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Stabilization and control of nonlinear partial differential equations

Mohamed Melek Jellouli

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Stabilisation et contrôle des équations
aux dérivées partielles non linéaires
*Stabilization and control of nonlinear
partial differential equations*

**Thèse de doctorat de l'université Paris-Saclay
et de l'université de Sousse**

École doctorale n°574 : Mathématiques Hadamard (EDMH)
Spécialité de doctorat : Mathématiques fondamentales
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Thèse préparée au LMV (Université Paris-Saclay, UVSQ, CNRS) sous la direction
de Luc ROBBIANO, professeur à l'université Paris-Saclay, et sous la co-direction
de Moez KHENISSI, professeur à l'université de Sousse

Thèse soutenue à Versailles, le 9 septembre 2022, par

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Titre : Stabilisation et contrôle des équations aux dérivées partielles non linéaires

Mots clés : Stabilisation, contrôle, EDP non linéaires, prolongement unique, équations dispersives.

Résumé : Cette thèse se décompose en trois parties basées sur des résultats de stabilisation et de contrôle des EDP non linéaires dispersives. Dans la première partie, nous considérons l'équation de KdV-BBM non-linéaire sur le tore avec un dissipateur localisé en espace. Nous montrons l'existence globale de la solution, ainsi que sa convergence en temps vers une fonction analytique. Cette propriété d'analyticité permet l'application de résultats de prolongement unique pour montrer que la fonction limite est une constante. Ensuite, nous en déduisons que l'approche que nous avons faite sur KdV-BBM est applicable pour l'équation de

BBM pour un certain type de dissipateur dépendant du temps. La première partie est donc un résultat de stabilisation par un dissipateur localisé en espace. La deuxième partie présente deux résultats de contrôlabilité en fréquences pour l'équation de BBM non linéaire sur le tore. Nous utilisons des contrôles trigonométriques prenant des valeurs dans un espace de dimension finie pour montrer que l'équation est approximativement contrôlable dans H^1 , et n'est pas exactement contrôlable dans H^s pour s compris entre 1 et 2. Dans la troisième partie, nous donnons un résultat de contrôlabilité approchée pour l'équation KP-I non linéaire sur le tore 2d. Nous utilisons le même type de contrôle trigonométrique considéré dans la seconde partie.

Title : Stabilization and control of nonlinear partial differential equations

Keywords : Stabilization, control, nonlinear PDE, unique continuation, dispersive equations.

Abstract : This thesis is divided into three parts based on results of stabilization and control of dispersive nonlinear PDEs. In the first part, we consider the nonlinear KdV-BBM equation on the torus with a localized damping. We show the global existence of the solution, as well as its convergence in time towards an analytical function. This property of analyticity allows the application of unique continuation results to show that the limit function is a constant. Then, we deduce that the approach we made on KdV-BBM is applicable for the BBM equation for a certain type of damping depending on time. The first part is therefore a result of sta-

bilization by a dissipator localized in space. The second part is through control terms localized in frequencies. We present two controllability results for the nonlinear BBM equation on the torus. We use trigonometric controls taking values in a finite dimensional space to show that the equation is approximately controllable in H^1 , and is not exactly controllable in H^s for s between 1 and 2. In the third part, we give an approximate controllability result for the nonlinear KP-I equation on the 2d torus. We use the same type of trigonometric control considered in the second part.

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Introduction

Plusieurs phénomènes dans la nature sont modélisés par des équations aux dérivées partielles (EDP). Parmi les questions importantes dans ce domaine :

Peut-on exercer sur un système d'EDP une force pour amener les trajectoires des solutions à un état souhaité ? Et peut-on contrôler le temps que prennent ces trajectoires pour atteindre l'état ?

Nous parlons ici de la stabilisation et du contrôle, et la réponse à ces questions nous permettent de mieux comprendre le comportement dynamique des systèmes d'EDP et d'avoir des prévisions. Parmi les exemples connus on peut citer l'équation des ondes

$$\partial_t^2 u - c^2 \Delta u = 0,$$

qui modélise la propagation d'une onde avec une vitesse c . En physique quantique l'équation de Schrödinger

$$i\partial_t \Psi + \Delta \Psi - V(t, x)\Psi = 0,$$

permet de trouver la probabilité pour qu'une particule dans un atome soit proche d'un certain état. En mécanique des fluides l'équation d'Euler

$$\partial_t u + \langle u, \nabla \rangle u + \nabla p = 0,$$

qui modélise l'écoulement d'un fluide parfait incompressible sous une pression intérieure p . L'équation de la chaleur

$$\partial_t u - \Delta u = f,$$

qui décrit la diffusion de la chaleur et l'évolution de la température en considérant une source f .

Nous intéressons dans la partie principale de cette thèse à l'équation de Benjamin-Bona-Mahony (BBM)

$$\partial_t u + \partial_x u - \partial_{xxt} u + u \partial_x u = 0.$$

Cette équation a été traitée en 1972 par Benjamin, Bona et Mahony dans [BBM72] comme une nouvelle version de l'équation de Korteweg-de Vries (KdV)

$$\partial_t u + \partial_{xxx} u + u \partial_x u = 0,$$

introduite par Korteweg et de Vries [KdV95] en 1895. Pour les détails sur les différences entre les équations de BBM et KdV nous renvoyons le lecteur à l'article d'Olver [Olv79].

Cette thèse se décompose de six chapitres. Dans ce premier chapitre nous donnons une introduction générale et des préliminaires sur les résultats principaux. Nous introduisons le problème que nous allons étudier dans chaque chapitre et nous présentons les travaux dans la littérature liés à notre cas, ainsi que la démarche détaillée qu'on considère dans les chapitres suivants.

Dans le deuxième chapitre, nous introduisons des outils mathématiques ainsi que le cadre fonctionnel dans lequel nous allons se placer. Nous définissons les espaces usuels et nous donnons des définitions et des résultats connus.

Dans le troisième chapitre nous présentons un résultat de stabilisation pour l'équation de KdV-BBM non-linéaire, où nous montrons l'existence globale de la solution, ainsi que sa convergence en temps vers une fonction analytique. Cette propriété d'analyticité permet d'appliquer des résultats de prolongement unique pour montrer que la fonction limite est une constante.

Nous étudions après la contrôlabilité de l'équation de BBM dans le quatrième chapitre. Plus précisément nous montrons qu'on peut contrôler approximativement dans H^1 l'équation en utilisant des termes de contrôles appartenant à des espaces de dimensions finis. Ensuite nous montrons la non contrôlabilité exacte de l'équation dans H^s lorsque $s \in [1, 2]$.

Dans le cinquième chapitre, nous appliquons la méthode de contrôle du quatrième chapitre à l'équation de Kadomtsev-Petviashvili I (KP-I) en dimension

2.

Dans le sixième chapitre nous démontrons quelques points techniques.

Définition 1 On dit qu'une EDP de la forme $\partial_t u + Au = 0$ est stable si la solution converge vers un état stationnaire quand t tend vers $+\infty$.

Définition 2 On dit qu'une EDP de la forme

$$\begin{cases} \partial_t u + Au = f \\ u(0) = u_0, \end{cases}$$

est exactement contrôlable en un temps T si pour toute fonction v , il existe f telle que la solution u vérifie $u(T) = v$. L'EDP est dite approximativement contrôlable en un temps T si pour toute fonction v et pour $\epsilon > 0$, il existe f telle que la solution u vérifie $\|u(T) - v\| \leq \epsilon$.

Le point de départ était la conjecture de Rosier [\[Ros16\]](#) suivante :

Conjecture de prolongement unique. Il existe une constante $\delta > 0$ telle que pour toute donnée initiale $v_0 \in H^1(\mathbb{T})$ vérifiant $\|v_0\|_{H^1(\mathbb{T})} < \delta$, si la solution v de

$$\begin{cases} \partial_t v + \partial_x v - \partial_{xxt} v + v \partial_x v = 0, & (t, x) \in]0, T[\times \mathbb{T} \\ v(\cdot, 0) = v_0, \end{cases} \quad (1.1)$$

vérifie $v(x, t) = 0$ pour tout $(x, t) \in \omega \times]0, T[$, où ω est un ouvert non vide de \mathbb{T} et $T > 0$, alors $v_0 = 0$, et ainsi, $v \equiv 0$.

Cette conjecture est utilisée pour montrer des résultats de stabilisation pour l'équation de BBM sur le tore en considérant différents types de dissipations. Pour montrer des résultats de stabilisation, nous n'avons pas besoin de démontrer cette conjecture pour toute solution v , mais seulement pour des solutions qui vérifient des propriétés supplémentaires permettant d'appliquer le théorème de Hale et Raugel [\[HR03\]](#).

Théorème 1 (Hale et Raugel) Soit Y un espace de Banach complexe. Soient $P_n \in \mathcal{L}(Y)$ une suite d'opérateurs continus linéaire et $Q_n = Id - P_n$. Notons par $A : \mathcal{D}(A) \rightarrow Y$ le générateur infinitésimal du semi-groupe $\{e^{tA}\}_{t \geq 0}$ et soit $G \in \mathcal{C}^1(Y)$. Supposons que V est une solution forte dans Y de l'équation

$$V'(t) = AV(t) + G(V(t)), \quad t \in \mathbb{R}.$$

On suppose que

1. $\{V(t); t \in \mathbb{R}\}$ est contenu dans un ensemble compact K de Y .
2. Pour tout $y \in Y$, $(P_n y)_n$ converge vers y quand $n \rightarrow +\infty$ et (P_n) et $(Q_n)_n$ sont des suites de $\mathcal{L}(Y)$ bornés par une constante K_0 .

3. L'opérateur A s'écrit $A = A_1 + B_1$ où B_1 est borné et A_1 commute avec P_n .
4. Il existe $M, c > 0$ tels que $\| \| e^{At} \| \|_{\mathcal{L}(Y)} \leq M e^{-ct}$ et $\| \| e^{(A_1 + Q_n B B^*)t} \| \|_{\mathcal{L}(Q_n Y, Y)} \leq M e^{-ct}$ pour tout $t \geq 0$.
5. G est analytique dans la boule $B_Y(0, r)$, où r vérifie $r \geq 4 \sup_{t \in \mathbb{R}} \|V(t)\|_Y$.
6. $\{DG(V(t))Z ; t \in \mathbb{R}, \|Z\|_Y \leq 1\}$ est relativement compact dans Y .

Alors la solution $V(t)$ est analytique en t .

Rappelons la définition d'une fonction analytique dans le cadre d'un espace de Banach.

Définition 3 Une fonction u est dite analytique sur un espace de Banach X si elle est développable en série entière en tout point de son domaine de définition. D'une manière équivalente, elle est analytique si elle est de classe \mathcal{C}^∞ et il existe $c > 0$ et $\sigma > 0$ tels que pour tout $\alpha \geq 0$,

$$\| \partial_t^\alpha u(\cdot, t) \|_X \leq c \alpha! \sigma^\alpha.$$

Le théorème de Hale et Raugel donne l'analyticité de la solution par rapport au temps. Ensuite, l'idée était de le combiner avec un autre théorème qui donne un résultat de prolongement unique à partir de l'analyticité. Le cas de l'équation de BBM était assez problématique. En effet, le comportement de la solution de l'équation de BBM ne permet pas d'avoir une décroissance exponentielle évidente de l'énergie quand nous considérons -comme dans le cas du papier de Rosier [Ros16]- un dissipateur localisé en espace. Plus précisément, la solution de l'équation de BBM n'a pas un caractère de transport, ce qui fait que lorsque le dissipateur est localisé en espace, il n'est pas évident de démontrer la décroissance exponentielle de l'énergie, qui elle, est une hypothèse dans le théorème de Hale et Raugel. Ce comportement peut être vu directement sur l'équation. En effet, l'équation

$$\partial_t v + \partial_x v - \partial_{xxt} v + v \partial_x v = 0,$$

est équivalente à

$$\partial_t v = Av,$$

avec $Av = -(1 - \partial_{xx})^{-1}(\partial_x v + v \partial_x v)$. Nous voyons ici que l'opérateur A est régularisant d'ordre 1, c'est-à-dire que si $v \in H^s(\mathbb{T})$ alors $Av \in H^{s+1}(\mathbb{T})$. En écrivant formellement

$$v(t, x) = v_0(x) + \int_0^t Av(s, x) ds, \quad (1.2)$$

nous pouvons voir que si $v_0 \in H^1(\mathbb{T})$ par exemple, et φ une fonction de troncature régulière vérifiant $\text{supp } v_0 \cap \text{supp } \varphi = \emptyset$, alors

$$\varphi v(t, x) = \int_0^t \varphi Av(s, x) ds \in H^2(\mathbb{T}).$$

Donc la fonction v n'a pas un caractère de transport car la partie peu régulière vient purement de v_0 . Autrement dit, la régularité de v à l'instant t va être la même que celle de v_0 puisque l'écriture (1.2) décompose l'expression de v à un instant t en une partie irrégulière $v_0(x)$ et une partie régulière $\int_0^t Av(s, x) ds$, par suite les singularités H^1 sont portées seulement par v_0 .

Nous avons décidé d'étudier le modèle de l'équation de KdV-BBM

$$\begin{cases} \partial_t u + \partial_x u - \partial_{xxt} u + \partial_{xxx} u - \partial_x(a(x)\partial_x u) + u\partial_x u = 0, & x \in \mathbb{T}, \\ u(., 0) = u_0 \in H^1(\mathbb{T}), & x \in \mathbb{T}, \end{cases} \quad (1.3)$$

où $a \geq 0$ est une fonction bornée de classe \mathcal{C}^∞ telle que $\{a > 0\} \neq \emptyset$. Nous verrons que les résultats sur KdV-BBM ont des conséquences sur l'équation BBM. Cette équation est une "combinaison" de l'équation de KdV:

$$\partial_t u + \partial_{xxx} u + u\partial_x u = 0,$$

avec l'équation de BBM:

$$\partial_t u + \partial_x u - \partial_{xxt} u + u\partial_x u = 0.$$

L'équation (1.3) est hyperbolique, en effet, en écrivant l'équation linéaire sous la forme

$$\partial_t u = Au, \quad (1.4)$$

où

$$A = -(1 - \partial_{xx})^{-1}(\partial_x + \partial_{xxx} - \partial_x(a(x)\partial_x)). \quad (1.5)$$

En appliquant la transformée de Fourier en x et en t à (1.4), la partie imaginaire du terme dominant est $\tau - (1 - \xi^2)^{-1}\xi^3$. Or

$$\tau - (1 - \xi^2)^{-1}\xi^3 = \tau - (1 - \xi^2)^{-1}(\xi(\xi^2 - 1 + 1)) = \tau + \xi - (1 - \xi^2)^{-1}\xi.$$

Ceci veut dire que le symbole principale est $\tau + \xi$. Autrement dit l'équation est de même type que l'équation de transport, d'où le caractère hyperbolique.

1.1 Première partie

Dans la littérature on peut trouver plusieurs modèles d'équations aux dérivées partielles décrivant le mouvement de l'eau dans des faibles profondeurs

suivant une propagation dans une seule direction, voir par exemple [BCPS18] et [CPS16]. En particulier l'équation de KdV-BBM sur le tore

$$\partial_t u + \partial_x u - \partial_{xxt} u + \partial_{xxx} u - \partial_x(a(x)\partial_x)u + u\partial_x u = 0, \quad (1.6)$$

où $a \geq 0$ est une fonction bornée de classe \mathcal{C}^∞ telle que $\{a > 0\} \neq \emptyset$.

À notre connaissance, l'équation de KdV-BBM a été traitée pour la première fois sous la forme d'un système dans [BCS02] par Bona, Chen et Saut. Cette équation a été récemment considérée dans [BNS17], [DP14] et [MR17] pour des résultats de simulations numériques. La stabilisation n'a pas été étudiée pour l'équation de KdV-BBM mais a été étudiée pour KdV dans [BCV19] par Baudouin et Crépeau, et dans [LRBY10] par Laurent, Rosier et Zhang. Il existe des résultats de stabilisations pour l'équation de BBM dans [AC19] par Ammari et Crépeau, et dans [Ros16] par Rosier.

Un résultat de prolongement unique pour l'équation de KdV-BBM peut être trouvé dans le papier de Rosier [Ros16] en utilisant des inégalités de Carleman. La particularité de l'équation KdV-BBM est qu'elle admet un opérateur non local et non borné d'ordre 1, ce qui introduit quelques difficultés dans l'analyse. Notre résultat principal est une propriété de stabilisation des solutions de (1.6) sur le tore.

Théorème 2 *Pour tout $u_0 \in H^1(\mathbb{T})$, il existe une unique solution $u = u(t, x)$ de (1.6) globale en temps telle que*

$$\lim_{t \rightarrow +\infty} u(t, \cdot) = \frac{1}{2\pi} \int_{\mathbb{T}} u_0(x) dx, \quad \text{dans } H^1(\mathbb{T}).$$

La vérification des hypothèses du théorème de Hale et Raugel fait l'objet des proposition [1], exprimant la compacité des trajectoires de la solution, ainsi que les théorèmes [3] et [8], exprimant deux différents types de décroissance exponentielles.

Donnons maintenant le plan de la preuve. Dans la Section [3.2] nous montrons l'existence de la solution du problème linéaire. Plus précisément nous utilisons le théorème de Lumer-Phillips et nous associons à l'opérateur de l'équation linéaire un groupe $\{S(t)\}_{t \in \mathbb{R}}$ construit à partir de deux semi-groupes $\{S_+(t)\}_{t \geq 0}$ et $\{S_-(t)\}_{t \leq 0}$, où le semi-groupe sur \mathbb{R}_+ est de contraction.

Dans la Section [3.3] nous obtenons un résultat de décroissance exponentielle pour le semi-groupe donné par le théorème suivant.

Théorème 3 *Il existe $\delta > 0$ et $M > 0$ tels que*

$$\|S(t)\|_{\mathcal{L}(H^1(\mathbb{T}))} \leq M e^{-\delta t}, \quad \forall t \geq 0. \quad (1.7)$$

Nous utilisons ici le résultat de Huang [\[Hua85\]](#).

Théorème 4 Soit H un espace de Hilbert, et $A : \mathcal{D}(A) \subset H \rightarrow H$ le générateur infinitésimal d'un semi-groupe $\{T(t)\}_{t \geq 0}$. Supposons que

1. Il existe $c_1 > 0$ tel que $\|T(t)\|_{\mathcal{L}(H)} \leq c_1, \forall t \geq 0$,
2. $A + i\mu$ est inversible pour tout μ dans \mathbb{R} ,
3. Il existe $c_2 > 0$ tel que $\|(A + i\mu)^{-1}\|_{\mathcal{L}(H)} \leq c_2$ pour tout μ dans \mathbb{R} .

Alors il existe $M > 0$ et $c > 0$ tels que

$$\|T(t)\|_{\mathcal{L}(H)} \leq Me^{-ct}, \forall t \geq 0. \quad (1.8)$$

Nous remarquons que le noyau de l'opérateur A donné par [\(1.5\)](#) contient les constantes, ce qui fait que la deuxième hypothèse du théorème de Huang n'est pas vérifiée quand $\mu = 0$. Nous se plaçons donc dans le sous-espace fermé de $H^1(\mathbb{T})$

$$\dot{H}^1(\mathbb{T}) = \left\{ u \in H^1(\mathbb{T}) ; (u, 1)_{L^2(\mathbb{T})} = 0 \right\},$$

muni de la norme $H^1(\mathbb{T})$. La troisième hypothèse du théorème de Huang consiste à estimer la norme de la résolvante avec une constante c_2 uniforme par rapport à μ . Pour démontrer cette estimation, nous raisonnons par l'absurde. Cela revient à construire une mesure semi-classique dont les différentes propriétés conduisent à une contradiction. Cette technique a été utilisée pour la stabilité de l'équation des ondes, voir [\[Bur02\]](#), [\[BG97\]](#), [\[GL93\]](#), [\[Gé91b\]](#), [\[Leb94\]](#), [\[Mil00\]](#) et [\[RZ09\]](#). La construction des mesures semi-classiques repose sur le théorème de Burq [\[Bur97\]](#) suivant.

Théorème 5 Soient $(h_k)_k$ une suite de nombres réels qui converge vers 0 et (v^{h_k}) une suite bornée dans $L^2(\mathbb{T})$ qui converge faiblement vers 0. Il existe une sous-suite de (v^{h_k}) -qu'on continue de la noter (v^{h_k}) - et une mesure de Radon positive ν définie sur $\mathbb{T} \times \mathbb{R}$ telle que pour tout $b \in C_0^\infty(\mathbb{T} \times \mathbb{R}; \mathbb{C})$

$$\lim_{h_k \rightarrow 0} (b(x, h_k \mathcal{D})v^{h_k}, v^{h_k})_{L^2(\mathbb{T})} = \int_{\mathbb{T} \times \mathbb{R}} b(x, \xi) d\nu(x, \xi).$$

ν est appelée mesure semi-classique associée à la suite (v^{h_k}) .

Dans la Section [3.4](#) à partir du théorème de point fixe de Picard, nous prouvons l'existence locale et l'unicité de la solution pour le problème non linéaire:

$$\begin{cases} \partial_t u + \partial_x u - \partial_{xxt} u + \partial_{xxx} u - \partial_x(a(x)\partial_x u) + u\partial_x u = 0, & x \in \mathbb{T}, \\ u(\cdot, 0) = u_0 \in H^1(\mathbb{T}). \end{cases} \quad (1.9)$$

Par une estimation a priori sur l'énergie, nous déduisons l'existence globale pour le problème non linéaire. Ainsi qu'une certaine continuité par rapport à la donnée initiale:

Théorème 6 *Il existe une unique solution de (1.9) définie sur tout \mathbb{R} . De plus, pour $T > 0$, l'application*

$$\begin{array}{ccc} H^1(\mathbb{T}) & \longrightarrow & \mathcal{C}\left([-T, T], \dot{H}^1(\mathbb{T})\right) \\ u_0 & \longmapsto & u, \end{array}$$

où u est la solution du problème non linéaire admettant u_0 comme donnée initiale, est Lipschitzienne.

Dans la Section 3.5 nous montrons que la solution converge vers une solution bornée pour tout t dans \mathbb{R} . Pour démontrer cela, nous prenons une suite croissante (t_n) de temps qui tend vers $+\infty$, et nous montrons que la suite $(u(t_n))_n$ est bornée dans $H^2(\mathbb{T})$. Donc elle converge faiblement -après extraction d'une sous suite- vers une fonction qu'on appelle u_∞ . Nous obtenons alors le résultat suivant.

Théorème 7 *Soient $(t_n)_n$ une suite croissante qui tend vers $+\infty$, u_0 une donnée initiale réelle dans $\dot{H}^1(\mathbb{T})$, et u est la solution correspondante de (1.9). Alors, il existe une sous-suite $(t_{\phi(n)})_n$ de $(t_n)_n$ et une fonction $u_\infty(\cdot) \in \mathcal{C}^0\left(\mathbb{R}, \dot{H}^1(\mathbb{T})\right)$ telles que*

$$\forall T > 0, \lim_{n \rightarrow +\infty} u(t_{\phi(n)} + \cdot) = u_\infty(\cdot) \text{ dans } \mathcal{C}^0\left([-T, T], \dot{H}^1(\mathbb{T})\right). \quad (1.10)$$

La fonction u_∞ admet des propriétés particulières qui permettent d'appliquer le théorème de Hale et Raugel contrairement à une solution quelconque du problème non linéaire. En effet, la proposition suivante montre que la solution u_∞ , qui elle, va jouer le rôle d'une donnée initiale dans la suite, est bornée sur \mathbb{R} . Ce qui permet de vérifier la première hypothèse du théorème de Hale et Raugel, car nous ne partons pas de n'importe quelle donnée initiale mais d'une donnée initiale qui est une limite de solutions.

Proposition 1 *Soit $u_\infty = u_\infty(0)$ la fonction obtenue par le Théorème 7 et $u_\infty(t)$ la solution correspondante. Il existe une constante $c > 0$ telle que*

$$\sup_{t \in \mathbb{R}} \|u_\infty(t)\|_{H^2(\mathbb{T})} \leq c. \quad (1.11)$$

L'hypothèse intéressante suivante est, dans un sens à préciser plus en détails, une décroissance exponentielle du semi-groupe linéaire sur les hautes fréquences.

Théorème 8 *Il existe $M > 0$ et $c > 0$ tels que pour tout $t \geq 0$ et $n \in \mathbb{N}$*

$$\| \| e^{(A_1 + Q_n B B^*)t} \| \|_{\mathcal{L}(Q_n H^1, H^1)} \leq M e^{-ct}. \quad (1.12)$$

Pour montrer cela, nous utilisons la décroissance exponentielle vue précédemment ainsi que le résultat de Haraux [Har89] qui énonce le théorème suivant. Rappelons le contexte du théorème, Soient les deux équations

$$\varphi'(t) + A_1\varphi(t) = 0, \quad (1.13)$$

et

$$y'(t) + A_1y(t) + BB^*y(t) = 0. \quad (1.14)$$

Théorème 9 *Les deux propriétés suivantes sont équivalentes*

1. Il existe $T_0 > 0$ et $c > 0$ tels que toute solution φ de (1.13) vérifie

$$\|\varphi(0)\|_H^2 \leq c \int_0^{T_0} \|B^*\varphi(s)\|_H^2 ds. \quad (1.15)$$

2. Il existe $T_1 > 0$ et $\delta > 0$ tels que toute solution y de (1.14) vérifie

$$\forall y_0 \in \mathcal{D}(A_1), \forall t \geq T_1, \|y(t)\|_H = \|S(t)y_0\|_H \leq e^{-\delta t} \|y_0\|_H, \quad (1.16)$$

où $\{S(t)\}_{t \geq 0}$ est le semi-groupe généré par $A_1 + BB^*$.

La raison de l'application du théorème 9 est la suivante, il est aisé de démontrer l'inégalité d'observabilité (1.15) associée à l'opérateur $A_1 + Q_n BB^*$ à partir de celle associée à l'opérateur $A_1 + BB^*$. Ensuite, nous montrons que la décroissance exponentielle associée au semi-groupe de l'opérateur $A_1 + Q_n BB^*$ est équivalente à l'inégalité d'observabilité pour $A_1 + BB^*$.

Les autres hypothèses du théorème de Hale et Raugel sont plus faciles à vérifier. Cette démarche pour montrer l'analyticité a été utilisée dans l'article de Joly et Laurent [JL13] qui l'on appliqué à l'équation des ondes. L'intérêt d'avoir une solution analytique est qu'elle permet d'appliquer le théorème de prolongement unique de Tataru [Tat95], voir aussi d'autres théorèmes dans l'article d'Hörmander [Hö97] et le papier de Robbiano et Zuily [RZ98].

Le prolongement unique consiste à démontrer qu'une solution nulle d'un côté d'une surface est nulle localement de l'autre côté. Les résultats de prolongement unique reposent sur la notion de pseudo-convexité des surfaces. Le théorème de Tataru est un résultat intermédiaire entre les théorèmes d'Hörmander et celui d'Holmgren. Plus précisément, dans le théorème d'Hörmander, les coefficients de l'opérateur sont supposés de classe \mathcal{C}^1 et doivent vérifier les conditions de la pseudo-convexité sur un certain ensemble. Holmgren suppose que les coefficients sont analytiques. Tataru donne un résultat intermédiaire entre les deux dans le sens où si le symbole principale de l'opérateur est de classe \mathcal{C}^1 et les coefficients d'ordre inférieurs sont analytiques en temps, alors nous

pouvons affaiblir les hypothèses d'Hörmander et vérifier les conditions de la pseudo-convexité sur un ensemble plus petit que celui d'Hörmander. Autrement dit, Tataru et Hörmander utilisent tous les deux une notion de pseudo-convexité mais la définition de Tataru est posée sur un ensemble strictement plus petit que celui considéré par Hörmander. Dans le résultat d'Holmgren, les coefficients sont supposés analytiques et les surfaces sont supposées non caractéristiques. Cette technique de déformation de surfaces dans le but d'avoir des résultats de prolongement unique peut être trouvée (avant Tataru et Hörmander) dans John [\[Joh49\]](#).

L'idée de la démonstration dans ce cas consiste à montrer d'abord que la fonction u_∞ est une constante (en temps et en espace) sur le support de a . Puis d'utiliser un argument de déformation de surfaces pour montrer que u_∞ est une constante sur une partie de la forme $]x^*, x^* + 2\pi[\times \{t = 0\}$, avec $x^* \in \mathbb{T}$. Ceci implique que la donnée initiale (qu'on peut noter $u_\infty(0)$) est égale à une constante c^* sur un intervalle en x de longueur 2π . Donc, en regardant l'équation vérifiée par $v := u_\infty - c^*$ nous trouvons que $v(0) = 0$ partout, et par unicité de la solution, $v(t) = 0$ partout et ainsi, u_∞ est une constante pour tout $(x, t) \in \mathbb{T} \times \mathbb{R}$.

À noter que le théorème standard d'Hörmander [\[Hö85\]](#) ne s'applique pas dans notre cas. La démonstration de l'analyticité en temps de la solution est donc cruciale pour l'utilisation du résultat de Tataru et cela conduit au résultat de stabilisation.

Dans la Section [3.7](#) nous déduisons que l'approche faite sur l'équation de KdV-BBM est applicable pour l'équation de BBM dans un cas particulier. En effet, la conjecture de prolongement unique que Rosier a énoncé dans [\[Ros16\]](#) est vraie avec un dissipateur a dépendant du temps et de la forme $a(x + ct)$ avec $c > 0$.

Nous considérons l'équation de BBM suivante

$$\begin{cases} \partial_t v + \partial_x v - \partial_{xxt} v - \partial_x(a(x + ct)\partial_x)v + v\partial_x v = 0, & x \in \mathbb{T}, t > 0, \\ v(\cdot, 0) = v_0, & x \in \mathbb{T}. \end{cases} \quad (1.17)$$

Nous montrons que modulo un changement de variable, l'étude faite sur l'équation de KdV-BBM reste valable pour l'équation [\(1.17\)](#). Ce type de dissipateur a été utilisé dans [\[CC18\]](#) pour montrer des résultats de contrôlabilité de l'équation de Boussinesq. Selon nos connaissances, le taux de décroissance de l'énergie de la solution du problème non linéaire de KdV-BBM est encore une question ouverte.

1.2 Deuxième partie

Dans le quatrième chapitre, nous étudions la contrôlabilité approchée et la non-contrôlabilité exacte de l'équation de BBM non-linéaire

$$\partial_t u + \partial_x u - \partial_{xxt} u + u \partial_x u = 0,$$

introduite par [BBM72], pour décrire un phénomène de propagation d'ondes longues unidirectionnelles de faibles amplitudes dans le cas dispersif non linéaire. Il a été prouvé dans [BT09] et [Ros16] que l'équation de BBM est bien posée lorsque la donnée initiale u_0 appartient à $H^s(\mathbb{T})$ avec $s \geq 0$, et avant cela, dans [BBM72] quand $s \in \mathbb{N}^*$. De plus, l'application $u_0 \mapsto u(t)$ est analytique. Nous savons aussi que l'équation est mal posée dans le cas $s < 0$ dans des sens différents (voir [BT09] et [Pan10] pour un énoncé précis). Dans cette partie nous intéressons à la contrôlabilité de l'équation de BBM sur le tore

$$\begin{cases} \partial_t u + \partial_x u - \partial_{xxt} u + u \partial_x u = \eta(t, x), & (t, x) \in]0, T[\times \mathbb{T}. \\ u(\cdot, 0) = u_0. \end{cases} \quad (1.18)$$

Micu [Mic01] a étudié la contrôlabilité de l'équation de BBM linéarisée sur $[0, 1]$ avec un terme de contrôle sur le bord. Il a montré que l'équation n'est pas spectralement contrôlable mais approximativement contrôlable avec un contrôle dans $L^2(0, T)$. Il a également montré que pour tout $N > 0$ et $T > 0$, il existe un contrôle tel que la projection de la solution sur l'espace de dimension finie généré par les $2N$ premières fonctions propres de l'opérateur de BBM est égale à zéro à l'instant $t = T$. Rosier et Zhang [Ros16] donnent un résultat contrôlabilité exacte pour l'équation de BBM avec un contrôle mobile pour un temps grand, ainsi que des estimations du terme de contrôle en fonction des données initiales et des données finales. Ils considèrent d'abord l'équation linéaire, puis ils déduisent le résultat sur l'équation non linéaire à l'aide d'un argument de point fixe. Zhang et Zuazua ont montré la contrôlabilité approximative de l'équation de BBM avec un terme de contrôle localisé, voir la Section 7 dans [ZZ03]. On peut aussi trouver la contrôlabilité approchée de l'équation de BBM en 1-d avec retard par Leiva [Lei17].

L'équation (1.18) préserve les constantes, nous introduisons donc pour $s \geq 0$ le sous-espace fermé de $H^s(\mathbb{T})$

$$\dot{H}^s(\mathbb{T}) = \left\{ u \in H^s(\mathbb{T}) ; (u, 1)_{L^2(\mathbb{T})} = 0 \right\},$$

muni de la norme $H^s(\mathbb{T})$. Nous appliquerons une méthode basée sur l'utilisation du terme non linéaire. Le type de contrôle que nous considérons est celui des fonctions trigonométriques. L'intérêt d'un tel choix de contrôle est que le terme non linéaire $u \partial_x u$ laisse les espaces engendrés par les cosinus et les sinus invariants. Nous partons d'une donnée initiale dans $\dot{H}^1(\mathbb{T})$ et d'un contrôle qui

appartient à un espace de dimension finie donné pour $I \subset \mathbb{Z}^*$ par

$$\mathcal{H}(I) = \text{vect}\{\cos(kx), \sin(kx) ; k \in I\}. \quad (1.19)$$

Définition 4 On dit que l'équation (1.18) est approximativement contrôlable dans $\dot{H}^1(\mathbb{T})$ par des valeurs dans $\mathcal{H}(I)$ si pour tout $T > 0$, $\epsilon > 0$ et tout $u_0, v \in \dot{H}^1(\mathbb{T})$, il existe un contrôle constant par morceaux à valeurs dans $\mathcal{H}(I)$ et une solution u de (1.18) telle que $\|u(T) - v\|_{H^1} \leq \epsilon$.

Définition 5 On dit que l'équation (1.18) est exactement contrôlable dans $\dot{H}^s(\mathbb{T})$ par des valeurs dans $\mathcal{H}(I)$ si pour tout $T > 0$ et tout $u_0, v \in \dot{H}^s(\mathbb{T})$, il existe un contrôle constant par morceaux à valeurs dans $\mathcal{H}(I)$ et une solution u de (1.18) telle que $u(T) = v$.

Théorème 10 L'équation (1.18) est approximativement contrôlable dans $\dot{H}^1(\mathbb{T})$ par des contrôles constants par morceaux à valeurs dans $\mathcal{H}(I)$ si et seulement si I est un générateur de \mathbb{Z} .

Théorème 11 L'équation (1.18) n'est pas exactement contrôlable dans $\dot{H}^s(\mathbb{T})$ par des contrôles constants par morceaux à valeurs dans $\mathcal{H}(I)$ lorsque $s \in [1, 2[$.

Le théorème 11 est donc un résultat qui dit que nous ne pouvons pas faire mieux que le théorème 10, dans le sens où, avec des contrôles de dimensions finie, l'équation ne peut être que contrôlable approximativement. L'idée de la preuve du théorème 10 est de commencer par considérer un contrôle dans $\mathcal{H}(I_0)$, et de montrer que la solution atteint un espace $\mathcal{H}(I_1)$ dans un temps t , où $I_1 \not\supseteq I_0$. Puis nous recommençons à partir de I_1 ce qui permet de construire un I_2 de la même façon, et ainsi de suite. Le but est de montrer que l'union de $\mathcal{H}(I_n)$ est dense en $\dot{H}^1(\mathbb{T})$, ce qui revient à montrer que l'union des I_n est \mathbb{Z} . La suite d'ensemble (I_n) joue le rôle des nouveaux modes de Fourier que l'on obtient après chaque étape. Il faut donc que l'inclusion de I_n dans I_{n+1} soit stricte, sinon la suite (I_n) sera stationnaire à un certain rang n et dans ce cas, l'espace atteignable serait de dimension finie.

Une remarque intéressante est de voir que pour atteindre à la fin un espace dense de $\dot{H}^1(\mathbb{T})$, il faut partir d'un contrôle dans $\mathcal{H}(I_0)$, où I_0 est générateur de \mathbb{Z} , c'est-à-dire que tout entier de \mathbb{Z} peut être écrit comme une combinaison linéaire avec des coefficients entiers d'éléments de I_0 . Remarquons que le contrôle repose sur un phénomène non linéaire où le terme $u\partial_x u$ génère de nouvelles fréquences, néanmoins, l'hypothèse sur I_0 est purement linéaire. En d'autres termes, en partant d'un I_0 qui génère linéairement \mathbb{Z} , nous définissons une suite (I_n) construite à partir d'un procédé non linéaire. Il est important de voir que dans ce raisonnement, nous ne perdons jamais d'informations puisqu'après chaque étape, nous montrons que le cible sera dans un espace plus riche en fréquences. En répétant cette procédure suffisamment de fois, nous

pouvons atteindre un espace dense dans $\dot{H}^1(\mathbb{T})$.

Nous commençons par utiliser un contrôle avec une très grande norme $H^1(\mathbb{T})$ en un temps petit, avec cela, nous montrons que la solution va converger vers une quantité dans laquelle le terme non linéaire de l'équation apparaît. Cette limite que l'on retrouve peut jouer le rôle d'une donnée initiale dans la suite. En appliquant à nouveau l'équation, le terme non linéaire donnera de nouvelles droites vectorielles qui peuvent être utilisées comme données initiales. Une fois que nous montrons que nous pouvons aller de n'importe quel point à n'importe quel point approximativement en un temps petit, nous pouvons généraliser cela pour conclure que le résultat est également vrai en un temps fixe.

Cette technique est inspirée de l'article de Nersesyan [Ner21], où il applique le même type de contrôle sur l'équation de la chaleur, et dans [Ner15] pour les systèmes de Navier-Stokes. Nous renvoyons également le lecteur aux travaux d'Agrachev et Sarychev [AS05, AS06] où ils utilisent des polynômes trigonométriques pour contrôler les systèmes de Navier-Stokes/Euler en 2-d et 3-d. Nersisyan [Ner10, Ner11] utilise cette méthode de contrôle avec des espaces de dimensions finies pour l'équation d'Euler. Voir aussi les travaux de Shirikyan [Shi06, Shi07, Shi14, Shi18] pour les équations de Burger et Navier-Stokes.

L'organisation du quatrième chapitre est la suivante. Dans la Section 4.2 nous étudions l'existence de la solution du problème non linéaire en utilisant un argument de point fixe sur l'équation de BBM modifiée suivante

$$\partial_t u + \partial_x(u + \delta^{-\frac{1}{2}}\varphi) - \partial_{xxt}u + (u + \delta^{-\frac{1}{2}}\varphi)\partial_x(u + \delta^{-\frac{1}{2}}\varphi) = \delta^{-1}f, \quad (1.20)$$

avec $u(0) = u_0$, $\delta > 0$ et φ une fonction régulière. L'intérêt d'ajouter la constante δ est de mesurer la taille du terme de contrôle qu'on va mettre à la première étape de notre raisonnement par récurrence, et de mesurer le temps d'existence de la solution avec un terme de contrôle sous la forme $\delta^{-1}f$. La fonction φ joue ici le rôle d'une deuxième donnée initiale injectée dans l'équation. La solution de (1.20) à l'instant t va être notée $u(t) = \mathcal{R}_t(u_0, \delta^{-\frac{1}{2}}\varphi, \delta^{-1}f)$. Nous établissons des estimations de continuité de la solution par rapport à la donnée initiale, à la fonction φ et du terme de contrôle. Nos résultats de continuité sont les suivants.

Proposition 2 *Il existe des constantes positives c_1 , c_2 et c_3 telles que pour $t \leq \delta$*

$$\begin{aligned} \left\| \mathcal{R}_t(u_0, \delta^{-\frac{1}{2}}\varphi, \delta^{-1}f) - \mathcal{R}_t(v_0, \delta^{-\frac{1}{2}}\psi, \delta^{-1}g) \right\|_{H^1} &\leq c_1 \|u_0 - v_0\|_{H^1} \\ &+ c_2 \|f - g\|_{L^1([0, \delta]; H^{-1})} + c_3 \|\varphi - \psi\|_{H^{-1}}. \end{aligned} \quad (1.21)$$

Lemme 1 *Soient $\epsilon > 0$, $v \in \dot{H}^1(\mathbb{T})$ et $u_0 \in B_{H^1}\left(v, \frac{\epsilon}{2}\right)$, alors il existe $C > 0$ et $t_* > 0$ telle que la solution $u(t_*) = \mathcal{R}_{t_*}(u_0, 0, 0) \in B_{H^1}(v, \epsilon)$ et*

pour tout $t \in [0, t_*]$

$$\|u(t) - u_0\|_{H^1} \leq Ct. \quad (1.22)$$

Nous posons

$$\mathcal{C}(I) = \left\{ \eta - (1 - \partial_{xx})^{-1} \sum_{i=1}^d \varphi_i \partial_x \varphi_i ; \eta, \varphi_i \in \mathcal{H}(I), \forall d \geq 1 \right\}.$$

Le résultat principal de la section [4.2](#) est la limite suivante.

