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On the endomorphism algebra of Gelfand–Graev representations and the unipotent ℓ-block of p-adic GL2 with $\ell \neq$ p

Tzu-Jan Li

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présentée par

Tzu-Jan LI

Sur l'algèbre d'endomorphismes des représentations de Gelfand–Graev et le ℓ-bloc unipotent de GL_2 p-adique avec $\ell \neq p$

---

dirigée par Jean-François DAT

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Sur l'algèbre d'endomorphismes des représentations de Gelfand–Graev et le ℓ-bloc unipotent de GL_2 p-adique avec $\ell \neq p$

Tzu-Jan LI

Thèse à l'École doctorale de Sciences Mathématiques de Paris Centre (année 2022)

Directeur de thèse : Jean-François DAT

Résumé. Inspiré par la conjecture de Langlands locale en familles de Dat, Helm, Kurinczuk et Moss, pour un groupe réductif connexe G défini sur $\mathbb{F}_q$, nous étudions les relations des trois anneaux suivants : (i) le $\mathbb{Z}$-modèle E_G des algèbres d'endomorphismes des représentations de Gelfand–Graev de $G(\mathbb{F}_q)$; (ii) l'anneau de Grothendieck K_{G^*} de la catégorie des représentations de $G^*(\mathbb{F}_q)$ de dimension finie sur $\overline{\mathbb{F}}_q$, avec G^* le dual de Deligne–Lusztig de G ; (iii) l'anneau des fonctions $\mathsf{B}_{G^\vee}$ du $\mathbb{Z}$-schéma $(T^\vee /\!\!/ W)^{F^\vee}$, avec $G^\vee$ le dual de Langlands (défini et déployé sur $\mathbb{Z}$) de G. Nous démontrons que $\mathbb{Z}[\frac{1}{pM}]\mathsf{E}_G \simeq \mathbb{Z}[\frac{1}{pM}]\mathsf{K}_{G^*}$ comme $\mathbb{Z}[\frac{1}{pM}]$-algèbres avec $p = \mathrm{char}(\mathbb{F}_q)$ et M le produit des nombres premiers mauvais pour G, et que $\mathsf{K}_{G^*} \simeq \mathsf{B}_{G^\vee}$ comme anneaux lorsque le groupe dérivé de $G^\vee$ est simplement connexe. Profitant de ces résultats, nous donnons ensuite une description explicite du ℓ-bloc unipotent de GL_2 p-adique avec $\ell \neq p$ et $\mathrm{ord}_\ell(q) = 2$. Les matériaux de ce travail, sauf § 4, proviennent principalement de mon article [Li2] et de mon autre article [LiSh] en collaboration avec J. Shotton.

Abstract. *(On the endomorphism algebra of Gelfand–Graev representations and the unipotent ℓ-block of p-adic* GL_2 *with $\ell \neq p$.)* Inspired by the conjecture of local Langlands in families of Dat, Helm, Kurinczuk and Moss, for a connected reductive group G defined over $\mathbb{F}_q$, we study the relations of the following three rings: (i) the $\mathbb{Z}$-model E_G of endomorphism algebras of Gelfand–Graev representations of $G(\mathbb{F}_q)$; (ii) the Grothendieck ring K_{G^*} of the category of representations of $G^*(\mathbb{F}_q)$ of finite dimension over $\overline{\mathbb{F}}_q$, with G^* the Deligne–Lusztig dual of G; (iii) the ring of functions $\mathsf{B}_{G^\vee}$ of the $\mathbb{Z}$-scheme $(T^\vee /\!\!/ W)^{F^\vee}$, with $G^\vee$ the Langlands dual (defined and split over $\mathbb{Z}$) of G. We show that $\mathbb{Z}[\frac{1}{pM}]\mathsf{E}_G \simeq \mathbb{Z}[\frac{1}{pM}]\mathsf{K}_{G^*}$ as $\mathbb{Z}[\frac{1}{pM}]$-algebras with $p = \mathrm{char}(\mathbb{F}_q)$ and M the product of bad primes for G, and that $\mathsf{K}_{G^*} \simeq \mathsf{B}_{G^\vee}$ as rings when the derived subgroup of $G^\vee$ is simply-connected. Benefiting from these results, we then give an explicit description of the unipotent ℓ-block of p-adic GL_2 with $\ell \neq p$ and $\mathrm{ord}_\ell(q) = 2$. The material of this work, except for § 4, mainly originates from my article [Li2] and from my other article [LiSh] in collaboration with J. Shotton.

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Introduction en français

Ce travail comprend deux aspects autour des représentations des groupes réductifs. D'une part, motivé par le travail en cours [DHKM] de Dat, Helm, Kurinczuk et Moss sur la conjecture de Langlands locale en familles, on étudie des modèles duaux de l'algèbre d'endomorphismes à coefficients entiers des représentations de Gelfand–Graev d'un groupe réductif fini ; d'autre part, on cherche à décrire explicitement le ℓ-bloc unipotent d'un groupe linéaire p-adique avec $\ell \neq p$, grâce au travail fondamental [Hel1] de Helm sur les ℓ-blocs des groupes linéaires p-adiques. Ces deux aspects sont reliés par le travail [Pa] de Paige sur l'algèbre d'endomorphismes d'une certaine enveloppe projective, comme on le verra plus loin dans cette introduction. On va maintenant élaborer ces perspectives.

Soit $\mathbf{F}$ un corps p-adique (une extension finie de $\mathbb{Q}_p$) dont le corps résiduel est le corps fini $\mathbb{F}_q$ avec q une puissance de p, et soit $O_{\mathbf{F}}$ l'anneau des entiers de $\mathbf{F}$. Soit $\mathcal{G}$ un groupe réductif connexe défini sur $O_{\mathbf{F}}$, et pour le moment on suppose que $\mathcal{G}$ est déployé sur $O_{\mathbf{F}}$. Soit $\mathcal{G}^\vee$ le groupe dual de Langlands (défini et déployé sur $\mathbb{Z}$) de $\mathcal{G}$. Alors l'espace de modules des paramètres modérés de Langlands pour $\mathcal{G}^\vee$ se trouve

$$\mathcal{Z}^1_{\mathrm{t}} = \{(s, \mathrm{Fr}) \in \mathcal{G}^\vee \times \mathcal{G}^\vee : \mathrm{Fr} \cdot s \cdot \mathrm{Fr}^{-1} = s^q\},$$

sur lequel $\mathcal{G}^\vee$ agit diagonalement ; soit $\mathcal{O}_{\mathcal{Z}^1_{\mathrm{t}} /\!\!/ \mathcal{G}^\vee}$ l'anneau des fonctions de $\mathcal{Z}^1_{\mathrm{t}} /\!\!/ \mathcal{G}^\vee$.

De l'autre côté, soit $G = \mathcal{G}(\overline{\mathbb{F}}_q)$, et soit F l'endomorphisme de Frobenius de G induit par une $\mathbb{F}_q$-structure de $\mathcal{G}$, de sorte que $G^F = \mathcal{G}(\mathbb{F}_q)$; notons que G^F est un groupe fini. Soit $\mathcal{U}$ le radical unipotent d'un sous-groupe Borel de $\mathcal{G}$, et soit $U = \mathcal{U}(\overline{\mathbb{F}}_q)$. En fixant un choix d'un caractère linéaire régulier (c'est-à-dire non dégénéré) $\psi : U^F \longrightarrow \overline{\mathbb{Z}}[\frac{1}{p}]^\times$, on peut considérer la représentation de Gelfand–Graev

$$\Gamma_{G,\psi} := \mathrm{Ind}_{U^F}^{G^F} \psi.$$

Soient $\mathbf{G} = \mathcal{G}(\mathbf{F})$ et $\mathbf{K} = \mathcal{G}(O_{\mathbf{F}})$, et considérons l'induction compacte

$$\mathcal{W}_0 := \mathrm{ind}_{\mathbf{K}}^{\mathbf{G}} \Gamma_{G,\psi}.$$

(Ici, $\Gamma_{G,\psi}$ est une représentation de $\mathbf{K}$ via la réduction $\mathbf{K} \twoheadrightarrow G^F$.) Il sera démontré dans [DHKM] que $\mathcal{W}_0$ est isomorphe à la partie de niveau zéro du modèle de Whittaker ("Gelfand–Graev p-adique") $\mathcal{W} = \mathrm{ind}_{\mathbf{U}}^{\mathbf{G}} \widetilde{\psi}$, où $\mathbf{U} = \mathcal{U}(\mathbf{F})$ et $\widetilde{\psi}$ est un caractère régulier compatible avec ψ. Il est connu que l'algèbre d'endomorphismes $\mathrm{End}_{\overline{\mathbb{Z}}[\frac{1}{p}]\mathbf{G}}(\mathcal{W})$ est commutative et réduite (voir par exemple [Ra]), donc $\mathrm{End}_{\overline{\mathbb{Z}}[\frac{1}{p}]\mathbf{G}}(\mathcal{W}_0)$ l'est aussi.

Avec ces préparations, la conjecture de Dat–Helm–Kurinczuk–Moss prédit l'existence d'un isomorphisme d'anneaux

$$\mathrm{LLIF} : \overline{\mathbb{Z}}[\tfrac{1}{p}].\mathcal{O}_{\mathcal{Z}^1_{\mathrm{t}} /\!\!/ \mathcal{G}^\vee} \xrightarrow{\sim} \mathrm{End}_{\overline{\mathbb{Z}}[\frac{1}{p}]\mathbf{G}}(\mathcal{W}_0)$$

compatible avec les conjectures de Langlands locales classiques. (Ici, "LLIF" est pour "local Langlands in families".) En posant

$$\overline{\mathbb{Z}}[\tfrac{1}{p}]\mathsf{E}_G = \mathrm{End}_{\overline{\mathbb{Z}}[\frac{1}{p}]G^F}(\Gamma_{G,\psi})$$

(cette algèbre $\overline{\mathbb{Z}}[\frac{1}{p}]\mathsf{E}_G$ est indépendante du choix de ψ à isomorphisme près) et en dénotant par $\mathsf{B}_{G^\vee}$ l'anneau des fonctions de $(\mathcal{G}^\vee /\!\!/ \mathcal{G}^\vee)^{(\cdot)^q}$ (le sous-schéma de $\mathcal{G}^\vee /\!\!/ \mathcal{G}^\vee$ fixé par l'endomorphisme $(\cdot)^q$; ici $\mathcal{G}^\vee$ agit sur lui-même par l'action adjointe), on obtient le diagramme suivant :

$$\begin{tikzcd} \overline{\mathbb{Z}}[\frac{1}{p}].\mathcal{O}_{\mathcal{Z}^1_{\mathrm{t}} /\!\!/ \mathcal{G}^\vee} \arrow[r, "\mathrm{LLIF}", "\sim"'] & \mathrm{End}_{\overline{\mathbb{Z}}[\frac{1}{p}]\mathbf{G}}(\mathcal{W}_0) \\ \overline{\mathbb{Z}}[\frac{1}{p}]\mathsf{B}_{G^\vee} \arrow[u, "i^{\mathsf{B}}"] \arrow[r, dashed, "?"] & \overline{\mathbb{Z}}[\frac{1}{p}]\mathsf{E}_G \arrow[u, hook, "i^{\mathsf{E}}"'] \end{tikzcd} \qquad (*)$$

Ici, i^{B} correspond au morphisme $\mathcal{Z}^1_{\mathrm{t}} /\!\!/ \mathcal{G}^\vee \longrightarrow (\mathcal{G}^\vee /\!\!/ \mathcal{G}^\vee)^{(\cdot)^q}$ induit par la première projection $\mathcal{G}^\vee \times \mathcal{G}^\vee \longrightarrow \mathcal{G}^\vee$, et i^{E} est induit par l'inclusion naturelle $\Gamma_{G,\psi} \hookrightarrow \mathcal{W}_0$. Lorsque le groupe dérivé de $G^\vee$ est simplement connexe, i^{B} est injectif, et l'on espère que $(*)$ identifiera les deux $\overline{\mathbb{Z}}[\frac{1}{p}]$-algèbres finies $\overline{\mathbb{Z}}[\frac{1}{p}]\mathsf{B}_{G^\vee}$ et $\overline{\mathbb{Z}}[\frac{1}{p}]\mathsf{E}_G$.

À ce stade, après l'extension à coefficients dans $\Lambda = \mathrm{Witt}(\overline{\mathbb{F}}_\ell)$ (l'anneau de vecteurs de Witt de $\overline{\mathbb{F}}_\ell$) avec $\ell \neq p$, le diagramme conjectural $(*)$ ainsi que l'isomorphisme d'anneaux $\Lambda\mathsf{B}_{G^\vee} \simeq \Lambda\mathsf{E}_G$ a été justifié pour tous $\mathcal{G} = \mathrm{GL}_n$ (avec $n \in \mathbb{N}^*$) par Helm et Moss (voir [Hel2] et [HeMo]). Leur méthode comprend des techniques dédiées aux GL_n et ne semble pas facile à se généraliser directement pour les autres groupes réductifs $\mathcal{G}$. En dépit de cette difficulté, Helm a observé que l'isomorphisme voulu $\Lambda\mathsf{B}_{G^\vee} \simeq \Lambda\mathsf{E}_G$ était "lui-même un résultat de la théorie des groupes finis" ; il a alors demandé une preuve directe de cet isomorphisme par la théorie des groupes finis (voir [Hel2, Rmk. 10.3]).

On présente dans ce travail, parmi d'autres choses, une réponse à la dernière question de Helm. On observe tout d'abord que pour étudier la relation entre $\Lambda\mathsf{B}_{G^\vee}$ et $\Lambda\mathsf{E}_G$, il n'est pas nécessaire d'évoquer le groupe p-adique $\mathcal{G}$. On commence donc directement par un groupe réductif connexe G défini sur $\mathbb{F}_q$, et l'on ne suppose plus que G est déployé sur $\mathbb{F}_q$. On identifie $G = G(\overline{\mathbb{F}}_q)$, et l'on note par $G^\vee$ le dual de Langlands de G (donc $G^\vee = \mathcal{G}^\vee$). On définit $\overline{\mathbb{Z}}[\frac{1}{p}]\mathsf{E}_G$ comme avant, tandis que l'on définit $\mathsf{B}_{G^\vee}$ comme $\mathcal{O}_{(T^\vee /\!\!/ W)^{F^\vee}}$, l'anneau des fonctions de $(T^\vee /\!\!/ W)^{F^\vee}$, où $T^\vee$ est un tore maximal de $G^\vee$ dual à un tore maximal F-stable T de G via l'identification $X(T^\vee) = Y(T)$ (ici $X(T^\vee) = \mathrm{Hom}_{\mathrm{alg}}(T^\vee, \mathbb{G}_m)$ est le groupe de caractères de $T^\vee$, et $Y(T) = \mathrm{Hom}_{\mathrm{alg}}(\mathbb{G}_m, T)$ est le groupe de cocaractères de T), $F^\vee$ est l'endomorphisme de $T^\vee$ induit par F via l'identification $X(T^\vee) = Y(T)$, et W désigne le groupe de Weyl de $(G^\vee, T^\vee)$. Alors

$$\mathsf{B}_{G^\vee} = \mathcal{O}_{(T^\vee /\!\!/ W)^{F^\vee}} = \mathbb{Z}[X(T^\vee)]^W / \langle f(F^\vee(\cdot)) - f : f \in \mathbb{Z}[X(T^\vee)]^W \rangle.$$

Lorsque G est déployé sur $\mathbb{F}_q$, l'anneau $\mathsf{B}_{G^\vee}$ coïncide avec celui précédemment défini (à savoir l'anneau des fonctions de $(G^\vee /\!\!/ G^\vee)^{(\cdot)^q}$), car dans ce cas $F^\vee = (\cdot)^q$, et l'inclusion $T^\vee \subset G^\vee$ induit un isomorphisme $T^\vee /\!\!/ W \simeq G^\vee /\!\!/ G^\vee$ (théorème de restriction de Chevalley) qui est $(\cdot)^q$-équivariant.

Notre stratégie pour relier $\Lambda\mathsf{B}_{G^\vee}$ et $\Lambda\mathsf{E}_G$ est d'introduire un troisième anneau

$$\mathsf{K}_{G^*} = \mathrm{Groth}(\overline{\mathbb{F}}_q G^{*F^*}\text{-mod}),$$

qui est l'anneau de Grothendieck de la catégorie des représentations de $G^{*F^*} = G^*(\mathbb{F}_q)$ de dimension finie sur $\overline{\mathbb{F}}_q$, où G^* est le dual de Deligne–Lusztig (défini sur $\mathbb{F}_q$) de G avec F^* l'endomorphisme de Frobenius de G^* associé. Grâce à cet anneau K_{G^*}, nous démontrerons dans ce travail le théorème suivant :

Théorème. — *Soit G un groupe réductif connexe défini (mais non nécessairement déployé) sur $\mathbb{F}_q$.*

(a) (§ 2.25 ; voir aussi [LiSh]) *Pour $\Lambda = \overline{\mathbb{Z}}[\frac{1}{pM}]$ avec M le produit des nombres premiers mauvais pour G au sens de* [Sp1, (2.10)], *$\Lambda\mathsf{E}_G \simeq \Lambda\mathsf{K}_{G^*}$ comme Λ-algèbres.*

(b) (§ 3.21 ; voir aussi [Li2]) *Si le groupe dérivé de $G^\vee$ est simplement connexe (c'est-à-dire si le centre de G est un tore), alors $\mathsf{K}_{G^*} \simeq \mathsf{B}_{G^\vee}$ comme anneaux.*

En particulier, lorsque $G = \mathrm{GL}_n$ (type A_{n-1}), il n'y a aucun nombre premier mauvais pour G, et le centre de G est un tore, donc ce théorème nous donnera un isomorphisme de $\overline{\mathbb{Z}}[\frac{1}{p}]$-algèbres $\overline{\mathbb{Z}}[\frac{1}{p}]\mathsf{E}_G \simeq \overline{\mathbb{Z}}[\frac{1}{p}]\mathsf{B}_{G^\vee}$ pour $G = \mathrm{GL}_n$ et répondra à la question de Helm.

Donnons maintenant les idées principales de la preuve de ce théorème. On commence par (a). Sur $\overline{\mathbb{Q}}$, il est connu (Deligne–Lusztig [DeLu] pour le cas où le centre de G est connexe, et Digne–Lehrer–Michel [DLM] pour le cas général) que les composantes irréductibles du $\overline{\mathbb{Q}}G^F$-module $\overline{\mathbb{Q}}\Gamma_{G,\psi}$ sont toutes de multiplicité un et sont paramétrisées par $G^{*F^*}_{\mathrm{ss}}/\sim$, où "ss" signifie les éléments semisimples et $\sim$ désigne la conjugaison ; le lemme de Schur donne alors un isomorphisme de $\overline{\mathbb{Q}}$-algèbres $\overline{\mathbb{Q}}\mathsf{E}_G \simeq \overline{\mathbb{Q}}^{G^{*F^*}_{\mathrm{ss}}/\sim}$. D'autre part, le caractère de Brauer donne un isomorphisme de $\overline{\mathbb{Q}}$-algèbres $\overline{\mathbb{Q}}\mathsf{K}_{G^*} \simeq \overline{\mathbb{Q}}^{G^{*F^*}_{\mathrm{ss}}/\sim}$. On peut donc identifier les $\overline{\mathbb{Q}}$-algèbres

$$\overline{\mathbb{Q}}\mathsf{E}_G = \overline{\mathbb{Q}}\mathsf{K}_{G^*} = \overline{\mathbb{Q}}^{G^{*F^*}_{\mathrm{ss}}/\sim}.$$

Afin d'identifier ensuite $\Lambda\mathsf{E}_G$ et $\Lambda\mathsf{K}_{G^*}$ pour un anneau Λ avec $\overline{\mathbb{Z}}[\frac{1}{p}] \subset \Lambda \subset \overline{\mathbb{Q}}$, notre stratégie sera d'utiliser la forme symétrisante sur $\Lambda\mathsf{E}_G$ étudiée dans l'article [BoKe] de Bonnafé et Kessar : en identifiant

$$\overline{\mathbb{Z}}[\tfrac{1}{p}]\mathsf{E}_G = e_\psi \overline{\mathbb{Z}}[\tfrac{1}{p}]G^F e_\psi \subset \overline{\mathbb{Z}}[\tfrac{1}{p}]G^F$$

avec $e_\psi = \frac{1}{|U^F|}\sum_{u\in U^F}\psi(u^{-1})u$ l'idempotent central primitif de $\overline{\mathbb{Z}}[\frac{1}{p}]U^F$ associé à ψ, cette dernière forme symétrisante, que l'on note par τ, se trouve

$$\tau = |U^F|\mathrm{ev}_{1_{G^F}} : \overline{\mathbb{Z}}[\tfrac{1}{p}]\mathsf{E}_G \longrightarrow \overline{\mathbb{Z}}[\tfrac{1}{p}],$$

avec $\mathrm{ev}_{1_{G^F}}$ l'évaluation en l'unité du groupe G^F. On démontrera en § 2.8 que l'extension à $\overline{\mathbb{Q}}$-coefficients de τ se restreint à une forme symétrisante sur $\overline{\mathbb{Z}}[\frac{1}{p}]\mathsf{K}_{G^*}$; cela fait, si l'on peut démontrer que

$$\tau(h\pi) \in \overline{\mathbb{Z}}[\tfrac{1}{p}] \text{ pour tout } (h,\pi) \in \overline{\mathbb{Z}}[\tfrac{1}{p}]\mathsf{E}_G \times \overline{\mathbb{Z}}[\tfrac{1}{p}]\mathsf{K}_{G^*}, \tag{$\dagger$}$$

alors on pourra identifier $\overline{\mathbb{Z}}[\frac{1}{p}]\mathsf{E}_G$ et $\overline{\mathbb{Z}}[\frac{1}{p}]\mathsf{K}_{G^*}$. Nous souhaitons que l'affirmation ($\dagger$) soit vraie, or jusqu'ici nous ne pouvons qu'établir ($\dagger$) en y inversant de plus les nombres premiers mauvais pour G ; cette condition supplémentaire est due à l'utilisation de presque-caractères de Lusztig, où les nombres premiers mauvais pour G apparaîtront en dénominateurs de certaines formules dont l'on aura besoin (voir § 2.22).

Pour (b), comme l'anneau K_{G^*} est réduit et sans torsion, l'anneau $\mathsf{B}_{G^\vee}$ devrait l'être aussi si l'on veut un isomorphisme d'anneaux $\mathsf{K}_{G^*} \simeq \mathsf{B}_{G^\vee}$. Lorsque le groupe dérivé de $G^\vee$ est simplement connexe, on démontrera en § 3.17 que $\mathsf{B}_{G^\vee}$ est d'une part un anneau réduit et sans torsion et d'autre part un $\mathbb{Z}$-module libre de rang égal à celui de K_{G^*} ; on considérera ensuite l'isomorphisme de caractère formel

$$\mathrm{ch} : \mathrm{Groth}(\overline{\mathbb{F}}_q G^*\text{-mod}) \xrightarrow{\sim} \mathbb{Z}[X(T^\vee)]^W,$$

puis on le réduira à l'isomorphisme voulu $\mathsf{K}_{G^*} \simeq \mathsf{B}_{G^\vee}$. Pour la raison de dimension, ce dernier isomorphisme ne serait pas vrai lorsque le groupe dérivé de $G^\vee$ n'est pas simplement connexe (i.e. lorsque le centre de G n'est pas un tore) : par exemple, pour $G = \mathrm{SL}_2$ défini sur $\mathbb{F}_q$ avec q impair (donc $G^* = \mathrm{PGL}_2$), le centre de G est $\{\pm \mathrm{id}_2\}$ et n'est donc pas un tore, et l'on vérifiera que $\dim \overline{\mathbb{Q}}\mathsf{K}_{G^*} = q+1$, alors que $\mathsf{B}_{G^\vee}$ est réduit avec $\dim \overline{\mathbb{Q}}\mathsf{B}_{G^\vee} = q$, donc $\mathsf{K}_{G^*} \not\simeq \mathsf{B}_{G^\vee}$.

Pendant l'étude de l'isomorphisme souhaité $\Lambda\mathsf{E}_G \simeq \Lambda\mathsf{B}_{G^\vee}$ (avec $\Lambda \supset \overline{\mathbb{Z}}[\frac{1}{p}]$), au moins deux autres questions se manifestent. D'un côté, comme la définition de $\Lambda\mathsf{E}_G$ dépend du choix de $\psi : U^F \longrightarrow \Lambda^\times$, on peut se demander s'il existe un $\mathbb{Z}$-modèle de $\Lambda\mathsf{E}_G$, c'est-à-dire un anneau dont la définition ne nécessite pas de choix de ψ et dont l'extension à Λ-coefficients coïncide avec $\Lambda\mathsf{E}_G$; dans ce travail, on construira un tel $\mathbb{Z}$-modèle que l'on notera par E_G (voir § 1.19). De l'autre côté, on a mentionné que l'anneau $\mathsf{B}_{G^\vee}$ est réduit lorsque le groupe dérivé de $G^\vee$ est simplement connexe, mais que se passe-il si le groupe dérivé de $G^\vee$ n'est pas simplement connexe ? Bien que nous n'ayons pas encore une réponse définitive, il semble que $\mathsf{B}_{G^\vee}$ soit toujours réduit — du moins, nous n'avons pas encore pu trouver un contre-exemple (voir § 3.4). Comme un exemple spécifique, on expliquera en § A que $\mathsf{B}_{G^\vee}$ est réduit lorsque $G = \mathrm{SO}_{2n}$ défini sur $\mathbb{F}_q$ avec q impair (dans ce cas, le groupe dérivé de $G^\vee$ et aussi SO_{2n} et n'est alors pas simplement connexe).

L'aspect sur l'algèbre d'endomorphismes des représentations de Gelfand–Graev étant élaboré, nous nous tournons maintenant vers la question de la description du ℓ-bloc unipotent d'un groupe linéaire p-adique avec $\ell \neq p$. Dans [Hel1], Helm a obtenu pour le groupe p-adique $\mathbf{G} = \mathrm{GL}_n(\mathbf{F})$ (avec $\mathcal{G} = \mathrm{GL}_n$) une décomposition en blocs de la catégorie $\mathrm{Rep}_{\overline{\mathbb{Z}}_\ell}(\mathbf{G})$ des $\overline{\mathbb{Z}}_\ell$-représentations lisses de $\mathbf{G}$ (avec $\ell \neq p$) :

$$\mathrm{Rep}_{\overline{\mathbb{Z}}_\ell}(\mathbf{G}) = \prod_{[\mathbf{M},\pi]} \mathrm{Rep}_{\overline{\mathbb{Z}}_\ell}(\mathbf{G})_{[\mathbf{M},\pi]},$$

où le produit parcourt les classes d'équivalences inertielles $[\mathbf{M},\pi]$. On appelle chaque $\mathrm{Rep}_{\overline{\mathbb{Z}}_\ell}(\mathbf{G})_{[\mathbf{M},\pi]}$ un ℓ-bloc de $\mathbf{G}$, et l'on appelle l'unique ℓ-bloc de $\mathbf{G}$ contenant $\mathbf{1}_{\mathbf{G}}$ (la représentation triviale de $\mathbf{G}$) le ℓ-bloc unipotent de $\mathbf{G}$, qui est en fait $\mathrm{Rep}_{\overline{\mathbb{Z}}_\ell}(\mathbf{G})_{[\mathbf{T},\mathbf{1}]}$ où $\mathbf{T}$ est le tore diagonal maximal de $\mathbf{G}$ et $\mathbf{1}$ est le caractère trivial de $\mathbf{T}$ à valeurs dans $\overline{\mathbb{F}}_\ell$. On veut décrire ce ℓ-bloc unipotent $\mathrm{Rep}_{\overline{\mathbb{Z}}_\ell}(\mathbf{G})_{[\mathbf{T},\mathbf{1}]}$.

Une possibilité pour décrire le ℓ-bloc unipotent de $\mathbf{G}$ est d'utiliser le progénérateur $\mathcal{P}$ de ce bloc construit en [Hel1] ; on peut alors identifier ce ℓ-bloc unipotent de $\mathbf{G}$ comme la catégorie des modules (à droite) de $\mathrm{End}(\mathcal{P})$, et la question se transforme en la description de $\mathrm{End}(\mathcal{P})$. Dans ce travail, nous développons cette approche en nous

bornant au cas de $n = 2$ (donc $\mathbf{G} = \mathrm{GL}_2(\mathbf{F})$) avec l'ordre de q dans $(\mathbb{Z}/\ell\mathbb{Z})^\times$ égal à $n = 2$. Dans ce cas, on peut prendre (voir [Hel1, Preuve du Cor. 11.19 ; Prop. 5.8])

$$\mathcal{P} = \mathrm{ind}_{\mathbf{I}}^{\mathbf{G}} \mathbf{1}_{\mathbf{I}} \oplus \mathrm{ind}_{\mathbf{K}}^{\mathbf{G}}(e_{[1]}^{G,\ell}.\overline{\mathbb{Z}}_\ell \Gamma_{G,\psi}),$$

où $\mathbf{I} = \begin{bmatrix} O_{\mathbf{F}}^\times & O_{\mathbf{F}} \\ \varpi O_{\mathbf{F}} & O_{\mathbf{F}}^\times \end{bmatrix}$ (avec ϖ une uniformisante de $O_{\mathbf{F}}$) est un sous-groupe d'Iwahori de $\mathbf{G}$, et $e_{[1]}^{G,\ell}$ désigne l'idempotent central de $\overline{\mathbb{Q}}G^F$ correspondant à la réunion des séries rationnelles de Lusztig $\mathcal{E}(G^F, [x])$ de G^F avec $[x] \in G_{\mathrm{ss},\ell}^{*F^*}$ (voir § 4.2 ; Broué et Michel ont démontré en [BrMi, Thm. 2.2] qu'en effet $e_{[1]}^{G,\ell} \in \overline{\mathbb{Z}}_\ell G^F$). On peut donc écrire

$$\begin{aligned} \mathrm{End}_{\overline{\mathbb{Z}}_\ell \mathbf{G}}(\mathcal{P}) &= \mathrm{End}_{\overline{\mathbb{Z}}_\ell \mathbf{G}}(\mathrm{ind}_{\mathbf{I}}^{\mathbf{G}} \mathbf{1}_{\mathbf{I}} \oplus \mathrm{ind}_{\mathbf{K}}^{\mathbf{G}}(e_{[1]}^{G,\ell}.\overline{\mathbb{Z}}_\ell \Gamma_{G,\psi})) \\ &= \begin{array}{c} \\ \mathrm{ind}_{\mathbf{I}}^{\mathbf{G}} \mathbf{1}_{\mathbf{I}} \\ \mathrm{ind}_{\mathbf{K}}^{\mathbf{G}}(e_{[1]}^{G,\ell}.\overline{\mathbb{Z}}_\ell \Gamma_{G,\psi}) \end{array} \begin{array}{c} \begin{array}{cc} \mathrm{ind}_{\mathbf{I}}^{\mathbf{G}} \mathbf{1}_{\mathbf{I}} & \mathrm{ind}_{\mathbf{K}}^{\mathbf{G}}(e_{[1]}^{G,\ell}.\overline{\mathbb{Z}}_\ell \Gamma_{G,\psi}) \end{array} \\ \begin{bmatrix} \mathsf{H} & \mathsf{M}_1 \\ \mathsf{N}_1 & \mathsf{E}_1 \end{bmatrix} \end{array}; \end{aligned}$$

notons que l'algèbre H est (anti-)isomorphe à l'algèbre d'Iwahori–Hecke $\overline{\mathbb{Z}}_\ell[\mathbf{I}\backslash\mathbf{G}/\mathbf{I}]$, et que l'algèbre E_1 est un facteur direct de l'algèbre $\mathrm{End}_{\overline{\mathbb{Z}}_\ell \mathbf{G}}(\mathcal{W}_0)$ dans le diagramme $(*)$ (tensorisé par $\overline{\mathbb{Z}}_\ell$). Dans ce travail, nous établirons la proposition suivante :

Proposition. — *On est dans le cas de $\mathbf{G} = \mathrm{GL}_2(\mathbf{F})$ (avec $\mathcal{G} = \mathrm{GL}_2$).*

(a) (§ 4.19) *L'algèbre $\mathrm{End}_{\overline{\mathbb{Z}}[\frac{1}{p}]\mathbf{G}}(\mathcal{W}_0)$ est filtrée, de sorte qu'elle est la limite (l'union) d'une chaîne croissante de ses sous-$\overline{\mathbb{Z}}_\ell$-modules $V^{(0)} \subset V^{(1)} \subset V^{(2)} \subset \cdots$ vérifiant $V^{(d)} \cdot V^{(d')} \subset V^{(d+d')}$ pour tous $d, d' \geq 0$. De plus, on peut trouver explicitement une partie $\{\Theta_\lambda : \lambda \in \mathbb{F}_q^\times\}$ de $V^{(1)} - V^{(0)}$ telle que, en identifiant*

$$\varpi = e_\psi(\varpi \mathrm{id}_2) e_\psi \in e_\psi.\overline{\mathbb{Z}}[\tfrac{1}{p}]\mathbf{G}.e_\psi = \mathrm{End}_{\overline{\mathbb{Z}}[\frac{1}{p}]\mathbf{G}}(\mathcal{W}_0),$$

on ait

$$\mathrm{End}_{\overline{\mathbb{Z}}[\frac{1}{p}]\mathbf{G}}(\mathcal{W}_0) = (\overline{\mathbb{Z}}[\tfrac{1}{p}]\mathsf{E}_G)[\varpi^{\pm 1}, \{\Theta_\lambda : \lambda \in \mathbb{F}_q^\times\}].$$

On suppose désormais que l'ordre de q dans $(\mathbb{Z}/\ell\mathbb{Z})^\times$ est égal à 2 ; il est donc pertinent de considérer la décomposition précédente de $\mathrm{End}(\mathcal{P})$.

(b) (§ 4.22) *On peut construire explicitement un $s_{\mathsf{M}_1} \in \mathsf{M}_1$ et un $s_{\mathsf{N}_1}^* \in \mathsf{N}_1$ tels que $\mathsf{M}_1 = \mathsf{H}.s_{\mathsf{M}_1}$ et $\mathsf{N}_1 = s_{\mathsf{N}_1}^*.\mathsf{H}$.*

(c) ([Pa] ou § 4.13) *On a $e_{[1]}^{G,\ell}.\overline{\mathbb{Z}}_\ell \mathsf{E}_G = \overline{\mathbb{Z}}_\ell[\theta]$ où θ est la composante dans $e_{[1]}^{G,\ell}.\overline{\mathbb{Z}}_\ell \mathsf{E}_G$ de l'élément de $\overline{\mathbb{Z}}_\ell \mathsf{E}_G$ correspondant à la fonction trace sur $(T^\vee /\!\!/ W)^{F^\vee}$ sous l'isomorphisme $\overline{\mathbb{Z}}_\ell \mathsf{E}_G \simeq \overline{\mathbb{Z}}_\ell \mathsf{B}_{G^\vee}$ donné par le théorème précédent.*

(d) (§ 4.23, voir aussi [Hel1, Prop. 9.9]) *En dénotant par ϖ_1 (resp. $\Theta_{\lambda,1}$) la composante dans E_1 de l'élément ϖ (resp. Θ_λ) de $\mathrm{End}_{\overline{\mathbb{Z}}_\ell \mathbf{G}}(\mathcal{W}_0)$, on a $\Theta_{\lambda,1} = \Theta_{1,1}$ pour tout $\lambda \in \mathbb{F}_q^\times$, et l'on a donc*

$$\mathsf{E}_1 = (e_{[1]}^{G,\ell}.\overline{\mathbb{Z}}_\ell \mathsf{E}_G)[\varpi_1^{\pm 1}, \{\Theta_{\lambda,1} : \lambda \in \mathbb{F}_q^\times\}] = \overline{\mathbb{Z}}_\ell[\theta, \varpi_1^{\pm 1}, \Theta_{1,1}].$$

(e) (§§ 4.21 - 4.24) *On peut calculer explicitement les produits parmi s_{M_1}, $s_{\mathsf{N}_1}^*$, les générateurs standards de H et les générateurs de E_1 en* (d).

On obtient ainsi une description de $\mathrm{End}(\mathcal{P})$ et donc du ℓ-bloc unipotent du groupe p-adique $\mathbf{G} = \mathrm{GL}_2(\mathbf{F})$; nous souhaitons que cette description puisse ouvrir une voie vers une description structurée du ℓ-bloc unipotent des groupes linéaires p-adiques.

L'organisation de ce travail est comme suit. En § 1, nous rappelons les études de Bonnafé–Kessar en [BoKe] sur l'algèbre $\overline{\mathbb{Z}}[\frac{1}{p}]\mathsf{E}_G$, on y introduit notamment le morphisme de Curtis et la forme symétrisante "standard" de $\overline{\mathbb{Z}}[\frac{1}{p}]\mathsf{E}_G$. En plus des rappels des travaux de Bonnafé–Kessar, nous construisons également un $\mathbb{Z}$-modèle E_G pour $\overline{\mathbb{Z}}[\frac{1}{p}]\mathsf{E}_G$, et nous analysons, comme une préparation de l'étude de la relation entre $\overline{\mathbb{Z}}[\frac{1}{p}]\mathsf{E}_G$ et K_{G^*} en § 2, le comportement de $\overline{\mathbb{Z}}[\frac{1}{p}]\mathsf{E}_G$ sous une extension centrale de G^*.

En § 2, nous étudions l'anneau K_{G^*}, expliquons comment la forme symétrisante sur $\overline{\mathbb{Z}}[\frac{1}{p}]\mathsf{E}_G$ considérée en § 1 induit une forme symétrisante sur $\overline{\mathbb{Z}}[\frac{1}{p}]\mathsf{K}_{G^*}$, puis nous établissons deux versions d'identifications des $\overline{\mathbb{Z}}[\frac{1}{p}]$-algèbres $\Lambda\mathsf{E}_G$ et $\Lambda\mathsf{K}_{G^*}$ pour $\Lambda \supset \overline{\mathbb{Z}}[\frac{1}{p}]$, la première version nécessitant l'inversion de $|W|$ dans Λ, tandis que la deuxième version ne nécessitant que l'inversion des nombres premiers mauvais pour G dans Λ. Pour examiner cette deuxième version, nous rappelons également la théorie des représentations des groupes algébriques.

En § 3, nous nous intéressons à l'anneau $\mathsf{B}_{G^\vee}$. Après des préparations préliminaires sur la combinatoire de donnée radicielle de G, nous démontrons que $\mathsf{B}_{G^\vee}$ est un anneau réduit, sans torsion et isomorphe à K_{G^*} lorsque le groupe dérivé de $G^\vee$ est simplement connexe. Par conséquent, lorsque le groupe dérivé de $G^\vee$ est simplement connexe et Λ est un anneau contenant $\overline{\mathbb{Z}}[\frac{1}{p}]$ et les inverses multiplicatifs des nombres premiers mauvais de G, nous obtenons un isomorphisme de Λ-algèbres $\Lambda\mathsf{E}_G \simeq \Lambda\mathsf{B}_{G^\vee}$. Nous incluons aussi une discussion sur la preuve directe de l'isomorphisme de $\overline{\mathbb{Z}}[\frac{1}{p}]$-algèbres $\overline{\mathbb{Z}}[\frac{1}{p}]\mathsf{E}_G \simeq \overline{\mathbb{Z}}[\frac{1}{p}]\mathsf{B}_{G^\vee}$ pour $G = \mathrm{GL}_2$.

En § 4, nous utilisons tout d'abord l'isomorphisme $\Lambda\mathsf{E}_G \simeq \Lambda\mathsf{B}_{G^\vee}$ pour généraliser le travail [Pa] de Paige sur l'anneau d'endomorphisme d'une enveloppe projective. Nous nous concentrons alors au cas du groupe p-adique $\mathbf{G} = \mathrm{GL}_2(\mathbf{F})$: dans ce cas, nous analysons la structure de l'algèbre d'endomorphismes de la partie de niveau zéro du modèle de Whittaker de $\mathbf{G}$, une structure que nous utilisons, avec le résultat de Paige que nous venons d'étudier, pour décrire enfin le ℓ-bloc unipotent de $\mathbf{G}$ avec $\ell \neq p$ sous la condition que $\mathrm{ord}_\ell(q) = 2$.

Dans l'appendice § A, nous étudions la structure de $\mathsf{B}_{G^\vee}$ dans le cas de $G^\vee = \mathrm{SO}_{2n}$ (dont le groupe dérivé n'est pas simplement connexe), où nous montrons notamment que $\mathsf{B}_{G^\vee}$ est encore un anneau réduit et sans torsion.

Les contenus des § 1, § 2, § 3 et § A s'agissent principalement d'une intégration de mon article [Li2] et de mon autre article [LiSh] en collaboration avec J. Shotton ; dans le présent travail, je ne citerai donc pas systématiquement ces deux derniers articles.

Introduction in English

This work consists of two aspects around representations of reductive groups. On one hand, inspired by the ongoing work [DHKM] of Dat, Helm, Kurinczuk and Moss on the conjecture of local Langlands in families, we study the dual models of the integral endomorphism algebra of Gelfand–Graev representations of a finite reductive group; on the other hand, thanks to Helm's fundamental work [Hel1] on the ℓ-blocks of p-adic general linear groups, we try to explicitly describe the unipotent ℓ-block of a p-adic general linear group with $\ell \neq p$. These two aspects are linked by Paige's work [Pa] on the endomorphism algebra of a certain projective cover, as we will see later in this introduction. We now elaborate on these perspectives.

Let $\mathbf{F}$ be a p-adic field (a finite extension of $\mathbb{Q}_p$) whose residue field is the finite field $\mathbb{F}_q$ with q a power of p, and let $O_{\mathbf{F}}$ be the ring of integers of $\mathbf{F}$. Let $\mathcal{G}$ be a connected reductive group defined over $O_{\mathbf{F}}$, and for the moment we suppose that $\mathcal{G}$ is split over $O_{\mathbf{F}}$. Let $\mathcal{G}^\vee$ be the Langlands dual group (defined and split over $\mathbb{Z}$) of $\mathcal{G}$. Then the moduli space of tame Langlands parameters for $\mathcal{G}^\vee$ is

$$\mathcal{Z}^1_{\mathrm{t}} = \{(s, \mathrm{Fr}) \in \mathcal{G}^\vee \times \mathcal{G}^\vee : \mathrm{Fr} \cdot s \cdot \mathrm{Fr}^{-1} = s^q\},$$

on which $\mathcal{G}^\vee$ acts diagonally; let $\mathcal{O}_{\mathcal{Z}^1_{\mathrm{t}} /\!\!/ \mathcal{G}^\vee}$ be the ring of functions of $\mathcal{Z}^1_{\mathrm{t}} /\!\!/ \mathcal{G}^\vee$.

On the other side, let $G = \mathcal{G}(\overline{\mathbb{F}}_q)$, and let F be the Frobenius endomorphism of G induced by an $\mathbb{F}_q$-structure of $\mathcal{G}$, in the way that $G^F = \mathcal{G}(\mathbb{F}_q)$; note that G^F is a finite group. Let $\mathcal{U}$ be the unipotent radical of a Borel subgroup of $\mathcal{G}$, and let $U = \mathcal{U}(\overline{\mathbb{F}}_q)$. By fixing a choice of regular (that is, nondegenerate) linear character $\psi : U^F \longrightarrow \overline{\mathbb{Z}}[\frac{1}{p}]^\times$, we can consider the Gelfand–Graev representation

$$\Gamma_{G,\psi} := \mathrm{Ind}_{U^F}^{G^F} \psi.$$

Let $\mathbf{G} = \mathcal{G}(\mathbf{F})$ and $\mathbf{K} = \mathcal{G}(O_{\mathbf{F}})$, and then consider the compact induction

$$\mathcal{W}_0 := \mathrm{ind}_{\mathbf{K}}^{\mathbf{G}} \Gamma_{G,\psi}.$$

(Here, $\Gamma_{G,\psi}$ is a representation of $\mathbf{K}$ via the reduction $\mathbf{K} \twoheadrightarrow G^F$.) It will be shown in [DHKM] that $\mathcal{W}_0$ is isomorphic to the depth-zero part of the Whittaker model ("p-adic Gelfand–Graev") $\mathcal{W} = \mathrm{ind}_{\mathbf{U}}^{\mathbf{G}} \widetilde{\psi}$, where $\mathbf{U} = \mathcal{U}(\mathbf{F})$ and $\widetilde{\psi}$ is a regular character compatible with ψ. It is known that the endomorphism algebra $\mathrm{End}_{\overline{\mathbb{Z}}[\frac{1}{p}]\mathbf{G}}(\mathcal{W})$ is commutative and reduced (see for example [Ra]), and hence so is $\mathrm{End}_{\overline{\mathbb{Z}}[\frac{1}{p}]\mathbf{G}}(\mathcal{W}_0)$.

With these preparations, the conjecture of Dat–Helm–Kurinczuk–Moss predicts the existence of a ring isomorphism

$$\mathrm{LLIF} : \overline{\mathbb{Z}}[\tfrac{1}{p}].\mathcal{O}_{\mathcal{Z}^1_{\mathrm{t}} /\!\!/ \mathcal{G}^\vee} \xrightarrow{\sim} \mathrm{End}_{\overline{\mathbb{Z}}[\frac{1}{p}]\mathbf{G}}(\mathcal{W}_0)$$

which is compatible with the classical local Langlands conjectures. (Here, "LLIF" is for "local Langlands in families.") Upon setting

$$\overline{\mathbb{Z}}[\tfrac{1}{p}]\mathsf{E}_G = \mathrm{End}_{\overline{\mathbb{Z}}[\frac{1}{p}]G^F}(\Gamma_{G,\psi})$$

(this algebra $\overline{\mathbb{Z}}[\frac{1}{p}]\mathsf{E}_G$ is independent of choice of ψ up to isomorphism) and upon denoting by $\mathsf{B}_{G^\vee}$ the ring of functions of $(\mathcal{G}^\vee /\!\!/ \mathcal{G}^\vee)^{(\cdot)^q}$ (the subscheme of $\mathcal{G}^\vee /\!\!/ \mathcal{G}^\vee$ fixed by the endomorphism $(\cdot)^q$; here $\mathcal{G}^\vee$ acts on itself by adjoint action), we obtain the following diagram:

$$\begin{tikzcd} \overline{\mathbb{Z}}[\frac{1}{p}].\mathcal{O}_{\mathcal{Z}_{\mathrm{t}}^1 /\!\!/ \mathcal{G}^\vee} \arrow[r, "\mathrm{LLIF}", "\sim"'] & \mathrm{End}_{\overline{\mathbb{Z}}[\frac{1}{p}]\mathbf{G}}(\mathcal{W}_0) \\ \overline{\mathbb{Z}}[\frac{1}{p}]\mathsf{B}_{G^\vee} \arrow[u, "i^{\mathsf{B}}"] \arrow[r, dashed, "?"] & \overline{\mathbb{Z}}[\frac{1}{p}]\mathsf{E}_G \arrow[u, hook, "i^{\mathsf{E}}"'] \end{tikzcd} \qquad (*)$$

Here, i^{B} corresponds to the morphism $\mathcal{Z}_{\mathrm{t}}^1 /\!\!/ \mathcal{G}^\vee \longrightarrow (\mathcal{G}^\vee /\!\!/ \mathcal{G}^\vee)^{(\cdot)^q}$ induced by the first projection $\mathcal{G}^\vee \times \mathcal{G}^\vee \longrightarrow \mathcal{G}^\vee$, and i^{E} is induced by the natural inclusion $\Gamma_{G,\psi} \hookrightarrow \mathcal{W}_0$. When the derived subgroup of $G^\vee$ is simply-connected, i^{B} is injective, and we hope that $(*)$ will identify the two finite $\overline{\mathbb{Z}}[\frac{1}{p}]$-algebras $\overline{\mathbb{Z}}[\frac{1}{p}]\mathsf{B}_{G^\vee}$ and $\overline{\mathbb{Z}}[\frac{1}{p}]\mathsf{E}_G$.

At the moment, after extension to coefficients in $\Lambda = \mathrm{Witt}(\overline{\mathbb{F}}_\ell)$ (the ring of Witt vectors of $\overline{\mathbb{F}}_\ell$) with $\ell \neq p$, the conjectural diagram $(*)$ as well as the ring isomorphism $\Lambda\mathsf{B}_{G^\vee} \simeq \Lambda\mathsf{E}_G$ has been justified for all $\mathcal{G} = \mathrm{GL}_n$ (with $n \in \mathbb{N}^*$) by Helm and Moss (see [Hel2] and [HeMo]). Their method consists of techniques dedicated to GL_n and does not seem easy to generalise to the other reductive groups $\mathcal{G}$. In spite of this difficulty, Helm observed that the wished isomorphism $\Lambda\mathsf{B}_{G^\vee} \simeq \Lambda\mathsf{E}_G$ was "a result in finite group theory in its own right"; he then asked for a direct proof of this isomorphism by the finite group theory (see [Hel2, Rmk. 10.3]).

We present in this work, among other things, a response to the last question of Helm. We first observe that to study the relation between $\Lambda\mathsf{B}_{G^\vee}$ and $\Lambda\mathsf{E}_G$ it is not necessary to involve the p-adic group $\mathcal{G}$. We thus start directly from a connected reductive group G defined over $\mathbb{F}_q$, and we suppose no longer that G is split over $\mathbb{F}_q$. We identify $G = G(\overline{\mathbb{F}}_q)$, and we denote by $G^\vee$ the Langlands dual of G (so $G^\vee = \mathcal{G}^\vee$). We define $\overline{\mathbb{Z}}[\frac{1}{p}]\mathsf{E}_G$ as before, while we define $\mathsf{B}_{G^\vee}$ as $\mathcal{O}_{(T^\vee /\!\!/ W)^{F^\vee}}$, the ring of functions of $(T^\vee /\!\!/ W)^{F^\vee}$, where $T^\vee$ is a maximal torus of $G^\vee$ dual to an F-stable maximal torus T of G via the identification $X(T^\vee) = Y(T)$ (here $X(T^\vee) = \mathrm{Hom}_{\mathrm{alg}}(T^\vee, \mathbb{G}_m)$ is the character group of $T^\vee$, and $Y(T) = \mathrm{Hom}_{\mathrm{alg}}(\mathbb{G}_m, T)$ is the cocaracter group of T), $F^\vee$ is the endomorphism of $T^\vee$ induced by F via the identification $X(T^\vee) = Y(T)$, and W denotes the Weyl group of $(G^\vee, T^\vee)$. Then

$$\mathsf{B}_{G^\vee} = \mathcal{O}_{(T^\vee /\!\!/ W)^{F^\vee}} = \mathbb{Z}[X(T^\vee)]^W / \langle f(F^\vee(\cdot)) - f : f \in \mathbb{Z}[X(T^\vee)]^W \rangle.$$

When G is split over $\mathbb{F}_q$, the ring $\mathsf{B}_{G^\vee}$ coincide with that previously defined (that is, the ring of functions of $(G^\vee /\!\!/ G^\vee)^{(\cdot)^q}$), since in this case $F^\vee = (\cdot)^q$, and the inclusion $T^\vee \subset G^\vee$ induces an isomorphism $T^\vee /\!\!/ W \simeq G^\vee /\!\!/ G^\vee$ (Chevalley restriction theorem) which is $(\cdot)^q$-equivariant.

Our strategy to relate $\Lambda\mathsf{B}_{G^\vee}$ and $\Lambda\mathsf{E}_G$ is to introduce a third ring

$$\mathsf{K}_{G^*} = \mathrm{Groth}(\overline{\mathbb{F}}_q G^{*F^*}\text{-mod}),$$

which is the Grothendieck ring of the category of finite $\overline{\mathbb{F}}_q$-dimensional representations of $G^{*F^*} = G^*(\mathbb{F}_q)$, where G^* is the Deligne–Lusztig dual (defined over $\mathbb{F}_q$) of G with F^* the associated Frobenius endomorphism of G^*. Thanks to this ring K_{G^*}, we will show in this work the following theorem:

THEOREM. — *Let G be a connected reductive group defined (but not necessarily split) over $\mathbb{F}_q$.*

(a) (§ 2.25; see also [LiSh]) *For $\Lambda = \overline{\mathbb{Z}}[\frac{1}{pM}]$ with M the product of bad prime numbers for G in the sense of* [Sp1, (2.10)], *$\Lambda\mathsf{E}_G \simeq \Lambda\mathsf{K}_{G^*}$ as Λ-algebras.*

(b) (§ 3.21; see also [Li2]) *If the derived subgroup of $G^\vee$ is simply-connected (that is, if the centre of G is a torus), then $\mathsf{K}_{G^*} \simeq \mathsf{B}_{G^\vee}$ as rings.*

In particular, when $G = \mathrm{GL}_n$ (type A_{n-1}), there is no bad prime number for G, and the centre of G is a torus, so that this theorem will give us a $\overline{\mathbb{Z}}[\frac{1}{p}]$-algebra isomorphism $\overline{\mathbb{Z}}[\frac{1}{p}]\mathsf{E}_G \simeq \overline{\mathbb{Z}}[\frac{1}{p}]\mathsf{B}_{G^\vee}$ for $G = \mathrm{GL}_n$ and will reply to Helm's question.

We now give the main ideas of the proof of this theorem. Let us start from (a). Over $\overline{\mathbb{Q}}$, it is known (Deligne–Lusztig [DeLu] for the case where the centre of G is connected, and Digne–Lehrer–Michel [DLM] for the general case) that the irreducible constituents of the $\overline{\mathbb{Q}}G^F$-module $\overline{\mathbb{Q}}\Gamma_{G,\psi}$ are all of mulpliticity one and are parametrised by $G^{*F^*}_{\mathrm{ss}}/\sim$, where "ss" signifies the semisimple elements and $\sim$ denotes the conjugation; Schur's lemma then gives a $\overline{\mathbb{Q}}$-algebra isomorphism $\overline{\mathbb{Q}}\mathsf{E}_G \simeq \overline{\mathbb{Q}}^{G^{*F^*}_{\mathrm{ss}}/\sim}$. On the other hand, Brauer character gives a $\overline{\mathbb{Q}}$-algebra isomorphism $\overline{\mathbb{Q}}\mathsf{K}_{G^*} \simeq \overline{\mathbb{Q}}^{G^{*F^*}_{\mathrm{ss}}/\sim}$. We can hence identify the $\overline{\mathbb{Q}}$-algebras

$$\overline{\mathbb{Q}}\mathsf{E}_G = \overline{\mathbb{Q}}\mathsf{K}_{G^*} = \overline{\mathbb{Q}}^{G^{*F^*}_{\mathrm{ss}}/\sim}.$$

Next, In order to identify $\Lambda\mathsf{E}_G$ and $\Lambda\mathsf{K}_{G^*}$ for a ring Λ with $\overline{\mathbb{Z}}[\frac{1}{p}] \subset \Lambda \subset \overline{\mathbb{Q}}$, our strategy will be to use the symmetrizing form $\Lambda\mathsf{E}_G$ studied in Bonnafé and Kessar's article [BoKe]: upon identifying

$$\overline{\mathbb{Z}}[\tfrac{1}{p}]\mathsf{E}_G = e_\psi\overline{\mathbb{Z}}[\tfrac{1}{p}]G^F e_\psi \subset \overline{\mathbb{Z}}[\tfrac{1}{p}]G^F$$

with $e_\psi = \frac{1}{|U^F|}\sum_{u\in U^F}\psi(u^{-1})u$ the primitive central idempotent of $\overline{\mathbb{Z}}[\frac{1}{p}]U^F$ associated to ψ, the last symmetrizing form, which we denote by τ, is

$$\tau = |U^F|\mathrm{ev}_{1_{G^F}} : \overline{\mathbb{Z}}[\tfrac{1}{p}]\mathsf{E}_G \longrightarrow \overline{\mathbb{Z}}[\tfrac{1}{p}],$$

with $\mathrm{ev}_{1_{G^F}}$ the evaluation at the identity element of G^F. We will show in § 2.8 that the extension to $\overline{\mathbb{Q}}$-coefficients of τ restricts to a symmetrizing form on $\overline{\mathbb{Z}}[\frac{1}{p}]\mathsf{K}_{G^*}$; this being done, if we can show that

$$\tau(h\pi) \in \overline{\mathbb{Z}}[\tfrac{1}{p}] \text{ for every } (h,\pi) \in \overline{\mathbb{Z}}[\tfrac{1}{p}]\mathsf{E}_G \times \overline{\mathbb{Z}}[\tfrac{1}{p}]\mathsf{K}_{G^*}, \tag{$\dagger$}$$

then we will be able to identify $\overline{\mathbb{Z}}[\frac{1}{p}]\mathsf{E}_G$ and $\overline{\mathbb{Z}}[\frac{1}{p}]\mathsf{K}_{G^*}$. We hope that the claim ($\dagger$) is true, yet so far we can only establish ($\dagger$) by moreover inverting therein the bad prime numbers for G; this additional condition is due to the use of Lusztig's almost characters, where the bad prime numbers for G will appear in the denominators of some formulae that we will need (see § 2.22).

For (b), as the ring K_{G^*} is reduced and without torsion, so should the ring $\mathsf{B}_{G^\vee}$ be if we want a ring isomorphism $\mathsf{K}_{G^*} \simeq \mathsf{B}_{G^\vee}$. When the derived subgroup of $G^\vee$ is simply-connected, we will show in § 3.17 that $\mathsf{B}_{G^\vee}$ is on one hand a reduced ring without torsion and on the other hand a free $\mathbb{Z}$-module of rank equal to that of K_{G^*}; we will then consider the formal character isomorphism

$$\mathrm{ch} : \mathrm{Groth}(\overline{\mathbb{F}}_q G^*\text{-mod}) \xrightarrow{\sim} \mathbb{Z}[X(T^\vee)]^W,$$

then we reduce it to the expected isomorphism $\mathsf{K}_{G^*} \simeq \mathsf{B}_{G^\vee}$. For dimension reasons, the last isomorphism would not be true when the derived subgroup of $G^\vee$ is not simply-connected (i.e. when the centre of G is not a torus): for example, for $G = \mathrm{SL}_2$ defined over $\mathbb{F}_q$ with q odd (so $G^* = \mathrm{PGL}_2$), the centre of G is $\{\pm \mathrm{id}_2\}$ and is thus not a torus, and one will be able to show that $\dim \overline{\mathbb{Q}}\mathsf{K}_{G^*} = q+1$, while $\mathsf{B}_{G^\vee}$ is reduced with $\dim \overline{\mathbb{Q}}\mathsf{B}_{G^\vee} = q$, whence $\mathsf{K}_{G^*} \not\simeq \mathsf{B}_{G^\vee}$.

Along with the study of the expected isomorphism $\Lambda\mathsf{E}_G \simeq \Lambda\mathsf{B}_{G^\vee}$ (with $\Lambda \supset \overline{\mathbb{Z}}[\frac{1}{p}]$), at least two other questions arise. On one side, as the definition of $\Lambda\mathsf{E}_G$ depends on the choice of $\psi : U^F \longrightarrow \Lambda^\times$, we may ask if there is a $\mathbb{Z}$-model of $\Lambda\mathsf{E}_G$, that is a ring whose definition does not depend on the choice of ψ and whose extension to Λ-coefficients coincides with $\Lambda\mathsf{E}_G$; in this work, we will construct such a $\mathbb{Z}$-model which we will denote by E_G (voir § 1.19). On the other side, we have mentioned that the ring $\mathsf{B}_{G^\vee}$ is reduced when the derived subgroup of $G^\vee$ is simply-connected, but what happens if the derived subgroup of $G^\vee$ is not simply-connected? Although we haven't had a definitive answer, it seems that $\mathsf{B}_{G^\vee}$ would always be reduced — at least, we haven't found any counterexample yet (see § 3.4). As a specific example, we will explain in § A that $\mathsf{B}_{G^\vee}$ is reduced when $G = \mathrm{SO}_{2n}$ defined over $\mathbb{F}_q$ with q odd (in this case, the derived subgroup of $G^\vee$ is also SO_{2n} and is then not simply-connected).

