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Damien Rivet

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Mémoire de Thèse

Représentations induites des groupes quantiques localement
compacts dans le cadre bornologique

Damien Rivet

Université Clermont Auvergne



Ecole doctorale des sciences fondamentales
Laboratoire de Mathématiques Blaise Pascal



Sous la direction de **Robert Yuncken**

Jury :

M. **Christian Voigt**, Reader, University of Glasgow

M. **Kenny De Commer**, Associate Professor, Vrije Universiteit
Brussel

Mme **Yulia Kuznetsova**, Maître de Conférences, Université de
Franche-Comté

M. **Uwe Franz**, Professeur, Université de Franche-Comté

M. **Roland Vergnioux**, Maître de Conférences, Université de Caen

M. **Julien Bichon**, Professeur, UCA

M. **Simon Riche**, Professeur, UCA

M. **Robert Yuncken**, Professeur, Université de Lorraine

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Introduction (Français)

Groupes quantiques localement compacts

Le terme *groupe quantique* peut faire référence à une grande variété d'objets mais dans chaque cas, il fait toujours référence à la généralisation de la notion de groupe, et dans chaque cas, un groupe quantique est systématiquement défini comme une algèbre de Hopf (ou un autre type d'objet similaire). Dans notre cas, les groupes quantiques sont une généralisation des groupes localement compacts et sont généralement appelés *groupes quantiques localement compacts*. L'une des motivations de cette généralisation provient de la dualité de Pontryagin. En effet, tout groupe abélien localement compact possède un groupe dual, lui-même localement compact et abélien. Cette dualité est une vaste abstraction de la transformée de Fourier et a été largement utilisée dans la théorie de l'analyse harmonique. Cependant, si l'on souhaite appliquer cet outil aux groupes non abéliens, on constate que l'objet dual d'un groupe n'a plus la structure d'un groupe, ce qui rompt la dualité. Or, la catégorie des groupes quantiques localement compacts est stable par une certaine dualité, qui coïncide avec la dualité de Pontryagin dans le cas des groupes abéliens *classiques*. En plus de donner un cadre solide et très général à la transformée de Fourier, la théorie des groupes quantiques localement compacts a donné lieu à de nombreux exemples. On cite le célèbre exemple de [PW90], où les auteurs définissent le groupe quantique $SL_q(2, \mathbb{C})$, une *déformation quantique*, avec le paramètre $0 < q < 1$, du groupe localement compact $SL(2, \mathbb{C})$. L'axiomatisation rigoureuse de cette théorie a présenté de nombreuses difficultés, en particulier parce que cette théorie fait un usage massif d'opérateurs non bornés, qui apportent des complications analytiques (ceci est discuté dans le chapitre 1). Mais une définition satisfaisante et complète a finalement été proposée par Kustermans et Vaes dans [KV00]. Ce travail, qui est maintenant reconnu comme la notion corrélative d'un groupe quantique localement compact, fournit une théorie extrêmement riche intégrant une grande variété d'exemples.

D'autre part, une axiomatique particulièrement simple a été développée par Van Daele [Dae98] via la notion de *groupe quantique algébrique*, qui comme son nom l'indique est presque exclusivement algébrique. Bien que simple, ce cadre permet d'exprimer clairement

la dualité de Pontryagin et de définir de nombreux exemples de groupes quantiques localement compacts, dont $SL_q(2, \mathbb{C})$, et de fait les groupes quantiques algébriques représentent la plupart des exemples traités dans cette thèse. Cependant, cette catégorie présente l'inconvénient majeur de ne pas contenir les groupes classiques localement compacts et semble donc trop restrictive pour servir de base à la théorie des groupes quantiques localement compacts.

Inspiré par les travaux de Van Daele et visant toujours à formaliser la dualité de Pontryagin dans un cadre général, Voigt a proposé dans [Voi08], la notion de groupe quantique bornologique. Bien que faisant appel à des notions analytiques avancées, ce formalisme s'avère en pratique particulièrement proche de celui de Van Daele (cette thèse se veut une illustration de ce fait) et répond au problème que nous venons de mentionner grâce aux résultats suivants.

Proposition 0.0.1. [Voi08, Proposition 9.2]. *Soit G un groupe de Lie. Alors l'algèbre $C_c^\infty(G)$ de fonctions lisses à support compact sur G est une algèbre de Hopf bornologique.*

Theorem 0.0.2. (Montgomery et Zippin, [MZ55]). *Soit G un groupe localement compact presque connecté, alors G est isomorphe à une limite projective des groupes de Lie.*

Ces deux résultats combinés suggèrent que, pour G un groupe localement compact presque connexe, l'algèbre $C_c^\infty(G)$ des fonctions lisses de Bruhat avec support compact sur G . à support compact sur G peut être dotée d'une structure de groupe quantique bornologique.

Groupes quantiques semisimples et induction parabolique

L'exemple le plus élémentaire d'un groupe quantique semisimple est donné par $SU_q(2)$. Il a été découvert par Woronowicz, dans le contexte actuel des algèbres de fonctions quantifiées. Cette découverte a été le point de départ de la théorie des groupes quantiques compacts. Comme mentionné précédemment, les groupes quantiques qui nous concernent ici sont précisément ceux qui ne sont pas compacts. Cependant, c'est à partir de $SU_q(2)$ que nous pouvons étudier notre exemple élémentaire, le groupe quantique $SL_q(2, \mathbb{C})$. En effet, la remarque décisive de [PW90] était la suivante : En considérant le dual de Pontryagin $\widehat{SU_q(2)}$ de $SU_q(2)$, nous observons qu'il existe une décomposition, utilisant la construction *produit croisé*

$$SL_q(2, \mathbb{C}) = SU_q(2) \rtimes \widehat{SU_q(2)}.$$

Cela généralise en quelque sorte la décomposition d'Iwasawa

$$SL(2, \mathbb{C}) = SU(2) \rtimes AN,$$

avec la différence majeure que dans le cas classique, il n'y a a priori aucun lien étroit entre les groupes $SU(2)$ et AN , alors que dans le cas quantique on peut écrire

$$AN_q = \widehat{SU_q(2)}.$$

De plus, étant le dual d'un groupe quantique compact, $\widehat{SU_q(2)}$ est un groupe quantique discret et est donc analytiquement assez simple (c'est en particulier un groupe quantique algébrique au sens de Van Daele). C'est la raison pour laquelle $SL_q(2, \mathbb{C})$, est algébrique, comme produit bicrossé de deux groupes quantiques algébriques. La notion de sous-groupe fermé est au cœur du principe d'induction, et dans cette thèse nous allons passer beaucoup de temps à définir et étudier cette notion dans le cas quantique. Dans notre exemple, $SU_q(2)$ admet pour sous-groupe fermé le tore $T = \mathbb{S}^1$, c'est-à-dire que par définition d'un sous-groupe quantique fermé, il existe un morphisme de groupe quantique entre les *algèbres de fonctions* sur $SU_q(2)$ et T .

$$\pi : \mathcal{A}(SU_q(2)) \rightarrow \mathcal{A}(T).$$

Notez que dans le cas classique, un tel morphisme entre les algèbres de fonctions correspond à la carte de restriction sur le sous-groupe concerné. En étudiant de plus près la structure bicrossée, on peut alors voir que la carte

$$\pi \otimes \text{id} : \mathcal{A}(SL_q(2, \mathbb{C})) \rightarrow \mathcal{A}(T \bowtie AN_q).$$

est également un morphisme *-Hopf et réalise ainsi $B_q = T \bowtie AN_q$ en tant que sous-groupe quantique de $SL_q(2, \mathbb{C})$. Bien sûr le choix de la notation B_q n'est pas anodin et ce sous-groupe possède toutes les propriétés pour être considéré comme un sous-groupe de Borel. De plus, puisque la dualité de Pontryaguin s'applique également aux morphismes, notre morphisme $\pi : \mathcal{A}(SU_q(2)) \rightarrow \mathcal{A}(T)$ donne lieu à un morphisme

$$\hat{\pi} : \mathcal{A}(\hat{T}) \rightarrow M(\mathcal{A}(\widehat{SU_q(2)})),$$

. où $M(\mathcal{A}(\widehat{SU_q(2)}))$ désigne le multiplicateur de l'algèbre non unitaire $\mathcal{A}(\widehat{SU_q(2)})$, voir chapitre 1. Il est bien connu que $\hat{T} \cong \mathbb{Z}$ et c'est précisément ce groupe, que nous notons A_q , qui joue le rôle de la partie non compacte du sous-groupe de Cartan. Dans le chapitre 5, nous verrons que A_q n'apparaît pas comme un sous-groupe quantique mais seulement comme un groupe "quotient" de AN_q . Nous avons découvert au cours de ce travail de thèse qu'il existe une algèbre qui a de bonnes raisons d'être dénommée $\mathcal{A}(G_q/N_q)$. En particulier cette algèbre est dotée d'une action à gauche par $SL_q(2, \mathbb{C})$ et d'une action à droite $L_q = T \times A_q$, qui prendra le rôle du facteur de Lévy. Cette observation conduit naturellement à l'émergence d'un $C^*(L_q)$ -module $\mathcal{E}(G_q/N_q)$, avec une représentation de $C^*(SL_q(2, \mathbb{C}))$.

L'inspiration pour cette construction vient de [CCH16], où les auteurs construisent le foncteur d'induction à partir de l'algèbre des fonctions sur l'espace homogène G/N . Nous montrerons qu'il implémente l'induction parabolique pour $SL_q(2, \mathbb{C})$ et que nous avons le résultat suivant.

Theorem 0.0.3. *En considérant $W = \mathbb{Z}_2$ le groupe de Weyl de $SL(2, \mathbb{C})$, on a*

$$C^*(SL_q(2, \mathbb{C})) \cong \mathfrak{K}(\mathcal{E}(SL_q(2, \mathbb{C})/N_q))^W,$$

où \mathfrak{K} indique l'algèbre des opérateurs compacts au sens des modules de Hilbert.

Cela donne un point de vue différent sur un résultat de Monk et Voigt [MV19]. Notons qu'il existe une grande catégorie de groupes quantiques semisimples, incluant les déformations $SL_q(n, \mathbb{C})$ pour tout $n \leq 2$. Les groupes quantiques semisimples complexes sont largement étudiés dans [VY20] et nous introduisons la définition et les résultats de base dans le chapitre 1. Un des résultats principaux de cette thèse est une généralisation du théorème précédent à tous les groupes quantiques semisimples complexes.

Le chapitre 1 détaille les généralités mentionnées ci-dessus et complète cette introduction. Le chapitre 2 présente les notions d'espaces vectoriels bornologiques, d'algèbres et enfin de groupes quantiques. Dans le chapitre 3 nous étudions le lien entre bornologique et groupe quantique localement compact dans le sens de [KV00], Ces résultats ont été publiés dans le preprint [RY21]. Dans le chapitre 4, nous développons un cadre général pour l'induction et établissons le lien avec les travaux de Vaes [Vae05]. Le chapitre 5 se concentre sur l'exemple des groupes quantiques semisimples. Tous les résultats de ces deux derniers chapitres sont présentés dans le preprint [Riv19].

Introduction

Locally compact quantum groups

The term *quantum group* can refer to a wide variety of objects but in each case it always refers to the generalization of the notion of group, and in each case, a quantum group is systematically defined as an Hopf algebra (or another similar type of object). In our case quantum groups are a generalization of locally compact groups and are usually called *locally compact quantum groups*. One of the motivations for this generalization comes from Pontryagin duality. Indeed, every locally compact abelian group has a dual group, itself locally compact and abelian. This duality is a vast abstraction of the Fourier transform and has been used extensively in the theory of harmonic analysis. However, if one wishes to apply this tool to non-abelian groups, one finds that the dual object of a group no longer has the structure of a group, breaking the duality. Now, the category of locally compact quantum groups is stable by a certain duality, which coincides with Pontryagin's duality in the case of abelian *classical* groups. In addition to giving a solid and very general framework to the Fourier transform, the theory of locally compact quantum groups has given rise to many examples. One cites the famous example of [PW90], where the authors define the quantum group $SL_q(2, \mathbb{C})$, a *quantum deformation*, with parameter $0 < q < 1$, of the locally compact group $SL(2, \mathbb{C})$. The rigorous axiomatization of this theory presented many difficulties, in particular because this theory makes massive use of unbounded operators, which bring analytical complications (this is discussed in Chapter 1). But a satisfactory and complete definition has finally been proposed by Kustermans and Vaes in [KV00]. This work, which is now recognized as the correct notion of a locally compact quantum group, provides an extremely rich theory incorporating a wide variety of examples.

On the other hand, a particularly simple axiomatic has been developed by Van Daele [Dae98] via the notion of *algebraic quantum group*, which as its name indicates is almost exclusively algebraic. Although simple, this framework allows to express Pontryagin duality clearly and to define many examples of locally compact quantum groups, including $SL_q(2, \mathbb{C})$, and in fact algebraic quantum groups represent most of the examples treated in

this thesis. However, this category has the major drawback of not containing the classical locally compact groups and thus seems too restrictive to be a basis for the theory of locally compact quantum groups.

Inspired by Van Daele's work and still aiming at formalizing Pontryagin's duality in a general framework, Voigt in [Voi08], proposed the notion of bornological quantum group. Although calling for advanced analytical notions, this formalism turns out in practice to be particularly close to Van Daele's (this thesis is intended to be an illustration of that fact) and it answers the problem just mentioned thanks to the following results.

Proposition 0.0.4. [Voi08, Proposition 9.2]. *Let G be a Lie group. Then the algebra $C_c^\infty(G)$ of smooth functions with compact support on G is a bornological Hopf algebra.*

Theorem 0.0.5. (Montgomery and Zippin, [MZ55]). *Let G be an almost connected locally compact group, then G is isomorphic to a projective limit of Lie groups.*

These two results combined suggest that, for G an almost connected locally compact group, the algebra $C_c^\infty(G)$ of Bruhat smooth functions with compact support on G can be endowed with a structure of bornological quantum group.

Semisimple quantum groups and parabolic induction The most elementary example of a semisimple quantum group is given by $SU_q(2)$. It has been discovered by Woronowicz, in the current context of quantized functions algebras. This discovery has been the starting point of the theory of compact quantum groups. As mentioned before, the quantum groups which concern us here are precisely those which are not compact. However, it is starting from $SU_q(2)$ that we can study our elementary example, the quantum group $SL_q(2, \mathbb{C})$. Indeed, the decisive remark of [PW90] was the following: Considering the Pontryagin dual $\widehat{SU_q(2)}$ of $SU_q(2)$, we observe that there exists a decomposition, using the *bicrossed product* construction

$$SL_q(2, \mathbb{C}) = SU_q(2) \bowtie \widehat{SU_q(2)}.$$

This generalizes in a way the Iwasawa decomposition

$$SL(2, \mathbb{C}) = SU(2) \bowtie AN,$$

with the major difference that in the classical case, there is a priori no close link between the groups $SU(2)$ and AN , while in the quantum case we can write

$$AN_q = \widehat{SU_q(2)}.$$

Moreover, being the dual of a compact quantum group $\widehat{SU_q(2)}$ is a discrete quantum group and thus is analytically quite simple (it is in particular an algebraic quantum group in the

Van Daele sense). This is the reason why $SL_q(2, \mathbb{C})$, is algebraic, as a bicrossed product of two algebraic quantum groups.

The notion of closed subgroup is at the heart of the induction principle, and in this thesis we will spend a lot of time defining and studying this notion in the quantum case. In our example $SU_q(2)$ admits for closed subgroup the torus $T = \mathbb{S}^1$, i.e., by definition of a closed quantum subgroup, there exists a quantum group morphism between the *algebras of functions* over $SU_q(2)$ and T

$$\pi : \mathcal{A}(SU_q(2)) \rightarrow \mathcal{A}(T).$$

Note that in the classical case, such a morphism between the algebras of functions corresponds to the restriction map over the concerned subgroup. By studying more closely the bicrossed structure, one can then see that the map

$$\pi \otimes \text{id} : \mathcal{A}(SL_q(2, \mathbb{C})) \rightarrow \mathcal{A}(T \bowtie AN_q)$$

is also a *-Hopf morphism and thus realizes $B_q = T \bowtie AN_q$ as a quantum subgroup of $SL_q(2, \mathbb{C})$. Of course the choice of the notation B_q is not insignificant and this subgroup has all the properties to be considered as a Borel subgroup. Furthermore, since Pontryaguin duality also applies to morphisms, our morphism $\pi : \mathcal{A}(SU_q(2)) \rightarrow \mathcal{A}(T)$ gives rise to a morphism

$$\hat{\pi} : \mathcal{A}(\hat{T}) \rightarrow M(\widehat{\mathcal{A}(SU_q(2))}),$$

where $M(\widehat{\mathcal{A}(SU_q(2))})$ denotes the multiplier of the non unitary algebra $\widehat{\mathcal{A}(SU_q(2))}$, see chapter 1.

It is well known that $\hat{T} \cong \mathbb{Z}$ and it is precisely this group, which we note A_q , which plays the role of the noncompact part of the Cartan subgroup. In Chapter 5 we will see that A_q does not appear as a quantum subgroup but only as a "quotient" group of AN_q . We discovered during this thesis work that their exists an algebra has good reasons to be denoted $\mathcal{A}(G_q/N_q)$. In particular this algebra is equipped with an action on the left by $SL_q(2, \mathbb{C})$ and an action on the right $L_q = T \times A_q$, which will take the role of the Lévy factor. This observation naturally leads to the emergence of a $C^*(L_q)$ module $\mathcal{E}(G_q/N_q)$, with a representation of $C_u^*(SL_q(2, \mathbb{C}))$. This idea of considering homogeneous spaces was first used by Woronowicz in [Wor00], where he constructs bundles over G_q/B_q to define principal series representations. The inspiration for the construction of a G_q/N_q comes from [CCH16], where the authors build the induction functor from the algebra of functions over the homogeneous space G/N . We will show that it implements the parabolic induction for $SL_q(2, \mathbb{C})$ and that we have the following result.

Theorem 0.0.6. *Consider $W = \mathbb{Z}_2$ the Weyl group of $SL(2, \mathbb{C})$, we have*

$$C_r^*(SL_q(2, \mathbb{C})) \cong \mathfrak{K}(\mathcal{E}(SL_q(2, \mathbb{C})/N_q))^W,$$

where \mathfrak{K} indicates the algebra of compact operators in the sense of Hilbert modules.

This gives a different point of view on a result of Monk and Voigt [MV19]. Let us note that there exists a large category of semisimple quantum groups, including the deformations $SL_q(n, \mathbb{C})$ for any $n \leq 2$. complex semisimple quantum groups are extensively studied in [VY20] and we introduce the definition and basic results in Chapter 1. One of the main result of this thesis is a generalization of Theorem 0.0.6 to all complex semisimple quantum groups.

Chapter 1 goes into detail on the generalities mentioned above and completes this introduction. Chapter 2 presents the notions of bornological vector spaces, algebras and finally, quantum groups. In Chapter 3 we study the link between bornological and locally compact quantum group in the sens of [KV00], These results have been published in the preprint [RY21]. In Chapter 4 we develop a general framework for induction and establish the link with works of Vaes [Vae05]. Chapter 5 focuses on the example of semisimple quantum groups. All the results of these two last chapters are presented in the preprint [Riv19].

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Chapter 1

Preliminaries and notations

In this chapter, we recall all the “basic notions” required to understand the context of this thesis and introduce all the “protagonists” that we will meet all along this manuscript. We progress from Hopf algebras to locally compact quantum groups, passing by discrete and compact groups.

Definition 1.0.1. *Let $0 < q < 1$ be a real number and n be an integer, we define the quantum number $[n]_q$ as*

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}.$$

Note that we have $\lim_{q \rightarrow 1} [n]_q = n$. The number $[n]_q$ can then be seen as a *quantum deformation* of the number n .

Remark 1.0.2. This notation only occurs at Example 15, but it seems important to us to introduce it for cultural reasons.

1.1 Hopf algebras, *-Hopf algebras and their representations

1.1.1 Hopf algebras. Examples.

We refer to [KS97, Section 1.2] for detailed definitions of algebras and coalgebras. This Section is largely based on that reference. We give a certain number of examples that are more or less closely related to $SL_q(2)$.

Definition 1.1.1. *Let H be an associative algebra over a field K , with multiplication $\mu : H \otimes H \rightarrow H$ and unit $\eta : K \rightarrow H$ such that H is also a coassociative algebra with comultiplication $\Delta : H \rightarrow H \otimes H$ and counit $\epsilon : H \rightarrow K$, such that Δ and ϵ are algebra*

homomorphism. We call H is a Hopf algebra if there exists a linear map $S : H \rightarrow H$ (called the antipode) such that the following diagram commutes.

$$\begin{array}{ccccc}
 & & H \otimes H & \xrightarrow{S \otimes \text{id}} & H \otimes H \\
 & \nearrow \Delta & & & \searrow \mu \\
 H & \xrightarrow{\epsilon} & K & \xrightarrow{\eta} & H \\
 & \searrow \Delta & & & \nearrow \mu \\
 & & H \otimes H & \xrightarrow{\text{id} \otimes S} & H \otimes H
 \end{array}$$

Throughout this thesis, K will be the field of complex numbers \mathbb{C} and all vector spaces will be over \mathbb{C} .

The fundamental examples of Hopf algebras are Hopf algebras of finite groups, as defined in the two following examples.

Example 1. Let G be a finite group and consider the group algebra $\mathbb{C}[G] = \text{Vect}\{[g], g \in G\}$ (where multiplication is given by $[g] * [k] = [gk]$). There exists a comultiplication Δ on $\mathbb{C}[G]$ such that

$$\Delta([g]) = [g] \otimes [g], \quad \forall g \in G.$$

Endowed with antipode S defined by $S([g]) = [g^{-1}]$ and counit such that $\epsilon([g]) = 1$, $\mathbb{C}[G]$ is a (cocommutative) Hopf algebra.

Remark 1.1.2. In reference to this example, an element x of a Hopf algebra H is called *group-like* if $\Delta(x) = x \otimes x$.

Example 2. One can also consider the algebra of functions over G , $C(G) = \{f : G \rightarrow \mathbb{C}\}$ with usual pointwise multiplication. There exists a comultiplication Δ on $C(G)$ such that,

$$\Delta(f)(g, h) = f(gh), \quad \forall f \in C(G), \quad \forall g, h \in G,$$

where we make the identification $C(G) \otimes C(G) \cong C(G \times G)$. Here we consider antipode S given by $S(f)(g) = f(g^{-1})$, $\forall g \in G$ and counit $\epsilon(f) = f(e)$, where e is the identity element of G . We then obtain a Hopf algebra structure on $C(G)$.

Denoting δ_g the Dirac function in element $g \in G$, the coproduct can be given explicitly via

$$\Delta(\delta_g) = \sum_{\substack{h, k \in G \\ hk = g}} \delta_h \otimes \delta_k.$$

This Hopf algebra is not cocommutative, as soon as G is not abelian.

Example 3. Let G be an infinite discrete group, then the algebra $\mathbb{C}[G]$ with comultiplication Δ and antipode S defined as in Example 1 is also a Hopf algebra. However $C(G)$ cannot be endowed with the structure defined in Example 2; indeed the identification $C(G) \otimes C(G) \cong C(G \times G)$ is false in this case. However $c_c(G)$ the algebra of finitely supported functions can “almost” be endowed with a Hopf algebra structure and this is the starting example of our next section.

Example 4. Let G be an algebraic group and consider $\mathcal{O}(G)$ the algebra of polynomials over G . Then $\mathcal{O}(G)$ can be turned into a Hopf algebra. For example, let us describe $G = SL(2, \mathbb{C})$ with $a = u_1^1, b = u_2^1, c = u_1^2$ and $d = u_2^2$ the matrix coefficients evaluations (where $u_1^1 \begin{pmatrix} x & y \\ z & t \end{pmatrix} = x$, etc). Then the coproduct given by

$$\Delta(a) = a \otimes a + b \otimes c, \quad \Delta(b) = a \otimes b + b \otimes d, \quad (1.1)$$

$$\Delta(c) = c \otimes a + d \otimes c, \quad \Delta(d) = c \otimes b + d \otimes d, \quad (1.2)$$

the counit is given by

$$\epsilon(a) = \epsilon(d) = 1, \quad \epsilon(b) = \epsilon(c) = 0, \quad (1.3)$$

and antipode given by

$$S(a) = d, \quad S(d) = a, \quad S(b) = -b, \quad S(c) = -c,$$

make $\mathcal{O}(SL(2, \mathbb{C}))$ a Hopf algebra.

Example 5. Let q a real number with $0 < q < 1$ and let $\mathcal{O}(SL_q(2))$ be the associative unital algebra with generators a, b, c and d and relations

$$\begin{aligned} ab &= qba, \quad ac = qca, \quad bd = qdb, \quad cd = qdc, \quad bc = cb, \\ ad - da &= (q - q^{-1})bc, \\ ad - qbc &= 1. \end{aligned}$$

This algebra is usually called the *algebra of polynomials over the quantum group $SL_q(2)$* . Endowed with the coproduct given by relations (1) and (2), counit with relations (3) and antipode

$$S(a) = d, \quad S(d) = a, \quad S(b) = -q^{-1}b, \quad S(c) = -qc.$$

With this structure, $\mathcal{O}_q(SL_q(2))$ is a Hopf algebra.

Example 6. Let \mathfrak{g} be a Lie algebra and $U(\mathfrak{g})$ be its enveloping algebra. Then there exists a Hopf algebra structure given by

$$\Delta(X) = X \otimes 1 + 1 \otimes X, \quad \epsilon(X) = 0, \quad S(X) = -X, \quad \forall X \in \mathfrak{g}.$$

Example 7. Let $U_q(\mathfrak{sl}_2)$ be the associative algebra with generators E, F and K such that K is invertible and with relations

$$KEK^{-1} = q^2E, \quad KFK^{-1} = q^{-2}F, \quad [E, F] = \frac{K - K^{-1}}{q - q^{-1}}.$$

There exists a Hopf algebra structure on $U_q(\mathfrak{sl}_2)$ given by

$$\begin{aligned} \Delta(E) &= E \otimes K + 1 \otimes E, \quad \Delta(F) = F \otimes 1 + K^{-1} \otimes F, \quad \Delta(K) = K \otimes K, \\ S(K) &= K^{-1}, \quad S(E) = -EK^{-1}, \quad S(F) = -KF, \quad \epsilon(K) = 1, \quad \epsilon(E) = \epsilon(F) = 0. \end{aligned}$$

This Hopf algebra is seen as a *quantum deformation* (with parameter q) of the classical $U(\mathfrak{sl}_2(\mathbb{C}))$. Roughly speaking one can start with E, F and H the usual generators of $U(\mathfrak{sl}_2(\mathbb{C}))$ and by setting the formal element $K = q^H$, one obtain the preceding relations and the element $\frac{K - K^{-1}}{q - q^{-1}} = \frac{q^H - q^{-H}}{q - q^{-1}}$ can be considered as “tending to H ” when q goes to 0.

Definition 1.1.3. *A Hopf morphism between two Hopf algebras \mathcal{A} and \mathcal{B} is an algebra and coalgebra homomorphism.*

Example 8. We denote by $\mathcal{O}(T)$ the algebra of polynomials over the circle T and z a generator, that is such that $\mathcal{O}(T) = \mathbb{C}[z, z^{-1}]$. We consider $\pi : \mathcal{O}(SL_q(2)) \rightarrow \mathcal{O}(\mathbb{T})$ the Hopf algebra homomorphism such that

$$\pi(a) = z, \quad \pi(d) = z^{-1}, \quad \pi(b) = \pi(c) = 0.$$

This morphism is surjective and we then say that it identifies T as a *quantum subgroup* of $SL_q(2)$.

1.1.2 *-structures and pairing

Definition 1.1.4. *A *-Hopf algebra is a Hopf algebra \mathcal{A} together with an involution $*$ such that \mathcal{A} is a *-algebra and the morphisms Δ and ϵ are *-morphisms, that is $\Delta(a^*) = \Delta(a)^*$ (where the involution in $H \otimes H$ is given by $(a \otimes b)^* = a^* \otimes b^*$ and $\epsilon(a^*) = \bar{a}$, for all $a \in H$).*

One can check that in a *-Hopf algebra \mathcal{A} we have for all $a \in \mathcal{A}$:

$$S(S(a)^*)^* = a.$$

Example 9. Let G be a finite group, the Hopf algebra $\mathbb{C}[G]$ endowed with involution simply given by $[g]^* = [g^{-1}]$ for all $g \in G$ makes $\mathbb{C}[G]$ into a *-Hopf algebra. One can also turn $C(G)$ into a *-Hopf algebra by defining

$$f^* = \bar{f}, \quad f \in C(G),$$

the pointwise conjugation.

Remark 1.1.5. By analogy with this classical case, we often refers to the *-involution via the notation \bar{f} for an element f of a Hopf algebra \mathcal{A} and this in particular when \mathcal{A} is seen as an algebra of “functions over a quantum group”.

Example 10. The Hopf algebra $\mathcal{O}(SU_q(2))$, endowed with *-structure given by

$$a^* = c, \quad b^* = -qc, \quad c^* = -q^{-1}c, \quad d^* = a,$$

is a *-Hopf algebra.

Remark 1.1.6. Later in this work we shall use the notation $\mathcal{A}(SU_q(2))$ instead of $\mathcal{O}(SU_q(2))$. What motivates this choice is the fact that $SU_q(2)$ is *compact*, i.e., $\mathcal{O}(SU_q(2))$ is unital. We will see that $\mathcal{A}(SU_q(2))$ can be then considered as an algebraic quantum group and we often refer to algebraic quantum groups via notation of type $\mathcal{A}(\mathbb{G})$.

Example 11. The Hopf algebra $U_q(\mathfrak{sl}_2)$ also admits different possible *-structures (in one to one correspondence with those of $\mathcal{O}(SL_q(2))$), but we just mention $U_q(\mathfrak{su}_2)$, with involution given by

$$E^* = FK, \quad F^* = K^{-1}E, \quad K^* = K.$$

Definition 1.1.7. Let $(\mathcal{A}, \Delta, S, \epsilon)$ with involution denoted $a \mapsto \bar{a}$ and $(\mathcal{B}, \hat{\Delta}, \hat{S}, \hat{\epsilon})$ with involution denoted $x \mapsto x^*$ be two *-Hopf algebras. A Hopf *-pairing between \mathcal{A} and \mathcal{B} is a bilinear map $(\cdot, \cdot) : \mathcal{B} \times \mathcal{A} \rightarrow \mathbb{C}$ such that for all $a, b \in \mathcal{A}$ and $x, y \in \mathcal{B}$ we have

$$\begin{aligned} (xy, a) &= (x \otimes y, \Delta(a)) & (\hat{\Delta}(x), a \otimes b) &= (x, ba) \\ \hat{\epsilon}(x) &= (x, 1) & (\hat{1}, a) &= \epsilon(a) \\ (\hat{S}(x), a) &= (x, S^{-1}(a)) & (\hat{S}^{-1}(x), a) &= (x, S(a)) \\ (x^*, a) &= \overline{(x, S(a))}, & (x, \bar{a}) &= \overline{(\hat{S}^{-1}(x)^*, a)}. \end{aligned}$$

Remark 1.1.8. We are using the convention of [VY20] in which the coproduct on \mathcal{B} is dual to the opposite product of \mathcal{A} . In [KS97], this is referred as a *skew-pairing*.

Example 12. Let G be a finite group. There exists a dual pairing between $C(G)$ and $\mathbb{C}[G]$ given by

$$([g], f) = f(g), \quad \forall g \in G, \quad f \in C(G).$$

Example 13. The $*$ -Hopf algebras $U_q(\mathfrak{su}_2)$ and $\mathcal{O}(SU_q(2))$ are dually paired via

$$(K, a) = q^{-1}, (K, d) = q, (E, c) = q^{1/2}, (F, b) = q^{-1/2},$$

the pairing between the other combinations of generators being 0.

Definition 1.1.9. A functional on \mathcal{A} is called *left invariant* (resp. *right invariant*) if it satisfies $(\iota \otimes \phi) \circ \Delta(a) = \phi(a)1$ (resp. $(\phi \otimes \iota) \circ \Delta(a) = \phi(a)1$) for all $a \in \mathcal{A}$.

If \mathcal{A} is a $*$ -Hopf algebra and

$$\phi(a^*a) > 0, \quad \forall a \in \mathcal{A}, \quad a \neq 0,$$

we say that ϕ is a *left* (resp. *right*) *Haar functional*.

Example 14. Consider $\mathcal{O}(T) = \mathbb{C}[z, z^{-1}]$ as in Example 8. We define ϕ_T as the functional such that $\phi_T(1) = 1$ and $\phi_T(z^n) = 0$ for all $n \neq 0$. This defines a Haar functional on $\mathcal{O}(T)$.

1.1.3 Modules and comodules

Let \mathcal{A} be a Hopf algebra

Definition 1.1.10. Let V be a vector space.

A (left) \mathcal{A} -module structure on V is given by a linear map $\rho : \mathcal{A} \otimes V \rightarrow V$ where we usually write $\rho(a \otimes v) = a \cdot v$, where $a \in \mathcal{A}, v \in V$ such that for $a, b \in \mathcal{A}$ and $v \in V$, we have $(ab) \cdot v = a \cdot (b \cdot v)$.

A (left) \mathcal{A} -comodule structure on V is given by a linear map $\alpha : V \rightarrow \mathcal{A} \otimes V$, called a *coaction*, such that

$$(\Delta \otimes \text{id})(\alpha(v)) = (\text{id} \otimes \alpha)(\alpha(v)), \quad \forall v \in V.$$

We usually use Sweedler notations for coactions by writing $\alpha(v) = v_{(-1)} \otimes v_{(0)}$.

Definition 1.1.11. Suppose now that \mathcal{A} is a $*$ -Hopf algebra and V is a Hilbert space with inner product $\langle \cdot, \cdot \rangle$.

The space V is said to be a *unitary module* if

$$\langle a \cdot v, w \rangle = \langle v, a^* \cdot w \rangle, \quad \forall a \in \mathcal{A}, \quad \forall v, w \in V.$$

The space V is said to be a *unitary comodule* if

$$v_{(-1)}^* \langle v_{(0)}, w \rangle = S^{-1}(w_{(-1)}) \langle v, w_{(0)} \rangle.$$

Remark 1.1.12. We generally call a module a *representation* of \mathcal{A} , and a comodule a *corepresentation* of \mathcal{A} .

Example 15. (Regular representation and corepresentation). Let \mathcal{A} be a Hopf algebra, then \mathcal{A} can be seen as a module with multiplication μ , where we have $a \cdot b = ab$. And a comodule with coaction Δ , where we have $a_{(-1)} \otimes a_{(0)} = \Delta(a)$.

Example 16. Consider the $*$ -Hopf algebra $U_q(\mathfrak{su}_2)$ defined in Example 11 with its generators E, F and K . Let then l be a nonnegative integer or half-integer and let $\omega \in \{+1, -1\}$. Let V_l be a $(2l + 1)$ -dimensional vector space with basis e_m , $m = -l, -l + 1, \dots, l$ where we set $e_{l+1} = e_{-l-1} = 0$. Define operators $T_{\omega l}(K), T_{\omega l}(E), T_{\omega l}(F)$ acting on V_l by

$$\begin{aligned} T_{\omega l}(K)e_m &= \omega q^{2m} e_m, & T_{\omega l}(E)e_m &= ([l - m]_q [l + m + 1]_q)^{1/2} e_{m+1}, \\ T_{\omega l}(F)e_m &= \omega ([l + m]_q [l - m + 1]_q)^{1/2} e_{m-1}. \end{aligned}$$

Then $T_{\omega l}$ defines an algebra homomorphism $T_{\omega l} : U_q(\mathfrak{su}_2) \rightarrow \mathcal{L}(V_l)$. Furthermore, endowed with inner product given by $\langle e_i, e_j \rangle = \delta_{i,j}$, V_l becomes a unitary representation of $U_q(\mathfrak{su}_2)$. The family $(V_l, T_{\omega l})_{l \in \frac{\mathbb{N}}{2}, \omega \in \{+1, -1\}}$ constitute all the irreducible representations of $U_q(\mathfrak{su}_2)$, up to unitary equivalence.

Proposition 1.1.13. *Let \mathcal{A} and \mathcal{B} be two paired $*$ -Hopf algebras and let α be a corepresentation of \mathcal{A} on a space V . There exists a representation of \mathcal{B} on V given by*

$$a \cdot v = v_{(0)}(a, S^{-1}(v_{(-1)})).$$

Remark 1.1.14. In some cases, there exists a way to build a corepresentation of \mathcal{B} from a representation of \mathcal{A} . In this case we say that the representation is *integrable*. In particular the representations $(T_l)_l$ are precisely the integrable representations of $U_q(\mathfrak{su}_2)$ with respect to its pairing with $\mathcal{O}(SL_q(2, \mathbb{R}))$.

