T = h_T = h$. By definition, we have that $\rho_{T-1}(V_{T-1} + \theta_{T-1}\Delta S_T - h_T) \leq 0$. We deduce that $V_{T-1} \geq P_{T-1}^*$ and, by induction, we get that $V_t \geq P_t^*$ for all $t \leq T$, since V_t is a risk-hedging price for $V_{t+1} \geq P_{t+1}^*$ at time $t+1$. In particular, $V_t \in \mathcal{P}_t(P_{t+1}^*) \neq \emptyset$ for all $t \in T - 1$.

3.3 No-arbitrage and pricing with risk-measures

An instantaneous profit is the possibility to super-replicate the zero contingent claim at a negative price, see [5].

Definition 3.3.1. *Absence of Instantaneous Profit (AIP) holds if, for any* $t \leq T$,

$$\mathcal{P}_t(0) \cap L^0(\mathbf{R}_-, \mathcal{F}_t) = \{0\}. \tag{3.3.4}$$

It is clear that AIP holds at time T since $\mathcal{P}_T(0) = L^0(\mathbf{R}_+, \mathcal{F}_T)$. We now formulate characterizations of the AIP condition in the multi-dimensional setting. We denote by $S(0,1)$ the set of all $z \in \mathbf{R}^d$ such that $|z| = 1$. We present our first result:

Theorem 3.3.2. *The following statements are equivalent:*

1. *AIP holds between time $t-1$ and t.*
2. *$\rho_{t-1}(x\Delta S_t) \geq 0$, for any $x \in \mathbf{R}^d$,* a.s..
3. *$\rho_{t-1}(z\Delta S_t) \geq 0$, for any $z \in S(0,1)$,* a.s..
4. *Let $x_{t-1} \in L^0(\mathbf{R}^d, \mathcal{F}_{t-1})$. If $x_{t-1}\Delta S_t$ is acceptable on some non null set $F_{t-1} \in \mathcal{F}_{t-1}$, then $\rho_{t-1}(x_{t-1}\Delta S_t) = 0$ on F_{t-1}.*

Proof. 1 $\iff$ 2. Consider $h_t = 0$ under AIP. As $P^*_{t-1} = \inf_{x\in\mathbf{R}^d} g^0_{t-1}(x) \geq 0$, we deduce that, for all $x \in \mathbf{R}^d$, $g^0_{t-1}(x) = xS_{t-1} + \rho_{t-1}(xS_t) = \rho_{t-1}(x\Delta S_t) \geq 0$. The equivalence 2 $\iff$ 3 is clear by homogeneity. Let us show that 2 $\implies$ 4. Suppose that $x_{t-1}\Delta S_t$ is acceptable on F_{t-1}, i.e. $\rho_{t-1}(x_{t-1}\Delta S_t) \leq 0$ on F_{t-1}. Then, by 2, we have $\rho_{t-1}(x_{t-1}\Delta S_t) = 0$ on F_{t-1}. Let us show that 4 implies 2. Consider the set $F_{t-1} = \{\rho_{t-1}(x_{t-1}\Delta S_t) < 0\} \in \mathcal{F}_{t-1}$. Then, $x_{t-1}\Delta S_t$ is acceptable on F_{t-1} hence by 4, $\rho_{t-1}(x_{t-1}\Delta S_t) = 0$ on F_{t-1}, which implies that $P(F_{t-1}) = 0$. Therefore, $\rho_{t-1}(x_{t-1}\Delta S_t) \geq 0$ a.s.. □

In the following, we consider a contingent claim $h_t \in L^0(\mathbf{R}, \mathcal{F}_t)$ and a jointly measurable version (see [67]) of the random function

$$g_{t-1}(\omega, x) := xS_{t-1}(\omega) + \rho_{t-1}(xS_t - h_t)(\omega) \tag{3.3.5}$$

which is associated to h_t. We then introduce two types of no-arbitrage conditions we comment below.

Definition 3.3.3. *We say that the Symmetric Risk Neutral condition SRN holds at time t if, almost surely, for any $z_t \in L^0(S(0,1), \mathcal{F}_t)$, $\rho_t(z_t\Delta S_{t+1}) = 0$ if and only if $\rho_t(-z_t\Delta S_{t+1}) = 0$. We say that SRN holds if it holds at any time.*

Observe that the SRN condition means that a zero cost position z_t is risk-neutral if and only if $-z_t$ is risk neutral.

Definition 3.3.4. *We say that the no-arbitrage NA condition holds at time t when both conditions AIP and SRN hold at time t. We say that NA holds if it holds at any time.*

Note that the NA condition depends on the risk-measure. In the usual case where $\rho_t(X) = -\operatorname{ess\,inf}_{\mathcal{F}_t} X$ or, equivalently, there is no risk measure in the sense that the acceptable positions are the non-negative random variables, then the NA condition above coincides with the usual one as claimed in the following new result, see the proof in Appendix:

Proposition 3.3.5. *Suppose that the risk-measure is $\rho_t(X) = -\operatorname{ess\,inf}_{\mathcal{F}_t} X$. Then, the NA condition coincides with the classical NA condition of frictionless models, i.e. it is equivalent to the existence of a risk-neutral probability measure.*

We recall that a function $f : \Omega \times \mathbf{R}^d \to \overline{\mathbf{R}}$ is an $\mathcal{F}_t$-normal integrand, if its epigraph is $\mathcal{F}_t$-measurable and closed. Since the probability space is complete, we know by [76, Corollary 14.34] that it is equivalent to suppose that $f(\omega, x)$ is $\mathcal{F}_t \otimes \mathcal{B}(\mathbf{R}^d)$-measurable and lower semi-continuous (l.s.c.) in x. Moreover, by [76, Theorem 14.37], we have:

Proposition 3.3.6. *If f is an $\mathcal{F}_t$-normal integrand, $\inf_{y\in\mathbf{R}^d} f(\omega, y)$ is $\mathcal{F}_t$-measurable and $\{(\omega, x) \in \Omega \times \mathbf{R}^d : f(\omega, x) = \inf_{y\in\mathbf{R}^d} f(\omega, y)\} \in \mathcal{F}_t \otimes \mathcal{B}(\mathbf{R}^d)$ is a measurable closed set.*

As we may choose a jointly measurable version of $g_t(\omega, x)$ when the payoff is $h_{t+1} = 0$, we consider a jointly measurable version of $\rho_t(\omega, x) := \rho_t(x\Delta S_{t+1})$ i.e. $\rho_t(\omega, x)$ is $\mathcal{F}_t \otimes \mathcal{B}(\mathbf{R}^d)$-measurable. Then, ρ_t is an $\mathcal{F}_t$-normal integrand. By Proposition 3.3.6, the set $\Gamma_t = \{z : \ \rho_t(z\Delta S_{t+1}) = \inf_{y\in S(0,1)} \rho_t(y\Delta S_{t+1})\}$ is $\mathcal{F}_t$-measurable. Moreover, each ω-section of Γ_t is non empty since ρ_t is l.s.c. and $S(0,1)$ is compact. Therefore, by a measurable selection argument, we may select $z_t \in L^0(S(0,1), \mathcal{F}_t)$ such that $\rho(z_t\Delta S_{t+1}) = \inf_{z\in S(0,1)} \rho_t(z\Delta S_{t+1})$ a.s..

Our first contribution is to show that, under NA, infimum super-hedging prices are minimal prices. To do so, we need the following new results which are proved in Appendix.

Theorem 3.3.7. *Suppose that AIP holds and consider $z_{t-1} \in L^0(S(0,1), \mathcal{F}_{t-1})$. Then, on the set $F_{t-1} = \{\rho_{t-1}(z_{t-1}\Delta S_t) = 0\} \cap \{\rho_{t-1}(-z_{t-1}\Delta S_t) = 0\}$, the random mapping $x \mapsto g_{t-1}(\omega, x)$ given by (3.3.5) is a.s. constant on the line $\mathbf{R}z_{t-1}$, i.e. $g_{t-1}(\omega, x_1) = g_{t-1}(\omega, x_2)$ for all $x_1, x_2 \in \mathbf{R}z_{t-1}(\omega)$ and $\omega \in F_{t-1}$.*

Theorem 3.3.8. *Let $h_t \in L^0(\mathbf{R}, \mathcal{F}_t)$ be a payoff such that $\rho_{t-1}(h_t) < \infty$ a.s..Consider the random function g_{t-1} associated to h_t given by (3.3.5). For any $z_{t-1} \in L^0(S(0,1), \mathcal{F}_{t-1})$, consider the random set*

$$F_{t-1} = \{\rho_{t-1}(z_{t-1}\Delta S_t) > 0\} \cap \{\rho_{t-1}(-z_{t-1}\Delta S_t) > 0\}.$$

We have:

$$\lim_{|r|\to\infty} g_{t-1}(\omega, rz_{t-1}) = +\infty, \quad \forall \omega \in F_{t-1}.$$

hence g_{t-1} admits a minimum on the line $\mathbf{R}z_{t-1}$ when $\omega \in F_{t-1}$.

Corollary 3.3.9. *Let $h_t \in L^0(\mathbf{R}, \mathcal{F}_t)$ be s.t. $\rho_{t-1}(h_t) < \infty$ and $\rho_{t-1}(-h_t) < \infty$ a.s.. Consider the function g_{t-1} associated to h_t given by (3.3.5). Suppose that $z_{t-1} \in L^0(S(0,1), \mathcal{F}_{t-1})$ is such that*

$$\rho_{t-1}(z_{t-1}\Delta S_t) = \inf_{z \in S(0,1)} \rho_t(z\Delta S_t).$$

Then, on the set $F_{t-1} = \{\rho_{t-1}(z_{t-1}\Delta S_t) > 0\} \cap \{\rho_{t-1}(-z_{t-1}\Delta S_t) > 0\}$, the random function g_{t-1} admits a minimum.

The following theorem is our first main contribution and shows that the set of all risk-hedging prices is closed under NA:

Theorem 3.3.10. *Suppose that NA holds at time $t \leq T$ and consider a payoff $h_{t+1} \in L^0(\mathbf{R}, \mathcal{F}_{t+1})$ such that $|\rho_t(h_{t+1})| + |\rho_t(-h_{t+1})| < \infty$ a.s.. Then, the minimal risk-hedging price P_t^* for the payoff h_{t+1} is a price.*

Notice that the proof of the theorem above (see Appendix) provides the existence of an optimal hedging strategy $\theta_t^* \in L^0(\mathbf{R}, \mathcal{F}_t)$ such that

$$P_t^* = g_t(\theta_t^*) = \theta_t^* S_t + \rho_t(\theta_t^* S_{t+1} - h_{t+1}) \in \mathcal{P}_t(h_{t+1}).$$

In the following, we say that a payoff h_{t+1} is not freely attainable at time t if it satisfies $\rho_t(-h_{t+1}) > 0$ a.s. and $|\rho_t(h_{t+1})| + |\rho_t(-h_{t+1})| < \infty$ a.s.. Note that if $\rho_t(-h_{t+1}) > 0$, then it is not possible to get the payoff h_{t+1} from nothing when writing $0 = h_{t+1} + (-h_{t+1})$ and letting aside $(-h_{t+1})$ since the latter is not acceptable. Notice that, if $\rho_t(X) = -\operatorname{ess\,inf}_{\mathcal{F}_t}(X)$ as in the usual case, $\rho_t(-h_{t+1}) > 0$ means that $\operatorname{ess\,sup}_{\mathcal{F}_t}(h_{t+1}) > 0$ and recall that h_{t+1} is acceptable if $h_{t+1} \geq 0$ a.s.. The following theorem gives an interpretation

of the NA condition. Precisely, NA means that the price of any no freely attainable and acceptable payoff is strictly positive. In the usual case, a no freely attainable and acceptable payoff is a non negative payoff which does not vanish on a non null $\mathcal{F}_t$-measurable set.

We then have a new financial interpretation of the NA condition, as proved in Appendix:

Theorem 3.3.11. *The NA condition holds at time $t \leq T$ if and only if the infinimum risk-hedging price P_t^* of any no freely attainable and acceptable payoff h_{t+1} at time t is strictly positive. Moreover, under NA, the infimum risk-hedging price P_t^* of any contingent claim $h_{t+1} \in L^0(\mathbf{R}^d, \mathcal{F}_{t+1})$ satisfies*

$$\rho_t(-h_{t+1}) \geq P_t^* \geq -\rho_t(h_{t+1}).$$

3.4 FTAP and dual representation for time-consistent risk measures.

Definition 3.4.1. *A dynamic risk-measure $(\rho_t)_{t\leq T}$ is said time-consistent if $\rho_{t+1}(X) = \rho_{t+1}(Y)$ implies $\rho_t(X) = \rho_t(Y)$ for $X, Y \in L^0(\mathbf{R}, \mathcal{F}_T)$ and $t \leq T-1$ (see Section 5 in [29]).*

The following result is very well known, see [2].

Lemma 3.4.2. *A dynamic risk-measure $(\rho_t)_{t\leq T}$ is time-consistent if and only if its family of acceptable sets $(\mathcal{A}_t)_{t\leq T}$ satisfies*

$$\mathcal{A}_{t,T} = \mathcal{A}_{t,t+1} + \mathcal{A}_{t+1,T}, \ \forall t \leq T-1. \tag{3.4.6}$$

Observe that, if $(\rho_t)_{t\leq T}$ is time-consistent, we may show by induction that $\rho_t(-\rho_{t+s}(\cdot)) = \rho_t(\cdot)$ for any $t \leq T$ and $s \geq 0$ such that $s + t \leq T$. In the following, we introduce another possible definition for the risk-hedging prices in the multi-period model, where the risk is only measured at time t.

Definition 3.4.3. *The contingent claim $h_T \in L^0(\mathbf{R}, \mathcal{F}_T)$ is said directly risk-hedged at time $t \leq T-1$ if there exists a (direct) price $P_t \in L^0(\mathbf{R}, \mathcal{F}_t)$ and a strategy $(\theta_u)_{u=t}^{T-1}$ such that that $P_t + \sum_{t\leq u\leq T-1} \theta_u \Delta S_{u+1} - h_T$ is acceptable at time t.*

The set of all direct risk-hedging prices at time t is then given by

$$\bar{\mathcal{P}}_t(h_T) \;=\; \left\{ \rho_t \left(\sum_{t \le u \le T-1} \theta_u \Delta S_{u+1} - h_T \right) : \; \theta_u \in L^0(\mathbf{R}^d, \mathcal{F}_u) \right\} + L^0(\mathbf{R}, \mathcal{F}_t).$$

and the infimum direct risk-hedging price is

$$\bar{P}_t^*(h_T) := \operatorname*{ess\,inf}_{(\theta_u)_{u=t}^{T-1}} \bar{\mathcal{P}}_t(h_T).$$

Remark 3.4.4. *A hedging strategy which is admissible at each step is a strategy that is considered as admissible because of the updated information and the updated risk-measure as well. Indeed, at each step, the acceptable positions are estimated through the time dependent risk-measure ρ_t and the information $\mathcal{F}_t$. On the other hand, a direct-hedge is only obtained at time 0 from the initial preferences we have on the acceptable positions, i.e. from ρ_0 and without information but $\mathcal{F}_0$. It is intuitively natural to prefer a strategy which is admissible at each step as this is coherent with the choice of a dynamic risk measure to take into account a change in time of preferences and information.*

The following result is proved in [67] and shows that the direct infimum risk-hedging prices may coincide with the infimum prices derived from the step by step backward procedure developed before, i.e. such that

$$P_t^*(h_T) = \operatorname*{ess\,inf}_{\theta_t \in L^0(\mathbf{R}, \mathcal{F}_t)} \mathcal{P}_t(P_{t+1}^*(h_T)),$$

where $P_T^*(h_T) = h_T$.

Theorem 3.4.5. *Suppose that the dynamic risk-measure $(\rho_t)_{t \le T}$ is time-consistent. Then, $\bar{P}_t^*(h_T) = P_t^*(h_T)$ for any $t \le T-1$. Moreover, the direct infimum risk-hedging prices are direct prices if and only if the infimum prices of the backward procedure are prices.*

Corollary 3.4.6. *Suppose that the dynamic risk-measure $(\rho_t)_{t \le T}$ is time-consistent. Then, $\bar{\mathcal{P}}_t(h_T) = \mathcal{P}_t(h_T)$ for all $t \le T$.*

3.4.1 Dual representation

As mentioned by Cherny [15, Theorem 2.2] and shown in [26], any time-consistent risk-measure ρ_t at time t, restricted to the set of all bounded

random variables, is characterized by a family $\mathcal{D}_t$ of absolutely continuous probability measures such that $\rho_t(X) = \text{ess sup}_{Q\in\mathcal{D}_t} E_Q(-X|\mathcal{F}_t)$. In the following, we consider the risk-measure ρ on L^0 as defined in this paper. The goal is to understand whether it is possible to get a dual characterization of ρ on the whole set L^0, at least under some conditions. For $X \in L^0$, we define $E_Q(-X|\mathcal{F}_t)$ as $E_Q(-X|\mathcal{F}_t) = E_Q(X^-|\mathcal{F}_t) - E_Q(X^+|\mathcal{F}_t)$ with the convention $\infty - \infty = \infty$. We say that a random variable X is $\mathcal{F}_t$-bounded from above if $X \leq c_t$ a.s. for some $c_t \in L^0(\mathbf{R}_+, \mathcal{F}_t)$. The proofs of the following new contributions are postponed in Appendix. They provide a dual representation of the risk-measure.

Proposition 3.4.7. *Let $(\rho_t)_{t=0,\cdots,T}$ be the coherent risk-measure as defined in Section 3.2. Then, there exists a family $\mathcal{D}_t$ of absolutely continuous probability measures such that, for every $\mathcal{F}_t$-bounded from above random variable X, we have:*

$$\rho_t(X) = \text{ess sup}_{Q\in\mathcal{D}_t} E_Q(-X|\mathcal{F}_t). \tag{3.4.7}$$

Unfortunately, it is unrealistic to expect that (3.4.7) may be extended in general from L^∞ to L^0, as mentioned by Cherny, [15]. The main problem is about the non negatives random variables as we shall see in the proof of the next proposition. Before, let us see a trivial example where we may meet some difficulties for non negative random variables.

Example 3.4.8. *We consider $\Omega = [0, 1]$ equipped with the Borel σ-algebra and the Lebesgue measure P. The random variable $X(\omega) = \omega^{-1}1_{(0,1]}(\omega)$ is non negative hence acceptable. Let us define the acceptable positions as the closure in L^0 of the random variables Z such that $E_P(Z) = E_P(Z^+) - E_P(Z^-) \geq 0$. We then define ρ on L^0 as in Section 2, see [67]. As $E_P(X) = \infty$, we deduce that $Z_\alpha := X - \alpha$ is acceptable for all $\alpha > 0$ if (3.4.7) holds. On the other hand, $P(Z_\alpha < 0) = 1 - \alpha^{-1}$ tends to 1 as $\alpha \to \infty$, which is unrealistic if Z_α is acceptable.*

Consider $Q \in \mathcal{D}_0$ and $Y = dQ/dP$. Suppose that $P(Y > 1) > 0$. We then choose $\alpha < 0$ and $\beta > 0$ such that $\alpha P(Y > 1) + \beta P(Y \leq 1) = 0$. Then, $X = \alpha 1_{\{Y>1\}} + \beta 1_{\{Y\leq 1\}}$ is acceptable as $E_P(X) = 0$. Therefore, by (3.4.7), $E_Q(X) \geq 0$. Actually,

$$E_Q(X) = E_P(XY) = E_P(\alpha Y 1_{\{Y>1\}} + \beta Y 1_{\{Y\leq 1\}}) \leq E_P(X) = 0$$

and $E_Q(X) = 0$ *if and only if* $\alpha Y 1_{\{Y>1\}} + \beta Y 1_{\{Y\leq 1\}} = X$. *In that case* $Y = 1$ *on* $\{Y > 1\}$ *hence a contradiction. We deduce that* $Y \leq 1$ *a.s.. At last, since* $Y \leq 1$ *a.s., we deduce that* $Y = 1$ *a.s.. We then deduce that* $\mathcal{D}_0 = \{P\}$.

Also, as another example, consider $X(\omega) = \omega^{-1}1_{(0,\alpha)}(\omega) + (\omega - 1)^{-1}1_{(\alpha,1)}(\omega)$, $\omega \in \Omega$. *Since* $E_P(X) = \infty - \infty = \infty$, *we deduce that* X *is acceptable. Nevertheless,* $P(X < 0) = 1 - \alpha$ *tends to* 1 *as* $\alpha \to 0$, *which is clearly unrealistic.*

In the following, we denote by $\mathcal{A}_t^{\infty,+}$ the set of all acceptable positions at time t which are $\mathcal{F}_t$-bounded from above.

Proposition 3.4.9. *Suppose that* $\mathcal{A}_t$ *is the closure of* $\mathcal{A}_t^{\infty,+} + L^0(\mathbf{R}^+, \mathcal{F}_T)$ *in* L^0 *and assume that, for some fixed* $\varepsilon > 0$, $\mathcal{A}_t^{\infty,+}$ *contains all the random variables* Z *which are* $\mathcal{F}_t$*-bounded from above and satisfy* $P(Z < 0) \leq \varepsilon$. *Let* $(\rho_t)_{t=0,\cdots,T}$ *be the coherent risk-measure as defined in Section 3.2. Then, there exists a family* $\mathcal{D}_t$ *of absolutely continuous probability measures such that we have.*

$$\rho_t(X) = \text{ess sup}_{Q\in\mathcal{D}_t} E_Q(-X|\mathcal{F}_t), \quad \forall X \in L^0. \tag{3.4.8}$$

The proof of the proposition above (see Appendix) shows that (3.4.8) holds as soon as it holds for any acceptable position which is the sum of an $\mathcal{F}_t$-bounded position plus a non negative one. By Proposition 3.4.7, (3.4.8) holds for any $\mathcal{F}_t$-bounded position. Therefore, the difficulty in proving (3.4.8) stems from the non negative random variables.

3.4.2 FTAP and dual description of the risk-hedging prices

We consider the set of all attainable claims $\mathcal{R}_t^T$ between t and T, when starting from the zero initial endowment, i.e.

$$\mathcal{R}_{t,T} := \left\{ \sum_{u=t+1}^{T} \theta_{u-1}\Delta S_u : \ \theta_u \in L^0(\mathbf{R}^d, \mathcal{F}_u),\ u \geq t \right\}.$$

We observe that $\bar{\mathcal{P}}_t(0) = (\mathcal{A}_{t,T} - \mathcal{R}_{t,T}) \cap L^0(\mathbf{R}, \mathcal{F}_t)$. In the following, we consider the sets $\mathcal{Z}_{t,T} := \mathcal{R}_{t,T} - \mathcal{A}_{t,T}$ and the sets

$$\mathcal{A}_{t,T}^0 = \{X \in L^0(\mathbf{R}, \mathcal{F}_T) : \ \rho_t(X) = \rho_t(-X) = 0\}.$$

Remark 3.4.10. *Note that $\mathcal{A}^0_{t,T} = \mathcal{A}_{t,T} \cap (-\mathcal{A}_{t,T})$. Indeed, first observe that $\mathcal{A}^0_{t,T} \subseteq \mathcal{A}_{t,T} \cap (-\mathcal{A}_{t,T})$. Reciprocally, if $x_{t,T} \in \mathcal{A}_{t,T} \cap (-\mathcal{A}_{t,T})$, we have:*

$$0 = \rho_t(x_{t,T} - x_{t,T}) \leq \rho_t(x_{t,T}) + \rho_t(-x_{t,T}) \leq 0.$$

This implies $\rho_t(x_{t,T}) = \rho_t(-x_{t,T}) = 0$ hence $x_{t,T} \in \mathcal{A}^0_{t,T}$.

The set $\mathcal{Z}_{t,T}$ is the family of all claims that are attainable up to an acceptable position at time t since every attainable claim $r_{t,T} \in \mathcal{R}^T_t$ may be written as $r_{t,T} = (r_{t,T} - a_{t,T}) + a_{t,T}$ where $a_{t,T} \in \mathcal{A}_{t,T}$ is let aside and $r_{t,T} - a_{t,T} \in \mathcal{Z}_{t,T}$.

We now formulate intermediate new results that we need to prove the FTAP theorem, which is the first contribution of this section.

Theorem 3.4.11. *Assume that the risk measure is time-consistent. Suppose that $\mathcal{R}_{t,T} \cap \mathcal{A}_{t,T} = \mathcal{A}^0_{t,T}$. Then, AIP holds and $\mathcal{Z}_{t,T}$ is closed in L^0 for every $t \leq T - 1$.*

Theorem 3.4.12. *Suppose that the risk-measure is time-consistent. Suppose that NA holds and $\mathcal{A}_{t,T} \cap L^0(\mathbf{R}_-, \mathcal{F}_T) = \{0\}$, for every $t \leq T$. Then, we have $\mathcal{Z}_{t,T} \cap L^0(\mathbf{R}_+, \mathcal{F}_T) = \{0\}$ and $\mathcal{R}_{t,T} \cap \mathcal{A}_{t,T} = \mathcal{A}^0_{t,T}$ for every t.*

Theorem 3.4.13 (FTAP)**.** *Suppose that the risk-measure is time-consistent and $\mathcal{A}_{t,T} \cap L^0(\mathbf{R}_-, \mathcal{F}_T) = \{0\}$ for every $t \leq T$. Then, the following statements are equivalent:*

1) NA

2) $\mathcal{R}_{t,T} \cap \mathcal{A}_{t,T} = \mathcal{A}^0_{t,T}$, for every $t \leq T$.

3) $\mathcal{R}_{0,T} \cap \mathcal{A}_{0,T} = \mathcal{A}^0_{0,T}$.

4) $\overline{\mathcal{Z}_{t,T}} \cap \mathcal{A}_{t,T} = \mathcal{A}^0_{t,T}$, for every $t \leq T$.

5) $\overline{\mathcal{Z}_{0,T}} \cap \mathcal{A}_{0,T} = \mathcal{A}^0_{0,T}$.

6) $\mathcal{Z}_{0,T} \cap \mathcal{A}_{0,T} = \mathcal{A}^0_{0,T}$ and $\mathcal{Z}_{0,T}$ is closed in L^0.

7) For all $t \leq T-1$, there exists $Q = Q^t \sim P$ with $dQ/dP \in L^\infty((0,\infty), \mathcal{F}_T)$ such that $(S_u)_{u=t}^T$ is a Q-martingale and, for all $t \leq T - 1$, for all X such that $E_Q(X^- | \mathcal{F}_t) < \infty$ a.s., $\rho_t(X) \geq -E_Q(X|\mathcal{F}_t)$.

Moreover, for all $x \in \mathcal{A}_{t,T} \setminus \mathcal{A}^0_{t,T}$, there exists such a $Q = Q^t_x$ such that $\mathrm{P}(E_Q(x|\mathcal{F}_t) \neq 0) > 0$.

The following result is the second main contribution of this section. It provides a dual description of the payoffs that can be super-hedged under NA. To do so, we denote by $\mathcal{Q}_t^e$ (and $\mathcal{Q}^e = \mathcal{Q}_0^e$) the set of equivalent martingale measures Q that satisfies $\rho_t(X) \geq -E_Q(X|\mathcal{F}_t)$, for all X such that $E_Q(X^-|\mathcal{F}_t) < \infty$ a.s.. We have $\mathcal{Q}_t^e \neq \emptyset$ under NA. We restrict the payoffs to the class $L_S(\mathbf{R}, \mathcal{F}_T)$ of random variables $h_T \in L^0(\mathbf{R}, \mathcal{F}_T)$ satisfying:

$$|h_T| \leq c^0 + \sum_{i=1}^{d} c^i S_T^i, \quad P - a.s.$$

for some constants $c^0, ..., c^d$ that may depend on h_T.

Theorem 3.4.14. *Suppose that the risk-measure is time-consistent and we have $\mathcal{A}_{t,T} \cap L^0(\mathbf{R}_-, \mathcal{F}_T) = \{0\}$ for every $t \leq T$. Consider the following sets:*

$$\Gamma_{0,T} := \mathcal{Z}_{0,T} \cap L_S(\mathbf{R}, \mathcal{F}_T),$$
$$\Theta_{0,T} := \left\{ h_T \in L_S(\mathbf{R}, \mathcal{F}_T), \sup_{Q \in \mathcal{Q}^e} E_Q(h_T) \leq 0 \right\}.$$

Then, under the NA condition, $\Gamma_{0,T} = \Theta_{0,T}$ and the minimal risk-hedging price $P_0^*(h_T)$ of any contingent claim $h_T \in L_S(\mathbf{R}, \mathcal{F}_T)$ is given by

$$P_0^*(h_T) = \sup_{Q \in \mathcal{Q}^e} E_Q(h_T).$$

3.4.3 Comparison with the No Good Deal condition

We recall that the No Good Deal condition (NGD) of Cherny [15] may be rephrased in our setting as follows:

Definition 3.4.15. *The NGD condition holds at any time $t \leq T$ if there is no $X_{t,T} \in \mathcal{R}_{t,T}$ such that $\rho_t(X_{t,T}) < 0$ on a non null set.*

In the setting of Cherny, we suppose that

$$\rho_t(X) = \text{ess sup}_{Q^t \in \mathcal{D}_t} E_{Q^t}(-X), \tag{3.4.9}$$

where $\mathcal{D}_t$ is a weakly compact subset of L^1 with respect to the $\sigma(L^1, L^\infty)$ topology and we use the definition $E_{Q^t}(-X) = E_{Q^t}(X^-) - E_{Q^t}(X^+)$ with the convention $\infty - \infty = +\infty$. Adapting [15, Theorem 3.4], we immediately get the following:

Corollary 3.4.16. *Suppose that the risk-measure is given by (3.4.9). Then, the NA condition and the NGD condition are equivalent to the existence of a probability measure $Q^t \in \mathcal{D}_t$ such that the price process $(S_u)_{u=t}^T$ is a Q^t-martingale for all $t \leq T-1$.*

3.5 Appendix: Proofs.

Proof of Theorem 3.3.5.

Proof. We know that the existence of a risk-neutral probability measure $Q \sim P$ implies AIP. Moreover, suppose that $\rho_t(z\Delta S_{t+1}) = 0$ on $F_t \in \mathcal{F}_t$ where $z \in S(0,1)$. Then, by definition of ρ_t, $1_{F_t} z\Delta S_{t+1} \geq 0$. As $E_Q(1_{F_t} z\Delta S_{t+1}) = 0$, we deduce that $1_{F_t} z\Delta S_{t+1} = 0$ hence $\rho_t(-z\Delta S_{t+1}) = 0$ on F_t. By symmetry, we deduce that SRN holds.

Reciprocally, suppose that AIP and SRN conditions hold. Let $\theta_t \in L^0(\mathbf{R}^d, \mathcal{F}_t)$ such that $\theta_t \Delta S_{t+1} \geq 0$ a.s.. Let us write $\theta_t = r_t z_t$ where $r_t \in L^0(\mathbf{R}, \mathcal{F}_t)$ and $z_t \in L^0(S(0,1), \mathcal{F}_t)$. On the set $F_t = \{r_t > 0\}$, $z_t \Delta S_{t+1} \geq 0$ hence $\operatorname{ess\,inf}_{\mathcal{F}_t}(z_t \Delta S_{t+1}) \geq 0$. By the AIP condition, $\rho_t(z_t \Delta S_{t+1}) \geq 0$. We deduce that $\operatorname{ess\,inf}_{\mathcal{F}_t}(z_t \Delta S_{t+1}) = 0 = \rho_t(z\Delta S_{t+1})$. Under SRN, we deduce that $\rho_t(-z\Delta S_{t+1}) = 0$ hence $z\Delta S_{t+1} \geq 0$ so that $z_t \Delta S_{t+1} = 0$. By a similar reasoning on the set $F_t = \{r_t < 0\}$, we also get that $z_t \Delta S_{t+1} = 0$ hence $\theta_t \Delta S_{t+1} = 0$. We then conclude by [55, Condition (g), p. 73, Section 2.1.1]. □

Proof of Theorem 3.3.7.

Proof. If $\lambda_{t-1} \in L^0(\mathbf{R}, \mathcal{F}_t)$, $g_{t-1}(\lambda_{t-1} z_{t-1} \Delta S_t) = |\lambda_{t-1}| g_{t-1}(\epsilon_{t-1} z_{t-1} \Delta S_t)$ for some $\epsilon_{t-1} \in L^0(\{-1,1\}, \mathcal{F}_{t-1})$. We deduce that $g_{t-1}(\lambda_{t-1} z_{t-1} \Delta S_t) = 0$ on F_{t-1}. Recall that $g_{t-1}(\lambda_{t-1} z_{t-1}) = \rho_{t-1}(\lambda_{t-1} z_{t-1} \Delta S_t - h_t)$ by Cash invariance. Using the triangular inequality, we then deduce on F_{t-1} that

$$\begin{aligned} g_{t-1}(0) = \rho_{t-1}(-h_t) &\leq \rho_{t-1}(-\lambda_{t-1} z_{t-1} \Delta S_t) + \rho_{t-1}(\lambda_{t-1} z_{t-1} \Delta S_t - h_t) \\ &\leq g_{t-1}(\lambda_{t-1} z_{t-1}). \end{aligned}$$

Similarly, we have

$$g_{t-1}(\lambda_{t-1} z_{t-1}) \leq \rho_{t-1}(\lambda_{t-1} z_{t-1} \Delta S_t) + \rho_{t-1}(-h_t) = \rho_{t-1}(-h_t).$$

We deduce that $g_{t-1}(\lambda_{t-1} z_{t-1}) = g_{t-1}(0)$ and this implies that g_{t-1} is a constant on the line $\mathbf{R} z_{t-1}$. Indeed, on the contrary case, the $\mathcal{F}_{t-1}$-measurable

set $\Gamma_{t-1}(\omega) = \{\alpha \in \mathbf{R} : \ g_{t-1}(\alpha z_{t-1}) \neq g_{t-1}(z_{t-1})\}$ is non empty on the non null set $G_{t-1} = \{\omega \in \Omega : \ \Gamma_{t-1}(\omega) \neq \emptyset\} \in \mathcal{F}_{t-1}$. We then deduce a measurable selection $\tilde{z}_t \in L^0(\mathbf{R}^d, \mathcal{F}_{t-1})$ such that $\tilde{z}_t = \alpha_t z_t$ and $\alpha_t \in \Gamma_{t-1}$ on the set G_{t-1} and we put $\tilde{z}_t = z_t$ on the complimentary set $\Omega \setminus G_{t-1}$. By the first part above, we deduce that $g_{t-1}(\tilde{z}_t) = g_{t-1}(z_t)$ a.s., which contradicts the fact that $\alpha_t \in \Gamma_{t-1}$ on F_{t-1}. □

Proof of Theorem 3.3.8.

Proof. If $\lambda_{t-1} \in L^0(\mathbf{R}, \mathcal{F}_t)$, $g_{t-1}(\lambda_{t-1} z_{t-1} \Delta S_t) = |\lambda_{t-1}| g_{t-1}(\epsilon_{t-1} z_{t-1} \Delta S_t)$, where $\epsilon_{t-1} \in L^0(\{-1, 1\}, \mathcal{F}_{t-1})$. Moreover, $g_{t-1}(\epsilon_{t-1} z_{t-1} \Delta S_t) > 0$ on F_{t-1}. By sub-additivity, we deduce that

$$|\lambda_{t-1}| g_{t-1}(\epsilon_{t-1} z_{t-1} \Delta S_t) \leq \rho_{t-1}(h_t) + g_{t-1}(\lambda_{t-1} z_{t-1}).$$

As $|\lambda_{t-1}|$ goes to $+\infty$, we conclude that $g_{t-1}(\lambda_{t-1} z_{t-1})$ tends to $+\infty$ on F_{t-1}.

Now, let us suppose that there is a non null set G_{t-1} of $\mathcal{F}_{t-1}$ such that $g_{t-1}(\omega, r z_{t-1})$ does not converge to $+\infty$ if $r \to \infty$ when $\omega \in G_{t-1}$. Note that $\omega \in G_{t-1}$ if and only if there exists $m(\omega) \in \mathbf{R}$ such that, for all $n \geq 1$, there exists $r_n(\omega) \geq n$ such that $g_{t-1}(\omega, r_n(\omega)) \leq m(\omega)$. Consider the following set

$$\Gamma_{t-1}(\omega) = \{(m, (r_n)_{n=1}^{\infty}) \in \mathbf{R} \times \mathbf{R}^{\mathbb{N}} : \ r_n \geq n \text{ and } g_t(\omega, r_n) \leq m, \ \forall n \geq 1\}.$$

The Borel σ-algebra $\mathcal{B}(\mathbf{R}^{\mathbb{N}})$ is defined as the smallest topology on $\mathbf{R}^{\mathbb{N}}$ such that the projection mappings $P^n : (r_j)_{j=1}^{\infty} \mapsto r_n$, $n \geq 1$, are continuous. Therefore, we deduce that Γ_{t-1} is $\mathcal{F}_{t-1}$-measurable. As Γ_{t-1} is non empty on G_{t-1}, we deduce a $\mathcal{F}_{t-1}$-measurable selection $(m, (r_n)_{n=1}^{\infty})$ of Γ_{t-1} on G_{t-1} that we extend to the whole space Ω by $m(\omega) = +\infty$ and $r_n(\omega) = n$, if $\omega \in \Omega \setminus G_{t-1}$. Since the $\mathcal{F}_{t-1}$-measurable sequence $(r_n)_{n=1}^{\infty}$ converges a.s. to $+\infty$, we deduce that $\lim_{n \to +\infty} g_{t-1}(r_n z_{t-1}) = +\infty$ on G_{t-1} by the first part of the proof. This is in contradiction with the property $g_t(\omega, r_n(\omega)) \leq m(\omega)$, for all $n \geq 1$, if $\omega \in G_{t-1}$.

Similarly, by symmetry, we may also prove that $\lim_{r \to -\infty} g_{t-1}(r z_{t-1}) = +\infty$ on F_t. As g_{t-1} is l.s.c., we finally deduce that g_{t-1} achieves a minimum on $\mathbf{R} z_{t-1}$. □

Proof of Corollary 3.3.9.

Proof. For any $z \in S(0, 1)$, we have $\rho_{t-1}(z \Delta S_t) > 0$ and $\rho_{t-1}(-z \Delta S_t) > 0$ by definition of F_{t-1} and z_{t-1}. By Theorem 3.3.8, there exists $r_{t-1} \in$

$L^0(\mathbf{R}_+, \mathcal{F}_{t-1})$ such that $\inf_{r\in\mathbf{R}} g_{t-1}(rz_{t-1}) = g_{t-1}(r_{t-1}z_{t-1})$. Notice that, by definition, we have $g_{t-1}(r_{t-1}z_{t-1}) \leq g_{t-1}(0) = \rho_{t-1}(-h_t)$. On the set $\{r_{t-1} > 0\}$, this is equivalent to

$$r_{t-1}\left(z_{t-1}S_{t-1} + \rho_{t-1}\left(z_{t-1}S_t - \frac{h_t}{r_{t-1}}\right)\right) \leq \rho_{t-1}(-h_t),$$
$$r_{t-1}\left(z_{t-1}S_{t-1} + \rho_{t-1}(z_{t-1}S_t) + \rho_{t-1}\left(z_{t-1}S_t - \frac{h_t}{r_{t-1}}\right) - \rho_{t-1}(z_{t-1}S_t)\right) \leq \rho_{t-1}(-h_t).$$

We observe that:

$$\rho_{t-1}\left(z_{t-1}S_t - \frac{h_t}{r_{t-1}}\right) - \rho_{t-1}(z_{t-1}S_t) \geq -\frac{1}{r_{t-1}}\rho_{t-1}(h_t).$$

Therefore, $r_{t-1}(z_{t-1}S_{t-1} + \rho_{t-1}(z_{t-1}S_t)) \leq \rho_{t-1}(-h_t) + \rho_{t-1}(h_t)$, i.e.

$$r_{t-1} \leq \frac{\rho_{t-1}(-h_t) + \rho_{t-1}(h_t)}{\rho_{t-1}(z_{t-1}\Delta S_t)}.$$

Similarly, on the set $\{r_{t-1} < 0\}$, we deduce that:

$$-r_{t-1} \leq \frac{\rho_{t-1}(-h_t) + \rho_{t-1}(h_t)}{\rho_{t-1}(-z_{t-1}\Delta S_t)}.$$

We finally deduce that, in any case, we have:

$$|r_{t-1}| \leq \max\left(\frac{|\rho_{t-1}(-h_t) + \rho_{t-1}(h_t)|}{\rho_{t-1}(z_{t-1}\Delta S_t)}, \frac{|\rho_{t-1}(-h_t) + \rho_{t-1}(h_t)|}{\rho_{t-1}(-z_{t-1}\Delta S_t)}\right) = M_{t-1} < \infty,$$

on F_{t-1}. At last, we deduce that for each $\omega \in F_{t-1}$:

$$\inf_{x\in\mathbf{R}^d} g_{t-1}(x) = \inf_{r\in[-M_{t-1},M_{t-1}]} \inf_{z\in S(0,1)} g_{t-1}(rz) = \inf_{x\in\bar{B}(0,M_{t-1})} g_{t-1}(x),$$

where $\bar{B}(0, M_{t-1})$ is the closed ball of radius M_{t-1} and centered at the origin. Since $\bar{B}(0, M_{t-1})$ is compact and g_{t-1} is l.s.c., we deduce that g_{t-1} admits a minimum on $\bar{B}(0, M_{t-1})$. By Proposition 3.3.6, observe that there exists a measurable version of an argmin, using a measurable selection argument. □

Proof of Theorem 3.3.10.

Proof. Suppose first that $d = 2$. Since ρ_t is l.s.c., there exists $z_t \in L^0(S(0,1), \mathcal{F}_t)$ such that $\inf_{z \in S(0,1)} \rho_t(z\Delta S_{t+1}) = \rho_t(z_t \Delta S_{t+1})$. By Corollary 3.3.9 and under SRN, g_t attains a minimum on $\mathbf{R}^2$ when $\omega \in F_t = \{\rho_t(z_t \Delta S_{t+1}) > 0\} \in \mathcal{F}_t$.

Let us now suppose that $\omega \in F_t^c = \{\rho_t(z_t \Delta S_{t+1}) = \rho_t(-z_t \Delta S_{t+1}) = 0\}$. We consider a line that is parallel to the line $\mathbf{R}z_t$. For any $z_1, z_2 \in L^0(\mathbf{R}^d, \mathcal{F}_t)$ on that line such that $z_1 - z_2 = r_t z_t \in \mathbf{R}z_t$, $r_t \in L^0(\mathbf{R}, \mathcal{F}_t)$, we have:

$$\begin{aligned} g_t(z_1) &= \rho_t((z_2 + r_t z_t)\Delta S_{t+1} - h_{t+1}) \\ &\leq \rho_t(z_2 \Delta S_{t+1} - h_{t+1}) + \rho_t(r_t z_t \Delta S_{t+1}) = g_t(z_2) \end{aligned}$$

By symmetry, we also have: $g_t(z_2) \leq g_t(z_1)$, hence $g_t(z_1) = g_t(z_2)$. Therefore, g_t is constant on any line which is parallel to $\mathbf{R}z_t$. Moreover,

$$\{(\omega, z_t^\perp) \in \Omega \times \mathbf{R}^2 : z_t^\perp z_t(\omega) = 0\} \in \mathcal{F}_t \otimes \mathcal{B}(\mathbf{R}^2).$$

By measurable selection argument, we may choose $z_t^\perp \in L^0(S(0,1), \mathcal{F}_t)$ such that the line $\mathbf{R}z_t^\perp$ is orthogonal to $\mathbf{R}z_t$. Since $d = 2$, for any $x \in \mathbf{R}^2$, there exist $\lambda \in \mathbf{R}$ such that $x - \lambda z_t^\perp \in \mathbf{R}z_t$. We then deduce from above that:

$$\inf_{x \in \mathbf{R}^2} g_t(x) = \inf_{\lambda \in \mathbf{R}} g_t(\lambda z_t^\perp).$$

On the set $\{\rho_t(z_t^\perp \Delta S_{t+1}) = 0\}$, we get that $\inf_{\lambda \in \mathbf{R}} g_t(\lambda z_t^\perp) = g_t(0)$ by Proposition 3.3.7. On the other hand, on the set $\{\rho_t(z_t^\perp \Delta S_{t+1}) > 0\}$, we get that $\lim_{|\lambda| \to \infty} g_t(\lambda z_t^\perp) = +\infty$ by Proposition 3.3.8 and SRN, hence g_t achieves a minimum on the line $\mathbf{R}z_t^\perp$.

Let us now prove the d-dimensional case by induction. Recall that there exists $z_t \in L^0(S(0,1), \mathcal{F}_t)$ such that $\rho_t(z_t \Delta S_{t+1}) = \inf_{z \in S(0,1)} \rho_t(z \Delta S_{t+1})$. On $F_t = \{\rho_t(z_t \Delta S_{t+1}) > 0\}$, by Corollary 3.3.9 and SRN, g_t attains a minimum on $\mathbf{R}^d$. On $F_t^c = \{\rho_t(z_t \Delta S_{t+1}) = 0\}$, consider a hyperplane I_{d-1} which is orthogonal to $\mathbf{R}z_t$ and admits an orthonormal basis $(z_1, z_2, ..., z_{d-1})$ such that for each $\omega \in \Omega$, $\hat{z} = (z_t, z_1, ..., z_{d-1})$ is an orthonormal basis for $\mathbf{R}^d$. Note that each z_i can be chosen in $L^0(S(0,1), \mathcal{F}_t)$. Indeed, similarly to the case $d = 2$, we first choose $z_1 \in L^0(S(0,1), \mathcal{F}_t)$ orthogonal to z_t. Recursively, for $i \in \{2, ..., d-1\}$, we have:

$$\{(\omega, z_i) \in \Omega \times \mathbf{R}^d : z_i z_j(\omega) = 0 \text{ for all } j = 0, ..., i-1\} \in \mathcal{F}_t \otimes \mathcal{B}(\mathbf{R}^d).$$

By measurable selection argument, we then choose $z_i \in L^0(S(0,1), \mathcal{F}_t)$. We denote by M_t the matrix such that $\hat{z}_i = M_t e_i$, for every $i \geq 1$, where $e_i =$

$(0, \cdots, 1, \cdots, 0) \in \mathbf{R}^d$. We recall the change of variable $x = M_t \tilde{x}$ where x and $\tilde{x}$ are the coordinates of an arbitrary vector of $\mathbf{R}^d$ in the basis $(e_i)_{i\geq 1}$ and $(\hat{z}_i)_{i\geq 1}$ respectively. The ith column vector of M_t coincides with $\hat{z}_i$ expressed in the basis $(e_i)_{i\geq 1}$, hence each entry of M_t belongs to $L^0(\mathbf{R}, \mathcal{F}_t)$ and so do the components of M_t^{-1}. We then define the adapted processes $\tilde{S}_u = M_t' S_u$, for $u = t, t+1$. We have:

$$g_t(x) = \rho_t(x\Delta S_{t+1} - h_{t+1}) = \rho_t(\tilde{x}\Delta\tilde{S}_{t+1} - h_{t+1}).$$

We observe that $\tilde{S}_{u=t,t+1}$ forms a new market model which also satisfies the NA condition between t and $t+1$. Indeed, for any $z \in S(0,1)$, we have:

$$\rho_t(z\Delta\tilde{S}_{t+1}) = \rho_t(zM_t'\Delta S_{t+1}),$$

hence $\rho_t(z\Delta\tilde{S}_{t+1}) = 0$ implies that $\rho_t(-zM_t'\Delta S_{t+1}) = 0$ by the NA condition satisfied in the market formed by S which, in turn, implies $\rho_t(-z\Delta\tilde{S}_{t+1}) = 0$.

Fix ω and, for any $x \in \mathbf{R}^d$, consider the orthogonal projection $\bar{x}$ of x onto I_{d-1}. We then have $g_t(x) = g_t(\bar{x})$. For $\bar{x} \in I_{d-1}$, we denote $\hat{x} = M_t^{-1}\bar{x}$, we have:

$$\bar{x}\Delta S_{t+1} = \hat{x}\Delta\tilde{S}_{t+1} := \sum_{i=1}^{d} \hat{x}^i \Delta\tilde{S}^i_{t+1} = \sum_{i=2}^{d} \hat{x}^i \Delta\tilde{S}^i_{t+1},$$

since the first coordinate of $\hat{x}$ equals 0 in the new basis. We deduce that:

$$\inf_{x\in\mathbf{R}^d} g_t(x) = \inf_{x\in I_{d-1}} \rho_t(x\Delta S_{t+1} - h_{t+1}) = \inf_{\hat{x}\in\mathbf{R}^{d-1}} \rho_t\left(\sum_{i=2}^{d} \hat{x}^i \Delta\tilde{S}^i_{t+1} - h_{t+1}\right)$$

This means that we have reduced the optimization problem to a market with only $d-1$ assets defined by $(\tilde{S}^2, ..., \tilde{S}^d)$. As it satisfies the NA condition, we deduce that $\inf_{x\in\mathbf{R}^d} g_t(x)$ is attained by induction. □

Proof of Theorem 3.3.11.

Proof. Suppose that NA holds. By Theorem 3.3.10, there is $z_t \in L^0(S(0,1), \mathcal{F}_t)$ and $r_t \in L^0(\mathbf{R}, \mathcal{F}_t)$ such that $P_t^* = \rho_t(r_t z_t \Delta S_{t+1} - h_{t+1})$. Suppose that $\rho_t(z_t\Delta S_{t+1})$ and $\rho_t(-z_t\Delta S_{t+1})$ are both equal to 0. Then, the function g_t associated to h_{t+1}, see (3.3.5), is constant on the line $\mathbf{R}z_t$ by Theorem

3.3.7. Therefore, $P_t^* = g_t(0) = \rho_t(-h_{t+1}) > 0$. Otherwise, under NA, $\rho_t(z_t\Delta S_{t+1}) > 0$ and $\rho_t(-z_t\Delta S_{t+1}) > 0$. Using triangular inequalities, and the assumption $\rho_t(h_{t+1}) \leq 0$, we then deduce that:

$$\begin{aligned}
P_t^* &= r_t z_t S_t + \rho_t(r_t z_t S_{t+1} - h_{t+1}), \\
&= \rho_t(-h_{t+1})1_{\{r_t=0\}} + r_t\rho_t\left(z_t\Delta S_{t+1} - \frac{h_{t+1}}{r_t}\right)1_{\{r_t>0\}} \\
&\quad -r_t\rho_t\left(-z_t\Delta S_{t+1} + \frac{h_{t+1}}{r_t}\right)1_{\{r_t<0\}}, \\
&\geq \rho_t(-h_{t+1})1_{\{r_t=0\}} + r_t\rho_t\left(z_t\Delta S_{t+1}\right)1_{\{r_t>0\}} - r_t\rho_t\left(-z_t\Delta S_{t+1}\right)1_{\{r_t<0\}}, \\
&> 0.
\end{aligned}$$

For the reverse implication, let us prove first that AIP holds. We fix h_{t+1} such that $\rho_t(-h_{t+1}) > 0$ and $\rho_t(h_{t+1}) \leq 0$. So, with the function g_t associated to h_{t+1}, see (3.3.5), we have $P_t^* = P_t^*(h_{t+1}) = \inf_{x\in\mathbf{R}} g_t(x) > 0$ by assumption and $g_t(rz) > 0$ for all $r \in \mathbf{R}$ and $z \in S(0,1)$. Let us show that the set $\{zS_t + \rho_t(zS_{t+1}) < 0\}$ is empty for all $z \in S(0,1)$ a.s.. In the contrary case, by measurable selection, we may construct $z_t \in L^0(\mathbf{R}^d, \mathcal{F}_t)$ such that we have $\mathrm{P}(z_tS_t + \rho_t(z_tS_{t+1}) < 0) > 0$. We then define

$$r_t := -\frac{\rho_t(-h_{t+1})}{\rho_t(z_t\Delta S_{t+1})}1_{\{\rho_t(z_t\Delta S_{t+1})<0\}} \geq 0.$$

We have

$$\begin{aligned}
g_t(r_tz_t) &= r_t z_t S_t + \rho_t(r_t z_t S_{t+1} - h_{t+1}), \\
&\leq r_t z_t S_t + \rho_t(r_t z_t S_{t+1}) + \rho_t(-h_{t+1}), \\
&\leq r_t\rho_t(z_t\Delta S_{t+1}) + \rho_t(-h_{t+1}), \\
&\leq \rho_t(-h_{t+1})1_{\{\rho_t(z_t\Delta S_{t+1})\geq 0\}}.
\end{aligned}$$

Therefore, $P_t^* \leq 0$ on the set $\{\rho_t(z_t\Delta S_{t+1}) < 0\}$ in contradiction with $P_t^* > 0$.

Let us show that $\rho_t(-z\Delta S_{t+1}) = 0$ if $\rho_t(z\Delta S_{t+1}) = 0$ for any $z \in S(0,1)$. Otherwise, by measurable selection argument, there exists $z_t \in L^0(S(0,1), \mathcal{F}_t)$ such that $\Lambda_t := \{\rho_t(z_t\Delta S_{t+1}) = 0\} \cap \{\rho_t(-z_t\Delta S_{t+1}) > 0\}$ satisfies $\mathrm{P}(\Lambda_t) > 0$. If $h_{t+1} = z_t\Delta S_{t+1}$, then $\rho_t(-h_{t+1}) = \rho_t(-z_t\Delta S_{t+1}) > 0$ on Λ_t. On the complimentary set, we fix $h_{t+1} = \gamma_t > 0$, $\gamma_t \in L^0((0,\infty), \mathcal{F}_t)$.

It follows that $\rho_t(-h_{t+1}) > 0$. Moreover, $\rho_t(h_{t+1}) = \rho_t(z_t\Delta S_{t+1}) = 0$ on Λ_t and, otherwise, $\rho_t(h_{t+1}) = -\gamma_t < 0$. Therefore, $\rho_t(h_{t+1}) \leq 0$. We deduce that $P_t^*(h_{t+1}) > 0$, by assumption. On the other hand, if $r \geq 1$, and $\omega \in \Lambda_t$,

$$P_t^*(h_{t+1}) \leq \rho_t(rz_t\Delta S_{t+1} - z_t\Delta S_{t+1}) = (r-1)\rho_t(z_t\Delta S_{t+1}) = 0.$$

It follows that $P_t^*(h_{t+1}) \leq 0$ on Λ_t, i.e. a contradiction. We conclude that $\rho_t(z\Delta S_{t+1}) = 0$ if and only if $\rho_t(-z\Delta S_{t+1}) = 0$ for any $z \in S(0,1)$.

At last, it is clear that $P_t^*(h_{t+1}) \leq g_t(0) = \rho_t(-h_{t+1})$. Moreover, for all $x \in \mathbf{R}^d$, $0 \leq \rho_t(x\Delta S_{t+1}) \leq \rho_t(x\Delta S_{t+1} - h_{t+1}) + \rho_t(h_{t+1})$. Taking the infimum in the r.h.s. of this inequality, we get that $0 \leq P_t^*(h_{t+1}) + \rho_t(h_{t+1})$ and we may conclude. □

Proof of Theorem 3.4.7.

Proof. By [2], [26], there exists $\mathcal{D}_t$ such that (3.4.7) holds if $X \in L^\infty$. By homogeneity, it is clear that (3.4.7) still holds if X is $\mathcal{F}_t$-bounded, i.e. $|X| \leq c_t$ where $c_t \in L^0(\mathbf{R}_+, \mathcal{F}_t)$. Let us show that (3.4.7) still holds for any random variable X such that $X \leq c_t$ a.s. for some $c_t \in L^0(\mathbf{R}_+, \mathcal{F}_t)$. Let us first suppose that X is acceptable. Let us define $X^M = X1_{\{X \geq -M\}}$ for any $M > 0$. Then, X^M is $\mathcal{F}_t$-bounded a.s.. As $X^M = X - X1_{\{X<-M\}}$ and $-X1_{\{X<-M\}} \geq 0$, then X^M is acceptable i.e. $\rho_t(X^M) \leq 0$. By (3.4.7), we deduce that $E_Q(X^M|\mathcal{F}_t) \geq 0$ for all $Q \in \mathcal{D}_t$. Thus, $E_Q((X^M)^+|\mathcal{F}_t) \geq E_Q((X^M)^-|\mathcal{F}_t)$ and, as $M \to \infty$, we get that $c_t \geq E_Q(X^+|\mathcal{F}_t) \geq E_Q(X^-|\mathcal{F}_t)$ hence $\infty > E_Q(X|\mathcal{F}_t) \geq 0$. More generally, for any X such that $X \leq c_t$ for some $c_t \in L^0(\mathbf{R}_+, \mathcal{F}_t)$, $\rho_t(X)+X$ is acceptable hence $\rho_t(X) \geq E_Q(-X|\mathcal{F}_t)$ for any $Q \in \mathcal{D}_t$. We deduce that the inequality $\rho_t(X) \geq \text{ess sup}_{Q\in\mathcal{D}_t} E_Q(-X|\mathcal{F}_t)$ holds.