The aspect on the endomorphism algebra of Gelfand–Graev representations being elaborated, we now turn to the question of the description of the unipotent ℓ-block of a p-adic general linear group with $\ell \neq p$. In [Hel1], Helm has obtained for the p-adic group $\mathbf{G} = \mathrm{GL}_n(\mathbf{F})$ (with $\mathcal{G} = \mathrm{GL}_n$) a decomposition in blocks of the category $\mathrm{Rep}_{\overline{\mathbb{Z}}_\ell}(\mathbf{G})$ of smooth $\overline{\mathbb{Z}}_\ell$-representations of $\mathbf{G}$ (with $\ell \neq p$) :

$$\mathrm{Rep}_{\overline{\mathbb{Z}}_\ell}(\mathbf{G}) = \prod_{[\mathbf{M},\pi]} \mathrm{Rep}_{\overline{\mathbb{Z}}_\ell}(\mathbf{G})_{[\mathbf{M},\pi]},$$

where the product runs over the inertial equivalence classes $[\mathbf{M},\pi]$. We call each $\mathrm{Rep}_{\overline{\mathbb{Z}}_\ell}(\mathbf{G})_{[\mathbf{M},\pi]}$ an ℓ-block of $\mathbf{G}$, and we call the unique ℓ-block of $\mathbf{G}$ containing $\mathbf{1}_\mathbf{G}$ (the trivial representation of $\mathbf{G}$) the unipotent ℓ-block of $\mathbf{G}$, which is in fact $\mathrm{Rep}_{\overline{\mathbb{Z}}_\ell}(\mathbf{G})_{[\mathbf{T},\mathbf{1}]}$ where $\mathbf{T}$ is the diagonal maximal torus of $\mathbf{G}$ and $\mathbf{1}$ is the $\overline{\mathbb{F}}_\ell$-valued trivial character of $\mathbf{T}$. We want to describe this unipotent ℓ-block $\mathrm{Rep}_{\overline{\mathbb{Z}}_\ell}(\mathbf{G})_{[\mathbf{T},\mathbf{1}]}$.

A possibility to describe the unipotent ℓ-block of $\mathbf{G}$ is to use the progenerator $\mathcal{P}$ of this block constructed in [Hel1]; we can then identify this unipotent ℓ-block of $\mathbf{G}$ as the category of (right) $\mathrm{End}(\mathcal{P})$-modules, and the question is transformed into the description of $\mathrm{End}(\mathcal{P})$. In this work, we develop this approach by restricting ourselves

in the case of $n = 2$ (so $\mathbf{G} = \mathrm{GL}_2(\mathbf{F})$) with the order of q in $(\mathbb{Z}/\ell\mathbb{Z})^\times$ equal to $n = 2$. In this case, we can take (see [Hel1, Proof of Cor. 11.19; Prop. 5.8])

$$\mathcal{P} = \mathrm{ind}_{\mathbf{I}}^{\mathbf{G}} \mathbf{1}_{\mathbf{I}} \oplus \mathrm{ind}_{\mathbf{K}}^{\mathbf{G}}(e_{[1]}^{G,\ell}.\overline{\mathbb{Z}}_\ell \Gamma_{G,\psi}),$$

where $\mathbf{I} = \begin{bmatrix} O_{\mathbf{F}}^\times & O_{\mathbf{F}} \\ \varpi O_{\mathbf{F}} & O_{\mathbf{F}}^\times \end{bmatrix}$ (with ϖ a uniformizer of $O_{\mathbf{F}}$) is an Iwahori subgroup of $\mathbf{G}$, and $e_{[1]}^{G,\ell}$ denotes the central idempotent of $\overline{\mathbb{Q}}G^F$ corresponding to the union of the rational Lusztig series $\mathcal{E}(G^F, [x])$ of G^F with $[x] \in G_{\mathrm{ss},\ell}^{*F^*}$ (see § 4.2; Broué and Michel showed in [BrMi, Thm. 2.2] that in fact $e_{[1]}^{G,\ell} \in \overline{\mathbb{Z}}_\ell G^F$). We may thus write

$$\begin{aligned} \mathrm{End}_{\overline{\mathbb{Z}}_\ell \mathbf{G}}(\mathcal{P}) &= \mathrm{End}_{\overline{\mathbb{Z}}_\ell \mathbf{G}}(\mathrm{ind}_{\mathbf{I}}^{\mathbf{G}} \mathbf{1}_{\mathbf{I}} \oplus \mathrm{ind}_{\mathbf{K}}^{\mathbf{G}}(e_{[1]}^{G,\ell}.\overline{\mathbb{Z}}_\ell \Gamma_{G,\psi})) \\ &= \begin{array}{c} \\ \mathrm{ind}_{\mathbf{I}}^{\mathbf{G}} \mathbf{1}_{\mathbf{I}} \\ \mathrm{ind}_{\mathbf{K}}^{\mathbf{G}}(e_{[1]}^{G,\ell}.\overline{\mathbb{Z}}_\ell \Gamma_{G,\psi}) \end{array} \begin{array}{c} \begin{array}{cc} \mathrm{ind}_{\mathbf{I}}^{\mathbf{G}} \mathbf{1}_{\mathbf{I}} & \mathrm{ind}_{\mathbf{K}}^{\mathbf{G}}(e_{[1]}^{G,\ell}.\overline{\mathbb{Z}}_\ell \Gamma_{G,\psi}) \end{array} \\ \begin{bmatrix} \mathsf{H} & \mathsf{M}_1 \\ \mathsf{N}_1 & \mathsf{E}_1 \end{bmatrix} \end{array}; \end{aligned}$$

note that the algebra H is (anti-)isomorphic to the Iwahori–Hecke algebra $\overline{\mathbb{Z}}_\ell[\mathbf{I}\backslash\mathbf{G}/\mathbf{I}]$, and that the algebra E_1 is a direct factor of the algebra $\mathrm{End}_{\overline{\mathbb{Z}}_\ell \mathbf{G}}(\mathcal{W}_0)$ in the diagram $(*)$ (tensored by $\overline{\mathbb{Z}}_\ell$). In this work, we will establish the following proposition:

Proposition. — *We are in the case of* $\mathbf{G} = \mathrm{GL}_2(\mathbf{F})$ *(with* $\mathcal{G} = \mathrm{GL}_2$*).*

(a) (§ 4.19) *The algebra* $\mathrm{End}_{\overline{\mathbb{Z}}[\frac{1}{p}]\mathbf{G}}(\mathcal{W}_0)$ *is filtered, in the way that it is the limit (the union) of an ascending chain of its* $\overline{\mathbb{Z}}_\ell$*-submodules* $V^{(0)} \subset V^{(1)} \subset V^{(2)} \subset \cdots$ *satisfying* $V^{(d)} \cdot V^{(d')} \subset V^{(d+d')}$ *for all* $d, d' \geq 0$*. Moreover, we can explicitly find a subset* $\{\Theta_\lambda : \lambda \in \mathbb{F}_q^\times\}$ *of* $V^{(1)} - V^{(0)}$ *such that, upon identifying*

$$\varpi = e_\psi(\varpi \mathrm{id}_2) e_\psi \in e_\psi.\overline{\mathbb{Z}}[\tfrac{1}{p}]\mathbf{G}.e_\psi = \mathrm{End}_{\overline{\mathbb{Z}}[\frac{1}{p}]\mathbf{G}}(\mathcal{W}_0),$$

we have

$$\mathrm{End}_{\overline{\mathbb{Z}}[\frac{1}{p}]\mathbf{G}}(\mathcal{W}_0) = (\overline{\mathbb{Z}}[\tfrac{1}{p}]\mathsf{E}_G)[\varpi^{\pm 1}, \{\Theta_\lambda : \lambda \in \mathbb{F}_q^\times\}].$$

We suppose from now on that the order of q *in* $(\mathbb{Z}/\ell\mathbb{Z})^\times$ *is equal to 2; it is thus appropriate to consider the previous decomposition of* $\mathrm{End}(\mathcal{P})$*.*

(b) (§ 4.22) *We can explicitly construct an* $s_{\mathsf{M}_1} \in \mathsf{M}_1$ *and an* $s_{\mathsf{N}_1}^* \in \mathsf{N}_1$ *such that* $\mathsf{M}_1 = \mathsf{H}.s_{\mathsf{M}_1}$ *and* $\mathsf{N}_1 = s_{\mathsf{N}_1}^*.\mathsf{H}$*.*

(c) ([Pa] or § 4.13) *We have* $e_{[1]}^{G,\ell}.\overline{\mathbb{Z}}_\ell \mathsf{E}_G = \overline{\mathbb{Z}}_\ell[\theta]$ *where* θ *is the component in* $e_{[1]}^{G,\ell}.\overline{\mathbb{Z}}_\ell \mathsf{E}_G$ *of the element of* $\overline{\mathbb{Z}}_\ell \mathsf{E}_G$ *corresponding to the trace function on* $(T^\vee /\!\!/ W)^{F^\vee}$ *under the isomorphism* $\overline{\mathbb{Z}}_\ell \mathsf{E}_G \simeq \overline{\mathbb{Z}}_\ell \mathsf{B}_{G^\vee}$ *given by the preceding theorem.*

(d) (§ 4.23; see also [Hel1, Prop. 9.9]) *Upon denoting by* ϖ_1 *(resp.* $\Theta_{\lambda,1}$*) the component in* E_1 *of the element* ϖ *(resp.* Θ_λ*) of* $\mathrm{End}_{\overline{\mathbb{Z}}_\ell \mathbf{G}}(\mathcal{W}_0)$*, we have* $\Theta_{\lambda,1} = \Theta_{1,1}$ *for every* $\lambda \in \mathbb{F}_q^\times$*, and we thus have*

$$\mathsf{E}_1 = (e_{[1]}^{G,\ell}.\overline{\mathbb{Z}}_\ell \mathsf{E}_G)[\varpi_1^{\pm 1}, \{\Theta_{\lambda,1} : \lambda \in \mathbb{F}_q^\times\}] = \overline{\mathbb{Z}}_\ell[\theta, \varpi_1^{\pm 1}, \Theta_{1,1}].$$

(e) (§§ 4.21 - 4.24) *We can explicitly calculate the products among* s_{M_1}*,* $s_{\mathsf{N}_1}^*$*, the standard generators of* H *and the generators of* E_1 *in* (d)*.*

We thus obtain a description of $\mathrm{End}(\mathcal{P})$ and hence of the unipotent ℓ-block of the p-adic group $\mathbf{G} = \mathrm{GL}_2(\mathbf{F})$; we hope that this description may open a way toward a structured description of unipotent ℓ-blocks of p-adic general linear groups.

The organisation of this work is as follows. In § 1, we review Bonnafé–Kessar's studies in [BoKe] on the algebra $\overline{\mathbb{Z}}[\frac{1}{p}]\mathsf{E}_G$, and we introduce in particular the Curtis homomorphism and the "standard" symmetrizing form on $\overline{\mathbb{Z}}[\frac{1}{p}]\mathsf{E}_G$. In addition to the review of Bonnafé–Kessar's work, we also construct a $\mathbb{Z}$-model E_G for $\overline{\mathbb{Z}}[\frac{1}{p}]\mathsf{E}_G$, and we analyse, as a preparation for the study of the relation between $\overline{\mathbb{Z}}[\frac{1}{p}]\mathsf{E}_G$ and K_{G^*} in § 2, the behaviour of $\overline{\mathbb{Z}}[\frac{1}{p}]\mathsf{E}_G$ under a central extension of G^*.

In § 2, we study the ring K_{G^*}, explain how the symmetrizing form on $\overline{\mathbb{Z}}[\frac{1}{p}]\mathsf{E}_G$ considered in § 1 induces a symmetrizing form on $\overline{\mathbb{Z}}[\frac{1}{p}]\mathsf{K}_{G^*}$, and then we establish two versions of identifications of the $\overline{\mathbb{Z}}[\frac{1}{p}]$-algebras $\Lambda\mathsf{E}_G$ and $\Lambda\mathsf{K}_{G^*}$ for $\Lambda \supset \overline{\mathbb{Z}}[\frac{1}{p}]$, the first version needing the inversion of $|W|$ in Λ, while the second version only needing the inversion of bad prime numbers for G in Λ. To examine this second version, we also review the representation theory of algebraic groups.

In § 3, we will deal with the ring $\mathsf{B}_{G^\vee}$. After preliminary preparations on the combinatorics of root datum of G, we show that $\mathsf{B}_{G^\vee}$ is a ring which is reduced, without torsion and isomorphic to K_{G^*} when the derived subgroup of $G^\vee$ is simply-connected. Consequently, when the derived subgroup of $G^\vee$ is simply-connected and Λ is a ring containing $\overline{\mathbb{Z}}[\frac{1}{p}]$ and the multiplicative inverses of bad prime numbers for G, we obtain a Λ-algebra isomorphism $\Lambda\mathsf{E}_G \simeq \Lambda\mathsf{B}_{G^\vee}$. We also include a discussion on the direct proof of the $\overline{\mathbb{Z}}[\frac{1}{p}]$-algebra isomorphism $\overline{\mathbb{Z}}[\frac{1}{p}]\mathsf{E}_G \simeq \overline{\mathbb{Z}}[\frac{1}{p}]\mathsf{B}_{G^\vee}$ for $G = \mathrm{GL}_2$.

In § 4, we first use the isomorphism $\Lambda\mathsf{E}_G \simeq \Lambda\mathsf{B}_{G^\vee}$ to generalise Paige's work [Pa] on the endomorphism ring of some projective cover. We then concentrate on the case of the p-adic group $\mathbf{G} = \mathrm{GL}_2(\mathbf{F})$: in this case, we analyse the structure of the endomorphism algebra of the depth-zero part of the Whittaker model of $\mathbf{G}$, a structure which we use, along with the Paige's result we just studied, to finally describe the unipotent ℓ-block of $\mathbf{G}$ with $\ell \neq p$ under the condition that $\mathrm{ord}_\ell(q) = 2$.

In the appendix § A, we study the structure of $\mathsf{B}_{G^\vee}$ in the case of $G^\vee = \mathrm{SO}_{2n}$ (whose derived subgroup is not simply-connected), where we show in particular that $\mathsf{B}_{G^\vee}$ is still a reduced ring without torsion.

The contents of § 1, § 2, § 3 and § A are mainly integrated from my article [Li2] and my other article [LiSh] in collaboration with J. Shotton; in the present work, I will thus not systematically cite these two articles.

1. Endomorphism algebras of Gelfand–Graev representations

Deligne–Lusztig dual groups

1.1. *Notation from finite group representation theory.* [Se1] — Let G be a finite group and let Λ be a ring. The group ring of G over Λ is denoted by $\Lambda\mathsf{G}$ or $\Lambda[\mathsf{G}]$; an element f of $\Lambda\mathsf{G}$ is thus a function $f : \mathsf{G} \longrightarrow \Lambda$, or equivalently a formal sum $f = \sum_{g\in\mathsf{G}} f(g)g$. The category of (left) $\Lambda\mathsf{G}$-modules of finite type over Λ is denoted by $\mathrm{Rep}_\Lambda(\mathsf{G})$. The set of isomorphism classes of objects of $\mathrm{Rep}_\Lambda(\mathsf{G})$ is denoted by $\mathrm{Irr}_\Lambda(\mathsf{G})$. For $\rho, \rho' \in \mathrm{Rep}_{\overline{\mathbb{Q}}}(\mathsf{G})$ with characters χ and χ' respectively, we also consider the scalar product

$$\langle \rho, \rho' \rangle_{\mathsf{G}} = \langle \chi, \chi' \rangle_{\mathsf{G}} = \dim_{\overline{\mathbb{Q}}} \mathrm{Hom}_{\overline{\mathbb{Q}}\mathsf{G}}(\rho, \rho') = \frac{1}{|\mathsf{G}|} \sum_{g\in\mathsf{G}} \chi(g^{-1})\chi'(g) \in \mathbb{Z}.$$

1.2. *Reductive groups.* [Sp3][DiMi] — Let G be a connected reductive group defined over a finite field $\mathbb{F}_q$ of characteristic p, and let F be the associated Frobenius endomorphism on G. We identify algebraic groups defined over $\overline{\mathbb{F}}_q$ with their $\overline{\mathbb{F}}_q$-points, so $G = G(\overline{\mathbb{F}}_q)$, and the F-fixed points of G form a finite group $G^F = G(\mathbb{F}_q)$.

We fix from now on a choice of F-stable Borel subgroup B of G, and consider its Levi decomposition $B = TU$ with T a Levi subgroup and U the unipotent radical; then U is F-stable, and T is an F-stable maximal torus of G.

Let S be an F-stable maximal torus of G, so that the Weyl group of (G, S) is $W_G(S) = N_G(S)/S$; we will often write $W_G(T)$ simply as W. By Lang–Steinberg's theorem, the set of G^F-conjugacy classes of F-stable maximal tori of G are in bijection with the set of F-conjugacy classes in $W_G(S)$; for $w \in W_G(S)$, we may thus choose a corresponding F-stable maximal torus S_w of G, in the way that $S_w = \mathrm{ad}(g)(S)$ with $g \in G$ and $g^{-1}F(g)S = wS \in W_G(S)$ — here $\mathrm{ad}(g)$ is the adjoint action given by $\mathrm{ad}(g)(x) = {}^g x = gxg^{-1}$.

1.3. *Deligne–Lusztig duals.* [DeLu][DiMi] — For our data (G, F), let us recall how to define its Deligne–Lusztig dual data (G^*, F^*).

It is convenient to start from an F-stable maximal torus S of G. To the pair (G, S) we associate a root datum $(X(S), Y(S), R(G, S), R^\vee(G, S))$, where $X(S) = \mathrm{Hom}_{\mathrm{alg}}(S, \mathbb{G}_m)$ is the group of characters, $Y(S) = \mathrm{Hom}_{\mathrm{alg}}(\mathbb{G}_m, S)$ is the group of cocharacters, $R(G, S) \subset X(S)$ is the set of roots and $R^\vee(G, S) \subset Y(S)$ is the set of coroots; the F-action on (G, S) induces an F-action on $(X(S), Y(S), R(G, S), R^\vee(G, S))$.

By classification of algebraic groups, there exists a reductive group G^* and a maximal torus S^* of G^* such that

$$(Y(S^*), X(S^*), R^\vee(G^*, S^*), R(G^*, S^*)) = (X(S), Y(S), R(G, S), R^\vee(G, S)).$$

This identification of root data induces a Frobenius endomorphism F^* on (G^*, S^*), so that G^* is defined over $\mathbb{F}_q$ and S^* is an F^*-stable maximal torus of G^*. The triple (G^*, S^*, F^*) is determined by (G, S, F) up to isomorphism, and this construction may be made independent of S, so (G^*, F^*) is determined by (G, F) up to isomorphism.

We fix hereafter a choice of (G^*, F^*). The set of semisimple elements of G^* will be denoted by G^*_{ss}, and the set of semisimple conjugacy classes in G^{*F^*} will be denoted by $G^{*F^*}_{\mathrm{ss}}/\sim$. Also, the Weyl group $W_{G^*}(T^*)$ will often be identified with $W = W_G(T)$.

1.4. *Duality of tori.* [DeLu][DiMi] — Henceforth, fix a choice of group isomorphism $(\mathbb{Q}/\mathbb{Z})_{p'} \simeq \overline{\mathbb{F}}_q^\times$ (choices of roots of unity) and a choice of embedding of multiplicative groups $\overline{\mathbb{F}}_q^\times \hookrightarrow \overline{\mathbb{Q}}^\times$. Then, given an F-stable maximal torus S of G we may consider its dual S^*, which is an F^*-stable maximal torus of G^* (see § 1.3), and then we have the toric duality

$$\mathrm{Irr}_{\overline{\mathbb{Q}}}(S^F) \simeq \mathrm{Irr}_{\overline{\mathbb{F}}_q}(S^F) \simeq S^{*F^*}.$$

Indeed, the first isomorphism is given by our choice of embedding $\overline{\mathbb{F}}_q^\times \hookrightarrow \overline{\mathbb{Q}}^\times$, and the second is determined by the following commutative diagram, where the two horizontal sequences are exact:

$$\begin{tikzcd}
0 \arrow[r] & X(S) \arrow[r, "F-1"] \arrow[d, equal] & X(S) \arrow[r, "\mathrm{Res}^S_{S^F}"] \arrow[d, equal] & \mathrm{Irr}_{\overline{\mathbb{F}}_q}(S^F) \arrow[r] \arrow[d, "\wr"] & 1 \\
0 \arrow[r] & Y(S^*) \arrow[r, "F^*-1"] & Y(S^*) \arrow[r, "N"] & S^{*F^*} \arrow[r] & 1
\end{tikzcd}$$

In the above diagram, $N : Y(S^*) \longrightarrow S^{*F^*}$ is defined as follows: choose a $d \in \mathbb{N}^*$ such that S^* is split over $\mathbb{F}_{q^d}$, set $\zeta := \iota^{-1}(\frac{1}{q^d-1}) \in \overline{\mathbb{F}}_q^\times$ (a primitive (q^d-1)-st root of unity), and consider the norm map $N_{F^{*d}/F^*} : S^* \longrightarrow S^*$ defined for every $z \in S^*$ by $N_{F^{*d}/F^*}(z) = z.F^*(z).\cdots.(F^*)^{d-1}(z)$; then $N(y) := N_{F^{*d}/F^*}(y(\zeta))$ for every $y \in Y(S^*)$.

1.5. *Deligne–Lusztig characters.* [DeLu][DiMi] — Let S be an F-stable maximal torus of G, let P be a Borel subgroup containing S, and let V be the unipotent radical of P. Then we have the Deligne–Lusztig variety

$$DL^G_{S\subset P} = \{gV \in G/V : g^{-1}F(g) \in V \cdot F(V)\},$$

which admits a (left) $G^F \times (S^F)^{\mathrm{op}}$-action. When there is no need to specify the chosen Borel subgroup P, we will write $DL^G_{S\subset P}$ simply as DL^G_S.

We consider the virtual ℓ-adic cohomology $H^*_c(\cdot) = \sum_{j\geq 0}(-1)^j H^j_c(\cdot, \overline{\mathbb{Q}}_\ell)$, for ℓ a prime distinct from p. For every character $\chi : S^F \longrightarrow \overline{\mathbb{Q}}^\times$, upon choosing a field embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_\ell$, we have the corresponding Deligne–Lusztig character

$$R^G_S(\chi)(-) := \mathrm{Tr}(-|H^*_c(DL^G_{S\subset P}) \otimes_{\overline{\mathbb{Q}}_\ell S^F} \chi) = \frac{1}{|S^F|}\sum_{s\in S^F} \mathrm{Tr}((-,s)|H^*_c(DL^G_{S\subset P}))\chi(s^{-1}),$$

which is independent of the choice of P and which takes values in $\overline{\mathbb{Q}}_\ell$ *a priori*; but by [DeLu, Prop. 3.3], for any $(g,s) \in G^F \times S^F$, the trace $\mathrm{Tr}((g,s)|H^*_c(DL^G_{S\subset P}))$ is an integer independent of ℓ, so in fact $R^G_S(\chi)$ takes values in $\overline{\mathbb{Q}}$, and it can be verified that $R^G_S(\chi)$ is independent of the choices of ℓ and of the embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_\ell$.

The endomorphism algebra $\overline{\mathbb{Z}}[\frac{1}{p}]\mathsf{E}_{G,\psi}$

1.6. *Regular linear characters.* [DLM, Sec. 2] — Recall the F-stable Levi decomposition $B = TU$ in § 1.2. Let R^+ (resp. Δ) be the set of positive roots (resp. of simple roots) associated to (G,T) (resp. to (G,T,B)). Denote by $U_\bullet$ the subgroup of U generated by the root subgroups of non-simple positive roots (roots in $R^+ - \Delta$); note that $U_\bullet = [U,U]$. Let Δ/F be the set of F-orbits in Δ. For each $i \in \Delta/F$, let U_i be the product of root subgroups of (simple) roots in i. The quotient group $U/U_\bullet$ is canonically isomorphic to the product group $\prod_{i\in\Delta/F} U_i$, and this isomorphism is F-stable, so that $U^F/U_\bullet^F \simeq \prod_{i\in\Delta/F} U_i^F$ as abelian groups. Then a linear character ψ on U^F is called regular if ψ is trivial $(=1)$ on $U_\bullet^F$ and is non-trivial on every U_i^F $(i \in \Delta/F)$.

Let Ψ be the set of $\overline{\mathbb{Z}}^\times$-valued regular linear characters of U^F; the group T^F acts on Ψ by adjoint action (for $t \in T^F$ and $\psi \in \Psi$, ${}^t\psi := \psi({}^{t^{-1}}(\cdot)) = \psi(t^{-1}(\cdot)t) \in \Psi$). Let Z be the centre of G, so the adjoint group of G is $G_{\mathrm{ad}} = G/Z$. Let $T_{\mathrm{ad}} = T/Z$ be the image of T in G_{ad}; the group T_{ad}^F also acts on Ψ by adjoint action (for $x \in T_{\mathrm{ad}}^F$ and $\psi \in \Psi$, choose a $t \in T$ such that $x = tZ$, and set ${}^x\psi := {}^t\psi$), and this T_{ad}^F-action on Ψ is regular (= free and transitive). Hence the first Galois cohomology group $H^1(F,Z)$, identified with T_{ad}^F/T^F, acts regularly on Ψ/T^F (the set of T^F-orbits in Ψ) by adjoint action.

1.7. *Gelfand–Graev representations.* [DiMi, Sec. 12.3] — Let $\psi : U^F \longrightarrow \overline{\mathbb{Z}}^\times$ be a regular linear character (so $\psi \in \Psi$ in the notation of § 1.6), and consider the primitive central idempotent

$$e_\psi := \frac{1}{|U^F|} \sum_{u\in U^F} \psi(u^{-1})u \in \overline{\mathbb{Z}}[\tfrac{1}{p}]U^F \tag{1.7.1}$$

of ψ ($|U^F|$ is a power of p). Let $(\overline{\mathbb{Z}}[\frac{1}{p}])_\psi$ be the $\overline{\mathbb{Z}}[\frac{1}{p}]U^F$-module $\overline{\mathbb{Z}}[\frac{1}{p}]$ on which U^F acts by ψ; we have $(\overline{\mathbb{Z}}[\frac{1}{p}])_\psi \simeq \overline{\mathbb{Z}}[\frac{1}{p}]U^F e_\psi = \overline{\mathbb{Z}}[\frac{1}{p}]e_\psi$ as $\overline{\mathbb{Z}}[\frac{1}{p}]U^F$-modules. Then

$$\Gamma_{G,\psi} := \mathrm{Ind}_{U^F}^{G^F}((\overline{\mathbb{Z}}[\tfrac{1}{p}])_\psi) \simeq \overline{\mathbb{Z}}[\tfrac{1}{p}]G^F e_\psi \in \mathrm{Rep}_{\overline{\mathbb{Z}}[\frac{1}{p}]}(G^F) \tag{1.7.2}$$

is called a Gelfand–Graev representation of G; the character of $\Gamma_{G,\psi}$ is $\mathrm{Ind}_{U^F}^{G^F}\psi$.

All $\Gamma_{G,\psi}$ $(\psi \in \Psi)$ are conjugate by elements of T_{ad}^F: indeed, for any $\psi, \psi' \in \Psi$, there is a unique $x \in T_{\mathrm{ad}}^F$ such that $\psi' = {}^x\psi$ (§ 1.6); then $e_{\psi'} = {}^x(e_\psi)$, and thus, upon identifying $\Gamma_{G,\psi} = \overline{\mathbb{Z}}[\frac{1}{p}]G^F e_\psi$ via (1.7.2),

$$\Gamma_{G,\psi'} = \overline{\mathbb{Z}}[\tfrac{1}{p}]G^F e_{\psi'} = {}^x(\overline{\mathbb{Z}}[\tfrac{1}{p}]G^F e_\psi) = {}^x(\Gamma_{G,\psi}). \tag{1.7.3}$$

If the centre Z of G is connected (that is, if $H^1(F,Z) = 0$), then $T_{\mathrm{ad}}^F = T^F/Z^F$, so all $\Gamma_{G,\psi}$ are conjugate by elements of T^F and are thus all isomorphic in $\mathrm{Rep}_{\overline{\mathbb{Z}}[\frac{1}{p}]}(G^F)$.

1.8. Theorem. [DeLu, Thm. 10.7][DiMi, Thm. 12.4.12] — *Let $\psi \in \Psi$. Then the $\overline{\mathbb{Q}}G^F$-module $\overline{\mathbb{Q}}\Gamma_{G,\psi}$ is multiplicity-free; more precisely, we have a $\overline{\mathbb{Q}}G^F$-module decomposition*

$$\overline{\mathbb{Q}}\Gamma_{G,\psi} = \bigoplus_{[x]\in G_{\mathrm{ss}}^{*F^*}/\sim} \rho_{\psi,[x]}$$

where for each $[x] \in G_{\mathrm{ss}}^{*F^*}/\sim$, $\rho_{\psi,[x]}$ *is an irreducible* $\overline{\mathbb{Q}}G^F$*-module whose character* $\chi_{\psi,[x]}$ *lies in the rational Lusztig series* $\mathcal{E}(G^F,[x])$.

The first reference deals with the case where the centre of G is connected, and the second reference deals with the general case.

1.9. *The endomorphism algebras* $\overline{\mathbb{Z}}[\frac{1}{p}]\mathsf{E}_{G,\psi}$ *and* $\overline{\mathbb{Q}}\mathsf{E}_{G,\psi}$. — By § 1.7, all endomorphism algebras

$$\overline{\mathbb{Z}}[\tfrac{1}{p}]\mathsf{E}_{G,\psi} := \mathrm{End}_{\overline{\mathbb{Z}}[\frac{1}{p}]G^F}(\Gamma_{G,\psi}) \quad (\psi \in \Psi)$$

are isomorphic as $\overline{\mathbb{Z}}[\frac{1}{p}]$-algebras. Let $\overline{\mathbb{Q}}\mathsf{E}_{G,\psi} := \overline{\mathbb{Q}} \otimes_{\overline{\mathbb{Z}}[\frac{1}{p}]} \overline{\mathbb{Z}}[\frac{1}{p}]\mathsf{E}_{G,\psi}$. By § 1.8 and Schur's lemma, we may decompose the $\overline{\mathbb{Q}}$-algebra $\overline{\mathbb{Q}}\mathsf{E}_{G,\psi}$ as:

$$\overline{\mathbb{Q}}\mathsf{E}_{G,\psi} = \mathrm{End}_{\overline{\mathbb{Q}}G^F}(\overline{\mathbb{Q}}\Gamma_{G,\psi}) \simeq \prod_{[x]\in G_{\mathrm{ss}}^{*F^*}/\sim} \mathrm{End}_{\overline{\mathbb{Q}}G^F}(\rho_{\psi,[x]}) \simeq \overline{\mathbb{Q}}^{G_{\mathrm{ss}}^{*F^*}/\sim}. \tag{1.9.1}$$

Thus $\overline{\mathbb{Q}}\mathsf{E}_{G,\psi}$ and $\overline{\mathbb{Z}}[\frac{1}{p}]\mathsf{E}_{G,\psi}$ are commutative reduced rings. In terms of algebraic varieties, (1.9.1) identifies $\mathrm{Specm}(\overline{\mathbb{Q}}\mathsf{E}_G)$ (consisting of maximal ideals of $\overline{\mathbb{Q}}\mathsf{E}_G$) with $G_{\mathrm{ss}}^{*F^*}/\sim$.

1.10. *Structures of* $\overline{\mathbb{Z}}[\frac{1}{p}]\mathsf{E}_{G,\psi}$ *and* $\overline{\mathbb{Q}}\mathsf{E}_{G,\psi}$ *via idempotents.* — Using the identification $\Gamma_{G,\psi} = \overline{\mathbb{Z}}[\frac{1}{p}]G^F e_\psi$ in (1.7.2), we shall identify

$$\overline{\mathbb{Z}}[\tfrac{1}{p}]\mathsf{E}_{G,\psi} = e_\psi \overline{\mathbb{Z}}[\tfrac{1}{p}]G^F e_\psi \subset \overline{\mathbb{Z}}[\tfrac{1}{p}]G^F$$

as $\overline{\mathbb{Z}}[\frac{1}{p}]$-algebras: indeed, there is a $\overline{\mathbb{Z}}[\frac{1}{p}]$-algebra anti-isomorphism

$$\overline{\mathbb{Z}}[\tfrac{1}{p}]\mathsf{E}_{G,\psi} = \mathrm{End}_{\overline{\mathbb{Z}}[\frac{1}{p}]G^F}(\overline{\mathbb{Z}}[\tfrac{1}{p}]G^F e_\psi) \xrightarrow{\sim} e_\psi \overline{\mathbb{Z}}[\tfrac{1}{p}]G^F e_\psi, \quad \theta \longmapsto \theta(e_\psi), \tag{1.10.1}$$

but $\overline{\mathbb{Z}}[\frac{1}{p}]\mathsf{E}_{G,\psi}$ is a commutative ring (§ 1.9), so (1.10.1) is also an isomorphism.

(a) By [CuRe, Prop. 11.30], $\overline{\mathbb{Z}}[\frac{1}{p}]\mathsf{E}_{G,\psi} = e_\psi \overline{\mathbb{Z}}[\frac{1}{p}]G^F e_\psi$ is a free $\overline{\mathbb{Z}}[\frac{1}{p}]$-module with the following $\overline{\mathbb{Z}}[\frac{1}{p}]$-linear basis: let $\{x_1, \cdots, x_r\} \subset G^F$ be a set of representatives of $U^F \backslash G^F / U^F$ (so G^F is the disjoint union of $\{U^F x_i U^F : i = 1, \cdots, r\}$), and let

$$J = \{j : 1 \le j \le r,\ {}^{x_j}\psi = \psi \text{ on } U^F \cap {}^{x_j}(U^F)\};$$

then $\{e_\psi x_j e_\psi : j \in J\}$ is a $\overline{\mathbb{Z}}[\frac{1}{p}]$-linear basis of $\overline{\mathbb{Z}}[\frac{1}{p}]\mathsf{E}_{G,\psi}$.

(b) By [Cu, Sec. 3], $\overline{\mathbb{Q}}\mathsf{E}_{G,\psi}$ may be described via idempotents: for $[x] \in G_{\mathrm{ss}}^{*F^*}/\sim$, set

$$e^G_{\psi,[x]} := \frac{\chi_{\psi,[x]}(1)}{|G^F|} \sum_{g\in G^F} \chi_{\psi,[x]}(g^{-1})g \in \overline{\mathbb{Q}}G^F \quad \text{and} \quad e^{\mathsf{E}}_{\psi,[x]} := e^G_{\psi,[x]} e_\psi \in \overline{\mathbb{Q}}\mathsf{E}_{G,\psi},$$

which are respectively primitive central idempotents of $\overline{\mathbb{Q}}G^F$ and of $\overline{\mathbb{Q}}\mathsf{E}_{G,\psi}$; then the inclusions $\overline{\mathbb{Q}}e^{\mathsf{E}}_{\psi,[x]} \subset \overline{\mathbb{Q}}\mathsf{E}_{G,\psi}$ induce a $\overline{\mathbb{Q}}$-algebra isomorphism

$$\overline{\mathbb{Q}}\mathsf{E}_{G,\psi} = \prod_{[x]\in G_{\mathrm{ss}}^{*F^*}/\sim} \overline{\mathbb{Q}}e^{\mathsf{E}}_{\psi,[x]}. \tag{1.10.2}$$

Combining (1.9.1) and (1.10.2), we obtain $\overline{\mathbb{Q}}$-algebra isomorphisms

$$\overline{\mathbb{Q}}\mathsf{E}_{G,\psi} \simeq \prod_{[x]\in G_{\mathrm{ss}}^{*F^*}/\sim} \overline{\mathbb{Q}}e^{\mathsf{E}}_{\psi,[x]} \simeq \overline{\mathbb{Q}}^{G_{\mathrm{ss}}^{*F^*}/\sim}, \tag{1.10.3}$$

where each $e^{\mathsf{E}}_{\psi,[x]}$ corresponds to the characteristic function $\mathbf{1}_{\{[x]\}}$ on $G_{\mathrm{ss}}^{*F^*}/\sim$.

Curtis homomorphisms and symmetrizing forms

1.11. *The Curtis homomorphism* Cur_S^G. [Cu, Sec. 4] — Hereafter, fix a $\psi \in \Psi$. For an F-stable maximal torus S of G, Curtis has constructed a $\overline{\mathbb{Q}}$-algebra homomorphism

$$\mathrm{Cur}_S^G : \overline{\mathbb{Q}}\mathsf{E}_{G,\psi} \longrightarrow \overline{\mathbb{Q}}S^F$$

(which he called f_S in his paper), which is the unique $\overline{\mathbb{Q}}$-algebra homomorphism from $\overline{\mathbb{Q}}\mathsf{E}_{G,\psi}$ to $\overline{\mathbb{Q}}S^F$ such that, for every irreducible character $\theta \in \mathrm{Irr}_{\overline{\mathbb{Q}}}(S^F)$ corresponding to $[x] \in G_{\mathrm{ss}}^{*F^*}/\sim$ under the duality $\mathrm{Irr}_{\overline{\mathbb{Q}}}(S^F) \simeq S^{*F^*}$ (§ 1.4), the following diagram of $\overline{\mathbb{Q}}$-algebras commutes:

$$\begin{tikzcd} \overline{\mathbb{Q}}\mathsf{E}_{G,\psi} \arrow[r, "\chi_{\psi,[x]}|_{\overline{\mathbb{Q}}\mathsf{E}_{G,\psi}}"] \arrow[d, "\mathrm{Cur}_S^G"'] & \overline{\mathbb{Q}} \\ \overline{\mathbb{Q}}S^F \arrow[ur, "\theta"'] & \end{tikzcd} \tag{1.11.1}$$

The homomorphism Cur_S^G is determined by § 1.10(a) and by the following formula:

Curtis' formula. For all $n \in G^F$, $\mathrm{Cur}_S^G(e_\psi n e_\psi) = \sum_{s \in S^F} \mathrm{Cur}_S^G(e_\psi n e_\psi)(s)s \in \overline{\mathbb{Q}}S^F$ with

$$\mathrm{Cur}_S^G(e_\psi n e_\psi)(s) = \frac{1}{\langle Q_S^G, \Gamma_{G,\psi}\rangle_{G^F}} \frac{1}{|U^F|} \frac{1}{|(C_G(s)^\circ)^F|} \sum_{\substack{g \in G^F,\, u \in U^F \\ (gung^{-1})_{\mathrm{ss}} = s}} \psi(u^{-1}) Q_S^{C_G(s)^\circ}((gung^{-1})_{\mathrm{u}}).$$

In the above formula, $Q_S^G : G^F \longrightarrow \mathbb{Z}$ is the Green function, which is the function supported on the set of unipotent elements of G^F defined by $Q_S^G(u) = R_S^G(\mathbf{1})(u)$ for unipotent elements u of G^F (see [DeLu, Def. 4.1]); $C_G(s)^\circ$ means the connected identity component of the centralizer of s in G; for $g \in G^F$, g_{ss} (resp. g_{u}) denotes its semisimple part (resp. unipotent part).

1.12. Lemma. — *Let S be an F-stable maximal torus of G. Then Cur_S^G is defined over $\overline{\mathbb{Z}}[\frac{1}{p}]$ in the sense that $\mathrm{Cur}_S^G(\overline{\mathbb{Z}}[\frac{1}{p}]\mathsf{E}_{G,\psi}) \subset \overline{\mathbb{Z}}[\frac{1}{p}]S^F$.*

Proof. Using the $\overline{\mathbb{Z}}[\frac{1}{p}]$-linear basis of $\overline{\mathbb{Z}}[\frac{1}{p}]\mathsf{E}_{G,\psi}$ in § 1.10(a), it suffices to show that $\mathrm{Cur}_S^G(e_\psi n e_\psi) \in \overline{\mathbb{Z}}[\frac{1}{p}]S^F$ for every $n \in G^F$. In Curtis' formula for $\mathrm{Cur}_S^G(e_\psi n e_\psi)$ (§ 1.11), observe that $\langle Q_S^G, \Gamma_{G,\psi}\rangle_{G^F} = \langle R_S^G(\mathbf{1}), \Gamma_{G,\psi}\rangle_{G^F} = \pm 1$ (§ 1.8) and that $|U^F|$ is a power of p, so to prove this lemma it remains to show that the number

$$C(s) := \frac{1}{|(C_G(s)^\circ)^F|} \sum_{\substack{g \in G^F,\, u \in U^F \\ (gung^{-1})_{\mathrm{ss}} = s}} \psi(u^{-1}) Q_S^{C_G(s)^\circ}((gung^{-1})_{\mathrm{u}})$$

lies in $\overline{\mathbb{Z}}[\frac{1}{p}]$ for all $s \in S^F$. If $(gung^{-1})_{\mathrm{ss}} = s$ ($g, n \in G^F$, $u \in U^F$), for every $c \in C_G(s)^\circ$ we have $((cg)un(cg)^{-1})_{\mathrm{ss}} = s$ and $((cg)un(cg)^{-1})_{\mathrm{u}} = c.(gung^{-1})_{\mathrm{u}}.c^{-1}$, so

$$C(s) = \sum_{\substack{g \in (C_G(s)^\circ)^F \backslash G^F,\, u \in U^F \\ (gung^{-1})_{\mathrm{ss}} = s}} \psi(u^{-1}) Q_S^{C_G(s)^\circ}((gung^{-1})_{\mathrm{u}}) \in \overline{\mathbb{Z}}[\tfrac{1}{p}]. \qquad \square$$

1.13. Lemma. [BoKe, Prop. 2.1] — *Let S be an F-stable maximal torus of G. For $z \in S^{*F^*}$, let $\widehat{z} : S^F \longrightarrow \overline{\mathbb{Q}}$ be its corresponding irreducible character of S^F obtained from the duality $S^{*F^*} \simeq \mathrm{Irr}_{\overline{\mathbb{Q}}}(S^F)$, and let $e_{\widehat{z}}^S := \dfrac{1}{|S^F|} \sum_{t \in S^F} \widehat{z}(t^{-1})t \in \overline{\mathbb{Q}}S^F$ be the primitive central idempotent associated to $\widehat{z}$. Then*

$$\mathrm{Cur}_S^G(e_{\psi,[x]}^{\mathsf{E}}) = \sum_{z \in [x] \cap S^{*F^*}} e_{\widehat{z}}^S \in \overline{\mathbb{Q}}S^F \quad \textit{for every } [x] \in G_{\mathrm{ss}}^{*F^*}/\sim . \tag{1.13.1}$$

As a consequence, Cur_S^G may be interpreted as a restriction map: using the identification $\overline{\mathbb{Q}}\mathsf{E}_{G,\psi} \simeq \overline{\mathbb{Q}}^{G_{\mathrm{ss}}^{*F^*}/\sim}$ (§ 1.10(b)) as well as a similar identification $\overline{\mathbb{Q}}S^F = \overline{\mathbb{Q}}^{S^{*F^*}}$ (such that $e_{\widehat{z}}^S$ corresponds to the characteristic function $\mathbf{1}_{\{z\}}$ on S^{*F^*}), we have the following commutative diagram of $\overline{\mathbb{Q}}$-algebras: ("Res" means the restriction map)

$$\begin{tikzcd} \overline{\mathbb{Q}}\mathsf{E}_{G,\psi} \arrow[r, leftrightarrow, "\sim"] \arrow[d, "\mathrm{Cur}_S^G"'] & \overline{\mathbb{Q}}^{G_{\mathrm{ss}}^{*F^*}/\sim} \arrow[d, "\mathrm{Res}"] \\ \overline{\mathbb{Q}}S^F \arrow[r, leftrightarrow, "\sim"] & \overline{\mathbb{Q}}^{S^{*F^*}} \end{tikzcd} \tag{1.13.2}$$

The formula (1.13.1), or equivalently the commutativity of (1.13.2), gives an alternative definition of Cur_S^G.

1.14. Corollary (of § 1.13; compare [BoKe, Prop. 3.2]). — *Let $W = W_G(T)$ (with T as in § 1.2). If for each $w \in W$ we choose an F-stable maximal torus T_w of G whose G^F-conjugacy class corresponds to the F-conjugacy class of w (§ 1.2), then the product map*

$$\mathrm{Cur}^G := (\mathrm{Cur}_{T_w}^G)_{w \in W} : \overline{\mathbb{Q}}\mathsf{E}_{G,\psi} \longrightarrow \prod_{w \in W} \overline{\mathbb{Q}}T_w^F$$

is an injective $\overline{\mathbb{Q}}$-algebra homomorphism.

1.15. Proposition. [BoKe, Prop. 2.5 revised] — *Let S be an F-stable maximal torus of G. Then for every $h \in \overline{\mathbb{Q}}\mathsf{E}_{G,\psi} \subset \overline{\mathbb{Q}}G^F$ (see § 1.10), we have*

$$\mathrm{Cur}_S^G(h) = \frac{\epsilon_G \epsilon_S}{|S^F|} \sum_{s \in S^F} \mathrm{Tr}((h,s)|H_c^*(DL_{S \subset P}^G))s^{-1} \in \overline{\mathbb{Q}}S^F.$$

Here we use the usual notation $\epsilon_G = (-1)^{\mathrm{rank}_{\mathbb{F}_q}(G)}$ for G any reductive group over $\mathbb{F}_q$.

1.16. *Symmetric algebras and symmetrizing forms.* [Br, Sec. 2] — Let Λ be a commutative ring with unity. By a symmetric Λ-algebra, we mean a pair (A, τ) where A is a Λ-algebra which is finitely generated and projective as a Λ-module, and where the map $\tau : A \longrightarrow \Lambda$ is Λ-linear such that the map

$$A \times A \longrightarrow \Lambda, \quad (a,b) \longmapsto \tau(ab),$$

is a perfect symmetric bilinear form ("perfect" means that the map $A \longrightarrow \mathrm{Hom}_\Lambda(A, \Lambda)$, $a \longmapsto \tau(a(-))$, is a Λ-module isomorphism). In this case, τ is called a symmetrizing form on A over Λ.

Now let (A,τ) be a symmetric algebra, let P be a finitely generated projective A-module and let $E=\mathrm{End}_A(P)$ be its endomorphism ring. Then τ induces a symmetrizing form $\tau_E : E \longrightarrow \Lambda$ via $\tau_E = [E \simeq \mathrm{Hom}_A(P,A)\otimes_A P \xrightarrow{\text{natural pairing}} A \xrightarrow{\tau} \Lambda]$, so that (E,τ_E) is a symmetric Λ-algebra.

Application to $\Lambda\mathsf{E}_{G,\psi}$. Let Λ be a ring containing $\overline{\mathbb{Z}}[\frac{1}{p}]$. The evaluation at identity $\mathrm{ev}_1 : \Lambda G^F \longrightarrow \Lambda$ is a symmetrizing form on ΛG^F; the Gelfand–Graev representation $\Gamma_{G,\psi} = \Lambda G^F e_\psi$ being a projective ΛG^F-module, the previous discussion implies that ev_1 induces a symmetrizing form τ on $\Lambda\mathsf{E}_{G,\psi} = \mathrm{End}_{\Lambda G^F}(\Gamma_{G,\psi})$; under the identification $\Lambda\mathsf{E}_{G,\psi} = e_\psi \Lambda G^F e_\psi \subset \Lambda G^F$, the form $\tau : \Lambda\mathsf{E}_{G,\psi} \longrightarrow \Lambda$ is just the restriction of ev_1 on $\Lambda\mathsf{E}_{G,\psi}$.

1.17. Lemma. [BoKe, Lem. 3.8] — *Let $\Lambda_1 \subset \Lambda_2$ be an inclusion of commutative rings with unity. Let (A,τ) be a symmetric Λ_1-algebra which is free as a Λ_1-module, and denote the Λ_2-linear extension of τ to $\Lambda_2 A$ again by τ. Let A' be a Λ_1-algebra which is free as Λ_1-module, such that $A \subset A' \subset \Lambda_2 A$ and $\tau(A') \subset \Lambda_1$. Then $A = A'$.*

1.18. Theorem. [BoKe, Thm. 3.7] — *The injective Curtis homomorphism*

$$\mathrm{Cur}^G = (\mathrm{Cur}^G_{T_w})_{w\in W} : \overline{\mathbb{Q}}\mathsf{E}_{G,\psi} \hookrightarrow \prod_{w\in W} \overline{\mathbb{Q}} T_w^F$$

in § 1.14 is saturated over $\overline{\mathbb{Z}}[\frac{1}{p|W|}]$ in the sense that

$$\mathrm{Cur}^G(\overline{\mathbb{Z}}[\tfrac{1}{p|W|}]\mathsf{E}_{G,\psi}) = \mathrm{Cur}^G(\overline{\mathbb{Q}}\mathsf{E}_{G,\psi}) \cap \left(\prod_{w\in W} \overline{\mathbb{Z}}[\tfrac{1}{p|W|}]T_w^F\right).$$

Remark. There are examples of G making Cur^G not saturated over $\overline{\mathbb{Z}}[\frac{1}{p}]$, such as $G = \mathrm{SL}_2(\overline{\mathbb{F}}_q)$ with q odd (see [BoKe, Rmk. 3.9]) or $G = \mathrm{GL}_2(\overline{\mathbb{F}}_q)$ with q odd (§ 3.29).

For later use, let us sketch a proof of this theorem. Consider the two $\overline{\mathbb{Z}}[\frac{1}{p|W|}]$-algebras $A := \mathrm{Cur}^G(\overline{\mathbb{Z}}[\frac{1}{p|W|}]\mathsf{E}_{G,\psi})$ and $A' := \mathrm{Cur}^G(\overline{\mathbb{Q}}\mathsf{E}_{G,\psi}) \cap \left(\prod_{w\in W} \overline{\mathbb{Z}}[\frac{1}{p|W|}]T_w^F\right)$; to prove that $A = A'$, the idea is to construct a symmetrizing form on A and then apply § 1.17.

We have $A \subset A'$ as each $\mathrm{Cur}^G_{T_w}$ is defined over $\overline{\mathbb{Z}}[\frac{1}{p}]$ (§ 1.12). As $\overline{\mathbb{Q}}A = \mathrm{Cur}^G(\overline{\mathbb{Q}}\mathsf{E}_{G,\psi})$, we get the inclusions $A \subset A' \subset \overline{\mathbb{Q}}A$. Consider

$$\tau^{\mathsf{E}} := |U^F|\mathrm{ev}_{1_{G^F}} : \overline{\mathbb{Z}}[\tfrac{1}{p}]\mathsf{E}_{G,\psi} \longrightarrow \overline{\mathbb{Z}}[\tfrac{1}{p}] \quad (\overline{\mathbb{Z}}[\tfrac{1}{p}]\mathsf{E}_{G,\psi} = e_\psi \overline{\mathbb{Z}}[\tfrac{1}{p}]G^F e_\psi),$$

so that $(\overline{\mathbb{Z}}[\frac{1}{p}]\mathsf{E}_{G,\psi}, \tau^{\mathsf{E}})$ is a symmetric $\overline{\mathbb{Z}}[\frac{1}{p}]$-algebra (§ 1.16). Denote the $\overline{\mathbb{Q}}$-linear extension of τ^{E} again by $\tau^{\mathsf{E}} : \overline{\mathbb{Q}}\mathsf{E}_{G,\psi} \longrightarrow \overline{\mathbb{Q}}$, and set

$$\tau^W := \frac{1}{|W|}\sum_{w\in W} \mathrm{ev}_{1_{T_w^F}} \circ \mathrm{pr}_{\overline{\mathbb{Q}}T_w^F} : \prod_{w\in W} \overline{\mathbb{Q}}T_w^F \longrightarrow \overline{\mathbb{Q}}$$

which is a symmetrizing form on $\prod_{w\in W} \overline{\mathbb{Q}}T_w^F$ and verifies the relation $\tau^{\mathsf{E}} = \tau^W \circ \mathrm{Cur}^G$ on $\overline{\mathbb{Q}}\mathsf{E}_G$ (see [BoKe, Sec. 3.B]).

The last relation and the injectivity of Cur^G together imply that $(A, \tau^W|_A)$ is a symmetric $\overline{\mathbb{Z}}[\frac{1}{p|W|}]$-algebra; on the other hand, since $|W|$ is invertible in $\overline{\mathbb{Z}}[\frac{1}{p|W|}]$, the map τ^W is defined over $\overline{\mathbb{Z}}[\frac{1}{p|W|}]$ and then we have $\tau^W(A') \subset \overline{\mathbb{Z}}[\frac{1}{p|W|}]$. So § 1.17 implies that $A = A'$, and this completes the proof of the theorem.

The $\mathbb{Z}$-model E_G

1.19. *Definition of the ring E_G.* — The decomposition $U^F/U^F_\bullet = \prod_{i\in\Delta/F} U^F_i$ (§ 1.6) identifies each U^F_i as a subgroup of $U^F/U^F_\bullet$. Consider the (left) $\mathbb{Z}U^F$-module

$$\mathcal{F}_0 = \{f : U^F/U^F_\bullet \longrightarrow \mathbb{Z} : \sum_{x\in U^F_i} f(x) = 0 \text{ for each } i \in \Delta/F\}$$

on which U^F acts by left translation: $u \cdot f := f(u^{-1}(\cdot))$ ($u \in U^F$, $f \in \mathcal{F}_0$). The groups T^F and T^F_{ad} both act on each U^F_i ($i \in \Delta/F$) by left adjoint action; this induces left T^F-actions and left T^F_{ad}-actions on $\mathcal{F}_0$, $\mathrm{Ind}^{G^F}_{U^F}(\mathcal{F}_0)$ and $\mathrm{End}_{\mathbb{Z}G^F}(\mathrm{Ind}^{G^F}_{U^F}(\mathcal{F}_0))$. The centre of the ring $\mathrm{End}_{\mathbb{Z}G^F}(\mathrm{Ind}^{G^F}_{U^F}(\mathcal{F}_0))$, denoted by $Z\left(\mathrm{End}_{\mathbb{Z}G^F}(\mathrm{Ind}^{G^F}_{U^F}(\mathcal{F}_0))\right)$, is fixed by the T^F-action on $\mathrm{End}_{\mathbb{Z}G^F}(\mathrm{Ind}^{G^F}_{U^F}(\mathcal{F}_0))$ and thus admits an $H^1(F,Z)$-action (recall that $H^1(F,Z) \simeq T^F_{\mathrm{ad}}/T^F$); we then set

$$\mathsf{E}_G := Z\left(\mathrm{End}_{\mathbb{Z}G^F}(\mathrm{Ind}^{G^F}_{U^F}(\mathcal{F}_0))\right)^{H^1(F,Z)}, \tag{1.19.1}$$

which consists of elements of $Z\left(\mathrm{End}_{\mathbb{Z}G^F}(\mathrm{Ind}^{G^F}_{U^F}(\mathcal{F}_0))\right)$ fixed by the $H^1(F,Z)$-action. This E_G is a ring and is also a free $\mathbb{Z}$-module of finite rank.

We shall show in § 1.22 that the ring E_G in (1.19.1) is consistent with the definition of $\overline{\mathbb{Z}}[\frac{1}{p}]\mathsf{E}_{G,\psi}$ in § 1.9.

1.20. Lemma. — *The natural inclusion $\mathcal{F}_0 \subset \mathbb{Z}U^F$ induces an embedding of $\overline{\mathbb{Z}}[\frac{1}{p}]$-modules $\mathrm{Ind}^{G^F}_{U^F}(\overline{\mathbb{Z}}[\frac{1}{p}]\mathcal{F}_0) \subset \overline{\mathbb{Z}}[\frac{1}{p}]G^F$, under which the actions of T^F and T^F_{ad} on $\mathrm{Ind}^{G^F}_{U^F}(\overline{\mathbb{Z}}[\frac{1}{p}]\mathcal{F}_0)$ (§ 1.19) are exactly the restrictions of their respective adjoint actions on $\overline{\mathbb{Z}}[\frac{1}{p}]G^F$. Moreover, we have a decomposition of $\overline{\mathbb{Z}}[\frac{1}{p}]G^F$-modules*

$$\mathrm{Ind}^{G^F}_{U^F}(\overline{\mathbb{Z}}[\tfrac{1}{p}]\mathcal{F}_0) = \bigoplus_{\psi\in\Psi} \overline{\mathbb{Z}}[\tfrac{1}{p}]G^F e_\psi = \bigoplus_{\psi\in\Psi} \Gamma_{G,\psi} \in \mathrm{Rep}_{\overline{\mathbb{Z}}[\frac{1}{p}]}(G^F),$$

where Ψ is as in § 1.6.

Proof. The description of the actions of T^F and T^F_{ad} on $\mathrm{Ind}^{G^F}_{U^F}(\overline{\mathbb{Z}}[\frac{1}{p}]\mathcal{F}_0)$ is clear from construction, so let us prove the decomposition formula directly.

(1) (Compare [DLM, (2.4.8)].) Each linear character $\psi : U^F/U^F_\bullet = \prod_{i\in\Delta/F} U^F_i \longrightarrow \overline{\mathbb{Z}}^\times$ is decomposed as $\psi = \prod_{i\in\Delta/F} \psi_i$ with $\psi_i = \psi|_{U^F_i}$ ($i \in \Delta/F$); such a ψ gives an element in Ψ if and only if each ψ_i is non-trivial.

(2) Using (1) and the orthogonality of characters, one can check that $\{e_\psi \,|\, \psi \in \Psi\}$ is a $\overline{\mathbb{Q}}$-linearly independent subset of $\overline{\mathbb{Q}}\mathcal{F}_0$. By definition of $\mathcal{F}_0$ and (1), we have

$$\dim_{\overline{\mathbb{Q}}}(\overline{\mathbb{Q}}\mathcal{F}_0) = \prod_{i\in\Delta/F} (|U^F_i| - 1) = |\Psi|,$$

so $\{e_\psi \,|\, \psi \in \Psi\}$ is in fact a $\overline{\mathbb{Q}}$-linear basis of $\overline{\mathbb{Q}}\mathcal{F}_0$; identifying $\overline{\mathbb{Q}}\mathcal{F}_0 \subset \overline{\mathbb{Q}}U^F$, we have the following decomposition of $\overline{\mathbb{Q}}\mathcal{F}_0$:

$$\overline{\mathbb{Q}}\mathcal{F}_0 = \bigoplus_{\psi\in\Psi} \overline{\mathbb{Q}}e_\psi; \quad \text{for } f \in \overline{\mathbb{Q}}\mathcal{F}_0,\ f = \sum_{\psi\in\Psi} f e_\psi \text{ with each } f e_\psi \in \overline{\mathbb{Q}}e_\psi. \tag{1.20.1}$$

As all e_ψ ($\psi \in \Psi$) lie in $\overline{\mathbb{Z}}[\frac{1}{p}]U^F$, the decomposition (1.20.1) also holds in $\overline{\mathbb{Z}}[\frac{1}{p}]$-coefficients, from which the desired decomposition of $\mathrm{Ind}_{U^F}^{G^F}(\overline{\mathbb{Z}}[\frac{1}{p}]\mathcal{F}_0)$ follows. □

1.21. Lemma. — *Let $\psi, \psi' \in \Psi$, and suppose that $\chi \in \mathrm{Irr}_{\overline{\mathbb{Q}}}(G^F)$ is a common irreducible character of the $\overline{\mathbb{Q}}G^F$-modules $\overline{\mathbb{Q}}\Gamma_{G,\psi}$ and $\overline{\mathbb{Q}}\Gamma_{G,\psi'}$. Then for the unique $y \in T_{\mathrm{ad}}^F$ such that $\psi' = {}^y\psi$ (§ 1.6), we have ${}^y\chi = \chi$.*

Proof. By § 1.8, χ lies in some rational Lusztig series $\mathcal{E}(G^F, [x])$; in other words, there is an F-stable maximal torus S and a $\theta \in \mathrm{Irr}_{\overline{\mathbb{Q}}}(S^F)$ such that θ corresponds to $[x]$ under the duality $\mathrm{Irr}_{\overline{\mathbb{Q}}}(S^F) \simeq S^{*F^*}$, and such that $\langle \chi, R_S^G(\theta)\rangle_{G^F} \neq 0$.