1.1.4 Structure of $SU_q(2)$

From now on, we always consider $SL_q(2)$ in its real form $SU_q(2)$. Let $l \in \frac{\mathbb{N}}{2}$, we write $T_l = T_{1l}$. Consider the functional on $U_q(\mathfrak{su}_2)$ of the form

$$X \mapsto \langle \xi', T_l(X)\xi \rangle,$$

where $\xi \in V_l$ and $\xi' \in V_l^*$. This functional is denoted by $\langle \xi' | \cdot | \xi \rangle$ and referred as the matrix coefficient of the representation T_l associated to ξ and ξ' . The space of all matrix coefficients of the representation T_l is naturally isomorphic to $V_l^* \otimes V_l$. Furthermore, this space can be endowed with the coalgebra structure given by duality with the product of $U_q(\mathfrak{su}_2)$. The element $\Delta(\langle \xi' | \cdot | \xi \rangle) : U_q(\mathfrak{su}_2) \otimes U_q(\mathfrak{su}_2) \rightarrow \mathbb{C}$ is such that

$$\Delta(\langle \xi' | \cdot | \xi \rangle)(X \otimes Y) = \langle \xi' | \cdot | \xi \rangle(XY).$$

One also has a counit given by $\langle \xi' | \cdot | \xi \rangle \rightarrow \langle \xi', \xi \rangle$. One now states the Peter-Weyl theorem for $SU_q(2)$.

Theorem 1.1.15. *We have coalgebra homomorphisms $\mathcal{A}(SU_q(2)) \rightarrow V_l^* \otimes V_l$ for all $l \in \frac{\mathbb{N}}{2}$ and the resulting map*

$$\mathcal{A}(SU_q(2)) \rightarrow \bigoplus_{l \in \frac{\mathbb{N}}{2}} V_l^* \otimes V_l$$

is an isomorphism.

One can endow $\bigoplus_{l \in \frac{\mathbb{N}}{2}} V_l^* \otimes V_l$ with the algebra structure of $\mathcal{A}(SU_q(2))$ through this isomorphism and it can also be recovered by duality with $U_q(\mathfrak{su}_2)$. Let $l_1, l_2 \in \frac{\mathbb{N}}{2}$ and $\langle \xi' | \cdot | \xi \rangle \in V_{l_1}^* \otimes V_{l_1}$, $\langle \eta' | \cdot | \eta \rangle \in V_{l_2}^* \otimes V_{l_2}$, for $X \in U_q(\mathfrak{su}_2)$ we have

$$\begin{aligned} (\langle \xi' | \cdot | \xi \rangle \cdot \langle \eta' | \cdot | \eta \rangle)(X) &= \langle \xi' \otimes \eta' | (T_{l_1} \otimes T_{l_2})(X) | \xi \otimes \eta \rangle \\ &= \langle \xi' \otimes \eta' | (T_{1l_1} \otimes T_{1l_2})(\Delta(X)) | \xi \otimes \eta \rangle. \end{aligned}$$

Remark 1.1.16. • $T_{l_1} \otimes T_{l_2}$ is by definition the morphism $U_q(\mathfrak{su}_2) \rightarrow \mathcal{L}(V_{l_1} \otimes V_{l_2})$ defined by $(T_{l_1} \otimes T_{l_2})(X) = (T_{l_1} \otimes T_{l_2})(\Delta(X))$.

- The fact that $(\langle \xi' | \cdot | \xi \rangle \cdot \langle \eta' | \cdot | \eta \rangle)$ is indeed a sum of matrix coefficients rests on the fact that the representation $V_{l_1} \otimes V_{l_2}$ of $U_q(\mathfrak{su}_2)$ can be decomposed as a direct sum of irreducible representations.

The antipode can be expressed for any matrix coefficient $\langle \xi' | \cdot | \xi \rangle$ by $S(\langle \xi' | \cdot | \xi \rangle) = \langle \xi' | S^{-1}(\cdot) | \xi \rangle$. To finish, we remark that from the element e_0 of V_0 one can build the matrix coefficient $X \mapsto \langle e_0, T_0(X)e_0 \rangle$. One see that its coincides with the counit of $U_q(\mathfrak{su}_2)$ and that this is a unital element with respect to the product mentionned above.

One can now easily build a Haar measure on $SU_q(2)$. For this one defines the functional $\phi_{SU_q(2)}$ on $\bigoplus_{l \in \frac{\mathbb{N}}{2}} V_l^* \otimes V_l$ defined by

- $\phi_{SU_q(2)}(1) = 1$,
- $\phi_{SU_q(2)}(f) = 0$, for all $f \in V_l^* \otimes V_l$ where $l \neq 0$.

The invariance property is immediate since $V_l^* \otimes V_l$ are coalgebras and thus $(\text{id} \otimes \phi_{SU_q(2)})$ always vanishes on $\Delta(V_l^* \otimes V_l)$ as soon as $l \neq 0$.

Theorem 1.1.17. *$\phi_{SU_q(2)}$ seen as a functional of $\mathcal{A}(SU_q(2))$, is a Haar functional.*

1.2 Algebraic quantum groups and Pontryagin duality

In Example 3, we considered a discrete group G and noticed that the suitable algebra to associate with G was $c_c(G)$. In this case we still have $c_c(G \times G) = c_c(G) \otimes c_c(G)$. However if one defines the coproduct Δ through

$$\Delta(f)(g, h) = f(gh), \quad g, h \in G,$$

we see that $\Delta(f)$ cannot be finitely supported (unless $f = 0$). For example if $f = \delta_e$ where e is the identity element then the support of $\Delta(f)$ is $\{(g, g^{-1}), g \in G\}$ which is infinite. To overcome this issue, we consider the notion of multiplier Hopf algebra, due to Van Daele [Dae98].

In this section we consider an associative and algebra \mathcal{A} with non-degenerate product. One also supposes that \mathcal{A} is essential i.e., the multiplication induces an isomorphism $\mathcal{A} \otimes_{\mathcal{A}} \mathcal{A}$.

Definition 1.2.1. *A left multiplier of \mathcal{A} is a linear map $\rho : \mathcal{A} \rightarrow \mathcal{A}$ such that $\rho(ab) = \rho(a)b \forall a, b \in \mathcal{A}$. A right multiplier of \mathcal{A} is a linear map $\rho : \mathcal{A} \rightarrow \mathcal{A}$ such that $\rho(ab) = a\rho(b) \forall a, b \in \mathcal{A}$. A multiplier of \mathcal{A} is a pair (ρ_1, ρ_2) of a left and right multiplier so that $\rho_2(a)b = a\rho_1(b) \forall a, b \in \mathcal{A}$.*

When ρ_1 and ρ_2 are linear maps on \mathcal{A} satisfying $\rho_2(a)b = a\rho_1(b) \forall a, b \in \mathcal{A}$, then already (ρ_1, ρ_2) is a multiplier. We denote by $M(\mathcal{A})$ the set of all multipliers of \mathcal{A} . The set $M(\mathcal{A})$ is endowed with its natural (unital) algebra structure.

Remark 1.2.2. Let \mathcal{B} be an essential and non-degenerate algebra. Any morphism $\pi : \mathcal{A} \rightarrow \mathcal{B}$ can be (uniquely) extended to a morphism $\tilde{\pi} : M(\mathcal{A}) \rightarrow M(\mathcal{B})$.

Example 17. If \mathcal{A} is unital then $M(\mathcal{A}) = \mathcal{A}$.

Example 18. Let X be a infinite discrete set. Then $M(c_c(X)) = c(X)$, the algebra of all functions on X .

Definition 1.2.3. *Let us consider the space $\mathcal{A} \otimes \mathcal{A}$ with is a natural algebra structure. A comultiplication is a homomorphism $\Delta : \mathcal{A} \rightarrow M(\mathcal{A} \otimes \mathcal{A})$ such that :*

1. $\Delta(a)(1 \otimes b)$ and $(a \otimes 1)\Delta(b)$ are in $\mathcal{A} \otimes \mathcal{A}$ for all $a, b \in \mathcal{A}$,
2. $(a \otimes 1 \otimes 1)(\Delta \otimes \iota)(\Delta(b)(1 \otimes c)) = (\iota \otimes \Delta)((a \otimes 1)\Delta(b))(1 \otimes 1 \otimes c)$ for all $a, b, c \in \mathcal{A}$.

Definition 1.2.4. *The following maps $\mathcal{A} \otimes M(\mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{A})$,*

$$\begin{aligned} \gamma_l : a \otimes b &\mapsto (\Delta a)(b \otimes 1), \\ \rho_r : a \otimes b &\mapsto (1 \otimes a)(\Delta b) \end{aligned}$$

and

$$\begin{aligned} \gamma_l : a \otimes b &\mapsto (\Delta a)(1 \otimes b), \\ \rho_l : a \otimes b &\mapsto (a \otimes 1)(\Delta b), \end{aligned}$$

are called the Galois maps associated to Δ .

Definition 1.2.5. *An algebra \mathcal{A} endowed with a comultiplication Δ and such that the Galois maps are isomorphisms of $\mathcal{A} \otimes \mathcal{A}$ is called a (regular) multiplier Hopf algebra.*

Remark 1.2.6. A multiplier Hopf algebra always has a counit ϵ and an antipode S . However we cannot define them as we did for Hopf algebras. In the context of multiplier Hopf algebras, the defining properties of the counit and the antipode are

$$\begin{aligned}(\epsilon \otimes \text{id})(\Delta(a)(1 \otimes b)) &= ab, \\ (\text{id} \otimes \epsilon)((a \otimes 1)\Delta(b)) &= ab,\end{aligned}$$

and

$$\begin{aligned}\mu(S \otimes \text{id})(\Delta(a)(1 \otimes b)) &= \epsilon(a)b, \\ \mu(\text{id} \otimes S)((a \otimes 1)\Delta(b)) &= \epsilon(b)a,\end{aligned}$$

for all $a, b \in \mathcal{A}$ and where $\mu : \mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{A}$ designates the multiplication.

One can directly extend the notion of Haar functional from Definition 1.1.9 to multiplier algebras. One can also easily the notion of multiplier $*$ -Hopf algebra with a definition similar to that of Definition 1.1.4.

Definition 1.2.7. *If \mathcal{A} is a multiplier $*$ -Hopf algebra equipped with a Haar functional, we say that \mathcal{A} is an algebraic quantum group.*

Definition 1.2.8. *Let \mathcal{A} be an algebraic quantum group. If \mathcal{A} is unital one says that \mathcal{A} is a compact quantum group. If \mathcal{A} contains an element δ such that*

$$\delta a = \epsilon(a)\delta, \quad \forall a \in \mathcal{A},$$

one says that \mathcal{A} is a discrete quantum group.

Remark 1.2.9. Compact (and discrete) quantum groups can also be presented in another formalism, developed by [Wor87], based on the utilisation of C^* -algebras. The algebras studied in this context are analogous to $C(G)$, the C^* -algebra of continuous functions over a compact group G .

Let \mathcal{A} be an algebraic quantum group with Haar functional ϕ . We define $\hat{\mathcal{A}}$ as the set of functionals on \mathcal{A} of the form $\phi(\cdot a)$, $a \in \mathcal{A}$. We have a pairing between \mathcal{A} and $\hat{\mathcal{A}}$ given by

$$(\phi(\cdot a), b) = \phi(ba), \quad \forall a, b \in \mathcal{A}.$$

Through this pairing, $\hat{\mathcal{A}}$ can be endowed with a multiplier $*$ -Hopf algebra structure, dual to that of \mathcal{A} . Furthermore, with respect to this structure, the functional on $\hat{\mathcal{A}}$ given by $\phi(\cdot a) \mapsto \epsilon(a)$ is a Haar functional. Thus $\hat{\mathcal{A}}$ is an algebraic quantum group, called the *dual quantum group* of \mathcal{A} .

Theorem 1.2.10. [Dae98] *The double quantum dual $\hat{\hat{\mathcal{A}}}$ is isomorphic to \mathcal{A} as an algebraic quantum group.*

Example 19. Let G be an abelian discrete group. The multiplier $*$ -Hopf algebra $c_c(G)$ endowed with the counting measure of G is an algebraic quantum group for which the quantum dual is $\mathcal{O}(\hat{G})$, where \hat{G} is the Pontryagin dual of G . In the context of algebraic quantum groups we will typically use the notation $\mathcal{A}(G) = c_c(G)$ and $\mathcal{A}(\hat{G}) = \mathcal{O}(\hat{G})$.

Example 20. The $*$ -Hopf algebra $\mathcal{A}(SU_q(2))$, endowed with its Haar functional defined in 1.1.17 directly satisfies the axioms of a multiplier Hopf algebra. Its quantum dual is denoted $\mathcal{A}(\widehat{SU_q(2)})$. Let us recall that

$$\mathcal{A}(SU_q(2)) \cong \bigoplus_{l \in \frac{\mathbb{N}}{2}} V_l^* \otimes V_l.$$

One can explicitly describe the algebra structure of $\mathcal{A}(\widehat{SU_q(2)})$ via the following isomorphism of algebras

$$\mathcal{A}(\widehat{SU_q(2)}) \cong \bigoplus_{l \in \frac{\mathbb{N}}{2}} \mathcal{L}(V_l).$$

The coproduct, however, is harder to describe explicitly.

The formalism of algebraic quantum groups is a simple and elegant alternative to the more powerful and technical framework of locally compact quantum groups (see Section 1.4). However it does not apply to many classical groups.

Example 21. Consider the group \mathbb{R} . There does not exist an algebra of functions over the locally compact group \mathbb{R} which can be endowed with an algebraic quantum group structure. The same applies for any locally compact group which is neither discrete nor compact. The notion of locally compact quantum group can be an answer to this issue but we will see that it can also be overcome with the notion of bornological quantum group.

1.3 Complex semisimple quantum groups

1.3.1 Compact semisimple quantum groups and their duals

Let \mathfrak{g} be a complex semisimple Lie algebra. We consider $U_q^{\mathbb{R}}(\mathfrak{k})$, the quantized universal enveloping algebra of the Lie algebra \mathfrak{k} of the compact real form K of the simply connected group G corresponding to \mathfrak{g} [VY20, Section 3.3]. We denote by \mathbf{P} the weight lattice associated to \mathfrak{g} . For all $\lambda \in \mathbf{P}$, the element $K_\lambda \in U_q^{\mathbb{R}}(\mathfrak{k})$ verifies

$$\Delta(K_\lambda) = K_\lambda \otimes K_\lambda, \quad K_\lambda^* = K_\lambda.$$

We denote by ρ the half-sum of positive roots. The Hopf $*$ -algebra generated by the $(K_\lambda)_{\lambda \in \mathbf{P}}$ is denoted by $U_q^{\mathbb{R}}(\mathfrak{t})$. The Hopf algebra $U_q^{\mathbb{R}}(\mathfrak{t})$ is commutative and cocommutative. In fact it is isomorphic to the algebraic quantum group $\mathbb{C}[\mathbf{P}]$. By Pontryagin duality it is isomorphic to $c_c(\hat{\mathbf{P}})$ where $\hat{\mathbf{P}}$ is a torus that we note T .

We recall $\mathcal{A}(K_q)$ is defined as the set of all matrix coefficients of finite dimensional $U_q^{\mathbb{R}}(\mathfrak{k})$ -modules. We have $\mathcal{A}(K_q) \subset U_q^{\mathbb{R}}(\mathfrak{k})^*$. Then $\mathcal{A}(K_q)$ is a $*$ -Hopf algebra and we have

$$\mathcal{A}(K_q) = \bigoplus_{\mu \in \mathbf{P}^+} \text{End}(V(\mu))^*,$$

where $V(\mu)$ is the highest weight $U_q^{\mathbb{R}}(\mathfrak{k})$ -module associated with μ .

The Hopf $*$ -algebra $\mathcal{A}(T) = \mathbb{C}[\mathbf{P}]$ is a sub Hopf algebra of $U_q^{\mathbb{R}}(\mathfrak{t})^*$. $\mathcal{A}(T)$ is spanned by the elements

$$e^\mu : K_\lambda \mapsto q^{(\mu, \lambda)},$$

where μ is an element of \mathbf{P} and (μ, λ) denotes the usual pairing on \mathfrak{h}^* , normalized so that $(\alpha, \alpha) = 2$ for all short root α . We write $(e^\mu, K_\lambda) = q^{(\mu, \lambda)}$. We denote by π_T the Hopf $*$ -morphism $\pi_T : \mathcal{A}(K_q) \rightarrow \mathcal{A}(T)$ induced by the restriction map $U_q^{\mathbb{R}}(\mathfrak{k})^* \rightarrow U_q^{\mathbb{R}}(\mathfrak{t})^*$. We denote by ϕ_T the Haar functional on $\mathcal{A}(T)$ such that $\phi_T(1) = 1$.

As in the previous section, one can define a Haar functional ϕ_{K_q} on $\mathcal{A}(K_q)$ that turns $\mathcal{A}(K_q)$ into an algebraic quantum group. The dual quantum group $\mathcal{A}(\widehat{K}_q)$ of $\mathcal{A}(K_q)$ has the following algebra structure

$$\mathcal{A}(\widehat{K}_q) = \bigoplus_{\mu \in \mathbf{P}^+} \text{End}(V(\mu)).$$

We denote by $\widehat{1}_{K_q}$ the element $\text{id} \in \text{End}(V(0))$. We have chosen this notation because this element happen to be the Fourier transform of the unital element $1_{K_q} \in \mathcal{A}(K_q)$. We have

$$\widehat{1}_{K_q} f = \epsilon_{\widehat{K}_q}(f) \widehat{1}_{K_q},$$

for all $f \in \mathcal{A}(\widehat{K}_q)$. We denote by $\phi_{\widehat{K}_q}$ the right invariant functional on $\mathcal{A}(\widehat{K}_q)$ such that $\phi_{\widehat{K}_q}(\widehat{1}_{K_q}) = 1$.

Remark 1.3.1. The functional $\phi_{\widehat{K}_q}$ can in fact be defined as $\phi_{\widehat{K}_q} : \mathfrak{F}(a) \mapsto \epsilon_{K_q}(a)$, where $a \in \mathcal{A}(K_q)$ and $\mathfrak{F} : \mathcal{A}(K_q) \rightarrow \mathcal{A}(\widehat{K}_q)$ denotes the Fourier transform. One also has

$$\epsilon_{\widehat{K}_q}(\mathfrak{F}(a)) = \phi_{K_q}(a).$$

We introduce some further notation. Note that $\mathcal{A}(T)$ can be identified as the algebra of polynomials over T a group isomorphic to a maximal torus in K . We denote by A_q its

Pontryagin dual \hat{T} , which is a classical group but we write A_q for a reason that will be detailed in Chapter 5. In fact there is a natural isomorphism $A_q \cong \mathbf{P}$. Finally, we write

$$L_q = T \times A_q.$$

We also write $\mathcal{A}(A_q)$ and $\mathcal{A}(A_q \times T)$ for the associated algebras of functions.

To finish we remark that we have $K_\lambda \in M(\mathcal{A}(\widehat{K}_q))$ for all $\lambda \in \mathbf{P}$. Explicitly let $x = \sum_{\mu \in \mathbf{P}^+} x_\mu \in$, one can set $K_\lambda x = \sum_{\mu \in \mathbf{P}^+} K_\lambda x_\mu$ and $x K_\lambda = \sum_{\mu \in \mathbf{P}^+} (x_\mu K_\lambda)$. One can also see that $K_\lambda \in M(\mathcal{A}(\hat{T}))$. More generally for any $\lambda \in \mathfrak{h}^*$ we can define $K_\lambda \in \mathcal{A}(T)^*$ by $(K_\lambda, e^\mu) = q^{-(\lambda, \mu)}$ and then $K_\lambda \in \mathcal{A}(K_q)^*$ by $(K_\lambda, a) = (K_\lambda, \pi_T(a))$ for all $a \in \mathcal{A}(K_q)$. Moreover we have $K_\lambda = K_{\lambda'}$ if and only if $\lambda - \lambda' \in i\hbar^{-1}Q^\vee$ where $\hbar = \frac{\log(q)}{2\pi}$ and Q^\vee denotes the coroot lattice, see [VY20, Section 3.3.1] for details. Thus we have a family of group-like elements $K_\lambda \in M(\mathcal{A}(\widehat{K}_q))$ indexed by $\lambda \in \mathfrak{h}_q^* := \mathfrak{h}^*/i\hbar^{-1}Q^\vee$.

1.3.2 Definition and structure of complex semisimple quantum groups

Let K_q be a compact semisimple quantum group. We will use the notation $\mathcal{A}(K_q)$ for the algebra of functions on K_q and $\mathcal{D}(K_q) = \mathcal{A}(\widehat{K}_q)$ for the dual algebraic quantum group. Through the Fourier map, we have an identification $\mathcal{D}(K_q) = \mathcal{A}(K_q)$. (A more specific meaning will be given to the notation $\mathcal{D}(\mathbb{G})$ in later chapters when we consider a bornological quantum group \mathbb{G}). We define the associated *complex semisimple quantum group* as the Drinfeld double

$$G_q = K_q \bowtie \widehat{K}_q,$$

with respect to the dual pairing between $\mathcal{A}(K_q)$ and $\mathcal{A}(\widehat{K}_q)$.

Precisely, this means that we define the algebra

$$\mathcal{A}(G_q) = \mathcal{A}(K_q) \otimes \mathcal{A}(\widehat{K}_q)$$

equipped with coproduct defined by

$$\Delta_{G_q} = \text{ad}(W_{K_q})_{32} \circ \Sigma_{23} \circ (\Delta_{K_q} \otimes \Delta_{\widehat{K}_q}),$$

where W_{K_q} designates the multiplicative unitary associated to $\mathcal{A}(K_q)$ and Σ is the flip map. One can also express the antipode and counit of $\mathcal{A}(G_q)$ as follows. Let $f \otimes x \in \mathcal{A}(K_q) \otimes \mathcal{D}(K_q)$,

$$\begin{aligned} S_{G_q}(f \otimes x) &= W_{\widehat{K}_q}^{-1}(S(f) \otimes \hat{S}(x))W_{K_q} = (S \otimes \hat{S})(W_{K_q}(f \otimes x)W_{\widehat{K}_q}^{-1}), \\ \epsilon_{G_q} &= \epsilon_{K_q} \otimes \widehat{\epsilon}_{\widehat{K}_q}. \end{aligned}$$

This structure can be made more explicit by the following consideration. Let $(u_{ij}^\sigma) \in \mathcal{A}(K_q)$ denote the matrix coefficient associated to a weight basis of an irreducible representation σ of K_q and let $(\omega_{ij}^\sigma) \in \mathcal{A}(\widehat{K}_q)$ denote the elements of the dual basis. We have

$$W_{K_q} = \sum_{i,j,\sigma} u_{ij}^\sigma \otimes \omega_{ij}^\sigma, \quad W_{K_q}^{-1} = \sum_{i,j,\sigma} S(u_{ij}^\sigma) \otimes \omega_{ij}^\sigma,$$

where the sums run over all equivalent classes of irreducible representations. In practice we only write $W_{K_q} = u_{ij}^\sigma \otimes \omega_{ij}^\sigma$. As a consequence, for an element $a \otimes f \in \mathcal{A}(G_q)$ we have

$$\Delta_{G_q}(a \otimes f) = a_{(1)} \otimes \omega_{ij}^\sigma f_{(1)} \omega_{rs}^\nu \otimes u_{ij}^\sigma a_{(2)} S(u_{rs}^\nu) \otimes f_{(2)}, \quad (1.4)$$

$$S_{G_q}(a \otimes f) = u_{ij}^\sigma S_{K_q}(a) S(u_{rs}^\nu) \otimes \omega_{ij}^\sigma S_{\widehat{K}_q}(f) \omega_{rs}^\nu. \quad (1.5)$$

Proposition 1.3.2. *Consider $\psi_{\widehat{K}_q}$ the right invariant functional on $\mathcal{A}(\widehat{K}_q)$ such that $\psi_{\widehat{K}_q}(\widehat{1}_{K_q}) = 1$. A positive left and right invariant functional on $\mathcal{A}(G_q)$ is given by*

$$\phi_{G_q}(f \otimes x) = \phi_{K_q}(f) \otimes \psi_{\widehat{K}_q}(x),$$

for $f \otimes x \in \mathcal{A}(K_q) \otimes \mathcal{D}(K_q)$.

For a proof, see [VY20, Proposition 4.19].

The dual $\mathcal{D}(G_q)$ of the algebraic quantum group $\mathcal{A}(G_q)$ is given by

$$\mathcal{D}(G_q) = \mathcal{D}(K_q) \bowtie \mathcal{A}(K_q),$$

equipped with tensor product comultiplication. The multiplication of two elements $x \otimes f, y \otimes g \in \mathcal{D}(K_q) \bowtie \mathcal{A}(K_q)$ is given by

$$(x \bowtie f) \cdot (y \bowtie g) = x(y_{(1)}, f_{(1)})y_{(2)} \bowtie f_{(2)}(\hat{S}(y_{(3)}), f_{(3)})g.$$

The pairing between $\mathcal{D}(G_q)$ and $\mathcal{A}(G_q)$ is given by

$$(y \bowtie g, f \otimes x) = (y, f)(x, S^{-1}(g)),$$

using the skew-pairing between $\mathcal{A}(K_q)$ and $\mathcal{D}(K_q)$, see [VY20, p. 219]. Furthermore, the functional $\phi_{\widehat{G}_q}$ on $\mathcal{D}(G_q)$ given by

$$\phi_{\widehat{G}_q}(x \otimes f) = \phi_{\widehat{K}_q}(x) \otimes \phi_{K_q}(f), \quad x \otimes f \in \mathcal{D}(K_q) \bowtie \mathcal{A}(K_q),$$

is a left Haar integral.

For reference, we record also that the C^* -algebra of functions on G_q is given by

$$C_0(G_q) = C(K_q) \hat{\otimes} C^*(K_q),$$

where the C^* -algebras $C(K_q)$ and $C^*(K_q)$ are obtained by closure with respect to the regular representations on the Hilbert space $L^2(K_q)$. We omit the details here, which will be discussed fully in Chapter 3. The structure of $C^*(G_q)$ is harder to describe because of the twisted product in $\mathcal{D}(K_q) \bowtie \mathcal{A}(K_q)$. In this thesis we will be only interested in tempered representations of G_q , which are precisely the principal series representations. In Chapter 5, we will give an explicit expression of the reduced C^* -algebra $C_r^*(G_q)$ in terms of operator algebras on those representations.

1.3.3 The Borel subgroup and its characters

The quantum Borel subgroup B_q of G_q is defined as $B_q = T \bowtie \widehat{K}_q$. Specifically, we consider the element $(\pi_T \otimes \text{id})(W_{K_q}) \in M(\mathcal{A}(T) \otimes \mathcal{D}(K_q))$. It allows us to define a twisted coproduct on the tensor algebra

$$\mathcal{A}(B_q) = \mathcal{A}(T) \otimes \mathcal{D}(K_q),$$

Explicitly we have

$$\begin{aligned} \Delta_{B_q}(a \otimes f) &= a_{(1)} \otimes \omega_{ii}^\sigma f_{(1)} \omega_{rr}^\nu \otimes \pi_T(u_{ii}^\sigma S(u_{rr}^\nu)) a_{(2)} \otimes f_{(2)}, \\ S_{B_q}(a \otimes f) &= \pi_T(u_{ii}^\sigma S(u_{rr}^\nu)) S_T(a) \otimes \omega_{ii}^\sigma S_T(f) \omega_{rr}^\nu, \end{aligned}$$

For all $a \otimes f \in \mathcal{A}(B_q)$. One can check that the map

$$\pi_T \otimes \text{id} : \mathcal{A}(G_q) \rightarrow \mathcal{A}(B_q)$$

is a surjective morphism of multiplier Hopf algebra. Thus, this map identifies B_q as a closed quantum subgroup of G_q .

Finally, let us recall what are the characters of B_q . We define $\mathfrak{h}_q^* = \mathfrak{h}^*/i\hbar^{-1}\mathbf{Q}^\vee$. The characters of $\mathcal{D}(B_q) = \mathcal{D}(T) \bowtie \mathcal{A}(K_q)$ are indexed by $(\mu, \lambda) \in \mathbf{P} \times \mathfrak{h}_q^*$ and defined as

$$\chi_{\mu, \lambda}(x \bowtie f) = (e^\mu, x)(K_\lambda, f).$$

1.3.4 Principal series Representations

As in the classical case, principal series representations of G_q are induced from the characters of the Borel subgroup B_q . Let $(\mu, \lambda) \in \mathbf{P} \times \mathfrak{t}^*$. In the quantum case, as in the classical case, one also has the two usual pictures for the induced representation associated to the parameters (μ, λ) .

The noncompact Picture. We first realise the space of the principal series representation associated to (μ, λ) as a subspace of $M(\mathcal{A}(G_q))$:

$$\text{Ind}_{B_q}^{G_q} \mathbb{C}_{\mu, \lambda} = \{\xi \in M(\mathcal{A}(G_q)) \mid (\text{id} \otimes \pi_{B_q}) \Delta_{G_q}(\xi) = \xi \otimes (e^\mu \otimes K_{2\rho+\lambda})\}.$$

Note that this is the analog of the “ B_q -equivariant functions over G_q ”, relative to the character given by (μ, λ) . As in the classical case, the added term $K_{2\rho}$ is here to ensure the unitarity of the representation.

The restriction of Δ_{G_q} to $\text{Ind}_{B_q}^{G_q} \mathbb{C}_{\mu, \lambda}$ induces a linear map $\text{Ind}_{B_q}^{G_q} \mathbb{C}_{\mu, \lambda} \rightarrow \text{Hom}_{\mathcal{A}(G_q)}(\mathcal{A}(G_q), \text{Ind}_{B_q}^{G_q} \mathbb{C}_{\mu, \lambda} \otimes \mathcal{A}(G_q))$ and this is called a left coaction of $\mathcal{A}(G_q)$ on $\text{Ind}_{B_q}^{G_q} \mathbb{C}_{\mu, \lambda}$. To endow $\text{Ind}_{B_q}^{G_q} \mathbb{C}_{\mu, \lambda}$ with a scalar product, we define the restriction map, denoted res ,

$$\text{res} \equiv \text{id} \otimes \hat{\varepsilon} : \text{Ind}_{B_q}^{G_q} \mathbb{C}_{\mu, \lambda} \rightarrow \mathcal{A}(K_q).$$

The scalar product of two elements $f, g \in \text{Ind}_{B_q}^{G_q} \mathbb{C}_{\mu, \lambda}$ is given by

$$\langle f, g \rangle = \phi(\overline{\text{res}(f)} \text{res}(g)).$$

This scalar product turns $\text{Ind}_{B_q}^{G_q} \mathbb{C}_{\mu, \lambda}$ into a unitary corepresentation of $\mathcal{A}(G_q)$.

The compact Picture. Note that the map res is injective ([VY20, Lemma 5.18]) and thus one can simplify the picture of this representation by considering it on the image of res . It turns out that this space is precisely

$$\Gamma(\mathcal{E}_{\mu, \lambda}) = \{\xi \in \mathcal{A}(K_q) \mid (\text{id} \otimes \pi_T)(\Delta(\xi)) = \xi \otimes e^\mu\}.$$

Once again, we refer to [VY20, Lemma 5.18]. One can also describe the inverse isomorphism $\text{ext} : \Gamma(\mathcal{E}_{\mu, \lambda}) \rightarrow \text{Ind}_{B_q}^{G_q} \mathbb{C}_{\mu, \lambda}$, $\text{ext}(\xi) = \xi \otimes K_{2\rho+\lambda}$. Then one can endow $\Gamma(\mathcal{E}_{\mu, \lambda})$ with a structure of unitary representation through this isomorphism. The coaction of $\mathcal{A}(G_q)$ on $\Gamma(\mathcal{E}_{\mu, \lambda})$ is given by

$$\xi \mapsto (\text{id} \otimes \text{id} \otimes \text{id} \otimes \hat{\varepsilon})(\Delta_{G_q}(\xi \otimes K_{2\rho+\lambda}))$$

Note that there is another approach to build this representation that we don't discuss here. The space $\Gamma(\mathcal{E}_{\mu, \lambda})$, is naturally endowed with a structure of *Yetter-Drinfeld module* for $\mathcal{A}(K_q)$ ([VY20, 5.4.1]), which corresponds to the above coaction of the Drinfeld double $\mathcal{A}(G_q)$ on $\Gamma(\mathcal{E}_{\mu, \lambda})$.

1.4 Locally compact quantum groups

In the section we briefly recall the definition of a locally compact quantum group that can be found in [KV00] and complements on weight theory can be found in [KV99]. Let G be

a locally compact group, as we said earlier in this section, there is no algebra of functions over G with a structure of algebraic quantum groups. Luckily, the C^* -algebra $C_0(G)$ has good properties which in particular allow to define a coproduct

$$\Delta : C_0(G) \rightarrow M(C_0(G) \hat{\otimes} C_0(G)),$$

where $\hat{\otimes}$ refers to the spatial tensor product of C^* -algebras and M designates the algebra of continuous multipliers.

However as soon as G is not compact, a Haar measure on G does not lead to a well defined functional over $C_0(G)$ and this issue is at the root of the difficulty of defining a locally compact quantum group. This problem is nevertheless circumvented using the notion of weight.

1.4.1 Definitions and basic results

Definition 1.4.1. Let A be a C^* -algebra, and let $A_{\geq 0}$ denote the set of positive elements of A . A weight on A is a function $\phi : A_{\geq 0} \rightarrow [0, \infty]$ such that

- $\phi(a_1 + a_2) = \phi(a_1) + \phi(a_2)$, $a_1, a_2 \in A_{\geq 0}$ and
- $\phi(r \cdot a) = r \cdot \phi(a)$, $r \in [0, \infty)$, $a \in A_{\geq 0}$.

Let ϕ be a weight on a C^* -algebra A . We use the following notation:

- $\mathcal{M}_\phi^+ := \{a \in A_{\geq 0} \mid \phi(a) < \infty\}$, which is called the set of all positive ϕ -integrable elements of A .
- $\mathcal{N}_\phi := \{a \in A \mid \phi(a^*a) < \infty\}$, which is called the set of all ϕ -square-integrable elements of A .
- $\mathcal{M}_\phi := \text{Span } \mathcal{M}_\phi^+ = \mathcal{N}_\phi^* \mathcal{N}_\phi$, which is called the set of all ϕ -integrable elements of A .

Let ϕ be a weight on a C^* -algebra A .

- We say that ϕ is *faithful* if and only if $\phi(a) \neq 0$ for each non-zero $a \in A_{\geq 0}$
- We say that ϕ is lower semi-continuous if and only if the set $\{a \in A_{\geq 0} \mid \phi(a) \leq \lambda\}$ is a closed subset of A for every $\lambda \in [0, \infty)$.
- We say that ϕ is *densely defined* if and only if \mathcal{M}_ϕ^+ is a dense subset of $A_{\geq 0}$, or equivalently, if and only if either \mathcal{N}_ϕ or \mathcal{M}_ϕ is a dense subset of A .
- We say that ϕ is *proper* if and only if it is non-zero, lower semi-continuous and densely defined.

Definition 1.4.2. Consider a weight ϕ on a C^* -algebra A . A GNS construction for ϕ is a triple $(H_\phi, \pi_\phi, \Lambda_\phi)$ such that

- H_ϕ is a Hilbert space
- Λ_ϕ is a linear map from \mathcal{N}_ϕ into H_ϕ such that
 1. $\Lambda_\phi(\mathcal{N}_\phi)$ is dense in H_ϕ
 2. We have for every $a, b \in \mathcal{N}_\phi$ that $\langle \Lambda_\phi(a), \Lambda_\phi(b) \rangle = \phi(b^*a)$
- π_ϕ is a representation of A on H_ϕ such that $\pi_\phi(a)\Lambda_\phi(b) = \Lambda_\phi(ab)$, for every $a \in A$ and $b \in \mathcal{N}_\phi$.

Definition 1.4.3. Let A be a C^* -algebra. A one-parameter group on A is a family $\alpha = (\alpha_t)_{t \in \mathbb{R}}$ of $*$ -automorphisms of A that satisfies $\alpha_s \circ \alpha_t = \alpha_{s+t}$ for all $s, t \in \mathbb{R}$. We also impose that for every $a \in A$, the mapping $\mathbb{R} \rightarrow A$ defined by $t \mapsto \alpha_t(a)$ is continuous.

Let us recall that such a one-parameter group α admits an analytic extension, that is there exists a family $\alpha = (\alpha_z)_{z \in \mathbb{C}}$ such that for all $z \in \mathbb{C}$, α_z is a map $D_z \subset A \rightarrow A$ which verifies that for all $a \in A$, $\alpha_z(a) = f(z)$, where $z \mapsto f(z)$ is the analytic extension of the map $t \mapsto \alpha_t(a)$. Furthermore the set $\bigcap_{z \in \mathbb{C}} D_z$ is dense in A .