For the reverse inequality, note that the random variable

$$\gamma^M = \text{ess sup}_{Q\in\mathcal{D}_t} E_Q(-X|\mathcal{F}_t) + X^M \in [-c_t + X^M, X^M]$$

is $\mathcal{F}_t$-bounded hence (3.4.7) holds for γ^M, as seen above. Moreover, we have $E_Q(-\gamma^M|\mathcal{F}_t) \leq E_Q(X|\mathcal{F}_t) - X^M = E_Q(X1_{X<-M}|\mathcal{F}_t) \leq 0$. We deduce by (3.4.7) that $\rho_{t-1}(\gamma^M) \leq 0$. Using the Cash invariance property, we deduce that $\rho_{t-1}(X^M) \leq \text{ess sup}_{Q\in\mathcal{D}_t} E_Q(-X|\mathcal{F}_t)$. As $\lim_{M\to\infty} X^M = X$, we then deduce that $\rho_{t-1}(X) \leq \liminf_{M\to\infty} \rho_{t-1}(X^M) \leq \text{ess sup}_{Q\in\mathcal{D}_t} E_Q(-X|\mathcal{F}_t)$ so that we may conclude that the equality (3.4.7) holds for any random variable that are $\mathcal{F}_t$-bounded form above. □

Proof of Theorem 3.4.9.

Proof. Suppose that $Z = X + \epsilon^+$ where X is $\mathcal{F}_t$-bounded from above and acceptable and $\epsilon^+ \geq 0$ a.s.. Then, $\mathcal{D}_t$ exists by Proposition 3.4.7 and, for all $Q \in \mathcal{D}_t$, $E_Q(Z|\mathcal{F}_t) \geq E_Q(X|\mathcal{F}_t) \geq 0$. As $\rho_t(Z) + Z$ admits the same form than Z, we deduce that $\rho_t(Z) + Z$ admits non negative conditional expectations under $Q \in \mathcal{D}_t$. Therefore, $\rho_t(Z) \geq E_Q(-Z|\mathcal{F}_t)$ for all $Z \in \mathcal{D}_t$ hence $\rho_t(Z) \geq \operatorname{ess\,sup}_{Q\in\mathcal{D}_t} E_Q(-Z|\mathcal{F}_t)$, at least when $\rho_t(Z) > -\infty$. Otherwise, when $\rho_t(Z) = -\infty$, $Z_\alpha = -\alpha + Z$ is acceptable for all $\alpha > 0$, hence $E_Q(Z_\alpha|\mathcal{F}_t) \geq 0$, i.e. $E_Q(Z|\mathcal{F}_t) \geq \alpha$ for all $\alpha > 0$. It follows that $E_Q(Z^-|\mathcal{F}_t) - E_Q(Z^+|\mathcal{F}_t) \leq -\alpha$ and finally, as $\alpha \to \infty$, we deduce that $\rho_t(Z) = \operatorname{ess\,sup}_{Q\in\mathcal{D}_t} E_Q(-Z|\mathcal{F}_t) = -\infty$.

Consider an acceptable position Z. Then, by assumption, $Z = \limsup_n Z^n$ where Z^n is of the form $Z^n = X^n + \epsilon_n^+$ with $\epsilon_n^+ \geq 0$ a.s. and X^n is $\mathcal{F}_t$-bounded from above. Note that $\sup_{k\leq n\leq m} X_n$ is still $\mathcal{F}_t$-bounded from above for all $m \geq k \geq 1$. Since $\sup_{n\geq k} Z_n \geq \sup_{k\leq n\leq m} Z_n \geq \sup_{k\leq n\leq m} X_n$, for all $m \geq k$, we deduce that $\sup_{n\geq k} Z_n$ is of the form $X_k + \epsilon_k^+$ where X_k is $\mathcal{F}_t$-bounded from above and acceptable while $\epsilon_k^+ \geq 0$ a.s.. It follows that any acceptable position is of the form $Z = \lim \downarrow Z_n$ where Z_n is of the form $Z_n = X_n + \epsilon_n^+$ and X_n is $\mathcal{F}_t$-bounded from above and acceptable while $\epsilon_n^+ \geq 0$ a.s.. As $Z \leq Z_n$, we deduce that $\rho_t(Z) \geq \rho_t(Z_n) \geq \operatorname{ess\,sup}_{Q\in\mathcal{D}_t} E_Q(-Z_n|\mathcal{F}_t)$ by virtue of the inequality we have shown in the first part. As $(-Z_n)$ is non decreasing we finally deduce that $\rho_t(Z) \geq E_Q(-Z|\mathcal{F}_t)$ for any $Q \in \mathcal{D}_t$, when $n \to \infty$. It follows that $\rho_t(Z) \geq \operatorname{ess\,sup}_{Q\in\mathcal{D}_t} E_Q(-Z|\mathcal{F}_t)$.

Moreover, suppose that (3.4.8) holds for any acceptable position Z_n of the form $Z_n = X_n + \epsilon_n^+$ where X_n is $\mathcal{F}_t$-bounded from above and acceptable and $\epsilon_n^+ \geq 0$ a.s.. By lower semi-continuity,

$$\rho_t(Z) \leq \liminf_n \rho_t(Z_n) = \liminf_n \operatorname{ess\,sup}_{Q\in\mathcal{D}_t} E_Q(-Z_n|\mathcal{F}_t).$$

As $Z \leq Z_n$, $E_Q(-Z_n|\mathcal{F}_t) \leq E_Q(-Z|\mathcal{F}_t)$, and we deduce the inequality $\rho_t(Z) \leq \operatorname{ess\,sup}_{Q\in\mathcal{D}_t} E_Q(-Z|\mathcal{F}_t)$. We then conclude that (3.4.8) holds for every acceptable position Z and, finally, for every $X \in L^0$ as $\rho_t(X) + X$ is acceptable.

It remains to show that (3.4.8) holds for $Z = X + \epsilon^+ \in \mathcal{A}_t^{\infty,+} + L^0(\mathbf{R}^+, \mathcal{F}_T)$. To get it, it is sufficient to prove that $\rho_t(Z) \leq \operatorname{ess\,sup}_{Q\in\mathcal{D}_t} E_Q(-Z|\mathcal{F}_t)$. Let us define $Z^n = X + \epsilon^+ 1_{\{\epsilon^+\leq n\}} + \alpha_n 1_{\{\epsilon^+>n\}} \in \mathcal{A}_t^{\infty,+}$ where $\alpha_n > 0$ is chosen large enough in such a way that $P(\alpha_n < \epsilon^+) < \varepsilon$. Then, $(\alpha_n - \epsilon^+)1_{\{\epsilon^+>n\}}$ is acceptable by hypothesis for $P((\alpha_n - \epsilon^+)1_{\{\epsilon^+>n\}} < 0) \leq P(\alpha_n < \epsilon^+) < \varepsilon$.

Since $Z^n \to Z$ a.s., we deduce that $\rho_t(Z) \leq \liminf_n \rho_t(Z^n)$. Recall that $\rho_t(Z^n) = \sup_{Q\in\mathcal{D}_t} E_Q(-Z^n|\mathcal{F}_t)$ by Proposition 3.4.7. Hence,

$$\rho_t(Z^n) \leq \text{ess sup}_{Q\in\mathcal{D}_t} E_Q(-Z|\mathcal{F}_t) + \text{ess sup}_{Q\in\mathcal{D}_t} E_Q(Z - Z^n|\mathcal{F}_t).$$

Moreover, since $Z^n - Z$ is $\mathcal{F}_t$-bounded from above, we have

$$\text{ess sup}_{Q\in\mathcal{D}_t} E_Q(Z - Z^n|\mathcal{F}_t) = \rho_t(Z^n - Z) = \rho_t((\alpha_n - \epsilon^+)1_{\{\epsilon^+>n\}}) \leq 0.$$

We then deduce that $\rho_t(Z) \leq \text{ess sup}_{Q\in\mathcal{D}_t} E_Q(-Z|\mathcal{F}_t)$ and the conclusion follows. □

Proof of Theorem 3.4.11.

Proof. Consider $\theta_t \in L^0(\mathbf{R}^d, \mathcal{F}_t)$. By Theorem 3.3.2, it suffices to show that $\rho_t(\theta_t\Delta S_{t+1}) \geq 0$ a.s.. Otherwise, the set $\Lambda_t = \{\rho_t(\theta_t\Delta S_{t+1}) < 0\}$ admits a positive probability and $\theta_t\Delta S_{t+1}1_{\Lambda_t} \in \mathcal{R}_{t,T} \cap \mathcal{A}_{t,T} = \mathcal{A}^0_{t,T}$. It follows that $\rho_t(\theta_t\Delta S_{t+1}1_{\Lambda_t}) = 0$ hence a contradiction. Therefore, AIP holds.

Let us show that $\overline{\mathcal{Z}_{t,T}} \subseteq \mathcal{Z}_{t,T}$. In the one step model, let us suppose that $\gamma^n = \theta^n_{T-1}\Delta S_T - \epsilon^n_{T-1,T} \in \mathcal{Z}_{T-1,T}$ converges to $\gamma^\infty \in L^0(\mathbf{R}, \mathcal{F}_T)$ in probability. We suppose that $\epsilon^n_{T-1,T} \in \mathcal{A}_{T-1,T}$. We need to show that $\gamma^\infty \in \mathcal{Z}_{T-1,T}$.

On the $\mathcal{F}_{T-1}$-measurable set $\Lambda_{T-1} := \{\liminf_n |\theta^n_{T-1}| < \infty\}$, by [55, Lemma 2.1.2], we may assume w.l.o.g. that θ^n_{T-1} is convergent to some θ^∞_{T-1} hence $\epsilon^n_{T-1,T}$ is also convergent and, finally, $\gamma^\infty 1_{\Lambda_{T-1}} \in \mathcal{Z}_{T-1,T}$.

Otherwise, on $\Omega \setminus \Lambda_{T-1}$, we use the normalized sequences,

$$\tilde{\theta}^n_{T-1} := \theta^n_{T-1}/(|\theta^n_{T-1}| + 1), \quad \tilde{\epsilon}^n_{T-1,T} := \epsilon^n_{T-1,T}/(|\theta^n_{T-1}| + 1).$$

By [55, Lemma 2.1.2], we may assume that a.s. $\tilde{\theta}^n_{T-1} \to \tilde{\theta}^\infty_{T-1}$, $\tilde{\epsilon}^n_{T-1,T} \to \tilde{\epsilon}^\infty_{T-1,T}$ and $\tilde{\theta}^\infty_{T-1}\Delta S_T - \tilde{\epsilon}^\infty_{T-1,T} = 0$ a.s.. Note that $|\tilde{\theta}^\infty_{T-1}| = 1$ a.s.. As $\tilde{\theta}^\infty_{T-1}\Delta S_T$ is acceptable ($\epsilon^\infty_{T-1,T} \in \mathcal{A}_{T-1,T}$) then $\tilde{\theta}^\infty_{T-1}\Delta S_T \in \mathcal{A}^0_{t,T}$ by assumption. We follow the recursive arguments on the dimension of [54]. Since $|\tilde{\theta}^\infty_{T-1}| = 1$, there exists a partition of $\Omega \setminus \Lambda_{T-1}$ into d disjoint subsets $G^i_{T-1} \in \mathcal{F}_{T-1}$ such that $\tilde{\theta}^{\infty,i}_{T-1} \neq 0$ on G^i_{T-1}. Define on G^i_{T-1}, $\tilde{\theta}^n_{T-1} := \theta^n_{T-1} - \beta^n_{T-1}\tilde{\theta}^\infty_{T-1}$ where $\beta^n_{T-1} := \theta^{n,i}_{T-1}/\tilde{\theta}^{\infty,i}_{T-1}$. Observe that $\gamma^n = \tilde{\theta}^n_{T-1}\Delta S_T - \tilde{\epsilon}^n_{T-1,T}$ where the position $\tilde{\epsilon}^n_{T-1,T} = \epsilon^n_{T-1,T} - \beta^n_{T-1}\tilde{\theta}^\infty_{T-1}\Delta S_T$ is acceptable since $\pm\tilde{\theta}^\infty_{T-1}\Delta S_T$ are acceptable. As $\tilde{\theta}^{n,i}_{T-1} = 0$ on G^i_{T-1}, we repeat the entire procedure on each G^i_{T-1} with the new expression $\gamma^n = \tilde{\theta}^n_{T-1}\Delta S_T - \tilde{\epsilon}^n_{T-1,T}$ such that the number of components of $\tilde{\theta}^n_{T-1}$ is reduced by one. We then conclude by recursion on

the number of non-zero components since the conclusion is trivial if all the coordinates vanish.

We now show the result in the multi-step models by induction. Fix some $s \in \{t, \dots, T-1\}$. We show that $\overline{\mathcal{Z}_{s+1}^T} \subseteq \mathcal{Z}_{s+1}^T$ implies the same property for s instead of $s+1$.

Since AIP holds, we get that $\mathcal{Z}_{s+1}^T \cap L^0(\mathbf{R}_+, \mathcal{F}_{s+1}) = \{0\}$ hence $\overline{\mathcal{Z}_{s+1}^T} \subseteq \mathcal{Z}_{s+1}^T$ implies that $\overline{\mathcal{Z}_{s+1}^T} \cap L^1(\mathbf{R}_+, \mathcal{F}_{s+1}) = \{0\}$. Using the Hahn-Banach separation theorem in L^1, we deduce $Q^{(s+1)} \ll P$ with $\frac{dQ^{(s+1)}}{dP} \in L^\infty$ such that $\rho_{s+1} := E_P(\frac{dQ^{(s+1)}}{dP}|\mathcal{F}_{s+1}) = 1$ a.s., $(S_u)_{u \geq s+1}$ is a martingale under $Q^{(s+1)}$ and $E_Q(a_{s+1,T}|\mathcal{F}_{s+1}) \geq 0$ for all $a_{s+1,T} \in \mathcal{A}_{s+1,T}$ such that $E_Q(|a_{s+1,T}||\mathcal{F}_{s+1}) < \infty$ a.s.. Suppose that

$$\gamma^n = \sum_{u=s+1}^{T} \theta_{u-1}^n \Delta S_u - \epsilon_{s,T}^n \in \mathcal{Z}_{s,T} \text{ converges to } \gamma^\infty \in L^0(\mathbf{R}, \mathcal{F}_T).$$

We suppose that $\epsilon_{s,T}^n \in \mathcal{A}_{s,T}$. By Lemma 3.4.2, $\epsilon_{s,T}^n = \epsilon_{s,s+1}^n + \epsilon_{s+1,T}^n$, where $\epsilon_{s,s+1}^n \in \mathcal{A}_{s,s+1}$ and $\epsilon_{s+1,T}^n \in \mathcal{A}_{s+1,T}$. As before, on the $\mathcal{F}_s$-measurable set $\Lambda_s := \{\liminf_n |\theta_s^n| < \infty\}$, we may assume w.l.o.g. that θ_s^n converges to θ_s^∞. Therefore, on Λ_s,

$$\sum_{u=s+2}^{T} \theta_{u-1}^n \Delta S_u - \epsilon_{s,T}^n = \gamma^n - \theta_s^n \Delta S_{s+1} \to \gamma^\infty - \theta_s^\infty \Delta S_{s+1}.$$

On the subset $\hat{\Lambda}_{s+1} := \{\liminf_n |\epsilon_{s,s+1}^n| = \infty\} \cap \Lambda_s \in \mathcal{F}_{s+1}$, we use the normalization procedure as previously, i.e. we divide by $|\epsilon_{s,s+1}^n|$, up to a subsequence, and, by the induction hypothesis, we obtain that

$$\sum_{u=s+2}^{T} \tilde{\theta}_{u-1}^n \Delta S_u - \tilde{\epsilon}_{s+1,T} = \tilde{\epsilon}_{s,s+1},$$

where $\tilde{\epsilon}_{s+1,T} \in \mathcal{A}_{s+1,T}$ and $\tilde{\epsilon}_{s,s+1} \in \mathcal{A}_{s,s+1}$ satisfies $|\tilde{\epsilon}_{s,s+1}| = 1$ a.s.. Moreover, by assumption, we may show that

$$E_{Q^{(s+1)}}\left(\sum_{u=s+2}^{T} \tilde{\theta}_{u-1}^n \Delta S_u | \mathcal{F}_{s+1}\right) = 0.$$

Moreover, still by assumption, $E_{Q^{(s+1)}}(\tilde{\epsilon}_{s+1,T}|\mathcal{F}_{s+1}) \geq 0$. We deduce that $\tilde{\epsilon}_{s,s+1} = E_{Q^{(s+1)}}(\tilde{\epsilon}_{s,s+1}|\mathcal{F}_{s+1}) \leq 0$. Therefore, $\tilde{\epsilon}_{s,s+1} = -1$ hence $\rho_s(\tilde{\epsilon}_{s,s+1}) =$

$\rho_s(-1) = 1$, which is in contradiction with $\rho_s(\tilde{\epsilon}_{s,s+1}) \leq 0$. Therefore, we may suppose, on Λ_s, that $\epsilon^n_{s,s+1}$ converges a.s. to some $\epsilon_{s,s+1} \in \mathcal{A}_{s,s+1}$. By the induction hypothesis, we then deduce that $\sum_{u=s+2}^{T} \theta^n_{u-1}\Delta S_u - \epsilon^n_{s+1,T}$ also converges to an element of $\mathcal{Z}^T_{s+1}$ and we conclude that $\gamma^\infty 1_{\Lambda_s} \in \mathcal{Z}^T_s$.
On $\Omega \setminus \Lambda_s$, we use the normalisation procedure as before, and deduce the equality

$$\sum_{u=s+1}^{T} \tilde{\theta}^\infty_{u-1}\Delta S_u - \tilde{\epsilon}^\infty_{s,T} = 0 \text{ a.s.}$$

for some $\tilde{\theta}^\infty_u \in L^0(\mathbf{R}, \mathcal{F}_u)$, $u \in \{s, \ldots, T-1\}$ and $\tilde{\epsilon}^\infty_{s,T} \in \mathcal{A}_{s,T}$. By Lemma 3.4.2, we write $\tilde{\epsilon}^\infty_{s,T} = \tilde{\epsilon}^\infty_{s,s+1} + \tilde{\epsilon}^\infty_{s+1,T}$ where $\tilde{\epsilon}^\infty_{s,s+1} \in \mathcal{A}_{s,s+1}$ and $\tilde{\epsilon}^\infty_{s+1,T} \in \mathcal{A}_{s+1,T}$. Moreover, $|\tilde{\theta}^\infty_s| = 1$ a.s.. We deduce that:

$$\tilde{\theta}^\infty_s \Delta S_{s+1} + \sum_{u=s+2}^{T} \tilde{\theta}^\infty_{u-1}\Delta S_u - \tilde{\epsilon}^\infty_{s+1,T} = \tilde{\epsilon}^\infty_{s,s+1} \text{ a.s..}$$

Taking the conditional expectation knowing $\mathcal{F}_{s+1}$ under $Q^{(t+1)}$, we deduce that $\tilde{\epsilon}^\infty_{s,s+1} \leq \tilde{\theta}^\infty_s \Delta S_{s+1}$. It follows that $\rho_s(\tilde{\theta}^\infty_s \Delta S_{s+1}) \leq \rho_s(\tilde{\epsilon}^\infty_{s,s+1}) \leq 0$ hence $\tilde{\theta}^\infty_s \Delta S_{s+1} \in \mathcal{A}^0_{s,T}$ by the assumption. Using the one step arguments based on the elimination of non-zero components of the sequence θ^n_s, we may replace θ^n_s by $\tilde{\theta}^n_s$ such that $\tilde{\theta}^n_s$ converges. We then repeat the same arguments on the set Λ_s to conclude that $\gamma^\infty 1_{\Omega\setminus\Lambda_s} \in \mathcal{Z}^T_s$. □

Proof of Theorem 3.4.12.

Proof. Let us consider $W_{t,T} \in \mathcal{R}_{t,T} \cap \mathcal{A}_{t,T}$ Then, $W_{t,T}$ is of the form:

$$W_{t,T} = \sum_{s=t+1}^{T} \theta_{s-1}\Delta S_s = \sum_{s=t+1}^{T} a_{s-1,s},$$

where $\theta_{s-1} \in L^0(\mathbf{R}, \mathcal{F}_{s-1})$ and $a_{s-1,s} \in \mathcal{A}_{s-1,s}$, for all $s = t+1, \cdots, T$. It follows that:

$$\theta_t \Delta S_{t+1} - a_{t,t+1} + \sum_{s=t+2}^{T} (\theta_{s-1}\Delta S_s - a_{s-1,s}) = 0. \tag{3.5.10}$$

Therefore, $p_t = \theta_t \Delta S_{t+1} - a_{t,t+1}$ is a (direct) price at time $s = t+1$ for the zero claim. Under AIP condition, we get that $\theta_t \Delta S_{t+1} \geq a_{t,t+1}$ hence

$\rho_t(\theta_t\Delta S_{t+1}) \leq 0$. As $\rho_t(\theta_t\Delta S_{t+1}) \geq 0$ by AIP, $\rho_t(\theta_t\Delta S_{t+1}) = 0$ and, by SRN, we get that $\rho_t(\theta_t\Delta S_{t+1}) = \rho_t(-\theta_t\Delta S_{t+1}) = 0$. We then deduce that $-p_t \in \mathcal{A}_{t,T} \cap L^0(\mathbf{R}_-, \mathcal{F}_T) = \{0\}$ hence $p_t = 0$ and $\theta_t\Delta S_{t+1} = a_{t,t+1} \in \mathcal{A}^0_{t,T}$. The equality (3.5.10) may be rewritten as:

$$\theta_{t+1}\Delta S_{t+2} - a_{t+1,t+2} + \sum_{s=t+3}^{T} (\theta_{s-1}\Delta S_s - a_{s-1,s}) = 0. \tag{3.5.11}$$

By induction, we finally deduce that $\theta_s\Delta S_{t+1} = a_{s,s+1} \in \mathcal{A}^0_{s,s+1}$ for all $s \geq t$. By Remark 3.4.10, we have $W_{t,T} \in \mathcal{A}^0_{t,T}$.

Consider now $\epsilon^+_T \in \mathcal{Z}_{t,T} \cap L^0(\mathbf{R}_+, \mathcal{F}_T)$. We may write $\epsilon^+_T = r_{t,T} - a_{t,T}$ where $r_{t,T} \in \mathcal{R}_{t,T}$ and $a_{t,T} \in \mathcal{A}_{t,T}$. We get that $r_{t,T} = a_{t,T} + \epsilon^+_T \in \mathcal{R}_{t,T} \cap \mathcal{A}_{t,T} = \mathcal{A}^0_{t,T}$ hence $-r_{t,T} \in \mathcal{A}_{t,T}$. It follows that $-\epsilon^+_T \in \mathcal{A}_{t,T} \cap L^0(\mathbf{R}_-, \mathcal{F}_T) = \{0\}$. □

Proof of Theorem 3.4.13.

Proof. Suppose that 1) holds. By Theorem 3.4.12, we deduce that 3) holds. Note that 2) and 3) are equivalent since the risk measure is time-consistent. Suppose that 3) holds. Since $-\mathcal{A}_{t,T} \subseteq \mathcal{Z}_{t,T}$, it follows that $\mathcal{A}^0_{t,T} \subseteq \mathcal{Z}_{t,T} \cap \mathcal{A}_{t,T}$. Reciprocally, consider $x_{t,T} = W_{t,T} - a_{t,T} \in \mathcal{Z}_{t,T} \cap \mathcal{A}_{t,T}$, where $W_{t,T} \in \mathcal{R}_{t,T}$ and $a_{t,T} \in \mathcal{A}_{t,T}$, then $W_{t,T} \in \mathcal{A}_{t,T}$ hence $W_{t,T} \in \mathcal{A}^0_{t,T}$ by 2). It follows that $x_{t,T} \in (-\mathcal{A}_{t,T})$ and we conclude that $\mathcal{Z}_{t,T} \cap \mathcal{A}_{t,T} = \mathcal{A}^0_{t,T}$. Moreover, by Theorem 3.4.11, $\mathcal{Z}_{t,T}$ is closed in probability hence 4) holds. Note that 4) and 5) are equivalent since the risk measure is time-consistent.

Assume that 4) holds. The existence of Q in 7) holds by standard arguments based on the Hahn-Banach separation theorem. In particular, NA holds under P' such that $P' \sim P$. We suppose w.l.o.g that S_t is integrable under P for every t. If $x \in L^1(\mathbf{R}, \mathcal{F}_T) \cap (\mathcal{A}_{t,T} \setminus \mathcal{A}^0_{t,T})$, $x \notin \mathcal{Z}_{t,T} \cap L^1(\mathbf{R}, \mathcal{F}_T)$. By the Hahn-Banach separation theorem, there exists $p_x \in L^\infty(\mathbf{R}, \mathcal{F}_T)$ and $c \in \mathbf{R}$ such that $E(p_x X) < c < E(xp_x)$, $\forall X \in \mathcal{Z}_{t,T}$. As $\mathcal{Z}_{t,T}$ is a cone, we get that $E(p_x X) \leq 0$ for all $X \in \mathcal{Z}_{t,T}$ and since $-L^0(\mathbf{R}_+, \mathcal{F}_T) \subseteq \mathcal{Z}_{t,T}$, we deduce that $p_x \geq 0$ a.s.. With $X = 0$, we get that $E(xp_x) > 0$ and, as $\mathcal{R}_{t,T}$ is a vector space, $E(p_x X) = 0$ for all $X \in \mathcal{R}_{t,T}$. As $P(p_x > 0) > 0$, we may renormalize and suppose that $||p_x||_\infty = 1$. Let us consider the family $G = (\Gamma_x)_{x \in I}$ where $I = L^1(\mathbf{R}, \mathcal{F}_T) \cap (\mathcal{A}_{t,T} \setminus \mathcal{A}^0_{t,T})$ and $\Gamma_x = \{p_x > 0\}$. For any $\Gamma \in \mathcal{F}_T$ such that $P(\Gamma) > 0$, $x = 1_\Gamma \in I$ since $\mathcal{A}_{t,T} \cap L^0(\mathbf{R}_-, \mathcal{F}_T) = \{0\}$. Therefore, $E(xp_x) = E(1_\Gamma p_x) > 0$ implies that $P(\Gamma_x \cap \Gamma) > 0$. By Lemma 2.1.3 in

[55], we deduce a countable family $(x_i)_{i=1}^{\infty}$ of I such that $\Omega = \bigcup_{i=1}^{\infty} \Gamma_{x_i}$. We define $p = \sum_{i=1}^{\infty} 2^{-i} p_{x_i}$. We have $p > 0$ a.s and we renormalize p such that $p \in L^{\infty}(\mathbf{R}_+, \mathcal{F}_T)$ and $E_P(p) = 1$. We define $Q \sim P$ such that $dQ/dP = p$. We have $E(pX) = 0$ for all $X \in \mathcal{R}_{t,T}$. Therefore, with $F_{u-1} \in \mathcal{F}_{u-1}$, $1_{F_{u-1}} \Delta S_u \in \mathcal{R}_{t,T}$ if $u \geq t+1$, so $E_Q(1_{F_{u-1}} \Delta S_u) = 0$. This implies that $E_Q(\Delta S_u | \mathcal{F}_{u-1}) = 0$, i.e $(S_u)_{u=t}^T$ is a Q-martingale.

Moreover, by the construction of Q above, for all $x \in \mathcal{A}_{t,T} \cap L^1(\mathbf{R}, \mathcal{F}_T)$, we have $E_Q(x|\mathcal{F}_t) \geq 0$. By truncature and homogeneity, we may extend this property to every x such that $E(|x||\mathcal{F}_t) < \infty$ a.s. since $x/(1 + E(|x||\mathcal{F}_t))$ is integrable. Finally, this also holds if $E_Q(x^-|\mathcal{F}_t) < \infty$ a.s.. At last, since $\rho_t(X) + X \in \mathcal{A}_{t,T}$, we may conclude that $\rho_t(X) \geq -E_Q(X|\mathcal{F}_t)$, for all X such that $E_Q(X^-|\mathcal{F}_t) < \infty$ a.s.. If $x \in \mathcal{A}_{t,T} \setminus \mathcal{A}^0_{t,T}$, it suffices to consider the probability measure $Q_x = \frac{1}{2}(Q + \tilde{Q})$ where $\tilde{Q}$ is defined by its density $d\tilde{Q}/d\mathrm{P} = p_x$. Indeed, since $E_{\tilde{Q}}(x) > 0$ and $E_Q(x) \geq 0$, this implies that $E_{Q_x}(x) > 0$ hence $\mathrm{P}(E_{Q_x}(x|\mathcal{F}_t) \neq 0) > 0$.

Assume that 7) holds. For some martingale measure $Q \sim P$ we have $\rho_t(\theta_t \Delta S_{t+1}) \geq -E_Q(\theta_t \Delta S_{t+1}|\mathcal{F}_t) = 0$, hence AIP holds. If $\rho_t(\theta_t \Delta S_{t+1}) = 0$ on some non null set Λ_t, we have $\rho_t(\theta_t \Delta S_{t+1} 1_{\Lambda_t}) = 0$. This implies $\theta_t \Delta S_{t+1} 1_{\Lambda_t}$ is acceptable. Moreover, if $\theta_t \Delta S_{t+1} 1_{\Lambda_t} \notin \mathcal{A}^0_{t,T}$, $E_{Q_x}(\theta_t \Delta S_{t+1} 1_{\Lambda_t}|\mathcal{F}_t) \neq 0$ by 7), which yields contradiction . Therefore, $\rho_t(\theta_t \Delta S_{t+1}) = \rho_t(-\theta_t \Delta S_{t+1}) = 0$ on Λ_t, i.e. SRN holds, and we deduce that 1) holds. Note that 5) and 6) are equivalent by Theorem 3.4.11. □

Proof of Theorem 3.4.14.

Proof. By Theorems 3.4.11 and 3.4.13, we know that $\Gamma_{0,T}$ is closed in probability. For any $h_T \in \Gamma_{0,T}$, there exists $\sum_{t=0}^{T} \theta_{t-1} \Delta S_t \in \mathcal{R}_{0,T}$ such that $\rho_0\left(\sum_{t=0}^{T} \theta_{t-1} \Delta S_t - h_T\right) \leq 0$. Since, $h_T \in L_S$, we suppose w.l.o.g that S_T and h_T are integrable under P.

Set $\gamma_t := \sum_{t=0}^{t} \theta_{t-1} \Delta S_t - h_T$ for every $t \leq T$. For any $Q \in \mathcal{Q}^e \neq \emptyset$, we have:

$$|\gamma_T| \leq \left| \sum_{t=0}^{T-1} \theta_{t-1} \Delta S_t \right| + |\theta_{T-1}||\Delta S_T| + |h_T|,$$

hence:

$$E_Q(|\gamma_T||\mathcal{F}_{T-1}) \leq \left| \sum_{t=0}^{T-1} \theta_{t-1} \Delta S_t \right| + |\theta_{T-1}| E_Q(|\Delta S_T||\mathcal{F}_{T-1}) + E_Q(|h_T||\mathcal{F}_{T-1}) < \infty \text{ a.s..}$$

By Statement 7) of Theorem 3.4.13 and the martingale property, we deduce that:

$$\rho_{T-1}(\gamma_T) \geq -E_Q(\gamma_{T-1}|\mathcal{F}_{T-1}). \tag{3.5.12}$$

At time $T-2$, by time-consistency of the risk measure and (3.5.12), we get that

$$\rho_{T-2}(\gamma_T) = \rho_{T-2}(-\rho_{T-1}(\gamma_T)) \geq \rho_{T-2}(E_Q(\gamma_{T-1}|\mathcal{F}_{T-1})).$$

Moreover, $E_Q(|E_Q(\gamma_{T-1}|\mathcal{F}_{T-1})||\mathcal{F}_{T-2}) \leq E_Q(|\gamma_{T-1}||\mathcal{F}_{T-2})$ and

$$\begin{aligned} E_Q(|\gamma_{T-1}||\mathcal{F}_{T-2}) \quad \leq \quad & \left|\sum_{t=0}^{T-2} \theta_{t-1}\Delta S_t\right| + |\theta_{T-2}|E_Q(|\Delta S_{T-1}||\mathcal{F}_{T-2}) \\ & + E_Q(|h_T||\mathcal{F}_{T-2}) < \infty \text{ a.s..} \end{aligned}$$

We deduce by Statement 7) of Theorem 3.4.13 that

$$\rho_{T-2}(E_Q(\gamma_{T-1}|\mathcal{F}_{T-1})) \geq -E_Q(\gamma_{T-1}|\mathcal{F}_{T-2}).$$

By the martingale property, we finally deduce that $\rho_{T-2}(\gamma_T) \geq -E_Q(\gamma_{T-2}|\mathcal{F}_{T-2})$. Recursively, we finally obtain:

$$0 \geq \rho_0\left(\sum_{t=0}^{T} \theta_{t-1}\Delta S_t - h_T\right) \geq -E_Q(\gamma_1|\mathcal{F}_0) \geq -E_Q(\theta_0\Delta S_1 - h_T) \geq E_Q(h_T). \tag{3.5.13}$$

This implies $\Gamma_{0,T} \subset \Theta_{0,T}$.

Reciprocally, assume that there is $\hat{h}_T \in \Theta_{0,T}\backslash\Gamma_{0,T}$. Since $\hat{h}_T \in L_S(\mathbf{R}, \mathcal{F}_T)$, $\hat{h}_T$ is integrable under $Q \in \mathcal{Q}^e$. Moreover, since $\Gamma_{0,T}$ is closed in probability, $\tilde{\Gamma}_{0,T} := \Gamma_{0,T} \cap L^1_Q(\mathbf{R}, \mathcal{F}_T)$ is closed in L^1. By the Hahn-Banach separation theorem, as $\hat{h}_T \notin \tilde{\Gamma}_{0,T}$, we deduce the existence of $Y \in L^\infty(\mathbf{R}, \mathcal{F}_T)$ such that:

$$\sup_{X \in \tilde{\Gamma}_{0,T}} E_Q(YX) < E_Q(Y\hat{h}_T).$$

Let H be the density Q w.r.t P, i.e. $H = dQ/dP$. We have:

$$\sup_{X \in \tilde{\Gamma}_{0,T}} E(HYX) < E(HY\hat{h}_T).$$

Since $\tilde{\Gamma}_{0,T}$ is a cone, we deduce that $E(HYX) \leq 0$ for all $X \in \tilde{\Gamma}_{0,T}$. Moreover, $E(HY\hat{h}_T) > 0$, $HY \geq 0$ a.s. and $E(HY) > 0$. Therefore, we deduce that $\hat{H} := HY/E(HY)$ defines the density of a probability measure $\hat{Q} \in \mathcal{Q}^a$.

We define $H^\epsilon := \epsilon H + (1-\epsilon)\hat{H}$. Since $E(\hat{H}\hat{h}_T) > 0$, we may choose $\epsilon \in (0,1)$ small enough so that $E(H^\epsilon \hat{h}_T) > 0$. Since H^ϵ defines the density of a probability measure $Q^\epsilon \in \mathcal{Q}^e$, we should have $E_{Q^\epsilon}\hat{h}_T = E(H^\epsilon \hat{h}_T) \leq 0$, as $\hat{h}_T \in \Theta_{0,T}$. This yields a contradiction. We conclude that $\Gamma_{0,T} = \Theta_{0,T}$.

At last, P_0 is a super-hedging price for h_T if and only if $h_T - P_0 \in \Gamma_{0,T}$. By the first part, we deduce that $P_0^* \geq \sup_{Q\in\mathcal{Q}^e} E_Q(h_T)$. Suppose there exists $\epsilon > 0$ such that $P_0^* - \epsilon \geq \sup_{Q\in\mathcal{Q}^e} E_Q(h_T)$. Then, $(h_T - P_0^* + \epsilon) \in \Theta_{0,T}$. Since $\Theta_{0,T} = \Gamma_{0,T}$, there exists $W_{0,T} \in \mathcal{R}_{0,T}$ such that $\rho_0(W_{0,T} - h_T + P_0^* - \epsilon) \leq 0$. This implies that $P_0^* - \epsilon \geq \rho_0(W_{0,T} - h_T)$. Since $\rho_0(W_{0,T} - h_T)$ is a super-hedging price for h_T, we also deduce that $\rho_0(W_{0,T} - h_T) \geq P_0^*$ which yields a contradiction. We conclude that $P_0^* = \sup_{Q\in\mathcal{Q}^e} E_Q(h_T)$. □

Chapter 4

Dynamic programming principle and computable prices in financial market models with transaction costs

Abstract

How to compute (super) hedging costs in rather general financial market models with transaction costs in discrete-time ? Despite the huge literature on this topic, most of results are characterizations of the super-hedging prices while it remains difficult to deduce numerical procedure to estimate them. We establish here a dynamic programming principle and we prove that it is possible to implement it under some conditions on the conditional supports of the price and volume processes for a large class of market models including convex costs such as order books but also non convex costs, e.g. fixed cost models.

4.1 Introduction

The problem of characterizing the set of all possible prices hedging a European claim has been extensively studied in the literature under classical no-arbitrage conditions. In discrete-time and without transaction costs, a dual characterization is deduced through dual elements, the equivalent martingale measures, whose existence characterizes the well known no-arbitrage

condition NA, see the FTAP theorem of [20]. In continuous time, similar characterizations are obtained under the NFLVR condition of Delbaen and Schachermayer [21], [22] for instance. The Black and Scholes model [7] is the canonical example of complete market in mathematical finance such that the equivalent probability measure is unique. The advantage of this simple model is that hedging prices are explicitly given. Unfortunately, for incomplete market models, it is difficult to establish numerical procedures to estimate the super-hedging prices from the dual characterization. This is why it is usual to specify a particular martingale measure, see [79], [36] and [47].

In presence of transaction costs, the financial market is a priori incomplete and computing the infimum super-hedging prices remains a challenge. In the Kabanov model with transaction costs [55], the main result is a dual characterization [55][Theorem 3.3] through the so-called consistent price systems (CPS) that characterize various kinds of no-arbitrage conditions for these models, see [55][Section 3.2]. Unfortunately, it is difficult to characterize the consistent price systems and deduce a numerical estimation of the prices. A first attempt (and the only one) is proposed in [68] for finite probability space. More generally, vector optimization methods are proposed for risk measures as in [16] still for finite probability spaces. Also, various asymptotic results are obtained for small transaction costs by Schachermayer [80], [40] and others [57], [72], still for conic models.

For non conic models, in the presence of an order book for instance, more generally with convex cost, or with fixed costs, few results are available in the literature. Well known papers such as [48], [74], [73] , [65], [66] only formulate characterizations of the super-hedging prices. The very question we aim to address in this paper is how to numerically compute the infimum super-hedging cost of a European claim.

To do so, we first provide a dynamic programming principle in a very general setting in discrete time, see Theorem 4.3.1. Notice that we do not need any no-arbitrage condition to formulate it. Secondly, we propose some conditions under which it is possible to implement the dynamic programming principle. Actually, we shall see that we only need to have an insight on the conditional supports of the increments of the process describing the financial market, mainly the price and volume process.

Our main results are formulated under some weak non-arbitrage conditions such that the minimal super-hedging costs are non negative for non negative payoffs, as in [17], [5]. These conditions avoid the unrealistic case of infinitely negative prices. The main problem is how to compute an essential

supremum and an essential infimum. We show that they may coincide with pointwise supremum and infimum respectively. This is sufficient to compute backwardly the hedging costs as solutions to pointwise (random) optimization problem.

The paper is organized as follows. The financial market is defined by a cost process, which is not necessarily convex, as described in Section 4.2. Then, the dynamic programming principle is established in Section 4.3, see Theorem 4.3.1. The last Section 4.4 is devoted to the implementation of the dynamic programming principle. Precisely, we formulate results that ensure the propagation of the lower semicontinuity to the minimal hedging cost at any time, e.g. with respect to the spot price, see Theorem 4.4.5, Corollary 4.4.9, Theorem 4.4.15, Theorem 4.4.17 and Theorem 4.4.27. In Subsection 4.4.3, fixed costs models are considered. Theorem 4.4.21 also states the propagation of the lower semicontinuity that allows to numerically compute the minimal hedging cost backwardly. It is formulated under a no-arbitrage condition on the enlarged market only composed of linear transaction costs in the spirit of [65] but also [73] in the context of utility maximization.

4.2 Financial market model defined by a cost process

We consider a stochastic basis in discrete-time $(\Omega, (\mathcal{F}_t)_{t=0}^T, P)$ where the filtration $(\mathcal{F}_t)_{t=0}^T$ is complete, i.e. $\mathcal{F}_0$ contains the negligible sets for P. By convention, we also define $\mathcal{F}_{-1} := \mathcal{F}_0$. If A is a random subset of $\mathbf{R}^d$, $d \geq 1$, we denote by $L^0(A, \mathbf{R}^d)$ the family of (equivalence classes of) all random variables X (defined up to a negligible set) such that $X(\omega) \in A(\omega)$, $P(\omega)$ a.s. It is well known that, if $A(\omega) \neq \emptyset$ $\mathrm{P}(\omega)$ a.s. and if A is graph-measurable, see [70], then $L^0(A, \mathbf{R}^d) \neq \emptyset$. When using this property, we refer it by saying *by measurable selection arguments*, as it is usual to do when claiming the existence of $X \in L^0(\mathbf{R}, \mathcal{F})$ such that $X \in A$ a.s..

We also adopt the following notations. We denote by $\mathrm{int}A$ the interior of any $A \subseteq \mathbf{R}^d$ and $\mathrm{cl}A$ is its closure. The positive dual of A is defined as $A^* := \{x \in \mathbf{R}^d : \ ax \geq 0, \forall a \in A\}$ where ax designates the Euclidean scalar product of $\mathbf{R}^d$. At last, if $r \geq 0$, we denote by $\bar{B}(0, r) \subseteq \mathbf{R}^d$ the closed ball of all $x \in \mathbf{R}^d$ such that the norm satisfies $|x| \leq r$.

We consider a financial market where transaction costs are charged when

the agents buy or sell risky assets. The typical case is a model defined by a bond whose discounted price is $S^1 = 1$ and $d-1$ risky assets that may be traded at some bid and ask discounted prices S^b and S^a, respectively, when selling or buying. We refer the readers to the huge literature on models with transactions costs, in particular see [55].

Our general model is defined by a set-valued process $(\mathbf{G}_t)_{t=0}^T$ adapted to the filtration $(\mathcal{F}_t)_{t=0}^T$. Precisely, we suppose that for all $t \leq T$, $\mathbf{G}_t$ is $\mathcal{F}_t$-measurable in the sense of the graph $\mathrm{Graph}(\mathbf{G}_t) = \{(\omega, x) : \ x \in \mathbf{G}_t(\omega)\}$ that belongs to $\mathcal{F}_t \otimes \mathcal{B}(\mathbf{R}^d)$, where $\mathcal{B}(\mathbf{R}^d)$ is the Borel σ-algebra on $\mathbf{R}^d$ and $d \geq 1$ is the number of assets.

We suppose that $\mathbf{G}_t(\omega)$ is closed for every $\omega \in \Omega$ and $\mathbf{G}_t(\omega) + \mathbf{R}^d_+ \subseteq \mathbf{G}_t(\omega)$, for all $t \leq T$. The cost value process $\mathrm{C} = (\mathrm{C}_t)_{t=0}^T$ associated to $\mathbf{G}$ is defined as:

$$\mathrm{C}_t(z) = \inf\{\alpha \in \mathbf{R} : \ \alpha e_1 - z \in \mathbf{G}_t\} = \min\{\alpha \in \mathbf{R} : \ \alpha e_1 - z \in \mathbf{G}_t\}, \quad z \in \mathbf{R}^d.$$

We suppose that the right hand side in the definition above is non empty a.s. and $-e_1$ does not belong to $\mathbf{G}_t$ a.s. where $e_1 = (1, 0, \cdots, 0) \in \mathbf{R}^d$. Moreover, by assumption, $\mathrm{C}_t(z)e_1 - z \in \mathbf{G}_t$ a.s. for all $z \in \mathbf{R}^d$. Note that $\mathrm{C}_t(z)$ is the minimal amount of cash one needs to get the financial position $z \in \mathbf{R}^d$ at time t. In particular, we suppose that $\mathrm{C}_t(0) = 0$.

Similarly, we may define the liquidation value process $\mathrm{Ł} = (\mathrm{Ł}_t)_{t=0}^T$ associated to $\mathbf{G}$ as:

$$\mathrm{Ł}_t(z) := \sup\{\alpha \in \mathbf{R} : z - \alpha e_1 \in \mathbf{G}_t\}, \quad z \in \mathbf{R}^d.$$

We observe that $\mathrm{Ł}_t(z) = -\mathrm{C}_t(-z)$ and $\mathbf{G}_t = \{z \in \mathbf{R}^d : \ \mathrm{Ł}_t(z) \geq 0\}$ so that our model is equivalently defined by Ł or C. Note that $\mathbf{G}_t$ is closed if and only if $\mathrm{Ł}_t(z)$ is upper semicontinuous (u.s.c.) in z, see [65], or equivalently $\mathrm{C}_t(z)$ is lower semicontinuous (l.s.c.) in z. Naturally, $\mathrm{C}_t(z) = \mathrm{C}_t(S_t, z)$ depends on the available quantities and prices for the risky assets, described by an exogenous vector-valued $\mathcal{F}_t$-measurable random variable S_t of $\mathbf{R}^m_+$, $m \geq d$, and on the quantities $z \in \mathbf{R}^d$ to be traded. Here, we suppose that $m \geq d$ as an asset may be described by several prices and quantities offered by the market, e.g. bid and ask prices, or several pair of bid and ask prices of an order book and the associated quantities offered by the market.

In the following, we suppose the following assumptions on the cost process C. For any $t \leq T$, the cost function C_t is a lower-semi continuous Borel

function defined on $\mathbf{R}^m \times \mathbf{R}^d$ such that

$$
\begin{aligned}
&\mathrm{C}_t(s,0) = 0,\ \forall s \in \mathbf{R}^m_+,\\
&\mathrm{C}_t(s, x + \lambda e_1) = \mathrm{C}_t(s,x) + \lambda,\ \lambda \in \mathbf{R},\ x \in \mathbf{R}^d,\ s \in \mathbf{R}^m_+ \ \text{(cash invariance)},\\
&\mathrm{C}_T(s,x_2) \geq \mathrm{C}_T(s,x_1),\ \forall x_1, x_2 \text{ s.t. } x_2 - x_1 \in \mathbf{R}^d_+ \ (\mathrm{C}_T \text{ is increasing w.r.t. } \mathbf{R}^d_+),\\
&|\mathrm{C}_t(s,x)| \leq h_t(s,x),
\end{aligned}
$$

where h_t is a deterministic continuous function. Note that C_T is increasing w.r.t. $\mathbf{R}^d_+$ is equivalent to $\mathbf{G}_T + \mathbf{R}^d_+ \subseteq \mathbf{G}_T$. Moreover, if δ is an increasing bijection from $[0, +\infty]$ to $[0, +\infty]$ such that $\delta(0) = 0$ and $\delta(\infty) = \infty$, we say that C_t is positively super δ-homogeneous if the following property holds:

$$
\mathrm{C}_t(s, \lambda x) \geq \delta(\lambda)\mathrm{C}_t(s,x), \forall \lambda \geq 1,\ s \in \mathbf{R}^m_+,\ x \in \mathbf{R}^d.
$$

A classical case is when $\delta(x) = x$ and the positive homogeneous property holds, e.g. for models with proportional transaction costs, as the solvency set process $\mathbf{G}$ is a positive cone, see [55]. More generally, if $\mathrm{C}_t(s,x)$ is convex in x and $\mathrm{C}_t(s,0) = 0$, it is clear that C_t is positively super δ-homogeneous with $\delta(x) = x$. Actually, in our definition, the domain of validity $\lambda \geq 1$ may be replaced by $\lambda \geq r$ where $r > 0$ is arbitrarily chosen. In that case, all the results we formulate in this paper are still valid. We now present a typical model that satisfies our assumptions:

Example 4.2.1 (Order book)**.** Suppose that the financial market is defined by an order book. In that case, we define S_t, at any time t, as

$$
S_t = ((S_t^{b,i,j}, S_t^{a,i,j}), (N_t^{b,i,j}, N_t^{a,i,j}))_{i=1,\cdots,d, j=1,\cdots,k},
$$

where k is the order book's depth and, for each $i = 1, \cdots, d$, $S_t^{b,i,j}, S_t^{a,i,j}$ are the bid and ask prices for asset i in the j-th line of the order book and $(N_t^{b,i,j}, N_t^{a,i,j}) \in (0,\infty)^2$ are the available quantities for these bid and ask prices. We suppose that $N_t^{b,i,k} = N_t^{a,i,k} = +\infty$ so that the market is somehow liquid. By definition of the order book, we have $S_t^{b,i,1} > S_t^{b,i,2} > \cdots > S_t^{b,i,k}$ and $S_t^{a,i,1} < S_t^{a,i,2} < \cdots < S_t^{a,i,k}$. We then define the cost function as

$$
\mathrm{C}_t(x) = x^1 + \sum_{i=2}^{d} \mathrm{C}_t^i(x^i), \quad x = (x^1, \cdots, x^d) \in \mathbf{R}^d.
$$

With the convention $\sum_{r=1}^{j} = 0$ if $j = 0$, we consider the cumulated quantities $Q_t^{a,i,j} := \sum_{r=1}^{j} N_t^{a,i,r}$, $j = 0, \cdots, k$, the same for $Q_t^{b,i,j}$. We have:

$$\mathrm{C}_t^i(y) = \sum_{r=1}^{j} N_t^{a,i,r} S_t^{a,i,r} + (y - Q_t^{a,i,j}) S_t^{a,i,j+1}, \quad \text{if } Q_t^{a,i,j} < y \leq Q_t^{a,i,j+1},$$

$$\mathrm{C}_t^i(y) = -\sum_{r=1}^{j} N_t^{b,i,r} S_t^{b,i,r} + (y + Q_t^{b,i,j}) S_t^{b,i,j+1}, \quad \text{if } -Q_t^{b,i,j+1} < y \leq -Q_t^{b,i,j}.$$

Note that the first expression of $\mathrm{C}_t^i(z)$ above corresponds to the case where we buy $y > 0$ units of asset i. The second expression is $\mathrm{C}_t^i(y) = -\mathrm{L}_t^i(-y)$ when $y < 0$ so that $-\mathrm{C}_t^i(y)$ is the liquidation value of the position $-y$, i.e. by selling the quantity $-y > 0$ at the bid prices. We observe that $\mathrm{C}_t^i(y)$ is a convex function in y satisfying the cash invariance, such that $\mathrm{C}_t^i(0) = 0$ and, at last, we show that C_t^i is positively super homogeneous as defined above.

To do so, we first consider $y > 0$ and we show that $\mathrm{C}_t^i(\lambda y) \geq \lambda \mathrm{C}_t^i(y)$ for $\lambda > 1$ by induction on the interval $]Q_t^{a,i,j}, Q_t^{a,i,j+1}]$ that contains y. For $j = 1$, $\mathrm{C}_t^i(y) = S_t^{a,i,1} y$ and $\mathrm{C}_t^i(\lambda y) = \mathrm{C}_t^i(Q_t^{a,i,j_\lambda}) + (\lambda y - Q_t^{a,i,j_\lambda}) S_t^{a,i,j_\lambda+1}$ where j_λ is such that $\lambda y \in]Q_t^{a,i,j_\lambda}, Q_t^{a,i,j_\lambda+1}]$. As $S_t^{a,i,1}$ is the smallest ask price, we get that $\mathrm{C}_t^i(Q_t^{a,i,j_\lambda}) \geq Q_t^{a,i,j_\lambda} S_t^{a,i,1}$ and $(y - Q_t^{a,i,j_\lambda}) S_t^{a,i,j_\lambda+1} \geq (\lambda y - Q_t^{a,i,j_\lambda}) S_t^{a,i,1}$. We deduce that $\mathrm{C}_t^i(\lambda y) \geq \lambda y S_t^{a,i,1}$ hence $\mathrm{C}_t^i(\lambda y) \geq \lambda \mathrm{C}_t^i(y)$. More generally, if $y \in]Q_t^{a,i,j}, Q_t^{a,i,j+1}]$, $\lambda y > \lambda Q_t^{a,i,j}$ hence $\mathrm{C}_t(\lambda y) \geq \mathrm{C}_t(\lambda Q_t^{a,i,j}) + (\lambda y - \lambda Q_t^{a,i,j}) S_t^{a,i,\tilde{j}}$ where $\tilde{j}$ is such that $Q_t^{a,i,\tilde{j}} < \lambda Q_t^{a,i,j} \leq Q_t^{a,i,\tilde{j}+1}$. Indeed, the extra quantity $\lambda y - \lambda Q_t^{a,i,j}$ is bought at a price larger than or equal to the maximal ask price $S_t^{a,i,\tilde{j}}$ when buying the quantity $\lambda Q_t^{a,i,j}$. As $\lambda Q_t^{a,i,j} > Q_t^{a,i,j}$, we deduce that $\tilde{j} \geq j + 1$. Using the induction hypothesis, we have $\mathrm{C}_t^i(\lambda Q_t^{a,i,j}) \geq \lambda \mathrm{C}_t^i(Q_t^{a,i,j})$ and we deduce that

$$\mathrm{C}_t^i(\lambda y) \geq \lambda C_t^i(Q_t^{a,i,j}) + (\lambda y - \lambda Q_t^{a,i,j}) S_t^{a,i,j+1} = \lambda \mathrm{C}_t^i(y).$$

By the same reasoning, $\mathrm{L}_t^i(\lambda y) \leq \lambda \mathrm{L}_t^i(y)$ if $y > 0$ with $\mathrm{L}_t^i(y) = -\mathrm{C}_t^i(-y)$. Therefore, we also get that $\mathrm{C}_t^i(\lambda y) \geq \lambda \mathrm{C}_t^i(y)$ for $\lambda > 1$ and $y < 0$.

We finally conclude that the cost process C satisfies the conditions we impose above. In particular, notice that $\mathrm{C}_t(s, z)$ is continuous in (s, z). $\triangle$

A portfolio process is by definition a stochastic process $(V_t)_{t=-1}^{T}$ where $V_{-1} \in \mathbf{R}e_1$ is the initial endowment expressed in cash that we may convert immediately into $V_0 \in \mathbf{R}^d$ at time $t = 0$. By definition, we suppose that

$$\Delta V_t = V_t - V_{t-1} \in -\mathbf{G}_t, \; a.s., \quad t = 0, \cdots, T.$$

This means that any position $V_{t-1} = V_t + (-\Delta V_t)$ may be changed into the new position V_t, letting aside the residual part $(-\Delta V_t)$ that can be liquidated without any debt, i.e. $Ł_t(-\Delta V_t) \geq 0$.

4.3 Dynamic programming principle for pricing

Let $\xi \in L^0(\mathbf{R}^d, \mathcal{F}_T)$ be a contingent claim. Our goal is to characterize the set of all portfolio processes $(V_t)_{t=-1}^T$ such that $V_T = \xi$, as defined in the last section. We are mainly interested by the infimum cost one needs to hedge ξ, i.e. the infimum value of the initial capitals $V_{-1}e_1 \in \mathbf{R}$ among the portfolios $(V_t)_{t=-1}^T$ replicating ξ.

In the following, we use the notation $z = (z^1, z^2, ..., z^d) \in \mathbf{R}^d$ and we denote $z^{(2)} = (z^2, ..., z^d)$. We shall heavily use the notion of $\mathcal{F}_t$-measurable conditional essential supremum (resp. infimum) of a family of random variables, i.e. the smallest (resp. largest) $\mathcal{F}_t$-measurable random variable that dominates (resp. is dominated by) the family with respect to the natural order between $[-\infty, \infty]$-valued random variables, i.e. $X \leq Y$ if $P(X \leq Y) = 1$, see [55, Section 5.3.1].