Proposition 3 Soit $u(t) = \mathcal{R}_t(u_0, \delta^{-\frac{1}{2}}\varphi, \delta^{-1}\eta_0)$ la solution de [\(1.20\)](#), alors

$$\lim_{\delta \rightarrow 0} u(\delta) = u_0 + (1 - \partial_{xx})^{-1} (\eta_0 - \varphi \partial_x \varphi), \text{ dans } H^1(\mathbb{T}). \quad (1.23)$$

Cette proposition dit que lorsque δ tend vers 0, nous pouvons atteindre approximativement l'espace $\mathcal{C}(I)$ où on voit le terme non linéaire de l'équation apparaître dans la limite.

Dans la Section [4.3](#) nous partons d'un ensemble I_0 qui génère \mathbb{Z} , puis nous définissons une suite d'ensembles (I_n) liés aux coefficients de Fourier donnés par le terme non linéaire, et nous prouvons que leur union forme un espace dense dans $\dot{H}^1(\mathbb{T})$. Ce qui revient au lemme suivant.

Lemme 2 Si I_0 est un générateur de \mathbb{Z} , alors

$$\bigcup_{n \in \mathbb{N}} I_n = \mathbb{Z}. \quad (1.24)$$

Ensuite, on fait le lien entre la suite (I_n) et l'espace $\mathcal{C}(I)$ à travers le résultat suivant.

Lemme 3 Pour tout $n \in \mathbb{N}$, on a

$$\mathcal{H}(I_{n+1}) \subset \mathcal{C}(I_n).$$

Autrement dit, ce qu'on peut atteindre en partant de φ et η dans $\mathcal{H}(I_n)$ va être une quantité dans un certain espace atteignable $\mathcal{C}(I)$.

Dans la Section [4.4](#) nous montrons que nous pouvons appliquer un contrôle pour passer d'une donnée initiale u_0 à un certain $u_0 + \mathcal{H}$ approximativement, où \mathcal{H} est un espace de dimension finie que nous allons enrichir jusqu'à obtenir un espace dense dans $\dot{H}^1(\mathbb{T})$. Ensuite, en utilisant un raisonnement par récurrence et une propriété de densité, nous montrons que nous pouvons atteindre approximativement n'importe quel point en un temps $T > 0$. Ceci conclut la preuve du Théorème [10](#).

Dans la Section [4.5](#), nous donnons une preuve du Théorème [11](#). Nous commençons par le cas le plus simple, la non contrôlabilité exacte dans $\dot{H}^1(\mathbb{T})$. Nous raisonnons par contradiction pour montrer avec le théorème de Baire que l'espace atteignable dans $\dot{H}^1(\mathbb{T})$ ne peut pas contenir un ouvert, donc son complémentaire est dense.

Le cas $s \in]1, 2[$ est plus technique. Nous utilisons un argument de l' ϵ -entropie pour compter le nombre de boules nécessaires pour recouvrir un certain ensemble. Plus précisément, nous estimons le nombre de boules de $\dot{H}^1(\mathbb{T})$ qu'il faut pour recouvrir la boule unité de $\dot{H}^s(\mathbb{T})$ lorsque $s \in]1, 2[$, puis nous montrons que l'application $\eta \mapsto u$ ne peut pas nous donner assez de boules pour couvrir une boule de $\dot{H}^s(\mathbb{T})$. Cela signifie que nous n'avons pas assez de contrôles pour obtenir une contrôlabilité exacte dans $\dot{H}^s(\mathbb{T})$.

Cette technique a été utilisée pour montrer que l'équation d'Euler n'est pas exactement contrôlable en 2-d par Shirikyan [\[Shi08\]](#). Dans les travaux de Nersisyan et Nersisyan on peut aussi trouver le raisonnement de l' ϵ -entropie pour prouver la non contrôlabilité exacte dans [\[NN12\]](#) pour l'équation de Schrödinger, et dans [\[Ner10\]](#) pour l'équation d'Euler en 3-d.

1.3 Troisième partie

Nous intéressons dans le cinquième chapitre à l'équation de KP-I en 2-d

$$\begin{cases} \partial_t u + \partial_{xxx} u + u \partial_x u - \partial_x^{-1} \partial_{yy} u = \eta, & (x, y) \in \mathbb{T}^2, t > 0, \\ u(\cdot, \cdot, 0) = u_0, & (x, y) \in \mathbb{T}^2. \end{cases} \quad (1.25)$$

Cette équation, vue comme une généralisation des équations de type KdV en dimensions supérieures, est introduite par Kadomtsev et Petviashvili [\[KP70\]](#). Kenig [\[Ken04\]](#) a montré que l'équation de KP-I est bien posée sur des espaces ayant une certaine régularité (voir aussi Bourgain [\[Bou93\]](#) et Saut [\[Sau93\]](#)). Un travail de Saut et Tzvetkov [\[ST99\]](#) montre que pour des espaces de Sobolev inhomogène avec une faible régularité par rapport à la variable x , l'équation de KP-I est mal posée.

La contrôlabilité a été étudiée pour l'équation de KP-II

$$\partial_t u + \partial_{xxx} u + u \partial_x u + \partial_x^{-1} \partial_{yy} u = 0,$$

dans l'article de Rivas et Sun [\[RS20\]](#). Un résultat de contrôlabilité pour l'équation de KP-I linéaire peut être trouvé dans l'article de Sun [\[Sun18\]](#).

Dans la Section 5.2 nous commençons par introduire les espaces

$$\tilde{H}^s(\mathbb{T}^2) = \left\{ u = \sum_{(k,n) \in \mathbb{Z}^* \times \mathbb{Z}} u_{k,n} e^{i(kx+ny)} \in L^2(\mathbb{T}^2); \right. \\ \left. \sum_{(k,n) \in \mathbb{Z}^* \times \mathbb{Z}} |u_{k,n}|^2 (1+k^2+n^2)^s < \infty \right\}, \quad (1.26)$$

et

$$\mathcal{Q}^s(\mathbb{T}^2) = \left\{ u = \sum_{(k,n) \in \mathbb{Z}^* \times \mathbb{Z}} u_{k,n} e^{i(kx+ny)} \in L^2(\mathbb{T}^2); \right. \\ \left. \sum_{(k,n) \in \mathbb{Z}^* \times \mathbb{Z}} |u_{k,n}|^2 \left(\frac{(k^4+n^2)^2}{k^2} + 1 \right) (1+k^2+n^2)^s < \infty \right\}, \quad (1.27)$$

dans lesquels nous allons étudier l'existence et les propriétés de la solution. L'espace \mathcal{Q}^s est le domaine de la partie linéaire vue comme opérateur non borné sur \tilde{H}^s . Nous considérons l'équation de KP-I avec un terme de contrôle $\eta = \eta(t, x, y)$ comme dans le problème étudié au quatrième chapitre.

L'idée pour montrer l'existence est de regarder directement l'équation non linéaire en remplaçant le terme $u\partial_x u$ par $v\partial_x u$ pour définir un semi-groupe associé à l'opérateur $A(v)$ et ensuite prendre $v = u$. Nous donnons également des estimations nécessaires pour le théorème d'existence.

Le résultat d'existence n'est pas optimale du point de vue de la régularité des données initiales, mais suffisent pour le problème du contrôle approché. Le résultat principal de la Section 5.3 est le théorème suivant.

Théorème 12 *Il existe une unique solution $u \in \mathcal{C}([0, T']; W) \cap \mathcal{C}^1([0, T']; H^s)$ de (1.25), où W est une boule ouverte (bornée) de \mathcal{Q}^s .*

L'opérateur $A(v)$ de l'équation de KP-I est un opérateur qui dépend du temps, l'équation (1.25) est donc équivalente à

$$\begin{cases} \partial_t u = A(u(t))u + f, \\ u(s) = 0, \end{cases} \quad (1.28)$$

où

$$A(v(\alpha)) = -(\partial_{xxx} u + v(\alpha)\partial_x u - \partial_x^{-1} \partial_{yy} u).$$

La preuve du théorème 12 est basée sur le théorème de Kato [Kat75, Theorem 6] suivant.

Théorème 13 Soient X un espace de Banach réflexif, $Y \subset X$ un autre espace de Banach réflexif qui s'injecte avec une injection dense dans X . Soit W une boule ouverte de Y contenant 0 , $T > 0$ et $S : Y \rightarrow X$ un isomorphisme. Supposons que

1. $A = A(v(\alpha))$ est le générateur infinitésimal d'un semi-groupe $\{e^{sA(v(\alpha))}\}_{s \geq 0}$ tel que pour tout $s > 0$

$$\| \| e^{-sA(v(\alpha))} \| \|_X \leq e^{\beta s}, \quad \forall (\alpha, v) \in [0, T] \times W.$$

2. Pour tout $(\alpha, v) \in [0, T] \times W$, l'opérateur $SA(\alpha, v)S^{-1} - A(\alpha, v)$ est uniformément borné sur X .
3. Pour tout $v \in W$, $\alpha \mapsto A(\alpha, v)$ est continue sur $\mathcal{L}(X, Y)$ et pour $\alpha \in [0, T]$, $v \mapsto A(\alpha, v)$ est Lipschitzienne.
4. Pour tout $v \in W$, $\alpha \mapsto f(\alpha, v)$ est continue de $[0, T]$ dans X , pour tout $\alpha \in [0, T]$, $v \mapsto f(\alpha, v)$ est Lipschitzienne et f est une fonction bornée de $[0, T] \times W$ dans Y .

Alors (1.28) admet une unique solution $u \in \mathcal{C}([0, T']; W) \cap \mathcal{C}^1([0, T']; X)$ pour un $T' \in [0, T]$.

Comme nous allons considérer un contrôle avec des fonctions trigonométriques. La première étape, après avoir montré l'existence, est de modifier l'équation en introduisant une constante $\delta > 0$ et $\varphi \in \mathcal{C}^\infty$. Nous regardons l'équation

$$\begin{cases} \partial_t u + \partial_{xxx}(u + \delta^{-\frac{1}{2}}\varphi) + (u + \delta^{-\frac{1}{2}}\varphi)\partial_x(u + \delta^{-\frac{1}{2}}\varphi) - \partial_x^{-1}\partial_{yy}(u + \delta^{-\frac{1}{2}}\varphi) \\ = \delta^{-1}\eta, \\ u(\cdot, \cdot, 0) = u_0. \end{cases} \quad (1.29)$$

Nous donnons des estimations de continuité pour la solution de (1.29). Ensuite, nous faisons tendre δ vers 0 dans H^s pour obtenir la limite suivante.

Proposition 4 La solution $u(t) = \mathcal{R}_t(u_0, \delta^{-\frac{1}{2}}\varphi, \delta^{-1}\eta_0)$ de (1.29) vérifie

$$\lim_{\delta \rightarrow 0} u(\delta) = u_0 + \eta_0 - \varphi \partial_x \varphi, \quad \text{dans } H^s(\mathbb{T}^2). \quad (1.30)$$

Cette proposition est l'équivalente de la proposition 3.

Dans la Section 5.4 Nous prouvons le résultat de contrôlabilité approchée donné par le théorème suivant. Soit $I_1 = \{(1, 1); (1, 0); (1, -1)\}$. Nous posons pour $I \subset \mathbb{Z}^* \times \mathbb{Z}$ l'espace

$$\mathcal{H}(I) = \text{vect}\{\cos(px + qy), \sin(px + qy); \text{ pour } (p, q) \in I\},$$

Théorème 14 *L'équation (1.25) est approximativement contrôlable dans \tilde{H}^s avec des contrôles constants par morceaux à valeurs dans $\mathcal{H}(I_1)$ pour $s > 2$. C'est-à-dire, pour tout $T > 0$, $\epsilon > 0$ et tout $u_0, v \in \tilde{H}^s$, il existe un contrôle $\eta(t, x, y)$ constant par morceaux en temps à valeurs dans $\mathcal{H}(I_1)$ et une solution u de (1.25) telle que*

$$\|u(T) - v\|_{H^s} \leq \epsilon.$$

Pour cela, nous énonçons un résultat analogue au lemme 3 pour conclure qu'avec une suite (I_N) définie à l'aide des produits des fonctions trigonométriques, nous pouvons, à partir du choix de I_1 , créer toutes les fréquences de $\mathbb{Z}^* \times \mathbb{Z}$. Cela conduit par suite à un résultat de contrôle approché.

Preliminaries

This chapter is a presentation of the mathematical notions and definitions that we will use in the following. Some results in this chapter are known in the context of a real or complex space. Since we are going to deal with equations in the torus

$$\mathbb{T} := \mathbb{R}/2\pi\mathbb{Z},$$

for the next chapters, we give equivalent versions of the known results in the case where the variable x is in the torus using the fact that there is a bijection between the functions defined on the torus and the 2π -periodic functions defined on \mathbb{R} .

Let $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$ be two Banach spaces.

2.1 Usual spaces

For $1 \leq p < \infty$, we denote by $\mathcal{L}^p(\mathbb{T}, \mathbb{C})$ the space of measurable functions $f : \mathbb{T} \rightarrow \mathbb{C}$ such that $\mu(|f|^p) < \infty$, where μ is a measure on \mathbb{T} . The Lebesgue space $L^p(\mathbb{T}, \mathbb{C})$ is defined as $L^p(\mathbb{T}, \mathbb{C}) = \mathcal{L}^p(\mathbb{T}, \mathbb{C})/\sim$, where \sim is the equivalence relation defined by $f \sim g$ if and only if $f = g$ almost everywhere in \mathbb{T} for the Lebesgue measure. For $f \in L^p(\mathbb{T}, \mathbb{C})$, the norm of f is

given by

$$\|f\|_{L^p} = \left(\int_{\mathbb{T}} |f(x)|^p dx \right)^{\frac{1}{p}}.$$

When $p = +\infty$, we define the space of bounded measurable functions $L^\infty(\mathbb{T}, \mathbb{C})$ equipped with the norm

$$\|f\|_{L^\infty} = \sup_{x \in \mathbb{T}} |f(x)|.$$

In a similar way, we define for $1 \leq p < \infty$ the space $\ell^p(\mathbb{Z})$ of sequences $(x_k)_k$ such that

$$\|(x_k)_k\|_{\ell^p} := \left(\sum_{k \in \mathbb{Z}} |x_k|^p \right)^{\frac{1}{p}} < \infty,$$

and the space $\ell^\infty(\mathbb{Z})$ the space of sequences $(x_k)_k$ such that

$$\|(x_k)_k\|_{\ell^\infty} := \sup_{k \in \mathbb{Z}} |x_k| < \infty.$$

The Sobolev spaces are defined for $s \in \mathbb{R}$ by

$$H^s(\mathbb{T}) = \left\{ u = \sum_{n \in \mathbb{Z}} u_n e^{inx} : \mathbb{T} \rightarrow \mathbb{C}; \sum_{n \in \mathbb{Z}} (1 + n^2)^s |u_n|^2 < \infty \right\}. \quad (2.1)$$

The space $H^s(\mathbb{T})$ is equipped by the inner product

$$\begin{aligned} \langle \cdot, \cdot \rangle_s : H^1(\mathbb{T}) \times H^1(\mathbb{T}) &\longrightarrow \mathbb{C} \\ (u, v) &\longmapsto \sum_{n \in \mathbb{Z}} (1 + n^2)^s u_n \overline{v_n} \end{aligned} \quad (2.2)$$

where u_n and v_n are the Fourier coefficients of u and v respectively.

We note $\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_1$.

Definition 1 1. A bounded operator from E to F is a linear continuous map from E to F .

2. An unbounded operator from E to F is the given of a vector subspace of E , denoted $D(A)$ and called domain of the operator, and of a continuous linear operator A from $D(A)$ in F .

$\mathcal{L}(E, F)$ denotes the space of bounded linear operators from E to F , equipped with the norm

$$\|A\|_{\mathcal{L}(E, F)} = \sup_{\|u\|_E \leq 1} \frac{\|Au\|_F}{\|u\|_E}.$$

For $k \in \mathbb{N}^*$, $\mathcal{C}^k(\mathbb{T})$ denotes the space of functions defined on the torus whose derivatives up to k exists and continuous. When $k = 0$, $\mathcal{C}^0(\mathbb{T})$ (or

$\mathcal{C}(\mathbb{T})$ denotes the space of the continuous functions on the torus. The space $\mathcal{C}(\mathbb{T})$ equipped with the norm $\|\cdot\|_\infty$ is a Banach space. The space \mathcal{C}^∞ is the space of functions infinitely differentiable, and \mathcal{C}_0^∞ when moreover the functions are with compact support.

We denote by $BV([0, T], X)$ the space of functions in $L^1([0, T], X)$ with bounded variations from $[0, T]$ with values in X , equipped with the norm

$$\|u\|_{BV} = \|u\|_{L^1} + \int_X |du|, \quad (2.3)$$

where $du(s) = u'(s)ds$ is the derivative of u in the sense of distributions that can assumed to be a Radon measure. See Chapter 1 in [Giu84] for more details.

2.2 Some definitions

We give in this subsection some definitions and well-known results. Let $p, q, r \geq 1$. We define the convolution $f * g$ between two functions $f \in L^p(\mathbb{T})$ and $g \in L^q(\mathbb{T})$ by

$$f * g(x) = \int_{\mathbb{T}} f(y)g(x - y) dy.$$

The function $f * g$ belongs to $L^r(\mathbb{T})$, where r satisfy the relation

$$\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}. \quad (2.4)$$

Moreover, we have the estimation

$$\|f * g\|_{L^r} \leq \|f\|_{L^p} \|g\|_{L^q}.$$

The same result holds true for sequences. The sequence $(x * y)_k$ is define in $\ell^r(\mathbb{Z})$ when $(x_k)_k \in \ell^p(\mathbb{Z})$ and $(y_k)_k \in \ell^q(\mathbb{Z})$ by

$$(x * y)_k = \sum_{n \in \mathbb{Z}} x_n y_{n-k},$$

where r satisfy (2.4), and we have

$$\|x * y\|_{\ell^r} \leq \|x\|_{\ell^p} \|y\|_{\ell^q}. \quad (2.5)$$

Definition 2 An operator A of $\mathcal{L}(E, F)$ is compact if the image by A of the unit ball of E has a compact closure in F .

We can see the compact property with using sequences.

Proposition 5 An operator A of $\mathcal{L}(E, F)$ is compact if and only if for any bounded sequence $(u_n)_n$ in E , the sequence $(Au_n)_n$ contains a converging subsequence in F .

Definition 3 Let (x_n) be a sequence in a Hilbert space H and x an element of H . We say that the sequence (x_n) converges weakly to x and we note $x_n \rightharpoonup x$ if

$$\forall h \in H, \lim_{n \rightarrow \infty} (x_n, h)_H = (x, h)_H.$$

Definition 4 We say that a family $\{S(t)\}_{t \geq 0}$ of bounded linear operators is a semigroup on E if

1. $S(0) = Id_E$,
2. $S(t_1 + t_2) = S(t_1)S(t_2)$, for all $t_1, t_2 \geq 0$.

If moreover the application $t \mapsto S(t)x$ is continuous in 0 for all x in E , then $\{S(t)\}_{t \geq 0}$ is called a C^0 -semigroup. And if these properties extend for $t \leq 0$, we talk about a group or a C^0 -group.

Definition 5 We call infinitesimal generator of a semigroup $\{S(t)\}_{t \geq 0}$ the unbounded linear operator $(A, D(A))$ defined by

$$D(A) = \left\{ u \in E; \lim_{t \rightarrow 0^+} \frac{S(t)u - u}{t} \text{ exists in } E \right\},$$

and for $u \in D(A)$

$$Au = \lim_{t \rightarrow 0^+} \frac{S(t)u - u}{t}.$$

Definition 6 (Fredholm operator) Let E and F be two Banach spaces. We say that an operator $A : E \rightarrow F$ is Fredholm if

1. $\dim \text{Ker } A < \infty$,
2. $\text{codim } \text{Im}(A) < \infty$.

The index of A is defined by $\text{Ind}(A) = \dim \text{Ker } A - \text{codim } \text{Im}(A)$.

Lemma 1 Let $\{S(t)\}_{t \geq 0}$ be a C^0 -semigroup, then, there exist $M \geq 1$ and $c \geq 0$ such that for all $t \geq 0$

$$\|S(t)\| \leq Me^{ct}.$$

If for all $t \geq 0$, $\|S(t)\| \leq 1$, we say that it is a semigroup of contraction.

Lemma 2 Let E and F be two Banach spaces. If $A : E \rightarrow F$ is a Fredholm operator of index $\text{Ind}(A)$ and $T : E \rightarrow F$ is a compact operator then $A+T$ is a Fredholm operator of index $\text{Ind}(A)$.

Theorem 1 (Picard) Let $A \subset X$ be a closed set and $\Phi : A \rightarrow A$ be a map of contraction, that is to say there exists $\theta \in]0, 1[$ such that for all $u, v \in A$

$$\|\Phi u - \Phi v\|_F \leq \theta \|u - v\|_E.$$

Then, there exists a unique $u \in A$ such that $\Phi u = u$.

Definition 7 (Dissipative operator) Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space. An operator $A : \mathcal{D}(A) \subset H \rightarrow H$ is called dissipative if for all $u \in \mathcal{D}(A)$,

$$\Re \langle Au, u \rangle \leq 0.$$

We will prove the existence of the solutions using the two following theorems.

Theorem 2 Let A be a linear operator. Then A is dissipative if and only if for all $u \in D(A)$ and $\lambda > 0$, we have

$$\|(\lambda - A)u\| \leq \lambda \|u\|.$$

Theorem 3 (Lumer-Phillips) Let A be a linear operator of dense domain in a Hilbert space H . If A is dissipative and if there exists $\lambda_0 > 0$ such that $\text{Im}(\lambda_0 I - A) = H$, then A is the infinitesimal generator of a \mathcal{C}^0 -semigroup of contraction.

See [Paz83] for a proof.

If the infinitesimal generator A of a \mathcal{C}^0 -semigroup $\{S(t)\}_{t \geq 0}$ verifies the assumptions of Theorem 3 then the solution of the equation

$$\begin{cases} \partial_t u = Au \\ u(0) = u_0, \end{cases}$$

is noted $u(t) = S(t)u_0$.

Theorem 4 (Hille-Yosida) We have an equivalence between the two assumptions.

1. A is the infinitesimal generator of a \mathcal{C}^0 -semigroup of contractions $\{S(s)\}_{s \geq 0}$.
2. A is closed, of domain dense and $\|R(\lambda, A(t))\| \leq \frac{1}{\lambda}$ for $\Re \lambda > 0$.

Theorem 5 (Banach) If $T \in \mathcal{L}(E, F)$ is bijective, then T^{-1} is continuous.

Definition 8 (Hamiltonian field) We call Hamiltonian field associated with a function $f \in \mathcal{C}^\infty$ the operator H_f defined by

$$H_f = \langle \partial_\xi f, \partial_x \rangle - \langle \partial_x f, \partial_\xi \rangle = \sum_{j=1}^n (\partial_{\xi_j} f \partial_{x_j} - \partial_{x_j} f \partial_{\xi_j}).$$

Definition 9 (Poisson bracket) Let $f, g : \mathbb{R}^{2n} \rightarrow \mathbb{R}$ be two functions in \mathcal{C}^1 . We define their Poisson bracket by

$$\{f, g\} = H_f g = \langle \partial_\xi f, \partial_x g \rangle - \langle \partial_x f, \partial_\xi g \rangle.$$

We denote by $\mathcal{D} = -i\partial_x$.

A differential operator P of order m is an operator defined by

$$P = \sum_{|m| \leq \alpha} \sigma_\alpha(x) \partial_x^\alpha,$$

where σ_α are regular functions. The symbol of P is the function $a : \mathbb{T} \times \mathbb{R} \rightarrow \mathbb{C}$ such that

$$a(x, \xi) = \sum_{|m| \leq \alpha} \sigma_\alpha(x) (i\xi)^\alpha.$$

This definition can be extended to other classes of operators in which the symbol is no longer necessarily a polynomial in ξ . In this case, we speak of pseudo-differential operators.

Definition 10 Let $a = a(x, \xi)$ be a symbol of $\mathcal{C}^\infty(\mathbb{T} \times \mathbb{R})$. The pseudo-differential operator of symbol a , noted $\text{Op}(a)$ is defined for all $u \in \mathcal{C}^\infty(\mathbb{T})$ by

$$\text{Op}(a)u(x) = \mathcal{F}^{-1}(a(x, \xi)\mathcal{F}u(\xi)) = \frac{1}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{T}} e^{i(x-y)\xi} a(x, \xi) u(y) dy d\xi. \quad (2.6)$$

Definition 11 Let $h > 0$. We define the semiclassical Fourier transform by

$$\mathcal{F}_h \varphi(\xi) = \int_{\mathbb{R}} e^{-\frac{i}{h}x\xi} \varphi(x) dx,$$

and its inverse

$$\mathcal{F}_h^{-1} \psi(x) = \frac{1}{2\pi h} \int_{\mathbb{R}} e^{\frac{i}{h}x\xi} \psi(\xi) d\xi.$$

Definition 12 Let $u(x) = \sum_{j \in \mathbb{Z}} u_j e^{ijx} \in L^2(\mathbb{T})$ and $a(x, \xi) = \sum_{j \in \mathbb{Z}} a_j(\xi) e^{ijx} \in \mathcal{C}_0^\infty(\mathbb{T} \times \mathbb{R})$ such that a is 2π -periodic on x and

$$a_j(\xi) = \frac{1}{2\pi} \int_{\mathbb{T}} a(x, \xi) e^{-ijx} dx.$$

Then the semi-classical pseudo-differential operator of a is defined by

$$\text{Op}_h(a)u = \frac{1}{2\pi} \int_{\mathbb{R}} e^{ix\xi} a(x, h\xi) \hat{u}(\xi) d\xi = \frac{1}{2\pi} \sum_{j, l \in \mathbb{Z}} u_j(x) a_{l-j}(h\xi) e^{ilx}. \quad (2.7)$$

We use here the standard quantification known in the literature. By an analogous calculation that in [HPS14], we show that $\text{Op}_h(a)$ is well defined. And by applying [HPS14, Lemma 6.4] and [HPS14, Proposition 6.5] we have that $\text{Op}_h(a)$ is a bounded operator in $L^2(\mathbb{T})$. See also [Zwo12, Section 5.3.1].

Theorem 6 *Let $a \in \mathcal{C}_0^\infty(\mathbb{T} \times \mathbb{R})$. The operator $\text{Op}_h(a)$ extends to a continuous operator on $\mathcal{L}^2(\mathbb{T})$, and there exists a constant $c > 0$ (which does not depend on h) such that*

$$\|\text{Op}_h(a)\|_{\mathcal{L}^2(\mathbb{T})} \leq c.$$

The character bounded on \mathcal{L}^2 is necessary to be able to write the existence theorem of semi-classical measure in the case where the space variable of the symbol belongs to the torus.

Stabilization of the KdV-BBM equation

3.1 Introduction

In the literature we can find several models of partial differential equations to describe the movement of water in shallow depths following a unidirectional propagation, see for instance [BCPS18], [BC99] and [CPS16]. In this work, we study the stabilization for the damped nonlinear KdV-BBM equation

$$\partial_t u + \partial_x u - \partial_{xxt} u + \partial_{xxx} u - \partial_x(a(x)\partial_x)u + u\partial_x u = 0, \quad (3.1)$$

where a , the damping, is non negative. This equation mixes the KdV equation $\partial_t u + \partial_{xxx} u + u\partial_x u = 0$, with the BBM equation $\partial_t u + \partial_x u - \partial_{xxt} u + u\partial_x u = 0$. To our knowledge the KdV-BBM equation was treated for the first time as a system in [BCS02] to study its properties. This equation has been recently considered in [BNS17], [DPI4] and [MRI7] from numerical analysis point of view. Stabilization was not considered for KdV-BBM equation but was considered for KdV in [BCV19] and [LRBY10], and for BBM equation in [AC19] and [Ros16].

The particularity of the KdV- BBM equation, is that it admits a nonlocal unbounded operator of order 1, that introduces some difficulties in the analysis. Our main result is a stabilization property for the solutions of (3.1) on the torus. The principal theorem is the following.

Theorem 7 For all $u_0 \in H^1(\mathbb{T})$, there exists a unique solution $u = u(t, x)$ of (3.1) global in time such that

$$\lim_{t \rightarrow +\infty} u(t, \cdot) = \frac{1}{2\pi} \int_{\mathbb{T}} u_0(x) dx, \text{ in } H^1(\mathbb{T}).$$

Now we give the outline of the proof. In Section 3.2 we show the existence of the solution of the linear problem. In Section 3.3 we obtain the exponential decay result on the semigroup by an estimation of the resolvent. This estimate comes from semiclassical measures and the technic was used for the stability of the wave equation. This method can be found for the wave equation, see [Bur02], [BG97], [GL93], [Ce91b], [Leb94], [Mi00] and [RZ09].

In Section 3.4 from the fixed point theorem we prove local existence and the uniqueness for the nonlinear problem. By an a priori estimate on the energy, we deduce global existence for the nonlinear problem.

In Section 3.5 from energy decay dynamical system technics, we prove first that solution converges to a bounded solution for all t in \mathbb{R} . Second, from result of [HR03] we prove that this particular solution is analytic in time. This allows to apply uniqueness results (see [Hö97], [Hö85], [RZ98] and [Tat95]) and we deduce that this particular solution is in fact a constant.

The result of unique continuation which we will arrive at in Section 3.6 is therefore a result which says that if the solution is a constant on an open set, then it is constant over the whole domain. The unique continuation gives stabilization at the end in the sense that some solutions will goes to a constant as t goes to infinity. This technique can be found in [UL13] where the authors consider the wave equation. They show that they can found an analytical solution in time and then use an unique continuation result to prove the stabilization.

In Section 3.7 we give a quick deduction that the approach we made on KdV-BBM is applicable for the equation of BBM in a particular case. More precisely we say that the unique continuation conjecture that Rosier has stated in [Ros16] is true with a dissipator a which depends on time. This type of dissipator was used in [CC18] to show controllability results for the Boussinesq equation.

According to our knowledge, the rate of decay of the energy of the KdV-BBM nonlinear problem solution is still an open question.

3.2 Linear equation

We consider the linear KdV-BBM equation

$$\begin{cases} \partial_t u + \partial_x u - \partial_{xxt} u + \partial_{xxx} u - \partial_x(a(x)\partial_x)u = 0, & x \in \mathbb{T}, t > 0, \\ u(\cdot, 0) = u_0 \in H^2(\mathbb{T}), & x \in \mathbb{T}, \end{cases} \quad (3.2)$$

where $a \geq 0$ is a bounded C^∞ function such that $\{a > 0\} \neq \emptyset$.

We start by showing the existence of the solution. The operator $(1 - \partial_{xx})^{-1}$ is defined as follow

$$(1 - \partial_{xx})^{-1} : \begin{array}{ccc} H^s(\mathbb{T}) & \longrightarrow & H^{s+2}(\mathbb{T}) \\ f & \longmapsto & u, \end{array}$$

the unique solution of the equation $(1 - \partial_{xx})u = f$, defined by its Fourier coefficients $u_n = \frac{1}{1 + n^2} f_n$. We have $u \in H^{s+2}(\mathbb{T})$ when $f \in H^s(\mathbb{T})$ since $\left((1 + n^2)^{\frac{s+2}{2}} u_n \right)^2 = (1 + n^2)^s |f_n|^2$, which have a finite sum. The undounded operator $(A, \mathcal{D}(A))$ in $H^1(\mathbb{T})$ is defined by $\mathcal{D}(A) = H^2(\mathbb{T})$ and

$$\begin{array}{ccc} A : H^2(\mathbb{T}) & \longrightarrow & H^1(\mathbb{T}) \\ u & \longmapsto & -(1 - \partial_{xx})^{-1}(\partial_x u + \partial_{xxx} u - \partial_x(a(x)\partial_x)u). \end{array} \quad (3.3)$$

The operator A has values in $H^1(\mathbb{T})$ since

$$\begin{aligned} (1 - \partial_{xx})^{-1} \partial_x u &\in H^3(\mathbb{T}), \quad (1 - \partial_{xx})^{-1} \partial_{xxx} u \in H^1(\mathbb{T}) \quad \text{and} \\ (1 - \partial_{xx})^{-1} \partial_x(a(x)\partial_x)u &\in H^1(\mathbb{T}). \end{aligned}$$

We prove now that A is a Fredholm operator.

Proposition 6 *The operator A -considered as operator on $H^2(\mathbb{T})$ with values in $H^1(\mathbb{T})$ - is a Fredholm operator of index 0.*

Proof Let $\delta > 0$, A can be written as sum of A_0 and B with

$$A_0 = -(1 - \partial_{xx})^{-1}(\partial_x + \partial_{xxx} + \delta)$$

and

$$B = -(1 - \partial_{xx})^{-1}(-\partial_x(a(x)\partial_x) - \delta).$$

Note that if $u(x, t) = \sum_{n \in \mathbb{Z}} u_n(t) e^{inx}$, then $A_0 u = \sum_{n \in \mathbb{Z}} \frac{-in + in^3 - \delta}{1 + n^2} u_n(t) e^{inx}$.

The operator A_0 is bijective from $H^2(\mathbb{T})$ to $H^1(\mathbb{T})$, indeed, the equation $A_0 u = f$ written with the Fourier coefficients is

$$\frac{-1}{1 + n^2} (in - in^3 + \delta) u_n = f_n.$$

So we can write u_n in respect of f_n

$$u_n = \frac{-(1+n^2)}{\delta + in(1-n^2)} f_n.$$

Note here that the denominator cannot be equal to 0, hence the interest in adding and subtracting δ in A . We thus have the injectivity and the surjectivity of A_0 , in other words

$$\dim \text{Ker } A_0 = \text{codim } \text{Im}(A_0) = 0.$$

This prove that A_0 is Fredholm of index 0.

The operator B is bounded from $H^2(\mathbb{T})$ in itself scince

$$\begin{aligned} \|Bu(t)\|_{H^2(\mathbb{T})} &\leq \|\partial_x(a\partial_x)u(t) + \delta u(t)\|_{L^2(\mathbb{T})} \\ &\leq \|a\partial_x u(t)\|_{H^1(\mathbb{T})} + \delta \|u(t)\|_{L^2(\mathbb{T})} \\ &\leq (\|a\|_\infty + \delta) \|u(t)\|_{H^2(\mathbb{T})}. \end{aligned} \quad (3.4)$$

So B is a compact operator from $H^2(\mathbb{T})$ in $H^1(\mathbb{T})$ thanks to the compact injection of $H^2(\mathbb{T})$ in $H^1(\mathbb{T})$. By applying Lemma 2 on A_0 And B , we obtain that A is a Fredholm of index 0. \square

Theorem 8 *The operator A generates a C^0 -semigroup $\{S_+(t)\}_{t \geq 0}$ of contraction.*

Proof We start by checking that A is dissipative. Note that $\mathcal{D}(A) = H^2(\mathbb{T})$ is dense in $H^1(\mathbb{T})$. For $\mu \in \mathbb{R}$ and $u \in H^2(\mathbb{T})$, we have

$$\begin{aligned} \langle (A + i\mu)u, u \rangle &= \langle (A + i\mu)u, (1 - \partial_{xx})u \rangle \\ &= -(\partial_x u, u) + (\partial_{xx} u, \partial_x u) - (a\partial_x u, \partial_x u) + i\mu \|u\|_{H^1}^2 \end{aligned} \quad (3.5)$$

The quantity $(\partial_x u, u)$ is pure imaginary since

$$\Re(\partial_x u, u) = \frac{1}{2} \int_{\mathbb{T}} \partial_x u \bar{u} + \overline{\partial_x u} u dx = \frac{1}{2} ((\partial_x u, u) + (u, \partial_x u)) = 0.$$

By the same way we show that $\Re(\partial_{xxx} u, u) = 0$. Then

$$\Re \langle (A + i\mu)u, u \rangle = - \int_{\mathbb{T}} a |\partial_x u|^2 dx \leq 0. \quad (3.6)$$

So A is dissipative.

Now it suffices to prove that $A - \lambda$ is injective for some $\lambda > 0$. Let $u \in H^2(\mathbb{T})$ such that $Au - \lambda u = 0$. We have

$$0 = \Re \langle Au - \lambda u, u \rangle = - \int_{\mathbb{T}} a |\partial_x u|^2 dx - \lambda \|u\|_{H^1}^2 \geq (-\|a\|_{L^\infty} - \lambda) \|u\|_{H^1}^2.$$

This give us that $A - \lambda$ is injective.

Since A is Fredholm of index 0, and $u \mapsto \lambda u$ is an operator compact from $H^2(\mathbb{T})$ in $H^1(\mathbb{T})$, then $A - \lambda$ is also Fredholm of index 0, since it is injective, it is therefore surjective. We apply Theorem [3](#) and the proof is complete. \square

We now prove that we have a semigroup on \mathbb{R}_- .

Theorem 9 *The operator $-A$ generates a C^0 -semigroup $\{S_-(t)\}_{t \geq 0}$.*

Proof Let $u \in H^2(\mathbb{T})$. For $\lambda > 1 + \|a\|_{L^\infty(\mathbb{T})}^2$ we have

$$\langle (-A - \lambda)u, u \rangle = (\partial_{xxx}u + \partial_x u - \partial_x(a\partial_x)u - \lambda(1 - \partial_{xx})u, u). \quad (3.7)$$

Thus

$$\Re \langle (-A - \lambda)u, u \rangle = \int_{\mathbb{T}} a |\partial_x u|_{H^1(\mathbb{T})} - \lambda \|u\|_{H^1(\mathbb{T})}^2 \leq 0. \quad (3.8)$$

By the same way as the previous proof, we show that the operator $A + \lambda$ is surjective. This proves that $-A - \lambda$ generates a contraction semigroup that we denote by $\{\tilde{S}_-(t)\}_{t \geq 0}$.

We pose now $S_- : t \mapsto e^{\lambda t} \tilde{S}_-(t)$.

S_- defines a semigroup with infinitesimal generator $-A$. Indeed, for $u \in H^1(\mathbb{T})$ we have

$$\begin{aligned} \frac{S_-(t)u - u}{t} &= \frac{e^{\lambda t} \tilde{S}_-(t)u - u}{t} \\ &= \frac{e^{\lambda t} \left(\tilde{S}_-(t)u - u \right)}{t} + \frac{e^{\lambda t}u - u}{t} \\ &\rightarrow (-A - \lambda)u + \lambda u = -Au, \end{aligned} \quad (3.9)$$

when $t \rightarrow 0$, and this was to be demonstrated. \square

We use now a result of [Paz83](#) to define a group from two semigroups on \mathbb{R}_+ and \mathbb{R}_- .

Theorem 10 *Let A and $-A$ be the generators of two semigroups $\{S_+(t)\}_{t \geq 0}$ and $\{S_-(t)\}_{t \geq 0}$ respectively, then the operator A generates a group $\{S(t)\}_{t \in \mathbb{R}}$ defined by*

$$S(t) = \begin{cases} S_+(t), & \text{if } t \geq 0 \\ S_-(-t), & \text{if } t \leq 0 \end{cases} \quad (3.10)$$

Proof We start by checking that $\{S(t)\}_{t \in \mathbb{R}}$ given by [\(3.10\)](#) is a group. Let us first show that for all $t \in \mathbb{R}$ we have

$$S(t)S(-t)u = u, \quad \forall u \in H^1(\mathbb{T}). \quad (3.11)$$

Let $t \geq 0$ and $u \in H^2(\mathbb{T})$. If $t = 0$ we have the equality by definition of S_+ . We calculate now the derivative of (3.11)

$$\begin{aligned} (S(t)S(-t)u)' &= (S_+(t)S_-(t)u)' \\ &= AS_+(t)S_-(t)u - S_+(t)AS_-(t)u \\ &= (AS_+(t) - S_+(t)A)S_-(t)u \\ &= 0, \end{aligned} \quad (3.12)$$

since A commutes with S_+ . If $t \leq 0$ we have by the same way

$$\begin{aligned} (S(t)S(-t)u)' &= (S_-(-t)S_+(-t)u)' \\ &= AS_-(-t)S_+(-t)u - S_-(-t)AS_+(-t)u \\ &= (AS_-(-t) - AS_-(-t))S_+(-t)u \\ &= 0. \end{aligned} \quad (3.13)$$

From the density of $H^2(\mathbb{T})$ in $H^1(\mathbb{T})$ we obtain (3.11). Let us now show that

$$S(s+t) = S(s)S(t), \forall s, t \in \mathbb{R}. \quad (3.14)$$

Several cases arise. Note that if $s > 0, t > 0, s+t > 0$ or $s < 0, t < 0, s+t < 0$ we have (3.14) from the definition of S_+ and S_- respectively.