Definition 1.4.4. Let A be a C^* -algebra and $\phi : A_{\geq 0} \rightarrow [0, \infty]$ a weight on A . We say that ϕ is a K.M.S. weight on A if and only if ϕ is a proper weight on A and there exists a norm-continuous one-parameter group $(\sigma_t)_{t \in \mathbb{R}}$ on A such that ϕ is invariant under σ , i.e., $\phi \circ \sigma_t = \phi$ for all $t \in \mathbb{R}$, and for every $a \in \text{Dom}(\sigma_{i/2})$, we have $\phi(a^*a) = \phi(\sigma_{i/2}(a)(\sigma_{i/2}(a))^*)$.

Definition 1.4.5. A C^* -algebraic (reduced) locally compact quantum group is a pair (A, Δ) , where A is a C^* -algebra and $\Delta : A \rightarrow M(A \hat{\otimes} A)$ is a non-degenerate $*$ -homomorphism, that satisfies the following conditions:

- The comultiplication Δ is coassociative.
- The sets $\{\omega \hat{\otimes} \text{id}(\Delta(a)) \mid \omega \in A^*, a \in A\}$ and $\{\text{id} \hat{\otimes} \omega(\Delta(a)) \mid \omega \in A^*, a \in A\}$ are dense linear subspaces of A .
- There exists a faithful K.M.S. weight ϕ on A that is left-invariant, i.e., $\phi(\omega \otimes \text{id}(\Delta(a))) = \bar{\omega}(1_{M(A)}) \cdot \phi(a)$ for all $\omega \in A^*$ and $a \in \mathcal{M}_\phi^+$. Similarly to the previous sections, ϕ is then called a left Haar state.
- There exists a faithful K.M.S. weight ψ on A that is right-invariant, i.e., $\psi(\text{id} \otimes \omega(\Delta(a))) = \bar{\omega}(1_{M(A)}) \cdot \psi(a)$ for all $\omega \in A^*$ and $a \in \mathcal{M}_\psi^+$; ψ is called a right Haar state.

The definition of a locally compact group can be weakened by introducing the notion of approximate KMS weight. Let $(H_\phi, \pi_\phi, \Lambda_\phi)$ a GNS construction for ϕ .

Definition 1.4.6. Consider a vector $v \in H_\phi$, then we say that v is right bounded with respect to $(H_\phi, \pi_\phi, \Lambda_\phi)$ if there exists a number $M \geq 0$ such that $\|\pi_\phi(x)v\| \leq M\|\Lambda_\phi(x)\|$ for all $x \in \mathcal{N}_\phi$.

Definition 1.4.7. We say that ϕ is approximately KMS is the subspace of right bounded elements is dense in H_ϕ .

Remark 1.4.8. If one replace KMS weight by approximately KMS in Definition 1.4.5, one also obtains a reduced locally compact quantum group.

Examples of locally compact quantum groups can be obtain from algebraic quantum groups. We recall that the main result of [Kus02] is the following. Consider an algebraic quantum group \mathcal{A} . One can build a Hilbert space H and a unitary representation of $m : \mathcal{A} \rightarrow B(H)$ such that the completion $C_0^r \mathcal{A}$ of $m(\mathcal{A})$ in $B(H)$ can be endowed with a comultiplication Δ which extends to comultiplication of \mathcal{A} such that:

Theorem 1.4.9. The pair $(C_0^r \mathcal{A}, \Delta)$ is a locally compact quantum group.

Remark 1.4.10. There also exist a definition of von Neumann algebraic locally compact quantum group very similar to the C^* -algebraic one and that we do not recall here (see [KV00]). These two definitions are thus equivalent theoretically, but each of these two formalisms can be useful for efficiently formulating definitions and results.

For the rest of this section (A, Δ) designates a locally compact quantum group, ϕ a left Haar weight on A , $(H_\phi, \pi_\phi, \Lambda_\phi)$ the GNS construction associated to ϕ and (M, Δ) the von Neumann algebraic quantum group associated to (A, Δ) . We recall that M is the strong closure in $B(H)$ of $\pi_\phi(A)$.

Definition 1.4.11. Let H be a Hilbert space. A multiplicative unitary is a unitary operator W of $H \hat{\otimes} H$ such that

$$W_{23}W_{12} = W_{12}W_{13}W_{23}.$$

Proposition 1.4.12. There exists a unique operator W on $H_\phi \hat{\otimes} H_\phi$ that satisfies

$$W(\Lambda_\phi \hat{\otimes} \Lambda_\phi)(\Delta(y)(x \otimes 1)) = \Lambda(x) \otimes \Lambda(y), \quad \forall x, y \in \mathcal{N}_\phi.$$

W is a multiplicative unitary.

Proposition 1.4.13. Let $x \in M$, we have $\Delta(x) = W^*(1 \otimes x)W$.

Proposition 1.4.14. *The C^* -algebra $\pi(A)$ can be recovered as the norm closure of*

$$\{(\text{id} \hat{\otimes} \omega)(W), \omega \in B(H)_*\}.$$

Definition 1.4.15. *We define the dual quantum group of (A, Δ) as the pair $(\hat{A}, \hat{\Delta})$ defined by*

- \hat{A} is the norm closure of $\{(\omega \hat{\otimes} \text{id})(W), \omega \in B(H)_*\}$,
- $\hat{\Delta}(x) = \Sigma W(x \otimes 1)W^* \Sigma, x \in \hat{A}$,

where Σ is the flip map on $H_\phi \hat{\otimes} H_\phi$.

Remark 1.4.16. There exists a third definition of quantum group, that starts with the definition of a multiplicative unitary. That is we first suppose the existence of such an operator, and we build an algebra with coproduct in a second step as in Proposition 1.4.14. See [Wor12].

The authors of [KV00] proved the Pontryagin duality in this setting. We mention the following result without specifying precisely what the term “isomorphic” refers to.

Theorem 1.4.17. *The double quantum dual $(\hat{\hat{A}}, \hat{\hat{\Delta}})$ is isomorphic to the original quantum group (A, Δ) .*

Definition 1.4.18. *Let B be a C^* -algebra. A unitary corepresentation of (A, Δ) on a C^* - B -module \mathcal{E} is a unitary element $X \in \mathcal{L}(A \hat{\otimes} \mathcal{E})$ satisfying*

$$(\Delta \hat{\otimes} \text{id})(X) = X_{13} X_{23}.$$

We finish this section by introducing the notion of *universal* C^* -algebra of a quantum group and the associated definitions. This notion has been developed in [Kus01].

Definition 1.4.19. *The Fourier algebra of A , denoted by $L^1(A)$ is a subspace of A^* and is defined the norm closure in $B(H)$ of*

$$\{a\phi b^* \mid a, b \in \mathcal{N}_\phi\},$$

where $(a\phi b^*)(x) = \phi(axb^*), x \in A$.

Theorem 1.4.20. *The algebra $L^1(A)$ admits a universal enveloping C^* -algebra denoted A_u and there exists a comultiplication Δ_u on A_u that turns (A_u, Δ_u) into a (full) locally compact quantum group.*

There is a surjective $*$ -homomorphism $\lambda : A^u \rightarrow A$, called the *regular representation* of A^u which intertwines the coproducts. Moreover there exists an element $W^u \in M(A \otimes \hat{A}^u)$ called the universal multiplicative unitary satisfying analogous properties to multiplicative unitary. For details see [Kus01, DKSS12].

Notation : For the rest of this manuscript, a locally compact quantum group will be referred to by a notation of type \mathbb{G} and we then will denote $C_0^r(\mathbb{G})$ its reduced C^* -algebraic version, $C_0^u(\mathbb{G})$ the universal one, $L^\infty(\mathbb{G})$ its von Neumann algebra and $L^1(\mathbb{G}) \cap L^1(\mathbb{G})^*$ the Fourier algebra. Furthermore, we introduce the notations $C_r^*(\mathbb{G})$, $C_u^*(\mathbb{G})$ and $\mathcal{L}(\mathbb{G})$ that stand respectively for $C_0^r(\hat{\mathbb{G}})$, $C_0^u(\hat{\mathbb{G}})$ and $L^\infty(\hat{\mathbb{G}})$.

1.4.2 Morphisms and closed quantum subgroups

Here we present the notion of morphism between locally compact quantum groups. We will see that there is a simpler notion of morphisms in the context of bornological quantum groups and that this definition is compatible with the following (see Section 2.3.5).

Definition-Proposition 1.4.21. [MRW12] *Let \mathbb{G} and \mathbb{H} be locally compact quantum groups. The following objects are in one to one correspondence :*

1. A **homomorphism** from \mathbb{G} and \mathbb{H} , that is, a morphism between the universal function algebras

$$\pi : C_0^u(\mathbb{G}) \rightarrow M(C_0^u(\mathbb{H}))$$

which intertwines the coproducts.

2. A **bicharacter** from \mathbb{G} and \mathbb{H} , that is

$$V \in M(C_0^r(\mathbb{H}) \otimes C_0^r(\hat{\mathbb{G}}))$$

satisfying

$$(\Delta_{\mathbb{H}} \hat{\otimes} \iota)V = V_{13}V_{23}, \quad (\iota \hat{\otimes} \Delta_{\hat{\mathbb{G}}})V = V_{13}V_{12}.$$

Note that in our conventions, the legs of the bicharacter are flipped with respect to those of [MRW12, DKSS12]. The following definition of a closed quantum subgroup is due to Vaes [Vae05]. There is another possible definition due to Woronowicz which is weaker than that of Vaes, see [DKSS12, Definition 3.2 and Theorem 3.5].

Definition 1.4.22. *Let \mathbb{G} be locally compact quantum groups. A closed quantum subgroup of \mathbb{G} in the sense of Vaes is a locally compact quantum group \mathbb{H} which fits into a commuting diagram*

$$\begin{array}{ccc} C_u^*(\mathbb{H}) & \xrightarrow{\hat{\pi}} & M(C_u^*(\mathbb{G})) \\ \lambda_{\mathbb{H}} \downarrow & & \downarrow \lambda_{\mathbb{G}} \\ \mathcal{L}(\mathbb{H}) & \xrightarrow{\hat{\pi}} & \mathcal{L}(\mathbb{G}) \end{array}$$

where the top arrow is an essential morphism of Hopf C^* -algebras, the bottom arrow is an injective normal unital $*$ -homomorphism, and the vertical maps are the regular representations.

Chapter 2

Bornological quantum groups

As we said in introduction, Voigt proposed and succeeded to use bornological analysis to overcome the issue raised in Example 21 and build an axiomatic for quantum groups almost as simple as these of algebraic quantum groups. In particular, by endowing the algebra $C_c^\infty(G)$ with the bornology coming from its structure of LF-space one can build what he defined as a bornological quantum group. Furthermore, the category of bornological quantum groups contains the algebraic ones. Voigt's framework is then an attractive generalisation of Van Daele's framework. However, the case where a bornological quantum group \mathcal{A} is endowed with a $*$ -structure hasn't been described yet. Thus we propose to slightly modify the definition of a bornological quantum group by adding the hypothesis of a $*$ -structure. This will allow us to study the corresponding locally compact quantum groups in the sense of [KV00].

If G is a general locally compact group, it is a priori not easy to find an algebra of functions over G that satisfies the axioms of a bornological quantum group. In particular algebras of type $C_c(G)$ are not compatible with the bornological tensor product (a notion very closed to the topological projective tensor product). However, we have the following structure theorem. This is cited in [Mey04a, Theorem 2.1] and due to Montgomery and Zippin [MZ55].

Theorem 2.0.1. *Let G be an almost connected locally compact group, then G is isomorphic to a projective limit of Lie groups.*

One benefit we have with the bornological framework is the compatibility with inductive limits. Since that for any Lie group H , one can endow $C_c^\infty(H)$ with the bornology associated to its LF-space structure, we can get a natural bornological structure on an algebra associated to G . We also refer to [Mey04b] for the various motivations for

the introduction of the bornological framework for group representation theory and non-commutative geometry.

2.1 Bornological Vector spaces

2.1.1 Basics

Definition 2.1.1. A bornology on a set X is a family \mathcal{B} of subsets of X such that

1. \mathcal{B} is a covering of X , i.e. $\bigcup_{B \in \mathcal{B}} B = X$,
2. \mathcal{B} is hereditary under inclusion, i.e. if $A \in \mathcal{B}$ and $B \subset A$, then $B \in \mathcal{B}$,
3. \mathcal{B} is stable under finite unions.

The pair (X, \mathcal{B}) is called a bornological set and the elements of \mathcal{B} are called the bounded subsets of X .

Definition 2.1.2. Let V be a vector space and A a subset of V . We say that A is circled if $\lambda A \subset A$ for all $\lambda < 1$. We say that A is a disk if it is both convex and circled.

Definition 2.1.3. Let V be a (complex) vector space, a bornology \mathcal{B} on V is said to be a vectorial bornology if

1. for all $A, B \in \mathcal{B}$ and $\lambda \in \mathbb{C}$, $A + B$ and λA are in \mathcal{B} ,
2. \mathcal{B} is stable under the formation of circled hulls, that is, for all $A \in \mathcal{B}$, $\bigcup_{\alpha \leq 1} \alpha A$ belongs to \mathcal{B} .

Let $A \subset V$, the convex hull of A is the smallest convex set of V that contains A . We say that the vector bornology \mathcal{B} is convex if it is stable under the formation of convex hulls.

Definition 2.1.4. A linear map $f : V \rightarrow W$ between two bornological vector spaces is called bounded if for all bounded subsets $B \subset V$, $f(B)$ is a bounded subset of W .

Example 22. Let V be a Banach space, then the family \mathcal{B} of bounded subsets of V (with respect to the norm of V) is a convex bornology on V . Thus, the two notions of bounded linear maps coincide. One can generalize this to any vector space V endowed with a family of seminorms.

Example 23. Let V be a locally convex vector space, a subset X of V is called bounded if it is absorbed by any neighborhood of 0 (that is, for any neighborhood U of 0, there exists $\lambda > 0$ such that $X \subset \lambda U$). The collection of all bounded subset of V is a convex bornology on V . This is called the Von Neumann bornology of V .

Remark 2.1.5. For any convex bornological vector space V one can build a locally convex topology on V , called the “bornological topology” ([HN77, Chapter IV]). In the case where V is a metrizable locally convex space, The study of V equipped with its Von Neumann bornology or with its topology are essentially equivalent (cf. [Mey04b]).

Example 24. Let V be a locally convex vector space, the family of precompact subset of V is a convex bornology on V , called the Precompact bornology. The vector space V endowed with this bornology is denoted $\mathbf{Comp}(V)$.

Example 25. Let V be any complex vector space, the family of compact subset of finite-dimensional vector spaces is a convex bornology on V . We refer to it as the **fine**-bornology and we will see that it allows to include Van Daele’s framework into the bornological one.

One can find many more examples in [HN77].

Let V and W be bornological spaces. We denote by $\text{Hom}(V, W)$ the set of bounded linear maps from V to W .

Definition 2.1.6. *Let L be a subset of $\text{Hom}(V, W)$. We said that L is equibounded if for all bounded set S in V , $\bigcup_{l \in L} l(S)$ is bounded in W .*

Proposition 2.1.7. *The family of all equibounded subsets of $\text{Hom}(V, W)$ is a convex bornology.*

In the rest of this thesis, $\text{Hom}(V, W)$ is always endowed with the equibounded bornology. In particular we will write V^* for $\text{Hom}(V, \mathbb{C})$

2.1.2 Convergence and Completeness

Definition 2.1.8. *Let V be a vector space and X a subset of V . We define the function $p_X : V \rightarrow [0, \infty]$ by*

$$p_X(v) = \inf\{r \in \mathbb{R} : r > 0 \mid r^{-1}v \in X\} \quad \text{for every } v \in V.$$

We call p_X the gauge of the set X . If X is convex, p_X is a semi-norm on V .

Definition 2.1.9. *Let V be a bornological vector space and A a bounded disk in V . We say that A is completant if the space $V_A = \text{Span}(A)$, equipped with the gauge semi-norm associated to A is a Banach space.*

Definition 2.1.10. *We say that V is complete if any bounded subset S is contained in a completant bounded disk.*

Definition 2.1.11. Let $(v_n)_{n \in \mathbb{N}}$ be a sequence in a bornological space V . We say that (v_n) converges bornologically towards a vector $v \in V$ if there exists a sequence $(\lambda_n)_{n \in \mathbb{N}} \in \mathbb{C}$ converging toward 0 and a bounded subset S of V such that $(v_n - v)$ belongs to $\lambda_n S$ for all $n \in \mathbb{N}$.

Definition-Proposition 2.1.12. Let V be a bornological space. There exists a complete bornological space V^c and a bounded linear map $\sharp : V \rightarrow V^c$ with the following universal property: For any complete bornological space W and bounded map $l : V \rightarrow W$, there exists a bounded map $l^c : V^c \rightarrow W$ such that $l = l^c \circ \sharp$.

The bornological space V^c is called the completion of V .

Remark 2.1.13. In all that follows, we only consider complete convex bornologies and refer to a *bornological vector space* as a complex vector space endowed with a complete convex vector bornology.

2.1.3 The category of bornological spaces

There exists a tensor product in the category of bornological spaces, defined similarly to the projective tensor product for locally convex spaces. More precisely for two bornological spaces V and W , one can endow the algebraic tensor product $V \otimes W$ with a bornology such that for any bornological space X , bounded bilinear maps $V \times W \rightarrow X$ correspond canonically to bounded linear maps $V \otimes W \rightarrow X$.

Definition 2.1.14. The bornological tensor product of V and W , denoted $V \hat{\otimes} W$ is defined as the bornological completion of $V \otimes W$.

Proposition 2.1.15. The bornological tensor product is associative and commutative and there is a natural adjunction isomorphism

$$\mathrm{Hom}(V \hat{\otimes} W, X) \cong \mathrm{Hom}(V, \mathrm{Hom}(W, X)),$$

for all bornological vector spaces V, W, X .

The bornological tensor product is compatible with the topological projective tensor product in the following sense (This result is formulated in the following way in [Voi08, Theorem 2.1] and is originally due to Grothendieck. [Gro54]).

Theorem 2.1.16. Let V and W be Fréchet spaces and let $V \hat{\otimes}_\pi W$ be their completed projective tensor product. Then there is a natural isomorphism

$$\mathbf{Comp}(V) \hat{\otimes} \mathbf{Comp}(W) \cong \mathbf{Comp}(V \hat{\otimes}_\pi W)$$

of bornological vector spaces.

The following result can be found in [HN77, Section 2:8].

Proposition 2.1.17. *Let I be a directed partially ordered set and $(V_i, u_{ji})_{i \leq j \in I}$ be an inductive system of bornological spaces. We denote by V the vector space limit of this system and $u_i : V_i \rightarrow V$ the associated maps. We denote \mathcal{B}_i the bornology on each V_i . Then $\mathcal{B} = \bigcup_{i \in I} u_i(\mathcal{B}_i)$ is a bornology on V and (V, \mathcal{B}) is called the bornological inductive limit.*

2.1.4 The approximation property

The approximation property originally refers to a property of certain locally convex spaces and a related notion has been introduced for bornological vector spaces. It is one of the hypotheses required to get a bornological quantum group. Before giving a definition, we introduce some terminology.

Let V be a bornological vector space. A subset S of V is *precompact* if there exists a compact disk A such that S is a precompact subset of the Banach space V_A . Recall that for any Banach space E , the space $\text{Hom}(E, V)$ carries the equibounded bornology ([HN77]).

Definition 2.1.18. *We say that V has the approximation property if for any compact disk A , there exists a sequence (f_n) of finite rank operators on V such that (f_n) converges to id in $\text{Hom}(V_A, V)$.*

Example 26. If V carries the **fine**-bornology one can see that a disk A can be compact only if it is contained in a finite dimensional vector space. Thus V_A is finite dimensional and we see that the approximation property is trivial for V .

In order to show that classical Lie groups are bornological, we cite the following theorem ([Mey04b, Theorem 5.11]).

Theorem 2.1.19. *Let V be a Fréchet space. The following are equivalent :*

1. V has the approximation property as a locally convex vector space.
2. V endowed with the Precompact bornology has the approximation property.

We also need the following result.

Proposition 2.1.20. *If V is the limit of an countable inductive system (V_i) of bornological spaces with the approximation property, then V has the approximation property.*

It is well known that for any compact Lie group K , the Fréchet space $C^\infty(K)$ has the approximation property, as well as $C_c^\infty(G)$ for any Lie group.

2.2 Bornological algebras and multipliers

Definition 2.2.1. A bornological algebra \mathcal{A} is a bornological vector space with a bounded multiplication $\mathcal{A} \hat{\otimes} \mathcal{A} \rightarrow \mathcal{A}$. If \mathcal{A} is a $*$ -algebra such that the $*$ -involution is bounded then one calls a bornological $*$ -algebra.

A bornological module is *essential* if every bounded set of \mathcal{V} is the image of a bounded set in $\mathcal{A} \hat{\otimes} \mathcal{V}$. A bornological algebra is *essential* if it is essential as a module over itself. A bounded morphism of bornological algebras $\phi : \mathcal{A} \rightarrow \mathcal{B}$ is *essential* if it makes \mathcal{B} into an essential \mathcal{A} -module.

A *multiplier* of a bornological algebra is a pair of bounded maps $c = (c_l, \cdot c_r)$ from \mathcal{A} to itself satisfying

$$c_l \cdot (ab) = (c_l \cdot a)b, \quad (ab) \cdot c_r = a(b \cdot c_r), \quad (a \cdot c_r)b = a(c_l \cdot b),$$

for all $a, b \in \mathcal{A}$. The multipliers form a bornological algebra $M(\mathcal{A})$, with the bornology restricted from $\text{End}(\mathcal{A}) \oplus \text{End}(\mathcal{A})$, and \mathcal{A} sits in $M(\mathcal{A})$ as an ideal. We may thus suppress the dots and the subscripts l and r in the notation. For details, see [Voi08].

Remark 2.2.2. The notation $M(\mathcal{A})$ could be confusing since it could refer to general multipliers and not only the bounded ones. However $M(\mathcal{A})$ will always refer to the set of bounded multipliers.

It is an important fact that if \mathcal{A} and \mathcal{B} are essential bornological algebras then any essential morphism $\phi : \mathcal{A} \rightarrow \mathcal{B}$ extends uniquely to a morphism on the multipliers $\phi : M(\mathcal{A}) \rightarrow M(\mathcal{B})$. We shall use this frequently without mention.

A bornological algebra equipped with a bounded antilinear involution is called a *bornological $*$ -algebra*. If \mathcal{A} is a bornological $*$ -algebra then so is $M(\mathcal{A})$.

We will make an unconventional choice of notation here. As we will describe later, the space of functions on a bornological quantum group is equipped with two distinct $*$ -structures, associated to the pointwise product and the convolution product, respectively, and it will be important to distinguish them. We will therefore use the notation $a \mapsto \bar{a}$ for the “pointwise adjoint” and $a \mapsto a^*$ for the “convolution adjoint”. In particular, the reader should keep in mind that $\overline{\bar{a}b} = \bar{b}a$.

We equip the tensor product $\mathcal{A} \hat{\otimes} \mathcal{B}$ of two bornological $*$ -algebras with the involution defined by

$$\overline{(a \otimes b)} = \bar{a} \otimes \bar{b}.$$

Definition 2.2.3. Let ϕ be functional on \mathcal{A} . We say that ϕ is positive if $\phi(a^*a) \geq 0$ for all $a \in \mathcal{A}$.

Proposition 2.2.4. *Let ϕ be a bounded positive linear functional on \mathcal{A} . We have that $\phi_{\mathbb{G}}(a^*) = \overline{\phi_{\mathbb{G}}(a)}$, for all $a \in \mathcal{A}(\mathbb{G})$ and we have the Cauchy-Schwartz inequality*

$$|\phi_{\mathbb{G}}(b^*c)|^2 \leq \phi_{\mathbb{G}}(b^*b)\phi_{\mathbb{G}}(c^*c),$$

for all $b, c \in \mathcal{A}(\mathbb{G})$.

Proof. For any $b, c \in \mathcal{A}(\mathbb{G})$ and $\lambda \in \mathbb{C}$ we calculate

$$0 \leq \phi_{\mathbb{G}}((b - \lambda c)^*(b - \lambda c)) = \phi_{\mathbb{G}}(b^*b) - \lambda\phi_{\mathbb{G}}(b^*c) - \bar{\lambda}\phi_{\mathbb{G}}(c^*b) + |\lambda|^2\phi_{\mathbb{G}}(c^*c).$$

Thus $\lambda\phi_{\mathbb{G}}(b^*c) + \bar{\lambda}\phi_{\mathbb{G}}(c^*b) \in \mathbb{R}$ for all $\lambda \in \mathbb{C}$ and thus $\phi_{\mathbb{G}}(b^*c) = \overline{\phi_{\mathbb{G}}(c^*b)}$. We replace c with c^* in this equality and use the essentialness of $\mathcal{A}(\mathbb{G})$ to conclude the first statement. For the Cauchy-Schwartz inequality just put $\lambda = \phi_{\mathbb{G}}(c^*b)/\phi_{\mathbb{G}}(c^*c)$. \square

Let \mathcal{A} be an essential bornological. We call a linear map $\rho : \mathcal{A} \rightarrow \mathcal{A}$ a left multiplier if $\rho(ab) = \rho(a)b$ for all $a, b \in \mathcal{A}$. A right multiplier is a linear map $\rho : \mathcal{A} \rightarrow \mathcal{A}$ a left multiplier if $\rho(ab) = a\rho(b)$ for all $a, b \in \mathcal{A}$. A multiplier of \mathcal{A} is a pair (ρ_1, ρ_2) such that ρ_1 is a left multiplier, ρ_2 is a right multiplier and $\rho_2(a)b = a\rho_1(b)$. We denote by $M(\mathcal{A})$ the set of all bounded multipliers of \mathcal{A} . The space $M(\mathcal{A})$ is an associative algebra and endowed with the bornology of $\text{Hom}(H, H)$, it becomes an essential bornological algebra. Furthermore we have a bounded natural inclusion of \mathcal{A} into $M(\mathcal{A})$ whose image is a two sided ideal.

Definition 2.2.5. *We say that a left multiplier $m : \mathcal{A} \rightarrow \mathcal{A}$ is adjointable if there exists a left multiplier $m^* : \mathcal{A} \rightarrow \mathcal{A}$ such that $(m^*a)^*b = a(mb)$, $\forall a, b \in \mathcal{A}$.*

Lemma 2.2.6. *Let $m : \mathcal{A} \rightarrow \mathcal{A}$ be a left multiplier of bornological $*$ -algebra. The following are equivalent*

1. m is adjointable.
2. m is a two-sided multiplier.

Proof. Given a left multiplier m , we define the right multiplier m' such that $m'(a) = (m^*(a^*))^*$. Then (m, m') is a two sided multiplier, which we again refer to as m . This proves (1) \implies (2). For the converse, the same formulas allows us to define m^* in terms of m' . \square

2.3 Bornological quantum groups

A *coproduct* on a bornological $*$ -algebra \mathcal{A} is an essential bounded $*$ -homomorphism $\Delta : \mathcal{A} \rightarrow M(\mathcal{A} \hat{\otimes} \mathcal{A})$ which is coassociative, meaning $(\text{id} \hat{\otimes} \Delta)\Delta = (\Delta \hat{\otimes} \text{id})\Delta$ as maps from \mathcal{A} to

$M(\mathcal{A} \hat{\otimes} \mathcal{A} \hat{\otimes} \mathcal{A})$, and such that the *Galois maps*

$$\gamma_l : a \otimes b \mapsto (\Delta a)(b \otimes 1), \quad \rho_r : a \otimes b \mapsto (1 \otimes a)(\Delta b)$$

are bounded linear maps from $\mathcal{A} \hat{\otimes} \mathcal{A}$ to itself. We write $\Delta^{\text{cop}} = \Sigma \circ \Delta$ for the co-opposite comultiplication, where Σ denotes the flip map. We note that Voigt [Voi08] does not impose the condition on the Galois maps in his definition of a bornological coproduct, although he does require it as a hypothesis in all his successive results.

We also define the maps

$$\gamma_r : a \otimes b \mapsto (\Delta a)(1 \otimes b), \quad \rho_l : a \otimes b \mapsto (a \otimes 1)(\Delta b),$$

as well as the variants γ_l^{op} , γ_l^{cop} , $\gamma_l^{\text{op,cop}}$, *etc.*, in which we replace the multiplication by m^{op} and/or the comultiplication by Δ^{cop} . The resulting sixteen maps from $\mathcal{A} \hat{\otimes} \mathcal{A} \rightarrow \mathcal{A} \hat{\otimes} \mathcal{A}$ will all be referred to as *Galois maps*. They all map $\mathcal{A} \hat{\otimes} \mathcal{A}$ into itself because they can all be related to γ_l and ρ_r via the flip maps and conjugation by the involution.

This condition on the Galois maps allows us to define, for any $a \in \mathcal{A}$ and $\omega \in \mathcal{A}^*$, a multiplier $(\text{id} \hat{\otimes} \omega)(\Delta(a)) \in M(\mathcal{A})$ by

$$\begin{aligned} b \cdot (\text{id} \hat{\otimes} \omega)(\Delta(a)) &= (\text{id} \hat{\otimes} \omega)((b \otimes 1)\Delta(a)), \\ (\text{id} \hat{\otimes} \omega)(\Delta(a)) \cdot b &= (\text{id} \hat{\otimes} \omega)(\Delta(a)(b \otimes 1)), \end{aligned}$$

where $b \in \mathcal{A}$. We can define the multiplier $(\omega \hat{\otimes} \text{id})(\Delta(a)) \in M(\mathcal{A})$ similarly.

Some notational remarks are in order. Firstly, if $\omega \in \mathcal{A}^*$ and $b, c \in \mathcal{A}$, we will use the notation $b\omega c$ for the linear functional $a \mapsto \omega(cab)$. This notation will be generalized to linear functionals on other algebras. Secondly, to simplify formulas, we will often use Sweedler notation for the coproduct, writing

$$\Delta(a) = a_{(1)} \otimes a_{(2)},$$

where $a \in \mathcal{A}$. For classical Hopf algebras, this can be understood as a summation convention, but here it is a purely formal notation. That is, the terms $a_{(1)}$ and $a_{(2)}$ have no meaning on their own, but are only placeholders for the position of a coproduct in the legs of the multipliers of the bornological tensor product $M(\mathcal{A} \hat{\otimes} \mathcal{A})$. Thus, for instance, we can write the Galois maps as

$$\gamma_l(a \otimes b) = a_{(1)} b \otimes a_{(2)}, \quad \gamma_l^{\text{cop}}(a \otimes b) = a_{(2)} b \otimes a_{(1)}, \quad \text{etc.}$$

We extend this to iterated coproducts in the usual way, writing

$$\Delta^{(n)}(a) = a_{(1)} \otimes \cdots \otimes a_{(n+1)}.$$

Thanks to the fact that all Galois maps have image in $\mathcal{A} \hat{\otimes} \mathcal{A}$, given any elements $a, b_1, \dots, b_{n-1} \in \mathcal{A}$ the product $b_1 a_{(1)} \otimes b_2 a_{(2)} \otimes \dots \otimes a_{(i)} \otimes \dots \otimes b_{n-1} a_{(n)}$ belongs to $\mathcal{A}^{\hat{\otimes} n}$, where exactly one of the legs $a_{(i)}$ is not multiplied by an element b_i of \mathcal{A} . The same is true if any number of the b_i is multiplied on the right instead of the left.

A coproduct Δ on a bornological $*$ -algebra \mathcal{A} is said to satisfy the *cancellation property* if the Galois maps γ_l and ρ_r are linear bornological isomorphisms from $\mathcal{A} \hat{\otimes} \mathcal{A}$ to itself. Once again, this implies all sixteen Galois maps are linear bornological isomorphisms from $\mathcal{A} \hat{\otimes} \mathcal{A}$ to itself.

A *left-invariant integral* on an essential bornological $*$ -algebra \mathcal{A} with coproduct is a bounded linear functional $\phi \in \mathcal{A}^*$ such that

$$(\text{id} \hat{\otimes} \phi)(\Delta(a)) = \phi(a)1$$

for all $a \in \mathcal{A}$. Similarly, a *right-invariant integral* is $\psi \in \mathcal{A}^*$ such that

$$(\psi \hat{\otimes} \text{id})(\Delta(a)) = \psi(a)1.$$

The following theorem is due to Voigt, see [Voi08, Section 3].

Theorem 2.3.1. *Let \mathcal{A} be a bornological $*$ -algebra equipped with a coproduct Δ and a positive faithful left invariant integral ϕ . The following are equivalent:*

- (i) Δ satisfies the cancellation property,
- (ii) there exists a bounded essential homomorphism $\epsilon : \mathcal{A} \rightarrow \mathbb{C}$, called the counit, and a bounded algebra antiautomorphism coalgebra antiautomorphism $S : \mathcal{A} \rightarrow \mathcal{A}$, called the antipode, satisfying the following Hopf-type axioms: For all $a, b \in \mathcal{A}(G)$,

$$(\epsilon \hat{\otimes} \text{id})(\Delta(a)) = a = (\text{id} \hat{\otimes} \epsilon)(\Delta(a))$$

and

$$\mu(S \hat{\otimes} \text{id})(\Delta(a)(1 \otimes b)) = \epsilon(a)b, \quad \mu(\text{id} \hat{\otimes} S)((a \otimes 1)\Delta(b)) = \epsilon(b)a.$$

In this case, the maps ϵ and S are uniquely defined and satisfy

$$\epsilon(\bar{a}) = \overline{\epsilon(a)}, \quad S(\bar{a}) = \overline{S^{-1}(a)}.$$

Proof. The only new point here is the compatibility with the involution. Define the map $\bar{\epsilon} : \mathcal{A} \rightarrow \mathbb{C}$ by $\bar{\epsilon}(a) = \overline{\epsilon(\bar{a})}$. Then

$$(\bar{\epsilon} \hat{\otimes} \text{id})(\Delta(a)) = \overline{(\epsilon \hat{\otimes} \text{id})(\Delta(\bar{a}))} = a = \overline{(\text{id} \hat{\otimes} \epsilon)(\Delta(\bar{a}))} = (\text{id} \hat{\otimes} \bar{\epsilon})(\Delta(a)).$$

By uniqueness of the counit we have $\bar{\epsilon} = \epsilon$. Similarly, if we define $\bar{S} : \mathcal{A} \rightarrow \mathcal{A}$ by $\bar{S}(a) = \overline{S^{-1}(\bar{a})}$ then \bar{S} satisfies the same properties as the counit S . \square

We note that the properties of ϵ and S cited in Theorem 2.3.1 extend to the situation where a is a multiplier and $b \in \mathcal{A}$.

We can now define a bornological quantum group (with involution) by adding an involution to Voigt's definition [Voi08], and requiring positivity of the invariant integral.

Definition 2.3.2. *A bornological quantum group algebra is a bornological $*$ -algebra \mathcal{A} satisfying the equivalent conditions of Theorem 2.3.1.*

As usual, we will use the notation $\mathcal{A} = \mathcal{A}(\mathbb{G})$ when the algebra is to be thought of as the algebra of functions on some quantum group \mathbb{G} .

Example 27. Let \mathcal{A} be an algebraic quantum group. If we endow \mathcal{A} with the **fine**-bornology, then \mathcal{A} is a bornological quantum group. Indeed in this case, every linear map is bounded and the bornological tensor product coincides with the algebraic one.

Example 28. Let K be a compact Lie group. As we have seen, $C^\infty(K)$, endowed with the precompact bornology associated to its Fréchet space structure, has the approximation property. Furthermore, because of the nuclearity of the Fréchet space $C^\infty(K)$, we have that

$$C^\infty(K) \hat{\otimes}_\pi C^\infty(K) \cong C^\infty(K \times K).$$

and thus it follows that

$$\mathbf{Comp}(C^\infty(K)) \hat{\otimes} \mathbf{Comp}(C^\infty(K)) \cong \mathbf{Comp}(C^\infty(K \times K)).$$

If G is a (non-compact) Lie group, we have

$$\mathbf{Comp}(C_c^\infty(G)) \hat{\otimes} \mathbf{Comp}(C_c^\infty(G)) \cong \mathbf{Comp}(C_c^\infty(G \times G)).$$

Proposition 2.3.3. *Let $(\mathcal{A}(\mathbb{G}_i), \Delta_{\mathbb{G}_i})_i$ an inductive system of bornological quantum groups. Then the limit bornological algebra, endowed with the limit application Δ is a bornological quantum group.*

2.3.1 Modular properties of the integral

Let $\mathcal{A} = \mathcal{A}(\mathbb{G})$ be a bornological quantum group. It is shown in [Voi08, Proposition 5.4], following [Dae98, Proposition 3.8], that there exists a unique multiplier $\delta_{\mathbb{G}} \in M(\mathcal{A})$, called the *modular element*, such that

$$(\phi \hat{\otimes} \text{id})(\Delta(a)) = \phi(a) \delta_{\mathbb{G}}$$

for all $a \in \mathcal{A}$. It is group-like, so that

$$\Delta(\delta_{\mathbb{G}}) = \delta_{\mathbb{G}} \otimes \delta_{\mathbb{G}}, \quad \epsilon(\delta_{\mathbb{G}}) = 1, \quad S(\delta_{\mathbb{G}}) = \delta_{\mathbb{G}}^{-1}.$$

Following the proof of [KvD97, Lemma 3.3], one sees that $\delta_{\mathbb{G}}$ is strictly positive in the sense that, for all nonzero $a \in \mathcal{A}$,

$$\phi(a^* \delta_{\mathbb{G}} a) > 0. \quad (2.1)$$

Hence $\delta_{\mathbb{G}} = \delta_{\mathbb{G}}^*$.