Let us show that ${}^y(R_S^G(\theta)) = R_S^G(\theta)$. Indeed, as $y \in T_{\mathrm{ad}}^F$, if we choose a $t \in T$ such that $y = tZ$, then $t^{-1}F(t) \in Z \subset S$, so the Lang–Steinberg theorem enables us to write $t^{-1}F(t) = sF(s)^{-1}$ for some $s \in S$; thus, upon setting $x := ts \in G^F$,

$$ {}^y(R_S^G(\theta)) = {}^t(R_S^G(\theta)) = R_{{}^tS}^G({}^t\theta) = R_{{}^xS}^G({}^x\theta) = {}^x(R_S^G(\theta)) = R_S^G(\theta). $$

By the above discussion, we have

$$ \langle {}^y\chi, R_S^G(\theta)\rangle_{G^F} = \langle {}^y\chi, {}^y(R_S^G(\theta))\rangle_{G^F} = \langle \chi, R_S^G(\theta)\rangle_{G^F} \neq 0. $$

On the other hand, with the help of (1.7.3),

$$ \langle {}^y\chi, \overline{\mathbb{Q}}\Gamma_{G,\psi'}\rangle_{G^F} = \langle {}^y\chi, {}^y(\overline{\mathbb{Q}}\Gamma_{G,\psi})\rangle_{G^F} = \langle \chi, \overline{\mathbb{Q}}\Gamma_{G,\psi}\rangle_{G^F} \neq 0. $$

Therefore, $\gamma = \chi$ and $\gamma = {}^y\chi$ both verify $\langle \gamma, R_S^G(\theta)\rangle_{G^F} \neq 0$ and $\langle \gamma, \overline{\mathbb{Q}}\Gamma_{G,\psi'}\rangle_{G^F} \neq 0$; but § 1.8 tells us that there is at most one such $\gamma \in \mathrm{Irr}_{\overline{\mathbb{Q}}}(G^F)$, so ${}^y\chi$ and χ must coincide. □

1.22. Proposition. — *For each $\psi \in \Psi$, $\overline{\mathbb{Z}}[\frac{1}{p}]\mathsf{E}_G \simeq \overline{\mathbb{Z}}[\frac{1}{p}]\mathsf{E}_{G,\psi}$ as $\overline{\mathbb{Z}}[\frac{1}{p}]$-algebras. Explicitly: upon identifying $\overline{\mathbb{Z}}[\frac{1}{p}]\mathsf{E}_{G,\psi} = e_\psi\overline{\mathbb{Z}}[\frac{1}{p}]G^Fe_\psi$ (§ 1.10) and identifying analogously*

$$ \mathrm{End}_{\overline{\mathbb{Z}}[\frac{1}{p}]G^F}(\mathrm{Ind}_{U^F}^{G^F}(\overline{\mathbb{Z}}[\tfrac{1}{p}]\mathcal{F}_0)) = \bigoplus_{\psi,\psi'\in\Psi} e_{\psi'}\overline{\mathbb{Z}}[\tfrac{1}{p}]G^Fe_\psi \subset \overline{\mathbb{Z}}[\tfrac{1}{p}]G^F, $$

we have a $\overline{\mathbb{Z}}[\frac{1}{p}]$-algebra isomorphism

$$ \overline{\mathbb{Z}}[\tfrac{1}{p}]\mathsf{E}_{G,\psi} \xrightarrow{\sim} \overline{\mathbb{Z}}[\tfrac{1}{p}]\mathsf{E}_G, \quad e_\psi f e_\psi \longmapsto \sum_{y\in T_{\mathrm{ad}}^F} e_{{}^y\psi}\cdot {}^yf\cdot e_{{}^y\psi} \ \ (f \in \overline{\mathbb{Z}}[\tfrac{1}{p}]G^F). $$

Proof. (1) An element in the centre of $\mathrm{End}_{\overline{\mathbb{Z}}[\frac{1}{p}]G^F}(\mathrm{Ind}_{U^F}^{G^F}(\overline{\mathbb{Z}}[\frac{1}{p}]\mathcal{F}_0))$ commutes with all e_ψ ($\psi \in \Psi$), so the orthogonality of idempotents implies that

$$ Z\left(\mathrm{End}_{\overline{\mathbb{Z}}[\frac{1}{p}]G^F}(\mathrm{Ind}_{U^F}^{G^F}(\overline{\mathbb{Z}}[\tfrac{1}{p}]\mathcal{F}_0))\right) \subset \bigoplus_{\psi\in\Psi} e_\psi\overline{\mathbb{Z}}[\tfrac{1}{p}]G^Fe_\psi. \tag{1.22.1} $$

(2) Let $y \in T_{\mathrm{ad}}^F$. For $\theta \in \mathrm{End}_{\overline{\mathbb{Z}}[\frac{1}{p}]G^F}(\mathrm{Ind}_{U^F}^{G^F}(\overline{\mathbb{Z}}[\frac{1}{p}]\mathcal{F}_0))$, its image $y\cdot\theta$ under the action of y on $\mathrm{End}_{\overline{\mathbb{Z}}[\frac{1}{p}]G^F}(\mathrm{Ind}_{U^F}^{G^F}(\overline{\mathbb{Z}}[\frac{1}{p}]\mathcal{F}_0))$ is given by $(y\cdot\theta)(\varphi) = {}^y(\theta({}^{y^{-1}}\varphi))$ ($\varphi \in \mathrm{Ind}_{U^F}^{G^F}(\overline{\mathbb{Z}}[\frac{1}{p}]\mathcal{F}_0)$). Then the subring $\bigoplus_{\psi\in\Psi} e_\psi\overline{\mathbb{Z}}[\frac{1}{p}]G^Fe_\psi$ is stable under this action of y: indeed, let $\theta \in$

$\bigoplus_{\psi\in\Psi} e_\psi\overline{\mathbb{Z}}[\frac{1}{p}]G^F e_\psi$ and write $\theta = \sum_{\psi\in\Psi}\theta_\psi$ where each $\theta_\psi = e_\psi\theta e_\psi \in e_\psi\overline{\mathbb{Z}}[\frac{1}{p}]G^F e_\psi$; then it can be checked that $y\cdot\theta = \sum_{\psi\in\Psi}(y\cdot\theta)_\psi$ with each $(y\cdot\theta)_\psi = y\cdot(\theta_{y^{-1}\psi}) = {}^y(\theta_{y^{-1}\psi}) \in e_\psi\overline{\mathbb{Z}}[\frac{1}{p}]G^F e_\psi$. We thus deduce from (1.22.1) that

$$\overline{\mathbb{Z}}[\tfrac{1}{p}]\mathsf{E}_G \subset \left(\bigoplus_{\psi\in\Psi} e_\psi\overline{\mathbb{Z}}[\tfrac{1}{p}]G^F e_\psi\right)^{T^F_{\mathrm{ad}}} =: \mathcal{A}. \tag{1.22.2}$$

(3) For each $\theta\in\mathcal{A}$, if we write $\theta = \sum_{\psi\in\Psi}\theta_\psi$ with each $\theta_\psi \in e_\psi\overline{\mathbb{Z}}[\frac{1}{p}]G^F e_\psi$ as in (2), then $\theta_\psi = {}^y(\theta_{y^{-1}\psi})$ for all $y\in T^F_{\mathrm{ad}}$; as T^F_{ad} acts regularly on Ψ (§ 1.6), we see that for every $\psi\in\Psi$ we have a ring isomorphism

$$\overline{\mathbb{Z}}[\tfrac{1}{p}]\mathsf{E}_{G,\psi} \xrightarrow{\sim} \mathcal{A}, \quad e_\psi f e_\psi \longmapsto \sum_{y\in T^F_{\mathrm{ad}}} e_{y\psi}\cdot {}^y f\cdot e_{y\psi} \ \ (f\in\overline{\mathbb{Z}}[\tfrac{1}{p}]G^F). \tag{1.22.3}$$

(4) By virtue of (1.22.2) and (1.22.3), in order to complete the proof of lemma, it remains to establish the inclusion

$$\overline{\mathbb{Q}}\mathcal{A} = \left(\bigoplus_{\psi\in\Psi} e_\psi\overline{\mathbb{Q}}G^F e_\psi\right)^{T^F_{\mathrm{ad}}} \subset Z\left(\bigoplus_{\psi,\psi'\in\Psi} e_{\psi'}\overline{\mathbb{Q}}G^F e_\psi\right). \tag{1.22.4}$$

For each $\psi,\psi'\in\Psi$, § 1.8 (with notations in § 1.10) tells us that the space $e_{\psi'}\overline{\mathbb{Q}}G^F e_\psi$ is the direct sum of $e_{\psi'}\overline{\mathbb{Q}}G^F e^G_{\psi,[x]}e_\psi$ for $[x]$ ranging over a (possibly empty) subset of $G^{*F^*}_{\mathrm{ss}}/\sim$. To show (1.22.4), it then suffices to show that

$$e_{y\psi} f e^G_{\psi,[x']} e_\psi \cdot e_\psi e^G_{\psi,[x]} e_\psi = e_{y\psi}({}^y e^G_{\psi,[x]}) e_{y\psi}\cdot e_{y\psi} f e^G_{\psi,[x']} e_\psi \tag{1.22.5}$$

for every $\psi\in\Psi$, $y\in T^F_{\mathrm{ad}}$, $[x],[x']\in G^{*F^*}_{\mathrm{ss}}/\sim$ and $f\in\overline{\mathbb{Q}}G^F$.

To prove (1.22.5), suppose that the element $e_{y\psi} f e^G_{\psi,[x']} e_\psi \in \mathrm{Hom}_{\overline{\mathbb{Q}}G^F}(\overline{\mathbb{Q}}\Gamma_{y\psi},\rho_{\psi,[x']})$ is not zero (as the opposite case is trivial). Then $\chi_{\psi,[x']}$ is a common irreducible character of $\overline{\mathbb{Q}}\Gamma_{y\psi}$ and $\overline{\mathbb{Q}}\Gamma_\psi$, so § 1.21 gives us ${}^y(\chi_{\psi,[x']}) = \chi_{\psi,[x']}$ and then ${}^y e^G_{\psi,[x']} = e^G_{\psi,[x']}$. We may also suppose that $[x]=[x']$, for otherwise both side (1.22.5) are zero because the idempotents $e^G_{\psi,[x]}$ are central and orthogonal in $\overline{\mathbb{Q}}G^F$. Under the above assumptions, the two sides of (1.22.5) are both $e_{y\psi} f e^G_{\psi,[x']} e_\psi$; this justifies (1.22.5).

The equality (1.22.5) being proved, we obtain the inclusion (1.22.4) as well as the equality $\overline{\mathbb{Z}}[\frac{1}{p}]\mathsf{E}_G = \mathcal{A}$; thus (1.22.3) gives the desired $\overline{\mathbb{Z}}[\frac{1}{p}]$-algebra isomorphism. □

1.23. *The Galois action on* $\overline{\mathbb{Q}}\mathsf{E}_G$. — The Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on the $\overline{\mathbb{Q}}$-coefficient of $\overline{\mathbb{Q}}\mathsf{E}_G$; this induces an action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on $\overline{\mathbb{Q}}\mathsf{E}_G$, and this action coincides with that induced by the identification

$$\overline{\mathbb{Q}}\mathsf{E}_G = Z\left(\mathrm{End}_{\overline{\mathbb{Q}}G^F}(\mathrm{Ind}^{G^F}_{U^F}(\overline{\mathbb{Q}}\mathcal{F}_0))\right)^{H^1(F,Z)}$$

and by the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on the $\overline{\mathbb{Q}}$-coefficient of $\overline{\mathbb{Q}}\mathcal{F}_0$.

In terms of idempotents, the above Galois action on $\overline{\mathbb{Q}}\mathsf{E}_G$ is, upon fixing a $\psi \in \Psi$ and identifying $\overline{\mathbb{Q}}\mathsf{E}_G = \overline{\mathbb{Q}}\mathsf{E}_{G,\psi} = e_\psi \overline{\mathbb{Q}} G^F e_\psi$ (§ 1.22):

$$\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}),\ e_\psi f e_\psi \in \overline{\mathbb{Q}}\mathsf{E}_{G,\psi}\ (f \in \overline{\mathbb{Q}}G^F) \Longrightarrow \sigma \cdot (e_\psi f e_\psi) = e_\psi \cdot \sigma({}^{y_\sigma^{-1}} f) \cdot e_\psi,$$

where y_σ is the unique element in T^F_{ad} such that $\sigma \cdot \psi = {}^{y_\sigma}\psi \in \Psi$ (σ acts on $\overline{\mathbb{Z}}^\times$ and hence on Ψ; then recall from § 1.6 that T^F_{ad} acts regularly on Ψ), and where, for each $h \in \overline{\mathbb{Q}}G^F$, $\sigma(h)$ denotes its image under the action of σ on the $\overline{\mathbb{Q}}$-coefficient of $\overline{\mathbb{Q}}G^F$.

Using § 1.24 below, the two vertical maps in the diagram (1.13.2) are $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-equivariant and are thus defined over $\mathbb{Q}$; the two horizontal maps in the same diagram, however, are not $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-equivariant.

1.24. Lemma. — *Identify $\overline{\mathbb{Z}}[\frac{1}{p}]\mathsf{E}_G = \overline{\mathbb{Z}}[\frac{1}{p}]\mathsf{E}_{G,\psi}$ (§ 1.22). Then $\mathrm{Cur}^G_S : \overline{\mathbb{Q}}\mathsf{E}_G \longrightarrow \overline{\mathbb{Q}}S^F$ is equivariant under the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ (§ 1.23). Thus (by § 1.12) Cur^G_S is defined over $\mathbb{Z}[\frac{1}{p}]$: $\mathrm{Cur}^G_S(\mathbb{Z}[\frac{1}{p}]\mathsf{E}_G) \subset \mathbb{Z}[\frac{1}{p}]S^F$.*

Proof. We have to show that

$$\sigma(\mathrm{Cur}^G_S(e_\psi n e_\psi)(s)) = \mathrm{Cur}^G_S(\sigma \cdot (e_\psi n e_\psi))(s) \tag{1.24.1}$$

for every $n \in G^F$, $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ and $s \in S^F$. We apply Curtis' formula in § 1.11: using § 1.23 and the notation therein, we have $\sigma \cdot \psi = {}^{y_\sigma}\psi$ with $y_\sigma \in T^F_{\mathrm{ad}}$; choose a $t \in T$ such that $y_\sigma = tZ$, and choose a $c \in S$ such that $x := t^{-1}c \in G^F$ (see § 1.21); then

$$\begin{aligned}
&\langle Q^G_S, \Gamma_{G,\psi}\rangle_{G^F} |U^F| |(C_G(s)^\circ)^F| \cdot \sigma(\mathrm{Cur}^G_S(e_\psi n e_\psi)(s)) \\
&= \sum_{\substack{g \in G^F,\, u \in U^F \\ (gung^{-1})_{\mathrm{ss}} = s}} \sigma(\psi(u^{-1})) Q_S^{C_G(s)^\circ}((gung^{-1})_{\mathrm{u}}) \\
&= \sum_{\substack{g \in G^F,\, u \in U^F \\ (gung^{-1})_{\mathrm{ss}} = s}} ({}^{y_\sigma}\psi)(u^{-1}) Q_S^{C_G(s)^\circ}((gung^{-1})_{\mathrm{u}}) \\
&= \sum_{\substack{g \in G^F,\, u \in U^F \\ (gu.{}^{y_\sigma^{-1}}n.g^{-1})_{\mathrm{ss}} = {}^{y_\sigma^{-1}}s}} \psi(u^{-1}) Q_S^{C_G(s)^\circ}({}^{y_\sigma}((gu.{}^{y_\sigma^{-1}}n.g^{-1})_{\mathrm{u}}))
\end{aligned}$$

(where we have performed $u \mapsto {}^{y_\sigma}u$ and $g \mapsto {}^{y_\sigma}g$)

$$= \sum_{\substack{g \in G^F,\, u \in U^F \\ {}^{x}((gu.{}^{y_\sigma^{-1}}n.g^{-1})_{\mathrm{ss}}) = {}^{t^{-1}}s}} \psi(u^{-1}) Q_S^{C_G(s)^\circ}({}^{y_\sigma x}((gu.{}^{y_\sigma^{-1}}n.g^{-1})_{\mathrm{u}}))$$

(where we have performed $g \mapsto xg$)

$$= \sum_{\substack{g \in G^F,\, u \in U^F \\ (gu.{}^{y_\sigma^{-1}}n.g^{-1})_{\mathrm{ss}} = s}} \psi(u^{-1}) Q_S^{C_G(s)^\circ}((gu.{}^{y_\sigma^{-1}}n.g^{-1})_{\mathrm{u}})$$

(by the proof of § 1.21, we have ${}^{x^{-1}y_\sigma^{-1}}(Q_S^{C_G(s)^\circ}) = Q_S^{C_G(s)^\circ}$)

$$= \langle Q^G_S, \Gamma_{G,\psi}\rangle_{G^F} |U^F| |(C_G(s)^\circ)^F| \cdot \mathrm{Cur}^G_S(\sigma \cdot (e_\psi n e_\psi))(s). \qquad \square$$

1.25. Corollary (of § 1.18 and § 1.24) — *Identify* $\overline{\mathbb{Z}}[\frac{1}{p}]\mathsf{E}_G = \overline{\mathbb{Z}}[\frac{1}{p}]\mathsf{E}_{G,\psi}$ *(§ 1.22). Then the map* Cur^G *is saturated over* $\mathbb{Z}[\frac{1}{p|W|}]$*:*

$$\mathrm{Cur}^G(\mathbb{Z}[\tfrac{1}{p|W|}]\mathsf{E}_G) = \mathrm{Cur}^G(\mathbb{Q}\mathsf{E}_G) \cap \left(\prod_{w\in W} \mathbb{Z}[\tfrac{1}{p|W|}]T_w^F\right).$$

Compatibility with central extensions

1.26. *Central extensions.* — For our group G, we can fit its Deligne–Lusztig dual G^* into an F^*-equivariant exact sequence of reductive groups

$$1 \longrightarrow Z^* \longrightarrow H^* \longrightarrow G^* \longrightarrow 1 \tag{1.26.1}$$

where the derived subgroup $H^*_{\mathrm{der}} = [H^*, H^*]$ of H^* is simply-connected and Z^* is a central torus in H^*.

We fix a choice of F-equivariant exact sequence of reductive groups

$$1 \longrightarrow G \longrightarrow H \xrightarrow{\kappa} Z \longrightarrow 1 \tag{1.26.2}$$

which is dual to (1.26.1). Let B_H be a Borel subgroup of H having unipotent radical V and containing an F-stable maximal torus T_H of H; then $T_G := \ker(\kappa|_{T_H} : T_H \twoheadrightarrow Z)$ is an F-stable maximal torus of G, and $B_G := \ker(\kappa|_{B_H} : B_H \twoheadrightarrow Z)$ is a Borel subgroup of G having unipotent radical V and containing T_G.

1.27. Lemma. [LiSh, Sec. 2] — *Notations as in § 1.26. Then:*

(a) *For* $(h,t) \in H^F \times (T_H^F)^{\mathrm{op}}$ *with* $\kappa(ht) \neq 1$*, we have* $\mathrm{Tr}((h,t)|H_c^*(DL^H_{T_H\subset B_H})) = 0$.

(b) *For* $(h,t) \in G^F \times (T_G^F)^{\mathrm{op}}$*, we have*

$$\mathrm{Tr}((h,t)|H_c^*(DL^H_{T_H\subset B_H})) = |Z^F| \cdot \mathrm{Tr}((h,t)|H_c^*(DL^G_{T_G\subset B_G})).$$

(c) *For every* $\chi \in \mathrm{Irr}_{\overline{\mathbb{Q}}}(T_H^F)$*, we have* $R^H_{T_H}(\chi)|_{G^F} = R^G_{T_G}(\chi|_{T_G^F})$.

Proof. For each $z \in Z^F$, set $(DL^H_{T_H\subset B_H})(z) := \{hV \in DL^H_{T_H\subset B_H} : \kappa(h) = z\}$, so that $DL^H_{T_H\subset B_H} = \bigsqcup_{z\in Z^F} (DL^H_{T_H\subset B_H})(z)$. As T_G is connected, we have $\kappa(T_H^F) = Z^F$, so for each $z \in Z^F$ we may choose a $\dot{z} \in T_H^F$ such that $\kappa(\dot{z}) = z$; then we have $DL^H_{T_H\subset B_H}(z) = DL^G_{T_G\subset B_G} \cdot \dot{z} \subset H/V$, so that $DL^H_{T_H\subset B_H}(z) \simeq DL^G_{T_G\subset B_G}$ as $(G^F \times (T_G^F)^{\mathrm{op}})$-varieties. In terms of virtual ℓ-adic cohomology, $H_c^*(DL^H_{T_H\subset B_H}) = \sum_{z\in Z^F} H_c^*(DL^H_{T_H\subset B_H}(z))$, and the $H^F \times (T_H^F)^{\mathrm{op}}$-action on $H_c^*(DL^H_{T_H\subset B_H})$ satisfies:

$$\begin{cases} \text{for every } (h,t) \in H^F \times (T_H^F)^{\mathrm{op}}, (h,t) \cdot H_c^*(DL^H_{T_H\subset B_H}(z)) \subset H_c^*(DL^H_{T_H\subset B_H}(\kappa(ht)z)); \\ \text{for every } z \in Z^F,\ H_c^*(DL^H_{T_H\subset B_H}(z)) \simeq H_c^*(DL^G_{T_G\subset B_G}) \text{ as } G^F \times (T_G^F)^{\mathrm{op}}\text{-modules.} \end{cases}$$

This proves (a) and (b), of which (c) is a corollary. □

1.28. Proposition. — *Notations as in § 1.26. Under the identification*

$$\overline{\mathbb{Z}}[\tfrac{1}{p}]\mathsf{E}_{G,\psi} = e_\psi\overline{\mathbb{Z}}[\tfrac{1}{p}]G^F e_\psi \hookrightarrow e_\psi\overline{\mathbb{Z}}[\tfrac{1}{p}]H^F e_\psi = \overline{\mathbb{Z}}[\tfrac{1}{p}]\mathsf{E}_{H,\psi}, \tag{1.28.1}$$

the following diagram of rings is commutative:

$$\begin{tikzcd} \overline{\mathbb{Q}}\mathsf{E}_{G,\psi} \arrow[r, "\sim", "(1.10.3)"'] \arrow[d, hook] & \overline{\mathbb{Q}}^{G^{*F^*}_{\mathrm{ss}}/\sim} \arrow[d, hook, "(1.26.1)"] \\ \overline{\mathbb{Q}}\mathsf{E}_{H,\psi} \arrow[r, "\sim", "(1.10.3)"'] & \overline{\mathbb{Q}}^{H^{*F^*}_{\mathrm{ss}}/\sim} \end{tikzcd} \tag{1.28.2}$$

Equivalently: for every $[x]_{G^{*F^*}} \in G^{*F^*}_{\mathrm{ss}}/\sim$, *we have*

$$e^{\mathsf{E}_G}_{\psi,[x]_{G^{*F^*}}} = \sum_{[y]_{H^{*F^*}} \mapsto [x]_{G^{*F^*}}} e^{\mathsf{E}_H}_{\psi,[y]_{H^{*F^*}}}, \tag{1.28.3}$$

where the sum ranges over all $[y]_{H^{*F^*}} \in H^{*F^*}_{\mathrm{ss}}/\sim$ *such that* $[y]_{H^{*F^*}} Z^{*F^*} = [x]_{G^{*F^*}}$.

First proof ([LiSh, Sec. 3]). In § 1.26, we may choose B_H to be F-stable (so B_G is also F-stable). Choose an F^*-stable maximal torus T^*_G of G^* dual to T_G (resp. T^*_H of H dual to T_H) such that $T^*_H/Z^* = T^*_G$. Then the Weyl groups of (G, T_G), (G^*, T^*_G), (H, T_H) and (H^*, T^*_H) are all the same, and we denote this common Weyl group by W. For each $w \in W$, choose an F-stable maximal torus $T_{G,w}$ of G whose G^F-conjugacy class corresponds to the F-conjugacy class of w in W (with respect to T_G, so that we may choose $T_{G,1} = T_G$); choose $T^*_{G,w} \subset G^*$, $T_{H,w} \subset H$ and $T^*_{H,w} \subset H^*$ in a similar way.

In the toric case where $(G, H) = (T_G, T_H)$, the commutativity of (1.28.2) follows from toric dualities.

For the general case of (G, H), we use the Curtis embeddings $\mathrm{Cur}^G = (\mathrm{Cur}^G_{T_{G,w}})_{w\in W}$ and $\mathrm{Cur}^H = (\mathrm{Cur}^H_{T_{H,w}})_{w\in W}$ to embed (1.28.2) into the following cubic diagram of rings:

$$\begin{tikzcd} & \prod_{w\in W} \overline{\mathbb{Q}} T^F_{G,w} \arrow[rr, "\sim"] \arrow[dd, hook] & & \prod_{w\in W} \overline{\mathbb{Q}}^{T^{*F^*}_{G,w}} \arrow[dd, hook] \\ \overline{\mathbb{Q}}\mathsf{E}_G \arrow[ur, hook, "\mathrm{Cur}^G"] \arrow[rr, "\sim"] \arrow[dd, hook] & & \overline{\mathbb{Q}}^{G^{*F^*}_{\mathrm{ss}}/\sim} \arrow[ur, hook, "\mathrm{Res}"] & \\ & \prod_{w\in W} \overline{\mathbb{Q}} T^F_{H,w} \arrow[rr, "\sim"] & & \prod_{w\in W} \overline{\mathbb{Q}}^{T^{*F^*}_{H,w}} \\ \overline{\mathbb{Q}}\mathsf{E}_H \arrow[ur, hook, "\mathrm{Cur}^H"] \arrow[rr, "\sim"] & & \overline{\mathbb{Q}}^{H^{*F^*}_{\mathrm{ss}}/\sim} \arrow[ur, hook, "\mathrm{Res}"] \arrow[uu, hook] & \end{tikzcd} \tag{1.28.4}$$

In (1.28.4), the right face is clearly commutative; the top and the bottom faces are commutative by § 1.13; the back face is the toric case of (1.28.2) and is hence commutative. So to prove the commutativity of (1.28.2), it remains to show that the left face in (1.28.4) is commutative.

Using § 1.15, the commutativity of the left face in (1.28.4) is equivalent to the property that, for all $h \in \overline{\mathbb{Q}}\mathsf{E}_G \subset \overline{\mathbb{Q}}G^F$ and all $w \in W$, we have

$$\frac{1}{|T^F_{H,w}|} \sum_{t\in T^F_{H,w}} \mathrm{Tr}((h,t)|H^*_c(DL^H_{T_{H,w}}))t^{-1} = \frac{1}{|T^F_{G,w}|} \sum_{t\in T^F_{G,w}} \mathrm{Tr}((h,t)|H^*_c(DL^G_{T_{G,w}}))t^{-1}. \tag{1.28.5}$$

By § 1.27(a)(b) and the equality $T^F_{H,w}/T^F_{G,w} = Z^F$, (1.28.5) is true for all $h \in G^F$, so the left face in (1.28.4) commutes. This completes the proof of the lemma. $\square$

Second proof. Let $e^G_{[x]_{G^{*F^*}}}$ be the central idempotent of $\overline{\mathbb{Q}}G^F$ cut by the rational Lusztig series $\mathcal{E}(G^F, [x]_{G^{*F^*}})$; that is, $e^G_{[x]_{G^{*F^*}}} = \sum_{\chi \in \mathcal{E}(G^F, [x]_{G^{*F^*}})} e^G_\chi$ where e^G_χ is the primitive central idempotent of $\overline{\mathbb{Q}}G^F$ associated to χ. Then we have $e^{\mathsf{E}_G}_{\psi,[x]_{G^{*F^*}}} = e_\psi e^G_{[x]_{G^{*F^*}}}$ because the Gelfand–Graev $\overline{\mathbb{Q}}G^F$-module $\overline{\mathbb{Q}}\Gamma_{G,\psi}$ is multiplicity-free. To prove (1.28.3), it then suffices to show that

$$e^G_{[x]_{G^{*F^*}}} = \sum_{[y]_{H^{*F^*}} \mapsto [x]_{G^{*F^*}}} e^H_{[y]_{H^{*F^*}}}. \tag{1.28.6}$$

Using § 1.27(c), we see that for each $\xi \in \mathcal{E}(H^F, [y]_{H^{*F^*}})$, all irreducible components of $\xi|_{G^F}$ belong to $\mathcal{E}(G^F, [x]_{G^{*F^*}})$ where $[y]_{H^{*F^*}} \mapsto [x]_{G^{*F^*}}$, so $e^G_{[x]_{G^{*F^*}}} e^H_{[y]_{H^{*F^*}}} = 0$ whenever $[y]_{H^{*F^*}} \not\mapsto [x]_{G^{*F^*}}$. As $\mathcal{E}(G^F, [x]_{G^{*F^*}})$ ($[x]_{G^{*F^*}} \in G^{*F^*}_{\mathrm{ss}}/\sim$) give a partition of $\mathrm{Irr}_{\overline{\mathbb{Q}}}(G^F)$, we have $1 = \sum_{[x]_{G^{*F^*}} \in G^{*F^*}_{\mathrm{ss}}/\sim} e^G_{[x]_{G^{*F^*}}}$; multiplying this equality by $e^H_{[y]_{H^{*F^*}}}$, we get $e^G_{[x]_{G^{*F^*}}} e^H_{[y]_{H^{*F^*}}} = e^H_{[y]_{H^{*F^*}}}$ whenever $[y]_{H^{*F^*}} \mapsto [x]_{G^{*F^*}}$. Multiplying an analogous equality $1 = \sum_{[y]_{H^{*F^*}} \in H^{*F^*}_{\mathrm{ss}}/\sim} e^H_{[y]_{H^{*F^*}}}$ by $e^G_{[x]_{G^{*F^*}}}$, we obtain (1.28.6). $\square$

2. Grothendieck rings of modular representations

The Grothendieck ring K_{G^*}

2.1. *The p-regular elements.* — Let $G^{*F^*}_{p'}$ be the set of p-regular elements of G^{*F^*}; recall that an element of G^{*F^*} is called p-regular if its order in G^{*F^*} is not divisible by p. We have $G^{*F^*}_{p'} = G^{*F^*}_{\mathrm{ss}}$ by the Jordan decomposition. Denote by $G^{*F^*}_{p'}/\sim$ the set of G^{*F^*}-conjugacy classes in $G^{*F^*}_{p'}$; thus $(G^{*F^*}_{p'}/\sim) = (G^{*F^*}_{\mathrm{ss}}/\sim)$.

2.2. *Definition of the ring K_{G^*}.* — Consider $\mathrm{Rep}_{\overline{\mathbb{F}}_q}(G^{*F^*})$, the category of finite-dimensional representations of the finite group G^{*F^*} over the field $\overline{\mathbb{F}}_q$, or equivalently the category of finitely generated $\overline{\mathbb{F}}_q G^{*F^*}$-modules. We define K_{G^*} to be the Grothendieck group of the category $\mathrm{Rep}_{\overline{\mathbb{F}}_q}(G^{*F^*})$. The tensor product on $\mathrm{Rep}_{\overline{\mathbb{F}}_q}(G^{*F^*})$ induces a multiplication $\otimes$ on K_{G^*}, and then we shall consider $\mathsf{K}_{G^*} = (\mathsf{K}_{G^*}, +, \otimes)$ as a ring.

Denote by $[\cdot] : \mathrm{Rep}_{\overline{\mathbb{F}}_q}(G^{*F^*}) \longrightarrow \mathsf{K}_{G^*}$ the natural map, which descends to a map $[\cdot]$ on the set of isomorphism classes in $\mathrm{Rep}_{\overline{\mathbb{F}}_q}(G^{*F^*})$. Let $\mathrm{Irr}_{\overline{\mathbb{F}}_q}(G^{*F^*})$ be the set of isomorphism classes of simple objects in $\mathrm{Rep}_{\overline{\mathbb{F}}_q}(G^{*F^*})$. Then K_{G^*} is a free $\mathbb{Z}$-module having the set $\{[M] \in \mathsf{K}_{G^*} : M \in \mathrm{Irr}_{\overline{\mathbb{F}}_q}(G^{*F^*})\}$ as a basis (see [Se1, Sec. 14.1]).

2.3. *The Brauer character.* [Se1, Sec. 18] — Recall that we have fixed an inclusion of multiplicative groups $\kappa : \overline{\mathbb{F}}_q^\times \hookrightarrow \overline{\mathbb{Q}}^\times$ (§ 1.4). Let $M \in \mathrm{Rep}_{\overline{\mathbb{F}}_q}(G^{*F^*})$ and $g \in G^{*F^*}_{p'}$ (§ 2.1). Then the action $g : M \longrightarrow M$ is diagonalizable over $\overline{\mathbb{F}}_q$ with eigenvalues $\lambda_1, \cdots, \lambda_N \in \overline{\mathbb{F}}_q^\times$ ($N := \dim_{\overline{\mathbb{F}}_q} M$), and we set $(\mathrm{br}\, M)(g) := \kappa(\lambda_1) + \cdots + \kappa(\lambda_N) \in \overline{\mathbb{Q}}$. We thus get a map $\mathrm{br}\, M : G^{*F^*}_{p'} \longrightarrow \overline{\mathbb{Q}}$ (the Brauer character of M), which descends to an element $\mathrm{br}\, M \in \overline{\mathbb{Q}}^{G^{*F^*}_{p'}/\sim}$. The map $M \longmapsto \mathrm{br}\, M$ then induces a ring homomorphism

$$\mathrm{br} : \mathsf{K}_{G^*} \longrightarrow \overline{\mathbb{Q}}^{G^{*F^*}_{p'}/\sim}.$$

The unique $\overline{\mathbb{Q}}$-linear extension of the map $\mathrm{br}(\cdot)$ is a $\overline{\mathbb{Q}}$-algebra isomorphism:

$$\mathrm{br} : \overline{\mathbb{Q}}\mathsf{K}_{G^*} \xrightarrow{\sim} \overline{\mathbb{Q}}^{G^{*F^*}_{p'}/\sim}.$$

Thus the rank of the free $\mathbb{Z}$-module K_{G^*} is

$$\mathrm{rank}_{\mathbb{Z}}\, \mathsf{K}_{G^*} = |\mathrm{Irr}_{\overline{\mathbb{F}}_q}(G^{*F^*})| = |G^{*F^*}_{p'}/\sim| = |G^{*F^*}_{\mathrm{ss}}/\sim|.$$

The above Brauer isomorphism and the canonical inclusion $\mathsf{K}_{G^*} \subset \overline{\mathbb{Q}}\mathsf{K}_{G^*}$ show that K_{G^*} is reduced and is embedded into $\overline{\mathbb{Q}}^{G^{*F^*}_{p'}/\sim}$ by the Brauer map $\mathrm{br}(\cdot)$.

2.4. *Projective objects in* $\mathrm{Rep}_{\overline{\mathbb{F}}_q}(G^{*F^*})$. [Se1, IIIe partie] — Let $\mathrm{Proj}_{\overline{\mathbb{F}}_q}(G^{*F^*})$ be the category of projective $\overline{\mathbb{F}}_q G^{*F^*}$-modules of finite dimension over $\overline{\mathbb{F}}_q$, and let P_{G^*} be the additive Grothendieck group of the category $\mathrm{Proj}_{\overline{\mathbb{F}}_q}(G^{*F^*})$. For $M \in \mathrm{Rep}_{\overline{\mathbb{F}}_q}(G^{*F^*})$, denote by $P_M \in \mathrm{Proj}_{\overline{\mathbb{F}}_q}(G^{*F^*})$ the projective cover of M (which is unique up to isomorphism). Then P_{G^*} is a free $\mathbb{Z}$-module having $\{[P_M] : M \in \mathrm{Irr}_{\overline{\mathbb{F}}_q}(G^{*F^*})\}$ as a basis (the image of $P \in \mathrm{Proj}_{\overline{\mathbb{F}}_q}(G^{*F^*})$ in P_{G^*} is denoted by $[P]$.) Two objects P and P' of $\mathrm{Proj}_{\overline{\mathbb{F}}_q}(G^{*F^*})$ are isomorphic if and only if $[P] = [P']$ in P_{G^*}. Furthermore:

(a) The pairing

$$\langle\cdot,\cdot\rangle_{\mathsf{P},\mathsf{K}} : \mathrm{Proj}_{\overline{\mathbb{F}}_q}(G^{*F^*}) \times \mathrm{Rep}_{\overline{\mathbb{F}}_q}(G^{*F^*}) \longrightarrow \mathbb{Z}, \quad (P,\pi) \longmapsto \dim_{\overline{\mathbb{F}}_q} \mathrm{Hom}_{\overline{\mathbb{F}}_q G^{*F^*}}(P,\pi)$$

descends to a $\mathbb{Z}$-bilinear perfect pairing $\langle\cdot,\cdot\rangle_{\mathsf{P},\mathsf{K}} : \mathsf{P}_{G^*} \times \mathsf{K}_{G^*} \longrightarrow \mathbb{Z}$ with $\{[P_M] : M \in \mathrm{Irr}_{\overline{\mathbb{F}}_q}(G^{*F^*})\}$ and $\{[M] : M \in \mathrm{Irr}_{\overline{\mathbb{F}}_q}(G^{*F^*})\}$ being dual bases with each other.

(b) Let $c : \mathsf{P}_{G^*} \longrightarrow \mathsf{K}_{G^*}$ be the Cartan homomorphism, which is the natural map induced by the inclusion $\mathrm{Proj}_{\overline{\mathbb{F}}_q}(G^{*F^*}) \subset \mathrm{Rep}_{\overline{\mathbb{F}}_q}(G^{*F^*})$. Then $c(\mathsf{P}_{G^*}) \supset |G^{*F^*}|_p \mathsf{K}_{G^*}$, the cokernel $\mathrm{coker}(c) = \mathsf{K}_{G^*}/c(\mathsf{P}_{G^*})$ is a finite p-group, and c is an injective map which identifies P_{G^*} as an ideal of the ring K_{G^*}.

On comparison between $\mathbb{Z}[\frac{1}{p}]\mathsf{K}_{G^*}$ and $\mathbb{Z}[\frac{1}{p}]\mathsf{E}_G$

2.5. *The identification $\overline{\mathbb{Q}}\mathsf{E}_G \simeq \overline{\mathbb{Q}}\mathsf{K}_{G^*}$.* — From now on, let us fix a $\psi \in \Psi$ (§ 1.6) and identify $\overline{\mathbb{Z}}[\frac{1}{p}]\mathsf{E}_G = \overline{\mathbb{Z}}[\frac{1}{p}]\mathsf{E}_{G,\psi}$ (§ 1.22).

Using the Brauer isomorphism in § 2.3, the equality $G^{*F^*}_{p'} = G^{*F^*}_{\mathrm{ss}}$ and (1.10.3), we obtain the following identifications of $\overline{\mathbb{Q}}$-algebras:

$$\left\{ \begin{array}{ccccccc} \overline{\mathbb{Q}}\mathsf{E}_G & \simeq & \overline{\mathbb{Q}}^{G^{*F^*}_{\mathrm{ss}}/\sim} & = & \overline{\mathbb{Q}}^{G^{*F^*}_{p'}/\sim} & \simeq & \overline{\mathbb{Q}}\mathsf{K}_{G^*} \\ e^{\mathsf{E}}_{\psi,[x]} & \leftrightarrow & \mathbf{1}_{\{[x]\}} & = & \mathbf{1}_{\{[x]\}} & \leftrightarrow & \mathrm{br}^{-1}(\mathbf{1}_{\{[x]\}}) \end{array} \right\}.$$

When $G = S$ is a torus, the identification $\overline{\mathbb{Q}}\mathsf{E}_S \simeq \overline{\mathbb{Q}}\mathsf{K}_{S^*}$ is already true over $\mathbb{Z}$: indeed, $\mathsf{E}_S = \mathbb{Z}S^F$, and the duality $S^F \simeq \mathrm{Irr}_{\overline{\mathbb{F}}_q}(S^{*F^*})$ (§ 1.4) yields $\mathbb{Z}S^F \simeq \mathsf{K}_{S^*}$.

In the general case, we first consider the coefficients $\overline{\mathbb{Z}}[\frac{1}{p}]$ for the sake of the structure theory of $\overline{\mathbb{Z}}[\frac{1}{p}]\mathsf{E}_G$, and it is expected that we still have a $\overline{\mathbb{Z}}[\frac{1}{p}]$-algebra isomorphism $\overline{\mathbb{Z}}[\frac{1}{p}]\mathsf{E}_G \simeq \overline{\mathbb{Z}}[\frac{1}{p}]\mathsf{K}_{G^*}$. The first idea is to reconstruct the expected isomorphism from the above toric case via the following commutative diagram of rings (where S is an F-stable maximal torus of G):

$$\begin{array}{ccc} \overline{\mathbb{Q}}\mathsf{E}_G & \xleftrightarrow{\sim} & \overline{\mathbb{Q}}\mathsf{K}_{G^*} \\ {\scriptstyle \mathrm{Cur}^G_S}\downarrow & & \downarrow{\scriptstyle \mathrm{Res}^{G^{*F^*}}_{S^{*F^*}}} \\ \overline{\mathbb{Q}}S^F & \xleftrightarrow{\sim} & \overline{\mathbb{Q}}\mathsf{K}_{S^*} \end{array}$$

2.6. Proposition. — *Let $\tau^{\mathsf{K}} : \overline{\mathbb{Q}}\mathsf{K}_{G^*} \longrightarrow \overline{\mathbb{Q}}$ be the $\overline{\mathbb{Q}}$-linear form induced by the symmetrizing form $\tau^{\mathsf{E}} = |U^F|\mathrm{ev}_{1_{G^F}} : \overline{\mathbb{Q}}\mathsf{E}_G \longrightarrow \overline{\mathbb{Q}}$ (§ 1.18) via the identification $\overline{\mathbb{Q}}\mathsf{E}_G \simeq \overline{\mathbb{Q}}\mathsf{K}_{G^*}$ in § 2.5. Then*

$$\tau^{\mathsf{K}}(\pi) = \frac{1}{|W|}\sum_{w\in W} \langle \mathbf{1}, \mathrm{Res}^{G^{*F^*}}_{T^{*F^*}_w}\pi\rangle_{\mathsf{P},\mathsf{K}} \quad (\pi \in \mathsf{K}_{G^*}),$$

where the perfect pairing $\langle\cdot,\cdot\rangle_{\mathsf{P},\mathsf{K}}$ is as in § 2.4(a), $W = W_G(T)$ and T_w are as in § 1.2, and $\mathbf{1}$ is the trivial representation over $\overline{\mathbb{F}}_q$

Note that, as $T^{*F^*}_w$ has order coprime to p, our trivial representation $\mathbf{1}$ is projective, and the pairing $\langle \mathbf{1}, \mathrm{Res}^{G^{*F^*}}_{T^{*F^*}_w}\pi\rangle_{\mathsf{P},\mathsf{K}}$ is simply the usual inner product of the characters for $\mathbf{1}$ and $\mathrm{Res}^{G^{*F^*}}_{T^{*F^*}_w}\pi$. (Nevertheless, the above expression of $\tau^{\mathsf{K}}(\pi)$ in terms of $\langle\cdot,\cdot\rangle_{\mathsf{P},\mathsf{K}}$ will be essential in § 2.8.)

Proof of proposition. Under the canonical identification $\overline{\mathbb{Q}}\mathsf{K}_{G^*} = \overline{\mathbb{Q}}\mathsf{E}_G$, $\pi \in \mathsf{K}_{G^*}$ corresponds to $\sum_{x \in G^{*F^*}_{\mathrm{ss}}/\sim} (\mathrm{br}\,\pi)(x) e^{\mathsf{E}}_{\psi,[x]} \in \overline{\mathbb{Q}}\mathsf{E}_G$. Using the formula $\tau^{\mathsf{E}} = \tau^W \circ \mathrm{Cur}^G$ (§ 1.18) as well as § 1.13, we have

$$\tau^{\mathsf{E}}(e^{\mathsf{E}}_{\psi,[x]}) = \frac{1}{|W|} \sum_{w \in W} \mathrm{Cur}^G_{T_w}(e^{\mathsf{E}}_{\psi,[x]})(1) = \frac{1}{|W|} \sum_{w \in W} \sum_{z \in [x] \cap T^{*F^*}_w} \frac{1}{|T^F_w|},$$

so that

$$\begin{aligned}
\tau^{\mathsf{K}}(\pi) &= \sum_{x \in G^{*F^*}_{\mathrm{ss}}/\sim} (\mathrm{br}\,\pi)(x) \cdot \tau^{\mathsf{E}}(e^{\mathsf{E}}_{\psi,[x]}) = \frac{1}{|W|} \sum_{x \in G^{*F^*}_{\mathrm{ss}}/\sim} \sum_{w \in W} \sum_{z \in [x] \cap T^{*F^*}_w} \frac{(\mathrm{br}\,\pi)(x)}{|T^F_w|} \\
&= \frac{1}{|W|} \sum_{w \in W} \frac{1}{|T^F_w|} \sum_{x \in G^{*F^*}_{\mathrm{ss}}/\sim} \sum_{z \in [x] \cap T^{*F^*}_w} (\mathrm{br}\,\pi)(z) \\
&= \frac{1}{|W|} \sum_{w \in W} \frac{1}{|T^F_w|} \sum_{z \in T^{*F^*}_w} (\mathrm{br}\,\pi)(z) = \frac{1}{|W|} \sum_{w \in W} \langle \mathbf{1}, \mathrm{Res}^{G^{*F^*}}_{T^{*F^*}_w} \pi \rangle_{\mathsf{P},\mathsf{K}}.
\end{aligned}$$

□

2.7. *The Steinberg character.* — Let $\mathrm{St}_{G^*} \in \mathrm{Rep}_{\overline{\mathbb{Q}}}(G^{*F^*})$ be the Steinberg character of G^{*F^*}, and let $\overline{\mathrm{St}}_{G^*}$ be its reduction modulo p (through an embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$). Recall that $\overline{\mathrm{St}}_{G^*}$ is an irreducible projective $\overline{\mathbb{F}}_q G^{*F^*}$-module and is isomorphic to its dual representation $\overline{\mathrm{St}}^\vee_{G^*}$. In addition:

(a) [DeLu, Cor. 7.15] In $\mathbb{Z}[\frac{1}{|W|}]\mathsf{K}_{G^*}$, we have (recall that $\epsilon_G = (-1)^{\mathrm{rank}_{\mathbb{F}_q}(G)}$):

$$\begin{aligned}
[\overline{\mathrm{St}}_{G^*}] &= \frac{1}{|W|} \sum_{w \in W} \epsilon_G \epsilon_{T_w} \left[\mathrm{Ind}^{G^{*F^*}}_{T^{*F^*}_w} \mathbf{1}\right]; \\
[\overline{\mathrm{St}}_{G^*}] \otimes [\overline{\mathrm{St}}_{G^*}] &= \frac{1}{|W|} \sum_{w \in W} \left[\mathrm{Ind}^{G^{*F^*}}_{T^{*F^*}_w} \mathbf{1}\right].
\end{aligned}$$

(b) [Lu1] The map $(\cdot) \otimes [\overline{\mathrm{St}}_{G^*}] : \mathsf{K}_{G^*} \longrightarrow \mathsf{P}_{G^*}$ is a $\mathbb{Z}$-module isomorphism.

2.8. Proposition. — *Let $\tau^{\mathsf{K}} : \overline{\mathbb{Q}}\mathsf{K}_{G^*} \longrightarrow \overline{\mathbb{Q}}$ be the $\overline{\mathbb{Q}}$-linear form in § 2.6.*

(a) *For every $\pi \in \mathsf{K}_{G^*}$ we have $\tau^{\mathsf{K}}(\pi) = \langle [\overline{\mathrm{St}}_{G^*}] \otimes [\overline{\mathrm{St}}_{G^*}], \pi \rangle_{\mathsf{P},\mathsf{K}} \in \mathbb{Z}$. Thus τ^{K} is defined over $\mathbb{Z}$, and we shall still denote the restriction of τ^{K} to K_{G^*} by $\tau^{\mathsf{K}} : \mathsf{K}_{G^*} \longrightarrow \mathbb{Z}$.*

(b) *The $\mathbb{Z}$-linear form $\tau^{\mathsf{K}} : \mathsf{K}_{G^*} \longrightarrow \mathbb{Z}$ obtained in (a) induces a $\mathbb{Z}$-bilinear form*

$$b^{\mathsf{K}} : \mathsf{K}_{G^*} \times \mathsf{K}_{G^*} \longrightarrow \mathbb{Z}, \quad b^{\mathsf{K}}(x, y) := \tau^{\mathsf{K}}(x \otimes y) \quad (x, y \in \mathsf{K}_{G^*}).$$

The discriminant $\mathrm{disc}\, b^{\mathsf{K}}$ of b^{K} is then an integer and satisfies

$$\left|\mathrm{disc}\, b^{\mathsf{K}}\right| = \left|\det\left[(\cdot) \otimes [\overline{\mathrm{St}}_{G^*}] : \mathsf{K}_{G^*} \longrightarrow \mathsf{K}_{G^*}\right]\right| = [\mathsf{K}_{G^*} : c(\mathsf{P}_{G^*})] = p^m$$

for some $m \in \mathbb{N}$. (Here $c : \mathsf{P}_{G^*} \longrightarrow \mathsf{K}_{G^*}$ is the Cartan homomorphism in § 2.4(b).)

(c) *$\tau^{\mathsf{K}} : \mathbb{Z}[\frac{1}{p}]\mathsf{K}_{G^*} \longrightarrow \mathbb{Z}[\frac{1}{p}]$ (well-defined by (a)) is a symmetrizing form on $\mathbb{Z}[\frac{1}{p}]\mathsf{K}_{G^*}$, so $(\mathbb{Z}[\frac{1}{p}]\mathsf{K}_{G^*}, \tau^{\mathsf{K}})$ is a symmetric $\mathbb{Z}[\frac{1}{p}]$-algebra.*

Proof. (a) The formula of $\tau^{\mathsf{K}}(\pi)$ follows from § 2.6, Frobenius reciprocity and § 2.7(a) (we extend $\langle\cdot,\cdot\rangle_{\mathsf{P},\mathsf{K}}$ $\overline{\mathbb{Q}}$-bilinearly):

$$\begin{aligned}\tau^{\mathsf{K}}(\pi) &= \frac{1}{|W|}\sum_{w\in W}\langle \mathbf{1}, \mathrm{Res}^{G^{*F^*}}_{T_w^{*F^*}}\pi\rangle_{\mathsf{P},\mathsf{K}} = \frac{1}{|W|}\sum_{w\in W}\langle \mathrm{Ind}^{G^{*F^*}}_{T_w^{*F^*}}\mathbf{1}, \pi\rangle_{\mathsf{P},\mathsf{K}} \\ &= \left\langle \frac{1}{|W|}\sum_{w\in W}\left[\mathrm{Ind}^{G^{*F^*}}_{T_w^{*F^*}}\mathbf{1}\right], \pi\right\rangle_{\mathsf{P},\mathsf{K}} = \langle[\overline{\mathrm{St}}_{G^*}]\otimes[\overline{\mathrm{St}}_{G^*}], \pi\rangle_{\mathsf{P},\mathsf{K}}.\end{aligned}$$

(b) By (a), adjunction and § 2.7, for all $x, y \in \mathsf{K}_{G^*}$ we have

$$b^{\mathsf{K}}(x,y) = \tau^{\mathsf{K}}(x\otimes y) = \langle[\overline{\mathrm{St}}_{G^*}]\otimes[\overline{\mathrm{St}}_{G^*}], x\otimes y\rangle_{\mathsf{P},\mathsf{K}} = \langle x^\vee\otimes[\overline{\mathrm{St}}_{G^*}], y\otimes[\overline{\mathrm{St}}_{G^*}]\rangle_{\mathsf{P},\mathsf{K}}.$$

Consider the basis $\beta := \{[x] : x \in \mathrm{Irr}_{\overline{\mathbb{F}}_q}(G^{*F^*})\}$ (resp. $P_\beta := \{[P_x] : x \in \mathrm{Irr}_{\overline{\mathbb{F}}_q}(G^{*F^*})\}$) for the free $\mathbb{Z}$-module K_{G^*} (resp. P_{G^*}); see § 2.4). Using the matrices associated to (bi-)linear forms, we have the decomposition

$$\left[b^{\mathsf{K}}\right]_{\beta\times\beta} = {}^tX\cdot Y\cdot Z$$

where:

$$\begin{aligned}X &:= \left[(\cdot)^\vee\otimes[\overline{\mathrm{St}}_{G^*}] : \mathsf{K}_{G^*}\longrightarrow \mathsf{P}_{G^*}\right]^{P_\beta}_{\beta};\\ Y &:= \left[\langle\cdot,\cdot\rangle_{\mathsf{P},\mathsf{K}} : \mathsf{P}_{G^*}\times\mathsf{K}_{G^*}\longrightarrow\mathbb{Z}\right]_{P_\beta,\beta};\\ Z &:= \left[(\cdot)\otimes[\overline{\mathrm{St}}_{G^*}] : \mathsf{K}_{G^*}\longrightarrow\mathsf{K}_{G^*}\right]^{\beta}_{\beta}.\end{aligned}$$

By § 2.7(b), $(\cdot)^\vee\otimes[\overline{\mathrm{St}}_{G^*}] : \mathsf{K}_{G^*}\longrightarrow\mathsf{P}_{G^*}$ is invertible, so $\det X\in\mathbb{Z}^\times = \{\pm1\}$; by § 2.4(a), $\det Y = 1$. We thus obtain:

$$\operatorname{disc} b^{\mathsf{K}} = \det\left(\left[b^{\mathsf{K}}\right]_{\beta\times\beta}\right) = \pm\det Z = \pm\det\left[(\cdot)\otimes[\overline{\mathrm{St}}_{G^*}] : \mathsf{K}_{G^*}\longrightarrow\mathsf{K}_{G^*}\right].$$

Recall that for a free $\mathbb{Z}$-module A of finite rank and for a $\mathbb{Z}$-module homomorphism $\varphi : A\longrightarrow A$, we have $|\det\varphi| = [A : \varphi(A)]$ (this can be deduced via the Smith normal form of φ). Using this fact, § 2.7(b) and § 2.4(b), we get, for some $m\in\mathbb{N}$,

$$\left|\det\left[(\cdot)\otimes[\overline{\mathrm{St}}_{G^*}] : \mathsf{K}_{G^*}\longrightarrow\mathsf{K}_{G^*}\right]\right| = [\mathsf{K}_{G^*} : c(\mathsf{P}_{G^*})] = p^m.$$

(c) By (b), $\operatorname{disc} b^{\mathsf{K}} = \pm p^m$ is invertible in $\mathbb{Z}[\frac{1}{p}]$, and this is equivalent to saying that $\tau^{\mathsf{K}} : \mathbb{Z}[\frac{1}{p}]\mathsf{K}_{G^*}\longrightarrow\mathbb{Z}[\frac{1}{p}]$ is a symmetrizing form on $\mathbb{Z}[\frac{1}{p}]\mathsf{K}_{G^*}$. □

Remark. The equality

$$\det\left[(\cdot)\otimes[\overline{\mathrm{St}}_{G^*}] : \mathsf{K}_{G^*}\longrightarrow\mathsf{K}_{G^*}\right] = \pm p^m$$

(a weaker version of (b) but will be sufficient for the proof of § 2.9) can be directly verified as follows: the map $(\cdot)\otimes[\overline{\mathrm{St}}_{G^*}] : \mathsf{K}_{G^*}\longrightarrow\mathsf{K}_{G^*}$ corresponds, under the Brauer isomorphism $\mathrm{br} : \overline{\mathbb{Q}}\mathsf{K}_{G^*}\xrightarrow{\sim}\overline{\mathbb{Q}}^{G^{*F^*}_{p'}/\sim}$ (§ 2.3), to the diagonalizable endomorphism

$$\overline{\mathbb{Q}}^{G^{*F^*}_{p'}/\sim}\longrightarrow\overline{\mathbb{Q}}^{G^{*F^*}_{p'}/\sim},\quad f\longmapsto f\cdot(\mathrm{br}\,\overline{\mathrm{St}}_{G^*}),$$

whose eigenvalues are all of the form $(\mathrm{br}\,\overline{\mathrm{St}}_{G^*})(x) = \mathrm{St}_{G^*}(x) = \pm|(C_G(x)^\circ)^F|_p$ with $x\in G^{*F^*}_{\mathrm{ss}}$; the desired equality then follows.

2.9. Theorem. — *Consider the following commutative diagram of $\overline{\mathbb{Q}}$-algebras introduced in § 2.5:*

$$\begin{tikzcd} & \overline{\mathbb{Q}}\mathsf{E}_G \arrow[r, leftrightarrow, "\sim"] \arrow[d, hook, "\mathrm{Cur}^G = (\mathrm{Cur}^G_{T_w})_{w\in W}"'] & \overline{\mathbb{Q}}\mathsf{K}_{G^*} \arrow[d, hook, "\mathrm{Res} := \left(\mathrm{Res}^{G^{*F^*}}_{T_w^{*F^*}}\right)_{w\in W}"] \\ & \prod_{w\in W} \overline{\mathbb{Q}}T_w^F \arrow[r, leftrightarrow, "\sim"] & \prod_{w\in W} \overline{\mathbb{Q}}\mathsf{K}_{T_w^*} \end{tikzcd} \tag{2.9.1}$$

This diagram restricts itself to the following commutative diagram of $\overline{\mathbb{Z}}[\frac{1}{p|W|}]$-algebras:

$$\begin{tikzcd} \overline{\mathbb{Z}}[\frac{1}{p|W|}]\mathsf{E}_G \arrow[r, leftrightarrow, "\sim"] \arrow[d, hook, "\mathrm{Cur}^G"'] & \overline{\mathbb{Z}}[\frac{1}{p|W|}]\mathsf{K}_{G^*} \arrow[d, hook, "\mathrm{Res}"] \\ \prod_{w\in W} \overline{\mathbb{Z}}[\frac{1}{p|W|}]T_w^F \arrow[r, leftrightarrow, "\sim"] & \prod_{w\in W} \overline{\mathbb{Z}}[\frac{1}{p|W|}]\mathsf{K}_{T_w^*} \end{tikzcd} \tag{2.9.2}$$

(By § 1.12, Cur^G is defined over $\overline{\mathbb{Z}}[\frac{1}{p}]$.) Moreover, the map $\mathrm{Res} : \overline{\mathbb{Q}}\mathsf{K}_{G^*} \longrightarrow \prod_{w\in W} \overline{\mathbb{Q}}\mathsf{K}_{T_w^*}$ is saturated over $\overline{\mathbb{Z}}[\frac{1}{p|W|}]$: $\mathrm{Res}(\overline{\mathbb{Z}}[\frac{1}{p|W|}]\mathsf{K}_{G^*}) = \mathrm{Res}(\overline{\mathbb{Q}}\mathsf{K}_{G^*}) \cap \left(\prod_{w\in W} \overline{\mathbb{Z}}[\frac{1}{p|W|}]\mathsf{K}_{T_w^*}\right)$.

In order to display the symmetry between the E-side and the K-side, we shall give two proofs of this theorem, in both of which the main idea is to use the symmetrizing form lemma § 1.17.

First proof. We first show that the map $\mathrm{Res} : \overline{\mathbb{Q}}\mathsf{K}_{G^*} \longrightarrow \prod_{w\in W} \overline{\mathbb{Q}}\mathsf{K}_{T_w^*}$ is saturated over $\overline{\mathbb{Z}}[\frac{1}{p|W|}]$. So consider the map

$$i := \frac{1}{|W|} \sum_{w\in W} \epsilon_{G^*}\epsilon_{T_w^*} \mathrm{Ind}^{G^{*F^*}}_{T_w^{*F^*}} : \left(\prod_{w\in W} \overline{\mathbb{Z}}[\tfrac{1}{p|W|}]\mathsf{K}_{T_w^*}\right) \longrightarrow \overline{\mathbb{Z}}[\tfrac{1}{p|W|}]\mathsf{K}_{G^*},$$

which is well-defined because $|W|$ is invertible in $\overline{\mathbb{Z}}[\frac{1}{p|W|}]$. Then the composition

$$\overline{\mathbb{Z}}[\tfrac{1}{p|W|}]\mathsf{K}_{G^*} \xrightarrow{\mathrm{Res}} \left(\prod_{w\in W} \overline{\mathbb{Z}}[\tfrac{1}{p|W|}]\mathsf{K}_{T_w^*}\right) \xrightarrow{i} \overline{\mathbb{Z}}[\tfrac{1}{p|W|}]\mathsf{K}_{G^*}$$

coincide with $[\overline{\mathrm{St}}_{G^*}] \otimes (\cdot) : \overline{\mathbb{Z}}[\frac{1}{p|W|}]\mathsf{K}_{G^*} \longrightarrow \overline{\mathbb{Z}}[\frac{1}{p|W|}]\mathsf{K}_{G^*}$, since for $\pi \in \overline{\mathbb{Z}}[\frac{1}{p|W|}]\mathsf{K}_{G^*}$ we have

$$\begin{aligned} (i \circ \mathrm{Res})(\pi) &= \frac{1}{|W|} \sum_{w\in W} \epsilon_{G^*}\epsilon_{T_w^*} \mathrm{Ind}^{G^{*F^*}}_{T_w^{*F^*}} \mathrm{Res}^{G^{*F^*}}_{T_w^{*F^*}} \pi \\ &= \frac{1}{|W|} \sum_{w\in W} \epsilon_{G^*}\epsilon_{T_w^*} \left(\mathrm{Ind}^{G^{*F^*}}_{T_w^{*F^*}} \mathbf{1}\right) \otimes \pi = [\overline{\mathrm{St}}_{G^*}] \otimes \pi \quad (\S\, 2.7(\mathrm{a})). \end{aligned}$$

§ 2.8(b) (or the remark of § 2.8) then implies: for some $m \in \mathbb{N}$,

$$\det(i \circ \mathrm{Res}) = \det\left[[\overline{\mathrm{St}}_{G^*}] \otimes (\cdot) : \overline{\mathbb{Z}}[\tfrac{1}{p}]\mathsf{K}_{G^*} \longrightarrow \overline{\mathbb{Z}}[\tfrac{1}{p}]\mathsf{K}_{G^*}\right] = \pm p^m;$$

thus $\det(i \circ \mathrm{Res}) \in \overline{\mathbb{Z}}[\frac{1}{p|W|}]^\times$. Hence $\mathrm{Res} : \overline{\mathbb{Q}}\mathsf{K}_{G^*} \longrightarrow \prod_{w\in W} \overline{\mathbb{Q}}\mathsf{K}_{T_w^*}$ is saturated over $\overline{\mathbb{Z}}[\frac{1}{p|W|}]$.