If $s > 0, t < 0, s+t > 0$ then

$$\begin{aligned} S(s)S(t) &= S_+(s)S_-(-t) \\ &= S_+(s+t-t)S_-(-t) \\ &= S_+(s+t)S_+(-t)S_-(-t) \\ &= S(s+t)S(-t)S(t) \\ &= S(s+t). \end{aligned}$$

If $s > 0, t < 0, s+t < 0$ then

$$\begin{aligned} S(s)S(t) &= S_+(s)S_-(-t) \\ &= S_+(s)S_-(s-s-t) \\ &= S_+(s)S_-(s)S_-(-(s+t)) \\ &= S(s)S(-s)S(s+t) \\ &= S(s+t). \end{aligned}$$

Likewise, if $t > 0, s < 0, s+t > 0$ then

$$\begin{aligned} S(s)S(t) &= S_-(-s)S_+(t) \\ &= S_-(-s)S_+(-s+s+t) \\ &= S_-(-s)S_+(-s)S_+(s+t) \\ &= S(s)S(-s)S(s+t) \\ &= S(s+t). \end{aligned}$$

If $t > 0, s < 0, s+t < 0$ then

$$\begin{aligned} S(s)S(t) &= S_-(-s)S_+(t) \\ &= S_-(-s-t+t)S_+(t) \\ &= S_-(-(s+t))S_-(t)S_+(t) \\ &= S(s+t)S(-t)S(t) \\ &= S(s+t). \end{aligned}$$

Hence we get (3.14). Now let $t, s \in \mathbb{R}$ and $u \in H^2(\mathbb{T})$. we have

$$\frac{S(t)u - u}{t}S(s) = \frac{S(t+s)u - S(s)u}{t} = S(s)\frac{S(t)u - u}{t}. \quad (3.15)$$

When t goes to 0, we obtain $AS(s) = S(s)A$. \square

We finish the subsection by giving this remark.

Remark 1 Taking $\lambda = 0$ in the proof of Theorem 8 we can see that A is the infinitesimal generator of a \mathcal{C}_0 semigroup of contractions $\{S(t)\}_{t \geq 0}$. Moreover, for u solution of (3.2), we have that $\|u(t)\|_{H^1(\mathbb{T})}^2$ is nonincreasing. Indeed

$$\begin{aligned} \frac{d}{dt} \left(\frac{1}{2} \|u(t)\|_{H^1}^2 \right) &= (\partial_t u, (1 - \partial_{xx})u)_{L^2} + (u, (1 - \partial_{xx})\partial_t u)_{L^2} \\ &= (-\partial_x u - \partial_{xxx} u + \partial_x(a\partial_x)u, u)_{L^2} \\ &\quad + (u, -\partial_x u - \partial_{xxx} u + \partial_x(a\partial_x)u)_{L^2} \\ &= -2 \int_{\mathbb{T}} a(x) |\partial_x u(t, x)|^2 dx \leq 0. \end{aligned} \quad (3.16)$$

3.3 Exponential stability

The goal of this subsection is to prove the exponential stability of $\{S(t)\}_{t \geq 0}$. More precisely, we will prove the following result.

Theorem 11 *There exists $\delta > 0$ and $M > 0$ such that*

$$\|S(t)\|_{\mathcal{L}(H^1(\mathbb{T}))} \leq Me^{-\delta t}, \quad \forall t \geq 0. \quad (3.17)$$

We will use the following theorem.

Theorem 12 *Let H be a Hilbert space, and $A : \mathcal{D}(A) \subset H \rightarrow H$ a generator infinitesimal of a semigroup $\{T(t)\}_{t \geq 0}$. Assume that*

1. *There exists $c_1 > 0$ such that $\|T(t)\|_{\mathcal{L}(H)} \leq c_1, \forall t \geq 0$,*
2. *$A + i\mu$ is invertible for all μ in \mathbb{R} ,*
3. *There exists $c_2 > 0$ such that $\|(A + i\mu)^{-1}\|_{\mathcal{L}(H)} \leq c_2$ for all μ in \mathbb{R} .*

Then, there exists $M > 0$ and $c > 0$ such that

$$\|T(t)\|_{\mathcal{L}(H)} \leq Me^{-ct}, \quad \forall t \geq 0. \quad (3.18)$$

See [Hua85, Theorem 3] for more details.

We cannot apply this theorem to A since it is not injective, the second assumption is not verified when $\mu = 0$. For that, we introduce the closed subspace of $H^1(\mathbb{T})$

$$\dot{H}^1(\mathbb{T}) = \left\{ u \in H^1(\mathbb{T}) ; (u, 1)_{L^2(\mathbb{T})} = 0 \right\}.$$

We equip $\dot{H}^1(\mathbb{T})$ and $H^2(\mathbb{T}) \cap \dot{H}^1(\mathbb{T})$ with the norms of $H^1(\mathbb{T})$ of $H^2(\mathbb{T})$ respectively.

Note that the space $\dot{H}^1(\mathbb{T})$ can be equipped by the norm $\|\partial_x \cdot\|_{L^2(\mathbb{T})}$ as shows the following proposition.

Proposition 7 *The norms $\|\cdot\|_{H^1(\mathbb{T})}$ and $\|\partial_x \cdot\|_{L^2(\mathbb{T})}$ are equivalent on $\dot{H}^1(\mathbb{T})$.*

See Annex for the proof.

We say that $Au \in \dot{H}^1(\mathbb{T})$ for $u \in H^1(\mathbb{T})$ since

$$\begin{aligned} (Au, 1)_{L^2(\mathbb{T})} &= (-(1 - \partial_{xx})^{-1} (\partial_{xxx}u + \partial_x u - \partial_x(a\partial_x u)), 1)_{L^2(\mathbb{T})} \\ &= (-\partial_x(1 - \partial_x)^{-1} (\partial_{xx}u + u - a\partial_x u), 1)_{L^2(\mathbb{T})} \\ &= ((1 - \partial_x)^{-1} (\partial_{xxx}u + u - a\partial_x u), \partial_x 1)_{L^2(\mathbb{T})} \\ &= 0. \end{aligned}$$

If $u(t, x) = \sum_{k \in \mathbb{Z}} u^k(t) e^{ikx} \in H^1(\mathbb{T})$, we can write $u(t, x) = u^0(t) + \dot{u}(t, x)$,

with $\dot{u} \in \dot{H}^1(\mathbb{T})$. It is easily checked that if $\partial_t u = Au$, then $u^0(t)$ is independent of t since

$$\partial_t u(t, x) = \partial_t u^0(t) + \partial_t \dot{u}(t, x) = Au(t, x).$$

Thus

$$(\partial_t u^0(t), 1)_{L^2(\mathbb{T})} + \underbrace{(\partial_t \dot{u}(t, x), 1)_{L^2(\mathbb{T})}}_{=0} = \underbrace{(Au(t, x), 1)_{L^2(\mathbb{T})}}_{=0}.$$

Then $\partial_t u^0(t) = 0$. So for $u \in H^1(\mathbb{T})$ we write $u(t, x) = u^0 + \dot{u}(t, x)$.

The space $\dot{H}^1(\mathbb{T})$ is then invariant by $\{S(t)\}_{t \geq 0}$, then it is also a semi-group of contraction on $\dot{H}^1(\mathbb{T})$. We thus define \dot{A} with $\mathcal{D}(\dot{A}) = H^2(\mathbb{T}) \cap \dot{H}^1(\mathbb{T})$ and $\dot{A}u = Au$ for $u \in \mathcal{D}(\dot{A})$. The operator \dot{A} is injective, indeed, for $u \in \dot{H}^1(\mathbb{T})$ such that $\dot{A}u = 0$, we can adapt the proof of Theorem [8](#) to obtain $\partial_x u = 0$ on \mathbb{T} . As $u \in \dot{H}^1(\mathbb{T})$, then $u = 0$. Furthermore, $\dot{A} : H^2(\mathbb{T}) \cap \dot{H}^1(\mathbb{T}) \rightarrow \dot{H}^1(\mathbb{T})$ is a Fredholm operator of index 0, so it is bijective. Then we have the second assumption of Theorem [12](#) for all $\mu \in \mathbb{R}$. Using bounded inverse theorem, A^{-1} is continuous. The third assumption of

Theorem [12](#) is thus checked for $\mu = 0$. We define now the resolvent of \dot{A} by

$$\mathcal{R}(\lambda) : \begin{array}{ccc} \dot{H}^1(\mathbb{T}) & \longrightarrow & \dot{H}^1(\mathbb{T}) \\ f & \longmapsto & \dot{u} \end{array} \quad (3.19)$$

solution of $(\dot{A} - \lambda)\dot{u} = f$.

Proposition 8 *There exists $c > 0$ such that for all $f \in \dot{H}^1(\mathbb{T})$ we have*

$$\forall \mu \in \mathbb{R}, \|\dot{u}(t)\|_{\dot{H}^1(\mathbb{T})} \leq c \|f\|_{\dot{H}^1(\mathbb{T})},$$

where \dot{u} is solution of $(\dot{A} + i\mu)\dot{u} = f$.

Proof Let $\mu_0 > 0$. We distinguish two cases.

1st case : $|\mu| \leq \mu_0$.

The resolvent $\mu \mapsto (A + i\mu)^{-1}$ is well defined from \mathbb{R} to $\mathcal{L}(\dot{H}^1(\mathbb{T}), \dot{H}^1(\mathbb{T}))$. Indeed, let u be such that $Au + i\mu u = 0$. Using the same arguments seen in [\(3.5\)](#) and [\(3.6\)](#) we obtain

$$\Re \langle (A + i\mu)u, u \rangle = - \int_{\mathbb{T}} a |\partial_x u|^2 dx = 0.$$

This give us that $\partial_x u = 0$ on $\{a > 0\}$. The equation $Au + i\mu u = 0$ becomes then

$$-\partial_{xxx}u - \partial_x u + i\mu u = 0. \quad (3.20)$$

Now let $x_0 \in \{a > 0\}$. The function $\partial_x \bar{u}$ is solution of [\(3.20\)](#) with the conditions

$$\partial_x u(x_0) = \partial_{xx}u(x_0) = \partial_{xxx}u(x_0) = 0,$$

which means that $\partial_x u \equiv 0$. So u is constant. Since u belongs to $\dot{H}^1(\mathbb{T})$, we deduce that $u \equiv 0$. Hence the operator $A + i\mu$ is injective. Since it is Fredholm of index 0, it is bijective. Furthermore, we know that the resolvent is continuous from \mathbb{R} to $\mathcal{L}(\dot{H}^1(\mathbb{T}), \dot{H}^1(\mathbb{T}))$. So there exists $c = c_{\mu_0} > 0$ such that

$$\forall \mu \in [-\mu_0, \mu_0], \left\| (A + i\mu)^{-1} \right\|_{\mathcal{L}(\dot{H}^1, \dot{H}^1)} \leq c.$$

2nd case : $|\mu| > \mu_0$.

Without loss of generality, we may consider from now on the operator A in $H^1(\mathbb{T})$ to simplify the notation instead of \dot{A} in $\dot{H}^1(\mathbb{T})$. By an argument of symmetry, we know that $\mathcal{R}(i\mu)\bar{f} = \overline{\mathcal{R}(-i\mu)f}$ for $f \in \dot{H}^1(\mathbb{T})$. Indeed, Let u such that $(A + i\mu)u = f$, with $\mu < -\mu_0$. On the one hand $\overline{(A + i\mu)u} = \bar{f}$, so $(A - i\mu)\bar{u} = \bar{f}$. Then $\bar{u} = \mathcal{R}(i\mu)\bar{f}$. On the other hand, $u = \mathcal{R}(-i\mu)f$. Thus $\bar{u} = \overline{\mathcal{R}(-i\mu)f}$, and hence $\|\mathcal{R}(-i\mu)f\| = \left\| \overline{\mathcal{R}(-i\mu)f} \right\| = \|\mathcal{R}(i\mu)\bar{f}\|$. So it

is sufficient to prove the result for $\mu > \mu_0$.

We will prove by contradiction the resolvente estimate. Assume that

$$\forall k \in \mathbb{N}, \exists f^k \in H^1(\mathbb{T}), \exists \mu_k \geq k / \|u^k\|_{H^1(\mathbb{T})} > k \|f^k\|_{H^1(\mathbb{T})}, \quad (3.21)$$

where u^k is solution of $(A + i\mu_k)u^k = f^k$. We can assume that $\|u^k\|_{H^1(\mathbb{T})} = 1$ and $(f^k)_{k \in \mathbb{N}} \rightarrow 0$ in $H^1(\mathbb{T})$, when $k \rightarrow +\infty$.

We notice that the sequence (μ_k) goes to $+\infty$. Indeed, suppose that (μ_k) stay bounded. We know that the map $\mu \rightarrow \|(A + i\mu)^{-1}\|_{\mathcal{L}(H^1)}$ is continuous, so $\|(A + i\mu)^{-1}\|_{\mathcal{L}(H^1)}$ is bounded. In this case we have

$$\|u^k\|_{H^1(\mathbb{T})} \leq \|(A + i\mu_k)^{-1}\|_{\mathcal{L}(H^1)} \|f^k\|_{H^1(\mathbb{T})} \rightarrow 0,$$

when k goes to $+\infty$, which contradicts the fact that $\|u^k\|_{H^1(\mathbb{T})} = 1$.

We consider the equation

$$(1 - \partial_{xx})^{-1} (\partial_{xxx} u^k + \partial_x u^k - \partial_x (a \partial_x u^k)) + i\mu_k u^k = f^k. \quad (3.22)$$

We note $(u^k)_k = \sum_{n \in \mathbb{Z}} u_n^k e^{inx}$, $(f^k)_k = \sum_{n \in \mathbb{Z}} f_n^k e^{inx}$ and $a = \sum_{n \in \mathbb{Z}} a_n e^{inx}$. We

know that if $a \partial_x u^k = \sum_{l \in \mathbb{Z}} (a \partial_x u^k)_l e^{ilx}$ then $(a \partial_x u^k)_l = \sum_{j \in \mathbb{Z}} i a_j (l - j) u_{l-j}^k$.

So we define \tilde{a}_n^k when $(1 - \partial_{xx})^{-1} \partial_x (a \partial_x u^k) = \sum_{n \in \mathbb{Z}} \tilde{a}_n^k e^{inx}$ by

$$\begin{aligned} \tilde{a}_n^k &= \{(1 - \partial_{xx})^{-1} (\partial_x (a \partial_x u^k))\}_n = \frac{in}{1 + n^2} (a \partial_x u^k)_n \\ &= \frac{-n}{1 + n^2} \sum_{j \in \mathbb{Z}} a_j (n - j) u_{n-j}^k. \end{aligned} \quad (3.23)$$

The equation (3.22) written with Fourier coefficients is

$$\frac{1}{1 + n^2} (-in^3 + in) u_n^k - \tilde{a}_n^k + i\mu_k u_n^k = f_n^k. \quad (3.24)$$

Let $\delta \in]0, \frac{1}{4}[$. Let us first take the frequencies n such that $|n - \mu_k| \geq \delta \mu_k$. We have

$$\frac{1}{1 + n^2} (-in^3 + in) + i\mu_k = i(-n + \mu_k) + \frac{2in}{1 + n^2}.$$

So we can write (3.24) under the form

$$u_n^k = \frac{-2n}{(-n + \mu_k)(1 + n^2)} u_n^k + \frac{\tilde{a}_n^k}{i(-n + \mu_k)} + \frac{f_n^k}{i(-n + \mu_k)}. \quad (3.25)$$

Since $\frac{1}{|n - \mu_k|} \leq \frac{1}{\delta\mu_k}$, we have

$$\begin{aligned} \left\| \sum_{|n-\mu_k| \geq \delta\mu_k} \frac{-2n}{(-n + \mu_k)(1 + n^2)} u_n^k e^{inx} \right\|_{H^1}^2 &= \sum_{|n-\mu_k| \geq \delta\mu_k} \frac{4(1 + n^2)}{(-n + \mu_k)^2} \underbrace{\frac{n^2}{(1 + n^2)^2}}_{\leq 1} |u_n^k|^2 \\ &\leq \frac{4}{(\delta\mu_k)^2} \sum_{|n-\mu_k| \geq \delta\mu_k} (1 + n^2) |u_n^k|^2 \\ &\leq \frac{4}{(\delta\mu_k)^2} \|u^k\|_{H^1(\mathbb{T})}^2 = \frac{4}{(\delta\mu_k)^2}. \end{aligned} \quad (3.26)$$

The operator $(1 - \partial_{xx})^{-1} (\partial_x(a\partial_x))$ is bounded on $H^1(\mathbb{T})$. Indeed

$$\begin{aligned} (1 + n^2) |\tilde{a}_n^k|^2 &= \frac{n^2}{1 + n^2} \left(\sum_{j \in \mathbb{Z}} a_j (n - j) |u_{n-j}^k| \right)^2 \\ &\lesssim \left(\sum_{j \in \mathbb{Z}} \frac{1}{(1 + j)^\sigma} (n - j) |u_{n-j}^k| \right)^2. \end{aligned}$$

This is true for all $\sigma > 1$ from the regularity of a . So $\left(\frac{1}{(1 + j)^\sigma} \right)_j \in \ell^1$ and $(nu_n)_n \in \ell^2$. From (2.5), we deduce that

$$\sum_{n \in \mathbb{Z}} (1 + n^2) |\tilde{a}_n^k|^2 \leq c.$$

Thus

$$\begin{aligned} \left\| \sum_{|n-\mu_k| \geq \delta\mu_k} \frac{\tilde{a}_n^k}{i(-n + \mu_k)} e^{inx} \right\|_{H^1}^2 &= \sum_{|n-\mu_k| \geq \delta\mu_k} \frac{1}{(-n + \mu_k)^2} (1 + n^2) |\tilde{a}_n^k|^2 \\ &\leq \frac{1}{(\delta\mu_k)^2} \sum_{n \in \mathbb{N}} (1 + n^2) |\tilde{a}_n^k|^2 \leq \frac{c}{(\delta\mu_k)^2}. \end{aligned} \quad (3.27)$$

We have also

$$\left\| \sum_{|n-\mu_k| \geq \delta\mu_k} \frac{f_n^k}{i(-n + \mu_k)} e^{inx} \right\|_{H^1}^2 = \sum_{|n-\mu_k| \geq \delta\mu_k} \frac{(1 + n^2)}{(-n + \mu_k)^2} |f_n^k|^2 \leq \frac{c}{(\delta\mu_k)^2}, \quad (3.28)$$

since $f \rightarrow 0$ in $H^1(\mathbb{T})$. Combining, (3.25), (3.26), (3.27) and (3.28), we obtain

$$\sum_{|n-\mu_k| \geq \delta\mu_k} (1 + n^2) |u_n^k|^2 \leq \frac{C}{(\delta\mu_k)^2} \rightarrow 0, \text{ when } k \rightarrow \infty. \quad (3.29)$$

We conclude that

$$\sum_{(1-\delta)\mu_k \leq n \leq (1+\delta)\mu_k} (1+n^2) |u_n^k|^2 \approx 1.$$

For the part where n is of the order of μ_k , we introduce a smooth fonction Ψ such that $0 \leq \Psi \leq 1$ and

$$\Psi(s) = \begin{cases} 1, & \text{if } |s-1| \leq \delta \\ 0, & \text{if } |s-1| \geq 2\delta \end{cases} \quad (3.30)$$

Let $h = \frac{1}{\mu_k}$. The operator $\Psi(-ih\partial_x)$ is defined by

$$\Psi(-ih\partial_x)u^k = \sum_{n \in \mathbb{Z}} \Psi(hn)u_n^k e^{inx}.$$

Note that $\Psi(hn) \neq 0$ for $n \in [(1-2\delta)\mu_k, (1+2\delta)\mu_k]$ since

$$\Psi(hn) = \begin{cases} 1, & \text{if } |n - \mu_k| \leq \delta\mu_k \\ 0, & \text{if } |n - \mu_k| \geq 2\delta\mu_k. \end{cases} \quad (3.31)$$

We can divide the frequencies which are of the order of μ_k and the frequencies which remain by the sum

$$u^k = (u^k - \Psi u^k) + \Psi u^k. \quad (3.32)$$

Indeed, when $|n - \mu_k| \leq \delta\mu_k$ we have

$$|n| \leq |n - \mu_k| + |\mu_k| \leq (1 + \delta)|\mu_k|,$$

and

$$|\mu_k| \leq |\mu_k - n| + |n| \leq \delta|\mu_k| + |n|.$$

Hence

$$(1 - \delta)|\mu_k| \leq |n| \leq (1 + \delta)|\mu_k|.$$

Since $\Psi(hn) = 1$ when $n \in [(1 - \delta)\mu_k, (1 + \delta)\mu_k]$, we have

$$\begin{aligned} \|u^k - \Psi(-ih\partial_x)u^k\|_{H^1(\mathbb{T})}^2 &= \sum_{n \in \mathbb{Z}} (1+n^2)(1 - \Psi(hn))^2 |u_n^k|^2 \\ &\leq \sum_{|n - \mu_k| \geq \delta\mu_k} (1+n^2) |u_n^k|^2 \leq \frac{C}{(\delta\mu_k)^2}. \end{aligned} \quad (3.33)$$

So for μ_k large enough, we can assume that $\|\Psi(-ih\partial_x)u^k\|_{H^1(\mathbb{T})} \geq \frac{1}{2}$.

Now let $v^k = \frac{\Psi(-ih\partial_x)u^k}{\|\Psi(-ih\partial_x)u^k\|_{L^2(\mathbb{T})}}$, and we have

$$\begin{aligned} & (1 - \partial_{xx})^{-1} (\partial_{xxx} + \partial_x - \partial_x(a\partial_x)) v^k + i\mu_k v^k \\ &= \frac{\Psi(-ih\partial_x)f^k}{\|\Psi(-ih\partial_x)u^k\|_{L^2}} + (1 - \partial_{xx})^{-1} \left(\partial_x [\Psi(-ih\partial_x), a] \frac{\partial_x u^k}{\|\Psi(-ih\partial_x)u^k\|_{L^2}} \right). \end{aligned} \quad (3.34)$$

We can easily check that

$$\|\Psi(-ih\partial_x)u^k\|_{L^2(\mathbb{T})} \approx h \|\Psi(-ih\partial_x)u^k\|_{H^1(\mathbb{T})} \approx h. \quad (3.35)$$

Indeed

$$\begin{aligned} 1 &\approx \|\Psi(-ih\partial_x)u^k\|_{H^1}^2 = \|\Psi(-ih\partial_x)u^k\|_{L^2}^2 + \|\partial_x(\Psi(-ih\partial_x)u^k)\|_{L^2}^2 \\ &= \|\Psi(-ih\partial_x)u^k\|_{L^2}^2 + h^{-2} \|h\partial_x(\Psi(-ih\partial_x)u^k)\|_{L^2}^2 \\ &= \sum_{hn \approx 1} (1 + n^2) |\Psi(hn)|^2 |u_n^k|^2 \\ &\approx \frac{1}{h^2} \sum_{hn \approx 1} |\Psi(hn)|^2 |u_n^k|^2 = \frac{1}{h^2} \|\Psi(-ih\partial_x)u^k\|_{L^2}^2. \end{aligned}$$

The term $\frac{\Psi(-ih\partial_x)f^k}{\|\Psi(-ih\partial_x)u^k\|_{L^2(\mathbb{T})}}$ tends towards 0 in $L^2(\mathbb{T})$ when k goes to $+\infty$ since

$$\frac{\|\Psi(-ih\partial_x)f^k\|_{L^2(\mathbb{T})}}{\|\Psi(-ih\partial_x)u^k\|_{L^2(\mathbb{T})}} \approx \frac{h \|f^k\|_{H^1(\mathbb{T})}}{\|\Psi(-ih\partial_x)u^k\|_{L^2(\mathbb{T})}} \approx \|f^k\|_{H^1(\mathbb{T})} \longrightarrow 0. \quad (3.36)$$

We apply $h^3(1 - \partial_{xx})$ to (3.34), we obtain

$$\begin{aligned} & (h\partial_x)^3 v^k + h^2 (h\partial_x) v^k - h (h\partial_x) (a (h\partial_x)) v^k - i (h\partial_x)^2 v^k + ih^2 v^k \\ &= \frac{h^3(1 - \partial_{xx})}{\|\Psi(-ih\partial_x)u^k\|_{L^2}} \Psi(-ih\partial_x)f^k + \frac{h (h\partial_x)}{\|\Psi(-ih\partial_x)u^k\|_{L^2}} [\Psi(-ih\partial_x), a] (h\partial_x) u^k. \end{aligned} \quad (3.37)$$

We will show now with the two following lemmas that we have

$$\begin{aligned} & \left\| \frac{h^3(1 - \partial_{xx})}{\|\Psi(-ih\partial_x)u^k\|_{L^2}} \Psi(-ih\partial_x)f^k \right. \\ & \quad \left. + \frac{h (h\partial_x)}{\|\Psi(-ih\partial_x)u^k\|_{L^2}} [\Psi(-ih\partial_x), a] (h\partial_x) u^k \right\|_{L^2} = o(h). \end{aligned} \quad (3.38)$$

Lemma 3 *There exists $c > 0$ such that*

$$\left\| \frac{h^3(1 - \partial_{xx})}{\|\Psi(-ih\partial_x)u^k\|_{L^2(\mathbb{T})}} \Psi(-ih\partial_x)f^k \right\|_{L^2(\mathbb{T})} \leq ch \|f^k\|_{H^1(\mathbb{T})}. \quad (3.39)$$

Proof Let $\tilde{\Psi} : s \mapsto -s^2\Psi(s)$. From (3.35) we have

$$\begin{aligned} \left\| \frac{h^3(1 - \partial_{xx})}{\|\Psi(-ih\partial_x)u^k\|_{L^2}} \Psi(-ih\partial_x)f^k \right\|_{L^2} &\lesssim \|(h^2 - h^2\partial_{xx})\Psi(-ih\partial_x)f^k\|_{L^2} \\ &\leq h^2 \|\Psi(-ih\partial_x)f^k\|_{L^2} + \|\tilde{\Psi}(h\mathcal{D})f^k\|_{L^2} \\ &\leq h^2 \|f^k\|_{L^2} + h \|\tilde{\Psi}(h\mathcal{D})f^k\|_{H^1} \\ &\leq ch \|f^k\|_{H^1}. \end{aligned}$$

□

Lemma 4 *There exists $c > 0$ such that for $z \in L^2(\mathbb{T})$*

$$\|[\Psi(-ih\partial_x), a]z\|_{H^1(\mathbb{T})} \leq c\|z\|_{L^2(\mathbb{T})}.$$

Proof Note that if $[\Psi(-ih\partial_x), a]u^k = \sum_{n \in \mathbb{Z}} \gamma_n^k e^{inx}$, then

$$\gamma_n^k = \Psi(hn) \sum_{j \in \mathbb{Z}} a_{n-j} u_j^k - \sum_{j \in \mathbb{Z}} a_{n-j} \Psi(hj) u_j^k = \sum_{j \in \mathbb{Z}} a_{n-j} (\Psi(hn) - \Psi(hj)) u_j^k.$$

We want to prove that

$$\begin{aligned} \sum_{n \in \mathbb{Z}} (1 + n^2) |\gamma_n^k|^2 &= \sum_{n \in \mathbb{Z}} (1 + n^2) \left(\sum_{j \in \mathbb{Z}} a_{n-j} |\Psi(hn) - \Psi(hj)| |z_j| \right)^2 \\ &\leq c \sum_{n \in \mathbb{Z}} |z_n|^2. \end{aligned} \tag{3.40}$$

The constant δ is the one used in (3.30).

We write $\mathbb{Z}^2 = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4 \cup \Gamma_5$, where

$$\Gamma_1 = \{(n, j) \in \mathbb{Z}^2; n \leq 0 \text{ and } |hj - 1| \leq 2\delta\},$$

$$\Gamma_2 = \{(n, j) \in \mathbb{Z}^2; 0 \leq hn \leq 1 - 3\delta \text{ and } |hj - 1| \leq 2\delta\},$$

$$\Gamma_3 = \{(n, j) \in \mathbb{Z}^2; hn \geq 1 + 3\delta \text{ and } |hj - 1| \leq 2\delta\},$$

$$\Gamma_4 = \{(n, j) \in \mathbb{Z}^2; |hn - 1| \leq 3\delta\},$$

and

$$\Gamma_5 = \{(n, j) \in \mathbb{Z}^2; |hn - 1| \geq 3\delta \text{ and } |hj - 1| \geq 2\delta\}.$$

In Γ_1 , we have $\Psi(hn) = 0$ and $|n - j| = |n| + j$. We know also that for all σ we have $|a_n| \leq \frac{1}{(1 + |n|)^\sigma}$. Thus

$$\begin{aligned} \sqrt{1 + n^2} |\gamma_n| &\leq c \sum_{j \in \mathbb{Z}} \underbrace{\frac{\sqrt{1 + n^2}}{1 + |n| + j}}_{\leq c} \frac{1}{(1 + |n - j|)^{\sigma-1}} \underbrace{|\Psi(hj)|}_{\leq 1} |z_j| \\ &\leq c \left(\sum_{j \in \mathbb{Z}} \frac{1}{(1 + |n - j|)^{\sigma-1}} \right)^{\frac{1}{2}} \left(\sum_{j \in \mathbb{Z}} \frac{1}{(1 + |n - j|)^{\sigma-1}} |z_j|^2 \right)^{\frac{1}{2}} \\ &\leq c \left(\sum_{j \in \mathbb{Z}} \frac{1}{(1 + |n - j|)^{\sigma-1}} |z_j|^2 \right)^{\frac{1}{2}}. \end{aligned}$$

Then

$$\sum_{n \in \mathbb{Z}} (1 + n^2) |\gamma_n|^2 \leq c \sum_{j \in \mathbb{Z}} |z_j|^2 \sum_{n \in \mathbb{Z}} \frac{1}{(1 + |n - j|)^{\sigma-1}}.$$

The term $\sum_{n \in \mathbb{Z}} \frac{1}{(1 + |n - j|)^{\sigma-1}}$ converges and it does not depend on j .

We deduce the estimate (3.40) on Γ_1 .

In Γ_2 , $\Psi(hn) = 0$, and we have

$$h + |hn - hj| \geq h + |hn - 1| - |hj - 1| \geq h + 3\delta - 2\delta \geq \delta.$$

It follows that $\frac{1}{1 + |n - j|} \leq ch$, thus

$$\sqrt{1 + n^2} |\gamma_n| \leq c \sum_{j \in \mathbb{Z}} \frac{h\sqrt{1 + n^2}}{(1 + |n - j|)^{\sigma-1}} |z_j|.$$

Since in Γ_2 we have $hn \leq 1 - 3\delta$, the term $h\sqrt{1 + n^2}$ is bounded independently of n . We obtain with the same Hölder inequality as in Γ_1 the estimate (3.40).

In Γ_3 , $\Psi(hn) = 0$ and we have also

$$h + |hn - hj| \geq h + hn - hj \geq h + 1 + 3\delta - (2\delta + 1) \geq \delta.$$

So

$$\frac{1}{1 + |n - j|} \lesssim h. \quad (3.41)$$

Furthermore

$$n \lesssim |n - j| + |j| \lesssim |n - j| + \frac{1}{h}. \quad (3.42)$$

By multiplying (3.41) and (3.42) we get

$$\frac{n}{1 + |n - j|} \lesssim h|n - j| + 1.$$

Hence we estimate

$$\sqrt{1 + n^2} |\gamma_n| \lesssim \sum_{j \in \mathbb{Z}} \frac{h|n - j| + 1}{(1 + |n - j|)^{\sigma-1}} |z_j| \lesssim \sum_{j \in \mathbb{Z}} \frac{1}{(1 + |n - j|)^{\sigma-2}} |z_j|,$$

and we use again Hölder inequality as in Γ_1 to obtain (3.40).

In Γ_4 , we use the mean value theorem to write

$$|\Psi(hn) - \Psi(hj)| \leq ch|n - j|.$$

Then

$$\sqrt{1 + n^2} |\gamma_n| \leq c \sum_{j \in \mathbb{Z}} \frac{|n - j|}{(1 + |n - j|)^\sigma} |z_j| \leq c \sum_{j \in \mathbb{Z}} \frac{1}{(1 + |n - j|)^{\sigma-1}} |z_j|.$$

As in Γ_2 , we obtain (3.40).

In Γ_5 , we have $\Psi(hn) = \Psi(hj) = 0$. The result follows directly. \square

We deduce from Lemma 4 taking $z = \partial_x u^k \in L^2(\mathbb{T})$ that

$$\begin{aligned} \left\| \frac{h^3}{\|\Psi(-ih\partial_x)u^k\|_{L^2}} \partial_x [\Psi(-ih\partial_x), a] \partial_x u^k \right\|_{L^2} &\leq h^2 \|\Psi(-ih\partial_x), a\|_{H^1} \|\partial_x u^k\|_{L^2} \\ &\leq ch^2 \|\partial_x u^k\|_{L^2} \\ &\leq ch^2 \|u^k\|_{H^1} = ch^2. \end{aligned} \quad (3.43)$$

Then (3.39) and (3.43) give (3.38). Furthermore

$$\|ih^2 v^k\|_{L^2(\mathbb{T})} = o(h), \text{ and } \|h^2(h\partial_x)v^k\|_{L^2(\mathbb{T})} = o(h), \quad (3.44)$$

since $\|v^k\|_{L^2(\mathbb{T})} = 1$.

We give also the following lemma.

Lemma 5 *There exists $c > 0$ such that*

$$\|(h\partial_x)(a(h\partial_x))v^k\|_{L^2(\mathbb{T})} \leq c. \quad (3.45)$$

Proof We recall that

$$\Psi(hn) = \begin{cases} 1, & \text{if } |hn - 1| \leq \delta \\ 0, & \text{if } |hn - 1| \geq 2\delta, \end{cases}$$

and for k large enough, $\|\Psi(-ih\partial_x)u^k\|_{L^2} \approx h$.

We have $(h\partial_x)(a(h\partial_x)) = a(h\partial_x)^2 + [h\partial_x, ah\partial_x]$. Thus

$$\|(h\partial_x)(a(h\partial_x))v^k\|_{L^2(\mathbb{T})} \leq \|a(h\partial_x)^2v^k\|_{L^2(\mathbb{T})} + \|[h\partial_x, ah\partial_x]v^k\|_{L^2(\mathbb{T})}$$

We estimate the first quantity.

$$\begin{aligned} \|a(h\partial_x)^2v^k\|_{L^2(\mathbb{T})}^2 &\leq c \|(h\partial_x)^2v^k\|_{L^2(\mathbb{T})}^2 \\ &\leq \sum_{n \in \mathbb{Z}} h^4 n^4 \frac{\psi^2(hn)(u_n^k)^2}{h^2} \\ &\leq c \sum_{n \in \mathbb{Z}} n^2 \underbrace{h^2 n^2 \psi^2(hn)}_{\leq c} (u_n^k)^2 \\ &\leq c \sum_{n \in \mathbb{Z}} n^2 (u_n^k)^2 \\ &\leq c \|u^k\|_{H^1(\mathbb{T})}^2 = c. \end{aligned}$$

Now we write

$$[h\partial_x, ah\partial_x] = h\partial_x(ah\partial_x) - a(h\partial_x)^2 = ha'h\partial_x + a(h\partial_x)^2 - a(h\partial_x)^2 = ha'h\partial_x.$$

Then

$$\begin{aligned} \|[h\partial_x, ah\partial_x]v^k\|_{L^2(\mathbb{T})}^2 &= \left\| ha'h\partial_x \frac{\Psi(-ih\partial_x)u^k}{\|\Psi(-ih\partial_x)u^k\|_{L^2(\mathbb{T})}} \right\|_{L^2(\mathbb{T})}^2 \\ &\leq c \sum_{n \in \mathbb{Z}} h^4 n^2 \frac{\psi^2(hn)(u_n^k)^2}{h^2} \\ &\leq c \sum_{n \in \mathbb{Z}} h^2 n^2 \psi^2(hn)(u_n^k)^2 \\ &\leq ch^2 \sum_{n \in \mathbb{Z}} n^2 (u_n^k)^2 \\ &\leq ch^2 \|u^k\|_{H^1(\mathbb{T})}^2 = ch^2. \end{aligned}$$

This give us [\(3.45\)](#). □

We can write $\|h(h\partial_x)(a(h\partial_x))v^k\|_{L^2(\mathbb{T})} = O(h)$. Now [\(3.37\)](#) becomes

$$(h\mathcal{D})^3v^k - (h\mathcal{D})^2v^k = hg^k, \quad (3.46)$$

where g is a bounded function in $L^2(\mathbb{T})$. We can now give the theorem of existence of semiclassical measure, which is a classical theorem and may be found in [\[Bur97, Theorem 2\]](#) or [\[Gé91a, Proposition 3.1\]](#).

Theorem 13 *Let $(h_k)_k$ be a sequence of reals which converges to 0 and (u^{h_k}) a bounded sequence in $L^2(\mathbb{T})$ which converges weakly to 0. There exists a*

subsequence from (u^{h_k}) -which we called also (u^{h_k}) - and a non negative Radon measure ν on $\mathbb{T} \times \mathbb{R}$ such that for every $b \in C_0^\infty(\mathbb{T} \times \mathbb{R}; \mathbb{C})$

$$\lim_{h_k \rightarrow 0} (b(x, h_k \mathcal{D})u^{h_k}, u^{h_k})_{L^2(\mathbb{T})} = \int_{\mathbb{T} \times \mathbb{R}} b(x, \xi) d\nu(x, \xi).$$

ν is called the semiclassical measure associated to the sequence (u^{h_k}) .

We will apply this theorem to the sequence (v^{h_k}) , so we check first that it converges weakly to 0.

Proposition 9 *The sequence (v^{h_k}) converges weakly to 0.*

Proof We know that (v^{h_k}) converges weakly since $\|v^{h_k}\|_{L^2(\mathbb{T})} = 1$.

We write $v^{h_k} = \sum_{n \in \mathbb{Z}} v_n^h e^{inx} = \sum_{n \in \mathbb{Z}} \frac{\Psi(hn)u_n^{h_k}}{\|\Psi(hn)u_n^{h_k}\|_{L^2}} e^{inx}$, and we pose

$$\varphi = \sum_{n \in \mathbb{Z}} \varphi_n e^{inx} \in C_0^\infty(\mathbb{T}).$$

So we have $|\varphi_n| \leq \frac{1}{(1+n^2)^\sigma}$ for all $\sigma > 0$. Let Θ be a C^∞ function such that $0 \leq \Theta \leq 1$ and

$$\Theta(s) = \begin{cases} 1, & \text{if } |s-1| \leq 2\delta \\ 0, & \text{if } |s-1| \geq 3\delta. \end{cases} \quad (3.47)$$

Note that $\Theta\Psi = \Psi$. We have

$$\begin{aligned} \left| \langle v^{h_k}, \varphi \rangle_{L^2, L^2} \right| &= \sum_{n \in \mathbb{Z}} v_n^h \overline{\varphi_n} \\ &= \sum_{n \in \mathbb{Z}} \Theta(hn) \Psi(hn) \frac{u_n^{h_k}}{\|\Psi(hn)u_n^{h_k}\|_{L^2}} \overline{\varphi_n} \\ &\lesssim \left(\sum_{n \in \mathbb{Z}} \Theta^2(hn) |\varphi_n|^2 \right)^{\frac{1}{2}} \underbrace{\left(\sum_{n \in \mathbb{Z}} |u_n^{h_k}|^2 \right)^{\frac{1}{2}}}_{=1} \\ &\lesssim \left(\sum_{n \in \mathbb{Z}} \underbrace{\Theta^2(hn)}_{\leq 1} \frac{1}{(1+n^2)^{2\sigma}} \right)^{\frac{1}{2}} \\ &\lesssim \frac{1}{\left(1 + \frac{1}{h^2}\right)^{\frac{2\sigma-1}{2}}} \underbrace{\left(\sum_{n \in \mathbb{Z}} \frac{1}{1+n^2} \right)^{\frac{1}{2}}}_{< \infty}, \end{aligned} \quad (3.48)$$

since n is of the order of $\frac{1}{h}$ when $\Theta(hn) \neq 0$. This quantity goes to 0 as $h \rightarrow 0$, and the proof is complete. \square

Remark 2 We can prove that the measure ν is bounded using Gårding inequality, more precisely

$$\int_{\mathbb{T} \times \mathbb{R}} d\nu(x, \xi) \leq 1. \quad (3.49)$$

Let us study where the measure ν is supported.

Proposition 10 We have

$$\nu(x, \xi) = \tilde{\nu}(x) \otimes \delta_{\xi=1}. \quad (3.50)$$

Proof We multiply the equation (3.46) by a function $b \in \mathcal{C}_0^\infty(\mathbb{T} \times \mathbb{R}; \mathbb{C})$

$$b(x, h\mathcal{D}) ((h\mathcal{D})^3 - (h\mathcal{D})^2) v^k = hb(x, h\mathcal{D})g^k = O(h).$$

Note that the symbol of $b(x, h\mathcal{D}) ((h\mathcal{D})^3 - (h\mathcal{D})^2)$ is $b(x, \xi) (\xi^3 - \xi^2)$ which belongs to $\mathcal{C}_0^\infty(\mathbb{T} \times \mathbb{R}; \mathbb{C})$. According to Theorem 13, we have on the one hand

$$(b(x, h\mathcal{D}) ((h\mathcal{D})^3 - (h\mathcal{D})^2) v^k, v^k)_{L^2} \longrightarrow \int_{\mathbb{T} \times \mathbb{R}} b(x, \xi) (\xi^3 - \xi^2) d\nu(x, \xi),$$

and on the other hand

$$(O(h), v^k)_{L^2(\mathbb{T})} \longrightarrow 0. \quad (3.51)$$

Then

$$\int_{\mathbb{T} \times \mathbb{R}} b(x, \xi) (\xi^3 - \xi^2) d\nu(x, \xi) = 0. \quad (3.52)$$

Now we want to show that

$$\text{supp}(\nu) \subset \mathbb{T} \times \{\xi = 1\}.$$

Let $b \in \mathcal{C}_0^\infty(\mathbb{R}; \mathbb{C})$ independent of x and $\chi \in \mathcal{C}^\infty(\mathbb{R})$ such that

$$\chi(\xi) = \begin{cases} 1, & \text{if } |\xi| \leq \delta \\ 0, & \text{if } |\xi| \geq 2\delta, \end{cases}$$

where the constant δ is the same one used in (3.30).