The Haar integral and the modular element are related by

$$\phi(a \delta_{\mathbb{G}}) = \phi(S(a))$$

for all $a \in \mathcal{A}$. The proof of this is essentially the same as for [Dae98, Proposition 3.10]. Applying this twice gives $\phi(S^2(a)) = \phi(\delta_{\mathbb{G}}^{-1} a \delta_{\mathbb{G}})$, and since $\phi \circ S^2$ is again a left-invariant integral, we have $\phi(S^2(a)) = \mu \phi(a)$ for some scalar $\mu \in \mathbb{C}$, called the *scaling constant*. It is possible to show that $|\mu| = 1$. De Commer and Van Daele have shown that we always have $\mu = 1$ in the case of algebraic quantum groups, see [DCVD10, Theorem, 3.4]. At present, we do not know if this is true for bornological quantum groups.

To simplify the exposition, **we will assume in this thesis that $\mu = 1$** , since all the examples we have in mind satisfy this assumption. As a consequence we have $\phi(\delta_{\mathbb{G}} a) = \phi(a \delta_{\mathbb{G}})$ for all $a \in \mathcal{A}$. The situation $\mu \neq 1$ would not add any particular difficulties, following the same methods as in [KvD97].

There is a unique bounded algebra automorphism $\sigma : \mathcal{A} \rightarrow \mathcal{A}$ such that $\phi(ab) = \phi(b\sigma(a))$ for all $a, b \in \mathcal{A}$, see [Dae98, Proposition 3.12] and [Voi08, Proposition 5.3]. This continues to hold when one of a or b is a multiplier, and by taking $b = 1$ we have that ϕ is invariant under σ . Our assumption that the scaling constant is 1 implies that

$$\sigma(\delta_{\mathbb{G}}) = \delta_{\mathbb{G}}.$$

We record some further basic properties of σ .

Proposition 2.3.4. *For all $a \in \mathcal{A}$ we have*

$$\begin{aligned} \sigma(S(\sigma(a))) &= \delta_{\mathbb{G}}^{-1} S(a) \delta_{\mathbb{G}}, & \sigma^{-1}(S(\sigma^{-1}(a))) &= \delta_{\mathbb{G}} S(a) \delta_{\mathbb{G}}^{-1}, \\ S^2(\sigma(a)) &= \sigma(S^2(a)), & \sigma(\bar{a}) &= \overline{\sigma^{-1}(a)}. \end{aligned}$$

Proof. For any $a, b \in \mathcal{A}$, we have

$$\begin{aligned} \phi(b\sigma(S(\sigma(a)))) &= \phi(S(\sigma(a))b) = \phi(S^{-1}(b)\sigma(a)\delta_{\mathbb{G}}) \\ &= \phi(a\delta_{\mathbb{G}}S^{-1}(b)) = \phi(b\delta_{\mathbb{G}}^{-1}S(a)\delta_{\mathbb{G}}). \end{aligned}$$

This proves the first equality. The second follows by pre- and post-composing with σ^{-1} , and the third then follows by composing the first two. The final equality follows from

$$\phi(b\sigma(\bar{a})) = \phi(\bar{a}b) = \overline{\phi(\bar{b}a)} = \overline{\phi(\sigma^{-1}(a)\bar{b})} = \phi(b\overline{\sigma^{-1}(a)}).$$

□

The map σ is not generally a coalgebra automorphism. Instead, we have the following property.

Proposition 2.3.5. *We have*

$$\Delta \circ \sigma = (S^2 \otimes \sigma) \circ \Delta = (\sigma \otimes \alpha) \circ \Delta,$$

where α is the bounded algebra automorphism defined by $\alpha(a) = \delta_{\mathbb{G}}^{-1} S^{-2}(a) \delta_{\mathbb{G}}$.

Proof. Let $a, b, c \in \mathcal{A}(\mathbb{G})$. Using the invariance of the Haar integral, we have

$$\begin{aligned} (\phi \hat{\otimes} \phi)((b \otimes c) \Delta(\sigma(a))) &= \phi(bS(c_{(1)})) \phi(c_{(2)} \sigma(a)) \\ &= \phi(bS(c_{(1)})) \phi(ac_{(2)}) \\ &= \phi(bS^2(a_{(1)})) \phi(a_{(2)}c) \\ &= \phi(bS^2(a_{(1)})) \phi(c\sigma(a_{(2)})), \end{aligned}$$

which proves the first equality. For the second, we calculate

$$\begin{aligned} \phi((b \otimes c) \Delta(\sigma(a))) &= \phi(b_{(1)} \sigma(a)) \phi(cS^{-1}(b_{(2)}) \delta_{\mathbb{G}}) \\ &= \phi(ab_{(1)}) \phi(b_{(2)} S(c)) \\ &= \phi(a_{(1)} b) \phi(S^{-1}(a_{(2)}) \delta_{\mathbb{G}} S(c)) \\ &= \phi(b\sigma(a_{(1)})) \phi(c\delta_{\mathbb{G}}^{-1} S^{-2}(a_{(2)}) \delta_{\mathbb{G}}). \end{aligned}$$

□

One can also consider the automorphism σ' associated to the right-invariant integral $\phi \circ S$, that is, $\phi(S(ab)) = \phi(S(b\sigma'(a)))$. We get immediately that

$$\sigma'(a) = \delta_{\mathbb{G}} \sigma(a) \delta_{\mathbb{G}}^{-1} = \sigma(\delta_{\mathbb{G}} a \delta_{\mathbb{G}}^{-1}) = S^{-1}(\sigma^{-1}(S(a))). \quad (2.2)$$

2.3.2 Pontryagin duality

Let $\mathcal{A} = \mathcal{A}(\mathbb{G})$ be a bornological quantum group. We write \hat{a} or $\mathfrak{F}(a)$ for the bounded linear functional $\hat{a} : b \mapsto \phi(ba)$. The Pontryagin dual, denoted $\hat{\mathcal{A}}$ or $\mathcal{A}(\hat{\mathbb{G}})$, is the space of bounded linear forms

$$\mathcal{A}(\hat{\mathbb{G}}) = \{\hat{a} \mid a \in \mathcal{A}(\mathbb{G})\} \subset \mathcal{A}(\mathbb{G})^*$$

equipped with the bornology inherited from the bijection $\mathfrak{F} : \mathcal{A} \rightarrow \hat{\mathcal{A}}$ and the Hopf operations defined by skew-duality, namely, for $a, b \in \mathcal{A}$ and $x, y \in \hat{\mathcal{A}}$

$$\begin{aligned} (xy, a) &= (x \otimes y, \Delta(a)) & (\hat{\Delta}(x), a \otimes b) &= (x, ba) \\ \hat{\epsilon}(x) &= (x, 1) & (\hat{1}, a) &= \epsilon(a) \\ (\hat{S}(x), a) &= (x, S^{-1}(a)) & (\hat{S}^{-1}(x), a) &= (x, S(a)) \\ (x^*, a) &= \overline{(x, S(a))}, & (x, \bar{a}) &= \overline{(\hat{S}^{-1}(x)^*, a)}. \end{aligned}$$

Note that we are using \bar{a} for the involution of $a \in \mathcal{A}(\mathbb{G})$ and x^* for the involution of $x \in \mathcal{A}(\hat{\mathbb{G}})$. The left Haar integral $\hat{\phi}$ on $\mathcal{A}(\hat{\mathbb{G}})$ is given by

$$\hat{\phi}(\mathfrak{F}(a)) = \epsilon(a).$$

The proof that $\mathcal{A}(\hat{\mathbb{G}})$ is indeed a bornological quantum group with this structure is done in [Voi08, Theorem 7.5], with the exception of the $*$ -structure. We will confirm that the $*$ -structure is compatible with the quantum group structure on $\mathcal{A}(\hat{\mathbb{G}})$ in Proposition 2.3.6 below.

Using the linear isomorphism \mathcal{F} we can transfer the Hopf operations from $\mathcal{A}(\hat{\mathbb{G}})$ to $\mathcal{A}(\mathbb{G})$, Specifically, we introduce the convolution product and convolution adjoint on $\mathcal{A}(\mathbb{G})$,

$$f * g := f_{(1)}\phi(S^{-1}(g)f_{(2)}) = \phi(S^{-1}(g_{(1)})f)g_{(2)}, \quad (2.3)$$

$$f^* = \overline{S(f)}\delta_{\mathbb{G}}. \quad (2.4)$$

Then one can verify the following formulas for the dual operations:

$$\begin{aligned} \mathfrak{F}(f)\mathfrak{F}(g) &= \mathfrak{F}(f * g), & \mathfrak{F}(f)^* &= \mathfrak{F}(f^*), \\ \hat{\epsilon}(\mathfrak{F}(f)) &= \phi(f), & \hat{S}(\mathfrak{F}(f)) &= \mathfrak{F}(\sigma(\delta_{\mathbb{G}}S(f))). \end{aligned}$$

Proposition 2.3.6. *The involution $*$ defined on $\mathcal{A}(\hat{\mathbb{G}})$ by the duality relations above makes $\mathcal{A}(\hat{\mathbb{G}})$ into a bornological quantum group in the sense of Definition 2.3.2.*

Proof. Using [Voi08, Theorem 7.5], we only need to check the compatibility of the involution. We see from the formula (2.4) that the convolution adjoint maps $\mathcal{A}(\mathbb{G})$ to $\mathcal{A}(\mathbb{G})$, so the involution is well-defined on $\mathcal{A}(\hat{\mathbb{G}})$. The fact that $x \mapsto x^*$ is a bounded involutive antilinear algebra anti-automorphism and coalgebra auto-morphism is straightforward, positivity and faithfulness of the left invariant integral $\hat{\phi}$ follows from the following well-known formula. \square

Lemma 2.3.7. *For any $f, g \in \mathcal{A}(\mathbb{G})$ we have $\epsilon(f^* * g) = \phi(\bar{f}g)$.*

Proof. We have $\epsilon(f^* * g) = \epsilon(g_{(2)})\phi(S^{-1}(g_{(1)})S^{-1}(\bar{f})\delta_{\mathbb{G}}) = \phi(S^{-1}(\bar{f}g)\delta_{\mathbb{G}}) = \phi(\bar{f}g)$. \square

We shall write $\mathcal{D}(\mathbb{G})$ for the linear space $\mathcal{A}(\mathbb{G})$ equipped with the Hopf operations pulled back from $\mathcal{A}(\hat{\mathbb{G}})$ via \mathcal{F} . In particular, as a $*$ -algebra, $\mathcal{D}(\mathbb{G})$ is equipped with the convolution product (2.3) and convolution adjoint (2.4) above, while the counit on $\mathcal{D}(\mathbb{G})$ is $\hat{\epsilon} = \phi$ and the antipode on $\mathcal{D}(\mathbb{G})$ is given by

$$\hat{S}(f) = \sigma(\delta_{\mathbb{G}}S(f)).$$

From this and Proposition 2.3.4, we get the notable formula

$$\hat{S}^2(f) = S^2(f). \quad (2.5)$$

We record the following compatibility between the pointwise coproduct and the convolution product.

Lemma 2.3.8. *For any $f, g \in \mathcal{A}(G)$ we have the formal equalities*

$$\Delta(f * g) = f_{(1)} \otimes (f_{(2)} * g) = (f * g_{(1)}) \otimes g_{(2)}.$$

More precisely, for any $a \in \mathcal{A}(G)$ we have

$$\begin{aligned} (a \otimes 1)\Delta(f * g) &= af_{(1)} \otimes (f_{(2)} * g) & \Delta(f * g)(a \otimes 1) &= f_{(1)}a \otimes (f_{(2)} * g) \\ (1 \otimes a)\Delta(f * g) &= (f * g_{(1)}) \otimes ag_{(2)} & \Delta(f * g)(1 \otimes a) &= (f * g_{(1)}) \otimes g_{(2)}a, \end{aligned}$$

where the right hand side of the first equation is understood by first applying a Galois map to $a \otimes f$ and then taking the convolution with g in the second leg, and similarly for the others.

Proof. We calculate

$$\begin{aligned} (a \otimes 1)\Delta(f * g) &= (a \otimes 1)\Delta(f_{(1)} \phi(S^{-1}(g)f_{(2)})) \\ &= (\text{id} \hat{\otimes} \text{id} \otimes \hat{\phi})(af_{(1)} \otimes f_{(2)} \otimes S^{-1}(g)f_{(3)}) \\ &= (af_{(1)} \otimes f_{(2)}) * (\hat{1} \otimes g), \end{aligned}$$

where $\hat{1}$ denotes the unit in the convolution algebra $M(\mathcal{D}(G))$. The other equalities are similar. \square

2.3.3 Modular properties of the dual quantum group and Radford's S^4 formula

From now on, we will write $\delta_{\mathbb{G}}, \sigma_{\mathbb{G}}$, etc. for the modular element and modular automorphism of \mathbb{G} , and $\delta_{\hat{\mathbb{G}}}, \sigma_{\hat{\mathbb{G}}}$ for those of $\mathcal{D}(\mathbb{G}) \cong \mathcal{A}(\hat{\mathbb{G}})$. We can give explicit formulas for the modular automorphisms of $\hat{\mathbb{G}}$.

Proposition 2.3.9. *Let $f \in \mathcal{D}(\mathbb{G})$. We have*

$$\sigma_{\hat{\mathbb{G}}}(f) = S^2(f)\delta_{\mathbb{G}}^{-1}, \quad \sigma'_{\hat{\mathbb{G}}}(f) = \delta_{\mathbb{G}}^{-1}S^{-2}(f)$$

Proof. Let $f, g \in \mathcal{D}(\mathbb{G})$. On the one hand we have

$$\phi_{\hat{\mathbb{G}}}(f * g) = \epsilon(f * g) = \phi_{\mathbb{G}}(S^{-1}(g)f)$$

and on the other hand

$$\begin{aligned} \phi_{\hat{\mathbb{G}}}(g * (S^2(f)\delta_{\mathbb{G}}^{-1})) &= \phi_{\mathbb{G}}(S^{-1}(S^2(f)\delta_{\mathbb{G}}^{-1})g) \\ &= \phi_{\mathbb{G}}(\delta_{\mathbb{G}}S(f)g) \\ &= \phi_{\mathbb{G}}(S^{-1}(g)f), \end{aligned}$$

which leads to the first equality. For the second equality we can dualize the identity (2.2) to obtain $\sigma'_{\hat{\mathbb{G}}} = \hat{S}^{-1}\sigma_{\mathbb{G}}^{-1}\hat{S}$. Hence,

$$\begin{aligned} \sigma'_{\hat{\mathbb{G}}}(f) &= S^{-1}(\delta_{\mathbb{G}}^{-1}\sigma_{\mathbb{G}}^{-1}(S^{-2}(\sigma_{\mathbb{G}}(\delta_{\mathbb{G}}S(f))\delta_{\mathbb{G}}))) \\ &= \delta_{\mathbb{G}}^{-1}S^{-3}(\delta_{\mathbb{G}}S(f))\delta_{\mathbb{G}} \\ &= \delta_{\mathbb{G}}^{-1}S^{-2}(f). \end{aligned}$$

□

Dualizing this formula and using Equation (2.5) yields the following.

Corollary 2.3.10. *For $f \in \mathcal{A}(\mathbb{G})$, we have*

$$\begin{aligned} \sigma_{\mathbb{G}}(f) &= S^2(f) * \delta_{\mathbb{G}}^{-1}, \\ \sigma'_{\hat{\mathbb{G}}}(f) &= \delta_{\hat{\mathbb{G}}}^{-1} * S^{-2}(f). \end{aligned}$$

Proposition 2.3.11. *The left and right actions of $\delta_{\mathbb{G}}$ and $\delta_{\hat{\mathbb{G}}}$ on $\mathcal{A}(\mathbb{G})$ by multiplication and convolution, respectively, all commute.*

Proof. The fact that left and right multiplication by $\delta_{\mathbb{G}}$ commute is obvious, as is the commutativity of left and right convolution by $\delta_{\hat{\mathbb{G}}}$. Using Corollary 2.3.10 we have, for all $f \in \mathcal{A}(\mathbb{G})$

$$\delta_{\hat{\mathbb{G}}} * f = S^{-2}(\sigma_{\mathbb{G}}^{-1}(f)), \quad f * \delta_{\hat{\mathbb{G}}} = S^2(\sigma_{\mathbb{G}}^{-1}(f)).$$

Therefore, noting that $\sigma_{\mathbb{G}}(\delta_{\mathbb{G}}) = \sigma'_{\mathbb{G}}(\delta_{\mathbb{G}}) = \delta_{\mathbb{G}}$, we obtain

$$\delta_{\hat{\mathbb{G}}} * (\delta_{\mathbb{G}}f) = S^{-2}(\sigma_{\mathbb{G}}^{-1}(\delta_{\mathbb{G}}f)) = \delta_{\mathbb{G}}(\delta_{\hat{\mathbb{G}}} * f).$$

The calculations for other combinations of actions are similar. □

Remark 2.3.12. If the scaling constant μ is not 1, these operators will commute up to a scalar, and moreover left and right convolution by $\hat{\delta}_{\mathbb{G}}$ will commute on the nose with the conjugation operator $f \mapsto \delta_{\mathbb{G}} f \delta_{\mathbb{G}}^{-1}$. This shows that the proof of the next theorem remains valid even if the scaling constant is not 1.

One can now generalize Radford's S^4 formula to bornological quantum group. See [DVDW06] for a discussion about this formula in the algebraic case.

Theorem 2.3.13. (*Radford's S^4 formula*) *Let $f \in \mathcal{A}(\mathbb{G})$, we have*

$$S^4(f) = \delta_{\mathbb{G}}(\delta_{\hat{\mathbb{G}}}^{-1} * f * \delta_{\hat{\mathbb{G}}})\delta_{\mathbb{G}}^{-1}.$$

Proof. Consider $g = S^2(f)$. We have $\sigma_{\mathbb{G}}(g) = \delta_{\mathbb{G}}\sigma'_{\mathbb{G}}(g)\delta_{\mathbb{G}}^{-1}$ and thus

$$S^4(f) * \delta_{\hat{\mathbb{G}}}^{-1} = \delta_{\mathbb{G}}(\delta_{\hat{\mathbb{G}}}^{-1} * f)\delta_{\mathbb{G}}^{-1}.$$

Since the actions of $\delta_{\hat{\mathbb{G}}}$ and $\delta_{\mathbb{G}}$ commute, we are done. \square

2.3.4 The bornological multiplicative unitary

Amongst the sixteen Galois maps and their inverses, one is particularly favoured. This choice, called the *multiplicative unitary*, comes from conventions fixed by Baaj and Skandalis in their foundational work on analytical quantum groups [BS93]. Here we give the bornological version.

Definition 2.3.14. *The bornological multiplicative unitary is the linear bornological isomorphism*

$$\begin{aligned} \mathcal{W} &= (\rho_l^{\text{op}})^{-1} : \mathcal{A}(G) \hat{\otimes} \mathcal{A}(G) \rightarrow \mathcal{A}(G) \hat{\otimes} \mathcal{A}(G) \\ a \otimes b &\mapsto S^{-1}(b_{(1)})a \otimes b_{(2)}, \end{aligned}$$

with inverse

$$\mathcal{W}^{-1} : a \otimes b \mapsto \Delta(b)(a \otimes 1) = b_{(1)}a \otimes b_{(2)}.$$

Proposition 2.3.15. *The bornological multiplicative unitary is a unitary multiplier of the algebra $\mathcal{A}(\mathbb{G}) \hat{\otimes} \mathcal{D}(\mathbb{G})$, in the sense that*

$$(\mathcal{W}(a \otimes b))^* \bullet (c \otimes d) = (a \otimes b)^* \bullet \mathcal{W}^{-1}(c \otimes d), \quad (2.6)$$

where \bullet and $*$ denote the product and involution in $\mathcal{A}(G) \hat{\otimes} \mathcal{D}(G)$.

Proof. First, we check that \mathcal{W}^{-1} is right $\mathcal{A}(G) \hat{\otimes} \mathcal{D}(G)$ -linear. Using Lemma 2.3.8, we calculate

$$\begin{aligned} \mathcal{W}^{-1}((a \otimes b) \bullet (c \otimes d)) &= \Delta(b * d)(ac \otimes 1) \\ &= (\Delta(b) \cdot (1 \otimes d))(ac \otimes 1) \\ &= \mathcal{W}^{-1}(a \otimes b) \bullet (c \otimes d). \end{aligned}$$

Thus \mathcal{W} is a left multiplier. Using the fact that $\phi(\cdot \delta)$ is a right invariant integral, we obtain

$$\begin{aligned} (\mathcal{W}^{-1}(a \otimes b))^* \bullet (\mathcal{W}^{-1}(c \otimes d)) &= (\Delta(b)(a \otimes 1))^* \bullet (\Delta(d)(c \otimes 1)) \\ &= \overline{ab}_{(1)} d_{(1)} c \otimes (S^{-1}(\overline{b}_{(2)}) \delta * d_{(2)}) \\ &= \overline{ab}_{(1)} d_{(1)} c \otimes \phi(S^{-1}(d_{(2)}) S^{-1}(\overline{b}_{(2)}) \delta) d_{(3)} \\ &= \overline{a} S(S^{-1}(\overline{b}_{(1)} d_{(1)})) c \otimes \phi(S^{-1}(\overline{b}_{(2)} d_{(2)}) \delta) d_{(3)} \\ &= \overline{a} c \otimes \phi(S^{-1}(\overline{b}_{(1)} d_{(1)}) \delta) d_{(2)} \\ &= \overline{a} c \otimes \phi(S^{-1}(d_{(1)}) b^*) d_{(2)} \\ &= \overline{a} c \otimes b^* * d \\ &= (a \otimes b)^* \bullet (c \otimes d). \end{aligned}$$

This proves that the left multiplier \mathcal{W}^{-1} admits \mathcal{W} as an adjoint in the sense of Equation (2.6). It follows that \mathcal{W} is a two-sided multiplier, since we can define the associated right multiplier by

$$(a \otimes b) \cdot \mathcal{W} = (\mathcal{W}^{-1} \cdot (a \otimes b)^*)^*.$$

This completes the proof. \square

The bornological multiplicative unitary \mathcal{W} satisfies the pentagonal equation

$$\mathcal{W}_{12} \mathcal{W}_{13} \mathcal{W}_{23} = \mathcal{W}_{23} \mathcal{W}_{12}$$

and the bicharacter properties

$$(\Delta \hat{\otimes} \text{id}) \mathcal{W} = \mathcal{W}_{13} \mathcal{W}_{23}, \quad (\text{id} \hat{\otimes} \hat{\Delta}) \mathcal{W} = \mathcal{W}_{13} \mathcal{W}_{12}, \quad (2.7)$$

Let us record two further relations concerning the bornological multiplicative unitary.

Lemma 2.3.16. *Considering \mathcal{W} as a linear automorphism of $\mathcal{A}(G) \hat{\otimes} \mathcal{A}(G)$, we have*

$$(\sigma \hat{\otimes} \sigma) \mathcal{W} = \mathcal{W}(\sigma \hat{\otimes} \alpha),$$

where α is the automorphism $\alpha : a \mapsto \delta^{-1} S^{-2}(a) \delta$ defined in Proposition 2.3.5. Moreover,

$$(\alpha \hat{\otimes} \alpha) \mathcal{W} = \mathcal{W}(\alpha \hat{\otimes} \alpha).$$

Proof. Let $a, b \in \mathcal{A}(G)$. According to Proposition 2.3.5, we have

$$\begin{aligned} \mathcal{W}^{-1}(\sigma \hat{\otimes} \sigma)(a \otimes b) &= \Delta(\sigma(b))(\sigma(a) \otimes 1) \\ &= (\sigma \hat{\otimes} \alpha)(\Delta(b)(a \otimes 1)) = (\sigma \hat{\otimes} \alpha)\mathcal{W}^{-1}(a \otimes b), \end{aligned}$$

which proves the first equality. The second follows from the fact that α is a Hopf morphism (though not a Hopf *-morphism). \square

2.3.5 Morphisms and closed subgroups

Definition 2.3.17. Let \mathbb{G} and \mathbb{H} be bornological quantum groups. A morphism of bornological quantum groups from \mathbb{H} to \mathbb{G} is an essential *-algebra morphism $\pi : \mathcal{A}(\mathbb{G}) \rightarrow M(\mathcal{A}(\mathbb{H}))$ which intertwines the coproducts:

$$\Delta_{\mathbb{H}} \circ \pi = (\pi \hat{\otimes} \pi) \circ \Delta_{\mathbb{G}}$$

If π maps $\mathcal{A}(\mathbb{G})$ surjectively onto $\mathcal{A}(\mathbb{H})$, then we call \mathbb{H} a closed quantum subgroup of \mathbb{G} . In this case we write $\pi = \pi_{\mathbb{H}}$ and refer to it as the restriction map.

Any morphism $\pi : \mathcal{A}(\mathbb{G}) \rightarrow M(\mathcal{A}(\mathbb{H}))$ of bornological quantum groups automatically respects the antipode and counit:

$$S_{\mathbb{H}} \circ \pi = \pi \circ S_{\mathbb{G}}, \quad \epsilon_{\mathbb{G}} = \epsilon_{\mathbb{H}} \circ \pi,$$

see Proposition 4.7 of [Voi08].

Proposition 2.3.18. For any morphism of bornological quantum groups π from \mathbb{H} to \mathbb{G} , there is a unique dual morphism $\hat{\pi}$ from $\hat{\mathbb{G}}$ to $\hat{\mathbb{H}}$ determined by

$$(\hat{\pi}(x), a) = (x, \pi(a))$$

for all $x \in \mathcal{A}(\hat{\mathbb{H}})$ and $a \in \mathcal{A}(\mathbb{G})$.

Proof. The well-definedness of $\hat{\pi}$ is Proposition 8.4 of [Voi08]. The compatibility of $\hat{\pi}$ with the involutions follows from duality with $\mathcal{A}(\mathbb{G})$ and $\mathcal{A}(\mathbb{H})$. \square

Chapter 3

From bornological to locally compact quantum groups

Our goal in this chapter is to make clear the compatibility of bornological quantum groups with the general framework of locally compact quantum group, as recalled in Section 1.4. That is, given a bornological quantum group as described in the previous chapter, we show that it gives rise to a uniquely determined locally compact C^* -algebraic quantum group. We also discuss the important issue of closed quantum subgroups.

The results from the early part of this chapter are mostly bornological generalisations of known results on algebraic quantum groups, particularly those in the article [KvD97]. We have however significantly altered their approach to the complex power of the modular element which, combined with Radford S^4 formula, are used to streamline the proofs. The final section on closed quantum subgroups is to our knowledge, new even in the context of algebraic quantum groups.

We maintain the notation of the previous chapter. In particular, the left invariant Haar functional on $\mathcal{A}(\mathbb{G})$ is denoted $\phi_{\mathbb{G}}$. We also recall that we are using $f \mapsto \bar{f}$ to denote the “pointwise involution” on $\mathcal{A}(\mathbb{G})$ and $f \mapsto f^* = \overline{S(f)}\delta_{\mathbb{G}}$ for the “convolution involution” on $\mathcal{D}(\mathbb{G}) = \mathcal{A}(\hat{\mathbb{G}})$.

3.1 The left regular representation : Construction of $C_0^r(\mathbb{G})$

We fix a GNS pair $(L^2(\mathbb{G}), \Lambda)$ associated to $\phi_{\mathbb{G}}$. This means that $L^2(\mathbb{G})$ is a Hilbert space with a linear map $\Lambda : \mathcal{A}(\mathbb{G}) \rightarrow L^2(\mathbb{G})$ such that $\Lambda(\mathcal{A}(\mathbb{G}))$ is a dense subspace and we have

$$\langle \Lambda(f), \Lambda(g) \rangle_{L^2(\mathbb{G})} = \phi_{\mathbb{G}}(\bar{f}g) = \epsilon(f^* * g), \quad \forall f, g \in \mathcal{A}(\mathbb{G}).$$

The second equality is from Lemma 2.3.7.

Remark 3.1.1. The map $\Lambda : \mathcal{A}(\mathbb{G}) \rightarrow L^2(\mathbb{G})$ is bounded with respect to the von Neumann bornology of $L^2(\mathbb{G})$. This is because the map $\|\cdot\| \circ \Lambda$, which maps $a \in \mathcal{A}(\mathbb{G})$ to $\phi_{\mathbb{G}}(a^*a)^{\frac{1}{2}}$, is bounded as the composition of bounded maps $a \mapsto a \otimes a^* \mapsto a^*a \mapsto \phi_{\mathbb{G}}(a^*a)^{\frac{1}{2}}$.

We denote by m the left action of $\mathcal{A}(\mathbb{G})$ on $\Lambda(\mathcal{A}(\mathbb{G})) \subset L^2(\mathbb{G})$ by multiplication and by λ the left action of $\mathcal{D}(\mathbb{G})$ by convolution, that is

- $m(f)\Lambda(g) = \Lambda(fg)$,
- $\lambda(f)\Lambda(g) = \Lambda(f * g)$.

Our first goal in this section is to show that densely defined operators $m(f)$, $f \in \mathcal{A}(\mathbb{G})$ extend to bounded operators on $L^2(\mathbb{G})$. This will be done by looking at the multiplicative unitary on $L^2(\mathbb{G}) \otimes L^2(\mathbb{G})$. First, note that $\Lambda \times \Lambda : \mathcal{A}(\mathbb{G}) \times \mathcal{A}(\mathbb{G}) \rightarrow L^2(\mathbb{G}) \otimes L^2(\mathbb{G})$ is a bounded bilinear map and thus extends to a bounded map $\Lambda \hat{\otimes} \Lambda : \mathcal{A}(\mathbb{G}) \hat{\otimes} \mathcal{A}(\mathbb{G}) \rightarrow L^2(\mathbb{G}) \otimes L^2(\mathbb{G})$.

Proposition 3.1.2. *There exists a unique unitary operator W of $L^2(\mathbb{G}) \otimes L^2(\mathbb{G})$ s.t. $W(\Lambda \hat{\otimes} \Lambda)(\Delta(g)(f \otimes 1)) = \Lambda(f) \otimes \Lambda(g)$, for all $f, g \in \mathcal{A}(\mathbb{G})$. It is a multiplicative unitary on $L^2(\mathbb{G})$ in the sense that $W_{12}W_{13}W_{23} = W_{23}W_{12}$.*

Proof. First, by the hypothesis on the Galois maps, it is clear that this operator W is well defined and invertible on $\Lambda \hat{\otimes} \Lambda(\mathcal{A}(\mathbb{G}) \hat{\otimes} \mathcal{A}(\mathbb{G}))$. To check the unitarity let $a \otimes b$, $c \otimes d \in \mathcal{A}(\mathbb{G}) \otimes \mathcal{A}(\mathbb{G})$ and observe that

$$\begin{aligned} \langle \Delta(b)(a \otimes 1), \Delta(d)(c \otimes 1) \rangle &= \phi_{\mathbb{G}} \hat{\otimes} \phi_{\mathbb{G}}((\bar{a} \otimes 1)\Delta(\bar{b}d)(c \otimes 1)) \\ &= \phi_{\mathbb{G}}(\bar{a}c)\phi_{\mathbb{G}}(\bar{b}d) \\ &= \langle a \otimes b, c \otimes d \rangle. \end{aligned}$$

Recall that the bornological multiplicative unitary \mathcal{W} belongs to $M(\mathcal{A}(\mathbb{G}) \hat{\otimes} \mathcal{D}(\mathbb{G}))$. From Lemma 2.3.7, the inner product on $L^2(\mathbb{G}) \otimes L^2(\mathbb{G})$ is given by

$$\langle \Lambda(a) \otimes \Lambda(b), \Lambda(c) \otimes \Lambda(d) \rangle = (\phi_{\mathbb{G}} \hat{\otimes} \epsilon)((\Lambda \hat{\otimes} \Lambda)((\bar{a} \otimes b^*) \bullet (c \otimes d))),$$

for all $a, c \in \mathcal{A}(\mathbb{G})$, $b, d \in \mathcal{D}(\mathbb{G})$, where \bullet denotes the product in $\mathcal{A}(\mathbb{G}) \otimes \mathcal{D}(\mathbb{G})$. It follows from Proposition 2.3.15 that the densely defined operator

$$(m \otimes \lambda)(\mathcal{W}) : \Lambda \hat{\otimes} \Lambda(a \otimes b) \mapsto \Lambda \hat{\otimes} \Lambda(\mathcal{W}(a \otimes b))$$

extends to a unitary operator W on $L^2(\mathbb{G})$ with the stated properties. \square

Given $\xi, \eta \in L^2(\mathbb{G})$, we denote by $\omega_{\xi, \eta}$ the state on $B(L^2(\mathbb{G}))$ given by

$$\omega_{\xi, \eta}(T) = \langle \xi, T\eta \rangle.$$

This will allow us to define the left and right slices of the multiplicative unitary.

Lemma 3.1.3. *For any $f, g \in \mathcal{A}(\mathbb{G})$, the endomorphism $m((\text{id} \hat{\otimes} \phi_{\mathbb{G}})(\mathcal{W}^{-1}(f \otimes g)))$ of $\Lambda(\mathcal{A}(\mathbb{G}))$ extends to a bounded operator of $L^2(\mathbb{G})$. Explicitly, it extends to the left slice $(\text{id} \hat{\otimes} \omega_{\Lambda(\bar{g}), \Lambda(f)})(W)$, where $\omega_{\Lambda(\bar{g}), \Lambda(f)} : T \mapsto \langle \Lambda(\bar{g}), T\Lambda(f) \rangle$, $T \in B(L^2(\mathbb{G}))$.*

Proof. A straightforward calculation, as in [KvD97, Lemma 2.3], shows that for all $h \in \mathcal{A}(\mathbb{G})$ we have

$$(\text{id} \hat{\otimes} \omega_{\Lambda(f), \Lambda(g)})(W)\Lambda(h) = \Lambda((\text{id} \hat{\otimes} \phi_{\mathbb{G}})(\Delta(\bar{f})(1 \otimes g))h). \quad (3.1)$$

On the right hand side we have $m(\mathcal{W}^{-1}(g \otimes \bar{f}))\Lambda(h)$ and on the other side the operator acting on $\Lambda(h)$ is $(\text{id} \hat{\otimes} \omega_{\Lambda(f), \Lambda(g)})(W)$, and this is a bounded operator. \square

Proposition 3.1.4. *The left regular representation $m : \mathcal{A}(\mathbb{G}) \rightarrow \text{End}(\Lambda(\mathcal{A}(\mathbb{G})))$ extends to a bounded $*$ -representation $m : \mathcal{A}(\mathbb{G}) \rightarrow B(L^2(\mathbb{G}))$.*

Proof. The bilinear map $(f, g) \mapsto (\text{id} \hat{\otimes} \omega_{\Lambda(f), \Lambda(g)})(W)$ from $\mathcal{A}(\mathbb{G}) \times \mathcal{A}(\mathbb{G})$ into $B(L^2(\mathbb{G}))$ is clearly bounded. Thus it extends to $\mathcal{A}(\mathbb{G}) \hat{\otimes} \mathcal{A}(\mathbb{G})$. Let $x \in \mathcal{A}(\mathbb{G})$ such that $\phi_{\mathbb{G}}(x) = 1$. For all $a \in \mathcal{A}(\mathbb{G})$, using Lemma 3.1.3, one can obtain $m(a)$ as the composition of bounded maps

$$a \mapsto a \otimes x \mapsto \mathcal{W}(a \otimes x) \xrightarrow{m \circ (\text{id} \hat{\otimes} \phi_{\mathbb{G}}) \circ \mathcal{W}^{-1}} m(a).$$

\square

Definition 3.1.5. *We define the reduced C^* -algebra of functions on \mathbb{G} , denoted $C_0^r(\mathbb{G})$, as the closure of $m(\mathcal{A}(\mathbb{G}))$ in $B(L^2(\mathbb{G}))$.*

Proposition 3.1.6. *We have that $\{(\text{id} \hat{\otimes} \omega_{\Lambda(f), \Lambda(g)})(W) \mid f, g \in \mathcal{A}(\mathbb{G})\} = m(\mathcal{A}(\mathbb{G}))$.*

Proof. We have that \mathcal{W}^{-1} is an isomorphism of $\mathcal{A}(\mathbb{G}) \hat{\otimes} \mathcal{A}(\mathbb{G})$ into itself and because $\text{id} \hat{\otimes} \phi_{\mathbb{G}} : \mathcal{A}(\mathbb{G}) \hat{\otimes} \mathcal{A}(\mathbb{G}) \rightarrow \mathcal{A}(\mathbb{G})$ is surjective we obtain that $\mathcal{A}(\mathbb{G}) = \{(\text{id} \hat{\otimes} \phi_{\mathbb{G}})((\Delta(\bar{a})(1 \otimes b)) \mid a, b \in \mathcal{A}(\mathbb{G})\}$. Thus the result follows from Equation (3.1). \square

We also derive from (3.1) the following result

Proposition 3.1.7. *The C^* -algebra $C_0^r(\mathbb{G})$ is the norm closure in $B(L^2(\mathbb{G}))$ of $\{(\text{id} \hat{\otimes} \omega)(W) \mid \omega \in B(L^2(\mathbb{G}))_*\}$.*

Definition 3.1.8. *We define the mapping Δ from $C_0^r(\mathbb{G})$ into $B(L^2(\mathbb{G}) \otimes L^2(\mathbb{G}))$ such that $\Delta(x) = W^*(1 \otimes x)W$.*

The proof of the following result can be readily adapted from the proof of the corresponding result in the algebraic case [KvD97, Theorem 2.11].