The diagram (2.9.1) gives the following commutative diagram:

$$\begin{tikzcd}
\overline{\mathbb{Z}}[\frac{1}{p|W|}]\mathsf{E}_G \arrow[rr, hook] \arrow[d, hook, "\mathrm{Cur}^G"'] & & \overline{\mathbb{Q}}\mathsf{K}_{G^*} \arrow[d, hook, "\mathrm{Res}"] \\
\prod_{w\in W} \overline{\mathbb{Z}}[\frac{1}{p|W|}]T_w^F \arrow[r, leftrightarrow, "\sim"] & \prod_{w\in W} \overline{\mathbb{Z}}[\frac{1}{p|W|}]\mathsf{K}_{T_w^*} \arrow[r, hook] & \prod_{w\in W} \overline{\mathbb{Q}}\mathsf{K}_{T_w^*}
\end{tikzcd}$$

The injective map $\overline{\mathbb{Z}}[\frac{1}{p|W|}]\mathsf{E}_G \hookrightarrow \overline{\mathbb{Q}}\mathsf{K}_{G^*}$ in this diagram identifies $\overline{\mathbb{Z}}[\frac{1}{p|W|}]\mathsf{E}_G \subset \overline{\mathbb{Q}}\mathsf{K}_{G^*}$. Using the commutativity of the same diagram as well as the fact that the injective map $\mathrm{Res} : \overline{\mathbb{Q}}\mathsf{K}_{G^*} \longrightarrow \prod_{w\in W} \overline{\mathbb{Q}}\mathsf{K}_{T_w^*}$ is saturated, we have $\overline{\mathbb{Z}}[\frac{1}{p|W|}]\mathsf{E}_G \subset \overline{\mathbb{Z}}[\frac{1}{p|W|}]\mathsf{K}_{G^*}$. Therefore:

(i) we have the inclusions of rings $\overline{\mathbb{Z}}[\frac{1}{p|W|}]\mathsf{E}_G \subset \overline{\mathbb{Z}}[\frac{1}{p|W|}]\mathsf{K}_{G^*} \subset \overline{\mathbb{Q}}\mathsf{E}_G$;

(ii) $(\overline{\mathbb{Z}}[\frac{1}{p|W|}]\mathsf{E}_G, \tau^{\mathsf{E}})$ in the proof of § 1.18 is a symmetric $\overline{\mathbb{Z}}[\frac{1}{p|W|}]$-algebra;

(iii) $\tau^{\mathsf{E}}(\overline{\mathbb{Z}}[\frac{1}{p|W|}]\mathsf{K}_{G^*}) \subset \overline{\mathbb{Z}}[\frac{1}{p|W|}]$ (§ 2.8(a)).

Thus § 1.17 implies that $\overline{\mathbb{Z}}[\frac{1}{p|W|}]\mathsf{E}_G = \overline{\mathbb{Z}}[\frac{1}{p|W|}]\mathsf{K}_{G^*}$ (under $\overline{\mathbb{Q}}\mathsf{E}_G = \overline{\mathbb{Q}}\mathsf{K}_{G^*}$). □

Second proof. This time we establish the $\overline{\mathbb{Z}}[\frac{1}{p|W|}]$-algebra isomorphism $\overline{\mathbb{Z}}[\frac{1}{p|W|}]\mathsf{E}_G \simeq \overline{\mathbb{Z}}[\frac{1}{p|W|}]\mathsf{K}_{G^*}$ first. The diagram (2.9.1) gives the following commutative diagram:

$$\begin{tikzcd}
\overline{\mathbb{Q}}\mathsf{E}_G \arrow[rrr, hook] \arrow[d, hook, "\mathrm{Cur}^G"'] & & & \overline{\mathbb{Z}}[\frac{1}{p|W|}]\mathsf{K}_{G^*} \arrow[d, hook, "\mathrm{Res}"] \\
\prod_{w\in W} \overline{\mathbb{Q}}T_w^F & \prod_{w\in W} \overline{\mathbb{Z}}[\frac{1}{p|W|}]T_w^F \arrow[l, hook'] \arrow[r, leftrightarrow, "\sim"] & \prod_{w\in W} \overline{\mathbb{Z}}[\frac{1}{p|W|}]\mathsf{K}_{T_w^*} &
\end{tikzcd}$$

We then identify $\overline{\mathbb{Z}}[\frac{1}{p|W|}]\mathsf{K}_{G^*} \subset \overline{\mathbb{Q}}\mathsf{E}_G$. As the map Cur^G is saturated over $\overline{\mathbb{Z}}[\frac{1}{p|W|}]$ (§ 1.18), the above diagram gives $\overline{\mathbb{Z}}[\frac{1}{p|W|}]\mathsf{K}_{G^*} \subset \overline{\mathbb{Z}}[\frac{1}{p|W|}]\mathsf{E}_G$. Therefore:

(i) we have the inclusions of rings $\overline{\mathbb{Z}}[\frac{1}{p|W|}]\mathsf{K}_{G^*} \subset \overline{\mathbb{Z}}[\frac{1}{p|W|}]\mathsf{E}_G \subset \overline{\mathbb{Q}}\mathsf{K}_{G^*}$;

(ii) $(\overline{\mathbb{Z}}[\frac{1}{p|W|}]\mathsf{K}_{G^*}, \tau^{\mathsf{K}})$ is a symmetric $\overline{\mathbb{Z}}[\frac{1}{p|W|}]$-algebra (§ 2.8(c));

(iii) $\tau^{\mathsf{K}}(\overline{\mathbb{Z}}[\frac{1}{p|W|}]\mathsf{E}_G) = \tau^{\mathsf{E}}(\overline{\mathbb{Z}}[\frac{1}{p|W|}]\mathsf{E}_G) \subset \overline{\mathbb{Z}}[\frac{1}{p|W|}]$.

Thus § 1.17 implies that $\overline{\mathbb{Z}}[\frac{1}{p|W|}]\mathsf{E}_G = \overline{\mathbb{Z}}[\frac{1}{p|W|}]\mathsf{K}_{G^*}$ (under $\overline{\mathbb{Q}}\mathsf{E}_G = \overline{\mathbb{Q}}\mathsf{K}_{G^*}$), and we also obtain the commutativity of (2.9.2). Using the saturatedness of Cur^G (§ 1.18) and the $\overline{\mathbb{Z}}[\frac{1}{p|W|}]$-algebra isomorphism $\overline{\mathbb{Z}}[\frac{1}{p|W|}]\mathsf{E}_G \simeq \overline{\mathbb{Z}}[\frac{1}{p|W|}]\mathsf{K}_{G^*}$, we see that the map $\mathrm{Res} : \overline{\mathbb{Q}}\mathsf{K}_{G^*} \longrightarrow \prod_{w\in W} \overline{\mathbb{Q}}\mathsf{K}_{T_w^*}$ is saturated over $\overline{\mathbb{Z}}[\frac{1}{p|W|}]$. □

2.10. Corollary (of § 2.9 and § 1.24). — *The commutative diagram (2.9.1) is equivariant under the action of* $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ *and induces by restriction the following com-*

mutative diagram of $\mathbb{Z}[\frac{1}{p|W|}]$*-algebras:*

$$\begin{tikzcd}
\mathbb{Z}[\frac{1}{p|W|}]\mathsf{E}_G \arrow[r, leftrightarrow, "\sim"] \arrow[d, "\mathrm{Cur}^G"'] & \mathbb{Z}[\frac{1}{p|W|}]\mathsf{K}_{G^*} \arrow[d, "\mathrm{Res}"] \\
\prod_{w\in W} \mathbb{Z}[\frac{1}{p|W|}]T_w^F \arrow[r, leftrightarrow, "\sim"] & \prod_{w\in W} \mathbb{Z}[\frac{1}{p|W|}]\mathsf{K}_{T_w^*}
\end{tikzcd}$$

Moreover, the map $\mathrm{Res} : \mathbb{Q}\mathsf{K}_{G^*} \longrightarrow \prod_{w\in W} \mathbb{Q}\mathsf{K}_{T_w^*}$ *is saturated over* $\mathbb{Z}[\frac{1}{p|W|}]$*:*

$$\mathrm{Res}(\mathbb{Z}[\tfrac{1}{p|W|}]\mathsf{K}_{G^*}) = \mathrm{Res}(\mathbb{Q}\mathsf{K}_{G^*}) \cap \left(\prod_{w\in W} \mathbb{Z}[\tfrac{1}{p|W|}]\mathsf{K}_{T_w^*} \right).$$

Algebraic representations

2.11. *Algebraic representations and formal characters.* [Ja, Ch. I.2 & II.2] — Let us denote by $\mathrm{Rep}_{\mathrm{alg}}(G^*)$ the category of algebraic $\overline{\mathbb{F}}_q G^*$-modules of finite $\overline{\mathbb{F}}_q$-dimension; an object of $\mathrm{Rep}_{\mathrm{alg}}(G^*)$ is identified with an algebraic (also called rational) representation of G^* over $\overline{\mathbb{F}}_q$ (that is, a G^*-representation over $\overline{\mathbb{F}}_q$ which is also a morphism of algebraic varieties). Let $\mathrm{Irr}_{\mathrm{alg}}(G^*)$ be the set of isomorphism classes of simple objects in $\mathrm{Rep}_{\mathrm{alg}}(G^*)$. Denote also by $\mathsf{K}(\mathrm{Rep}_{\mathrm{alg}}(G^*))$ the Grothendieck group of $\mathrm{Rep}_{\mathrm{alg}}(G^*)$; with multiplication given by the tensor product, $\mathsf{K}(\mathrm{Rep}_{\mathrm{alg}}(G^*))$ is thus a ring.

For $M \in \mathrm{Rep}_{\mathrm{alg}}(G^*)$, its formal character $\mathrm{ch}\, M$ with respect to a maximal torus A^* of G^* is defined as follows. The A^*-action on M gives a weight space decomposition

$$M = \bigoplus_{\lambda\in X(A^*)} M_\lambda,$$

where the weight spaces $M_\lambda := \{m \in M : zm = \lambda(z)m \text{ for all } z \in A^*\}$. (We call $\lambda \in X(A^*)$ a weight of M if $M_\lambda \neq \{0\}$.) Then we set

$$\mathrm{ch}\, M := \sum_{\lambda\in X(A^*)} \dim_{\overline{\mathbb{F}}_q} M_\lambda \cdot e(\lambda) \in \mathbb{Z}[X(A^*)]^{W_{G^*}(A^*)},$$

where $e : X(A^*) \hookrightarrow \mathbb{Z}[X(A^*)]$ is the canonical inclusion. Indeed, $M_{w\lambda} = wM_\lambda$ for every $w \in W_{G^*}(A^*)$, so $\mathrm{ch}\, M$ is $W_{G^*}(A^*)$-invariant. It is known that the map $M \longmapsto \mathrm{ch}\, M$ induces a ring isomorphism

$$\mathrm{ch} : \mathsf{K}(\mathrm{Rep}_{\mathrm{alg}}(G^*)) \xrightarrow{\sim} \mathbb{Z}[X(A^*)]^{W_{G^*}(A^*)}.$$

2.12. *The subring* $\mathsf{K}^\circ_{G^*}$ *of* K_{G^*}. — Let $\mathsf{K}^\circ_{G^*}$ be the image of the restriction map

$$\mathrm{Res}^{G^*}_{G^{*F^*}} : \mathsf{K}(\mathrm{Rep}_{\mathrm{alg}}(G^*)) \longrightarrow \mathsf{K}_{G^*}.$$

In general, $\mathsf{K}^\circ_{G^*}$ need not be the whole K_{G^*} (that is, $\mathrm{Res}^{G^*}_{G^{*F^*}}$ need not be surjective). For example: when $G^* = \mathrm{PGL}_2(\overline{\mathbb{F}}_q)$ with q odd, we have $G^* = [G^*, G^*]$, so all linear characters of G^* are trivial; but $G^{*F^*} \supsetneq [G^{*F^*}, G^{*F^*}]$, so G^{*F^*} admit a non-trivial linear character, which cannot be the restriction of a linear character of G^*.

2.13. THEOREM. [St1, Thm. 7.4][Her, Thm. 3.10] — *When the derived subgroup G^*_{der} of G^* is simply-connected (that is, when the centre of G is a torus), the restriction map $\mathrm{Res}^{G^*}_{G^{*F^*}} : \mathsf{K}(\mathrm{Rep}_{\mathrm{alg}}(G^*)) \longrightarrow \mathsf{K}_{G^*}$ is surjective, or equivalently $\mathsf{K}^\circ_{G^*} = \mathsf{K}_{G^*}$.*

The first reference proves the case where G^* is simply-connected, and the second reference generalize it to the case where G^*_{der} is simply-connected.

On comparison between $\mathbb{Z}[\frac{1}{p}]\mathsf{K}_{G^*}$ and $\mathbb{Z}[\frac{1}{p}]\mathsf{E}_G$ (part II)

2.14. *Comparison through the symmetrizing form τ_G.* [LiSh, Sec. 0] — From now on (and until the end of § 2), let Λ be a subring of $\overline{\mathbb{Q}}$ containing $\overline{\mathbb{Z}}[\frac{1}{p}]$, identify $\overline{\mathbb{Q}}\mathsf{E}_G = \overline{\mathbb{Q}}\mathsf{K}_{G^*}$ as in § 2.5, identify $\Lambda\mathsf{E}_G \subset \Lambda G^F$ via § 1.10, and consider the symmetrizing form

$$\tau_G := \tau^{\mathsf{E}} = \tau^{\mathsf{K}} = |U^F|\mathrm{ev}_{1_{G^F}} : \overline{\mathbb{Q}}\mathsf{E}_G = \overline{\mathbb{Q}}\mathsf{K}_{G^*} \longrightarrow \overline{\mathbb{Q}},$$

which induces a perfect symmetric bilinear form

$$b_G : \overline{\mathbb{Q}}\mathsf{E}_G \times \overline{\mathbb{Q}}\mathsf{K}_{G^*} \longrightarrow \overline{\mathbb{Q}}, \quad b_G(h,\pi) \longmapsto \tau_G(h\pi).$$

By § 1.16 and § 2.8(c), the restrictions $\tau_G : \Lambda\mathsf{E}_G \longrightarrow \Lambda$ and $\tau_G : \Lambda\mathsf{K}_{G^*} \longrightarrow \Lambda$ are both well-defined and are both symmetrizing forms. Thus the equality $\Lambda\mathsf{E}_G = \Lambda\mathsf{K}_{G^*}$ will hold if

$$b_G(\Lambda\mathsf{E}_G \times \Lambda\mathsf{K}_{G^*}) \subset \Lambda. \tag{2.14.1}$$

2.15. *Behaviour of K_{G^*} under central extensions.* — From now on (and until the end of § 2), we adopt the construction in § 1.26; in particular, G^* fits into an F^*-equivariant central extension

$$1 \longrightarrow Z^* \longrightarrow H^* \longrightarrow G^* \longrightarrow 1$$

where H^*_{der} is simply-connected and Z^* ia a torus central in H^*. The association of each irreducible $\overline{\mathbb{F}}_q H^{*F^*}$-module to the restriction to Z^{*F^*} of its character then induces a $\widehat{Z^{*F^*}}$-graded decomposition

$$\mathsf{K}_{H^*} = \bigoplus_{\lambda \in \widehat{Z^{*F^*}}} (\mathsf{K}_{H^*})_\lambda \ \text{ with } \ \mathsf{K}_{G^*} = (\mathsf{K}_{H^*})_0. \tag{2.15.1}$$

Here $\widehat{Z^{*F^*}}$ is the group of $\overline{\mathbb{F}}_q^\times$-valued linear characters of Z^{*F^*}. In particular, we have a ring inclusion $\mathsf{K}_{G^*} \subset \mathsf{K}_{H^*}$, and it is evident that the following diagram of rings is commutative:

$$\begin{array}{ccc} \overline{\mathbb{Q}}\mathsf{K}_{G^*} & \xrightarrow[\sim]{\mathrm{br}} & \overline{\mathbb{Q}}^{G^{*F^*}_{\mathrm{ss}}/\sim} \\ \big\downarrow & & \big\downarrow \\ \overline{\mathbb{Q}}\mathsf{K}_{H^*} & \xrightarrow[\sim]{\mathrm{br}} & \overline{\mathbb{Q}}^{H^{*F^*}_{\mathrm{ss}}/\sim} \end{array} \tag{2.15.2}$$

2.16. LEMMA. [LiSh, Sec. 4] — *Identify $\Lambda\mathsf{E}_G \subset \Lambda\mathsf{E}_H$ via (1.28.1), and identify $\Lambda\mathsf{K}_{G^*} \subset \Lambda\mathsf{K}_{H^*}$ via (2.15.1). Then $b_H = b_G$ on $\Lambda\mathsf{E}_G \times \Lambda\mathsf{K}_{G^*}$.*

This lemma shows that to verify the condition (2.14.1) for G, it suffices to verify the same condition for H, that is to verify the condition

$$b_H(\Lambda\mathsf{E}_H \times \Lambda\mathsf{K}_{H^*}) \subset \Lambda. \tag{2.16.1}$$

As H^*_{der} is simply-connected, § 2.13 shows that $\mathsf{K}_{H^*} = \mathsf{K}^\circ_{H^*}$, so the condition (2.16.1) is the same as the condition $b_H(\Lambda\mathsf{E}_H \times \Lambda\mathsf{K}^\circ_{H^*}) \subset \Lambda$. This suggest that to prove (2.14.1) for G it may be useful to show first that $b_G(\Lambda\mathsf{E}_G \times \Lambda\mathsf{K}^\circ_{G^*}) \subset \Lambda$.

Proof of lemma. The identification $\overline{\mathbb{Q}}\mathsf{E}_G = \overline{\mathbb{Q}}\mathsf{K}_{G^*}$ (§ 2.5) and the two compatibility diagrams (1.28.2) and (2.15.2) give us the following commutative diagram of rings:

$$\begin{tikzcd}
\Lambda\mathsf{E}_G \arrow[r, hook] \arrow[d, hook, "\iota^{\mathsf{E}}"'] & \overline{\mathbb{Q}}\mathsf{E}_G \arrow[d, hook, "\iota^{\mathsf{E}}"'] & \overline{\mathbb{Q}}\mathsf{K}_{G^*} \arrow[l, "\jmath^{\mathsf{E}}", "\sim"'] \arrow[r, hook] \arrow[d, hook, "\iota^{\mathsf{K}}"] & \Lambda\mathsf{K}_{G^*} \arrow[d, hook, "\iota^{\mathsf{K}}"] \\
\Lambda\mathsf{E}_H \arrow[r, hook] & \overline{\mathbb{Q}}\mathsf{E}_H & \overline{\mathbb{Q}}\mathsf{K}_{H^*} \arrow[l, "\jmath^{\mathsf{K}}", "\sim"'] \arrow[r, hook] & \Lambda\mathsf{K}_{H^*}
\end{tikzcd} \tag{2.16.2}$$

For $h \in \Lambda\mathsf{E}_G$ and $\pi \in \Lambda\mathsf{K}_{G^*}$, the commutativity of the diagram (2.16.2) implies that

$$b_H(h,\pi) = |U^F|\mathrm{ev}_{1_{H^F}}(\iota^{\mathsf{E}}(h)\cdot\jmath^{\mathsf{K}}(\iota^{\mathsf{K}}(\pi))) = |U^F|\mathrm{ev}_{1_{G^F}}(h\cdot\jmath^{\mathsf{E}}(\pi)) = b_G(h,\pi). \qquad \square$$

2.17. Lemma. [LiSh, Sec. 5] — *The bilinear form $b_G : \overline{\mathbb{Q}}\mathsf{E}_G \times \overline{\mathbb{Q}}\mathsf{K}_{G^*} \longrightarrow \overline{\mathbb{Q}}$ is extended to the bilinear form*

$$\widetilde{b}_G : \overline{\mathbb{Q}}G^F \times \overline{\mathbb{Q}}\mathsf{K}_{G^*} \longrightarrow \overline{\mathbb{Q}}$$

defined by (one of) the following formulae for $h \in \overline{\mathbb{Q}}G^F$ and $\pi \in \overline{\mathbb{Q}}\mathsf{K}_{G^*}$:

$$\widetilde{b}_G(h,\pi) = \frac{1}{|W|}\sum_{w\in W}\frac{\epsilon_G\epsilon_{T_w}}{|T_w^F|}\sum_{t\in T_w^F}\sum_{\chi\in\mathrm{Irr}_{\overline{\mathbb{Q}}}(T_w^F)} R^G_{T_w}(\chi)(h)\cdot\chi(\mathrm{Cur}^G_{T_w}(\pi)). \tag{2.17.1}$$

$$= \frac{1}{|G^F|}\sum_{S\in\mathcal{T}_G}\epsilon_G\epsilon_S\sum_{\chi\in\mathrm{Irr}_{\overline{\mathbb{Q}}}(S^F)} R^G_S(\chi)(h)\cdot\chi(\mathrm{Cur}^G_S(\pi)). \tag{2.17.2}$$

In the above, $\mathcal{T}_G$ denotes the set of F-stable maximal tori of G.

Proof. For $h \in \overline{\mathbb{Q}}\mathsf{E}_G$ and $\pi \in \overline{\mathbb{Q}}\mathsf{K}_{G^*}$, we have:

$$\begin{aligned}
\tau_G(h\pi) &= \frac{1}{|W|}\sum_{w\in W}\mathrm{ev}_{1_{T_w^F}}(\mathrm{Cur}^G_{T_w}(h\pi)) \quad (\text{by [BoKe, Eq. 3.5]})\\
&= \frac{1}{|W|}\sum_{w\in W}\mathrm{ev}_{1_{T_w^F}}(\mathrm{Cur}^G_{T_w}(h)\cdot\mathrm{Cur}^G_{T_w}(\pi)) \quad (\mathrm{Cur}^G_{T_w} \text{ is a ring homomorphism})\\
&= \frac{1}{|W|}\sum_{w\in W}\sum_{t\in T_w^F}\mathrm{Cur}^G_{T_w}(h)(t^{-1})\cdot\mathrm{Cur}^G_{T_w}(\pi)(t)\\
&= \frac{1}{|W|}\sum_{w\in W}\sum_{t\in T_w^F}\frac{\epsilon_G\epsilon_{T_w}}{|T_w^F|}\cdot\mathrm{Tr}((h,t)|H^*_c(DL^G_{T_w}))\cdot\mathrm{Cur}^G_{T_w}(\pi)(t) \quad (\text{by § 1.15})\\
&= \frac{1}{|W|}\sum_{w\in W}\frac{\epsilon_G\epsilon_{T_w}}{|T_w^F|}\sum_{t\in T_w^F}\sum_{\chi\in\mathrm{Irr}_{\overline{\mathbb{Q}}}(T_w^F)} R^G_{T_w}(\chi)(h)\cdot\chi(t)\cdot\mathrm{Cur}^G_{T_w}(\pi)(t) \quad (\text{trace formula})\\
&= \frac{1}{|W|}\sum_{w\in W}\frac{\epsilon_G\epsilon_{T_w}}{|T_w^F|}\sum_{\chi\in\mathrm{Irr}_{\overline{\mathbb{Q}}}(T_w^F)} R^G_{T_w}(\chi)(h)\cdot\chi(\mathrm{Cur}^G_{T_w}(\pi))\\
&= \sum_{S\in\mathcal{T}_G/G^F}\frac{\epsilon_G\epsilon_S}{|W_G(S)^F|}\frac{1}{|S^F|}\sum_{\chi\in\mathrm{Irr}_{\overline{\mathbb{Q}}}(S^F)} R^G_S(\chi)(h)\cdot\chi(\mathrm{Cur}^G_S(\pi))
\end{aligned}$$

$$= \frac{1}{|G^F|} \sum_{S \in \mathcal{T}_G} \epsilon_G \epsilon_S \sum_{\chi \in \mathrm{Irr}_{\overline{\mathbb{Q}}}(S^F)} R_S^G(\chi)(h) \cdot \chi(\mathrm{Cur}_S^G(\pi)).$$

In the above, $\mathcal{T}_G/G^F$ is the set of G^F-conjugacy classes in $\mathcal{T}_G$. □

2.18. Lemma. [LiSh, Sec. 6] — *Let Λ_0 be an integral domain, let $h = su \in G^F$ where $s \in G^F$ (resp. $u \in G^F$) is the semisimple (resp. unipotent) part in the Jordan decomposition of h.*

(a) $\widetilde{b}_G(h, \pi) = \dfrac{1}{|C_G(s)^{\circ F}|} \displaystyle\sum_{S \in \mathcal{T}_{C_G(s)^\circ}} \epsilon_G \epsilon_S |S^F| \cdot Q_S^{C_G(s)^\circ}(u) \cdot \mathrm{Cur}_S^G(\pi)(s^{-1}).$

(b) *Consider the following statement:*

$$\widetilde{b}_G(h, \pi) \in \Lambda_0 \quad \textit{for all } \pi \in \mathsf{K}_{G^*}^\circ. \tag{2.18.1}$$

Suppose that (2.18.1) is true when G therein is replaced by $C_G(s)^\circ$ (by [DiMi, Prop. 3.5.3], $u \in C_G(s)^\circ$ and hence $h \in C_G(s)^{\circ F}$). Then (2.18.1) is true for G.

As s is central in $C_G(s)^\circ$, (b) shows that the study of the condition $\widetilde{b}_G(h, \pi) \in \Lambda_0$ is reduced to the case where the semisimple part of h is central in G.

Proof of lemma. (a) Recall Deligne–Lusztig's character formula [DeLu, Thm. 4.2] (for each F-stable maximal torus S of G):

$$R_S^G(\chi)(h) = \frac{1}{|C_G(s)^{\circ F}|} \sum_{\substack{g \in G^F \\ g^{-1}sg \in S^F}} Q_{\mathrm{ad}(g)S}^{C_G(s)^\circ}(u) \cdot ({}^g\chi)(s). \tag{2.18.2}$$

Substituting (2.18.2) into (2.17.2), we obtain:

$$\begin{aligned}
\widetilde{b}_G(h, \pi) &= \frac{1}{|G^F|} \sum_{S \in \mathcal{T}_G} \epsilon_G \epsilon_S \sum_{\chi \in \mathrm{Irr}_{\overline{\mathbb{Q}}}(S^F)} \frac{1}{|C_G(s)^{\circ F}|} \sum_{\substack{g \in G^F \\ g^{-1}sg \in S^F}} Q_{\mathrm{ad}(g)S}^{C_G(s)^\circ}(u) \cdot \chi({}^{g^{-1}}s \cdot \mathrm{Cur}_S^G(\pi)) \\
&= \frac{1}{|G^F|} \frac{1}{|C_G(s)^{\circ F}|} \sum_{S \in \mathcal{T}_G} \epsilon_G \epsilon_S |S^F| \sum_{\substack{g \in G^F \\ g^{-1}sg \in S^F}} Q_{\mathrm{ad}(g)S}^{C_G(s)^\circ}(u) \cdot \mathrm{Cur}_S^G(\pi)({}^{g^{-1}}(s^{-1}))
\end{aligned}$$

(where we have applied the orthogonality of characters)

$$= \frac{1}{|G^F|} \frac{1}{|C_G(s)^{\circ F}|} \sum_{g \in G^F} \sum_{\substack{S \in \mathcal{T}_G \\ (\mathrm{ad}(g)S)^F \ni s}} \epsilon_G \epsilon_S |S^F| \cdot Q_{\mathrm{ad}(g)S}^{C_G(s)^\circ}(u) \cdot \mathrm{Cur}_{\mathrm{ad}(g)S}^G(\pi)(s^{-1})$$

(where we have used $\mathrm{Cur}_{\mathrm{ad}(g)S}^G(\pi)({}^g x) = \mathrm{Cur}_S^G(\pi)(x)$ for $g \in G^F$)

$$\begin{aligned}
&= \frac{1}{|C_G(s)^{\circ F}|} \sum_{\substack{S \in \mathcal{T}_G \\ S^F \ni s}} \epsilon_G \epsilon_S |S^F| \cdot Q_S^{C_G(s)^\circ}(u) \cdot \mathrm{Cur}_S^G(\pi)(s^{-1}) \quad (S \longmapsto \mathrm{ad}(g^{-1})S) \\
&= \frac{1}{|C_G(s)^{\circ F}|} \sum_{S \in \mathcal{T}_{C_G(s)^\circ}} \epsilon_G \epsilon_S |S^F| \cdot Q_S^{C_G(s)^\circ}(u) \cdot \mathrm{Cur}_S^G(\pi)(s^{-1}),
\end{aligned}$$

where the last equality holds because for $S \in \mathcal{T}_G$, $s \in S^F$ if and only if $S \subset C_G(s)^\circ$.

(b) Choose an F-stable maximal torus A of G containing s, so that A is also an F-stable maximal torus of $C_G(s)^\circ$. For $\lambda \in X(A^*)$, set

$$r_{G,\lambda} := \sum_{\mu \in W\lambda} e(\mu) \in \mathbb{Z}[X(A^*)]^{W_{G^*}(A^*)} \quad \text{and} \quad \pi_{G,\lambda} := \mathrm{ch}^{-1}(r_{G,\lambda})|_{G^{*F^*}} \in \mathsf{K}_{G^*},$$

where $W\lambda$ denotes the W-orbit of λ and $\mathrm{ch}(\cdot)$ is the formal character isomorphism

$$\mathrm{ch} : \mathsf{K}(\mathrm{Rep}_{\mathrm{alg}}(G^*)) \xrightarrow{\sim} \mathbb{Z}[X(A^*)]^{W_{G^*}(A^*)}.$$

in § 2.11. As $\{r_{G,\lambda} : \lambda \in X(A^*)\}$ generates the $\mathbb{Z}$-module $\mathbb{Z}[X(A^*)]^{W_{G^*}(A^*)}$, we see that $\{\pi_{G,\lambda} : \lambda \in X(A^*)\}$ generates the $\mathbb{Z}$-module $\mathsf{K}^\circ_{G^*}$ in § 2.12. To prove the lemma, it then suffices to show that $\widetilde{b}_G(h, \pi_{G,\lambda}) \in \Lambda_0$ for all $\lambda \in X(A^*)$.

Let S be an F-stable maximal torus of G. Choose an F^*-stable maximal torus S^* of G^* dual to S and with a duality $\widehat{\cdot} : S^F \xrightarrow{\sim} \mathrm{Irr}_{\overline{\mathbb{F}}_q}(S^{*F^*})$, and fix a choice of $g \in G^*$ such that $S^* = {}^gA^*$. This duality and the fixed embedding $\overline{\mathbb{F}}_q^\times \hookrightarrow \overline{\mathbb{Q}}^\times$ allow us to identify $\overline{\mathbb{Q}}S^F$ with $\overline{\mathbb{Q}}^{S^{*F^*}}$. For each $\mu \in X(A^*)$, set $\mu_{S^*} = {}^g\mu \in X(S^*)$, and define $\phi_S(\mu) \in S^F$ by the relation $\mu_{S^*}|_{S^{*F^*}} = \widehat{\phi_S(\mu)} \in \mathrm{Irr}_{\overline{\mathbb{F}}_q}(S^{*F^*})$. We then have a map $\phi_S : X(A^*) \longrightarrow S^F$ which extends to a ring homomorphism

$$\phi_S : \overline{\mathbb{Q}}[X(A^*)] \longrightarrow \overline{\mathbb{Q}}S^F = \overline{\mathbb{Q}}^{S^{*F^*}}.$$

The following diagram then commutes:

$$\begin{tikzcd}
\overline{\mathbb{Q}}\mathsf{K}(\mathrm{Rep}_{\mathrm{alg}}(G^*)) \arrow[r, "\mathrm{ch}", "\sim"'] \arrow[d, "\mathrm{Res}^{G^*}_{G^{*F^*}}"'] & \overline{\mathbb{Q}}[X(A^*)]^{W_{G^*}(A^*)} \arrow[d, "\phi_S"] \\
\overline{\mathbb{Q}}\mathsf{K}_{G^*} \arrow[r, "\mathrm{br}", "\sim"'] & \overline{\mathbb{Q}}^{G^{*F^*}} \arrow[r, "\mathrm{Res}^{G^{*F^*}}_{S^{*F^*}}"] & \overline{\mathbb{Q}}^{S^{*F^*}}
\end{tikzcd}$$

Combining this with (1.13.2), we see that the following diagram of rings also commutes:

$$\begin{tikzcd}
\overline{\mathbb{Q}}\mathsf{K}(\mathrm{Rep}_{\mathrm{alg}}(G^*)) \arrow[r, "\mathrm{Res}^{G^*}_{G^{*F^*}}"] \arrow[d, "\mathrm{ch}"', "\wr"] & \overline{\mathbb{Q}}\mathsf{K}_{G^*} \arrow[r, equal, "\S\,2.5"] & \overline{\mathbb{Q}}\mathsf{E}_G \arrow[r, "\mathrm{Cur}^G_S"] & \overline{\mathbb{Q}}S^F \\
\overline{\mathbb{Q}}[X(A^*)]^{W_{G^*}(A^*)} \arrow[urrr, "\phi_S"'] & & &
\end{tikzcd} \tag{2.18.3}$$

The commutative diagram (2.18.3) gives the relation

$$\mathrm{Cur}^G_S(\pi_{G,\lambda}) = \phi_S(r_{G,\lambda}), \tag{2.18.4}$$

for all $\lambda \in X(A^*)$. Via the identifications

$$W_{C_G(s)^{\circ *}}(A^*) = W_{C_G(s)^\circ}(A) \leq W_G(A) = W_{G^*}(A^*),$$

we may write $W_{G^*}(A^*)\lambda = \bigsqcup_{\lambda'\in\Omega} W_{C_G(s)^{\circ *}}(A^*)\lambda'$ for some finite subset Ω of $W_{G^*}(A^*)\lambda$, so that $r_{G,\lambda} = \sum_{\lambda'\in\Omega} r_{C_G(s)^\circ,\lambda'}$ and then $\mathrm{Cur}_S^G(\pi_{G,\lambda}) = \sum_{\lambda'\in\Omega} \mathrm{Cur}_S^{C_G(s)^\circ}(\pi_{C_G(s)^\circ,\lambda'})$ by (2.18.4). Applying (a) to $\pi = \pi_{G,\lambda}$, we thus deduce that

$$\widetilde{b}_G(h,\pi_{G,\lambda}) = \epsilon_G\epsilon_{C_G(s)^\circ}\sum_{\lambda'\in\Omega}\widetilde{b}_{C_G(s)^\circ}(h,\pi_{C_G(s)^\circ,\lambda'}), \tag{2.18.5}$$

from which $\widetilde{b}_G(h,\pi_{G,\lambda}) \in \Lambda_0$ for all $\lambda \in X(A^*)$. □

2.19. Lemma. [LiSh, Sec. 7] — *Let $h = su \in G^F$ be as in § 2.18, and suppose that s lies in the centre of G. Let $\pi \in \mathsf{K}_{G^*}$, and let $\widetilde{\pi}$ be any extension of the Brauer character of π to an ordinary $\overline{\mathbb{Q}}$-valued virtual character of G^{*F^*} (see [Se1, Thm. 33]). Then*

$$\widetilde{b}_G(h,\pi) = \langle \widetilde{\pi}, \gamma \otimes \widehat{s^{-1}} \otimes \mathrm{St}_{G^*}\rangle_{G^{*F^*}}$$

where

$$\gamma = \frac{1}{|W|}\sum_{w\in W} Q_{T_w}^G(u) R_{T_w^*}^{G^*}(\mathbf{1}) \tag{2.19.1}$$

and where $\widehat{s^{-1}}$ is a $\overline{\mathbb{Q}}^\times$-valued multiplicative character of G^{*F^*} via toric duality (§ 1.4).

Proof. We have $C_G(s)^\circ = G$, so (2.18.2) becomes $R_S^G(\chi)(h) = Q_S^G(u)\chi(s)$, and then (2.17.1) yields

$$\begin{aligned}\widetilde{b}_G(h,\pi) &= \frac{1}{|W|}\sum_{w\in W}\frac{\epsilon_G\epsilon_{T_w}}{|T_w^F|}\sum_{\chi\in\mathrm{Irr}_{\overline{\mathbb{Q}}}(T_w^F)} Q_{T_w}^G(u)\cdot\chi(s)\cdot\chi(\mathrm{Cur}_{T_w}^G(\pi))\\ &= \frac{1}{|W|}\sum_{w\in W}\epsilon_G\epsilon_{T_w}Q_{T_w}^G(u)\langle \pi|_{T_w^{*F^*}}, \widehat{s^{-1}}\rangle_{T_w^{*F^*}} \quad \text{(orthogonality of characters)}\\ &= \langle\widetilde{\pi},\gamma'\rangle_{G^{*F^*}}\end{aligned}$$

where (using [DeLu, Prop. 7.3])

$$\begin{aligned}\gamma' &= \frac{1}{|W|}\sum_{w\in W}\epsilon_G\epsilon_{T_w}Q_{T_w}^G(u)\mathrm{Ind}_{T_w^{*F^*}}^{G^{*F^*}}\widehat{s^{-1}}\\ &= \frac{1}{|W|}\sum_{w\in W}Q_{T_w}^G(u)R_{T_w^*}^{G^*}\widehat{s^{-1}}\otimes\mathrm{St}_{G^*} = \gamma\otimes\widehat{s^{-1}}\otimes\mathrm{St}_{G^*}.\end{aligned}$$ □

2.20. *Almost characters.* [Lu2, Ch. 3 - 4] (see also [Ca, Sec. 7.3]) — Let $\langle\mathsf{F}\rangle$ be the infinite cyclic group generated by the letter F; the action of $\langle\mathsf{F}\rangle$ on W given by $\mathsf{F}\cdot w = F(w)$ ($w\in W$), where F is the Frobenius action on W, gives rise to a semi-direct product $\widehat{W} := W \rtimes \langle\mathsf{F}\rangle$. Let c be the order of the automorphism F on W (when G is split, we have $c = 1$), and denote by $\widehat{W}_{\mathrm{ex}}$ the set of all $\phi \in \mathrm{Irr}_{\mathbb{Q}}(W)$ which can be extended to a $\mathbb{Q}$-valued irreducible character of $\widehat{W}$ (by [Sp3, Cor. 1.15], every irreducible representation of W over a characteristic 0 field is defined over $\mathbb{Q}$); for each $\phi\in\widehat{W}_{\mathrm{ex}}$, there exists such an extension (in fact, exactly two) $\widetilde{\phi}\in\mathrm{Irr}_{\mathbb{Q}}(\widehat{W})$. Fixing a choice of such $\widetilde{\phi}$, we then call

$$R_{\widetilde{\phi}}^{G^*} := \frac{1}{|W|}\sum_{w\in W}\widetilde{\phi}(wF)R_{T_w^*}^{G^*}(\mathbf{1})$$

an *almost character* of G^{*F^*}.

2.21. *Bad primes.* [Sp1] — Denoting by R the root system of G, a prime number ℓ is called *bad* for G if one of the following three conditions holds: (i) $\ell = 2$, and R has an irreducible factor not of type A; (ii) $\ell = 3$, and R has an irreducible factor of exceptional type (G_2, F_4, E_6, E_7, or E_8); (iii) $\ell = 5$, and R has an irreducible factor of type E_8. From now on (and until the end of § 2), we set

$$M_G = \text{product of all bad primes for } G.$$

(If G is of type A, there is no bad prime for G and we set $M_G = 1$.) Then we have $M_G = M_{G^*}$ by classification of root data, and we also have $M_G = M_H$ as G and H have the same type of root data.

2.22. Theorem. [LiSh, Sec. 7] — *Each almost character $R_{\widetilde{\phi}}^{G^*}$ is a $\overline{\mathbb{Z}}[\frac{1}{M_G}]$-linear combination of irreducible $\overline{\mathbb{Q}}$-valued unipotent characters of G^{*F^*}.*

Proof. The case where G^* has connected centre is proved in [Lu2, Ch. 4]. Indeed, in this case, [Lu2, Thm. 4.23] expresses $R_{\widetilde{\phi}}$ as a linear combination of unipotent characters of G^{*F^*}. By [Lu2, (4.21.7)], the denominators divide the orders of certain groups $\mathcal{G}_{\mathcal{F}}$ of the form $\prod \mathcal{G}_{\mathcal{F}_i}$ where the product is over the irreducible factors of the root system of G^*. Each $\mathcal{G}_{\mathcal{F}_i}$ is defined in a case-by-case fashion in [Lu2, 4.4 - 4.13] in a way depending only on the corresponding irreducible factor of the root system, and has order divisible only by bad primes for that factor.

If G^* does not have connected centre, we choose a short exact sequence

$$1 \to G^* \to (H')^* \to (Z')^* \to 1$$

as in (1.26.2) (with the roles of G^* and G reversed). Extending the chosen maximal F^*-stable torus and Borel from G^* to $(H')^*$ as in Section 1.26, we may identify the Weyl groups of G^* and $(H')^*$. Using § 1.27(c) (with $\chi = \mathbf{1}$ therein), we then have

$$R_{\widetilde{\phi}}^{(H')^*}|_{G^{*F^*}} = R_{\widetilde{\phi}}^{G^*},$$

whence $R_{\widetilde{\phi}}^{G^*}$ is $\overline{\mathbb{Z}}[\frac{1}{M_G}]$-linear combination of restrictions to G^{*F^*} of unipotent characters of $(H')^{*F^*}$. However, the restriction to G^{*F^*} of a unipotent character of $(H')^{*F^*}$ is a unipotent character by [DiMi, Prop. 11.3.8], so the theorem is proved. □

2.23. Lemma. [LiSh, Sec. 7] — *The sum γ in (2.19.1) is a finite $\overline{\mathbb{Z}}[\frac{1}{M_G}]$-linear combination of almost characters of G^{*F^*}.*

Proof. We have

$$\begin{aligned}
\gamma &= \frac{1}{|W|} \sum_{w \in W} R_{T_w}^G(\mathbf{1})(u) R_{T_w^*}^{G^*}(\mathbf{1}) \\
&= \frac{1}{|W|} \sum_{w \in W} R_{T_w}^G(\mathbf{1})(u) \sum_{\phi \in \widehat{W}_{\mathrm{ex}}} \widetilde{\phi}(wF) R_{\widetilde{\phi}}^{G^*} \quad \text{(see [Ca, p. 76])} \\
&= \sum_{\phi \in \widehat{W}_{\mathrm{ex}}} R_{\widetilde{\phi}}^G(u) R_{\widetilde{\phi}}^{G^*};
\end{aligned}$$

by § 2.22 and the fact that character values of representations of finite groups are algebraic integers, all $R_{\widetilde{\phi}}^G$ must take values in $\overline{\mathbb{Z}}[\frac{1}{M_G}]$. $\square$

Remark. Although not needed in the sequel, γ is in fact a finite $\overline{\mathbb{Z}}$-linear combination of almost characters of G^{*F^*}. Let us now give a proof of this fact.

By [Sh, Thm. 5.5] (see also [DiMi, Thm. 13.2.3] and [Lau]),

$$Q_{T_w}^G(u) = \mathrm{Tr}(wF|H_c^*(\mathcal{B}_u)) \ \ (w \in W) \tag{2.23.1}$$

where $\mathcal{B}_u$ is the variety of Borel subgroups of G containing u. Upon considering the composition series of the finite-$\overline{\mathbb{Q}}_\ell$-dimensional $\overline{\mathbb{Q}}_\ell\widehat{W}$-module $H_c^*(\mathcal{B}_u)$ ($\ell \neq p$), the trace $\mathrm{Tr}(wF|H_c^*(\mathcal{B}_u))$ in (2.23.1) is the finite sum of $\mathrm{Tr}(wF|V)$ where V runs through the composition factors of the preceding composition series (so each V is a finite-$\overline{\mathbb{Q}}_\ell$-dimensional irreducible $\overline{\mathbb{Q}}_\ell\widehat{W}$-module). Take such a composition factor V.

(i) Suppose that $V|_W$ is not an irreducible $\overline{\mathbb{Q}}_\ell W$-module. Then Clifford's theory says that $V|_W = V' \oplus FV' \oplus \cdots \oplus F^m V'$ for some $V' \in \mathrm{Irr}_{\overline{\mathbb{Q}}_\ell}(W)$ and some $m \in \mathbb{N}^*$, so the matrix of $wF \in \mathrm{End}_{\overline{\mathbb{Q}}_\ell}(V|_W)$ has zero diagonal, whence $\mathrm{Tr}(wF|V) = 0$.

(ii) Suppose that $V|_W$ is an irreducible $\overline{\mathbb{Q}}_\ell W$-module. As every irreducible $\overline{\mathbb{Q}}_\ell W$-module is defined over $\mathbb{Q}$ [Sp2, Cor. 1.15], there is an irreducible $\mathbb{Q}W$-module V_1 such that $V|_W = \overline{\mathbb{Q}}_\ell V_1$. By definition, $V|_W$ is extended to the $\overline{\mathbb{Q}}_\ell W$-module V, so V_1 can be extended to a $\mathbb{Q}\widehat{W}$-module $\widetilde{V_1}$ on which F^c acts trivially (compare [Ca, Sec. 7.3]), and then

$$R_{\widetilde{V_1}} := \frac{1}{|W|} \sum_{w \in W} \mathrm{Tr}(wF|\widetilde{V_1}) R_{T_w^*}^{G^*}(\mathbf{1})$$

is an almost character of G^{*F^*}. As $V|_W = (\overline{\mathbb{Q}}_\ell \widetilde{V_1})|_W$ as irreducible $\overline{\mathbb{Q}}_\ell W$-modules, we have $\mathrm{Tr}(wF^i|V) = \mathrm{Tr}(wF^i|\overline{\mathbb{Q}}_\ell\widetilde{V_1}) \cdot \chi_V(F^i)$ ($w \in W$, $i \in \mathbb{Z}$) for some linear character $\chi_V : \langle F \rangle \longrightarrow \overline{\mathbb{Q}}_\ell^\times$, so that

$$\frac{1}{|W|} \sum_{w \in W} \mathrm{Tr}(wF|V) R_{T_w^*}^{G^*}(\mathbf{1}) = \chi_V(F) \cdot R_{\widetilde{V_1}}.$$

By [De, Lem. 1.7], the eigenvalues of the endomorphism F on $H_c^*(\mathcal{B}_u)$ all lie in $\overline{\mathbb{Z}}$, so χ_V takes values in $\overline{\mathbb{Z}}^\times$.

In conclusion,

$$\gamma = \frac{1}{|W|} \sum_{V|_W \in \mathrm{Irr}(W)} \mathrm{Tr}(wF|V) R_{T_w^*}^{G^*}(\mathbf{1}) = \sum_{V|_W \in \mathrm{Irr}(W)} \chi_V(F) \cdot R_{\widetilde{V_1}}$$

is a $\overline{\mathbb{Z}}$-linear combination of almost characters of G^{*F^*}.

2.24. Corollary (of § 2.19 and § 2.23; see also [LiSh, Sec. 7]). — *For all $h \in G^F$ whose semisimple part is central in G and for all $\pi \in \mathsf{K}_{G^*}$, we have $\widetilde{b}_G(h, \pi) \in \overline{\mathbb{Z}}[\frac{1}{M_G}]$.*

2.25. Theorem. [LiSh] — *We have $\overline{\mathbb{Z}}[\frac{1}{pM_G}]\mathsf{E}_G = \overline{\mathbb{Z}}[\frac{1}{pM_G}]\mathsf{K}_{G^*}$ (under the identification $\overline{\mathbb{Q}}\mathsf{E}_G = \overline{\mathbb{Q}}\mathsf{K}_{G^*}$ in § 2.5.)*

Proof. Observe that a prime number that is bad for $C_G(x)^\circ$ with x a semisimple element of G^F is also bad for G; indeed, this follows from the definition of bad primes (§ 2.21) and from the following two facts: (i) if G is simple of type A (resp. of classical type), then the centralizer of every semisimple element of G has only factors of type A (resp. of classical type); (ii) if G is simple of type G_2, F_4, E_6 or E_7, then the centralizer of every semisimple element of G cannot have factors of type E_8 (for dimensional reasons). Thus § 2.18(b) and § 2.24 together imply that $\widetilde{b}_G(h,\pi) \in \overline{\mathbb{Z}}[\frac{1}{M_G}]$ for all $h \in G^F$ and all $\pi \in \mathsf{K}^\circ_{G^*}$. As $\widetilde{b}_G$ is an extension of b_G (§ 2.17), we thus have

$$b_G(\Lambda[\tfrac{1}{M_G}]\mathsf{E}_G \times \Lambda[\tfrac{1}{M_G}]\mathsf{K}^\circ_{G^*}) \subset \Lambda[\tfrac{1}{M_G}]. \tag{2.25.1}$$

(Recall from § 2.14 that Λ denotes a subring of $\overline{\mathbb{Q}}$ containing $\overline{\mathbb{Z}}[\frac{1}{p}]$.) Applying (2.25.1) to H and using that $M_H = M_G$ (§ 2.21) and $\mathsf{K}^\circ_{H^*} = \mathsf{K}_{H^*}$ (§ 2.13; recall that H^*_{der} is simply-connected), we get

$$b_H(\Lambda[\tfrac{1}{M_G}]\mathsf{E}_H \times \Lambda[\tfrac{1}{M_G}]\mathsf{K}_{H^*}) \subset \Lambda[\tfrac{1}{M_G}],$$

so § 2.16 implies that

$$b_G(\Lambda[\tfrac{1}{M_G}]\mathsf{E}_G \times \Lambda[\tfrac{1}{M_G}]\mathsf{K}_{G^*}) \subset \Lambda[\tfrac{1}{M_G}],$$

which is the condition (2.14.1) with Λ replaced by $\Lambda[\frac{1}{M_G}]$. We therefore conclude from § 2.14 that $\Lambda[\frac{1}{M_G}]\mathsf{E}_G = \Lambda[\frac{1}{M_G}]\mathsf{K}_{G^*}$. □

2.26. Corollary (of § 2.25). — *The diagram in § 2.10 descends to the following commutative diagram of $\mathbb{Z}[\frac{1}{pM}]$-algebras:*

$$\begin{tikzcd}
\mathbb{Z}[\frac{1}{pM_G}]\mathsf{E}_G \arrow[r, leftrightarrow, "\sim"] \arrow[d, "\mathrm{Cur}^G"'] & \mathbb{Z}[\frac{1}{pM_G}]\mathsf{K}_{G^*} \arrow[d, "\mathrm{Res}"] \\
\prod_{w\in W} \mathbb{Z}[\frac{1}{pM_G}]T_w^F \arrow[r, leftrightarrow, "\sim"] & \prod_{w\in W} \mathbb{Z}[\frac{1}{pM_G}]\mathsf{K}_{T_w^*}
\end{tikzcd}$$

3. Algebras from the invariant theory

Langlands dual groups

3.1. *The Langlands dual.* — Recall the pair (G,T) and the associated root datum $(X(T), R(G,T), Y(T), R^\vee(G,T))$ from §§ 1.2 - 1.3. By definition, a Langlands dual of (G,T) is a pair $(G^\vee, T^\vee)$ defined and split over $\mathbb{Z}$ and obtained by assigning its character group (resp. cocharacter group, resp. set of roots, resp. set of coroots) as $Y(T)$ (resp. $X(T)$, resp. $R^\vee$, resp. R). From now on, we fix such a Langlands dual $(G^\vee, T^\vee)$ (all choices are isomorphic).

Note that the Deligne–Lusztig dual pair (G^*, T^*) (§ 1.3) may be obtained from $(G^\vee, T^\vee)$ through the base change $\mathrm{Spec}(\overline{\mathbb{F}}_q) \longrightarrow \mathrm{Spec}(\mathbb{Z})$. We have the identifications $X(T^\vee) = X(T^*) = Y(T)$ and $Y(T^\vee) = Y(T^*) = X(T)$; moreover, the Weyl groups of $(G^\vee, T^\vee)$ and of (G^*, T^*) are both identified with the Weyl group W of (G,T).

3.2. *The automorphism $\tau^\vee$ and the endomorphism $F^\vee$.* — Let τ^* be the automorphism on $X(T^*) = \mathrm{Hom}_{\mathrm{alg}}(T^*, \mathbb{G}_m)$ induced by the arithmetic Frobenius endomorphisms φ on T^* and $\mathbb{G}_m$; more precisely, $\tau^*(\lambda) := \varphi^{-1} \circ \lambda \circ \varphi$ for $\lambda \in X(T^*)$. Via the identification $X(T^\vee) = X(T^*)$ we obtain an automorphism $\tau^\vee$ on $X(T^\vee)$ and hence on $T^\vee$. Using the same identification $X(T^\vee) = X(T^*)$, the Frobenius endomorphism F^* on $X(T^*)$ induces an endomorphism $F^\vee$ on $X(T^\vee)$ and then on $T^\vee$. We have the following properties (compare [DiMi, Ch. 4]):

(a) On both $X(T^\vee)$ and $T^\vee$, the endomorphisms $\tau^\vee \circ F^\vee$ and $F^\vee \circ \tau^\vee$ are equal; these endomorphisms are the multiplication by q on $X(T^\vee)$ and are $(\cdot)^q$ on $T^\vee$.

(b) The following statements are equivalent: (i) G is split over $\mathbb{F}_q$; (ii) G^* is split over $\mathbb{F}_q$; (iii) $\tau^\vee = \mathrm{id}$ on $X(T^\vee)$; (iv) $F^\vee = q$ on $X(T^\vee)$; (v) $F^\vee = (\cdot)^q$ on $T^\vee$.

The ring $\mathsf{B}_{G^\vee}$ of the $\mathbb{Z}$-scheme $(T^\vee /\!\!/ W)^{F^\vee}$

3.3. *Definition of the ring $\mathsf{B}_{G^\vee}$.* — As $X(T^\vee)$ is an abelian group of finite rank, the group ring $\mathbb{Z}[X(T^\vee)]$ is a finitely generated commutative $\mathbb{Z}$-algebra, and we know that $T^\vee = \mathrm{Spec}(\mathbb{Z}[X(T^\vee)])$. The Weyl group $W \simeq N_{G^\vee}(T^\vee)/T^\vee$ acts on $T^\vee$ by conjugation and hence on $\mathbb{Z}[X(T^\vee)]$ (by adjoint action). The ring $\mathbb{Z}[X(T^\vee)]^W$, consisting of elements of $\mathbb{Z}[X(T^\vee)]$ fixed by the W-action, is also finitely generated (see [Se2, pf. of Prop. III.18]). We then consider the categorical quotient $T^\vee /\!\!/ W := \mathrm{Spec}(\mathbb{Z}[X(T^\vee)]^W)$, which is an affine $\mathbb{Z}$-scheme.

The endomorphism $F^\vee$ on $X(T^\vee)$ induces an endomorphism $F^\vee$ on $\mathbb{Z}[X(T^\vee)]^W$ (because $F^\vee(W) = W$) and thus an $F^\vee$-action on $T^\vee /\!\!/ W$, so we have the fixed-point subscheme $(T^\vee /\!\!/ W)^{F^\vee}$, which is also an affine $\mathbb{Z}$-scheme. We define $\mathsf{B}_{G^\vee}$ as the ring of functions of the affine scheme $(T^\vee /\!\!/ W)^{F^\vee}$, so that

$$(T^\vee /\!\!/ W)^{F^\vee} = \mathrm{Spec}(\mathsf{B}_{G^\vee}) \quad \text{and} \quad \mathsf{B}_{G^\vee} = \mathbb{Z}[X(T^\vee)]^W / I$$

where I is the ideal of $\mathbb{Z}[X(T^\vee)]^W$ generated by the subset $\{F^\vee f - f : f \in \mathbb{Z}[X(T^\vee)]^W\}$; then $\mathsf{B}_{G^\vee}$ is a finitely generated commutative $\mathbb{Z}$-algebra.

3.4. *A reducedness problem and the ring $\mathsf{B}_{G^\vee,\mathrm{red}}$.* — In the algebro-geometric viewpoint, we wish to know whether the scheme $(T^\vee /\!\!/ W)^{F^\vee}$ is reduced or not; that is,

whether the ring $\mathsf{B}_{G^\vee}$ is reduced or not. We shall see later (§ 3.17) that $\mathsf{B}_{G^\vee}$ is reduced when the derived subgroup $G^\vee_{\mathrm{der}}$ of $G^\vee$ is simply-connected.

Beyond the case of simply-connected $G^\vee_{\mathrm{der}}$, so far we haven't developed a general theory to determine if $\mathsf{B}_{G^\vee}$ is reduced or not, though we have verified directly that $\mathsf{B}_{G^\vee}$ is reduced when $G = \mathrm{SO}_{2n}(\overline{\mathbb{F}}_q)$ for q odd (using combinatorics of the root system of SO_{2n}; see § A for the details), when $G = \mathrm{SL}_2(\overline{\mathbb{F}}_q)$ for all q (simple computation), or when $G = \mathrm{SL}_3(\overline{\mathbb{F}}_q)$ for $q \in \{2, 3\}$ (direct calculation via Gröbner basis) — it is thus hoped that a uniform approach can be found even just for all $G = \mathrm{SL}_3(\overline{\mathbb{F}}_q)$!

At the moment, let us denote by $\mathsf{B}_{G^\vee,\,\mathrm{red}}$ the reduced ring derived from $\mathsf{B}_{G^\vee}$; we have $\mathsf{B}_{G^\vee,\mathrm{red}} = \mathbb{Z}[X(T^\vee)]^W/\sqrt{I}$, where $\sqrt{I} = \{f \in \mathbb{Z}[X(T^\vee)]^W : f^m \in I \text{ for some } m \in \mathbb{N}^*\}$ is the radical of I. So $\mathsf{B}_{G^\vee}$ is reduced if and only if $\mathsf{B}_{G^\vee,\,\mathrm{red}}$ is equal to $\mathsf{B}_{G^\vee}$. Besides, $\mathsf{B}_{G^\vee,\,\mathrm{red}}$ is also a finitely generated commutative $\mathbb{Z}$-algebra (see § 3.3).