We want to prove that

$$\int_{\mathbb{T} \times \mathbb{R}} b(\xi)(1 - \xi) d\nu(x, \xi) = 0. \quad (3.53)$$

We write $b = b_1 + b_2$ with $b_1 = \chi b$ and $b_2 = (1 - \chi)b$. Clearly

$$\text{supp}(b_1) \subset \{\mathbb{R} \setminus \text{supp}(\Psi)\}. \quad (3.54)$$

On the one hand

$$\begin{aligned} & ((1 - h\mathcal{D})b_1(h\mathcal{D})v^k, v^k)_{L^2} = \\ & \left((1 - h\mathcal{D}) \underbrace{b_1(h\mathcal{D})\Psi(h\mathcal{D})}_{=0} \frac{u^k}{\|\Psi(h\mathcal{D})u^k\|}, \Psi(h\mathcal{D}) \frac{u^k}{\|\Psi(h\mathcal{D})u^k\|} \right)_{L^2} = 0. \end{aligned} \quad (3.55)$$

From Theorem 13 we have

$$\int_{\mathbb{T} \times \mathbb{R}} (1 - \xi)b_1(\xi)d\nu(x, \xi) = 0, \quad (3.56)$$

for every b_1 such that (3.54).

On the other hand

$$\begin{aligned} \int_{\mathbb{T} \times \mathbb{R}} b_2(\xi)(1 - \xi)d\nu(x, \xi) &= \int_{\mathbb{T} \times \mathbb{R}} (1 - \chi(\xi))b(\xi)(1 - \xi)d\nu(x, \xi) \\ &= \int_{\mathbb{T} \times \mathbb{R}} \frac{(1 - \chi(\xi))b(\xi)}{\xi^2} (\xi^3 - \xi^2)d\nu(x, \xi) \\ &= 0, \end{aligned} \quad (3.57)$$

from (3.52) by replacing $b(x, \xi)$ with $\frac{(1 - \chi(\xi))b(\xi)}{\xi^2}$. This give us (3.53) for every $b \in \mathcal{C}_0^\infty(\mathbb{R}; \mathbb{C})$. Then $(1 - \xi)\nu = 0$ and hence we obtain (3.50). \square

Proposition 11 *The measure $\nu \equiv 0$ everywhere.*

Proof We first show that ν is an uniform measure on $\{\xi = 1\}$, i.e. $\nu(x, \xi) = \tilde{\nu} \otimes \delta_{\xi=1}$, that is $\tilde{\nu}$ does not depend on x . We start by giving the following lemma.

Lemma 6 *Let a_1 and a_2 be two symbols in \mathcal{C}^∞ , then*

$$[\text{Op}_h(a_1), \text{Op}_h(a_2)] = \frac{h}{i} \text{Op}_h(\{a_1, a_2\}) + h^2 O(1). \quad (3.58)$$

See [Zwo12, Theorem 4.12] for a proof.

We come back to the equation (3.37). From (3.38) and (3.44) we have

$$(h\mathcal{D})^3 v^k - (h\mathcal{D})^2 v^k + ih(h\mathcal{D})(a(x)(h\mathcal{D})v^k) = o(h), \quad (3.59)$$

where $o(h)$ is a function going to 0 in $L^2(\mathbb{T})$ when h goes to 0.

We first prove that $\nu = 0$ on $\{a > 0\}$. Let $b \in \mathcal{C}_0^\infty(\mathbb{T} \times \mathbb{R}; \mathbb{R})$ such that $b(x, \xi) = b(\xi)$. Since b is real and does not depend on x , we know from [Zwo12, Theorem 4.1] that the operator $b(h\mathcal{D})$ is self adjoint, and the operator $h\mathcal{D}$ is also self adjoint. Taking the imaginary part of the inner product of (3.59) with $b(h\mathcal{D})v^k$ we get

$$h\Im \left(ib(h\mathcal{D})(h\mathcal{D}) \left(a(x)(h\mathcal{D})v^k \right), v^k \right)_{L^2(\mathbb{T})} = \Im \left(o(h), v^k \right)_{L^2(\mathbb{T})}.$$

Thus

$$\Im \left(ib(h\mathcal{D})(h\mathcal{D}) \left(a(x)(h\mathcal{D})v^k \right), v^k \right)_{L^2(\mathbb{T})} \longrightarrow 0.$$

By Theorem 13 we know that

$$\left(b(h\mathcal{D})(h\mathcal{D}) \left(a(x)(h\mathcal{D})v^k \right), v^k \right)_{L^2} \longrightarrow \int_{\mathbb{T} \times \mathbb{R}} a(x)\xi^2 b(\xi) d\nu(x, \xi) = 0.$$

That means $\nu = 0$ in $\{a > 0\}$.

Now let $b = b(x, \xi) \in \mathcal{C}_0^\infty(\mathbb{T} \times \mathbb{R}; \mathbb{R})$, $P = (h\mathcal{D})^3 - (h\mathcal{D})^2 + ih(h\mathcal{D}) \left(a(x)(h\mathcal{D}) \right)$, and $p = \xi^3 - \xi^2$ its principal symbol. We have

$$\begin{aligned} & \left((Pb(x, h\mathcal{D}) - b(x, h\mathcal{D})P)v^k, v^k \right) \\ &= \left(b(x, h\mathcal{D})v^k, P^*v^k \right) - \left(b(x, h\mathcal{D})Pv^k, v^k \right) \\ &= \left(b(x, h\mathcal{D})v^k, \left((h\mathcal{D})^3 - (h\mathcal{D})^2 - ih(h\mathcal{D}) \left(a(x)(h\mathcal{D}) \right) \right) v^k \right) \\ & \quad - \left(b(x, h\mathcal{D})Pv^k, v^k \right) \\ &= \left(b(x, h\mathcal{D})v^k, \left((h\mathcal{D})^3 - (h\mathcal{D})^2 + ih(h\mathcal{D}) \left(a(x)(h\mathcal{D}) \right) \right) v^k \right) \\ & \quad - \left(b(x, h\mathcal{D})v^k, 2ih(h\mathcal{D}) \left(a(x)(h\mathcal{D}) \right) v^k \right) - \left(b(x, h\mathcal{D})Pv^k, v^k \right). \end{aligned}$$

On the one hand

$$\frac{1}{h} \left((Pb(x, h\mathcal{D}) - b(x, h\mathcal{D})P)v^k, v^k \right) = \frac{1}{i} \left(\text{Op}_h(\{p, b\})v^k, v^k \right) + o(1),$$

according to (3.58), and converges to $\frac{1}{i} \int_{\mathbb{T} \times \mathbb{R}} \{p, b\} d\nu(x, \xi)$ by Theorem 13.

On the other hand

$$\begin{aligned} & \frac{1}{h} \left(b(x, h\mathcal{D})Pv^k, v^k \right) \\ &= \frac{1}{h} \left(\underbrace{b(x, h\mathcal{D}) \left((h\mathcal{D})^3 - (h\mathcal{D})^2 + ih(h\mathcal{D}) \left(a(x)(h\mathcal{D}) \right) \right) v^k}_{=o(h)}, v^k \right) \longrightarrow 0. \end{aligned} \tag{3.60}$$

The term

$$\frac{1}{h} \left(b(x, h\mathcal{D})v^k, 2ih(h\mathcal{D}) \left(a(x)(h\mathcal{D}) \right) v^k \right) \rightarrow -2i \int_{\mathbb{T} \times \mathbb{R}} a(x)\xi^2 b(x, \xi) d\nu(x, \xi).$$

Then

$$\frac{1}{i} \int_{\mathbb{T} \times \mathbb{R}} \{p, b\} d\nu(x, \xi) = 2i \int_{\mathbb{T} \times \mathbb{R}} a(x) \xi^2 b(x, \xi) d\nu(x, \xi) = 0. \quad (3.61)$$

Now we have

$$\begin{aligned} 0 &= \int_{\mathbb{T} \times \mathbb{R}} \{p, b\} d\nu = \langle \nu, H_p b \rangle = - \langle H_p \nu, b \rangle = - \langle \partial_\xi p \partial_x \nu - \partial_x p \partial_\xi \nu, b \rangle \\ &= - \langle (3\xi^2 - 2\xi) \partial_x \nu, b \rangle. \end{aligned} \quad (3.62)$$

This is true for every b . Thus $(3\xi^2 - 2\xi) \partial_x \nu = 0$. Since ν is supported in $\{\xi = 1\}$ and $(3\xi^2 - 2\xi)|_{\xi=1} = 1$, we get $\partial_x \nu = 0$. So, $\tilde{\nu}(x)$ does not depend of x . Since $\nu = 0$ on the support on a , we deduce that $\nu \equiv 0$ everywhere. \square

To finish we verify at the same time that $\nu \neq 0$. For that, we recall that

$$v_h^k = \frac{\Psi(h\mathcal{D})u_h^k}{\|\Psi(h\mathcal{D})u_h^k\|_{L^2(\mathbb{T})}}.$$

Let Θ be a function in \mathcal{C}^∞ such that $0 \leq \Theta \leq 1$ and

$$\Theta(s) = \begin{cases} 1, & \text{if } |s - 1| \leq 2\delta \\ 0, & \text{if } |s - 1| \geq 3\delta. \end{cases} \quad (3.63)$$

Note that $\Theta\Psi = \Psi$. On the one hand

$$\lim_{h \rightarrow 0} (\text{Op}_h(\Theta)v_h^k, v_h^k)_{L^2(\mathbb{T})} = \int_{\mathbb{T} \times \mathbb{R}} \Theta(\xi) d\nu(x, \xi),$$

and on the other hand

$$\begin{aligned} (\text{Op}_h(\Theta)v_h^k, v_h^k)_{L^2(\mathbb{T})} &= \frac{1}{\|\Psi(h\mathcal{D})u_h^k\|_{L^2}} (\text{Op}_h(\Theta)\text{Op}_h(\Psi)u_h^k, v_h^k)_{L^2(\mathbb{T})} \\ &= \frac{1}{\|\Psi(h\mathcal{D})u_h^k\|_{L^2}} (\text{Op}_h(\Psi)u_h^k, v_h^k)_{L^2(\mathbb{T})} \\ &= \|v_h^k\|_{L^2(\mathbb{T})}^2 = 1. \end{aligned}$$

Thus $\int_{\mathbb{T} \times \mathbb{R}} \Theta(\xi) d\nu(x, \xi) = 1$. This means that ν cannot be identically 0. This is absurd, so the assumption [\(3.21\)](#) is false.

We can now return to the operator \hat{A} . From the above, we have

$$\forall \mu \neq 0, \forall f \in H^1(\mathbb{T}), \exists u \in \mathcal{D}(A); (A + i\mu)u = f.$$

Furthermore, there exists $c > 0$ which does not depend on μ such that

$$\|u(t)\|_{H^1(\mathbb{T})} \leq c \|f\|_{H^1(\mathbb{T})}.$$

In particular, if $f \in \dot{H}^1(\mathbb{T})$, we have $u \in \dot{H}^1(\mathbb{T})$ since

$$i\mu(u, 1)_{L^2} = (f, 1)_{L^2} - (Au, 1)_{L^2} = 0.$$

In this case, we can write for all $\mu \neq 0$

$$\|u(t)\|_{\dot{H}^1(\mathbb{T})} = \|u(t)\|_{H^1(\mathbb{T})} \leq c\|f\|_{H^1(\mathbb{T})} = c\|f\|_{\dot{H}^1(\mathbb{T})}.$$

So $(\dot{A} + i\mu)^{-1} = (A + i\mu)|_{\dot{H}^1}^{-1}$. From Theorem [12](#), we have for all $t \geq 0$,

$$\|S(t)\|_{\mathcal{L}(H^1, H^1)} \leq Me^{-\delta t}.$$

□

We finish this subsection by showing the exponential stability in $H^2(\mathbb{T})$, which will be useful later.

Lemma 7 *There exist $M, \delta > 0$ such that for all $t \geq 0$ we have*

$$\|S(t)\|_{\mathcal{L}(\dot{H}^2)} \leq Me^{-\delta t}. \quad (3.64)$$

Proof Let $u(t) \in H^2(\mathbb{T}) \cap \dot{H}^1(\mathbb{T})$ and $t \geq 0$. We have

$$\begin{aligned} \|S(t)u\|_{H^2(\mathbb{T})} &\simeq \|AS(t)u\|_{H^1(\mathbb{T})} + \|S(t)u\|_{H^1(\mathbb{T})} \\ &= \|S(t)Au\|_{H^1(\mathbb{T})} + \|S(t)u\|_{H^1(\mathbb{T})} \\ &\leq Me^{-\delta t}\|Au\|_{H^1(\mathbb{T})} + Me^{-\delta t}\|u\|_{H^1(\mathbb{T})} \\ &\leq Me^{-\delta t}\|u\|_{H^2(\mathbb{T})}. \end{aligned}$$

□

3.4 Nonlinear equation

We consider now the nonlinear equation

$$\begin{cases} \partial_t u + \partial_x u - \partial_{xxt} u + \partial_{xxx} u - \partial_x(a(x)\partial_x u) + u\partial_x u = 0, & x \in \mathbb{T}, \\ u(., 0) = u_0 \in H^1(\mathbb{T}), & x \in \mathbb{T}. \end{cases} \quad (3.65)$$

We show in this paragraph that this equation admits a unique solution defined over all \mathbb{R} . The principal theorem of this subsection is the following result.

Theorem 14 *Let $R > 0$. There is a unique solution of [\(3.65\)](#) that exists on \mathbb{R} . Moreover, for $T > 0$, the map*

$$\begin{array}{ccc} H^1(\mathbb{T}) & \longrightarrow & \mathcal{C}\left([-T, T], \dot{H}^1(\mathbb{T})\right) \\ u_0 & \longmapsto & u, \end{array}$$

where u is the solution of the nonlinear problem with the initial data u_0 , is Lipschitz on $\overline{B_{\dot{H}^1(\mathbb{T})}(0, R)}$.

The first equation of (3.65) is equivalent to $\partial_t u = Au - (1 - \partial_{xx})^{-1} u \partial_x u$. The local existence is a consequence of the Picard fixed point theorem. Let $R > 0$ and $0 < T \leq 1$. We define

$$B_{R,T} = \left\{ u \in \mathcal{C}([-T, T], \dot{H}^1(\mathbb{T})) ; \sup_{s \in [-T, T]} \|u(s)\|_{\dot{H}^1(\mathbb{T})} \leq R \right\}.$$

We equip $\mathcal{C}([-T, T], \dot{H}^1(\mathbb{T}))$ with the distance

$$d(u, v) = \sup_{s \in [-T, T]} \|u(s) - v(s)\|_{\dot{H}^1(\mathbb{T})}.$$

The space $(\mathcal{C}([-T, T], \dot{H}^1(\mathbb{T})), d)$ is a complete metric space. For u_0 in $\dot{H}^1(\mathbb{T})$ and $t \in [-T, T]$, we introduce

$$\begin{aligned} \Phi : \mathcal{C}([-T, T], \dot{H}^1(\mathbb{T})) &\longrightarrow \mathcal{C}([-T, T], \dot{H}^1(\mathbb{T})) \\ u &\longmapsto S(t)u_0 \\ &\quad - \int_0^t S(t-s)(1 - \partial_{xx})^{-1} u(s) \partial_x u(s) ds. \end{aligned}$$

It is clear that $\Phi(u) \in \mathcal{C}([-T, T], \dot{H}^1(\mathbb{T}))$ when $u \in \mathcal{C}([-T, T], \dot{H}^1(\mathbb{T})) \subset$

$\mathcal{C}([-T, T], L^\infty(\mathbb{T}))$ since $\partial_x u \in \mathcal{C}([-T, T], L^2(\mathbb{T}))$, then $u \partial_x u \in \mathcal{C}([-T, T], L^2(\mathbb{T}))$, which implies

$$(1 - \partial_{xx})^{-1} u \partial_x u \in \mathcal{C}([-T, T], H^2(\mathbb{T}) \cap \dot{H}^1(\mathbb{T})).$$

So the quantities $S(t)u_0$ and $S(t-s)(1 - \partial_{xx})^{-1} u \partial_x u \in \mathcal{C}([-T, T], \dot{H}^1(\mathbb{T}))$.

Lemma 8 *There exists $C_0 > 0$ such that*

$$\forall M > 0, \exists R, T > 0 ; RT < C_0, \forall \|u_0\|_{\dot{H}^1(\mathbb{T})} \leq M, \Phi : B_{R,T} \longrightarrow B_{R,T},$$

is a contraction.

Proof It is easy to show that for every $g \in \mathcal{C}([-T, T], L^2(\mathbb{T}))$ we have for every $t \in [-T, T]$

$$\left\| \int_0^t S(t-s)(1 - \partial_{xx})^{-1} g(s) ds \right\|_{\dot{H}^1(\mathbb{T})} \leq cT e^\delta \sup_{s \in [-T, T]} \|g(s)\|_{L^2(\mathbb{T})}.$$

Let $u, v \in B_{R,T}$. We have

$$\begin{aligned} \Phi(u)(x, t) - \Phi(v)(x, t) &= - \int_0^t S(t-s)(1 - \partial_{xx})^{-1} (u(s)\partial_x u(s) \\ &\quad - v(s)\partial_x v(s)) ds \\ &= - \int_0^t S(t-s)(1 - \partial_{xx})^{-1} \{ (u(s) - v(s)) \partial_x u(s) \\ &\quad + (\partial_x u(s) - \partial_x v(s)) v(s) \} ds. \end{aligned}$$

As $\dot{H}^1(\mathbb{T}) \hookrightarrow L^\infty(\mathbb{T})$ continuously, we have

$$\begin{aligned} \|\Phi(u)(t) - \Phi(v)(t)\|_{H^1} &\leq cTe^\delta \sup_{s \in [-T, T]} (\|u(s) - v(s)\|_{L^\infty} \|\partial_x u(s)\|_{L^2} \\ &\quad + \|\partial_x u(s) - \partial_x v(s)\|_{L^2} \|v(s)\|_{L^\infty}) \\ &\leq cTe^\delta \sup_{s \in [-T, T]} (\|u(s) - v(s)\|_{H^1} \|\partial_x u(s)\|_{L^2} \\ &\quad + \|\partial_x u(s) - \partial_x v(s)\|_{L^2} \|v(s)\|_{H^1}) \\ &\leq 2cTRe^\delta \sup_{s \in [-T, T]} \|u(s) - v(s)\|_{H^1} \\ &= 2cTRe^\delta d(u, v). \end{aligned} \tag{3.66}$$

Taking the supremum on t we get

$$d(\Phi(u), \Phi(v)) \leq 2cTRe^\delta d(u, v).$$

So, for $T \leq \min \left\{ \frac{e^{-\delta}}{4cR}, 1 \right\}$, Φ is a contraction.

Let us now show that if $u \in B_{R,T}$ then $\Phi(u) \in B_{R,T}$. From (3.66) and taking $v = 0$ we obtain

$$\begin{aligned} \left| \|\Phi(u)(t)\|_{H^1} - \|\Phi(0)(t)\|_{H^1} \right| &\leq \|\Phi(u)(t) - \Phi(0)(t)\|_{H^1} \\ &\leq 2cTRe^\delta \sup_{s \in [-T, T]} \|u(s)\|_{H^1}. \end{aligned}$$

Let $M > 0$ and $u_0 \in H^1(\mathbb{T})$ be such that $\|u_0\|_{H^1(\mathbb{T})} \leq M$. We can write

$$\|\Phi(u)(t)\|_{H^1(\mathbb{T})} \leq e^\delta \|u_0\|_{H^1(\mathbb{T})} + 2cTR^2 e^\delta \leq (M + 2cTR^2) e^\delta \tag{3.67}$$

Choosing $R \geq 4Me^\delta + \frac{e^{-\delta}}{4c}$ we get $\|\Phi(u)(t)\|_{H^1(\mathbb{T})} < R$. Since $R > \frac{e^{-c}}{4c}$, this estimate is valid when $T \leq \frac{e^{-\delta}}{4cR}$, and the proof is complete. \square

Before showing the global existence, we give the following result of the semigroup.

Proposition 12 *There exists $c > 0$ which depends only on the function $a(x)$ such that the solution u of the nonlinear problem which exists on $] - T, T[$ verifies*

$$\|u(t)\|_{H^1(\mathbb{T})} \leq \|u_0\|_{H^1(\mathbb{T})} e^{\delta|t|}, \quad \forall t \in] - T, T[. \tag{3.68}$$

Proof Let $t \in]-T, 0[$, we write

$$\begin{aligned} \|u(t)\|_{H^1(\mathbb{T})}^2 - \|u(0)\|_{H^1(\mathbb{T})}^2 &= \int_0^t \partial_t \|u(\sigma)\|_{H^1(\mathbb{T})}^2 d\sigma \\ &= -4 \int_0^t \int_{\mathbb{T}} a(x) |\partial_x u(\sigma)|^2 dx d\sigma \\ &\leq 4 \|a\|_{L^\infty(\mathbb{T})} \int_t^0 \|u(\sigma)\|_{H^1(\mathbb{T})}^2 d\sigma. \end{aligned}$$

Then

$$\|u(t)\|_{H^1(\mathbb{T})}^2 \leq c \int_t^0 \|u(\sigma)\|_{H^1(\mathbb{T})}^2 d\sigma + \|u(0)\|_{H^1(\mathbb{T})}^2.$$

Using Gronwall inequality

$$\|u(t)\|_{H^1(\mathbb{T})}^2 \leq \|u(0)\|_{H^1(\mathbb{T})}^2 e^{-ct}, \quad \forall t \in]-T, 0[.$$

We can repeat the same arguments for $t \in [0, T[$. This was to be demonstrated.

□

We can now prove Theorem [14](#)

Proof (of Theorem [14](#)) Let u and v be two solutions of [\(3.65\)](#) admitting the same initial data. The function $w = u - v$ verifies

$$\partial_t w + \partial_x w - \partial_{xxt} w + \partial_{xxx} w - \partial_x(a(x)\partial_x)w + w\partial_x u + v\partial_x w = 0.$$

We know that

$$w(t) = \int_0^t S(t-s)(1 - \partial_{xx})^{-1} (w(s)\partial_x u(s) + v(s)\partial_x w(s)) ds.$$

According to [\(3.68\)](#) we can deduce that $\|\partial_x u\|_{L^2(\mathbb{T})} \leq c_1$ and $\|v(s)\|_{L^2(\mathbb{T})} \leq c_2$. It follows for $t \geq 0$

$$\begin{aligned} \|w(t)\|_{H^1} &= \left\| \int_0^t S(t-s)(1 - \partial_{xx})^{-1} (w(s)\partial_x u(s) + v(s)\partial_x w(s)) ds \right\|_{H^1} \\ &\leq \int_0^t \max\{c_1, c_2\} \|w(s)\|_{H^1} ds. \end{aligned}$$

By Gronwall's inequality, we obtain $\|w(t)\|_{H^1(\mathbb{T})} = 0$. We can do the same calculus for $t \leq 0$, so that we have uniqueness.

To prove that the solution is global, we recall that for the local existence of the solution, by taking R and T such that $RT \leq \frac{e^{-\delta}}{4c}$, we found a solution which exists on $[0, T]$.

Now let $T^* = \sup\{t \geq 0 ; u \text{ exists on } [0, t]\}$ and $0 < T_1 < T^*$.

Suppose that $T^* < \infty$. From Proposition [12](#) if

$$\|u(T_1)\|_{H^1(\mathbb{T})} \leq \|u_0\|_{H^1(\mathbb{T})} e^{\delta T_1} = R_1,$$

then we have a solution which exists on $[T_1, T]$ as soon as $(T - T_1)R_1 \leq \frac{e^{-\delta}}{4c}$. That means

$$T \leq \frac{e^{-\delta}}{4cR_1} + T_1 = \frac{e^{-\delta}}{4c\|u_0\|_{H^1} e^{\delta T_1}} + T_1 = f(T_1).$$

It is clear that $f(T^*) > T^*$, since f is continuous, there exists $T < T^*$ such that $f(T) > T^*$. This means that the solution starting from T_1 will exist beyond T^* , which is absurd. This implies $T^* = +\infty$.

The same argument can be repeated for $t \leq 0$.

Once we know that the solution of the nonlinear problem exists on all \mathbb{R} , we can repeat the same proof as that of Proposition [12](#) with all $T > 0$ to write

$$\|u(t)\|_{H^1(\mathbb{T})} \leq \|u_0\|_{H^1(\mathbb{T})} e^{c|t|}, \quad \forall t \in \mathbb{R}, \quad (3.69)$$

where u is the solution of the nonlinear problem which exists on all \mathbb{R} . Now we finish the proof by proving that the map $u_0 \mapsto u$ is locally Lipschitz continuous from $H^2(\mathbb{T}) \cap \dot{H}^1(\mathbb{T})$ to $\mathcal{C}([-T, T], \dot{H}^1(\mathbb{T}))$. Let u and v be two solutions of the nonlinear problem with initial data u_0 and v_0 respectively, and $T > 0$. We have $\forall t \in [-T, T]$,

$$\begin{aligned} u(t) - v(t) &= S(t)(u_0 - v_0) \\ &\quad - \int_0^t S(t-s) \underbrace{(1 - \partial_{xx})^{-1} (u(s)\partial_x u(s) - v(s)\partial_x v(s))}_{=(Fu-Fv)(s)} ds. \end{aligned}$$

But

$$\begin{aligned} \|(Fu - Fv)(s)\|_{H^1} &= \|((u(s) - v(s))\partial_x u(s) - (\partial_x u(s) - \partial_x v(s))v(s))\|_{L^2} \\ &\leq c(\|u(s) - v(s)\|_{L^\infty} \|\partial_x u(s)\|_{L^2} \\ &\quad + \|\partial_x u(s) - \partial_x v(s)\|_{L^2} \|v(s)\|_{L^\infty}) \\ &\leq c(\|u(s)\|_{H^1} + \|v(s)\|_{H^1}) \|u(s) - v(s)\|_{H^1}. \end{aligned} \quad (3.70)$$

Then there exists $c > 0$ which depend on T such that

$$\begin{aligned} \|u(t) - v(t)\|_{H^1} &\leq c\|u_0 - v_0\|_{H^1} \\ &\quad + c \left| \int_0^t (\|u(s)\|_{H^1} + \|v(s)\|_{H^1}) \|u(s) - v(s)\|_{H^1} ds \right| \\ &\leq c\|u_0 - v_0\|_{H^1} \\ &\quad + c(\|u_0\|_{H^1} + \|v_0\|_{H^1}) \left| \int_0^t \|u(s) - v(s)\|_{H^1} ds \right|. \end{aligned}$$

By Gronwall inequality

$$\|u(t) - v(t)\|_{H^1(\mathbb{T})} \leq C\|u_0 - v_0\|_{H^1(\mathbb{T})}, \quad (3.71)$$

with $C = ce^{\delta T(\|u_0\|_{H^1} + \|v_0\|_{H^1})}$. \square

3.5 Convergence of the solutions

We start by introducing a function called u_∞ , which is a limit in a certain way of the solution given in the following theorem. The advantage of u_∞ is that it is an analytic function in time, this property will be used later to show that u_∞ is a constant.

Theorem 15 *Let $(t_n)_n$ be a nondecreasing sequence of times which goes to $+\infty$, and u_0 a real valued initial data in $\dot{H}^1(\mathbb{T})$ with u the corresponding solution of (3.65). Then, there exists a subsequence $(t_{\phi(n)})_n$ and an analytic function in time u_∞ such that*

$$\forall T > 0, \lim_{n \rightarrow +\infty} u(t_{\phi(n)} + \cdot) = u_\infty(\cdot) \text{ in } \mathcal{C}^0([-T, T], \dot{H}^1(\mathbb{T})). \quad (3.72)$$

We can easily prove the convergence of $u(t_{\phi(n)})$ towards a function u_∞ . We start by giving this remark.

Remark 3 *If u_0 real valued, the corresponding solution u of (3.65) is bounded for $t \geq 0$. Moreover*

$$\|u(t)\|_{H^1(\mathbb{T})} \leq \|u_0\|_{H^1(\mathbb{T})}, \quad \forall t \geq 0. \quad (3.73)$$

Indeed, note that u is real when u_0 is real. We recall that the norm $\|\cdot\|_{H^1(\mathbb{T})}^2$ of the solution of the linear problem is nonincreasing. Since $\Re(u\partial_x u, u) = 0$, the same calculus as in Remark 1 gives us that $\|u(t)\|_{H^1(\mathbb{T})}$ is nonincreasing where $u(t)$ is the solution of nonlinear problem.

Now let $(t_n)_n$ be a sequence which goes to $+\infty$, according to Duhamel's formula

$$u(t_n) = S(t_n)u_0 - \int_0^{t_n} S(s)(1 - \partial_{xx})^{-1}u(t_n - s)\partial_x u(t_n - s)ds.$$

When $t_n \rightarrow +\infty$, the term $S(t_n)u_0$ goes to 0 from (3.17).

Since $u \in L^\infty([0, +\infty[, \dot{H}^1(\mathbb{T})) \subset L^\infty([0, +\infty[\times \mathbb{T})$, and $\partial_x u \in L^\infty([0, +\infty[, L^2(\mathbb{T}))$, then $u\partial_x u \in L^\infty([0, +\infty[, L^2(\mathbb{T}))$. Which implies $(1 - \partial_{xx})^{-1}u\partial_x u \in L^\infty([0, +\infty[, H^2(\mathbb{T}) \cap \dot{H}^1(\mathbb{T}))$. From (9) we have thus

$$\begin{aligned} \left\| \int_0^{t_n} S(s)(1 - \partial_{xx})^{-1}u(t_n - s)\partial_x u(t_n - s)ds \right\|_{H^2} &\leq c \int_0^{t_n} \|S(s)\|_{\mathcal{L}(H^2)} ds \\ &\leq c \int_0^{+\infty} e^{-\delta s} ds < \infty. \end{aligned}$$

The sequence $\left(\int_0^{t_n} S(s)(1 - \partial_{xx})^{-1}u(t_n - s)\partial_x u(t_n - s)ds \right)_n$ is uniformly bounded in $H^2(\mathbb{T})$, and then, it converges weakly in $H^2(\mathbb{T})$ up to a subsequence $(\phi_n)_n$ and so, strongly in $H^1(\mathbb{T})$ to u_∞ .

Now let $\left\{ \tilde{S}(t) \right\}_{t \in \mathbb{R}}$ be the nonlinear semigroup, in other words $u(t) = \tilde{S}(t)u_0$. On the one hand, for $s \in \mathbb{R}$ we have

$$\tilde{S}(s)u(t_{\phi_n}) = \tilde{S}(t_{\phi_n} + s)u_0 = u(t_{\phi_n} + s).$$

On the other hand, we use the continuity property of u seen in Theorem [14](#) to write

$$\lim_{n \rightarrow +\infty} \tilde{S}(s)u(t_{\phi_n}) = \tilde{S}(s)u_\infty = u_\infty(s).$$

This give us the limit [\(3.72\)](#).

Remark 4 The function u_∞ given by Theorem [15](#) will be now considered as initial data. We will write it down $u_\infty(0)$, and we write down u_∞ the corresponding solution.

Now let us prove the analyticity of u_∞ . To do that we shall apply the following theorem, see [\[HR03, Theorem 2.20\]](#) with assumptions (H3mod) and (H5).

Theorem 16 Let Y be a complex Banach space. Let $P_n \in \mathcal{L}(Y)$ be a sequence of continuous linear maps and let $Q_n = Id - P_n$. Let $A : \mathcal{D}(A) \rightarrow Y$ be the generator of a continuous semigroup $\{e^{tA}\}_{t \geq 0}$ and let $G \in \mathcal{C}^1(Y)$. We assume that V is a complete mild solution in Y of

$$V'(t) = AV(t) + G(V(t)), \quad t \in \mathbb{R}. \quad (3.74)$$

We further assume that

1. $\{V(t), t \in \mathbb{R}\}$ is contained in a compact set K of Y .
2. For any $y \in Y$, $(P_n y)_n$ converges to y when $n \rightarrow +\infty$ and (P_n) and $(Q_n)_n$ are sequences of $\mathcal{L}(Y)$ bounded by K_0 .
3. The operator A splits in $A = A_1 + B_1$ where B_1 is bounded and A_1 commutes with P_n .
4. There exists $M, \lambda > 0$ such that $\|e^{At}\|_{\mathcal{L}(Y)} \leq Me^{-\lambda t}$ and $\|e^{(A_1 + Q_n B B^*)t}\|_{\mathcal{L}(Q_n Y, Y)} \leq Me^{-\lambda t}$ for all $t \geq 0$.
5. G is analytic in the ball $B_Y(0, r)$, where r is such that $r \geq 4 \sup_{t \in \mathbb{R}} \|V(t)\|_Y$.
6. $\{DG(V(t))Z ; t \in \mathbb{R}, \|Z\|_Y \leq 1\}$ is relatively compact set of Y .

Then, the solution $V(t)$ is analytic from $t \in \mathbb{R}$ into Y .

We use this theorem taking

$$A : u \longmapsto -(1 - \partial_{xx})^{-1}(\partial_x u + \partial_{xxx} u - \partial_x(a(x)\partial_x)u)$$

and

$$G : u \longmapsto -(1 - \partial_{xx})^{-1}(u\partial_x u).$$

Note that u_∞ verifies the same equation as u . We deduce from this the existence of u_∞ on all \mathbb{R} as well as the first part of the assumption 4.

We check the rest of the assumptions.

Proposition 13 *Let $u_\infty(0)$ be the function obtained by Theorem 15 and $u_\infty(t)$ the corresponding solution. Then there exists $c > 0$ such that*

$$\sup_{t \in \mathbb{R}} \|u_\infty(t)\|_{H^1(\mathbb{T})} \leq c. \quad (3.75)$$

Proof Let $t \in \mathbb{R}$ and $(t_{\phi(n)})_n$ the subsequence given by Theorem 15. Using Proposition 12, there exists $N_1 > 0$ be such that for all $n \geq N_1$ we have $\|u_\infty(t) - u(t_{\phi(n)} + t)\|_{H^1} \leq 1$. On the other hand, there exist $N_2 > 0$ such that for $n \geq N_2$ we have $t_{\phi(n)} + t > 0$. We deduce from (3.69) that for $n \geq N_1 + N_2$

$$\begin{aligned} \|u_\infty(t)\|_{H^1} &\leq \|u_\infty(t) - u(t_{\phi(n)} + t)\|_{H^1} + \|u(t_{\phi(n)} + t)\|_{H^1} \\ &\leq 1 + \|u(0)\|_{H^1} \\ &\leq c. \end{aligned}$$

It follows the estimate (3.75). \square

Proposition 14 *Let $u_\infty(0)$ be the function obtained by Theorem 15 and $u_\infty(t)$ the corresponding solution. Then there exist $c > 0$ such that*

$$\sup_{t \in \mathbb{R}} \|u_\infty(t)\|_{H^2(\mathbb{T})} \leq c. \quad (3.76)$$

Proof Let $t \in \mathbb{R}$, $n \in \mathbb{N}$ and v a solution of the nonlinear problem such that $v(\cdot) = u_\infty(\cdot + t - n)$. We can write

$$v(\sigma) = S(\sigma)v(0) - \int_0^\sigma S(\sigma - s)(1 - \partial_{xx})^{-1}v(s)\partial_x v(s)ds.$$

Note that $v(n - t) = u_\infty(0)$. Thus, we can use a simple change of variables to get

$$u_\infty(t) = S(n)u_\infty(t - n) + \int_0^n S(\tau)(1 - \partial_{xx})^{-1}u_\infty(t - \tau)\partial_x u_\infty(t - \tau)d\tau. \quad (3.77)$$

We know that $\|S(n)u_\infty(t-n)\|_{H^1(\mathbb{T})} \leq ce^{-\delta n} \|u_\infty(t-n)\|_{H^1(\mathbb{T})} \leq Ce^{-\delta n}$. Then, when n goes to $+\infty$, (3.77) becomes

$$u_\infty(t) = \int_0^{+\infty} S(\tau)(1 - \partial_{xx})^{-1}u_\infty(t-\tau)\partial_x u_\infty(t-\tau)d\tau.$$

Since the operator $u \mapsto (1 - \partial_{xx})^{-1}u\partial_x u$ is bounded from $\dot{H}^1(\mathbb{T})$ into $H^2(\mathbb{T}) \cap \dot{H}^1(\mathbb{T})$, using Lemma 7 and Proposition 13 we obtain

$$\|u_\infty(t)\|_{H^2(\mathbb{T})} \leq c \int_0^{+\infty} e^{-\delta\tau} \|u_\infty(t-\tau)\|_{H^1(\mathbb{T})}^2 d\tau \leq c \int_0^{+\infty} e^{-\delta\tau} d\tau \leq C.$$

$\{u_\infty(t) ; t \in \mathbb{R}\}$ is then a bounded set of $H^2(\mathbb{T}) \cap \dot{H}^1(\mathbb{T})$, and consequently it is a relatively compact set of $\dot{H}^1(\mathbb{T})$. \square

We deduce that u_∞ satisfies the first assumption of Theorem 16 with $K = \overline{\{u_\infty(t), t \in \mathbb{R}\}}$.

We define $(P_n)_n$ and $(Q_n)_n$ for $u = \sum_{k \in \mathbb{Z}^*} u_k e^{ikx} \in \dot{H}^1(\mathbb{T})$ with

$$P_n(u) = \sum_{|k| \leq n} u_k e^{ikx} \quad \text{and} \quad Q_n(u) = \sum_{|k| > n} u_k e^{ikx}.$$

Clearly $\|P_n\|_{\mathcal{L}(H^1, H^1)} \leq 1$ and $\|Q_n\|_{\mathcal{L}(H^1, H^1)} \leq 1$, as well as $(P_n(u))_n$ converges towards u for all $u \in \dot{H}^1(\mathbb{T})$.

We denote by

$$A_1 = -(1 - \partial_{xx})^{-1}(\partial_x + \partial_{xxx})$$

and

$$B = (1 - \partial_{xx})^{-1}\partial_x a^{\frac{1}{2}}(1 - \partial_{xx})^{\frac{1}{2}}.$$

It is clear too that A_1 commutes with P_n . The assumption 2 and 3 of Theorem 16 are verified. We can easily check that the adjoint of B in $H^1(\mathbb{T})$ is given by $B^* = -(1 - \partial_{xx})^{-\frac{1}{2}}a^{\frac{1}{2}}\partial_x$. Furthermore, $BB^* = -(1 - \partial_{xx})^{-1}\partial_x a \partial_x$. We can write $A = A_1 + BB^*$. The operator BB^* is nonnegative and $(BB^*u, u)_{H^1} = \int_{\mathbb{T}} a(x)|\partial_x u(t, x)|^2 dx$.

We will now prove the following result.

Theorem 17 *There exists $M > 0$ and $c > 0$ be such that for every $t \geq 0$ and for every $n \in \mathbb{N}$*

$$\| \| e^{(A_1 + Q_n BB^*)t} \| \|_{\mathcal{L}(Q_n H^1, H^1)} \leq M e^{-ct}. \quad (3.78)$$

Let us start by giving an abstract result following the same approach as [Har89]. We consider a complex Hilbert space H , A_1 an unbounded skew-adjoint, m -dissipative linear operator on H and B a bounded linear operator

such as $BB^* \geq 0$.

We also consider these two equations

$$\varphi'(t) + A_1\varphi(t) = 0, \quad (3.79)$$

and

$$y'(t) + A_1y(t) + BB^*y(t) = 0. \quad (3.80)$$

Theorem 18 *The following properties are equivalent.*

1. There exists $T_0 > 0$ and $c > 0$ such that every solution φ of (3.79) satisfies

$$\|\varphi(0)\|_H^2 \leq c \int_0^{T_0} \|B^*\varphi(s)\|_H^2 ds. \quad (3.81)$$

2. There exists $T_1 > 0$ and $\delta > 0$ such that every solution y of (3.80) satisfies

$$\forall y_0 \in \mathcal{D}(A_1), \forall t \geq T_1, \|y(t)\|_H = \|S(t)y_0\|_H \leq e^{-\delta t} \|y_0\|_H, \quad (3.82)$$

where $\{S(t)\}_{t \geq 0}$ is the semigroup generated by $A_1 + BB^*$.

See Annex for the proof.