Theorem 3.1.9. *We have that $C_0^r(\mathbb{G})$ is a non-degenerate C^* -subalgebra of $B(L^2(\mathbb{G}))$ and Δ is a non-degenerate injective $*$ -homomorphism from $C_0^r(\mathbb{G})$ to $M(C_0^r(\mathbb{G}) \otimes C_0^r(\mathbb{G}))$ such that:*

- $(\Delta \hat{\otimes} \text{id}) \circ \Delta = (\text{id} \hat{\otimes} \Delta) \circ \Delta$.
- *The vector spaces $\Delta(C_0^r(\mathbb{G}))(C_0^r(\mathbb{G}) \otimes 1)$ and $\Delta(C_0^r(\mathbb{G}))(1 \otimes C_0^r(\mathbb{G}))$ are dense subsets of $C_0^r(\mathbb{G}) \otimes C_0^r(\mathbb{G})$.*

A similar construction yields the regular representation λ of $\mathcal{D}(\mathbb{G})$, as follows.

Proposition 3.1.10. *For any $x \in \mathcal{D}(\mathbb{G})$, $\lambda(x)$ extends to a bounded operator on $L^2(\mathbb{G})$. Explicitly, if $f, g \in \mathcal{A}(\mathbb{G})$ we have*

$$(\omega_{\Lambda(f), \Lambda(g)} \hat{\otimes} \text{id})W = \lambda(g \sigma_{\mathbb{G}}(\bar{f})).$$

The resulting map $\lambda : \mathcal{D}(\mathbb{G}) \rightarrow B(L^2(\mathbb{G}))$ is a bounded $$ -representation.*

Proof. This is another standard calculation. For any $a, b \in \mathcal{A}(\mathbb{G})$ we have

$$\begin{aligned} \langle \lambda(a), ((\omega_{\Lambda(f), \Lambda(g)} \hat{\otimes} \text{id})W)\Lambda(b) \rangle &= \langle \Lambda \hat{\otimes} \Lambda(f \otimes a), W\Lambda \hat{\otimes} \Lambda(g \otimes b) \rangle \\ &= (\phi_{\mathbb{G}} \hat{\otimes} \phi_{\mathbb{G}})(S^{-1}(b_{(1)})g\sigma_{\mathbb{G}}(\bar{f}) \otimes \bar{a}b_{(2)}) \\ &= \langle \Lambda(a), \Lambda(g\sigma_{\mathbb{G}}(\bar{f}) * b) \rangle, \end{aligned}$$

which proves the displayed formula. Since $\mathcal{A}(\mathbb{G})$ is essential, it follows that $\lambda(x)$ extends to a bounded operator for every $x \in \mathcal{D}(\mathbb{G})$. \square

Note that, from the definition of W in Proposition 3.1.2, the bornological and C^* -algebraic multiplicative unitaries can now be related by $W = (m \otimes \lambda)(\mathcal{W})$.

Definition 3.1.11. *We define the C^* -algebra $C_r^*(G)$ as the norm closure of $\{(\omega \hat{\otimes} \text{id})(W) \mid \omega \in B(L^2(\mathbb{G}))_*\}$.*

Proposition 3.1.12. *The bornological multiplicative unitary for the Pontryagin dual $\hat{\mathbb{G}}$ is given by $\hat{\mathcal{W}} = \Sigma(\mathcal{W}^*)$, where $*$ denotes the involution of $\mathcal{A}(\mathbb{G}) \hat{\otimes} \mathcal{A}(\hat{\mathbb{G}}) \cong \mathcal{A}(\mathbb{G}) \hat{\otimes} \mathcal{D}(\mathbb{G})$.*

Proof. Let $f, g \in \mathcal{D}(\mathbb{G})$ and $a, b \in \mathcal{A}(\mathbb{G})$. We recall that the counit ϵ of $\mathcal{A}(\mathbb{G})$ is the identity

element of $M(\mathcal{D}(\mathbb{G}))$. We have

$$\begin{aligned}
(\hat{\mathcal{W}}^{-1}(f \otimes g), a \otimes b) &= (\hat{\Delta}(g) * (f \otimes \epsilon), a \otimes b) \\
&= (\hat{\Delta}(g) \otimes (f \otimes \epsilon), a_{(1)} \otimes b_{(1)} \otimes a_{(2)} \otimes b_{(2)}) \\
&= (g, ba_{(1)})(f, a_{(2)}) \\
&= \phi_{\mathbb{G}}(ba_{(1)}g)\phi_{\mathbb{G}}(a_{(2)}f) \\
&= \phi_{\mathbb{G}}(bS^{-1}(f_{(1)})g)\phi_{\mathbb{G}}(af_{(2)}) \\
&= (f_{(2)} \otimes S^{-1}(f_{(1)})g, a \otimes b) \\
&= (\Sigma(\mathcal{W}^{-1}(\Sigma(f \otimes g))), a \otimes b) \\
&= (\Sigma(\mathcal{W}^{-1})(f \otimes g), a \otimes b),
\end{aligned}$$

and we know from Proposition 2.3.15 that $\mathcal{W}^{-1} = \mathcal{W}^*$. \square

The map $\mathcal{F} : \mathcal{A}(\mathbb{G}) \mapsto \mathcal{A}(\hat{\mathbb{G}})$ extends to an isometric isomorphism of $L^2(\mathbb{G})$ with $L^2(\hat{\mathbb{G}})$ thanks to Lemma 2.3.7. Using this, we obtain the following result, which should be no surprise.

Proposition 3.1.13. *We have that $C_r^*(\mathbb{G}) = C_0^r(\hat{\mathbb{G}})$.*

3.2 The modular element at the C^* -algebraic level

In order to extend $\delta_{\mathbb{G}}$ to a positive operator on $L^2(\mathbb{G})$ we shall introduce another GNS construction. For the inspiration here, see [KvD97, Section 3].

Recall from Equation (2.1) that $\delta_{\mathbb{G}} \in M(\mathcal{A}(\mathbb{G}))$ is strictly positive:

$$\phi_{\mathbb{G}}(\bar{a}\delta_{\mathbb{G}}a) > 0 \quad \text{for all nonzero } a \in \mathcal{A}(\mathbb{G}).$$

We can therefore define a Hilbert space $L^2(\mathbb{G})_{\delta}$ together with an injective linear map Λ_{δ} from $\mathcal{A}(\mathbb{G})$ to $L^2(\mathbb{G})_{\delta}$ such that

1. Λ_{δ} has dense range in $L^2(\mathbb{G})_{\delta}$,
2. $\langle \Lambda_{\delta}(f), \Lambda_{\delta}(g) \rangle = \phi_{\mathbb{G}}(\bar{f}\delta_{\mathbb{G}}g)$ for all $f, g \in \mathcal{A}(\mathbb{G})$.

We now define the closed operator L from $L^2(\mathbb{G})$ to $L^2(\mathbb{G})_{\delta}$ with core $\Lambda(\mathcal{A}(\mathbb{G}))$ such that for every $f \in \mathcal{A}(\mathbb{G})$ we have $L\Lambda(f) = \Lambda_{\delta}(f)$. Then

$$\langle Lv, \Lambda_{\delta}(f) \rangle = \langle v, \Lambda(\delta_{\mathbb{G}}f) \rangle$$

for any $v \in \text{Dom}(L)$ and $f \in \mathcal{A}(\mathbb{G})$. It follows that $\Lambda_{\delta}(\mathcal{A}(\mathbb{G}))$ is a subset of $\text{Dom}(L^*)$ and that $L^*\Lambda_{\delta}(f) = \Lambda(\delta_{\mathbb{G}}f)$.

Definition 3.2.1. We set $\delta = L^*L$, so that δ is a positive unbounded operator on $L^2(\mathbb{G})$. Note that for all $f \in \mathcal{A}(\mathbb{G})$ we have

$$\delta\Lambda(f) = \Lambda(\delta_{\mathbb{G}}f).$$

We denote by $\hat{\delta}$ the operator associated to $\delta_{\mathbb{G}}$ via the analogous construction.

We now recall a technical lemma that will be used regularly in the rest of this chapter. For the proof see [KvD97, Lemma 3.7].

Lemma 3.2.2. Consider Hilbert spaces K_1, K_2, H_1, H_2 , a unitary operator U from K_1 to H_2 , a unitary operator V from H_1 to K_2 , a closed linear operator F from within K_1 into H_1 , a closed linear operator G from within H_2 into K_2 . Suppose there exists a core C for F such that $U(C)$ is a core for G and such that $V(F(v)) = G(U(v))$ for every $v \in C$. Then we have that $VF = GU$.

Lemma 3.2.3. The operator U from $L^2(\mathbb{G}) \hat{\otimes} L^2(\mathbb{G})_{\delta}$ to $L^2(\mathbb{G})_{\delta} \hat{\otimes} L^2(\mathbb{G})_{\delta}$ such that $U(\Lambda(f) \otimes \Lambda_{\delta}(g)) = (\Lambda_{\delta} \hat{\otimes} \Lambda_{\delta})(\Delta(g)(f \otimes 1))$ is well defined and unitary.

Proof. Let $f, g, a, b \in \mathcal{A}(\mathbb{G})$. We have

$$\begin{aligned} \langle U(\Lambda(f) \otimes \Lambda_{\delta}(g)), U(\Lambda(f) \otimes \Lambda_{\delta}(g)) \rangle &= \phi_{\mathbb{G}}(\bar{f}\bar{g}_{(1)}\delta_{\mathbb{G}}b_{(1)}a)\phi_{\mathbb{G}}(\bar{g}_{(2)}\delta_{\mathbb{G}}b_{(2)}) \\ &= \phi_{\mathbb{G}}(\overline{f(\delta_{\mathbb{G}}g)}_{(1)}b_{(1)}a)\phi_{\mathbb{G}}(\overline{(\delta_{\mathbb{G}}g)}_{(2)}b_{(2)}) \\ &= \phi_{\mathbb{G}}(\bar{f}a)\phi_{\mathbb{G}}(\overline{\delta_{\mathbb{G}}gb}_{(2)}) \\ &= \langle \Lambda(f) \otimes \Lambda_{\delta}(g), \Lambda(f) \otimes \Lambda_{\delta}(g) \rangle. \end{aligned}$$

□

Lemma 3.2.4. We have $(1 \otimes \delta)W = W(\delta \otimes \delta)$.

Proof. Let $f, g \in \mathcal{A}(\mathbb{G})$, we have

$$\begin{aligned} (L \hat{\otimes} L)W^*(\Lambda(f) \otimes \Lambda(g)) &= (L \hat{\otimes} L)(\Lambda \hat{\otimes} \Lambda)(\Delta(g)(f \otimes 1)) \\ &= (\Lambda_{\delta} \hat{\otimes} \Lambda_{\delta})(\Delta(g)(f \otimes 1)) \\ &= U(1 \hat{\otimes} L)(\Lambda(f) \otimes \Lambda(g)). \end{aligned}$$

Using Lemma 3.2.2 we deduce that $(L \hat{\otimes} L)W^* = U(1 \hat{\otimes} L)$. Composing this with its adjoint, the result follows. □

Proposition 3.2.5. We have that δ is a strictly positive element affiliated with $C_0^*(\mathbb{G})$ in the C^* -algebraic sense. Furthermore, $\Delta(\delta) = \delta \otimes \delta$.

Proof. Our proof is similar to the proofs of [KvD97, Propositions 8.5 and 8.6]. From the preceding lemma, we obtain that $W^*(1 \otimes \delta) = (\delta \otimes \delta)W^*$ and thus

$$(1 \otimes \delta^{-it})W^*(1 \otimes \delta^{it}) = (\delta^{it} \otimes 1)W^*,$$

for all $t \in \mathbb{R}$. Let $\omega \in \mathcal{K}(L^2(\mathbb{G}))^*$. Applying $\text{id} \hat{\otimes} \omega$ to this equality we get

$$\delta^{it}(\text{id} \hat{\otimes} \omega)W^* = \text{id} \hat{\otimes} \delta^{-it} \omega \delta^{it}(W^*),$$

where the notation $\delta^{-it} \omega \delta^{it}$ refers to the functional $\omega(\delta^{-it} \cdot \delta^{it})$. Thus, by Proposition 3.1.7 (which remains true if we replace W by W^*) we conclude that $\delta^{it}C_0^r(\mathbb{G}) \subset C_0^r(\mathbb{G})$ for all $t \in \mathbb{R}$. By definition this says that δ is an unbounded element affiliated to $C_0^r(\mathbb{G})$.

We also derive from Lemma 3.2.4 that $W^*(1 \otimes \delta)W = \delta \otimes \delta$, i.e. $\Delta(\delta) = \delta \otimes \delta$. \square

By induction on n , one can deduce the following lemma.

Lemma 3.2.6. *Consider $f \in \mathcal{A}(\mathbb{G})$ and $n \in \mathbb{Z}$. Then $\Lambda(f)$ belongs to $\text{Dom}(\delta^n)$ and $\delta^n \Lambda(f) = \Lambda(\delta_{\mathbb{G}}^n f)$.*

Lemma 3.2.7. *Consider $f \in \mathcal{A}(\mathbb{G})$ and $z \in \mathbb{C}$. Then $\Lambda(f)$ belongs to $\text{Dom}(\delta^z)$.*

Proof. We already saw in Proposition 3.2.5 that $\Lambda(f) \in \text{Dom}(\delta^{it})$ for all $t \in \mathbb{R}$, so an interpolation using the previous lemma proves the result. \square

Similarly, the proof of [KvD97, Lemma 8.9] is still valid for the following two propositions. Here we are writing $\text{Dom}(T) \subseteq C_0^r(\mathbb{G})$ for the domain of a positive element T affiliated to the C^* -algebra $C_0^r(\mathbb{G})$.

Proposition 3.2.8. *For every $n \in \mathbb{Z}$ and $f \in \mathcal{A}(\mathbb{G})$, we have that $m(f)$ belongs to $\text{Dom}(\delta_{\mathbb{G}}^n)$ and $\delta^n m(f) = m(\delta_{\mathbb{G}}^n f)$.*

Proposition 3.2.9. *For every $z \in \mathbb{C}$ and $f \in \mathcal{A}(\mathbb{G})$, we have that $m(f)$ belongs to $\text{Dom}(\delta^z)$.*

As in the algebraic framework, we will prove more: that the complex powers δ^z of the C^* -algebraic modular element multiply the bornological subalgebra $m(\mathcal{A}(\mathbb{G}))$ into itself and moreover defines bounded multipliers of $\mathcal{A}(\mathbb{G})$ in the bornological sense. To do so, we need a series of technical lemmas.

Firstly, we observe that by Pontryagin duality, elements of the bornological dual $\mathcal{A}(\hat{\mathbb{G}})$ also give elements of the pre-dual of the von Neumann closure $L^\infty(\mathbb{G})$.

Lemma 3.2.10. *For every $f \in \mathcal{A}(\mathbb{G})$, the linear functional $\hat{f} = \mathcal{F}(f) \in \mathcal{A}(\mathbb{G})^*$ extends to a normal linear functional on $L^\infty(\mathbb{G})$ and we obtain a bounded linear map $\mathcal{A}(\mathbb{G}) \rightarrow L^\infty(\mathbb{G})_*$.*

Proof. The linear map

$$\begin{aligned} \rho : \mathcal{A}(\mathbb{G}) \otimes \mathcal{A}(\mathbb{G}) &\longrightarrow B(L^2(\mathbb{G}))_* \\ a \otimes b &\longmapsto \widehat{ab} = \omega_{\Lambda(\sigma_{\mathbb{G}}(b^*)), \Lambda(a)} \end{aligned}$$

is bounded so extends to $\mathcal{A}(\mathbb{G}) \widehat{\otimes} \mathcal{A}(\mathbb{G})$. Let $a, b, f \in \mathcal{A}(\mathbb{G})$, we have that

$$\begin{aligned} \widehat{ab}(m(f)) &= \phi_{\mathbb{G}}(\sigma_{\mathbb{G}}(b^*)^* f a) \\ &= \phi_{\mathbb{G}}(fab). \end{aligned}$$

Thus elements of the form $ac \otimes b - a \otimes cb$, $a, b, c \in \mathcal{A}(\mathbb{G})$ belong to the kernel of ρ . As a consequence, ρ descends to a map on the balanced tensor product $\mathcal{A}(\mathbb{G}) \otimes_{\mathcal{A}(\mathbb{G})} \mathcal{A}(\mathbb{G})$ and we obtain a bounded map $\mathcal{A}(\mathbb{G}) \widehat{\otimes}_{\mathcal{A}(\mathbb{G})} \mathcal{A}(\mathbb{G}) \rightarrow B(L^2(\mathbb{G}))_*$, that is a bounded map $\mathcal{A}(\mathbb{G}) \rightarrow B(L^2(\mathbb{G}))_*$, using the essentialness of $\mathcal{A}(\mathbb{G})$. \square

The next Lemma is the bornological analogue of [KvD97, Lemma 7.6]. It essentially says that slices of the C^* -algebraic coproduct by elements of the bornological dual yield bornological multipliers.

Lemma 3.2.11. *Consider $f, g \in \mathcal{A}(\mathbb{G})$ and $x \in M(C_0^r(\mathbb{G}))$, then $(\text{id} \otimes \widehat{f})(\Delta(x))m(g)$ belongs to $m(\mathcal{A}(\mathbb{G}))$.*

Proof. Let $x \in M(C_0^r(\mathbb{G}))$ and consider the bilinear map $L_x : \mathcal{A}(\mathbb{G}) \times \mathcal{A}(\mathbb{G}) \rightarrow B(L^2(\mathbb{G}))$ defined by $(f, g) \mapsto (\text{id} \widehat{\otimes} \widehat{f})(\Delta(x))m(g)$. On the one hand the map $f \mapsto \text{id} \otimes \widehat{f}$ from $\mathcal{A}(\mathbb{G})$ to $B(L^2(\mathbb{G})) \widehat{\otimes} B(L^2(\mathbb{G}))_*$ is bounded, according Lemma 3.2.10. On the other hand $g \mapsto m(g)$ is bounded too. Finally L_x is bounded since the evaluation map $B(L^2(\mathbb{G}))_*$ and the product map $B(L^2(\mathbb{G})) \times B(L^2(\mathbb{G})) \rightarrow B(L^2(\mathbb{G}))$ are bounded. One can thus consider the linear map $L_x : \mathcal{A}(\mathbb{G}) \widehat{\otimes} \mathcal{A}(\mathbb{G}) \rightarrow B(L^2(\mathbb{G}))$.

Let $q, r \in \mathcal{A}(\mathbb{G})$ and consider $\Delta(q)(r \otimes 1) \in \mathcal{A}(\mathbb{G}) \widehat{\otimes} \mathcal{A}(\mathbb{G})$. For every $y \in \mathcal{A}(\mathbb{G})$, we have

$$\begin{aligned} L_{m(y)}(\Delta(q)(r \otimes 1)) &= m((\text{id} \widehat{\otimes} \phi_{\mathbb{G}})(\Delta(y)\Delta(q)(r \otimes 1))) \\ &= \phi_{\mathbb{G}}(yq)m(r). \end{aligned}$$

Using Lemma 3.2.10, and because $m(\mathcal{A}(\mathbb{G}))$ is strictly dense in $M(C_0^r(\mathbb{G}))$, one can replace $m(y)$ by any $x \in M(C_0^r(\mathbb{G}))$ in this equality. Therefore $L_x \circ \gamma_l$ is bounded and extends to $\mathcal{A}(\mathbb{G}) \widehat{\otimes} \mathcal{A}(\mathbb{G})$. Thus L_x maps $\mathcal{A}(\mathbb{G}) \widehat{\otimes} \mathcal{A}(\mathbb{G})$ into $\mathcal{A}(\mathbb{G})$ as required. \square

Now we return to the complex powers of the modular element.

Lemma 3.2.12. *Let $z \in \mathbb{C}$ and $f \in \mathcal{A}(\mathbb{G})$, The linear functional $\widehat{f} \circ \delta^z \circ m : \mathcal{A}(\mathbb{G}) \rightarrow \mathbb{C}$ is well-defined and bounded.*

Proof. Consider the linear map $\mathcal{A}(\mathbb{G}) \otimes \mathcal{A}(\mathbb{G}) \rightarrow \mathcal{A}(\mathbb{G})^*$ defined by $a \otimes b \mapsto \widehat{ab} \circ \delta^z \circ m$. We must show that this map is bounded. Let $a, b \in \mathcal{A}(\mathbb{G})$. For all $x \in \mathcal{A}(\mathbb{G})$ we have

$$\begin{aligned} (\widehat{ab} \circ \delta^z \circ m)(x) &= \langle \Lambda(\sigma_{\mathbb{G}}(b^*)), \delta^z \Lambda(xa) \rangle \\ &= \langle \delta^{\bar{z}} \Lambda(\sigma_{\mathbb{G}}(b^*)), \Lambda(xa) \rangle \\ &= \omega_{\delta^{\bar{z}} \Lambda(\sigma_{\mathbb{G}}(b^*)), \Lambda(a)}(m(x)). \end{aligned}$$

Thus $\widehat{ab} \circ \delta^z = \omega_{\delta^{\bar{z}} \Lambda(\sigma_{\mathbb{G}}(b^*)), \Lambda(a)}$. The result follows.

It is sufficient to check that $b \mapsto \widehat{f}(\delta^z m(b))$ is bounded. For that observe that, since $\delta^z m(b) \in m(\mathcal{A}(\mathbb{G}))$ according to Proposition 3.2.9, the map $\widehat{f} \circ \delta^z \circ m$ can be expressed as $b \mapsto \phi_{\mathbb{G}}(\delta^z m(b)m(f))$ where here $\phi_{\mathbb{G}}$ is seen as the functional on $m(\mathcal{A}(\mathbb{G}))$ such that $\phi_{\mathbb{G}}(m(a)) = \phi_{\mathbb{G}}(a)$, $a \in \mathcal{A}(\mathbb{G})$. We have $\phi_{\mathbb{G}}(\delta^z m(b)m(f)) = \phi_{\mathbb{G}}(m(\sigma_{\mathbb{G}}^{-1}(f))\delta^z m(b))$. Since $m(\sigma_{\mathbb{G}}^{-1}(f))\delta^z$ also belongs to $m(\mathcal{A}(\mathbb{G}))$ then the result follows. \square

Next we recall Lemma 8.11 of [KvD97], for which the proof also remains valid.

Lemma 3.2.13. *Consider an element α affiliated with $C_0^r(\mathbb{G})$ and elements $x \in \text{Dom}(\alpha)$, $y \in C_0^r(\mathbb{G})$. Then $\Delta(x)(1 \otimes y)$ belongs to $\text{Dom}(\Delta(\alpha))$ and $\Delta(\alpha)\Delta(x)(1 \otimes y) = \Delta(\alpha(x))(1 \otimes y)$.*

Proposition 3.2.14. *Let $z \in \mathbb{C}$. Then $\delta^z m(\mathcal{A}(\mathbb{G})) \subset m(\mathcal{A}(\mathbb{G}))$ and δ^z is a bounded multiplier of $m(\mathcal{A}(\mathbb{G}))$, where $m(\mathcal{A}(\mathbb{G}))$ is endowed with the bornology inherited from $\mathcal{A}(\mathbb{G})$ through the injective linear map m .*

Proof. In this proof we adapt the arguments of the proof of [KvD97, Proposition 8.12]. Let $p, q \in \mathcal{A}(\mathbb{G})$ and $f \in \mathcal{A}(\mathbb{G})$. We consider the element $a = (\text{id} \hat{\otimes} (\widehat{f} \circ \delta^z \circ m))(\Delta(p)(q \otimes 1))$ of $\mathcal{A}(\mathbb{G})$. We have

$$\begin{aligned} \delta^z m(a) &= (\text{id} \hat{\otimes} \widehat{f})((\delta^z \otimes \delta^z)(\Delta(m(p))(m(q) \otimes 1))) \\ &= (\text{id} \hat{\otimes} \widehat{f})(\Delta(\delta^z m(p)))(m(q) \otimes 1). \end{aligned}$$

Since $m(p)$ is an analytic element for δ , we have that $\delta^z m(p) \in M(C_0^r(\mathbb{G}))$. By Lemma 3.2.11, it follows that $\delta^z m(a)$ belongs to $m(\mathcal{A}(\mathbb{G}))$. Now, because of the boundedness of the map $\text{id}(\hat{\otimes} \widehat{f} \circ \delta^z \circ m)$, see Lemma 3.2.12, we can apply this method to any element $a = \text{id} \hat{\otimes} (\widehat{f} \delta^z \circ m)(X)$, $X \in \mathcal{A}(\mathbb{G}) \hat{\otimes} \mathcal{A}(\mathbb{G})$, that is, to any element a of $\mathcal{A}(\mathbb{G})$. Thus one can now define $\delta_{\mathbb{G}}^z$ as the unique multiplier of $\mathcal{A}(\mathbb{G})$ such that

$$m(\delta_{\mathbb{G}}^z a) = \delta^z m(a).$$

To prove that it is indeed a bounded multiplier, let $a \in \mathcal{A}(\mathbb{G})$ and consider an element $b \in \mathcal{A}(\mathbb{G})$ such that $\phi_{\mathbb{G}}(\delta_{\mathbb{G}}^{-z}b) = 1$. We have $\Delta(\delta_{\mathbb{G}}^{-z}b) = (\delta_{\mathbb{G}}^{-z} \otimes \delta_{\mathbb{G}}^{-z})\Delta(b)$. It follows that

$$a = a\phi_{\mathbb{G}}(\delta_{\mathbb{G}}^{-z}b) = (\text{id} \hat{\otimes} \phi_{\mathbb{G}})(\Delta(\delta_{\mathbb{G}}^{-z}b)(a \otimes 1)) = \delta_{\mathbb{G}}^{-z}(\text{id} \hat{\otimes} \phi_{\mathbb{G}}\delta_{\mathbb{G}}^{-z})(\Delta(b)(a \otimes 1)),$$

where $\phi_{\mathbb{G}}\delta_{\mathbb{G}}^{-z}$ denotes the linear functional $g \mapsto \phi_{\mathbb{G}}(\delta_{\mathbb{G}}^{-z}g)$. Thus, the multiplier $\delta_{\mathbb{G}}^z$ can be expressed as the composition map

$$a \mapsto a \otimes b \mapsto \Delta(b)(a \otimes 1) \mapsto (\text{id} \hat{\otimes} \phi_{\mathbb{G}}\delta_{\mathbb{G}}^{-z})(\Delta(b)(a \otimes 1)).$$

It remains to show that the last map in this composition is well defined and bounded. Let x, y and f in $\mathcal{A}(\mathbb{G})$, we have

$$\begin{aligned} (\text{id} \otimes \phi_{\mathbb{G}}\delta_{\mathbb{G}}^{-z})(xf \otimes y) &= (\text{id} \otimes \phi_{\mathbb{G}}\sigma^{-1}(f)\delta_{\mathbb{G}}^{-z})(x \otimes y) \\ &= (\text{id} \otimes (\widehat{\sigma^{-1}(f)} \circ \delta^z \circ m))(x \otimes y) \end{aligned}$$

We then deduce the boundedness of $(x, y) \mapsto (\text{id} \otimes \phi_{\mathbb{G}}\delta_{\mathbb{G}}^{-z})(x \otimes y)$ using the essentiality of $\mathcal{A}(\mathbb{G})$. \square

With the above proposition, the following theorem is now straightforward, compare [KvD97, Section 8].

Theorem 3.2.15. *For all $z \in \mathbb{C}$, there exists a unique bounded multiplier of $\mathcal{A}(\mathbb{G})$ denoted $\delta_{\mathbb{G}}^z$ such that for all $a \in \mathcal{A}(\mathbb{G})$,*

$$\delta^z m(a) = m(\delta_{\mathbb{G}}^z a).$$

Furthermore, we have the following properties :

1. For any $z \in \mathbb{C}$, $\overline{\delta_{\mathbb{G}}^z} = \delta_{\mathbb{G}}^{\bar{z}}$
2. For any $y, z \in \mathbb{C}$, $\delta_{\mathbb{G}}^y \delta_{\mathbb{G}}^z = \delta_{\mathbb{G}}^{y+z}$,
3. For any $t \in \mathbb{R}$, $\delta_{\mathbb{G}}^{it}$ is unitary in $M(\mathcal{A}(\mathbb{G}))$,
4. For any $t \in \mathbb{R}$, $\delta_{\mathbb{G}}^t$ is a positive element, in the sense that $\delta_{\mathbb{G}}^t = \delta_{\mathbb{G}}^{t/2} \delta_{\mathbb{G}}^{t/2}$ and $\delta_{\mathbb{G}}^{t/2}$ is a self adjoint element.

With our assumption that the scaling constant is 1, we obtain the following.

Proposition 3.2.16. *The right Haar functional $\phi_{\mathbb{G}} \circ S$ of $\mathcal{A}(\mathbb{G})$ is positive.*

Proof. Let $f \in \mathcal{A}(\mathbb{G})$, we have

$$\phi_{\mathbb{G}}(S(\bar{f}f)) = \phi_{\mathbb{G}}(\bar{f}f\delta_{\mathbb{G}}) = \phi_{\mathbb{G}}(\overline{f\delta_{\mathbb{G}}^{1/2}} f\delta_{\mathbb{G}}^{1/2}) > 0,$$

where we use that $\delta_{\mathbb{G}}^{1/2}$ is self-adjoint. \square

3.3 Preliminary remarks on the modular group

A central point in the framework of locally compact quantum groups is a good understanding of the modular theory of the associated operator algebras. Let us briefly recall the main definitions.

First, we define the closed operator T on $L^2(\mathbb{G})$ as the closed antilinear operator with core $\Lambda(\mathcal{A}(\mathbb{G}))$ such that $T\Lambda(f) = \Lambda(\bar{f})$ for all $f \in \mathcal{A}(\mathbb{G})$. We have

$$\begin{aligned} \langle T\Lambda(f), \Lambda(g) \rangle &= \phi_{\mathbb{G}}(fg) \\ &= \overline{\phi_{\mathbb{G}}(\bar{g}\bar{f})} \\ &= \overline{\phi_{\mathbb{G}}(\bar{f}\sigma_{\mathbb{G}}(\bar{g}))}, \end{aligned}$$

for all $f, g \in \mathcal{A}(\mathbb{G})$. Then we have $T^*\Lambda(f) = \Lambda(\sigma_{\mathbb{G}}(\bar{f}))$ for all $f \in \mathcal{A}(\mathbb{G})$. Hence the modular operator $\nabla = T^*T$ satisfies

$$\nabla\Lambda(f) = \Lambda(\sigma_{\mathbb{G}}(f)).$$

We denote by J the anti-unitary component of the polar decomposition of T , so that $T = J\nabla^{\frac{1}{2}} = \nabla^{-\frac{1}{2}}J$.

Definition 3.3.1. *Let $x \in B(L^2(\mathbb{G}))$. We define $\sigma_t(x) = \nabla^{it}x\nabla^{-it}$. The family $(\sigma_t)_{t \in \mathbb{R}}$ is called the modular group associated to $C_0^*(\mathbb{G})$.*

Classically, the study of the modular group is undertaken using the unitary antipode and the scaling group $\tau_t(x) = M^{it}xM^{-it}$, where M is the positive operator in the polar decomposition $G = IM^{\frac{1}{2}}$ of the closed antilinear operator G with $G\Lambda(f) = \Lambda(S(\bar{f}))$, see [KvD97, KV00].

Remark 3.3.2. Kustermans and Vaes [KV00] use N for the operator M , but since we are mainly following [KvD97] here, we will stick with their notation.

In order to study the stability properties of an algebraic quantum group with respect to these operator algebraic automorphism groups, Kustermans and Van Daele proceed as for the complex powers of the modular element δ in the previous section, namely they seek out commutation relations between the positive operators M, ∇ (and other auxiliary operators) and the multiplicative unitary W , in order to obtain similar relations for the associated automorphism groups.

We shall follow the same general strategy, but with a change of focus. Note that, by Proposition 2.3.9, the modular operator $\hat{\nabla}$ for the Pontryagin dual satisfies

$$\hat{\nabla}\Lambda(f) = \Lambda(S^2(f)\delta_{\mathbb{G}}^{-1}), \quad (3.2)$$

for all $f \in \mathcal{D}(\mathbb{G})$. This indicates that one can relate the modular group of the Pontryagin dual $(\hat{\sigma}_t)_{t \in \mathbb{R}}$ to the complex powers of the modular element $\delta_{\mathbb{G}}$, which we have already studied, and the automorphism group associated to a closure N of the operator S^2 . This can then be related to the usual scaling group and unitary antipode by the above formula for $\hat{\nabla}$, or its dual version.

The advantage of this approach is that the operator S^2 is both an algebra and a coalebra automorphism of $\mathcal{A}(\mathbb{G})$, so has very nice algebraic properties.

3.4 The automorphism group associated to S^2

We denote by $\psi_{\mathbb{G}}$ the functional $\phi_{\mathbb{G}} \circ S$. We have seen at Proposition 3.2.16 that $\psi_{\mathbb{G}}$ was positive and one can thus consider $(\Lambda', L^2(\mathbb{G})')$ the GNS construction associated to $(\mathcal{A}(\mathbb{G}), \psi_{\mathbb{G}})$.

In order to build a positive operator associated with S^2 we introduce the following operator.

Definition 3.4.1. *We define K as the closed unbounded antilinear operator from $L^2(\mathbb{G})$ to $L^2(\mathbb{G})'$ such that $\Lambda(\mathcal{A}(\mathbb{G}))$ is a core for K and $K\Lambda(f) = \Lambda'(S(\bar{f}))$.*

Lemma 3.4.2. *Let $f \in \mathcal{A}(\mathbb{G})$. We have that $K^*\Lambda'(f) = \Lambda(\overline{S(f)})$*

Proof. Let $f, g \in \mathcal{A}(\mathbb{G})$, we have

$$\begin{aligned} \langle K\Lambda(f), \Lambda'(g) \rangle &= \phi_{\mathbb{G}}(S(\overline{S(f)})g) \\ &= \phi_{\mathbb{G}}(S(g)f) \\ &= \langle \Lambda(\overline{S(g)}), \Lambda(f) \rangle \end{aligned}$$

□

Definition 3.4.3. *We set $N = K^*K$.*

Thus N is a positive operator on $L^2(\mathbb{G})$ such that $N\Lambda(f) = \Lambda(S^2(f))$ for all $f \in \mathcal{A}(\mathbb{G})$. We remark (again) that this N differs from the operator N in [KV00], which corresponds to the operator denoted by M here and in [KvD97].

Lemma 3.4.4. *The operator V from $L^2(\mathbb{G})' \hat{\otimes} L^2(\mathbb{G})'$ to $L^2(\mathbb{G})' \hat{\otimes} L^2(\mathbb{G})'$ such that $V^*(\Lambda'(f) \otimes \Lambda'(g)) = (\Lambda' \hat{\otimes} \Lambda')(\Delta^{\text{op}}(g)(f \otimes 1))$ is well defined and unitary.*

Proof. Using the right invariance of $\psi_{\mathbb{G}}$ we directly get

$$\begin{aligned} \langle (\Lambda' \hat{\otimes} \Lambda')(\Delta^{\text{op}}(g)(f \otimes 1)), (\Lambda' \hat{\otimes} \Lambda')(\Delta^{\text{op}}(b)(a \otimes 1)) \rangle &= \psi_{\mathbb{G}}(\bar{f}\bar{g}_{(2)}b_{(2)}a)\psi_{\mathbb{G}}(\bar{g}_{(1)}b_{(1)}) \\ &= \langle \Lambda'(f) \otimes \Lambda'(g), \Lambda'(a) \otimes \Lambda'(b) \rangle, \end{aligned}$$

for all $f, g, a, b \in \mathcal{A}(\mathbb{G})$. □

Using the fact that S^2 is both an algebra and coalgebra automorphisms, it follows immediately that $S^2 \hat{\otimes} S^2$ commutes with all Galois maps. This gives a formal justification for the next lemma although one needs to be careful when passing to unbounded extensions.