4.3.1 The one step hedging problem

Recall that $V_{T-1} \geq_{\mathbf{G}_T} V_T$ by definition of a portfolio process. Then, the hedging problem $V_T = \xi$ [1] is equivalent at time $T-1$ to:

$$\begin{aligned} Ł_T(V_{T-1} - \xi) \geq 0 \quad &\Longleftrightarrow V_{T-1}^1 \geq \xi^1 - Ł_T((0, V_{T-1}^{(2)})), \\ &\Longleftrightarrow V_{T-1}^1 \geq \text{ess sup}_{\mathcal{F}_{T-1}} \left(\xi^1 - Ł_T((0, V_{T-1}^{(2)} - \xi^{(2)})) \right), \\ &\Longleftrightarrow V_{T-1}^1 \geq \text{ess sup}_{\mathcal{F}_{T-1}} \left(\xi^1 + \mathrm{C}_T((0, \xi^{(2)} - V_{T-1}^{(2)})) \right), \\ &\Longleftrightarrow V_{T-1}^1 \geq F_{T-1}^{\xi}(V_{T-1}^{(2)}), \end{aligned}$$

where

[1] The problem $V_T \geq_{\mathbf{G}_T} \xi$ is equivalent to our one if $\mathbf{G}_T + \mathbf{G}_T \subseteq \mathbf{G}_T$. In general, any V_T such that $V_T \geq_{\mathbf{G}_T} \xi$ may be changed into ξ through an additional cost. So, the formulation $V_T = \xi$ is chosen as we are interested in minimal costs.

$$F^{\xi}_{T-1}(y) \;:=\; \operatorname{ess\,sup}_{\mathcal{F}_{T-1}} \left(\xi^1 + \mathrm{C}_T((0, \xi^{(2)} - y))\right). \tag{4.3.1}$$

By virtue of Proposition 4.5.7 in Appendix, we may suppose that $F^{\xi}_{T-1}(\omega, y)$ is jointly $\mathcal{F}_{T-1} \otimes \mathcal{B}(\mathbf{R}^{d-1})$-measurable, l.s.c. as a function of y and convex if $\mathrm{C}_T(s, y)$ is convex in y. As $\mathcal{F}_{T-1}$ is supposed to be complete, we conclude that F^{ξ}_{T-1} is an $\mathcal{F}_{T-1}$ normal integrand, see Definition 4.5.1 and [76].

4.3.2 The multi-step hedging problem

We denote by $\mathcal{P}_t(\xi)$ the set of all portfolio processes starting at time $t \leq T$ that replicates ξ at the terminal date T:

$$\mathcal{R}_t(\xi) := \left\{(V_s)_{s=t}^T, -\Delta V_s \in L^0(\mathbf{G}_s, \mathcal{F}_s),\, \forall s \geq t+1, V_T = \xi\right\}.$$

The set of replicating prices of ξ at time t is

$$\mathcal{P}_t(\xi) := \left\{V_t = (V_t^1, V_t^{(2)}) : (V_s)_{s=t}^T \in \mathcal{R}_t(\xi)\right\}.$$

The infimum replicating cost is then defined as:

$$c_t(\xi) := \operatorname{ess\,inf}_{\mathcal{F}_t} \left\{\mathrm{C}_t(V_t),\, V_t \in \mathcal{P}_t(\xi)\right\}.$$

By the previous section, we know that $V_{T-1} \in \mathcal{P}_{T-1}(\xi)$ if and only if

$$V^1_{T-1} \geq \operatorname{ess\,sup}_{\mathcal{F}_{T-1}} \left(\xi^1 + \mathrm{C}_T(0, \xi^{(2)} - V^{(2)}_{T-1})\right) \text{ a.s.}$$

Similarly, $V_{T-2} \in \mathcal{R}_{T-2}(\xi)$ if and only if there exists $V^{(2)}_{T-1} \in L^0(\mathbf{R}^{d-1}, \mathcal{F}_{T-1})$ such that

$$V^1_{T-2} \geq \operatorname{ess\,sup}_{\mathcal{F}_{T-2}} \left(\operatorname{ess\,sup}_{\mathcal{F}_{T-1}} \left(\xi^1 + \mathrm{C}_T(0, \xi^{(2)} - V^{(2)}_{T-1})\right) + \mathrm{C}_{T-1}(0, V^{(2)}_{T-1} - V^{(2)}_{T-2})\right).$$

By the tower property satisfied by the conditional essential supremum, we deduce that $V_{T-2} \in \mathcal{R}_{T-2}(\xi)$ if and only if there is $V^{(2)}_{T-1} \in L^0(\mathbf{R}^{d-1}, \mathcal{F}_{T-1})$ such that

$$V^1_{T-2} \geq \operatorname{ess\,sup}_{\mathcal{F}_{T-2}} \left(\xi^1 + \mathrm{C}_T(0, \xi^{(2)} - V^{(2)}_{T-1}) + \mathrm{C}_{T-1}(0, V^{(2)}_{T-1} - V^{(2)}_{T-2})\right).$$

Recursively, we get that $V_t \in \mathcal{P}_t(\xi)$ if and only if, for some $V_s^{(2)} \in L^0(\mathbf{R}^{d-1}, \mathcal{F}_s)$, $s = t+1, \cdots, T-1$, and $V_T^{(2)} = \xi^{(2)}$, we have

$$V_t^1 \geq \text{ess sup}_{\mathcal{F}_t} \left(\xi^1 + \sum_{s=t+1}^{T} \mathrm{C}_s(0, V_s^{(2)} - V_{s-1}^{(2)}) \right).$$

In the following, for $u \leq T-1$, $\xi_{u-1} \in L^0(\mathbf{R}^d, \mathcal{F}_{u-1})$, and $\xi \in L^0(\mathbf{R}^d, \mathcal{F}_T)$, we introduce the sets

$$\Pi_u^T(\xi_{u-1}, \xi) := \{\xi_{u-1}^{(2)}\} \times \Pi_{s=u}^{T-1} L^0(\mathbf{R}^{d-1}, \mathcal{F}_s) \times \{\xi^{(2)}\}$$

of all families $(V_s^{(2)})_{s=u-1}^{t+1}$ such that $V_{u-1}^{(2)} = \xi_{u-1}^{(2)}$, $V_s^{(2)} \in L^0(\mathbf{R}^{d-1}, \mathcal{F}_s)$ for all $s = u, \cdots, T-1$ and $V_T^{(2)} = \xi^{(2)}$. We set $\Pi_u^T(\xi) := \Pi_u^T(0, \xi) = \Pi_u^T(\xi_{u-1}, \xi)$ when $\xi_{u-1}^{(2)} = 0$. When $u = T$, we set $\Pi_T^T(\xi_{T-1}, \xi) := \{\xi_{T-1}^{(2)}\} \times \{\xi^{(2)}\}$. Therefore, the infimum replicating cost at time 0 is given by

$$c_0(\xi) = \underset{V^2 \in \Pi_0^T(\xi)}{\text{ess inf}_{\mathcal{F}_0}} \text{ess sup}_{\mathcal{F}_0} \left(\xi^1 + \sum_{s=0}^{T} \mathrm{C}_s(0, V_s^2 - V_{s-1}^2) \right).$$

For $0 \leq t \leq T$ and $V_{t-1} \in L^0(\mathbf{R}^d, \mathcal{F}_t)$, we define $\gamma_t^\xi(V_{t-1})$ as:

$$\gamma_t^\xi(V_{t-1}) := \underset{V^{(2)} \in \Pi_t^T(V_{t-1}, \xi)}{\text{ess inf}_{\mathcal{F}_t}} \text{ess sup}_{\mathcal{F}_t} \left(\xi^1 + \sum_{s=t}^{T} \mathrm{C}_s(0, V_s^{(2)} - V_{s-1}^{(2)}) \right).$$

Note that $\gamma_t^\xi(V_{t-1})$ is the infimum cost to replicate the payoff ξ when starting from the initial risky position $(0, V_{t-1}^{(2)})$ at time t. Observe that $\gamma_t^\xi(V_{t-1})$ does not depend on the first component V_{t-1}^1. Moreover,

$$\gamma_T^\xi(V_{T-1}) = \xi^1 + \mathrm{C}_T(0, \xi^{(2)} - V_{T-1}^{(2)}).$$

As $\mathbf{G}_T + \mathbf{R}_+^d \subseteq \mathbf{G}_T$, we also observe that $\gamma_T^\xi(V_{T-1}) \geq \gamma_T^0(V_{T-1})$. At last, observe that $c_0(\xi) = \gamma_0^\xi(0)$. Therefore, the main goal of our paper is to study the random functions $(\gamma_t^\xi)_{t=0,1,\cdots,T}$ and to propose conditions under which it is possible to compute them backwardly so that we may estimate $c_0(\xi)$. The main contribution of this section is the following:

Theorem 4.3.1 (Dynamic Programming Principle)**.** *For any $0 \leq t \leq T-1$ and $V_{t-1} \in L^0(\mathbf{R}^d, \mathcal{F}_{t-1})$, we have*

$$\gamma_t^\xi(V_{t-1}) = \underset{V_t \in L^0(\mathbf{R}^d, \mathcal{F}_t)}{\text{ess inf}_{\mathcal{F}_t}} \text{ess sup}_{\mathcal{F}_t} \left(\mathrm{C}_t(0, V_t^{(2)} - V_{t-1}^{(2)}) + \gamma_{t+1}^\xi(V_t) \right). \quad (4.3.2)$$

Proof. We denote the right hand side of (4.3.2) by $\bar{\gamma}_t^{\xi}(V_{t-1})$. We first verify (4.3.2) for $t = T-1$. Recall that $\gamma_T^{\xi}(V_{T-1}) = \xi^1 + \mathrm{C}_T(0, \xi^{(2)} - V_{T-1}^{(2)})$ if V_{T-1} belongs to $L^0(\mathbf{R}^d, \mathcal{F}_{T-1})$. It is clear that (4.3.2) holds for $t = T-1$ by definition of $\gamma_{T-1}^{\xi}(V_{T-1})$. By induction, let us show that (4.3.2) holds at time t if this holds at time $t+1$. Let us define

$$f_t(V_{t-1}, V_t) := \mathrm{ess\,sup}_{\mathcal{F}_t} \left(\mathrm{C}_t(0, V_t^{(2)} - V_{t-1}^{(2)}) + \gamma_{t+1}^{\xi}(V_t) \right), t \leq T-1.$$

We observe that the collection of random variables

$$\Gamma_t = \{ f_t(V_{t-1}, V_t) : \ V_t \in L^0(\mathbf{R}^d, \mathcal{F}_t) \}$$

is directed downward, i.e. if $f_t^j = f_t(V_{t-1}, V_t^j) \in \Gamma_t$, $j = 1, 2$, then there exists $f_t \in \Gamma_t$ such that $f_t \leq f_t^1 \wedge f_t^2$. Indeed, to see it, it suffices to consider $f_t = f_t(V_{t-1}, V_t)$ where $V_t = V_t^1 1_{\{f_t^1 \leq f_t^2\}} + V_t^2 1_{\{f_t^1 > f_t^2\}}$. Therefore, there exists a sequence $(V_t^n)_{n \geq 1} \in L^0(\mathbf{R}^d, \mathcal{F}_t)$ such that $\bar{\gamma}_t^{\xi}(V_{t-1}) = \inf_n f_t(V_{t-1}, V_t^n)$, see [55, Section 5.3.1]. We deduce for any $\epsilon > 0$, the existence of $\tilde{V}_t \in L^0(\mathbf{R}^d, \mathcal{F}_t)$ such that $\bar{\gamma}_t^{\xi}(V_{t-1}) + \epsilon \geq f_t(V_{t-1}^{(2)}, \tilde{V}_t^{(2)})$. Similarly, by forward iteration, using the induction hypothesis $\gamma_r^{\xi}(\tilde{V}_{r-1}) = \bar{\gamma}_r^{\xi}(\tilde{V}_{r-1})$, $r \geq t+1$, we obtain the existence of $\tilde{V}_r \in L^0(\mathbf{R}^d, \mathcal{F}_r)$ such that $\gamma_r^{\xi}(\tilde{V}_{r-1}) + \epsilon \geq f_r(\tilde{V}_{r-1}^{(2)}, \tilde{V}_r^{(2)})$, for all $r = t+1, \cdots, T-1$. With $\tilde{V}_{t-1} = V_{t-1}$ and $\tilde{V}_T = \xi$, we deduce that

$$\bar{\gamma}_t^{\xi}(V_{t-1}) + \epsilon T \geq \mathrm{ess\,sup}_{F_t} \left(\xi^1 + \sum_{s=t}^{T} \mathrm{C}_s(0, \tilde{V}_s^{(2)} - \tilde{V}_{s-1}^{(2)}) \right) \geq \gamma_t^{\xi}(V_{t-1}).$$

As ϵ goes to 0, we conclude that $\bar{\gamma}_t^{\xi}(V_{t-1}) \geq \gamma_t^{\xi}(V_{t-1})$. The reverse inequality is easily obtained by induction and using the assumption that $\bar{\gamma}_r^{\xi}$ and γ_t^{ξ} coincide if $r \geq t$ with the tower property. The conclusion follows. □

4.4 Computational feasibility of the dynamic programming principle

The dynamic programming principle (4.3.2) allows to get $\gamma_t^{\xi}(V_{t-1})$ from the cost function C_t and from γ_{t+1}^{ξ}. In this section, our first main contribution is to formulate some results allowing to compute ω-wise the essential supremum and the essential infimum of (4.3.2).

As the term $\mathrm{C}_t(0, V_t^{(2)} - V_{t-1}^{(2)})$ in (4.3.2) is $\mathcal{F}_t$-measurable, it is sufficient to consider the conditional supremum

$$\theta_t^\xi(V_t) := \operatorname{ess\,sup}_{\mathcal{F}_t} \gamma_{t+1}^\xi(V_t)$$

to compute the essential supremum of (4.3.2). In the following, we shall use the following notations:

$$D_t^\xi(V_{t-1}, V_t) = \mathrm{C}_t((0, V_t^{(2)} - V_{t-1}^{(2)})) + \theta_t^\xi(V_t), \tag{4.4.3}$$

$$D_t^\xi(S_t, V_{t-1}, V_t) = \mathrm{C}_t(S_t, (0, V_t^{(2)} - V_{t-1}^{(2)})) + \theta_t^\xi(S_t, V_t). \tag{4.4.4}$$

The second notation is used when we stress the dependence on S_t.

4.4.1 Computational feasibility for convex costs

The following first result ensures the propagation of the lower semicontinuity and convexity of the random function γ_{t+1}^ξ to γ_t^ξ as we shall see in Theorem 4.4.5. This is a crucial property to compute pointwise the essential infimum in (4.3.2).

Proposition 4.4.1. *Suppose that there exists a random $\mathcal{F}_{t+1}$-measurable lower semi-continuous function $\bar{\gamma}_{t+1}^\xi$ defined on $\mathbf{R}^d$ such that $\gamma_{t+1}^\xi(V_t) = \bar{\gamma}_{t+1}^\xi(V_t)$ for all $V_t \in L^0(\mathbf{R}^d, \mathcal{F}_t)$. Then, there exists a random $\mathcal{F}_t$-measurable lower semi-continuous function $\bar{\theta}_t^\xi$ defined on $\mathbf{R}^d$ such that $\theta_t^\xi(V_t) = \bar{\theta}_t^\xi(V_t)$ for all $V_t \in L^0(\mathbf{R}^d, \mathcal{F}_t)$. Moreover, the random function $y \mapsto \bar{\theta}_t^\xi(y)$ is a.s. convex if $y \mapsto \bar{\gamma}_{t+1}^\xi(y)$ is a.s. convex.*

Proof. We consider the random function

$$f(z) = z^1 + \bar{\gamma}_{t+1}^\xi((0, z^{(2)})) = z^1 + f((0, z^{(2)})), \quad z \in \mathbf{R}^d.$$

We have $\gamma_{t+1}^\xi(V_t) = f((0, V_t^{(2)}))$ so it suffices to apply Proposition 4.5.7. □

In order to numerically compute the minimal costs, we need to impose the finiteness of $\gamma_t^\xi(V_{t-1})$, i.e. $\gamma_t^\xi(V_{t-1}) > -\infty$, at any time t, and for all $V_{t-1} \in L^0(\mathbf{R}^d, \mathcal{F}_{t-1})$. This is why we introduce the following condition:

Definition 4.4.2. *We say that the financial market satisfies the Absence of Early Profit condition (AEP) if, at any time $t \leq T$, and for all $V_t \in L^0(\mathbf{R}^d, \mathcal{F}_t)$, $\gamma_t^0(V_t) > -\infty$ a.s..*

Remark 4.4.3.
1.) Let us comment the condition AEP. Suppose that AEP does not hold, i.e. there is $V_t \in L^0(\mathbf{R}^d, \mathcal{F}_t)$ such that $\Lambda_t = \{\gamma_t^0(V_t) = -\infty\}$ satisfies $P(\Lambda_t) > 0$. Any arbitrarily chosen amount of cash $-n < 0$ allows to hedge the zero payoff at time t on Λ_t when starting from the initial position $(0, V_t^2)$ by definition of $\gamma_t^0(V_t) = -\infty$. Then, at time t, we may obtain an arbitrarily large profit on Λ_t as follows: We write $0 = ((0, V_t^2) - ne_1)\, 1_{\Lambda_t} + a_{t-1}^n$ where $a_{t-1}^n = (ne_1 - (0, V_t^2))\, 1_{\Lambda_t}$. The position $(0, V_t^2) - ne_1$ allows to get the zero claim at time T. Moreover, $Ł_t(a_{t-1}^n) = n1_{\Lambda_t} + Ł_t((0, V_t^2))1_{\Lambda_t}$ tends to $+\infty$ as $n \to \infty$ on Λ_t, i.e. it is possible to make an early profit at time t, as large as possible.
2.) If $\xi \in L^0(\mathbf{R}_+^d, \mathcal{F}_T)$, then $\gamma_t^\xi(V_{t-1}) \geq \gamma_t^0(V_{t-1}) > -\infty$ under AEP.
3.) Under Assumptions 4 and 5 below, condition AEP holds by Lemma 4.5.22. $\triangle$

Assumption 1. *The payoff ξ is hedgeable, i.e. there exists a portfolio process $(V_u^\xi)_{u=0}^T$ such that $\xi = V_T^\xi$.*

Lemma 4.4.4. *Under Assumption 1, $\gamma_t^\xi(V_{t-1}) < \infty$ for all $V_{t-1} \in L^0(\mathbf{R}^d, \mathcal{F}_t)$.*

Proof. We observe that the amount of capital $\alpha_t = \mathrm{C}_t(V_t^\xi - (0, V_{t-1}^{(2)}))$ allows one to get the position $V_t^\xi - (0, V_{t-1}^{(2)})$. Therefore, starting from the initial position $(0, V_{t-1}^{(2)})$, the capital $\mathrm{C}_t(V_t^\xi - (0, V_{t-1}^{(2)}))$ is enough to get V_t^ξ and then ξ at time T since $V_T^\xi = \xi$. We then deduce that

$$\gamma_t^\xi(V_{t-1}) \leq \alpha_t \leq h_t(S_t, V_t^\xi - (0, V_{t-1}^{(2)})) < \infty.$$

□

The following theorem states that convexity and lower semicontinuity propagates backwardly from γ_{t+1}^ξ to γ_t^ξ.

Theorem 4.4.5. *Suppose that Assumption 1 and condition AEP hold. Suppose that there exists a random $\mathcal{F}_{t+1}$-normal convex integrand $\bar{\gamma}_{t+1}^\xi$ defined on $\mathbf{R}^d$ such that $\gamma_{t+1}^\xi(V_t) = \bar{\gamma}_{t+1}^\xi(V_t)$ for all $V_t \in L^0(\mathbf{R}^d, \mathcal{F}_t)$. Suppose that the cost function $\mathrm{C}_t(s, z)$ is convex in z. Then, there exists a random $\mathcal{F}_t$-normal convex integrand $\bar{\gamma}_t^\xi$ defined on $\mathbf{R}^d$ such that $\gamma_t^\xi(V_{t-1}) = \bar{\gamma}_t^\xi(V_{t-1})$ for all $V_{t-1} \in L^0(\mathbf{R}^d, \mathcal{F}_t)$ and we have:*

$$\bar{\gamma}_t^\xi(v_{t-1}) = \inf_{y \in \mathbf{R}^d} \left(\mathrm{C}_t(0, y^{(2)} - v_{t-1}^{(2)}) + \bar{\theta}_t^\xi(y) \right),$$

where $\bar{\theta}_t^\xi$ is given by Proposition 4.4.1. In particular, $\bar{\gamma}_t^\xi(\omega, .) \in \mathbf{R}$ a.s. thus continuous a.s.

Proof. By Proposition 4.4.1, we deduce that $\theta_t^\xi(V_t) = \bar{\theta}_t^\xi(V_t)$ a.s. for every $V_t \in L^0(\mathbf{R}^d, \mathcal{F}_t)$ where $\bar{\theta}_t^\xi$ is an $\mathcal{F}_t$-normal convex integrand. Therefore, $\bar{D}_t(v_{t-1}, v_t) := \mathrm{C}_t(0, v_t^{(2)} - v_{t-1}^{(2)}) + \bar{\theta}_t^\xi(v_t)$ is an $\mathcal{F}_t$-normal integrand, convex in (v_{t-1}, v_t). By Lemma 4.5.5, we have $\bar{\gamma}_t^\xi(V_{t-1}) = \gamma_t^\xi(V_{t-1})$ a.s. for any $V_{t-1} \in L^0(\mathbf{R}^d, \mathcal{F}_t)$.

We claim that the function defined by $\bar{\gamma}_t^\xi(v_{t-1})$ is $\mathcal{F}_t \otimes \mathcal{B}(\mathbf{R}^d)$-measurable. Indeed, since $\bar{D}_t$ is convex and admits finite values in $\mathbf{R}$, we necessarily have $\inf_{v_t \in \mathbf{R}^d} \bar{D}_t(v_{t-1}, v_t) = \inf_{v_t \in \mathbf{Q}^d} \bar{D}_t(v_{t-1}, v_t)$, the measurability then follows. Next, we show that $\bar{\gamma}_t^\xi(\omega, \cdot) \in \mathbf{R}$ a.s. First, $\bar{\gamma}_t^\xi(\omega, \cdot) > -\infty$ a.s.. Otherwise, by a measurable selection argument, we may find an $\mathcal{F}_t$-measurable selection V_{t-1} such that $-\infty = \bar{\gamma}_t^\xi(V_{t-1}) = \gamma_t^\xi(V_{t-1})$ on a non null set. This is in contradiction with the AEP condition. Similarly, by Lemma 4.4.4, we deduce that $\bar{\gamma}_t^\xi(\omega, \cdot) < \infty$ a.s.. Therefore, the random function $\bar{\gamma}_t^\xi(\omega, \cdot)$ only takes finite values a.s.

We finally conclude that $\bar{\gamma}_t^\xi(v_{t-1})$ is a real-valued random convex function. In particular, $\bar{\gamma}_t^\xi$ is continuous. □

Remark 4.4.6. Suppose that the cost functions $\mathrm{C}_t(s, z)$, $t \leq T$, are convex in z. Under Assumption 1, as $\gamma_T^\xi(V_{T-1}) = \xi^1 + \mathrm{C}_T(0, \xi^{(2)} - V_{T-1}^{(2)}))$ is l.s.c. and convex in V_{T-1}, we deduce that Theorem 4.4.5 applies backwardly step by step. In particular, it is possible to compute $\gamma_t^\xi(v_{t-1})$ at any time t as a ω-wise infimum. △

In the following, we consider conditions under which it is possible to compute ω-wise the essential supremum θ_t^ξ. The main ingredient is the knowledge of the conditional support $\mathrm{supp}_{\mathcal{F}_t} S_{t+1}$ of S_{t+1} knowing $\mathcal{F}_t$. Recall that $\mathrm{supp}_{\mathcal{F}_t} S_{t+1}$ is the smallest $\mathcal{F}_t$-measurable random closed set that contains $S_{t+1}(\omega)$ a.s., see [31].

Assumption 2. *For each $t \leq T-1$, there exists a family of Borel functions $(\alpha_t^m)_{m \geq 1}$ defined on $\mathbf{R}^m$ such that $\mathrm{supp}_{\mathcal{F}_t} S_{t+1}$ admits the Castaing representation $(\alpha_t^m(S_t))_{m \geq 1}$, i.e. $\mathrm{supp}_{\mathcal{F}_t} S_{t+1} = \mathrm{cl}(\alpha_t^m(S_t))_{m \geq 1}$.*

Proposition 4.4.7. *Suppose that there exists a lower semi-continuous function $\tilde{\gamma}_{t+1}^\xi$ defined on $\mathbf{R}^m \times \mathbf{R}^d$ such that $\gamma_{t+1}^\xi(V_t) = \tilde{\gamma}_{t+1}^\xi(S_{t+1}, V_t)$ for all $V_t \in L^0(\mathbf{R}^d, \mathcal{F}_t)$. Then, $\theta_t^\xi(V_t) = \sup_{z \in supp_{\mathcal{F}_t} S_{t+1}} \tilde{\gamma}_{t+1}^\xi(z, V_t)$. Moreover, under*

Assumption 2, there exists a function $\tilde{\theta}_t^\xi(s,v)$ *defined on* $(s,v) \in \mathbf{R}^m \times \mathbf{R}^d$, *which is l.s.c. in* v, *such that* $\theta_t^\xi(V_t) = \tilde{\theta}_t^\xi(S_t, V_t)$ *for all* $V_t \in L^0(\mathbf{R}^d, \mathcal{F}_t)$ *and we have:*

$$\tilde{\theta}_t^\xi(s,v) := \sup_m \tilde{\gamma}_{t+1}^\xi(\alpha_m(s), v) \quad (s,v) \in \mathbf{R}^m \times \mathbf{R}^d.$$

At last, $\tilde{\theta}_t^\xi(s,v)$ *is l.s.c. in* (s,v) *if the functions* $(\alpha_m)_{m\geq 1}$ *are continuous and, if* $\tilde{\gamma}_{t+1}^\xi(s,v)$ *is convex in* v, *then* $\tilde{\theta}_t^\xi(s,v)$ *is convex in* v.

Proof. The proof is immediate by Proposition 4.5.6 and Lemma 4.5.8. □

Assumption 3. *For each* $t \leq T-1$, *there exists a family of Borel functions* $(\alpha_t^m)_{m\geq 1}$ *such that* $S_{t+1} \in \{\alpha_t^m(S_t) : m \geq 1\}$ *a.s. and* $\mathrm{P}(S_{t+1} = \alpha_t^m(S_t)|\mathcal{F}_t) > 0$ *a.s. for all* $m \geq 1$.

Proposition 4.4.8. *Suppose that there exists a Borel function* $\tilde{\gamma}_{t+1}^\xi$ *defined on* $\mathbf{R}^m \times \mathbf{R}^d$ *such that* $\gamma_{t+1}^\xi(V_t) = \tilde{\gamma}_{t+1}^\xi(S_{t+1}, V_t)$ *for all* $V_t \in L^0(\mathbf{R}^d, \mathcal{F}_t)$. *Then, under Assumption 3 , there exists a Borel function* $\tilde{\theta}_t^\xi(s,v)$ *defined on* $(s,v) \in \mathbf{R}^m \times \mathbf{R}^d$ *such that* $\theta_t^\xi(V_t) = \tilde{\theta}_t^\xi(S_t, V_t)$ *for all* $V_t \in L^0(\mathbf{R}^d, \mathcal{F}_t)$ *and we have:*

$$\tilde{\theta}_t^\xi(s,v) := \sup_m \tilde{\gamma}_{t+1}^\xi(\alpha_m(s), v) \quad (s,v) \in \mathbf{R}^m \times \mathbf{R}^d.$$

Proof. The proof is immediate by Lemma 4.5.19. Note that we do not suppose that C_t is convex to obtain this result. □

Corollary 4.4.9. *Assume that the assumptions of Proposition 4.4.7 or Proposition 4.4.8 hold and Condition AEP holds. Suppose that* $\tilde{\gamma}_{t+1}^\xi(s,v)$ *is convex in* v. *Then,* $\gamma_t^\xi(V_{t-1}) = \tilde{\gamma}_t^\xi(S_t, V_{t-1})$ *where* $\tilde{\gamma}_t^\xi(s,v)$ *is an* $\mathcal{F}_t$*-normal integrand, convex in* v. *Moreover,*

$$\tilde{\gamma}_t^\xi(s,v) = \inf_{y\in\mathbf{R}^d} \left(\mathrm{C}_t(s, (0, y^{(2)} - v^{(2)})) + \sup_m \tilde{\gamma}_{t+1}^\xi(\alpha_m(s), y) \right).$$

Proof. Under our assumptions, $\theta_t^\xi(V_t) = \tilde{\theta}_t^\xi(S_t, V_t)$ for all $V_t \in L^0(\mathbf{R}^d, \mathcal{F}_t)$ where $\tilde{\theta}_t^\xi(s,v) = \sup_m \tilde{\gamma}_{t+1}^\xi(\alpha_m(s), v)$ by Proposition 4.4.7 or Proposition 4.4.8. As a supremum, $\tilde{\theta}_t^\xi(s,v)$ is convex in v if $\tilde{\gamma}_{t+1}^\xi(s,v)$ is. As $\mathrm{C}_t(s,y)$ is also convex in y, we deduce that $D_t^\xi(y,v) = \mathrm{C}_t(s, (0, y^{(2)} - v^{(2)})) + \tilde{\theta}_t^\xi(s,y)$ is convex in (y,v). Now, by arguing similarly to the proof of Theorem 4.4.5, under AEP, $\tilde{\gamma}_t^\xi(v_{t-1})$ is a real- valued convex function a.s. □

4.4.2 Computational feasibility under strong AIP no-arbitrage condition

The results of Section 4.4.1 are not a priori sufficient to compute backwardly θ^{ξ}_{t-1} as we need $\gamma^{\xi}_t(s,v)$ be l.s.c. in s, see Proposition 4.4.7. This is why, we introduce the following conditions.

Assumption 4. *The payoff function ξ is of the form $\xi = g(S_T)$, where $g \in \mathbf{R}^d_+$ is continuous. Moreover, ξ is hedgeable, i.e. there exists a portfolio process $(V^{\xi}_u)_{u=0}^T$ such that $\xi = V^{\xi}_T$.*

Assumption 5. *The conditional support is such that $\mathrm{supp}_{\mathcal{F}_t} S_{t+1} = \phi_t(S_t)$ where ϕ_t is a set-valued lower hemicontinuous function, see Definition 4.5.11, with compact values such that $\phi_t(S_t) \subseteq \bar{B}(0, R_t(S_t))$ where R_t is a continuous function on $\mathbf{R}^m$.*

Note that under Assumption 2, $\phi_t(S_t) = \mathrm{cl}\{\alpha_m(S_t) : \ m \geq 1\}$ defines a set-valued lower hemicontinuous function if the functions $(\alpha_m)_{m\geq 1}$ are continuous, see Lemma 4.5.15.

Definition 4.4.10. *We say that the condition AIP holds at time t if the minimal cost $c_t(0) = \gamma^0_t(0)$ of the European zero claim $\xi = 0$ is 0 at time $t \leq T$. We say that AIP holds if AIP holds at any time.*

The condition AIP has been introduced for the first time in the paper [5]. This is a weak no-arbitrage condition which is clearly satisfied in the real financial markets i.e. the price of a non negative payoff is non negative.

Lemma 4.4.11. *Suppose that the cost functions are either sub-additive or super-additive. Then, AIP implies AEP.*

Proof. We prove it in the case where the cost function is sub-additive, the supper-additive case is similar. Suppose that AIP holds and $\mathrm{C}_t(s,v)$ is sub-additive in v. For any $V_t, \tilde{V}_t \in L^0(\mathbf{R}^d, \mathcal{F}_t)$, we have:

$$\begin{aligned} D^0_t(S_t, V_t, \tilde{V}_t) &= \mathrm{C}_t(S_t, \tilde{V}_t - V_t) + \theta^0_t(S_t, \tilde{V}_t), \\ &\geq \mathrm{C}_t(S_t, \tilde{V}_t) + \theta^0_t(S_t, \tilde{V}_t) - \mathrm{C}_t(S_t, V_t), \\ &= D^0_t(S_t, 0, \tilde{V}_t) - \mathrm{C}_t(S_t, V_t). \end{aligned}$$

Under AIP, $D^0_t(S_t, 0, \tilde{V}_t) \geq 0$ hence $D^0_t(S_t, V_t, \tilde{V}_t) \geq -\mathrm{C}_t(S_t, V_t)$. We deduce that $\gamma^0_t(V_t) = \mathrm{ess\ inf}_{\tilde{V}_t} D^0_t(S_t, V_t, \tilde{V}_t) \geq -\mathrm{C}_t(S_t, V_t) > -\infty$. □

Definition 4.4.12. *We say that the condition SAIP (Strong AIP condition) holds at time t if AIP holds at time t and, for any $Z_t \in L^0(\mathbf{R}^d, \mathcal{F}_t)$, we have $D^0_t(S_t, 0, Z_t) = 0$ if and only if $Z^{(2)}_t = 0$ a.s.. We say that SAIP holds if SAIP holds at any time.*

Recall that $D^0_t(S_t, 0, Z_t)$ is given by (4.4.4) and it is the minimal cost expressed in cash that is needed at time t to hedge the zero payoff when we start from the initial strategy $V_t = (\theta^0_t(Z_t), Z^{(2)}_t)$, initial value of a portfolio process $(V_u)_{t\leq u\leq T}$ such that $V_T = 0$. Therefore, the condition SAIP states that the minimal cost of the zero payoff is 0 at time t and this minimal cost is only attained by the zero strategy $V_t = 0$. This is intuitively clear as soon as any non null transaction implies positive costs.

The following proposition shows that the classical Robust No Arbitrage NA^{r} ([55, Chapter 3]) used to characterize the super hedging prices in the Kabanov model with proportional transaction costs is stronger than the SAIP condition.

Proposition 4.4.13. *Suppose that* $\operatorname{int} \mathbf{G}^*_{\mathrm{t}} \neq \emptyset$ *for any* $t \leq T$. *Then,* NA^{r} *implies SAIP.*

Proof. Recall that NA^{r} is equivalent to the existence of a martingale $(K_s)_{s\leq T}$ such that $K_s \in \operatorname{int} \mathbf{G}^*_{\mathrm{s}}$, [55, Theorem 3.2.1]. Consider $Z_{T-1} \in L^0(\mathbf{R}^d, \mathcal{F}_{T-1})$. As $D_{T-1}(0, Z_{T-1}) = D_{T-1}(0, (0, Z^{(2)}_{T-1}))$, we may suppose that $Z_{T-1} = (0, Z^{(2)}_{T-1})$. By the definition of C_u, there exists $\tilde{g}_u \in L^0(\mathbf{G}_u, \mathcal{F}_u)$, $u = T-1, T$, such that:

$$\mathrm{C}_{T-1}((0, Z^{(2)}_{T-1}))e^1 - g_{T-1} = (0, Z^{(2)}_{T-1})$$
$$\mathrm{C}_T((0, -Z^{(2)}_{T-1}))e^1 - \tilde{g}_T = (0, -Z^{(2)}_{T-1}).$$

Adding these equalities, we get that $D_{T-1}(0, Z_{T-1})e^1 = g_{T-1} + g_T$ for some $g_T \in L^0(\mathbf{G}_T, \mathcal{F}_T)$, see (4.4.3). So, we get that $K_T D_{T-1}(0, Z_{T-1})e^1 \geq K_T g_{T-1}$ and, taking the generalized conditional expectation w.r.t $\mathcal{F}_{T-1}$, we deduce that $K_{T-1} D_{T-1}(0, Z_{T-1})e^1 \geq K_{T-1} g_{T-1} \geq 0$. Since $K_{T-1}e^1 = K^1_{T-1} > 0$, AIP holds at time $T-1$. Moreover, $g_{T-1} \neq 0$ a.s. as soon as $Z^{(2)}_{T-1} \neq 0$. Since $K_{T-1} \in \operatorname{int} \mathbf{G}^*_{T-1}$, we finally deduce that

$$K_{T-1} D^0_{T-1}(S_t, 0, Z_{T-1})e^1 \geq K_{T-1} g_{T-1} > 0$$

as soon as $Z^{(2)}_{T-1} \neq 0$, which means that SAIP holds at time $T-1$.

Suppose that we have already shown SAIP for $s \geq t+1$. For a given $Z_t \in L^0(\mathbf{R}^d, \mathcal{F}_t)$, we consider $g_t \in L^0(\mathbf{G}_t, \mathcal{F}_t)$ such that

$$\mathrm{C}_t((0, Z_t^{(2)}))e^1 - g_t = (0, Z_t^{(2)}). \tag{4.4.5}$$

Since AIP holds at time $t+1$, by Lemma 4.4.11, we have $\gamma_{t+1}(Z_t) > -\infty$ under AEP. Since the family $\{D_{t+1}^0(Z_t, Z_{t+1}), Z_{t+1} \in L^0(\mathbf{R}^d, \mathcal{F}_{t+1})\}$ is directed downward, we deduce the existence of a sequence $Z_{t+1}^n \in L^0(\mathbf{R}^d, \mathcal{F}_{t+1})$, $n \in \mathbb{N}$ such that

$$\gamma_{t+1}^0(Z_t) = \operatorname{ess\,inf}_{Z_{t+1} \in L^0(\mathbf{R}^d, \mathcal{F}_{t+1})} D_{t+1}^0(Z_t, Z_{t+1}) = \inf_n D_{t+1}^0(Z_t, Z_{t+1}^n) > -\infty \text{ a.s.}$$

We deduce that, for any $\epsilon > 0$, there exists $Z_{t+1}^\epsilon \in L^0(\mathbf{R}^d, \mathcal{F}_{t+1})$ such that $\gamma_{t+1}^0(Z_t) + \epsilon \geq D_{t+1}^0(Z_t, Z_{t+1}^\epsilon)$. Proceeding forward with the induction hypothesis, we construct a sequence $g_s^\epsilon \in L^0(\mathbf{G}_s, \mathcal{F}_s), s \geq t+1$, such that

$$(D_t^0(0, Z_t) + \epsilon T)e^1 = g_t + \sum_{s=t+1}^{T} g_s^\epsilon.$$

Therefore, multiplying by $K_T \in \mathbf{G}_T^*$ and then taking the (generalized) conditional expectation knowing $\mathcal{F}_{T-1}$, we get that

$$\begin{aligned} K_T(D_t^0(0, Z_t) + \epsilon T)e^1 &\geq K_T \left(g_t + \sum_{s=t+1}^{T-1} g_s^\epsilon \right), \\ K_{T-1}(D_t^0(0, Z_t) + \epsilon T)e^1 &\geq K_{T-1} \left(g_t + \sum_{s=t+1}^{T-1} g_s^\epsilon \right). \end{aligned}$$

By successive iterations, we finally get that $K_t(D_t^0(0, Z_t) + \epsilon T)e^1 \geq K_t g_t$. Since g_t does not depend on ϵ, see its definition in (4.4.5), we deduce as $\epsilon \to 0$, that $K_t D_t^0(0, Z_t)e^1 \geq K_t g_t \geq 0$ and $K_t D_t^0(0, Z_t)e^1 > 0$ if $g_t \neq 0$ when $Z_t^{(2)} \neq 0$. Therefore, SAIP holds at time t and we may conclude. □

The following result is the last main contribution of this section: It states that the minimal cost function γ_t^ξ is a l.s.c. function of S_t and V_{t-1}, i.e. γ_t^ξ inherits from the lower-semicontinuity of γ_{t+1}^ξ, under Assumption 4 and 5, if SAIP holds as we shall see. We introduce the notation

$$S^{d-1}(0, 1) = \{z \in \mathbf{R}^d : \ z^1 = 0 \,\text{and}\, |z| = 1\}.$$

The following Lemma will be used in our next Theorem.

Lemma 4.4.14. *We denote* $\mathrm{cl}f$ *the l.s.c. regularization of the function* $f : \mathbf{R}^k \to \mathbf{R}$ *(i.e. the greatest l.s.c. function dominated by* f*). Suppose that* f *is l.s.c. on some open set* $O \subset \mathbf{R}^k$*, then* $f(\bar{x}) = \mathrm{cl}f(\bar{x})$ *for any* $\bar{x} \in O$*.*

Proof. We define $g(x) := \mathrm{cl}f(x)1_{O^c}(x) + f(x)1_O(x)$. As $\mathrm{cl}f \leq f$ and O is open, we deduce that g is l.s.c. and $g \leq f$. By definition of $\mathrm{cl}f$, we have $g \leq \mathrm{cl}f$. This implies that $f(\bar{x}) \leq \mathrm{cl}f(\bar{x}) \leq f(\bar{x})$ for any $\bar{x} \in O$. The conclusion follows. □

Theorem 4.4.15. *Suppose that* C_t *is positively super* δ*-homogeneous. Suppose that there exists a* $\mathcal{F}_{t+1}$*-normal integrand* $\tilde{\gamma}^{\xi}_{t+1}$ *defined on* $\Omega \times \mathbf{R}^m \times \mathbf{R}^d$ *such that* $\gamma^{\xi}_{t+1}(V_t) = \tilde{\gamma}^{\xi}_{t+1}(S_{t+1}, V_t)$ *for all* $V_t \in L^0(\mathbf{R}^d, \mathcal{F}_t)$*. Assume that Assumption 4 and Assumption 5 hold. Suppose that the cost function* $\mathrm{C}_t(s, z)$ *is* $\mathcal{F}_t$*-normal integrand and* C_t *is either super-additive or sub-additive. Then, if* $\inf_{z \in S^{d-1}(0,1)} D^0_t(S_t, 0, z) > 0$*,* $\gamma^{\xi}_t(V_{t-1}) = \tilde{\gamma}^{\xi}_t(S_t, V_{t-1})$ *where* $\tilde{\gamma}^{\xi}_t(s, v_{t-1})$ *is* $\mathcal{F}_t$*-normal integrand.*

Proof. Since $\tilde{\gamma}^{\xi}_{t+1}(s, v)$ is l.s.c. in s, we deduce that $\theta^{\xi}_t(V_t) = \tilde{\theta}^{\xi}_t(S_t, V_t)$ by Proposition 4.5.6, for all $V_t \in L^0(\mathbf{R}^d, \mathcal{F}_t)$, where by Assumption 5

$$\tilde{\theta}^{\xi}_t(s, v) = \sup_{z \in \phi_t(S_t)} \tilde{\gamma}^{\xi}_{t+1}(z, v).$$

As ϕ_t is lower hemicontinuous by assumption, we deduce by [1, Lemma 17.29] that $\tilde{\theta}^{\xi}_t(s, v)$ is l.s.c. in (s, v). Therefore, the function

$$D^{\xi}_t(s, v_{t-1}, v_t) = \mathrm{C}_t(s, (0, v^{(2)}_t - v^{(2)}_{t-1})) + \tilde{\theta}^{\xi}_t(s, v_t)$$

is l.s.c. in (s, v_{t-1}, v_t) by assumption on C_t. By Lemma 4.5.5, we get that $\gamma^{\xi}_t(V_{t-1}) = \tilde{\gamma}^{\xi}_t(S_t, V_{t-1})$ where $\tilde{\gamma}^{\xi}_t(s, v_{t-1}) = \inf_{v_t \in \mathbf{R}^d} D^{\xi}_t(s, v_{t-1}, v_t)$. The next step is to show that $\tilde{\gamma}^{\xi}_t(s, v_{t-1}) = \inf_{v_t \in \tilde{\phi}_t(s, v_{t-1})} D^{\xi}_t(s, v_{t-1}, v_t)$ where $\tilde{\phi}_t$ is a set-valued upper hemicontinuous function, see Definition 4.5.10, with compact values. We then conclude that $\tilde{\gamma}^{\xi}_t(s, v_{t-1})$ is l.s.c. in (s, v_{t-1}) by Proposition 4.5.17.

To obtain $\tilde{\phi}_t$, first observe that $\gamma^{\xi}_t(V_{t-1}) \leq D^{\xi}_t(s, v_{t-1}, 0)$ hence we get that $\gamma^{\xi}_t(V_{t-1}) = \tilde{\gamma}^{\xi}_t(S_t, V_{t-1})$ where $\tilde{\gamma}^{\xi}_t(s, v_{t-1}) = \inf_{v_t \in K_t(s, v_{t-1})} D^{\xi}_t(s, v_{t-1}, v_t)$ and

$$K_t(s, v_{t-1}) = \left\{ v_t \in \mathbf{R}^d : \ D^{\xi}_t(s, v_{t-1}, v_t) \leq D^{\xi}_t(s, v_{t-1}, 0) \right\}.$$

Since C_T is increasing w.r.t. $\mathbf{R}^d_+$, we deduce that $D^\xi_t(s, v_{t-1}, v_t) \geq D^0_t(s, v_{t-1}, v_t)$. Moreover,

$$D^0_t(s, v_{t-1}, v_t) = \mathrm{C}_t(s, (0, v^{(2)}_t - v^{(2)}_{t-1})) + \theta^0_t(s, v_t) \geq \mathrm{C}_t(s, (0, -v^{(2)}_{t-1})) + D^0_t(s, 0, v_t)$$

in the case where C_t is super-additive and, if C_t is sub-additive, we have

$$D^0_t(s, v_{t-1}, v_t) = \mathrm{C}_t(s, (0, v^{(2)}_t - v^{(2)}_{t-1})) + \theta^0_t(s, v_t) \geq -\mathrm{C}_t(s, (0, v^{(2)}_{t-1})) + D^0_t(s, 0, v_t).$$

As C_t is dominated by a continuous function by hypothesis, we get that $D^0_t(s, v_{t-1}, v_t) \geq \tilde{h}_t(s, v_{t-1}) + D^0_t(s, 0, v_t)$ where $\tilde{h}_t$ is a continuous function. Moreover, by Lemma 4.5.20, if $|v_t| \geq 1$,

$$D^0_t(s, 0, v_t) \geq \delta(|v_t|) D^0_t(s, 0, v_t/|v_t|) \geq \delta(|v_t|) \inf_{z \in S^{d-1}(0,1)} D^0_t(s, 0, z). \quad (4.4.6)$$

By Lemma 4.5.21, $|D^\xi_t(s, v_{t-1}, 0)| \leq \hat{h}^\xi_t(s, v_{t-1})$ for some continuous function $\hat{h}^\xi_t \geq 0$. Recall that $\inf_{z \in S^{d-1}(0,1)} D^0_t(S_t, 0, z) > 0$ a.s. by assumption. It follows that $K_t(s, v_{t-1}) \subseteq \tilde{\phi}_t(s, v_{t-1}) := \bar{B}_t(0, r_t(s, v_{t-1}) + 1)$ where

$$\begin{aligned} r_t(s, v_{t-1}) &:= \delta^{-1}\left(\frac{\lambda_t(s, v_{t-1})}{i_t(s)}\right), \\ i_t(s) &:= \inf_{z \in S^{d-1}(0,1)} D^0_t(s, 0, z),\ \lambda_t(s, v_{t-1}) = |\tilde{h}_t(s, v_{t-1})| + \hat{h}^\xi_t(s, v_{t-1}). \end{aligned}$$

Since λ_t is continuous and i_t is l.s.c. by Proposition 4.5.17, we deduce that λ_t / i_t is u.s.c. on the open set $\mathcal{O}_t := \{(s, v_{t-1}) \in \mathbf{R}^m \times \mathbf{R}^d : \ i_t(s, v_{t-1}) > 0\}$. As δ^{-1} is continuous and increasing, we finally get that r_t is also u.s.c. in $(s, v_{t-1}) \in \mathcal{O}_t$. By Lemma 4.5.12, we deduce that the function $\tilde{\phi}_t$ is upper hemicontinuous in $(s, v_{t-1}) \in \mathcal{O}_t$. Therefore, $\tilde{\gamma}^\xi_t(s, v_{t-1}) = \inf_{v_t \in \tilde{\phi}_t(s, v_{t-1})} D^\xi_t(s, v_{t-1}, v_t)$ is l.s.c. on $\mathcal{O}_t$ by Proposition 4.5.17. Observe that $(S_t, z) \in \mathcal{O}_t$ a.s. for all $z \in S(0, 1)$ a.s. under our hypothesis.

Consider the mapping $p^\xi_t(s, v_{t-1}) := \inf_{v_t \in \mathbf{R}^d} D^\xi_t(s, v_{t-1}, v_t)$ and its l.s.c. regularization $\mathrm{cl}(p^\xi_t)(s, v_{t-1})$. Since D^ξ_t is $\mathcal{F}_t$-normal integrand by our assumption, we deduce by [76, Theorem 14.47] that $\mathrm{cl}(p^\xi_t)(s, v_{t-1})$ is $\mathcal{F}_t$-normal integrand. Moreover, we know that on the open set $\mathcal{O}_t$, $\tilde{\gamma}^\xi_t(s, v_{t-1})$ is l.s.c. hence coincides with $\mathrm{cl}(p^\xi_t)(s, v_{t-1})$ by Lemma 4.4.14. Therefore, we deduce that $\mathrm{cl}(p^\xi_t)(S_t, v_{t-1}) = \tilde{\gamma}^\xi_t(S_t, v_{t-1})$ a.s.. The conclusion follows.

□

The following result asserts that the SAIP condition and the condition $\inf_{z\in S^{d-1}(0,1)} D_t^0(S_t,0,z)>0$, both with AIP, are actually equivalent.

Theorem 4.4.16. *Assume that Assumption 4 holds. Suppose that either Assumption 5 holds or the cost functions $\mathrm{C}_t(s,z)$ are convex in z. Suppose that the cost functions $\mathrm{C}_t(s,z)$ are l.s.c. in (s,z) and $\mathrm{C}_t(s,z)$ are either super-additive or sub-additive, for any $t\leq T$. Then, the following statements are equivalent:*

1.) SAIP.

2.) AIP holds and $\inf_{z\in S^{d-1}(0,1)} D_t^0(S_t,0,z)>0$ a.s..

Proof. Let us show that 1.) implies 2.). Suppose first that Assumption 5 holds. As $\gamma_T^0(Z_T)=\mathrm{C}_T(0,-Z_T^{(2)})$ is $\mathcal{F}_T$-normal integrand, we deduce by Proposition 4.4.1 that $\theta_{T-1}^0(Z_{T-1})$ is $\mathcal{F}_{T-1}$-normal integrand. Therefore, the function $D_{T-1}^0(S_{T-1},Z_{T-2},Z_{T-1})$ is $\mathcal{F}_{T-1}$-normal integrand. Then by lower-semicontinuity on the compact set $S^{d-1}(0,1)$ and by a measurable selection argument, there exists $\hat{Z}_{T-1}\in L^0(\mathbf{R}^d,\mathcal{F}_{T-1})$ such that

$$\inf_{z\in S^{d-1}(0,1)} D_{T-1}^0(S_{T-1},0,z)=D_{T-1}^0(S_{T-1},0,\hat{Z}_{T-1}).$$

Moreover, $D_{T-1}^0(S_{T-1},0,\hat{Z}_{T-1})>0$, i.e. $\inf_{z\in S^{d-1}(0,1)} D_{T-1}^0(S_{T-1},0,z)>0$ under SAIP. By Theorem 4.4.15, we deduce that $\gamma_{T-1}^0(S_{T-1},Z_{T-2})$ is $\mathcal{F}_{T-1}$-normal integrand. By Proposition 4.4.1, we deduce that $\theta_{T-2}^0(Z_{T-2})$ is $\mathcal{F}_{T-1}$-normal integrand. Therefore, $D_{T-2}^0(S_{T-2},Z_{T-3},Z_{T-2})$ is $\mathcal{F}_{T-2}$-normal integrand and, as previously, we deduce that $\inf_{z\in S^{d-1}(0,1)} D_{T-2}^0(S_{T-2},0,z)>0$ under SAIP. Then, we may proceed by induction by virtue of Theorem 4.4.15 and Proposition 4.4.1.

At last, if the cost functions are convex, recall that AEP holds by Lemma 4.4.11. Then, it suffices to apply Theorem 4.4.5 and Proposition 4.4.1 to deduce that for fixed $S_t\in L^0(\mathbf{R}^d,\mathcal{F}_t)$, $D_t^0(S_t,0,z)$ is $\mathcal{F}_t$-normal integrand as a function of z so that we may conclude similarly.

Let us show that 2.) implies 1.) Suppose that $D_t^0(S_t,0,Z_t)=0$ for some $Z_t\in L^0(\mathbf{R}^d\setminus\{0\},\mathcal{F}_t)$. By Lemma 4.5.20,

$$D_t^0(S_t,0,Z_t)\geq\delta(|Z_t|)D_t^0(S_t,0,Z_t/|Z_t|)\geq\delta(|Z_t|)\inf_{z\in S^{d-1}(0,1)} D_t^0(S_t,0,z)>0.$$

This yields a contradiction hence the conclusion follows under Assumption 5. □

We then conclude that, under SAIP, the dynamic programming principle allows to compute $\tilde{\gamma}_t^\xi$ backwardly so that it is possible to deduce the minimal hedging price $c_0(\xi) = \gamma_0^\xi(0)$.

Theorem 4.4.17. *Assume that Assumption 4 and Assumption 5 hold. Suppose that the cost functions are normal integrands and either super-additive of sub-additive. Then, under the condition SAIP, there exists $\mathcal{F}_t$-normal integrand $\tilde{\gamma}_t^\xi$ defined on $\Omega \times \mathbf{R}^m \times \mathbf{R}^m$ such that, for all $V_{t-1} \in L^0(\mathbf{R}^d, \mathcal{F}_{t-1})$, we have $\gamma_t^\xi(V_{t-1}) = \tilde{\gamma}_t^\xi(S_t, V_{t-1})$. Moreover, the dynamic programming principle 4.3.2 is computable ω-wise as:*

$$\gamma_t^\xi(S_t, V_{t-1}) = \inf_{y \in \mathbf{R}} \left(\mathrm{C}_t(S_t, (0, y^{(2)} - V_{t-1}^{(2)})) + \sup_{s \in \phi_t(S_t)} \gamma_{t+1}^\xi(s, y) \right),$$

where $\phi_t(S_t) = \mathrm{supp}_{\mathcal{F}_t} S_{t+1}$. Also, the infimum hedging cost of ξ at any time t is reached, i.e. $\gamma_t^\xi(V_{t-1})$ is a mimimal cost.

4.4.3 The case of fixed transaction costs

In the case of fixed costs, the cost functions C_t, $t \leq T$, are not convex in general. Moreover, C_t is a priori positively lower homogeneous, i.e. for any $\lambda \geq 1$, $\mathrm{C}_t(\lambda z) \leq \lambda \mathrm{C}_t(z)$. Then, C_t does not satisfy the assumptions we impose in this paper. Nevertheless, we shall see in this section that we may also implement the dynamic programming principle under a robust SAIP condition imposed on the enlarged market with only proportional transaction costs.

To do so, recall that for a l.s.c. function g, the horizon function (see [76, Section 3.C]) g^∞ of g is defined as:

$$g^\infty(y) := \liminf_{\alpha \to \infty} \frac{g(\alpha y)}{\alpha}.$$

Recall that g^∞ is positively homogeneous and l.s.c. in y. We then define the *horizon* cost function as

$$\hat{\mathrm{C}}_t(s, y) = \mathrm{C}_t^\infty(s, y) = \liminf_{\alpha \to \infty} \frac{\mathrm{C}_t(s, \alpha y)}{\alpha}. \tag{4.4.7}$$

The liquidation value associated to the cost function $\hat{\mathrm{C}}_t$ is then given by

$$\hat{Ł}_t(s,y) = \limsup_{\alpha\to\infty} \frac{Ł_t(s,\alpha y)}{\alpha}.$$

Note that in the case where $\hat{C}_t(s,y) = \lim_{\alpha\to\infty} \frac{C_t(s,\alpha y)}{\alpha}$, then $\hat{Ł}_t = Ł_t^\infty$. Moreover, if $\hat{C}_t$ is subadditive, we deduce that

$$\hat{\mathbf{G}}_t(\omega) := \{z :\ \hat{Ł}_t(S_t(\omega), z) \geq 0\}$$

is an $\mathcal{F}_t$-measurable random positive closed cone. We then deduce that the enlarged market defined by the solvency sets $(\hat{\mathbf{G}}_t)_{t\in[0,T]}$ corresponds to a model with proportional transaction costs, as defined in [55, Section 3]. The cash invariance property propagates from C_t to $\hat{C}_t$. In that case, we may verify that $\hat{Ł}_t(s,z) = \max\{\alpha \in \mathbf{R} :\ z - \alpha e_1 \in \hat{\mathbf{G}}_t\}$ and similarly, we have $\hat{C}_t(s,z) = \min\{\alpha \in \mathbf{R} :\ \alpha e_1 - z \in \hat{\mathbf{G}}_t\}$. We then deduce the following:

Lemma 4.4.18. *Suppose that C_t is cash invariant. Then, $\mathbf{G}_t \subseteq \hat{\mathbf{G}}_t$ if and only if $\hat{C}_t(S_t,z) \leq C_t(S_t,z)$ for any z a.s..*

Proof. First suppose that $\mathbf{G}_t \subseteq \hat{\mathbf{G}}_t$. As $C_t(S_t,z)e_1 - z \in \mathbf{G}_t$, then we get that $C_t(S_t,z)e_1 - z \in \hat{\mathbf{G}}_t$. Therefore, we deduce that

$$\hat{C}_t(s,z) = \min\{\alpha \in \mathbf{R} :\ \alpha e_1 - z \in \hat{\mathbf{G}}_t\} \leq C_t(S_t,z).$$

Reciprocally, if $\hat{C}_t \leq C_t$, then $\hat{Ł}_t \geq Ł_t$ hence $\mathbf{G}_t \subseteq \hat{\mathbf{G}}_t$. □

Note that in [65], such an enlarged model $(\hat{\mathbf{G}}_t)_{t\in[0,T]}$ is studied and $\hat{Ł}_t$ is the liquidation value of the closed conic hull K_t of $\mathbf{G}_t$, i.e. $\hat{\mathbf{G}}_t = K_t$.