Proposition 15 *Let $u \in \dot{H}^1(\mathbb{T})$, and let v be a solution of*

$$\begin{cases} \partial_t v = (A_1 + Q_n BB^*) v \\ v(0) = Q_n u \end{cases}$$

and w a solution of

$$\begin{cases} \partial_t w = Q_n (A_1 + BB^*) Q_n w \\ w(0) = Q_n u. \end{cases}$$

Then $v = w$.

Proof Let $z = Q_n w$. We have

$$\partial_t z = Q_n \partial_t w = Q_n (Q_n (A_1 + BB^*) Q_n w) = Q_n (A_1 + BB^*) Q_n z.$$

Since $z(0) = w(0)$, by uniqueness $w = z = Q_n w$, and we have

$$\begin{aligned} \partial_t w &= Q_n (A_1 + BB^*) Q_n w \\ &= Q_n A_1 Q_n w + Q_n BB^* Q_n w \\ &= A_1 Q_n Q_n w + Q_n BB^* Q_n w \\ &= (A_1 + Q_n BB^*) w \end{aligned}$$

Then $w = v$. □

We prove now Theorem [17](#).

Proof According to Proposition [15](#), it suffices to show that we have an exponential decrease for the semigroup associated with the operator $Q_n (A_1 + BB^*) Q_n$. For this, we will use Theorem [18](#). So we must prove that

$$\|\varphi(0)\|_{H^1}^2 \leq c_1 \int_0^{T_0} \|Q_n B^* \varphi(s)\|_{H^1}^2 ds, \quad (3.83)$$

where φ is solution of

$$\begin{cases} \varphi'(t) = Q_n A_1 Q_n \varphi(t) \\ \varphi(0) = Q_n \varphi(0). \end{cases}$$

We consider the problem

$$\begin{cases} \psi'(t) = A_1 \psi(t) \\ \psi(0) = Q_n \varphi(0). \end{cases}$$

From [\(3.18\)](#) and Theorem [18](#), ψ satisfies

$$\|\psi(0)\|_{H^1}^2 \leq c_2 \int_0^{T_0} \|B^* \psi(s)\|_{H^1}^2 ds.$$

Note that the constant c_2 depends only on the operator A_1 . So the estimate is uniform in n . Since Q_n commutes with A_1 , we can take $\psi = Q_n \varphi$. Thus we have [\(3.83\)](#) and then [\(3.78\)](#). □

The two last assumptions of Theorem [16](#) are satisfied with the following proposition.

Proposition 16 *The map G is holomorphic from $\dot{H}^1(\mathbb{T})$ to itself. Moreover, the set $\{DG(u_\infty(t))h ; t \in \mathbb{R}, \|h\|_{\dot{H}^1(\mathbb{T})} \leq 1\}$ is a bounded set in $\dot{H}^2(\mathbb{T})$.*

Proof We recall that for $u \in \dot{H}^1(\mathbb{T})$, $G(u) = -(1 - \partial_{xx})^{-1}(u \partial_x u)$. We know that $G(u) \in \dot{H}^1(\mathbb{T})$ since

$$\begin{aligned} (G(u), 1)_{L^2} &= -((1 - \partial_{xx})^{-1}(u \partial_x u), 1)_{L^2} \\ &= -\frac{1}{2} ((1 - \partial_x)^{-1} \partial_x (u^2), 1)_{L^2} \\ &= 0. \end{aligned}$$

We give the following definition of a holomorphic function on a Banach space. See [\[Har04\]](#) for more details.

Definition 13 Let X and Y be complex Banach spaces and let \mathcal{D} be an open subset of X . A function $G : \mathcal{D} \rightarrow Y$ is holomorphic if for each $u \in \mathcal{D}$, there exists a continuous complex-linear mapping $DG(u) : X \rightarrow Y$ such that

$$\lim_{h \rightarrow 0} \frac{\|G(u+h) - G(u) - DG(u)h\|_Y}{\|h\|_X} = 0. \quad (3.84)$$

Let u and h be two functions in $\dot{H}^1(\mathbb{T})$. We have

$$\begin{aligned} G(u+h) &= -(1 - \partial_{xx})^{-1}((u+h)\partial_x(u+h)) \\ &= G(u) - (1 - \partial_{xx})^{-1}(u\partial_x h + h\partial_x u) + G(h) \end{aligned}$$

The map $DG(u) : h \rightarrow -(1 - \partial_{xx})^{-1}(u\partial_x h + h\partial_x u)$ is linear and continuous, indeed

$$\begin{aligned} \|DG(u)h\|_{H^1(\mathbb{T})} &\leq \|DG(u)h\|_{H^2(\mathbb{T})} \\ &\leq \|u\|_{L^\infty(\mathbb{T})} \|h\|_{H^1(\mathbb{T})} + \|h\|_{L^\infty(\mathbb{T})} \|\partial_x u\|_{L^2(\mathbb{T})} \quad (3.85) \\ &\leq 2\|u\|_{H^1(\mathbb{T})} \|h\|_{H^1(\mathbb{T})}. \end{aligned}$$

Furthermore

$$\|G(u+h) - G(u) - DG(u)h\|_{H^1} = \|G(h)\|_{H^1} \leq \|h\|_{L^\infty} \|\partial_x h\|_{L^2} \leq \|h\|_{H^1}^2.$$

We have then (3.84). Now let u_∞ be the solution of the nonlinear problem with the initial data $u_\infty(0)$, and $h \in \dot{H}^1(\mathbb{T})$ such that $\|h\|_{H^1(\mathbb{T})} \leq 1$. We know that $u_\infty \in L^\infty(\mathbb{R}, \dot{H}^1(\mathbb{T}))$ from Proposition 13. Estimation (3.85) shows that $\{DG(u_\infty(t))h ; t \in \mathbb{R}, \|h\|_{\dot{H}^1(\mathbb{T})} \leq 1\}$ is a bounded set in $\dot{H}^2(\mathbb{T})$.

Thanks to the compact injection $\dot{H}^2(\mathbb{T}) \hookrightarrow \dot{H}^1(\mathbb{T})$, we deduce the assumption 6 of Theorem 16. \square

All the assumptions of Theorem 16 are verified, the solution u_∞ of the nonlinear problem is an analytical function.

3.6 Unique continuation

We will show in this last part that u_∞ is constant for all $(x, t) \in \mathbb{T} \times \mathbb{R}$.

Proposition 17 There exists $c^* \geq 0$ such that for all $(x, t) \in \mathbb{T} \times \mathbb{R}$,

$$u_\infty(x, t) = c^*.$$

Let $-2\pi < \alpha < \beta < 0$ be two real numbers such that $]\alpha, \beta[\subset \{a > 0\}$. We first give the following proposition.

Proposition 18 There exists $c^* \geq 0$ such that for all $(x, t) \in]\alpha, \beta[\times \mathbb{R}$,

$$u_\infty(x, t) = c^*.$$

Proof Let u be a solution of the non linear problem. From Remark 3, $\|u(t)\|_{H^1(\mathbb{T})}^2$ is nonincreasing to a constant $c \geq 0$, and we have with the notation of Theorem 15

$$\lim_{n \rightarrow +\infty} \|u(t_n + t) - u_\infty(t)\|_{H^1(\mathbb{T})} \longrightarrow 0, \quad \forall t \in \mathbb{R},$$

so $\|u_\infty(t)\|_{H^1(\mathbb{T})}^2 = c$ for all $t \in \mathbb{R}$. Now

$$\|u_\infty(t)\|_{H^1(\mathbb{T})}^2 - \|u_\infty(0)\|_{H^1(\mathbb{T})}^2 = -2 \int_0^t a(x) |\partial_x u_\infty(x, \sigma)|^2 d\sigma, \quad \forall t \in \mathbb{R}.$$

Then

$$\int_0^t a(x) |\partial_x u_\infty(x, \sigma)|^2 d\sigma = 0, \quad \forall t \in \mathbb{R}.$$

This implies that $u_\infty(x, t)$ is a constant on x for $x \in]\alpha, \beta[$.

Now, since u_∞ satisfies

$$(1 - \partial_{xx})\partial_t u_\infty + \partial_{xxx} u_\infty + \partial_x u_\infty + u_\infty \partial_x u_\infty = 0,$$

and $\partial_x u_\infty = 0$, we deduce that we have also $\partial_t u_\infty = 0$. \square

The function $v = u_\infty - c^*$ verifies the equation

$$\begin{cases} (1 - \partial_{xx})\partial_t v + \partial_{xxx} v + (1 + u_\infty)\partial_x v = 0, & (x, t) \in \mathbb{T} \times \mathbb{R} \\ v(x, 0) = v_0 \in H^1(\mathbb{T}), \\ v(x, t) = 0, & (x, t) \in]\alpha, \beta[\times \mathbb{R}. \end{cases} \quad (3.86)$$

Let $P(x, t, \partial_x, \partial_t) = \partial_{xxx} - \partial_{xx}\partial_t + \partial_t + w(t, x)\partial_x$, where $w = 1 + u_\infty$. The principal symbol is given by $p(x, t, \xi, \tau) = \xi^2(\xi - \tau)$. let $x^* \in]\alpha, \beta[$. We denote by

$$\psi : (x, t) \longmapsto (x - x^*)^2 - t^2,$$

and

$$\Gamma = \{(x, t, \xi, 0) ; x \in \mathbb{T}, t, \xi \in \mathbb{R}\}.$$

We recall the following definition (see [Lat95, Definition 1.2])

Definition 14 (Pseudo-convex surface) Let P be a differential operator of order m , with a principal symbol p , S a level set of a smooth function ψ , and $(x_0, t_0) \in S$ such that $\nabla\psi(x_0, t_0) \neq 0$. We say that S is strongly pseudo-convex in (x_0, t_0) with respect to P on Γ if

1. $\Re\{\bar{p}, \{p, \psi\}\}(x_0, t_0, \xi, 0) > 0$, on

$$\{(x_0, t_0, \xi, 0) \in \Gamma ; p(x_0, t_0, \xi, 0) = \{p, \psi\}(x_0, t_0, \xi, 0) = 0, \text{ with } \xi \neq 0\}$$

$$2. \frac{1}{\gamma i} \{\bar{p}(x, t, \xi - i\gamma\psi'_x, \tau - i\gamma\psi'_t), p(x, t, \xi + i\gamma\psi'_x, \tau + i\gamma\psi'_t)\} (x_0, t_0, \xi, 0) > c(\xi^2 + \gamma^2)^{m-1}, \text{ on}$$

$$\{(x_0, t_0, \xi, 0) \in \Gamma; p(x_0, t_0, \xi + i\gamma\psi'_x, i\gamma\psi'_t) = \{p(x, t, \xi + i\gamma\psi'_x, \tau + i\gamma\psi'_t), \psi\}(x_0, t_0, \xi, 0) = 0, \text{ with } \gamma > 0\}.$$

We will use this theorem given in [Cat95, Theorem 2]

Theorem 19 *Let K be an open set of \mathbb{R}^n and P a differential operator of order m such that*

1. *The principal symbol of P is real and whose coefficients are independent of t and assumed to be in $C^1(\mathbb{T})$,*
2. *The coefficients of lower order terms of P are analytic from $I \subset \mathbb{R}$ into $L^\infty(\mathbb{T})$,*

Let $(x_0, t_0) \in \mathbb{T} \times \mathbb{R}$ and ψ a smooth function such that $\nabla\psi(x_0, t_0) \neq 0$. Assume that the level surface $\{\psi(x, t) = \psi(x_0, t_0)\}$ is strongly pseudo-convex in (x_0, t_0) with respect to P on Γ . Then there exists an open neighbourhood V of (x_0, t_0) such that if u is solution of $P(x, \mathcal{D})u = 0$ in K and $u = 0$ in $\{\psi > \psi(x_0, t_0)\}$, then $u = 0$ in V .

Clearly P verifies the first assumption of the theorem. The second assumption is also verified due to Proposition 13 and Theorem 15. We will show now that the level surfaces of the function ψ are strongly pseudo-convex on Γ .

Proposition 19 *Let $\lambda_1 = \psi(\alpha + 2\pi, 0) = (\alpha + 2\pi - x^*)^2$. The level surfaces $\{\psi = \lambda\}$ for $\lambda \in]0, \lambda_1]$ are strongly pseudo-convex with respect to P on Γ .*

Proof We will show that the sets

$$\{(x_0, t_0, \xi, 0) \in \Gamma; p(x_0, t_0, \xi, 0) = \{p, \psi\}(x_0, t_0, \xi, 0) = 0, \text{ with } \xi \neq 0\},$$

and

$$\{(x_0, t_0, \xi, 0) \in \Gamma; p(x_0, t_0, \xi + i\gamma\psi'_x, i\gamma\psi'_t) = \{p(x, t, \xi + i\gamma\psi'_x, \tau + i\gamma\psi'_t), \psi\}(x_0, t_0, \xi, 0) = 0, \text{ with } \gamma > 0\}$$

are empty. For the first set, we already notice that the case $p(x_0, t_0, \xi, 0) = \xi^3 = 0$ give necessarily $\xi = 0$, which is impossible. For the second one, we have $\gamma > 0$, then $p(x_0, t_0, \xi + i\gamma\psi'_x, i\gamma\psi'_t) = (\xi + 2i\gamma(x - x^*))^2(\xi + 2i\gamma(t + x - x^*))$ is equal to 0 when $\xi = 0$ and $x = x^*$, or when $\xi = 0$ and $t + x - x^* = 0$. The first case is impossible since $x = x^*$ means that $\psi = -t^2 < 0$, but we want that the surfaces $\{\psi = \lambda\}$ are strongly pseudo-convex for $\lambda \in]0, \lambda_1]$. For the same reason, the second one is also impossible since $t + x - x^* = 0$ correspond for all $t \in \mathbb{R}$ to $\{\psi = 0\}$. \square

The assumptions of Theorem 19 are satisfied. So we can use it to show the following result.

Theorem 20 Let \mathcal{T} be the triangle of vertices

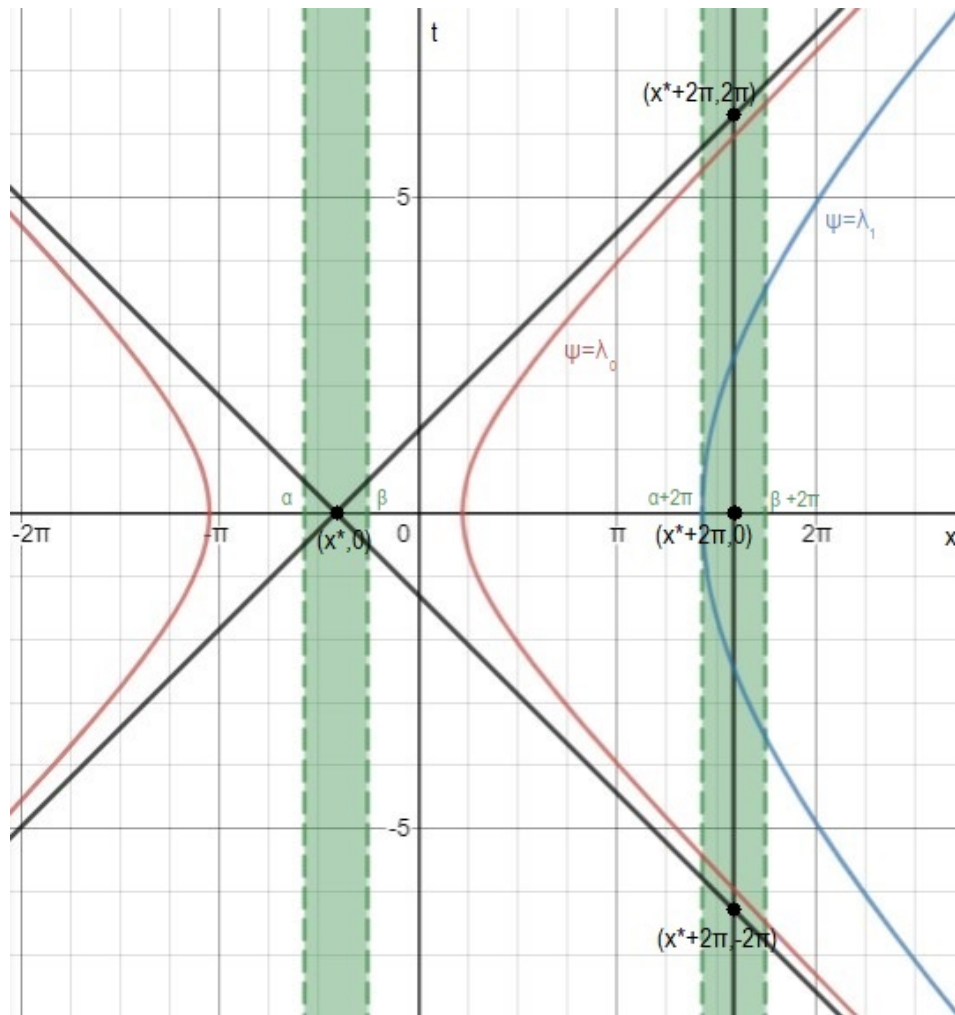
$$\{(x^*, 0); (x^* + 2\pi, 2\pi); (x^* + 2\pi, -2\pi)\},$$

then the function v is identically zero on \mathcal{T} .

Proof Let

$$\lambda_0 = \inf \{ \lambda \in]0, (2\pi)^2 [; v(x, t) = 0, \forall (x, t) \in \{ \psi > \lambda \} \cap \mathcal{T} \}.$$

We give the following figure.



Here the green part corresponds to a connex part of $\{a > 0\}$, on which we know that $v = 0$. The idea is to use this information to show that $v = 0$ on a set slightly larger than $\{a > 0\}$ using a surface deformation technique. It is sufficient to prove that $\lambda_0 = 0$. Assume that $\lambda_0 > 0$.

We know that $v = 0$ on $\{\psi \geq \lambda_1\} \cap \mathcal{T}$ from Remark [5](#). By applying Theorem [19](#) to each point $(x, t) \in \{\psi = \lambda_0\} \cap \mathcal{T} =: \mathcal{T}_0$, there exists an open neighborhood $B(x, t)$ of (x, t) such that $v = 0$ on $B(x, t)$.

So, if $\mathcal{T}_0 \subset \bigcup_{(x_i, t_i) \in \mathcal{T}_0} B(x_i, t_i)$, there exists $B(x_1, t_1), \dots, B(x_p, t_p)$ by a

compactness argument such that $\mathcal{T}_0 \subset \bigcup_{i=1}^p B(x_i, t_i)$.

We will show that there exists $\lambda \in [0, \lambda_0[$ such that $\{\psi = \lambda\} \cap \mathcal{T} \subset \bigcup_{i=1}^p B(x_i, t_i)$.

Suppose that for all $n \in \mathbb{N}^*$, $\left\{\psi = \lambda_0 - \frac{1}{n}\right\} \cap \mathcal{T} \not\subset \bigcup_{i=1}^p B(x_i, t_i)$.

Let $(y_n, s_n) \in \left\{\psi = \lambda_0 - \frac{1}{n}\right\} \cap \mathcal{T}$. We know that for all $n \in \mathbb{N}^*$ $(y_n, s_n) \in \text{supp}(v)$. Since $\left\{\psi = \lambda_0 - \frac{1}{n}\right\} \cap \mathcal{T}$ is a compact set, there exists $(y_{\varphi(n)}, s_{\varphi(n)})$ a subsequence of (y_n, s_n) such that $y_{\varphi(n)} \rightarrow y$ and $s_{\varphi(n)} \rightarrow s$. This give us that $\psi(y, s) = \lambda_0$. Hence, there exists $i \in \{1, \dots, p\}$ such that $(y, s) \in B(x_i, t_i)$. So for n large enough, we have $(y_{\varphi(n)}, s_{\varphi(n)}) \in B(x_i, t_i)$, and then, $(y_{\varphi(n)}, s_{\varphi(n)}) \notin \text{supp}(v)$, which reach a contradiction. We deduce that $\lambda_0 = 0$ and then $v = 0$ on \mathcal{T} . \square

Remark 5 The constant λ_1 is chosen so that

$$(\{\psi \geq \lambda_1\} \cap \mathcal{T}) \subset (\{a > 0\} \times \mathbb{R}).$$

So combining this with the last line of equation [\(3.86\)](#), we have clearly that $v = 0$ on $\{\psi \geq \lambda_1\} \cap \mathcal{T}$.

The proof of Proposition [17](#) is easy now.

Proof We conclude from the preceding result that v is equal to zero everywhere since $v = 0$ on $]x^*, x^* + 2\pi] \times \{t = 0\}$, and then, the function v_0 in [\(3.86\)](#) is equal to 0. Thus we obtain $v(x, t) = u_\infty(x, t) - c^* = 0$ for all $(x, t) \in \mathbb{T} \times \mathbb{R}$. \square

Remark 6 Note that [\[Hö85, Theorem 28.3.4\]](#) is not applicable here since the level surfaces of ψ are not strongly pseudo-convex with respect to P in the sense of the theorem. Indeed, by a simple calculus we can prove that the assumption

$$\Re \{\bar{p}, \{p, \psi\}\}(x_0, t_0, \xi, \tau) > 0,$$

in the definition of pseudo-convexity in [Hö85] is not verified on the set

$$\left\{ (\xi, \tau) \neq (0, 0) \in \mathbb{R}^2 ; p(x_0, t_0, \xi, \tau) = \langle p'_{(\xi, \tau)}(x_0, t_0, \xi, \tau), \psi'_{(x, t)}(x_0, t_0) \rangle = 0 \right\},$$

since the choice $\xi = 0$ and $\tau \neq 0$ vanish the two quantities

$\langle p'_{(\xi, \tau)}(x_0, t_0, \xi, \tau), \psi'_{(x, t)}(x_0, t_0) \rangle$, $p(x_0, t_0, \xi, \tau)$ and vanish also the term $\Re \{ \bar{p}, \{p, \psi\} \}$.

We give finally a proof of Theorem 7

Proof Let $u = \sum_{k \in \mathbb{Z}} u^k(t) e^{ikx}$ be the solution of the nonlinear problem. Combining Theorem 15 and Proposition 17 we can write for all $T > 0$

$$\lim_{n \rightarrow +\infty} \sup_{s \in [-T, T]} \|u(t_n + s) - c^*\|_{H^1(\mathbb{T})} = 0.$$

Thus

$$\lim_{n \rightarrow +\infty} \sup_{s \in [-T, T]} \left((u^0(t_n + s) - c^*)^2 + \sum_{k \in \mathbb{Z}^*} (1 + k^2) (u^k(t_n + s))^2 \right)^{\frac{1}{2}} = 0.$$

We already know that $u^0(t_n + s)$ is constant in time, in particular $u^0(t_n + s) = u^0(0)$, and since the term $\sum_{k \in \mathbb{Z}^*} (1 + k^2) (u^k(t_n + s))^2$ goes to 0, we deduce that

$c^* = u^0(0)$. We finish by proving that $\lim_{t \rightarrow +\infty} u(t, \cdot) = c^*$. Suppose that there exists $(s_n)_n$ a sequence of times which goes to $+\infty$ and $\|u(s_n) - c^*\|_{H^1(\mathbb{T})} \geq \epsilon$, for some $\epsilon > 0$. From Theorem 15 there exists a subsequence $(s_{\phi(n)})_n$ such that $\lim_{n \rightarrow +\infty} \|u(s_{\phi(n)}) - c^*\|_{H^1(\mathbb{T})} = 0$, which is absurd. \square

3.7 A remark on the BBM equation

We consider this BBM equation

$$\begin{cases} \partial_t v + \partial_x v - \partial_{xxt} v - \partial_x(a(x + ct)\partial_x)v + v\partial_x v = 0, & x \in \mathbb{T}, t > 0, \\ v(\cdot, 0) = v_0, & x \in \mathbb{T}, \end{cases} \quad (3.87)$$

where $c > 0$ and $a \geq 0$ is assumed to be a bounded function in $\mathcal{C}^\infty(\mathbb{T})$ such that $\{a > 0\} \neq \emptyset$.

We can prove the result of stabilization by following the same approach in the preceding paragraphs. After a change of variables we can write the equation under the form

$$\partial_t v - c \partial_{xxx} v - \partial_{xxt} v + (c+1) \partial_x v - \partial_x(a(x) \partial_x) v + v \partial_x v = 0.$$

We prove that if v is the solution of the linear equation then

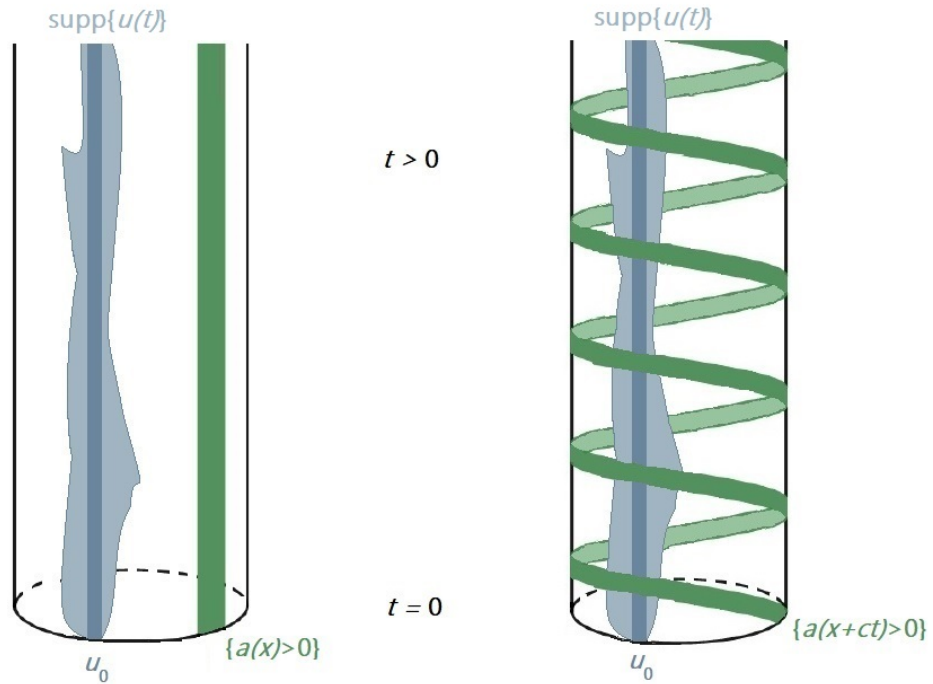
$$\frac{d}{dt} \left(\frac{1}{2} \|v(t)\|_{H^1}^2 \right) = -2 \int_{\mathbb{T}} a(x) |\partial_x v(t, x)|^2 dx \leq 0. \quad (3.88)$$

Except that there are different constants that appear in the equation compared to what we did previously, the results are essentially the same, and we give the following theorem.

Theorem 21 For all $v_0 \in H^1(\mathbb{T})$, there exists a unique solution $v = v(t, x)$ of (3.87) global in time such that

$$\lim_{t \rightarrow +\infty} v(t, \cdot) = \frac{1}{2\pi} \int_{\mathbb{T}} v_0(x) dx, \text{ in } H^1(\mathbb{T}).$$

We can see in this figure the difference between considering a dissipator of the form $a(x)$ and another of the form $a(x+ct)$.



Remark 7 *As we saw in the introduction chapter, the mass of the solution of the BBM equation will remain around the initial data. This is why when we put a dissipator $a(x)$ localized in space, the intersection between the solution and the dissipator will be negligible. The result becomes different when the dissipator is localized in space but of the form $a(x + ct)$. In this case, even if the solution remains around a single place, the dissipator can move over time to touch the solution sufficiently in order to obtain the decreasing of energy (3.88).*

Controllability of the BBM equation

4.1 Introduction

The nonlinear BBM equation $\partial_t u + \partial_x u - \partial_{xxt} u + u \partial_x u = 0$, introduced by [BBM72], seen as a regularized counterpart of the nonlinear KdV equation $\partial_t u + \partial_x u + \partial_{xxx} u + u \partial_x u = 0$, is a model to describe a propagation phenomenon of one-dimensional unidirectional small amplitude long waves in nonlinear dispersive case. It was proven in [BT09] and [Ros16] that the BBM equation is well posed when the initial data u_0 belongs to $H^s(\mathbb{T})$ with $s \geq 0$, and before that, in [BBM72] when $s \in \mathbb{N}^*$. Furthermore, the map $u_0 \mapsto u(t)$ is real analytic. It is also known that the equation is ill-posed in the case $s < 0$ in different senses (see [BT09] and [Pan10] for precise statement).

In this chapter, we are interested in the approximate controllability of the BBM equation on the torus

$$\begin{cases} \partial_t u + \partial_x u - \partial_{xxt} u + u \partial_x u = \eta(t, x), & (t, x) \in]0, T[\times \mathbb{T}. \\ u(\cdot, 0) = u_0. \end{cases} \quad (4.1)$$

Micu [Mic01] studied the controllability of the linearized BBM equation on $[0, 1]$ with a control term on the boundary. He showed that the equation is not spectrally controllable but approximately controllable with a control in $L^2(0, T)$. He also showed that for all $N > 0$ and $T > 0$, we can find a

control such that the projection of the solution on the finite dimensional space generated by the first $2N$ eigenfunctions of the BBM operator is equal to zero at $t = T$. Rosier and Zhang [Ros16] give an exact controllability result for the BBM equation with a moving control for a large time, as well as estimates of the control term in combination of the initial data and the final data, by considering first the linearized equation, then deduce the result on the nonlinear equation using a fixed point argument. Zhang and Zuazua have shown the approximate controllability for the BBM equation with a localized control term, see Section 7 in [ZZ03]. We can also find the approximate controllability for the equation of BBM in 1-d with delay in [Lei17].

The equation (4.1) preserves the constants, so we introduce the closed subspace of $H^1(\mathbb{T})$

$$\dot{H}^1(\mathbb{T}) = \left\{ u \in H^1(\mathbb{T}) ; (u, 1)_{L^2(\mathbb{T})} = 0 \right\},$$

equipped with $H^1(\mathbb{T})$ -norm. We will apply a method based on the use of the nonlinear term. The type of control we consider is trigonometric functions. The interest of such a choice of control is that our nonlinear term $u\partial_x u$ leaves the cosines and the sines invariant. We start with an initial data in $\dot{H}^1(\mathbb{T})$ and a control which belongs to a finite dimension space given for $I \subset \mathbb{Z}^*$ by

$$\mathcal{H}(I) = \text{span}\{\cos(kx), \sin(kx); \text{ for } k \in I\}. \quad (4.2)$$

Definition 15 We say that the equation (4.1) is approximately controllable in $\dot{H}^1(\mathbb{T})$ by values in $\mathcal{H}(I)$ if for any $T > 0$, $\epsilon > 0$ and any $u_0, v \in \dot{H}^1(\mathbb{T})$, there is a piecewise constant control with values in $\mathcal{H}(I)$ and a solution u of (4.1) such that

$$\|u(T) - v\|_{H^1} \leq \epsilon.$$

Definition 16 We say that the equation (4.1) is exactly controllable in $\dot{H}^s(\mathbb{T})$ by values in $\mathcal{H}(I)$ if for any $T > 0$ and any $u_0, v \in \dot{H}^s(\mathbb{T})$, there is a piecewise constant control with values in $\mathcal{H}(I)$ and a solution u of (4.1) such that

$$u(T) = v.$$

Our two main results are the following theorems.

Theorem 22 The equation (4.1) is approximately controllable in $\dot{H}^1(\mathbb{T})$ by piecewise constant controls with values in $\mathcal{H}(I)$ if and only if I is a generator of \mathbb{Z} .

Theorem 23 The equation (4.1) is not exactly controllable in $\dot{H}^s(\mathbb{T})$ by piecewise constant controls with values in $\mathcal{H}(I)$ when $s \in [1, 2]$.

The idea of the proof of Theorem [22](#) is to start by considering a control in $\mathcal{H}(I_0)$, and show that the solution reaches a space $\mathcal{H}(I_1)$ in a time t , where $I_1 \supsetneq I_0$. Then we start again from I_1 which allows us to construct an I_2 , and so on. The goal is to show that the union of $\mathcal{H}(I_n)$ is dense in $\dot{H}^1(\mathbb{T})$, which amounts to show that the union of I_n is \mathbb{Z} . The sequence (I_n) plays the role of the new Fourier modes that we obtain after each step. So it is necessary that the inclusion of I_n in I_{n+1} is strict, because otherwise the sequence (I_n) will stop at a certain moment and we cannot reach a dense space, since the goal is to have an infinite dimension space from finite dimension spaces. Another interesting remark is to see that to reach at the end a dense space in $\dot{H}^1(\mathbb{T})$, we must start from a control in $\mathcal{H}(I_0)$, where I_0 is generator of \mathbb{Z} , i.e. any integer of \mathbb{Z} can be written as a linear combination with integer coefficients of elements of I_0 . So in fact, we can say that a nonlinear phenomenon gives a linear phenomenon in the sense that it is the term $u\partial_x u$ which generates new frequencies. In other words, we have that I_0 generates \mathbb{Z} and we define a sequence (I_n) constructed from with linear combinations while the product $u\partial_x u$ is not linear. It is important to see that in the reasoning, we never lose informations since after each step, we show that the target will be in a larger space. By repeating this procedure enough times, we can reach a dense space in $\dot{H}^1(\mathbb{T})$. We start by using a control with a very large norm $H^1(\mathbb{T})$ in a small time, with that, we show that the solution will converge to a quantity in which the nonlinear term of the equation appears. This limit that we find can play the role of an initial data in the following. By applying the equation again, the nonlinear term will give new vector lines that can be used as initial data and so on. Once we show that we can go from any point to any point approximately in a small time, we can generalize that to conclude that the result is also true in a fixed time.

This technique is inspired by the paper of Nersesyan [\[Ner21\]](#), where he applies the same type of control on the heat equation, and in [\[Ner15\]](#) for Navier-Stokes systems. We refer the reader also to the works of Agrachev and Sarychev [\[AS05\]](#) [\[AS06\]](#) where they use trigonometric polynomials as a control over the equation of Navier-Stokes/Euler systems in 2-d and 3-d. Nersisyan [\[Ner10\]](#) [\[Ner11\]](#) uses this method of controlling with spaces of finite dimensions for the Euler equation. See as well works of Shirikyan [\[Shi06\]](#) [\[Shi07\]](#) [\[Shi14\]](#) [\[Shi18\]](#) for the Burger and Navier-Stokes equations.

The organization of the chapter is as follows. In Section [4.2](#) we study the existence of the solution of the nonlinear problem using a fixed point argument. Then we establish some estimations of continuity of the solution with respect to the initial data and to the control term.

In Section [4.3](#) we start from a set I_0 which generates \mathbb{Z} , then we define a sequence of sets (I_n) related to the Fourier coefficients given by nonlinear term,

and we prove that their union forms a dense space in $\dot{H}^1(\mathbb{T})$.

In Section 4.4 we prove that we can apply a control to go from an initial data u_0 to some $u_0 + \mathcal{H}$ approximately, where \mathcal{H} is a finite dimension space that we will enrich until we have a dense space in $\dot{H}^1(\mathbb{T})$. Then, arguing by induction and density, we show that we can reach approximately any point for any time $T > 0$.

In Section 4.5 we give a proof of Theorem 2. We start with the easiest case, the non exact controllability in $\dot{H}^1(\mathbb{T})$. We argue by contradiction to show with the Baire's theorem that the attainable space in $\dot{H}^1(\mathbb{T})$ cannot contain an open set, therefore its complement is dense. The case $s \in]1, 2[$ is more technical. We use an ϵ -entropy argument to count the number of balls needed to cover a certain set. More precisely, we estimate the number of balls it takes to cover the unit ball of $\dot{H}^s(\mathbb{T})$ when $s \in]1, 2[$, then we show that the application $\eta \mapsto u$ cannot give us enough balls to cover a ball of $\dot{H}^s(\mathbb{T})$. This means that we do not have enough controls to achieve exact controllability in $\dot{H}^s(\mathbb{T})$. This technique is inspired by Shirikyan in [Shi08] who used it to show that the Euler equation is not exactly controllable in 2-d. We can also find the reasoning of ϵ -entropy to prove the non exact controllability in [NN12] for the Schrödinger equation, and in [Ner10] for the Euler equation in 3-d.

4.2 Existence and properties

For technical reasons that we will see below, we introduce a function $\varphi \in \dot{H}^1(\mathbb{T})$ in the equation. So we consider the following nonlinear BBM equation on the torus

$$\partial_t u + \partial_x(u + \varphi) - \partial_{xxt}u + (u + \varphi)\partial_x(u + \varphi) = \tilde{\eta}(t, x), \quad (t, x) \in]0, T[\times \mathbb{T}, \quad (4.3)$$

where $\tilde{\eta}(t) \in L^1(\mathbb{T})$. Note that the function $v = u + \varphi$ is solution of (4.1) with the initial condition $u_0 + \varphi$. For $u_0 \in \dot{H}^1(\mathbb{T})$, the solution of (4.3) at time t , with a control term $\tilde{\eta}$, is noted $u(t) = \mathcal{R}_t(u_0, \varphi, \tilde{\eta})$. The equation (4.3) can be written under the form

$$\partial_t u = \eta + A(u + \varphi) - (1 - \partial_{xx}^2)^{-1}(u + \varphi)\partial_x(u + \varphi), \quad (4.4)$$

where $A = -(1 - \partial_{xx}^2)^{-1}\partial_x$ and $\eta = (1 - \partial_{xx}^2)^{-1}\tilde{\eta}$. We know that since A is bounded then it is the infinitesimal generator of a uniformly continuous semigroup $\{e^{tA}\}_{t \geq 0}$.

We start by studying the existence of the solution as well as some estimations that we will use in the following. Let $\delta > 0$, φ and f two smooth

functions, we consider the equation

$$\begin{cases} \partial_t u + \partial_x(u + \delta^{-\frac{1}{2}}\varphi) - \partial_{xxt}u + (u + \delta^{-\frac{1}{2}}\varphi)\partial_x(u + \delta^{-\frac{1}{2}}\varphi) = \delta^{-1}f, \\ u(\cdot, 0) = u_0 \in \dot{H}^1(\mathbb{T}). \end{cases} \quad (4.5)$$

Lemma 9 For all $u_0, \varphi, f \in \dot{H}^1(\mathbb{T})$, there exists $R > 1$ and $\delta' \in]0, 1[$ such that for all $\delta < \delta'$, there exists a unique solution $u \in \mathcal{C}([0, 2\delta], \dot{H}^1(\mathbb{T}))$ of (4.5) satisfying

$$\|u(t)\|_{\dot{H}^1(\mathbb{T})} \leq R, \quad \forall t \in [0, 2\delta]. \quad (4.6)$$

Proof For $R > 1$ and $0 < T \leq 1$, we define

$$B_{R,T} = \left\{ u \in \mathcal{C}([0, T], \dot{H}^1(\mathbb{T})) ; \sup_{s \in [0, T]} \|u(s)\|_{\dot{H}^1(\mathbb{T})} \leq R \right\}.$$

The space $\mathcal{C}([0, T], \dot{H}^1(\mathbb{T}))$ equipped with the distance

$$d(u, v) = \sup_{s \in [0, T]} \|u(s) - v(s)\|_{\dot{H}^1}$$

is a complete metric space. For $\delta > 0$, $u_0 \in \dot{H}^1(\mathbb{T})$ and $t \in [0, T]$, we introduce

$$\begin{aligned} \Phi : \mathcal{C}([0, T], \dot{H}^1(\mathbb{T})) &\longrightarrow \mathcal{C}([0, T], \dot{H}^1(\mathbb{T})) \\ u &\longmapsto e^{tA}u_0 + \int_0^t e^{(t-s)A}(1 - \partial_{xx})^{-1} \left(\delta^{-1}f - \delta^{-\frac{1}{2}}\partial_x\varphi \right. \\ &\quad \left. - (u(s) + \delta^{-\frac{1}{2}}\varphi)\partial_x(u(s) + \delta^{-\frac{1}{2}}\varphi) \right) ds. \end{aligned}$$

Let $u, v \in B_{R,T}$. We have

$$\begin{aligned} (\Phi(u) - \Phi(v))(x, t) &= - \int_0^t e^{(t-s)A}(1 - \partial_{xx})^{-1} \left\{ (u(s) + \delta^{-\frac{1}{2}}\varphi)\partial_x(u(s) + \delta^{-\frac{1}{2}}\varphi) \right. \\ &\quad \left. - (v(s) + \delta^{-\frac{1}{2}}\varphi)\partial_x(v(s) + \delta^{-\frac{1}{2}}\varphi) \right\} ds \\ &= - \int_0^t e^{(t-s)A}(1 - \partial_{xx})^{-1} \left\{ (u(s) - v(s))\partial_x(u(s) + \delta^{-\frac{1}{2}}\varphi) \right. \\ &\quad \left. + (\partial_x u(s) - \partial_x v(s))(v(s) + \delta^{-\frac{1}{2}}\varphi) \right\} ds. \quad (4.7) \end{aligned}$$

Thus, since the norm of A is bounded by 1, we have for $t < 2\delta$

$$\begin{aligned} \|(\Phi(u) - \Phi(v))(t)\|_{H^1} &\leq cTe^T \sup_{s \in [0, T]} \left(\|u(s) - v(s)\|_{L^\infty} \left\| \partial_x \left(u(s) + \delta^{-\frac{1}{2}} \varphi \right) \right\|_{L^2} \right. \\ &\quad \left. + \|\partial_x u(s) - \partial_x v(s)\|_{L^2} \left\| v(s) + \delta^{-\frac{1}{2}} \varphi \right\|_{L^\infty} \right) \\ &\leq 2CT \left(R + \delta^{-\frac{1}{2}} \|\varphi\|_{H^1} \right) \sup_{s \in [0, T]} \|u(s) - v(s)\|_{H^1} \\ &\leq 4C\delta^{\frac{1}{2}} \left(\delta^{\frac{1}{2}} R + \|\varphi\|_{H^1} \right) \sup_{s \in [0, T]} \|u(s) - v(s)\|_{H^1}. \end{aligned}$$

Taking the supremum on t we get

$$d(\Phi(u), \Phi(v)) \leq 4C\delta^{\frac{1}{2}} \left(\delta^{\frac{1}{2}} R + \|\varphi\|_{H^1(\mathbb{T})} \right) d(u, v).$$

We choose δ' such that $d(\Phi(u), \Phi(v)) \leq \frac{1}{2}d(u, v)$.