3.5. *Decompositions of the algebras $k\mathsf{B}_{G^\vee}$ and $k\mathsf{B}_{G^\vee,\,\mathrm{red}}$.* — Let k be an algebraically closed field. The set of k-points of $(T^\vee /\!\!/ W)^{F^\vee} = \mathrm{Spec}(\mathsf{B}_{G^\vee})$ may be identified as:

$$(T^\vee /\!\!/ W)^{F^\vee}(k) = \mathrm{Specm}(k\mathsf{B}_{G^\vee}) = (T^\vee(k)/W)^{F^\vee} = \left(\bigcup_{w\in W} T^\vee(k)^{wF^\vee}\right)\Big/W.$$

(Notation: when a set X is equipped with a (left) W-action, we denote by X/W the set of W-orbits in X.) Each $T^\vee(k)^{wF^\vee}$ may be identified as $T^{*wF^*} \simeq T^{*F^*}_w$ (with respect to an embedding $\overline{\mathbb{F}}_q^\times \hookrightarrow k^\times$) and is thus a finite set (compare § 3.6 below); the set $\mathrm{Specm}(k\mathsf{B}_{G^\vee})$ is thus finite, so $k\mathsf{B}_{G^\vee}$ is a finite-dimensional vector space over k and in particular an artinian k-algebra. Therefore, if we denote by $(k\mathsf{B}_{G^\vee})_{\mathfrak{m}}$ the localization of $k\mathsf{B}_{G^\vee}$ at the maximal ideal $\mathfrak{m} \in (T^\vee /\!\!/ W)^{F^\vee}(k) = \mathrm{Specm}(k\mathsf{B}_{G^\vee})$, then the map

$$k\mathsf{B}_{G^\vee} \longrightarrow \prod_{\mathfrak{m}\in(T^\vee /\!\!/ W)^{F^\vee}(k)} (k\mathsf{B}_{G^\vee})_{\mathfrak{m}}, \quad f \longmapsto (f)_{\mathfrak{m}\in(T^\vee /\!\!/ W)^{F^\vee}(k)}$$

is a k-algebra isomorphism, where each $(k\mathsf{B}_{G^\vee})_{\mathfrak{m}} \simeq (k\mathsf{B}_{G^\vee})/\mathfrak{m}^N$ as k-algebras for some $N \in \mathbb{N}^*$ depending on $\mathfrak{m}$. We have the following equivalent conditions:

$$\begin{aligned} k\mathsf{B}_{G^\vee} \text{ is reduced} &\iff \text{each } (k\mathsf{B}_{G^\vee})_{\mathfrak{m}} \simeq (k\mathsf{B}_{G^\vee})/\mathfrak{m} = k \text{ as } k\text{-algebras} \\ &\iff k\mathsf{B}_{G^\vee} \simeq k^{(T^\vee /\!\!/ W)^{F^\vee}(k)} \text{ as } k\text{-algebras} \\ &\iff \dim_k(k\mathsf{B}_{G^\vee}) = |(T^\vee /\!\!/ W)^{F^\vee}(k)|. \end{aligned}$$

The above discussion also applies to the reduced version $(T^\vee /\!\!/ W)^{F^\vee}_{\mathrm{red}} = \mathrm{Spec}(\mathsf{B}_{G^\vee,\,\mathrm{red}})$. When $k = \overline{\mathbb{Q}}$ (or other fields of characteristic zero), the reducedness of $\mathsf{B}_{G^\vee,\,\mathrm{red}}$ imply that of $\overline{\mathbb{Q}}\mathsf{B}_{G^\vee,\,\mathrm{red}}$, so that:

(a) $\overline{\mathbb{Q}}\mathsf{B}_{G^\vee,\,\mathrm{red}} \simeq \overline{\mathbb{Q}}^{(T^\vee /\!\!/ W)^{F^\vee}(\overline{\mathbb{Q}})}$ as $\overline{\mathbb{Q}}$-algebras;

(b) $\dim_{\overline{\mathbb{Q}}}(\overline{\mathbb{Q}}\mathsf{B}_{G^\vee,\,\mathrm{red}}) = |(T^\vee /\!\!/ W)^{F^\vee}(\overline{\mathbb{Q}})|$.

Remark. In general, $k\mathsf{B}_{G^\vee,\,\mathrm{red}}$ need not be reduced. For example, let $G^\vee = T^\vee = \mathbb{G}_m$ with $F^\vee = (\cdot)^q$, so $\mathsf{B}_{G^\vee} = \mathbb{Z}[X^{\pm 1}]/(X^{q-1} - 1) = \mathsf{B}_{G^\vee,\,\mathrm{red}}$ (X an indeterminate); if ℓ is a prime number dividing $q - 1$, then $\overline{\mathbb{F}}_\ell\mathsf{B}_{G^\vee,\,\mathrm{red}} = \overline{\mathbb{F}}_\ell[X^{\pm 1}]/((X^{\frac{q-1}{\ell}} - 1)^\ell)$ is not reduced.

3.6. Lemma. — *Denote by $G^*_{\mathrm{ss}}/\sim$ the set of G^*-conjugacy classes of semisimple elements of G^*, and define $G^\vee(\overline{\mathbb{Q}})_{\mathrm{ss}}/\sim$ in a similar way. Then*

$$|(T^\vee /\!\!/ W)^{F^\vee}(\overline{\mathbb{Q}})| = |(T^* /\!\!/ W)^{F^*}(\overline{\mathbb{F}}_q)| = |(G^\vee(\overline{\mathbb{Q}})_{\mathrm{ss}}/\sim)^{F^\vee}| = |(G^*_{\mathrm{ss}}/\sim)^{F^*}|.$$

Proof. These equalities come from the chosen embedding $\kappa : \overline{\mathbb{F}}_q^\times \hookrightarrow \overline{\mathbb{Q}}^\times$ and the following observations (stated for G^* but also valid for $G^\vee(\overline{\mathbb{Q}})$): (i) G^*_{ss} is the union of all maximal tori of G^*; (ii) for $z \in T^*$, $Wz \in (T^* /\!\!/ W)^{F^*}(\overline{\mathbb{F}}_q)$ if and only if $z \in T^{*wF^*}$ for some $w \in W$; (iii) elements of each T^{*wF^*} are of finite order prime to p; (iv) The canonical bijection $(G^*_{\mathrm{ss}}/\sim) \simeq T^* /\!\!/ W$ induced by the diagonalization is compatible with respect to F^* (compare [Lan, Sec. 3.1 & App. B]). □

3.7. Lemma. — *The $\mathbb{Z}$-module $\mathsf{B}_{G^\vee,\mathrm{red}}$ has no nonzero $\mathbb{Z}$-torsion elements.*

Proof. Let $0 \neq f \in \mathsf{B}_{G^\vee,\mathrm{red}}$. As $\mathsf{B}_{G^\vee,\mathrm{red}}$ is a finitely generated $\mathbb{Z}$-algebra (§ 3.4), it is a Jacobson ring, so its Jacobson radical (intersection of maximal ideals) is equal to its nilradical (intersection of prime ideals), which is zero because $\mathsf{B}_{G^\vee,\mathrm{red}}$ is reduced. Thus there is a maximal ideal $\mathfrak{m}$ of $\mathsf{B}_{G^\vee,\mathrm{red}}$ such that $f \notin \mathfrak{m}$. Consider the field $k = \mathsf{B}_{G^\vee,\mathrm{red}}/\mathfrak{m}$, which is a finite field because $\mathsf{B}_{G^\vee,\mathrm{red}}$ is a finitely-generated ring; for the canonical quotient map $t : \mathsf{B}_{G^\vee,\mathrm{red}} \longrightarrow k$ which represents a k-point of $\mathrm{Spec}(\mathsf{B}_{G^\vee,\mathrm{red}})$, we then have $f(t) = t(f) \neq 0 \in k$. We may and we shall replace k by its algebraic closure, so we shall write $k = \overline{\mathbb{F}}_d$ for some prime number d.

Let us show that the $\overline{\mathbb{F}}_d$-point $t : \mathsf{B}_{G^\vee,\mathrm{red}} \longrightarrow \overline{\mathbb{F}}_d$ can be canonically lifted to a $\overline{\mathbb{Z}}_d$-point $t' : \mathsf{B}_{G^\vee,\mathrm{red}} \longrightarrow \overline{\mathbb{Z}}_d$. As $t \in \mathrm{Specm}(\overline{\mathbb{F}}_d\mathsf{B}_{G^\vee,\mathrm{red}}) = (T^\vee /\!\!/ W)^{F^\vee}(\overline{\mathbb{F}}_d)$, we may write $t = Ws$ where $s \in T^\vee(\overline{\mathbb{F}}_d)^{wF^\vee}$ for some $w \in W$ (see § 3.5). Using the canonical lifting $i : \overline{\mathbb{F}}_d^\times \hookrightarrow \overline{\mathbb{Z}}_d^\times$ of d'-th roots of unity (Hensel's lemma), our $s \in T^\vee(\overline{\mathbb{F}}_d) = \mathrm{Hom}(X(T^\vee), \overline{\mathbb{F}}_d^\times)$ can be canonically lifted to some $s' \in T^\vee(\overline{\mathbb{Z}}_d) = \mathrm{Hom}(X(T^\vee), \overline{\mathbb{Z}}_d^\times)$, and it can be checked that $s' \in T^\vee(\overline{\mathbb{Z}}_d)^{wF^\vee}$, so $t' := Ws' \in (T^\vee /\!\!/ W)^{F^\vee}(\overline{\mathbb{Z}}_d)$; this t' is a $\overline{\mathbb{Z}}_d$-point $t' : \mathsf{B}_{G^\vee,\mathrm{red}} \longrightarrow \overline{\mathbb{Z}}_d$ which lifts the $\overline{\mathbb{F}}_d$-point t, in the sense that the following diagram is commutative (where r_d is the standard reduction map):

$$\begin{tikzcd} & \overline{\mathbb{Z}}_d \arrow[d, two heads, "r_d"] \\ \mathsf{B}_{G^\vee,\mathrm{red}} \arrow[ur, "t'"] \arrow[r, "t"'] & \overline{\mathbb{F}}_d \end{tikzcd}$$

Now, for our nonzero element $f \in \mathsf{B}_{G^\vee,\mathrm{red}}$ at the beginning of the proof, we have seen that $t(f) \neq 0 \in \overline{\mathbb{F}}_d$, so $t'(f) \neq 0 \in \overline{\mathbb{Z}}_d$ thanks to the above lifting diagram. Suppose that $n \cdot f = 0 \in \mathsf{B}_{G^\vee,\mathrm{red}}$ for some $n \in \mathbb{Z}$. Then $n \cdot t'(f) = t'(n \cdot f) = 0 \in \overline{\mathbb{Z}}_d$ while $t'(f) \neq 0$, whence $n = 0$. This shows that f is not a $\mathbb{Z}$-torsion element of $\mathsf{B}_{G^\vee,\mathrm{red}}$. □

3.8. Corollary (of § 3.7). — *The natural map $\mathsf{B}_{G^\vee,\mathrm{red}} \longrightarrow \overline{\mathbb{Q}}\mathsf{B}_{G^\vee,\mathrm{red}}$ is injective; combining this with the identification $\overline{\mathbb{Q}}\mathsf{B}_{G^\vee,\mathrm{red}} = \overline{\mathbb{Q}}^{(T^\vee /\!\!/ W)^{F^\vee}(\overline{\mathbb{Q}})}$ in § 3.5(a), we obtain a canonical injection of $\mathsf{B}_{G^\vee,\mathrm{red}}$ into $\overline{\mathbb{Q}}^{(T^\vee /\!\!/ W)^{F^\vee}(\overline{\mathbb{Q}})}$.*

3.9. Proposition. — *The $\mathbb{Z}$-module $\mathsf{B}_{G^\vee,\mathrm{red}}$ is free of rank*

$$\mathrm{rank}_{\mathbb{Z}}\, \mathsf{B}_{G^\vee,\mathrm{red}} = |(T^\vee /\!\!/ W)^{F^\vee}(\overline{\mathbb{Q}})| = |(G^*_{\mathrm{ss}}/\sim)^{F^*}|.$$

Remark. Having just seen that $\mathsf{B}_{G^\vee,\mathrm{red}}$ is without nonzero $\mathbb{Z}$-torsion (§ 3.7), the proposition will be proved once we have the finite-generacy of $\mathsf{B}_{G^\vee,\mathrm{red}}$ as a $\mathbb{Z}$-module; however, it seems difficult to prove this finite-generacy directly.

Proof of proposition, admitting § 3.17. Let us admit for the moment the results in § 3.17, which say that this proposition is true when the derived group $G^\vee_{\mathrm{der}}$ of $G^\vee$ is simply-connected. Let us adopt the construction in § 1.26 for G. The exact sequence (1.26.2) then induces an exact sequence of reductive groups over $\mathbb{Z}$,

$$1 \longrightarrow Z^\vee \longrightarrow H^\vee \longrightarrow G^\vee \longrightarrow 1,$$

with $H^\vee_{\mathrm{der}}$ simply-connected and with $Z^\vee$ a central torus in $H^\vee$; then the Weyl group of $H^\vee$ is also W. Let $T^\vee_H$ denote the $F^\vee$-stable maximal torus of $H^\vee_{\mathrm{der}}$ which maps onto $T^\vee$. We then have a surjective morphism of reduced $\mathbb{Z}$-schemes, $(T^\vee_H /\!\!/ W)^{F^\vee}_{\mathrm{red}} \longrightarrow (T^\vee /\!\!/ W)^{F^\vee}_{\mathrm{red}}$, so that in the level of ring we obtain an inclusion of rings $\mathsf{B}_{G^\vee,\mathrm{red}} \subset \mathsf{B}_{H^\vee,\mathrm{red}}$. As $H^\vee_{\mathrm{der}}$ is simply-connected, § 3.17 tells us that $\mathsf{B}_{H^\vee,\mathrm{red}}$ is a free $\mathbb{Z}$-module of finite rank; thus its $\mathbb{Z}$-submodule $\mathsf{B}_{G^\vee,\mathrm{red}}$ is also free of finite rank. This obtained, the rank of $\mathsf{B}_{G^\vee,\mathrm{red}}$ may be calculated from §§ 3.5 - 3.6. □

On the reducedness of $\mathsf{B}_{G^\vee}$

3.10. *The derived subgroup G_{der}.* (Compare [Her, App.].) — We shall freely use the following properties of the derived subgroup G_{der} of G:

(a) we have $G = R(G).G_{\mathrm{der}}$ with $R(G) \cap G_{\mathrm{der}}$ being a finite set (here $R(G)$ is the reductive radical of G; it is the connected identity component of the centre of G);

(b) $T_{\mathrm{der}} := T \cap G_{\mathrm{der}}$ is a maximal torus of G_{der}, and $\overline{T} := G/G_{\mathrm{der}}$ is a torus on which the Weyl group $W = N_G(T)/T$ acts trivially; the Weyl group of $(G_{\mathrm{der}}, T_{\mathrm{der}})$ is identified with W;

(c) we have a canonical exact sequence of tori $1 \longrightarrow T_{\mathrm{der}} \longrightarrow T \longrightarrow \overline{T} \longrightarrow 1$, which induces the following exact sequences of groups:

$$1 \longrightarrow (T_{\mathrm{der}})^{wF} \longrightarrow T^{wF} \longrightarrow \overline{T}^{wF} (= \overline{T}^F) \longrightarrow 1 \quad (w \in W);$$
$$0 \longrightarrow X(\overline{T}) \longrightarrow X(T) \longrightarrow X(T_{\mathrm{der}}) \longrightarrow 0;$$

the last exact sequence gives the identifications $X(\overline{T}) = X(T)^W = X^0(T)$ where $X^0(T) := \{\lambda \in X(T) : \langle \lambda, \alpha^\vee \rangle = 0 \text{ for all } \alpha \in \Delta\}$.

We shall apply these results on the dual sides $G^*_{\mathrm{der}} := (G^*)_{\mathrm{der}}$ and $G^\vee_{\mathrm{der}} := (G^\vee)_{\mathrm{der}}$. Observe that G^*_{der} is simply-connected if and only if $G^\vee_{\mathrm{der}}$ is simply-connected, and if and only if the centre of G is a torus.

3.11. Theorem. [St1, pf. of Lem. 3.9][St2, Cor. 8.5] — *If G_{der} is simply-connected, then the centralizer $C_G(x)$ of every semisimple element x of G is connected.*

The citations here address the case where G is simply-connected, while the case of simply-connected G_{der} can be deduced as a corollary via the following observation: for every semisimple element $x = zy \in G$ with $z \in R(G)$ and $y \in G_{\mathrm{der}}$ (§ 3.10), we have $C_G(x) = R(G).C_{G_{\mathrm{der}}}(y)$.

3.12. Lemma. — *Recall that $G^*_{\mathrm{ss}}/\sim$ denotes the set of G^*-conjugacy classes of semisimple elements of G^* (§ 3.6). Then:*

(a) [DiMi, Cor. 4.2.15] *the map* $(G_{\mathrm{ss}}^{*F^*}/\sim) \longrightarrow (G_{\mathrm{ss}}^*/\sim)^{F^*}$ *induced by the set inclusion* $G_{\mathrm{ss}}^{*F^*} \subset G_{\mathrm{ss}}^*$ *is surjective.*

(b) *Assume that* G_{der}^* *is simply-connected. Then the surjection in (a) is a bijection* $(G_{\mathrm{ss}}^{*F^*}/\sim) \xrightarrow{\sim} (G_{\mathrm{ss}}^*/\sim)^{F^*}$, *and we have* $|(T^* /\!\!/ W)^{F^*}(\overline{\mathbb{F}}_q)| = q^{\operatorname{rank} G_{\mathrm{der}}^*} \cdot |\overline{T}^{*F^*}|$.

Similar results hold for the Langlands dual side $(G^\vee(\overline{\mathbb{Q}}), T^\vee(\overline{\mathbb{Q}}))$.

Proof of (b). To prove the injectivity of the map $(G_{\mathrm{ss}}^{*F^*}/\sim) \longrightarrow (G_{\mathrm{ss}}^*/\sim)^{F^*}$, suppose that $x, y \in G_{\mathrm{ss}}^{*F^*}$ are G^*-conjugate, say $y = gxg^{-1}$ for some $g \in G^*$; then $g^{-1}F^*(g) \in C_{G^*}(x)$; as G_{der}^* is simply-connected, $C_{G^*}(x)$ is connected (§ 3.11), so the Lang-Steinberg theorem gives us an $h \in C_{G^*}(x)$ such that $h^{-1}F^*(h) = g^{-1}F^*(g)$; thus $y = gh^{-1}x(gh^{-1})^{-1}$ with $gh^{-1} \in G^{*F^*}$. (See also [DiMi, Cor. 4.2.15(ii)].)

By [St2, § 14.7], $|(T^* /\!\!/ W)^{F^*}(\overline{\mathbb{F}}_q)| = \dfrac{1}{|W|}\displaystyle\sum_{w\in W} |T^{*wF^*}|$. From the exact sequences of finite groups $1 \longrightarrow (T_{\mathrm{der}}^*)^{wF^*} \longrightarrow T^{*wF^*} \longrightarrow \overline{T}^{*F^*} \longrightarrow 1$ for every $w \in W$ (§ 3.10), we get $|T^{*wF^*}| = |(T_{\mathrm{der}}^*)^{wF^*}| \cdot |\overline{T}^{*F^*}|$ for every $w \in W$, so that the above formula of $|(T^* /\!\!/ W)^{F^*}(\overline{\mathbb{F}}_q)|$ gives the equality $|(T^* /\!\!/ W)^{F^*}(\overline{\mathbb{F}}_q)| = |(T_{\mathrm{der}}^* /\!\!/ W)^{F^*}(\overline{\mathbb{F}}_q)| \cdot |\overline{T}^{*F^*}|$. As G_{der}^* is simply-connected, we have $|(T_{\mathrm{der}}^* /\!\!/ W)^{F^*}(\overline{\mathbb{F}}_q)| = |(G_{\mathrm{der}}^{*F^*})_{\mathrm{ss}}/\sim\,| = q^{\operatorname{rank} G_{\mathrm{der}}^*}$: the first equality comes from (a) and § 3.6, and the second equality has been proved in [St1, Lem. 7.3]. □

3.13. *Combinatorics of root data.* — From now on, let $\Delta^\vee \subset Y(T)$ be the set of simple coroots determined by (G, T, B) in § 1.2; identifying $Y(T) = X(T^*) = X(T^\vee)$, we have $\Delta^\vee \subset X(T^\vee)$, and $\Delta^\vee$ is a basis for the root system of $(G^\vee, T^\vee)$.

Elements of $X(T^\vee)$ are also called weights. Inside $X(T^\vee)$, let us consider:

$$X^+(T^\vee) := \{\lambda \in X(T^\vee) : \langle \lambda, \alpha^\vee\rangle \geq 0 \text{ for all } \alpha \in \Delta^\vee\} \quad \text{(dominant weights);}$$
$$X_q^+(T^\vee) := \{\lambda \in X(T^\vee) : 0 \leq \langle \lambda, \alpha^\vee\rangle < q \text{ for all } \alpha \in \Delta^\vee\} \quad (q\text{-restricted weights}).$$

Note that $X^+(T^\vee)$ is identified with the space $X(T^\vee)/W$ of W-orbits in $X(T^\vee)$, in the way that every W-orbit in $X(T^\vee)$ contains exactly one element of $X^+(T^\vee)$.

When $G^\vee$ is semisimple, $\Delta^\vee$ is a $\mathbb{Q}$-linear basis of $X(T^\vee)_{\mathbb{Q}} := X(T^\vee) \otimes_{\mathbb{Z}} \mathbb{Q}$, from which we introduce two additional notions:

(a) let $\{\omega_\alpha : \alpha \in \Delta^\vee\} \subset X(T^\vee)_{\mathbb{Q}}$ be the set of fundamental weights of $(G^\vee, T^\vee, B^\vee)$, characterised by the relations $\langle \omega_\alpha, \beta^\vee\rangle = \left\{\begin{array}{ll} 1 & \text{if } \alpha = \beta \\ 0 & \text{if } \alpha \neq \beta \end{array}\right\}$ for all $\alpha, \beta \in \Delta^\vee$;

(b) let $\mathrm{ht} : X(T^\vee) \longrightarrow \mathbb{Q}$ be the height function with respect to $\Delta^\vee$, defined for every $\lambda \in X(T^\vee)$ by $\mathrm{ht}(\lambda) = \displaystyle\sum_{\alpha\in\Delta^\vee} m_\alpha$ where $\lambda = \displaystyle\sum_{\alpha\in\Delta^\vee} m_\alpha\alpha$ with all $m_\alpha \in \mathbb{Q}$.

3.14. Lemma. — *Suppose that* $G^\vee$ *is semisimple, so that we have the height function* $\mathrm{ht} : X(T^\vee) \longrightarrow \mathbb{Q}$ *with respect to* $\Delta^\vee$ *(§ 3.13). Then:*

(a) *for* $\lambda \in X^+(T^\vee)$, *we have* $\mathrm{ht}(\lambda') < \mathrm{ht}(\lambda)$ *for every* $\lambda \neq \lambda' \in W\lambda$;

(b) *for every* $0 \neq \lambda \in X^+(T^\vee)$, *we have* $\mathrm{ht}(\lambda) > 0$.

Proof. (a) Choose a W-invariant inner product $(\cdot|\cdot)$ on the $\mathbb{R}$-vector space $X(T^\vee)_{\mathbb{R}} =$

$X(T^\vee)\otimes_{\mathbb{Z}}\mathbb{R}$. Set $\widetilde{\rho} := \sum_{\alpha\in\Delta^\vee} \frac{2\omega_\alpha}{(\alpha|\alpha)} \in X(T^\vee)_{\mathbb{R}}$. Then one can show that $\mathrm{ht}(\mu) = (\mu|\widetilde{\rho})$ for every $\mu \in X(T^\vee)$. For every $\mu \in X(T^\vee)$, fix a $\sigma_\mu \in W$ such that $\mathrm{ht}(\sigma_\mu\mu) = \max_{\sigma\in W} \mathrm{ht}(\sigma\mu)$. Then $\sigma_\mu\mu \in X^+(T^\vee)$: for every $\alpha \in \Delta^\vee$, we have $\langle \sigma_\mu\mu, \alpha^\vee\rangle \geq 0$ because

$$(\sigma_\mu\mu|\widetilde{\rho}) \geq (s_\alpha\sigma_\mu\mu|\widetilde{\rho}) = (\sigma_\mu\mu|s_\alpha\widetilde{\rho}) = \left(\sigma_\mu\mu\middle|\widetilde{\rho} - \frac{2\alpha}{(\alpha|\alpha)}\right) = (\sigma_\mu\mu|\widetilde{\rho}) - \langle\sigma_\mu\mu, \alpha^\vee\rangle.$$

Now let $\lambda \in X^+(T^\vee)$ and suppose that $w \in W$ is such that $\lambda' := w\lambda \neq \lambda$. The previous discussion tells us that $\mathrm{ht}(\lambda') \leq \mathrm{ht}(\sigma_\lambda\lambda)$ and that $\sigma_\lambda\lambda \in X^+(T^\vee)\cap W\lambda = \{\lambda\}$, so $\sigma_\lambda\lambda = \lambda$ and $\mathrm{ht}(\lambda') \leq \mathrm{ht}(\lambda)$. But $\mathrm{ht}(\lambda') \neq \mathrm{ht}(\lambda)$, for otherwise the last paragraph (with (μ, σ_μ) therein replaced by (λ, w)) would show that $\lambda' \in X^+(T^\vee) \cap W\lambda = \{\lambda\}$ and then $\lambda' = \lambda$, contradicting our hypothesis. Thus $\mathrm{ht}(\lambda') < \mathrm{ht}(\lambda)$ as desired.

(b) Recall that $\{s_\alpha : \alpha \in \Delta^\vee\}$ determines a length function $W \to \mathbb{N}$. Let $w_\circ$ be the longest element of W (which maximize the length function); then it is a fact that the action $-w_\circ : X(T^\vee) \longrightarrow X(T^\vee)$ permutes the elements of $\Delta^\vee$, so that for every $\lambda \in X(T^\vee)$ we have $\mathrm{ht}(\lambda) = \mathrm{ht}(-w_\circ\lambda) = -\mathrm{ht}(w_\circ\lambda)$.

Now let $0 \neq \lambda \in X^+(T^\vee)$, so $w_\circ\lambda \neq \lambda$ and hence (a) implies that $\mathrm{ht}(w_\circ\lambda) < \mathrm{ht}(\lambda)$. But by the last paragraph we have $\mathrm{ht}(\lambda) = -\mathrm{ht}(w_\circ\lambda)$, whence $\mathrm{ht}(\lambda) > 0$. □

3.15. *The canonical $\mathbb{Z}$-basis of $\mathbb{Z}[X(T^\vee)]^W$.* — For each $\lambda \in X(T^\vee)$, denote by $e(\lambda)$ its image in the group algebra $\mathbb{Z}[X(T^\vee)]$; thus $e(\lambda)$ is identified with the characteristic function $\mathbf{1}_{\{\lambda\}} : X(T^\vee) \longrightarrow \mathbb{Z}$, and $\{e(\lambda) : \lambda \in X(T^\vee)\}$ is a $\mathbb{Z}$-linear basis for $\mathbb{Z}[X(T^\vee)]$. For $\lambda \in X(T^\vee)$, let $W\lambda \subset X(T^\vee)$ be the W-orbit of λ, let $W_\lambda \subset W$ be the stabilizer of λ under the W-action on $X(T^\vee)$, and set

$$r(\lambda) := \frac{1}{|W_\lambda|}\sum_{w\in W} e(w\lambda) = \sum_{\mu\in W\lambda} e(\mu) \in \mathbb{Z}[X(T^\vee)]^W.$$

The identification $X(T^\vee)/W = X^+(T^\vee)$ implies that $\{r(\lambda) : \lambda \in X^+(T^\vee)\}$ is a $\mathbb{Z}$-linear basis for $\mathbb{Z}[X(T^\vee)]^W$.

Observe also that $F^\vee(r(\lambda)) = r(F^\vee\lambda)$ for every $\lambda \in X(T^\vee)$ (because $F^\vee(W) = W$).

3.16. Lemma. — *Let $\pi : X(T^\vee) \twoheadrightarrow X(T^\vee_{\mathrm{der}})$ be the canonical surjection (§ 3.10), and let $\mathrm{ht} : X(T^\vee_{\mathrm{der}}) \longrightarrow \mathbb{Q}$ be the height function with respect to $\Delta^\vee$ (recall that $G^\vee_{\mathrm{der}}$ is semisimple, and that $\Delta^\vee$ is a set of simple roots for the root system of $(G^\vee_{\mathrm{der}}, T^\vee_{\mathrm{der}})$).*

(a) *For every $\lambda \in X(T^\vee)$, the restriction of π to $W\lambda$ is injective.*

(b) *For every $\lambda_1, \lambda_2 \in X^+(T^\vee)$, in $\mathbb{Z}[X(T^\vee)]^W$ we have*

$$r(\lambda_1)\cdot r(\lambda_2) = r(\lambda_1+\lambda_2) + \sum_{\substack{\mu\in\Omega(\lambda_1,\lambda_2)\\ \mu\neq\lambda_1+\lambda_2}} c_\mu r(\mu)$$

where $\Omega(\lambda_1, \lambda_2) = (W\lambda_1 + W\lambda_2) \cap X^+(T^\vee)$ and $c_\mu \in \mathbb{N}^*$ for all μ. Moreover, for every $\mu \in \Omega(\lambda_1, \lambda_2)$ with $\mu \neq \lambda_1 + \lambda_2$, we have $\mathrm{ht}(\pi(\mu)) < \mathrm{ht}(\pi(\lambda_1 + \lambda_2))$.

Proof. (a) If $w_1, w_2 \in W$ are such that $\pi(w_1\lambda) = \pi(w_2\lambda)$, then $\nu := w_1\lambda - w_2\lambda$ lies in $\ker(\pi)$; as $\ker(\pi) = X(T^\vee)^W$ (§ 3.10), we have

$$\nu = \frac{1}{|W|}\sum_{w\in W} w\nu = \frac{1}{|W|}\left(\sum_{w\in W} ww_1\lambda - \sum_{w\in W} ww_2\lambda\right) = 0,$$

whence $w_1\lambda = w_2\lambda$.

(b) By definition of $r(\lambda)$, it is clear that

$$r(\lambda_1)\cdot r(\lambda_2) = \sum_{\mu\in\Omega(\lambda_1,\lambda_2)} c_\mu r(\mu)$$

with all coefficients $c_\mu \in \mathbb{N}^*$. For each $\mu \in \Omega(\lambda_1, \lambda_2)$, we have $\mu = \mu_1 + \mu_2$ where $(\mu_1, \mu_2) \in W\lambda_1 \times W\lambda_2$; if $\mu \neq \lambda_1 + \lambda_2$, then $(\mu_1, \mu_2) \neq (\lambda_1, \lambda_2)$, so (a) implies that $(\pi(\mu_1), \pi(\mu_2)) \neq (\pi(\lambda_1), \pi(\lambda_2))$ in $W.\pi(\lambda_1) \times W.\pi(\lambda_2)$, and hence § 3.14(a) shows that

$$\mathrm{ht}(\pi(\mu)) = \mathrm{ht}(\pi(\mu_1)) + \mathrm{ht}(\pi(\mu_2)) < \mathrm{ht}(\pi(\lambda_1)) + \mathrm{ht}(\pi(\lambda_2)) = \mathrm{ht}(\pi(\lambda_1 + \lambda_2)).$$

It then remains to prove that $c_{\lambda_1+\lambda_2} = 1$. Indeed, $c_{\lambda_1+\lambda_2}$ is the number of pairs (μ_1, μ_2) in $W\lambda_1 \times W\lambda_2$ such that $\mu_1 + \mu_2 = \lambda_1 + \lambda_2$. For such a pair (μ_1, μ_2), we have

$$\mathrm{ht}(\pi(\mu_1)) + \mathrm{ht}(\pi(\mu_2)) = \mathrm{ht}(\pi(\lambda_1)) + \mathrm{ht}(\pi(\lambda_2)),$$

so the argument of the last paragraph tells us that $(\mu_1, \mu_2) = (\lambda_1, \lambda_2)$. This proves the equality $c_{\lambda_1+\lambda_2} = 1$ and completes the proof of the lemma. □

3.17. Theorem. — *Suppose that $G^\vee_{\mathrm{der}}$ is simply-connected, so that the fundamental weights ω'_α ($\alpha \in \Delta^\vee$) of $G^\vee_{\mathrm{der}}$ all lie in $X^+(T^\vee_{\mathrm{der}})$. Let $\pi : X^+(T^\vee) \twoheadrightarrow X^+(T^\vee_{\mathrm{der}})$ be the canonical surjection, and identify $X(\overline{T}^\vee) \subset X(T^\vee)$ (§ 3.10). For each $\alpha \in \Delta^\vee$, choose a lifting of ω'_α to $\omega_\alpha \in X^+(T^\vee)$ via π, so that $\pi(\omega_\alpha) = \omega'_\alpha$. Let also $\mathfrak{A} \subset X(\overline{T}^\vee)$ be a set of representatives of the $\mathbb{Z}$-module $X(\overline{T}^\vee)/(F^\vee - \mathrm{id})X(\overline{T}^\vee)$. Then $\mathsf{B}_{G^\vee} = \mathbb{Z}[X(T^\vee)]^W/I$ is a free $\mathbb{Z}$-module having the set*

$$\mathfrak{F} := \left\{ r\left(\mu + \sum_{\alpha\in\Delta^\vee} b_\alpha\omega_\alpha\right) + I : \mu \in \mathfrak{A},\ b_\alpha \in \{0, 1, \cdots, q-1\}\ (\alpha \in \Delta^\vee)\right\},$$

as its basis, and the rank of $\mathsf{B}_{G^\vee}$ over $\mathbb{Z}$ is

$$\mathrm{rank}_{\mathbb{Z}}\, \mathsf{B}_{G^\vee} = |\mathfrak{F}| = q^{\mathrm{rank}\, G^\vee_{\mathrm{der}}} \cdot |\overline{T}^\vee(\overline{\mathbb{Q}})^{F^\vee}| = |(T^\vee /\!\!/ W)^{F^\vee}(\overline{\mathbb{Q}})| = |G^{*F^*}_{\mathrm{ss}}/\sim|.$$

Moreover, $\mathsf{B}_{G^\vee}$ is a reduced ring, so $I = \sqrt{I}$ and $\mathsf{B}_{G^\vee} = \mathsf{B}_{G^\vee,\mathrm{red}}$.

Remark. In the special case where $G^\vee$ is simply-connected (so that $G^\vee_{\mathrm{der}} = G^\vee$), we have $\pi = \mathrm{id}$, $\overline{T}^\vee = 1$ and $X(\overline{T}^\vee) = 0$; we may choose $\omega_\alpha = \omega'_\alpha$, so $\{\omega_\alpha\}_{\alpha\in\Delta^\vee}$ is a $\mathbb{Z}$-basis of $X(T^\vee)$; thus $\mathsf{B}_{G^\vee}$ is a reduced ring and is also a free $\mathbb{Z}$-module having $\mathfrak{F} = \{r(\lambda) + I : \lambda \in X^+_q(T^\vee)\}$ as its $\mathbb{Z}$-linear basis; the $\mathbb{Z}$-rank of $\mathsf{B}_{G^\vee}$ is $q^{\mathrm{rank}\, G^\vee}$.

Proof of theorem. (Compare [Hu, Sec. 5.6 - 5.7].) Let us consider the height function $\mathrm{ht} : X(T^\vee_{\mathrm{der}}) \longrightarrow \mathbb{Q}$ as in § 3.16. For each $f \in \mathbb{Z}[X(T^\vee)]^W$, write $\overline{f} := f + I \in \mathsf{B}_{G^\vee}$, so

that $\{\overline{r(\lambda)} : \lambda \in X^+(T^\vee)\}$ generates $\mathsf{B}_{G^\vee}$ as a $\mathbb{Z}$-module. In $\mathsf{B}_{G^\vee}$, we have (using § 3.15 and the relation $q = F^\vee \circ \tau^\vee$ from § 3.2(a))

$$\overline{r(F^\vee\lambda)} = \overline{r(\lambda)} \quad \text{and} \quad \overline{r(q\lambda)} = \overline{r(\tau^\vee\lambda)} \ \text{ for every } \lambda \in X(T^\vee).$$

(1) We first prove that $\mathsf{B}_{G^\vee}$ is generated by $\{\overline{r(\lambda)} : \lambda \in X_q^+(T^\vee)\}$ as a $\mathbb{Z}$-module. Suppose that $\lambda \in X^+(T^\vee)$ but $\lambda \notin X_q^+(T^\vee)$. Then there exists an $\alpha \in \Delta^\vee$ such that $\langle \lambda, \alpha^\vee \rangle \geq q$. Consider $\lambda' := \lambda - q\omega_\alpha \in X(T^\vee)$; by hypothesis on λ, we have in fact $\lambda' \in X^+(T^\vee)$. We may then use § 3.16(b) to expand the product $r(\lambda')r(q\omega_\alpha)$ as

$$r(\lambda')r(q\omega_\alpha) = r(\lambda) + \sum_{\lambda \neq \mu \in \Omega(\lambda', q\omega_\alpha)} c_\mu r(\mu) \in \mathbb{Z}[X(T^\vee)]^W,$$

with $c_\mu \in \mathbb{N}^*$ and $\mathrm{ht}(\pi(\mu)) < \mathrm{ht}(\pi(\lambda))$ for all $\lambda \neq \mu \in \Omega(\lambda', q\omega_\alpha)$. Passing this expansion into $\mathsf{B}_{G^\vee}$ and using the relation $\overline{r(q\omega_\alpha)} = \overline{r(\tau^\vee\omega_\alpha)}$ in $\mathsf{B}_{G^\vee}$, we get

$$\overline{r(\lambda')}\,\overline{r(\tau^\vee\omega_\alpha)} = \overline{r(\lambda)} + \sum_{\lambda \neq \mu \in \Omega(\lambda', q\omega_\alpha)} c_\mu \overline{r(\mu)} \in \mathsf{B}_{G^\vee}.$$

As $\tau^\vee\omega_\alpha \in X^+(T^\vee)$ (T is contained in the F-stable Borel subgroup B, so $X^+(T^\vee)$ is $\tau^\vee$-invariant), we may use § 3.16(b) again to expand the product $r(\lambda')r(\tau^\vee\omega_\alpha)$ as

$$r(\lambda')r(\tau^\vee\omega_\alpha) = r(\lambda' + \tau^\vee\omega_\alpha) + \sum_{\lambda' + \tau^\vee\omega_\alpha \neq \nu \in \Omega(\lambda', \tau^\vee\omega_\alpha)} c'_\nu r(\nu) \in \mathbb{Z}[X(T^\vee)]^W,$$

with $c'_\nu \in \mathbb{N}^*$ and $\mathrm{ht}(\pi(\nu)) < \mathrm{ht}(\pi(\lambda' + \tau^\vee\omega_\alpha))$ for all $\lambda' + \tau^\vee\omega_\alpha \neq \nu \in \Omega(\lambda', \tau^\vee\omega_\alpha)$. Moreover, as $\tau^\vee$ preserves the height function on $X(T^\vee_{\mathrm{der}})$ ($\tau^\vee$ permutes elements of $\Delta^\vee$), we have $\mathrm{ht}(\pi(\tau^\vee\omega_\alpha)) = \mathrm{ht}(\tau^\vee\omega'_\alpha) = \mathrm{ht}(\omega'_\alpha) = \mathrm{ht}(\pi(\omega_\alpha))$, so by § 3.14(b) we have

$$\mathrm{ht}(\pi(\lambda' + \tau^\vee\omega_\alpha)) < \mathrm{ht}(\pi(\lambda' + q\omega_\alpha)) = \mathrm{ht}(\pi(\lambda)).$$

We thus obtain the relation

$$\overline{r(\lambda)} = \overline{r(\lambda' + \tau^\vee\omega_\alpha)} + \sum_{\lambda' + \tau^\vee\omega_\alpha \neq \nu \in \Omega(\lambda', \tau^\vee\omega_\alpha)} c'_\nu \overline{r(\nu)} - \sum_{\lambda \neq \mu \in \Omega(\lambda', q\omega_\alpha)} c_\mu \overline{r(\mu)} \in \mathsf{B}_{G^\vee},$$

which expresses $\overline{r(\lambda)}$ in terms of a $\mathbb{Z}$-linear combination of some $\overline{r(\gamma)}$ where $\gamma \in X^+(T^\vee)$ with $\mathrm{ht}(\pi(\gamma)) < \mathrm{ht}(\pi(\lambda))$. On the other hand, as $X(T^\vee)$ is a free $\mathbb{Z}$-module of finite rank, we see that $\mathrm{ht}(\pi(X^+(T^\vee))) \subset h^{-1}\mathbb{N}$ for some $h \in \mathbb{N}^*$. We can thus repeat the above reduction process of $\overline{r(\lambda)}$, so that for every $\lambda \in X^+(T^\vee)$ we can eventually express $\overline{r(\lambda)}$ as a $\mathbb{Z}$-linear combination of those $\overline{r(\mu)}$ with $\mu \in X_q^+(T^\vee)$.

(2) Let us use (1) to prove that $\mathsf{B}_{G^\vee}$ is generated by $\mathfrak{F}$ as a $\mathbb{Z}$-module. From the exact sequence $0 \longrightarrow X(\overline{T}^\vee) \longrightarrow X(T^\vee) \longrightarrow X(T^\vee_{\mathrm{der}}) \longrightarrow 0$ (§ 3.10), each $\lambda \in X_q^+(T^\vee)$ may be expressed as $\lambda = \mu + \sum_{\alpha \in \Delta^\vee} b_\alpha \omega_\alpha$ where $\mu \in X(\overline{T}^\vee)$ and each $b_\alpha \in \{0, 1, \cdots, q-1\}$. Furthermore, for each $\mu \in X(\overline{T}^\vee) \subset X(T^\vee)$, we have $r(\mu) = e(\mu) \in \mathbb{Z}[X(T^\vee)]^W$ and therefore $e((F^\vee - \mathrm{id})\mu) - 1 = e(-\mu)(e(F^\vee\mu) - e(\mu)) \in I$. These observations and (1) together imply that the $\mathbb{Z}$-module is $\mathsf{B}_{G^\vee}$ is generated by $\mathfrak{F}$.

(3) Let us now prove that $\mathsf{B}_{G^\vee}$ is a free $\mathbb{Z}$-module having $\mathfrak{F}$ as its basis and having the desired rank formulae. In (2) we have seen that $\mathfrak{F}$ generates the $\mathbb{Z}$-module $\mathsf{B}_{G^\vee}$, so that $\mathfrak{F}$ also generates the $\overline{\mathbb{Q}}$-linear vector space $\overline{\mathbb{Q}}\mathsf{B}_{G^\vee}$. By § 3.5, we have

$$\dim_{\overline{\mathbb{Q}}} \overline{\mathbb{Q}}\mathsf{B}_{G^\vee} \geq |(T^\vee /\!\!/ W)^{F^\vee}(\overline{\mathbb{Q}})|;$$

on the other hand, we have (compare the part of duality of tori in § 1.4)

$$|\mathfrak{A}| = [X(\overline{T}^\vee) : (F^\vee - \mathrm{id})X(\overline{T}^\vee)] = |\mathrm{Irr}_{\overline{\mathbb{Q}}}(\overline{T}^\vee(\overline{\mathbb{Q}})^{F^\vee})| = |\overline{T}^\vee(\overline{\mathbb{Q}})^{F^\vee}|,$$

which implies, together with § 3.12 and § 3.6, that

$$\dim_{\overline{\mathbb{Q}}} \overline{\mathbb{Q}}\mathsf{B}_{G^\vee} \leq |\mathfrak{F}| = q^{\mathrm{rank}\, G^\vee_{\mathrm{der}}} \cdot |\overline{T}^\vee(\overline{\mathbb{Q}})^{F^\vee}| = |(T^\vee /\!\!/ W)^{F^\vee}(\overline{\mathbb{Q}})| = |G^{*F^*}_{\mathrm{ss}}/\sim|.$$

Therefore, $\dim_{\overline{\mathbb{Q}}} \overline{\mathbb{Q}}\mathsf{B}_{G^\vee} = |(T^\vee /\!\!/ W)^{F^\vee}(\overline{\mathbb{Q}})|$, and $\mathfrak{F}$ is in fact a basis for the $\overline{\mathbb{Q}}$-linear vector space $\overline{\mathbb{Q}}\mathsf{B}_{G^\vee}$; it follows that $\mathsf{B}_{G^\vee}$ is a free $\mathbb{Z}$-module of basis $\mathfrak{F}$.

(4) Finally, let us prove the reducedness of $\mathsf{B}_{G^\vee}$. We have just proved in (3) that $\mathsf{B}_{G^\vee}$ is a free $\mathbb{Z}$-module, and this implies that the natural map $\mathsf{B}_{G^\vee} \longrightarrow \overline{\mathbb{Q}}\mathsf{B}_{G^\vee}$ is injective. In (3) we have also shown that $\dim_{\overline{\mathbb{Q}}} \overline{\mathbb{Q}}\mathsf{B}_{G^\vee} = |(T^\vee /\!\!/ W)^{F^\vee}(\overline{\mathbb{Q}})|$, so the discussion in § 3.5 implies that $\overline{\mathbb{Q}}\mathsf{B}_{G^\vee}$ is a reduced ring. Hence $\mathsf{B}_{G^\vee}$ is also reduced. □

Comparison between $\mathsf{B}_{G^\vee}$ and K_{G^*}

3.18. *Highest weights.* [Ja, Ch. II.2] — Identify $X(T^*) = X(T^\vee)$, so for $\lambda \in X(T^*)$ we may define $r(\lambda) \in \mathbb{Z}[X(T^*)]^W$ as in § 3.15. Let $\leq$ be the standard partial ordering on $X(T^*) = X(T^\vee)$ determined by $\Delta^\vee$. Using the notation of algebraic representations in § 2.11 (in particular, ch is the formal character with respect to the T^*-action):

(a) every $M \in \mathrm{Irr}_{\mathrm{alg}}(G^*)$ admits a unique highest weight λ_M with respect to the partial ordering $\leq$; we have $\lambda_M \in X^+(T^*)$ and $\dim_{\overline{\mathbb{F}}_q} M_{\lambda_M} = 1$; also, every weight $\lambda \in X(T^*)$ of M satisfies $\lambda \leq \lambda_M$; as a result, for each $M \in \mathrm{Irr}_{\mathrm{alg}}(G^*)$, we have $\mathrm{ch}\, M \in r(\lambda_M) + \sum_{\lambda \in X^+(T^*),\, \lambda < \lambda_M} \mathbb{Z}.r(\lambda)$;

(b) for every $\lambda \in X^+(T^*)$, there is a unique $M \in \mathrm{Irr}_{\mathrm{alg}}(G^*)$ which admits λ as its highest weight in the sense of (a); we shall denote this unique M by $L(\lambda)$.

Observe that the formal character isomorphism $\mathrm{ch} : \mathsf{K}(\mathrm{Rep}_{\mathrm{alg}}(G^*)) \xrightarrow{\sim} \mathbb{Z}[X(T^*)]^W$ in § 2.11 is a corollary of (a) and (b) here.

3.19. Lemma. — *Let $\lambda \in X^+(T^*)$. Consider the Frobenius twist $L(\lambda)^{[F^*]}$: it is the G^*-module whose underlying set is $L(\lambda)$ and whose G^*-action is given by the composition $G^* \xrightarrow{F^*} G^* \xrightarrow{\rho} \mathrm{GL}(L(\lambda))$ where ρ denotes the G^*-action on $L(\lambda)$. Then $L(F^*\lambda) \simeq L(\lambda)^{[F^*]}$ in $\mathrm{Rep}_{\mathrm{alg}}(G^*)$.*

Proof. The G^*-module $L(\lambda)^{[F^*]}$ is irreducible by [St1, Thm. 5.1]. As the highest weight of $L(\lambda)^{[F^*]}$ is $F^*\lambda$, we see from § 3.18 that $L(F^*\lambda) \simeq L(\lambda)^{[F^*]}$ as G^*-modules. □

3.20. Proposition. — *Identify $X(T^*) = X(T^\vee)$ and let $\pi : \mathbb{Z}[X(T^*)]^W \twoheadrightarrow \mathsf{B}_{G^\vee,\mathrm{red}}$ be the induced quotient map. Observe that the maps*

$$(T^\vee /\!\!/ W)^{F^\vee}(\overline{\mathbb{Q}}) \xrightarrow[\sim]{\S\, 3.6} (G^*_{\mathrm{ss}}/\sim)^{F^*} \xrightarrow{\S\, 3.12} (G^{*F^*}_{\mathrm{ss}}/\sim) \overset{\S\, 2.1}{=\!=} (G^{*F^*}_{p'}/\sim)$$

induce an injection of $\overline{\mathbb{Q}}$*-algebras*

$$j : \overline{\mathbb{Q}}^{(T^\vee /\!\!/ W)^{F^\vee}(\overline{\mathbb{Q}})} \hookrightarrow \overline{\mathbb{Q}}^{G^{*F^*}_{p'}/\sim}.$$

Then there is an injective ring homomorphism $\overline{\gamma} : \mathsf{B}_{G^\vee,\mathrm{red}} \hookrightarrow \mathsf{K}_{G^*}$ *making the following diagram commutative (recall the formal character isomorphism* ch *from § 2.11):*

$$\begin{tikzcd}
\mathsf{K}(\mathrm{Rep}_{\mathrm{alg}}(G^*)) \arrow[r, "\mathrm{ch}", "\sim"'] \arrow[d, "\mathrm{Res}^{G^*}_{G^{*F^*}}"'] & \mathbb{Z}[X(T^*)]^W \arrow[d, two heads, "\pi"] \\
\mathsf{K}_{G^*} \arrow[d, hook, "(\S\,2.3)\ \mathrm{br}"'] & \mathsf{B}_{G^\vee,\mathrm{red}} \arrow[l, hook, "\overline{\gamma}"'] \arrow[d, hook, "\S\,3.8"] \\
\overline{\mathbb{Q}}^{G^{*F^*}_{p'}/\sim} & \overline{\mathbb{Q}}^{(T^\vee /\!\!/ W)^{F^\vee}(\overline{\mathbb{Q}})} \arrow[l, hook, "j"']
\end{tikzcd} \tag{3.20.1}$$

Proof. (1) Let $\lambda \in X^+(T^*)$. Then $L(F^*\lambda) \simeq L(\lambda)^{[F^*]}$ in $\mathrm{Rep}_{\mathrm{alg}}(G^*)$ (§ 3.19). As F^* acts trivially on G^{*F^*}, we have $\mathrm{Res}^{G^*}_{G^{*F^*}}(L(\lambda)^{[F^*]}) = \mathrm{Res}^{G^*}_{G^{*F^*}}(L(\lambda))$ in $\mathrm{Rep}_{\overline{\mathbb{F}}_q}(G^{*F^*})$, so $\mathrm{Res}^{G^*}_{G^{*F^*}} L(F^*\lambda) = \mathrm{Res}^{G^*}_{G^{*F^*}} L(\lambda)$ in $\mathrm{Rep}_{\overline{\mathbb{F}}_q}(G^{*F^*})$. Therefore, if we denote by J the ideal of $\mathbb{Z}[X(T^*)]^W$ generated by $\{\mathrm{ch}\, L(F^*\lambda) - \mathrm{ch}\, L(\lambda) : \lambda \in X^+(T^*)\}$, then J lies in the kernel of the composition $\gamma := \mathrm{Res}^{G^*}_{G^{*F^*}} \circ \mathrm{ch}^{-1} : \mathbb{Z}[X(T^*)]^W \longrightarrow \mathsf{K}_{G^*}$. As K_{G^*} is reduced (§ 2.3), γ descends to $\overline{\gamma} : \mathbb{Z}[X(T^*)]^W/\sqrt{J} \longrightarrow \mathsf{K}_{G^*}$, and we obtain the following commutative diagram of rings:

$$\begin{tikzcd}
\mathsf{K}(\mathrm{Rep}_{\mathrm{alg}}(G^*)) \arrow[r, "\mathrm{ch}", "\sim"'] \arrow[d, "\mathrm{Res}^{G^*}_{G^{*F^*}}"'] & \mathbb{Z}[X(T^*)]^W \arrow[d, two heads, "\pi"] \\
\mathsf{K}_{G^*} & \mathbb{Z}[X(T^*)]^W/\sqrt{J} \arrow[l, "\overline{\gamma}"]
\end{tikzcd} \tag{3.20.2}$$

(2) We now show that under the identification $\mathbb{Z}[X(T^*)]^W = \mathbb{Z}[X(T^\vee)]^W$ (induced by the identification $X(T^*) = X(T^\vee)$), the ideal $J \subset \mathbb{Z}[X(T^*)]^W$ corresponds to the ideal $I \subset \mathbb{Z}[X(T^\vee)]^W$ appearing in the definition of $\mathsf{B}_{G^\vee,\mathrm{red}} = \mathbb{Z}[X(T^\vee)]^W/\sqrt{I}$ (§ 3.4). By § 3.3 and § 3.15, the ideal I is generated by $\{r(F^\vee\lambda) - r(\lambda) : \lambda \in X^+(T^\vee)\}$. For $\lambda \in X^+(T^*)$, by § 3.18 we may write $\mathrm{ch}\, L(\lambda) = r(\lambda) + \sum_{\mu \in X^+(T^*),\, \mu<\lambda} c_\mu r(\mu)$ where only a finite number of c_μ is not zero. Then $\mathrm{ch}\, L(F^*\lambda) = r(F^*\lambda) + \sum_{\mu \in X^+(T^*),\, \mu<\lambda} c_\mu r(F^*\mu)$: in fact, again by § 3.19 we have $L(F^*\lambda) \simeq L(\lambda)^{[F^*]}$ in $\mathrm{Rep}_{\mathrm{alg}}(G^*)$, thus all the weights of $L(F^*\lambda)$ are of the form $F^*\mu$ where μ is a weight of $L(\lambda)$, and moreover $L(F^*\lambda)_{F^*\mu} \simeq L(\lambda)_\mu$ for all $\mu \in X(T^*)$, whence the desired expression for $\mathrm{ch}\, L(F^*\lambda)$. We thus have:

$$\mathrm{ch}\, L(F^*\lambda) - \mathrm{ch}\, L(\lambda) = (r(F^*\lambda) - r(\lambda)) + \sum_{\mu \in X^+(T^*),\, \mu<\lambda} c_\mu.(r(F^*\mu) - r(\mu)) \quad (\lambda \in X^+(T^*)).$$

This implies that the transition matrix from $\{\mathrm{ch}\, L(F^*\lambda) - \mathrm{ch}\, L(\lambda) : \lambda \in X^+(T^*)\}$ to $\{r(F^*\lambda) - r(\lambda) : \lambda \in X^+(T^*)\}$ is triangular with all diagonal elements being 1 (with respect to the partial ordering $\leq$ on $X^+(T^*)$), whence $J = I$.

(3) The equality $J = I$ established in (2) implies that $\mathbb{Z}[X(T^*)]^W/\sqrt{J} = \mathsf{B}_{G^\vee,\,\mathrm{red}}$ under the identification $X(T^*) = X(T^\vee)$, so (3.20.2) is exactly the upper part of (3.20.1), where we still need to show the injectivity of $\overline{\gamma}$.

(4) To establish (3.20.1) (in particular the injectivity of $\overline{\gamma}$), it remains to prove that the outermost rectangle diagram in (3.20.1) commutes, that is to prove the following diagram commutes:

$$\begin{tikzcd}
\mathsf{K}(\mathrm{Rep}_{\mathrm{alg}}(G^*)) \arrow[r, "\mathrm{ch}", "\sim"'] \arrow[d, "\mathrm{br}\circ\mathrm{Res}^{G^*}_{G^{*F^*}}"'] & \mathbb{Z}[X(T^*)]^W \arrow[d, two heads, "\text{canonical}"] \\
\overline{\mathbb{Q}}^{G^{*F^*}_{p'}/\sim} & \overline{\mathbb{Q}}^{(T^*/\!\!/W)^{F^*}(\overline{\mathbb{F}}_q)} \arrow[l, hook, "j"']
\end{tikzcd} \tag{3.20.3}$$

(We have identified $(T^\vee /\!\!/ W)^{F^\vee}(\overline{\mathbb{Q}}) = (T^* /\!\!/ W)^{F^*}(\overline{\mathbb{F}}_q)$.) To do this, we need the help of the following properties:

(i) $G^{*F^*}_{p'} = G^{*F^*}_{\mathrm{ss}}$ is the union of all S^{*F^*} where S^* runs through elements of $\mathcal{T}^* := \{F^*\text{-stable maximal tori of } G^*\}$;

(ii) $(T^* /\!\!/ W)^{F^*}(\overline{\mathbb{F}}_q)$ is the set of W-orbits in $\bigcup_{w\in W} T^{*wF^*}$;

(iii) For each F^*-stable maximal torus S^* of G^*, we can find a $g^* \in G^*$ such that $g^*S^*(g^*)^{-1} = T^*$, so that the map $x^* \longmapsto g^*x^*(g^*)^{-1}$ establishes an isomorphism $S^{*F^*} \xrightarrow{\sim} T^{*wF^*}$ where $w \in W$ is the quotient image of $g^*F^*(g^*)^{-1} \in N_{G^*}(T^*)$.

These properties enable us to integrate (3.20.3) into the following cubic diagram:

$$\begin{tikzcd}
 & \prod_{S^*\in\mathcal{T}^*} \mathsf{K}(\mathrm{Rep}_{\mathrm{alg}}(S^*)) \arrow[rr, "\sim", "\mathrm{ch}"'] \arrow[dd, "\mathrm{br}\circ\mathrm{Res}"] & & \prod_{S^*\in\mathcal{T}^*} \mathbb{Z}[X(S^*)] \arrow[dd, "\text{canonical}"] \\
\mathsf{K}(\mathrm{Rep}_{\mathrm{alg}}(G^*)) \arrow[ur, "\mathrm{Res}"] \arrow[rr, "\sim", "\mathrm{ch}"'] \arrow[dd, "\mathrm{br}\circ\mathrm{Res}^{G^*}_{G^{*F^*}}"'] & & \mathbb{Z}[X(T^*)]^W \arrow[ur, "\text{(iii)}"'] \arrow[dd, two heads] & \\
 & \prod_{S^*\in\mathcal{T}^*} \overline{\mathbb{Q}}^{S^{*F^*}} \arrow[rr, equal] & & \prod_{S^*\in\mathcal{T}^*} \overline{\mathbb{Q}}^{S^{*F^*}} \\
\overline{\mathbb{Q}}^{G^{*F^*}_{p'}/\sim} \arrow[ur, hook, "\mathrm{Res}"] \arrow[rr, hook, "j"'] & & \overline{\mathbb{Q}}^{(T^*/\!\!/W)^{F^*}(\overline{\mathbb{F}}_q)} \arrow[ur, hook, "\text{(iii)}"'] \arrow[uu, phantom, "\text{canonical}"] &
\end{tikzcd}$$

(Here "Res" means the natural restriction maps, and the maps "(iii)" on the right face are the natural maps induced by the bijection in (iii) above.) In the above cubic diagram, it can be checked that all the five faces other than the front face (3.20.3) are commutative diagrams; thus the front face (3.20.3) is also commutative. □

3.21. Theorem. — *If G^*_{der} is simply-connected, then the formal character isomorphism* $\mathrm{ch} : \mathsf{K}(\mathrm{Rep}_{\mathrm{alg}}(G^*)) \xrightarrow{\sim} \mathbb{Z}[X(T^*)]^W$ *(§ 3.18) and the bijection of finite sets* $(G^{*F^*}_{p'}/\sim) \simeq (T^\vee /\!\!/ W)^{F^\vee}(\overline{\mathbb{Q}})$ *(§ 2.1, § 3.6 and § 3.12) both induce, via the commutative diagram in § 3.20, the same ring isomorphism* $\mathsf{B}_{G^\vee} \simeq \mathsf{K}_{G^*}$.