Lemma 3.4.5. *We have that $(N \hat{\otimes} N)W = W(N \hat{\otimes} N)$.*

Proof. First we prove that $(K \hat{\otimes} K)W^* = V^*(K \hat{\otimes} K)$:

Let $f, g \in \mathcal{A}(\mathbb{G})$, we have

$$\begin{aligned} (K \hat{\otimes} K)W^*(\Lambda(f) \otimes \Lambda(g)) &= (K \hat{\otimes} K)(\Lambda \hat{\otimes} \Lambda)(\Delta(g)(f \otimes 1)) \\ &= (\Lambda' \hat{\otimes} \Lambda')(\Delta^{\text{op}}(S(\bar{g})(\bar{f} \otimes 1)) \\ &= V^*(K \hat{\otimes} K)(\Lambda'(f) \otimes \Lambda'(g)). \end{aligned}$$

Using Lemma 3.2.2 we get that $(K \hat{\otimes} K)W^* = V^*(K \hat{\otimes} K)$. Similarly we also get that $(K^* \hat{\otimes} K^*)W^* = V^*(K^* \hat{\otimes} K^*)$ and the result follows. \square

Definition 3.4.6. *Let $(\rho_t)_{t \in \mathbb{R}}$ designate the one parameter group of automorphisms $B(L^2(\mathbb{G}))$ generated by N , that is, for all $t \in \mathbb{R}$ and $x \in B(L^2(\mathbb{G}))$, we define $\rho_t(x) = N^{it}xN^{-it}$.*

We will see that the automorphism group $(\rho_t)_{t \in \mathbb{R}}$ is closely related to the scaling group $(\tau_t)_{t \in \mathbb{R}}$.

Proposition 3.4.7. *For all $t \in \mathbb{R}$ we have that $\rho_t(C_0^r(\mathbb{G})) \subset C_0^r(\mathbb{G})$.*

Proof. From $(N \hat{\otimes} N)W = W(N \hat{\otimes} N)$ we obtain

$$(N^{it} \hat{\otimes} 1)W(N^{-it} \hat{\otimes} 1) = (1 \hat{\otimes} N^{-it})W(1 \hat{\otimes} N^{it})$$

Let $\omega \in B(L^2(\mathbb{G}))_*$. Applying $\text{id} \hat{\otimes} \omega$ to this equality we get

$$N^{it}(\text{id} \hat{\otimes} \omega)(W)N^{-it} = (\text{id} \hat{\otimes} N^{-it} \omega N^{it})(W),$$

so the result follows from Proposition 3.1.7. \square

Lemma 3.4.8. *The operators δ and $\hat{\delta}$ strongly commute with N .*

Proof. Thanks to Lemma 3.2.2, in order to show that δ and N strongly commute it is enough to show that δ^{it} and N commute on $\Lambda(\mathcal{A}(\mathbb{G}))$ for any $t \in \mathbb{R}$. Since $\delta_{\mathbb{G}}^{it}$ is a group-like element of $M(\mathcal{A}(\mathbb{G}))$ we have that $S^2(\delta_{\mathbb{G}}^{it}) = \delta_{\mathbb{G}}^{it}$, and thus for all $f \in \mathcal{A}(\mathbb{G})$ we have

$$\begin{aligned} N \delta^{it} \Lambda(f) &= \Lambda(S^2(\delta_{\mathbb{G}}^{it} f)) \\ &= \Lambda(\delta_{\mathbb{G}}^{it} S^2(f)) \\ &= \delta^{it} N \Lambda(f). \end{aligned}$$

A similar argument applies for $\hat{\delta}$. Note that $\hat{S}^2(f) = S^2(f)$ for all $f \in \mathcal{D}(\mathbb{G})$. \square

We define δ' as the unbounded operator $\delta' = J\delta J$. This is merely a convenient way to introduce the appropriate unbounded closure of the operator of right multiplication by $\delta_{\mathbb{G}}$, since one can show that for all $f \in \mathcal{A}(\mathbb{G})$,

$$\delta' \Lambda(f) = \Lambda(f \delta_{\mathbb{G}}).$$

Dually, we make the analogous definition of the operator $\hat{\delta}'$, so that

$$\hat{\delta}' \Lambda(f) = \Lambda(f * \delta_{\mathbb{G}}).$$

The following then follows in an analogous fashion to Lemma 3.4.8.

Lemma 3.4.9. *The operators δ' and $\hat{\delta}'$ strongly commute with N .*

In order to define ρ_z at the bornological level, which informally can be understood as the operator $(S^2)^{z/2}$, we will generalize Radford's S^4 formula. For this we need the following lemmas.

Lemma 3.4.10. *There exists a constant $\nu > 0$ such that $\sigma(\delta_{\mathbb{G}}^z) = \nu^{iz} \delta_{\mathbb{G}}^z$ for all $z \in \mathbb{C}$.*

Proof. First, applying Proposition 2.3.5 to $\delta_{\mathbb{G}}^z$, we derive that

$$\sigma(\delta_{\mathbb{G}}^z) \otimes \delta_{\mathbb{G}}^z = \delta_{\mathbb{G}}^z \otimes \sigma(\delta_{\mathbb{G}}^z)$$

and thus there exists $c(z) \in \mathbb{C}$ such that $\sigma(\delta_{\mathbb{G}}^z) = c(z) \delta_{\mathbb{G}}^z$. Clearly, $c : \mathbb{C} \rightarrow \mathbb{C}^\times$ is a homomorphism.

Now consider $f, g \in \mathcal{A}(\mathbb{G})$. Since $\phi_{\mathbb{G}}$ is invariant by σ we have

$$\begin{aligned} \langle \delta^{-z} \Lambda(f), \Lambda(g) \rangle &= \phi_{\mathbb{G}}(\bar{f} \delta^z g) \\ &= c(z) \phi_{\mathbb{G}}(\sigma(\bar{f}) \delta^z \sigma(g)). \end{aligned}$$

The function $z \mapsto \langle \delta^{-z} \Lambda(f), \Lambda(g) \rangle$ is holomorphic, then so is $z \mapsto c(z)$. Using Proposition 2.3.4, for $t \in \mathbb{R}$ we have

$$\begin{aligned} \sigma(\delta_{\mathbb{G}}^{it}) &= \sigma\left(\overline{\delta_{\mathbb{G}}^{-it}}\right) \\ &= \overline{\sigma^{-1}(\delta_{\mathbb{G}}^{-it})} \end{aligned}$$

and thus $c(it) = \overline{c(it)}$. The result follows. \square

Note that the scaling constant is given by $\mu = \nu^i$.

Lemma 3.4.11. *The operators $\delta \delta'^{-1}$ and $\hat{\delta} \hat{\delta}'^{-1}$ on $L^2(\mathbb{G})$ strongly commute.*

Proof. In fact, we will prove a stronger statement, namely that $\hat{\delta}$ and $\hat{\delta}'$ both strongly commute with $\delta\delta'^{-1}$. From the proof of Proposition 2.3.11 we have that $f * \delta_{\mathbb{G}} = S^2(\sigma_{\mathbb{G}}^{-1}(f))$. Now, using the preceding lemma, let $t \in \mathbb{R}$ and observe that

$$\begin{aligned} \hat{\delta}'\delta^{it}\Lambda(f) &= \Lambda((\delta_{\mathbb{G}}^{it}f) * \delta_{\mathbb{G}}) \\ &= \Lambda(S^2(\sigma_{\mathbb{G}}^{-1}(\delta_{\mathbb{G}}^{it}f))) \\ &= \nu^t\Lambda(\delta_{\mathbb{G}}^{it}S^2(\sigma_{\mathbb{G}}^{-1}(f))) \\ &= \nu^t\delta^{it}\hat{\delta}'\Lambda(f). \end{aligned}$$

In the same way, we have

$$\hat{\delta}'\delta'^{-it}\Lambda(f) = \nu^{-t}\delta'^{-it}\hat{\delta}'\Lambda(f).$$

Combining these, we see that $\hat{\delta}'$ strongly commutes with $\delta\delta'^{-1}$. A similar argument shows that $\hat{\delta}$ strongly commutes with $\delta\delta'^{-1}$. \square

Theorem 3.4.12. *Let $z \in \mathbb{C}$, for any $f \in \mathcal{A}(\mathbb{G})$ we have that $\rho_z(m(f))$ belongs to $m(\mathcal{A}(\mathbb{G}))$. More precisely we have*

$$\rho_z(m(f)) = m(\delta_{\mathbb{G}}^{-iz/2}(\delta_{\mathbb{G}}^{iz/2} * f * \delta_{\mathbb{G}}^{-iz/2})\delta_{\mathbb{G}}^{iz/2}).$$

Proof. Considering Radford's S^4 formula, Theorem 2.3.13, and Lemma 3.4.10, we deduce that for any $g \in \mathcal{A}(\mathbb{G})$,

$$N^z\Lambda(g) = \Lambda(\delta_{\mathbb{G}}^{iz/2}(\delta_{\mathbb{G}}^{-iz/2} * g * \delta_{\mathbb{G}}^{iz/2})\delta_{\mathbb{G}}^{-iz/2}).$$

The result follows. \square

Remark 3.4.13. We note that the formula for $N^z\Lambda(g)$ in the proof is self-dual, up to a sign. It follows that the bornological subalgebra $\lambda(\mathcal{D}(\mathbb{G})) \subset C_r^*(\mathbb{G})$ is also stable with respect to the automorphism group $(\rho_z)_{z \in \mathbb{C}}$.

3.5 The modular groups of $C_r^*(\mathbb{G})$ and $C_0^r(\mathbb{G})$

As mentioned above, our approach for the construction of the modular group of $C_0^r(\mathbb{G})$ is somewhat different from that of [KvD97]. We start by building the the modular group of $C_r^*(\mathbb{G})$ and then apply duality to get that of $C_0^r(\mathbb{G})$. The motivation for this is Equation (3.2), which gives a formula for $\hat{\nabla}$ in terms of the strongly commuting operators N and δ' , explicitly, $\hat{\nabla} = \delta'^{-1}N$. Thus, the modular group can be expressed in terms of the automorphism groups (ρ_z) and (δ'^z) , both of which we have already shown to stabilize $\mathcal{A}(\mathbb{G})$.

Recall that we use $\lambda : \mathcal{D}(\mathbb{G}) \rightarrow C_r^*(\mathbb{G})$ to denote the regular representation. We may extend $(\hat{\sigma}_t)$ to a complex 1-parameter group on analytic elements.

Proposition 3.5.1. *Let $f \in \mathcal{D}(\mathbb{G})$, and $n \in \mathbb{Z}$. Then $\hat{\sigma}_{in}(\lambda(f)) = \lambda(S^{-2n}(f)\delta_{\mathbb{G}}^n)$.*

Proof. A direct calculation using the above formula for $\hat{\nabla}$ shows that, for all $g \in \mathcal{A}(\mathbb{G})$,

$$\begin{aligned} \hat{\sigma}_i(\lambda(f))\Lambda(g) &= \hat{\nabla}^{-1}\lambda(f)\hat{\nabla}\Lambda(g) \\ &= \Lambda(S^{-2}(f_{(1)})\delta_{\mathbb{G}})\phi_{\mathbb{G}}(\delta_{\mathbb{G}}S(g)f_{(2)}) \\ &= \Lambda(S^{-2}(f_{(1)})\delta_{\mathbb{G}})\phi_{\mathbb{G}}(S^{-1}(g)S^{-2}(f_{(2)})\delta_{\mathbb{G}}) \\ &= \lambda(S^{-2}(f)\delta_{\mathbb{G}})\Lambda(g). \end{aligned}$$

The result follows by induction. □

A standard interpolation argument allows us to conclude that the elements of $\lambda(\mathcal{D}(\mathbb{G}))$ are analytic for the modular group $(\hat{\sigma}_z)_{z \in \mathbb{C}}$. Moreover, since N and δ' strongly commute, we have

$$\sigma_z(\lambda(f)) = \delta'^{-iz} \rho_z(\lambda(f)) \delta'^{iz},$$

for all $f \in \mathcal{D}(\mathbb{G})$. Applying Theorem 3.2.15 for the dual group $\hat{\mathbb{G}}$ and Remark 3.4.13, we obtain the following result.

Proposition 3.5.2. *We have $\hat{\sigma}_z(\lambda(\mathcal{D}(\mathbb{G}))) \subset \lambda(\mathcal{D}(\mathbb{G}))$.*

By duality, one can deduce the analogous result for σ_z .

Proposition 3.5.3. *We have $\sigma_z(m(\mathcal{A}(\mathbb{G}))) \subset m(\mathcal{A}(\mathbb{G}))$.*

Finally, although we shall not need it here, let us remark that the bornological algebra $\mathcal{A}(\mathbb{G})$ is preserved by the scaling group $(\tau_z)_{z \in \mathbb{C}}$. Indeed, the scaling group is given by $\tau_t(x) = M^{-it}xM^{it}$, where M is defined as a closure of the operator $\Lambda(f) \mapsto \Lambda(S^2(f)\delta) = \delta'N\Lambda(f)$. Therefore, using the strong commutativity of the operators N and δ' , the stability of $\mathcal{A}(\mathbb{G})$ by τ_t follows from the results above.

3.6 A Left Haar weight for $(C_0^r(\mathbb{G}), \Delta)$

Remark 3.6.1. In this section 3.6 we will use an approximate unit of $\mathcal{A}(\mathbb{G})$, that is, a sequence $(e_n)_n \in \mathcal{A}(\mathbb{G})$ which converges toward 1 in the bornology of $M(\mathcal{A}(\mathbb{G}))$. We impose as an hypothesis for the rest of this Chapter that there exists an approximate unit $(e_n)_n$ such that $(m(e_n))_n$ and $(m(\sigma_{i/2}(e_n)))_n$ are both uniformly bounded in $B(L^2(\mathbb{G}))$ (where $\sigma_{i/2}$ is considered in its bornological version). One could remark that, from the above, $(\sigma_{i/2}(e_n))_n$ is an approximate unit of $\mathcal{A}(\mathbb{G})$ since it converges toward $\sigma_{i/2}(1) = 1$ in $M(\mathcal{A}(\mathbb{G}))$.

The fact that the Haar functional $\phi_{\mathbb{G}}$ can be extended into a Haar weight of $C_0^r(\mathbb{G})$ is not trivial and, in the algebraic case, this is the whole consideration of [KvD97, Section 6]. Here we follow the ideas of that section.

To begin we recall the following result of [KvD97, Section 6]. This result can be directly applied in our case because it uses only common properties shared by algebraic and bornological quantum groups. This result uses the standard machinery of Hilbert algebras, which we will appeal to without comment. For details, we refer the reader to [KvD97] and to the books [Dix81, Tak70].

Proposition 3.6.2. *There exists a faithful lower semi-continuous weight of $C_0^r(\mathbb{G})$, denoted ϕ , such that $m(\mathcal{A}(\mathbb{G}))$ is a subset of \mathcal{N}_ϕ and $\phi(m(f)) = \phi_{\mathbb{G}}(f)$ for all $f \in \mathcal{A}(\mathbb{G})$. Moreover we have that ϕ is invariant under σ and more generally, $\Lambda_\phi(\sigma_t(x)) = \nabla^{it}\Lambda_\phi(x)$, we $\Lambda_\phi : \mathcal{N}_\phi \mapsto L^2(\mathbb{G})$ is the GNS map.*

Next, we relate this construction more specifically with the bornological structure.

Remark 3.6.3. One of our main motivation to study z -th powers of the modular automorphism σ and show that σ^z stabilizes our bornological algebra was to simplify the proof of the following theorem, which is the analogous of [KvD97, Theorem 6.12].

Theorem 3.6.4. *The set $m(\mathcal{A}(\mathbb{G}))$ is a core for Λ_ϕ .*

Proof. The linear map $\Lambda_0 : m(\mathcal{A}(\mathbb{G})) \rightarrow L^2(\mathbb{G})$, $m(f) \mapsto \Lambda(f)$ satisfies $\Lambda_0 \leq \Lambda_\phi$. Thus it is closable and we again denote Λ_0 its closure, with domain denoted by A_0 . Our goal is to show that $A_0 = \mathcal{N}_\phi$.

First, we observe that A_0 is a left ideal. Let $a \in A_0$ and $x \in C_0^r(\mathbb{G})$. Because of the closedness of Λ_0 , one can choose a sequence $a_n \in m(\mathcal{A}(\mathbb{G}))$ such that (a_n) converges to a in $C_0^r(\mathbb{G})$ and $(\Lambda_0(a_n))_n$ converges to $\Lambda_0(a)$. We also consider a sequence $x_n \in m(\mathcal{A}(\mathbb{G}))$ that converges to x . The sequence $(x_n a_n)$ converges to xa and for all n we have $\Lambda_0(x_n a_n) = x_n \Lambda_\phi(a_n) = x_n \Lambda_0(a_n)$. Thus $(\Lambda_0(x_n a_n))_n$ is convergent and so xa belongs to A_0 .

Let us now consider an approximate unit $(e_n)_n$ in $\mathcal{A}(\mathbb{G})$ such that $(m(e_n))_n$ satisfying hypothesis of Remark 3.6.1. Since $(m(e_n))_n$ is uniformly bounded, the sequence $(m(e_n))_n$ converges toward 1 in the strict topology of $M(C_0^r(\mathbb{G}))$, that is, $(xm(e_n))_n$ converges toward x for all $x \in C_0^r(\mathbb{G})$. Let $x \in \mathcal{N}_\phi$, each $xm(e_n)$ belongs to A_0 and we have

$$\Lambda_0(xm(e_n)) = \Lambda_\phi(xm(e_n)) = J\sigma_{i/2}(m(e_n^*))J(\Lambda_\phi(x)),$$

where we recall that J denotes the anti-unitary component of the polar decomposition of T such that $T\Lambda(f) = \Lambda(\bar{f})$, see Section 3.5. From Remark 3.6.1, we deduce that $(\Lambda_0(xm(e_n)))$ converges toward $\Lambda_\phi(x)$. Thus $A_0 = \mathcal{N}_\phi$, so $m(\mathcal{A}(\mathbb{G}))$ is a core for Λ_ϕ . \square

Lemma 3.6.5. *Consider $x, y \in m(\mathcal{A}(\mathbb{G}))$ and $\omega \in C_0^r(\mathbb{G})^*$. We have that $(y\omega \hat{\otimes} \text{id})(\Delta(x))$ belongs to $m(\mathcal{A}(\mathbb{G}))$ and $(y\omega \hat{\otimes} \phi)(\Delta(x)) = \omega(y)\phi(x)$, where we are using the notation $y\omega$ for the functional $y\omega : a \mapsto \omega(ay)$.*

Proof. Let $f, g \in \mathcal{A}(\mathbb{G})$. We have

$$\begin{aligned} (m(g)\omega \hat{\otimes} \text{id})(\Delta(m(f))) &= (\omega \hat{\otimes} \text{id})(\Delta(m(f))(m(g) \otimes 1)) \\ &= (\omega \hat{\otimes} \text{id})(m \hat{\otimes} m)(\Delta(f)(g \otimes 1)) \\ &= m((\omega \circ m \hat{\otimes} \text{id})(\Delta(f)(g \otimes 1))). \end{aligned}$$

Note that the last equality rests on the fact that $(\omega \circ m \otimes \text{id})$ is a bounded map and thus $(\omega \circ m \otimes \text{id})(\Delta(f)(g \hat{\otimes} 1))$ is well defined and belongs to $\mathcal{A}(\mathbb{G})$. One can then conclude using the left invariance of $\phi_{\mathbb{G}}$. \square

This lemma is a preliminary version of the left-invariance of the Haar weight ϕ . To convert it into the desired result, given the technical result Theorem 3.6.4, we can appeal exactly to the proof of [KvD97, Theorem 6.13].

Theorem 3.6.6. *Let $x \in M_\phi$ and $\omega \in C_0^r(\mathbb{G})^*$. We have that $(\omega \hat{\otimes} \text{id})(\Delta(x))$ belongs to M_ϕ and $(\omega \hat{\otimes} \phi_{\mathbb{G}})(\Delta(x)) = \omega(1)\phi(x)$.*

3.7 $C_0^r(\mathbb{G})$ as a reduced C^* -algebraic quantum group

It remains to show that the left Haar weight is KMS. In the context of algebraic quantum groups, Kustermans and Van Daele [KvD97] show the KMS property directly. Kustermans and Vaes [KV00] have since showed that approximately KMS suffices. By Definition 1.4.7, this means we must show that for a dense subset of elements $v \in L^2(\mathbb{G})$, there is a constant $M = M_v$ such that $\|xv\|_{L^2(\mathbb{G})} \leq M\|\Lambda(x)\|_{L^2(\mathbb{G})}$ for all $x \in \mathcal{N}_\phi$.

Proposition 3.7.1. *The Haar state ϕ is an approximate KMS state.*

Proof. Let $a \in \mathcal{A}(\mathbb{G})$. For all $x \in \mathcal{N}_\phi$ and $w \in L^2(\mathbb{G})$ we have

$$\begin{aligned} \langle \Lambda_\phi(xa), w \rangle &= \langle T\Lambda_\phi(a^*x^*), w \rangle \\ &= \langle J\sigma_{i/2}(m(a^*))J\Lambda_\phi(x), w \rangle. \end{aligned}$$

Using Proposition 3.5.2, we know that $\sigma_{i/2}(m(a^*)) = m(\sigma_{i/2}(a^*))$ and thus is a bounded operator and we have $\|x\Lambda(a)\| \leq \|\sigma_{i/2}(m(a^*))\|_{B(L^2(\mathbb{G}))}\|\Lambda_\phi(x)\|$. The result follows. \square

As usual, we use the notation R for the unitary antipode, that is, the unitary closure of the densely defined operator $\tau_{i/2} \circ S$. Note that $\phi \circ R = \phi \circ S$ on $m(\mathcal{A}(\mathbb{G}))$ and because

$\phi \circ S$ is positive on that dense subspace and $\phi \circ R$ is a well defined weight, we see that $\phi \circ R$ is a positive right Haar weight on $(C_0^r(\mathbb{G}), \Delta)$.

With the same arguments as in the preceding proof, we obtain the following proposition.

Proposition 3.7.2. *The weight right Haar weight $\phi \circ R$ of $(C_0^r(\mathbb{G}), \Delta)$ satisfies the approximate KMS condition.*

We now know that our quantum group $(C_0^r(\mathbb{G}), \Delta)$ satisfies the definition of a reduced C^* -algebraic quantum group.

Theorem 3.7.3. *The pair $(C_0^r(\mathbb{G}), \Delta)$ is a reduced C^* -algebraic quantum group.*

3.8 Von Neumann, Fourier and universal algebras

With the reduced C^* -algebraic quantum group $C_0^r(\mathbb{G})$ built from a bornological quantum group \mathbb{G} comes the algebras $L^\infty(\mathbb{G})$ and $L^1(\mathbb{G})$. It should be no surprise that the bornological algebra $\mathcal{A}(\mathbb{G})$ is dense in each of these, for the appropriate topologies. To complete this section, we make the necessary remarks to confirm this.

The fact that $\mathcal{A}(\mathbb{G})$ is weak operator dense in $L^\infty(\mathbb{G})$ is obvious, since $L^\infty(\mathbb{G})$ is the weak operator closure of $C_r^*(\mathbb{G})$.

For the Fourier algebra, defined in Section 1.4, we start with the Fourier algebra of the dual $\mathcal{A}(\hat{\mathbb{G}})$, $L^1(\mathbb{G})$, which is defined as the predual of $L^\infty(\mathbb{G})$.

Proposition 3.8.1. *The convolution algebra $\mathcal{D}(\mathbb{G})$ is a dense in the convolution algebra $L^1(\mathbb{G}) \cap L^1(\mathbb{G})^*$. Explicitly, for every $x \in \mathcal{D}(\mathbb{G})$, the linear functional*

$$\hat{x} : \mathcal{A}(\mathbb{G}) \rightarrow \mathbb{C}; \quad \hat{x} : a \mapsto \phi(ax)$$

extends to an ultraweakly continuous linear functional on $L^\infty(\mathbb{G})$, and the map $x \mapsto \hat{x}$ is a bounded $$ -algebra homomorphism of $\mathcal{D}(\mathbb{G})$ into $L^1(\mathbb{G})$ with dense range in the Banach topology.*

Proof. First consider $x = fg$ where $f, g \in \mathcal{A}(\mathbb{G})$ and the product is the pointwise product of $\mathcal{A}(\mathbb{G})$. Then

$$\hat{x}(a) = \phi(afg) = \phi(\sigma^{-1}(g)af) = \langle \Lambda(\sigma(\bar{g})), m(a)\Lambda(f) \rangle$$

for all $a \in \mathcal{A}(\mathbb{G})$. This obviously extends to an element of the predual of $L^\infty(\mathbb{G})$. Moreover, the sequence of maps

$$\begin{array}{ccccc} \mathcal{A}(\mathbb{G}) \times \mathcal{A}(\mathbb{G}) & \longrightarrow & L^2(\mathbb{G}) \otimes L^2(\mathbb{G}) & \longrightarrow & L^\infty(\mathbb{G})_* \\ f \otimes g & \longmapsto & \Lambda(\sigma(\bar{g})) \otimes \Lambda(f) & \longmapsto & \langle \Lambda(\sigma(\bar{g})), m(\bullet)\Lambda(f) \rangle \end{array}$$

induces a bounded map $\mathcal{A}(\mathbb{G}) \hat{\otimes}_{\mathcal{A}(\mathbb{G})} \mathcal{A}(\mathbb{G}) \cong \mathcal{A}(\mathbb{G}) \rightarrow L^1(\mathbb{G})$, where we use the same argument as in the proof of Lemma 3.2.10.

Suppose now that $a \in L^\infty(\mathbb{G})$ is such that $\hat{x}(a) = 0$ for all $x \in \mathcal{D}(\mathbb{G})$. Then the above calculations show that $\langle \Lambda(g), m(a)\Lambda(f) \rangle = 0$ for all $f, g \in \mathcal{A}(\mathbb{G})$. But $\Lambda(\mathcal{A}(\mathbb{G}))$ is dense in $L^2(\mathbb{G})$, so we get $a = 0$. This proves that the image of $\mathcal{D}(\mathbb{G})$ in $L^1(\mathbb{G})$ is dense.

The convolution products on $\mathcal{D}(\mathbb{G})$ and its image in $L^1(\mathbb{G})$ clearly coincide because both are dual to the product in $\mathcal{A}(\mathbb{G}) \subset L^\infty(\mathbb{G})$. \square

By duality, the bornological algebra $\mathcal{A}(\mathbb{G})$ is dense in the Fourier algebra $A(\mathbb{G}) = L^\infty(\hat{\mathbb{G}})_*$. We will frequently use the notation $x \mapsto \tilde{x}$ for the inclusion of $\mathcal{D}(\mathbb{G})$ into $L^1(\mathbb{G})$, and likewise $a \mapsto \tilde{a}$ for the inclusion of $\mathcal{A}(\mathbb{G})$ into $A(\mathbb{G})$.

Finally, the universal C^* -algebraic quantum group $C_0^u(\mathbb{G})$ is the enveloping C^* -algebra of $A(\mathbb{G})$. We will write $m_{\mathbb{G}}^u$ for the universal representation of $\mathcal{A}(\mathbb{G})$, namely,

$$m_{\mathbb{G}}^u : \mathcal{A}(\mathbb{G}) \rightarrow A(\mathbb{G}) \rightarrow C_0^u(\mathbb{G}).$$

This is an injective $*$ -algebra map with dense range. Dually, we write

$$\lambda_{\mathbb{G}}^u : \mathcal{D}(\mathbb{G}) \rightarrow C_u^*(\mathbb{G})$$

for the universal representation of the convolution algebra. We may also consider elements of the Fourier algebra $A(\mathbb{G})$, or indeed the dense subalgebra $\mathcal{A}(\mathbb{G})$, as forms on $C_u^*(\mathbb{G})$, by precomposing with the regular representation $C_u^*(\mathbb{G}) \rightarrow C_r^*(\mathbb{G})$.

3.9 Homomorphisms and closed quantum subgroups

One of the major advantages of bornological quantum groups is the simplicity of the notion of a quantum subgroup. In this section we define closed quantum subgroups of bornological quantum groups, and show that they give rise to closed quantum subgroups of the corresponding locally compact quantum groups.

Our goal in the next section is to show the compatibility of this definition with that given at the locally compact level in 1.4.21. The proof of the equivalence of these definitions relies on lifting bicharacters to the universal algebras. Explicitly, there is a bicharacter $W^u \in M(C_0^u(\mathbb{G}) \hat{\otimes} C_u^*(\mathbb{G}))$, called the universal multiplicative unitary, which is uniquely characterized by the fact that its image under the regular representations is the usual multiplicative unitary $W \in M(C_r^*(\mathbb{G}) \hat{\otimes} C_r^*(\mathbb{G}))$. The universal bicharacter can be defined by $V^u = (\pi \hat{\otimes} \text{id})(W^u)$.

To make the connection with bornological quantum groups, we have the following simple construction.

Proposition 3.9.1. *Let $\pi : \mathcal{A}(\mathbb{G}) \rightarrow M(\mathcal{A}(\mathbb{H}))$ be a morphism of bornological quantum groups. The element*

$$(m_{\mathbb{H}} \circ \pi \hat{\otimes} \lambda_{\mathbb{G}})(\mathcal{W}) \in M(C_0^r(\mathbb{H}) \hat{\otimes} C_r^*(\mathbb{G}))$$

is a bicharacter from \mathbb{G} to \mathbb{H} as locally compact quantum groups.

Proof. Given a morphism π of bornological quantum groups from \mathbb{H} to \mathbb{G} as above, we define a *bornological bicharacter*

$$\mathcal{V} = (\pi \hat{\otimes} \text{id})\mathcal{W} \in M(\mathcal{A}(\mathbb{H}) \hat{\otimes} \mathcal{D}(\mathbb{G})).$$

It satisfies the properties

$$(\Delta_{\mathbb{H}} \hat{\otimes} \text{id})\mathcal{V} = \mathcal{V}_{13}\mathcal{V}_{23}, \quad (\text{id} \hat{\otimes} \Delta_{\mathbb{G}})\mathcal{V} = \mathcal{V}_{13}\mathcal{V}_{12}, \quad (3.3)$$

thanks to the analogous properties of \mathcal{W} , see Equation (2.7).

Moreover, it is a unitary multiplier in the same sense as \mathcal{W} from Proposition 2.3.15. Therefore, it maps under the regular representations $m_{\mathbb{G}} \hat{\otimes} \lambda_{\mathbb{H}}$ to a unitary Hilbert space operator V on $L^2(\mathbb{G}) \hat{\otimes} L^2(\mathbb{H})$ which is therefore a unitary bicharacter in the C^* -algebraic sense. \square

Combining Proposition 3.9.1 with Definition-Proposition 1.4.21, the above proposition yields a Hopf $*$ -morphism

$$\tilde{\pi} : C_0^u(\mathbb{G}) \rightarrow M(C_0^u(\mathbb{H}))$$

associated to any morphism π of bornological quantum groups. As mentioned above, this morphism is obtained by passing via the universal bicharacter V^u , which can be made explicit as follows.

Lemma 3.9.2. *The element*

$$V^u = (m_{\mathbb{H}}^u \circ \pi \hat{\otimes} \lambda_{\mathbb{G}}^u)(\mathcal{W})$$

is the universal bicharacter associated to the bicharacter V from Proposition 3.9.1, where $m_{\mathbb{H}}^u$ and $\lambda_{\mathbb{G}}^u$ denote universal representations of $\mathcal{A}(\mathbb{H})$ and $\mathcal{D}(\mathbb{G})$, respectively.

Proof. The proof is essentially the same as that of Proposition 3.9.1. The operator $(m_{\mathbb{H}}^u \circ \pi \hat{\otimes} \lambda_{\mathbb{G}}^u)(\mathcal{W})$ is a densely defined multiplier of $C_0^u(\mathbb{G}) \hat{\otimes} C_u^*(\mathbb{G})$. It is unitary on its domain, so extends to a bounded multiplier, and again satisfies the bicharacter properties. Applying the regular representations, V^u maps to the reduced bicharacter $V \in C_0^r(\mathbb{H}) \hat{\otimes} C_r^*(\mathbb{G})$ from the previous lemma. This characterizes the universal bicharacter uniquely, see [MRW12]. \square

We are now in a position to directly compare the bornological and C^* -algebraic maps arising from a homomorphism of bornological quantum groups.

Theorem 3.9.3. *Let $\pi : \mathcal{A}(\mathbb{G}) \rightarrow M(\mathcal{A}(\mathbb{H}))$ be a morphism of bornological quantum groups from \mathbb{H} to \mathbb{G} . We have a commuting diagram*

$$\begin{array}{ccc} \mathcal{A}(\mathbb{G}) & \xrightarrow{\pi} & \pi(\mathcal{A}(\mathbb{G})) \subseteq M(\mathcal{A}(\mathbb{H})) \\ m_{\mathbb{G}}^u \downarrow & & \downarrow m_{\mathbb{H}}^u \\ C_0^u(\mathbb{G}) & \xrightarrow{\tilde{\pi}} & M(C_0^u(\mathbb{H})) \end{array}$$

where the vertical arrows are the natural inclusions.

Remark 3.9.4. Note that one cannot define the right-hand vertical map directly on the bornological multipliers in $M(\mathcal{A}(\mathbb{H}))$, since these will generally map to unbounded multipliers of $C_0^u(\mathbb{H})$.

The extension of the universal representation $m_{\mathbb{H}}^u : \mathcal{A}(\mathbb{H}) \rightarrow C_0^u(\mathbb{H})$ to $\pi(\mathcal{A}(\mathbb{G}))$ is made explicit in the proof below.

Proof. It is a consequence of Proposition 3.1.6 that any element $a \in \mathcal{A}(\mathbb{G})$ can be written as $a = (\text{id} \hat{\otimes} \omega) \mathcal{W}_{\mathbb{G}}$ for some $\omega \in \mathcal{A}(\mathbb{G}) \subseteq \mathcal{D}(\mathbb{G})^*$. Then we have

$$\pi(a) = (\pi \hat{\otimes} \omega) \mathcal{W}_{\mathbb{G}} = (\text{id} \hat{\otimes} \omega) \mathcal{V},$$

where $\mathcal{V} = (\pi \hat{\otimes} \text{id}) \mathcal{W}$ is the bornological bicharacter associated to the morphism π , as above.

We can then define

$$m_{\mathbb{H}}^u(\pi(a)) = ((m_{\mathbb{H}}^u \circ \pi) \hat{\otimes} \omega)(\mathcal{W}_{\mathbb{G}}) = (\text{id} \hat{\otimes} \tilde{\omega})(V^u),$$

where the second equality uses Lemma 3.9.2 and $\tilde{\omega}$ denotes the image of ω in the Fourier algebra $A(\mathbb{G})$, see Proposition 3.8.1 and the remarks that follow it. This map is well-defined because if $\pi(a) = 0$ then $(\pi \otimes \omega) \mathcal{W}_{\mathbb{G}} = 0$ and so the expression defining $m_{\mathbb{H}}^u(\pi(a))$ is zero.

To check that the diagram commutes, we note that the image of $a = (\text{id} \otimes \omega) \mathcal{W}_{\mathbb{G}}$ under $m_{\mathbb{G}}^u$ is $(\text{id} \hat{\otimes} \tilde{\omega}) \mathcal{W}_{\mathbb{G}}^u$, so that $\tilde{\pi} \circ m_{\mathbb{G}}^u(a) = (\text{id} \hat{\otimes} \tilde{\omega})(V^u)$, as desired. \square

Theorem 3.9.5. *Let \mathbb{H} be a closed quantum subgroup of a bornological quantum group \mathbb{G} . Then the corresponding locally compact quantum group \mathbb{H} is a closed quantum subgroup of the locally compact quantum group \mathbb{G} in the sense of Vaes (and hence also the sense of Woronowicz).*

Proof. Explicitly, we will show that there is a commuting diagram

$$\begin{array}{ccc}
\mathcal{D}(\mathbb{H}) & \xrightarrow{\hat{\pi}} & \hat{\pi}(\mathcal{D}(\mathbb{H})) \subseteq M(\mathcal{D}(\mathbb{G})) \\
\lambda_{\mathbb{H}}^u \downarrow & & \downarrow \lambda_{\mathbb{G}}^u \\
C_u^*(\mathbb{H}) & \xrightarrow{\hat{\pi}} & M(C_u^*(\mathbb{G})) \\
\downarrow & & \downarrow \\
\mathcal{L}(\mathbb{H}) & \xrightarrow{\hat{\pi}} & \mathcal{L}(\mathbb{G}).
\end{array}$$

Recall, from Definition 1.4.22 that we need to prove the commutativity of the bottom square as well as the injectivity of the bottom arrows. The top square is the dual of the commuting square from Theorem 3.9.3 (we are suppressing the tilde from the horizontal C^* -algebra map $\tilde{\pi}$ to simplify the notation). For the von Neumann morphism, for any $x \in \mathcal{D}(\mathbb{H})$ and $b \in \mathcal{A}(\mathbb{G})$ we have $(\hat{\pi}(x), b) = (x, \pi(b))$, and it follows that $\hat{\pi}$ is ultraweakly continuous, so can be extended to a normal unital $*$ -homomorphism. The outer rectangle is commutative and thus, by the density of $\mathcal{D}(\mathbb{H})$ in $C_u^*(\mathbb{H})$, the bottom square is also commutative. The crucial point is to prove that the von Neumann algebra map is injective.