So for $\delta' < \min \left\{ \frac{1}{8CR}, \frac{1}{(8C\|\varphi\|_{H^1})^2} \right\}$, Φ is a contraction. Now we show that if $u \in B_{R, T}$ then $\Phi(u) \in B_{R, T}$. Taking $v = 0$, and using the fact that $\|u(t)\|_{H^1} \leq R$ we obtain

$$\begin{aligned} \left| \|\Phi(u)(t)\|_{H^1} - \|\Phi(0)(t)\|_{H^1} \right| &\leq \|\Phi(u)(t) - \Phi(0)(t)\|_{H^1} \\ &\leq 2CTR \left(R + \delta^{-\frac{1}{2}} \|\varphi\|_{H^1} \right). \end{aligned}$$

Since $t \leq 2\delta \leq 2\delta'$ we see that

$$\begin{aligned} \|\Phi(0)\|_{H^1} &\leq c\|u_0\|_{H^1} + ct \left(\delta^{-1} \|f\|_{L^2} + \delta^{-\frac{1}{2}} \|\varphi\|_{H^1} + \left(R + \delta^{-\frac{1}{2}} \|\varphi\|_{H^1} \right)^2 \right) \\ &\leq c\|u_0\|_{H^1} + 2c \left(\|f\|_{L^2} + \delta^{\frac{1}{2}} \|\varphi\|_{H^1} + \left(\delta^{\frac{1}{2}} R + \|\varphi\|_{H^1} \right)^2 \right) \end{aligned} \tag{4.8}$$

So we can choose R large enough such that we estimate each term in (4.8) by $\frac{R}{8}$ and the term $2CTR \left(R + \delta^{-\frac{1}{2}} \|\varphi\|_{H^1} \right)$ by $\frac{R}{2}$. Thus we obtain

$$\|\Phi(u)(t)\|_{H^1} \leq \frac{R}{2} + \frac{R}{8} + \frac{R}{8} + \frac{R}{8} + \frac{R}{8} = R.$$

□

We give now an estimation of continuity.

Proposition 20 *For all $u_0, v_0 \in \dot{H}^1(\mathbb{T})$, $\varphi, \psi \in H^{-1}(\mathbb{T})$ and $f, g \in L^1([0, \delta]; H^{-1}(\mathbb{T}))$, there are positive constants $\delta'' \leq 1$, $c_1, c_2 = c_2(R)$*

and $c_3 = c_3(\|\varphi\|_{H^{-1}}, \|\psi\|_{H^{-1}}, R)$ such that if $\delta < \delta''$, then for all $t \leq \delta$

$$\begin{aligned} \left\| \mathcal{R}_t(u_0, \delta^{-\frac{1}{2}}\varphi, \delta^{-1}f) - \mathcal{R}_t(v_0, \delta^{-\frac{1}{2}}\psi, \delta^{-1}g) \right\|_{H^1} &\leq c_1 \|u_0 - v_0\|_{H^1} \\ &+ c_2 \|f - g\|_{L^1([0, \delta]; H^{-1})} + c_3 \|\varphi - \psi\|_{H^{-1}}. \end{aligned} \quad (4.9)$$

Proof Let $u(t) = \mathcal{R}_t(u_0, \delta^{-\frac{1}{2}}\varphi, \delta^{-1}f)$ and $v(t) = \mathcal{R}_t(v_0, \delta^{-\frac{1}{2}}\psi, \delta^{-1}g)$. We have

$$\begin{aligned} \partial_t(u-v) &= (1 - \partial_{xx})^{-1} \left(\delta^{-1}(f-g) - \left(u + \delta^{-\frac{1}{2}}\varphi + 1 \right) \partial_x \left(u + \delta^{-\frac{1}{2}}\varphi \right) \right. \\ &\quad \left. + \left(v + \delta^{-\frac{1}{2}}\psi + 1 \right) \partial_x \left(v + \delta^{-\frac{1}{2}}\psi \right) \right). \end{aligned} \quad (4.10)$$

We can see that

$$\begin{aligned} &\left(u + \delta^{-\frac{1}{2}}\varphi + 1 \right) \partial_x \left(u + \delta^{-\frac{1}{2}}\varphi \right) - \left(v + \delta^{-\frac{1}{2}}\psi + 1 \right) \partial_x \left(v + \delta^{-\frac{1}{2}}\psi \right) \\ &= (u-v)\partial_x u + \delta^{-1}(\varphi - \psi)\partial_x \varphi + (1+v)\partial_x(u-v) + \delta^{-\frac{1}{2}}\left(1 + \delta^{-\frac{1}{2}}\psi\right)\partial_x(\varphi - \psi) \\ &\quad + \delta^{-\frac{1}{2}}\partial_x((u-v)\varphi + (\varphi - \psi)v). \end{aligned} \quad (4.11)$$

So, by integrating [\(4.10\)](#) between 0 and t , we obtain for all $t \leq \delta''$

$$\begin{aligned} \|u(t) - v(t)\|_{H^1} &\leq c_1 \|u_0 - v_0\|_{H^1} + 2c \left(\|f - g\|_{L^1([0, \delta]; H^{-1})} \right. \\ &+ 2\delta''(1+R)\|u(t) - v(t)\|_{H^1} + \left(\|\varphi\|_{H^{-1}} + \|\psi\|_{H^{-1}} + \delta''^{\frac{1}{2}}(1+R) \right) \|\varphi - \psi\|_{H^{-1}} \\ &\quad \left. + \delta''^{\frac{1}{2}} \|\varphi\|_{H^{-1}} \|u(t) - v(t)\|_{H^1} \right). \end{aligned} \quad (4.12)$$

We can choose δ'' small enough such that $1 - 4c\delta''(1+R) - 2c\delta''^{\frac{1}{2}}\|\varphi\|_{H^{-1}} > 0$, which complete the proof. \square

From now on, we take $\delta_* = \min\{\delta', \delta''\}$. Using the same estimations as in the previous proposition, we have the following corollary.

Corollary 1 For all $u_0 \in \dot{H}^1(\mathbb{T})$ and $f, g \in L^1([0, \delta_*]; H^{-1})$, there is a positive constant $c = c(R)$ such that for all $t \leq \delta_*$

$$\left\| \mathcal{R}_t(u_0, 0, f) - \mathcal{R}_t(u_0, 0, g) \right\|_{H^1} \leq c \|f - g\|_{L^1([0, \delta_*]; H^{-1})}. \quad (4.13)$$

Remark 8 From uniqueness of the solution, we have the equality for all $t \in [0, \delta_*]$

$$\mathcal{R}_t(u_0, \varphi, \eta) = \mathcal{R}_t(u_0 + \varphi, 0, \eta) - \varphi. \quad (4.14)$$

We will also use the continuity in time given by the following lemma.

Lemma 10 Let $\epsilon > 0$, $v \in \dot{H}^1(\mathbb{T})$ and $u_0 \in B_{H^1}\left(v, \frac{\epsilon}{2}\right)$, then there exists $C = C(\epsilon, \|v\|_{H^1}) > 0$ and $t_* > 0$ such that the solution $u(t_*) = \mathcal{R}_{t_*}(u_0, 0, 0) \in B_{H^1}(v, \epsilon)$ and for all $t \in [0, t_*]$ we have

$$\|u(t) - u_0\|_{H^1} \leq Ct. \quad (4.15)$$

Proof Since A is a bounded operator, it is sufficient to estimate u in the expression $\partial_t u = Au - (1 - \partial_{xx}^2)^{-1}u\partial_x u$, and we get

$$\begin{aligned} \|u(t) - u_0\|_{H^1} &\leq ct \left(\sup_{s \in [0, t]} \|u(s)\|_{H^1} + \sup_{s \in [0, t]} \|u(s)\|_{H^1}^2 \right) \\ &\leq ct \left(\frac{3\epsilon}{2} + \|v\|_{H^1} + \left(\frac{3\epsilon}{2} + \|v\|_{H^1} \right)^2 \right) \leq Ct. \end{aligned} \quad (4.16)$$

□

Let $\mathcal{H}(I)$ defined in (4.2). We set

$$\mathcal{C}(I) = \left\{ \eta - (1 - \partial_{xx})^{-1} \sum_{i=1}^d \varphi_i \partial_x \varphi_i ; \eta, \varphi_i \in \mathcal{H}(I), \forall d \geq 1 \right\}.$$

The following proposition shows that the nonlinear term of the equation appears in the limit of the solution as t goes to 0. In other words, we can approximately reach the elements of $\mathcal{C}(I)$. Let us set

$$\Theta(u_0, \varphi, \delta_*) = \left\{ \eta \in L^1([0, \delta_*]; H^{-1}) ; \mathcal{R}_t(u_0, \delta^{-\frac{1}{2}}\varphi, \delta^{-1}\eta) \text{ exists and continuous for } t \in [0, \delta_*] \right\}.$$

Proposition 21 For $u_0 \in \dot{H}^1(\mathbb{T})$, $\varphi \in H^{-1}(\mathbb{T})$ and $\eta_0 \in \Theta(u_0, \varphi, \delta_*)$, the solution $u(t) = \mathcal{R}_t(u_0, \delta^{-\frac{1}{2}}\varphi, \delta^{-1}\eta_0)$ of (4.5) verifies

$$\lim_{\delta \rightarrow 0} u(\delta) = u_0 + (1 - \partial_{xx})^{-1}(\eta_0 - \varphi \partial_x \varphi), \text{ in } H^1(\mathbb{T}). \quad (4.17)$$

Proof For $t \in [0, \delta_*]$, let $v(t) = u(\delta t)$, where u is the solution of (4.5), then v satisfies

$$\partial_t v + \delta \partial_x v - \partial_{xxt} v + \delta v \partial_x v = \delta^{-1} \eta_0.$$

So

$$\partial_t v = -\delta(1 - \partial_{xx})^{-1} \left(\partial_x(v + \delta^{-\frac{1}{2}}\varphi) + (v + \delta^{-\frac{1}{2}}\varphi) \partial_x(v + \delta^{-\frac{1}{2}}\varphi) - \delta^{-1}\eta_0 \right).$$

Let $w(t) = u_0 + t(1 - \partial_{xx})^{-1}(\eta_0 - \varphi \partial_x \varphi)$ and $z = v - w$. We have

$$\begin{aligned} \partial_t z &= -\delta(1 - \partial_{xx})^{-1} \left(\partial_x(z + w + \delta^{-\frac{1}{2}}\varphi) + (z + w) \partial_x(z + w + \delta^{-\frac{1}{2}}\varphi) \right. \\ &\quad \left. + \delta^{-\frac{1}{2}}\varphi \partial_x(z + w) \right). \end{aligned} \quad (4.18)$$

Taking the inner product in $H^1(\mathbb{T})$ with z we have

$$\begin{aligned} \frac{1}{2} \partial_t (\|z(t)\|_{H^1}^2) &= -\delta \left\{ \left(\partial_x \left(z + w + \delta^{-\frac{1}{2}} \varphi \right), z \right)_{L^2} \right. \\ &\left. + \left((z + w) \partial_x \left(z + w + \delta^{-\frac{1}{2}} \varphi \right), z \right)_{L^2} + \delta^{-\frac{1}{2}} (\varphi \partial(z + w), z)_{L^2} \right\}. \end{aligned} \quad (4.19)$$

We claim

$$\begin{aligned} \frac{1}{2} \partial_t (\|z(t)\|_{H^1}^2) &\leq \delta \left(C_1 + C_2 \delta^{\frac{1}{2}} \right) (\|z(t)\|_{H^1} + \|z(t)\|_{H^1}^2) \\ &\leq C \delta (1 + \|z(t)\|_{H^1}^2). \end{aligned} \quad (4.20)$$

Indeed, as $\int_{\mathbb{T}} z^k \partial_x z \, dx = 0$ for $k = 1, 2$, we estimate the quantities

$$\begin{aligned} \left| \int_{\mathbb{T}} z \partial_x \left(z + w + \delta^{-\frac{1}{2}} \varphi \right) \, dx \right| &= \|w(t)\|_{H^1} \|z(t)\|_{L^2} + \delta^{-\frac{1}{2}} \|\varphi\|_{H^1} \|z(t)\|_{L^2} \\ &\leq \left(\|w(t)\|_{H^1} + \delta^{-\frac{1}{2}} \|\varphi\|_{H^1} \right) \|z(t)\|_{H^1}, \end{aligned} \quad (4.21)$$

$$\begin{aligned} &\left| \int_{\mathbb{T}} z(z + w) \partial_x \left(z + w + \delta^{-\frac{1}{2}} \varphi \right) \, dx \right| \\ &\leq \left| \int_{\mathbb{T}} z^2 \partial_x \left(w + \delta^{-\frac{1}{2}} \varphi \right) \, dx \right| + \left| \int_{\mathbb{T}} w z \partial_x z \, dx \right| + \left| \int_{\mathbb{T}} w z \partial_x w \, dx \right| \\ &\quad + \delta^{-\frac{1}{2}} \left| \int_{\mathbb{T}} w z \partial_x \varphi \, dx \right| \\ &\leq 2 \|z(t)\|_{\infty} \|w(t)\|_{L^2} \|z(t)\|_{H^1} + 2 \delta^{-\frac{1}{2}} \|\varphi\|_{L^2} \|z(t)\|_{H^1}^2 \\ &+ \|w(t)\|_{L^2} \|z(t)\|_{H^1}^2 + \|w(t)\|_{H^1}^2 \|z(t)\|_{H^1} + \delta^{-\frac{1}{2}} \|w(t)\|_{\infty} \|z(t)\|_{H^1} \|\varphi\|_{H^1} \\ &\leq \left(C_1 + C_2 \delta^{-\frac{1}{2}} \right) (\|z(t)\|_{H^1} + \|z(t)\|_{H^1}^2), \end{aligned} \quad (4.22)$$

and

$$\left| \int_{\mathbb{T}} \varphi z \partial_x (z + w) \, dx \right| \leq \|\varphi\|_{H^1} \|z(t)\|_{H^1} (\|z(t)\|_{H^1} + \|w(t)\|_{H^1}),$$

which give us (4.20). By integrating it between 0 and t we have

$$\|z(t)\|_{H^1}^2 \leq 2C \left(\delta t + \int_0^t \|z(s)\|_{H^1}^2 \, ds \right).$$

By Gronwall estimation

$$\|z(t)\|_{H^1}^2 \leq \delta C_1 e^{C_2 t}.$$

Since $t \leq 1$, we have $\|z(t)\|_{H^1}^2 \leq \delta C_1 e^{C_2} \rightarrow 0$ as $\delta \rightarrow 0$.

In particular, $u(\delta) = v(1) \rightarrow w(1)$ as $\delta \rightarrow 0$, and then we obtain (4.17). \square

4.3 Some algebraic results

Let $I_0 \subset \mathbb{Z}$ be a symmetric set. We define by induction for all $n \in \mathbb{N}$

$$I_{n+1} = \{j + p ; \text{ for } j, p \in I_n\}.$$

This sequence of sets will play a role in the proof of the main theorem. More precisely, we claim that the union of (I_n) will be equal to \mathbb{Z} , which will correspond afterwards to showing that we can reach all the Fourier modes to achieve the approximate controllability.

Lemma 11 *If I_0 is a generator of \mathbb{Z} , then*

$$\bigcup_{n \in \mathbb{N}} I_n = \mathbb{Z}. \quad (4.23)$$

Proof For $n \in \mathbb{N}^*$, we define

$$\Gamma_n = \left\{ \sum_{j=1}^d p_j k_j ; \text{ for } p_j \in \mathbb{Z}, k_j \in I_0 \text{ such that } \sum_{j=1}^d |p_j| \leq n, \forall d \geq 1 \right\}.$$

Since I_0 is a generator of \mathbb{Z} , it is clear that

$$\bigcup_{n \in \mathbb{N}^*} \Gamma_n = \mathbb{Z}.$$

Now we will show by induction that for all $n \in \mathbb{N}^*$, $\Gamma_n \subset I_{n-1}$.

As I_0 is symmetric, $\Gamma_1 = \{\pm k / k \in I_0\} = I_0$.

Suppose that $\Gamma_n \subset I_{n-1}$ for some $n \in \mathbb{N}^*$, and let $q = \sum_{j=1}^d p_j k_j \in \Gamma_{n+1}$.

If $\sum_{j=1}^d |p_j| \leq n$, then $q \in I_n$ and the result follows directly. If $\sum_{j=1}^d |p_j| = n + 1$, note that we can suppose $p_j \neq 0$ for all $j = 1, \dots, d$. Let $q' = (p_1 \pm 1)k_1 + \sum_{j=2}^d p_j k_j$ be such that $|p_1 \pm 1| = |p_1| - 1$. We see that $q' \in \Gamma_n$

since $|p_1 \pm 1| + \sum_{j=2}^d |p_j| = n$. According to the induction hypothesis we have

$q' \in I_{n-1}$, and we know that $k_1 \in I_0 \subset I_{n-1}$. Hence, from the definition of I_n , we have

$$I_n \ni k_1 \mp q' = k_1 \mp (p_1 \pm 1)k_1 + \sum_{j=2}^d p_j k_j = q.$$

So $\Gamma_n \subset I_{n-1}$ for all $n \in \mathbb{N}^*$, thus $\mathbb{Z}^* \subset \bigcup_{n \in \mathbb{N}} I_n$, which completes the proof. \square

We can deduce from the previous lemma that if I_0 is a generator of \mathbb{Z} , the space $\mathcal{H}_\infty := \bigcup_{n \in \mathbb{N}} \mathcal{H}(I_n)$ is dense in $\dot{H}^1(\mathbb{T})$. Since for all

$$z = \sum_{k=1}^{\infty} a_k \cos(kx) + b_k \sin(kx) \in \dot{H}^1(\mathbb{T}),$$

we can find N large enough and $n \in \mathbb{N}$ such that $\sum_{k=1}^N a_k \cos(kx) + b_k \sin(kx) \in \mathcal{H}(I_n)$ and

$$\left\| z - \sum_{k=1}^N a_k \cos(kx) + b_k \sin(kx) \right\|_{H^1} < \epsilon.$$

Now we show that the elements of space $\mathcal{H}(I)$ formed by trigonometric polynomials are in fact elements of space $\mathcal{C}(I)$ that we have already shown that we can reach according to Proposition [23](#).

Lemma 12 *For all $n \in \mathbb{N}$, we have the inclusion*

$$\mathcal{H}(I_{n+1}) \subset \mathcal{C}(I_n).$$

Proof Let $n \in \mathbb{N}$ and $\eta_0 \in \mathcal{H}(I_{n+1})$. From [\(4.2\)](#), η_0 is a linear combination of terms $\xi_{j,p}$ of the following form

$$\begin{aligned} \xi_{j,p} = a \cos(jx) + b \sin(jx) + c \cos((j+p)x) + d \sin((j+p)x) + e \cos((j-p)x) \\ + f \cos((j-p)x), \end{aligned}$$

where $j, p \in I_n$.

Case 1 : $j \pm p \neq 0$

We look for $\eta, \varphi_1, \dots, \varphi_d$ in $\mathcal{H}(I_n)$ such that $\xi_{j,p} = \eta - (1 - \partial_{xx})^{-1} \sum_{i=1}^d \varphi_i \partial_x \varphi_i$.

We notice that the two terms $a \cos(jx) + b \sin(jx)$ in expression of $\xi_{j,p}$ are in $\mathcal{H}(I_n)$, so we take $\eta = a \cos(jx) + b \sin(jx)$. Now we set

$$\varphi_1 = \alpha \cos(jx) + \beta \sin(jx) + \cos(px) + \sin(px),$$

$$\varphi_2 = -\beta \cos(jx) + \alpha \sin(jx) - \cos(px) + \sin(px),$$

$$\varphi_3 = \alpha' \cos(jx) + \beta' \sin(jx) + \cos(px) + \sin(px),$$

and

$$\varphi_4 = -\beta' \cos(jx) + \alpha' \sin(jx) + \cos(px) - \sin(px).$$

The functions $\varphi_1, \varphi_2, \varphi_3$ and φ_4 are in $\mathcal{H}(I_n)$, and we obtain

$$\begin{aligned} & \varphi_1 \partial_x \varphi_1 + \varphi_2 \partial_x \varphi_2 + \varphi_3 \partial_x \varphi_3 + \varphi_4 \partial_x \varphi_4 \\ &= (p-j) \{(\alpha-\beta) \cos((j-p)x) + (\alpha+\beta) \sin((j-p)x)\} \\ &+ (p+j) \{(\beta'+\alpha') \cos((j+p)x) + (\beta'-\alpha') \sin((j+p)x)\}. \end{aligned} \quad (4.24)$$

Since the map

$$(\alpha, \beta, \alpha', \beta') \mapsto ((p-j)(\alpha-\beta), (p-j)(\alpha+\beta), (p+j)(\beta'+\alpha'), (p+j)(\beta'-\alpha'))$$

is bijective (since the determinant of the associated matrix is equal to $4(p-j)^2(p+j)^2 \neq 0$), then we can write the quantities $c \cos((j+p)x) + d \sin((j+p)x) + e \cos((j-p)x) + f \sin((j-p)x)$ in the expression of $\xi_{j,p}$ under the form $\varphi_1 \partial_x \varphi_1 + \varphi_2 \partial_x \varphi_2 + \varphi_3 \partial_x \varphi_3 + \varphi_4 \partial_x \varphi_4$. We deduce that $\eta_0 \in \mathcal{C}(I_n)$.

Case 2 : $j \pm p = 0$

Without loss of generalities we can suppose that $j = p > 0$.

In this case, $\xi_{j,p}$ take the following form

$$\xi_{j,p} = \xi_j = a \cos(jx) + b \sin(jx) + c \cos(2jx) + d \sin(2jx), \text{ where } j \in I_n.$$

Note that we can suppose $(c, d) \neq (0, 0)$ (if not, $\eta_0 \in \mathcal{H}(I_n)$). Let $\varphi_1 = \alpha \cos(jx) + \beta \sin(jx)$. We have $\varphi_1 \partial_x \varphi_1 = j\alpha\beta \cos(2jx) + \frac{j}{2}(\beta^2 - \alpha^2) \sin(2jx)$.

We can check that a solution of

$$\begin{cases} j\alpha\beta = c \\ \frac{j}{2}(\beta^2 - \alpha^2) = d, \end{cases}$$

is $(\alpha, \beta) = \left(\frac{c}{\sqrt{j} \sqrt{d + \sqrt{c^2 + d^2}}}, \frac{1}{\sqrt{j}} \sqrt{d + \sqrt{c^2 + d^2}} \right)$. As in the first case, we can write $a \cos(jx) + b \sin(jx) = \eta \in \mathcal{H}(I_n)$ and $c \cos(2jx) + d \sin(2jx) = \varphi_1 \partial_x \varphi_1$. Hence $\eta_0 \in \mathcal{C}(I_n)$. \square

We finish this section by giving an algebraic property of \mathcal{R}_t .

Lemma 13 Let $\mathcal{R}_t(u_0, 0, \eta)$ be the solution of (4.3), where η is given by

$$\eta(s) = \begin{cases} \eta_1(s), & s \in [0, t_1] \\ \eta_2(s), & s \in [t_1, t_2] \\ \eta_3(s), & s \in [t_2, t_3]. \end{cases}$$

For all $t_1, t_2, t_3 \geq 0$, we have the equality

$$\mathcal{R}_{t_1+t_2+t_3}(u_0, 0, \eta) = \mathcal{R}_{t_3}(\mathcal{R}_{t_2}(\mathcal{R}_{t_1}(u_0, 0, \eta_1(\cdot)), 0, \eta_2(\cdot-t_1)), 0, \eta_3(\cdot-t_2-t_1)). \quad (4.25)$$

Proof We denote by $\hat{\mathcal{R}}(t, s, v, \eta)$ the solution of (4.3) at the instant t , when $\varphi = 0$ and with initial data $\hat{\mathcal{R}}(s, s, v, \eta) = v$. That means $\mathcal{R}_t(u_0, 0, \eta) = \hat{\mathcal{R}}(t, 0, u_0, \eta)$. From the uniqueness of solutions, we can see that for all $\sigma \geq 0$

$$\hat{\mathcal{R}}(t, \sigma, \hat{\mathcal{R}}(\sigma, s, u_0, \eta), \eta) = \hat{\mathcal{R}}(t, s, u_0, \eta). \quad (4.26)$$

Using (4.26) we can write

$$\begin{aligned} & \hat{\mathcal{R}}(t_1 + t_2 + t_3, 0, u_0, \eta) \\ &= \hat{\mathcal{R}}(t_1 + t_2 + t_3, t_1 + t_2, \hat{\mathcal{R}}(t_1 + t_2, t_1, \hat{\mathcal{R}}(t_1, 0, u_0, \eta_1(\cdot)), \eta_2(\cdot - t_1)), \eta_3(\cdot - t_2 - t_1)) \\ &= \hat{\mathcal{R}}(t_3, 0, \hat{\mathcal{R}}(t_1 + t_2, t_1, \hat{\mathcal{R}}(t_1, 0, u_0, \eta_1(\cdot)), \eta_2(\cdot - t_1)), \eta_3(\cdot - t_2 - t_1)) \\ &= \mathcal{R}_{t_3}(\hat{\mathcal{R}}(t_1 + t_2, t_1, \hat{\mathcal{R}}(t_1, 0, u_0, \eta_1(\cdot)), \eta_2(\cdot - t_1)), 0, \eta_3(\cdot - t_2 - t_1)) \\ &= \mathcal{R}_{t_3}(\mathcal{R}_{t_2}(\mathcal{R}_{t_1}(u_0, 0, \eta_1(\cdot)), 0, \eta_2(\cdot - t_1)), 0, \eta_3(\cdot - t_2 - t_1)). \end{aligned} \quad (4.27)$$

□

4.4 Proof of Theorem 22

We can now give a proof of Theorem 22. For $I \subset \mathbb{Z}$ and $k \in \mathbb{N}^*$, we set

$$\mathcal{C}_k(I) = \left\{ \eta - (1 - \partial_{xx})^{-1} \sum_{i=1}^k \varphi_i \partial_x \varphi_i ; \eta, \varphi_i \in \mathcal{H}(I) \right\}.$$

Proof Let $v \in \dot{H}^1(\mathbb{T})$. Suppose I_0 is generator of \mathbb{Z}^* . This means that \mathcal{H}_∞ is dense in $\dot{H}^1(\mathbb{T})$. For all $\epsilon > 0$ there exist $n > 0$ and $\hat{v} \in u_0 + \mathcal{H}(I_{n+1})$ such that $\|v - \hat{v}\|_{H^1} \leq \epsilon$. So in the rest of the proof, we will call w the element which belongs to $\mathcal{H}(I_{n+1}) \subset \mathcal{C}(I_n)$.

Step 1: Controllability in small time. We prove by induction the following property

$$\begin{aligned} \mathcal{P}_k^n : & \forall u_0 \in \dot{H}^1(\mathbb{T}), \forall \epsilon > 0, \forall \sigma > 0, \forall w \in \mathcal{C}_k(I_n), \exists t \in]0, \sigma], \\ & \exists \eta \in \Theta(u_0, \varphi, t) \cap \mathcal{H}(I_0) \text{ s.t. } \|\mathcal{R}_t(u_0, 0, \eta) - (u_0 + w)\|_{H^1} \leq \epsilon. \end{aligned}$$

The property \mathcal{P}_k^0 follows from Proposition 23. We give a double induction on both index k and n . Suppose that \mathcal{P}_k^n is true for some $k \in \mathbb{N}^*$ and $n \in \mathbb{N}^*$. Let $u_0 \in \dot{H}^1(\mathbb{T})$, $\epsilon > 0$, $\sigma > 0$, $w \in \mathcal{C}_k(I_n)$ and

$$v = w - (1 - \partial_{xx})^{-1} \varphi \partial_x \varphi \in \mathcal{C}_{k+1}(I_n),$$

where $\varphi \in \mathcal{H}(I_n)$.

We know from Proposition 23 that $\mathcal{R}_\delta(u_0, \delta^{-\frac{1}{2}}\varphi, 0) \rightarrow u_0 - (1 - \partial_{xx})^{-1}\varphi\partial_x\varphi$, when $\delta \rightarrow 0$. Combing this with (4.14), we can found $\delta_1 > 0$ small enough such that

$$\begin{aligned} \left\| \mathcal{R}_{\delta_1}(u_0 + w + \delta_1^{-\frac{1}{2}}\varphi, 0, 0) - (u_0 + w - (1 - \partial_{xx})^{-1}\varphi\partial_x\varphi + \delta_1^{-\frac{1}{2}}\varphi) \right\|_{H^1} \\ \leq \frac{\epsilon}{3c_1}, \end{aligned} \quad (4.28)$$

where c_1 is the constant given by Proposition 20. Since $w + \delta_1^{-\frac{1}{2}}\varphi \in \mathcal{C}_k(I_n)$, we use the fact that \mathcal{P}_k^n is true to found $t_1 > 0$ and $\eta_1 \in \Theta(u_0, \varphi, t_1 + \delta_1) \cap \mathcal{H}(I_0)$ such that

$$\left\| \mathcal{R}_{t_1}(u_0, 0, \eta_1) - (u_0 + w + \delta_1^{-\frac{1}{2}}\varphi) \right\|_{H^1} \leq \frac{\epsilon}{3c_1^2}. \quad (4.29)$$

Thus, using the property of the nonlinear semigroup in Lemma 13 with t_1 and δ_1 we have

$$\begin{aligned} \left\| \mathcal{R}_{t_1+\delta_1}(u_0, 0, \mathbb{1}_{[0,t_1]}\eta_1) - (u_0 + w - (1 - \partial_{xx})^{-1}\varphi\partial_x\varphi + \delta_1^{-\frac{1}{2}}\varphi) \right\|_{H^1} \\ \leq \left\| \mathcal{R}_{\delta_1}(\mathcal{R}_{t_1}(u_0, 0, \eta_1), 0, 0) - \mathcal{R}_{\delta_1}(u_0 + w + \delta_1^{-\frac{1}{2}}\varphi, 0, 0) \right\|_{H^1} \\ + \left\| \mathcal{R}_{\delta_1}(u_0 + w + \delta_1^{-\frac{1}{2}}\varphi, 0, 0) - (u_0 + w - (1 - \partial_{xx})^{-1}\varphi\partial_x\varphi + \delta_1^{-\frac{1}{2}}\varphi) \right\|_{H^1}. \end{aligned} \quad (4.30)$$

From the continuity of \mathcal{R}_{δ_1} with respect to the initial data seen in Proposition 20 and from (4.29), we can estimate the first term on the right by $\frac{\epsilon}{3c_1}$. By the same method applied to the second one using (4.28), we can write

$$\left\| \mathcal{R}_{t_1+\delta_1}(u_0, 0, \mathbb{1}_{[0,t_1]}\eta_1) - (u_0 + w - (1 - \partial_{xx})^{-1}\varphi\partial_x\varphi + \delta_1^{-\frac{1}{2}}\varphi) \right\|_{H^1} \leq \frac{2\epsilon}{3c_1}. \quad (4.31)$$

This means that starting from u_0 , after a time $t_1 + \delta_1$ and using the control $\mathbb{1}_{[0,t_1]}\eta_1$ we can reach $u_0 + w - (1 - \partial_{xx})^{-1}\varphi\partial_x\varphi + \delta_1^{-\frac{1}{2}}\varphi$. Now we take this quantity as initial data and we apply \mathcal{P}_k^n again, there exist $t_2 > 0$ and $\eta_2 \in \Theta(u_0, \varphi, t_2 + t_1 + \delta_1) \cap \mathcal{H}(I_0)$ such that

$$\left\| \mathcal{R}_{t_2}(u_0 + w - (1 - \partial_{xx})^{-1}\varphi\partial_x\varphi + \delta_1^{-\frac{1}{2}}\varphi, 0, \eta_2) - (u_0 + v) \right\|_{H^1} \leq \frac{\epsilon}{3}. \quad (4.32)$$

Let $\eta : s \mapsto \mathbb{1}_{[0,t_1]}\eta_1(s) + \mathbb{1}_{[t_1+\delta_1, t_1+\delta_1+t_2]}\eta_2(s - t_1 - \delta_1)$. As in (4.30), we use Lemma 13, (4.31), (4.32) and the continuity of \mathcal{R}_{t_2} with respect to the initial

data, we obtain

$$\begin{aligned}
& \|\mathcal{R}_{t_2+t_1+\delta_1}(u_0, 0, \eta) - (u_0 + v)\|_{H^1} \\
& \leq \|\mathcal{R}_{t_2}(\mathcal{R}_{t_1+\delta_1}(u_0, 0, \mathbf{1}_{[0,t_1]}\eta_1), 0, \eta_2) \\
& \quad - \mathcal{R}_{t_2}(u_0 + w - (1 - \partial_{xx})^{-1}\varphi\partial_x\varphi + \delta_1^{-\frac{1}{2}}\varphi, 0, \eta_2)\|_{H^1} \\
& + \|\mathcal{R}_{t_2}(u_0 + w - (1 - \partial_{xx})^{-1}\varphi\partial_x\varphi + \delta_1^{-\frac{1}{2}}\varphi, 0, \eta_2) - (u_0 + v)\|_{H^1} \\
& \leq \frac{2\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.
\end{aligned}$$

Note that we can always take δ_1, t_1 and t_2 small enough such that

$$t_2 + t_1 + \delta_1 < \sigma.$$

This proves that \mathcal{P}_{k+1}^n is true. So we have that \mathcal{P}_k^0 is true for all $k \in \mathbb{N}$. From Lemma [12](#), \mathcal{P}_0^1 is true. We can repeat the same reasoning in this first step of the proof to deduce that \mathcal{P}_k^1 is true for all $k \in \mathbb{N}$, and then, \mathcal{P}_0^2 is true from Lemma [12](#) and so on. In the end, \mathcal{P}_k^n is true for all $k, n \in \mathbb{N}$.

Step 2: Controllability in fixed time. Once we have approximate controllability in a small time, we can generalize this for any $T > 0$. Indeed, we know from what precedes that for all $v \in \mathcal{C}(I_n)$, starting from any u_0 , the solution will be in a neighborhood of v in a time shorter than σ . We use the same notation $\hat{\mathcal{R}}$ seen in the proof of Lemma [13](#).

Let $T > 0$, from the first step, there exist $t_1 < T$ and $\eta_1 \in \mathcal{H}(I_0)$ such that $\|u_1 - v\|_{H^1} \leq \frac{\epsilon}{2}$, where $u_1 = \mathcal{R}_{t_1}(u_0, 0, \eta_1)$. Now we set

$$t_2 = \inf \left\{ t \in [t_1, T[; \|\hat{\mathcal{R}}(t, t_1, u_1, 0) - v\|_{H^1} \geq \epsilon \right\}.$$

Suppose that t_2 exists (if not, the proof is complete).

We know that if $\|\hat{\mathcal{R}}(t, t_1, u_1, 0) - v\|_{H^1} = \epsilon$ then $t \geq \frac{\epsilon}{2C}$, indeed, we can use Lemma [10](#) to write

$$\begin{aligned}
\|\hat{\mathcal{R}}(t, t_1, u_1, 0) - v\|_{H^1} & \leq \|\hat{\mathcal{R}}(t, t_1, u_1, 0) - u_1\|_{H^1} + \|u_1 - v\|_{H^1} \\
& \leq Ct + \frac{\epsilon}{2}.
\end{aligned}$$

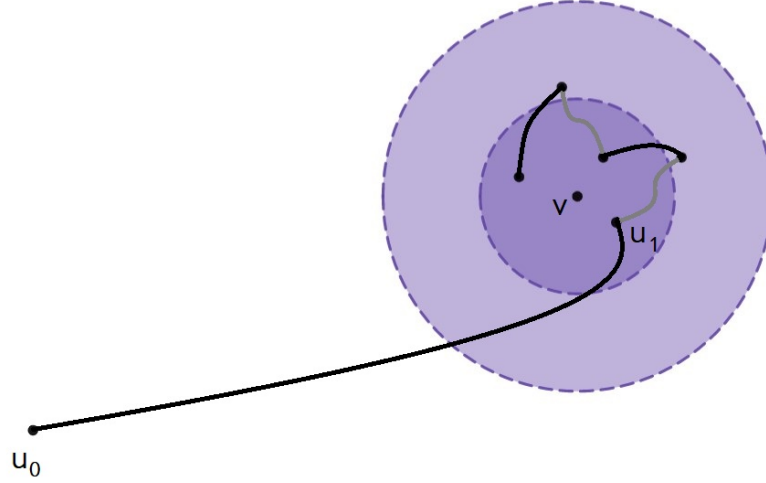
At $t = t_2$, we use the first step taking $\sigma = \frac{T - t_2}{2}$, so we can find t_3 and η_3 such that

$$\|\hat{\mathcal{R}}(t_3, t_2, \hat{\mathcal{R}}(t_2, t_1, u_1, 0), \eta_3) - v\|_{H^1} \leq \frac{\epsilon}{2}.$$

We can introduce t_4 by the same way as t_2

$$t_4 = \inf \left\{ t \in [t_3, T[; \|\hat{\mathcal{R}}(t, t_3, \hat{\mathcal{R}}(t_3, t_2, \hat{\mathcal{R}}(t_2, t_1, u_1, 0), \eta_3), 0) - v\|_{H^1} \geq \epsilon \right\},$$

and repeat the same process until we reach T .



Note that the sequence (t_j) will not converge since the “waiting time” inside the ball $B(v, \epsilon)$ is always greater than $\frac{\epsilon}{2C}$.

Step 3: Necessity of the condition on I_0 . Suppose that I_0 is not a generator of \mathbb{Z} , and let u_0 be an initial data in $\overline{\mathcal{H}_\infty}$. As the product $u\partial_x u$ leaves the space $\overline{\mathcal{H}_\infty}$ invariant, then we can apply the argument of the fixed point seen in Lemma 9 in $\mathcal{C}([0, T], \mathcal{H}_\infty)$. By uniqueness of the solution we will have $u(t) \in \overline{\mathcal{H}_\infty}$. This means that since I_0 is not a generator of \mathbb{Z} we can find a set $I_1 \subset \mathbb{Z}$ such that $\mathcal{H}(I_1)$ is orthogonal to \mathcal{H}_∞ , so \mathcal{H}_∞ cannot be dense in $\dot{H}^1(\mathbb{T})$. \square

4.5 Proof of Theorem 23

We prove in this section that we cannot do exact controllability of the BBM equation in $\dot{H}^s(\mathbb{T})$ for $s \in [1, 2]$.

Definition 17 Let A be a set of a Banach space X and $\epsilon > 0$. We say that a family of sets $(A_i)_{i=1, \dots, n}$ is an ϵ -covering of A if $A \subset \bigcup_{i=1}^n A_i$ and $\text{diam}(A_i) < 2\epsilon$, for all $i = 1, \dots, n$. The ϵ -entropy of A for the topology of X is the number H_ϵ given by the logarithm of the minimal number N_ϵ of sets

A_i so that it forms an ϵ -covering of A . So we write

$$H_\epsilon(A, X) = \ln(N_\epsilon(A, X)).$$

Remark 9 It is known that if $A = \prod_{k=1}^N A_k$ then

$$H_\epsilon(A, X) \leq \sum_{k=1}^N H_{\frac{\epsilon}{\sqrt{N}}}(A_k, X).$$

See inequality (7) in Section 10.1 of [Lor86] for details.

Proposition 22 Let $J = [0, T]$, E a finite dimensional vector space and B a closed ball in $BV(J, E)$. Then there exists $c > 0$ such that

$$H_\epsilon(B, L^1(J, E)) \leq c \frac{1}{\epsilon} \ln\left(\frac{1}{\epsilon}\right). \quad (4.33)$$

See Annex for the proof.

We will use the following Lemma (Lemma 2.1. in [Shi08]).