Proof. The simple-connectedness of G^*_{der} has two consequences: (i) the restriction map $\mathrm{Res}^{G^*}_{G^{*F^*}} : \mathsf{K}(\mathrm{Rep}_{\mathrm{alg}}(G^*)) \longrightarrow \mathsf{K}_{G^*}$ is surjective (§ 2.13), so the injective map $\overline{\gamma} : \mathsf{B}_{G^\vee,\mathrm{red}} \longrightarrow \mathsf{K}_{G^*}$ in § 3.20 is also surjective and is therefore a ring isomorphism; (ii) $\mathsf{B}_{G^\vee} = \mathsf{B}_{G^\vee,\mathrm{red}}$ (§ 3.17). Thus the map $\overline{\gamma}$, coming from the formal character map, establishes a ring isomorphism $\mathsf{B}_{G^\vee} \simeq \mathsf{K}_{G^*}$. □

3.22. Corollary (of § 2.25 and § 3.21). — *The maps*

$$(G^{*F^*}_{\mathrm{ss}}/\sim) \;=\!=\; (G^{*F^*}_{p'}/\sim) \xrightarrow[\S\,3.12]{\S\,3.6} (T^\vee /\!\!/ W)^{F^\vee}(\overline{\mathbb{Q}})$$

induce $\overline{\mathbb{Q}}$-algebra homomorphisms

$$\overline{\mathbb{Q}}\mathsf{E}_G \xleftrightarrow{\sim} \overline{\mathbb{Q}}\mathsf{K}_{G^*} \hookleftarrow (\overline{\mathbb{Q}}\mathsf{B}_{G^\vee})_{\mathrm{red}} \overset{\S\,3.5}{=\!=} \overline{\mathbb{Q}}\mathsf{B}_{G^\vee,\mathrm{red}} \twoheadleftarrow \overline{\mathbb{Q}}\mathsf{B}_{G^\vee}$$

which descend to $\mathbb{Z}[\frac{1}{pM_G}]$-algebra homomorphisms

$$\mathbb{Z}[\tfrac{1}{pM_G}]\mathsf{E}_G \xleftrightarrow{\sim} \mathbb{Z}[\tfrac{1}{pM_G}]\mathsf{K}_{G^*} \hookleftarrow \mathbb{Z}[\tfrac{1}{pM_G}]\mathsf{B}_{G^\vee,\mathrm{red}} \twoheadleftarrow \mathbb{Z}[\tfrac{1}{pM_G}]\mathsf{B}_{G^\vee}, \tag{3.22.1}$$

where M_G be the product of bad primes for G (§ 2.21). If G^*_{der} is simply-connected, all maps in (3.22.1) are $\mathbb{Z}[\frac{1}{pM_G}]$-algebra isomorphisms:

$$\mathbb{Z}[\tfrac{1}{pM_G}]\mathsf{E}_G \simeq \mathbb{Z}[\tfrac{1}{pM_G}]\mathsf{K}_{G^*} \simeq \mathbb{Z}[\tfrac{1}{pM_G}]\mathsf{B}_{G^\vee,\mathrm{red}} = \mathbb{Z}[\tfrac{1}{pM_G}]\mathsf{B}_{G^\vee}.$$

3.23. *Remark on toric graduations.* — For a connected reductive group $G^\vee$ over $\mathbb{Z}$, as in the proof of § 3.9 we fit $G^\vee$ into an exact sequence of reductive groups over $\mathbb{Z}$,

$$1 \longrightarrow Z^\vee \longrightarrow H^\vee \longrightarrow G^\vee \longrightarrow 1,$$

with $H^\vee_{\mathrm{der}}$ simply-connected and with $Z^\vee$ a central torus in $H^\vee$. Denote by $G^\vee /\!\!/ G^\vee$ be the categorical quotient induced by the adjoint action of $G^\vee$ on itself; note that $G^\vee /\!\!/ G^\vee \simeq T^\vee /\!\!/ W$ by Chevalley's restriction theorem. The multiplication action of $Z^{\vee F^\vee}$ on $(H^\vee /\!\!/ H^\vee)^{F^\vee} = \mathrm{Spec}(\mathsf{B}_{H^\vee})$ induces a $Z^{\vee F^\vee}$-action on $\mathsf{B}_{H^\vee}$, from which the ring $\mathsf{B}_{H^\vee}$ admits a $\widehat{Z^{\vee F^\vee}}$-graded decomposition $\mathsf{B}_{H^\vee} = \bigoplus_{\lambda \in \widehat{Z^{\vee F^\vee}}} (\mathsf{B}_{H^\vee})_\lambda$. Observe that $(\mathsf{B}_{H^\vee})_0 = (\mathsf{B}_{H^\vee})^{Z^{\vee F^\vee}}$, so the canonical surjection $(H^\vee /\!\!/ H^\vee)^{F^\vee} /\!\!/ Z^{\vee F^\vee} \twoheadrightarrow (G^\vee /\!\!/ G^\vee)^{F^\vee}$ induces an inclusion of rings $\mathsf{B}_{G^\vee,\mathrm{red}} \hookrightarrow (\mathsf{B}_{H^\vee})_0$.

For the K-side, the ring K_{H^*} also admits an analogous $\widehat{Z^{*F^*}}$-graded structure (2.15.1). Under the identification $\widehat{Z^{\vee F^\vee}} \simeq \widehat{Z^{*F^*}}$, the ring isomorphism $\mathsf{B}_{H^\vee} \simeq \mathsf{K}_{H^*}$ in § 3.21 respects the above graded structures and induces a ring isomorphism $(\mathsf{B}_{H^\vee})_0 \simeq (\mathsf{K}_{H^*})_0$.

In summary, we have the following commutative diagram of rings:

$$\begin{tikzcd}
\mathsf{B}_{G^\vee,\mathrm{red}} \arrow[r, hook] \arrow[d, hook, "\S\,3.20"'] & (\mathsf{B}_{H^\vee})_0 \arrow[r, hook] \arrow[d, leftrightarrow, "\wr"] & \mathsf{B}_{H^\vee} \arrow[d, leftrightarrow, "\wr"', "\text{graded } (\S\,3.21)"] \\
\mathsf{K}_{G^*} \arrow[r, leftrightarrow, "\sim"] & (\mathsf{K}_{H^*})_0 \arrow[r, hook] & \mathsf{K}_{H^*}
\end{tikzcd}$$

On direct identification of $\mathbb{Z}[\frac{1}{p}]\mathsf{E}_G$ and $\mathbb{Z}[\frac{1}{p}]\mathsf{B}_{G^\vee}$

3.24. *Curtis homomorphisms on the* B*-side.* — We suppose that G^*_{der} is simply-connected. Then $\mathsf{B}_{G^\vee} = \mathbb{Z}[X(T^\vee)]^W/I$ is a reduced ring (§ 3.17), and we have a bijection $(G^{*F^*}_{\mathrm{ss}}/\sim) \simeq (T^\vee /\!\!/ W)^{F^\vee}$ (§ 3.6 and § 3.12) which induces a $\overline{\mathbb{Q}}$-algebra isomorphism $\overline{\mathbb{Q}}\mathsf{E}_G \simeq \overline{\mathbb{Q}}\mathsf{B}_{G^\vee}$ (via (1.10.3) and § 3.5(a)); together with the Curtis homomorphism Cur^G and the canonical isomorphisms $(T_w)^F \simeq T^{wF}$ $(w \in W)$, we define the $\overline{\mathbb{Q}}$-algebra homomorphism

$$\Phi = (\Phi_w)_{w\in W} : \overline{\mathbb{Q}}\mathsf{B}_{G^\vee} \longrightarrow \prod_{w\in W} \overline{\mathbb{Q}}T^{wF}$$

in the way that the following diagram is commutative:

$$\begin{tikzcd}
 & \overline{\mathbb{Q}}\mathsf{E}_G \arrow[r, leftrightarrow, "\sim"] \arrow[d, hook, "\mathrm{Cur}^G = (\mathrm{Cur}^G_{T_w})_{w\in W}"'] & \overline{\mathbb{Q}}\mathsf{B}_{G^\vee} \arrow[d, hook, "\Phi = (\Phi_w)_{w\in W}"] \\
 & \prod_{w\in W} \overline{\mathbb{Q}}(T_w)^F \arrow[r, leftrightarrow, "\sim"] & \prod_{w\in W} \overline{\mathbb{Q}}T^{wF}
\end{tikzcd} \tag{3.24.1}$$

Observe that Φ is injective as Cur^G is.

For $w \in W$, $\lambda \in X(T^\vee)$ and $t \in T^{wF}$, we have

$$\Phi_w(r(\lambda) + I)(t) = \frac{1}{|W_\lambda|} \sum_{\sigma\in W} \big\langle (\sigma\lambda)|_{T^{*wF^*}}, \widehat{t}\,\big\rangle_{T^{*wF^*}} = \sum_{\sigma\in W/W_\lambda} \big\langle (\sigma\lambda)|_{T^{*wF^*}}, \widehat{t}\,\big\rangle_{T^{*wF^*}} \in \mathbb{Z}. \tag{3.24.2}$$

In the above formula, $\lambda \in X(T^\vee)$ is identified with a map $\lambda : T^* \longrightarrow \overline{\mathbb{Q}}^\times$ via the identifications $T^\vee(\overline{\mathbb{F}}_q) = T^*$ and $\overline{\mathbb{F}}_q^\times \hookrightarrow \overline{\mathbb{Q}}^\times$, and $\widehat{\cdot} : T^{wF} \simeq \mathrm{Irr}_{\overline{\mathbb{Q}}}(T^{*wF^*})$ is the chosen duality (see § 1.4). As a consequence:

(a) the map Φ is in fact defined over $\mathbb{Z}$:

$$\Phi = (\Phi_w)_{w\in W} : \mathsf{B}_{G^\vee} \longrightarrow \prod_{w\in W} \mathbb{Z}T^{wF}; \tag{3.24.3}$$

(b) the commutative diagram (3.24.1) is equivariant under the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ (using (a) and § 1.24) and hence induces by restriction a commutative diagram of $\mathbb{Q}$-algebras, obtained by replacing all $\overline{\mathbb{Q}}$-coefficients in (3.24.1) by $\mathbb{Q}$-coefficients.

The diagram (3.24.1) ("E-B diagram") is similar to (2.9.1) ("E-K diagram"), the map Φ here being the counterpart of the restriction map on the K-side. The E-K diagram was well-studied in § 2.9 and has led to results over integral coefficients. However, an analogous study seems to be difficult for the E-B diagram, as a general study of the map Φ on the B-side would involve complicated combinatorics; for the moment, we only know how to do the case where G is $\mathrm{GL}_2(\overline{\mathbb{F}}_q)$ or $\mathrm{PGL}_2(\overline{\mathbb{F}}_q)$, in which case the combinatorics will be traceable and we will be able to establish a $\mathbb{Z}[\frac{1}{p}]$-algebra isomorphism $\mathbb{Z}[\frac{1}{p}]\mathsf{E}_G \simeq \mathbb{Z}[\frac{1}{p}]\mathsf{B}_{G^\vee}$.

3.25. *Toric duality in the case of* $G = \mathrm{GL}_2(\overline{\mathbb{F}}_q)$. — Let $G = \mathrm{GL}_2(\overline{\mathbb{F}}_q)$ (with the standard q-Frobenius endomorphism F on G via $F((g_{ij})_{i,j}) = (g_{ij}^q)_{i,j}$), let T be the

diagonal maximal torus of G, and let $W = \{1, s\}$ be the Weyl group of G (where s is the image of $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \in N_G(T)$ in W).

Let $\iota : \overline{\mathbb{F}}_q^\times \xrightarrow{\sim} (\mathbb{Q}/\mathbb{Z})_{p'}$ be the chosen isomorphism (§ 1.4). Set $\xi := \iota^{-1}(\frac{1}{q^2-1})$ which is a primitive (q^2-1)-st root of unity generating the multiplicative group $\mathbb{F}_{q^2}^\times$, and set $\zeta := \iota^{-1}(\frac{1}{q-1}) = \xi^{q+1}$ which is a primitive $(q-1)$-st root of unity generating the multiplicative group $\mathbb{F}_q^\times$. Then we have the following group isomorphisms:

$$\theta_1 : \mathbb{Z}/(q-1)\mathbb{Z} \times \mathbb{Z}/(q-1)\mathbb{Z} \xrightarrow{\sim} T^F, \qquad (a,b) \longmapsto \theta_1(a,b) = \begin{bmatrix} \zeta^a & 0 \\ 0 & \zeta^b \end{bmatrix};$$
$$\theta_s : \mathbb{Z}/(q^2-1)\mathbb{Z} \xrightarrow{\sim} T^{sF}, \qquad c \longmapsto \theta_s(c) = \begin{bmatrix} \xi^c & 0 \\ 0 & \xi^{qc} \end{bmatrix}.$$

Denoting by $\kappa : \overline{\mathbb{F}}_q^\times \hookrightarrow \overline{\mathbb{Q}}^\times$ the chosen embedding (§ 1.4), the dual elements $\widehat{t}$ in § 3.24 may be described as follows (G being autodual, we have $T^{*F^*} \simeq T^F$ and $T^{*sF^*} \simeq T^{sF}$):

$$t = \theta_1(a,b) \in T^F \implies \widehat{t} : T^{*F^*} \longrightarrow \overline{\mathbb{Q}}^\times, \ \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} \longmapsto \kappa(x^a y^b);$$
$$t = \theta_s(c) \in T^{sF} \implies \widehat{t} : T^{*sF^*} \longrightarrow \overline{\mathbb{Q}}^\times, \ \begin{bmatrix} z & 0 \\ 0 & z^q \end{bmatrix} \longmapsto \kappa(z^c).$$

For the Langlands dual side, we also know that $G^\vee = \mathrm{GL}_2$ with $T^\vee$ being its diagonal maximal torus, and that $X(T^\vee) = \mathbb{Z}\varepsilon_1 \oplus \mathbb{Z}\varepsilon_2$ where

$$\varepsilon_i : T^\vee \longrightarrow \mathbb{G}_m, \quad \varepsilon_i(\mathrm{diag}(t_1,t_2)) := t_i \quad (i = 1,2).$$

For $\lambda = a\varepsilon_1 + b\varepsilon_2 \in X(T^\vee)$ $(a,b \in \mathbb{Z})$, $s\lambda = b\varepsilon_1 + a\varepsilon_2$; identifying λ with $\lambda : T^* \longrightarrow \overline{\mathbb{Q}}^\times$ as in § 3.24, we have $\lambda\left(\begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix}\right) = \kappa(x^a y^b)$ and $\lambda\left(\begin{bmatrix} z & 0 \\ 0 & z^q \end{bmatrix}\right) = \kappa(z^{a+qb})$.

3.26. Proposition. — *Let $G = \mathrm{GL}_2(\overline{\mathbb{F}}_q)$ and use the setting in § 3.25, so that the map Φ in (3.24.3) is*

$$\Phi = (\Phi_1, \Phi_s) : \mathsf{B}_{G^\vee} \longrightarrow \mathbb{Z}T^F \times \mathbb{Z}T^{sF}.$$

Denote again the linear extensions of this map by Φ. Let Z be the centre of G, so that $Z^F = T^F \cap T^{sF}$. Let Λ be an integral domain and let K be the field of fractions of Λ. Then Φ is saturated over Λ (with respect to K):

$$\Phi(\Lambda\mathsf{B}_{G^\vee}) = \Phi(K\mathsf{B}_{G^\vee}) \cap \{(f_1, f_s) \in \Lambda T^F \times \Lambda T^{sF} : f_1|_{Z^F} - f_s|_{Z^F} \in 2\Lambda Z^F\}.$$

Proof. Notation as in § 3.25. By § 3.17, the $\mathbb{Z}$-module $\mathsf{B}_{G^\vee} = \mathbb{Z}[X(T^\vee)]^W/I$ is free of rank $q(q-1)$ and admits a basis

$$\mathfrak{F} = \{r_{i,j} := r((i+j)\varepsilon_1 + j\varepsilon_2) + I : i,j \in \mathbb{N},\ 0 \le i \le q-1,\ 0 \le j \le q-2\}.$$

By (3.24.2), for $0 \le i \le q-1$ and $0 \le j \le q-2$, we have

$$\Phi_w(r_{i,j})(t) = \frac{1}{|W_{(i+j)\varepsilon_1 + j\varepsilon_2}|} \sum_{\sigma \in W} \left\langle (\sigma((i+j)\varepsilon_1 + j\varepsilon_2))|_{T^{*wF^*}}, \widehat{t} \right\rangle_{T^{*wF^*}} \in \mathbb{Z}.$$

For $m \in \mathbb{Z}$, we write $\overline{m} := m + (q-1)\mathbb{Z} \in \mathbb{Z}/(q-1)\mathbb{Z}$, we then have :

(i) $w = 1$, $t = \theta_1(a,b) \in T^F$:

$$\text{if } a = b \in \mathbb{Z}/(q-1)\mathbb{Z}\text{: } \Phi_1(r_{i,j})(\theta_1(a,a)) = \begin{cases} 1 & \text{if } (i,\bar{j}) = (0,a); \\ 2 & \text{if } (i,\bar{j}) = (q-1,a); \\ 0 & \text{otherwise;} \end{cases}$$

$$\text{if } a \neq b \in \mathbb{Z}/(q-1)\mathbb{Z}\text{: } \Phi_1(r_{i,j})(\theta_1(a,b)) = \begin{cases} 1 & \text{if } (\bar{i},\bar{j}) = (a-b,b) \text{ or } (b-a,a); \\ 0 & \text{otherwise.} \end{cases}$$

(ii) $w = s$, $t = \theta_s(c) \in T^{sF}$: for every $c \in \mathbb{Z}/(q^2-1)\mathbb{Z}$, lift it to $\dot{c} \in \mathbb{Z}$ with $0 \leq \dot{c} \leq q^2-2$ ($\dot{c}$ is uniquely determined by c), write $\dot{c} = (q+1)\dot{u} + \dot{v}$ where $\dot{u}, \dot{v} \in \mathbb{Z}$ with $0 \leq \dot{v} \leq q$ and $0 \leq \dot{u} \leq q-2$ (division of $\dot{c}$ by $(q+1)$); set also $u = \dot{u} + (q-1)\mathbb{Z} \in \mathbb{Z}/(q-1)\mathbb{Z}$ and $v = \dot{v} + (q-1)\mathbb{Z} \in \mathbb{Z}/(q-1)\mathbb{Z}$; then:

$$\text{if } \dot{v} = 0\text{: } \Phi_s(r_{i,j})(\theta_s(c)) = \begin{cases} 1 & \text{if } (i,j) = (0,\dot{u}); \\ 0 & \text{otherwise;} \end{cases}$$

$$\text{if } \dot{v} = 1 \text{ or } q\text{: } \Phi_s(r_{i,j})(\theta_s(c)) = \begin{cases} 1 & \text{if } (i,j) = (1,\dot{u}); \\ 0 & \text{otherwise;} \end{cases}$$

$$\text{if } 2 \leq \dot{v} \leq q-1\text{: } \Phi_s(r_{i,j})(\theta_s(c)) = \begin{cases} 1 & \text{if } (i,\bar{j}) = (v,u) \text{ or } (q+1-\dot{v}, u+v-1); \\ 0 & \text{otherwise.} \end{cases}$$

We are now ready to prove the proposition (one may compare the following argument with the example of $q = 3$ after this proof). We know that $\mathfrak{F}$ is a basis for the K-vector space $K\mathsf{B}_{G^\vee}$. Let $f = \sum_{i,j} f_{i,j} r_{i,j} \in K\mathsf{B}_{G^\vee}$ ($f_{i,j} \in K$) and suppose that this f fulfils the following condition $(*)$:

$$(*): \quad \Phi_1(f) \in \Lambda T^F,\ \Phi_s(f) \in \Lambda T^{sF} \text{ and } \Phi_1(f)|_{Z^F} - \Phi_s(f)|_{Z^F} \in \Lambda Z^F.$$

Then the proposition will be proved if we can show that $f_{i,j} \in \Lambda$ for all i and j, which is equivalent to saying that the following property (P_i) holds for every $0 \leq i \leq q-1$:

$$(P_i): \quad f_{i,j} \in \Lambda \text{ for every } 0 \leq j \leq q-2.$$

To prove the properties (P_i), notice first that the condition $(*)$ tells us:

(1) $\displaystyle\sum_{i,j} \Phi_1(r_{i,j})(\theta_1(a,b)) f_{i,j} \in \Lambda$ for every $a, b \in \mathbb{Z}/(q-1)\mathbb{Z}$;

(2) $\displaystyle\sum_{i,j} \Phi_s(r_{i,j})(\theta_s(c)) f_{i,j} \in \Lambda$ for every $c \in \mathbb{Z}/(q^2-1)\mathbb{Z}$;

(3) $\displaystyle\sum_{i,j} \left[\Phi_1(r_{i,j})(\theta_1(a,a)) - \Phi_s(r_{i,j})(\theta_s((q+1)a))\right] f_{i,j} \in 2\Lambda$ for every $a \in \mathbb{Z}/(q-1)\mathbb{Z}$.

The coefficients involved in the above conditions (1), (2) and (3) have been previously calculated in (i) and (ii). We then perform the following iteration scheme:

- (P_0) is true: using (2) with $c = (q+1)a$ $(0 \leq a \leq q-2)$, we get $f_{0,a} \in \Lambda$.
- (P_1) is true: using (2) with $c = (q+1)a + 1$ $(0 \leq a \leq q-2)$, we get $f_{1,a} \in \Lambda$.
- (P_{q-1}) is true: for $0 \leq a \leq q-2$, by (3) we have $2f_{q-1,a} \in 2\Lambda$, so $f_{q-1,a} \in \Lambda$.
- (P_2) is true: using (2) with $c = (q+1)a + 2$ $(0 \leq a \leq q-2)$, we see that $f_{2,a} + f_{q-1,[(a+1) \bmod (q-1)]} \in \Lambda$; as (P_{q-1}) is true, we have $f_{q-1,[(a+1) \bmod (q-1)]} \in \Lambda$, so we get $f_{2,a} \in \Lambda$.

- If (P_h) is true for some $1 \leq h \leq q-3$, then (P_{q-1-h}) and (P_{h+2}) are true: suppose that (P_h) is true for some $1 \leq h \leq q-3$; for $0 \leq b \leq q-2$, putting $a = b+h$ in (1), we get $f_{h,b} + f_{q-1-h,[(b+h) \bmod (q-1)]} \in \Lambda$, so $f_{q-1-h,[(b+h) \bmod (q-1)]} \in \Lambda$ by (P_h), and hence (P_{q-1-h}) is true; furthermore, for $0 \leq a \leq q-2$, using (2) with $c = (q+1)a + h'$ and $h' = q-1-h$, we get $f_{q-1-h,a} + f_{h+2,[(a+h'-1) \bmod (q-1)]} \in \Lambda$, so $f_{h+2,[(a+h'-1) \bmod (q-1)]} \in \Lambda$ by (P_{q-1-h}), and thus (P_{h+2}) is true.

We conclude by the above iteration that (P_i) is true for every $0 \leq i \leq q-1$. $\square$

Example. When $q = 3$, the matrices of Φ_1 and Φ_s are :

$$[\Phi_1] = \begin{array}{c} \\ \theta_1(0,0) \\ \theta_1(1,1) \\ \theta_1(1,0) \\ \theta_1(1,0) \end{array} \begin{array}{c} \begin{array}{cccccc} r_{0,0} & r_{0,1} & r_{1,0} & r_{1,1} & r_{2,0} & r_{2,1} \end{array} \\ \begin{bmatrix} 1 & & & & 2 & \\ & 1 & & & & 2 \\ & & 1 & 1 & & \\ & & 1 & 1 & & \end{bmatrix} \end{array}$$

$$[\Phi_s] = \begin{array}{c} \\ \theta_s(0) \\ \theta_s(4) \\ \theta_s(1) \\ \theta_s(5) \\ \theta_s(2) \\ \theta_s(6) \\ \theta_s(3) \\ \theta_s(7) \end{array} \begin{array}{c} \begin{array}{cccccc} r_{0,0} & r_{0,1} & r_{1,0} & r_{1,1} & r_{2,0} & r_{2,1} \end{array} \\ \begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & 1 & 1 \\ & & & & 1 & 1 \\ & & 1 & & & \\ & & & 1 & & \end{bmatrix} \\ \begin{array}{cccccc} r_{0,0} & r_{0,1} & r_{1,0} & r_{1,1} & r_{2,0} & r_{2,1} \end{array} \end{array}$$

3.27. Proposition. — *Let $G = \mathrm{GL}_2(\overline{\mathbb{F}}_q)$. Let T and $W = \{1, s\}$ be as in § 3.26, so that the Curtis homomorphism Cur^G is*

$$\mathrm{Cur}^G = (\mathrm{Cur}^G_T, \mathrm{Cur}^G_{T_s}) : \overline{\mathbb{Q}}\mathsf{E}_G \longrightarrow \overline{\mathbb{Q}}T^F \times \overline{\mathbb{Q}}T_s^F.$$

Let Z be the centre of G (so $Z^F = T^F \cap T_s^F$). Then

$$\mathrm{Cur}^G(\overline{\mathbb{Z}}[\tfrac{1}{p}]\mathsf{E}_G) = \mathrm{Cur}^G(\overline{\mathbb{Q}}\mathsf{E}_G) \cap \{(f_1, f_s) \in \overline{\mathbb{Z}}[\tfrac{1}{p}]T^F \times \overline{\mathbb{Z}}[\tfrac{1}{p}]T_s^F : f_1|_{Z^F} - f_s|_{Z^F} \in 2\overline{\mathbb{Z}}[\tfrac{1}{p}]Z^F\}.$$

Proof. As in the proof of § 3.26, the idea is to choose a $\overline{\mathbb{Z}}[\frac{1}{p}]$-linear basis of $\overline{\mathbb{Z}}[\frac{1}{p}]\mathsf{E}_G$, calculate their images under Cur^G and then invert the resulting system. We choose the Borel subgroup $B = \begin{bmatrix} \overline{\mathbb{F}}_q^\times & \overline{\mathbb{F}}_q \\ 0 & \overline{\mathbb{F}}_q^\times \end{bmatrix}$ of G with its unipotent radical $U = \begin{bmatrix} 1 & \overline{\mathbb{F}}_q \\ 0 & 1 \end{bmatrix}$. Let $\psi_0 : \mathbb{F}_q \longrightarrow \overline{\mathbb{Z}}[\frac{1}{p}]^\times$ be a non-trivial linear character; composing ψ_0 by the canonical additive group isomorphism $U^F \simeq \mathbb{F}_q$, we get a regular linear character $\psi : U \longrightarrow \overline{\mathbb{Z}}[\frac{1}{p}]^\times$. By § 1.10(a), a $\overline{\mathbb{Z}}[\frac{1}{p}]$-linear basis of $\overline{\mathbb{Z}}[\frac{1}{p}]\mathsf{E}_G$ is

$$\{c_a := e_\psi n_a e_\psi : a \in \mathbb{F}_q^\times\} \sqcup \{c'_{a,b} := q e_\psi n'_{a,b} e_\psi : a, b \in \mathbb{F}_q^\times\}$$

where $n_a := \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$ and $n'_{a,b} := \begin{bmatrix} 0 & b \\ -a^{-1} & 0 \end{bmatrix}$. With the help of Curtis' formula (§ 1.11) and reasoning as in [Cu, Sec. 5], one may obtain the following formulae:

(i) $\mathrm{Cur}_T^G(c_a)(t) = \begin{cases} 1 & (t = n_a); \\ 0 & (\text{otherwise}); \end{cases}$

$\mathrm{Cur}_T^G(c'_{a,b})(t) = \begin{cases} \psi_0(a.\mathrm{tr}(t)) & (\det(t) = \det(n'_{a,b})); \\ 0 & (\text{otherwise}); \end{cases}$

(ii) $\mathrm{Cur}_{T_s}^G(c_a)(t) = \begin{cases} 1 & (t = n_a); \\ 0 & (\text{otherwise}). \end{cases}$

$\mathrm{Cur}_{T_s}^G(c'_{a,b})(t) = \begin{cases} -\psi_0(a.\mathrm{tr}(t)) & (\det(t) = \det(n'_{a,b})); \\ 0 & (\text{otherwise}). \end{cases}$

Now let $f \in \overline{\mathbb{Q}}\mathsf{E}_G$ and write $f = \sum_a f_a c_a + \sum_{a,b} f'_{a,b} c'_{a,b}$ ($f_a, f'_{a,b} \in \overline{\mathbb{Q}}$). Suppose that $\mathrm{Cur}^G(f) \in \overline{\mathbb{Z}}[\frac{1}{p}]T^F \times \overline{\mathbb{Z}}[\frac{1}{p}]T^{sF}$ and that $\mathrm{Cur}_T^G(f)|_{Z^F} - \mathrm{Cur}_{T_s}^G(f)|_{Z^F} \in 2\overline{\mathbb{Z}}[\frac{1}{p}]Z^F$. Using (i) and (ii) above, the conditions imposed on f imply the following: for $(\tau, \delta) \in \mathbb{F}_q \times \mathbb{F}_q^\times$,

(1) if $(\tau, \delta) = (2v, v^2)$ for some $v \in \mathbb{F}_q^\times$, then $f_v + \sum_{a \in \mathbb{F}_q^\times} \psi_0(\tau a) f'_{a,\delta} \in \overline{\mathbb{Z}}[\frac{1}{p}]$,

$f_v - \sum_{a \in \mathbb{F}_q^\times} \psi_0(\tau a) f'_{a,\delta} \in \overline{\mathbb{Z}}[\frac{1}{p}]$, and $2 \sum_{a \in \mathbb{F}_q^\times} \psi_0(\tau a) f'_{a,\delta} \in \overline{\mathbb{Z}}[\frac{1}{p}] \in 2\overline{\mathbb{Z}}[\frac{1}{p}]$;

(2) if $(\tau, \delta) \neq (2v, v^2)$ for every $v \in \mathbb{F}_q^\times$, then $\sum_{a \in \mathbb{F}_q^\times} \psi_0(\tau a) f'_{a,\delta} \in \overline{\mathbb{Z}}[\frac{1}{p}]$.

By (1) and (2), we have $[\psi_0(\tau a)]_{\tau, a \in \mathbb{F}_q^\times} \cdot \left[f'_{a,\delta}\right]_{a,\delta \in \mathbb{F}_q^\times} \in \mathrm{M}_{q-1}(\overline{\mathbb{Z}}[\frac{1}{p}])$; as $\psi_0 : \mathbb{F}_q \longrightarrow \overline{\mathbb{Z}}[\frac{1}{p}]^\times$ is a non-trivial linear character, we have

$$[\psi_0(\tau a)]_{\tau, a \in \mathbb{F}_q^\times} [\psi_0(-\tau a)]_{\tau, a \in \mathbb{F}_q^\times} = q \cdot \mathrm{id}_{q-1} - J_{q-1} = \begin{bmatrix} q-1 & -1 & \cdots & -1 \\ -1 & q-1 & \cdots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \cdots & q-1 \end{bmatrix},$$

where J_{q-1} is the $(q-1) \times (q-1)$ matrix whose coefficients are all 1. It is easy to check that the eigenvalues of $q \cdot \mathrm{id}_{q-1} - J_{q-1}$ are q (with multiplicity $q-2$) and 1 (with multiplicity 1), so $q \cdot \mathrm{id}_{q-1} - J_{q-1}$ is invertible over $\overline{\mathbb{Z}}[\frac{1}{p}]$ and thus $f'_{a,\delta} \in \overline{\mathbb{Z}}[\frac{1}{p}]$ for all $a, \delta \in \mathbb{F}_q^\times$. Using (1) again, we deduce that $f_v \in \overline{\mathbb{Z}}[\frac{1}{p}]$ for all $v \in \mathbb{F}_q^\times$. Hence $f \in \overline{\mathbb{Z}}[\frac{1}{p}]\mathsf{E}_G$. The proof is thus complete. □

3.28. Corollary (of §§ 3.24 - 3.27). — *For $G = \mathrm{GL}_2(\overline{\mathbb{F}}_q)$, the diagram (3.24.1) induces (by restriction) $\mathbb{Z}[\frac{1}{p}]$-algebra isomorphism $\mathbb{Z}[\frac{1}{p}]\mathsf{E}_G \simeq \mathbb{Z}[\frac{1}{p}]\mathsf{B}_{G^\vee}$.*

This isomorphism is also a corollary of § 3.22 since for our $G = \mathrm{GL}_2(\overline{\mathbb{F}}_q)$ (type A_1), no prime number is bad and $G^*_{\mathrm{der}} = \mathrm{SL}_2(\overline{\mathbb{F}}_q)$ is simply-connected. The point here is that the arguments in §§ 3.24 - 3.27 provide a direct proof of the isomorphism $\mathbb{Z}[\frac{1}{p}]\mathsf{E}_G \simeq \mathbb{Z}[\frac{1}{p}]\mathsf{B}_{G^\vee}$ without using the ring K_{G^*}, and that we have also obtained finer structures of Curtis homomorphisms (see also § 3.29 below).

3.29. *On the non-saturatedness of Curtis homomorphisms.* — Still consider the case of $G = \mathrm{GL}_2(\overline{\mathbb{F}}_q)$ (so that $|W| = 2$) and suppose that q is odd. In this case, Cur^G and Φ are both not saturated over $\overline{\mathbb{Z}}[\frac{1}{p}]$ (with respect to $\overline{\mathbb{Q}}$): indeed, by the $\overline{\mathbb{Z}}[\frac{1}{p}]$-algebra isomorphism $\overline{\mathbb{Z}}[\frac{1}{p}]\mathsf{E}_G \simeq \overline{\mathbb{Z}}[\frac{1}{p}]\mathsf{B}_{G^\vee}$ in § 3.28 and the E-B commutative diagram in § 3.24, it suffices to show that Φ is not saturated over $\overline{\mathbb{Z}}[\frac{1}{p}]$; using the notation in the proof of

§ 3.26, if we consider the element $f = \sum_{i,j} f_{i,j} r_{i,j} \in \overline{\mathbb{Q}}\mathsf{B}_{G^\vee}$ (the sum is over $0 \leq i \leq q-1$ and $0 \leq j \leq q-2$) where $f_{i,j} = 1/2$ if $i = 2, 4, 6, \cdots, q-1$ and $f_{i,j} = 0$ otherwise, then the calculations made in the proof of § 3.26 will show that

$$f \notin \overline{\mathbb{Z}}[\tfrac{1}{p}]\mathsf{B}_{G^\vee} \text{ but } \Phi(f) \in \prod_{w \in W} \overline{\mathbb{Z}}[\tfrac{1}{p}]T^{wF};$$

thus Φ is not saturated over $\overline{\mathbb{Z}}[\frac{1}{p}]$. The same argument also shows that neither Φ nor Cur^G is saturated over $\mathbb{Z}[\frac{1}{p}]$ (with respect to $\mathbb{Q}$).

3.30. *The example* $G = \mathrm{PGL}_2(\overline{\mathbb{F}}_q)$. — For $G = \mathrm{PGL}_2(\overline{\mathbb{F}}_q)$, we know that its dual is $G^* = \mathrm{SL}_2(\overline{\mathbb{F}}_q)$, which is already simply-connected. Using a similar discussion as in the previous GL_2-case, it can be shown that the conclusions of §§ 3.26 - 3.28 all hold for $G = \mathrm{PGL}_2(\overline{\mathbb{F}}_q)$; we omit the details here.

4. Sur le ℓ-bloc unipotent de GL_2 p-adique avec $\ell \neq p$

La composante ℓ-unipotente de $\overline{\mathbb{Z}}_\ell \mathsf{E}_G$

4.1. *Notation sur les parties ℓ-unipotentes et ℓ-régulières.* — Soient G un groupe fini et ℓ un nombre premier. Pour tout $g \in \mathsf{G}$, on rappelle que $g_\ell \in \mathsf{G}$ (resp. $g_{\ell'} \in \mathsf{G}$) désigne la partie ℓ-unipotente (resp. ℓ-régulière) de g. Si A est une partie de G, on pose $A_\ell = \{g \in A : g_{\ell'} = 1\}$ et $A_{\ell'} = \{g \in A : g_\ell = 1\}$. Les notations A_ℓ et $A_{\ell'}$ s'étendent de façon évidente au cas où A se compose d'orbites d'une action adjointe.

4.2. *Idempotents centraux associés aux ℓ-séries.* — Dorénavant, revenons au groupe réductif G défini sur $\mathbb{F}_q$ en § 1.2, et soit ℓ un nombre premier différent de $p = \mathrm{char}(\mathbb{F}_q)$. On se rappelle que chaque $\chi \in \mathrm{Irr}_{\overline{\mathbb{Q}}}(G^F)$ correspond à un idempotent central primitif $e_\chi^G = \dfrac{\chi(1)}{|G^F|} \displaystyle\sum_{g \in G^F} \chi(g^{-1})g$ de $\overline{\mathbb{Q}}G^F$, et que chaque classe rationnelle $[x] \in G_{\mathrm{ss}}^{*F^*}/\sim$ détermine une série rationnelle de Lusztig $\mathcal{E}(G^F, [x]) \subset \mathrm{Irr}_{\overline{\mathbb{Q}}}(G^F)$.

Pour tout $[s] \in (G_{\mathrm{ss},\ell'}^{*F^*}/\sim) = (G_{\mathrm{ss}}^{*F^*}/\sim)_{\ell'}$, on pose

$$\mathcal{E}_\ell(G^F, [s]) = \bigcup_{[x] \in G_{\mathrm{ss}}^{*F^*}/\sim,\, x_{\ell'} = s} \mathcal{E}(G^F, [x]),$$

et l'on considère l'idempotent central $e_{[s]}^{G,\ell}$ de $\overline{\mathbb{Q}}G^F$ défini par

$$e_{[s]}^{G,\ell} = \sum_{\chi \in \mathcal{E}_\ell(G^F, [s])} e_\chi^G.$$

Notons que $e_{[s]}^{\mathsf{E},\ell} := e_{[s]}^{G,\ell} e_\psi$ est un idempotent central de $\overline{\mathbb{Q}}\mathsf{E}_G = \overline{\mathbb{Q}}\mathsf{E}_{G,\psi} = e_\psi \overline{\mathbb{Q}}G^F e_\psi$ (comparer § 1.10 et § 1.22), et l'on a $e_{[s]}^{G,\ell}.\overline{\mathbb{Q}}\mathsf{E}_G = e_{[s]}^{\mathsf{E},\ell}.\overline{\mathbb{Q}}\mathsf{E}_G$.

4.3. Théorème. [BrMi, Thm. 2.2] — *On a $e_{[s]}^{G,\ell} \in \overline{\mathbb{Z}}_\ell G^F$ pour tout $[s] \in G_{\mathrm{ss},\ell'}^{*F^*}/\sim$.*

4.4. Corollaire (de § 4.3). — *On a $e_{[s]}^{\mathsf{E},\ell} \in \overline{\mathbb{Z}}_\ell \mathsf{E}_G$ pour tout $[s] \in G_{\mathrm{ss},\ell'}^{*F^*}/\sim$, et la $\overline{\mathbb{Z}}_\ell$-algèbre $\overline{\mathbb{Z}}_\ell \mathsf{E}_G$ se décompose comme suit :*

$$\overline{\mathbb{Z}}_\ell \mathsf{E}_G = \prod_{[s] \in G_{\mathrm{ss},\ell'}^{*F^*}/\sim} e_{[s]}^{\mathsf{E},\ell}.\overline{\mathbb{Z}}_\ell \mathsf{E}_G.$$

On appelle $e_{[1]}^{\mathsf{E},\ell}.\overline{\mathbb{Z}}_\ell \mathsf{E}_G$ la composante ℓ-unipotente de $\overline{\mathbb{Z}}_\ell \mathsf{E}_G$.

4.5. *Structure de $e_{[1]}^{\mathsf{E},\ell}.\overline{\mathbb{Q}}\mathsf{E}_G$.* — On a un isomorphisme de $\overline{\mathbb{Q}}$-algèbres

$$e_{[1]}^{\mathsf{E},\ell}.\overline{\mathbb{Q}}\mathsf{E}_G \xrightarrow[\sim]{\S\, 1.8} \overline{\mathbb{Q}}^{G_{\mathrm{ss},\ell}^{*F^*}/\sim}. \tag{4.5.1}$$

Sous l'isomorphisme $\overline{\mathbb{Q}}\mathsf{E}_G \simeq \overline{\mathbb{Q}}^{G_{\mathrm{ss}}^{*F^*}/\sim}$ en (1.10.3), l'idempotent $e_{[1]}^{\mathsf{E},\ell}$ de $\overline{\mathbb{Q}}\mathsf{E}_G$ correspond à la fonction caractéristique $\mathbf{1}_{G_{\mathrm{ss},\ell}^{*F^*}/\sim} : (G_{\mathrm{ss}}^{*F^*}/\sim) \longrightarrow \overline{\mathbb{Q}}$.

4.6. Lemme. — *Sous l'isomorphisme de $\overline{\mathbb{Q}}$-algèbres $\overline{\mathbb{Q}}\mathsf{B}_{G^\vee,\mathrm{red}} \simeq \overline{\mathbb{Q}}^{(T^\vee /\!\!/ W)^{F^\vee}(\overline{\mathbb{Q}})}$ en § 3.5(a), soit $e_{[1]}^{\mathsf{B},\ell}$ l'idempotent de $\overline{\mathbb{Q}}\mathsf{B}_{G^\vee,\mathrm{red}}$ correspondant à la fonction caractéristique $\mathbf{1}_{(T^\vee /\!\!/ W)^{F^\vee}(\overline{\mathbb{Q}})_\ell} : (T^\vee /\!\!/ W)^{F^\vee}(\overline{\mathbb{Q}}) \longrightarrow \overline{\mathbb{Q}}$. Alors $e_{[1]}^{\mathsf{B},\ell} \in \mathbb{Z}_\ell\mathsf{B}_{G^\vee,\mathrm{red}}$.*

Preuve. Remarquons tout d'abord que $(T^\vee /\!\!/ W)^{F^\vee}(\overline{\mathbb{Q}})$ est l'ensemble des W-orbites dans $\bigcup_{w\in W} T^\vee(\overline{\mathbb{Q}})^{wF^\vee}$. Soit $d = \max_{w\in W}(\mathrm{val}_\ell|T^\vee(\overline{\mathbb{Q}})_\ell^{wF^\vee}|)$, et soit $m \in \mathbb{N}^*$ le plus petit commun multiple des ordres des éléments de $\bigcup_{w\in W} T^\vee(\overline{\mathbb{Q}})_{\ell'}^{wF^\vee}$ (donc ℓ ne divise pas m). Soit l'inclusion naturelle $e(\cdot) : X(T^\vee) \hookrightarrow \mathbb{Z}[X(T^\vee)]$ et fixons une $\mathbb{Z}$-base $\{\lambda_1, \cdots, \lambda_n\}$ de $X(T^\vee)$, puis considérons la fonction

$$\varphi = \prod_{j=1}^{n}\prod_{w\in W}\left(\frac{1}{m}\sum_{h=0}^{m-1} e(w\lambda_j)^{\ell^d h}\right) : T^\vee(\overline{\mathbb{Q}})/W \longrightarrow \overline{\mathbb{Q}}.$$

Observons que

$$\sum_{h=0}^{m-1} e(w\lambda_j)^{\ell^d h} = \frac{e(w\lambda_j)^{\ell^d m} - 1}{e(w\lambda_j)^{\ell^d} - 1}\ ;$$

d'autre part, pour tout $t \in \bigcup_{w\in W} T^\vee(\overline{\mathbb{Q}})^{wF^\vee}$, on a $t^{\ell^d m} = 1$ et l'équivalence suivante :

$$t^{\ell^d} = 1 \Longleftrightarrow t \in \bigcup_{w\in W} T^\vee(\overline{\mathbb{Q}})_\ell^{wF^\vee}.$$

On en vérifie que la restriction de φ sur $(T^\vee /\!\!/ W)^{F^\vee}(\overline{\mathbb{Q}}) = (T^\vee(\overline{\mathbb{Q}})/W)^{F^\vee}$ est la fonction caractéristique $\mathbf{1}_{(T^\vee /\!\!/ W)^{F^\vee}(\overline{\mathbb{Q}})_\ell}$. Par conséquent, $e_{[1]}^{\mathsf{B},\ell}$ est l'image de $\varphi \in \mathbb{Z}[\frac{1}{m}][X(T^\vee)]^W$ dans $\overline{\mathbb{Q}}\mathsf{B}_{G^\vee,\mathrm{red}} = \overline{\mathbb{Q}}[X(T^\vee)]^W/\sqrt{I}$ (§ 3.4), d'où $e_{[1]}^{\mathsf{B},\ell} \in \mathbb{Z}[\frac{1}{m}]\mathsf{B}_{G^\vee,\mathrm{red}} \subset \mathbb{Z}_\ell\mathsf{B}_{G^\vee,\mathrm{red}}$. □

4.7. *Le $\mathbb{Z}$-schéma $(T^\vee /\!\!/ W)_\ell^{F^\vee}$.* — On se rappelle que l'anneau des fonctions du $\mathbb{Z}$-schéma $(T^\vee /\!\!/ W)^{F^\vee}$ est $\mathsf{B}_{G^\vee} = \mathbb{Z}[X(T^\vee)]^W/I$ avec $I = \langle F^\vee f - f : f \in \mathbb{Z}[X(T^\vee)]^W\rangle$ (§ 3.3), et que $(T^\vee /\!\!/ W)^{F^\vee}(\overline{\mathbb{Q}})_\ell$ est l'ensemble des W-orbites dans $\bigcup_{w\in W} T^\vee(\overline{\mathbb{Q}})_\ell^{wF^\vee}$.

Soit $d = \max_{w\in W}(\mathrm{val}_\ell|T^\vee(\overline{\mathbb{Q}})_\ell^{wF^\vee}|)$. Posons

$$I_\ell = \langle \lambda^{\ell^d} - \lambda : \lambda \in X(T^\vee)\rangle_{\mathbb{Z}[X(T^\vee)]} \cap \mathbb{Z}[X(T^\vee)]^W$$

et définissons

$$(T^\vee /\!\!/ W)_\ell^{F^\vee} = \mathrm{Spec}(\mathsf{B}_{G^\vee}^{(\ell)}) \quad \text{où} \quad \mathsf{B}_{G^\vee}^{(\ell)} = \mathbb{Z}[X(T^\vee)]^W/(I + I_\ell).$$

On a alors $(T^\vee /\!\!/ W)_\ell^{F^\vee}(\overline{\mathbb{Q}}) = (T^\vee /\!\!/ W)^{F^\vee}(\overline{\mathbb{Q}})_\ell$.

On introduit aussi la version réduite de $(T^\vee /\!\!/ W)_\ell^{F^\vee}$:

$$(T^\vee /\!\!/ W)_{\ell,\mathrm{red}}^{F^\vee} = \mathrm{Spec}(\mathsf{B}_{G^\vee,\mathrm{red}}^{(\ell)}) \quad \text{avec} \quad \mathsf{B}_{G^\vee,\mathrm{red}}^{(\ell)} = \mathbb{Z}[X(T^\vee)]^W/\sqrt{I + I_\ell}.$$

En plus de la propriété évidente $(T^\vee /\!\!/ W)_{\ell,\mathrm{red}}^{F^\vee}(\overline{\mathbb{Q}}) = (T^\vee /\!\!/ W)^{F^\vee}(\overline{\mathbb{Q}})_\ell$, l'évaluation en $\overline{\mathbb{Q}}$-points par localisation (§ 3.5) nous donne un isomorphisme de $\overline{\mathbb{Q}}$-algèbres

$$\overline{\mathbb{Q}}\mathsf{B}_{G^\vee,\mathrm{red}}^{(\ell)} \simeq \overline{\mathbb{Q}}^{(T^\vee /\!\!/ W)^{F^\vee}(\overline{\mathbb{Q}})_\ell}. \tag{4.7.1}$$

4.8. Lemme. — *La surjection canonique*

$$\mathbb{Z}_\ell \mathsf{B}_{G^\vee,\mathrm{red}} = \mathbb{Z}_\ell[X(T^\vee)]^W/\sqrt{I} \twoheadrightarrow \mathbb{Z}_\ell[X(T^\vee)]^W/\sqrt{I+I_\ell} = \mathbb{Z}_\ell \mathsf{B}^{(\ell)}_{G^\vee,\mathrm{red}}$$

(voir § 4.7) se restreint à un isomorphisme d'anneaux

$$e^{\mathsf{B},\ell}_{[1]}.\mathbb{Z}_\ell \mathsf{B}_{G^\vee,\mathrm{red}} \xrightarrow{\sim} \mathbb{Z}_\ell \mathsf{B}^{(\ell)}_{G^\vee,\mathrm{red}},$$

où $e^{\mathsf{B},\ell}_{[1]}$ est l'idempotent de $\mathbb{Z}_\ell \mathsf{B}_{G^\vee,\mathrm{red}}$ en § 4.6.

Preuve. Soit $\pi : \overline{\mathbb{Q}}\mathsf{B}_{G^\vee,\mathrm{red}} \twoheadrightarrow \overline{\mathbb{Q}}\mathsf{B}^{(\ell)}_{G^\vee,\mathrm{red}}$ l'extension à $\overline{\mathbb{Q}}$-coefficients de la surjection canonique dans l'énoncé ; π correspond à l'inclusion $(T^\vee /\!\!/ W)(\overline{\mathbb{Q}})^{F^\vee}_\ell \subset (T^\vee /\!\!/ W)(\overline{\mathbb{Q}})^{F^\vee}$. Dénotons par $\widetilde{e}^{\mathsf{B},\ell}_{[1]} \in \mathbb{Z}_\ell[X(T^\vee)]^W$ la fonction φ dans la preuve de § 4.6 ; alors l'image de $\widetilde{e}^{\mathsf{B},\ell}_{[1]}$ dans $\mathbb{Z}_\ell \mathsf{B}_{G^\vee,\mathrm{red}}$ est $e^{\mathsf{B},\ell}_{[1]}$. Comme $e^{\mathsf{B},\ell}_{[1]}$ correspond à la fonction caractéristique $\mathbf{1}_{(T^\vee /\!\!/ W)^{F^\vee}(\overline{\mathbb{Q}})_\ell} : (T^\vee /\!\!/ W)^{F^\vee}(\overline{\mathbb{Q}}) \longrightarrow \overline{\mathbb{Q}}$, on voit que la multiplication par $\widetilde{e}^{\mathsf{B},\ell}_{[1]}$ donne un morphisme d'anneaux bien défini

$$\widetilde{e}^{\mathsf{B},\ell}_{[1]} : \overline{\mathbb{Q}}\mathsf{B}^{(\ell)}_{G^\vee,\mathrm{red}} \longrightarrow e^{\mathsf{B},\ell}_{[1]}.\overline{\mathbb{Q}}\mathsf{B}_{G^\vee,\mathrm{red}},$$

et ce dernier morphisme d'anneaux est en effet un isomorphisme d'inverse la restriction de π sur $e^{\mathsf{B},\ell}_{[1]}.\overline{\mathbb{Q}}\mathsf{B}_{G^\vee,\mathrm{red}}$; le lemme en découle en se rappelant que $\widetilde{e}^{\mathsf{B},\ell}_{[1]} \in \mathbb{Z}_\ell[X(T^\vee)]^W$. □

4.9. Lemme. — *L'anneau $\mathsf{B}^{(\ell)}_{G^\vee}$ est un facteur direct de l'anneau $\mathsf{B}_{G^\vee}$.*

Preuve. Rappelons que $(T^\vee /\!\!/ W)^{F^\vee}_\ell(\overline{\mathbb{Q}}) = (\bigcup_{w\in W} T^\vee(\overline{\mathbb{Q}})^{wF^\vee}_\ell)/W$. Soit N le plus petit commun multiple des éléments de $\{|T^{\vee F^\vee}(\overline{\mathbb{Q}})^{wF^\vee}_\ell| : w \in W\}$, et écrivons $N = \ell^d m$ où $d \in \mathbb{N}$ et où $m \in \mathbb{N}^*$ n'est pas un multiple de ℓ. Les deux entiers ℓ^d et m étant premiers entre eux, on peut trouver $a, b \in \mathbb{Z}$ tels que $a\ell^d + bm = 1$; l'application

$$(\cdot)^{bm} : (T^\vee /\!\!/ W)^{F^\vee}(\overline{\mathbb{Q}}) \longrightarrow (T^\vee /\!\!/ W)^{F^\vee}_\ell(\overline{\mathbb{Q}})$$

envoie alors chaque élément de $(T^\vee /\!\!/ W)^{F^\vee}(\overline{\mathbb{Q}})$ à sa partie ℓ-unipotente et induit ainsi une section de la surjection canonique $\mathsf{B}_{G^\vee} \twoheadrightarrow \mathsf{B}^{(\ell)}_{G^\vee}$. □

4.10. Corollaire (des § 4.9 et § 3.17). — *Lorsque $G^\vee_{\mathrm{der}}$ est simplement connexe, l'anneau $\mathsf{B}^{(\ell)}_{G^\vee}$ est réduit (c'est-à-dire que $\mathsf{B}^{(\ell)}_{G^\vee,\mathrm{red}} = \mathsf{B}^{(\ell)}_{G^\vee}$).*

4.11. Proposition. — *On suppose que $G^\vee_{\mathrm{der}}$ est simplement connexe et que le nombre premier ℓ n'est pas mauvais pour G (de sorte que ℓ ne divise pas l'entier M_G en § 2.21). Alors l'isomorphisme de $\overline{\mathbb{Z}}_\ell$-algèbres*

$$\overline{\mathbb{Z}}_\ell \mathsf{E}_G \simeq \overline{\mathbb{Z}}_\ell \mathsf{B}_{G^\vee} \tag{4.11.1}$$

donné par § 3.22 induit un isomorphisme de $\overline{\mathbb{Z}}_\ell$-algèbres

$$e^{\mathsf{E},\ell}_{[1]}.\overline{\mathbb{Z}}_\ell \mathsf{E}_G \simeq \overline{\mathbb{Z}}_\ell \mathsf{B}^{(\ell)}_{G^\vee}. \tag{4.11.2}$$

Preuve. Sur $\overline{\mathbb{Q}}$, l'isomorphisme (4.11.1) devient $\overline{\mathbb{Q}}\mathsf{E}_G \simeq \overline{\mathbb{Q}}\mathsf{B}_{G^\vee}$, qui correspond à la bijection $(G^{*F^*}_{\mathrm{ss}}/\sim) \xrightarrow{\sim} (T^\vee /\!\!/ W)^{F^\vee}(\overline{\mathbb{Q}})$ induite par la diagonalisation (§ 3.12 et § 3.6).

Cette dernière bijection induit une bijection $(G^{*F^*}_{\mathrm{ss},\ell}/\sim) \xrightarrow{\sim} (T^\vee /\!\!/ W)^{F^\vee}(\overline{\mathbb{Q}})_\ell$, qui donne au niveau d'anneaux un isomorphisme

$$e^{\mathsf{E},\ell}_{[1]}.\overline{\mathbb{Q}}\mathsf{E}_G \simeq e^{\mathsf{B},\ell}_{[1]}.\overline{\mathbb{Q}}\mathsf{B}_{G^\vee} \tag{4.11.3}$$

grâce aux propriétés des idempotents $e^{\mathsf{E},\ell}_{[1]}$ et $e^{\mathsf{B},\ell}_{[1]}$ (§ 4.5 et § 4.6). En intersectant les deux isomorphismes (4.11.1) et (4.11.3), on obtient l'isomorphisme

$$e^{\mathsf{E},\ell}_{[1]}.\overline{\mathbb{Z}}_\ell\mathsf{E}_G \simeq e^{\mathsf{B},\ell}_{[1]}.\overline{\mathbb{Z}}_\ell\mathsf{B}_{G^\vee},$$

d'où l'isomorphisme (4.11.2) compte tenu des § 4.8 et § 4.10. □

4.12. *Relation de $e^{\mathsf{E},\ell}_{[1]}.\overline{\mathbb{Z}}_\ell\mathsf{E}_G$ avec une enveloppe projective.* [Hel1, Sec. 5][Pa] — Soit $G = \mathrm{GL}_n(\overline{\mathbb{F}}_q)$ avec l'action de q-Frobenius standard $F((g_{ij})_{i,j}) = (g^q_{ij})_{i,j}$, et soit T le tore maximal diagonal de G. On suppose que $\mathrm{ord}_\ell(q) = n$ (c'est-à-dire q est d'ordre n dans $(\mathbb{Z}/\ell\mathbb{Z})^\times$). Il existe alors une unique représentation irréductible $\overline{\pi} \in \mathrm{Irr}_{\overline{\mathbb{F}}_\ell}(G^F)$ qui est cuspidale, non supercuspidale et à support supercuspidal $(T^F, \mathbf{1})$. Notons $\mathcal{P}_{\overline{\pi}}$ le relèvement à $\overline{\mathbb{Z}}_\ell$-coefficients de l'enveloppe projective de $\overline{\pi}$. Alors

$$\mathcal{P}_{\overline{\pi}} \simeq e^{G,\ell}_{[1]}.\overline{\mathbb{Z}}_\ell\Gamma_{G,\psi}$$

comme $\overline{\mathbb{Z}}_\ell G^F$-modules, d'où un isomorphisme d'anneaux

$$\mathrm{End}_{\overline{\mathbb{Z}}_\ell G^F}(\mathcal{P}_{\overline{\pi}}) \simeq e^{\mathsf{E},\ell}_{[1]}.\overline{\mathbb{Z}}_\ell\mathsf{E}_G. \tag{4.12.1}$$

4.13. *Application aux travaux de Paige sur $\mathrm{End}_{\overline{\mathbb{Z}}_\ell G^F}(\mathcal{P}_{\overline{\pi}})$.* — Sous le contexte de § 4.12, Paige a obtenu une description concrète de l'algèbre $\mathrm{End}_{\overline{\mathbb{Z}}_\ell G^F}(\mathcal{P}_{\overline{\pi}})$:

Théorème. [Pa] — *Notations et hypothèses comme en § 4.12. Soit $r = \mathrm{val}_\ell(q^n - 1)$ (on a $r \geq 1$). Soit A la sous-algèbre de $\mathbb{Z}[x]/(x^{\ell^r} - 1)$ (avec x une indéterminée) fixée par l'action $x \longmapsto x^q$, et soit θ l'image de $x + x^q + \cdots + x^{q^{n-1}}$ dans A.*

(a) *La $\overline{\mathbb{Z}}_\ell$-algèbre $\overline{\mathbb{Z}}_\ell\mathsf{A}$ est mono-engendrée par θ.*

(b) *Il existe un isomorphisme de $\overline{\mathbb{Z}}_\ell$-algèbres $\overline{\mathbb{Z}}_\ell\mathsf{A} \simeq \mathrm{End}_{\overline{\mathbb{Z}}_\ell G^F}(\mathcal{P}_{\overline{\pi}})$ tel que, sous les identifications*

$$\mathrm{End}_{\overline{\mathbb{Z}}_\ell G^F}(\mathcal{P}_{\overline{\pi}}) \overset{(4.12.1)}{=\!=} e^{\mathsf{E},\ell}_{[1]}.\overline{\mathbb{Z}}_\ell\mathsf{E}_G \subset e^{\mathsf{E},\ell}_{[1]}.\overline{\mathbb{Q}}\mathsf{E}_G \overset{(4.5.1)}{=\!=} \overline{\mathbb{Q}}^{G^{*F^*}_{\mathrm{ss},\ell}/\sim},$$

l'élément $\theta \in \overline{\mathbb{Z}}_\ell\mathsf{A}$ correspond à la fonction

$$[\,(G^{*F^*}_{\mathrm{ss},\ell}/\sim) \xrightarrow[\sim]{\S\,3.12 + \S\,3.6} ((T^\vee /\!\!/ W)^{F^\vee}(\overline{\mathbb{Q}}))_\ell \xrightarrow{\mathrm{tr}} \overline{\mathbb{Q}}\,] \in \overline{\mathbb{Q}}^{G^{*F^*}_{\mathrm{ss},\ell}/\sim}. \tag{4.13.1}$$

Paige a établi (b) à l'aide des calculs délicats des caractères centraux de $\mathrm{GL}_n(\mathbb{F}_q)$; ici on propose de donner une nouvelle preuve de (b) à l'aide de § 4.11.