Example 4.4.19. The market is composed of one bond whose price is $B_t = 1$ and $d-1$ risky assets, $d \geq 2$, whose prices are described by a family of bid and ask prices and fixed costs $S = ((S^{b,i}, S^{a,i}, c^i))_{i=2,\cdots,d}$. In the following, we denote by $s = ((s^{b,i}, s^{a,i}, c^i))_{i=2,\cdots,d}$ any element of $\mathbf{R}^{3(d-1)}$. We consider the fixed costs model defined by the following liquidation process:

$$\begin{aligned} Ł_t(s,y) &:= y^1 + \sum_{i=2}^{d} Ł_t^i(s^{b,i}, s^{a,i}, c^i, y^i),\ (s,y) \in \mathbf{R}^{3(d-1)} \times \mathbf{R}^d, \\ Ł_t^i(s^{b,i}, s^{a,i}, c^i, y^i) &:= \left(y^i s^{b,i} - c_t^i\right)^+ 1_{y^i>0} + \left(y^i s^{a,i} - c_t^i\right) 1_{y^i<0}. \end{aligned}$$

Note that the $(c^i)_{i=2,\cdots,d}$ are interpreted as fixed costs while $(s^{b,i}, s^{a,i})_{i=2,\cdots,d}$ are bid and ask prices for the risky assets. We may of course generalize this model to an order book with several bid and ask prices for each asset, as in Example 4.2.1. Recall that by definition $\mathrm{C}_t(s,y) = -\mathrm{Ł}_t(s,-y)$ and we may verify that $\mathrm{C}_t(s,y)$ is l.s.c. in every (s,y) such that $(c^i)_{i=2,\cdots,d} \in \mathbf{R}_+^{d-1}$. To see it, it suffices to observe that $\mathrm{Ł}_t^i(s,y)$ is continuous at each point (s,y) such that $y \neq 0$. At last, if $y = 0$, $\mathrm{Ł}_t(s,y) = 0$ and $\liminf_{r\to s, y\to 0} \mathrm{Ł}_t(r,y) \leq 0$ since $c_t^i \geq 0$. Therefore, $\mathrm{Ł}_t^i$ is u.s.c. Moreover, $\mathrm{C}_t(s,y)$ subadditive in y. A direct computation yields that $\hat{\mathrm{Ł}}_t(s,y) = y^1 + \sum_{i=2}^d \hat{\mathrm{Ł}}_t^i(s^{b,i}, s^{a,i}, y^i)$ where

$$\hat{\mathrm{Ł}}_t^i(s^{b,i}, s^{a,i}, y^i) = (y^i)^+ s^{b,i} - (y^i)^- s^{a,i}.$$

Note that $\hat{\mathrm{Ł}}_t = \mathrm{Ł}_t^\infty$ and we have $\hat{\mathrm{C}}_t(s,y) = y^1 + \sum_{i=2}^d \hat{\mathrm{C}}_t^i(s^{b,i}, s^{a,i}, y^i)$ where

$$\hat{\mathrm{C}}_t^i(s^{b,i}, s^{a,i}, y^i) = (y^i)^+ s^{a,i} - (y^i)^- s^{b,i}.$$

Observe that $\hat{\mathrm{Ł}}_t$ and $\hat{\mathrm{C}}_t$ are continuous in (s,y). Moreover, $\hat{\mathrm{C}}_t \leq \mathrm{C}_t$ and $\hat{\mathrm{C}}_t$ is super δ-homogeneous with $\delta(x) = x$. $\triangle$

In the following, we adapt the notations of Section 4.3 to the enlarged model $(\hat{\mathbf{G}}_t)_{t\in[0,T]}$ as follows: We set

$$\hat{\gamma}_T(S_T, V_{T-1}) = g^1(S_T) + \hat{\mathrm{C}}_T(S_T, (0, g^{(2)}(S_T) - V_{T-1}^{(2)})),$$

and we define recursively

$$\begin{aligned} \hat{\theta}_t^\xi(V_t) &:= \operatorname{ess\,sup}_{\mathcal{F}_t} \hat{\gamma}_{t+1}^\xi(V_t), \\ \hat{D}_t^\xi(S_t, V_{t-1}, V_t) &:= \hat{\mathrm{C}}_t(S_t, (0, V_t^{(2)} - V_{t-1}^{(2)})) + \hat{\theta}_t^\xi(S_t, V_t). \end{aligned}$$

Definition 4.4.20. *We say that the robust no-arbitrage condition RSAIP holds at time t if the SAIP condition holds at time t for the enlarged model $(\hat{\mathbf{G}}_t)_{t\in[0,T]}$. We say that RSAIP holds if it holds at any time.*

Theorem 4.4.21. *Suppose that the enlarged market satisfies $\hat{\mathrm{C}}_t \leq \mathrm{C}_t$, $\hat{\mathrm{C}}$ is super δ-homogeneous and is either sub-additive or super-additive. Suppose that there exists an $\mathcal{F}_{t+1}$-normal integrand $\tilde{\gamma}_{t+1}^\xi$ defined on $\mathbf{R}^m \times \mathbf{R}^d$ such that $\gamma_{t+1}^\xi(V_t) = \tilde{\gamma}_{t+1}^\xi(S_{t+1}, V_t)$ for all $V_t \in L^0(\mathbf{R}^d, \mathcal{F}_t)$. Assume that Assumption 4 and Assumption 5 hold. Suppose that the cost function $\mathrm{C}_t(s,z)$ is an $\mathcal{F}_t$-normal integrand and C_t is either super-additive or sub-additive. Then, if $\inf_{z\in S^{d-1}(0,1)} \hat{D}_t^0(S_t, 0, z) > 0$, $\gamma_t^\xi(V_{t-1}) = \tilde{\gamma}_t^\xi(S_t, V_{t-1})$ where $\tilde{\gamma}_t^\xi(s, v_{t-1})$ is an $\mathcal{F}_t$-normal integrand.*

Proof. As $\hat{\mathrm{C}}_t(x) \leq \mathrm{C}_t(x)$, we deduce by induction that $\hat{D}_t^0(s,0,v_t) \leq D_t^0(s,0,v_t)$. We adapt the main arguments of the proof of Theorem 4.4.15. Recall that $D_t^0(s,v_{t-1},v_t) \geq \tilde{h}_t(s,v_{t-1}) + D_t^0(s,0,v_t)$ where $\tilde{h}_t$ is a continuous function. By Lemma 4.5.20, we have for $|v_t| \geq 1$,

$$D_t^0(s,0,v_t) \geq \hat{D}_t^0(s,0,v_t) \geq \delta(|v_t|)\hat{D}_t^0(s,0,v_t/|v_t|) \geq \delta(|v_t|) \inf_{z \in S^{d-1}(0,1)} \hat{D}_t^0(s,0,z).$$

Therefore, we also get that $\tilde{\gamma}_t^\xi(s,v_{t-1}) = \inf_{v_t \in K_t(s,v_{t-1})} D_t^\xi(s,v_{t-1},v_t)$ where $K_t(s,v_{t-1}) \subseteq \phi_t(s,v_{t-1}) := \bar{B}_t(0, r_t(s,v_{t-1})+1)$ and

$$\begin{aligned} r_t(s,v_{t-1}) &:= \delta^{-1}\left(\frac{\lambda_t(s,v_{t-1})}{i_t(s)}\right), \\ i_t(s) &:= \inf_{z \in S^{d-1}(0,1)} \hat{D}_t^0(s,0,z),\ \lambda_t(s,v_{t-1}) = |\tilde{h}_t(s,v_{t-1})| + \hat{h}_t^\xi(s,v_{t-1}). \end{aligned}$$

Applying Theorem 4.4.15 by induction to the enlarged market, we deduce that $\hat{D}_t^0(s,0,z)$ is l.s.c. in (s,z), see the proof of Theorem 4.4.15. We then conclude as in the proof of Theorem 4.4.15.

□

Remark 4.4.22. Recall that the condition $\inf_{z \in S^{d-1}(0,1)} \hat{D}_t^0(S_t,0,z) > 0$ we impose in the theorem above holds under the RSAIP condition by Theorem 4.4.16. For a fixed costs model, this means that SAIP holds for the enlarged market, a priori without fixed cost. Moreover, the other conditions we impose are also satisfied in the fixed costs model of Example 4.4.19. △

4.4.4 Computational feasibility under a weaker SAIP no-arbitrage condition

In this section, we consider a no-arbitrage condition called LAIP, weaker than SAIP, but still sufficient to deduce that the essential infimum in the dynamic programming principle (4.3.1) is a pointwise infimum so that it can be numerically computed.

Lemma 4.4.23. *Suppose that C_t is sub-additive for any $t \leq T$. Then, for any payoff $\xi \in L^0(\mathbf{R}^d, \mathcal{F}_T)$, the function D_t^ξ defined by (4.4.3) satisfies the following inequality:*

$$D_t^\xi(V_{t-1}+\bar{V}_{t-1}, V_t+\bar{V}_t) \leq D_t^\xi(V_{t-1},V_t) + D_t^0(\bar{V}_{t-1},\bar{V}_t).$$

Proof. By definition with the sub-additivity of C_T, we have:

$$\begin{aligned}\gamma_T^{\xi}(V_{T-1}+\bar{V}_{T-1}) &= \xi^1+\mathrm{C}_T((0,\xi^{(2)}-V_{T-1}^{(2)}-\bar{V}_{T-1}^{(2)})),\\ &= \xi^1+\mathrm{C}_T((0,-V_{T-1}^{(2)}))+\mathrm{C}_T((0,-\bar{V}_{T-1}^{(2)})),\\ &\leq \gamma_T^{\xi}(V_{T-1})+\gamma_T^0(\bar{V}_{T-1}).\end{aligned}$$

We deduce that $\theta_{T-1}^{\xi}(V_{T-1}+\bar{V}_{T-1}) \leq \theta_{T-1}^{\xi}(V_{T-1})+\theta_{T-1}^0(\bar{V}_{T-1})$ and, since $D_{T-1}^{\xi}(V_{T-2},V_{T-1}) = \mathrm{C}_{T-1}((0,V_{T-1}-V_{T-2}))+\theta_T^{\xi}(V_{T-1})$, we get that:

$$D_{T-1}^{\xi}(V_{T-2}+\bar{V}_{2-1},V_{T-1}+\bar{V}_{T-1}) \leq D_{T-1}^{\xi}(V_{T-2},V_{T-1})+D_{T-1}^0(\bar{V}_{T-2},\bar{V}_{T-1}).$$

Taking the essential infimum with respect to V_{T-1} and $\bar{V}_{T-1}$, we get that

$$\gamma_{T-1}^{\xi}(V_{T-2}+\bar{V}_{T-2}) \leq \gamma_{T-1}^{\xi}(V_{T-2})+\gamma_{T-1}^0(\bar{V}_{T-2}).$$

We may pursue by induction and conclude. □

We now introduce the LAIP condition. By Proposition 4.5.7, we may suppose that the function $D_t^0(y,z)$ defined by (4.4.3) is l.s.c. in (y,z) and it is $\mathcal{F}_t\otimes\mathcal{B}(\mathbf{R}^d)\otimes\mathcal{B}(\mathbf{R}^d)$ measurable w.r.t. (ω,y,z). Note that, under AIP, the family of random variables $\mathcal{N}_t := \left\{Z_t\in L^0(\mathbf{R}^d,\mathcal{F}_t), Z_t^1=0,\ \ D_t^0(0,Z_t)=0\right\}$ coincides with $\left\{Z_t\in L^0(\mathbf{R}^d,\mathcal{F}_t), Z_t^1=0,\ \ D_t^0(0,Z_t)\leq 0\right\}$. Therefore, by lower semicontinuity, $\mathcal{N}_t$ is a closed subset of $L^0(\mathbf{R}^d,\mathcal{F}_t)$. Moreover, $\mathcal{N}_t$ is $\mathcal{F}_t$- decomposable, see [55, Section 5.4]. Therefore, by [55, Proposition 5.4.3], there exists an $\mathcal{F}_t$-measurable random set N_t such that $\mathcal{N}_t = L^0(\mathrm{N}_t,\mathcal{F}_t)$.

Definition 4.4.24. *We say that the condition LAIP (Linear AIP condition) holds at time t if AIP holds at time t and $\mathcal{N}_t$ is a linear vector space, or equivalently N_t is a.s. a linear subspace of $\mathbf{R}^d$. We say that LAIP holds if LAIP holds at any time.*

Note that if $\mathcal{N}_t=\{0\}$, then SAIP, AIP and LAIP are equivalent. In general, SAIP implies LAIP. The following result gives a financial interpretation of LAIP. If LAIP holds, the cost to hedge the zero payoff from an initial risky position $Z_t = V_t^{(2)}\in L^0(\mathbf{R}^{d-1},\mathcal{F}_t)$ is zero if and only if the cost is also zero for the position $-Z_t$. This symmetric property is related to the SRN condition in Chapter 3.

Lemma 4.4.25. *Suppose that C_t is sub-additive and is positively super δ-homogeneous, for any $t\leq T$. The following statements are equivalent:*

1.) LAIP holds.

2.) AIP holds and, if $Z_t \in L^0(\mathbf{R}^d, \mathcal{F}_t)$, then $D_t^0(0, Z_t) = 0$ if and only if $D_t^0(0, -Z_t) = 0$, $t \leq T$.

Proof. The implication 1.) $\implies$ 2.) is immediate. Reciprocally, suppose that 2.) holds. Let us show that $\mathcal{N}_t$ is stable under addition. We consider $Z_t^1, Z_t^2 \in \mathcal{N}_t$. By Proposition 4.4.23, we get under AIP that

$$0 \leq D_t^0(0, Z_t^1 + Z_t^2) \leq D_t^0(0, Z_t^1) + D_t^0(0, Z_t^2) \leq 0.$$

We deduce that $Z_t^1 + Z_t^2 \in \mathcal{N}_t$. By induction, we then deduce that for any integer n, $n\mathcal{N}_t \subseteq \mathcal{N}_t$. Moreover, by Lemma 4.5.20, if $\lambda_t \in L^0((0,1], \mathcal{F}_t)$,

$$D_t^0(0, V_t) = D_t^0(0, \lambda_t(\lambda_t)^{-1} V_t) \geq \delta((\lambda_t)^{-1}) D_t^0(0, \lambda_t V_t) \geq 0.$$

So $V_t \in \mathcal{N}_t$ implies that $\lambda_t V_t \in \mathcal{N}_t$ if $\lambda_t \in L^0((0,1], \mathcal{F}_t)$. Finally, as $\mathbb{N}\mathcal{N}_t \subseteq \mathcal{N}_t$, $\lambda_t V_t \in \mathcal{N}_t$ for every $\lambda_t \geq 0$. Moreover, $\mathcal{N}_t$ is symmetric by assumption. The conclusion follows. □

In the following, let us consider $\mathrm{N}_t^\perp := \{z \in \mathbf{R}^d : \ zx = 0, \forall x \in \mathrm{N}_t\}$, the random $\mathcal{F}_t$-measurable linear subspace orthogonal to N_t.

Lemma 4.4.26. *Suppose that C_t is sub-additive and LAIP holds. Then, for all $V_{t-1} \in L^0(\mathbf{R}^d, \mathcal{F}_t)$, there exists $V_t^2 \in L^0(\mathrm{N}_t^\perp, \mathcal{F}_t)$ such that*

$$D_t^\xi(V_{t-1}, V_t) = D_t^\xi(V_{t-1}, V_t^2) \ a.s..$$

Proof. By a measurable selection argument, it is possible to decompose any $V_t \in L^0(\mathbf{R}^d, \mathcal{F}_t)$ into $V_t = V_t^1 + V_t^2$, where $V_t^1 \in L^0(\mathrm{N}_t, \mathcal{F}_t)$, $V_t^2 \in L^0(\mathrm{N}_t^\perp, \mathcal{F}_t)$. By Lemma 4.4.23, we have

$$D_t^\xi(V_{t-1}, V_t) \leq D_t^\xi(V_{t-1}, V_t^2) + D_t^0(0, V_t^1) = D_t^\xi(V_{t-1}, V_t^2).$$

On the other hand, as $V_t^2 = V_t - V_t^1$ and $-V_t^1 \in \mathcal{N}_t$ under LAIP, we also have

$$D_t^\xi(V_{t-1}, V_t^2) \leq D_t^\xi(V_{t-1}, V_t) + D_t^0(0, -V_t^1) = D_t^\xi(V_{t-1}, V_t).$$

The conclusion follows. □

In the following, we assume the following condition.

Assumption 6. *For any $t \leq T$, $|\mathrm{C}_t((0, x^{(2)}))| < \bar{h}_t(x)$, where $\bar{h}_t$ is a random function $\bar{h}_t : (\omega, x) \in \Omega \times \mathbf{R}^d \mapsto \bar{h}_t(\omega, x) \in \mathbf{R}$ which is $\mathcal{F}_t \otimes \mathcal{B}(\mathbf{R}^d)$-measurable and continuous a.s. in x.*

Note that the condition above holds under our initial hypothesis with $\bar{h}_t(x) = h_t(S_t, x)$ but, here, we do not stress a dependence of C_t on S_t.

Theorem 4.4.27. *Suppose that there exists an $\mathcal{F}_{t+1}$-normal integrand function $\tilde{\gamma}^{\xi}_{t+1}$ defined on $\mathbf{R}^d$. Assume that Assumption 6 holds. Suppose that the cost function $\mathrm{C}_t(z)$ is an $\mathcal{F}_t$-normal integrand and C_t is sub-additive, positively super δ-homogeneous. If LAIP holds, then $\gamma^{\xi}_t(V_{t-1}) = \tilde{\gamma}^{\xi}_t(V_{t-1})$ where $\tilde{\gamma}^{\xi}_t(v_{t-1})$ is an $\mathcal{F}_t$-normal integrand*

Proof. By Lemma 4.4.26, we get that

$$\operatorname*{ess\,inf}_{\substack{\mathcal{F}_t \\ V_t \in L^0(\mathbf{R}^d, \mathcal{F}_t)}} D^{\xi}_t(V_{t-1}, V_t) = \operatorname*{ess\,inf}_{\substack{\mathcal{F}_t \\ V_t \in L^0(\mathrm{N}^{\perp}_t, \mathcal{F}_t)}} D^{\xi}_t(V_{t-1}, V_t).$$

Since $\mathrm{N}^{\perp}_t$ is an $\mathcal{F}_t$-measurable random closed set, by Proposition 4.5.7 and Lemma 4.5.5, we have

$$\operatorname*{ess\,inf}_{\substack{\mathcal{F}_t \\ V_t \in L^0(\mathrm{N}^{\perp}_t, \mathcal{F}_t)}} D^{\xi}_t(V_{t-1}, V_t) = \inf_{y \in \mathrm{N}^{\perp}_t} D^{\xi}_t(V_{t-1}, y).$$

On $\left\{\omega : \ \mathrm{N}^{\perp}_t(\omega) = \{0\}\right\} \in \mathcal{F}_t$, we have $\gamma^{\xi}_t(V_{t-1}) = D^{\xi}_t(V_{t-1}, 0)$. On the complementary set, $\left\{\mathrm{N}^{\perp}_t \neq \{0\}\right\} \in \mathcal{F}_t$, under LAIP, we have $\inf_{z \in M_t} D^0_t(0, z) > 0$, where $M_t = \mathrm{N}^{\perp}_t \cap S^{d-1}(0, 1) \neq \emptyset$. We now adapt the notations and the main arguments in the proof of Theorem 4.4.15 with $V_t \in \mathrm{N}^{\perp}_t$. In our case, we use Assumption 6 in order to dominate the cost function by a continuous function. By Lemma 4.5.20, for all $v_t \in \mathrm{N}^{\perp}_t$, we may suppose w.l.o.g. that $v^1_t = 0$ and we get that

$$D^0_t(0, v_t) \geq \delta(|v_t|) D^0_t(0, v_t/|v_t|) \geq \delta(|v_t|) \inf_{z \in M_t} D^0_t(0, z).$$

Moreover, by Assumption 6, we have:

$$D_t(v_{t-1}, 0) = \mathrm{C}_t((0, v^{(2)}_{t-1})) + \theta^{\xi}_t(0) \leq \bar{h}_t(v_{t-1}) + \theta^{\xi}_t(0).$$

Therefore, we deduce that $\tilde{\gamma}^{\xi}_t(v_{t-1}) = \inf_{v_t \in \phi_t(v_{t-1})} D^{\xi}_t(v_{t-1}, v_t)$ where ϕ is the set-valued mapping $\phi_t(v_{t-1}) := \bar{B}_t(0, r_t(v_{t-1}) + 1)$ and

$$\begin{aligned} r_t(v_{t-1}) &:= \delta^{-1}\left(\frac{\lambda_t(v_{t-1})}{i_t}\right), \\ i_t &:= \inf_{z \in M_t} D^0_t(0, z), \ \lambda_t(v_{t-1}) = \tilde{h}_t(v_{t-1}) + \bar{h}_t(v_{t-1}) + \theta^{\xi}_t(0). \end{aligned}$$

By Corollary 4.5.3, $i_t > 0$ is $\mathcal{F}_t$-measurable while $\lambda_t(\omega, v_{t-1})$ is $\mathcal{F}_t \otimes \mathcal{B}(\mathbf{R}^d)$-measurable and continuous in v_{t-1}. Therefore, $r_t(\omega, v_{t-1})$ is $\mathcal{F}_t \otimes \mathcal{B}(\mathbf{R}^d)$-measurable and continuous in v_{t-1}. We deduce that $\bar{B}_t(0, r_t(v_{t-1}))$ is a continuous set-valued mapping by Corollary 4.5.14. We then conclude by Proposition 4.5.17. □

Note that the theorem above states that, under LAIP, $\gamma_t^\xi(V_{t-1})$ is a lower-semicontinuous function of V_{t-1}. Therefore, by Lemma 4.5.5, $\gamma_t^\xi(V_{t-1})$ may be computed pointwise as $\gamma_t^\xi(V_{t-1}) = \inf_{y \in \mathbf{R}^d} \left(\mathrm{C}_t((0, y^{(2)} - V_{t-1}^{(2)})) + \theta_t^\xi(y) \right)$. Moreover, the infimum is reached so that $\gamma_t^\xi(V_{t-1})$ is a minimal cost.

4.5 Appendix

4.5.1 Normal integrands

Definition 4.5.1. *Let $\mathcal{F}$ be a complete σ-algebra. We say that the function $(\omega, x) \in \Omega \times \mathbf{R}^k \mapsto f(\omega, x) \in \mathbf{R}$ is an $\mathcal{F}$-normal integrand if f is $\mathcal{F} \otimes \mathcal{B}(\mathbf{R}^k)$-measurable and lower semi-continuous in x. If $Z \in L^0(\mathbf{R}^k, \mathcal{F})$, we use the notation $f(Z) : \omega \mapsto f(Z(\omega)) = f(\omega, Z(\omega))$. If f is $\mathcal{F} \otimes \mathcal{B}(\mathbf{R}^k)$-measurable then $f(Z) \in L^0(\mathbf{R}^k, \mathcal{F})$.*

By [76, Theorem 14.37], we have:

Proposition 4.5.2. *If f is an $\mathcal{F}$-normal integrand, $\inf_{y \in \mathbf{R}^d} f(\omega, y)$ is $\mathcal{F}$-measurable and $\{(\omega, x) \in \Omega \times \mathbf{R}^d : f(\omega, x) = \inf_{y \in \mathbf{R}^d} f(\omega, y)\} \in \mathcal{F} \otimes \mathcal{B}(\mathbf{R}^d)$ is a measurable closed set.*

Corollary 4.5.3. *For any $\mathcal{F}$ normal integrand $f : \Omega \times \mathbf{R}^d \to \overline{\mathbf{R}}$ and any $\mathcal{F}$-measurable random set A, let $p(\omega) = \inf_{x \in A} f(\omega, x)$. Then the function $p : \Omega \to \overline{\mathbf{R}}$ is $\mathcal{F}$-measurable.*

Proof. Let us define $\delta_{A(\omega)}(x) = +\infty$ if $x \notin A(\omega)$ and $\delta_{A(\omega)}(x) = 0$ otherwise. Then, the function $g(\omega, x) := f(\omega, x) + \delta_{A(\omega)}(x)$ is an $\mathcal{F}$-normal integrand since A is closed and $\mathcal{F}$-measurable. Moreover, we observe that $p(\omega) = \inf_{x \in A(\omega)} g(\omega, x)$. The conclusion follows from Proposition 4.5.2. □

Corollary 4.5.4. *If f is an $\mathcal{F}$-normal integrand, and if K is an $\mathcal{F}$-measurable set-valued compact set, then $\inf_{y \in K(\omega)} f(\omega, y)$ is $\mathcal{F}$-measurable. Moreover,*

$M(\omega) = \{x \in K(\omega) : f(\omega, x) = \inf_{y \in K(\omega)} f(\omega, y)\} \in \mathcal{F} \otimes \mathcal{B}(\mathbf{R}^d)$ *is a non-empty* $\mathcal{F}$*-measurable closed set. In particular,* $\inf_{y \in K(\omega)} f(\omega, y) = f(\omega, y)$ *for all* $y \in L^0(M, \mathcal{F}) \neq \emptyset$.

Proof. It suffices to extend the function f to $\mathbf{R}^d$ by setting $f = +\infty$ on $\mathbf{R}^d \backslash K(\omega)$ so that f is still l.s.c. on $\mathbf{R}^d$. Then, we may apply Proposition 4.5.2. Notice that $M(\omega) \neq \emptyset$ a.s. by compactness argument so that $L^0(M, \mathcal{F}) \neq \emptyset$ by a measurable selection argument. □

In the following, we use the abuse of notation $f(y) = f(\omega, y)$ for any $f : \Omega \times \mathbf{R}^d \to \overline{\mathbf{R}}$.

Lemma 4.5.5. *For any* $\mathcal{F}$ *normal integrand* $f : \Omega \times \mathbf{R}^d \to \overline{\mathbf{R}}$ *such that* f *is bounded below a.s. by a random variable, and any non-empty* $\mathcal{F}$*-measurable closed set* A*, we have:*

$$\operatorname{ess\,inf}_{\mathcal{F}} \left\{ f(a),\ a \in L^0(A, \mathcal{F}) \right\} = \inf_{a \in A} f(a) \ \text{ a.s.}$$

Proof. We first prove that

$$\operatorname{ess\,inf}_{\mathcal{F}} \left\{ f(a), a \in L^0(A, \mathcal{F}) \right\} \leq \inf_{a \in A} f(a).$$

Recall that f is $\mathcal{F}$-normal integrand and $\inf_{a \in A} f(a)$ is $\mathcal{F}$-measurable by Corollary 4.5.3. Therefore, the set

$$\{(\omega, a) : a \in A(\omega), \inf_{x \in A} f(x) \leq f(a) < \inf_{x \in A} f(x) + 1/n\}$$

is $\mathcal{F}$-measurable and has non-empty ω sections for each $n \in \mathbb{N}$. By measurable selection argument, we deduce $a^n \in L^0(A, \mathcal{F})$ such that

$$\inf_{a \in A} f(a) \leq f(a^n) < \inf_{a \in A} f(a) + 1/n.$$

This implies that $\lim_n f(a^n) = \inf_{a \in A} f(a)$. Therefore,

$$\inf_{a \in A} f(a) = \inf_n f(a^n) \geq \operatorname{ess\,inf}_{\mathcal{F}} \left\{ f(a), a \in L^0(A, \mathcal{F}) \right\}.$$

For the reversed inequality, for each $a \in L^0(A, \mathcal{F})$, $f(a) \geq \inf_{a \in A} f(a)$ and, since $\inf_{a \in A} f(a)$ is $\mathcal{F}$-measurable by Corollary 4.5.3, we deduce by definition of conditional essential infimum that

$$\operatorname{ess\,inf}_{\mathcal{F}} \left\{ f(a), a \in L^0(A, \mathcal{F}) \right\} \geq \inf_{a \in A} f(a) \text{ a.s..}$$

□

We recall a result from [5] which characterizes a conditional essential supremum as a pointwise supremum on a random set. Let $\mathcal{H}$ and $\mathcal{F}$ be two complete sub-σ-algebras of $\mathcal{F}_T$ such that $\mathcal{H} \subseteq \mathcal{F}$. The conditional support of $X \in L^0(\mathbf{R}^d, \mathcal{F})$ with respect to $\mathcal{H}$ is the smallest $\mathcal{H}$-graph measurable random set $\mathrm{supp}_{\mathcal{H}} X$ containing the singleton $\{X\}$ a.s., see [5].

Proposition 4.5.6. *Let $h : \Omega \times \mathbf{R}^k \to \mathbf{R}$ be a $\mathcal{H} \otimes \mathcal{B}(\mathbf{R}^k)$-measurable function which is l.s.c. in x. Then, for all $X \in L^0(\mathbf{R}^k, \mathcal{F})$,*

$$\operatorname{ess\,sup}_{\mathcal{H}} h(X) = \sup_{x \in supp_{\mathcal{H}} X} h(x) \quad a.s.$$

Proposition 4.5.7. *Fix $\xi^1 \in L^0(\mathbf{R}, \mathcal{F})$ and $d \geq 2$. Let us consider a random function $f : \Omega \times \mathbf{R}^d \to \mathbf{R}$ that satisfies $f(z) = z^1 + f(0, z^{(2)})$, for any $z = (z^1, z^{(2)}) \in \mathbf{R}^d$. Suppose that $z \mapsto f(z)$ is l.s.c. a.s.. Then, there exists a $\mathcal{F}_{t-1} \otimes \mathcal{B}(\mathbf{R}^{d-1})$-measurable random function $F^*_{t-1}(\omega, y)$ such that, for any $Y_{t-1} \in L^0(\mathbf{R}^{d-1}, \mathcal{F}_{t-1})$,*

$$F^*_{t-1}(Y_{t-1}) = \operatorname{ess\,sup}_{\mathcal{F}_{t-1}} \left(\xi^1 + f(0, Y_{t-1})\right) =: F^{\xi^1, f}_{t-1}(Y_{t-1}), \text{ a.s.}$$

Moreover, $F^*_{t-1}(\omega, y)$ is l.s.c. in y and if, in addition, $y \in \mathbf{R}^{d-1} \mapsto f(0, y)$ is a.s. convex, then $y \mapsto F^*_{t-1}(\omega, y)$ is a.s. convex.

Proof. Consider the family of random variables:

$$\begin{aligned} \Lambda_{t-1} &= \left\{(x_{t-1}, y_{t-1}) \in L^0(\mathbf{R}^d, \mathcal{F}_{t-1}) : f(-x_{t-1}, y_{t-1}) \leq -\xi^1\right\} \\ &= \left\{(x_{t-1}, y_{t-1}) \in L^0(\mathbf{R}^d, \mathcal{F}_{t-1}) : x_{t-1} \geq F^{\xi^1, f}_{t-1}(y_{t-1})\right\}. \end{aligned}$$

Notice that Λ_{t-1} is closed in L^0 since f is l.s.c.. Moreover, Λ_{t-1} is $\mathcal{F}_{t-1}$-decomposable, i.e. $g^1_{t-1} 1_{A_{t-1}} + g^2_{t-1} 1_{A^c_{t-1}} \in \Lambda_{t-1}$ if g^1_{t-1} and g^2_{t-1} belong to Λ_{t-1} and $A_{t-1} \in \mathcal{F}_{t-1}$. By [63, Corollary 2.5], there exists an $\mathcal{F}_{t-1}$-measurable random closed set Γ_{t-1} such that $\Lambda_{t-1} = L^0(\Gamma_{t-1}, \mathcal{F}_{t-1})$. Moreover, there is a Castaing representation, i.e. a countable family $(z^n_{t-1})_{n \geq 1} \in \Lambda_{t-1}$ such that $\Gamma_{t-1}(\omega) = \mathrm{cl}\{z^n_{t-1}(\omega) : n \geq 1\}$, $\omega \in \Omega$. We define

$$F^*_{t-1}(\omega, y) := \inf\{x \in \mathbf{R} : (x, y) \in \Gamma_{t-1}(\omega)\}.$$

We claim that $F^*_{t-1}(\omega, y) = \inf\left\{x \in \mathbb{Q} : (x, y) \in \Gamma_{t-1}(\omega)\right\}$. Indeed, first we have $F^*_{t-1}(\omega, y) \leq \inf\left\{x \in \mathbb{Q} : (x, y) \in \Gamma_{t-1}(\omega)\right\}$. Moreover, in the case where $F^*_{t-1}(\omega, y) > -\infty$, for every $\epsilon > 0$, there exist $x \in \mathbf{R}$ such that

$(x,y) \in \Gamma_{t-1}$ and $F^*_{t-1}(\omega,y) + \epsilon \geq x$. Choose $\tilde{x} \in \mathbb{Q} \cap [x, x+\epsilon]$. Observe that $(\tilde{x},y) \in \Gamma_{t-1}$ as the y-sections of Λ_{t-1} are upper sets. We then have:

$$\begin{aligned} &F^*_{t-1}(\omega,y) + 2\epsilon \geq x + \epsilon \geq \tilde{x}, \\ &F^*_{t-1}(\omega,y) \geq \tilde{x} - 2\epsilon \geq \inf\big\{x \in \mathbb{Q} : (x,y) \in \Gamma_{t-1}(\omega)\big\} - 2\epsilon. \end{aligned}$$

Since ϵ is arbitrary chosen, we conclude that

$$F^*_{t-1}(\omega,y) = \inf\big\{x \in \mathbb{Q} : (x,y) \in \Gamma_{t-1}(\omega)\big\}.$$

Notice that when $F^*_{t-1}(\omega,y) = -\infty$, then we may choose $x \to -\infty$ so that we also have $\tilde{x} \to -\infty$ and we conclude similary. We then deduce that $F^*_{t-1}(\omega,y)$ is $\mathcal{F}_{t-1} \otimes \mathcal{B}(\mathbf{R}^{d-1})$-measurable. Indeed, for every $c < +\infty$, we have:

$$\big\{(\omega,y) : F^*_{t-1}(\omega,y) \geq c\big\} = \bigcap_{x \in \mathbb{Q}} \big\{(\omega,y) : x 1_{(\omega,x,y) \in \mathrm{Graph}\Gamma_{t-1}} \geq c 1_{(\omega,x,y) \in \mathrm{Graph}\Gamma_{t-1}}\big\}.$$

Since Γ_{t-1} is graph-measurable, $\big\{(\omega,y) : F^*_{t-1}(\omega,y) \geq c\big\} \in \mathcal{F}_{t-1} \otimes \mathcal{B}(\mathbf{R}^{d-1})$. We then conclude that F^*_{t-1} is $\mathcal{F}_{t-1} \otimes \mathcal{B}(\mathbf{R}^{d-1})$-measurable. Moreover, if f_t is convex, Γ_{t-1} is convex a.s. and we deduce that $F^*_{t-1}(\omega,y)$ is convex in y a.s.

Consider a sequence $y^n \in \mathbf{R}^{d-1}$ which converges to y and let us denote $\beta^n := F^*_{t-1}(\omega,y^n)$. We have $(\beta^n,y^n) \in \Gamma_{t-1}$ if $\beta^n > -\infty$. If $\inf_n \beta^n = -\infty$, then, up to a subsequence, $F^*_{t-1}(\omega,y) - 1 > \beta^n$ for n large enough, hence $(F^*_{t-1}(\omega,y) - 1, y^n) \in \Gamma_{t-1}(\omega)$ since the y^n-sections of Γ_{t-1} are upper sets. As $n \to \infty$, we deduce that $(F^*_{t-1}(\omega,y) - 1, y) \in \Gamma_{t-1}(\omega)$, which contradicts the definition of F^*_{t-1}. Moreover it is trivial that $F^*_{t-1}(\omega,y) \leq \liminf_n \beta^n$ if $\liminf_n \beta^n = \infty$. Otherwise, $\beta^\infty := \liminf_n \beta^n < \infty$ and $(\beta^\infty, y) \in \Gamma_{t-1}$ since Γ_{t-1} is closed. It follows that $F^*_{t-1}(\omega,y) \leq \beta^\infty = \liminf_n \beta^n$ by the definition of F^*_{t-1}. We conclude that $F^*_{t-1}(\omega,x)$ is l.s.c. in x.

We show that $F^{\xi^1,f}_{t-1}(Y_{t-1}) = F^*_{t-1}(Y_{t-1})$ a.s. for all $Y_{t-1} \in L^0(\mathbf{R}^{d-1}, \mathcal{F}_{t-1})$. We first restrict Ω to the $\mathcal{F}_{t-1}$-measurable set $\{\omega : \Gamma_{t-1}(\omega) \neq \emptyset\}$. We may then consider a measurable selection $(\tilde{x}_{t-1}, \tilde{y}_{t-1}) \in \Gamma_{t-1} \neq \emptyset$ a.s.. By definition, we have $\tilde{x}_{t-1} \geq F^*_{t-1}(\tilde{y}_{t-1})$. We deduce that $F^*_{t-1}(\tilde{y}_{t-1}) < \infty$ a.s. We define:

$$\widehat{Y}_{t-1} = \tilde{y}_{t-1} 1_{F^*_{t-1}(Y_{t-1}) = \infty} + Y_{t-1} 1_{F^*_{t-1}(Y_{t-1}) < \infty}.$$

Then:

$$F^*_{t-1}(\widehat{Y}_{t-1}) = F^*_{t-1}(\tilde{y}_{t-1}) 1_{F^*_{t-1}(Y_{t-1}) = \infty} + F^*_{t-1}(Y_{t-1}) 1_{F^*_{t-1}(Y_{t-1}) < \infty}.$$

Observe that on the set $\{F^*_{t-1}(Y_{t-1}) < \infty\}$, $(F^*_{t-1}(\widehat{Y}_{t-1}), \widehat{Y}_{t-1}) \in \Gamma_{t-1}$ a.s. since Γ_{t-1} is closed. Therefore, $(F^*_{t-1}(\widehat{Y}_{t-1}), \widehat{Y}_{t-1}) \in \Lambda_{t-1} = L^0(\Gamma_{t-1}, \mathcal{F}_{t-1})$ and we deduce that $F^*_{t-1}(\widehat{Y}_{t-1}) \geq F^{\xi^1,f}_{t-1}(\widehat{Y}_{t-1})$ a.s.. We conclude that on the set $\{F^*_{t-1}(Y_{t-1}) < \infty\}$, $F^*_{t-1}(Y_{t-1}) \geq F^{\xi^1,f}_{t-1}(Y_{t-1})$ while the inequality is trivial on the complementary set. On the other hand, let us define

$$\begin{aligned}
\widehat{X}_{t-1} &= F^{\xi^1,f}_{t-1}(Y_{t-1}) 1_{F^{\xi^1,f}_{t-1}(Y_{t-1}) < \infty} + F^{\xi^1,f}_{t-1}(\tilde{y}_{t-1}) 1_{F^{\xi^1,f}_{t-1}(Y_{t-1}) = \infty}, \\
\widehat{Y}_{t-1} &= Y_{t-1} 1_{F^{\xi^1,f}_{t-1}(Y_{t-1}) < \infty} + \tilde{y}_{t-1} 1_{F^{\xi^1,f}_{t-1}(Y_{t-1}) = \infty}.
\end{aligned}$$

Observe that $(\widehat{X}_{t-1}, \widehat{Y}_{t-1}) \in \Lambda_{t-1}$ hence $F^*_{t-1}(\widehat{Y}_{t-1}) \leq \widehat{X}_{t-1}$ by definition of F^*_{t-1}. Then, $F^*_{t-1}(Y_{t-1}) \leq \widehat{X}_{t-1} = F^{\xi^1,f}_{t-1}(Y_{t-1})$ on $\{F^{\xi^1,f}_{t-1}(Y_{t-1}) < \infty\}$. The inequality is trivial on the complementary set so that we may conclude.

On the set $\{\omega : \Gamma_{t-1}(\omega) = \emptyset\}$, we have $F^*_{t-1}(Y_{t-1}) = +\infty$. Moreover, if $F^{\xi^1,f}_{t-1}(Y_{t-1}) < \infty$, we deduce that $(F^{\xi^1,f}_{t-1}(Y_{t-1}), Y_{t-1}) \in \Gamma_{t-1} = \emptyset$ since $\xi^1 + f(0, Y_{t-1}) \leq F^{\xi^1,f}_{t-1}(Y_{t-1})$. This is a contradiction hence $F^{\xi^1,f}_{t-1}(Y_{t-1}) = +\infty$ and the conclusion follows. □

Lemma 4.5.8. *Suppose that Assumption 2 holds and consider an $\mathcal{F}_{t-1}$-normal integrand $\gamma_t : (\omega, s, y) : \Omega \times \mathbf{R}^m \times \mathbf{R}^d \mapsto \gamma_t(\omega, s, y)$. Then, for any $V_{t-1} \in L^0(\mathbf{R}^d, \mathcal{F}_{t-1})$, we have:*

$$\operatorname{ess\,sup}_{\mathcal{F}_{t-1}} \gamma_t(S_t, V_{t-1}) = \sup_{s \in supp_{\mathcal{F}_{t-1}} S_t} \gamma_t(s, V_{t-1}) = \sup_{m \geq 1} \gamma_t(\alpha^m_{t-1}(S_{t-1}), V_{t-1}).$$

Proof. As $(\omega, s) \mapsto \gamma_t(\omega, s, V_{t-1}(\omega))$ is an $\mathcal{F}_{t-1}$-normal integrand under our assumptions, the first equality holds by Theorem 4.5.6. It remains to observe that, if $s \in \operatorname{supp}_{\mathcal{F}_{t-1}} S_t$, then $s = \lim_m \alpha^m_{t-1}(S_{t-1})$ for a subsequence and, by lower semicontinuity, we deduce that

$$\gamma_t(s, V_{t-1}) \leq \liminf_m \gamma^\xi_t(\alpha^m_{t-1}(S_{t-1})), V_{t-1}) \leq \sup_{m \geq 1} \gamma^\xi_t(\alpha^m_{t-1}(S_{t-1})), V_{t-1}).$$

It follows that $\sup_{s \in \operatorname{supp}_{\mathcal{F}_{t-1}} S_t} \gamma_t(s, V_{t-1}) \leq \sup_{m \geq 1} \gamma_t(\alpha^m_{t-1}(S_{t-1}), V_{t-1})$ and, finally, the equality holds. □

4.5.2 Continuous set-valued functions

For two topological vector spaces X, Y, consider a set-valued function $\phi : X \twoheadrightarrow Y$. We recall the definition of hemicontinuous set-valued mappings as formulated in [1].

Definition 4.5.9. *We say that ϕ is **lower hemicontinuous** at x if for every open set $U \subset Y$ such that $\phi(x) \cap U \neq \emptyset$, there exits a neighborhood V of x such that $z \in V$ implies $\phi(x) \cap U \neq \emptyset$.*

Definition 4.5.10. *We say that ϕ is **upper hemicontinuous** at x if for every open set $U \subset Y$ such that $\phi(x) \subseteq U$, there is a neighborhood V of x such that $z \in V$ implies $\phi(z) \subset U$.*

Definition 4.5.11. *We say that ϕ is **continuous** at x if it is both upper and lower hemicontinuous at x. It is continuous if it is continuous at any point.*

Lemma 4.5.12. *Let $f : \mathbf{R}^k \to \mathbf{R}_+$ be an upper semicontinuous function. Then, the mapping $x \mapsto \bar{B}(0, f(x))$ is upper hemicontinuous in the sense of definition 4.5.10.*

Proof. The upper hemicontinuity is simple to check. Indeed, consider an open set in $U \subseteq \mathbf{R}^k$, such that $\phi(x) = \bar{B}(0, f(x)) \subset U$. We may suppose that U is bounded w.l.o.g. and we deduce $\epsilon > 0$ such that $\bar{B}(0, f(x)+\epsilon) \subset U$. By upper semicontinuity, there exists an open set V containing x such that $z \in V$ implies $f(z) \leq f(x) + \epsilon$ hence $\phi(z) \subseteq U$. □

Lemma 4.5.13. *Let $f : \mathbf{R}^k \to \mathbf{R}_+$ be a lower semicontinuous function. Then, the mapping $x \mapsto \bar{B}(0, f(x))$ is lower hemicontinuous in the sense of definition 4.5.9.*

Proof. For any ball $B(y, r) \in \mathbf{R}^k$, we have $\bar{B}(0, f(x)) \cap B(y, r) \neq \emptyset$ if and only if $f(x)+r > |y|$. We also have $f(x)-\epsilon+r > |y|$ for some small $\epsilon > 0$. As f is l.s.c., we deduce that $f(z) \geq f(x) - \epsilon$ for every z in some neighborhood V of x. This implies that $f(z) + r > |y|$, i.e. $\bar{B}(0, f(x)) \cap B(y, r) \neq \emptyset$ for every $z \in V$. The conclusion follows. □

Corollary 4.5.14. *Let $f : \mathbf{R}^k \to \mathbf{R}_+$ be a continuous function. Then, the mapping $x \mapsto \bar{B}(0, f(x))$ is continuous in the sense of definition 4.5.11.*

Lemma 4.5.15. *Consider the set-valued mapping $\alpha : \mathbf{R}^m \twoheadrightarrow \mathbf{R}^m$ defined by $\alpha(s) = \mathrm{cl}\{\alpha^m(s), m \in \mathbb{N}\}$ where $(\alpha^m)_{m\geq 1}$ are continuous functions. Then, α is lower hemicontinuous.*

Proof. Consider $\omega \in \Omega$ and some open set $U \in \mathbf{R}^d$. We have $\alpha_t(\omega, z) \cap U \neq \emptyset$ if and only if there is $m \in \mathbb{N}$ such that $\alpha_t^m(\omega, z) \in U$. Since $\alpha_t^m(\omega, .)$ is continuous, we deduce that there exists an open neighborhood V of z such that $\alpha_t^m(\omega, x) \in U$ for any $x \in V$. The conclusion follows. □

We recall a result from [1][Theorem 17.31].

Proposition 4.5.16. *Let $\phi : \mathbf{R}^k \twoheadrightarrow \mathbf{R}^m$ be a continuous set-valued mapping with nonempty compact values and suppose that $f : \mathbf{R}^k \times \mathbf{R}^m \to \mathbf{R}$ is continuous. Then, the function $m(x) = \inf_{y\in\phi(x)} f(x,y)$ and the function $M(x) = \sup_{y\in\phi(x)} f(x,y)$ are continuous.*

Proposition 4.5.17. *Let $\phi : \mathbf{R}^k \twoheadrightarrow \mathbf{R}^m$ be an upper hemicontinous set-valued mapping with nonempty compact values and suppose that $f : \mathbf{R}^k \times \mathbf{R}^m \to \mathbf{R}$ is lower semicontinuous. Then, the function $m(x) = \inf_{y\in\phi(x)} f(x,y)$ is l.s.c.*

Proof. We have $m(x) = -\sup_{y\in\phi(x)} g(x,y)$ where $g = -f$ is upper semicontinuous. By [1][Lemma 17.30], the mapping $x \mapsto \sup_{y\in\phi(x)} g(x,y)$ is upper semicontinuous hence m is l.s.c. □

Lemma 4.5.18. *Let O be an open subset of $\mathbf{R}^k$, if $\gamma : O \to \mathbf{R}$ is l.s.c. and $\gamma \geq g$ on O for some l.s.c. function $g : \mathbf{R}^k \to \mathbf{R}$. Then, there exists a l.s.c. function $\tilde{\gamma} : \mathbf{R}^k \to \mathbf{R}$ such that $\gamma = \tilde{\gamma}$ on O.*

Proof. It suffices to consider $\tilde{\gamma} = \gamma 1_O + g 1_{\Omega\setminus O}$. □

4.5.3 Auxiliary results

Lemma 4.5.19. *Suppose that there is a family of $\mathcal{F}_{t-1}$-measurable random variables $(\alpha_{t-1}^m)_{m\geq 1}$ such that $S_t \in \{\alpha_{t-1}^m : \ m \geq 1\}$ a.s. and suppose that $P(S_t = \alpha_{t-1}^m | \mathcal{F}_{t-1}) > 0$ a.s. for all $m \geq 1$. Then, for any $\mathcal{F}_{t-1}$-measurable random function $f : \Omega \times \mathbf{R}^d \to \mathbf{R}$,*

$$\operatorname{ess\,sup}_{\mathcal{F}_{t-1}} f(S_t) = \sup_{m\geq 1} f(\alpha_{t-1}^m).$$

Proof. It is clear that $\operatorname{ess\,sup}_{\mathcal{F}_{t-1}} f(S_t) \leq \sup_{m\geq 1} f(\alpha_{t-1}^m)$ a.s. since S_t belongs to $\{\alpha_{t-1}^m : \ m \geq 1\}$ and $\sup_{m\geq 1} f(\alpha_{t-1}^m)$ is $\mathcal{F}_{t-1}$-measurable by assumption. On the other hand, consider $\Gamma_t^m := \{S_t \in \alpha_{t-1}^m\} \in \mathcal{F}_t$. We have:

$$\operatorname{ess\,sup}_{\mathcal{F}_{t-1}} f(S_t) 1_{\Gamma_t^m} \geq f(S_t) 1_{\Gamma_t^m} \geq f(\alpha_{t-1}^m) 1_{\Gamma_t^m} \text{ a.s.}$$

Taking the conditional expectation, we get that

$$\begin{aligned} &E(\operatorname{ess\,sup}_{\mathcal{F}_{t-1}} f(S_t) 1_{\Gamma_t^m} | \mathcal{F}_{t-1}) \geq E(f(\alpha_{t-1}^m) 1_{\Gamma_t^m} | \mathcal{F}_{t-1}) \text{ a.s.}, \\ &\operatorname{ess\,sup}_{\mathcal{F}_{t-1}} f(S_t) P(\Gamma_t^m | \mathcal{F}_{t-1})) \geq f(\alpha_{t-1}^m) P(\Gamma_t^m | \mathcal{F}_{t-1})) \text{ a.s.} \end{aligned}$$

As $P(\Gamma_t^m|\mathcal{F}_{t-1})) > 0$ by assumption, we get that ess $\sup_{\mathcal{F}_{t-1}} f(S_t) \geq f(\alpha_{t-1}^m)$ a.s. for any $m \geq 1$ so that the reverse inequality holds. □

Lemma 4.5.20. *Let D^0 given by (4.4.3) with $\xi = 0$. Suppose that* C *is positively super δ-homogeneous. For any $t \leq T$, and any $\lambda_t \in L^0([1,\infty), \mathcal{F}_t)$, we have $D_t^0(\lambda_t V_{t-1}, \lambda_t V_t) \geq \delta(\lambda_t) D_t^0(V_{t-1}, V_t)$ and $\gamma_t^0(\lambda_t V_{t-1}) \geq \delta(\lambda_t)\gamma_t^0(V_{t-1})$ for all $(V_{t-1}, V_t) \in L^0(\mathbf{R}^d, \mathcal{F}_t) \times L^0(\mathbf{R}^d, \mathcal{F}_t)$.*

Proof. For $t = T$, we have by assumption:

$$\gamma_T^0(\lambda_T V_{T-1}) = \mathrm{C}_T((0, -\lambda_T V_{T-1}^{(2)}) \geq \delta(\lambda_T)\mathrm{C}_T((0, -V_{T-1}^{(2)}) = \delta(\lambda_T)\gamma_T^0(V_{T-1}).$$

We deduce that

$$\begin{aligned} \theta_{T-1}^0(\lambda_{T-1}V_{T-1}) &= \text{ess sup}_{\mathcal{F}_{T-1}}\, \gamma_T^0(\lambda_{T-1}V_{T-1}), \\ &\geq \delta(\lambda_{T-1})\, \text{ess sup}_{\mathcal{F}_{T-1}}\, \gamma_T^0(V_{T-1}), \\ &\geq \delta(\lambda_{T-1})\theta_{T-1}^0(V_{T-1}). \end{aligned}$$

As we also have

$$\mathrm{C}_{T-1}((0, \lambda_{T-1}V_{T-1}^{(2)} - \lambda_{T-1}V_{T-2}^{(2)})) \geq \delta(\lambda_{T-1})\mathrm{C}_{T-1}((0, V_{T-1}^{(2)} - V_{T-2}^{(2)})),$$

we deduce that

$$\begin{aligned} & D_{T-1}(\lambda_{T-1}V_{T-2}, \lambda_{T-1}V_{T-1}) \\ = \;& \mathrm{C}_{T-1}((0, \lambda_{T-1}V_{T-1}^{(2)} - \lambda_{T-1}V_{T-2}^{(2)})) + \theta_{T-1}^0(\lambda_{T-1}V_{T-1}), \\ \geq \;& \delta(\lambda_{T-1})\mathrm{C}_{T-1}((0, V_{T-1}^{(2)} - V_{T-2}^{(2)})) + \delta(\lambda_{T-1})\theta_{T-1}^0(V_{T-1}), \\ \geq \;& \delta(\lambda_{T-1})D_{T-1}(V_{T-2}, V_{T-1}). \end{aligned}$$

Therefore, as $\lambda_{T-1} \geq 1$,

$$\begin{aligned} \gamma_{T-1}^0(\lambda_{T-1}V_{T-2}) &= \text{ess inf}_{V_{T-1}\in L^0(\mathbf{R}^d, \mathcal{F}_{T-1})}\, D_{T-1}(\lambda_{T-1}V_{T-2}, \lambda_{T-1}V_{T-1}), \\ &\geq \delta(\lambda_{T-1})\, \text{ess inf}_{V_{T-1}\in L^0(\mathbf{R}^d, \mathcal{F}_{T-1})}\, D_{T-1}(V_{T-2}, V_{T-1}), \\ &\geq \delta(\lambda_{T-1})\gamma_{T-1}^0(V_{T-2}). \end{aligned}$$

We then conclude by induction. □

Lemma 4.5.21. *Suppose that Assumption 4 and Assumption 5 hold. For every $t \leq T$, there exists a continuous function $\hat{h}_t \geq 0$ such that the function D_t^ξ given by (4.4.4) satisfies $|D_t^\xi(s, v_{t-1}, 0)| \leq \hat{h}_t^\xi(s, v_{t-1})$.*

Proof. Recall that $\gamma_T^\xi(V_T) = g^1(S_T) + \mathrm{C}_T(S_T, (0, g^2(S_T) - V_T^{(2)}))$. By assumption on C_T and g, we deduce that $\gamma_T^\xi(V_T) \leq f_T(S_T, V_T)$ where f_T is continuous. Therefore, by Proposition 4.5.6,

$$\begin{aligned}\theta_{T-1}^\xi(V_{T-1}) &= \text{ess sup}_{\mathcal{F}_{T-1}} \gamma_T^\xi(V_{T-1}) \leq \text{ess sup}_{\mathcal{F}_{T-1}} f_T(S_T, V_{T-1}),\\ &\leq \sup_{z \in \text{supp}_{\mathcal{F}_{T-1}} S_T} f_T(z, V_{T-1}) \leq \sup_{z \in \bar{B}(0, R_{T-1}(S_{T-1}))} f_T(z, V_{T-1}).\end{aligned}$$

As R_{T-1} is continuous, we deduce by Corollary 4.5.14 and Proposition 4.5.16 that $\bar{\theta}_{T-1}^\xi(S_{T-1}, V_{T-1}) = \sup_{z \in \bar{B}(0, R_{T-1}(S_{T-1}))} f_T(z, V_{T-1})$ is a continuous function in (S_{T-1}, V_{T-1}). Recall that $\mathrm{C}_{T-1}(S_{T-1}, (0, -V_{T-1}^{(2)}) \leq h_{T-1}(S_{T-1}, V_{T-1})$ where h_{T-1} is continuous. As

$$D_{T-1}^\xi(S_{T-1}, V_{T-1}, 0) = \mathrm{C}_{T-1}(S_{T-1}, (0, -V_{T-1}^{(2)}) + \theta_{T-1}^\xi(V_{T-1}),$$

we deduce that $D_{T-1}^\xi(S_{T-1}, V_{T-1}, 0) \leq \hat{h}_{T-1}^\xi(S_{T-1}, V_{T-1})$ where $\hat{h}_{T-1}^\xi$ is given by $\hat{h}_{T-1}^\xi(S_{T-1}, V_{T-1}) = \bar{\theta}_{T-1}^\xi(S_{T-1}, V_{T-1}) + h_{T-1}(S_{T-1}, V_{T-1})$, i.e. $\hat{h}_{T-1}^\xi$ is continuous. Since $\gamma_{T-1}^\xi(S_{T-1}, V_{T-1}) \leq D_{T-1}^\xi(S_{T-1}, V_{T-1}, 0)$, we deduce that $\gamma_{T-1}^\xi(S_{T-1}, V_{T-1}) \leq \hat{h}_{T-1}^\xi(S_{T-1}, V_{T-1}) = f_{T-1}(S_{T-1}, V_{T-1})$ and we may proceed by induction to conclude. □

Following the same arguments, we also deduce the following:

Lemma 4.5.22. *Suppose that Assumption 4 and Assumption 5 hold. For every $t \leq T$, there exists a continuous function $\bar{h}_t$ such that $\gamma_t^\xi(V_t) \geq \bar{h}_t(S_t, V_t)$.*

Chapter 5

Limit theorems for the super-hedging prices in general models with transaction costs

Abstract

We propose numerical methods that provide estimations of super-hedging prices of European claims in financial market models with transaction costs. The transaction costs we consider are functions of the traded volumes and prices. Contrarily to the usual models of the literature, the transaction costs are not necessary proportional to the traded volumes, neither convex. The particular case of fixed cost is also considered. Limit theorem are established and allow to numerically compute the infimum super-hedging prices.