Lemma 14 Let X, Y be two Banach spaces, $K \subset X$ a compact set and $f : X \rightarrow Y$ a function such that

$$\|f(u) - f(v)\|_Y \leq L \|u - v\|_X.$$

Then for all $\epsilon > 0$ we have

$$H_\epsilon(f(K), Y) \leq H_{\frac{\epsilon}{L}}(K, X).$$

Proof (of Theorem 23) **Case 1** : $s = 1$. Let Q be an open set in $\dot{H}^1(\mathbb{T})$ and $(B_n)_{n \in \mathbb{N}}$ a family of closed balls with center 0 and radius n in $BV([0, \delta_*], \mathcal{H}(I_0))$. Suppose that we have exact controllability in $\dot{H}^1(\mathbb{T})$, then

$$Q \subset \bigcup_{n \in \mathbb{N}} \overline{\mathcal{R}_t(u_0, 0, B_n)}.$$

We recall that $\mathcal{R}_t(u_0, 0, \cdot) : BV([0, \delta_*], \mathcal{H}(I_0)) \rightarrow \dot{H}^1(\mathbb{T})$. Since $\mathcal{H}(I_0)$ is a finite dimension space, the injection $BV([0, \delta_*], \mathcal{H}(I_0)) \hookrightarrow L^1([0, \delta_*], \mathcal{H}(I_0))$ is compact (see Theorem 1.19 in [Giu84] for the proof). So from the continuity result of Corollary 1 we can deduce that the family $(\mathcal{R}_t(u_0, 0, B_n))_{n \in \mathbb{N}}$ is relatively compact in $\dot{H}^1(\mathbb{T})$. This is absurd because the interior of $\bigcup_{n \in \mathbb{N}} \overline{\mathcal{R}_t(u_0, 0, B_n)}$

is empty from Baire's theorem, so it cannot contain Q .

Case 2 : $s \in]1, 2[$. We say that $x \sim y$ if there are $c_1, c_2 > 0$ such that $c_1 y \leq x \leq c_2 y$. We start by giving another version of the ϵ -entropy definition then we will give the link between them.

Definition 18 Let A_1, A_2 two Banach spaces, $T \in \mathcal{L}(A_1, A_2)$ and

$$U_{A_i} = \left\{ a \in A_i; \|a\|_{A_i} \leq 1 \right\}.$$

The k -th entropy of T is defined by

$$e_k(T) = \inf \left\{ \epsilon > 0; T(U_{A_1}) \subset \bigcup_{j=1}^{2^{k-1}} (b_j + \epsilon U_{A_2}), b_1, \dots, b_{2^{k-1}} \in A_2 \right\}.$$

This definition describe the same thing as Definition [17](#). Indeed, Definition [17](#) says that we need at least $N_\epsilon(A, X)$ balls of size ϵ to cover a set A . In Definition [18](#) (Definition 1 in [ET96](#)) we fix the radius of the balls $e_k = \epsilon$, and we say that we need 2^{k-1} balls, in other words $N_\epsilon \sim 2^{k-1}$.

We have the following Lemma.

Lemma 15 Let B be a ball of $\dot{H}^s(\mathbb{T})$ for $s \in]1, 2[$, then

$$H_\epsilon(B, H^1(\mathbb{T})) \sim \epsilon^{\frac{-1}{s-1}}. \quad (4.34)$$

Proof We know from Theorem 2 of the Section 3.3.3. in [ET96](#) that if we take $T = id : \dot{H}^s(\mathbb{T}) \rightarrow \dot{H}^1(\mathbb{T})$ then $e_k(T) \sim k^{-(s-1)}$. So for k large enough, if we set $\epsilon = k^{-(s-1)}$, we have $N_\epsilon(B, H^1(\mathbb{T})) \sim 2^{k-1}$ which give us [\(4.34\)](#). \square

From Lemma [14](#) and Propostion [22](#) we have

$$H_\epsilon(\mathcal{R}_t(u_0, 0, B), H^1) \leq H_{\frac{\epsilon}{c}}(B, L^1) \leq C \frac{1}{\epsilon} \ln \left(\frac{1}{\epsilon} \right).$$

This means that the solution $\mathcal{R}_t(u_0, 0, \cdot)$ cannot create more than the order $\frac{1}{\epsilon} \ln \left(\frac{1}{\epsilon} \right)$ of balls in H^1 . But from Lemma [15](#) we see that we need the order $\epsilon^{\frac{-1}{s-1}}$ of balls in H^1 to cover a ball in \dot{H}^s . So if $s \in]1, 2[$ and $B \subset \dot{H}^s$, we have

$$H_\epsilon(\mathcal{R}_t(u_0, 0, B), H^1) < c H_\epsilon(B, H^1)$$

and thus, the exact controllability result is not possible. \square

Controllability of the KP-I equation

5.1 Introduction

We consider the Kadomtsev–Petviashvili I equation on the torus

$$\begin{cases} \partial_t u + \partial_{xxx} u + u \partial_x u - \partial_x^{-1} \partial_{yy} u = \eta, & (x, y) \in \mathbb{T}^2, t > 0, \\ u(\cdot, \cdot, 0) = u_0, & (x, y) \in \mathbb{T}^2, \end{cases} \quad (5.1)$$

where the operator ∂_x^{-1} is defined for $u = \sum_{(k,n) \in \mathbb{Z}^* \times \mathbb{Z}} u_{k,n} e^{i(kx+ny)}$ by

$$\partial_x^{-1} u = - \sum_{(k,n) \in \mathbb{Z}^* \times \mathbb{Z}} \frac{i}{k} u_{k,n} e^{i(kx+ny)}.$$

This equation, seen as a generalization of KdV type equations in higher dimensions, is introduced by Kadomtsev and Petviashvili [KP70]. Kenig [Ken04] showed that the KP-I equation is well posed on spaces with a certain regularity (see also Bourgain [Bou93] and Saut [Sau93]). A work by Saut and Tzvetkov [ST99] shows that for inhomogeneous Sobolev spaces with weak regularity with respect to the variable x , the KP-I equation is ill-posed. A controllability result for the linear KP-I equation can be found in Sun’s article [Sun18].

We define for $s > 1$ the spaces

$$\tilde{H}^s(\mathbb{T}^2) = \left\{ u = \sum_{(k,n) \in \mathbb{Z}^* \times \mathbb{Z}} u_{k,n} e^{i(kx+ny)} \in L^2(\mathbb{T}^2); \right. \\ \left. \sum_{(k,n) \in \mathbb{Z}^* \times \mathbb{Z}} |u_{k,n}|^2 (1+k^2+n^2)^s < \infty \right\}, \quad (5.2)$$

and \mathcal{Q}^s , the domain of the operator $\partial_{xxx} - \partial_x^{-1} \partial_{yy}$ is given by

$$\mathcal{Q}^s(\mathbb{T}^2) = \left\{ u = \sum_{(k,n) \in \mathbb{Z}^* \times \mathbb{Z}} u_{k,n} e^{i(kx+ny)} \in L^2(\mathbb{T}^2); \right. \\ \left. \sum_{(k,n) \in \mathbb{Z}^* \times \mathbb{Z}} |u_{k,n}|^2 \left(\frac{(k^4+n^2)^2}{k^2} + 1 \right) (1+k^2+n^2)^s < \infty \right\}. \quad (5.3)$$

We consider the KP-I equation with a control term $\eta = \eta(t, x, y)$ as in the problem studied in the fourth chapter. The idea to show the existence is to look directly at the nonlinear equation by replacing the term $u \partial_x u$ by $v \partial_x u$ to define a semi-group associated with the operator $A(v)$ and then take $v = u$. We also give necessary estimates for the existence theorem. The existence result is not optimal from the point of view of the regularity of the initial data, but is sufficient for the approximate control problem.

We write the equation (5.1) under the form

$$\begin{cases} \partial_t u = A(u)u + \eta, \\ u(0) = u_0, \end{cases} \quad (5.4)$$

where $A(v)$ is defined for $v \in \mathcal{Q}^s(\mathbb{T}^2)$ by

$$A(v) : \mathcal{Q}^s(\mathbb{T}^2) \longrightarrow \tilde{H}^s(\mathbb{T}^2) \\ u \longmapsto -(\partial_{xxx} u + v \partial_x u - \partial_x^{-1} \partial_{yy} u).$$

For a proof that $A(v)$ operates from \mathcal{Q}^s to \tilde{H}^s , we refer to Lemma 16 and Lemma 18. The operator $A(v)$ can be written as the sum $A(v) = A_1 + A_2(v)$ with

$$A_1 u = -\partial_{xxx} u + \partial_x^{-1} \partial_{yy} u + 1,$$

and

$$A_2(v)u = -v \partial_x u - 1.$$

If $u = \sum_{(k,n) \in \mathbb{Z}^* \times \mathbb{Z}} u_{k,n} e^{i(kx+ny)} \in \mathcal{Q}^s(\mathbb{T}^2)$, then the operator A_1 is given by

$$(A_1 u)_{k,n} = \left(\frac{i(k^4 + n^2)}{k} + 1 \right) u_{k,n},$$

where $((A_1 u)_{k,n})$ are the coefficients of $A_1 u$. We see that the space \mathcal{Q}^s is made so that A_1 is bijective, and A_1^{-1} is given by

$$(A_1^{-1} u)_{k,n} = \frac{k}{i(k^4 + n^2) + k} u_{k,n}.$$

We give the principal theorem of this chapter.

Let $I_1 = \{(1, 1); (1, 0); (1, -1)\}$. We define for $I \subset \mathbb{Z}^* \times \mathbb{Z}$ the space

$$\mathcal{H}(I) = \text{span}\{\cos(px + qy), \sin(px + qy); \text{ for } (p, q) \in I\},$$

Theorem 24 *The equation (5.1) is approximately controllable in $\tilde{H}^s(\mathbb{T}^2)$ by piecewise constant controls with values in $\mathcal{H}(I_1)$ for $s > 2$. In other words, for all $T > 0$, $\epsilon > 0$ and all $u_0, v \in \tilde{H}^s$, there exist a control $\eta(t, x, y)$ piecewise constant in time with values in $\mathcal{H}(I_1)$ and a solution u of (5.1) such that*

$$\|u(T) - v\|_{H^s} \leq \epsilon.$$

5.2 Some estimations

We give some useful estimations that we will need in the next section.

Lemma 16 *We have the injection*

$$\mathcal{Q}^s(\mathbb{T}^2) \hookrightarrow H^{s+\frac{3}{2}}(\mathbb{T}^2). \quad (5.5)$$

Moreover, if $u \in \mathcal{Q}^s(\mathbb{T}^2)$, then for $\alpha \in \{0, 1, 2, 3\}$

$$\partial_x^\alpha u \in H^{s+\frac{3-\alpha}{2}}, \quad \partial_y u \in H^{s+\frac{1}{2}}, \quad \partial_x \partial_y u \in H^s \quad \text{and} \quad \partial_x^{-1} \partial_{yy} u \in H^s. \quad (5.6)$$

Proof Let $u \in \mathcal{Q}^s(\mathbb{T}^2)$. We write

$$\begin{aligned} \|\partial_x^\alpha u\|_{H^{s+\frac{3-\alpha}{2}}}^2 &= \sum_{(k,n) \in \mathbb{Z}^* \times \mathbb{Z}} k^{2\alpha} |u_{k,n}|^2 (1 + k^2 + n^2)^{s+\frac{3-\alpha}{2}} \\ &= \sum_{(k,n) \in \mathbb{Z}^* \times \mathbb{Z}} \left(1 + \frac{(k^4 + n^2)^2}{k^2} \right) |u_{k,n}|^2 (1 + k^2 + n^2)^s \frac{k^{2+2\alpha} (1 + k^2 + n^2)^{\frac{3-\alpha}{2}}}{k^2 + (k^4 + n^2)^2}. \end{aligned}$$

Now it is all about estimating the term $\frac{k^{2+2\alpha}}{k^2 + (k^4 + n^2)^2} (1 + k^2 + n^2)^{\frac{3-\alpha}{2}}$. We see that

$$\frac{k^{2+2\alpha}(1 + k^2 + n^2)^{\frac{3-\alpha}{2}}}{k^2 + (k^4 + n^2)^2} \lesssim \begin{cases} \frac{k^{2+2\alpha+3-\alpha}}{k^8} = k^{-3+\alpha} \leq C, & \text{if } k^2 \geq n^2, \\ \frac{k^{2+2\alpha}n^{3-\alpha}}{k^8} \leq n^{-3+\alpha}n^{3-\alpha} \leq C, & \text{if } k^2 < n^2 < k^4, \\ \frac{k^{2+2\alpha}n^{3-\alpha}}{n^4} \leq n^{1+\alpha}n^{-1-\alpha} \leq C, & \text{if } n^2 \geq k^4. \end{cases}$$

So $\|\partial_x^\alpha u\|_{H^{s+\frac{3-\alpha}{2}}}^2 \leq C\|u\|_{Q^s}^2$ and hence we obtain (5.5) and $\partial_x^\alpha u \in H^{s+\frac{3-\alpha}{2}}(\mathbb{T}^2)$. We prove the second statement of (5.6).

$$\begin{aligned} \|\partial_y u\|_{H^{s+\frac{1}{2}}}^2 &= \sum_{(k,n) \in \mathbb{Z}^* \times \mathbb{Z}} n^2 |u_{k,n}|^2 (1 + k^2 + n^2)^{s+\frac{1}{2}} \\ &= \sum_{(k,n) \in \mathbb{Z}^* \times \mathbb{Z}} \left(1 + \frac{(k^4 + n^2)^2}{k^2}\right) |u_{k,n}|^2 (1 + k^2 + n^2)^s \frac{n^2 k^2 (1 + k^2 + n^2)^{\frac{1}{2}}}{k^2 + (k^4 + n^2)^2}. \end{aligned}$$

As previously, we estimate

$$\frac{n^2 k^2 (1 + k^2 + n^2)^{\frac{1}{2}}}{k^2 + (k^4 + n^2)^2} \lesssim \begin{cases} k^{-3} \leq C, & \text{if } k^2 \geq n^2, \\ n^3 k^{-6} \leq C, & \text{if } k^2 < n^2 < k^4, \\ n^{-1} k^2 \leq C, & \text{if } n^2 \geq k^4. \end{cases}$$

Thus $\partial_y u \in H^{s+\frac{1}{2}}(\mathbb{T}^2)$. For the third one

$$\begin{aligned} \|\partial_x \partial_y u\|_{H^s}^2 &= \sum_{(k,n) \in \mathbb{Z}^* \times \mathbb{Z}} k^2 n^2 |u_{k,n}|^2 (1 + k^2 + n^2)^s \\ &= \sum_{(k,n) \in \mathbb{Z}^* \times \mathbb{Z}} \left(1 + \frac{(k^4 + n^2)^2}{k^2}\right) |u_{k,n}|^2 \frac{(1 + k^2 + n^2)^s k^4 n^2}{k^2 + (k^4 + n^2)^2}. \end{aligned}$$

We estimate

$$\frac{k^4 n^2}{k^2 + (k^4 + n^2)^2} \lesssim \begin{cases} k^{-4} n^2 \leq C, & \text{if } k^4 \geq n^2, \\ k^4 n^{-2} \leq C, & \text{if } n^2 \geq k^4. \end{cases}$$

Then $\partial_x \partial_y u \in H^s(\mathbb{T}^2)$, and for the last one

$$\begin{aligned} \|\partial_x^{-1} \partial_{yy} u\|_{H^s}^2 &= \sum_{(k,n) \in \mathbb{Z}^* \times \mathbb{Z}} \frac{n^4}{k^2} |u_{k,n}|^2 (1 + k^2 + n^2)^s \\ &= \sum_{(k,n) \in \mathbb{Z}^* \times \mathbb{Z}} \left(1 + \frac{(k^4 + n^2)^2}{k^2}\right) |u_{k,n}|^2 \frac{(1 + k^2 + n^2)^s n^4}{k^2 + (k^4 + n^2)^2}. \end{aligned}$$

We have

$$\frac{n^4}{k^2 + (k^4 + n^2)^2} \lesssim \begin{cases} n^4 k^{-8} \leq C, & \text{if } k^4 \geq n^2, \\ C, & \text{if } n^2 \geq k^4. \end{cases}$$

Then $\partial_x^{-1} \partial_{yy} u \in H^s(\mathbb{T}^2)$, which completes the proof of (5.6). \square

Lemma 17 *Let $s_0 > 1$ and $w = \sum_{k,n \in \mathbb{Z}^2} w_{k,n} e^{i(kx+ny)} \in H^{s_0}(\mathbb{T}^2)$. Then there exists $c = c(s_0) > 0$ such that*

$$\|w_{k,n}\|_{\ell^1} \leq c \|w\|_{H^{s_0}}.$$

Proof From Cauchy-Schwarz inequality we have

$$\sum_{k,n \in \mathbb{Z}^2} |w_{k,n}| \leq \left(\sum_{k,n \in \mathbb{Z}^2} |w_{k,n}| (1 + |k|^2 + |n|^2)^{\frac{s_0}{2}} \right)^{\frac{1}{2}} \left(\sum_{k,n \in \mathbb{Z}^2} \frac{1}{(1 + |k|^2 + |n|^2)^{\frac{s_0}{2}}} \right)^{\frac{1}{2}} < \infty.$$

\square

Lemma 18 *Let $s \geq s_0 > 1$ and f, g in $H^s(\mathbb{T}^2)$. There exists $c = c(s, s_0) > 0$ such that*

$$\|fg\|_{H^s} \leq c (\|f\|_{H^s} \|g\|_{H^{s_0}} + \|f\|_{H^{s_0}} \|g\|_{H^s}). \quad (5.7)$$

Proof We write $\|fg\|_{H^s}^2 = \sum_{k,n \in \mathbb{Z}^2} q_{k,n}^2$ with

$$q_{k,n} = \sum_{l_1, l_2 \in \mathbb{Z}^2} f_{k-l_1, n-l_2} g_{l_1, l_2} (1 + |k|^2 + |n|^2)^{\frac{s}{2}}.$$

Now we see that

$$\begin{aligned} |q_{k,n}| &\leq \sum_{l_1, l_2 \in \mathbb{Z}^2} |f_{k-l_1, n-l_2}| |g_{l_1, l_2}| (1 + |k-l_1|^2 + |n-l_2|^2)^{\frac{s}{2}} \\ &\quad + \sum_{l_1, l_2 \in \mathbb{Z}^2} |f_{k-l_1, n-l_2}| |g_{l_1, l_2}| (1 + |l_1|^2 + |l_2|^2)^{\frac{s}{2}}. \end{aligned} \quad (5.8)$$

By setting $a_{k,n} = |f_{k,n}| (1 + |k|^2 + |n|^2)^{\frac{s}{2}}$ and $b_{k,n} = |g_{k,n}|$, we can deduce that $(a_{k,n}) \in \ell^2$ since $f \in H^s$ and $(b_{k,n}) \in \ell^1$ from Lemma 17. Moreover, $\|a_{k,n}\|_{\ell^2} = \|f\|_{H^s}$ and $\|b_{k,n}\|_{\ell^1} \leq c \|g\|_{H^{s_0}}$. This means that the two sums in the right side of (5.8), which are actually $a_{k,n} * b_{k,n}$ each, are in ℓ^2 and the proof is complete. \square

Lemma 19 *If $v, w \in \mathcal{Q}^s$, then there exists a constant $c = c(s) > 0$ such that*

$$\|[\partial_{yy}\partial_x^{-1}, v]\partial_x w\|_{H^s} \leq c\|v\|_{H^s}\|w\|_{H^s}.$$

Proof We use the equality $[\partial_x^{-1}, v] = -\partial_x^{-1}(\partial_x v)\partial_x^{-1}$ to write

$$\begin{aligned} [\partial_{yy}\partial_x^{-1}, v]\partial_x w &= \partial_{yy}\partial_x^{-1}(v\partial_x w) - v\partial_{yy}w \\ &= \partial_x^{-1}(\partial_{yy}v\partial_x w + 2\partial_y v\partial_{yx}w + v\partial_{yyx}w) - v\partial_{yy}w \\ &= \partial_x w\partial_x^{-1}\partial_{yy}v + [\partial_x^{-1}, \partial_x w]\partial_{yy}v + 2\partial_x^{-1}(\partial_y v\partial_{yx}w) \\ &\quad + v\partial_{yy}w + [\partial_x^{-1}, v]\partial_{yyx}w - v\partial_{yy}w \\ &= \partial_x w\partial_x^{-1}\partial_{yy}v - \partial_x^{-1}(\partial_{xx}w\partial_x^{-1}\partial_{yy}v) + 2\partial_x^{-1}(\partial_y v\partial_{yx}w) \\ &\quad - \partial_x^{-1}(\partial_x v\partial_{yy}w) \\ &= \partial_x w\partial_x^{-1}\partial_{yy}v - \partial_x^{-1}(\partial_{xx}w\partial_x^{-1}\partial_{yy}v) + 2\partial_x^{-1}(\partial_y v\partial_{yx}w) \\ &\quad - \partial_x v\partial_x^{-1}\partial_{yy}w - [\partial_x^{-1}, \partial_x v]\partial_{yy}w \\ &= \partial_x w\partial_x^{-1}\partial_{yy}v - \partial_x^{-1}(\partial_{xx}w\partial_x^{-1}\partial_{yy}v) + 2\partial_x^{-1}(\partial_y v\partial_{yx}w) \\ &\quad - \partial_x v\partial_x^{-1}\partial_{yy}w + \partial_x^{-1}(\partial_{xx}v\partial_x^{-1}\partial_{yy}w). \end{aligned}$$

Now we use Lemma 16 to say that both terms $\partial_x w\partial_x^{-1}\partial_{yy}v$ and $\partial_x v\partial_x^{-1}\partial_{yy}w$ belong to H^s . The other terms belong at least to H^s since they have ∂_x^{-1} which is bounded in H^s . So all the quantities are bounded in H^s and the proof is complete. \square

Lemma 20 *Let $s > 1$, $v, z \in \mathcal{Q}^s$, and $\Lambda^s = (1 - \Delta)^{\frac{s}{2}}$. There exists $c = c(s) > 0$ such that*

$$\|[\Lambda^s, v]\Lambda^{1-s}z\|_{L^2} \leq c\|v\|_{H^{s+1}}\|z\|_{L^2}.$$

Proof Let $[\Lambda^s, v]\Lambda^{1-s}z = \sum_{k,n \in \mathbb{Z}^2} \gamma_{k,n}$. Writing $[\Lambda^s, v]\Lambda^{1-s}z = \Lambda^s(v\Lambda^{1-s}z) -$

$v\Lambda^1 z$, we have

$$\{\Lambda^s(v\Lambda^{1-s}z)\}_{k,n} = \sum_{l_1, l_2 \in \mathbb{Z}^2} (1 + k^2 + n^2)^{\frac{s}{2}} v_{k-l_1, n-l_2} (1 + l_1^2 + l_2^2)^{\frac{1-s}{2}} z_{l_1, l_2},$$

and

$$\{v\Lambda^1 z\}_{k,n} = \sum_{l_1, l_2 \in \mathbb{Z}^2} v_{k-l_1, n-l_2} (1 + l_1^2 + l_2^2)^{\frac{1}{2}} z_{l_1, l_2}.$$

Thus

$$\gamma_{k,n} = \sum_{l_1, l_2 \in \mathbb{Z}^2} (1 + l_1^2 + l_2^2)^{\frac{1-s}{2}} ((1 + k^2 + n^2)^{\frac{s}{2}} - (1 + l_1^2 + l_2^2)^{\frac{s}{2}}) v_{k-l_1, n-l_2} z_{l_1, l_2}$$

We set

$$\Gamma_1 = \left\{ (l_1, l_2) \in \mathbb{Z}^2 ; |k - l_1| + |n - l_2| \leq \frac{|l_1| + |l_2|}{2} \right\},$$

$$\Gamma_2 = \left\{ (l_1, l_2) \in \mathbb{Z}^2 ; |k - l_1| + |n - l_2| > \frac{|l_1| + |l_2|}{2} \text{ and } |k| + |n| > |l_1| + |l_2| \right\},$$

and

$$\Gamma_3 = \left\{ (l_1, l_2) \in \mathbb{Z}^2 ; |k - l_1| + |n - l_2| > \frac{|l_1| + |l_2|}{2} \text{ and } |k| + |n| \leq |l_1| + |l_2| \right\}.$$

On Γ_1 we estimate

$$\begin{aligned} |\gamma_{k,n}| &\leq c \sum_{l_1, l_2 \in \mathbb{Z}^2} (1 + |l_1|^2 + |l_2|^2)^{\frac{1-s}{2}} |n - k| (1 + |l_1|^2 + |l_2|^2)^{\frac{s-1}{2}} |v_{k-l_1, n-l_2}| |z_{l_1, l_2}| \\ &\leq c \sum_{l_1, l_2 \in \mathbb{Z}^2} |n - k| |v_{k-l_1, n-l_2}| |z_{l_1, l_2}|. \end{aligned}$$

So, from Lemma [17](#) we can write

$$\begin{aligned} \|(\gamma_{k,n})_{k,n}\|_{\ell^2(\Gamma_1)} &= \left\| \left(\sum_{l_1, l_2 \in \mathbb{Z}^2} |n - k| v_{k-l_1, n-l_2} z_{l_1, l_2} \right)_{k,n} \right\|_{\ell^2} \\ &\leq c \| (kv_{k,n})_{k,n} \|_{\ell^1} \| (z_{k,n})_{k,n} \|_{\ell^2} \\ &\leq c \|v\|_{H^{s+1}} \|z\|_{L^2} \end{aligned}$$

On Γ_2 we estimate

$$\begin{aligned} |\gamma_{k,n}| &\leq c \sum_{l_1, l_2 \in \mathbb{Z}^2} (1 + |l_1|^2 + |l_2|^2)^{\frac{1-s}{2}} (1 + |k|^2 + |n|^2)^{\frac{s}{2}} |v_{k-l_1, n-l_2}| |z_{l_1, l_2}| \\ &\leq c \sum_{l_1, l_2 \in \mathbb{Z}^2} (1 + |k - l_1|^2 + |n - l_2|^2)^{\frac{s}{2}} |v_{k-l_1, n-l_2}| |z_{l_1, l_2}|. \end{aligned}$$

So

$$\begin{aligned} \|(\gamma_{k,n})_{k,n}\|_{\ell^2(\Gamma_2)} &= \left\| \left(\sum_{l_1, l_2 \in \mathbb{Z}^2} (1 + |k - l_1|^2 + |n - l_2|^2)^{\frac{s}{2}} v_{k-l_1, n-l_2} z_{l_1, l_2} \right)_{k,n} \right\|_{\ell^2} \\ &\leq c \|((1 + k^2 + n^2)^{\frac{s}{2}} v_{k,n})_{k,n}\|_{\ell^1} \| (z_{k,n})_{k,n} \|_{\ell^2} \\ &\leq C \|v\|_{H^s} \|z\|_{L^2} \end{aligned}$$

And on Γ_3 we estimate

$$\begin{aligned} |\gamma_{k,n}| &\leq c \sum_{l_1, l_2 \in \mathbb{Z}^2} (1 + |l_1|^2 + |l_2|^2)^{\frac{1-s}{2}} (1 + |l_1|^2 + |l_2|^2)^{\frac{s}{2}} |v_{k-l_1, n-l_2}| |z_{l_1, l_2}| \\ &\leq c \sum_{l_1, l_2 \in \mathbb{Z}^2} (1 + |l_1|^2 + |l_2|^2)^{\frac{1}{2}} |v_{k-l_1, n-l_2}| |z_{l_1, l_2}| \\ &\leq c \sum_{l_1, l_2 \in \mathbb{Z}^2} (1 + |k - l_1|^2 + |n - l_2|^2)^{\frac{1}{2}} |v_{k-l_1, n-l_2}| |z_{l_1, l_2}|, \end{aligned}$$

and we obtain

$$\|(\gamma_{k,n})_{k,n}\|_{\ell^2(\Gamma_3)} \leq C \|v\|_{H^{s+1}} \|z\|_{L^2},$$

following the argument of proof on Γ_2 . \square

Lemma 21 *There exists $c > 0$ such that for $a \in H^{s+1}$ and $z \in H^s$ with $s > 1$, we have*

$$|\Re(\Lambda^s(a\partial_x z), \Lambda^s z)_{L^2}| \leq (c \|a\|_{H^{s+1}} + c \|a\|_{H^{2+\epsilon}}) \|z\|_{H^s}^2. \quad (5.9)$$

Proof We know that

$$\Re(\Lambda^s(a\partial_x z), \Lambda^s z)_{L^2} = \Re([\Lambda^s, a]\partial_x z, \Lambda^s z)_{L^2} + \Re(a\partial_x \Lambda^s z, \Lambda^s z)_{L^2}.$$

We use Lemma 20 to estimate the first term

$$|([\Lambda^s, a]\partial_x z, \Lambda^s z)_{L^2}| = |([\Lambda^s, a]\Lambda^{1-s}\Lambda^{s-1}\partial_x z, \Lambda^s z)_{L^2}| \leq c \|a\|_{H^{s+1}} \|z\|_{H^s}^2. \quad (5.10)$$

For the second term, we write

$$\begin{aligned} (a\partial_x \Lambda^s z, \Lambda^s z)_{L^2} &= ([\Lambda^s, a]\partial_x z, \Lambda^s z)_{L^2} + (\partial_x(a\Lambda^s z), \Lambda^s z)_{L^2} \\ &= ([\Lambda^s, a]\partial_x z, \Lambda^s z)_{L^2} - (\Lambda^s z, \partial_x a \Lambda^s z)_{L^2}. \end{aligned}$$

So

$$2\Re(a\partial_x \Lambda^s z, \Lambda^s z)_{L^2} = \Re([\Lambda^s, a]\partial_x z, \Lambda^s z)_{L^2},$$

and it is easy to see that

$$|([a, \partial_x]\Lambda^s z, \Lambda^s z)_{L^2}| \leq c \|a'\|_{L^\infty} \|\Lambda^s z\|_{L^2}^2 \leq c \|a\|_{H^{2+\epsilon}} \|z\|_{H^s}^2. \quad (5.11)$$

From (5.10) and (5.11) we deduce (5.9). \square

Lemma 22 *Let $s_0 > 1$, $s \geq s_0 + 1$, $v, z \in \mathcal{Q}^s$. Then*

$$\|[\Lambda^s, v]\partial_x z\|_{L^2} \lesssim \|v\|_{H^s} \|z\|_{H^{s_0+1}} + \|v\|_{H^{s_0+1}} \|z\|_{H^s}. \quad (5.12)$$

In particular

$$\|[\Lambda^s, z]\partial_x z\|_{L^2} \lesssim \|z\|_{H^s} \|z\|_{H^{s_0+1}} \lesssim \|z\|_{H^s}^2. \quad (5.13)$$

Proof The proof is similar to the proof of Lemma 20. We write

$$[\Lambda^s, v]\partial_x z = \Lambda^s(v\partial_x z) - v\Lambda^s\partial_x z.$$

We know that

$$\{\Lambda^s(v\partial_x z)\}_{k,n} = \sum_{l_1, l_2 \in \mathbb{Z}^2} (1 + k^2 + n^2)^{\frac{s}{2}} v_{k-l_1, n-l_2} i l_1 z_{l_1, l_2},$$

and

$$\{v\Lambda^s\partial_x z\}_{k,n} = \sum_{l_1, l_2 \in \mathbb{Z}^2} v_{k-l_1, n-l_2} (1+l_1^2+l_2^2)^{\frac{s}{2}} i l_1 z_{l_1, l_2}.$$

So

$$\{[\Lambda^s, v]\partial_x z\}_{k,n} = \sum_{l_1, l_2 \in \mathbb{Z}^2} ((1+k^2+n^2)^{\frac{s}{2}} - (1+l_1^2+l_2^2)^{\frac{s}{2}}) v_{k-l_1, n-l_2} i l_1 z_{l_1, l_2}.$$

We use the same sets Γ_1 , Γ_2 and Γ_3 seen in the proof of Lemma [20](#). On Γ_1 we have

$$|(1+k^2+n^2)^{\frac{s}{2}} - (1+l_1^2+l_2^2)^{\frac{s}{2}}| \lesssim (|k-l_1| + |n-l_2|) (1+l_1^2+l_2^2)^{\frac{s-1}{2}}.$$

So

$$|\{[\Lambda^s, v]\partial_x z\}_{k,n}| \lesssim \sum_{l_1, l_2 \in \mathbb{Z}^2} (|k-l_1| + |n-l_2|) |v_{k-l_1, n-l_2}| (1+l_1^2+l_2^2)^{\frac{s}{2}} |z_{l_1, l_2}|.$$

The sequence $((|k| + |n|) v_{k,n})_{k,n}$ belongs to ℓ^1 if $v \in H^{s_0+1}$, and the sequence $((1+l_1^2+l_2^2)^{\frac{s}{2}} z_{l_1, l_2})_{l_1, l_2}$ belongs to ℓ^2 if $z \in H^s$. Then

$$\begin{aligned} |\{[\Lambda^s, v]\partial_x z\}_{k,n}| &\lesssim \|((|k| + |n|) v_{k,n})_{k,n} * ((1+l_1^2+l_2^2)^{\frac{s}{2}} z_{l_1, l_2})_{l_1, l_2}\|_{\ell^2} \\ &\lesssim \|v_{k,n}\|_{\ell^1} \|z_{k,n}\|_{\ell^2} \\ &\lesssim \|v\|_{H^{s_0+1}} \|z\|_{H^s}. \end{aligned}$$

On Γ_2 we have

$$\begin{aligned} |(1+k^2+n^2)^{\frac{s}{2}} - (1+l_1^2+l_2^2)^{\frac{s}{2}} l_1| &\lesssim (1+k^2+n^2)^{\frac{s}{2}} |l_1| \\ &\lesssim (1+|k-l_1|^2 + |n-l_2|^2)^{\frac{s}{2}} |l_1|. \end{aligned}$$

Hence

$$\begin{aligned} |\{[\Lambda^s, v]\partial_x z\}_{k,n}| &\lesssim \sum_{l_1, l_2 \in \mathbb{Z}^2} (1+|k-l_1|^2 + |n-l_2|^2)^{\frac{s}{2}} |v_{k-l_1, n-l_2}| |l_1| |z_{l_1, l_2}| \\ &\lesssim \|v\|_{H^s} \|z\|_{H^{s_0+1}}. \end{aligned}$$

And on Γ_3 we have

$$\begin{aligned} |(1+k^2+n^2)^{\frac{s}{2}} - (1+l_1^2+l_2^2)^{\frac{s}{2}} l_1| &\lesssim (1+l_1^2+l_2^2)^{\frac{s}{2}} |l_1| \\ &\lesssim (1+|k-l_1|^2 + |n-l_2|^2)^{\frac{s}{2}} |l_1|, \end{aligned}$$

and the estimation of $\{[\Lambda^s, v]\partial_x z\}_{k,n}$ is the same as on Γ_2 , which complete the proof of [\(5.12\)](#). \square

5.3 Existence

We consider the equation

$$\begin{cases} \partial_t u + \partial_{xxx} u + u \partial_x u - \partial_x^{-1} \partial_{yy} u = 0, & (x, y) \in \mathbb{T}^2, t > 0, \\ u(\cdot, \cdot, t = \sigma) = u_0, & (x, y) \in \mathbb{T}^2, \end{cases} \quad (5.14)$$

We recall that we write this equation under the form

$$\begin{cases} \partial_t u = A(u(t))u, \\ u(\sigma) = u_0, \end{cases} \quad (5.15)$$

where

$$\begin{aligned} A(v(\alpha)) : \mathcal{Q}^s(\mathbb{T}^2) &\longrightarrow \tilde{H}^s(\mathbb{T}^2) \\ u &\longmapsto -(\partial_{xxx} u + v(\alpha) \partial_x u - \partial_x^{-1} \partial_{yy} u). \end{aligned}$$

The semigroup associated with the operator $A(v(\alpha))$ will be noted $\{e^{tA(v(\alpha))}\}_{t \geq 0}$, and the solution of the equation (5.14) at time t will be noted $S(t, \sigma)u_0$.

Here the parameter t describe at what time we look at the solution and the parameter σ describe at what time the initial data is taken. For the existence of the solution of (5.15) we shall apply the following theorem. (See [Kat75, Theorem 6]).

Theorem 25 *Let X be a reflexive Banach space, $Y \subset X$ another reflexive Banach space continuously and densely embedded in X . Let W be an open ball in Y which contains 0, $T > 0$ and $S : Y \rightarrow X$ an isomorphism. Assume that*

1. $A = A(v(\alpha))$ is a generator of a semigroup $\{e^{sA(v(\alpha))}\}_{s \geq 0}$ such that for all $s > 0$

$$\|e^{-sA(v(\alpha))}\|_X \leq e^{\beta s}, \quad \forall (\alpha, v) \in [0, T] \times W.$$

2. For each $(\alpha, v) \in [0, T] \times W$, the operator $SA(v(\alpha))S^{-1} - A(v(\alpha))$ is uniformly bounded in X .
3. For each $v \in W$, $\alpha \mapsto A(v(\alpha))$ is continuous in $\mathcal{L}(X, Y)$ and for each $t \in [0, T]$, $v \mapsto A(v(\alpha))$ is Lipschitz-continuous.
4. For each $v \in W$, $\alpha \mapsto f(\alpha, v)$ is continuous from $[0, T]$ to X , for each $\alpha \in [0, T]$, $v \mapsto f(\alpha, v)$ is Lipschitz continuous and f is a bounded function on $[0, T] \times W$ to Y .

Then (5.15) has a unique solution $u \in \mathcal{C}([0, T']; W) \cap \mathcal{C}^1([0, T']; X)$ for some $T' \in [0, T]$.

We recall the two following theorems to prove the existence.

Theorem 26 (Lumer-Phillips) *Let A be a linear operator of dense domain in a Hilbert space H . If A is dissipative and if there exists $\lambda_0 > 0$ such that $\text{Im}(\lambda_0 I - A) = H$, then A is the infinitesimal generator of a C^0 -semigroup of contractions.*

Theorem 27 (Hille-Yosida) *We have an equivalence between the two assumptions.*

1. A is the infinitesimal generator of a C^0 -semigroup of contractions $\{S(s)\}_{s \geq 0}$.
2. A is closed, of domain dense and $\|R(\lambda, A(t))\| \leq \frac{1}{\lambda}$ for $\Re \lambda > 0$.

Theorem 28 *There exists a unique solution $u \in C([0, T']; W) \cap C^1([0, T']; H^s)$ of (5.4), where W is a bounded open set of \mathcal{Q}^s .*

Proof We use Theorem 25 taking $X = H^s$, $Y = \mathcal{Q}^s$ and $S = A_1$. The operator $A_1 : \mathcal{Q}^s \rightarrow H^s$ is invertible, it is thus Fredholm of index 0.

The operator $A_2(v) : \mathcal{Q}^s \rightarrow H^s$ (when $v \in \mathcal{Q}^s$) is compact since $v \partial_x u \in H^{s+1} \hookrightarrow H^s$ from (5.6). So $A(v) = A_1 + A_2(v)$ is also Fredholm of index 0.

To prove that A is dissipatif, let $\lambda > 0$ and λ_0 large enough. As $\partial_{xxx} - \partial_x^{-1} \partial_{yy}$ is skew-adjoint since $\Re \langle \partial_{xxx} u - \partial_x^{-1} \partial_{yy} u, u \rangle_{H^s} = 0$, we have

$$\begin{aligned}
\Re \langle (\lambda_0 + \lambda - A)u, u \rangle_{H^s} &= (\lambda_0 + \lambda) \|u\|_{H^s}^2 + \Re \langle v \partial_x u, u \rangle_{H^s} \\
&= (\lambda_0 + \lambda) \|u\|_{H^s}^2 + \Re (\Lambda^s (v \partial_x u), \Lambda^s u)_{L^2} \\
&= (\lambda_0 + \lambda) \|u\|_{H^s}^2 + \Re ([\Lambda^s, v] \partial_x u, \Lambda^s u)_{L^2} \\
&\quad - \frac{1}{2} \Re (\partial_x v \Lambda^s u, \Lambda^s u)_{L^2} \\
&\geq \left(\lambda_0 + \lambda - c \|v\|_{H^s} - \frac{1}{2} \|\partial_x v\|_{L^\infty} \right) \|u\|_{H^s}^2 \\
&\geq (\lambda_0 + \lambda - C \|v\|_{H^s}) \|u\|_{H^s}^2.
\end{aligned} \tag{5.16}$$

Hence A is dissipative. Now we estimate

$$\begin{aligned}
\|(\lambda_0 + \lambda - A)u\|_{L^2} \|u\|_{L^2} &\geq \Re ((\lambda_0 + \lambda - A)u, u)_{L^2} \\
&= (\lambda_0 + \lambda) \|u\|_{L^2}^2 + \Re (v \partial_x u, u)_{L^2} \\
&= (\lambda_0 + \lambda) \|u\|_{L^2}^2 - \frac{1}{2} \int_{\mathbb{T}^2} \partial_x v \cdot |u|^2 dx dy \\
&= \left(\lambda_0 + \lambda - \frac{1}{2} \|\partial_x v\|_{L^\infty} \right) \|u\|_{L^2}^2 \geq \lambda \|u\|_{L^2}^2.
\end{aligned}$$

This mean that $\lambda_0 + \lambda - A$ is injective, since it is a Fredholm operator of index 0, it is also surjective, then invertible. From Theorem 26 $\lambda_0 - A$

is a generator of a C^0 -semigroup of contractions. In other words, for $\lambda_0 > C \|v\|_{H^s}$, we have

$$\|e^{-tA}\|_{H^s} \leq e^{C\|v\|_{H^s}t}. \quad (5.17)$$

This give us the first assumption of Theorem 25 with $\beta = C \|v\|_{H^s}$. To prove now the second assumption, let $u \in H^s(\mathbb{T}^2)$. Since the terms ∂_{xxx} and $\partial_x^{-1}\partial_{yy}$ commute with A_1 , we note $\tilde{w} = A_1^{-1}\partial_x u$ (which belongs to \mathcal{Q}^{s-1}) and we have

$$\begin{aligned} A_1 A A_1^{-1} u - A u &= A_1 v \partial_x A_1^{-1} u - v \partial_x u \\ &= [A_1, v] \tilde{w} \\ &= -[\partial_{xxx}, v] \tilde{w} - [\partial_x^{-1} \partial_{yy}, v] \tilde{w} \\ &= -(\partial_{xxx} v) \tilde{w} - 3 \partial_{xx} v \partial_x \tilde{w} - 3 \partial_x v \partial_{xx} \tilde{w} - [\partial_x^{-1} \partial_{yy}, v] \tilde{w}. \end{aligned}$$

We use results from Lemma 16 and Lemma 19 to estimate each quantity.

$$\begin{aligned} \|A_1 A A_1^{-1} u - A u\|_{H^s} &\lesssim \|v\|_{H^s} \|u\|_{H^{s-1}} + \|v\|_{H^{s+\frac{1}{2}}} \|u\|_{H^{s+\frac{1}{2}}} + \|v\|_{H^{s+1}} \|u\|_{H^s} \\ &\quad + \|v\|_{H^s} \|u\|_{H^s} \\ &\lesssim \|v\|_{\mathcal{Q}^s} \|u\|_{H^s}. \end{aligned}$$

So the operator $A_1 A A_1^{-1} - A$ is bounded in H^s .