Preuve de (b). Pour notre $G = \mathrm{GL}_n(\overline{\mathbb{F}}_q)$ (type A_{n-1}), il n'y a aucun nombre premier mauvais (§ 2.21). De plus, on a $G^\vee = \mathrm{GL}_n$ et puis $G^\vee_{\mathrm{der}} = \mathrm{SL}_n$ est simplement connexe. On a donc l'isomorphisme (4.11.2), d'où les identifications suivantes :

$$\begin{array}{ccccc} \mathrm{End}_{\overline{\mathbb{Z}}_\ell G^F}(\mathcal{P}_{\overline{\pi}}) & \overset{(4.12.1)}{=\!=} & e^{\mathsf{E},\ell}_{[1]}.\overline{\mathbb{Z}}_\ell\mathsf{E}_G & \hookrightarrow & e^{\mathsf{E},\ell}_{[1]}.\overline{\mathbb{Q}}\mathsf{E}_G \overset{(4.5.1)}{=\!=} \overline{\mathbb{Q}}^{G^{*F^*}_{\mathrm{ss},\ell}/\sim} \\ & & \downarrow{\scriptstyle (4.11.2)}\ \wr & & \\ & & \overline{\mathbb{Z}}_\ell\mathsf{B}^{(\ell)}_{G^\vee} & & \end{array} \tag{4.13.2}$$

Dans notre cas, on a $T^\vee \simeq \mathbb{G}_m^n$ est le tore maximal diagonal de $G^\vee = \mathrm{GL}_n$, donc $\mathbb{Z}[X(T^\vee)] = \mathbb{Z}[t_1^{\pm 1}, \cdots, t_n^{\pm 1}]$ (avec t_i la i-ième coordonnée sur $T^\vee$), $W = \mathfrak{S}_n = \mathfrak{S}_{\{t_1,\cdots,t_n\}}$ et puis $\mathbb{Z}[X(T^\vee)]^W = \mathbb{Z}[\sigma_1, \cdots, \sigma_{n-1}, \sigma_n^{\pm 1}]$ où

$$\sigma_i = \sigma_i(t_1, \cdots, t_n) := (t_1 t_2 \cdots t_i) + (t_1 t_2 \cdots t_{i-1} t_{i+1}) + \cdots + (t_{n-i+1} t_{n-i+2} \cdots t_n)$$

est le i-ième polynôme symétrique élémentaire de $\{t_1, \cdots, t_n\}$. D'autre part, comme $F^\vee = (\cdot)^q$ sur $T^\vee$, l'ensemble des $\overline{\mathbb{Q}}$-points du schéma affine $\mathrm{Spec}(\mathsf{B}_{G^\vee}^{(\ell)})$ est décrit par $(T^\vee /\!\!/ \mathfrak{S}_n)^{(\cdot)^q}(\overline{\mathbb{Q}})_\ell = (\bigcup_{w \in \mathfrak{S}_n} T^\vee(\overline{\mathbb{Q}})_\ell^{w.(\cdot)^q}))/\mathfrak{S}_n$ (§ 4.7). Vu notre hypothèse $\mathrm{ord}_\ell(q) = n$, on peut vérifier que $T^\vee(\overline{\mathbb{Q}})_\ell^{w.(\cdot)^q} = 1$ lorsque $w \in \mathfrak{S}_n = \mathfrak{S}_{\{t_1,\cdots,t_n\}}$ n'est pas conjugué au n-cycle $w_n := (t_1, t_2, \cdots, t_n)$. En posant $\mu_j = \{a \in \overline{\mathbb{Q}}^\times : a^j = 1\}$ pour $j \in \mathbb{N}^*$, on a $T^\vee(\overline{\mathbb{Q}})^{w_n.(\cdot)^q} = \{\mathrm{diag}(t, t^q, \cdots, t^{q^{n-1}}) : t \in \mu_{q^n-1}\}$ et puis

$$(T^\vee /\!\!/ \mathfrak{S}_n)^{(\cdot)^q}(\overline{\mathbb{Q}})_\ell = T^\vee(\overline{\mathbb{Q}})_\ell^{w_n.(\cdot)^q}/\mathfrak{S}_n = T^\vee(\overline{\mathbb{Q}})_\ell^{w_n.(\cdot)^q}/(\cdot)^q \xrightarrow{\sim} \mu_{\ell^r}/(\cdot)^q$$

où $r := \mathrm{val}_\ell(q^n - 1) \geq 1$ et la dernière bijection est $\mathrm{diag}(t, t^q, \cdots, t^{q^{n-1}}) \longmapsto t$. L'ensemble $\mu_{\ell^r}/(\cdot)^q$ étant en bijection avec les $\overline{\mathbb{Q}}$-points de $\mathrm{Spec}(\mathsf{A})$, on obtient un isomorphisme de $\overline{\mathbb{Q}}$-algèbres $\overline{\mathbb{Q}}\mathsf{B}_{G^\vee}^{(\ell)} \xrightarrow{\sim} \overline{\mathbb{Q}}\mathsf{A}$ explicitement donné par l'association $\sigma_i(t_1, \cdots, t_n) \longmapsto \sigma_i(x, x^q, \cdots, x^{q^{n-1}})$; ce dernier isomorphisme, étant défini sur $\mathbb{Z}$, se restreint à un isomorphisme d'anneau

$$\mathsf{B}_{G^\vee}^{(\ell)} \xrightarrow{\sim} \mathsf{A}, \quad \sigma_i(t_1, \cdots, t_n) \longmapsto \sigma_i(x, x^q, \cdots, x^{q^{n-1}}). \tag{4.13.3}$$

Sous les identifications (4.13.2) et (4.13.3), on a un isomorphisme de $\overline{\mathbb{Z}}_\ell$-algèbres $\overline{\mathbb{Z}}_\ell\mathsf{A} \simeq \mathrm{End}_{\overline{\mathbb{Z}}_\ell G^F}(\mathcal{P}_{\overline{\pi}})$; l'élément $\theta = \sigma_1(x, x^q, \cdots, x^{q^{n-1}}) \in \mathsf{A}$ correspond à l'élément $\sigma_1(t_1, \cdots, t_n) = \mathrm{tr}(\mathrm{diag}(t_1, \cdots, t_n)) \in \mathsf{B}_{G^\vee}^{(\ell)}$ et ainsi à la fonction (4.13.1). □

La partie de niveau zéro du modèle de Whittaker

4.14. *Groupes réductifs p-adiques et leurs representations.* [Vi] — Soit $\mathbf{F}$ un corps p-adique (une extension finie de $\mathbb{Q}_p$) dont le corps résiduel est $\mathbb{F}_q$. Soit $O_{\mathbf{F}}$ l'anneau des entiers de $\mathbf{F}$, et fixons une uniformisante ϖ de $O_{\mathbf{F}}$; alors $O_{\mathbf{F}}/\varpi O_{\mathbf{F}} \simeq \mathbb{F}_q$. Soit aussi $\mathrm{val}_\varpi : \mathbf{F} \longrightarrow \mathbb{Z} \cup \{+\infty\}$ la valuation discrète par rapport à ϖ.

Soit $\mathcal{G}$ un groupe réductif connexe défini sur $O_{\mathbf{F}}$, et soit $\mathbf{G} = \mathcal{G}(\mathbf{F})$. Soit $\mathbf{K} = \mathcal{G}(O_{\mathbf{F}})$, qui est un sous-groupe maximal ouvert compact de $\mathbf{G}$. On aura besoin des sous-groupes $\mathbf{K}^1 \subset \mathbf{I}^1 \subset \mathbf{I} \subset \mathbf{K}$, où $\mathbf{I}$ est un sous-groupe d'Iwahori de $\mathbf{G}$ avec son pro-p-radical unipotent $\mathbf{I}^1$, et $\mathbf{K}^1$ est le pro-p-radical de $\mathbf{K}$. Soit $G = \mathcal{G}(\overline{\mathbb{F}}_q)$ avec l'endomorphisme de Frobenius $F : G \longrightarrow G$ induit par une $\mathbb{F}_q$-structure de $\mathcal{G}$; alors $G^F = \mathcal{G}(\mathbb{F}_q) = \mathbf{K}/\mathbf{K}^1$, et l'on peut choisir un sous-groupe de Borel B de G tel que $B^F = \mathbf{I}/\mathbf{K}^1$ et tel que le radical unipotent U de B vérifie $U^F = \mathbf{I}^1/\mathbf{K}^1$.

Les représentations d'un groupe p-adique seront toujours supposées lisses, et l'on notera $\mathrm{ind}_{(\cdot)}^{(\cdot)}$ l'induction compacte. Pour un anneau Λ, l'algèbre de Hecke de $\mathbf{G}$ à Λ-coefficients sera notée $\Lambda\mathbf{G}$; elle se constitue des fonctions $f : \mathbf{G} \longrightarrow \Lambda$ lisses et à supports compacts sur $\mathbf{G}$; une telle $f \in \Lambda\mathbf{G}$ sera écrite comme une somme formelle $f = \sum_{g \in G} f(g)g$. La multiplication de $\Lambda\mathbf{G}$ est donnée par la convolution.

4.15. *Sur l'anneau* $\mathrm{End}_{\overline{\mathbb{Z}}[\frac{1}{p}]\mathbf{G}}(\mathcal{W}_0)$. — Notations comme en § 4.14. La représentation de Gelfand–Graev $\Gamma_{G,\psi} = \mathrm{Ind}_{U^F}^{G^F}\psi$ à $\overline{\mathbb{Z}}[\frac{1}{p}]$-coefficients (§ 1.7) est une représentation (lisse) de $\mathbf{K}$ via l'inflation $\mathbf{K} \twoheadrightarrow \mathbf{K}/\mathbf{K}^1 = G^F$, on peut alors considérer l'induction compacte

$$\mathcal{W}_0 := \mathrm{ind}_{\mathbf{K}}^{\mathbf{G}}\Gamma_{G,\psi} = \mathrm{ind}_{\mathbf{I}^1}^{\mathbf{G}}\psi.$$

Il sera démontré dans [DHKM] que $\mathcal{W}_0$ est isomorphe à la partie de niveau zéro du modèle de Whittaker (la représentation de Gelfand–Graev p-adique) $\mathcal{W} = \mathrm{ind}_{\mathbf{U}}^{\mathbf{G}}\widetilde{\psi}$, où $\mathcal{U}$ est un radical unipotent d'un sous-groupe Borel de $\mathcal{G}$ tel que $\mathcal{U}(O_{\mathbf{F}})/(\mathbf{K}^1 \cap \mathcal{U}(O_{\mathbf{F}})) = U^F$, et $\widetilde{\psi}$ est un caractère régulier (= non dégénéré) compatible avec ψ. En particulier, $\mathcal{W}_0$ est un facteur direct de $\mathcal{W}$. Comme l'algèbre d'endomorphismes $\mathrm{End}_{\overline{\mathbb{Z}}[\frac{1}{p}]\mathbf{G}}(\mathcal{W})$ est commutative et réduite (voir [Ra]), l'algèbre d'endomorphismes $\mathrm{End}_{\overline{\mathbb{Z}}[\frac{1}{p}]\mathbf{G}}(\mathcal{W}_0)$ l'est aussi.

Pour ce qui suit, on s'intéressera à la structure de l'algèbre d'endomorphismes $\mathrm{End}_{\overline{\mathbb{Z}}[\frac{1}{p}]\mathbf{G}}(\mathcal{W}_0)$ pour $\mathcal{G} = \mathrm{GL}_2$; dans ce cas, on exhibera que cet anneau admettra une structure filtrée, une structure qui est espérée de se généraliser aux autres $\mathcal{G}$.

4.16. Lemme. — *On identifie* $\mathcal{W}_0 = \mathrm{ind}_{\mathbf{I}^1}^{\mathbf{G}}\psi = \overline{\mathbb{Z}}[\frac{1}{p}]\mathbf{G}.e_\psi$, *où* e_ψ *est l'idempotent central primitif de* $\overline{\mathbb{Z}}[\frac{1}{p}]U^F$ *associé à* $\psi : U^F \longrightarrow \overline{\mathbb{Z}}[\frac{1}{p}]^\times$ *en (1.7.1) et où on regarde* $e_\psi \in \overline{\mathbb{Z}}[\frac{1}{p}]\mathbf{I}^1$ *via* $\pi : \mathbf{I}^1 \twoheadrightarrow \mathbf{I}^1/\mathbf{K}^1 = U^F$. *Pour tout* $g \in \mathbf{G}$, *soit* $g_{\mathsf{E}} \in \mathrm{End}_{\overline{\mathbb{Z}}[\frac{1}{p}]\mathbf{G}}(\mathcal{W}_0)$ *défini par*

$$g_{\mathsf{E}} : \mathcal{W}_0 \longrightarrow \mathcal{W}_0, \quad f \longmapsto [\mathbf{I}^1 : \mathbf{I}^1 \cap {}^g(\mathbf{I}^1)]f.ge_\psi.$$

Alors $\mathrm{End}_{\overline{\mathbb{Z}}[\frac{1}{p}]\mathbf{G}}(\mathcal{W}_0)$ *est un* $\overline{\mathbb{Z}}[\frac{1}{p}]$*-module libre de base* $\{g_{\mathsf{E}} : g \in \Upsilon\}$, *où* $\Upsilon \subset \mathbf{G}$ *est un choix arbitraire d'ensemble de représentants de*

$$\overline{\Upsilon} := \{g \in \mathbf{I}^1\backslash\mathbf{G}/\mathbf{I}^1 : \psi = {}^g\psi \text{ sur } \mathbf{I}^1 \cap {}^g\mathbf{I}^1\}$$

(on regarde ψ *comme* $\psi : \mathbf{I}^1 \longrightarrow \overline{\mathbb{Z}}[\frac{1}{p}]^\times$ *via* π*). De plus,* $g_{\mathsf{E}} = 0$ *si* $\mathbf{I}^1 g \mathbf{I}^1 \notin \overline{\Upsilon}$.

Preuve. (Comparer la preuve de § 1.10(a) en [CuRe, Prop. 11.30].) La réciprocité de Frobenius nous dit que $\mathrm{End}_{\overline{\mathbb{Z}}[\frac{1}{p}]\mathbf{G}}(\mathcal{W}_0) \simeq \mathrm{Hom}_{\overline{\mathbb{Z}}[\frac{1}{p}]\mathbf{I}^1}(\psi, \mathrm{Res}_{\mathbf{I}^1}^{\mathbf{G}}\mathcal{W}_0)$, et la décomposition de Mackey nous donne une décomposition de $\overline{\mathbb{Z}}[\frac{1}{p}]\mathbf{I}^1$-modules

$$\mathrm{Res}_{\mathbf{I}^1}^{\mathbf{G}}\mathcal{W}_0 = \bigoplus_{g\in\mathbf{I}^1\backslash\mathbf{G}/\mathbf{I}^1} \mathrm{ind}_{\mathbf{I}^1\cap{}^g\mathbf{I}^1}^{\mathbf{I}^1}\mathrm{Res}_{\mathbf{I}^1\cap{}^g\mathbf{I}^1}^{{}^g\mathbf{I}^1}{}^g\psi.$$

qui correspond à la décomposition

$$\overline{\mathbb{Z}}[\tfrac{1}{p}]\mathbf{G}.e_\psi = \bigoplus_{g\in\mathbf{I}^1\backslash\mathbf{G}/\mathbf{I}^1} \overline{\mathbb{Z}}[\tfrac{1}{p}]\mathbf{I}^1.ge_\psi.$$

On a alors

$$\begin{aligned}\mathrm{End}_{\overline{\mathbb{Z}}[\frac{1}{p}]\mathbf{G}}(\mathcal{W}_0) &\simeq \bigoplus_{g\in\mathbf{I}^1\backslash\mathbf{G}/\mathbf{I}^1} \mathrm{Hom}_{\overline{\mathbb{Z}}[\frac{1}{p}]\mathbf{I}^1}(\psi, \mathrm{ind}_{\mathbf{I}^1\cap{}^g\mathbf{I}^1}^{\mathbf{I}^1}\mathrm{Res}_{\mathbf{I}^1\cap{}^g\mathbf{I}^1}^{{}^g\mathbf{I}^1}{}^g\psi)\\ &\simeq \bigoplus_{g\in\mathbf{I}^1\backslash\mathbf{G}/\mathbf{I}^1} \mathrm{Hom}_{\overline{\mathbb{Z}}[\frac{1}{p}](\mathbf{I}^1\cap{}^g\mathbf{I}^1)}(\mathrm{Res}_{\mathbf{I}^1\cap{}^g\mathbf{I}^1}^{\mathbf{I}^1}\psi, \mathrm{Res}_{\mathbf{I}^1\cap{}^g\mathbf{I}^1}^{{}^g\mathbf{I}^1}{}^g\psi).\end{aligned}$$

La preuve du lemme est donc complétée si on observe que

$$\mathrm{Hom}_{\overline{\mathbb{Z}}[\frac{1}{p}](\mathbf{I}^1\cap{}^g\mathbf{I}^1)}(\mathrm{Res}^{\mathbf{I}^1}_{\mathbf{I}^1\cap{}^g\mathbf{I}^1}\psi, \mathrm{Res}^{{}^g\mathbf{I}^1}_{\mathbf{I}^1\cap{}^g\mathbf{I}^1}{}^g\psi) = \begin{cases} \overline{\mathbb{Z}}[\frac{1}{p}] & \text{si } \psi = {}^g\psi \text{ sur } \mathbf{I}^1\cap{}^g\mathbf{I}^1 \ ; \\ 0 & \text{sinon.} \end{cases} \qquad \square$$

4.17. *Le cas de $\mathcal{G} = \mathrm{GL}_2$.* — Dans ce cas, on prend les groupes mentionnés en § 4.14 comme suit. On a $\mathbf{G} = \mathrm{GL}_2(\mathbf{F})$, $\mathbf{K} = \mathrm{GL}_2(O_{\mathbf{F}})$ et puis $\mathbf{K}^1 = \mathrm{id}_2 + \varpi.\mathrm{M}_2(O_{\mathbf{F}})$. Prenons $\mathbf{I} = \begin{bmatrix} O_{\mathbf{F}}^\times & O_{\mathbf{F}} \\ \varpi O_{\mathbf{F}} & O_{\mathbf{F}}^\times \end{bmatrix}$, puis $\mathbf{I}^1 = \begin{bmatrix} 1+\varpi O_{\mathbf{F}} & O_{\mathbf{F}} \\ \varpi O_{\mathbf{F}} & 1+\varpi O_{\mathbf{F}} \end{bmatrix}$. On a $G = \mathrm{GL}_2(\overline{\mathbb{F}}_q)$ avec $F((g_{ij})_{i,j}) = (g_{ij}^q)_{i,j}$ l'action de Frobenius standard, puis $G^F = \mathrm{GL}_2(\mathbb{F}_q) = \mathbf{K}/\mathbf{K}^1$. On choisit $B = \begin{bmatrix} \overline{\mathbb{F}}_q^\times & \overline{\mathbb{F}}_q \\ 0 & \overline{\mathbb{F}}_q^\times \end{bmatrix}$ et donc $U = \begin{bmatrix} 1 & \overline{\mathbb{F}}_q \\ 0 & 1 \end{bmatrix}$; alors $B^F = \begin{bmatrix} \mathbb{F}_q^\times & \mathbb{F}_q \\ 0 & \mathbb{F}_q^\times \end{bmatrix} = \mathbf{I}/\mathbf{K}^1$ et $U^F = \begin{bmatrix} 1 & \mathbb{F}_q \\ 0 & 1 \end{bmatrix} = \mathbf{I}^1/\mathbf{K}^1$. Soit aussi le tore maximal diagonal $T = \begin{bmatrix} \mathbb{F}_q^\times & 0 \\ 0 & \mathbb{F}_q^\times \end{bmatrix}$ de G ; alors $T = B/U$ et $T^F = B^F/U^F = \mathbf{I}/\mathbf{I}^1$.

À l'aide de la décomposition d'Iwahori $\mathbf{I}\backslash\mathbf{G}/\mathbf{I} = W_{\mathbf{G}}(\mathbf{T}) \ltimes X(\mathbf{T})$ (avec $\mathbf{T}$ le tore diagonal maximal de $\mathbf{G}$) et de l'identification $T^F = \mathbf{I}/\mathbf{I}^1$, on peut vérifier qu'un ensemble de représentants dans $\mathbf{G}$ pour le double quotient $\mathbf{I}^1\backslash\mathbf{G}/\mathbf{I}^1$ se trouve

$$\left\{ \begin{bmatrix} \dot\lambda\varpi^a & 0 \\ 0 & \dot\mu\varpi^b \end{bmatrix}, \begin{bmatrix} 0 & \dot\lambda\varpi^a \\ \dot\mu\varpi^b & 0 \end{bmatrix} : a, b \in \mathbb{Z} \ ; \ \lambda, \mu \in \mathbb{F}_q^\times \right\},$$

où pour tout $x \in \mathbb{F}_q = O_{\mathbf{F}}/\varpi O_{\mathbf{F}}$ on a fixé un choix de $\dot x \in O_{\mathbf{F}}$ relevant x de sorte que $\dot x + \varpi O_{\mathbf{F}} = x$. On fixe un choix de caractère linéaire non trivial $\psi_0 : \mathbb{F}_q \longrightarrow \overline{\mathbb{Z}}[\frac{1}{p}]^\times$, puis on choisit ψ comme :

$$\psi : U^F \longrightarrow \overline{\mathbb{Z}}[\tfrac{1}{p}]^\times, \quad \begin{bmatrix} 1 & u \\ 0 & 1 \end{bmatrix} \longmapsto \psi_0(u).$$

La partie Υ de $\mathbf{G}$ en § 4.16 peut alors être choisie comme

$$\Upsilon = \left\{ \begin{bmatrix} \dot\lambda\varpi^a & 0 \\ 0 & \dot\lambda\varpi^a \end{bmatrix}, \begin{bmatrix} 0 & \dot\lambda\varpi^a \\ \dot\mu\varpi^b & 0 \end{bmatrix} : a, b \in \mathbb{Z} \ ; \ a \geq b \ ; \ \lambda, \mu \in \mathbb{F}_q^\times \right\}.$$

4.18. Lemme. — *Soit $\mathcal{G} = \mathrm{GL}_2$ (donc $\mathbf{G} = \mathrm{GL}_2(\mathbf{F})$) et maintenons les notations en §§ 4.16 - 4.17. Pour tous $\varphi_1, \varphi_2 \in \mathrm{End}_{\overline{\mathbb{Z}}[\frac{1}{p}]\mathbf{G}}(\mathcal{W}_0)$, on écrira la composition $\varphi_2 \circ \varphi_1$ par $\varphi_2\varphi_1$ ou $\varphi_2 \cdot \varphi_1$.*

(a) *On a $(g_2)_{\mathsf{E}} \cdot (g_1)_{\mathsf{E}} = (g_1g_2)_{\mathsf{E}} = (g_2g_1)_{\mathsf{E}}$ si g_1 ou g_2 est central dans $\mathbf{G}$.*

(b) *Pour tous $a, b \in \mathbb{N}$ et tous $\lambda, \mu \in \mathbb{F}_q^\times$, on a*

$$\begin{aligned} &\begin{bmatrix} 0 & \dot\mu\varpi^b \\ 1 & 0 \end{bmatrix}_{\mathsf{E}} \cdot \begin{bmatrix} 0 & \dot\lambda\varpi^a \\ 1 & 0 \end{bmatrix}_{\mathsf{E}} \\ &= \beta_{a,b}^{\lambda,\mu} \cdot (\dot\lambda\varpi^a \mathrm{id}_2)_{\mathsf{E}} + \sum_{\substack{0\leq c\leq \min(a,b) \\ \nu\in\mathbb{F}_q^\times}} \gamma_{a,b,c}^{\lambda,\mu,\nu} \cdot \begin{bmatrix} 0 & -\dot\lambda\dot\mu\dot\nu^{-1}\varpi^{a+b-c} \\ \dot\nu\varpi^c & 0 \end{bmatrix}_{\mathsf{E}} \end{aligned}$$

où $\begin{cases} \beta_{a,b}^{\lambda,\mu} &= q^{a+1}\delta_{a=b}\delta_{\lambda=\mu} \ ; \\ \gamma_{a,b,c}^{\lambda,\mu,\nu} &= q^{3c} \cdot \psi_0(\delta_{c=0} \cdot (-\dot\nu) + \dot\nu^{-1}(\delta_{a=c} \cdot \dot\lambda + \delta_{b=c} \cdot \dot\mu)). \end{cases}$

(Pour un énoncé E, on écrit $\delta_E = 1$ si E est vrai et $\delta_E = 0$ sinon.)

Observons que (a) et (b) déterminent la structure multiplicative de $\mathrm{End}_{\overline{\mathbb{Z}}[\frac{1}{p}]\mathbf{G}}(\mathcal{W}_0)$.

Preuve. (a) Si g_1 est central dans $\mathbf{G}$, alors pour tout $f \in \mathcal{W}_0 = \overline{\mathbb{Z}}[\frac{1}{p}]\mathbf{G}.e_\psi$ on peut calculer $((g_2)_\mathsf{E} \cdot (g_1)_\mathsf{E})(f) = f g_1 e_\psi g_2 e_\psi = f e_\psi g_1 g_2 e_\psi = f g_1 g_2 e_\psi = (g_1 g_2)_\mathsf{E}(f)$, d'où la formule $(g_2)_\mathsf{E} \cdot (g_1)_\mathsf{E} = (g_1 g_2)_\mathsf{E}$.

(b) Soient $g_1 = \begin{bmatrix} 0 & \dot{\lambda}\varpi^a \\ 1 & 0 \end{bmatrix}_\mathsf{E}$ et $g_2 = \begin{bmatrix} 0 & \dot{\mu}\varpi^b \\ 1 & 0 \end{bmatrix}_\mathsf{E}$, et l'on veut calculer le produit $(g_2)_\mathsf{E} \cdot (g_1)_\mathsf{E}$. Notons d'abord que le support de $((g_2)_\mathsf{E} \cdot (g_1)_\mathsf{E})(e_\psi) = e_\psi g_1 e_\psi g_2 e_\psi$ est contenu dans $\mathbf{I}^1 g_1 \mathbf{I}^1 g_2 \mathbf{I}^1$. Par des calculs directs, on peut montrer que

$$\mathbf{I}^1 g_1 \mathbf{I}^1 g_2 \mathbf{I}^1 = \mathbf{I}^1 g_1 g_2 \mathbf{I}^1 \sqcup \bigsqcup_{\substack{0 \le c \le \min(a,b) \\ \nu \in \mathbb{F}_q^\times}} \mathbf{I}^1 h_{c,\nu} \mathbf{I}^1 \quad \text{avec} \quad h_{c,\nu} = \begin{bmatrix} 0 & -\dot{\lambda}\dot{\mu}\dot{\nu}^{-1}\varpi^{a+b-c} \\ \dot{\nu}\varpi^c & 0 \end{bmatrix}.$$

On en déduit que

$$(g_2)_\mathsf{E} \cdot (g_1)_\mathsf{E} = \beta \cdot (g_1 g_2)_\mathsf{E} + \sum_{\substack{0 \le c \le \min(a,b) \\ \nu \in \mathbb{F}_q^\times}} \gamma_{c,\nu} \cdot (h_{c,\nu})_\mathsf{E}$$

avec les constantes β et $\gamma_{c,\nu}$ à déterminer. Comme $h_{c,\nu} \in \Upsilon$, on peut calculer

$$\gamma_{c,\nu} = \frac{(e_\psi g_1 e_\psi g_2 e_\psi)(h_{c,\nu})}{(e_\psi h_{c,\nu} e_\psi)(h_{c,\nu})},$$

et par des calculs directs on vérifiera que $\gamma_{c,\nu}$ coïncide avec $\gamma_{a,b,c}^{\lambda,\mu,\nu}$; d'autre part, l'élément $g_1 g_2 = \begin{bmatrix} \dot{\lambda}\varpi^a & 0 \\ 0 & \dot{\mu}\varpi^b \end{bmatrix}$ appartient à Υ si et seulement si $a = b$ et $\lambda = \mu$, et dans le cas où $g_1 g_2 \in \Upsilon$, on a $g_1 g_2 = \dot{\lambda}\varpi^a \mathrm{id}_2$ et l'on pourra calculer

$$\beta = \frac{(e_\psi g_1 e_\psi g_2 e_\psi)(\dot{\lambda}\varpi^a \mathrm{id}_2)}{(e_\psi \dot{\lambda}\varpi^a \mathrm{id}_2 e_\psi)(\dot{\lambda}\varpi^a \mathrm{id}_2)} = q^{a+1}. \qquad \square$$

4.19. Proposition. — *Soit* $\mathcal{G} = \mathrm{GL}_2$ *(donc* $\mathbf{G} = \mathrm{GL}_2(\mathbf{F})$*) et gardons les notations en §§ 4.16 - 4.18. L'application* $\mathbb{F}_q^\times \ni x \longmapsto \dot{x} \in O_\mathbf{F}^\times$ *s'étend* $\overline{\mathbb{Z}}[\frac{1}{p}][\varpi^{\pm 1}]$*-linéairement à une application* $\overline{\mathbb{Z}}[\frac{1}{p}][\mathbb{F}_q^\times, \varpi^{\pm 1}] \longrightarrow O_\mathbf{F}$ *encore notée* $x \longmapsto \dot{x}$*. Alors l'application*

$$e : \overline{\mathbb{Z}}[\tfrac{1}{p}][\mathbb{F}_q^\times, \varpi^{\pm 1}] \longrightarrow \mathrm{End}_{\overline{\mathbb{Z}}[\frac{1}{p}]\mathbf{G}}(\mathcal{W}_0), \quad x \longmapsto (\dot{x}.\mathrm{id}_2)_\mathsf{E},$$

est un monomorphisme de $\overline{\mathbb{Z}}[\frac{1}{p}]$*-algèbres et identifie* $\overline{\mathbb{Z}}[\frac{1}{p}][\mathbb{F}_q^\times, \varpi^{\pm 1}]$ *comme une sous-*$\overline{\mathbb{Z}}[\frac{1}{p}]$*-algèbre de* $\mathrm{End}_{\overline{\mathbb{Z}}[\frac{1}{p}]\mathbf{G}}(\mathcal{W}_0)$*. Considérée comme une* $\overline{\mathbb{Z}}[\frac{1}{p}][\mathbb{F}_q^\times, \varpi^{\pm 1}]$*-algèbre,* $\mathrm{End}_{\overline{\mathbb{Z}}[\frac{1}{p}]\mathbf{G}}(\mathcal{W}_0)$ *est filtrée au sens suivant : pour tout* $d \in \mathbb{N}$*, posons*

$$V^{(d)} = \sum_{\substack{0 \le a \le d \\ \lambda \in \mathbb{F}_q^\times}} \overline{\mathbb{Z}}[\tfrac{1}{p}][\mathbb{F}_q^\times, \varpi^{\pm 1}] \cdot \begin{bmatrix} 0 & \dot{\lambda}\varpi^a \\ 1 & 0 \end{bmatrix}_\mathsf{E} ;$$

alors $V^{(0)} \subset V^{(1)} \subset V^{(2)} \subset \cdots \subset \mathrm{End}_{\overline{\mathbb{Z}}[\frac{1}{p}]\mathbf{G}}(\mathcal{W}_0)$*,* $\mathrm{End}_{\overline{\mathbb{Z}}[\frac{1}{p}]\mathbf{G}}(\mathcal{W}_0) = \bigcup_{d \ge 0} V^{(d)}$*, et pour tous* $d, d' \in \mathbb{N}$ *on a* $V^{(d)} \cdot V^{(d')} \subset V^{(d+d')}$*. De plus, on a*

$$\mathrm{End}_{\overline{\mathbb{Z}}[\frac{1}{p}]\mathbf{G}}(\mathcal{W}_0) = \overline{\mathbb{Z}}[\tfrac{1}{p}][\mathbb{F}_q^\times, \varpi^{\pm 1}] \left[\left\{ \begin{bmatrix} 0 & \dot{\lambda} \\ 1 & 0 \end{bmatrix}_\mathsf{E}, \begin{bmatrix} 0 & \dot{\lambda}\varpi \\ 1 & 0 \end{bmatrix}_\mathsf{E} : \lambda \in \mathbb{F}_q^\times \right\} \right], \tag{4.19.1}$$

de sorte que l'anneau $\mathrm{End}_{\overline{\mathbb{Z}}[\frac{1}{p}]\mathbf{G}}(\mathcal{W}_0)$ *est engendré par* $V^{(1)}$.

Remarque. En observant que

$$\overline{\mathbb{Z}}[\tfrac{1}{p}]\mathsf{E}_G \simeq \mathrm{End}_{\overline{\mathbb{Z}}[\frac{1}{p}]\mathbf{K}}(\mathrm{ind}_{\mathbf{I}^1}^{\mathbf{K}}\psi) \simeq \overline{\mathbb{Z}}[\tfrac{1}{p}][\mathbb{F}_q^\times]\left[\left\{\begin{bmatrix} 0 & \dot{\lambda} \\ 1 & 0 \end{bmatrix}_{\mathsf{E}} : \lambda \in \mathbb{F}_q^\times\right\}\right],$$

l'égalité (4.19.1) nous donne un isomorphisme de $\overline{\mathbb{Z}}[\frac{1}{p}]$-algèbres

$$\mathrm{End}_{\overline{\mathbb{Z}}[\frac{1}{p}]\mathbf{G}}(\mathcal{W}_0) \simeq \overline{\mathbb{Z}}[\tfrac{1}{p}]\mathsf{E}_G\left[\varpi^{\pm 1}, \left\{\begin{bmatrix} 0 & \dot{\lambda}\varpi \\ 1 & 0 \end{bmatrix}_{\mathsf{E}} : \lambda \in \mathbb{F}_q^\times\right\}\right]. \tag{4.19.2}$$

Preuve de la proposition. Toutes les affirmations autres que (4.19.1) résultent de § 4.18. Pour établir (4.19.1), l'idée est de montrer par récurrence que $V^{(d)} \subset \mathbb{Z}[V^{(1)}]$ pour tout $d \in \mathbb{N}^*$. Dans ce qui suit, on identifie $\overline{\mathbb{Z}}[\frac{1}{p}][\mathbb{F}_q^\times]$ avec son image dans $\mathrm{End}_{\overline{\mathbb{Z}}[\frac{1}{p}]\mathbf{G}}(\mathcal{W}_0)$ sous le monomorphisme e.

Compte tenu de § 4.18, pour tout $d \in \mathbb{N}^*$ et tout $\lambda \in \mathbb{F}_q^\times$, on a

$$\begin{bmatrix} 0 & \varpi \\ 1 & 0 \end{bmatrix}_{\mathsf{E}} \cdot \begin{bmatrix} 0 & \dot{\lambda}\varpi^d \\ 1 & 0 \end{bmatrix}_{\mathsf{E}} = \beta_{d,1}^{\lambda,1} \cdot e(\lambda\varpi^d) + \sum_{\substack{0\le c\le 1 \\ \nu\in\mathbb{F}_q^\times}} \gamma_{d,1,c}^{\lambda,1,\nu} \cdot \begin{bmatrix} 0 & -\dot{\lambda}\dot{\nu}^{-1}\varpi^{d+1-c} \\ \dot{\nu}\varpi^c & 0 \end{bmatrix}_{\mathsf{E}}$$

et donc, en posant $f_\gamma := \sum_{\alpha\in\mathbb{F}_q^\times,\, \alpha^2=\gamma} \psi_0(-\alpha)e(\alpha) \in \overline{\mathbb{Z}}[\frac{1}{p}][\mathbb{F}_q^\times]$ pour tout $\gamma \in (\mathbb{F}_q^\times)^2$:

$$\sum_{\gamma\in(\mathbb{F}_q^\times)^2} f_\gamma \cdot \begin{bmatrix} 0 & -\dot{\lambda}\dot{\gamma}^{-1}\varpi^{d+1} \\ 1 & 0 \end{bmatrix}_{\mathsf{E}} \in \mathbb{Z}[V^{(d)}]. \tag{4.19.3}$$

On suppose pour la suite de la preuve que q est impair (le cas de q pair pourra être traité de façon analogue et donc sera omis). Comme q est impair, $[\mathbb{F}_q^\times : (\mathbb{F}_q^\times)^2] = 2$; en fixant un $\alpha_0 \in \mathbb{F}_q^\times - (\mathbb{F}_q^\times)^2$, on a $\mathbb{F}_q^\times = (\mathbb{F}_q^\times)^2 \sqcup (\mathbb{F}_q^\times)^2\alpha_0$. En considérant les deux cas $\lambda \in (\mathbb{F}_q^\times)^2$ et $\lambda \in (\mathbb{F}_q^\times)^2\alpha_0$ dans (4.19.3), on voit que

$$\sum_{\gamma\in(\mathbb{F}_q^\times)^2} f_{\gamma\lambda} \cdot \begin{bmatrix} 0 & -\dot{\epsilon}\dot{\gamma}^{-1}\varpi^{d+1} \\ 1 & 0 \end{bmatrix}_{\mathsf{E}} \in \mathbb{Z}[V^{(d)}]. \tag{4.19.4}$$

pour tout $\lambda \in (\mathbb{F}_q^\times)^2$ et tout $\epsilon \in \{1, \alpha_0\}$. Si la matrice $(f_{\gamma\lambda})_{\gamma,\lambda\in(\mathbb{F}_q^\times)^2}$ est inversible sur $\overline{\mathbb{Z}}[\frac{1}{p}][\mathbb{F}_q^\times]$, alors (4.19.4) impliquera que

$$\begin{bmatrix} 0 & -\dot{\epsilon}\dot{\gamma}^{-1}\varpi^{d+1} \\ 1 & 0 \end{bmatrix}_{\mathsf{E}} \in \mathbb{Z}[V^{(d)}]$$

pour tout $\gamma \in (\mathbb{F}_q^\times)^2$ et tout $\epsilon \in \{1, \alpha_0\}$; on aura alors $V^{(d+1)} \subset \mathbb{Z}[V^{(d)}]$ pour tout $d \in \mathbb{N}^*$, donc $\mathbb{Z}[V^{(d)}] = \mathbb{Z}[V^{(1)}]$ pour tout $d \in \mathbb{N}^*$; ainsi $\mathrm{End}_{\overline{\mathbb{Z}}[\frac{1}{p}]\mathbf{G}}(\mathcal{W}_0) = \mathbb{Z}[V^{(1)}]$. Pour démontrer l'égalité voulue (4.19.1), il suffit donc de démontrer que

$$\det((f_{\gamma\lambda})_{\gamma,\lambda\in(\mathbb{F}_q^\times)^2}) \in (\overline{\mathbb{Z}}[\tfrac{1}{p}][\mathbb{F}_q^\times])^\times. \tag{4.19.5}$$

Écrivons $q = 2m+1$ avec $m \in \mathbb{N}$, et soit η un générateur du groupe cyclique $\mathbb{F}_q^\times$ (donc $\eta^{2m} = 1$ et $\eta^m = -1 \neq 1$ dans $\mathbb{F}_q^\times$). En posant

$$f'_j = f_{\eta^{2j}} = \psi_0(\eta^{m+j})e(\eta)^j + \psi_0(\eta^j)e(\eta)^{m+j} \quad (j \in \mathbb{Z}),$$

on a $\det((f_{\gamma\lambda})_{\gamma,\lambda\in(\mathbb{F}_q^\times)^2}) = \pm\det((f'_{i+j})_{0\leq i,j<m})$, donc (4.19.5) est équivalent à

$$\det((f'_{i+j})_{0\leq i,j<m}) \in (\overline{\mathbb{Z}}[\tfrac{1}{p}][\mathbb{F}_q^\times])^\times. \tag{4.19.6}$$

On va maintenant démontrer (4.19.6). Rappelons tout d'abord la factorisation du déterminant circulant

$$\begin{vmatrix} x_0 & x_1 & x_2 & \cdots & x_{n-1} \\ x_1 & x_2 & x_3 & \cdots & x_0 \\ x_2 & x_3 & x_4 & \cdots & x_1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_{n-1} & x_0 & x_1 & \cdots & x_{n-2} \end{vmatrix} = \prod_{k=0}^{n-1}(x_0 + \omega_n^k x_1 + \omega_n^{2k}x_2 + \cdots + \omega_n^{(n-1)k}x_{n-1}) \tag{4.19.7}$$

pour tous $x_0, \cdots, x_{n-1}$ dans un même anneau commutatif, où $\omega_n = e^{2\pi\sqrt{-1}/n} \in \overline{\mathbb{Z}}^\times$ (ω_n est une n-ième racine de l'unité primitive, de plus on a les compatibilités $\omega_{nn'}^{n'} = \omega_n$). D'après (4.19.7), on a alors

$$\det((f'_{i+j})_{0\leq i,j<m}) = \prod_{k=0}^{m-1} F_k \tag{4.19.8}$$

où chaque $F_k := f'_0 + \omega_m^k f'_1 + \omega_m^{2k} f'_2 + \cdots + \omega_m^{(m-1)k} f'_{m-1} \in \overline{\mathbb{Z}}[\frac{1}{p}][\mathbb{F}_q^\times]$. L'égalité (4.19.8) nous dit donc que la condition (4.19.6) est équivalente à la condition que

$$F_k \in (\overline{\mathbb{Z}}[\tfrac{1}{p}][\mathbb{F}_q^\times])^\times \text{ pour tout } 0 \leq k < m. \tag{4.19.9}$$

Pour continuer l'analyse, on se rappelle que la représentation régulière du groupe fini $\mathbb{F}_q^\times = \langle\eta\rangle$ induit une inclusion de $\overline{\mathbb{Z}}[\frac{1}{p}]$-algèbres

$$\iota : \overline{\mathbb{Z}}[\tfrac{1}{p}][\mathbb{F}_q^\times] \hookrightarrow \mathrm{M}_{q-1}(\overline{\mathbb{Z}}[\tfrac{1}{p}]) = \mathrm{M}_{2m}(\overline{\mathbb{Z}}[\tfrac{1}{p}]), \quad \eta \mapsto \left[\begin{array}{cccc|c} & & & & 1 \\ \hline 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \end{array}\right].$$

L'image de chaque élément de $\overline{\mathbb{Z}}[\frac{1}{p}][\mathbb{F}_q^\times]$ sous l'inclusion ι est donc une matrice circulante ; puisque l'inverse d'une matrice circulante inversible est aussi circulante, on a

$$\iota^{-1}(\mathrm{GL}_{2m}(\overline{\mathbb{Z}}[\tfrac{1}{p}])) = (\overline{\mathbb{Z}}[\tfrac{1}{p}][\mathbb{F}_q^\times])^\times. \tag{4.19.10}$$

Pour tout $0 \leq k < m$, la matrice $\iota(F_k) \in \mathrm{M}_{2m}(\overline{\mathbb{Z}}[\frac{1}{p}])$ étant circulante, on utilise la formule (4.19.7) à nouveau pour obtenir son déterminant :

$$\det \iota(F_k) = \prod_{h=0}^{2m-1}\sum_{j=0}^{2m-1} \omega_{2m}^{jh}\omega_m^{jk}\psi_0(\eta^{m+j}) \ ; \tag{4.19.11}$$

en faisant $h \mapsto h-2k$ dans (4.19.11), on a $\det \iota(F_k) = \det \iota(F_0)$ pour tout $0 \leq k < m$. À l'aide de (4.19.10), la condition (4.19.9) est donc équivalente à la condition que

$$\det \iota(F_0) \in \overline{\mathbb{Z}}[\tfrac{1}{p}]^\times. \tag{4.19.12}$$

D'après (4.19.11), on a

$$\det \iota(F_0) = \prod_{h=0}^{2m-1} u_h \quad \text{où} \quad u_h = \sum_{j=0}^{2m-1} \omega_{2m}^{jh} \psi_0(\eta^{m+j}).$$

On vérifie directement que $u_0 = -1$ et que $u_h u_{2m-h} = (-1)^h q$ pour tout $1 \leq h \leq m$; en particulier, $u_m^2 = (-1)^m q$ et puis $u_m^{-1} = (-1)^m q^{-1} u_m \in \overline{\mathbb{Z}}[\frac{1}{p}]$. On en déduit que

$$\det \iota(F_0) = u_m q^{n-1} (-1)^{m(m+1)/2} \in \overline{\mathbb{Z}}[\tfrac{1}{p}]^\times.$$

La condition (4.19.12) est ainsi vérifiée, puis les implications

$$(4.19.12) \Longleftrightarrow (4.19.9) \Longleftrightarrow (4.19.6) \Longleftrightarrow (4.19.5) \Longrightarrow (4.19.1)$$

nous permettent d'obtenir l'égalité voulue (4.19.1). □

Le ℓ-bloc unipotent de GL_2 p-adique avec $\ell \neq p$

4.20. *Le ℓ-bloc unipotent de* $\mathbf{G} = \mathrm{GL}_2(\mathbf{F})$. — Jusqu'à la fin de § 4, on continue de considérer le groupe $\mathcal{G} = \mathrm{GL}_2$ (donc $\mathbf{G} = \mathrm{GL}_2(\mathbf{F})$) et d'utiliser les notations en § 4.17, et l'on se souvient que ℓ désigne un nombre premier qui ne divise pas q. Grâce au travail [Hel1] de Helm, la catégorie $\mathrm{Rep}_{\overline{\mathbb{Z}}_\ell}(\mathbf{G})$ des représentations lisses de $\mathbf{G}$ à valeurs dans $\overline{\mathbb{Z}}_\ell$ admet une décomposition en un produit de blocs

$$\mathrm{Rep}_{\overline{\mathbb{Z}}_\ell}(\mathbf{G}) = \prod_{[\mathbf{M},\pi]} \mathrm{Rep}_{\overline{\mathbb{Z}}_\ell}(\mathbf{G})_{[\mathbf{M},\pi]}$$

où le produit parcourt les classes d'équivalences inertielles $[\mathbf{M}, \pi]$ ($\mathbf{M}$ est un sous-groupe de Levi de $\mathbf{G}$; π est un caractère supercuspidal irréductible de $\mathbf{M}$ à valeurs dans $\overline{\mathbb{F}}_\ell$). Soit $\mathbf{T}$ le tore diagonal maximal de $\mathbf{G}$ et soit $\mathbf{1}$ le caractère trivial de $\mathbf{T}$ à valeurs dans $\overline{\mathbb{F}}_\ell$. On appelle alors $\mathrm{Rep}_{\overline{\mathbb{Z}}_\ell}(\mathbf{G})_{[\mathbf{T},\mathbf{1}]}$ le ℓ-bloc unipotent de $\mathbf{G}$.

Désormais (et jusqu'à la fin de § 4), on suppose que $\mathrm{ord}_\ell(q) = 2$ (c'est le cas de $n = 2$ en § 4.12). Alors [Hel1, Preuve du Cor. 11.19] implique que le bloc $\mathrm{Rep}_{\overline{\mathbb{Z}}_\ell}(\mathbf{G})_{[\mathbf{T},\mathbf{1}]}$ admet un progénérateur

$$\mathcal{P} := \mathrm{ind}_{\mathbf{K}}^{\mathbf{G}}(\mathrm{ind}_{B^F}^{G^F} \mathbf{1}_{B^F} \oplus \mathcal{P}_{\overline{\pi}}) \simeq \mathrm{ind}_{\mathbf{I}}^{\mathbf{G}} \mathbf{1}_{\mathbf{I}} \oplus \mathrm{ind}_{\mathbf{K}}^{\mathbf{G}}(e_{[1]}^{G,\ell}.\overline{\mathbb{Z}}_\ell \Gamma_{G,\psi}), \tag{4.20.1}$$

où $\mathbf{1}_{B^F}$ (resp. $\mathbf{1}_{\mathbf{I}}$) est le caractère trivial de B^F (resp. de $\mathbf{I}$) à valeurs dans $\overline{\mathbb{Z}}_\ell$, $e_{[1]}^{G,\ell}$ est défini en § 4.2, et $\mathcal{P}_{\overline{\pi}} \simeq e_{[1]}^{G,\ell}.\overline{\mathbb{Z}}_\ell \Gamma_{G,\psi}$ comme en § 4.12. Le foncteur $\mathrm{Hom}(\mathcal{P}, -)$ établit donc une équivalence des catégories de la catégorie $\mathrm{Rep}_{\overline{\mathbb{Z}}_\ell}(\mathbf{G})_{[\mathbf{T},\mathbf{1}]}$ à la catégorie des $\mathrm{End}_{\overline{\mathbb{Z}}_\ell \mathbf{G}}(\mathcal{P})$-modules à droite.

Dans ce qui suit, on se propose de décrire l'anneau $\mathrm{End}_{\overline{\mathbb{Z}}_\ell \mathbf{G}}(\mathcal{P})$.

4.21. *Études préliminaires sur* $\mathrm{End}_{\overline{\mathbb{Z}}_\ell \mathbf{G}}(\mathcal{P})$. — On est dans le cas de $\mathcal{G} = \mathrm{GL}_2$ ($\mathbf{G} = \mathrm{GL}_2(\mathbf{F})$) avec $\mathrm{ord}_\ell(q) = 2$. Vu la décomposition de $\mathcal{P}$ en (4.20.1), on peut identifier

$$\begin{aligned}\mathrm{End}_{\overline{\mathbb{Z}}_\ell \mathbf{G}}(\mathcal{P}) &= \mathrm{End}_{\overline{\mathbb{Z}}_\ell \mathbf{G}}(\mathrm{ind}_{\mathbf{I}}^{\mathbf{G}} \mathbf{1}_{\mathbf{I}} \oplus \mathrm{ind}_{\mathbf{K}}^{\mathbf{G}}(e_{[1]}^{G,\ell}.\overline{\mathbb{Z}}_\ell \Gamma_{G,\psi})) \\ &= \begin{array}{c} \\ \mathrm{ind}_{\mathbf{I}}^{\mathbf{G}} \mathbf{1}_{\mathbf{I}} \\ \mathrm{ind}_{\mathbf{K}}^{\mathbf{G}}(e_{[1]}^{G,\ell}.\overline{\mathbb{Z}}_\ell \Gamma_{G,\psi}) \end{array} \begin{array}{c} \begin{array}{cc} \mathrm{ind}_{\mathbf{I}}^{\mathbf{G}} \mathbf{1}_{\mathbf{I}} & \mathrm{ind}_{\mathbf{K}}^{\mathbf{G}}(e_{[1]}^{G,\ell}.\overline{\mathbb{Z}}_\ell \Gamma_{G,\psi}) \end{array} \\ \begin{bmatrix} \mathsf{H} & \mathsf{M}_1 \\ \mathsf{N}_1 & \mathsf{E}_1 \end{bmatrix} \end{array},\end{aligned}$$

où on pose

$$\begin{aligned}\mathsf{H} &= \mathrm{End}_{\overline{\mathbb{Z}}_\ell \mathbf{G}}(\mathrm{ind}_{\mathbf{I}}^{\mathbf{G}} \mathbf{1}_{\mathbf{I}}) \ ; \\ \mathsf{M}_1 &= \mathrm{Hom}_{\overline{\mathbb{Z}}_\ell \mathbf{G}}(\mathrm{ind}_{\mathbf{K}}^{\mathbf{G}}(e_{[1]}^{G,\ell}.\overline{\mathbb{Z}}_\ell \Gamma_{G,\psi}), \mathrm{ind}_{\mathbf{I}}^{\mathbf{G}} \mathbf{1}_{\mathbf{I}}) \ ; \\ \mathsf{N}_1 &= \mathrm{Hom}_{\overline{\mathbb{Z}}_\ell \mathbf{G}}(\mathrm{ind}_{\mathbf{I}}^{\mathbf{G}} \mathbf{1}_{\mathbf{I}}, \mathrm{ind}_{\mathbf{K}}^{\mathbf{G}}(e_{[1]}^{G,\ell}.\overline{\mathbb{Z}}_\ell \Gamma_{G,\psi})) \ ; \\ \mathsf{E}_1 &= \mathrm{End}_{\overline{\mathbb{Z}}_\ell \mathbf{G}}(\mathrm{ind}_{\mathbf{K}}^{\mathbf{G}}(e_{[1]}^{G,\ell}.\overline{\mathbb{Z}}_\ell \Gamma_{G,\psi})).\end{aligned}$$

L'algèbre H est (anti-)isomorphe à l'algèbre d'Iwahori-Hecke $\overline{\mathbb{Z}}_\ell[\mathbf{I}\backslash\mathbf{G}/\mathbf{I}]$ dont la structure est connue (voir [Vi, I.3.14]). En identifiant $\mathrm{ind}_{\mathbf{I}}^{\mathbf{G}} \mathbf{1}_{\mathbf{I}} = \overline{\mathbb{Z}}_\ell \mathbf{G}.e_{\mathbf{I}}$ où $e_{\mathbf{I}}$ est l'idempotent de $\overline{\mathbb{Z}}_\ell \mathbf{G}$ dont le support est $\mathbf{I}$, et en profitant des générateurs standards de l'algèbre $\overline{\mathbb{Z}}_\ell[\mathbf{I}\backslash\mathbf{G}/\mathbf{I}]$ (on rappelle la décomposition d'Iwahori $\mathbf{I}\backslash\mathbf{G}/\mathbf{I} = W_{\mathbf{G}}(\mathbf{T}) \ltimes X(\mathbf{T})$), on a

$$\mathsf{H} = \overline{\mathbb{Z}}_\ell[s_{\mathsf{H}}, t_{\mathsf{H}}]$$

où $s = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \in \mathbf{G}$, $t = \begin{bmatrix} 0 & 1 \\ \varpi & 0 \end{bmatrix} \in \mathbf{G}$ et pour tout $g \in \mathbf{G}$ on pose $g_{\mathsf{H}} \in \mathsf{H}$ via

$$g_{\mathsf{H}} : \mathrm{ind}_{\mathbf{I}}^{\mathbf{G}} \mathbf{1}_{\mathbf{I}} \longrightarrow \mathrm{ind}_{\mathbf{I}}^{\mathbf{G}} \mathbf{1}_{\mathbf{I}}, \quad f \longmapsto [\mathbf{I} : \mathbf{I} \cap {}^g\mathbf{I}] f.g e_{\mathbf{I}}.$$

On a les relations $(s_{\mathsf{H}})^2 = q + (q-1)s_{\mathsf{H}}$ et $s_{\mathsf{H}}(t_{\mathsf{H}})^2 = (t_{\mathsf{H}})^2 s_{\mathsf{H}}$.

Soient

$$\begin{aligned}\mathsf{M} &:= \mathrm{Hom}_{\overline{\mathbb{Z}}_\ell \mathbf{G}}(\overline{\mathbb{Z}}_\ell \mathcal{W}_0, \mathrm{ind}_{\mathbf{I}}^{\mathbf{G}} \mathbf{1}_{\mathbf{I}}) = \mathrm{Hom}_{\overline{\mathbb{Z}}_\ell \mathbf{G}}(\mathrm{ind}_{\mathbf{K}}^{\mathbf{G}}(\overline{\mathbb{Z}}_\ell \Gamma_{G,\psi}), \mathrm{ind}_{\mathbf{I}}^{\mathbf{G}} \mathbf{1}_{\mathbf{I}}) \ ; \\ \mathsf{N} &:= \mathrm{Hom}_{\overline{\mathbb{Z}}_\ell \mathbf{G}}(\mathrm{ind}_{\mathbf{I}}^{\mathbf{G}} \mathbf{1}_{\mathbf{I}}, \overline{\mathbb{Z}}_\ell \mathcal{W}_0) = \mathrm{Hom}_{\overline{\mathbb{Z}}_\ell \mathbf{G}}(\mathrm{ind}_{\mathbf{I}}^{\mathbf{G}} \mathbf{1}_{\mathbf{I}}, \mathrm{ind}_{\mathbf{K}}^{\mathbf{G}}(\overline{\mathbb{Z}}_\ell \Gamma_{G,\psi})).\end{aligned}$$

En identifiant $e_{[1]}^{G,\ell}$ à l'action de multiplication à droite

$$(-) \cdot e_{[1]}^{G,\ell} : \mathrm{ind}_{\mathbf{K}}^{\mathbf{G}}(\overline{\mathbb{Z}}_\ell \Gamma_{G,\psi}) = \overline{\mathbb{Z}}_\ell \mathbf{G}.e_\psi \longrightarrow \overline{\mathbb{Z}}_\ell \mathbf{G}.e_\psi e_{[1]}^{G,\ell} = \mathrm{ind}_{\mathbf{K}}^{\mathbf{G}}(e_{[1]}^{G,\ell}.\overline{\mathbb{Z}}_\ell \Gamma_{G,\psi}),$$

On a les deux isomorphismes de $\overline{\mathbb{Z}}_\ell$-modules

$$(-) \cdot e_{[1]}^{G,\ell} : \mathsf{M} \xrightarrow{\sim} \mathsf{M}_1 \quad \text{et} \quad e_{[1]}^{G,\ell} \cdot (-) : \mathsf{N} \xrightarrow{\sim} \mathsf{N}_1,$$

dont les inverses sont les inclusions naturelles $\mathsf{M}_1 \subset \mathsf{M}$ et $\mathsf{N}_1 \subset \mathsf{N}$. Ces deux isomorphismes se déduisent de la réciprocité de Frobenius, de la décomposition de Mackey, et de la décomposition of $\overline{\mathbb{Q}}_\ell \Gamma_{G,\psi}$ en § 1.8. Il s'ensuit que $f \cdot e_{[1]}^{G,\ell} = f$ pour tout $f \in \mathsf{M}$ et que $e_{[1]}^{G,\ell} \cdot f = f$ pour tout $f \in \mathsf{N}$.

Enfin, l'anneau E_1 se trouve un facteur direct de l'anneau

$$\mathsf{E} := \mathrm{End}_{\overline{\mathbb{Z}}_\ell \mathbf{G}}(\mathrm{ind}_{\mathbf{K}}^{\mathbf{G}}(\overline{\mathbb{Z}}_\ell \Gamma_{G,\psi})) = \mathrm{End}_{\overline{\mathbb{Z}}_\ell \mathbf{G}}(\overline{\mathbb{Z}}_\ell \mathcal{W}_0)$$

($\mathcal{W}_0 = \mathrm{ind}_{\mathbf{K}}^{\mathbf{G}} \Gamma_{G,\psi}$; voir § 4.15), et l'on a étudié la structure de E en §§ 4.17 - 4.19.

4.22. Proposition. — *Notations comme en § 4.21 (donc* $\mathrm{ind}_{\mathbf{I}}^{\mathbf{G}}\mathbf{1}_{\mathbf{I}} = \overline{\mathbb{Z}}_\ell\mathbf{G}.e_{\mathbf{I}}$*), et identifier* $\overline{\mathbb{Z}}_\ell\mathcal{W}_0 = \overline{\mathbb{Z}}_\ell\mathbf{G}.e_\psi$ *(§ 4.16). Pour tout* $g \in \mathbf{G}$*, soient* $g_{\mathsf{M}} \in \mathsf{M}$ *et* $g_{\mathsf{N}}^* \in \mathsf{N}$ *via*

$$\begin{aligned} g_{\mathsf{M}} &: \overline{\mathbb{Z}}_\ell\mathcal{W}_0 \longrightarrow \mathrm{ind}_{\mathbf{I}}^{\mathbf{G}}\mathbf{1}_{\mathbf{I}}, \quad f \longmapsto f.ge_\psi, \\ g_{\mathsf{N}}^* &: \mathrm{ind}_{\mathbf{I}}^{\mathbf{G}}\mathbf{1}_{\mathbf{I}} \longrightarrow \overline{\mathbb{Z}}_\ell\mathcal{W}_0, \quad f \longmapsto f.g^{-1}e_{\mathbf{I}}, \end{aligned}$$

puis posons $g_{\mathsf{M}_1} = g_{\mathsf{M}} \cdot e_{[1]}^{G,\ell} \in \mathsf{M}_1$ *et* $g_{\mathsf{N}_1}^* = e_{[1]}^{G,\ell} \cdot g_{\mathsf{N}}^* \in \mathsf{N}_1$ *(sous les inclusions* $\mathsf{M}_1 \subset \mathsf{M}$ *et* $\mathsf{N}_1 \subset \mathsf{N}$*, on a* $g_{\mathsf{M}_1} = g_{\mathsf{M}}$ *et* $g_{\mathsf{N}_1}^* = g_{\mathsf{N}}^*$*). Soit aussi*

$$\Omega = \left\{ \begin{bmatrix} \varpi^{a+1} & 0 \\ 0 & \varpi^b \end{bmatrix}, \begin{bmatrix} 0 & \varpi^a \\ \varpi^b & 0 \end{bmatrix} : a, b \in \mathbb{Z} \; ; \; a \geq b \right\} \subset \mathbf{G}.$$

Alors :

(a) *Le* $\overline{\mathbb{Z}}_\ell$*-module* M *(resp.* N*) est engendré par* $\{g_{\mathsf{M}} : g \in \Omega\}$ *(resp.* $\{g_{\mathsf{N}}^* : g \in \Omega\}$*) ; on a* $g_{\mathsf{M}} = 0$ *(resp.* $g_{\mathsf{N}}^* = 0$*) lorsque* $\mathbf{I}g\mathbf{I}$ *n'appartient pas à l'image de* Ω *dans* $\mathbf{I}\backslash\mathbf{G}/\mathbf{I}$*.*

(b) *Rappelons que* $t = \begin{bmatrix} 0 & 1 \\ \varpi & 0 \end{bmatrix}$ *et* $s = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$*. On a*

$$(t_{\mathsf{H}})^m = (t^m)_{\mathsf{H}} \text{ pour tout } m \in \mathbb{Z} \; ; \; t_{\mathsf{H}} \cdot g_{\mathsf{M}} = (gt)_{\mathsf{M}} \text{ pour tout } g \in \Omega. \tag{4.22.1}$$

De plus :

$$\begin{aligned} s_{\mathsf{H}} \cdot \begin{bmatrix} \varpi^{a+1} & 0 \\ 0 & \varpi^b \end{bmatrix}_{\mathsf{M}} &= q \cdot \begin{bmatrix} 0 & \varpi^{a+1} \\ \varpi^b & 0 \end{bmatrix}_{\mathsf{M}} ; \\ s_{\mathsf{H}} \cdot \begin{bmatrix} 0 & \varpi^{a+1} \\ \varpi^b & 0 \end{bmatrix}_{\mathsf{M}} &= \begin{bmatrix} \varpi^{a+1} & 0 \\ 0 & \varpi^b \end{bmatrix}_{\mathsf{M}} + (q-1) \cdot \begin{bmatrix} 0 & \varpi^{a+1} \\ \varpi^b & 0 \end{bmatrix}_{\mathsf{M}} ; \\ s_{\mathsf{H}} \cdot \begin{bmatrix} 0 & \varpi^a \\ \varpi^a & 0 \end{bmatrix}_{\mathsf{M}} &= - \begin{bmatrix} 0 & \varpi^a \\ \varpi^a & 0 \end{bmatrix}_{\mathsf{M}}. \end{aligned} \tag{4.22.2}$$

On a des formules analogues concernant l'action de H *sur* N *(voir le début de la preuve de (c) ci-dessous).*

(c) *On a* $\mathsf{M} = \mathsf{H}.s_{\mathsf{M}}$ *et* $\mathsf{N} = s_{\mathsf{N}}^*.\mathsf{H}$*.*

Ces résultats restent vrais lorsque la paire (M, N) *y est remplacée par* $(\mathsf{M}_1, \mathsf{N}_1)$*.*

Preuve. (a) La preuve suit celle de § 4.16, et cette fois il convient de remarquer que $\mathbf{I}^1\backslash\mathbf{G}/\mathbf{I} = \mathbf{I}\backslash\mathbf{G}/\mathbf{I} = \mathbf{I}\backslash\mathbf{G}/\mathbf{I}^1$ (car $\mathbf{I}/\mathbf{I}^1 = T^F$).