5.1 Introduction

Computing the super-hedging prices of a European option in presence of transaction costs is a difficult task. Indeed, the classical results of the literature focus on linear transaction costs and only dual characterizations of the super-hedging prices are formulated, see the FTAP theorems (Fundamental Theorem of Asset Pricing) by [40], [39], [55] among others. These results are formulated under rather strong no-arbitrage conditions (see [41], [30]) and the super-hedging prices are estimated through dual characterizations based on the so-called consistent price systems, see [13], [28].

The interesting question is how to implement the FTAP theorem and deduce numerical estimation of the prices. Few attempts have been achieved in that direction, e.g. [68] in the case of a finite probability space. The general case is difficult as we have first to identify the dual elements, i.e. the consistent price systems, which are martingales evolving in the positive duals of the solvency cones. The second step is to propose a numerical procedure to evaluate the possible super-hedging prices. There is no such a numerical method in the literature. Moreover, if the transaction costs are non linear, there is a priori no dual elements characterizing the no-arbitrage condition.

The methods we develop in this paper are based on Chapter 4 where the super-hedging prices are characterized for a large class of transaction cost models which are not necessary linear. In Chapter 4, the results are merely theoretical, we do not provide algorithms to compute the super-hedging costs in practice. In this Chapter, we address this problem. To be more precise, we consider financial markets with transaction costs defined by a cost process $(\mathrm{C}_t)_{0\leq t\leq T}$ depending on traded volumes and a process $(S_t)_{0\leq t\leq T}$ that includes the asset prices. We shall consider the case of countably infinite t-conditional supports for S_{t+1} where an exact characterization of the super-hedging costs is given. The randomized procedure we propose is based on the simulation of conditionally identically distributed random variables which share the same conditional support as the price process $(S_t)_{0\leq t\leq T}$. We formulate a limit theorem, see Theorem 5.3.15, that proves the efficiency of our method.

This Chapter is organized as follows. In Section 5.3, we describe the numerical scheme and the main convergence theorems. We present in Section 5.4 the special case of a model with one risky asset and a piecewise cost process $(\mathrm{C}_t)_{0\leq t\leq T}$. In Section 5.5, we also give the exact solution of the super-hedging cost in the models with proportional costs and with and without fixed cost. Finally, in Section 5.6, we prove a limit theorem for a sequence of financial markets defined by convex cost processes.

5.2 The model

Let $\xi \in L^0(\mathbf{R}^d, \mathcal{F}_T)$ be a contingent claim. Our goal is to characterize the set of all self-financing portfolio processes $(V_t)_{t=-1}^T$ such that $V_T = \xi$. We use the same notations and definitions in Chapter 4. For convenience, we recall the following result from Chapter 4:

Proposition 5.2.1 (Dynamic Programming Principle)**.** *For any* $0 \leq t \leq T-1$ *and* $V_{t-1} \in L^0(\mathbf{R}^d, \mathcal{F}_{t-1})$*, we have*

$$\gamma_t^\xi(V_{t-1}) = \underset{V_t \in L^0(\mathbf{R}^d, \mathcal{F}_t)}{\operatorname{ess\,inf}_{\mathcal{F}_t}} \operatorname{ess\,sup}_{\mathcal{F}_t} \left(\mathrm{C}_t(0, V_t^{(2)} - V_{t-1}^{(2)}) + \gamma_{t+1}^\xi(V_t) \right). \quad (5.2.1)$$

Assumption 7. *The payoff* ξ *is hedgeable, i.e. there exists a portfolio process* $(V_u^\xi)_{u=0}^T$ *such that* $\xi = V_T^\xi$*.*

We briefly recall here the defition of some important functions:

$$\theta_t^\xi(V_t) := \operatorname{ess\,sup}_{\mathcal{F}_t} \gamma_{t+1}^\xi(V_t)$$

and

$$D_t^\xi(V_{t-1}, V_t) = \mathrm{C}_t((0, V_t^{(2)} - V_{t-1}^{(2)})) + \theta_t^\xi(V_t), \quad (5.2.2)$$

$$D_t^\xi(S_t, V_{t-1}, V_t) = \mathrm{C}_t(S_t, (0, V_t^{(2)} - V_{t-1}^{(2)})) + \theta_t^\xi(S_t, V_t). \quad (5.2.3)$$

The second notation is used when we stress the dependence on S_t. Observe that $\gamma_t^\xi(V_{t-1}) = \underset{V_t \in L^0(\mathbf{R}^d, \mathcal{F}_t)}{\operatorname{ess\,inf}_{\mathcal{F}_t}} D_t^\xi(V_{t-1}, V_t)$.

In order to numerically compute the minimal costs, we need to impose the finiteness of $\gamma_t^\xi(V_{t-1})$, i.e. $\gamma_t^\xi(V_{t-1}) > -\infty$ a.s., at any time t and for all $V_{t-1} \in L^0(\mathbf{R}^d, \mathcal{F}_{t-1})$. This is why, we recall the following condition, see Chapter 4:

Definition 5.2.2. *We say that the financial market satisfies the Absence of Early Profit condition (AEP) if, at any time* $t \leq T$*, and for all* $V_t \in L^0(\mathbf{R}^d, \mathcal{F}_t)$*,* $\gamma_t^0(V_t) > -\infty$ *a.s..*

5.3 Numerical schemes

In the following, we suppose the following assumptions on the cost process C. For any $t \leq T$, the cost function C_t is a lower-semi continuous Borel function defined on $\mathbf{R}^k \times \mathbf{R}^d$ such that

$\mathrm{C}_t(s, 0) = 0$, $\forall s \in \mathbf{R}_+^k$,

$\mathrm{C}_t(s, x + \lambda e_1) = \mathrm{C}_t(s, x) + \lambda$, $\lambda \in \mathbf{R}$, $x \in \mathbf{R}^d$, $s \in \mathbf{R}_+^k$ (cash invariance),

$\mathrm{C}_T(s, x_2) \geq \mathrm{C}_T(s, x_1)$, $\forall x_1, x_2$ s.t. $x_2 - x_1 \in \mathbf{R}_+^d$ (C_T is increasing w.r.t. $\mathbf{R}_+^d$).

Note that C_T is increasing w.r.t. $\mathbf{R}_+^d$ is equivalent to $\mathbf{G}_T + \mathbf{R}_+^d \subseteq \mathbf{G}_T$. Moreover, for some $a \geq 0$, we say that C_t is a- super homogeneous if the following property holds:

$$\mathrm{C}_t(s, \lambda x) \geq \lambda \mathrm{C}_t(s, x), \forall \lambda \geq a,\ s \in \mathbf{R}_+^k,\ x \in \mathbf{R}^d.$$

5.3.1 The one period model

In this section, we consider two complete sub σ-algebras $\mathcal{F}_t$ and $\mathcal{F}_{t+1}$ such that $\mathcal{F}_t \subset \mathcal{F}_{t+1} \subset \mathcal{F}$ and an adapted price process $(S_s)_{s=t,t+1}$ satisfying the following assumption.

Assumption 8. *Suppose that there is a family of $\mathcal{F}_t$-measurable random variables $(\alpha_t^m)_{m\geq 1}$ such that $S_{t+1} \in \{\alpha_t^m :\ m \geq 1\}$ a.s. and suppose that $P(S_{t+1} = \alpha_t^m | \mathcal{F}_t) > 0$ a.s. for all $m \geq 1$. Moreover, we suppose that there exists continuous functions on $\mathbf{R}^m$, that we still denote by α_t^m with an abuse of notation, such that $\alpha_t^m = \alpha_t^m(S_t)$.*

In Chapter 4, we have shown the following:

Lemma 5.3.1. *Suppose that Assumption 8 holds. Then, for any Borel function $f : \mathbf{R}^d \to \mathbf{R}$, we have*

$$\text{ess sup}_{\mathcal{F}_t} f(S_{t+1}) = \sup_{m\geq 1} f(\alpha_t^m), \text{ a.s..}$$

Definition 5.3.2. *The random variables $\{b_{t+1}^i, i \geq 1\}$, $b_{t+1}^i \in L^0(\mathbf{R}^k, \mathcal{F}_{t+1})$, are said independent and identically distributed conditionally to $\mathcal{F}_t$ (for short $\mathcal{F}_t$-i.i.d.) if, for all finite set $J \subset \mathbb{N}$, and Borel sets $B, B_j, j \in J$:*

$$P\left[b_{t+1}^i \in B | \mathcal{F}_t\right] = P\left[b_{t+1}^j \in B | \mathcal{F}_t\right], \text{ a.s. } \forall i, j \geq 1,$$
$$P\left[\bigcap_{j\in J} \left\{b_{t+1}^j \in B_j\right\} \middle| \mathcal{F}_t\right] = \prod_{j\in J} P\left[b_{t+1}^j \in B_j \middle| \mathcal{F}_t\right], \text{ a.s..}$$

Lemma 5.3.3. *Consider a family of $\mathcal{F}_t$-i.i.d. random variables b_{t+1}^i, $i \geq 1$ and $\theta_t \in L^0(\mathbf{R}^m, \mathcal{F}_t)$. Let $f^j : \mathbf{R}^k \times \mathbf{R}^m \to \mathbf{R}$, $j = 1, \cdots, n$ be $n \geq 1$ measurable functions such that $E\left[|f^j(b_{t+1}^1, \theta_t)||\mathcal{F}_t\right] < \infty$ a.s. (resp. f^j is*

non negative), for all $j \leq n$. Then, for any finite set $J \subset \mathbb{N}$ of cardinality n, we have:

$$E\left[f^k(b_{t+1}^i, \theta_t)|\mathcal{F}_t\right] = E\left[f^k(b_{t+1}^j, \theta_t)|\mathcal{F}_t\right], \text{ a.s., } i, j, k \geq 1,$$
$$E\left[\prod_{j\in J} f^j(b_{t+1}^j, \theta_t)\big|\mathcal{F}_t\right] = \prod_{j\in J} E\left[f^j(b_{t+1}^j, \theta_t)\big|\mathcal{F}_t\right], \text{ a.s..}$$

Proof. We prove the result by induction on n. Suppose that $f^j = 1_{D_j}$ where $D_j = B_j \times A_j$ and $B_j \in \mathcal{B}(\mathbf{R}^k), A_j \in \mathcal{B}(\mathbf{R}^m)$. Then, the claim holds by definition of the $\mathcal{F}_t$-i.i.d. random variables for all $n \geq 1$ and the $\mathcal{F}_t$-measurability of θ_t. By the monotone class argument, this holds for any $D_1 \in \mathcal{B}(\mathbf{R}^k)\otimes\mathcal{B}(\mathbf{R}^m)$ if $n = 1$. If $n > 1$, we expand the product in the second claim and we use the induction hypothesis. Then, we repeat the arguments for $D_2 \in \mathcal{B}(\mathbf{R}^k) \otimes \mathcal{B}(\mathbf{R}^m)$ and so on. By linearity, and the induction argument after having expanding the product, we also deduce that the claim holds when $f^j = \sum_{h=1}^n c_h^j 1_{C_h^j}$ and for any $c_h^j \in \mathbf{R}, C_h^j \in \mathcal{B}(\mathbf{R}^k) \otimes \mathcal{B}(\mathbf{R}^m), h \geq 1$. By standard increasing approximations, we conclude in the case where $f^j \geq 0$. Otherwise, we write $f^j = (f^j)^+ - (f^j)^-$. In particular, we get that

$$E\left[|f^j(b_{t+1}^i, \theta_t)||\mathcal{F}_t\right] = E\left[|f^j(b_{t+1}^1, \theta_t)||\mathcal{F}_t\right] < \infty, \text{ a.s.}$$

in the case where $E\left[|f^j(b_{t+1}^1, \theta_t)||\mathcal{F}_t\right] < \infty$. □

Lemma 5.3.4. *Consider a Borel function $f : \mathbf{R}^k \to \mathbf{R}$ and a family of $\mathcal{F}_t$-i.i.d. random variables $(b_{t+1}^m)_{m\geq 1}$ with values in $\mathbf{R}^k$ and $\mathcal{F}_{t+1}$-measurable. Suppose that there exists $\mathcal{F}_t$-measurable random variables $(\alpha_t^n)_{n\geq 1}$ such that $b_{t+1}^m \in \{\alpha_t^n, n \geq 1\}$ a.s. and $P(b_{t+1}^m = \alpha_t^n|\mathcal{F}_t) > 0$ a.s. for all $n, m \geq 1$.*

Let us define $\theta_t := \sup_{m\geq 1} f(\alpha_t^m) = \operatorname{ess\,sup}_{\mathcal{F}_t} f(S_{t+1})$ (by Lemma 5.3.1) and $\theta_t^m := \max_{i\leq m} f(b_{t+1}^i)$. The following holds:

$$\theta_t^m \to \theta_t, \text{ a.s. as } m \to \infty.$$

In particular, $\sup_m \theta_t^m = \theta_t$ a.s.

Proof. We may suppose w.l.o.g. that $\theta_t < \infty$. Indeed, we may consider $g(\theta_t)$ and the sequence $(g(\theta_t^m))_{m\geq 1}$ where g is a bounded strictly increasing continuous function in the contrary case. By Lemma 5.3.1, we get that

$\text{ess sup}_{\mathcal{F}_t} f(b^1_{t+1}) = \sup_{m\geq 1} f(\alpha^m_t) = \theta_t$ a.s. For any $\epsilon > 0$, we deduce by assumption that

$$\begin{aligned} P[\theta_t - \theta^m_t > \epsilon|\mathcal{F}_t] &= P[\theta_t - \max_{i\leq m} f(b^i_{t+1}) > \epsilon|\mathcal{F}_t] \\ &= P[\theta_t - f(b^i_{t+1}) > \epsilon, \forall i \leq m|\mathcal{F}_t] \\ &= E\left[\prod_{i=1}^m 1_{\{\theta_t - f(b^i_{t+1})>\epsilon\}}\big|\mathcal{F}_t\right], \text{ a.s..} \end{aligned}$$

By Lemma 5.3.3, we deduce that

$$\begin{aligned} P[\theta_t - \theta^m_t > \epsilon|\mathcal{F}_t] &= P[\theta_t - f(b^1_{t+1}) > \epsilon|\mathcal{F}_t]^m \\ &= P[\text{ess sup}_{\mathcal{F}_t} f(b^1_{t+1}) - f(b^1_{t+1}) > \epsilon|\mathcal{F}_t]^m, \text{ a.s.} \end{aligned}$$

We claim that $P[\text{ess sup}_{\mathcal{F}_t} f(b^1_{t+1}) - f(b^1_{t+1}) > \epsilon|\mathcal{F}_t] < 1$ a.s. Indeed, assume on the contrary that $P[\text{ess sup}_{\mathcal{F}_t} f(b^1_{t+1}) - f(b^1_{t+1}) > \epsilon|\mathcal{F}_t] = 1$ on some non null set $\Lambda_t \in \mathcal{F}_t$. In other words, we have

$$E\left[1_{\{\text{ess sup}_{\mathcal{F}_t} f(b^1_{t+1})>f(b^1_{t+1})+\epsilon\}}\big|\mathcal{F}_t\right] 1_{\Lambda_t} = 1_{\Lambda_t}.$$

Taking the expectation, we deduce that:

$$E\left[1_{\{\text{ess sup}_{\mathcal{F}_t} f(b^1_{t+1})>f(b^1_{t+1})+\epsilon\}} 1_{\Lambda_t}\right] = E\left[1_{\Lambda_t}\right]$$

We then deduce that $1_{\{\text{ess sup}_{\mathcal{F}_t} f(b^1_{t+1})>f(b^1_{t+1})+\epsilon\}} 1_{\Lambda_t} = 1_{\Lambda_t}$ a.s. We now define $\hat{\theta}_t := \text{ess sup}_{\mathcal{F}_t} f(b^1_{t+1}) 1_{\Omega\setminus\Lambda_t} + (\text{ess sup}_{\mathcal{F}_t} f(b^1_{t+1}) - \epsilon) 1_{\Lambda_t}$. Observe that $\hat{\theta}_t$ is $\mathcal{F}_t$-measurable and $\hat{\theta}_t \geq f(b^1_{t+1})$ a.s. However, $\hat{\theta}_t < \text{ess sup}_{\mathcal{F}_t} f(b^1_{t+1})$ on the non null set Λ_t, in contradiction with the definition of the conditional essential supremum. Therefore,

$$\lim_{m\to\infty} P[\theta_t - \theta^m_t > \epsilon|\mathcal{F}_t] = 0, \text{ a.s.}$$

Finally, by the dominated convergence theorem, we have

$$\begin{aligned} \lim_{m\to\infty} P[\theta_t - \theta^m_t > \epsilon] &= \lim_{m\to\infty} E\left[E[1_{\{\theta_t - \theta^m_t>\epsilon\}}|\mathcal{F}_t]\right] \\ &= E\left[\lim_{m\to\infty} E[1_{\{\theta_t - \theta^m_t>\epsilon\}}|\mathcal{F}_t]\right] \\ &= 0. \end{aligned}$$

Hence θ^m_t increasingly tends to θ_t in probability, i.e. $\sup_m \theta^m_t = \theta_t$ a.s.. □

Assumption 9. *The payoff function ξ is of the form $\xi = g(S_T)$, where $g \in \mathbf{R}_+^k$ is continuous. Moreover, ξ is hedgeable, i.e. there exists a portfolio process $(V_u^\xi)_{u=-1}^T$ such that $\xi = V_T^\xi$.*

We recall two weak no-arbitrage conditions introduced in Chapter 4:

Definition 5.3.5. *We say that the condition AIP holds at time t if the minimal cost $c_t(0) = \gamma_t^0(0)$ of the European zero claim $\xi = 0$ is 0 at time $t \leq T$. We say that AIP holds if AIP holds at any time.*

The following condition is more technical.

Definition 5.3.6. *We say that the condition SAIP (Strong AIP condition) holds at time t if AIP holds at time t and, for any $Z_t \in L^0(\mathbf{R}^d, \mathcal{F}_t)$, we have $D_t^0(S_t, 0, Z_t) = 0$ if and only if $Z_t^{(2)} = 0$ a.s.. We say that SAIP holds if SAIP holds at any time.*

We now introduce the sequence of functions which is defined recursively as follows:

$$\begin{aligned}
&\tilde{\gamma}_T^\xi(s, v_{T-1}) := \xi^1 + \mathrm{C}_T(s, (0, \xi^{(2)} - v_{T-1}^{(2)})), \quad v_{t-1}, \xi \in \mathbf{R}^d,\ s \in \mathbf{R}^k,\\
&\tilde{\theta}_t^\xi(s, v_t) := \sup_m \tilde{\gamma}_{t+1}^\xi(\alpha_t^m(s), v_t),\ t \leq T-1,\ v_t \in \mathbf{R}^d,\\
&\tilde{D}_t^\xi(s, v_{t-1}, v_t) := \tilde{\theta}_t^\xi(s, v_t) + \mathrm{C}_t(s, (v_t^{(2)} - v_{t-1}^{(2)})),\\
&\tilde{\gamma}_t^\xi(s, v_{t-1}) := \mathrm{cl}\left(\inf_{v_t \in \mathbf{R}^d} \tilde{D}_t^\xi(s, v_{t-1}, v_t)\right). \qquad (5.3.4)
\end{aligned}$$

Here, the notation $\mathrm{cl}(f)$ designates the l.s.c. regularization of f. In this paper, we will impose later in the sequel a condition under which we have $\tilde{\gamma}_t^\xi(s, v_{t-1}) := \inf_{v_t \in \mathbf{R}^d} \tilde{D}_t^\xi(s, v_{t-1}, v_t)$.

The introduction of the functions above is motivated by the following result proved in Chapter 4.

Theorem 5.3.7. *Suppose that either AIP holds and $\mathrm{C}_t(s, .)$ is convex for fixed s or SAIP holds. Then, we have $\gamma_t^\xi(S_t, V_t) = \tilde{\gamma}_t^\xi(S_t, V_t)$ a.s. and, also, $\theta_t^\xi(S_t, V_t) = \tilde{\theta}_t^\xi(S_t, V_t)$ a.s. and $D_t^\xi(S_t, V_{t-1}, V_t) = \tilde{D}_t^\xi(S_t, V_{t-1}, V_t)$ for any $V_{t-1}, V_t \in L^0(\mathbf{R}^d, \mathcal{F}_t)$. Moreover, $\tilde{\gamma}_t^\xi(s, v)$ is l.s.c. on $\mathbf{R}^k \times \mathbf{R}^d$ and convex in v when $\mathrm{C}_t(s, .)$ is convex.*

Recall that the family of $\mathcal{F}_t$-measurable random variables $(\alpha_t^n(S_t))_{n\geq 1}$ is defined in Assumption 8. We now consider an $\mathcal{F}_t$-i.i.d. sample of random variables $\{b_{t+1}^i, i \geq 1\}$ that satisfies $P[b_{t+1}^1 = \alpha_t^n(S_t)|\mathcal{F}_t] > 0$ a.s. for all $n \geq 1$ and $b_{t+1}^1 \in \{\alpha_t^n(S_t), n \geq 1\}$ a.s. Now, let us define the (random) functions

$$\begin{aligned}
\bar{D}_T^\xi(s,x,y) &:= \tilde{\gamma}_T^\xi(s,y), \\
\bar{D}_t^\xi(s,x,y) &:= \mathrm{C}_t(s,(0,y^{(2)}-x^{(2)})) + \tilde{\gamma}_{t+1}^\xi(s,y), \\
\bar{D}_T^n(\omega,x,y) &:= \bar{D}_T^\xi(s,x,y) \\
\bar{D}_t^n(\omega,x,y) &:= \max_{i\leq n} \bar{D}_t^\xi(b_{t+1}^i(\omega),x,y).
\end{aligned} \tag{5.3.5}$$

Since $\tilde{\gamma}_{t+1}^\xi(s,x)$ is l.s.c. in s, it is Borel in s for fixed x. Then, by Lemma 5.3.4, we deduce that:

$$\lim_{n\to\infty} \max_{i\leq n} \tilde{\gamma}_{t+1}^\xi(b_{t+1}^i(\omega),y) = \sup_n \tilde{\gamma}_{t+1}^\xi(\alpha_t^n(S_t(\omega)),y) = \tilde{\theta}_t^\xi(S_t(\omega),y), \text{a.s.}$$

In particular, $\lim_{n\to\infty} \bar{D}_t^n(\omega,x,y) = \tilde{D}_t^\xi(S_t(\omega),x,y)$. We now investigate the question whether $\inf_{y\in\mathbf{R}^d} \bar{D}_t^n(\omega,x,y)$ converge a.s.(ω) to $\inf_{y\in\mathbf{R}^d} \tilde{D}_t^\xi(\omega,x,y)$ as $n\to\infty$. To do so, we first recall the definition of epi-convergence, see [70, Chapter 3] or [76, Chapter 7]. In the following, the notation $B(x,r)$ designates the closed ball of $\mathbf{R}^d$, where $d \geq 1$ depends on the context, centered a point $x \in \mathbf{R}^d$ and of radius $r \geq 0$.

Definition 5.3.8. *Let $f_n : \mathbf{R}^k \to \overline{\mathbf{R}}$, $n \geq 1$, be a sequence of functions. The **epi-limit inferior** $\mathrm{li}_e f_n$ and **epi-limit superior** $\mathrm{ls}_e f_n$ of $(f_n)_{n\geq 1}$ are defined as:*

$$\begin{aligned}
\mathrm{li}_e[(f_n)_{n\geq 1}](u) &:= \sup_{k\geq 1} \liminf_{n\to\infty} \inf_{v\in B(u,1/k)} f_n(v), \\
\mathrm{ls}_e[(f_n)_{n\geq 1}](u) &:= \sup_{k\geq 1} \limsup_{n\to\infty} \inf_{v\in B(u,1/k)} f_n(v).
\end{aligned}$$

The sequence $(f_n)_{n\geq 1}$ is said to be epi-convergent at point u if

$$\mathrm{li}_e[(f_n)_{n\geq 1}](u) = \mathrm{ls}_e[(f_n)_{n\geq 1}](u).$$

We also introduce the definition of almost sure epi-convergence for random functions.

Definition 5.3.9. *If $(f_n)_{n\geq 1}$, is a sequence of functions $f_n : \Omega \times \mathbf{R}^k \to \overline{\mathbf{R}}$ such that f_n is $\mathcal{F}_t \otimes \mathcal{B}(\mathbf{R}^d)$-measurable for each n, we say that f_n epi-converges to f almost surely (notation $f_n \xrightarrow{epi} f$ a.s.) if, for any ω outside a P-null set, and for all u: $\mathrm{li}_e[(f_n(\omega,\cdot))_{n\geq 1}](u) = \mathrm{ls}_e[(f_n(\omega,\cdot))_{n\geq 1}](u) = f(\omega,u)$.*

Theorem 5.3.10. *Suppose that AIP holds and $\mathrm{C}_t(s,y)$ is convex in y. We then have $\bar{D}_t^n(\omega,.,.) \vee \left(-\mathrm{C}_t(S_t(\omega),(0,x^{(2)}))\right) \xrightarrow{\text{epi}} \tilde{D}_t^\xi(S_t(\omega),.,.)$ a.s.(ω), as $n \to \infty$.*

Suppose that for any t, we have $\mathrm{C}_t(s,v_t^1) \geq \mathrm{C}_t(s,v_t^2)$ if $v_t^1 \geq_{\mathbf{R}_+^d} v_t^2$. Then, $\bar{D}_t^n(\omega,.,.) \xrightarrow{\text{epi}} \tilde{D}_t^\xi(S_t(\omega),.,.)$ a.s..

Proof. We first consider the case where AIP holds and $\mathrm{C}_t(s,y)$ is convex in y. Let us define $\bar{L}_t^\xi(\omega,x,y) := \bar{D}_t^n(\omega,x,y) \vee \left(-\mathrm{C}_t(S_t(\omega),(0,x^{(2)}))\right)$. Observe that $\bar{L}_t^n(\omega,x,y)$ is l.s.c. in (x,y) as a maximum of two l.s.c. functions. As the sequence $(\bar{L}_t^n)_{n\geq 1}$ is also non decreasing, we deduce by [76, Proposition 7.4], that for any ω:

$$\mathrm{li}_e[(\bar{L}_t^n(\omega,\cdot,\cdot)_{n\geq 1})](x,y) = \mathrm{ls}_e[(\bar{L}_t^n(\omega,\cdot,\cdot)_{n\geq 1})](x,y) = \sup_n \bar{L}_t^n(\omega,x,y).$$

We now prove that there exists a negligible set H such that for any $\omega \in \Omega \setminus H$ and $x,y \in \mathbf{R}^d \times \mathbf{R}^d$ the following holds:

$$\sup_n \bar{L}_t^n(\omega,x,y) = \tilde{D}_t^\xi(\omega,x,y). \tag{5.3.6}$$

By assumption on $(\mathrm{C}_t)_{t\geq 0}$, we get by induction that $\theta_t^\xi(V_t) \geq \theta_t^0(V_t)$ a.s. for any $V_t \in L^0(\mathbf{R}^d,\mathcal{F}_t)$. We deduce that $D_t^\xi(V_{t-1},V_t) \geq -\mathrm{C}_t(S_t(\omega),(0,V_{t-1}^{(2)}))$ for any for any $V_{t-1},V_t \in L^0(\mathbf{R}^d,\mathcal{F}_t)$. Indeed, under AIP, $D_t^0(0,V_t) \geq 0$ a.s. hence

$$\begin{aligned}
D_t^\xi(V_{t-1},V_t) &= \theta_t^\xi(V_t) + \mathrm{C}_t(S_t,(0,V_t^{(2)} - V_{t-1}^{(2)})) \\
&\geq \theta_t^\xi(V_t) + \mathrm{C}_t(S_t,(0,V_t^{(2)})) - \mathrm{C}_t(S_t,(0,V_{t-1}^{(2)})), \text{ (by subadditivity)} \\
&\geq \theta_t^0(V_t) + \mathrm{C}_t(S_t,(0,V_t^{(2)})) - \mathrm{C}_t(S_t,(0,V_{t-1}^{(2)})) \\
&\geq D_t^0(0,V_t) - \mathrm{C}_t(S_t,(0,V_{t-1}^{(2)})) \geq -\mathrm{C}_t(S_t,(0,V_{t-1}^{(2)})), \text{ a.s.}
\end{aligned}$$

for any $V_{t-1},V_t \in L^0(\mathbf{R}^d,\mathcal{F}_t)$.

We now deduce that $\tilde{D}_t^\xi(S_t(\omega), x, y) \geq -\mathrm{C}_t\left(S_t(\omega), (0, x^{(2)})\right)$ for every x, y a.s.(ω). Indeed, suppose on the contrary that the $\mathcal{F}_t$-measurable set

$$\Gamma_t(\omega) := \left\{(x, y) \in \mathbf{R}^d \times \mathbf{R}^d : \tilde{D}_t^\xi(S_t(\omega), x, y) < -\mathrm{C}_t\left(S_t(\omega), (0, x^{(2)})\right)\right\}$$

is non-empty on the non-null set $G_t := \{\omega : \Gamma_t(\omega) \neq \emptyset\}$. We then deduce a measurable selection $(\bar{V}_{t-1}, \bar{V}_t) \in L^0(\mathbf{R}^d, \mathcal{F}_t) \times L^0(\mathbf{R}^d, \mathcal{F}_t)$ such that we have $\tilde{D}_t^\xi(S_t, \bar{V}_t, \bar{V}_{t-1}) < -\mathrm{C}_t(S_t, (0, \bar{V}_{t-1}^{(2)}))$ on G_t and we extend to the whole space by putting $\bar{V}_{t-1} = 0 = \bar{V}_t$ on the complementary set $\Omega \setminus G_t$. Moreover, by Theorem 5.5.5, we then deduce that $D_t^\xi(\bar{V}_t, x) < -\mathrm{C}_t(S_t, (0, \bar{V}_{t-1}^{(2)})$ on the non-null set G_t, which is a contradiction.

Similarly, under AEP and Assumption 1, we have that $D_t^\xi(V_{t-1}, V_t) \in \mathbf{R}$ a.s. for any $V_{t-1}, V_t \in L^0(\mathbf{R}^d, \mathcal{F}_t)$, see Chapter 4. Then, by a measurable selection argument, using the fact that $D_t^\xi(V_{t-1}, V_t) = \tilde{D}_t^\xi(S_t, V_{t-1}, V_t)$ a.s., we deduce that $\tilde{D}_t^\xi(S_t(\omega), x, y) \in \mathbf{R}$ for any x, y, for any ω outside a negligible set.

By Lemma 5.3.4, $\bar{L}_t^n(\omega, x, y) \to \tilde{D}_t^\xi(S_t(\omega), x, y) \vee (-\mathrm{C}_t(S_t(\omega), (0, x^{(2)}))$ as $n \to \infty$ for any ω outside a negligible set $N(x, y)$. Moreover, by the discussion above, we deduce a negligible set M such that for any $\omega \in \Omega \setminus M$, we have $\tilde{D}_t^\xi(S_t(\omega), x, y) \geq -\mathrm{C}_t(S_t(\omega), (0, x^{(2)}))$ and $\tilde{D}_t^\xi(S_t(\omega), x, y) \in \mathbf{R}$ for any x, y. We set $H := \cup_{y \in \mathbf{Q}^d} N(x, y) \cup M$, we claim that for any $\omega \in \Omega \setminus H$, $\sup_n \bar{L}_t^n(\omega, x, y) = \tilde{D}_t^\xi(S_t(\omega), x, y)$ for all $x, y \in \mathbf{R}^d$. Indeed, by the definition of H, we deduce that (5.3.6) holds for any $y \in \mathbf{Q}^d$. Now, since $\tilde{D}_t^\xi(S_t(\omega), ., .)$ is convex and takes values in $\mathbf{R}$, it is continuous for any $\omega \in \Omega \setminus H$. Moreover, we claim that $\sup_n \bar{L}_t^n(\omega, x, y) < \infty$ for any $x, y \in \mathbf{R}^d$ and $\omega \in \Omega \setminus H$. Indeed, by lower semicontinuity, we have:

$$\sup_n \bar{L}_t^n(\omega, x, y) \leq \liminf_k \sup_n \bar{L}_t^n(\omega, x_k, y_k)$$

for any sequence $x_k, y_k \in \mathbf{Q}^d$ such that $x_k \to x$ and $y_k \to y$. Moreover, by the definition of H and the continuity of $\tilde{D}_t^\xi(S_t(\omega), ., .)$ for any $\omega \in \Omega \setminus H$, we have $\liminf_k \sup_n \bar{L}_t^n(\omega, x_k, y_k) = \liminf_k \tilde{D}_t^\xi(S_t(\omega), x_k, y_k) = D_t^\xi(S_t(\omega), x, y) \in \mathbf{R}$. We deduce that $\sup_n \bar{L}_t^n(\omega, x, y) \in \mathbf{R}$ for any $x, y \in \mathbf{R}^d$, and $\omega \in \Omega \setminus H$. Moreover, $\sup_n \bar{L}_t^n(\omega, ., .)$ also convex as a supremum of convex functions, it is then also continuous. We then deduce by continuity that (5.3.6) holds for any $y \in \mathbf{R}^d$.

Now, we consider the second case where $\mathrm{C}_t(s, v_t^1) \geq \mathrm{C}_t(s, v_t^2)$ for any $v_t^1, v_t^2 \in \mathbf{R}^d$ such that $v_t^1 \geq_{\mathbf{R}_+^d} v_t^2$. Similarly to the first case, we only need to

prove $\sup_n \bar{D}_t^n(\omega, x, y) = \tilde{D}_t^\xi(S_t(\omega), x, y)$ for all x, y and ω outside a negligible set. By the definition of $\tilde{\gamma}_t^\xi$ and $\tilde{\theta}_t^\xi$, we can show by induction an by Lemma 5.3.11 that the mappings $y \mapsto \tilde{\theta}_t^\xi(s, y)$ and $y \mapsto \tilde{\gamma}_t^\xi(s, y)$ are decreasing with respect to $\mathbf{R}_+^d$.

Recall the definition of $N(x, y)$, we also denote $H := \cup_{y \in \mathbf{Q}^d} N(x, y) \cup M$ and claim that for any $\omega \in \Omega \setminus H$, $\sup_n \tilde{\gamma}_{t+1}^\xi(b_{t+1}^n(\omega), y) = \tilde{\theta}_t^\xi(S_t(\omega), y)$, for all $y \in \mathbf{R}^d$. Indeed, fix some $y \in \mathbf{R}^d$ and a sequence $(y_k)_{k \geq 1}$ in $\mathbf{Q}^d$ such that $y_k \to y$ and $y_k \geq_{\mathbf{R}_+^d} y$. By lower semicontinuity and the discussion above, we have for any $\omega \in \Omega \setminus H$:

$$\tilde{\theta}_t^\xi(S_t(\omega), y) \leq \liminf_k \tilde{\theta}_t^\xi(S_t(\omega), y_k) \leq \tilde{\theta}_t^\xi(S_t(\omega), y), \text{ and}$$
$$\sup_n \tilde{\gamma}_{t+1}^\xi(b_{t+1}^n(\omega), y) \leq \liminf_k \sup_n \tilde{\gamma}_{t+1}^\xi(b_{t+1}^n(\omega), y_k) \leq \sup_n \tilde{\gamma}_{t+1}^\xi(b_{t+1}^n(\omega), y).$$

Then, we have

$$\begin{aligned} \tilde{\theta}_t^\xi(S_t(\omega), y) &= \liminf_k \tilde{\theta}_t^\xi(S_t(\omega), y_k), \\ \sup_n \tilde{\gamma}_{t+1}^\xi(b_{t+1}^n(\omega), y) &= \liminf_k \sup_n \tilde{\gamma}_{t+1}^\xi(b_{t+1}^n(\omega), y_k). \end{aligned}$$

Moreover, by the definition of H, we have $\sup_n \tilde{\gamma}_{t+1}^\xi(b_{t+1}^n(\omega), y_k) = \tilde{\theta}_t^\xi(S_t(\omega), y_k)$ for any $\omega \in \Omega \setminus H$. We then deduce that $\sup_n \tilde{\gamma}_{t+1}^\xi(b_{t+1}^n(\omega), y) = \tilde{\theta}_t^\xi(S_t(\omega), y)$ for any $\omega \in \Omega \setminus H$. At last, by the definition of $\tilde{D}_t^\xi$ and $\bar{D}_t^n$, we conclude that $\sup_n \bar{D}_t^n(\omega, x, y) = \tilde{D}_t^\xi(S_t(\omega), x, y)$ for any x, y and $\omega \in \Omega \setminus H$. □

In the Proof of Theorem 5.3.10, we have used the following result:

Lemma 5.3.11. *Let $f : \mathbf{R}^k \to \mathbf{R}$ be a function such that f that is non increasing with respect to the partial order $\geq_{\mathbf{R}_+^k}$. Consider $\mathrm{cl}(f)$ the lower semicontinuous regularization of f. Then, $\mathrm{cl}(f)$ is non increasing w.r.t. the partial order $\geq_{\mathbf{R}_+^k}$.*

Proof. From [76, Lemma 1.7], we have the following representation of the l.s.c. closure:

$$\mathrm{cl}(f)(x) = \liminf_{y \to x} f(x) = \min \left\{ \alpha \in \overline{\mathbf{R}} : \ \exists (x_n)_{n \geq 1},\ x_n \to x, \lim_n f(x_n) = \alpha \right\}.$$

Consider $x^1, x^2 \in \mathbf{R}^d$ such that $x^1 \geq_{\mathbf{R}_+^d} x^2$ and a sequence $(x_n)_{n \geq 1}$ such that $x_n \to x^2$ and $f(x_n) \to \mathrm{cl}(f)(x^2)$ as $n \to \infty$. Observe that $x_n + x^1 - x^2 \to x^1$

as $n \to \infty$. We then have $f(x_n + x^1 - x^2) \leq f(x_n)$ by our hypothesis. We deduce that

$$\mathrm{cl}(f)(x^1) \leq \liminf_n f(x_n + x^1 - x^2) \leq \lim_n f(x_n) = \mathrm{cl}(f)(x^2).$$

□

Definition 5.3.12. *We say that a set-valued mapping $K_t : \mathbf{R}_+^k \times \mathbf{R}^d \twoheadrightarrow \mathbf{R}^d$ is a reachability set at time $t \leq T$ for the super-hedging problem if K_t has compact set values and satisfies:*

$$\inf_{y \in \mathbf{R}^d} \tilde{D}_t^\xi(S_t(\omega), x, y) = \inf_{y \in K_t(S_t(\omega), x)} \tilde{D}_t^\xi(S_t(\omega), x, y), \text{ a.s..}$$

Moreover, we suppose that $K_t(s, x)$ is upper hemicontinuous in (s, x).

Remark 5.3.13. *By Chapter 4, under SAIP, the determining set $K_t(s, x)$ is constructed for $s = S_t(\omega)$ as a closed ball $\overline{B}(0, r_t(s, x) + 1)$, where $r_t(s, x)$ is an u.s.c. function. We shall see later in the model with one risky asset how to characterize $K_t(s, x)$ explicitely for **every** $(s, x) \in \mathbf{R} \times \mathbf{R}$ such that $K_t(s, x)$ is compact for all (s, x) and upper hemicontinuous. Moreover, By [1, Lemma 17.29], the upper hemicontinuity of K implies that*

$$\tilde{\gamma}_t^\xi(s, v_{t-1}) := \inf_{v_t \in \mathbf{R}^d} \tilde{D}_t^\xi(s, v_{t-1}, v_t). \tag{5.3.7}$$

Theorem 5.3.14. *Suppose that SAIP holds and $\mathrm{C}_t(s, v_t^1) \geq \mathrm{C}_t(s, v_t^2)$ for any $v_t^1, v_t^2 \in \mathbf{R}^d$ such that $v^1 \geq_{\mathbf{R}_+^d} v^2$. Then, we have:*

$$\lim_{n \to \infty} \inf_{y \in K_t(S_t(\omega), x)} \bar{D}_t^n(\omega, x, y) = \inf_{y \in K_t(S_t(\omega), x)} \tilde{D}_t^\xi(S_t(\omega), x, y), \quad \forall x, y, \text{ a.s.} \tag{5.3.8}$$

Moreover, for each fixed $x_t \in L^0(\mathbf{R}^d, \mathcal{F}_t)$ such that the random set $K_t(S_t, x_t)$ is $\mathcal{F}_t$-measurable, there exists a sequence $(\hat{y}_{t+1}^n)_{n \geq 1}$ of $L^0(\mathbf{R}^d, \mathcal{F}_{t+1})$ such that $\hat{y}_{t+1}^n \in \arg\min_{K_t(S_t, x_t)}(\bar{D}_t^n(\omega, x_t, .))$ a.s. and $\hat{y}_{t+1}^n \to \hat{y}_{t+1}^0 \in L^0(\mathbf{R}^d, \mathcal{F}_{t+1})$ along a random $\mathcal{F}_{t+1}$-measurable subsequence where $\hat{y}_{t+1}^0 \in \arg\min(\tilde{D}_t^\xi(S_t, x_t, .))$.

In the case where $\mathrm{C}_t(s, y)$ is convex in y, the same conclusion holds if we replace $\bar{D}_t^n(\omega, x, y)$ by $\bar{D}_t^n(\omega, x, y) \vee \big(-\mathrm{C}_t(S_t(\omega), (0, x^{(2)}))\big)$. Moreover, in that case, if $K_t(S_t, x_t)$ is also convex, for fixed $x_t \in L^0(\mathbf{R}^d, \mathcal{F}_t)$ such that the random set $K_t(S_t, x_t)$ is $\mathcal{F}_t$-measurable, $\hat{y}_t^n = E(\hat{y}_{t+1}^n | \mathcal{F}_t) \in K_t(S_t, x_t)$ a.s. and converges a.s. to $\hat{y}_t^0 = E(\hat{y}_{t+1}^0 | \mathcal{F}_t) \in \arg\min(\tilde{D}_t^\xi(S_t, x_t, .))$.

Proof. We prove the claim in the first case, the second case is deduced similarly using Theorem 5.3.10.

Consider the negligible set H in the proof of Theorem 5.3.10 such that $\bar{D}_t^n(\omega,x,y) \leq \tilde{D}_t^\xi(\omega,x,y)$, for all x,y and for any $\omega \in \Omega \setminus H$ and $n \geq 1$. We then have:

$$\lim_{n\to\infty} \inf_{y\in K_t(S_t(\omega),x)} \bar{D}_t^n(\omega,x,y) \leq \inf_{y\in K_t(S_t(\omega),x)} \tilde{D}_t^\xi(S_t(\omega),x,y),\ \forall x, \tag{5.3.9}$$

for any $\omega \in \Omega \setminus H$. We now establish the reversed inequality. Since each $\bar{D}_t^n$ is an $\mathcal{F}$-normal integrand, then by [76, Theorem 13.37], we deduce that $\inf_{y\in K_t(S_t(\omega),x)} \bar{D}_t^n(\omega,x,y)$ is almost surely attained at some $\hat{y}_t^n(\omega,x)$. In other words, we have $\hat{y}_t^n(\omega,x) \in \arg\min_{K_t(S_t(\omega),x)}(\bar{D}_t^n(\omega,x,.))$ for any ω outside a negligible set N such that $H \subset N$.

Since $K_t(s,x)$ is compact, for any $\omega \in \Omega \setminus N$ and $x \in \mathbf{R}^d$, there is a random subsequence $\{\hat{y}_t^{n_k}(\omega,x), k \geq 1\}$ of $\{\hat{y}_t^n(\omega,x), n \geq 1\}$ converging to some $\hat{y}_t^0(\omega,x) \in K_t(S_t(\omega),x)$. Since $\bar{D}_t^n(\omega,.,.) \xrightarrow{\text{epi}} \tilde{D}_t^\xi(S_t(.),.,.)$ a.s.(ω) by Theorem 5.3.10, we deduce by [76, Proposition 7.2] that:

$$\liminf_{k\to\infty} \bar{D}_t^k(\omega,x,\hat{y}_t^k(\omega,x)) \geq \tilde{D}_t^\xi(S_t(\omega),x,\hat{y}_t^0(\omega,x)) \tag{5.3.10}$$

for any $\omega \in \Omega \setminus N$. As $\tilde{D}_t^\xi(S_t(\omega),x,\hat{y}_t^0(\omega,x)) \geq \inf_{y\in K_t(S_t(\omega),x)} \tilde{D}_t^\xi(S_t(\omega),x,y)$, we deduce that for any $\omega \in \Omega \setminus N$:

$$\liminf_{k\to\infty} \bar{D}_t^k(\omega,x,\hat{y}_t^k(\omega,x)) \geq \inf_{y\in K_t(S_t(\omega),x)} \tilde{D}_t^\xi(S_t(\omega),x,y). \tag{5.3.11}$$

We deduce from (5.3.9) and (5.3.11) and, finally (5.3.10), that

$$\liminf_{k\to\infty} \bar{D}_t^k(\omega,x,\hat{y}_t^k(\omega,x)) = \inf_{y\in K_t(S_t(\omega),x)} \tilde{D}_t^\xi(S_t(\omega),x,y) = \tilde{D}_t^\xi(S_t(\omega),x,\hat{y}_t^0(\omega,x))$$

We then deduce that $\hat{y}_t^0(\omega,x) \in \arg\min_{K_t(S_t(\omega),x)}(\tilde{D}_t^\xi(S_t(\omega),x,.))$ for any $\omega \in \Omega \setminus N$, i.e. (5.3.8) holds. Using the definition of the reachability set-valued mapping K_t, we conclude that $\hat{y}_t^0(\omega,x) \in \arg\min(\tilde{D}_t^\xi(S_t(\omega),x,.))$ outside a negligeable set.

Recall that $\inf_{y\in K_t(S_t(\omega),x_t)} \bar{D}_t^n(\omega,x_t,y)$ is $\mathcal{F}_{t+1}$-measurable. Therefore, by a measurable selection argument, we may deduce the existence of $\hat{y}_{t+1}^n \in L^0(\mathbf{R}^d,\mathcal{F}_{t+1})$ such that $\bar{D}_t^n(\omega,x_t,\hat{y}_{t+1}^n) = \inf_{y\in K_t(S_t(\omega),x_t)} \bar{D}_t^n(\omega,x_t,y)$ and $\hat{y}_{t+1}^n \in K_t(S_t,x_t)$ a.s.. By [55, Lemma 2.1.2], we may suppose that $\hat{y}_{t+1}^n \in K_t(S_t,x_t)$

is convergent for some random subsequence towards a $\mathcal{F}_{t+1}$-measurable limit $\hat{y}_{t+1}^0 \in K_t(S_t, x_t)$. Moreover, by the first step $\hat{y}_{t+1}^0 \in \arg\min_{K_t(S_t,x_t)}(\tilde{D}_t^\xi(S_t, x_t, .))$.

If $K_t(S_t, x_t)$ is $\mathcal{F}_t$-measurable, consider a Castaing representation $(z_t^m)_{m\geq 1}$ of $K_t(S_t, x_t)$. The generalized conditional expectation $E(\hat{y}_{t+1}^n|\mathcal{F}_t)$ exists as $\hat{y}_{t+1}^n \in K_t(S_t, x_t)$ is $\mathcal{F}_t$-bounded. Note that $\hat{y}_{t+1}^n$ may be approximated by a sequence of $\mathcal{F}_{t+1}$-measurable random variables in the set $\{z_t^m : \ m \geq 1\}$. We deduce that $E(\hat{y}_{t+1}^n|\mathcal{F}_t) \in K_t(S_t, x_t)$ if $K_t(S_t, x_t)$ is convex. It is clear that $E(\hat{y}_{t+1}^n|\mathcal{F}_t)$ converges to $E(\hat{y}_{t+1}^0|\mathcal{F}_t) \in K_t(S_t, x_t)$.

When the cost function is convex, $\bar{D}_t^n(\omega, x_t, y)$ is convex. Using the Jensen inequality for conditional expectations, we get that

$$\begin{aligned}
\tilde{D}_t^\xi(S_t, x_t, E(\hat{y}_{t+1}^0|\mathcal{F}_t)) &\leq E\left(\tilde{D}_t^\xi(S_t, x_t, \hat{y}_{t+1}^0)|\mathcal{F}_t\right), \\
&\leq E\left(\inf_{y\in\mathbf{R}^d} \tilde{D}_t^\xi(S_t, x_t, y)|\mathcal{F}_t\right), \\
&\leq \inf_{y\in\mathbf{R}^d} \tilde{D}_t^\xi(S_t, x_t, y).
\end{aligned}$$

The last inequality holds since $\inf_{y\in\mathbf{R}^d} \tilde{D}_t^\xi(S_t, x_t, y)$ is $\mathcal{F}_t$-measurable. This implies that $E(\hat{y}_{t+1}^0|\mathcal{F}_t) \in \arg\min(\tilde{D}_t^\xi(S_t, x_t, .))$.

□

5.3.2 Multi-period framework

In this section, we consider the multi-period setting $t = 0, \cdots, T$. Our goal is to determine the infimum super-hedging cost of $\xi := g(S_T) = (g^1(S_T), g^{(2)}(S_T))$ at time 0, where $g : \mathbf{R}_+^k \to \mathbf{R}_+^d$ is a deterministic continuous function. To do so, we apply the dynamic programming principle of Proposition 5.2.1 to recursively compute $\gamma_t^\xi(V_{t-1})$ for $t = 0, \cdots, T$. Moreover, since $\gamma_0^\xi(0) = \tilde{\gamma}_0^\xi(S_0, 0)$ under the weak no-arbitrage condition we suppose, it is then sufficient to compute $\tilde{\gamma}_0(S_0, V_0)$ for $V_0 = 0$. We work under the following assumption:

Assumption 10. *For each t, suppose that there is a reachability set-valued mapping $K_t : \mathbf{R}_+^k \times \mathbf{R}^d \twoheadrightarrow \mathbf{R}^d$ such that $K_t(s, v_{t-1})$ is a compact upper hemicontinuous set-valued mapping, i.e.*

$$\inf_{y\in\mathbf{R}^d} \tilde{D}_t^\xi(s, x, y) = \inf_{y\in K_t(s,x)} \tilde{D}_t^\xi(s, x, y), \text{ a.s..}$$

For simplicity, we consider the model where the price process satisfies

$$\mathrm{supp}_{\mathcal{F}_t}(S_{t+1}) = \{a_t S_t : \ a_t \in \Theta\}, \, t \leq T-1,$$

such that $P\left[S_{t+1} = a_t S_t | \mathcal{F}_t\right] > 0$ a.s. for all $a_t \in \Theta$, where $\Theta = \{a_t^n, n \geq 1\}$ is a deterministic sequence of positive numbers. Consider a sequence of random variables $\{b_t^i, i \in J_t, t = 0, \cdots, T\}$ in $\mathbf{R}^{k \times T}$ generated by the following procedure:

1) $b_0^i = S_0$ for all $i \in J_0 = \mathbb{N} \setminus \{0\}$.

2) For given $t \geq 0$, we denote $\tilde{\mathcal{F}}_t = \sigma(b_u^k : \ k \in J_u, u \leq t)$ where $(b_u^k)_{k \in J_u}$ are the random variables constructed at time t. Then, for time $t+1$, and for each $i \in J_t$, we generate a sequence of i.i.d. random variables α_{t+1}^j, $j \geq 1$, independent of $\mathcal{F}_t$ such that $\alpha_{t+1}^j \in L^0(\Theta, \mathcal{F}_{t+1})$ for each j. Moreover, $\mathrm{supp}_{\mathcal{F}_t} \alpha_{t+1}^j = \Theta$. We then define for each $i \in J_t$ and $j \geq 1$, $b_{t+1}^{i,j} = \alpha_{t+1}^j b_t^i$. Then, $J_{t+1} = \{(i,j) : \ i \in J_t, \, j \geq 1\}$.

To compute $\tilde{\gamma}_0^{\xi}(S_0, 0)$, we approximate $\tilde{\gamma}_t^{\xi}(b_t^i, v_{t-1})$ by the randomization method considered in the last section that we extend to the multi-period setting.

We denote $\mathbf{n}^1 = (\mathbf{n}_u^1)_{u=1,\cdots,T}$ a generic element in $\mathbb{N}^T$ and, for $t = 1, \cdots, T$, we define $\mathbf{n}^t = (\mathbf{n}_u^t)_{u=t,\cdots,T} \in \mathbb{N}^{T-t+1}$. If $b_t^i \in \{\alpha_t^k b_{t-1}^j; \, j \in J_{t-1}, k \geq 1\}$, $i \in J_t$, we set:

$$\hat{\theta}_{T-1}^{\mathbf{n}^T}(b_{T-1}^i, v_{T-1}) := \max_{m \leq \mathbf{n}_T^T} \tilde{\gamma}_T^{\xi}(\alpha_T^m b_{T-1}^i, v_{T-1}),$$

$$\hat{\theta}_t^{\mathbf{n}^{t+1}}(b_t^i, v_t) := \max_{m \leq \mathbf{n}_{t+1}^{t+1}} \hat{\gamma}_{t+1}^{\mathbf{n}^{t+2}}(\alpha_{t+1}^m b_t^i, v_t), \, \mathbf{n}^{t+2} = (\mathbf{n}_u^{t+1})_{u=t+2,\cdots,T}, \, t \leq T-1,$$

$$\hat{D}_t^{\mathbf{n}^{t+1}}(b_t^i, v_{t-1}, v_t) := \hat{\theta}_t^{\mathbf{n}^{t+1}}(b_t^i, v_t) + \mathrm{C}_t(b_t^i, (0, v_t^{(2)} - v_{t-1}^{(2)})), \, t \leq T-1,$$

$$\hat{\gamma}_t^{\mathbf{n}^{t+1}}(b_t^i, v_{t-1}) := \inf_{v_t \in K_t(b_t^i, v_{t-1})} \hat{D}_t^{\mathbf{n}^{t+1}}(b_t^i, v_{t-1}, v_t), \, t \leq T-1.$$

Note that by assumption

$$\tilde{\gamma}_T^{\xi}(s, v_{T-1}) := g^1(s) + \mathrm{C}_T(s, (0, g^2(s) - v_{T-1}^{(2)})).$$

Therefore, $\tilde{\gamma}_T^{\xi}$ is l.s.c. Since K_t is an upper hemicontinuous compact set-valued mapping by assumption (see Lemma 4.4.14 and Theorem 4.4.15 in Chapter 4), and $\hat{D}_t^{\mathbf{n}_{t+1}}$ is l.s.c. by induction, $\hat{\gamma}_t^{\mathbf{n}_{t+1}}(b_t^i, v_{t-1})$ is l.s.c. in b_t^i and v_{t-1} by [1, Lemma 17.29].

The following theorem is our main contribution of this section. We use the convention that $\mathbf{n}^1 \to \infty$, $\mathbf{n}^1 \in \mathbb{N}^T$, if and only if $\mathbf{n}_i^1 \to \infty, \forall i = 1, \cdots, T$.