From Proposition 3.8.1, $\mathcal{A}(\mathbb{H}) = \mathcal{D}(\hat{\mathbb{H}})$ embeds as a dense subspace of the predual $\mathcal{L}(\mathbb{H})_*$. Explicitly, we identify $a \in \mathcal{A}(\mathbb{H})$ with the functional $\tilde{a} \in \mathcal{L}(\mathbb{H})_*$ where

$$\tilde{a}(x) = \hat{\phi}_{\mathbb{H}}(\mathcal{F}(x)\mathcal{F}(a)) = \epsilon_{\mathbb{H}}(x * a)$$

for $a \in \mathcal{A}(\mathbb{H})$, $x \in \mathcal{D}(\mathbb{H})$. Choose $b \in \mathcal{A}(\mathbb{G})$ with $\pi(b) = a$. Using Proposition ??, we have

$$\tilde{a}(x) = \epsilon_{\mathbb{H}}(x * \pi(b)) = \epsilon_{\mathbb{H}}(\pi(\hat{\pi}(x) * b)) = \epsilon_{\mathbb{G}}(\hat{\pi}(x) * b) = \hat{b}(\hat{\pi}(x)),$$

for all $x \in \mathcal{D}(\mathbb{H})$, and hence $\tilde{a}(x) = \hat{b}(\hat{\pi}(x))$ for all $x \in \mathcal{L}(\mathbb{G})$ by ultraweak continuity. Therefore, if $x \in \mathcal{L}(\mathbb{H})$ is in the kernel of $\hat{\pi}$ then x is annihilated by all of the functionals \tilde{a} with $a \in \mathcal{A}(\mathbb{H})$. These are dense in the $\mathcal{L}(\mathbb{H})_*$ so $x = 0$. \square

Chapter 4

Rieffel induction for bornological quantum groups

Our main purpose in this thesis is to study induced representations of quantum groups. Methods of induction for locally compact quantum groups are discussed in [Kus02] and [Vae05]. In Chapter 1, we have seen an ad hoc construction of parabolically induced representations for complex semisimple quantum groups, which has been developed by [Ara14] and [VY20].

Let us briefly recall the classical construction. Let G be a locally compact group and H a closed subgroup with a unitary representation α on a Hilbert space V . The unitary induction procedure developed by Mackey in [Mac52] works as follows. One build a space of H -equivariant functions on G valued in the Hilbert space V . Mackey showed that this space is naturally a unitary representation of G . In the case of parabolic induction for a semisimple Lie group, this construction yields the unitary principal series representations, which are generically irreducible.

Mackey's method is one of the fundamental operations in representation theory of locally compact groups. After Mackey's publication, it was realised that this process can be generalized. Rieffel proposed a framework for induced representations of C^* -algebras in [Rie74]. The ideas of Mackey are formulated in a geometrical setting and inspired by algebraic methods and Rieffel made them compatible with C^* -algebras. In short, if one starts with a C^* -algebra A and a C^* -subalgebra B and given a $*$ -representation V of B , one obtains a representation of A on the space

$$A \hat{\otimes}_B V.$$

By using C^* -Hilbert module techniques [Lan95], one can turn $A \hat{\otimes}_B V$ into a Hilbert space and even more generally, a C^* -Hilbert module for any C^* -algebra C if V is also chosen to

be a C -Hilbert module.

Of course, one of the main motivations of Rieffel was to apply this procedure to group C^* -algebras in order to generalize Mackey's induction. This approach has been adapted to quantum groups; the main references for this are [Vae05][Kus02][KKSS]. On the one hand in [Vae05], Vaes succeeded to get a very general and elegant framework for quantum group induction where he proved the various versions of the imprimitivity theorem. Nevertheless, this work is formalized in the Von Neumann algebraic setting which does not make the calculations easy. On the other hand, Kalantar et al. in [KKSS] suggested something more computable but limited to the case where one has a quantum subgroup \mathbb{H} of a locally compact quantum group \mathbb{G} which is both closed and open. The problem is that, as noticed by Rieffel in the classical case, the C^* -algebra $C^*(\mathbb{H})$ does not sit as a subalgebra of $C^*(\mathbb{G})$ in general.

In [Rie74, Section 4], Rieffel suggested to work with the algebras $C_c(G)$ and $C_c(H)$ and consider the restriction map $C_c(G) \rightarrow C_c(H)$ as a kind of (non-continuous) conditional expectation. So far, we have no way to build a similar algebra out of a general locally compact quantum group \mathbb{G} . This is why we propose to use the bornological framework in order to start with an appropriate dense convolution algebra, which then allows us to imitate the Rieffel construction. The goal of this chapter is to show that this translation is compatible with Vaes' approach when one considers the C^* -algebraic quantum group associated to the starting bornological quantum groups, in the sense of the previous chapter.

Throughout this chapter we consider bornological quantum groups $\mathcal{A}(\mathbb{G})$ and $\mathcal{A}(\mathbb{B})$ such that there exists a morphism $\pi : \mathcal{A}(\mathbb{G}) \rightarrow \mathcal{A}(\mathbb{B})$ that identifies $\mathcal{A}(\mathbb{B})$ as a closed quantum subgroup of $\mathcal{A}(\mathbb{G})$. The choice of the notation \mathbb{B} is motivated by the application to semisimple quantum groups where \mathbb{B} will be the Borel subgroup. It is further assumed that \mathbb{B} is amenable that is we have $C_u^*(\mathbb{B}) = C_r^*(\mathbb{B})$. We will simply denote this C^* -algebra $C^*(\mathbb{B})$.

Remark 4.0.1. In general we have $\sigma_{\mathbb{B}}(\pi(\delta_{\mathbb{G}})) = \mu\pi(\delta_{\mathbb{G}})$ for some complex number μ with modulus 1 (this can be proved exactly as Lemma 3.4.10). As for the scaling constant, **we make the hypothesis that this constant equals 1**. In particular we have $\sigma_{\mathbb{B}}(\pi(\delta_{\mathbb{G}}^{\frac{1}{2}})) = \pi(\delta_{\mathbb{G}}^{\frac{1}{2}})$.

4.1 The generalized conditional expectation

The convolution algebras $\mathcal{D}(\mathbb{G})$ and $\mathcal{D}(\mathbb{B})$ are, by definition, identified as linear spaces with the spaces $\mathcal{A}(\mathbb{G})$ and $\mathcal{A}(\mathbb{B})$. Therefore the map $\pi : \mathcal{A}(\mathbb{G}) \rightarrow \mathcal{A}(\mathbb{B})$ can also be seen as a map from $\mathcal{D}(\mathbb{G})$ to $\mathcal{D}(\mathbb{B})$. However as it stands, this map does not have the properties

of what we will call a *generalized conditional expectation*. Instead, we first define

$$\gamma = \pi(\delta_{\mathbb{G}}^{-\frac{1}{2}})\delta_{\mathbb{B}}^{\frac{1}{2}} \in M(\mathcal{A}(\mathbb{B})),$$

which is a group-like element. And now we modify the map π into

$$E : \mathcal{D}(\mathbb{G}) \rightarrow \mathcal{D}(\mathbb{B}), \quad E(f) = \pi(f)\gamma.$$

In order to describe the relevant properties of E , we must start with some preliminaries concerning the action of $\mathcal{D}(\mathbb{B})$ on $\mathcal{D}(\mathbb{G})$. We consider the morphism $\hat{\pi} : \mathcal{D}(\mathbb{B}) \rightarrow M(\mathcal{D}(\mathbb{G}))$, defined in Section 2.3.5 and set for all f in $\mathcal{D}(\mathbb{G})$ and for all $h \in \mathcal{D}(\mathbb{B})$

$$f \cdot h = f * \hat{\pi}(h\gamma).$$

Proposition 4.1.1. *The map $f \mapsto f \cdot h$ defines a right action of the algebra $\mathcal{D}(\mathbb{B})$ on the space $\mathcal{D}(\mathbb{G})$.*

Proof. Let $h, k \in \mathcal{D}(\mathbb{B})$. Since γ is group-like we have $(h * k)\gamma = h\gamma * k\gamma$ and thus for $f \in \mathcal{D}(\mathbb{G})$ we have

$$\begin{aligned} f \cdot (h * k) &= f * \hat{\pi}(h\gamma) * \hat{\pi}(k\gamma) \\ &= (f \cdot h) \cdot k. \end{aligned}$$

□

We are going to prove that E preserves the *-involution and has a “conditional expectation” property with respect to this action.

Lemma 4.1.2. *The two multipliers $\delta_{\mathbb{B}}$ and $\pi(\delta_{\mathbb{G}})$ commute.*

Proof. We know that we have

$$\phi_{\mathbb{B}}(S(h)) = \phi_{\mathbb{B}}(h\delta_{\mathbb{B}}),$$

for all $h \in \mathcal{A}(\mathbb{B})$. By our hypothesis in Remark 4.0.1 we also have that $\sigma_{\mathbb{B}}(\pi(\delta_{\mathbb{G}}^{-1})) = \pi(\delta_{\mathbb{G}}^{-1})$. Let then $h \in \mathcal{A}(\mathbb{B})$. We have $\phi_{\mathbb{B}}(S(\pi(\delta_{\mathbb{G}})h)) = \phi_{\mathbb{B}}(S(h\pi(\delta_{\mathbb{G}})))$. On the one hand this gives

$$\begin{aligned} \phi_{\mathbb{B}}(S(\pi(\delta_{\mathbb{G}})h)) &= \phi_{\mathbb{B}}(\pi(\delta_{\mathbb{G}})h\delta_{\mathbb{B}}) \\ &= \phi_{\mathbb{B}}(h\delta_{\mathbb{B}}\pi(\delta_{\mathbb{G}})), \end{aligned}$$

and on the other

$$\phi_{\mathbb{B}}(S(h\pi(\delta_{\mathbb{G}}))) = \phi_{\mathbb{B}}(h\pi(\delta_{\mathbb{G}})\delta_{\mathbb{B}}).$$

Therefore $\delta_{\mathbb{B}}\pi(\delta_{\mathbb{G}}) = \pi(\delta_{\mathbb{G}})\delta_{\mathbb{B}}$. □

Proposition 4.1.3. *Let f in $\mathcal{D}(\mathbb{G})$ and $h \in \mathcal{D}(\mathbb{B})$. We have seen in Proposition ?? that*

$$\begin{aligned}\hat{\pi}(h) * f &= f_{(2)}\phi_{\mathbb{B}}(\pi(S^{-1}(f_{(1)}))h), \\ f * \hat{\pi}(h) &= f_{(1)}\phi_{\mathbb{B}}(S^{-1}(h)\pi(f_{(2)})\gamma^2).\end{aligned}$$

Note that since $\pi(\delta_{\mathbb{G}})$ and $\delta_{\mathbb{B}}$ commute, we have $\pi(\delta_{\mathbb{G}}^{-1})\delta_{\mathbb{B}} = \gamma^2$.

Proposition 4.1.4. *The map $E : \mathcal{D}(\mathbb{G}) \rightarrow \mathcal{D}(\mathbb{B})$, $E(f) = \pi(f)\gamma$, has the two following properties :*

1. $E(f^*) = E(f)^*$, for all $f \in \mathcal{D}(\mathbb{G})$,
2. $E(f \cdot h) = E(f) * h$. for all $f \in \mathcal{D}(\mathbb{G})$ and $h \in \mathcal{D}(\mathbb{B})$.

The map E is the generalized conditional expectation we were looking to build.

Proof. Let $f \in \mathcal{D}(\mathbb{G})$. We have

$$\begin{aligned}E(f^*) &= E(\overline{S(f)}\delta_{\mathbb{G}}) \\ &= \overline{S(\pi(f))}\pi(\delta_{\mathbb{G}})\gamma \\ &= \overline{S(\pi(f))}\pi(\delta_{\mathbb{G}}^{\frac{1}{2}})\delta_{\mathbb{B}}^{\frac{1}{2}} \\ &= \overline{S(\pi(f)\gamma)}\delta_{\mathbb{B}} = E(f)^*.\end{aligned}$$

Now let $h \in \mathcal{D}(\mathbb{B})$. Using that $\sigma(\gamma^{-1}) = \gamma^{-1}$ we get

$$\begin{aligned}E(f \cdot h) &= E(f * \hat{\pi}(h\gamma)) \\ &= (\text{id} \hat{\otimes} \phi_{\mathbb{B}})((1 \otimes \gamma^{-1}S^{-1}(h))(\pi \hat{\otimes} \pi)(\Delta(f))(1 \otimes \gamma^2))\gamma \\ &= (\text{id} \hat{\otimes} \phi_{\mathbb{B}})((1 \otimes S^{-1}(h))(\pi \hat{\otimes} \pi)(\Delta(f))(\gamma \otimes \gamma)) \\ &= E(f) * h.\end{aligned}$$

□

4.2 The induction module $\mathcal{E}(\mathbb{G})$

The goal of this section is to define a Hilbert $C^*(\mathbb{B})$ -module with a left $C_u^*(\mathbb{G})$ -action by completing $\mathcal{D}(\mathbb{G})$. We equip the space $\mathcal{D}(\mathbb{G})$ with the right action of $\mathcal{D}(\mathbb{B})$ defined as in Proposition 4.1.1.

Definition 4.2.1. *Let \mathcal{V} be a right $\mathcal{D}(\mathbb{B})$ -module. A $\mathcal{D}(\mathbb{B})$ -valued inner product on \mathcal{V} will mean a sesquilinear map $\langle \cdot, \cdot \rangle : \mathcal{V} \times \mathcal{V} \rightarrow \mathcal{D}(\mathbb{B})$ such that for all $v, w \in \mathcal{V}$ and $h \in \mathcal{D}(\mathbb{B})$ we have*

1. $\langle v, w \cdot h \rangle = \langle v, w \rangle * h$,
2. $\langle v, w \rangle^* = \langle w, v \rangle$.
3. $\lambda_{\mathbb{B}}(\langle v, v \rangle)$ is a positive element of $C^*(\mathbb{B})$ and $\langle v, v \rangle = 0 \Leftrightarrow v = 0$.

Such a module \mathcal{V} endowed with a $\mathcal{D}(\mathbb{B})$ -valued inner product will be called a $\mathcal{D}(\mathbb{B})$ -inner product space.

Remark 4.2.2. The fact that we have to call on the regular representation $\lambda_{\mathbb{B}}$ is not very aesthetic but this is because the notion of positivity in the bornological quantum group $\mathcal{D}(\mathbb{B})$ cannot be defined intrinsically. Furthermore, in the case where \mathbb{B} is not amenable, we should use the universal representation $\lambda_{\mathbb{B}}^u$ to ensure that \mathcal{V} leads to a $C_u^*(\mathbb{B})$ -module. However, to build the induction module it seems necessary to work with the regular representation, in the spirit of Lemma 4.2.6 below. In the classical case Rieffel [Rie74] used the existence of a Bruhat section, and one could imagine adding the hypothesis that a quantum analogue of a Bruhat section exists in our case as well. Nevertheless, assuming that a Bruhat section exists seems technically problematic, since it is not clear if one can suppose its existence in $\mathcal{A}(\mathbb{G})$ or if one must work in $C_0^u(\mathbb{G})$. We will not pursue this line of research further in the present work.

Proposition 4.2.3. *The sesquilinear map $\langle \cdot, \cdot \rangle_{\mathcal{D}(\mathbb{B})}$ defined for $f, g \in \mathcal{D}(\mathbb{G})$ by*

$$\langle f, g \rangle_{\mathcal{D}(\mathbb{B})} = E(f^* * g),$$

defines a $\mathcal{D}(\mathbb{B})$ -valued inner product.

To prove the previous proposition we note first that the $\mathcal{D}(\mathbb{B})$ -linearity and compatibility with the involution of the above sesquilinear map follow immediately from Proposition 4.1.4. It only remains to check the strict positivity, which will be a consequence of Proposition 4.2.6 below.

Remark 4.2.4. Let $f, g \in \mathcal{D}(\mathbb{G})$. We have

$$\begin{aligned} E(f^* * g) &= \phi_{\mathbb{G}}(S^{-1}(g_{(1)})\overline{S(f)}\delta_{\mathbb{G}})\pi(g_{(2)})\gamma \\ &= \phi_{\mathbb{G}}(\bar{f}g_{(1)})\pi(g_{(2)})\gamma. \end{aligned}$$

Remark 4.2.5. In what follows we will often use the maps $\Lambda_{\mathbb{G}} : \mathcal{A}(\mathbb{G}) \rightarrow L^2(\mathbb{G})$, $\lambda_{\mathbb{G}} : \mathcal{D}(\mathbb{G}) \rightarrow B(L^2(\mathbb{G}))$ defined in Chapter 3 and the analogous maps $\Lambda_{\mathbb{B}}$ and $\lambda_{\mathbb{B}}$, but we will only write Λ and λ . Their relation to \mathbb{B} or \mathbb{G} will depend on the context.

Proposition 4.2.6. *The linear map $\rho_\bullet : \mathcal{D}(\mathbb{G}) \rightarrow B(L^2(\mathbb{B}), L^2(\mathbb{G}))$ defined by $f \mapsto \rho_f$ where*

$$\rho_f(\Lambda(\eta)) = \Lambda(f \cdot \eta), \quad \forall f \in \mathcal{D}(\mathbb{G}), \forall \eta \in \mathcal{D}(\mathbb{B}),$$

satisfies

$$\lambda_{\mathbb{B}}(\langle f, g \rangle_{\mathcal{D}(\mathbb{B})}) = \rho_f^* \rho_g.$$

Proof. First we claim that, as an operator from $L^2(\mathbb{G})$ to $L^2(\mathbb{B})$, $(\rho_f)^*$ acts on elements of $\Lambda(\mathcal{A}(\mathbb{G}))$ as

$$(\rho_f)^* : \Lambda(g) \mapsto \Lambda(\langle f, g \rangle_{\mathcal{D}(\mathbb{B})}).$$

For this, note that using $\epsilon_{\mathbb{B}}(E(x)) = \epsilon_{\mathbb{G}}(x)$, for any $x \in \mathcal{D}(\mathbb{G})$, we obtain $\epsilon_{\mathbb{B}}(E(x * y)) = \epsilon_{\mathbb{G}}(x^* * y) = \langle x, y \rangle_{L^2(\mathbb{G})}$ for any $x, y \in \mathcal{D}(\mathbb{G})$. Therefore, for all $\eta \in \mathcal{A}(\mathbb{B})$ and $\xi \in \mathcal{A}(\mathbb{G})$ we have

$$\begin{aligned} \langle \rho_f \Lambda(\eta), \Lambda(\xi) \rangle_{L^2(\mathbb{G})} &= \langle \Lambda(f \cdot \eta), \Lambda(\xi) \rangle_{L^2(\mathbb{G})} \\ &= \epsilon_{\mathbb{B}}(\langle f \cdot \eta, \xi \rangle_{\mathcal{D}(\mathbb{B})}) \\ &= \epsilon_{\mathbb{B}}(\eta^* * \langle f, \xi \rangle_{\mathcal{D}(\mathbb{B})}) \\ &= \langle \Lambda(\eta), \Lambda(\langle f, \xi \rangle) \rangle_{L^2(\mathbb{B})}. \end{aligned}$$

We therefore have :

$$\begin{aligned} \Lambda(\langle f, g \rangle_{\mathcal{D}(\mathbb{B})} * \eta) &= \Lambda(\langle f, g \cdot \eta \rangle_{\mathcal{D}(\mathbb{B})}) \\ &= \rho_f^* \rho_g \Lambda(\eta). \end{aligned}$$

□

This concludes the proof of Proposition 4.2.3. We also record the explicit formula

$$\langle f, g \rangle_{\mathcal{D}(\mathbb{B})} = (\phi_{\mathbb{G}} \hat{\otimes} \text{id})((\bar{f} \otimes 1) \Delta(g) (1 \otimes \gamma)).$$

Definition 4.2.7. *The Hilbert $C^*(\mathbb{B})$ -module obtained by completing $\mathcal{D}(\mathbb{G})$ with respect to the inner product above is denoted $\mathcal{E}(\mathbb{G})$ and we call it the induction module (associated to \mathbb{B}).*

See [Lan95] for details about the completion. The space $\mathcal{E}(\mathbb{G})$ is innately equipped with a left $C_u^*(\mathbb{G})$ -action, which commutes with the right $C^*(\mathbb{B})$ -action. We then get our induction bi-module

$$C_u^*(\mathbb{G}) \mathcal{E}(\mathbb{G})_{C^*(\mathbb{B})}.$$

Now, for α a representation of $C^*(\mathbb{B})$ on an A -Hilbert module K (where A is any C^* -algebra) we consider, following Rieffel's definition for induced representations in [Rie74], the A -Hilbert module

$$\text{Ind}_{\mathbb{B}}^{\mathbb{G}} V = \mathcal{E}(\mathbb{G}) \hat{\otimes}_{C^*(\mathbb{B})} V,$$

where the tensor product is completed with respect to the interior inner product [Lan95, Proposition 4.5].

4.3 Link with Vaes' approach to induction

We consider in this Section our bornological quantum groups \mathbb{G} and \mathbb{B} as locally compact quantum groups and we **assume** that \mathbb{B} is a closed quantum subgroup of \mathbb{G} in the sens of 1.4.22. We illustrate in what follows the induction procedure of [Vae05]. Let us recall that we have at the bornological level a map $\hat{\pi} : \mathcal{D}(B) \rightarrow M(\mathcal{D}(\mathbb{G}))$ defined in Section 2.3.5, and its operator algebraic version $\hat{\pi} : \mathcal{L}(B) \rightarrow \mathcal{L}(\mathbb{G})$, encountered in Theorem 3.9.5. We recall that those two maps are linked via the relation $\lambda(\hat{\pi}(f)) = \hat{\pi}(\lambda(f))$, for all $f \in \mathcal{D}(\mathbb{G})$. Throughout this section we will be using the definitions and notations of Section 1.4.

We recall first what Vaes' induction process consists of. We begin with some definitions and results from [Vae05, Section 3]. We consider (A, Δ) a locally compact quantum group with von Neumann algebra M and GNS Hilbert space H . We also fix a C^* -algebra B . If \mathcal{V} is a C^* - B -module we write $\mathcal{L}(\mathcal{V})$ for the $*$ -algebra of adjointable B -linear operators.

Definition 4.3.1. *Let N be a von Neumann algebra and \mathcal{V} a C^* - B -module. A unital $*$ -homomorphism $\beta : N \rightarrow \mathcal{L}(\mathcal{V})$ is said to be strict (or normal) if it is strong $*$ continuous on the unit ball of N .*

Definition 4.3.2. *Let M and N be von Neumann algebras. We say that a C^* - B -module \mathcal{V} is a B -correspondence from N to M if we have*

- a strict $*$ -homomorphism $\beta_l : M \rightarrow \mathcal{L}(\mathcal{V})$,
- a strict $*$ -antihomomorphism $\beta_r : N \rightarrow \mathcal{L}(\mathcal{V})$, such that $\beta_l(M)$ and $\beta_r(N)$ commute.

Remark 4.3.3. In [Vae05] the notation π is used instead of β . Here we keep π to designate the morphism from $\mathcal{A}(\mathbb{G})$ to $\mathcal{A}(\mathbb{B})$.

We will denote $x \cdot v = \beta_l(x)v$ and $v \cdot y = \beta_r(y)v$ for all $x \in M$, $y \in N$ and $v \in \mathcal{V}$ and this correspondence will be denoted as $M \boxed{\mathcal{V}}_N$.

Proposition 4.3.4. ([Vae05, Proposition 3.4]). *Let $X \in \mathcal{L}(A \hat{\otimes} \mathcal{V})$ be a unitary corepresentation on a C^* - B -module \mathcal{E} . There is a B -correspondence $\hat{M} \boxed{H \hat{\otimes} \mathcal{V}}_{\hat{M}}$ given by*

$$x \cdot v = X(x \otimes 1)X^*v \text{ and } v \cdot y = (\hat{J}_{\mathbb{G}}y^*\hat{J}_{\mathbb{G}} \otimes 1)v \text{ for } x, y \in \hat{M}, v \in H \hat{\otimes} \mathcal{V}.$$

Definition 4.3.5. ([Vae05, Definition 3.5]). Let ${}_{\hat{M}}\boxed{\mathcal{F}}_{\hat{M}}$ be a B -correspondence from \hat{M} to \hat{M} and suppose $\beta : M' \rightarrow \mathcal{L}(\mathcal{F})$ is a strict $*$ -homomorphism. We say that β is bicovariant when

$$\begin{aligned}(\beta_l \hat{\otimes} \text{id})(\hat{\Delta}(x)) &= (\beta \hat{\otimes} \text{id})(\hat{V})(\beta_l(x) \otimes 1)(\beta \hat{\otimes} \text{id})(\hat{V}^*) \quad \text{and} \\(\beta_r \hat{\otimes} \hat{R})(\hat{\Delta}(x)) &= (\beta \hat{\otimes} \text{id})(\hat{V})(\beta_l(x) \otimes 1)(\beta \hat{\otimes} \text{id})(\hat{V}^*),\end{aligned}$$

where $\hat{V} = (J \hat{\otimes} J)W(J \hat{\otimes} J)$ and \hat{R} denotes the unitary antipode of \hat{M} , see [Vae05, Preliminaries]. In this case we call \mathcal{F} a bicovariant B -correspondence and we write ${}_{\hat{M}}\boxed{\mathcal{F}}_{\hat{M}}^{M'}$.

Remark 4.3.6. We give this definition because we will need to deal with bicovariant B -correspondences. However its technical aspect does not concern us directly. The core of this section is to show the equivalence between two different bicovariant B -correspondences, where their structure is already provided by the results of [Vae05]. Showing such an equivalence is simply a matter of showing that the morphisms satisfy the right commutation relations.

According to [Vae05, Remark 3.6], we have a structure of bicovariant B -correspondence ${}_{\hat{M}}\boxed{H \hat{\otimes} \mathcal{V}}_{\hat{M}}^{M'}$ where the B -correspondence is given by Proposition 4.3.4 and $\beta : M' \rightarrow \mathcal{L}(H \hat{\otimes} \mathcal{V})$ is given by $\beta(x) = x \otimes 1$.

Remark 4.3.7. It should be noted that there is a slight difference in conventions between the current work and the article of Vaes. Namely the skew-pairing between $\mathcal{A}(\hat{\mathbb{G}})$ and $\mathcal{A}(\mathbb{G})$ is such that the coproduct on $\mathcal{A}(\hat{\mathbb{G}})$ is reversed in our conventions, while it is the multiplication which is reversed in Vaes' conventions. Given that the modules discussed here are defined primarily in terms of $\mathcal{D}(\mathbb{G})$ -actions, this means that the action of the function algebra $M' = L^\infty(\mathbb{G})'$ in the bicovariant modules we define below will be intertwined by the unitary antipode R . This forces us to slightly modify the definition of the morphism β so that $\beta(x) = R(Jx^*J) \otimes 1$.

In practice, this means the following. If $a \in \mathcal{A}(\mathbb{G})$ then the action of $m'(a) \in M'$ on the GNS space $H = L^2(\mathbb{G})$ in our conventions needs to be defined as

$$m'(a) \cdot \Lambda(\xi) = \Lambda(R(a)\xi),$$

where $\xi \in \mathcal{A}(\mathbb{G})$ and R designates the unitary antipode of M . The fact that R stabilizes the bornological algebra $\mathcal{A}(\mathbb{G})$ is a consequence of Chapter 3.

The following proposition will be the key result that we use to establish the equivalence between our approach to induction and Vaes'.

Proposition 4.3.8. ([Vae05, Proposition 3.7]) *If $\hat{\mathcal{F}}_{\hat{M}}^{M'}$ is a bicovariant B -correspondence, there exists a canonically determined C^* - B -module \mathcal{E} and a corepresentation $X \in \mathcal{L}(A \hat{\otimes} \mathcal{E})$, unique up to equivalence, such that*

$$\hat{\mathcal{F}}_{\hat{M}}^{M'} \cong \hat{\mathcal{H}}_{\hat{M}}^{M'} \hat{\otimes} \mathcal{E}$$

as bicovariant correspondences. So, we get a bijective relation between unitary corepresentations on C^* - B -module and bicovariant B -correspondences.

Note that from the corepresentation $X \in \mathcal{L}(A \hat{\otimes} \mathcal{V})$, we obtain a $*$ -morphism $\alpha : \hat{A}^u \rightarrow \mathcal{L}(\mathcal{V})$ which verifies

$$(\text{id} \hat{\otimes} \alpha)(W^u) = X,$$

where W^u designates the universal multiplicative unitary of the quantum group (A, Δ) .

We now set $A = C_0^r(\mathbb{G})$ and thus we have $H = L^2(\mathbb{G})$, $M = L^\infty(\mathbb{G})$ and $\hat{M} = \mathcal{L}(\mathbb{G})$. Let $X \in \mathcal{L}(C_0^r(\mathbb{G}) \hat{\otimes} \mathcal{V})$ be a corepresentation of \mathbb{G} on a Hilbert B -module \mathcal{V} . We still denote by α the corresponding $*$ -morphism $\alpha : C_u^*(\mathbb{G}) \rightarrow \mathcal{L}(\mathcal{V})$, as well as its bornological version, $\alpha : \mathcal{D}(\mathbb{B}) \rightarrow \mathcal{L}(\mathcal{V})$, which can be defined by restriction of the original α to $\lambda^u(\mathcal{D}(\mathbb{G}))$.

Remark 4.3.9. One can describe explicitly the structure of the bicovariant B -correspondence $\mathcal{L}(\mathbb{G}) \hat{\otimes}_{L^\infty(\mathbb{G})'} L^2(\mathbb{G}) \hat{\otimes} \mathcal{V} \mathcal{L}(\mathbb{G})$. Let $f \in \mathcal{D}(\mathbb{G})$, $\xi \in \mathcal{A}(\mathbb{G})$ and $v \in \mathcal{V}$. We have

- $\lambda(f) \cdot (\Lambda(\xi) \otimes v) = (\lambda \otimes \alpha)(\hat{\Delta}(f))(\Lambda(\xi) \otimes v)$,
- $(\Lambda(\xi) \otimes v) \cdot \lambda'(f) = \Lambda(\xi * f) \otimes v$,
- $\beta(m'(f))(\Lambda(\xi) \otimes v) = \Lambda(R(f)\xi) \otimes v$.

Let us remark that if our conventions were coherent with those of Vaes we would have a flipped coproduct $\hat{\Delta}^{\text{op}}$ in first point. This is because in Proposition 4.3.4, the left action is defined by $x \cdot v = X(x \otimes 1)X^*v$, for $x \in \hat{M}$, $v \in H \hat{\otimes} \mathcal{V}$ and we have $W(x \otimes 1)W^* = \hat{\Delta}^{\text{op}}(x)$.

From now we consider $X \in \mathcal{L}(C_0^r(\mathbb{B}) \hat{\otimes} \mathcal{V})$ a corepresentation of \mathbb{B} on a Hilbert B -module \mathcal{V} , accompanied by the $*$ -morphism $\alpha : C^*(\mathbb{B}) \rightarrow \mathcal{L}(\mathcal{V})$. The aim of the next paragraphs is to build the induced corepresentation of \mathcal{V} with Vaes' technique. Following [Vae05, Lemma 4.5] we consider the B -correspondence $\mathcal{L}(\mathbb{B}) \hat{\otimes}_{L^\infty(\mathbb{G})} L^2(\mathbb{G}) \hat{\otimes} \mathcal{V} \mathcal{L}(\mathbb{G})$.

Remark 4.3.10. The morphisms in this structure of B -correspondence can be made explicit as is the previous remark. Let $f \in \mathcal{D}(\mathbb{G})$, $h \in \mathcal{D}(\mathbb{B})$, $\xi \in \mathcal{A}(\mathbb{G})$ and $v \in \mathcal{V}$. We have

- $\lambda(h) \cdot (\Lambda(\xi) \otimes v) = (\lambda \circ \hat{\pi} \otimes \alpha)(\hat{\Delta}(h))(\Lambda(\xi) \otimes v)$, $h \in \mathcal{D}(\mathbb{B})$,
- $(\xi \otimes v) \cdot \lambda'(f) = (\xi * f) \otimes v$.

The second point does not differ from the formula in Remark 4.3.9. The first point requires justification. It is claimed in [Vae05, Lemma 4.5] that the morphism $\beta_l : \mathcal{L}(\mathbb{B}) \rightarrow \mathcal{L}(L^2(\mathbb{G}) \hat{\otimes} \mathcal{V})$ is characterized by the property

$$\beta_l(a)(u \otimes 1)\xi = (u \otimes 1)X(a \otimes 1)X^*\xi,$$

for every $a \in \mathcal{L}(\mathbb{B})$, $\xi \in L^2(\mathbb{B}) \hat{\otimes} \mathcal{V}$ and $u \in B(L^2(\mathbb{B}), L^2(\mathbb{G}))$ satisfying $ux = \hat{\pi}(x)u$ for all $x \in \mathcal{L}(\mathbb{B})$. Let then $u \in B(L^2(\mathbb{B}), L^2(\mathbb{G}))$ satisfying $ux = \hat{\pi}(x)u$ for all $x \in \mathcal{L}(\mathbb{B})$ and let $h \in \mathcal{D}(\mathbb{B})$, $\eta \in \mathcal{A}(\mathbb{B})$ and $v \in \mathcal{V}$. According to Remark 4.3.9 we have

$$X(a \otimes 1)X^*(\Lambda_{\mathbb{B}}(\eta) \otimes v) = (\lambda_{\mathbb{B}} \otimes \alpha)(\hat{\Delta}(h))(\Lambda_{\mathbb{B}}(\xi) \otimes v).$$

Therefore, since $\lambda(h) \in \mathcal{L}(\mathbb{B})$, we have

$$\begin{aligned} [(u \otimes 1)X(\lambda(h) \otimes 1)X^*](\Lambda(\xi) \otimes v) &= (u \otimes 1)((\lambda \otimes \alpha)(\hat{\Delta}(h)))(\Lambda(\xi) \otimes v) \\ &= (\hat{\pi} \circ \lambda \otimes \alpha)(\hat{\Delta}(h))(u \otimes 1)(\Lambda(\xi) \otimes v) \\ &= (\lambda \circ \hat{\pi} \otimes \alpha)(\hat{\Delta}(h))(u \otimes 1)(\Lambda(\xi) \otimes v) \\ &= (\lambda \circ \hat{\pi} \otimes \alpha)(\hat{\Delta}(h))[(u \otimes 1)(\Lambda(\xi) \otimes v)] \end{aligned}$$

and thus it coincides with what we claimed. Finally we note that we also have a *-morphism $\beta_{L^2(\mathbb{G}) \hat{\otimes} \mathcal{V}} : L^\infty(\mathbb{G})' \rightarrow \mathcal{L}(L^2(\mathbb{G}) \hat{\otimes} \mathcal{V})$ given by $\beta_{L^2(\mathbb{G}) \hat{\otimes} \mathcal{V}}(m'(f))(\Lambda(\xi) \otimes v) = \Lambda(R(f)\xi) \otimes v$.

We introduce the space \mathcal{I} from [Vae05, Definition 4.2]:

$$\mathcal{I} = \{u \in B(L^2(\mathbb{B}), L^2(\mathbb{G})), ux = \hat{\pi}'(x)u \ \forall x \in \mathcal{L}(\mathbb{B})'\},$$

where $\hat{\pi}'$ refers to the natural action of $\mathcal{L}(\mathbb{B})'$ on $L^2(\mathbb{G})$ given by

$$\hat{\pi}'(x) = \hat{J}_{\mathbb{G}}\hat{\pi}(\hat{J}_{\mathbb{B}}x\hat{J}_{\mathbb{B}})\hat{J}_{\mathbb{G}}.$$

the space \mathcal{I} is endowed with

- its natural $\mathcal{L}(\mathbb{G})$ left action by composition,
- its natural $\mathcal{L}(\mathbb{B})$ right action by composition,
- an $\mathcal{L}(\mathbb{B})$ -inner product given by $\langle u, v \rangle_{\mathcal{L}(\mathbb{B})} = u^*v$, for all $u, v \in \mathcal{I}$.
- a *-morphism $\beta_{\mathcal{I}} : L^\infty(\mathbb{G}) \rightarrow \mathcal{L}(\mathcal{I})$ given by $\beta_{\mathcal{I}}(m'(f))u = m(R(f))u$, for all $f \in \mathcal{A}(\mathbb{G})$ and $u \in \mathcal{I}$.

With this structure the space \mathcal{I} is a bicovariant W^* -bimodule (see [Vae05, Section 3.2]).