(b) Les relations dans (4.22.1) s'obtiennent du fait que t normalise $\mathbf{I}$, tandis que les autres relations sont obtenues par calculs directs.

(c) On observe d'abord que pour tout $h \in \mathbf{G}$ et tout $g \in \Omega$, si $h_{\mathsf{H}} \cdot g_{\mathsf{M}} = \sum_{x \in \Omega} c_x x_{\mathsf{M}}$ ($c_x \in \overline{\mathbb{Z}}_\ell$), alors $g_{\mathsf{N}}^* \cdot (h^{-1})_{\mathsf{H}} = \sum_{x \in \Omega} c_x x_{\mathsf{N}}^*$. Ainsi $\mathsf{M} = \mathsf{H}.s_{\mathsf{M}}$ si et seulement si $\mathsf{N} = s_{\mathsf{N}}^*.\mathsf{H}$. On va donc établir seulement l'égalité $\mathsf{M} = \mathsf{H}.s_{\mathsf{M}}$.

Écrivons $\Omega = \Omega_1 \cup \Omega_2$ où

$$\Omega_1 = \left\{ \begin{bmatrix} \varpi^{a+1} & 0 \\ 0 & \varpi^b \end{bmatrix} : a, b \in \mathbb{Z} \; ; \; a \geq b \right\} \text{ et } \Omega_2 = \left\{ \begin{bmatrix} 0 & \varpi^a \\ \varpi^b & 0 \end{bmatrix} : a, b \in \mathbb{Z} \; ; \; a \geq b \right\}.$$

Comme $\Omega_1 \cdot t = \Omega_2$, (4.22.1) implique que M est un H-module (à gauche) engendré par $\{g_{\mathsf{M}} : g \in \Omega_1\}$. De plus, d'après (4.22.1) et (4.22.2), on a pour tous $a \geq b$:

$$\begin{aligned} t_{\mathsf{H}} s_{\mathsf{H}} \cdot \begin{bmatrix} \varpi^{a+1} & 0 \\ 0 & \varpi^{b} \end{bmatrix}_{\mathsf{M}} &= q \cdot \begin{bmatrix} \varpi^{a+2} & 0 \\ 0 & \varpi^{b} \end{bmatrix}_{\mathsf{M}} ; \\ (t^2)_{\mathsf{H}} \cdot \begin{bmatrix} \varpi^{a+1} & 0 \\ 0 & \varpi^{b} \end{bmatrix}_{\mathsf{M}} &= \begin{bmatrix} \varpi^{a+2} & 0 \\ 0 & \varpi^{b+1} \end{bmatrix}_{\mathsf{M}} ; \\ (t^{-2})_{\mathsf{H}} \cdot \begin{bmatrix} \varpi^{a+1} & 0 \\ 0 & \varpi^{b} \end{bmatrix}_{\mathsf{M}} &= \begin{bmatrix} \varpi^{a} & 0 \\ 0 & \varpi^{b-1} \end{bmatrix}_{\mathsf{M}} . \end{aligned}$$

Ces trois dernières relations impliquent que $\mathsf{H}.s_{\mathsf{M}} \supset \{g_{\mathsf{M}} : g \in \Omega_1\}$, d'où $\mathsf{M} = \mathsf{H}.s_{\mathsf{M}}$. □

4.23. Proposition. — *Notations et hypothèses comme en §§ 4.21 - 4.22. Posons*

$$g_{\mathsf{E}_1} = e^{G,\ell}_{[1]} \cdot g_{\mathsf{E}} \cdot e^{G,\ell}_{[1]} \in \mathsf{E}_1$$

pour tout $g \in \mathbf{G}$, et écrivons $\varpi_{\mathsf{E}_1} = (\varpi.\mathrm{id}_2)_{\mathsf{E}_1}$. Alors :

(a) *Pour tout $\lambda \in \mathbb{F}_q^\times$, on a*

$$\begin{aligned} s_{\mathsf{M}} \cdot \begin{bmatrix} 0 & \dot{\lambda}\varpi \\ 1 & 0 \end{bmatrix}_{\mathsf{E}_1} &= q \cdot \begin{bmatrix} \varpi & 0 \\ 0 & 1 \end{bmatrix}_{\mathsf{M}} + (q-1) \cdot \begin{bmatrix} 0 & \varpi \\ 1 & 0 \end{bmatrix}_{\mathsf{M}} \\ &= (q.t_{\mathsf{H}} + q^{-1}(q-1).s_{\mathsf{H}} t_{\mathsf{H}}) s_{\mathsf{M}} ; \end{aligned}$$

Ces formules restent vraies lorsque M y est remplacé par M_1.

(b) *Pour tout $\lambda \in \mathbb{F}_q^\times$, on a* $\begin{bmatrix} 0 & \dot{\lambda}\varpi \\ 1 & 0 \end{bmatrix}_{\mathsf{E}_1} = \begin{bmatrix} 0 & \varpi \\ 1 & 0 \end{bmatrix}_{\mathsf{E}_1}$.

(c) (Comparer [Hel1, Prop. 9.9 et Rmk. 9.10].) *Identifions $e^{\mathsf{E},\ell}_{[1]}.\overline{\mathbb{Z}}_\ell \mathsf{E}_G \subset \mathsf{E}_1$ à l'aide de (4.19.2). En dénotant par $\theta \in e^{\mathsf{E},\ell}_{[1]}.\overline{\mathbb{Z}}_\ell \mathsf{E}_G \subset \mathsf{E}_1$ l'élément correspondant à la fonction trace sur $(T^\vee /\!\!/ W)^{F^\vee}_\ell$ sous l'isomorphisme $e^{\mathsf{E},\ell}_{[1]}.\overline{\mathbb{Z}}_\ell \mathsf{E}_G \simeq \overline{\mathbb{Z}}_\ell \mathsf{B}^{(\ell)}_{G^\vee}$ en (4.11.2) (notre θ ici correspond à celui en § 4.13), on a*

$$\mathsf{E}_1 = \overline{\mathbb{Z}}_\ell \left[\theta, \varpi_{\mathsf{E}_1}^{\pm 1}, \begin{bmatrix} 0 & \varpi \\ 1 & 0 \end{bmatrix}_{\mathsf{E}_1} \right].$$

Preuve. Les formules en (a) s'obtiennent par des calculs directs, tandis que (c) provient de (4.19.2), de § 4.13 et de (b). Il reste alors à démontrer (b).

Démontrons maintenant (b). D'après [Hel1, Cor. 5.14], on a une décomposition de $\overline{\mathbb{Q}}_\ell G^F$-modules

$$\overline{\mathbb{Q}}_\ell \mathcal{P}_{\overline{\pi}} = e^{G,\ell}_{[1]}.\overline{\mathbb{Q}}_\ell \Gamma_{G,\psi} = \mathrm{St}_G \oplus C$$

avec C une représentation cuspidale de G^F. (En effet, [DeLu, Thm. 10.7] nous donne $C = \bigoplus_{\chi \in (\widehat{T_s^F})_\ell} [-R^G_{T_s}(\chi)]$.) On a alors une décomposition de $\overline{\mathbb{Q}}_\ell$-algèbres

$$\overline{\mathbb{Q}}_\ell \mathsf{E}_1 = \mathrm{End}_{\overline{\mathbb{Q}}_\ell \mathbf{G}}(\mathrm{ind}^{\mathbf{G}}_{\mathbf{K}}(\overline{\mathbb{Q}}_\ell \mathcal{P}_{\overline{\pi}})) = \mathrm{End}_{\overline{\mathbb{Q}}_\ell \mathbf{G}}(\mathrm{ind}^{\mathbf{G}}_{\mathbf{K}} \mathrm{St}_G) \oplus \mathrm{End}_{\overline{\mathbb{Q}}_\ell \mathbf{G}}(\mathrm{ind}^{\mathbf{G}}_{\mathbf{K}} C). \tag{4.23.1}$$

Pour tout $\lambda \in \mathbb{F}_q^\times$, soit $g_\lambda = \begin{bmatrix} 0 & \dot{\lambda}\varpi \\ 1 & 0 \end{bmatrix} \in \mathbf{G}$. En identifiant l'élément $(g_\lambda)_{\mathsf{E}_1}$ de $\overline{\mathbb{Q}}_\ell \mathsf{E}_1 = \mathrm{End}_{\overline{\mathbb{Q}}_\ell \mathbf{G}}(\mathrm{ind}_{\mathbf{K}}^{\mathbf{G}}(\overline{\mathbb{Q}}_\ell \mathcal{P}_{\overline{\pi}}))$ avec une fonction $(\mathbf{K} \times \mathbf{K}^{\mathrm{op}})$-équivariante

$$(g_\lambda)_{\mathsf{E}_1} : \mathbf{G} \longrightarrow \mathrm{End}_{\overline{\mathbb{Q}}_\ell}(\overline{\mathbb{Q}}_\ell \mathcal{P}_{\overline{\pi}}),$$

le support de la fonction $(g_\lambda)_{\mathsf{E}_1}$ est inclus dans $\mathbf{K} g_\lambda \mathbf{K}$; sous la décomposition (4.23.1), la composante $(g_\lambda)'_{\mathsf{E}_1}$ de $(g_\lambda)_{\mathsf{E}_1}$ dans $\mathrm{End}_{\overline{\mathbb{Q}}_\ell \mathbf{G}}(\mathrm{ind}_{\mathbf{K}}^{\mathbf{G}} C)$ s'identifie avec une fonction

$$(g_\lambda)'_{\mathsf{E}_1} : C \longrightarrow {}^{g_\lambda} C$$

qui est $(\mathbf{K} \cap {}^{g_\lambda}\mathbf{K})$-équivariante (comparer la preuve de § 4.16). Comme g_λ n'est pas central dans $\mathbf{G}$, on a $(g_\lambda)'_{\mathsf{E}_1} = 0$ d'après [Hel1, Prop. 6.7] : en effet, on a

$$\mathbf{K} \cap {}^{g_\lambda}\mathbf{K} = {}^s\mathbf{I} = \begin{bmatrix} O_{\mathbf{F}}^\times & \varpi O_{\mathbf{F}} \\ O_{\mathbf{F}} & O_{\mathbf{F}}^\times \end{bmatrix} \supset {}^s\mathbf{U}(O_{\mathbf{F}}) = \begin{bmatrix} 1 & 0 \\ O_{\mathbf{F}} & 1 \end{bmatrix},$$

et l'on peut vérifier directement que ${}^s\mathbf{U}(O_{\mathbf{F}})$ agit trivialement sur ${}^{g_\lambda}C$, donc la fonction $(g_\lambda)'_{\mathsf{E}_1}$ se factorise par l'espace des ${}^s\mathbf{U}(O_{\mathbf{F}})$-coinvariants de C, alors que ce dernier espace de coinvariants est nul puisque C est cuspidale, d'où $(g_\lambda)'_{\mathsf{E}_1} = 0$. On peut donc regarder chaque $(g_\lambda)_{\mathsf{E}_1}$ comme un élément de $\mathrm{End}_{\overline{\mathbb{Q}}_\ell \mathbf{G}}(\mathrm{ind}_{\mathbf{K}}^{\mathbf{G}} \mathrm{St}_G)$.

Ensuite, on observe que l'action de $\overline{\mathbb{Q}}\mathsf{E}_1$ sur $\overline{\mathbb{Q}}\mathsf{M}_1$ se restreint à une action fidèle de $\mathrm{End}_{\overline{\mathbb{Q}}_\ell \mathbf{G}}(\mathrm{ind}_{\mathbf{K}}^{\mathbf{G}} \mathrm{St}_G)$ sur $\overline{\mathbb{Q}}\mathsf{M}_1$: en effet, comme $\mathrm{ind}_{B^F}^{G^F} \mathbf{1}_{B^F} = \mathbf{1}_{G^F} \oplus \mathrm{St}_G$, on peut montrer que $\mathrm{ind}_{\mathbf{K}}^{\mathbf{G}} \mathrm{St}_G$ est un objet du bloc principal $\mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}(\mathbf{G})_{[\mathbf{T},\mathbf{1}]}$ (le bloc de Bernstein de $\mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}(\mathbf{G})$ contenant le caractère trivial de $\mathbf{G}$) ; d'autre part, on a une équivalence de catégories

$$\mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}(\mathbf{G})_{[\mathbf{T},\mathbf{1}]} \xrightarrow{\sim} \text{Mod-}\overline{\mathbb{Q}}_\ell \mathsf{H}, \quad V \longmapsto V^{\mathbf{I}}$$

(voir [Li1, Sec. 5]), donc l'action de $\mathrm{End}_{\overline{\mathbb{Q}}_\ell \mathbf{G}}(\mathrm{ind}_{\mathbf{K}}^{\mathbf{G}} \mathrm{St}_G)$ sur $\mathrm{Hom}_{\overline{\mathbb{Q}}_\ell \mathbf{G}}(\mathrm{ind}_{\mathbf{K}}^{\mathbf{G}} \mathrm{St}_G, \mathrm{ind}_{\mathbf{I}}^{\mathbf{G}} \mathbf{1}_{\mathbf{I}})$ correspond à l'action de $\mathrm{End}_{\overline{\mathbb{Q}}_\ell \mathsf{H}}((\mathrm{ind}_{\mathbf{K}}^{\mathbf{G}} \mathrm{St}_G)^{\mathbf{I}})$ sur $\mathrm{Hom}_{\overline{\mathbb{Q}}_\ell \mathsf{H}}((\mathrm{ind}_{\mathbf{K}}^{\mathbf{G}} \mathrm{St}_G)^{\mathbf{I}}, (\mathrm{ind}_{\mathbf{I}}^{\mathbf{G}} \mathbf{1}_{\mathbf{I}})^{\mathbf{I}})$; vu que $(\mathrm{ind}_{\mathbf{I}}^{\mathbf{G}} \mathbf{1}_{\mathbf{I}})^{\mathbf{I}} = \overline{\mathbb{Q}}_\ell \mathsf{H}$ et que $(\mathrm{ind}_{\mathbf{K}}^{\mathbf{G}} \mathrm{St}_G)^{\mathbf{I}} = \varepsilon.\overline{\mathbb{Q}}_\ell \mathsf{H}$ où ε est un idempotent de $\overline{\mathbb{Q}}_\ell \mathsf{H}$ induit par le caractère de signature, cette dernière action se trouve l'action de multiplication (à droite) de $\varepsilon.\overline{\mathbb{Q}}_\ell \mathsf{H}.\varepsilon$ sur $\overline{\mathbb{Q}}_\ell \mathsf{H}.\varepsilon$, une action qui est bien fidèle ; on en déduit que $\mathrm{End}_{\overline{\mathbb{Q}}_\ell \mathbf{G}}(\mathrm{ind}_{\mathbf{K}}^{\mathbf{G}} \mathrm{St}_G)$ agit fidèlement sur $\mathrm{Hom}_{\overline{\mathbb{Q}}_\ell \mathbf{G}}(\mathrm{ind}_{\mathbf{K}}^{\mathbf{G}} \mathrm{St}_G, \mathrm{ind}_{\mathbf{I}}^{\mathbf{G}} \mathbf{1}_{\mathbf{I}})$ et donc sur $\overline{\mathbb{Q}}\mathsf{M}_1$. Cette dernière action fidèle induit une inclusion d'anneaux

$$i : \mathrm{End}_{\overline{\mathbb{Q}}_\ell \mathbf{G}}(\mathrm{ind}_{\mathbf{K}}^{\mathbf{G}} \mathrm{St}_G) \hookrightarrow \mathrm{End}_{\overline{\mathbb{Q}}_\ell \mathsf{H}}(\overline{\mathbb{Q}}_\ell \mathsf{M}_1).$$

Comme $\mathsf{M}_1 = \mathsf{H}.s_{\mathsf{M}_1}$ (§ 4.22(c)), les formules obtenues en (a) nous disent que tous les $i((g_\lambda)_{\mathsf{E}_1})$ $(\lambda \in \mathbb{F}_q^\times)$ coïncident puisque tous les $s_{\mathsf{M}_1} \cdot (g_\lambda)_{\mathsf{E}_1}$ $(\lambda \in \mathbb{F}_q^\times)$ le sont ; ainsi tous les $(g_\lambda)_{\mathsf{E}_1}$ $(\lambda \in \mathbb{F}_q^\times)$ coïncident. □

4.24. *Sur la description de* $\mathrm{End}_{\overline{\mathbb{Z}}_\ell \mathbf{G}}(\mathcal{P})$. — Notations comme en §§ 4.21 - 4.23. On a vu que

$$\mathrm{End}_{\overline{\mathbb{Z}}_\ell \mathbf{G}}(\mathcal{P}) = \begin{bmatrix} \mathsf{H} & \mathsf{M}_1 \\ \mathsf{N}_1 & \mathsf{E}_1 \end{bmatrix} = \begin{bmatrix} \mathsf{H} & \mathsf{H}.s_{\mathsf{M}_1} \\ s_{\mathsf{N}_1}^*.\mathsf{H} & \mathsf{E}_1 \end{bmatrix},$$

que $\mathsf{H} = \overline{\mathbb{Z}}_\ell[s_{\mathsf{H}}, t_{\mathsf{H}}]$ et que $\mathsf{E}_1 = \overline{\mathbb{Z}}_\ell[\theta, \varpi_{\mathsf{E}_1}^{\pm 1}, \Theta]$ avec $\Theta = \begin{bmatrix} 0 & \varpi \\ 1 & 0 \end{bmatrix}_{\mathsf{E}_1}$.

On peut vérifier que $s_{\mathsf{M}_1} \cdot s^*_{\mathsf{N}_1} = q^{-1} - q^{-2} s_{\mathsf{H}} \in \mathsf{H}$. Cet élément $s_{\mathsf{M}_1} \cdot s^*_{\mathsf{N}_1}$ n'est pas inversible dans H : en effet, on a d'abord $s_{\mathsf{M}_1} \cdot s^*_{\mathsf{N}_1} \in \overline{\mathbb{Z}}_\ell[s_{\mathsf{H}}]$; en identifiant

$$\overline{\mathbb{Z}}_\ell[s_{\mathsf{H}}] = \mathrm{End}_{\overline{\mathbb{Z}}_\ell \mathbf{K}}(\mathrm{ind}_{\mathbf{I}}^{\mathbf{K}} \mathbf{1}_{\mathbf{I}}) = \mathrm{End}_{\overline{\mathbb{Z}}_\ell G^F}(\mathrm{ind}_{B^F}^{G^F} \mathbf{1}_{B^F})$$

et en se rappelant que $\overline{\mathbb{Q}}_\ell \mathrm{ind}_{B^F}^{G^F} \mathbf{1}_{B^F} = \mathbf{1}_{G^F} \oplus \mathrm{St}_G$, on a alors les identifications

$$\overline{\mathbb{Z}}_\ell[s_{\mathsf{H}}] \subset \mathrm{End}_{\overline{\mathbb{Q}}_\ell G^F}(\overline{\mathbb{Q}}_\ell \mathrm{ind}_{B^F}^{G^F} \mathbf{1}_{B^F}) = \mathrm{End}_{\overline{\mathbb{Q}}_\ell G^F}(\mathbf{1}_{G^F}) \times \mathrm{End}_{\overline{\mathbb{Q}}_\ell G^F}(\mathrm{St}_G) = \overline{\mathbb{Q}}_\ell \times \overline{\mathbb{Q}}_\ell,$$

sous lesquelles $s_{\mathsf{H}} \in \overline{\mathbb{Z}}_\ell[s_{\mathsf{H}}]$ correspond à $(q, -1) \in \overline{\mathbb{Q}}_\ell \times \overline{\mathbb{Q}}_\ell$, et donc $s_{\mathsf{M}_1} \cdot s^*_{\mathsf{N}_1} \in \overline{\mathbb{Z}}_\ell[s_{\mathsf{H}}]$ correspond à $q^{-1}(1,1) - q^{-2}(q,-1) = (0, q^{-2}(q-1)) \in \overline{\mathbb{Q}}_\ell \times \overline{\mathbb{Q}}_\ell$ qui n'est pas inversible dans $\overline{\mathbb{Q}}_\ell \times \overline{\mathbb{Q}}_\ell$.

On peut aussi calculer que

$$s^*_{\mathsf{N}_1} \cdot s_{\mathsf{M}_1} = \frac{1}{q(q-1)^2} \sum_{\lambda \in \mathbb{F}_q^\times} e(\lambda) + \frac{1}{(q-1)^2} \sum_{\mu, \lambda, \alpha \in \mathbb{F}_q^\times} \psi_0(-\alpha + \alpha^{-1}\mu) e(\lambda) \cdot \begin{bmatrix} 0 & \dot{\mu} \\ 1 & 0 \end{bmatrix}_{\mathsf{E}_1},$$

que $s_{\mathsf{M}_1} \cdot \varpi_{\mathsf{E}_1}^{\pm 1} = (\varpi^{\pm 1} s)_{\mathsf{M}_1}$, et que $s_{\mathsf{M}_1} \cdot \theta = 2 s_{\mathsf{M}_1}$. On sait donc comment E_1 agit sur M_1. On peut aussi décrire l'action de E_1 sur N_1 de manière analogue, et l'on obtient ainsi une description de la $\overline{\mathbb{Z}}_\ell$-algèbre $\mathrm{End}_{\overline{\mathbb{Z}}_\ell \mathbf{G}}(\mathcal{P})$. Il serait alors intéressant si l'on parvient à généraliser, de manière structurée, les descriptions de $\mathrm{End}_{\overline{\mathbb{Z}}[\frac{1}{p}]\mathbf{G}}(\mathcal{W}_0)$ et du ℓ-bloc unipotent de $\mathbf{G}$ au-delà du cas de $\mathbf{G} = \mathrm{GL}_2(\mathbf{F})$ étudié dans ce travail.

A. La structure de $\mathsf{B}_{\mathrm{SO}_{2n}}$

A.1. *Le groupe* $G = \mathrm{SO}_{2n}(\overline{\mathbb{F}}_q)$. — Pour tout corps (commutatif) $\mathbb{K}$, on pose

$$\mathrm{SO}_{2n}(\mathbb{K}) = \{A \in \mathrm{GL}_{2n}(\mathbb{K}) : {}^tA \cdot J \cdot A = J\} \quad \text{où} \quad J = \begin{bmatrix} & & & 1 \\ & & \iddots & \\ & \iddots & & \\ 1 & & & \end{bmatrix} \in \mathrm{GL}_{2n}(\mathbb{K}).$$

Désormais, soit $G = \mathrm{SO}_{2n}(\overline{\mathbb{F}}_q)$ avec q impair et $n \geq 4$. L'action de Frobenius standard $(g_{ij})_{i,j} \longmapsto (g_{ij}^q)_{i,j}$ sur $\mathrm{GL}_{2n}(\overline{\mathbb{F}}_q)$ se restreint à une action de Frobenius F sur G ; (G, F) est alors déployé sur $\mathbb{F}_q$, et $G^F = \mathrm{SO}_{2n}(\mathbb{F}_q)$. Les matrices diagonales (resp. les matrices triangulaires supérieures) de G forment un tore maximal (resp. un sous-groupe de Borel) F-stable de G. Le dual de Langlands de (G, T) se trouve $(G^\vee, T^\vee)$ avec $G^\vee = \mathrm{SO}_{2n}$ (sur $\mathbb{Z}$) et avec

$$T^\vee = \{\theta(z_1, \cdots, z_n) := \mathrm{diag}(z_1, \cdots, z_n, z_n^{-1}, \cdots, z_1^{-1}) : z_1, \cdots, z_n \in \mathbb{G}_m\} \simeq \mathbb{G}_m^n$$

le tore diagonal maximal de $G^\vee$. Le sous-groupe de Borel B de G correspond à un sous-groupe de Borel $B^\vee$ de $G^\vee$ décrivant les matrices triangulaires supérieures.

Pour chaque $1 \leq i \leq n$, on considère les deux morphismes suivants :

$$\begin{cases} \varepsilon_i : T^\vee \longrightarrow \mathbb{G}_m, & \theta(z_1, \cdots, z_n) \longmapsto z_i \ ; \\ \eta_i : \mathbb{G}_m \longrightarrow T^\vee, & z \longmapsto \theta(1, \cdots, z, \cdots, 1) \quad (z \text{ à la } i\text{-ième place}). \end{cases}$$

Il s'ensuit que $X(T^\vee)$ (resp. $Y(T^\vee)$) est un $\mathbb{Z}$-module libre ayant pour base $\{\varepsilon_1, \cdots, \varepsilon_n\}$ (resp. $\{\eta_1, \cdots, \eta_n\}$) ; de plus, les deux bases $\{\varepsilon_1, \cdots, \varepsilon_n\}$ et $\{\eta_1, \cdots, \eta_n\}$ sont duales pour l'accouplement parfait $\langle \cdot, \cdot \rangle : X(T^\vee) \times Y(T^\vee) \longrightarrow \mathbb{Z}$.

L'ensemble des racines simples déterminé par $(G^\vee, T^\vee, B^\vee)$ est $\Delta^\vee = \{\alpha_1, \cdots, \alpha_n\} \subset X(T^\vee)$ où $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ pour $1 \leq i \leq n-1$ et $\alpha_n = \varepsilon_{n-1} + \varepsilon_n$. Le système de racines de $G^\vee$ est de type D_n. Le groupe $G^\vee$ est semisimple (autrement dit, le groupe dérivé de $G^\vee$ coïncide avec $G^\vee$), mais il n'est ni adjoint, ni simplement connexe.

Le groupe de Weyl $W = N_{G^\vee}(T^\vee)/T^\vee$ de $(G^\vee, T^\vee)$ peut être décrit comme suit. Soit $\mathfrak{S}_{\{\pm\varepsilon_1, \cdots, \pm\varepsilon_n\}}$ le groupe symétrique de $\{\pm\varepsilon_1, \cdots, \pm\varepsilon_n\}$, et considérons les deux inclusions de groupes suivantes :

$$\begin{array}{ll} i_1 : \mathfrak{S}_n \hookrightarrow \mathfrak{S}_{\{\pm\varepsilon_1, \cdots, \pm\varepsilon_n\}}, & \sigma \mapsto [\pm\varepsilon_i \mapsto \pm\varepsilon_{\sigma(i)} \ \ (1 \leq i \leq n)]; \\ i_2 : \{\pm 1\}^n \hookrightarrow \mathfrak{S}_{\{\pm\varepsilon_1, \cdots, \pm\varepsilon_n\}}, & (j_1, \cdots, j_n) \mapsto [\pm\varepsilon_i \mapsto \pm j_i\varepsilon_i \ \ (1 \leq i \leq n)]; \end{array}$$

Soit $D \subset \{\pm 1\}^n$ l'ensemble des éléments $(j_1, \cdots, j_n) \in \{\pm 1\}^n$ tels que (-1) apparaisse un nombre pair de fois dans $j_1, \cdots, j_n$. Alors $W \simeq i_1(\mathfrak{S}_n).i_2(D) \subset \mathfrak{S}_{\{\pm\varepsilon_1, \cdots, \pm\varepsilon_n\}}$ (un produit semi-direct), puis $|W| = |\mathfrak{S}_n| \cdot |D| = n! \cdot 2^{n-1}$. De plus, le groupe W est engendré par $\{s_1, \cdots, s_n\}$ où chaque s_i est la réflexion orthogonale par rapport à α_i ; sous l'identification $W \subset \mathfrak{S}_{\{\pm\varepsilon_1, \cdots, \pm\varepsilon_n\}}$, on a :

$$\begin{cases} s_i = (\varepsilon_i, \varepsilon_{i+1})(-\varepsilon_i, -\varepsilon_{i+1}) \in \mathfrak{S}_{\{\pm\varepsilon_1, \cdots, \pm\varepsilon_n\}} \quad (1 \leq i \leq n-1) \ ; \\ s_n = (\varepsilon_{n-1}, -\varepsilon_n)(\varepsilon_n, -\varepsilon_{n-1}) \in \mathfrak{S}_{\{\pm\varepsilon_1, \cdots, \pm\varepsilon_n\}}. \end{cases}$$

A.2. *Notations pour l'anneau* $\mathbb{Z}[X(T^\vee)]$. — Privilégiant l'addition dans $\mathbb{Z}[X(T^\vee)]$ pour la notation additive, désormais on écrit multiplicativement la loi de composition du groupe $X(T^\vee)$: via les identifications $X(T^\vee) \simeq \mathbb{Z}\varepsilon_1 \oplus \cdots \oplus \mathbb{Z}\varepsilon_n \simeq t_1^{\mathbb{Z}} \cdots t_n^{\mathbb{Z}}$ ($\varepsilon_i \leftrightarrow t_i$; en notation de § 3.15, $t_i = e(\varepsilon_i)$), un élément de $X(T^\vee)$ s'écrit $t_1^{a_1} \cdots t_n^{a_n}$, et la loi du groupe $X(T^\vee)$ s'écrit multiplicativement comme $(t_1^{a_1} \cdots t_n^{a_n}) \cdot (t_1^{b_1} \cdots t_n^{b_n}) = t_1^{a_1+b_1} \cdots t_n^{a_n+b_n}$. Un élément de $\mathbb{Z}[X(T^\vee)]$ est de la forme $\displaystyle\sum_{j=1}^{N} c_j t_1^{a_{j,1}} \cdots t_n^{a_{j,n}}$ ($c_j \in \mathbb{Z}, a_{j,i} \in \mathbb{Z}$).

L'identification $W \subset \mathfrak{S}_{\{\pm\varepsilon_1, \cdots, \pm\varepsilon_n\}} \simeq \mathfrak{S}_{\{t_1^{\pm 1}, \cdots, t_n^{\pm 1}\}}$ donne l'action de W sur $X(T^\vee)$; les générateurs $s_1, \cdots, s_n \in W$ agissent sur $X(T^\vee)$ par :

$$\begin{cases} s_i = (t_i, t_{i+1})(t_i^{-1}, t_{i+1}^{-1}) \in \mathfrak{S}_{\{t_1^{\pm 1}, \cdots, t_n^{\pm 1}\}} \quad (1 \leq i \leq n-1) \ ; \\ s_n = (t_{n-1}, t_n^{-1})(t_n, t_{n-1}^{-1}) \in \mathfrak{S}_{\{t_1^{\pm 1}, \cdots, t_n^{\pm 1}\}}. \end{cases}$$

A.3. *Structure linéaire de* $\mathbb{Z}[X(T^\vee)]^W$. — Par définition, $\mathbb{Z}[X(T^\vee)]^W$ se compose des éléments de $\mathbb{Z}[X(T^\vee)]$ invariants sous l'action de W (§ 3.3). La théorie générale (§ 3.15) nous dit que $\mathbb{Z}[X(T^\vee)]^W$ admet une base $\mathbb{Z}$-linéaire $\{r(\lambda) : \lambda \in X^+(T^\vee)\}$ où $r(\lambda) = \displaystyle\sum_{\mu \in W\lambda} \mu \in \mathbb{Z}[X(T^\vee)]^W$ pour $\lambda \in X(T^\vee)$; ici, $X^+(T^\vee)$ est l'ensemble des poids dominants (§ 3.13), et $W\lambda$ est la W-orbite de λ. Dans notre cas particulier :

(a) Les éléments de $X^+(T^\vee)$ sont soit de la forme $t_1^{a_1} \cdots t_n^{a_n}$ avec $a_1 \geq \cdots \geq a_n \geq 0$, soit de la forme $t_1^{a_1} \cdots t_{n-1}^{a_{n-1}} t_n^{-a_n}$ avec $a_1 \geq \cdots \geq a_n > 0$.

(b) L'élément $r(\lambda) \in \mathbb{Z}[X(T^\vee)]^W$ associé à $\lambda = t_1^{a_1} \cdots t_n^{a_n} \in X(T^\vee)$ s'explicite :

$$r(t_1^{a_1} \cdots t_n^{a_n}) = \sum_{\sigma \in \mathfrak{S}_n / \mathfrak{S}_{n,(a_1, \cdots, a_n)}} \ \sum_{\substack{\gamma : \{1, \cdots, n\} \to \{\pm 1\} \\ |\gamma^{-1}(-1)| = 0 \,(\mathrm{mod}\, 2)}} t_{\sigma(1)}^{\gamma(1) a_1} \cdots t_{\sigma(n)}^{\gamma(n) a_n}$$

où $\mathfrak{S}_{n,(a_1, \cdots, a_n)} \subset \mathfrak{S}_n$ est le stabilisateur de $(a_1, \cdots, a_n) \in \mathbb{Z}^n$ de l'action de $\mathfrak{S}_n$ sur $\mathbb{Z}^n$ via permutations des coefficients.

A.4. Proposition. — *Posons* $\Sigma_i = r(t_1 t_2 \cdots t_i) \in \mathbb{Z}[X(T^\vee)]^W$ *pour* $1 \leq i \leq n$ *et posons* $\Sigma'_n = r(t_1 t_2 \cdots t_{n-1} t_n^{-1}) \in \mathbb{Z}[X(T^\vee)]^W$. *Alors* $\mathbb{Z}[X(T^\vee)]^W = \mathbb{Z}[\Sigma_1, \cdots, \Sigma_n, \Sigma'_n]$.

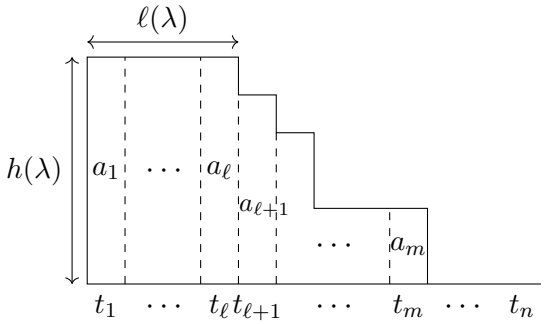
Preuve. La famille $\{r(\lambda) : \lambda \in X^+(T^\vee)\}$ étant une base $\mathbb{Z}$-linéaire de $\mathbb{Z}[X(T^\vee)]^W$ (§ A.3), il suffit de montrer que $r(\lambda) \in \mathbb{Z}[\Sigma_1, \cdots, \Sigma_n, \Sigma'_n]$ pour tout $\lambda \in X^+(T^\vee)$.

Pour tout $\lambda \in X^+(T^\vee)$, ou bien $\lambda = t_1^{a_1} \cdots t_{n-1}^{a_{n-1}} t_n^{a_n}$ pour un unique $(a_1, \cdots, a_n) \in \mathbb{Z}^n$ avec $a_1 \geq \cdots \geq a_n \geq 0$, ou bien $\lambda = t_1^{a_1} \cdots t_{n-1}^{a_{n-1}} t_n^{-a_n}$ pour un unique $(a_1, \cdots, a_n) \in \mathbb{Z}^n$ avec $a_1 \geq \cdots \geq a_n > 0$ (§ A.3) ; quel que soit le cas :

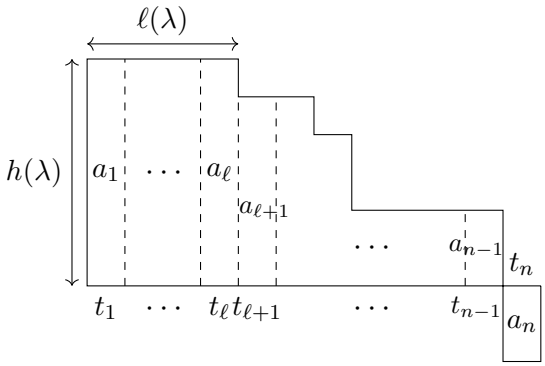
- on définit $h(\lambda) = a_1 \in \mathbb{N}$;
- on définit $\ell(\lambda) = \#\{i \in \{1, \cdots, n\} : a_i = h(\lambda)\} \in \mathbb{N}^*$; $\ell(\lambda)$ est donc l'unique élément de $\{1, \cdots, n\}$ vérifiant $a_1 = \cdots = a_{\ell(\lambda)} > a_{\ell(\lambda)+1} \geq \cdots$;
- on définit aussi $d(\lambda) = \displaystyle\sum_{i \in \{\ell(\lambda)+1, \cdots, n\},\, a_i \geq 1} (a_i - 1) \in \mathbb{N}$.

Il convient d'illustrer $h(\lambda)$ et $\ell(\lambda)$ par le dessin du "bâtiment de λ" :

$$\boxed{\lambda = t_1^{a_1} \cdots t_m^{a_m}}$$
$$(a_1 = \cdots = a_\ell > a_{\ell+1} \geq \cdots \geq a_m \geq 0)$$



$$\boxed{\lambda = t_1^{a_1} \cdots t_{n-1}^{a_{n-1}} t_n^{-a_n}}$$
$$(a_1 = \cdots = a_\ell > a_{\ell+1} \geq \cdots \geq a_n > 0)$$



On revient à démontrer que $r(\lambda) \in \mathbb{Z}[\Sigma_1, \cdots, \Sigma_n, \Sigma'_n]$ pour tout $\lambda \in X^+(T^\vee)$. La preuve se fait par récurrence sur $h(\lambda) \geq 0$ comme suivant. Tout d'abord, si $h(\lambda) = 0$ alors $r(\lambda) = 1 \in \mathbb{Z}[\Sigma_1, \cdots, \Sigma_n, \Sigma'_n]$ sans problème. Dans ce qui suit, on utilisera la notation suivante de § 3.16(b) : pour tous $\lambda_1, \lambda_2 \in X^+(T^\vee)$, on pose

$$\Omega(\lambda_1, \lambda_2) = (W\lambda_1 + W\lambda_2) \cap X^+(T^\vee).$$

Soit $\lambda \in X^+(T^\vee)$ avec $h(\lambda) = h > 0$. On va montrer que $r(\lambda) \in \mathbb{Z}[\Sigma_1, \cdots, \Sigma_n, \Sigma'_n]$ sous l'hypothèse que pour tout $\mu \in X(T^\vee)$ avec $h(\mu) < h$ on ait $r(\mu) \in \mathbb{Z}[\Sigma_1, \cdots, \Sigma_n, \Sigma'_n]$.

(1) Supposons que $\ell(\lambda) = 1$. On sépare la discussion en deux cas suivants.

1a/ Le cas de $\lambda = t_1^h t_2^{a_2} \cdots t_m^{a_m}$ avec $1 \leq m \leq n$ et avec $h > a_2 \geq \cdots \geq a_m > 0$. On considère $\lambda' = \lambda/(t_1 \cdots t_m) \in X^+(T^\vee)$ et développe le produit $r(\lambda') \cdot \Sigma_1$ via § 3.16(b) :

$$r(\lambda') \cdot \Sigma_1 = r(\lambda) + \sum_{\mu \in \Omega,\, \mu \neq \lambda} c_\mu \cdot r(\mu) \in \mathbb{Z}[\Sigma_1, \cdots, \Sigma_n, \Sigma'_n]$$

avec $\Omega = \Omega(\lambda', t_1) \subset X^+(T^\vee)$ et avec les $c_\mu \in \mathbb{N}^*$. On peut vérifier que $h(\mu) < h$ pour tout $\mu \in \Omega$; l'hypothèse de récurrence entraîne alors

$$r(\lambda) = r(\lambda') \cdot \Sigma_1 - \sum_{\mu \in \Omega,\, \mu \neq \lambda} c_\mu \cdot r(\mu) \in \mathbb{Z}[\Sigma_1, \cdots, \Sigma_n, \Sigma'_n].$$

1b/ Le cas de $\lambda = t_1^h t_2^{a_2} \cdots t_{n-1}^{a_{n-1}} t_n^{-a_n}$ avec $h > a_2 \geq \cdots \geq a_{n-1} \geq a_n > 0$. Cette fois on pose $\lambda' = \lambda/(t_1 \cdots t_{n-1} t_n^{-1}) \in X^+(T^\vee)$ et développe le produit $r(\lambda') \cdot \Sigma'_n$ via § 3.16(b). Pour $\Omega = \Omega(\lambda', t_1 \cdots t_{n-1} t_n^{-1}) \subset X^+(T^\vee)$, on a $r(\lambda') \cdot \Sigma'_n = r(\lambda) + \sum_{\mu \in \Omega,\, \mu \neq \lambda} c_\mu \cdot r(\mu)$ avec les $c_\mu \in \mathbb{N}^*$; on peut en plus vérifier que :

- pour tout $\mu \in \Omega$, on a toujours $h(\mu) \leq h$;
- si $\mu \in \Omega$ avec $h(\mu) = h$, alors : soit μ entre dans le cas 1a/ (de sorte que $\mu = t_1^h t_2^{b_2} \cdots t_n^{b_n}$ avec $h > b_2 \geq \cdots \geq b_n \geq 0$), soit μ reste dans ce cas ($\mu = t_1^h t_2^{b_2} \cdots t_{n-1}^{b_{n-1}} t_n^{-b_n}$, $h > b_2 \geq \cdots \geq b_n > 0$) avec $d(\mu) < d(\lambda)$.

Avec une récurrence supplémentaire sur $d(\lambda) \geq 0$, on peut raisonner comme en 1a/ et donc déduire que $r(\lambda) \in \mathbb{Z}[\Sigma_1, \cdots, \Sigma_n, \Sigma'_n]$.

(2) Supposons maintenant que $\ell(\lambda) = \ell > 0$. On va montrer que $r(\lambda)$ appartient à $\mathbb{Z}[\Sigma_1, \cdots, \Sigma_n, \Sigma'_n]$ sous l'hypothèse supplémentaire que $r(\mu) \in \mathbb{Z}[\Sigma_1, \cdots, \Sigma_n, \Sigma'_n]$ lorsque $\mu \in X(T)$ vérifie $h(\mu) = h$ et $\ell(\mu) < \ell$.

2a/ Le cas de $\lambda = t_1^{a_1} \cdots t_m^{a_m}$ avec $1 \leq m \leq n$ et avec $h = a_1 = \cdots = a_\ell > a_{\ell+1} \geq \cdots \geq a_m > 0$. Ici, on pose $\lambda' = \lambda/(t_1 \cdots t_\ell) \in X^+(T^\vee)$, développe le produit $r(\lambda') \cdot \Sigma_\ell$ et montrera comme en 1a/ que $r(\lambda) \in \mathbb{Z}[\Sigma_1, \cdots, \Sigma_n, \Sigma'_n]$.

2b/ Le cas de $\lambda = t_1^{a_1} \cdots t_{n-1}^{a_{n-1}} t_n^{-a_n}$ avec $h = a_1 = \cdots = a_\ell > a_{\ell+1} \geq \cdots \geq a_n > 0$. Cette fois on pose $\lambda' = \lambda/(t_1 \cdots t_{n-1} t_n^{-1})$ et l'on développe le produit $r(\lambda') \cdot \Sigma'_n$; comme en 1b/, il faudra faire une récurrence supplémentaire sur $d(\lambda)$, mais tout ira bien de reste et l'on aura $r(\lambda) \in \mathbb{Z}[\Sigma_1, \cdots, \Sigma_n, \Sigma'_n]$.

D'après (1) et (2) ci-dessus (il s'agissait d'une récurrence sur $\ell(\lambda) \geq 0$), on a donc $r(\lambda) \in \mathbb{Z}[\Sigma_1, \cdots, \Sigma_n, \Sigma'_n]$ lorsque $h(\lambda) = h$. La récurrence sur h est donc complétée. □

A.5. Lemme. — *Soit l'anneau $\mathsf{B}_{G^\vee} = \mathsf{B}_{\mathrm{SO}_{2n}} = \mathbb{Z}[X(T^\vee)]^W/I$ défini comme en § 3.3 ; en idéal I on utilisera la notation $F^\vee$, et dans notre cas nous avons $F^\vee = (\cdot)^q$ sur $T^\vee$. Considérons les trois parties disjointes suivantes de $\mathsf{B}_{G^\vee}$:*

- *S_1 se compose des $r(t_1^{a_1} \cdots t_n^{a_n}) + I \in \mathsf{B}_{G^\vee}$ vérifiant les conditions suivantes : $a_1 \geq \cdots \geq a_n \geq 0$, $a_i - a_{i+1} < q$ $(1 \leq i \leq n-1)$ et $a_n < q$;*
- *S_2 se compose des $r(t_1^{a_1} \cdots t_{n-1}^{a_{n-1}} t_n^{-a_n}) + I \in \mathsf{B}_{G^\vee}$ vérifiant les conditions suivantes : $a_1 \geq \cdots \geq a_n > 0$, $a_i - a_{i+1} < q$ $(1 \leq i \leq n-2)$, $a_{n-1} < q$ et $a_n < q$;*
- *S'_2 se compose des $r(t_1^{a_1} \cdots t_{n-1}^{a_{n-1}} t_n^{-a_n}) + I \in \mathsf{B}_{G^\vee}$ vérifiant les conditions suivantes : $a_1 \geq \cdots \geq a_n > 0$, $a_i - a_{i+1} < q$ $(1 \leq i \leq n-2)$ et $a_n < a_{n-1} - q < q$.*

Alors $S := S_1 \cup S_2 \cup S'_2$ est une famille génératrice du $\mathbb{Z}$-module $\mathsf{B}_{G^\vee}$.

Preuve. (Comparer l'étape (1) de la preuve de § 3.17.) Pour $\lambda \in X^+(T^\vee)$, en plus des quantités $h(\lambda)$, $\ell(\lambda)$ et $d(\lambda)$ introduites en § A.4, on définit deux nouvelles quantités $v_i(\lambda)$ et $\Delta_h(\lambda)$ ici. Pour $\lambda \in X^+(T^\vee)$, soit on a $\lambda = t_1^{a_1} \cdots t_n^{a_n}$ avec $a_1 \geq \cdots \geq a_n \geq 0$, soit on a $\lambda = t_1^{a_1} \cdots t_{n-1}^{a_{n-1}} t_n^{-a_n}$ avec $a_1 \geq \cdots \geq a_{n-1} \geq a_n > 0$ (§ A.3) ;

- dans les deux cas de λ, on pose $v_i(\lambda) = a_i - a_{i+1} \in \mathbb{N}$ pour $1 \leq i \leq n-2$ et $v_n(\lambda) = a_n \in \mathbb{N}$; dans le premier cas où $\lambda = t_1^{a_1} \cdots t_n^{a_n}$, on pose $v_{n-1}(\lambda) = a_{n-1} - a_n \in \mathbb{N}$; dans le deuxième cas où $r\lambda = t_1^{a_1} \cdots t_{n-1}^{a_{n-1}} t_n^{-a_n}$, on pose $v_{n-1}(\lambda) = a_{n-1} \in \mathbb{N}$;
- dans les deux cas de λ, pour $h \in \mathbb{N}$ on pose $\Delta_h(\lambda) = \displaystyle\sum_{1 \leq i \leq n,\, h < a_i < h(\lambda)} (a_i - h) \in \mathbb{N}$.

  On remarque que $\Delta_h(\lambda) = \displaystyle\sum_{\ell(\lambda) < i \leq n,\, a_i > h} (a_i - h)$ et que $d(\lambda) = \Delta_1(\lambda)$.

Comme $\{r(\lambda) : \lambda \in X^+(T^\vee)\}$ est une base $\mathbb{Z}$-linéaire de $\mathbb{Z}[X(T^\vee)]^W$, il s'agit de montrer que l'élément $r(\lambda) + I \in \mathsf{B}_{G^\vee}$ est une combinaison $\mathbb{Z}$-linéaire des éléments de S pour tout $\lambda \in X^+(T^\vee)$. Dans ce qui suit, on notera $\Sigma_i^{(q)} = r(t_1^q \cdots t_i^q)$ et $\Sigma_n'^{(q)} = r(t_1^q \cdots t_{n-1}^q t_n^{-q})$.

Soit donc $\lambda = t_1^{a_1} \cdots t_{n-1}^{a_{n-1}} t_n^{\pm a_n} \in X^+(T^\vee)$ avec $a_1 \geq \cdots \geq a_n \geq 0$. Considérons les réductions suivantes.

(1) On réduit tout d'abord $v_n(\lambda)$. Supposons que $v_n(\lambda) \geq q$.

1a/ Dans le cas où $\lambda = t_1^{a_1} \cdots t_n^{a_n}$ avec $a_1 \geq \cdots \geq a_n \geq 0$, posons $\lambda' = \lambda/(t_1 \cdots t_n)^q = t_1^{a_1 - q} \cdots t_n^{a_n - q} \in X^+(T^\vee)$ et $h = q$. On developpe les produits $r(\lambda')\Sigma_n^{(q)}$ et $r(\lambda')\Sigma_n$ dans

$\mathbb{Z}[X(T^\vee)]^W$ en combinaison $\mathbb{Z}$-linéaire des éléments de la base $\{r(\mu) : \mu \in X^+(T^\vee)\}$ (comme en § A.4), puis on les égale dans $\mathsf{B}_{G^\vee}$ via la relation $r(\lambda')\Sigma_n^{(q)} + I = r(\lambda')\Sigma_n + I$, et l'on obtiendra une réduction $r(\lambda) + I = \left(\sum_{\mu\in\Omega} c_\mu r(\mu)\right) + I$ où :

- $\Omega \subset X^+(T^\vee)$, $\lambda \notin \Omega$, $0 \neq c_\mu \in \mathbb{Z}$ $(\mu \in \Omega)$;
- pour tout $\mu \in \Omega$ tel que $v_n(\mu) \geq q$, on a ou bien $h(\mu) < h(\lambda)$, ou bien $h(\mu) = h(\lambda)$ mais $\ell(\mu) < \ell(\lambda)$, ou bien $h(\mu) = h(\lambda)$ et $\ell(\mu) = \ell(\lambda)$ mais $\Delta_h(\mu) < \Delta_h(\lambda)$.

Par récurrence sur $h(\lambda)$ (et sur $\ell(\lambda)$ ainsi que sur $\Delta_h(\lambda)$ le cas échéant), on pourra réduire $r(\lambda) + I$ dans $\mathsf{B}_{G^\vee}$ jusqu'à ce que l'on puisse supposer que $v_n(\lambda) < q$ (si λ reste dans ce cas).

1b/ Dans le cas où $\lambda = t_1^{a_1} \cdots t_{n-1}^{a_{n-1}} t_n^{-a_n}$ avec $a_1 \geq \cdots \geq a_n > 0$, on peut employer la même réduction en 1a/ en y remplaçant λ' par $\lambda/(t_1 \cdots t_{n-1} t_n^{-1})^q$ et Σ_n (resp. $\Sigma_n^{(q)}$) par Σ'_n (resp. $\Sigma_n'^{(q)}$), on réduira donc l'élément $r(\lambda) + I \in \mathsf{B}_{G^\vee}$ de manière que l'on pourra supposer que $v_n(\lambda) < q$.

(2) On réduit ensuite $v_{n-1}(\lambda)$. Supposons que $v_{n-1}(\lambda) \geq q$.

2a/ Dans le cas où $\lambda = t_1^{a_1} \cdots t_n^{a_n}$ avec $a_1 \geq \cdots \geq a_n \geq 0$, on fera à nouveau la réduction en 1a/ en y remplaçant λ' par $\lambda/(t_1 \cdots t_{n-1})^q$, $h = q$ par $h = a_{n-1}$ et Σ_n (resp. $\Sigma_n^{(q)}$) par Σ_{n-1} (resp. $\Sigma_{n-1}^{(q)}$). On réduira alors $r(\lambda) + I \in \mathsf{B}_{G^\vee}$ de sorte que l'on pourra supposer que $v_{n-1}(\lambda) < q$.

2b/ Le cas de $\lambda = t_1^{a_1} \cdots t_{n-1}^{a_{n-1}} t_n^{-a_n}$ avec $a_1 \geq \cdots \geq a_n > 0$ est plus subtil. Dans ce cas, on réduira $r(\lambda)+I$ comme en 2a/ mais on ne pourra pas baisser $v_{n-1}(\lambda)$ strictement au-dessous de q : par exemple, pour $\lambda = t_1^{q+2} \cdots t_{n-1}^{q+2} t_n^{-1}$ on a $r(t_1^2 \cdots t_{n-1}^2 t_n^{-1})\Sigma_{n-1}^{(q)} = r(\lambda) + r(t_1^{q+2} \cdots t_{n-2}^{q+2} t_{n-1}^{q+1} t_n^{-2}) +$ (les autres termes réduits). Cependant, en effectuant la même réduction en 2a/ et en utilisant les réductions faites en (1) lorsque l'on rencontre le cas de $v_n(\lambda) \geq q$, on parviendra à pouvoir supposer que $v_{n-1}(\lambda) < 2q$, que $v_n(\lambda) < q$ et que $v_{n-1}(\lambda) - q > v_n(\lambda)$ (si λ reste dans ce cas).

(3) On réduit enfin les $v_i(\lambda)$ pour $1 \leq i \leq n-2$. Les réductions se feront comme en 1a/, sauf que l'on devra y remplacer λ' par $\lambda/(t_1 \cdots t_i)^q$, $h = q$ par $h = a_i$ et Σ_n (resp. $\Sigma_n^{(q)}$) par Σ_i (resp. $\Sigma_i^{(q)}$). On pourra ainsi réduire $r(\lambda) + I$ par ordre décroissant des indices i (donc commencer par $i = n-2$ et puis $i = n-3, n-4$, etc.) jusqu'à ce que l'on ait $v_i(\lambda) < q$ pour tout $n-2 \geq i \geq 1$.

La preuve du lemme est donc complétée par les réductions (1), (2) et (3). □

A.6. Corollaire (de § A.5). — *On a*

$$\dim_{\overline{\mathbb{Q}}} \overline{\mathbb{Q}}\mathsf{B}_{G^\vee} \leq |S| = |S_1| + |S_2| + |S'_2| = 2q^n - 2q^{n-1} + q^{n-2}.$$

(Notons que $|S_1| = q^n$, $|S_2| = q^{n-2} \cdot \binom{q-1}{2}$ et $|S'_2| = q^{n-2} \cdot (q - 1 + \binom{q-1}{2})$.)

A.7. Lemme. — *On a* $|(T^\vee /\!\!/ W)^{(\cdot)^q}| = 2q^n - 2q^{n-1} + q^{n-2}$.

Preuve. Comme $|(T^\vee /\!\!/ W)^{(\cdot)^q}| = |(T^*/W)^{(\cdot)^q}|$ (§ 3.6), il suffit de montrer que $|(T^*(\overline{\mathbb{F}}_q)/W)^{(\cdot)^q}| = 2q^n - 2q^{n-1} + q^{n-2}$. Observons de § A.1 que

$$T^*(\overline{\mathbb{F}}_q) = \{\mathrm{diag}(z_1, \cdots, z_n, z_n^{-1}, \cdots, z_1^{-1}) : z_1, \cdots, z_n \in \overline{\mathbb{F}}_q^\times\}.$$

Soit D l'ensemble des matrices diagonales de taille $n \times n$ et à coefficients dans $\overline{\mathbb{F}}_q$. Le groupe symmétrique $\mathfrak{S}_n$ agit sur D par permutation des coefficients sur le diagonal, puis on peut considérer $(D/\mathfrak{S}_n)^{(\cdot)^q} = \{\mathfrak{S}_n x \in D/\mathfrak{S}_n : x \in D,\ \mathfrak{S}_n x^q = \mathfrak{S}_n x\}$. Grâce au fait que le corps $\overline{\mathbb{F}}_q$ est de caractéristique p, l'application

$$\begin{array}{rccc} \pi : & (T^*(\overline{\mathbb{F}}_q)/W)^{(\cdot)^q} & \longrightarrow & (D/\mathfrak{S}_n)^{(\cdot)^q} \\ & \mathrm{diag}(z_1, \cdots, z_n, z_n^{-1}, \cdots, z_1^{-1}) & \mapsto & \mathrm{diag}(z_1 + z_1^{-1}, \cdots, z_n + z_n^{-1}) \end{array}$$

est bien définie (pour tout $z \in \overline{\mathbb{F}}_q$, on a $(z + z^{-1})^q = z^q + z^{-q}$). L'application π est clairement surjective, et l'application

$$\begin{array}{rccc} \rho : & (D/\mathfrak{S}_n)^{(\cdot)^q} & \longrightarrow & \mathfrak{P} := \{f \in \mathbb{F}_q[X] : f \text{ unitaire de degré } n \text{ en } X\} \\ & x & \mapsto & \text{le polynôme caractéristique (unitaire) de } x \end{array}$$

est une bijection. On a donc

$$|(T^*(\overline{\mathbb{F}}_q)/W)^{(\cdot)^q}| = \sum_{\mathfrak{S}_n x \in (D/\!\!/\mathfrak{S}_n)^F} |\pi^{-1}(x)| = \sum_{f \in \mathfrak{P}} |\pi^{-1}(\rho^{-1}(f))|.$$

Ensuite, pour $x = \mathrm{diag}(x_1, \cdots, x_n) \in D$, on peut vérifier que :

- si $x_i \notin \{\pm 2\}$ pour tout $1 \leq i \leq n$, alors $|\pi^{-1}(\mathfrak{S}_n x)| = 2$;
- si $x_i \in \{\pm 2\}$ pour un $1 \leq i \leq n$, alors $|\pi^{-1}(\mathfrak{S}_n x)| = 1$.

On en obtient :

$$\begin{aligned} |(T^*(\overline{\mathbb{F}}_q)/W)^{(\cdot)^q}| &= \sum_{f \in \mathfrak{P}} |\pi^{-1}\rho^{-1}(f))| = \sum_{f \in \mathfrak{P},\ f(2)f(-2) \neq 0} 2 + \sum_{f \in \mathfrak{P},\ f(2)f(-2) = 0} 1 \\ &= 2\#\mathfrak{P} - \#\{f \in \mathfrak{P} : f(2)f(-2) = 0\} \\ &= 2\#\mathfrak{P} - \#\{f \in \mathfrak{P} : f(2) = 0\} - \#\{f \in \mathfrak{P} : f(-2) = 0\} \\ &\quad + \#\{f \in \mathfrak{P} : f(2) = f(-2) = 0\} \\ &= 2q^n - q^{n-1} - q^{n-1} + q^{n-2} \\ &= 2q^n - 2q^{n-1} + q^{n-2}. \end{aligned}$$

□

A.8. Théorème. — *L'anneau $\mathsf{B}_{G^\vee}$ est réduit ; en tant qu'un $\mathbb{Z}$-module, $\mathsf{B}_{G^\vee}$ est libre de rang $2q^n - 2q^{n-1} + q^{n-2}$ et de base S définie comme en § A.5.*

Preuve. La preuve se poursuit comme les étapes (3) et (4) de la preuve de § 3.17, mais dans le cas présent on utilisera §§ A.5 - A.7 pour déterminer $\dim_{\overline{\mathbb{Q}}} \overline{\mathbb{Q}}\mathsf{B}_{G^\vee}$. □

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