Theorem 5.3.15. *Suppose that Assumption 10 holds and suppose that* C_t *satisfies* $\mathrm{C}_t(s, v_t^1) \geq \mathrm{C}_t(s, v_t^2)$ *whenever* $v^1 \geq_{\mathbf{R}_+^d} v_t^2$. *Then:*

$$\lim_{\mathbf{n}^1 \to \infty} \hat{\gamma}_0^{\mathbf{n}^1}(S_0, 0) = \tilde{\gamma}_0^{\xi}(S_0, 0), \text{ a.s..}$$

Proof. By Remark 5.3.13, Assumption 10 implies that

$$\tilde{\gamma}_0^{\xi}(S_0, 0) = \inf_{v_1 \in K_0(S_0,0)} \tilde{D}_0^{\xi}(S_0, 0, v_1)$$

where $K_0(S_0, 0)$ is a compact set-valued mapping. Moreover, since $\tilde{\gamma}_{t+1}^{\xi}(., v_t)$ is l.s.c. hence Borel, Theorem 5.3.14 applies when we replace S_t by each random variable $b_t^i \in \left\{\alpha_t^k b_{t-1}^j;\ j \in J_{t-1}, k \geq 1\right\}$. Precisely, in accordance with (5.3.5), we shall consider:

$$\begin{aligned}
&\bar{D}_t^{\mathbf{n}_{t+1}^{t+1}}(b_t^i, v_{t-1}, v_t) = \sup_{n \leq n_{t+1}^s} \tilde{\gamma}_{t+1}^{\xi}(\alpha_t^n b_t^i, v_t) + \mathrm{C}_t(\alpha_t^n b_t^i, (0, v_t^{(2)} - v_{t-1}^{(2)})), t \leq T-1,\\
&\bar{\gamma}_t^{\mathbf{n}_{t+1}^{t+1}}(b_t^i, v_{t-1}) := \inf_{v_t \in K_t(b_t^i, v_{t-1})} \bar{D}_t^{\mathbf{n}_{t+1}^{t+1}}(b_t^i, v_{t-1}, v_t),\ t \leq T-1,\\
&\sup_{\mathbf{n}_{t+1}^{t+1}} \bar{\gamma}_t^{\mathbf{n}_{t+1}^{t+1}}(b_t^i, v_{t-1}) = \tilde{\gamma}_t^{\xi}(b_t^i, v_{t-1}),\ t \leq T-1, \text{ by Theorem 5.3.14.} \qquad (5.3.12)
\end{aligned}$$

We now prove by induction that $\lim_{\mathbf{n}^1 \to \infty} \hat{\gamma}_0^{\mathbf{n}^1}(S_0, 0) = \tilde{\gamma}_0^{\xi}(S_0, 0)$ a.s. Observe that, at time $T-1$, $\mathbf{n}^T =: n^T \in N$ and $\hat{\gamma}_{T-1}^{\mathbf{n}^T}(b_{T-1}^i, v_{T-1})$ and $\bar{\gamma}_{T-1}^{n^T}(b_{T-1}^i, v_{T-1})$ coincide. So, by Theorem 5.3.14, we have

$$\lim_{\mathbf{n}^T \to \infty} \hat{\gamma}_{T-1}^{\mathbf{n}^T}(b_{T-1}^i, v_{T-2}) = \lim_{n^T \to \infty} \bar{\gamma}_{T-1}^{n^T}(b_{T-1}^i, v_{T-2}) = \tilde{\gamma}_{T-1}^{\xi}(b_{T-1}^i, v_{T-2})$$

Now, we suppose that $\sup_{\mathbf{n}^{t+2} \in \mathbb{N}^{T-t-1}} \hat{\gamma}_{t+1}^{\mathbf{n}^{t+2}}(b_{t+1}^i, v_t) = \tilde{\gamma}_{t+1}^{\xi}(b_{t+1}^i, v_t)$ for any $b_{t+1}^i \in \{\alpha_{t+1}^k b_t^j;\ j \in J_t, k \geq 1\}$. We have by definition:

$$\begin{aligned}
\hat{D}_t^{\mathbf{n}^{t+1}}(b_t^i, (0, v_t^{(2)} - v_{t-1}^{(2)})) &= \hat{\theta}_t^{\mathbf{n}^{t+1}}(b_t^i, v_t) + \mathrm{C}_t(b_t^i, (0, v_t^{(2)} - v_{t-1}^{(2)}))\\
&= \max_{m \leq n_{t+1}^{t+1}} \hat{\gamma}_{t+1}^{\mathbf{n}^{t+2}}(\alpha_t^m b_t^i, v_t) + \mathrm{C}_t(b_t^i, (0, v_t^{(2)} - v_{t-1}^{(2)})),\\
&\quad \mathbf{n}^{t+2} = (n_u^{t+1})_{u=t+2,\cdots,T}.
\end{aligned}$$

Consider the directed set of all $\mathbf{n}^{t+1} \in \mathbb{N}^{T-t}$ endowed with the partial order $\mathbf{n}^{t+1} \geq \mathbf{m}^{t+1}$ if and only if $\mathbf{n}_i^{t+1} \geq \mathbf{m}_i^{t+1}$ for all $t+1 \leq i \leq T$. By

construction and by induction, it is easy to check that $(\hat{D}_t^{\mathbf{n}})_{\mathbf{n}\in N^{[t+1,T]}}$ is increasing, i.e. $\bar{D}_t^{\mathbf{n}} \geq \bar{D}_t^{\mathbf{m}}$ whenever $\mathbf{n} \geq \mathbf{m}$. Also, we may show by induction that $\hat{D}_t^{\mathbf{n}}(b_t^i, .)$ is l.s.c. for all $\mathbf{n}$. By Lemma 5.3.16 that allows us to exchange the supremum and infimum in the following first equality, plus the induction hypothesis, we deduce that

$$\begin{aligned}
\sup_{\mathbf{n}^{t+1}} \hat{\gamma}_t^{\mathbf{n}_{t+1}}(b_t^i, v_{t-1}) &= \sup_{\mathbf{n}^{t+1}} \inf_{v_t \in K_t(b_t^i, v_{t-1})} \hat{D}_t^{\mathbf{n}^{t+1}}(b_t^i, v_{t-1}, v_t) \\
&= \inf_{v_t \in K_t(b_t^i, v_{t-1})} \sup_{\mathbf{n}^{t+1}} \hat{D}_t^{\mathbf{n}^{t+1}}(b_t^i, v_{t-1}, v_t) \\
&= \inf_{v_t \in K_t(b_t^i, v_{t-1})} \sup_{\mathbf{n}_{t+1}^{t+1} \in \mathbb{N}} \sup_{\mathbf{n}^{t+2}} \hat{D}_t^{\mathbf{n}^{t+1}}(b_t^i, v_{t-1}, v_t) \\
&= \inf_{v_t \in K_t(b_t^i, v_{t-1})} \sup_{\mathbf{n}_{t+1}^{t+1} \in \mathbb{N}} \sup_{\mathbf{n}^{t+2}} \Bigg(\max_{m \leq \mathbf{n}_{t+1}^{t+1}} \hat{\gamma}_{t+1}^{\mathbf{n}^{t+2}}(\alpha_{t+1}^m b_t^i, v_t) \\
&\qquad + \mathrm{C}_t(b_t^i, (0, v_t^{(2)} - v_{t-1}^{(2)})) \Bigg) \\
&= \inf_{v_t \in K_t(b_t^i, v_{t-1})} \sup_{\mathbf{n}_{t+1}^{t+1} \in \mathbb{N}} \max_{m \leq \mathbf{n}_{t+1}^{t+1}} \Bigg(\sup_{\mathbf{n}^{t+2}} \hat{\gamma}_{t+1}^{\mathbf{n}^{t+2}}(\alpha_{t+1}^m b_t^i, v_t) \\
&\qquad + \mathrm{C}_t(b_t^i, (0, v_t^{(2)} - v_{t-1}^{(2)})) \Bigg) \\
&= \inf_{v_t \in K_t(b_t^i, v_{t-1})} \sup_{\mathbf{n}_{t+1}^{t+1} \in \mathbb{N}} \max_{m \leq \mathbf{n}_{t+1}^{t+1}} \Bigg(\tilde{\gamma}_{t+1}^{\xi}(\alpha_{t+1}^m b_t^i, v_t) \\
&\qquad + \mathrm{C}_t(b_t^i, (0, v_t^{(2)} - v_{t-1}^{(2)})) \Bigg) \\
&= \inf_{v_t \in K_t(b_t^i, v_{t-1})} \tilde{D}_t^{\xi}(b_t^i, , v_{t-1}, v_t) = \tilde{\gamma}_t^{\xi}(b_t^i, v_{t-1}).
\end{aligned}$$

To deduce the last two equalities, we use the definition of $\tilde{\theta}_t^{\xi}(b_t^i, , v_{t-1}, v_t)$ and $\tilde{D}_t^{\xi}(b_t^i, , v_{t-1}, v_t)$, see (5.3.4) but also (5.3.7) in Remark 5.3.13. The conclusion follows by induction. □

In the proof above, we have used the following lemma:

Lemma 5.3.16 (Dini-Cartan)**.** *Consider a family of l.s.c. functions $(f_n)_{n\in I}$, $f_n : \mathbf{R}^d \to \overline{\mathbf{R}}$ such that for every finite set $J \subset I$, there is $n_0 \in I$ with*

$\sup_{j\in J} f_j \le f_{n_0}$. *Consider a compact set G, then the following holds:*

$$\sup_n \inf_{x\in G} f_n(x) = \inf_{x\in G} \sup_n f_n(x).$$

Proof. By considering an increasing homeomorphism from $[-\infty, +\infty]$ onto $[0, 1]$, we then restrict ourselves to the case $\sup_n f_n$ is bounded. It is clear that $\sup_n \inf_{x\in G} f_n(x) \le \inf_{x\in G} \sup_n f_n(x)$ so that the inequality holds if the second term is $-\infty$. For the reverse inequality, consider any $a < \inf_{x\in G} \sup_n f_n(x)$. For all $x \in G$, we have $a < \sup_n f_n(x)$. Then, there exists some $k = k_x$ such that $a < f_k(x)$. Note that the set $O_k := \{x : a < f_k(x)\}$ is open since f_k is l.s.c. By compactness argument, we deduce a finite covering of G by some O_{k_i}, $j = 1, \cdots, N$. By our hypothesis, there exists n_0 such that $a \le f_{k_i}(x) \le f_{n_0}(x)$, for all $x \in G$ and $i = 1, \cdots, N$ hence we have $a \le \inf_{x\in G} f_{n_0}(x) \le \sup_n \inf_{x\in G} f_n(x)$. □

Lemma 5.3.17. *For all t , for all $j \in J_{t+1}$, consider $b^j_{t+1} = \alpha^k_{t+1} b^i_t$ where $i \in J_t$ and $k \ge 1$. Then, $b^j_{t+1} \in \{a^n_t b^i_t, n \ge 1\}$ a.s. and $P[b^j_{t+1} = a^n_t b^i_t | \mathcal{F}_t] > 0$ a.s. Moreover, $\{b^j_{t+1}, j \in J_{t+1}\}$ are $\mathcal{F}_t$-i.i.d.*

Proof. For all $n \ge 1$, we have almost surely :

$$P\left[b^j_{t+1} = a^n_t b^i_t | \mathcal{F}_t\right] = P\left[\alpha^k_{t+1} b^i_t = a^n_t b^i_t | \mathcal{F}_t\right] \ge P\left[\alpha^k_{t+1} = a^n_t | \mathcal{F}_t\right] > 0.$$

The last statement follows directly from Lemma 5.3.3 as $(\alpha^j_{t+1})_{j\ge1}$ are $\mathcal{F}_t$-i.i.d. by assumption.

□

5.4 Model with one risky asset and piecewise linear costs

As we may observe in the previous section, the reachability set-valued mapping plays an important role in propagating the lower semicontinuity which, in turn, propagates the convergence property. We consider in this section a special case of convex cost functions and provide explicit expressions for the minimal super-hedging costs. In particular, under SAIP condition, we obtain an explicit expression of the reachability set $K_t(s, v_{t-1})$ when the payoff is of linear growth, i.e. $\xi = (\xi^1, \xi^2) \le_{\mathbf{R}^2_+} (aS_T + b, c)$ for some $a, b, c \in \mathbf{R}_+$.

We suppose the market consists of one risk-free asset and one risky asset denoted by $(S_t)_{0\leq t\leq T}$. We impose the following assumption for the conditional support of the price and cost processes.

Assumption 11. *The price process satisfies $S_{t+1} \in \{a_t^n S_t, n \geq 1\}$ where the sequence $(a_t^n)_{n\geq 1}$ is deterministic and satisfies $a_t^1 = \min_n a_t^n = k_t^d \geq 0$, $a_t^2 = \max_n a_t^n = k_t^u \in \mathbf{R}_+$, where k_t^d, k_t^u are deterministic. The cost process C_t is given by $\mathrm{C}_t(S_t,(x,v_t)) = x + S_t\tilde{\mathrm{C}}_t(v_{t-1}^2)$ for some deterministic piecewise linear function $\tilde{\mathrm{C}}_t : \mathbf{R} \to \mathbf{R}$.*

We recall the AEP condition

Definition 5.4.1. *We say that the financial market satisfies the Absence of Early Profit condition (AEP) if, at any time $t \leq T$, and for all $V_t \in L^0(\mathbf{R}^d, \mathcal{F}_t)$, $\gamma_t^0(V_t) > -\infty$ a.s..*

By Lemma 4.4.11 in Chapter 4, AIP implies AEP if the cost function C_t is either sub-additive or super-additive. Moreover, by Theorem 4.4.5 in Chapter 4, AEP implies that $\tilde{\gamma}_t^{\xi}(S_t,.) > -\infty$ a.s. This property will be used in the proof of the following result.

Proposition 5.4.2. *Suppose that Condition AEP and Assumption 11 hold. Then the minimal hedging cost of the payoff $\xi = (mS_T + G, K)$, $m, G, K \in \mathbf{R}$, is given by $\tilde{\gamma}_t^{\xi}(S_t, v_{t-1}) = G + S_t h_t(v_{t-1}^2)$, where $h_t : \mathbf{R} \to \mathbf{R}$ is a deterministic piecewise linear function.*

Moreover, $\tilde{D}_t(S_t, v_t, v_{t-1}) = S_t\tilde{h}_t(v_t, v_{t-1})$ for some deterministic piecewise linear function $\tilde{h}_t : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$.

Proof. We first show by induction that, if $\tilde{\gamma}_{t+1}^{\xi}(S_{t+1}, v_t) = S_{t+1}\tilde{f}_{t+1}(v_t^2)$ where $\tilde{f}_{t+1} : \mathbf{R} \to \mathbf{R}$ is a piecewise linear function, then $\tilde{\gamma}_t^{\xi}(S_t, v_{t-1}) = S_t\tilde{f}_t(v_{t-1}^2)$ for some piecewise linear function $\tilde{f}_t : \mathbf{R} \to \mathbf{R}$. To do so, observe that:

$$\begin{aligned}\tilde{\theta}_t^{\xi}(S_t, v_t) = \sup_{s\in\{a_t^n S_t, n\geq 1\}} \left(s\tilde{f}_{t+1}(v_t^2)\right) &= \max\left\{k_t^d S_t\tilde{f}_{t+1}(v_t^2), k_t^u S_t\tilde{f}_{t+1}(v_t^2)\right\} \\ &= S_t \max\left\{k_t^d\tilde{f}_{t+1}(v_t^2), k_t^u\tilde{f}_{t+1}(v_t^2)\right\}.\end{aligned}$$

Since $\tilde{f}_{t+1}$ is piecewise linear function by the hypothesis, we deduce that $\tilde{g}_t(v_t^2) := \max\{k_t^d\tilde{f}_{t+1}(v_t^2), k_t^u\tilde{f}_{t+1}(v_t^2)\}$ is also piecewise linear by [76, Propo-

sition 3.55]. Therefore,

$$\tilde{\gamma}_t^{\xi}(S_t, v_{t-1}) = \inf_{v^2 \in \mathbf{R}} \tilde{D}_t^{\xi}(S_t, v_{t-1}, v_t) = \inf_{v^2 \in \mathbf{R}} \left(\tilde{\theta}_t^{\xi}(S_t, v_t) + \mathrm{C}_t(S_t, v_t^2 - v_{t-1}^2) \right)$$
$$= S_t \inf_{v_t^2 \in \mathbf{R}} \left(\tilde{g}_t(v_t^2) + \tilde{\mathrm{C}}_t(v_t^2 - v_{t-1}^2) \right).$$

By [76, Proposition 3.55], we also deduce that $\tilde{g}_t(v_t^2) + \tilde{\mathrm{C}}_t(v_t^2 - v_{t-1}^2)$ is a piecewise linear function in (v_t^2, v_{t-1}^2). Moreover, under AEP, we know that $\tilde{\gamma}_t^{\xi}(S_t, v_{t-1}) > -\infty$ a.s.. Therefore, by [76, Proposition 3.55],

$$\tilde{f}_t(v_{t-1}^2) := \inf_{v^2 \in \mathbf{R}} \left(\tilde{g}_t(v_t^2) + \tilde{\mathrm{C}}_t(v_t^2 - v_{t-1}^2) \right)$$

is a piecewise linear function in v_{t-1}^2.

If the payoff is $\xi = (mS_T + G, K)$, then $\tilde{\gamma}_T^{\xi}(S_T, v_{T-1}) = G + S_T \tilde{f}_T(v_{T-1}^2)$ where $\tilde{f}_T(v_{T_1}^2) := m + \tilde{\mathrm{C}}_T(K - v_{T-1}^2)$ is a piecewise linear function by assumption on C_T. We then argue by induction as previously done to deduce that $\gamma_{t-1}^{\xi}(S_{t-1}, v_{t-2}) = G + S_{t-1}\tilde{f}_{t-1}(v_{t-2}^2)$ for some piecewise linear function $\tilde{f}_{t-1}$.

At last, since $D_t^{\xi}(S_t, v_t, v_{t-1}) = \tilde{\theta}_t(S_t, v_t) + \mathrm{C}_t(S_t, (0, v_t^{(2)} - v_{t-1}^{(2)}))$, the conclusion on $\tilde{D}_t$ follows. □

The following is our main result of this section. It states the existence of the reachability set under SAIP.

Proposition 5.4.3. *Suppose that the payoff $\xi = (g^1(S_T), g^2(S_T))$ satisfies $g^1(S_T) \leq aS_T + b$ and $g^2(S_T) \leq c$ for some $a, b, c \in \mathbf{R}_+$. We also suppose that $\mathrm{C}_t(s, v^1) \geq \mathrm{C}_t(s, v^2)$ whenever $v^1 \geq_{\mathbf{R}_+^2} v^2$ and suppose that $\mathrm{C}_t(s, .)$ is subadditive and 1-homogeneous.*

*Under the no-arbitrage condition SAIP, the reachability set $K_t(s, v_{t-1})$ is defined for **every** $(s, v_{t-1}) \in \mathbf{R} \times \mathbf{R}$ and is explicitly given by:*

$$K_t(s, v_{t-1}) = \bar{B}_t(0, r_t(s, v_{t-1}) + 1)$$

where $r_t(s, v_{t-1}) = sf_t(v_{t-1})/g_t(s)$ and f_t, g_t are deterministic piecewise linear functions such that $g_t(s) > 0$ for all $s > 0$.

Proof. We define $\tilde{\xi} := (aS_T + b, c)$ so that $\xi \leq_{\mathbf{R}_+^2} \tilde{\xi}$. We show by induction that $\tilde{D}_t^0(s, v_{t-1}, v_t) \leq \tilde{D}_t^{\xi}(s, v_{t-1}, v_t) \leq \tilde{D}_t^{\tilde{\xi}}(s, v_{t-1}, v_t)$. By the proof of Theorem 4.4.15 in Chapter 4, we get that

$$K_t(s, v_{t-1}) \subseteq \left\{ v_t : \tilde{D}_t^{\xi}(s, v_{t-1}, v_t) \leq \tilde{D}_t^{\tilde{\xi}}(s, v_{t-1}, 0) \right\}$$

Moreover, by sub-additivity and 1-homogeneity.

$$\tilde{D}^0_t(s, v_{t-1}, v_t) = \mathrm{C}_t(s, (0, v_t^2 - v_{t-1}^{(2)})) + \tilde{\theta}^0_t(s, v_t) \geq -\mathrm{C}_t(s, (0, v_{t-1}^2)) + \tilde{D}^0_t(s, 0, v_t)$$
$$\tilde{D}^0_t(s, 0, v_t) \geq |v_t| \tilde{D}^0_t(s, 0, v_t/|v_t|) \geq |v_t| \min_{z\in\{-1,1\}} \tilde{D}^0_t(s, 0, z), \ \forall\ |v_t| \geq 1.$$

We deduce that $K_t(S_t, v_{t-1}) \subseteq \bar{B}(0, r_t(S_t, v_{t-1})+1)$, where the radius $r_t(S_t, v_{t-1})$ is given by

$$r_t(S_t, v_{t-1}) := \frac{\tilde{D}^{\tilde{\xi}}_t(S_t, v_{t-1}, 0) + \mathrm{C}_t(S_t, (0, v_{t-1}^2))}{\min_{z\in\{-1,1\}} \tilde{D}^0_t(S_t, 0, z)} =: \frac{S_t f_t(v_{t-1}^2)}{g_t(S_t)}.$$

Note that by Proposition 5.4.2, $f_t : \mathbf{R} \to \mathbf{R}$ and $g_t : \mathbf{R} \to \mathbf{R}$ are deterministic piecewise linear functions. Moreover, we have $g_t(S_t) = S_t \inf_{z\in\{-1,1\}} a_t(z)$ for some deterministic piecewise linear function a_t. Since SAIP holds, we deduce that $\inf_{z\in\{-1,1\}} a_t(z) > 0$. We then define $g_t(s) := s \inf_{z\in\{-1,1\}} a_t(z) > 0$ for all $s > 0$. The conclusion follows. □

5.5 Examples

In this section, we consider two classical examples. The first one corresponds to the market with proportional transaction cost and the second one is with fixed cost. We provide the explicit expression of the reachability set-valued mapping K_t for the Put option. Then, as a by-product, the minimal super-hedging cost for Put option is computed.

For a sake of simplicity, we consider the binomial market model, i.e. the price process satisfies $\mathrm{supp}_{\mathcal{F}_t} S_{t+1} = \left\{k_t^d S_t, k_t^u S_t\right\}$, where $k_t^d, k_t^u \in \mathbf{R}_+$.

5.5.1 Market model with proportional transaction costs

We consider a particular case of section 5.4 where

$$\mathrm{C}_t(S_t, v) = v^1 + (1+\epsilon_t) S_t v^2 1_{v^2\geq 0} + (1-\epsilon_t) S_t v^2 1_{v^2\leq 0}. \tag{5.5.13}$$

for some deterministic coefficient $\epsilon_t \in \mathbf{R}_+$. By a direct computation, see Appendix, we obtain the following

Proposition 5.5.1. *If $v_{t-1} \in \mathbf{R}^2$, the following holds:*

$$\begin{aligned}\tilde{\theta}^0_{t-1}(S_{t-1}, v) &= -(1-\epsilon_t)k^d_{t-1}S_{t-1}v^2 1_{v^2\geq 0} - (1+\epsilon_t)k^u_{t-1}S_{t-1}v^2 1_{v^2\leq 0}\\ \tilde{D}^0_{t-1}(S_{t-1}, 0, v) &= ((1+\epsilon_{t-1})S_{t-1} - (1-\epsilon_t)k^d_{t-1}S_{t-1})v^2 1_{v^2\geq 0}\\ &\quad + ((1-\epsilon_{t-1})S_{t-1} - (1+\epsilon_t)k^u_{t-1}S_{t-1})v^2 1_{v^2\leq 0}\end{aligned}$$

Moreover, AIP_{t-1} *holds if and only if:*

$$k^d_{t-1} \leq \frac{1+\epsilon_{t-1}}{1-\epsilon_t} \text{ and } k^u_{t-1} \geq \frac{1-\epsilon_{t-1}}{1+\epsilon_t}. \tag{5.5.14}$$

Moreover, SAIP_{t-1} *holds if and only if the above inequalities are strict. If* AIP_{t-1} *holds, we then deduce that:*

$$\begin{aligned}\inf_{v^2\in\{-1,1\}} \tilde{D}^0_{t-1}(S_{t-1}, 0, v) = S_{t-1}\min\big\{&(1+\epsilon_{t-1}) - (1-\epsilon_t)k^d_{t-1},\\ &(1+\epsilon_t)k^u_{t-1} - (1-\epsilon_{t-1})\big\}.\end{aligned}$$

Proof. Recall that AIP_{t-1} holds if and only if $\tilde{D}^0_{t-1}(S_{t-1}, 0, v) \geq 0$ for any $v \in \mathbf{R}^d$ which is equivalent to (5.5.14). Moreover, suppose that SAIP_{t-1} holds. If $k^d_{t-1} = \dfrac{1+\epsilon_{t-1}}{1-\epsilon_t}$, $D^0_{t-1}(S_{t-1}, 0, v) = 0$ for any $v^2 > 0$, i.e. SAIP_{t-1} fails. Similarly, we get that $k^u_{t-1} > (1-\epsilon_{t-1})/(1+\epsilon_t)$. At last, suppose that the inequalities in (5.5.14) are strict. Since $S_{t-1} > 0$ a.s.,

$$\inf_{v^2\in\{-1,1\}} \tilde{D}^0_{t-1}(S_{t-1}, 0, v) > 0, \text{ a.s.}$$

so that SAIP_{t-1} holds by Theorem 4.4.16 in Chapter 4. □

We apply the result above at time T and we proceed by induction, see Appendix, to deduce the following result at time $T-2$.

Proposition 5.5.2. *Assume that* $1+\epsilon_{T-1} \leq (1+\epsilon_T)k^u_{T-1}$ *and* $1-\epsilon_{T-1} \geq (1-\epsilon_T)k^d_{T-1}$*, we have:*

$$\begin{aligned}\tilde{\theta}^0_{T-2}(S_{T-2}, z) &= -(1+\epsilon_{T-1})k^d_{T-2}S_{T-2}z^2 1_{z^2\geq 0} - (1-\epsilon_T)k^d_{T-1}k^u_{T-2}S_{T-2}z^2 1_{z^2\leq 0},\\ \tilde{D}^0_{T-2}(S_{T-2}, 0, z) &= \left((1+\epsilon_{T-2})S_{T-2} - (1+\epsilon_{T-1})k^d_{T-2}S_{T-2}\right) z^2 1_{z^2\geq 0}\\ &\quad + \left((1-\epsilon_{T-2})S_{T-2} - (1-\epsilon_T)k^d_{T-1}k^u_{T-2}S_{T-2}\right) z^2 1_{z^2\leq 0}.\end{aligned}$$

and AIP_{T-2} *holds if and only if:*

$$k^d_{T-2} \le \frac{1+\epsilon_{T-2}}{1+\epsilon_{T-1}} \text{ and } k^u_{T-2} \ge \frac{1-\epsilon_{T-2}}{(1-\epsilon_T)k^d_{T-1}}.$$

Moreover, SAIP_{T-2} *holds if and only if the above inequalities are strict. Moreover, under* SAIP_{T-2}*, we have:*

$$\inf_{v^2\in\{-1,1\}} \tilde{D}^0_{T-2}(S_{T-2},0,v) = S_{T-2}\min\big\{((1+\epsilon_{T-2})-(1-\epsilon_{T-1})k^d_{T-2}), \\ -((1-\epsilon_{T-2})-(1+\epsilon_T)k^d_{T-1}k^u_{T-2})\big\}.$$

The assumptions of Proposition 5.5.2 are chosen for a sake of simplification. The computations for $t < T-2$ are similar. In particular, for a Put option with payoff $(K-S_T)^+$, $K>0$, we obtain a simple formula for the reachability set.

Lemma 5.5.3. *Suppose that SAIP holds and* $\xi = (g(S_T),0)$ *where* g *is a continuous function bounded from above by a constant* $M \in \mathbf{R}_+$*. Then, there exists a reachability set* $K_t(s,v_{t-1}) = \bar{B}_t(0, r_t(s,v_{t-1})+1)$*,* $t \le T-1$*, closed ball of radius* $r_t(s,v_{t-1}) := \lambda_t(s,v_{t-1})/i_t(s)$ *where the functions*

$$i_t(s) := \inf_{v^2\in\{-1,1\}} \tilde{D}^0_t(s,0,v),$$
$$\lambda_t(s,v_{t-1}) := \mathrm{C}_t(s,(0,v^2_{t-1})) + M + \mathrm{C}_t(s,(0,-v^2_{t-1})),$$

are explicitely given by Proposition 5.5.1 and Proposition 5.5.2. In particular, we have $i_t(s) > 0$ *for all* $s > 0$*.*

We illustrate the results above by a numerical example. We consider the put option payoff $g(S_T) := (K-S_T)^+$ at time $T=2$. We suppose that the proportional cost coefficients $\epsilon_1 = \epsilon_2 = 0.02$. We assume that SAIP condition holds and choose $k^d_2 = 0.9, k^u_2 = 1.1$, $k^d_1 = 0.9, k^u_1 = 1.2$. The price function at time $t=0$ is presented in Figure 5.1.

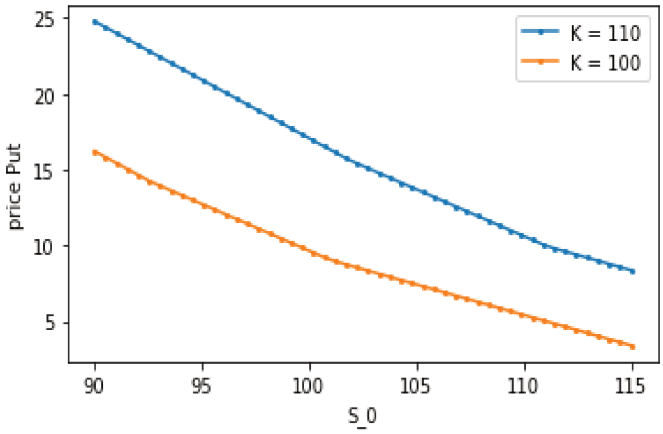

Figure 5.1: Price of Put option

We also visualize the ratio of put option to asset price S_0

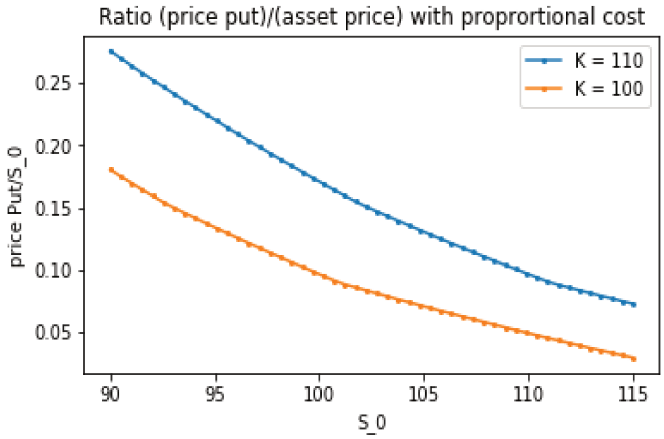

Figure 5.2: Ratio Put option/Asset Price

5.5.2 Market model with fixed cost

In this section, we recall a financial market model in presence of both proportional and fixed costs modeled by the following liquidation and cost functions:

$$\mathrm{L}_t(S_t, v_t) := v_t^1 + (v_t^2(1-\epsilon_t)S_t - c_t)^+ 1_{v_t>0} + (v_t^2(1+\epsilon_t)S_t - c_t)1_{v_t<0}$$
$$\mathrm{C}_t(S_t, v_t) := -\mathrm{L}_t(S_t, -v_t).$$

for some deterministic constant $c_t > 0$ representing the fixed cost we need to pay to obtain a non-null position.

In Chapter 4, we have introduced the *horizon* cost function defined as follows:

$$\mathrm{C}_t^{\infty}(s, y) := \liminf_{\alpha \to \infty} \frac{\mathrm{C}_t(s, \alpha y)}{\alpha}. \tag{5.5.15}$$

Definition 5.5.4. *We say that the robust no-arbitrage condition RSAIP holds at time t if the SAIP condition holds at time t for the enlarged model defined by C_t^{∞}. We say that RSAIP holds if it holds at any time.*

In Chapter 4, we have proved the following theorem:

Theorem 5.5.5. *Suppose that the condition RSAIP holds. Then, we have $\gamma_t^{\xi}(S_t, V_t) = \tilde{\gamma}_t^{\xi}(S_t, V_t)$ a.s., $\theta_t^{\xi}(S_t, V_t) = \tilde{\theta}_t^{\xi}(S_t, V_t)$ a.s. and, also, we have $D_t^{\xi}(S_t, V_{t-1}, V_t) = \tilde{D}_t^{\xi}(S_t, V_{t-1}, V_t)$ a.s. for any $V_{t-1}, V_t \in L^0(\mathbf{R}^d, \mathcal{F}_t)$, where $\tilde{\theta}_t^{\xi}, \tilde{D}_t^{\xi}$ are given by (5.3.4).*

As the horizon cost function coincides with the cost function (5.5.13) without fixed costs, the results stated in Propositions 5.5.14 and 5.5.2 allows us to characterize the reachability set-valued mapping K_t for this market. In particular, since $\mathrm{C}_t \leq \mathrm{C}_t^{\infty} + c_t$, by a straightforward computation, we deduce a simple formula of K_t for the Put option:

Lemma 5.5.6. *Suppose that $\xi = (g(S_T), 0)$ where g is a continuous function bounded from above by $M \in \mathbf{R}_+$. Then, a reachability set $K_t(s, v_{t-1})$ is explicitly given at any time $t \leq T-1$ by $K_t(s, v_{t-1}) = \bar{B}_t(0, r_t(s, v_{t-1}) + 1)$, closed ball of radius $r_t(s, v_{t-1}) := \lambda_t(s, v_{t-1})/i_t(s)$ where*

$$i_t(s) := \inf_{v^2 \in \{-1,1\}} D_t^{0,\infty}(s, 0, v),$$

$$\lambda_t(s, v_{t-1}) := \mathrm{C}_t^{\infty}(s, (0, v_{t-1}^2)) + M + \mathrm{C}_t^{\infty}(s, (0, -v_{t-1}^2)) + \sum_{s=t}^{T} c_s,$$

and $D_t^{0,\infty}$ is given in the model without fixed cost given by Proposition 5.5.1 or Proposition 5.5.2. In particular, we have $i_t(s) > 0$ for all $s > 0$.

As a numerical example, we also consider the put option payoff $(K - S_T)^+$ at time $T = 2$. We consider the binomial tree model as previously. In the

case where the conditional support $\text{supp}_{\mathcal{F}_t} S_t$ is countable, we can use the randomized method established in section 5.3.

We use the same parameters as in Section 5.5.1 and we consider fixed costs $c_1 = c_2 = 0.8$. The price function is illustrated in Figure 5.3.

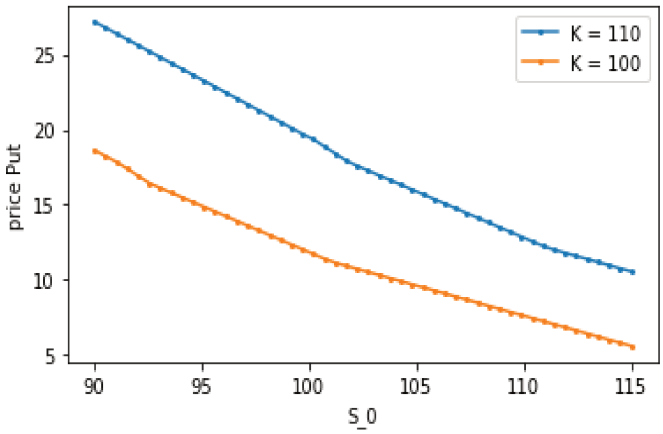

Figure 5.3: Price of put option with fixed costs.

We also visualize the ratio of put price to asset price S_0

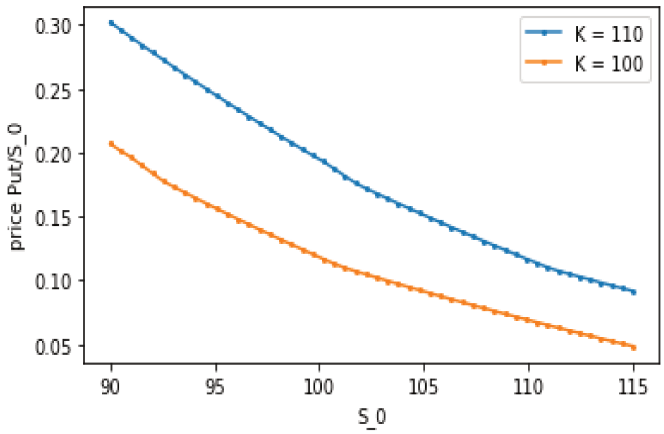

Figure 5.4: Ratio price of put to asset price with fixed costs.

We also compare the price of put option with and without fixed costs.

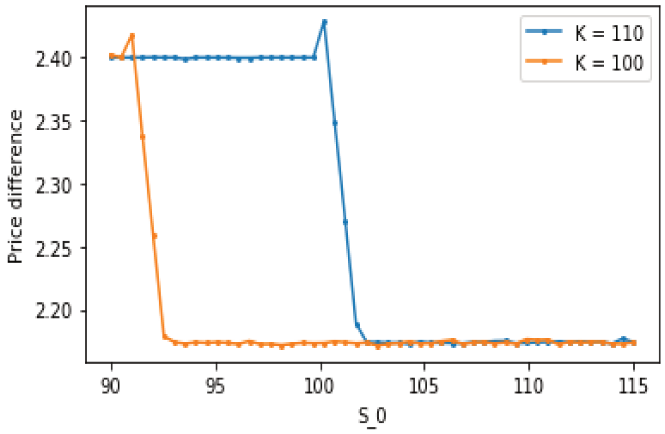

Figure 5.5: Price difference between two cases

5.6 Limit theorem for convex markets

In the literature, there is few results providing limit theorems for financial market models with transaction costs, see [38] and [4], but also [56] and [46] without transaction costs. In this section, we consider a sequence of markets defined by convex cost functions $\{\mathrm{C}_t^n(S_t, x), n \geq 1\}$ such that $\mathrm{C}_t^n(S_t, x) \downarrow \mathrm{C}_t(S_t, x)$ as $n \to \infty$ for some convex function C_t. We associate to each C_t^n a dynamic programming scheme deduced by our general analysis:

$$\begin{aligned}
\gamma_T^{\xi,n}(S_T, V_{T-1}) &:= g^1(S_T) + \mathrm{C}_T^n(S_T, (0, g^{(2)}(S_T) - V_{T-1})),\\
\theta_t^{\xi,n}(S_t, v_t) &:= \operatorname{ess\,sup}_{\mathcal{F}_t} \gamma_{t+1}^{\xi,n}(S_{t+1}, V_t),\\
D_t^{\xi,n}(S_t, V_{t-1}, V_t) &:= \theta_t^{\xi,n}(S_t, V_t) + \mathrm{C}_t^n(S_t, (0, V_t^{(2)} - V_{t-1}^{(2)})),\\
\gamma_t^{\xi,n}(S_t, V_{t-1}) &:= \operatorname*{ess\,inf}_{\mathcal{F}_t,\ V_t \in L^0(\mathbf{R}^d, \mathcal{F}_t)} D_t^{\xi,n}(S_t, V_{t-1}, V_t).
\end{aligned}$$

Assumption 12. *We suppose that* $\operatorname{supp}_{\mathcal{F}_t} S_{t+1} = \phi_t(S_t) = \operatorname{conv}\{\phi_t^1(S_t), \ldots \phi_t^J(S_t)\}$ *where* $\phi_t^j : \mathbf{R}^d \to \mathbf{R}^d$, $j \leq J$, *are piecewise linear mappings in the sense of Definition 5.7.1.*

We define $\tilde{\gamma}_t^{\xi,n} : \mathbf{R}^d \times \mathbf{R}^d \to \mathbf{R}$ recursively as follows:

$$\begin{aligned}
&\tilde{\gamma}_T^{\xi,n}(s, v_{T-1}) := \gamma_T^{\xi,n}(s, v_{T-1}),\\
&\tilde{\theta}_{T-1}^{\xi,n}(s, v_{T-1}) := \max_{j \le J} \tilde{\gamma}_T^{\xi,n}(\phi_{T-1}^j(s), v_{t-1}),\\
&\tilde{D}_t^{\xi,n}(s, v_{t-1}, v_t) := \tilde{\theta}_t^{\xi}(s, v_t) + \mathrm{C}_t^n(s, v_t^{(2)} - v_{t-1}^{(2)}),\\
&\tilde{\gamma}_t^{\xi,n}(s, v_{t-1}) := \mathrm{cl}\left(\inf_{v_t \in \mathbf{R}^d} \tilde{D}_t^{\xi,n}(s, v_{t-1}, v_t)\right).
\end{aligned}$$

Assumption 13. *Suppose that for any $t \le T-1$, $\inf_{v_t \in S^{d-1}(0,1)} \tilde{D}_t^0(s, 0, v_t) > 0$ for all $s \in \mathbf{R}_+^k$, so that there is a upper hemicontinuous reachability set-valued mapping $K_t(s, v_{t-1})$ for the super-hedging problem in the market defined by C_t. Moreover, we suppose that K_t is a **universal reachability** set in the sense that it satisfies for all $n \ge 1$ and (s, v_{t-1}):*

$$\tilde{\gamma}_t^{\xi,n}(s, v_{t-1}) = \inf_{v_t \in K_t(s, v_{t-1})} \tilde{D}_t^{\xi,n}(s, v_{t-1}, v_t).$$

Remark 5.6.1. *Consider the case where C, C^n and S_t satisfy the assumptions specified in section 5.4. Since $\mathrm{C} \le \mathrm{C}^n$ for all $n \ge 1$ by assumption, we deduce that $\inf_{v_t \in S^{d-1}(0,1)} \tilde{D}_t^0(s, 0, v_t) > 0$ implies $\inf_{v_t \in S^{d-1}(0,1)} \tilde{D}_t^{0,n}(s, 0, v_t) > 0$ for all n. By the proof of Proposition 5.4.3, it is sufficient to suppose that SAIP holds for the market defined by C. If we suppose that $\mathrm{C}_t(s, v_t), \mathrm{C}_t^n(s, v_t)$ are bounded above by $|h_t(s, v_t)|$ for some continuous function h_t, by the same argument as in Lemma 4.5.21 in Chapter 4, we deduce that the quantities $\tilde{D}_t^0(s, v_{t-1}, 0)$ and $\tilde{D}_t^{0,n}(s, v_{t-1}, 0)$ are bounded above by a continuous function $\hat{h}_t(s, v_{t-1})$. Hence, a universal reachability set exists as $K_t(s, v_{t-1}) = \bar{B}(0, r_t(s, v_{t-1}) + 1)$ where*

$$r_t(s, v_{t-1}) = \frac{\hat{h}_t(s, v_{t-1}) + |h_t(s, v_{t-1})|}{\inf_{v_t \in S^{d-1}(0,1)} \tilde{D}_t^0(s, 0, v_t)}.$$

Since r_t is u.s.c., we deduce by Lemma that K_t is upper hemicontinuous.

Theorem 5.6.2. *Suppose that the functions $\phi_t^j : \mathbf{R}_+^k \to \mathbf{R}_+^k$, $j \le J$ satisfy Assumption 12. Suppose that Assumption 13 holds. Then, for any $t \le T-1$ and for any $v_{t-1} \in \mathbf{R}^d$, $\lim_{n\to\infty} \tilde{\gamma}_t^{\xi,n}(s, v_{t-1}) = \tilde{\gamma}_t^{\xi}(s, v_{t-1})$. Moreover, SAIP condition holds for the markets defined by C^n and $\lim_{n\to\infty} \gamma_t^{\xi,n}(S_t, V_t) = \gamma_t^{\xi}(S_t, V_t)$ a.s. as $n \to \infty$ for any $V_t \in L^0(\mathbf{R}^d, \mathcal{F}_t)$ and $t \le T$.*

Proof. We first observe that $\tilde{\gamma}_t^{\xi,n}$ is convex in (s, v_{t-1}) for any n. We now prove that $\tilde{D}_t^{\xi,n}(s, v_{t-1}, .) \xrightarrow{epi} \tilde{D}_t^{\xi}(s, v_{t-1}, .)$. Indeed, by the definition of $\tilde{\gamma}_T^{\xi,n}$ we have that $\tilde{\gamma}_T^{\xi,n}(s, .) \downarrow \tilde{\gamma}_T^{\xi}(s, .)$. Since $\tilde{\gamma}_T^{\xi}(s, .)$ is convex and takes values in $\mathbf{R}$, it is continuous. We deduce by [76, Proposition 7.4(c)] that $\tilde{\gamma}_T^{\xi,n}(s, .) \xrightarrow{epi} \tilde{\gamma}_T^{\xi}(s, .)$. Moreover, by convexity and by assumption, we get that

$$\begin{aligned}\tilde{\theta}_{T-1}^{\xi,n}(s, v_{T-1}) &= \max_{j\leq J} \tilde{\gamma}_T^{\xi,n}(\phi_{T-1}^j(s), v_{t-1}),\\ \tilde{\theta}_{T-1}^{\xi}(s, v_{T-1}) &= \max_{j\leq J} \tilde{\gamma}_T^{\xi}(\phi_j(s), v_{t-1}).\end{aligned}$$

Under Assumption 12 holds, the mapping $(s, v_{t-1}) \mapsto (\phi_j(s), v_{t-1})$ is piecewise linear in the sense of Definition 5.7.1. Since, $\tilde{\gamma}_T^{\xi,n}$ is convex, we deduce by [76, Exercies 2.20] that $\tilde{\gamma}_T^{\xi,n}(\phi_j(.), .)$ is jointly convex. Moreover, since we have $\lim_{n\to\infty} \tilde{\gamma}_T^{\xi,n}(\phi_j(s), .) \overset{epi}{=} \tilde{\gamma}_T^{\xi}(\phi_j(s), .)$, for any $j \leq J$, we deduce by [76, Proposition 7.48] that:

$$\tilde{\theta}_{T-1}^{\xi,n}(s, .) = \max_{j\leq J} \tilde{\gamma}_T^{\xi,n}(\phi_j(s), .) \xrightarrow{epi} \max_{j\in J} \tilde{\gamma}_T^{\xi}(\phi_j(s), .) = \tilde{\theta}_{T-1}^{\xi}(s, .),\ n \to \infty.$$

Since $\mathrm{C}_{T-1}^n(s, .) \downarrow \mathrm{C}_{T-1}(s, .)$ and $\mathrm{C}_{T-1}(s, .)$ is continuous, we deduce by the Dini theorem that the convergence is uniform on any compact subset K of $\mathbf{R}^d$. By [76, Theomrem 7.14], we deduce that $\mathrm{C}_{T-1}^n(s, .)$ converges continuously to $\mathrm{C}_{T-1}(s, .)$ in the sense that $\mathrm{C}_{T-1}^n(s, x^n) \to \mathrm{C}_{T-1}(s, x)$ whenever $x^n \to x$. We then deduce by [76, Theorem 7.46] that

$$\tilde{D}_{T-1}^{\xi,n}(s, v_{T-2}, .) \xrightarrow{epi} \tilde{D}_{T-1}^{\xi,n}(s, v_{T-2}, .),\ n \to \infty.$$

Suppose that $\lim_{n\to\infty} \tilde{D}_{t+1}^{\xi,n}(s, v_t, .) \overset{epi}{=} \tilde{D}_{t+1}^{\xi,n}(s, v_t, .)$ and, by induction, let us show that $\lim_{n\to\infty} \tilde{D}_t^{\xi,n}(s, v_{t-1}, .) \overset{epi}{=} \tilde{D}_t^{\xi,n}(s, v_{t-1}, .)$. Since $K_{t+1}(s, .)$ is compact, we deduce that $\tilde{\gamma}_{t+1}^{\xi,n}(s, .) \downarrow \tilde{\gamma}_{t+1}^{\xi}(s, .)$. Since $\tilde{\gamma}_{t+1}^{\xi}(s, .)$ is convex and takes real values, it is also continuous. We deduce by [76, Proposition 7.4] that $\lim_{n\to\infty} \tilde{\gamma}_{t+1}^{\xi,n}(s, .) \overset{epi}{=} \tilde{\gamma}_{t+1}^{\xi}(s, .)$. As in the case $t = T - 1$, we deduce by induction that $\lim_{n\to\infty} \tilde{D}_t^{\xi,n}(s, v_{t-1}, .) \overset{epi}{=} \tilde{D}_t^{\xi}(s, v_{t-1}, .)$.

At last, since $\inf_{v_t\in S^{d-1}(0,1)} \tilde{D}_t^0(s, 0, v_t) > 0$, SAIP holds for the market defined by C_t, see Theorem 4.4.16 in Chapter 4. By Theorem 5.5.5, we have $\tilde{\gamma}_t^{\xi}(S_t, V_t) = \gamma_t^{\xi}(S_t, V_t)$ a.s. for any $V_t \in L^0(\mathbf{R}^d, \mathcal{F}_t)$. Moreover, since $\tilde{D}_t^{0,n}(s, 0, v_t) \geq \tilde{D}_t^0(s, 0, v_t)$, we deduce that SAIP also holds for market defined by C_t^n and, similarly, we have $\tilde{\gamma}_t^{\xi,n}(S_t, V_t) = \gamma_t^{\xi,n}(S_t, V_t)$ a.s. for any $V_t \in L^0(\mathbf{R}^d, \mathcal{F}_t)$. The conclusion follows. □

5.7 Appendix

We recall from [76] the definiton of piecewise linear function:

Definition 5.7.1. *A mapping $F : D \to \mathbf{R}^m$ defined on a set $D \in \mathbf{R}^n$ is piecewise linear on D if D is the union of finitely many polyhedral sets $(P_i)_{i\in J}$ such that, for all $x \in P_i$, $F(x) = A_i x + B_i$, for some matrix $A_i \in \mathbf{R}^{m\times n}$ and $B_i \in \mathbf{R}^m$. A function $f : \mathbf{R}^n \to \bar{\mathbf{R}}$ is piecewise linear if it is a real-valued piecewise linear function on its domain.*

We now provide the complement to Section 5.4. Recall that the model is defined by one risk-free asset and one risky asset denoted by S. The cost function is given by

$$\mathrm{C}_t(S_t, v) = v^1 + S_t \tilde{\mathrm{C}}_t(v^2), \tag{5.7.16}$$

where $\tilde{\mathrm{C}}_t : \mathbf{R} \to \mathbf{R}$ is a piecewise linear function. By Proposition 4.5.6 in Chapter 4, we have:

$$\begin{aligned}
\theta^0_{T-1}(S_{T-1}, v) &:= \operatorname{ess\,sup}_{\mathcal{F}_{T-1}} \mathrm{C}_T(S_T, (0, -v^2)) = \sup_{s\in \operatorname{supp}_{\mathcal{F}_{T-1}} S_T} \mathrm{C}_T(s, (0, -v^2)) \\
&= \sup_{s\in \operatorname{supp}_{\mathcal{F}_{T-1}} S_T} \big(-(1+\epsilon_T) s v^2 1_{v^2\le 0} - (1-\epsilon_T) s v^2 1_{v^2\ge 0}\big) \\
&= \sup_{s\in [k^d_{T-1} S_{T-1}, k^u_{T-1} S_{T-1}]} \big(-(1+\epsilon_T) s v^2 1_{v^2\le 0} - (1-\epsilon_T) s v^2 1_{v^2\ge 0}\big) \\
&= \max\big\{ -(1+\epsilon_T) k^d_{T-1} S_{T-1} v^2 1_{v^2\le 0} - (1-\epsilon_T) k^d_{T-1} S_{T-1} v^2 1_{v^2\ge 0}, \\
&\quad -(1+\epsilon_T) k^u_{T-1} S_{T-1} v^2 1_{v^2\le 0} - (1-\epsilon_T) k^u_{T-1} S_{T-1} v^2 1_{v^2\ge 0}\big\} \\
&= -(1-\epsilon_T) k^d_{T-1} S_{T-1} v^2 1_{v^2\ge 0} - (1+\epsilon_T) k^u_{T-1} S_{T-1} v^2 1_{v^2\le 0}.
\end{aligned}$$

and

$$\begin{aligned}
&\mathrm{C}_{T-1}(S_{T-1}, (0, v^2 - z^2)) \\
= \;& (1+\epsilon_{T-1}) S_{T-1} v^2 1_{v^2-z^2\ge 0} + (1-\epsilon_{T-1}) S_{T-1} v^2 1_{v^2-z^2\le 0} \\
& -(1+\epsilon_{T-1}) S_{T-1} z^2 1_{v^2-z^2\ge 0} + (1-\epsilon_{T-1}) S_{T-1} z^2 1_{v^2-z^2\le 0}.
\end{aligned}$$

We then have:

$$\begin{aligned}
D^0_{T-1}(S_{T-1}, 0, v) &= \theta^0_{T-1}(S_{T-1}, v) + \mathrm{C}_{T-1}(S_{T-1}, (0, v^2)) \\
&= ((1+\epsilon_{T-1}) S_{T-1} - (1-\epsilon_T) k^d_{T-1} S_{T-1}) v^2 1_{v^2\ge 0} \\
&\quad + ((1-\epsilon_{T-1}) S_{T-1} - (1+\epsilon_T) k^u_{T-1} S_{T-1}) v^2 1_{v^2\le 0}
\end{aligned}$$

More generally:

$$\begin{aligned} D^0_{T-1}(S_{T-1},z,v) &= \theta^0_{T-1}(S_{T-1},v)+\mathrm{C}_{T-1}(S_{T-1},(0,v-z)) \\ &= (1+\epsilon_{T-1})S_{T-1}v^2 1_{v^2-z^2\geq 0}+(1-\epsilon_{T-1})S_{T-1}v^2 1_{v^2-z^2\leq 0} \\ &\quad -(1+\epsilon_{T-1})S_{T-1}z^2 1_{v^2-z^2\geq 0}+(1-\epsilon_{T-1})S_{T-1}z^2 1_{v^2-z^2\leq 0} \\ &\quad -(1-\epsilon_T)k^d_{T-1}S_{T-1}v^2 1_{v^2\geq 0}-(1+\epsilon_T)k^u_{T-1}S_{T-1}v^2 1_{v^2\leq 0}. \end{aligned}$$

In the following, we assume that $1+\epsilon_{T-1}\leq (1+\epsilon_T)k^u_{T-1}$ and, also, that $1-\epsilon_{T-1}\geq (1-\epsilon_T)k^d_{T-1}$. We shall use the usual convention that $\inf\emptyset=\infty$. We get that:

$$\gamma^0_{T-1}(z)=\inf_{v\in\mathbf{R}^2} D^0_{T-1}(S_{T-1},z,v)=\min_{i=1,\cdots,4} D^{0,i}_{T-1}(S_{T-1},z,v),$$

where:

$$\begin{aligned} D^{0,1}_{T-1} &= \inf_{v^2:v^2\geq z^2,v^2\geq 0}\left((1+\epsilon_{T-1})S_{T-1}(v^2-z^2)-(1-\epsilon_T)k^d_{T-1}S_{T-1}v^2\right) \\ &= -(1-\epsilon_T)k^d_{T-1}S_{T-1}z^2 1_{z^2\leq 0}-(1+\epsilon_{T-1})S_{T-1}z^2 1_{z^2\geq 0}. \end{aligned}$$

$$\begin{aligned} D^{0,2}_{T-1} &= \inf_{v^2:v^2\geq z^2,v^2\leq 0}\left((1+\epsilon_{T-1})S_{T-1}(v^2-z^2)-(1+\epsilon_T)k^u_{T-1}S_{T-1}v^2\right) \\ &= \infty 1_{z^2>0}-(1+\epsilon_{T-1})S_{T-1}z^2 1_{z^2\leq 0}. \end{aligned}$$

$$\begin{aligned} D^{0,3}_{T-1} &= \inf_{v^2:v^2\leq z^2,v^2\geq 0}\left((1-\epsilon_{T-1})S_{T-1}(v^2-z^2)-(1-\epsilon_T)k^d_{T-1}S_{T-1}v^2\right) \\ &= \infty 1_{z^2<0}-(1-\epsilon_{T-1})S_{T-1}z^2 1_{z^2\geq 0}. \end{aligned}$$

$$\begin{aligned} D^{0,4}_{T-1} &= \inf_{v^2:v^2\leq z^2,v^2\leq 0}\left((1-\epsilon_{T-1})S_{T-1}(v^2-z^2)-(1+\epsilon_T)k^u_{T-1}S_{T-1}v^2\right) \\ &= -(1-\epsilon_{T-1})S_{T-1}z^2 1_{z\geq 0}-(1+\epsilon_T)k^u_{T-1}S_{T-1}z^2 1_{z^2\leq 0}. \end{aligned}$$

We deduce that

$$\begin{aligned} \gamma^0_{T-1}(S_{T-1},z) &= \inf_{v\in\mathbf{R}^2} D^0_{T-1}(S_{T-1},z,v) \\ &= -(1+\epsilon_{T-1})S_{T-1}z^2 1_{z^2\geq 0}-(1-\epsilon_T)k^d_{T-1}S_{T-1}z^2 1_{z^2\leq 0}. \end{aligned}$$

We now compute $D^0_{T-2}(S_{T-1}, 0, z)$. We have:

$$\begin{aligned}
& \theta^0_{T-2}(S_{T-2}, z) \\
= \;& \text{ess sup}_{\mathcal{F}_{T-2}} \gamma^0_{T-1}(S_{T-1}, z) \\
= \;& \sup_{s \in [k^d_{T-2} S_{T-2}, k^u_{T-2} S_{T-2}]} \gamma^0_{T-1}(s, z) \\
= \;& \sup_{s \in [k^d_{T-2} S_{T-2}, k^u_{T-2} S_{T-2}]} \left(-(1+\epsilon_{T-1}) s z^2 1_{z^2 \geq 0} - (1-\epsilon_T) k^d_{T-1} s z^2 1_{z^2 \leq 0} \right) \\
= \;& -(1+\epsilon_{T-1}) k^d_{T-2} S_{T-2} z^2 1_{z^2 \geq 0} - (1-\epsilon_T) k^d_{T-1} k^u_{T-2} S_{T-2} z^2 1_{z^2 \leq 0}.
\end{aligned}$$

and

$$\begin{aligned}
& D^0_{T-2}(S_{T-2}, 0, z) \\
= \;& \theta^0_{T-2}(S_{T-2}, z) + \mathrm{C}_{T-2}(S_{T-2}, (0, z^2)) \\
= \;& -(1+\epsilon_{T-1}) k^d_{T-2} S_{T-2} z^2 1_{z^2 \geq 0} - (1-\epsilon_T) k^d_{T-1} k^u_{T-2} S_{T-2} z^2 1_{z^2 \leq 0} \\
& + (1+\epsilon_{T-2}) S_{T-2} z^2 1_{z^2 \geq 0} + (1-\epsilon_{T-2}) S_{T-2} z^2 1_{z^2 \leq 0} \\
= \;& \left((1+\epsilon_{T-2}) S_{T-2} - (1+\epsilon_{T-1}) k^d_{T-2} S_{T-2} \right) z^2 1_{z^2 \geq 0} \\
& + \left((1-\epsilon_{T-2}) S_{T-2} - (1-\epsilon_T) k^d_{T-1} k^u_{T-2} S_{T-2} \right) z^2 1_{z^2 \leq 0}.
\end{aligned}$$

We then get the following:

Proposition 5.7.2. *AIP holds at time $T-2$ if and only if the following holds:*

$$k^d_{T-2} \leq \frac{1+\epsilon_{T-2}}{1+\epsilon_{T-1}} \text{ and } k^u_{T-2} \geq \frac{1-\epsilon_{T-2}}{(1-\epsilon_T) k^d_{T-1}}.$$

Chapter 6

Portfolio optimization under credit risk constraints

6.1 The model

Consider a financial market model defined on a stochastic basis $(\Omega, (\mathcal{F}_t)_{t\in[0,T]}, P)$ satisfying the usual assumptions. We denote by S^0 the risk-free asset of the market and we suppose without loss of generality that $S^0 = 1$ so that the risk-free interest rate is $r^0 = 0$. In the following, we consider at any time $t \in [0, T]$ a firm characterized by it debts $(D_r)_{r\in[t,T]}$ and its asset $(A_r)_{r\in[t,T]}$ so that the equity is given by $(E_r)_{r\in[t,T]}$ such that $E = (A - D)^+$. We suppose that D satisfies the SDE:

$$dD_u = r_u D_u du - K_u du, \quad u \in [t, T], \tag{6.1.1}$$

where $r \geq 0$ is the debt interest rate (interpreted as a risk premium since $r^0 = 0$) and K_u is the amount of the firm reimbursement per unit of time. We suppose that $K_u := k_u D_u$ for some process k. Asset A of the firm satisfies by assumption $A_r = \theta^0_r S^0_r + \theta_r S_r$ where θ^0 and θ are quantities invested in asset S^0 and some risky assets $S = (S_1, \cdots, S_d)$, $d \geq 1$, held by the firm. In this model, we suppose that $\theta^0 \geq 0$ and $\theta \geq 0$ and $d = 1$. We suppose the following self-financing condition:

$$dA_r = \theta_r dS_r - c_r dr, \quad r \in [t, T], \tag{6.1.2}$$

where c is a cash process such that $c \geq K$. We interpret $c_t - K_t$ as the amount of dividends distributed at time t. We only consider admissible strategies θ

such that $A_r \geq \kappa^\theta$ for all $r \in [t,T]$, for some $\kappa^\theta \in \mathbf{R}$. Liquidation value of the asset firm at time $u \geq t$ is defined as $L_u := A_u - D_u$ so that we have $E = L^+$. Note that the dynamics of L is :

$$dL_u = \theta_u dS_u - d_u du - r_u D_u du, \quad u \in [t,T], \tag{6.1.3}$$

where $d_t = c_t - k_t D_t$ is the amount of dividends. The dynamics above shows that the liquidation value of the firm's financial position is naturally controlled by the investment strategy θ but it is adversely impacted by the dividends $d \geq 0$ paid to the share holders and by the credit risk premium r as well. In particular, apart the risk provided by the risky asset S, there is a risk generated by the investment strategy θ such that an increase of the credit risk premium r may decrease the liquidation value L, which may leads to a bankruptcy when $L = 0$.