For the third assumption, we take v_1 and v_2 in \mathcal{Q}^s , and it is sufficient to use Lemma 18 to write

$$\begin{aligned} \|A(v_1)u - A(v_2)u\|_{H^s} &\lesssim \|v_1 - v_2\|_{H^s} \|\partial_x u\|_{H^{s_0}} + \|v_1 - v_2\|_{H^{s_0}} \|\partial_x u\|_{H^s} \\ &\lesssim \|u\|_{\mathcal{Q}^s} \|v_1 - v_2\|_{\mathcal{Q}^s}. \end{aligned}$$

Since we will use trigonometric function to be the control term f , which do not depends on v , the last assumption of Theorem 25 is verified. \square

We give the following theorem, which is a particular case of the [Paz83, Theorem 3.1] in chapter 5.

Theorem 29 *Let $T > 0$ and $A(t)$ be the infinitesimal generator of a C^0 -semigroup $\{e^{sA(t)}\}_{s \geq 0}$ on a Banach space X for $t \in [0, T]$. Assume that*

1. *The domaine of $A(t)$ does not depend on t ,*
2. *There exists $c \in \mathbb{R}$ such that $]c, +\infty[\subset \rho(A(t))$,*
3. *There exists $M \geq 1$ such that $\|\prod_{j=1}^k R(\lambda, A(t_j))\| \leq \frac{M}{\lambda^k}$, for $0 \leq t_1 \leq \dots \leq t_k \leq T$.*

Then, for $t > s$, there exists a unique solution $u(t, s)$ of

$$\begin{cases} \partial_t u = A(t)u \\ u(s) = u_0, \end{cases}$$

such that

$$\|u(t, s)\| \leq M e^{c(t-s)}. \quad (5.18)$$

Remark 10 The assumptions of [Paz83] Theorem 3.1] in chapter 5 are satisfied in our case since we take (under the notations of [Paz83]) $\tilde{A}(t) = A(t)$, $Y = \mathcal{D}(A(t)) = \mathcal{Q}^s$ and $t \mapsto A(t)$ is continuous from the continuity of the function u . From Theorem 27 we can deduce that $M = 1$. We know also that using the arguments seen in the proof of Theorem 28 we have $\lambda_0 - A$ is inversible for all $\lambda_0 > C \|v\|_{H^s}$. This means that there are no eigenvalues of A on $]C \|v\|_{H^s}, +\infty[$, which gives us the second assumption of Theorem 29 with $c = C \|v\|_{H^s}$. So the solution of

$$\begin{cases} \partial_t u = A(v(t))u, \\ u(s) = u_0, \end{cases}$$

verifies

$$\|u(t, s)\| \leq e^{C\|v\|_{H^s}(t-s)}. \quad (5.19)$$

Note that to return to the nonlinear equation, we can take $v = u$ (since we know from Theorem 28 that u exists) and we obtain

$$\|u(t, s)\| \leq e^{C\|u\|_{H^s}(t-s)}. \quad (5.20)$$

Let $\delta > 0$ and $\varphi \in \mathcal{C}^\infty$. We consider the equation

$$\begin{cases} \partial_t u + \partial_{xxx}(u + \delta^{-\frac{1}{2}}\varphi) + (u + \delta^{-\frac{1}{2}}\varphi)\partial_x(u + \delta^{-\frac{1}{2}}\varphi) - \partial_x^{-1}\partial_{yy}(u + \delta^{-\frac{1}{2}}\varphi) \\ \quad = \delta^{-1}\eta, \\ u(\cdot, \cdot, 0) = u_0, \end{cases} \quad (5.21)$$

Remark 11 Let \tilde{u} be the solution of (5.21) and $u(t)$ the solution of

$$\begin{cases} \partial_t u + \partial_{xxx}u + u\partial_x u - \partial_x^{-1}\partial_{yy}u = \delta^{-1}\eta, \\ u(\cdot, \cdot, 0) = u_0, \end{cases} \quad (5.22)$$

Then $\tilde{u}(t) = u(t) + \delta^{-\frac{1}{2}}\varphi$.

It is easy to see from Theorem 25 that the function

$$u(t) = S(t, 0)u_0 + \int_0^t S(t, \sigma)\delta^{-1}\eta d\sigma,$$

is the unique solution of

$$\begin{cases} \partial_t u + \partial_{xxx}u + v(t)\partial_x u - \partial_x^{-1}\partial_{yy}u = \delta^{-1}\eta, & (x, y) \in \mathbb{T}^2, t > 0, \\ u(\cdot, \cdot, t = \sigma) = u_0, & (x, y) \in \mathbb{T}^2, \end{cases}$$

on an interval $[0, T_\delta]$. The following Lemma gives an estimation of continuity on u .

Lemma 23 *Let $c_1 > 0$. There exists $\delta_0 > 0$ such that for all $\delta \in]0, \delta_0[$, if the initial data u_0 satisfies $\|u_0\|_{H^s} \leq c_1 \delta^{-\frac{1}{2}}$, then there exist a solution u of (5.21) on $[0, 2\delta]$ such that*

$$\|u(t)\|_{H^s} \leq 4c_1 \delta^{-\frac{1}{2}}. \quad (5.23)$$

Proof Let

$$T = \sup \left\{ t_1 \in [0, 2\delta] ; u \text{ exists and } \|u(\sigma)\|_{H^s} \leq 4c_1 \delta^{-\frac{1}{2}} \text{ for } \sigma \in [0, t_1] \right\}. \quad (5.24)$$

We prove that if $t \leq T$, then $\|u(t)\|_{H^s} \leq 2c_1 \delta^{-\frac{1}{2}}$. From the continuity of u , there exists $\epsilon > 0$ such that if $t \leq T + \epsilon$, then $\|u(t)\|_{H^s} \leq 4c_1 \delta^{-\frac{1}{2}}$, and hence $T = 2\delta$.

From Duhamel formula we can write

$$u(t) = S(t, 0)u_0 + \int_0^t S(t, \sigma) \delta^{-1} \eta d\sigma.$$

So using (5.20) and the estimation of u in (5.24)

$$\begin{aligned} \|u(t)\|_{H^s} &\leq e^{4C_1 \delta^{-\frac{1}{2}} t} \|u_0\|_{H^s} + \int_0^t \delta^{-1} e^{4C_1 \delta^{-\frac{1}{2}} (t-\sigma)} \|\eta\|_{H^s} d\sigma \\ &\leq c_1 e^{4C_1 \delta^{-\frac{1}{2}} t} \delta^{-\frac{1}{2}} + \delta^{-1} \|\eta\|_{H^s} \int_0^t e^{4C_1 \delta^{-\frac{1}{2}} y} dy \\ &\leq c_1 e^{4C_1 \delta^{-\frac{1}{2}} t} \delta^{-\frac{1}{2}} + \delta^{-1} \|\eta\|_{H^s} \frac{e^{4C_1 \delta^{-\frac{1}{2}} t} - 1}{4C_1 \delta^{-\frac{1}{2}}}. \end{aligned}$$

Since $t \leq 2\delta$ we have

$$\|u(t)\|_{H^s} \leq c_1 e^{8C_1 \delta^{\frac{1}{2}}} \delta^{-\frac{1}{2}} + \frac{\|\eta\|_{H^s}}{4C_1} \delta^{-\frac{1}{2}} (e^{8C_1 \delta^{\frac{1}{2}}} - 1).$$

We choose δ_0 small enough so that

$$e^{8C_1 \delta_0^{\frac{1}{2}}} \leq \frac{3}{2}, \text{ and } \frac{\|\eta\|_{H^s}}{4C_1} (e^{8C_1 \delta_0^{\frac{1}{2}}} - 1) \leq \frac{c_1}{2}.$$

So for all $\delta \in]0, \delta_0[$ we obtain

$$\|u(t)\|_{H^s} \leq 2c_1 \delta^{-\frac{1}{2}}. \quad (5.25)$$

So we can extend the time over which the solution u satisfies (5.25) from T to $T + \epsilon$. From this we can deduce that $T = 2\delta$. \square

5.4 Approximate controllability

We note the solution of (5.21) at time t by $u(t) = \mathcal{R}_t(u_0, \delta^{-\frac{1}{2}}\varphi, \delta^{-1}\eta)$. We want to have a limit in H^s of $\mathcal{R}_t(u_0, \delta^{-\frac{1}{2}}\varphi, \delta^{-1}\eta)$ when δ goes to 0, for s large enough, of the solution in small time. Let $\delta_* \in [0, 2\delta]$ and Θ the space defined by

$$\Theta(u_0, \varphi, \delta_*) = \left\{ \eta \in L^1([0, T]; H^s) ; \mathcal{R}_t(u_0, \delta^{-\frac{1}{2}}\varphi, \delta^{-1}\eta) \right. \\ \left. \text{exists and continuous for } t \in [0, \delta_*] \right\}.$$

Proposition 23 For $s > 1$, $u_0 \in H^s(\mathbb{T}^2)$, $\varphi \in \mathcal{C}^\infty(\mathbb{T}^2)$ and $\eta_0 \in \Theta(u_0, \varphi, \delta_*)$, the solution $u(t) = \mathcal{R}_t(u_0, \delta^{-\frac{1}{2}}\varphi, \delta^{-1}\eta_0)$ of (5.21) verifies

$$\lim_{\delta \rightarrow 0} u(\delta) = u_0 + \eta_0 - \varphi \partial_x \varphi, \text{ in } H^s(\mathbb{T}^2). \quad (5.26)$$

Proof For $t \in [0, \delta_*]$, let $v(t) = u(\delta t)$, where u is the solution of (5.1), then v satisfies

$$\partial_t v = -\delta \left(\partial_{xxx}(v + \delta^{-\frac{1}{2}}\varphi) + (v + \delta^{-\frac{1}{2}}\varphi) \partial_x (v + \delta^{-\frac{1}{2}}\varphi) \right. \\ \left. - \partial_x^{-1} \partial_{yy}(v + \delta^{-\frac{1}{2}}\varphi) \right) + \eta_0.$$

Let $w(t) = u_0 + t(\eta_0 - \varphi \partial_x \varphi)$ and $z = v - w$. We have

$$\partial_t z = -\delta \left(\partial_{xxx}(z + w + \delta^{-\frac{1}{2}}\varphi) + (z + w) \partial_x (z + w + \delta^{-\frac{1}{2}}\varphi) \right. \\ \left. + \delta^{-\frac{1}{2}}\varphi \partial_x (z + w) - \partial_x^{-1} \partial_{yy}(z + w + \delta^{-\frac{1}{2}}\varphi) \right). \quad (5.27)$$

Taking the inner product in $H^s(\mathbb{T}^2)$ with z we have

$$\frac{1}{2} \partial_t (\|z(t)\|_{H^s}^2) = -\delta \left\{ \Re \left(\Lambda^s \partial_{xxx} \left(z + w + \delta^{-\frac{1}{2}}\varphi \right), \Lambda^s z \right)_{L^2} \right. \\ \left. + \Re \left(\Lambda^s (z + w) \partial_x \left(z + w + \delta^{-\frac{1}{2}}\varphi \right), \Lambda^s z \right)_{L^2} + \delta^{-\frac{1}{2}} \Re \left(\Lambda^s \varphi \partial_x (z + w), \Lambda^s z \right)_{L^2} \right. \\ \left. - \Re \left(\Lambda^s \partial_x^{-1} \partial_{yy} (z + w + \delta^{-\frac{1}{2}}\varphi), \Lambda^s z \right)_{L^2} \right\}. \quad (5.28)$$

We claim

$$\frac{1}{2} \partial_t (\|z(t)\|_{H^s}^2) \leq C \delta^{\frac{1}{2}} (1 + \|z(t)\|_{H^s}^3). \quad (5.29)$$

We know that $\Re (\Lambda^s \partial_{xxx} z, \Lambda^s z) = \Re (\Lambda^s \partial_x^{-1} \partial_{yy}^2 z, \Lambda^s z) = 0$. So we estimate the quantities

$$\left| \Re \left(\Lambda^s \partial_{xxx} \left(z + w + \delta^{-\frac{1}{2}}\varphi \right), \Lambda^s z \right)_{L^2} \right| \lesssim \left(\|w\|_{H^{s+3}} + \delta^{-\frac{1}{2}} \|\varphi\|_{H^{s+3}} \right) \|z\|_{H^s} \\ \lesssim \delta^{-\frac{1}{2}} \left(1 + \|z\|_{H^s}^3 \right),$$

$$\begin{aligned} & \left| \Re \left(\Lambda^s \left(w \partial_x \left(w + \delta^{-\frac{1}{2}} \varphi \right) \right), \Lambda^s z \right)_{L^2} \right| \\ & \lesssim \left(\|w\|_{H^{s+1}} + \delta^{-\frac{1}{2}} \|\varphi\|_{H^{s+1}} \right) \|w\|_{H^s} \|z\|_{H^s} \\ & \lesssim \delta^{-\frac{1}{2}} \left(1 + \|z\|_{H^s}^3 \right), \end{aligned}$$

$$\begin{aligned} & \left| \Re \left(\Lambda^s \left(z \partial_x \left(w + \delta^{-\frac{1}{2}} \varphi \right) \right), \Lambda^s z \right)_{L^2} \right| \lesssim \left(\|w\|_{H^{s+1}} + \delta^{-\frac{1}{2}} \|\varphi\|_{H^{s+1}} \right) \|z\|_{H^s}^2 \\ & \lesssim \delta^{-\frac{1}{2}} \left(1 + \|z\|_{H^s}^3 \right), \end{aligned}$$

$$\begin{aligned} & \left| \Re \left(\Lambda^s \partial_x^{-1} \partial_{yy} \left(z + w + \delta^{-\frac{1}{2}} \varphi \right), \Lambda^s z \right)_{L^2} \right| \lesssim \left(\|w\|_{\mathcal{Q}^s} + \delta^{-\frac{1}{2}} \|\varphi\|_{\mathcal{Q}^s} \right) \|z\|_{H^s} \\ & \lesssim \delta^{-\frac{1}{2}} \left(1 + \|z\|_{H^s}^3 \right), \end{aligned}$$

$$\begin{aligned} & \left| \Re \left(\Lambda^s \left(\varphi \partial_x (z + w) \right), \Lambda^s z \right)_{L^2} \right| \lesssim \|z\|_{H^s}^2 \\ & \lesssim \delta^{-\frac{1}{2}} \left(1 + \|z\|_{H^s}^3 \right), \end{aligned}$$

and

$$\left| \Re \left(\Lambda^s \left(z \partial_x z \right), \Lambda^s z \right)_{L^2} \right| \leq \left| \left([\Lambda^s, z] \partial_x z, \Lambda^s z \right)_{L^2} \right| + \left| \left(z \Lambda^s \partial_x z, \Lambda^s z \right)_{L^2} \right|.$$

The term $\left([\Lambda^s, z] \partial_x z, \Lambda^s z \right)_{L^2}$ can be estimated by $\|z\|_{H^s}^3$ from (5.13). And for the term $\left(z \Lambda^s \partial_x z, \Lambda^s z \right)_{L^2}$ we can see that

$$\begin{aligned} 2\Re \left(z \partial_x \Lambda^s z, \Lambda^s z \right)_{L^2} &= \left(z \partial_x \Lambda^s z, \Lambda^s z \right)_{L^2} + \left(\Lambda^s z, z \partial_x \Lambda^s z \right)_{L^2} \\ &= - \left(\Lambda^s z, \partial_x (z \Lambda^s z) \right)_{L^2} + \left(\Lambda^s z, z \partial_x \Lambda^s z \right)_{L^2} \\ &= - \left(\Lambda^s z, \partial_x (z) \Lambda^s z \right)_{L^2} - \left(\Lambda^s z, z \partial_x \Lambda^s z \right)_{L^2} \\ & \quad + \left(\Lambda^s z, z \partial_x \Lambda^s z \right)_{L^2} \\ &= - \left(\Lambda^s z, \partial_x (z) \Lambda^s z \right)_{L^2}. \end{aligned}$$

So

$$\left| \left(\Lambda^s \left(z \partial_x z \right), \Lambda^s z \right)_{L^2} \right| \lesssim \|z\|_{H^s}^2 + \|z\|_{H^s}^3 \lesssim 1 + \|z\|_{H^s}^3,$$

which give us (5.29).

Integrating it between 0 and t , we have

$$\|z(t)\|_{H^s}^2 \leq C \delta^{\frac{1}{2}} \int_0^t \left(1 + \|z(s)\|_{H^s}^3 \right) ds = C \delta^{\frac{1}{2}} + C \delta^{\frac{1}{2}} \int_0^t \|z(s)\|_{H^s}^3 ds.$$

Note that the inequality (5.23) holds also for z (with a different constant), so

$$\|z(t)\|_{H^s}^2 \leq C \delta^{\frac{1}{2}} + C \int_0^t \|z(s)\|_{H^s}^2 ds.$$

By Gronwall inequality

$$\|z(t)\|_{H^s}^2 \leq C\delta^{\frac{1}{2}}e^{Ct} \rightarrow 0,$$

as $\delta \rightarrow 0$. In particular, $u(\delta) = v(1) \rightarrow w(1)$ in H^s as $\delta \rightarrow 0$, and then we obtain (5.26). \square

To define a sequence (I_N) of sets as we did in section 4.3, we use the nonlinear term $u\partial_x u$ to find out which frequencies can be reached.

For $I \subset \mathbb{Z}^* \times \mathbb{Z}$ we set

$$\mathcal{H}(I) = \text{span}\{\cos(px + qy), \sin(px + qy); \text{ for } (p, q) \in I\},$$

and

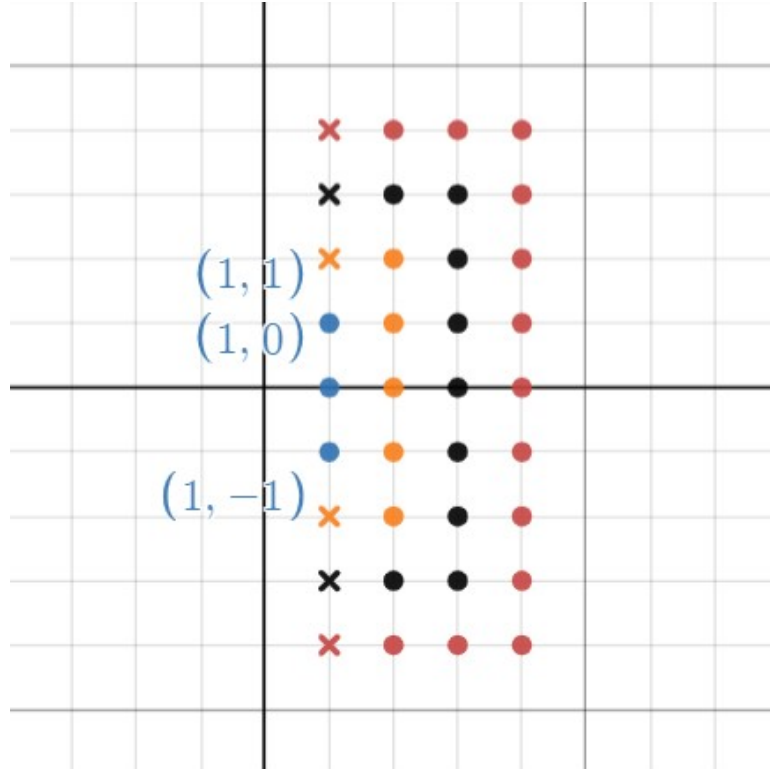
$$\mathcal{C}(I) = \left\{ \eta - \sum_{i=1}^d \varphi_i \partial_x \varphi_i ; \eta, \varphi_i \in \mathcal{H}(I), d \geq 1 \right\}.$$

Let $I_1 = \{(1, 1); (1, 0); (1, -1)\}$, and for $N \geq 2$ we set

$$I_N = \{(p, q) \in \mathbb{Z}^* \times \mathbb{Z} ; 0 < p \leq N \text{ and } |q| \leq N - 1\},$$

and

$$G_N = \{(p, q) \in \mathbb{Z}^* \times \mathbb{Z} ; 0 < p \leq N + 1 \text{ and } |q| \leq N\} \setminus \{(1, \pm N)\}.$$



Lemma 24 For $N \geq 2$ we have the three inclusions

$$\mathcal{H}(G_N) \subset \mathcal{C}(I_N), \quad (5.30)$$

$$\mathcal{H}(I_{N+1}) \subset \mathcal{C}(G_N), \quad (5.31)$$

and

$$\mathcal{H}(I_2) \subset \mathcal{C}(I_1). \quad (5.32)$$

Proof The idea to prove (5.30) here is to advance in p and q by making products of the form $\varphi \partial_x \varphi$. Let $f \in \mathcal{H}(G_N)$ and for $(a, b) \neq (\pm 1, \mp 1)$

$$\varphi_1(x, y) = \sin(x + y) + \cos(x + y) + c \sin(px + qy) + d \cos(px + qy),$$

and

$$\varphi_2(x, y) = a \sin(x + y) + b \cos(x + y) + d \sin(px + qy) - c \cos(px + qy).$$

By symmetry arguments, we can consider just the case $q \geq 0$.

We want to write f under the form $f = \eta - \sum_{i=1}^d \varphi_i \partial_x \varphi_i$ where $\eta, \varphi_i \in \mathcal{H}(I_N)$.

We have

$$\begin{aligned} f(x, y) &= \sum_{(p,q) \in G_N} \alpha \cos(px + qy) + \beta \sin(px + qy) \\ &= \sum_{(p,q) \in I_N} \alpha \cos(px + qy) + \beta \sin(px + qy) \\ &+ \sum_{\{(N+1,q); |q| \leq N\}} \alpha \cos(px + qy) + \beta \sin(px + qy) \\ &+ \sum_{\{(p,q); 2 \leq p \leq N, |q|=N\}} \alpha \cos(px + qy) + \beta \sin(px + qy) \end{aligned} \quad (5.33)$$

Note that the numbers α and β depend of p and q .

We call

$$\eta_1 = \sum_{(p,q) \in I_N} \alpha \cos(px + qy) + \beta \sin(px + qy) \in \mathcal{H}(I_N). \quad (5.34)$$

Now we prove that

$$\begin{aligned} \varphi_1 \partial_x \varphi_1 + \varphi_2 \partial_x \varphi_2 &= \frac{1}{2} ((b^2 - a^2) \sin(2x + 2y) + 2(1 + ab) \cos(2x + 2y) \\ &+ (p + 1)(c(1 + b) + d(1 - a)) \sin((p + 1)x + (q + 1)y) \\ &+ (p + 1)(c(1 - a) + d(1 + b)) \cos((p + 1)x + (q + 1)y) \\ &+ (1 - p)(c(1 - b) + d(1 + a)) \sin((p - 1)x + (q - 1)y) \\ &+ (1 - p)(-c(1 + a) + d(1 - b)) \cos((p - 1)x + (q - 1)y)). \end{aligned} \quad (5.35)$$

By replacing p with N and q with $q - 1$ in (5.35), and since the map

$$(c, d) \mapsto (c(1 - a) + d(1 + b), c(1 + b) + d(1 - a)),$$

is bijective, we can take

$$\frac{1}{2}(N + 1)(c(1 - a) + d(1 + b)) = \alpha, \quad \frac{1}{2}(N + 1)(c(1 + b) + d(1 - a)) = \beta,$$

and

$$\begin{aligned} \eta_2 = & -\frac{1}{2}((b^2 - a^2) \sin(2x + 2y) + 2(1 + ab) \cos(2x + 2y)) \\ & + (1 - N)(c(1 - b) + d(1 + a)) \sin((N - 1)x + (q - 2)y) \\ & + (1 - N)(-c(1 + a) + d(1 - b)) \cos((N - 1)x + (q - 2)y). \end{aligned}$$

We can see that

$$\alpha \cos(px + qy) + \beta \sin(px + qy) = \eta_2 + \varphi_1 \partial_x \varphi_1 + \varphi_2 \partial_x \varphi_2. \quad (5.36)$$

For the third sum of (5.33), we replace p with $p + 1$ and q with $N - 1$ in (5.35), and we take

$$\frac{1}{2}p(c(1 - a) + d(1 + b)) = \alpha, \quad \frac{1}{2}p(c(1 + b) + d(1 - a)) = \beta,$$

and

$$\begin{aligned} \eta_3 = & -\frac{1}{2}((a^2 - b^2) \sin(2x + 2y) + 2(1 + ab) \cos(2x + 2y)) \\ & + (2 - p)(c(1 - b) + d(1 + a)) \sin((p - 1)x + (N - 2)y) \\ & + (2 - p)(-c(1 + a) + d(1 - b)) \cos((p - 1)x + (N - 2)y), \end{aligned}$$

to obtain

$$\alpha \cos(px + qy) + \beta \sin(px + qy) = \eta_3 + \varphi_1 \partial_x \varphi_1 + \varphi_2 \partial_x \varphi_2. \quad (5.37)$$

Combining (5.34), (5.36) and (5.37), we deduce that $f \in \mathcal{C}(I_N)$.

To prove the inclusion (5.31), let

$$g = \sum_{(p,q) \in \mathcal{H}(I_{N+1})} \alpha \cos(px + qy) + \beta \sin(px + qy) \in \mathcal{H}(I_{N+1}),$$

$$\varphi_1(x, y) = \cos(x) + \sin(x) + c \cos(2x + qy) + d \sin(2x + qy),$$

and

$$\varphi_2(x, y) = a \cos(x) - b \sin(x) - d \cos(2x + qy) + c \sin(2x + qy).$$

We prove that

$$\begin{aligned} \varphi_1 \partial_x \varphi_1 + \varphi_2 \partial_x \varphi_2 &= \frac{1}{2} \left((b^2 - a^2) \sin(2x) + 2(1 + ab) \cos(2x) \right. \\ &\quad + 3(c(b - 1) + d(1 + a)) \sin(3x + qy) \\ &\quad + 3(c(1 + a) + d(1 - b)) \cos(3x + qy) \\ &\quad - (c(1 + b) + d(1 - a)) \sin(x + qy) \\ &\quad \left. + (c(a - 1) + d(1 + b)) \cos(x + qy) \right). \end{aligned} \quad (5.38)$$

Now we take

$$\frac{1}{2}(c(a - 1) + d(1 + b)) = \alpha, \quad -\frac{1}{2}(c(1 + b) + d(1 - a)) = \beta,$$

and

$$\begin{aligned} \eta &= -\frac{1}{2} \left((b^2 - a^2) \sin(2x) + 2(1 + ab) \cos(2x) \right. \\ &\quad + 3(c(b - 1) + d(1 + a)) \sin(3x + qy) \\ &\quad \left. + 3(c(1 + a) + d(1 - b)) \cos(3x + qy) \right). \end{aligned}$$

Hence, $g = \eta + \varphi_1 \partial_x \varphi_1 + \varphi_2 \partial_x \varphi_2 \in \mathcal{C}(G_N)$.

For the inclusion [\(5.32\)](#). Let $u_0 = a \cos(x) + b \sin(x)$. Then

$$u_0 \partial_x u_0 = ab \cos(2x) + \frac{1}{2}(b^2 - a^2) \sin(2x).$$

Now let

$$\varphi_1(x, y) = \cos(x) + \sin(x) + c \cos(x + y) + d \sin(x + y),$$

and

$$\varphi_2(x, y) = \cos(x) - \sin(x) - d \cos(x + y) + c \sin(x + y).$$

We prove that

$$\varphi_1 \partial_x \varphi_1 + \varphi_2 \partial_x \varphi_2 = 2(d - c) \sin(2x + y) + 2(d + c) \cos(2x + y).$$

Since the two maps $(a, b) \mapsto \left(ab, \frac{1}{2}(b^2 - a^2) \right)$ and $(c, d) \mapsto (2(d - c), 2(c + d))$ are bijective, we obtain [\(5.32\)](#) by the same way as the two first inclusions. \square

We can now prove Theorem [24](#).

Proof For $J \subset \mathbb{Z}^* \times \mathbb{Z}$, let

$$\mathcal{C}_k(J) = \left\{ \eta - \sum_{i=1}^k \varphi_i \partial_x \varphi_i ; \eta, \varphi_i \in \mathcal{H}(J) \right\}.$$

We prove the following property

$$\begin{aligned} \mathcal{P}_k(J) : & \forall u_0 \in \mathcal{Q}^s(\mathbb{T}^2), \forall \epsilon > 0, \forall \sigma > 0, \forall w \in \mathcal{C}_k(J), \exists t \in]0, \sigma], \\ & \exists \eta \in \Theta(u_0, \varphi, t) \cap \mathcal{H}(I_1) \text{ s.t. } \|\mathcal{R}_t(u_0, 0, \eta) - (u_0 + w)\|_{H^1} \leq \epsilon. \end{aligned}$$

We show the result by induction on the variable k . The proof is essentially the same as we did to prove Theorem 22. We recall that in the first stage of the proof, we reasoned by induction to prove that if \mathcal{P}_k^n is true, then \mathcal{P}_{k+1}^n is true. We keep the property (4.14) and Lemma 13 in this proof.

By repeating the same arguments as in the first step of the proof of Theorem 22 we can say that we have $\mathcal{P}_k(J)$ is true for all $k \in \mathbb{N}$. This is due to Proposition 23. In other words

$$\mathcal{P}_k(J) \implies \mathcal{P}_{k+1}(J). \quad (5.39)$$

Taking $J = I_1$, we can deduce from (5.32), that $\mathcal{P}_0(I_2)$ is true. Now we use (5.39) again with $J = I_2$ to obtain that $\mathcal{P}_k(I_2)$ is true for all k . From 5.30, we have $\mathcal{P}_0(G_2)$ is true. So applying (5.39) with $J = G_2$ we deduce that $\mathcal{P}_k(G_2)$ is true for all k . Now we use (5.31) to say that $\mathcal{P}_0(I_3)$ is true, and so on, we can start again to write

$$\mathcal{P}_0(I_N) \implies \mathcal{P}_0(I_{N+1}),$$

for all $N \in \mathbb{N}^*$. So if $\mathcal{P}_0(I_N)$ is true, then $\mathcal{P}_0(I_{N+1})$ is true, and from (5.39), $\mathcal{P}_k(I_{N+1})$ is true for all k . Finally, the fact that $\bigcup_{N \in \mathbb{N}^*} I_N = \mathbb{Z}^* \times \mathbb{Z}$ give us the approximate controllability in small time. The second step of the proof of Theorem 22 also remains valid for having controllability in a fixed time. \square

Proof (of Proposition 7) Let $\dot{u} \in \dot{H}^1(\mathbb{T})$. We know that $\int_0^{2\pi} \dot{u}(x) dx = 0$. Now let $x, y \in [0, 2\pi]$, we have

$$\dot{u}(x) = \dot{u}(y) + \int_y^x \partial_x \dot{u}(\sigma) d\sigma.$$

By integrating in y between 0 and 2π

$$2\pi \dot{u}(x) = \int_0^{2\pi} \int_y^x \partial_x \dot{u}(\sigma) d\sigma dy.$$

We can estimate

$$2\pi |\dot{u}(x)| \leq \int_0^{2\pi} \int_0^{2\pi} |\partial_x \dot{u}(\sigma)| d\sigma dy = 2\pi \int_0^{2\pi} |\partial_x \dot{u}(\sigma)| d\sigma.$$

From Hölder inequality

$$|\dot{u}(x)| \leq \sqrt{2\pi} \left(\int_0^{2\pi} |\partial_x \dot{u}(\sigma)|^2 d\sigma \right)^{\frac{1}{2}}.$$

By taking the square and then integrating with respect to x we find

$$\|\dot{u}\|_{L^2(\mathbb{T})}^2 \leq 4\pi^2 \|\partial_x \dot{u}\|_{L^2(\mathbb{T})}^2.$$

Hence for all $\dot{u} \in \dot{H}^1(\mathbb{T})$ we have

$$\|\partial_x \dot{u}\|_{L^2(\mathbb{T})} \leq \|\dot{u}\|_{H^1(\mathbb{T})} \leq (2\pi + 1) \|\partial_x \dot{u}\|_{L^2(\mathbb{T})}.$$

□

Proof (of Theorem 18) We suppose (3.81). Let $y(t) = S(t)y_0$ be a solution of (3.80) and $E(t) = \frac{1}{2} \|y(t)\|_H^2$.

We have for every $t \geq 0$

$$(y'(t), y(t))_H + (A_1 y(t), y(t))_H = -(BB^* y(t), y(t))_H.$$

Since A_1 is skew-adjoint, we obtain

$$E'(t) = -\|B^* y(t)\|_H^2 \leq 0. \quad (6.1)$$

Thus E is nonincreasing. Now let $T = kT_0$, where k is to be chosen further, by integrating (6.1) between 0 and T we get

$$\frac{1}{2} \|y(T)\|_H^2 - \frac{1}{2} \|y(0)\|_H^2 = -\int_0^T \|B^* y(s)\|_H^2 ds.$$

Then

$$\int_0^T \|B^* y(s)\|_H^2 ds \leq E(0). \quad (6.2)$$

This implies that there exists $p \in \{0, \dots, k-1\}$ such that

$$\int_{pT_0}^{(p+1)T_0} \|B^* y(s)\|_H^2 ds \leq \frac{1}{k} E(0). \quad (6.3)$$

Let v be the solution of

$$\begin{cases} v'(t) + A_1 v(t) = 0 \\ v(pT_0) = y(pT_0). \end{cases} \quad (6.4)$$

The function $w = y - v$ satisfy $w'(t) + A_1 w(t) = -BB^* y(t)$, and $w(pT_0) = 0$. Thus

$$w(t) = -\int_{pT_0}^t V(t-s)BB^* y(s) ds,$$

where $\{V(t)\}_{t \geq 0}$ is the semigroup generated by A_1 , which is a semigroup of contractions by the Hille-Yosida Theorem.

We set $M = \|B\|_{\mathcal{L}(H)}$. For $t \in [pT_0, (p+1)T_0]$ we have

$$\begin{aligned} \|w(t)\|_H &\leq M \int_{pT_0}^{(p+1)T_0} \|B^* y(s)\|_H ds \\ &\leq M \left(T_0 \int_{pT_0}^{(p+1)T_0} \|B^* y(s)\|_H^2 ds \right)^{\frac{1}{2}} \\ &\leq M \sqrt{\frac{T_0 E(0)}{k}}. \end{aligned}$$

It follows

$$\int_{pT_0}^{(p+1)T_0} \|B^*w(s)\|_H^2 ds \leq M^2 \int_{pT_0}^{(p+1)T_0} \|w(s)\|_H^2 ds \leq M^4 T_0^2 \frac{E(0)}{k}. \quad (6.5)$$

As we have

$$\|B^*v(t)\|_H^2 = \|B^*y(t) - B^*w(t)\|_H^2 \leq 2(\|B^*y(t)\|_H^2 + \|B^*w(t)\|_H^2),$$

combinning (6.3) and (6.5) we get

$$\int_{pT_0}^{(p+1)T_0} \|B^*v(s)\|_H^2 ds \leq 2(1 + M^4 T_0^2) \frac{E(0)}{k}. \quad (6.6)$$

Since v verifies (6.4), we have from (3.81) and (6.6)

$$2E(pT_0) = \|v(pT_0)\|_H^2 \leq 2c(1 + M^4 T_0^2) \frac{E(0)}{k}.$$

Since E is nonincreasing, then

$$E(kT_0) \leq c(1 + M^4 T_0^2) \frac{E(0)}{k}.$$

So, for $k > 4c(1 + M^4 T_0^2)$ and $T = kT_0$ we obtain $\|y(T)\|_H \leq \frac{1}{2} \|y_0\|_H$, and it is classical that this implies (3.82).

Reciprocally, suppose that we have (3.82). Since E decreases towards 0, we can find $T_0 > 0$ such that every solution y of (3.80) verifies

$$\begin{aligned} \int_0^{T_0} \|B^*y(s)\|_H^2 ds &= E(0) - E(T_0) \\ &> \frac{1}{2} E(0) \\ &= \frac{1}{4} \|y_0\|_H^2. \end{aligned} \quad (6.7)$$

Now let φ be a solution of (3.79) and y a solution of (3.80) with the initial data $\varphi(0) = y(0)$. The function $z = y - \varphi$ verifies $z'(t) + A_1 z + BB^*z = -BB^*\varphi$. Then

$$z(t) = - \int_0^t S(t-s) BB^* \varphi(s) ds.$$

For all $t \in [0, T_0]$ we have

$$\|z(t)\|_H^2 \leq c \int_0^{T_0} \|B^*\varphi(s)\|_H^2 ds \quad (6.8)$$

Since $\|B^*y(s)\|_H \leq \|B^*\varphi(s)\|_H + \|B^*z(s)\|_H$, using (6.8) and the fact that B^* is bounded

$$\int_0^{T_0} \|B^*y(s)\|_H^2 ds \leq c \int_0^{T_0} \|B^*\varphi(s)\|_H^2 ds.$$

From (6.7), we get (3.81). \square

Proof (of Proposition 22) We recall that the norm on BV is given by (2.3). To simplify we take $J = [0, 1]$, and without loss of generalities, we can suppose from Remark 9 that $\dim(E) = 1$.

Let $\epsilon > 0$ and $B \subset L^1(J)$ a closed ball of radius 1. Let also $L, M \in \mathbb{N}^*$ be two constants to choose later. We define $t_k = \frac{k}{L}$ and the intervals $I_k = [t_{k-1}, t_k[$. We set \mathcal{F} a finite family of functions f of $L^1(J)$ constant on each I_k which cover B such that every f is worth one of the values $\frac{2j}{M}$ for $j = -M, \dots, M$ on each I_k . The number of elements of \mathcal{F} is given by $(2M + 1)^L$. For $t \in I_k$, we have

$$|u(t) - u(t_{k-1})| \leq \int_{t_{k-1}}^t |du(\sigma)|.$$

Thus

$$\begin{aligned} \sum_{k=1}^L \int_{I_k} |u(t) - u(t_{k-1})| dt &\leq \sum_{k=1}^L \int_{I_k} \int_{t_{k-1}}^t |du(\sigma)| dt \\ &\leq \sum_{k=1}^L \int_{I_k} \int_{\sigma}^{t_k} dt |du(\sigma)| \\ &\leq \frac{1}{L} \sum_{k=1}^L \int_{I_k} |du(\sigma)| \leq \frac{1}{L}. \end{aligned}$$

We know that there exists $f \in \mathcal{F}$ such that $|f(t) - u(t_{k-1})| \leq \frac{2}{M}$, so we can write

$$\begin{aligned} \int_0^1 |u(t) - f(t)| dt &\leq \sum_{k=1}^L \int_{I_k} |u(t) - u(t_{k-1})| + |u(t_{k-1}) - f(t)| dt \\ &\leq \frac{1}{L} + \frac{2L}{M}. \end{aligned}$$

Now we choose L and M so that the quantities $\frac{1}{L}$ and $\frac{2L}{M}$ have the same

order of magnitude. Hence we take $L = \left\lceil \frac{2}{\epsilon} \right\rceil + 1$ and $M = \left\lceil \frac{4L}{\epsilon} \right\rceil + 1$. So the number of element of \mathcal{F} is

$$(2M + 1)^L \leq \left(\frac{c_1}{\epsilon^2} \right)^{\frac{c_2}{\epsilon}} \leq \exp \left(c \frac{1}{\epsilon} \ln \left(\frac{1}{\epsilon} \right) \right).$$

We take the logarithm and the proof is complete. \square



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