From (6.1.2) and (6.1.1), suppose that $E_t \geq 0$, then we have:

$$A_T \geq D_T \Leftrightarrow E_t + \int_t^T \theta_u dS_u - \int_t^T d_u du \geq \int_t^T r_u D_u du. \tag{6.1.4}$$

We suppose in the following that the credit risk premium and the reimbursement rate are constant denoted respectively r and k. Then, in the case where the inequality above holds, it also holds for $r = 0$. Moreover, as D is an increasing function of r, the inequality is violated as soon as r is large enough. Therefore, it is possible for the debt holders to deliberately make the firm insolvent by increasing the credit risk premium.

6.2 Valuation of the risk premium under risk-neutral measure

In this section, we consider the problem of evaluation of the Fair Credit Risk Premium. We suppose that the market is complete and we shall restrict ourselves to strategies θ such that $A \geq 0$. We suppose without loss of generality that P is the risk-neutral probability measure for S. Taking into account a possible default, the payoff delivered to the credit holders is as the Merton model:

$$h_T^D := \int_t^T kD_u du + A_T \wedge D_T$$

and the payoff delivered to the equity holders is:

$$h_T^E := \int_t^T d_u du + (A_T - D_T)^+.$$

Observe that, when $A_T \geq 0$, we have:

$$\begin{aligned} h_T^D + h_T^E &= \int_t^T kD_u du + \int_t^T d_u du + A_T \\ &= E_t + D_t + \int_t^T \theta_u dS_u. \end{aligned} \tag{6.2.5}$$

Lemma 6.2.1. *Suppose that the process $N : r \mapsto \int_t^r \theta_u dS_u$ is a P-martingale on $[t, T]$ and $A_T \geq 0$. Then, $E(h_T^D|\mathcal{F}_t) = D_t$ and $E(h_T^E|\mathcal{F}_t) = E_t$ if and only if $E(h_T^D|\mathcal{F}_t) = D_t$. This condition is called market pricing (MP) at time t.*

Proof. By (6.2.5), we deduce that $E(h_T^D|\mathcal{F}_t) + E(h_T^E|\mathcal{F}_t) = D_t + E_t$. This equality implies that $E(h_T^D|\mathcal{F}_t) = D_t$ and $E(h_T^E|\mathcal{F}_t) = E_t$ as soon as $E(h_T^D|\mathcal{F}_t) = D_t$. □

We recall that for all $s \geq t$:

$$\begin{aligned} A_s &= A_t + \int_t^s \theta_u dS_u - \int_t^s d_u du - \int_t^s kD_u du, \\ D_s &= D_t + \int_t^s (r-k)D_u du. \end{aligned}$$

It follows that

$$\begin{aligned} A_T \geq D_T &\iff A_t + \int_t^T \theta_u dS_u - \int_t^T rD_u du \geq D_t + \int_t^T d_u du, \\ &\iff E_t + \int_t^T \theta_u dS_u - \int_t^T d_u du \geq \int_t^T rD_u du. \end{aligned} \tag{6.2.6}$$

Therefore, when $A_T \geq 0$, we deduce that:

$$\begin{aligned} h_T^D &= h_T^D(r) = \min\left(A_t + \int_t^T \theta_u dS_u - \int_t^T d_u du, D_t + \int_t^T rD_u du\right) \quad (6.2.7) \\ &= \min\left(D_t + L_T + \int_t^T rD_u du, D_t + \int_t^T rD_u du\right), \\ h_T^E &= h_T^E(r) = \max\left(\int_t^T d_u du, E_t + \int_t^T \theta_u dS_u - \int_t^T rD_u du\right). \quad (6.2.8) \end{aligned}$$

Note that (6.2.8) also holds when $A_T \leq 0$.

Lemma 6.2.2. *The function $r \mapsto \int_t^T rD_r(r)du$ is increasing on $[0,\infty)$ and $\int_t^T rD_u(r)du \geq 0$. Moreover, $\int_t^T rD_u(r)du = 0$ if and only if $r = 0$.*

Proof. By a direct computation, we have:

$$D_u = D_t e^{(r-k)(u-t)}$$
$$\int_t^T rD_u(r)du = \frac{D_t r\left(e^{(r-k)(T-t)} - 1\right)}{r-k}$$

with the convention $(e^X - 1)/X = 1$ if $X = 0$. We then conclude. □

Lemma 6.2.3. *Suppose that $A_T \geq 0$ a.s.. The condition $h_T^D(r) \geq D_t$ holds a.s. if and only if $A_t + \int_t^T \theta_u dS_u - \int_t^T d_u du \geq D_t$ a.s. and, under this condition, (MP) holds at time t if and only if $r = 0$.*

Proof. Suppose that $h_T^D \geq D_t$. We use (6.2.7) to deduce that $h_T^D = A_t + \int_t^T \theta_u dS_u - \int_t^T d_u du$, then the inequality $h_T^D \geq D_t$ implies that $A_t + \int_t^T \theta_u dS_u - \int_t^T d_u du \geq D_t$. Otherwise, using (6.2.7), we deduce that

$$A_t + \int_t^T \theta_u dS_u - \int_t^T d_u du \geq h_T^D = D_t + \int_r^T rD_u du \geq D_t.$$

The reverse implication is trivial. At last, under this last condition, since $h_T^D(r) \geq D_t$ and (MP) means that $E_Q(h_T^D(r)|\mathcal{F}_t) = D_t$, we deduce that (MP) holds if and only if $h_T^D(r) = D_t$ or equivalently $r = 0$. □

Remark 6.2.4. *Let us suppose that the inequality $h_T^D(0) = D_t$ is not satisfied a.s. Therefore, $P(h_T^D(0) < D_t) > 0$. Since $h_T^D(0) \leq D_t$ a.s., we deduce that $E_Q(h_T^D(0)|\mathcal{F}_t) < D_t$ on a non null set. Therefore, it is necessary to increase the credit risk premium r for Condition (MP) to be satisfied.*

The following result says that, if the dividend plan d is too large, then the firm faces a bankruptcy at time T.

Proposition 6.2.5. *Suppose that $N : r \mapsto \int_t^r \theta_u dS_u$ is a P-martingale on $[t,T]$. Suppose that $A_T \geq 0$ a.s. and $L_t \geq 0$. Suppose that $E\left(\int_t^T d_r dr|\mathcal{F}_t\right) \geq E_t$ on a non null set $\Lambda_t \in \mathcal{F}_t$. Then, on Λ_t, $E\left(h_T^D(r)|\mathcal{F}_t\right) \geq D_t$ a.s. if and only if we have $E\left(\int_t^T d_r dr|\mathcal{F}_t\right) = E_t$ and $L_T = A_T - D_T \leq 0$, i.e.*

$$E_t + \int_t^T \theta_u dS_u - \int_t^T d_u du \leq \int_t^T rD_u du, \text{ a.s.}$$

Under the equivalent conditions above, we then have $E(h_T^D(r)|\mathcal{F}_t) = D_t$ *on* Λ_t. *Moreover,* h_T^D *does not depend on* r *and* $E(h_T^D(r)|\mathcal{F}_t) = D_t$.

Proof. By (6.2.8), $E_Q(h_T^D(r)|\mathcal{F}_t) \geq D_t$ if and only if

$$E(L_T^-|\mathcal{F}_t) \leq E\left(\int_t^T rD_u du|\mathcal{F}_t\right)$$

or, equivalently, if

$$E\left(\left(\int_t^T rD_u du - \gamma\right)^+ |\mathcal{F}_t\right) \leq E\left(\int_t^T rD_u du|\mathcal{F}_t\right),$$

where

$$\gamma = L_t + \int_t^T \theta_u dS_u - \int_t^T d_u du = L_T + \int_t^T rD_u du.$$

Note that the first equality above comes from (6.1.3) and γ does not depend on r. Since $x^+ \geq x$, we deduce that

$$E\left(\left(\int_t^T rD_u du - \gamma\right)^+ |\mathcal{F}_t\right) \geq E\left(\int_t^T rD_u du|\mathcal{F}_t\right) - E(\gamma|\mathcal{F}_t).$$

Since $L_t = L_t^+ = E_t$ and $E\left(\int_t^T d_u du|\mathcal{F}_t\right) \geq E_t$, then we deduce that $E(\gamma|\mathcal{F}_t) \leq 0$. It follows that $E(h_T^D(r)|\mathcal{F}_t) \geq D_t$ if and only if $E(\gamma|\mathcal{F}_t) = 0$, i.e. $E\left(\int_t^T d_u du|\mathcal{F}_t\right) = E_t$, and

$$E\left(\left(\int_t^T rD_u du - \gamma\right)^+ |\mathcal{F}_t\right) = E\left(\int_t^T rD_u du|\mathcal{F}_t\right).$$

Therefore, $E(h_T^D(r)|\mathcal{F}_t) \geq D_t$ if and only if $E(\gamma|\mathcal{F}_t) = 0$ and

$$\left(\int_t^T rD_u du - \gamma\right)^+ = \int_t^T rD_u du - \gamma,$$

as the difference between the l.h.s. and the r.h.s. of the equality above is non negative with a zero expectation. This implies that

$$\gamma \leq \int_0^T rD_u du, \quad \text{a.s,}$$

i.e. $L_T = A_T - D_T \leq 0$ a.s. Reciprocally, if $A_T \leq D_T$, by (6.2.8) we get that

$$\begin{aligned} h_T^D(r) &= A_T + \int_t^T k D_u du, \\ &= E_t + D_t + \int_t^T \theta_u dS_u - \int_t^T d_u du. \end{aligned}$$

Therefore, $h_T^D(r)$ does not depend on r and satisfies $E(h_T^D(r)|\mathcal{F}_t) = D_t$. □

Proposition 6.2.6. *Suppose that $N : r \mapsto \int_t^r \theta_u dS_u$ is a P-martingale on $[t,T]$. Suppose that $E\left(\int_t^T d_r dr|\mathcal{F}_t\right) < E_t$ a.s. and $A_T \geq 0$ a.s. There exists a unique credit prime $r^* \in L^0(\mathbf{R}_+, \mathcal{F}_t)$ such that (MP) holds.*

Proof. We use the same notations as in the proof of Proposition 6.2.5 where the initial time is t instead of 0. Let us introduce random function

$$\phi_t : r \in \mathbf{R}_+ \mapsto E(h_T^D(r)|\mathcal{F}_t) - D_t,$$

where a regular version of the conditional probability measure $P(\cdot|\mathcal{F}_t)$ is considered. We have

$$\phi_t(r) = E\left(\int_t^T r D_u(r) du - \left(\int_t^T r D_u(r) du - \gamma\right)^+ |\mathcal{F}_t\right).$$

Note that the function $\delta(x) = x - (x-\gamma)^+$ is non decreasing and for $x \geq 0$, $|\delta(x)| \leq |\gamma|$. As γ is conditionally integrable, we deduce by the dominated convergence theorem that $\phi_t(\infty) = E(\gamma|\mathcal{F}_t)$. In particular, since we have $E(\gamma|\mathcal{F}_t) = E_t - E\left(\int_t^T d_u du|\mathcal{F}_t\right)$, we get that $\phi_t(\infty) > 0$ a.s.. Moreover, $\phi_t(0) = -E(\gamma^-|\mathcal{F}_t) \leq 0$. Therefore, as $r \mapsto \phi_t(r)$ is continuous a.s. and non decreasing, there exists $r^* = r^*(\omega) \in \mathbf{R}^+$ such that $\phi_t(r^*) = 0$. By continuity, we actually get that $\phi_t = \phi_t(\omega, r)$ is a normal integrand, see [76], so that ϕ_t is $\mathcal{F}_t \otimes \mathcal{B}(\mathbf{R})$-measurable. Therefore, the set $\Gamma_t = \{(\omega, r) \in \Omega \times \mathcal{B}(\mathbf{R})_+ : \phi_t(\omega, r) = 0\}$ belongs to $\mathcal{F}_t \otimes \mathcal{B}(\mathbf{R})$. Since the ω-sections of Γ_t are not empty, we deduce by a measurable selection argument the existence of $r^* \in L^0(\mathbf{R}_+, \mathcal{F}_t)$ such that $\phi_t(r^*) = 0$.

Suppose that there are distinct $r_1, r_2 \in L^0(\mathbf{R}_+, \mathcal{F}_t)$ such that $\phi_t(r_1) = \phi_t(r_2) = 0$. Then, the same holds for $r_1 \wedge r_2$ and $r_1 \vee r_2$ so that we may

assume without loss of generality that $r_1 \leq r_2$. Since δ is strictly increasing on $(-\infty, \gamma)$ and constant on $[\gamma, \infty)$, we obtain that

$$\delta\left(\int_t^T r_1 D_u(r_1)du\right) \leq \delta\left(\int_t^T r_2 D_u(r_2)du\right)$$

and, finally, the equality holds due to the assumption. Therefore, we necessarily have

$$\int_t^T r_2 D_u(r_2)du \geq \gamma,$$

at least when r_1 and r_2 are distinct, since δ is strictly increasing on $(-\infty, \gamma)$. We deduce that $\phi_t(r_2) \geq E(\gamma|\mathcal{F}_t)$ where $E(\gamma|\mathcal{F}_t) > 0$ by assumption. This yields a contradiction. □

Note that, when $\phi_t(0) = 0$, then $\gamma = L_T \geq 0$ a.s. hence $A_T \geq D_T$ a.s. so that, by the proposition below, $r^* = 0$ is the only risk premium satisfying $\phi_t(r^*) = 0$ under (MP).

6.3 Optimization problem for the firm

We consider a utility function U which is strictly concave, strictly increasing and of class C^1. We consider the model of Section 6.1 and we assume that there is a unique risk neutral probability measure P, see Section 6.2. At time $t \leq T$, $L_t \in L^0(\mathbf{R}, \mathcal{F}_t)$ and $D_t \in L^0(\mathbf{R}, \mathcal{F}_t)$ are given and we introduce the function $J^0(t, x, y, \hat{\theta}, c) := E\left(U\left(\int_t^T d_u du + L_T^{\hat{\theta},c}\right)|(D_t, L_t) = (x, y)\right)$, where $L_T^{\hat{\theta},c}$ starts from the initial value L_t at time t and D starts from the intial value D_t. By Condition Z2 in [82], we have

$$J^0(t, x, y, \hat{\theta}, c) := E\left(U\left(\int_t^T d_u du + L_T^{x,y,\hat{\theta},c}\right)\right) \tag{6.3.9}$$

where d is the dividend's plan of the firm, $y = L_t$ is the initial value of the liquidation value, $x = D_t$ is the initial value of the debt, and $L_T^{x,y,\hat{\theta},c}$ is defined by (6.1.3) with the initial values x, y at time t defining respectively D_t and L_t. Moreover, the control is $c = (d, k)$ such that (MP) holds, i.e. $E(h_T^D(r)|(D_t, L_t) = (x, y)) = E(h_T^D(r)) = x$ where the process $h^D(r)$ in the last expectation is defined from the initial value $x = D_t$. Here, we suppose

that the reimbursement plan k is a real-valued constant. The constant credit risk prime r is real-valued constant given by Proposition 6.2.6.

We require that $\hat{\theta}S$ is a square integrable martingale on $[t,T]$ and $\hat{\theta}^0, \hat{\theta} \geq 0$ is such that $A_T = A_T(\hat{\theta}) \geq 0$ a.s. This is equivalent to say that $A_u \geq \hat{\theta}_u S_u$ for every $u \in [t,T]$ and $A_T \geq 0$. So we impose that $A_u \geq \hat{\theta}_u S_u$ for every $u \in [t,T]$. Here, for technical reason, we suppose that $A_u \geq \hat{\theta}_{u+} S_u$ where $\hat{\theta}_{u+}$ is the right limit of $\hat{\theta}$. As A is continuous, this implies that $\hat{\theta}^0_{u+}, \hat{\theta}^0_{u-} \geq 0$. It is enough for our purposes since $\int_0^t \hat{\theta}_u dS_u = \int_0^t \hat{\theta}_{u+} dS_u = \int_0^t \hat{\theta}_{u-} dS_u$ as soon as S is continuous. In particular, if $\hat{\theta}$ is an optimal strategy, this is also the case for $\hat{\theta}_+$. The goal of the problem is to maximize $J^0(t,x,y,z,\hat{\theta},c)$ over all strategies $\hat{\theta}$ and control c satisfying the constraints described previously.

Remark 6.3.1. *In the case where* $E\left(\int_t^T d_r dr | \mathcal{F}_t\right) \geq E_t$, *we deduce by Proposition 6.2.5 that the condition* $E(h_T^D|\mathcal{F}_t) = D_t$ *implies that we have* $E\left(\int_t^T d_r dr | \mathcal{F}_t\right) = E_t$ *and the equality* $E(h_T^D(r)|\mathcal{F}_t) = D_t$ *holds whatever the credit risk prime* r *since* h_T^D *does not depend on* r. *As the mapping* $r \mapsto L_T^{x,y,\hat{\theta},c}$ *in non increasing and* U *is increasing, the maximization leads to choose* $r = 0$.

Let us introduce the more general function $J(t,x,y,\hat{\theta},c,r)$ defined as:

$$J(t,x,y,\hat{\theta},c,r) := E\left(U\left(\int_t^T d_u du + L_T^{x,y,\hat{\theta},c}\right) + x - h_T^D(r)\right) \quad (6.3.10)$$

Lemma 6.3.2. *The initial maximization problem of (6.3.9) is equivalent to maximize* $J(t,x,y,\hat{\theta},c,p)$ *of (6.3.10) over all strategies* $\hat{\theta}$ *such that* $A_u \geq \hat{\theta}_{u+} S_u$ *for all* $u \geq t$, *controls* c *and credit risk primes* $r \geq 0$, *under the target constraint* $E(h_T^D(r)) \geq x$ *where* $h_T^D(r)$ *is defined from the initial values* x, y.

Proof. Under the target constraint $E(h_T^D(r)) \geq x$, $x - E(h_T^D(r))$ is necessarily non positive. Recall that $E(h_T^D(r)|\mathcal{F}_t) = \phi_t(r) + D_t$ where ϕ_t, given in the proof of Proposition 6.2.6, in non decreasing in r. Therefore, the mapping $r \mapsto x - E(h_T^D(r))$ is non increasing. Also, $r \mapsto L_T(r)$ is non increasing by (6.1.3). Therefore, it suffices to decrease r to increase $J(t,x,y,\hat{\theta},c,r)$. Precisely, we consider the smallest and unique r^* such that $E(h_T^D(r^*)) = x$, see proposition 6.2.6. □

We define $\mathcal{U}^t$ as the collection of $\nu = (\hat{\theta}, d, k, r)$ such that $(\hat{\theta}, d)$ are progressively measurable processes with values in $\mathbf{R} \times \mathbf{R}$ and $k, r \in \mathbf{R}_+$. We

aim to solve the following stochastic target problem:

$$V(t,z) := \sup \Big\{ J(t,x,y,\hat{\theta},c,r), \nu = (\hat{\theta},c,r) \in \mathcal{U}_{t,z} \Big\}$$

where $\mathcal{U}_{t,z}$ is given by:

$$\mathcal{U}_{t,z} := \{\nu \in \mathcal{U}^t,\ E(h_T^D(r)) \geq x, A_u \geq \hat{\theta}_{u+} S_u,\ \forall u \geq t \text{ a.s.}\}.$$

Following the idea in [10], we add an extra control to the initial problem so that we may rewrite the problem. To be precise, let $\mathcal{A}$ be the set of $\mathbb{F}$-progressively measurable real-valued square integrable process. To each $\alpha \in \mathcal{A}$, we associate a controlled process:

$$P_{t,x}^{\alpha}(r) = x + \int_t^r \alpha_u dW_u, \quad r \in [t,T].$$

Instead of considering the controls $(\hat{\theta},c,r)$ such that $E(h_T^D(r)) \geq x$, we then work with the controls $(\hat{\theta},c,r,\alpha)$ such that $g(h_T^D(r), P_{t,x}^{\alpha}(T)) \geq 0$ where $g(Z,P) = Z - P$. Therefore, the equivalent problem is to solve:

$$\max_{\theta,c,r,\alpha} J(t,x,y,\hat{\theta},c,r,\alpha) \tag{6.3.11}$$

under the stochastic target constraint $g(h_T^D(r), P_{t,x}^{\alpha}(T)) \geq 0$. To see it, we observe that $h_T^D(r)$ is square integrable so that we may apply the predictable representation theorem.

In the following, we suppose that the price S of the risky asset starts from an initial point s at time t and satisfies the dynamics:

$$S_{t,s}(r) = s + \int_t^r S_{t,s}(u)\sigma(u, S_{t,s}(u))dW_u, \quad r \in [t,T],$$

where W is a 1-dimensional standard Brownian motion and σ is a positive Lipschitz function defined on $[0,T] \times \mathbf{R}$ such that σ is uniformly bounded from below and above by positive constants. We deduce that the liquidation process L satisfies:

$$L_u = y + \int_t^u \hat{\theta}_v S_{t,s}(v)\sigma(v, S_{t,s}(v))dW_v - \int_t^u d_v dv - \int_t^u r D_v dv,$$

Note that $D_u = xe^{(r-k)(u-t)}$ if $u \in [t,T]$ and by direct computation, we get:

$$\begin{aligned} h_T^D &= x + \int_t^T rD_u du + \min(L_T, 0) \\ &= x + \min(L_T, 0) + \frac{xr_t(e^{(r-k)(T-t)} - 1)}{r-k} \end{aligned}$$

Here, we use the convention that $(e^X - 1)/X = 1$ if $X = 0$.

In the following, we use the notations of [10]. Precisely, let $\mathcal{U}^t$ be the set of all controls $\nu = (\hat{\theta}, d, k, r, \alpha) = (\nu^i)_{i=1,\dots,5}$ where we require $\hat{\theta}, d, k, r \geq 0$. We define the state process $Z_{t,z}^\nu = (Z_{t,z}^{\nu,i})_{i=1,\dots,6} = (X_{t,z}^\nu, Y_{t,z}^\nu)$ where $X_{t,z}^\nu = (X_{t,z}^{\nu,i})_{i=1,\dots,6}$ takes values in $\mathbf{R}^6$ while $Y_{t,z}^\nu \in \mathbf{R}$. They are defined as follows: for $z = (x,y) = (s_t, db_t, cd_t, cr_t, p_t, l_t)$ and $v \in [t,T]$,

$$\begin{aligned} X_{t,z}^{\nu,1}(v) &= s_t + \int_t^v \sigma_X^1(u, X_{t,z}^\nu(u), \nu_u) dW_u, \\ X_{t,z}^{\nu,2}(v) &= db_t + \int_t^v \mu_X^2(X_{t,z}^\nu(u), \nu_u) du, \\ X_{t,z}^{\nu,3}(v) &= cd_t + \int_t^v \mu_X^3(X_{t,z}^\nu(u), \nu_u) du, \\ X_{t,z}^{\nu,4}(v) &= cr_t + \int_t^v \mu_X^4(X_{t,z}^\nu(u), \nu_u) du, \\ X_{t,z}^{\nu,5}(v) &= dr_t + \int_t^v \mu_X^5(X_{t,z}^\nu(u), \nu_u) du, \\ X_{t,z}^{\nu,6}(v) &= p_t + \int_t^v \sigma_X^6(X_{t,z}^\nu(u), \nu_u) dW_u, \end{aligned}$$

where $\sigma_X^1(u,x,\nu) = x^1\sigma(u,x^1)$, i.e. $X_{t,z}^{\nu,1} = S_{t,s}$, $\mu_X^2(x,\nu) = (\nu^4 - \nu^3)x^2$ i.e. $X_{t,z}^{\nu,2} = D$, $\mu_X^3(x,\nu) = \nu^2$, i.e. $X_{t,z}^{\nu,3}(v) = cd_t + \int_t^v d_u du$, $\mu_X^4(x,\nu) = \nu^3 x^2$, i.e. $X_{t,z}^{\nu,4}(v) = cr_t + \int_t^v kD_u du$, $\mu_X^5(x,\nu) = \nu^4 x^2$, i.e. $X_{t,z}^{\nu,5}(v) = dr_t + \int_t^v rD_u du$, $\sigma_X^6(x,\nu) = \nu^6$, i.e. $X_{t,z}^{\nu,6} = p_t + P_{t,0}^\alpha$. At last,

$$Y_{t,z}^\nu(v) = y + \int_t^v \sigma_Y(u, Z_{t,z}^\nu(u), \nu_u) dW_u + \int_t^v \mu_Y(Z_{t,z}^\nu(u), \nu_u) du.$$

with $\sigma_Y(u,z,\nu) = \nu^1 z^1 \sigma(u, z^1)$, $\mu_Y(z,\nu) = -\nu^4 z^2 - \nu^2$, i.e. $Y_{t,z}^\nu = L$. For our problem, we are interested in the case where $cd_t = cr_t = dr_t = 0$ and $p_t =$

db_t. It follows that the optimization problem reads as $\sup_{\nu\in\mathcal{U}^t} E[f(Z^\nu_{t,z}(T))]$, where we define

$$f(Z) := U(Z^3 + Z^6) + [Z^6]^- - Z^5$$

Recall that $h^D_T = \int_t^T kD_u du + A_T \wedge D_T$ and $A = L + D$. Therefore, the target constraint reads as $g(Z^\nu_{t,z}) \geq 0$ a.s. where

$$g(Z) = g(X, Y) = X^4 + (X^2 + Y) \wedge X^2 - X^6.$$

6.3.1 Dynamic Programming Principle for optimal control under pathwise constraint

Let us define the auxiliary value function:

$$w(t, x) := \inf\{y \in \mathbf{R} : (t, x, y) \in \mathcal{D}\}$$

where the domain $\mathcal{D}$ is defined as $\mathcal{D} = \{(t, z) : \mathcal{U}_{t,z} \neq \emptyset\}$. Recall that the set of admissible controls is now written as:

$$\mathcal{U}_{t,z} := \left\{\nu : g(Z^\nu_{t,z}(T)) \geq 0 \,\text{and}\, Z^{\nu,2}_{t,z}(u) + Z^{\nu,7}_{t,z}(u) \geq \nu^1_{u+} Z^{\nu,1}_{t,z}(u)\ \forall u \in [t, T] \text{ a.s.}\right\}.$$

We define $Z^\nu_{t,z}(u)$ for $u < t$ as $Z^\nu_{t,z}(u) = 0$. We denote by $\mathcal{Q}^T$ the set of all rational numbers of $[0, T]$ completed with the terminal date T. We write $\mathcal{Q}^T$ as $\mathcal{Q}^T = (T_n)_{n\geq 0}$ with $T_0 = T$. By right continuity, we deduce that

$$\mathcal{U}_{t,z} := \left\{\nu \in \mathcal{U}^t : g^n(Z^\nu_{t,z}(T_n), \nu(T_n+)) \geq 0,\ \forall n \geq 0\right\}, \tag{6.3.12}$$

where $(g^n)_{n\geq 0}$ are continuous functions such that $y \mapsto g^n(x, y)$ is non-decreasing and $g^n(0, u) = 0$ for all $n \geq 0$. Precisely, we have $g^0(x, y, u) = g(x, y)$ while $g^n(z, u) = g^1(z, u) = x^2 + y - u^1 x^1$.

Lemma 6.3.3. *For all $t \in [t, T]$, we have*

$$\begin{aligned} w(t, x) &= +\infty, \quad \text{if } x^6 > x^2, \\ w(t, x) &= -x^2 + (x^6)^+, \quad \text{if } x^6 \leq x^2. \end{aligned}$$

Proof. We use the notation from above, i.e. $z = (x, y) = (s_t, db_t, cd_t, cr_t, dr_t, p_t, l_t)$. If $t = T$, we have $A_T = A_t = D_t + L_t = db_t + y$. Therefore, the constraint $A \geq \hat{\theta}_+ S$ reads as $y + db_t \geq \hat{\theta}_{t+} S_t$, i.e. $y \geq \hat{\theta}_{t+} x^1 - x^2$, $\hat{\theta}_t = \nu^1_t$. We also have

$h_T^D = db_t \wedge (db_t + y)$ hence the target constraint is $x^2 \wedge (x^2 + y) \geq x^6$. Therefore, $w(T,x) = +\infty$ if $x^6 > x^2$. If $x^6 \leq x^2$, it suffices that $y \geq \hat{\theta}_{t+} S_t - db_t$ and $y \geq x^6 - x^2$ for some control $\hat{\theta}_t$. Choosing $\hat{\theta}_t = 0$, we deduce that $w(T,x) = \max(x^6 - x^2, -x^2) = -x^2 + (x^6)^+$.

If $x^6 > x^2$, we get that $w(t,x) = \infty$ by the same reasoning. If $x^6 \leq x^2$, choose the control $\nu = 0$ and we get that $(t,x,y) \in \mathcal{D}$ with $y = -x^2 + (x^6)^+$ hence $w(t,x) \leq -x^2 + (x^6)^+$. On the other hand, consider y such that $(t,x,y) \in \mathcal{D}$. Then, from $A_u \geq \hat{\theta}_{u+} S_u$ a.s. for all $u \in [t,T]$, we deduce that $A_t \geq \hat{\theta}_{t+} S_t$. On the other hand, the target constraint is

$$h_T^D = \int_t^T k_u D_u du + A_T \wedge D_T \geq x^6 + \int_t^T \alpha_u dW_u = P^{\alpha}_{t,z^6}(T).$$

We use the fact that $A_T \wedge D_T \leq A_T$ a.s. and the expression of A_T deduced from its dynamics (6.1.2) to deduce that

$$y + db_t \geq x^6 - \int_t^T \theta_u dS_u + \int_t^T \alpha_u dW_u.$$

Taking the expectation in both sides of the inequality above, one has $y + db_t \geq x^6$. We then deduce, as in the case $t = T$, that $w(t,x) \geq -x^2 + (x^6)^+$. The conclusion follows. □

Lemma 6.3.4. *$\mathcal{D}$ is an upper set in y.*

Proof. Consider $(t,x,y) \in \mathcal{D}$. We aim to show that $(t,x,\tilde{y}) \in \mathcal{D}$ if $\tilde{y} \geq y$. First, we observe that by changing (t,x,y) into $(t,x,\tilde{y})$, we do not change $X^\nu_{t,z} = X^\nu_{t,x}$, i.e. $X^\nu_{t,(x,y)} = X^\nu_{t,(x,\tilde{y})} = X^\nu_{t,x}$. Indeed, $X^\nu_{t,x}$ does not depend on $Y^\nu_{t,(x,y)}$. On the other hand, $Y^\nu_{t,(x,y)}$ only depends on y and $X^\nu_{t,x}$. This implies that the process $-(\tilde{y} - y) + Y^\nu_{t,(x,\tilde{y})}$ satisfies the same SDE as $Y_{t,(x,y)}$. We then deduce by uniqueness that $Y^\nu_{t,(x,\tilde{y})} = Y_{t,(x,y)} + \tilde{y} - y$. Therefore, if $\tilde{y} \geq y$, we get that $Y^\nu_{t,(x,\tilde{y})} \geq Y_{t,(x,y)}$. As $g(x,y)$ is increasing in y, we deduce that $g(X^\nu_{t,x}, Y^\nu_{t,(x,\tilde{y})}) \geq g(X^\nu_{t,x}, Y^\nu_{t,(x,y)}) \geq 0$ a.s. Moreover, we have

$$\begin{aligned} A^\nu_{t,(x,\tilde{y})} &= L^\nu_{t,(x,\tilde{y})} + D^\nu_{t,(x,\tilde{y})} = Y^\nu_{t,(x,\tilde{y})} + X^{\nu,2}_{t,x} \\ &\geq Y^\nu_{t,(x,y)} + X^{\nu,2}_{t,x} = A^\nu_{t,(x,y)} \geq \theta_+ S. \end{aligned}$$

We then deduce that $(t,x,\tilde{y}) \in \mathcal{D}$. □

Lemma 6.3.5. *We have:*

$$\mathrm{cl}(\mathcal{D}) = \{(t,x,y) \in [0,T] \times \mathbf{R}^6 \times \mathbf{R} : \quad y \geq w(t,x)\}.$$

Proof. Consider (t,x,y) such that $y \geq w(t,x)$. Then, $w(t,x) < \infty$ and, for any $\epsilon > 0$, $y+\epsilon > w(t,x)$ implies that $y+\epsilon > \tilde{y}$ for some $\tilde{y}$ such that $(t,x,\tilde{y}) \in \mathcal{D}$. As $\mathcal{D}$ is an upper set in y, we deduce that $(t,x,y+\epsilon) \in \mathcal{D}$. As $\epsilon \to 0$, we deduce that $(t,x,y) \in \mathrm{cl}(\mathcal{D})$. Reciprocally, consider $(t,x,y) \in \mathrm{cl}(\mathcal{D})$. We have $(t,x,y) = \lim_{n\to\infty}(t_n,x_n,y_n)$ where $(t_n,x_n,y_n) \in \mathcal{D}$. This implies that $y_n \geq w(t_n,x_n)$ by definition of w. In particular, $w(t_n,x_n) = -x_n^2 + (x_n^6)^+ < \infty$ for all $n \geq 1$ hence $x_n^6 \leq x_n^2$. As $n \to \infty$, we deduce that $x^6 \leq x^2$ and $y \geq -x^2 + (x^6)^+ = w(t,x)$. The conclusion follows. □

We recall the famous assumption Z5 in [82], whichs holds in our model.

Assumption $Z5$: For any $u \leq T$, the map:

$$(t,z,\nu) \in \mathbf{R}_+ \times \mathbf{R}^7 \times \mathcal{U}^t \mapsto Z^\nu_{t,z}(u)$$

is Borel measurable.

Lemma 6.3.6. *Suppose that Assumption $Z5$ holds. Then, the set*

$$B := \left\{(t,z,\nu) \in \mathbf{R}_+ \times \mathbf{R}^7 \times \mathcal{U}^t : \ \nu \in \mathcal{U}_{t,z}\right\}$$

is Borel-measurable. In particular, it is analytically measurable. Moreover, for each $\epsilon > 0$, there exists an universally measurable map $\hat{\nu}^\epsilon : \mathbf{R}_+ \times \mathbf{R}^7 \to \mathcal{U}$ such that $\hat{\nu}^\epsilon(t,z) \in \mathcal{U}_{t,z}$ and

$$J(t,z,\nu^\epsilon(t,z)) \geq V(t,z) - \epsilon, \text{ if } V(t,z) < \infty,$$
$$J(t,z,\nu^\epsilon(t,z)) \geq \epsilon^{-1}, \text{ if } V(t,z) = \infty.$$

Proof. We first recall that the set

$$\tilde{B} = \left\{(t,z,\nu) \in \mathbf{R}_+ \times \mathbf{R}^7 \times \mathcal{U} : \ \nu \in \mathcal{U}^t\right\}$$

is closed, see for example the proof of [10, Lemma A.1]. Therefore, $\tilde{B}$ is a Borel set. By (6.3.12), we have $B = \cap_{n\geq 1} B^n \cap \tilde{B}$ where

$$B^n = \left\{(t,z,\nu) \in \mathbf{R}_+ \times \mathbf{R}^7 \times \mathcal{U}^t : \ g^n(Z^\nu_{t,z}(T_n), \nu(T_n+)) \geq 0\right\}.$$

The projection mapping $\nu \mapsto \nu(\tilde{T})$ is Borel-measurable for all fixed $\tilde{T} \in [0,T]$. Therefore, for fixed $n \geq 1$, the mapping

$$\nu \mapsto \nu(T_n+) = \lim_{m\to\infty} \nu(T_n + 1/m)$$

is also Borel-measurable. Using Assumption $Z5$, we then deduce that the mapping $(t,z,\nu) \mapsto (Z^\nu_{t,z}(T_n), \nu(T_n+))$ is Borel-measurable for every $n \geq 1$ hence so does $(t,z,\nu) \mapsto g^n(Z^\nu_{t,z}(T_n), \nu(T_n+))$ since the function g^n is continuous. We then conclude that B is Borel-measurable, hence a priori analytically measurable.

By the Fubini theorem, we also deduce that $J : \mathbf{R}_+ \times \mathbf{R}^7 \times \mathcal{U}^t \to \mathbf{R}$ defined as $J(t,z,\nu) = E[f(Z^\nu_{t,z}(T))]$ is Borel-measurable. In particular, it is upper semianalytic. By [7, Theorem 7.50], we now deduce the existence of the desired $\nu^\epsilon(t,z)$. □

Theorem 6.3.7. *Let θ be a stopping time with values in $[t,T]$, $z \in \mathbf{R}^7$, and $\nu \in \mathcal{U}^t$. The following equivalence holds:*

There exists $\tilde{\nu} \in \mathcal{U}_{t,z}$ such that $\nu = \tilde{\nu}$ on $[t,\theta]$ if and only if, for all $u \in [t,\theta]$, $g^1(Z^\nu_{t,z}(u), \nu(u+)) \geq 0$, a.s. and $(\theta, Z^\nu_{t,z}(\theta)) \in \mathcal{D}$ a.s..

Proof. We first suppose that $\nu = \tilde{\nu}$ on $[t,\theta]$. Then, if $u \in [t,\theta]$, we have $g^1(Z^\nu_{t,z}(u), \nu(u+)) = g^1(Z^{\tilde{\nu}}_{t,z}(u), \tilde{\nu}(u+)) \geq 0$, since $\tilde{\nu} \in \mathcal{U}_{t,z}$. Moreover, following the proof of [82, Theorem 3.1], we may show that $(\theta, Z^\nu_{t,z}(\theta)) \in D$ a.s. Indeed, it suffices to follow the same arguments if we replace, for every $n \geq 0$, the terminal date T by $T_n \vee \theta$.

Reciprocally, using the ideas of [82, Theorem 3.1] and recall the set B defined in Lemma 6.3.6, we first construct a Borel-measurable mapping ϕ such that $(t', z', \phi(t', z')) \in B$ μ a.s. where μ is the distribution of $(\theta, Z^\nu_{t,z}(\theta))$. Therefore, we consider the control of the form $\phi(\theta, Z^\nu_{t,z}(\theta))$ and by [82, A.2], we deduce some $\nu^1 \in \mathcal{U}$ such that $\nu^1(\omega,u) = \phi(\theta(\omega), Z^\nu_{t,z}(\theta)(\omega))(\omega,u)$ for all $u \geq \theta(\omega)$. We then define $\hat{\nu}$ the concatenation between ν and ν^1, see [82]. By the flow property, we then get that, for all $n \geq 0$,

$$\begin{aligned} g^n(Z^{\hat{\nu}}_{t,z}(T_n \vee \theta), \hat{\nu}(T_n \vee \theta+)) &= g^n(Z^{\hat{\nu}}_{\theta, Z^{\hat{\nu}}_{t,z}(\theta)}(T_n \vee \theta), \hat{\nu}(T_n \vee \theta+)) \\ &= g^n(Z^{\hat{\nu}}_{\theta, Z^{\nu}_{t,z}(\theta)}(T_n \vee \theta), \hat{\nu}(T_n \vee \theta+)) \\ &= g^n(Z^{\nu^1}_{\theta, Z^{\nu}_{t,z}(\theta)}(T_n \vee \theta), \nu^1(T_n \vee \theta+)). \end{aligned}$$

By assumption on ϕ, $g^n(Z^{\phi(t',z')}_{t',z'}(T_n \vee t'), \phi(t',z')(T_n \vee t'+)) \geq 0$ a.s. for all $n \geq 0$ and μ a.s. (t',z'). Taking the conditional expectation knowing $(\theta, Z^\nu_{t,z}(\theta))$, we deduce that the following chain of equalities:

$$\begin{aligned}
&E\left(1_{\left\{g^n(Z^{\nu^1}_{\theta,Z^\nu_{t,z}(\theta)}(T_n\vee\theta),\nu^1(T_n\vee\theta+))\geq 0\right\}}\right)\\
=&E\left(1_{\left\{g^n(Z^{\phi(\theta,Z^\nu_{t,z}(\theta))}_{\theta,Z^\nu_{t,z}(\theta)}(T_n\vee\theta),\phi(\theta,Z^\nu_{t,z}(\theta))(T_n\vee\theta+)))\geq 0\right\}}\right)\\
=&\int_{[0,T]\times\mathbf{R}^7} E\left(1_{\left\{g^n(Z^{\phi(t',z')}_{t',z'}(T_n\vee t'),\phi(t',z')(T_n\vee t'+))\geq 0\right\}}\right)\mu(dt',dz')\\
=&1.
\end{aligned}$$

We then deduce that $g^n(Z^{\nu^1}_{\theta,Z^\nu_{t,z}(\theta)}(T_n \vee \theta), \nu^1(T_n \vee \theta+)) \geq 0$ a.s. and finally, $g^n(Z^{\hat{\nu}}_{t,z}(T_n \vee \theta), \hat{\nu}(T_n \vee \theta+)) \geq 0$ a.s.

On the other hand, $g^n(Z^{\hat{\nu}}_{t,z}(T_n \wedge \theta)) = g^n(Z^\nu_{t,z}(T_n \wedge \theta)) \geq 0$ by assumption as $\hat{\nu}$ and ν coincide on $[t,\theta]$. We then conclude that $\hat{\nu} \in \mathcal{U}_{t,z}$.

□

Remark 6.3.8. *If $\nu \in \mathcal{U}_{t,z}$, then $z^2 + z^7 \geq \nu^1(t+)z^1$.*

We then deduce the following:

Lemma 6.3.9. *For any $(t,x,y) \in [0,T) \times \mathbf{R}^6 \times \mathbf{R}$, $\nu \in \mathcal{U}^t$ and $(t,T]$-valued stopping time θ, we have:*

1. If $Y^\nu_{t,x,y}(\theta) > w(\theta, X^\nu_{t,x,y}(\theta))$ and $g^1((X^\nu_{t,x,y}(u), Y^\nu_{t,x,y}(u)), \nu(u+)) \geq 0$, for all $u \in [t,\theta]$, then there exists a control $\tilde{\nu} \in \mathcal{U}_{t,x,y}$ such that $\nu = \tilde{\nu}$ on $[t,\theta)$.

2. If there exists a control $\tilde{\nu} \in \mathcal{U}_{t,x,y}$ such that $\nu = \tilde{\nu}$ on $[t,\theta)$, then $Y^\nu_{t,x,y}(\theta) \geq w(\theta, X^\nu_{t,x,y}(\theta))$ and $g^1((X^\nu_{t,x,y}(u), Y^\nu_{t,x,y}(u)), \nu(u+)) \geq 0$, for all $u \in [t,\theta]$.

We are now in a position to prove the Dynamic Programming Principle. Suppose w.l.o.g. that we work with the space $\Omega = C([0,T], \mathbf{R})$ equipped with a Wiener measure P. The corresponding Brownian motion is $W(\omega) = (\omega_t)_{t\geq 0}$, and the filtration $\mathbb{F} := \{\mathcal{F}_t,\ t \geq 0\}$ is the P-augmentation of the right-continuous filtration generated by W.

Since the elements of Ω are path, we can define the stopped process $\omega^r := (\omega_{t\wedge r})_{t\leq T}$ and the shifted process $\mathbb{T}_s(\omega) := \omega_\cdot \mapsto \omega_{s+\cdot} - \omega_s$. We also define concatenation operator:

$$g : \mathbf{R}_+ \times \Omega \times \Omega \to \Omega, \quad g_t(s,\omega,\tilde{\omega}) = \omega_t 1_{[0,s)}(t) + (\tilde{\omega}_{t-s} + \omega_s)1_{[s,T]}(t)$$

we then have $g_t(s,\omega^s_\cdot,\mathbb{T}_s(\omega)) = \omega_t$ so that $\nu(\omega) = \nu(g(s,\omega^s_\cdot,\mathbb{T}_s(\omega)))$ for any s. Here, we have used the notation $\omega^s = W^s(\omega)$. We deduce the following *weak* version of Dynamic Programming Principle.

Theorem 6.3.10. *Fix $(t,z) \in \text{int}(\mathcal{D})$ and let $\{\theta^\nu, \nu \in \mathcal{U}\}$ be a family of stopping times with values in $[t,T]$. Then,*

$$V(t,z) \leq \sup_{\nu\in\mathcal{U}_{t,z}} E\left[f^*(Z^\nu_{t,z}(\theta^\nu))1_{\theta^\nu=T} + V^*(\theta^\nu, Z^\nu_{t,z}(\theta^\nu))1_{\theta^\nu<T}\right]$$
$$V(t,z) \geq \sup_{\nu\in\mathcal{U}_{t,z}} E\left[f_*(Z^\nu_{t,z}(\theta^\nu))1_{\theta^\nu=T} + V_*(\theta^\nu, Z^\nu_{t,z}(\theta^\nu))1_{\theta^\nu<T}\right]$$

where V^* and V_* are respectively denote the u.s.c. and l.s.c. envelope of V defined as:

$$V_*(t,z) := \sup\{v_*(t,z) :\ v^* \leq V \text{ and, } v_* \text{ l.s.c.}\},$$
$$V^*(t,z) := \inf\{v^*(t,z) :\ v^* \geq V \text{ and, } v^* \text{ u.s.c.}\},$$

f^* and f_* are defined analogously.

Proof. 1. We show the first inequality. For any stopping time θ with value in $[t,T]$:

$$E\left[f(Z^\nu_{t,z}(T))\right] = E\left[E\left[f(Z^\nu_{t,z}(T))\middle|\mathcal{F}_\theta\right]\right]$$

By the strong Markov property, we know that $\mathbb{T}_\theta = \mathbb{T}_{\theta(\cdot)}(\cdot)$ is a Brownian motion independent of $\mathcal{F}_\theta$. Recall that, for any $\mathcal{F}_\theta$-measurable random variable η, $\mathbb{T}_\theta(.)$ and η are independent and, for any Borel-measurable function h, we have $E[h(\mathbb{T}_\theta,\eta)|\mathcal{F}_\theta] = \phi^h(\eta)$ a.s. where

$$\phi^h(x) = E[h(W,x)]. \tag{6.3.13}$$

In the following, we use the fact that $Z^\nu_{t,z}(r) = h(t,z,\nu,r,W^r)$ for some measurable function h. In particular, we have for any stopping time θ valued

in $[t,T]$, $Z^{\nu}_{t,z}(\theta)(\omega) = h(t,z,\nu(\omega),\theta(\omega),W^{\theta}(\omega))$ and, by the flow property, we also deduce that

$$\begin{aligned} Z^{\nu}_{t,z}(T) &= Z^{\nu}_{\theta,Z^{\nu}_{t,z}(\theta)}(T) = h(\theta, Z^{\nu}_{t,z}(\theta), \nu, T, W) \\ &= h(\theta, Z^{\nu}_{t,z}(\theta), \nu(g(\theta, W^{\theta}, \mathbb{T}_{\theta})), T, g(\theta, W^{\theta}, \mathbb{T}_{\theta})), \text{ a.s.} \end{aligned}$$

By (6.3.13), we then deduce that $E\left[f(Z^{\nu}_{t,z}(T))\middle|\mathcal{F}_{\theta}\right] = \gamma(\theta, Z^{\nu}_{t,z}(\theta), W^{\theta}))$ for some Borel measurable mapping γ defined by

$$\gamma(t',z',\omega) = E\left(f \circ h(t',z',\nu(g(t',\omega^{t'},W)),T,g(t',\omega^{t'},W))\right)$$

for all $t' \geq t, z' \in \mathbf{R}^d, \omega \in C([0,T],\mathbf{R})$.

Note that for $u \geq t'$, $g_u(t',\omega^{t'},W) = W_u$, we then have

$$h(t',z',\nu(g(t',\omega^{t'},W)),T,g(t',\omega^{t'},W)) = Z^{\nu\circ g(t',\omega^{t'},W)}_{t',z'}(T)(W) = Z^{\nu(W)}_{t',z'}(T)(W).$$

We deduce that $E\left[f(Z^{\nu}_{t,z}(T))\middle|\mathcal{F}_{\theta}\right] = Ef(Z^{\nu}_{t',z'}(T))$ with $(t',z') = (\theta, Z^{\nu}_{t,z}(\theta))$. Therefore, $E\left[f(Z^{\nu}_{t,z}(T))\middle|\mathcal{F}_{\theta}\right] = J(\theta, Z^{\nu}_{t,z}(\theta), \nu)$. Moreover, $\nu \in \mathcal{U}_{t',z'} \neq \emptyset$ for almost every (t',z') in the support of $(\theta, Z^{\nu}_{t,z}(\theta))$. The proof in [82, Theorem 3.1] implies the property $(\theta, Z^{\nu}_{t,z}(\theta)) \in \mathcal{D}$ a.s.. As $J \leq V \leq V^*$ and $J(T,.) = f$, it follows that

$$E\left[f(Z^{\nu}_{t,z}(T))\right] = E\left[J(\theta, Z^{\nu}_{t,z}(\theta), \nu)\right] \leq E\left[V^*(\theta, Z^{\nu}_{t,z}(\theta))1_{\theta<T} + f(Z^{\nu}_{t,z}(T))1_{\theta=T}\right].$$

2. The second inequality

If $V(t,z) = \infty$, there is nothing to prove. Suppose that $V(t,z) < \infty$, then for any $\epsilon > 0$, by Lemma 6.3.6, there is a universally measurable mapping $\hat{\nu}^{\epsilon} : \mathbf{R}_+ \times \mathbf{R}^7 \ni (t,z) \mapsto \hat{\nu}^{\epsilon}(t,z) \in \mathcal{U}_{t,z}$ such that $J(t,z,\nu^{\epsilon}(t,z)) \geq V(t,z) - \epsilon$. Moreover, it follows from [7, Lemma 7.27] that, for any probability measure m defined on $\mathbf{R}_+ \times \mathbf{R}^7$, there exists a Borel mapping $\hat{\nu}^{\epsilon}_m : \mathbf{R}_+ \times \mathbf{R}^7 \ni (t,z) \mapsto \hat{\nu}^{\epsilon}_m(t,z) \in \mathcal{U}_{t,z}$ such that $J(t,z,\hat{\nu}^{\epsilon}(t,z)) \geq V(t,z) - \epsilon \geq V_*(t,z) - \epsilon$ m-a.e.(t,z). We now fix $\nu_1 \in \mathcal{U}_{t_0,z_0}$ for some $(t_0,z_0) \in \text{int}(\mathcal{D})$ and θ be a stopping time with values in $[t_0,T]$. Let m be the distribution of $(\theta, Z^{\nu_1}_{t_0,z_0}(\theta))$. By Theorem 6.3.7, we deduce that $(\theta, Z^{\nu_1}_{t_0,z_0}(\theta)) \in \mathcal{D}$ P-a.s.. Moreover, we have:

$$\hat{\nu}^{\epsilon}_m(\theta, Z^{\nu_1}_{t_0,z_0}(\theta)) \in \mathcal{U}_{\theta, Z^{\nu_1}_{t_0,z_0}}, \; J(\theta, Z^{\nu_1}_{t_0,z_0}, \hat{\nu}^{\epsilon}_m(\theta, Z^{\nu_1}_{t_0,z_0}(\theta))) \geq V_*(\theta, Z^{\nu_1}_{t_0,z_0}(\theta)) - \epsilon, \text{ a.s.}$$

Now, by [82, Lemma 2.1], there exists $\nu^{\epsilon}_2 \in \mathcal{U}$ such that:

$$\nu^{\epsilon}_2(\omega,t)1_{[\theta(\omega),T]}(t) := \nu^{\epsilon}_m(\theta(\omega), Z^{\nu_1}_{t_0,z_0}(\theta(\omega)))1_{[\theta(\omega),T]}(t), \; (dP \times dt)(\omega,t) \text{ a.e.}$$

We then define the concatenated control $\nu^\epsilon := \nu_1 1_{[t_0,\theta)} + \nu_2^\epsilon 1_{[\theta,T]}$. We claim that $\nu^\epsilon \in \mathcal{U}_{t_0,z_0}$. To do so, we first observe by the flow property and the causality condition (resp. Z3 and Z4 in [82]) that :

$$Z^{\nu^\epsilon}_{t_0,z_0}(T) = Z^{\nu^\epsilon}_{\theta,Z^{\nu^\epsilon}_{t_0,z_0}(\theta)}(T) = Z^{\nu^\epsilon}_{\theta,Z^{\nu_1}_{t_0,z_0}(\theta)}(T) = Z^{\nu_2^\epsilon}_{\theta,Z^{\nu_1}_{t_0,z_0}(\theta)}(T) = Z^{\nu_m^\epsilon(\theta,Z^{\nu_1}_{t_0,z_0}(\theta))}_{\theta,Z^{\nu_1}_{t_0,z_0}(\theta)}(T).$$

This implies that $g(Z^{\nu^\epsilon}_{t_0,z_0}(T)) \geq 0$ a.s. Moreover, on the set $\{T_n \geq \theta\}$, we also have $Z^{\nu^\epsilon}_{t_0,z_0}(T_n) = Z^{\nu_2^\epsilon}_{\theta,Z^{\nu_1}_{t_0,z_0}(\theta)}(T_n) = Z^{\nu_m^\epsilon(\theta,Z^{\nu_1}_{t_0,z_0}(\theta))}_{\theta,Z^{\nu_1}_{t_0,z_0}(\theta)}(T_n)$ a.s. and, moreover, $\nu^\epsilon(T_n+) = \nu_2^\epsilon(T_n+) = \nu_m^\epsilon(\theta, Z^{\nu_1}_{t_0,z_0}(\theta))(T_n+)$. On the set $\{T_n < \theta\}$, we have $Z^{\nu^\epsilon}_{t_0,z_0}(T_n) = Z^{\nu_1}_{t_0,z_0}(T_n)$ a.s. and $v^\epsilon(T_n+) = \nu_1(T_n+)$. We then deduce that $g^n(Z^{\nu^\epsilon}_{t_0,z_0}(T_n), \nu^\epsilon(T_n+)) \geq 0$ a.s. for all $n \geq 1$. We conclude that $\nu^\epsilon \in \mathcal{U}_{t_0,z_0}$. As in Step 1, we have

$$\begin{aligned} E\left[f(Z^{\nu^\epsilon}_{t_0,z_0}(T))\middle|\mathcal{F}_\theta\right] = E\left[f(Z^{\nu_2^\epsilon}_{\theta,Z^{\nu_1}_{t_0,z_0}(\theta)}(T))\middle|\mathcal{F}_\theta\right] &= J(\theta, Z^{\nu_1}_{t_0,z_0}(\theta), \nu_2^\epsilon) \\ &= J\left(\theta, Z^{\nu_1}_{t_0,z_0}(\theta), \hat{\nu}^\epsilon_m(\theta, Z^{\nu_1}_{t_0,z_0}(\theta))\right) \\ &\geq V_*(\theta, Z^{\nu_1}_{t_0,z_0}(\theta)) - \epsilon, \text{ a.s.} \end{aligned}$$

We finally deduce that:

$$\begin{aligned} V(t_0,z_0) &\geq E\left[E\left[f(Z^{\nu^\epsilon}_{t_0,z_0}(T))\middle|\mathcal{F}_\theta\right]\right] \\ &\geq E\left[V_*(\theta, Z^{\nu_1}_{t_0,z_0}(\theta))1_{\theta<T} + f_*(Z^{\nu_1}_{t_0,z_0}(T))1_{\theta=T}\right] - \epsilon \end{aligned}$$

Since ϵ is arbitrarily chosen, the second inequality follows.

□

6.3.2 PDE characterization of the value function

In this section, we provide the PDE characterization for the problem 6.3.11. We shall follow the main lines of [10].

With $z = (x, y)$ we introduce

$$\mu_X(u,z) = \begin{bmatrix} 0 \\ (u^4 - u^3)x^2 \\ u^2 \\ u^3x^2 \\ u^4x^2 \\ 0 \end{bmatrix}, \sigma_X(u,z) = \begin{bmatrix} x^1\sigma(u,x^1) \\ 0 \\ 0 \\ 0 \\ 0 \\ u^5 \end{bmatrix},$$

$$\mu_Y(u,z) = -u^4x^2 - u^2, \ \sigma_Y(u,z) = u^1x^1\sigma(u,x^1).$$

and $\mu_Z(u,z) = \begin{bmatrix}\mu_X(u,z)\\ \mu_Y(u,z)\end{bmatrix}$, $\sigma_Z(u,z) = \begin{bmatrix}\sigma_X(u,z)\\ \sigma_Y(u,z)\end{bmatrix}$.

For each fixed u and a smooth function φ, we consider the operator $\mathcal{L}_Z^u$ defined for each u:

$$\mathcal{L}_Z^u\varphi := \partial_t\varphi + \mu_Z\nabla\varphi + \frac{1}{2}\mathrm{Tr}\left[\sigma_Z\sigma_Z^T D^2\varphi\right].$$

The Hamiltonian H is given by

$$\begin{aligned} H^u(u,z,q,A) &:= -\langle\mu_Z(u,z),q\rangle - \frac{1}{2}\mathrm{Tr}(\sigma_Z(u,z)\sigma_Z^T(u,z)A)\\ H(u,z,q,A) &:= \inf_{u\in\mathcal{C}_z} H^u(u,z,q,A), \end{aligned}$$

where $\mathcal{C}_z = \{u\in\mathbf{R}^5 :\ z^2+z^7 > u^1z^1\}$ and $H^u\varphi(t,z) := H^u(t,z,D\varphi(t,z),D^2\varphi(t,z))$ and similarly for $H\varphi(t,z)$.

Consider a real-valued function f, we define the lower semicontinuous envelope f_* (respectively, upper semicontinuous f^*) of a function f as:

$$\begin{aligned} f_*(\bar{x}) &:= \liminf_{x\to\bar{x}} f(x) := \lim_{\delta\downarrow 0}\inf_{x\in B(\bar{x},\delta)} f(x) = \sup_{\delta>0}\inf_{x\in B(\bar{x},\delta)} f(x),\\ f^*(\bar{x}) &:= \limsup_{x\to\bar{x}} f(x) := \lim_{\delta\downarrow 0}\sup_{x\in B(\bar{x},\delta)} f(x) = \inf_{\delta>0}\sup_{x\in B(\bar{x},\delta)} f(x). \end{aligned}$$

We denote by $H_*\varphi(t,z)$, $H^*\varphi(t,z)$ the l.s.c. (respectively, u.s.c.) envelope of $H\varphi(t,z)$.

In the following, the expression *around* z means *in a neighborhood of* z where we adopt the notation $z=(x,y)$. We recall the notaions in [10]:

$$\begin{aligned} \mathcal{W}^*(t,x) &:= \left\{\phi\in C^{1,2}([0,T]\times\mathbf{R}^6) :\ \phi - w > (\phi-w)(t,x) = 0 \text{ around } (t,x)\right\}\\ \mathcal{W}_*(t,x) &:= \left\{\phi\in C^{1,2}([0,T]\times\mathbf{R}^6) :\ \phi - w < (\phi-w)(t,x) = 0 \text{ around } (t,x)\right\}\\ N^u(t,z,q) &:= \sigma_Y(t,z,u) - \sigma_X(t,z,u)^T q,\\ \mathcal{N}_\delta(t,z,q) &:= \left\{u\in\mathcal{C}_z :\ |N^u(t,z,q)|\le\delta\right\},\\ \mathcal{U}_{\delta,\gamma}(t,z,\phi) &:= \left\{u\in\mathcal{N}_\delta(t,x,y,\nabla\phi(t,x)) :\ \mu_Y(t,z,u) - \mathcal{L}_X^u\phi(t,x)\ge\gamma\right\},\\ F^\phi_{\delta,\gamma}(t,z,q,A) &:= \inf_{\nu\in\mathcal{U}_{\delta,\gamma}(t,z,\phi)}\left\{-\mu_Z(t,z,u)^Tq - \mathrm{Tr}\left[(\sigma_Z\sigma_Z^T)(t,z,u)A\right]\right\}\\ F^{\phi*}(t,z,q,A) &:= \limsup_{\substack{(t',z',q',A')\to(t,z,q,A)\\(\delta,\gamma)\to(0,0)}} F^\phi_{\delta,\gamma}(t',z',q',A') \end{aligned}$$

Lemma 6.3.11. *Consider $u_0 \in \mathcal{N}_0(t_0, z_0, \psi(t_0, x_0))$ where $z_0 = (x_0, y_0)$ is such that $x_0 > 0$ and ψ is a locally Lipschitz function with values in $[0, T] \times \mathbf{R}^7$. Then, there exits a neighborhood $\mathcal{O}$ of (t_0, z_0) and a locally Lipschitz map $\hat{\nu}$ defined on $\mathcal{O}$ such that $\hat{\nu}(t_0, z_0) = u_0$ and $\hat{\nu}(t, z) \in \mathcal{N}_0(t, z, \psi(t, z))$ for all $(t, z) \in \mathcal{O}$.*

Proof. We have by definition:

$$\begin{aligned}\mathcal{N}_0(t, z, \psi(t, x)) &= \left\{u \in \mathbf{R}^5 :\ \sigma_Y(t, z, u) = \sigma_X(t, z, u)^T \psi(t, x)\right\} \\ &= \left\{u \in \mathbf{R}^5 :\ u^1 x^1 \sigma(t, x^1) = x^1 \sigma(t, x^1)\psi^1(t, x) + u^5 \psi^6(t, x)\right\} \\ &= \left\{u \in \mathbf{R}^5 :\ u^T \Psi(t, x) = x^1 \sigma(t, x^1)\psi^1(t, x)\right\},\end{aligned}$$

where $\Psi(t, x) = (x^1\sigma(t, x^1), 0, 0, 0, -\psi^6(t, x))^T \in \mathbf{R}^5$. Notice that, if $x_0 > 0$, we may suppose that $|\Psi(t, x)| > 0$ in a neighborhood of (t_0, z_0). It follows that $\nu \in \mathcal{N}_0(t, z, \psi(t, x))$ if and only if $u = u(t, x) = \gamma(t, x) + \Psi^\perp$ where $\Psi^\perp$ is any vector such that $(\Psi^\perp)^T \Psi(t, x) = 0$ and

$$\gamma(t, x) = x^1 \sigma(t, x_1)\psi^1(t, x) \frac{\Psi(t, x)}{|\Psi(t, x)|^2}.$$

We observe that γ is locally Lipschitz as a product of locally Lipschitz (and bounded) functions. Since $u_0 \in \mathcal{N}_0(t_0, z_0, \psi(t_0, x_0))$, we have $u_0 = \gamma(t_0, x_0) + w_0$ where w_0 is orthogonal to $\Psi(t_0, x_0)$. Let us define $\hat{u}(t, x) = \gamma(t, x) + g(t, x)$ where

$$g(t, x) = w_0 - (w_0^T \Psi(t, x)) \frac{\Psi(t, x)}{|\Psi(t, x)|^2}.$$

Since $g(t, x)^T \Psi(t, x) = 0$, $\hat{u}(t, x) \in \mathcal{N}_0(t, z, \psi(t, x))$ for all (t, x, z) in a neighborhood of (t_0, z_0) such that $|\Psi(t, x)| > 0$. Moreover, when $(t, x) \to (t_0, x_0)$, $g(t, x) \to w_0 - (w_0^T \Psi(t_0, x_0)) \frac{\Psi(t_0, x_0)}{|\Psi(t_0, x_0)|^2} = g(t_0, x_0) = w_0$ as $w_0^T \Psi(t_0, x_0) = 0$. Since $g(t, x)$ is also locally Lipschitz around (t_0, z_0), the conclusion follows.

Lemma 6.3.12. *We have*

$$F^{\phi *}(t, z, q, A) = \limsup_{\substack{(t', z', q', A') \to (t, z, q, A) \\ \gamma \to 0}} F^\phi_{0,\gamma}(t', z', q', A').$$

Proof. Recall that by definition

$$F^{\phi *}(t, z, q, A) = \lim_{r \to 0} \sup_{(\delta, \gamma, t', z', q', A') \in B_r(0, 0, t, z, q, A)} F^\phi_{\delta,\gamma}(t', z', q', A'),$$

where $B_r(\delta,\gamma,t,z,q,A)$ designates the ball of center $(0,0,t,z,q,A)$ and radius $r>0$. Since $\sup_{(\delta,\gamma,t',z',q',A')\in B_r(0,0,t,z,q,A)} F^{\phi}_{\delta,\gamma}(t',z',q',A') \geq F^{\phi}_{0,\gamma}(t',z',q',A')$, for all $(\gamma,t',z',q',A') \in B_r(0,t,z,q,A)$, we deduce that

$$F^{\phi *}(t,z,q,A) \geq \limsup_{\substack{(t',z',q',A')\to(t,z,q,A)\\ \gamma\to 0}} F^{\phi}_{0,\gamma}(t',z',q',A').$$

On the other hand, for all $\delta \geq 0$ and $\gamma' \geq \gamma$, we have the inclusion $\mathcal{U}_{0,\gamma'}(t,z,\phi) \subseteq \mathcal{U}_{\delta,\gamma}(t,z,\phi)$. It follows that for all (t',z',q',A'), $\delta \geq 0$ and $\gamma' \geq \gamma$, we have

$$F^{\phi}_{0,\gamma'}(t',z',q',A') \geq F^{\phi}_{\delta,\gamma'}(t',z',q',A').$$

Therefore,

$$\limsup_{\substack{(t',z',q',A')\to(t,z,q,A)\\ \gamma\to 0}} F^{\phi}_{0,\gamma}(t',z',q',A') \geq F^{\phi *}(t,z,q,A).$$

The conclusion follows. □

We use the following notations in [10]:

$$\begin{aligned}
\mathrm{int}_p(D) &:= \left\{(t,x,y)\in[0,T)\times\mathbf{R}^7 :\ y > w(t,x)\right\}\\
\partial_Z D &:= \partial D \cap ([0,T)\times\mathbf{R}^7) = \left\{(t,x,y)\in[0,T)\times\mathbf{R}^7 :\ y = w(t,x)\right\}\\
\partial_T D &:= \partial D \cap ([0,T)\times\mathbf{R}^7) = \left\{(t,x,y)\in\{T\}\times\mathbf{R}^7 :\ y \geq w(t,x)\right\}\\
\partial_p D &:= \partial_Z D \cup \partial_T D.
\end{aligned}$$

Theorem 6.3.13. *The value function is a solution to the following PDE in viscosity sense:*

1. V_* is a viscosity supersolution on cl(D) *of:*

$$\begin{cases}
(-\partial_t\varphi + H^*\varphi)(t,x,y) \geq 0 \quad \text{if} \quad (\mathrm{t},\mathrm{x},\mathrm{y}) \in \mathrm{int_p(D)}\\
\forall\phi\in\mathcal{W}^*(t,x), (-\partial_t\varphi + F^{\phi *}\varphi(t,x,y)) \geq 0 \quad \text{if} \quad (\mathrm{t},\mathrm{x},\mathrm{y}) \in \partial_{\mathrm{Z}}\mathrm{D},
\end{cases}$$

$$\begin{cases}
\varphi(T,x,y) \geq f_*(x,y) \quad \text{if} \quad (t,x,y)\in\partial_{\mathrm{T}}\mathrm{D},\\
\qquad\qquad\qquad\qquad\quad y > w(T,x),\ H^*\varphi(T,x,y) < \infty
\end{cases}$$

2. V^* is a viscosity subsolution on cl(D) *of*

$$\begin{cases}
(-\partial_t\varphi + H_*\varphi)(t,x,y) \leq 0 \quad \text{if} \quad (\mathrm{t},\mathrm{x},\mathrm{y}) \in \mathrm{int_p(D)}\cup\partial_{\mathrm{Z}}(\mathrm{D})\\
\varphi(T,x,y) \leq f^*(x,y) \quad \text{if} \quad (t,x,y)\in\partial_T D,\ H_*\varphi(T,x,y) > -\infty
\end{cases}$$

Proof. We split the proof into several steps.

1) Supersolution inequality on $\text{int}_p(D)$**.**

Consider some $(t_0, x_0) \in \text{int}_p(D)$ and let φ be a smooth function such that $V_* - \varphi > (V_* - \varphi)(t_0, z_0) = 0$. We will argue using contradiction by assuming that $(-\partial_t\varphi + H^*\varphi)(t_0, z_0) < 0$.

As $H^* \geq H$, we get that $(-\partial_t\varphi + H\varphi)(t_0, z_0) < 0$ and, by definition of H, there exists $\hat{u} \in \mathcal{C}_{z_0}$ such that$(-\partial_t\varphi + H^{\hat{u}}\varphi)(t_0, z_0) < 0$. This implies that $-\mathcal{L}_Z^{\hat{u}}\varphi(t_0, z_0) < 0$. By the continuity of φ, w and the coefficients μ_Z, σ_Z, we deduce that $\mathcal{L}_Z^{\hat{u}}\varphi(t, z) > 0$ for every (t, z) in an open ball B of center (t_0, z_0) such that $\text{cl}(B) \subseteq \text{int}_p(D)$. Moreover, since $g^1(., \hat{u})$ is continuous and $g^1(z_0, \hat{u}) > 0$, we may suppose w.l.o.g. that $g^1(z, \hat{u}) > 0$ for any $(t, z) \in B$.

By definition of V_*, there exists a sequence $(t_n, z_n) \in B$ such that (t_n, z_n) converges to (t_0, z_0) and $V_*(t_0, z_0) = \lim_{n\to\infty} V(t_n, z_n)$. Consider the process $\hat{Z}^n(t) := Z^{\hat{u}}_{t_n,z_n}(t)$, $t \geq t_n$ and let us define with the convention $\inf\emptyset = T$:

$$\theta_n := \inf\left\{t \in [t_n, T] : \ (t, \hat{Z}^n(t)) \notin B\right\}.$$

Since $(t_n, z_n) \in B$, we deduce that $\theta_n \in (t_n, T]$, in particular $\theta_n > t_n$ a.s.. Choose $\epsilon > 0$ such that $t_0 < T - \epsilon$ and replace θ_n by $\theta_n \wedge (T - \epsilon)$. We may suppose w.l.o.g. that $B := (t_0 - \epsilon, t_0 + \epsilon) \times O$ for some bounded open neighborhood O of z_0. Moreover, we can also choose ϵ and O such that $\text{cl}(B) \subset \text{int}_p(\mathcal{D})$.

We have by continuity $(\theta_n, \hat{Z}^n(\theta_n)) \in \text{cl}(B) \subseteq \text{int}_p\mathcal{D}$ and $g^1(\hat{Z}^n(t), \hat{u}) \geq 0$ for all $t \in [t_n, \theta_n]$. By Lemma 6.3.9, we deduce the existence of $\nu^n \in \mathcal{U}_{t_n,z_n}$ such that $\nu^n = \hat{u}$ on $[t_n, \theta_n)$. Let us define $Z^n = Z^{\nu^n}_{t_n,z_n}$. We then have $Z^n = \hat{Z}^n$ on $[t_n, \theta_n]$ by continuity of the trajectories. We apply Itô's Lemma for $M_t = \varphi(\theta_n \wedge t, Z^n(\theta_n \wedge t))$, $t \in [t_n, T]$ to deduce that $M_t = \varphi(t_n, z_n) - N_t + P_t$ where

$$\begin{aligned} N_t &= -\int_{t_n}^{\theta_n\wedge t} \sigma_Z(u, \hat{Z}^n(u))\nabla\varphi(u, \hat{Z}^n(u))dW_u, \\ P_t &= \int_{t_n}^{\theta_n\wedge t} \mathcal{L}_Z^{\hat{\nu}}\varphi(u, \hat{Z}^n(u))du \geq 0. \end{aligned}$$

It follows that $N_t \geq \varphi(t_n, z_n) - M_t$. Since, $\nabla\varphi(r, \hat{Z}^n(r))$ is bounded for any $r \in [t_n, \theta_n \wedge t]$, we get that N is a martingale, the following holds:

$$\varphi(t_n, z_n) \leq E\left[\varphi(\theta_n, Z^n(\theta_n))\right].$$

We define a subset K of $\mathrm{cl}(B)$ as follows:

$$K := [t_0 - \epsilon, t_0 + \epsilon] \times \partial O \cup \{t_0 + \epsilon\} \times \mathrm{cl}(O) \subset \mathrm{cl}(B).$$

By continuity we deduce that $(\theta_n, Z^n(\theta_n)) \in K$ a.s.. We then deduce that there exists $\delta := \min_K(V_* - \varphi) > 0$ does not depend on n such that the following holds

$$\varphi(t_n, z_n) \leq E\left[\varphi(\theta_n, Z^n(\theta_n))\right] \leq E\left[V_*(\theta_n, Z^n(\theta_n))\right] - \delta. \tag{6.3.14}$$

Now, as $\varphi(t_n, z_n) - V(t_n, z_n) \to \varphi(t_0, z_0) - V_*(t_0, z_0) = 0$, in the inequality (6.3.14), we may replace $\varphi(t_n, z_n)$ by $V(t_n, z_n)$ and δ by $\delta/2$ for n large enough. This yields a contradiction by the first inequality in Theorem 6.3.10 since $E\left[V_*(\theta_n, Z^n(\theta_n))\right] \geq V(t_n, z_n)$ as long as $\theta_n < T$.

2) Supersolution inequality on $\partial_Z D$:

For $(t_0, z_0) \in \partial_Z D$, we suppose that there exists a function $\phi \in \mathcal{W}^*(t_0, x_0)$ such that $(-\partial_t \varphi + F^{\phi^*}\varphi)(t_0, x_0, y_0) < 0$. We denote by $\tilde{\mathcal{O}}$ a closed neighborhood of (t_0, x_0) such that $\phi - w > (\phi - w)(t_0, x_0) = 0$ on $\tilde{\mathcal{O}}$. By Lemma 6.3.12, we deduce that

$$\limsup_{\substack{(t,z)\to(t_0,z_0)\\ \gamma\to 0}} \inf_{u \in \mathcal{U}_{0,\gamma}(t,z,\phi)} (-\mathcal{L}_Z^u \varphi(t, z)) < 0. \tag{6.3.15}$$

This implies that $\mathcal{U}_{0,\gamma}(t, z, \phi) \neq \emptyset$ for every point (t, z) in a neighborhood $\mathcal{O}$ of (t_0, z_0). In particular, $\mathcal{N}_0(t_0, z_0, \nabla\psi(t_0, z_0)) \neq \emptyset$. We deduce from (6.3.15) the existence of $\gamma > 0$ small enough and a compact neighborhood $\mathcal{O}$ of (t_0, z_0) such that, for every $(t, z) \in \mathcal{O}$, $\inf_{u \in \mathcal{U}_{0,\gamma}(t,z,\phi)} (-\mathcal{L}_Z^u \varphi(t, z)) < 0$. Therefore, there exists $\hat{\nu}_{(t_0,z_0)} \in \mathcal{U}_{0,\gamma}(t_0, z_0, \phi)$ such that $-\mathcal{L}_Z^{\hat{\nu}_{(t_0,z_0)}} \varphi(t_0, z_0) < 0$. Moreover, by Lemma 6.3.11, that there exists a Lipschitz map $\hat{\nu}_{(t_0,z_0)}$ defined on a neighborhood $\mathcal{O}_{t_0,z_0} \subseteq \mathcal{O}$ of (t_0, z_0) such that $\hat{\nu}_{(t_0,z_0)}(r, a) \in \mathcal{N}_0(r, a, \nabla\varphi(r, a))$ for all $(r, a) \in \mathcal{O}_{t_0,z_0}$ and $\hat{\nu}_{(t_0,z_0)}(t_0, z_0) = \hat{\nu}_{(t_0,z_0)}$. Since $-\mathcal{L}_Z^{\hat{\nu}_{(t_0,z_0)}} \varphi(t_0, z_0) < 0$ and $\hat{\nu}_{(t_0,z_0)}$ is continuous, we may reduce $\mathcal{O}_{t_0,z_0}$ so that $-\mathcal{L}_Z^{\hat{\nu}_{(t_0,z_0)}(r,a)} \varphi(r, a) < 0$ for all $(r, a) \in \mathcal{O}_{t_0,z_0}$. As $\hat{\nu}^{(t_0,z_0)} \in \mathcal{U}_{0,\gamma}(t_0, z_0, \phi)$, we also have $\mu_Y(t_0, z_0, \hat{\nu}^{(t_0,z_0)}) - \mathcal{L}_X^{\hat{\nu}^{(t_0,z_0)}} \phi(t_0, z_0) \geq \gamma$. Still by continuity, we may also suppose that $\mu_Y(t, z, \hat{\nu}_{(t_0,z_0)}(t, z)) - \mathcal{L}_X^{\hat{\nu}_{(t_0,z_0)}} \phi(t, z) > 0$ for all $(t, z) \in \mathcal{O}_{t_0,z_0}$. Note that we may also shrink $\mathcal{O}_{t_0,z_0}$ further so that $(t, z) \in \mathcal{O}_{t_0,z_0}$ with $z = (x, y)$ implies $(t, x) \in \tilde{\mathcal{O}}$.

We claim that there is a Lipschitz function defined on $[0,T]\times\mathbf{R}^7$ such that $\eta(t,z)=\hat{\nu}_{t_0,z_0}(t,z)$ for every $(t,z)\in\mathcal{O}_{t_0,z_0}$. Indeed, without loss of generality, we suppose that $\mathcal{O}_{t_0,z_0}$ is closed and convex. Consider the projection π onto $\mathcal{O}_{t_0,z_0}$. In Hilbert spaces, we know that π is 1-Lipschitz. We then define $\eta:=\hat{\nu}_{t_0,z_0}\circ\pi$ so that η is Lipschitz on $[0,T]\times\mathbf{R}^7$.

Consider a sequence $(t_n,z_n)_n$ in a bounded open set $B\subset\mathcal{O}_{t_0,z_0}\cap\mathrm{int}_pD$ such that $(t_n,z_n)\to(t_0,z_0)$ and $V(t_n,z_n)\to V_*(t_0,z_0)$. Since g^1 and η are continuous, $g^1(z_0,\eta(t_0,z_0))>0$ by definition of $\mathcal{N}_0(t_0,z_0,\nabla\phi(t_0,z_0))$, we may suppose w.l.o.g. that $g^1(z,\eta(t,z))>0$ for any $(t,z)\in B$.

Let $Z^n=(X^n,Y^n)_n$ be the process defined on $[t_n,T]$ as the unique solution to the equation:

$$dZ_t^n=\sigma_Z(t,Z_t^n,\eta(t,Z_t^n))dW_t+\mu_Z(t,Z_t^n,\eta(t,Z_t^n))dt,\quad Z_{t_n}^n=z_n.$$

By Lipschitz property of the σ_Z and μ_Z, Z^n is uniquely defined. It is clear that $Z^n=Z^{\hat{\nu}^n}$ where $\hat{\nu}^n(\omega,t)=\eta(t,Z_t^n(\omega))$. We define the following stopping time:

$$\theta_n:=\inf\left\{t\in[t_n,T]:\ (t,Z^n(t))\notin B\right\}.$$

We replace θ_n by $\theta_n\wedge(t_0+\epsilon)$ where ϵ is small enough so that $t_0+\epsilon<T$ and we may suppose that $\theta_n\in(t_n,t_0+\epsilon]$.

We now apply the Ito formula for ϕ. Knowing that we have the equality $\sigma_X(t,x,\nu)^T\nabla\phi(t,x)=\sigma_Y(t,y,\nu)$ for all (t,z) such that $\nu\in\mathcal{N}_0(t,z,\nabla\varphi(t,z))$, and $\mathcal{L}_X^{\hat{\nu}_{(t_0,z_0)}}\phi(t,z)\le\mu_Y(t,z,\hat{\nu}_{(t_0,z_0)}(t,z))$ for all $(t,z)\in\mathcal{O}_{t_0,z_0}$, we then get the following:

$$\begin{aligned}
\phi(\theta_n,X^n(\theta_n)) &= \phi(t_n,x_n)+\int_0^{\theta_n}\sigma_X(t,X^n(t),\hat{\nu}_t^n)\nabla\phi(t,X^n(t))dW_t\\
&\quad+\int_0^{\theta_n}\mathcal{L}_X^{\hat{\nu}^n}\phi(t,X^n(t))dt\\
&\le\phi(t_n,z_n)+\int_0^{\theta_n}\sigma_Y(t,X^n(t),\hat{\nu}_t^n)dW_t\\
&\quad+\int_0^{\theta_n}\mu_Y(t,X^n(t),\hat{\nu}_t^n)dt\\
&\le\phi(t_n,x_n)-y_n+Y^n(\theta_n).
\end{aligned}$$

Therefore, we have

$$\begin{aligned} Y^n(\theta_n) &\geq \phi(\theta_n, X^n_{t_n}(\theta_n)) + y_n - \phi(t_n, x_n) \\ &\geq \big(\phi(\theta_n, X^n_{t_n}(\theta_n)) - w(\theta_n, X^n_{t_n}(\theta_n))\big) \\ &\quad + w(\theta_n, X^n_{t_n}(\theta_n)) + y_n - \phi(t_n, x_n). \end{aligned}$$

Now, we suppose w.l.o.g. that B has the form $B = (t_0 - \epsilon, t_0 + \epsilon) \times O$ for some open set $O \in \mathbf{R}^7$. We define the compact set $K := [t_0 - \epsilon, t_0 + \epsilon] \times \partial O \cup \{t_0 + \epsilon\} \times \mathrm{cl}(O) \subset \mathrm{cl}(B)$.

By the definition of $\mathcal{W}^*(t_0, x_0)$, we deduce that there is a positive constant κ such that $\phi - w \geq \kappa$ on $\tilde{K}$, where the compact set $\tilde{K}$ is the image of K under the projection mapping $(t, x, y) \mapsto (t, x)$. Moreover, recall that by construction and continuity, $(\theta_n, X^n_{t_n}(\theta_n)) \in \tilde{K}$ a.s. and $\lim_{n\to\infty}(y_n - \phi(t_n, x_n)) = y_0 - \phi(t_0, x_0)) = y_0 - w(t_0, x_0) = 0$. Therefore, for n large enough, we have $Y^n(\theta_n) \geq w(\theta_n, Z^n_{t_n}(\theta_n))$. Moreover, since $g^1(Z^n(t), \hat{\nu}^n_t) \geq 0$ for any $t \in [t_n, \theta_n)$, by Lemma 6.3.9, we deduce the existence of $\nu^n \in \mathcal{U}_{t_n,z_n}$ such that $\nu^n = \hat{\nu}^n$ on $[t_n, \theta_n)$. We define $\hat{Z}^n = Z^{\nu^n}_{t_n,z_n}$ so that $\hat{Z}^n = Z^n$ on $[t_n, \theta_n]$ by continuity of both processes.

Now, by a similar argument as in the first case, we then deduce that $E\left[V_*(\theta_n, Z^n(\theta_n))\right] - \delta > E\left[\varphi(\theta_n, Z^n(\theta_n))\right]$ for some $\delta > 0$ does not depend on n. We then proceed to conclude.

3) Subsolution property:

Let φ be a smooth function and $(t_0, x_0) \in \mathrm{int}_p D \cup \partial_Z D$ such that $V^* - \varphi < (V^* - \varphi)(t_0, z_0) = 0$. We assume that the subsolution property does not holds at (t_0, z_0) for φ:

$$-\partial_t \varphi + H_* \varphi(t_0, z_0) > 0$$

This implies that for all $u \in \mathcal{C}_{z_0}$ we have $-\mathcal{L}^u_Z \varphi(t_0, z_0) > 0$. Moreover, by continuity of the coefficients, we can find $\epsilon > 0$ and a bounded open set $O \in \mathbf{R}^7$ such that $-\mathcal{L}^u_Z \varphi(t, z) \geq 0$ for every $u \in \mathcal{C}_{z_0}$ and for every $(t, z) \in \mathcal{O} := (t_0 - \epsilon, t_0 + \epsilon) \times O$, $t_0 + \epsilon < T$. Consider a sequence (t_n, z_n) be a sequence in $\mathcal{O} \cap \mathrm{int}_p D$ such that $(t_n, z_n) \to (t_0, z_0)$ and $V(t_n, z_n) \to V^*(t_0, z_0)$. For each n, there exists $\nu_n \in \mathcal{U}_{t_n,z_n}$ since $(t_n, z_n) \in D$. We then set $Z^n(t) := Z^{\nu_n}_{t_n,z_n}(t)$. We define θ_n as:

$$\theta_n := \inf\left\{t \in [t_n, T] : \ (t, Z^n(t)) \notin \mathcal{O} \cap (\mathrm{int}_p D \cup \partial_Z D)\right\}$$

By a similar argument as in the case *Supersolution inequality on* $\text{int}_p(D)$, we deduce that $E\left[V^*(\theta_n, Z^n(\theta_n))\right] < E\left[\varphi(\theta_n, Z^n(\theta_n))\right]$.

By Theorem 6.3.7, we then deduce that $(s, Z^n(s)) \in \text{int}_p D \cup \partial_Z D$ for all $s \in [t_n, T)$ so that $(\theta_n, Z^n(\theta_n)) \in K := [t_0 - \epsilon, t_0 + \epsilon] \times \partial O \cup \{t_0 + \epsilon\} \times \text{cl}(O)$ by continuity. By Ito lemma, we obtain:

$$\varphi(t_n, z_n) \geq E[\varphi(\theta_n, Z^n(\theta_n)] \geq E[V^*(\theta_n, Z^n(\theta_n))] + \zeta$$

where $-\zeta := \max_K(V^* - \varphi) < 0$ does not depend on n. Since $(\varphi - V)(t_n, z_n) \to (\varphi - V^*)(t_0, z_0)$, then for n large enough similar to the case supersolution, we get the contradiction to 6.3.10.

4) Terminal condition, supersolution:

We consider $z_0 = (x_0, y_0)$ and a test function φ such that $y_0 > w(x_0, T)$ and (T, x_0) is a strict minimum of $V_* - \varphi$ on $\text{cl}(D)$. We also suppose that $V_*(T, z_0) - \varphi(T, z_0) = 0$.

We argue by contradiction by first supposing that $V_*(T, z_0) < f_*(z_0)$. By lower semicontinuity, we deduce that there are some $r, \eta > 0$ such that $\varphi \leq f_* - \eta$ on $(\{T\} \times B_r(z_0)) \cap \text{cl}(D)$. Let $(t_n, z_n)_n$ be a sequence in $\text{int}_p(D)$ such that $V(t_n, z_n) \to V_*(t_0, z_0)$ and $(t_n, z_n) \to (T, z_0)$. We cosider the modified test function $\tilde{\varphi} := \varphi - (T - t)^{1/2}$. We observe that (T, z_0) is also the strict minimum of $V_* - \tilde{\varphi}$.

Since $-\partial_t \tilde{\varphi} = -\partial_t \varphi - 1/2(T - t)^{-1/2}$, $H^*\varphi(T, z_0) = H^*\tilde{\varphi}(T, z_0) < \infty$ and $(T - t)^{-1/2} \to \infty$ when $t \to T$, we can choose $r > 0$ and $\bar{u} \in \mathcal{C}_{z_0}$ such that $-\mathcal{L}_Z^{\bar{u}} \leq 0$ on a bounded open set $\mathcal{O} \subset \text{cl}(D)$. We suppose w.l.o.g. that $\mathcal{O}$ is of the form $\mathcal{O} := [T - r, T) \times B_r(z_0)$. Since $g^1(., \bar{u})$ is continuous and $g^1(z_0, \bar{u}) > 0$, we also suppose w.l.o.g. that $g^1(z, \bar{u}) \geq 0$ on $\text{cl}(\mathcal{O})$.

Since w is continuous, we can choose r small enough such that $\text{cl}(\mathcal{O}) \subset \{(t, z) \in D : y \geq w(t, x) + r/2\} \subset D$. Set $Z^n := Z^{\bar{u}}_{t_n, z_n}$, where $\bar{u}$ is a constant control in $\mathcal{U}^{t_n}$. We consider the stopping time θ_n defined as:

$$\theta_n := \inf\{t \in [t_n, T] : \ (t, Z^n(t)) \notin \mathcal{O}\}$$

From Lemma 6.3.9, we deduce that there exists a control $\hat{\nu}^n \in \mathcal{U}_{t_n, z_n}$ such that $\hat{\nu}^n = \bar{u}$ on $[t_n, \theta_n)$. We set $\hat{Z}^n := Z^{\nu^n}_{t_z, z_n}$, since $-\mathcal{L}_Z^{\bar{u}} \leq 0$ on $\mathcal{O}$, we deduce from Itô's Lemma that:

$$\tilde{\varphi}(t_n, z_n) \leq E[\tilde{\varphi}(\theta_n, \hat{Z}^n(\theta_n))]$$

By the definition of z_0, we can choose some $\zeta > 0$ such that $V_* - \zeta \geq \tilde{\varphi}$ on the compact set $\partial\mathcal{O} := [T-r, T] \times \partial B_r(z_0) \cup \{T\} \times \bar{B}_r(z_0)$ does not contain (T, z_0). On the set $\{\theta_n < T\}$, we have, $\tilde{\varphi}(\theta_n, Z^n(\theta_n)) \leq V_*(\theta_n, Z^n(\theta_n)) - \zeta$, where $\zeta := \min_{[T-r,T]\times\partial B_r(z_0)}(V_* - \tilde{\varphi}) > 0$. Moreover, recall the definition of η, we deduce that:

$$\tilde{\varphi}(t_n, z_n) \leq E\left[(f_*(Z^n(\theta_n)) - \eta)1_{\theta_n = T} + (V_*(\theta_n, Z^n(\theta_n)) - \zeta)1_{\theta_n < T}\right]$$

By sending $n \to \infty$, we deduce a contradition with Theorem 6.3.10.

5) Terminal condition, subsolution:

We consider $z_0 = (x_0, y_0)$ and a test function φ such that $y_0 \geq w(x_0, T)$ and (T, x_0) is a strict maximum of $V^* - \varphi$ on $\mathrm{cl}(D)$. We also suppose that $V^*(T, z_0) - \varphi(T, z_0) = 0$ and $H_*\varphi(T, z_0) > -\infty$.

We argue by contradiction by first supposing that $V^*(T, z_0) > f^*(z_0)$, by upper semicontinuity, we deduce that there are some $r, \eta > 0$ such that $\varphi \geq f^* - \eta$ on $(\{T\} \times B_r(z_0)) \cap \mathrm{cl}(D)$. Let $(t_n, z_n)_n$ be a sequence in $\mathrm{int}_p(D)$ such that $V(t_n, z_n) \to V_*(t_0, z_0)$ and $(t_n, z_n) \to (T, z_0)$. We consider the modified test function $\tilde{\varphi} := \varphi + (T-t)^{1/2}$. We observe that (T, z_0) is also the strict maximum of $V_* - \tilde{\varphi}$.

Since $-\partial_t\tilde{\varphi} = -\partial_t\varphi + 1/2(T-t)^{-1/2}$ and $(T-t)^{-1/2} \to \infty$ when $t \to T$, we can choose $r > 0$ such that for all $u \in \mathcal{C}_{z_0}$, $-\mathcal{L}_Z^u \geq 0$ on the set $\mathcal{O} := [T-r, T) \times B_r(z_0)$. Without loss of generality, we suppose that $(t_n.z_n) \in \mathcal{O}$ for all n. For each n, there exists $\nu_n \in \mathcal{U}_{t_n,z_n}$ since $(t_n.z_n) \in D$. We now set $Z^n := Z^{\nu_n}_{t_n,z_n}$. We define θ_n as:

$$\theta_n := \inf\{t \in [t_n, T] : \ (t, Z^n(t)) \notin \mathcal{O}\}$$

By Theorem 6.3.7, we then deduce that $(s, Z^n(s)) \in D$ for all $s \in [t_n, T]$ so that $(\theta, Z^n(\theta)) \in \partial\mathcal{O} := [T-r, T] \times \partial B_r(z_0) \cup \{T\} \times \bar{B}_r(z_0)$. We deduce from Itô's Lemma that:

$$\tilde{\varphi}(t_n, z_n) \geq E[\tilde{\varphi}(\theta_n, Z^n(\theta_n))].$$

On the set $\{\theta_n < T\}$, we have, $\tilde{\varphi}(\theta_n, Z^n(\theta_n)) \geq V^*(\theta_n, Z^n(\theta_n)) - \zeta$, where $\zeta := \max_{[T-r,T]\times\partial B_r(z_0)}(V^* - \tilde{\varphi}) > 0$. Moreover, recall the definition of η, we deduce that:

$$\tilde{\varphi}(t_n, z_n) \geq E\left[(f^*(Z^n(\theta_n)) - \eta)1_{\theta_n = T} + (V^*(\theta_n, Z^n(\theta_n)) - \zeta)1_{\theta_n < T}\right].$$

By sending $n \to \infty$, we deduce a contradition with Theorem 6.3.10.

Chapter 7

Future perspectives

The main texts of this thesis is to discuss some new pricing techniques in financial markets with transaction costs or in the presence of risk measures. In this chapter, we will elaborate some ideas for future researches.

7.1 No arbitrage of the first kind and market viability

Consider a market defined by a price process $S := (S_t)_{0\leq t\leq T}$, see the classical setting in Chapter 2. We recall the definition of $V_t^{x,\theta}$, the value of a portfolio at time t using trading strategy θ and starting from initial capital x. In this section, we say that a trading strategy θ is *admissible* if $V_t^{1,\theta} \geq 0$ a.s. for all t. We denote by Θ_{adm} the set of all admissible trading strategies. In this section, we suppose that market participants also face trading restrictions so that $\theta \in \Theta_c$ for some predictable set-valued process $\Theta_c := (\Theta_{c,t})_{t=1,...,T}$ such that $\Theta_{c,t}(\omega)$ is a convex closed subset of $\mathbf{R}^d$ for all $(\omega, t) \in \Omega \times \{1, ..., T\}$. We consider the family of trading strategies Θ defined by $\Theta := \Theta_{\text{adm}} \cap \Theta_c$.

We now recall the definition of the super-hedging price of ξ as:

$$p(\xi) := \inf \left\{ p \in \mathbf{R} : \ \exists\, \theta \in \Theta \text{ s.t. } V_T^{p,\theta} \geq \xi, \text{ a.s.} \right\}.$$

We consider the following notion of arbitrage, see [58], [34] or [35] or [53]:

Definition 7.1.1. *A random variable $\xi \in L^0(\mathbf{R}_+, \mathcal{F}_T)$ with $P(\xi > 0) > 0$ is an arbitrage of the first kind if $p(\xi) = 0$. No arbitrage of the first kind (NA_1) holds if, for every $\xi \in L^0(\mathbf{R}_+, \mathcal{F}_T)$, $p(\xi) = 0$ implies that $\xi = 0$ a.s.*

We denote $\mathcal{U}$ the set of all utility functions $U : \Omega \times \mathbf{R}_+ \mapsto \mathbf{R} \cup \{-\infty\}$ such that $U(.,x)$ is $\mathcal{F}_T$-measurable and bounded from below, and, for every $x > 0$, $U(\omega, .)$ is continuous, strictly increasing and concave for a.s. ω. We say that the market is *viability* if for every utility function $U \in \mathcal{U}$ such that $\sup_{\theta\in\Theta} E\left(U^+(V_T^{1,\theta})\right) < \infty$, there exists a strategy $\theta^* \in \Theta$ such that:

$$E\left(U^+(V_T^{1,\theta^*})\right) = \sup_{\theta\in\Theta} E\left(U^+(V_T^{1,\theta})\right)$$

The following Theorem in [34] gives an economic interpretation of NA_1. Roughly speaking, it states that NA_1 is equivalent to the solvability of utility optimization problems.

Theorem 7.1.2. *The following are equivalent:*

1) NA_1 *holds,*

2) Market is viability.

In [34], the authors also show that NA_1 condition is sufficient to give a dual characterization for the super-hedging price of the payoff $\xi \in L^0(\mathbf{R}_+, \mathcal{F}_T)$.

Definition 7.1.3. *An adapted stochastic process $Z = (Z_t)_{t=0,\dots,T}$ such that $Z_t > 0$ a.s. for all t and $Z_0 = 1$ is said to be a supermartingale deflator if $ZV^{1,\theta}$ is a supermartingale, for all $\theta \in \Theta$. The set of all supermartingale deflators is denoted by $\mathcal{D}$.*

Theorem 7.1.4. *The following are equivalent:*

1) NA_1 *holds,*

2) $\mathcal{D} \neq \emptyset$.

Moreover, suppose that NA_1 *holds and let $\xi \in L^0(\mathbf{R}^d, \mathcal{F}_T)$, then*

$$p(\xi) = \sup_{Z\in\mathcal{D}} E[Z_T\xi].$$

In [34], the authors state that under *conic* trading restrictions (Θ_t is a cone for any t), no classical arbitrage holds if and only if there are no arbitrage of the first kind. We have seen in Chapter 3, our NA condition in the presence of risk measure (AIP + SRN) is a generalization of NA_1. Indeed, if $\Theta_t = \mathbf{R}^d$, we

showed by Theorem 3.3.5 in Chapter 3 that when the risk measure $\rho_t(X) = -\operatorname{ess\,inf}_{\mathcal{F}_t} X$, the classical NA and our NA conditions coincide. Moreover, the equivalence between NA_1 and our NA was established in Theorem 3.3.5.

Our next step is to understand the meaning of NA_1 in financial market models with transaction costs. Then, we wish to relate it to market viability and the set of supermartingale deflators in the spirit of Theorems 7.1.2 and 7.1.4. Moreover, one possible development is to extend the concepts of our weak NA conditions (AIP, SAIP, LAIP) introduced in Chapter 4 to a continuous time setting.

7.2 Super-hedging cost under model uncertainty

The aim of this section is to discuss the pricing problem when there are *model uncertainty* and transaction costs. In particular, we consider a dynamic programming approach, which is a direct extension to [19]. To do so, we first need to recall the multi-prior setting introduced in [12].

Given a measurable space $(\Omega, \mathcal{A})$, we denote by $\mathfrak{B}(\Omega)$ the set of all probability measures on $\mathcal{A}$. If Ω is a topological space, $\mathcal{B}(\Omega)$ denotes its Borel σ-algebra. We always endow $\mathfrak{B}(\Omega)$ with the topology of weak convergence, it is well known that $\mathfrak{B}(\Omega)$ is Polish whenever Ω is Polish. Given a family of measure $\mathcal{P} \subset \mathfrak{B}(\Omega)$, a subset $A \subset \Omega$ is called $\mathcal{P}$-polar if $A \subset A'$ for some $A' \in \mathcal{A}$ satisfying $P(A') = 0$ for all $P \in \mathcal{P}$, and a property is said to hold $\mathcal{P}$-quasi surely or $\mathcal{P}$-q.s. if it holds outside a $\mathcal{P}$-polar set.

Let $T \in \mathbb{N}$ be the time horizon and let Ω_1 be the Polish space. For $t = \{1, ..., T\}$, let Ω^t be the t-fold Cartesian product, with the convention that Ω_0 is a singleton, i.e.

$$\Omega^t = \Omega_1 \times ... \times \Omega_1 \ (t \text{ times}).$$

We denote by $\mathcal{F}_t$ the universal completion of $\mathcal{B}(\Omega^t)$ and write $(\Omega, \mathcal{F})$ for $(\Omega^T, \mathcal{F}_T)$. For each $t \in \{1, ..., T\}$ and $\omega^t \in \Omega^t$ we are given a nonempty convex set $\mathcal{P}_t(\omega^t) \in \mathfrak{B}(\Omega_1)$ of probability measures. One financial interpretation for this set-up is that at time t, when realizes ω^t, an agent faces *model risk* in the market, where a possible model represented by a probability measure $P \in \mathcal{P}_t(\omega^t)$. We assume that for each t:

$$\mathrm{Graph}(\mathcal{P}_t) := \left\{(\omega^t, P) : \omega^t \in \Omega^t, P \in \mathcal{P}_t(\omega^t)\right\} \subseteq \Omega^t \times \mathfrak{B}(\Omega_1)$$

is an analytic set in sense of Definition 2.6.2. This assumption is sufficient to invoke the Jankov-von Neumann theorem (see Chapter 2) to deduce the existence of a universally measurable selector kernel $P_t : \Omega^t \to \mathfrak{B}(\Omega_1)$ such that $P_t(\omega^t) \in \mathcal{P}_t(\omega^t)$ for all $\omega^t \in \Omega^t$. We can then define $\mathcal{P}$ the set of probability measures on Ω by Fubini's theorem, i.e:

$$\mathcal{P} := \{P_0 \otimes ... \otimes P_{T-1},\ P_t(.) \in \mathcal{P}_t(.), t = 0, 1, ..., T-1\}$$

In the following, for a fixed σ-algebra $\mathcal{A}$, we denote $L^0(\mathbf{R}^d, \mathcal{A})$ the set of $\mathcal{A}$-measurable random variables valued in $\mathbf{R}^d$.

We now consider a general set-up for models with transaction costs. For each trading date t, we consider the Borel-measurable random set $\mathbf{G}_t : \Omega^t \twoheadrightarrow \mathbf{R}^d$, it represents the positions that are solvent. We suppose that $\mathbf{G}_t(\omega^t)$ is closed for every $\omega^t \in \Omega^t$ and that $\mathbf{G}_t(\omega^t) + \mathbf{R}^d_+ \subseteq \mathbf{G}_t(\omega^t)$, for all $t \leq T$. Now, the cost value process $\mathrm{C} = (\mathrm{C}_t)_{t=0}^T$ associated to $\mathbf{G}$ is defined ω^t-wise as:

$$\mathrm{C}_t(\omega^t, z) = \inf\{\alpha \in \mathbf{R} :\ \alpha e_1 - z \in \mathbf{G}_t(\omega^t)\} = \min\{\alpha \in \mathbf{R} :\ \alpha e_1 - z \in \mathbf{G}_t(\omega^t)\}.$$

It is simple to verify that $\mathrm{C}_t(\omega^t, .)$ is a lower semicontinuous and C_t is Borel as a function of (ω^t, z). We consider the super-hedging problem for a random payoff $\xi \in L^0(\mathbf{R}^d, \mathcal{F}_T)$. We denote by $\mathcal{R}_t(\xi)$ the set of all portfolio processes starting at time $t \leq T$ that replicates ξ at the terminal date T. In particular, $\mathcal{R}_t(\xi)$ is defined as follows:

$$\mathcal{R}_t(\xi) := \left\{(V_s)_{s=t}^T \in \mathcal{A}_t(\mathbb{F}), -\Delta V_s \in \mathbf{G}_s\ \mathcal{P} - \text{q.s}, \forall s \geq t+1, V_T = \xi,\ \mathcal{P} - \text{q.s.}\right\}$$

where $\mathcal{A}_t(\mathbb{F})$ is a stochastic process starting from t defined as follows:

$$\mathcal{A}_t(\mathbb{F}) := \left\{(\theta_s)_{s=t}^T :\ \theta_s \in L^0(\mathbf{R}^d, \mathcal{F}_s) \text{ for all } t \leq s \leq T\right\}.$$

The set of replicating prices of ξ at time t is given by:

$$\mathcal{H}_t(\xi) := \left\{V_t = (V_t^1, V_t^{(2)}) \in L^0(\mathbf{R}^d, \mathcal{F}_t) :\ (V_s)_{s=t}^T \in \mathcal{R}_t(\xi)\right\}.$$

The infimum replicating cost is then defined as:

$$c_0(\xi) := \inf_{V_0 \in \mathbf{R}^d} \{\mathrm{C}_0(V_0),\ V_0 \in \mathcal{H}_0(\xi)\}$$

Our key of observation is that for any stochastic process $(V_t)_{t=1}^T$, it is a portfolio process if and only if $V_{t-1} - V_t \in \mathbf{G}_t$, $\mathcal{P}$-q.s. or equivalently the following holds:

$$V_{T-1}^1 \geq \xi^1 + \mathrm{C}_T(0, (\xi^{(2)} - V_{T-1}^{(2)})),\ \mathcal{P} - \text{q.s.}$$
$$V_{t-1}^1 \geq V_t^1 + \mathrm{C}_t(0, (V_t^{(2)} - V_{t-1}^{(2)})),\ \mathcal{P} - \text{q.s.},\ \forall\ t \leq T-1.$$

Recursively, we deduce that:

$$V_0^1 \geq \sum_{t=1}^{T-1} \mathrm{C}_t((0, X_t^{(2)} - X_{t-1}^{(2)})) + \xi^1 + \mathrm{C}_T((0, \xi^{(2)} - X_{T-1}^{(2)})),\ \mathcal{P} - \text{q.s.} \quad (7.2.1)$$

And the problem of finding $c_0(\xi)$ amounts to minimize over all $(V_t)_{0\leq t\leq T-1}$ such that V_0 satisfies 7.2.1. To do so, a natural idea is to establish a dynamic programming principle. We already established in Chapter 4 a dynamic programming principle for the mono-prior case, i.e. $\mathcal{P}$ is a singleton. In [19] and [12], the authors claimed a dynamic programming principle for the market without frictions, i.e. $\mathrm{C}_t(x) = S_t x$ for some Borel measurable price process $(S_t)_{0\leq t\leq T}$. The following dynamic programming procedure is a straightforward combination of the ideas from Chapter 4 and [19].

DPP:

$$\gamma_T^\xi(\omega^T, y) := \xi^1(\omega^T) + \mathrm{C}_T(\omega^T, (0, \xi^2(\omega^T) - y^{(2)})),$$
$$\theta_t^\xi(\omega^t, y) := \inf\left\{z \in \mathbf{R}, z \geq \gamma_{t+1}^\xi((\omega^t, .), y),\ \mathcal{P}_t(\omega^t) - \text{q.s.}\right\},\ t \leq T-1,$$
$$D_t^\xi(\omega^t, x, y) := \mathrm{C}_t(\omega^t, (0, y^{(2)} - x^{(2)})) + \theta_t^\xi(\omega^t, y),\ t \leq T-1,$$
$$\gamma_t^\xi(\omega^t, x) := \inf_{y\in\mathbf{R}^d} D_t^\xi(\omega^t, x, y)),\ t \leq T-1.$$

One of the main goals is to find an appropriate no arbitrage condition such that $\gamma_t^\xi(\omega^0, 0) = c_0(\xi)$. To do so, we first need to overcome some measurability issues, for example, the function $\gamma_t^\xi : \Omega^t \times \mathbf{R}^d \to \overline{\mathbf{R}}$ in **DPP** is not upper semianalytic for free. In [19] and [12], the authors use a classical no arbitrage condition to claim the validity of the dynamic programming principle for the frictionless markets. Furthermore, we want to know whether it is possible to compute γ_t^ξ when the cost process C_t depends on some Borel measurable price process S_t. In other words, we want to answers the computability question in the same fashion as Chapter 4 but in multi-prior setting.

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RÉSUMÉ

Cette thèse présente quatre problèmes d'évaluation et d'optimisation en mathématiques financières. Dans la première partie, nous considérons le problème de couverture en présence de mesures de risque dynamiques définies sur l'espace général des variables aléatoires. Dans la seconde partie, nous résolvons un problème classique de caractérisation des prix des options européennes dans des modèles de marchés financiers avec coûts de transaction. Dans la troisième partie, nous appliquons le résultat théorique établi dans la deuxième partie en fournissant un algorithme pour calculer les prix de sur-réplication en pratique. En particulier, les prix exacts seront déduits pour le cas du coût de transaction proportionnel et le cas du coût fixe. Dans la dernière partie, nous présentons nos avancées actuelles sur le problème d'optimisation de portefeuille sous contrainte de risque de crédit.

MOTS CLÉS

Marchés avec coûts de transactions, ensembles aléatoires, non arbitrage, sur-réplication.

ABSTRACT

This thesis presents four problems of pricing and optimization in financial mathematics. In the first part, we consider a hedging problem in the presence of dynamic risk measures defined on the general space of random variables. In the second part, we resolve a classical pricing problem for European options in financial markets with transaction costs. In the third part, we apply the theory established in the second part by providing an algorithm to calculate the super-hedging prices in practice. In particular, the exact prices can be deduced for the cases of proportional and fixed transaction costs. In the last part, we present some recent advances for the portfolio optimization problem under credit risk constraint.

KEYWORDS

Markets with transaction costs, random sets, no arbitrage, super-hedging.