Let K be a B -Hilbert module endowed with a left $\mathcal{L}(\mathbb{B})$ -action. One can consider the space $\mathcal{I} \otimes_{\mathcal{L}(\mathbb{B})} K$ which is a B -Hilbert module when it is endowed with the *interior inner product* ([Vae05, Section 12.3]) as follows. Let $u, v \in \mathcal{I}$ and $x, y \in K$. The interior tensor product is given by

$$\langle u \otimes x, v \otimes y \rangle_B = \left\langle x, \langle u, v \rangle_{\mathcal{L}(\mathbb{B})} \cdot y \right\rangle_B.$$

Now, following Vaes' induction procedure, we set $K = L^2(\mathbb{G}) \hat{\otimes} \mathcal{V}$. Vaes build a bicovariant B -correspondence

$$\mathcal{L}(\mathbb{G}) \left[\begin{array}{c} L^\infty(\mathbb{G})' \\ \mathcal{I} \otimes_{\mathcal{L}(\mathbb{B})} (L^2(\mathbb{G}) \hat{\otimes} \mathcal{V}) \end{array} \right]_{\mathcal{L}(\mathbb{G})}.$$

Remark 4.3.11. On this balanced tensor product, the left action of $\mathcal{L}(\mathbb{G})$ is done via the left action of $\mathcal{L}(\mathbb{G})$ on \mathcal{I} . The right action of $\mathcal{L}(\mathbb{G})$ via its right action on $(L^2(\mathbb{G}) \hat{\otimes} \mathcal{V})$, as specified in Remark 4.3.10. Finally the morphism $\beta : L^\infty(\mathbb{G})' \rightarrow \mathcal{L}(\mathcal{I} \otimes_{\mathcal{L}(\mathbb{B})} (L^2(\mathbb{G}) \hat{\otimes} \mathcal{V}))$ is given by $\beta = (\beta_{\mathcal{I}} \hat{\otimes} \beta_{L^2(\mathbb{G}) \hat{\otimes} \mathcal{V}}) \circ \Delta$. Specifically, let $f \in \mathcal{D}(\mathbb{G})$, $\xi \in \mathcal{A}(\mathbb{G})$ and $v \in \mathcal{V}$. We have

- $\lambda(g) \cdot (\iota(f) \otimes \Lambda(\xi) \otimes v) = \iota(g * f) \otimes \Lambda(\xi) \otimes v$,
- $(\iota(f) \otimes \Lambda(\xi) \otimes v) \cdot \lambda'(g) = \iota(f) \otimes \Lambda(\xi * g) \otimes v$,
- $\beta(m'(g))(\iota(f) \otimes \Lambda(\xi) \otimes v) = \iota(R(g_{(2)})f) \otimes \Lambda(R(g_{(1)})\xi) \otimes v$.

Then, using Proposition 4.3.8 we have the existence of a corepresentation of $C_0^r(\mathbb{G})$ on a B -Hilbert module $\text{Ind } \mathcal{V}$ such that there is an isomorphism of B -correspondences

$$\mathcal{L}(\mathbb{G}) \left[\begin{array}{c} L^\infty(\mathbb{G})' \\ \mathcal{I} \hat{\otimes}_{\mathcal{L}(\mathbb{B})} (L^2(\mathbb{G}) \otimes \mathcal{V}) \end{array} \right]_{\mathcal{L}(\mathbb{G})} \cong \mathcal{L}(\mathbb{G}) \left[\begin{array}{c} L^\infty(\mathbb{G})' \\ L^2(\mathbb{G}) \hat{\otimes} \text{Ind } \mathcal{V} \end{array} \right]_{\mathcal{L}(\mathbb{G})}.$$

The aim of this section is therefore to establish an equivalence of corepresentations

$$\text{Ind } \mathcal{V} \cong \mathcal{E}(\mathbb{G}) \hat{\otimes}_{C^*(\mathbb{B})} \mathcal{V}.$$

According to Proposition 4.3.4, there exists a structure of bicovariant B -correspondence

$\mathcal{L}(\mathbb{G}) \left[\begin{array}{c} L^\infty(\mathbb{G})' \\ L^2(\mathbb{G}) \otimes \mathcal{E}(\mathbb{G}) \otimes_{\mathcal{D}(\mathbb{B})} \mathcal{V} \end{array} \right]_{\mathcal{L}(\mathbb{G})}$. Specifically, let $g \in \mathcal{D}(\mathbb{G})$, $\xi, f \in \mathcal{A}(\mathbb{G})$ and $v \in \mathcal{V}$. We have

- $\lambda(g) \cdot (\Lambda(\xi) \otimes f \otimes v) = (\Lambda \otimes \text{id})(\hat{\Delta}(g) * (\xi \otimes f)) \otimes v$,
where $*$ refers to the product of $\mathcal{D}(\mathbb{G}) \hat{\otimes} \mathcal{D}(\mathbb{G})$,
- $(\Lambda(\xi) \otimes f \otimes v) \cdot \lambda'(g) = \Lambda(\xi * g) \otimes f \otimes v$,

- $\beta(m'(g))(\Lambda(\xi) \otimes f \otimes v) = \Lambda(R(g)\xi) \otimes f \otimes v$.

Proposition 4.3.12. *We have an equivalence of bicovariant B-correspondences*

$$\mathcal{L}(\mathbb{G}) \left[\begin{array}{c} L^\infty(\mathbb{G}') \\ \mathcal{I} \hat{\otimes}_{\mathcal{L}(\mathbb{B})} (L^2(\mathbb{G}) \hat{\otimes} \mathcal{V}) \end{array} \right]_{\mathcal{L}(\mathbb{G})} \cong \mathcal{L}(\mathbb{G}) \left[\begin{array}{c} L^\infty(\mathbb{G}') \\ L^2(\mathbb{G}) \hat{\otimes} \mathcal{E}(\mathbb{G}) \hat{\otimes}_{C^*(\mathbb{B})} \mathcal{V} \end{array} \right]_{\mathcal{L}(\mathbb{G})}.$$

To prove this we need several results.

Lemma 4.3.13. *Let $h \in \mathcal{D}(\mathbb{B})$. We have that $\hat{\pi}(h)\delta_{\mathbb{G}}^{\frac{1}{2}}$ is a well defined element of $M(\mathcal{D}(\mathbb{G}))$ and we have $\hat{\pi}(h)\delta_{\mathbb{G}}^{\frac{1}{2}} = \hat{\pi}(h\pi(\delta_{\mathbb{G}}^{\frac{1}{2}}))$.*

Proof. First, observe that, since $\delta_{\mathbb{G}}^{\frac{1}{2}}$ is group-like, $f \mapsto f\delta_{\mathbb{G}}^{\frac{1}{2}}$ is a bijective homomorphism of the algebra $\mathcal{D}(\mathbb{G})$. As a consequence this map extends to a map $M(\mathcal{D}(\mathbb{G})) \rightarrow M(\mathcal{D}(\mathbb{G}))$ defined for $m \in M(\mathcal{D}(\mathbb{G}))$ and $f \in \mathcal{D}(\mathbb{G})$ by

$$(m\delta_{\mathbb{G}}^{\frac{1}{2}}) * f = (m * (f\delta_{\mathbb{G}}^{-\frac{1}{2}}))\delta_{\mathbb{G}}^{\frac{1}{2}}.$$

Then, setting $m = \hat{\pi}(h)$, we get

$$\begin{aligned} (\hat{\pi}(h)\delta_{\mathbb{G}}^{\frac{1}{2}}) * f &= (\hat{\pi}(h) * (f\delta_{\mathbb{G}}^{-\frac{1}{2}}))\delta_{\mathbb{G}}^{\frac{1}{2}} \\ &= \phi_{\mathbb{B}}(\pi(S^{-1}(f_{(2)}\delta_{\mathbb{G}}^{-\frac{1}{2}}))h)f_{(1)}\delta_{\mathbb{G}}^{-\frac{1}{2}}\delta_{\mathbb{G}}^{\frac{1}{2}} \\ &= \phi_{\mathbb{B}}(\pi(S^{-1}(f_{(2)}))h\sigma_{\mathbb{B}}(\pi(\delta_{\mathbb{G}}^{\frac{1}{2}})))f_{(1)} \\ &= \hat{\pi}(h\pi(\delta_{\mathbb{G}}^{\frac{1}{2}})) * f \end{aligned}$$

where the last equality follows from the hypothesis we made at Remark 4.0.1 which gives $\sigma_{\mathbb{B}}(\pi(\delta_{\mathbb{G}}^{\frac{1}{2}})) = \pi(\delta_{\mathbb{G}}^{\frac{1}{2}})$. □

Lemma 4.3.14. *Let $h \in \mathcal{D}(\mathbb{B})$ and $\xi \in \mathcal{D}(\mathbb{G})$. We have that $\hat{\pi}'(\lambda'(h))\Lambda(\xi) = \Lambda(\xi) \cdot h$.*

Proof. Let $h \in \mathcal{D}(\mathbb{B})$. We recall that we have the polar decomposition of the operator $\hat{T}_{\mathbb{B}} : \Lambda(f) \mapsto \Lambda(f^*)$ as $\hat{T}_{\mathbb{B}} = \hat{J}_{\mathbb{B}}\hat{\nabla}_{\mathbb{B}}^{\frac{1}{2}} = \hat{\nabla}_{\mathbb{B}}^{-\frac{1}{2}}\hat{J}_{\mathbb{B}}$, so

$$\hat{J}_{\mathbb{B}}\lambda'(h)\hat{J}_{\mathbb{B}} = \hat{\nabla}_{\mathbb{B}}^{\frac{1}{2}}\lambda'(h^*)\hat{\nabla}_{\mathbb{B}}^{-\frac{1}{2}}.$$

Recall also that $\hat{\nabla}_{\mathbb{B}}\Lambda(\eta) = \Lambda(S^2(\eta)\delta_{\mathbb{B}}^{-1})$. Using the strong commutation of the operator extensions of S^2 and $\delta_{\mathbb{G}}$ from Section 3.4, we obtain

$$\hat{J}_{\mathbb{B}}\lambda'(h)\hat{J}_{\mathbb{B}} = \lambda(|S|(h)^*\delta_{\mathbb{B}}^{\frac{1}{2}})$$

where $|S| : \mathcal{A}(\mathbb{B}) \rightarrow \mathcal{A}(\mathbb{B})$ is the automorphism induced by N , the operator defined in Section 3.4. Of course the same result stay true if we replace \mathbb{B} by \mathbb{G} . We use in the next

calculation that the automorphisms $|S|$ are intertwined by $\hat{\pi}$ and that $|S|(\delta_{\mathbb{B}}) = \delta_{\mathbb{B}}$. We also have

$$\begin{aligned} (h\delta_{\mathbb{B}}^{\frac{1}{2}})^* &= \overline{S(h\delta_{\mathbb{B}}^{\frac{1}{2}})\delta_{\mathbb{B}}} \\ &= h^* \delta_{\mathbb{B}}^{-\frac{1}{2}}. \end{aligned}$$

. One can now calculate

$$\begin{aligned} \hat{\pi}'(\lambda'(h)) &= \hat{J}_{\mathbb{G}} \hat{\pi}(\lambda(|S|(h)^* \delta_{\mathbb{B}}^{-\frac{1}{2}})) \hat{J}_{\mathbb{G}} \\ &= \hat{J}_{\mathbb{G}} \hat{\pi}(\lambda(|S|(h\delta_{\mathbb{B}}^{\frac{1}{2}})^*)) \hat{J}_{\mathbb{G}} \\ &= \hat{J}_{\mathbb{G}} \lambda(|S|(\hat{\pi}(h\delta_{\mathbb{B}}^{\frac{1}{2}})^*)) \hat{J}_{\mathbb{G}} \\ &= \hat{J}_{\mathbb{G}} \lambda(|S|(\hat{\pi}(h\delta_{\mathbb{B}}^{\frac{1}{2}}))^* \delta_{\mathbb{G}}^{\frac{1}{2}} \delta_{\mathbb{G}}^{-\frac{1}{2}}) \hat{J}_{\mathbb{G}} \\ &= \hat{J}_{\mathbb{G}} \lambda(|S|(\hat{\pi}(h\delta_{\mathbb{B}}^{\frac{1}{2}}) \delta_{\mathbb{G}}^{-\frac{1}{2}})^* \delta_{\mathbb{G}}^{-\frac{1}{2}}) \hat{J}_{\mathbb{G}} \\ &\stackrel{(*)}{=} \hat{J}_{\mathbb{G}} \lambda(|S|(\hat{\pi}(h\gamma)^* \delta_{\mathbb{G}}^{-\frac{1}{2}})) \hat{J}_{\mathbb{G}} \\ &= \hat{J}_{\mathbb{G}} \hat{J}_{\mathbb{G}} \lambda'(\hat{\pi}(h\gamma)) \hat{J}_{\mathbb{G}} \hat{J}_{\mathbb{G}} \\ &= \lambda'(\hat{\pi}(h\gamma^{-1})). \end{aligned}$$

Where for (*) we used Lemma 4.3.13. The result follows from the definition of the right action. \square

Lemma 4.3.15. *The map ρ_{\bullet} from Proposition 4.2.6 defines an injection with dense image $\mathcal{E}(\mathbb{G}) \rightarrow \mathcal{I}$ (with respect to the weak topology of $B(L^2(\mathbb{B}), L^2(\mathbb{G}))$). Its image is denoted \mathcal{I}_0 .*

Proof. Let $f \in \mathcal{D}(\mathbb{G})$, $h \in \mathcal{D}(\mathbb{B})$ and $\eta \in \mathcal{D}(\mathbb{B})$. We have

$$\begin{aligned} \rho_f(\lambda'(h)\Lambda(\eta)) &= \Lambda(f \cdot (\eta * h)) \\ &= \Lambda((f \cdot \eta) \cdot h) \\ &= \hat{\pi}'(\lambda'(h))\rho_f(\Lambda(\eta)). \end{aligned}$$

Thus the operator ρ_f belongs to \mathcal{I} . It remains to show that the image of $\mathcal{E}(\mathbb{G})$ in \mathcal{I} is dense. Let $\xi \in L^2(\mathbb{G})$ and $\eta \in L^2(\mathbb{B})$. Suppose we have

$$\langle \Lambda(\xi), \rho_f(\Lambda(\eta)) \rangle = 0$$

for all $f \in \mathcal{E}(\mathbb{G})$. Let then $u \in \mathcal{I}$, we want to obtain that $\langle \xi, u(\Lambda(\eta)) \rangle = 0$. Let $\varepsilon > 0$, there exist

1. $a \in \mathcal{A}(\mathbb{B})$ s.t. $\|\Lambda(\eta - a)\|_{L^2(\mathbb{B})} \leq \varepsilon$ (density of $\mathcal{A}(\mathbb{B})$),

2. $b \in \mathcal{A}(\mathbb{B})$ s.t. $\|\Lambda(b * a - a)\|_{L^2(\mathbb{B})} \leq \varepsilon$ (essentialness),
3. $c \in \mathcal{A}(\mathbb{G})$ s.t. $\|u(\Lambda(b)) - \Lambda(c)\|_{L^2(\mathbb{G})} \leq \varepsilon$ (density of $\mathcal{A}(\mathbb{G})$).

Now, there exist $k_1, k_2, k_3 > 0$ (depending only on the norms of $\Lambda(\xi)$, $\Lambda(\eta)$ and u such that

1. $|\langle \Lambda(\xi), u(\Lambda(\eta)) \rangle - \langle \Lambda(\xi), u(\Lambda(a)) \rangle| \leq k_1 \varepsilon$,
2. $|\langle \Lambda(\xi), u(\Lambda(a)) \rangle - \langle \Lambda(\xi), u(\Lambda(b * a)) \rangle| \leq k_2 \varepsilon$, and we note that $u(\Lambda(b * a)) = \lambda'(a)u(\Lambda(b))$;
3. $|\langle \Lambda(\xi), u(\Lambda(b)) \cdot \lambda(a) \rangle - \langle \Lambda(\xi), \Lambda(c \cdot \eta) \rangle| \leq k_3 \varepsilon$,

Finally, since $\langle \xi, c \cdot \eta \rangle = 0$ we have

$$|\langle \xi, u(\eta) \rangle| \leq (k_1 + k_2 + k_3) \varepsilon,$$

So $\langle \xi, u(\eta) \rangle = 0$ and we are done. \square

Lemma 4.3.16. *Let \mathcal{V} be a representation of $C_u^*(\mathbb{G})$ on any Hilbert module. One can endow \mathcal{V} with its von Neumann bornology and consider the bornological space*

$$\mathcal{V}^\infty = \mathcal{D}(\mathbb{G}) \hat{\otimes}_{\mathcal{D}(\mathbb{G})} \mathcal{V},$$

equipped with the left convolution action of $\mathcal{D}(\mathbb{G})$ is a bornological $\mathcal{D}(\mathbb{G})$ -module and defines a dense subspace of \mathcal{V} .

Proof. First, from the associativity of the bornological tensor product we have that

$$\mathcal{D}(\mathbb{G}) \hat{\otimes}_{\mathcal{D}(\mathbb{G})} (\mathcal{D}(\mathbb{G}) \hat{\otimes}_{\mathcal{D}(\mathbb{G})} \mathcal{V}) = \mathcal{D}(\mathbb{G}) \hat{\otimes}_{\mathcal{D}(\mathbb{G})} \mathcal{V},$$

and thus \mathcal{V}^∞ is a bornological $\mathcal{D}(\mathbb{G})$ -module.

Now consider the linear map $\mathcal{D}(\mathbb{G}) \otimes_{\mathcal{D}(\mathbb{G})} \mathcal{V} \rightarrow C_u^*(\mathbb{G}) \otimes \mathcal{V}$ defined by $f \otimes v \mapsto \lambda(f) \otimes v$. This map is bounded since bounded subspaces of \mathcal{V} are precisely bounded subspaces of \mathcal{V} with respect to its Hilbert topology. Furthermore this map leads to an injective map $\mathcal{D}(\mathbb{G}) \hat{\otimes}_{\mathcal{D}(\mathbb{G})} \mathcal{V} \rightarrow C_u^*(\mathbb{G}) \hat{\otimes}_{C_u^*(\mathbb{G})} \mathcal{V} \cong \mathcal{V}$ which has dense range. \square

Remark 4.3.17. Using the duality between modules and comodules at the bornological level, we obtain that \mathcal{V}^∞ is also a comodule.

Lemma 4.3.18. *Let $h \in \mathcal{D}(\mathbb{B})$, $\eta \in \mathcal{A}(\mathbb{G})$ and $w \in \mathcal{V}^\infty$. We have*

$$\lambda(h) \cdot (\Lambda(\eta) \otimes w) = \Lambda(\eta_{(2)}) \otimes \pi(S^{-1}(\eta_{(1)}))h \cdot w,$$

where “ \cdot ” on the left hand side stands for the action of $\mathcal{D}(\mathbb{B})$ on \mathcal{V}^∞ .

Proof. $h \in \mathcal{D}(\mathbb{B})$, $\eta \in \mathcal{A}(\mathbb{G})$ and $w \in \mathcal{V}^\infty$. We have

$$\begin{aligned}
\lambda(h) \cdot (\Lambda(\eta) \otimes w) &= \lambda(h) \cdot (\xi \otimes v) = (\lambda \circ \hat{\pi} \otimes \alpha)(\hat{\Delta}(h))(\Lambda(\eta) \otimes v) \\
&\stackrel{(*)}{=} ((\hat{\pi} \otimes \text{id})(\hat{\Delta}(h)), S^{-1}(\eta_{(1)}) \otimes S^{-1}(w_{(-1)}))\Lambda(\eta_{(2)}) \otimes w_{(0)} \\
&= (h, S^{-1}(w_{(-1)})\pi(S^{-1}(\eta_{(1)})))\Lambda(\eta_{(2)}) \otimes w_{(0)} \\
&= \phi_{\mathbb{B}}(S^{-1}(w_{(-1)})(S^{-1}(\pi(\eta_{(1)}))h))\Lambda(\eta_{(2)}) \otimes w_{(0)} \\
&= \Lambda(\eta_{(2)}) \otimes \pi(S^{-1}(\eta_{(1)}))h \cdot w,
\end{aligned}$$

where in equality (*) we apply the definition of the action associated to a coaction, with the same conventions as in Proposition 1.1.13. \square

Proof of Proposition 4.3.12. Lemma 4.3.15 allows us to consider the linear map

$$\begin{aligned}
\Psi : \mathcal{A}(\mathbb{G}) \otimes \mathcal{D}(\mathbb{G}) \otimes \mathcal{V}^\infty &\rightarrow \mathcal{I} \otimes (L^2(\mathbb{G}) \otimes \mathcal{V}) \\
\xi \otimes f \otimes v &\mapsto (\iota \hat{\otimes} \Lambda)(\Delta(\xi)(f \otimes 1)) \otimes v
\end{aligned}$$

where ι stands for the injection $\mathcal{D}(\mathbb{G}) \rightarrow \mathcal{I}$ from Proposition 4.2.6.

Let $\xi, \eta \in \mathcal{A}(\mathbb{G})$, $f, g \in \mathcal{D}(\mathbb{G})$ and $v, w \in \mathcal{V}^\infty$. We consider the elements $[\iota(\xi_{(1)}f) \otimes \Lambda(\xi_{(2)}) \otimes v]$ and $[\iota(\eta_{(1)}g) \otimes \Lambda(\eta_{(2)}) \otimes w]$ of $\mathcal{I} \otimes_{\mathcal{L}(\mathbb{B})} (L^2(\mathbb{G}) \otimes \mathcal{V})$ (where we use the notation $[\cdot]$ to refer to a class of elements in the balanced tensor product). Using the Lemma 4.3.18 in the equality (*) below and the definition of the interior inner product we obtain

$$\begin{aligned}
&\langle [\iota(\xi_{(1)}f) \otimes \Lambda(\xi_{(2)}) \otimes v], [\iota(\eta_{(1)}g) \otimes \Lambda(\eta_{(2)}) \otimes w] \rangle_{\mathcal{I} \otimes_{\mathcal{L}(\mathbb{B})} (L^2(\mathbb{G}) \otimes \mathcal{V})} \\
&= \left\langle \Lambda(\xi_{(2)}) \otimes v, \lambda(\langle \xi_{(1)}f, \eta_{(1)}g \rangle_{\mathcal{D}(\mathbb{B})}) \cdot (\Lambda(\eta_{(2)}) \otimes w) \right\rangle_{L^2(\mathbb{G}) \otimes \mathcal{V}} \\
&= \left\langle \Lambda(\xi_{(2)}) \otimes v, \phi_{\mathbb{G}}(\overline{\xi_{(1)}f} \eta_{(1)} g_{(1)}) \lambda(\pi(\eta_{(2)} g_{(2)} \gamma)) \cdot (\Lambda(\eta_{(3)}) \otimes w) \right\rangle_{L^2(\mathbb{G}) \otimes \mathcal{V}} \\
&\stackrel{(*)}{=} \left\langle \Lambda(\xi_{(2)}) \otimes v, \phi_{\mathbb{G}}(\overline{\xi_{(1)}f} \eta_{(1)} g_{(1)}) \Lambda(\eta_{(4)}) \otimes (\pi(S^{-1}(\eta_{(3)})) \pi(\eta_{(2)} g_{(2)} \gamma)) \cdot w \right\rangle_{L^2(\mathbb{G}) \otimes \mathcal{V}} \\
&= \left\langle \Lambda(\xi_{(2)}) \otimes v, \phi_{\mathbb{G}}(\overline{f} \overline{\xi_{(1)}} \eta_{(1)} g_{(1)}) \Lambda(\eta_{(2)}) \otimes (\pi(g_{(2)} \gamma) \cdot w) \right\rangle_{L^2(\mathbb{G}) \otimes \mathcal{V}} \\
&= \phi_{\mathbb{G}}(\overline{\xi_{(2)}} \eta_{(2)}) \langle v, \phi_{\mathbb{G}}(\overline{f} \overline{\xi_{(1)}} \eta_{(1)} g_{(1)}) (\pi(g_{(2)} \gamma) \cdot w) \rangle_{\mathcal{V}} \\
&= \left\langle \Lambda(\xi) \otimes v, \Lambda(\eta) \otimes \langle f, g \rangle_{\mathcal{D}(\mathbb{B})} \cdot w \right\rangle_{L^2(\mathbb{G}) \otimes \mathcal{V}} \\
&= \langle \Lambda(\xi) \otimes [f \otimes v], \Lambda(\eta) \otimes [g \otimes w] \rangle_{L^2(\mathbb{G}) \otimes \mathcal{E}(\mathbb{G}) \otimes_{\mathcal{D}(\mathbb{B})} \mathcal{V}}.
\end{aligned}$$

In particular, this shows that elements in the kernel of the quotient $\mathcal{A}(\mathbb{G}) \otimes \mathcal{D}(\mathbb{G}) \otimes \mathcal{V}^\infty \rightarrow \mathcal{A}(\mathbb{G}) \otimes \mathcal{D}(\mathbb{G}) \otimes_{\mathcal{D}(\mathbb{B})} \mathcal{V}$ have null image in $\mathcal{I} \otimes_{\mathcal{L}(\mathbb{B})} (L^2(\mathbb{G}) \otimes \mathcal{V})$ so the map Ψ descends to a unitary map to the balanced tensor product.

Now we can consider the associated map

$$\tilde{\Psi} : L^2(\mathbb{G}) \otimes \mathcal{E}(\mathbb{G}) \otimes_{C^*(\mathbb{B})} \mathcal{V} \rightarrow \mathcal{I} \otimes_{\mathcal{L}(\mathbb{B})} (L^2(\mathbb{G}) \otimes \mathcal{V}).$$

Let us check that Ψ intertwines the bicovariant B -correspondence structure. We start with the left action of $\mathcal{L}(\mathbb{G})$. Let $g \in \mathcal{D}(\mathbb{G})$ and $\xi \otimes f \otimes v \in \mathcal{A}(\mathbb{G}) \otimes \mathcal{D}(\mathbb{G}) \otimes \mathcal{V}^\infty$. We have

$$\begin{aligned} \lambda(g) \cdot (\Lambda(\xi) \otimes [f \otimes v]) &= ((\lambda \otimes \text{id})(\hat{\Delta}(g) \otimes \text{id}))(\Lambda(\xi) \otimes [f \otimes v]) \\ &= \phi_{\mathbb{G}}(S^{-1}(\xi_{(1)}f_{(1)})g)\Lambda(\xi_{(2)}) \otimes [f_{(2)} \otimes v], \end{aligned}$$

and

$$\begin{aligned} \phi_{\mathbb{G}}(S^{-1}(\xi_{(1)}f_{(1)})g)\tilde{\Psi}(\Lambda(\xi_{(2)}) \otimes [f_{(2)} \otimes v]) &= \phi_{\mathbb{G}}(S^{-1}(\xi_{(1)}f_{(1)})g)[\iota(\xi_{(2)}f_{(2)}) \otimes \Lambda(\xi_{(3)}) \otimes v] \\ &= [\iota(g * \xi_{(1)}f) \otimes \Lambda(\xi_{(2)}) \otimes v] \\ &= \lambda(g) \cdot [\iota(\xi_{(1)}f) \otimes \Lambda(\xi_{(2)}) \otimes v] \end{aligned}$$

For the right action of $\mathcal{L}(\mathbb{G})'$ consider again $g \in \mathcal{D}(\mathbb{G})$ and $\xi \otimes f \otimes v \in \mathcal{A}(\mathbb{G}) \otimes \mathcal{D}(\mathbb{G}) \otimes \mathcal{V}^\infty$. We have

$$\begin{aligned} \tilde{\Psi}(\Lambda(\xi * g) \otimes [f \otimes v]) \cdot \lambda(g) &= [\iota(\xi_{(1)}f) \otimes \Lambda(\xi_{(2)} * g) \otimes v] \\ &= \Psi(\Lambda(\xi) \otimes [f \otimes v]) \cdot \lambda'(g), \end{aligned}$$

where we use Lemma 2.3.8. Let now $g \in L^\infty(\mathbb{G})'$. We have

$$\begin{aligned} \tilde{\Psi}(\beta(g)(\Lambda(\xi) \otimes [f \otimes v])) &= \tilde{\Psi}(\Lambda(R(g)\xi) \otimes [f \otimes v]) \\ &= \iota(R(g_{(2)})\xi_{(1)}f) \otimes \Lambda(R(g_{(1)})\xi_{(2)}) \\ &= \beta(g)\tilde{\Psi}(\Lambda(\xi) \otimes [f \otimes v]), \end{aligned}$$

where we use that R is an anti coalgebra homomorphism.

We finish with the surjectivity of $\tilde{\Psi}$. Let $f \in \mathcal{D}(\mathbb{G})$, $g \in \mathcal{A}(\mathbb{G})$ and $v \in \mathcal{V}^\infty$ and consider the element $\iota(f) \otimes [\Lambda(g) \otimes v]$ of $\mathcal{I} \otimes_{\mathcal{L}(\mathbb{B})} (L^2(\mathbb{G}) \otimes \mathcal{V})$. We observe that the element $[\Lambda(g_{(2)}) \otimes S^{-1}(g_{(1)})f \otimes v]$ of $L^2(\mathbb{G}) \otimes \mathcal{E}(\mathbb{G}) \otimes_{C^*(\mathbb{B})} \mathcal{V}$ is an antecedent of $\iota(f) \otimes [\Lambda(g) \otimes v]$ for $\tilde{\Psi}$. We conclude with a density argument. \square

Theorem 4.3.19. *The representations $\text{Ind } \mathcal{V}$ and $\mathcal{E}(\mathbb{G}) \otimes_{C^*(\mathbb{B})} \mathcal{V}$ are equivalent.*

Proof. This follows directly from Proposition 4.3.12 and Proposition 4.3.8. \square

Chapter 5

The parabolic induction module

The goal of this Chapter is twofold. First it allows us to illustrate the general construction of the preceding chapter. Secondly we use this construction to express in a simple way the parabolic induction functor for a complex semisimple quantum group. By means of that, we give an explicit expression for the reduced C^* -algebra of such a quantum group. As we have seen in Chapter 1, the construction of principal series representations are very similar to the classical case. In particular those representations are induced from an analog of the classical Borel subgroup. We thus propose to adapt the ideas of P. Clare et al. [CCH16]. Let G be a semisimple Lie group (here we suppose it complex in order to avoid certain subtleties of the general real case) and let $G = KAN$, $B = MAN$ be an Iwasawa decomposition and the associated Borel subgroup. All the characters of B involved in the parabolic induction are characters of the Levy factor $L = MA$ extended trivially to B . Thus the parabolic induction (associated to the chosen Iwasawa decomposition) realizes a functor between the category of characters of L and the category of unitary representations of G . In [CCH16], following an original construction of P. Clare, the authors adapt Rieffel's ideas [Rie74] to build a $C^*(G)$ - $C^*(L)$ -correspondence, that is, a $C^*(L)$ -Hilbert module, denoted $\mathcal{E}(G/N)$, with a $*$ -representation of $C^*(G)$. They show that the parabolic induction functor is exactly $\mathcal{E}(G/N) \otimes_{C^*(L)} -$. In the quantum case we have an analog L_q of L , but there does not seem to exist an analog the unipotent radical N , at least not with a full quantum group structure. Despite this issue, we can build a $C^*(G_q)$ - $C^*(L_q)$ correspondence with the expected property. Furthermore, thanks to the structure of a semisimple quantum group $G_q = K_q \rtimes \widehat{K}_q$ and the discreteness of \widehat{K}_q , certain computations are made easier than in the classical case.

We remark that the quantum groups in this chapter are all algebraic quantum groups, so that questions of bornology disappear.

We also remark that we have $\delta_{G_q} = 1$ so the hypothesis we have made on $\pi_{B_q}(\delta_{G_q})$ is

trivially verified and we have that B_q is amenable thus all the results of Chapter 4 can be applied to implement the induction from B_q to G_q .

5.1 The quotient map

In the classical case, with $G = KAN$, principal series representations are induced from characters of the the Borel subgroup $B = MAN$. Explicitly, we choose first a character μ of M and λ of A and then the identification $MA = B/N$ allows us to extend $\mu \otimes \lambda$ to a character of B . In this way we obtain the principal series representation

$$\text{Ind}_B^G \mu \otimes \lambda.$$

In the quantum case we do not have an analog for the subgroup N . But, as we now explain, we do have a “quotient” map

$$\widehat{K}_q \twoheadrightarrow A_q.$$

Let us make this explicit. There are two versions of the map π_T . First with the canonical identification of $*$ -algebras $\mathcal{A}(K_q) = \mathcal{D}(\widehat{K}_q)$ and $\mathcal{A}(T) = \mathcal{D}(A_q)$, one can consider

$$\pi_T : \mathcal{D}(\widehat{K}_q) \rightarrow \mathcal{D}(A_q),$$

which is a $*$ -morphism and comes with its dual morphism $\hat{\pi}_T : \mathcal{A}(A_q) \rightarrow M(\mathcal{A}(\widehat{K}_q))$. Secondly, using the identifications of vector spaces $\mathcal{A}(\widehat{K}_q) \cong \mathcal{D}(\widehat{K}_q)$ and $\mathcal{A}(A_q) \cong \mathcal{D}(A_q)$ the same map can be interpreted as a map

$$\pi_T : \mathcal{A}(\widehat{K}_q) \rightarrow \mathcal{A}(A_q).$$

This is a conditional expectation in the sense of Proposition 4.1.4, observing that K_q and T are unimodular. In particular $\pi(fK_\lambda) = \pi(f)K_\lambda$ for all $f \in \mathcal{A}(\widehat{K}_q)$, $\lambda \in \mathbf{P}$. This is the map $\pi_T : \mathcal{A}(\widehat{K}_q) \rightarrow \mathcal{A}(A_q)$ that we call the *quotient map*. This map has also the notable property

$$\phi_{\widehat{K}_q}(f) = \phi_{A_q}(\pi_T(f)).$$

Indeed we have for all $a \in \mathcal{A}(K_q)$

$$\begin{aligned} \phi_{\widehat{K}_q}(\mathcal{F}_{K_q}(a)) &= \epsilon_{K_q}(a) \\ &= \epsilon_T(\pi_T(a)) \\ &= \phi_{A_q}(\pi_T(\mathcal{F}_{K_q}(a))). \end{aligned}$$

Remark 5.1.1. In the rest of this Chapter we extensively use Sweedler notations. Since one considers both $\mathcal{A}(K_q)$ and $\mathcal{A}(\widehat{K}_q)$, this can be confusing. The convention is as follows. If we write $f \in \mathcal{A}(G_q)$ or $f \in \mathcal{D}(G_q)$ (where $G_q = K_q, \widehat{K}_q, T$ or A_q) then $f_{(1)} \otimes f_{(2)}$ always refers to the coproduct of $\mathcal{A}(G_q)$.

Lemma 5.1.2. *Let $f \in \mathcal{A}(\widehat{K}_q)$. We have*

$$\pi_T(f_{(2)}) \otimes f_{(1)} = \pi_T(f)_{(2)} \otimes \hat{\pi}_T(\pi_T(f)_{(1)}).$$

In particular this means that the map $\mathcal{A}(A_q) \rightarrow M(\mathcal{A}(\widehat{K}_q) \otimes \mathcal{A}(A_q))$ given by $\pi_T(f) \mapsto f_{(1)} \otimes \pi_T(f_{(2)})$ is well defined.

Proof. Let $f, g \in \mathcal{D}(\widehat{K}_q)$. On the one hand we have

$$\begin{aligned} \pi_T(g * f) &= \pi_T(f_{(2)}) \phi_{\widehat{K}_q}(S^{-1}(f_{(1)})g) \\ &= \pi_T(f_{(2)})(g, S^{-1}(f_{(1)})). \end{aligned}$$

And on the other hand

$$\begin{aligned} \pi_T(g) * \pi_T(f) &= \pi_T(f)_{(2)} \phi_{A_q}(S^{-1}(\pi_T(f)_{(1)})\pi_T(g)) \\ &= \pi_T(f)_{(2)}(g, \hat{\pi}_T(S^{-1}(\pi_T(f)_{(1)}))) \\ &= \pi_T(f)_{(2)}(g, S^{-1}(\hat{\pi}_T(\pi_T(f)_{(1)}))) \end{aligned}$$

One can thus identify the legs and we obtain

$$\pi_T(f_{(2)}) \otimes f_{(1)} = \pi_T(f)_{(2)} \otimes \hat{\pi}_T(\pi_T(f)_{(1)}).$$

□

We denote by $\alpha_{A_q} : \mathcal{A}(A_q) \rightarrow M(\mathcal{A}(\widehat{K}_q)) \otimes \mathcal{A}(A_q)$ the $\mathcal{A}(\widehat{K}_q)$ coaction we obtain on $\mathcal{A}(A_q)$. For $h \in \mathcal{A}(A_q)$ we have

$$\alpha_{A_q}(h) = \hat{\pi}_T(h_{(1)}) \otimes h_{(2)},$$

and for $f \in \mathcal{A}(\widehat{K}_q)$ one can also write

$$\alpha_{A_q}(\pi_T(f)) = f_{(1)} \otimes \pi_T(f_{(2)}).$$

5.2 The parabolic induction module

In this section we use all the notation introduced in Section 1.3. The goal here is to build a Hilbert module which implements the parabolic induction functor. We define this module in this section as a balanced tensor product $\mathcal{E}(G_q) \otimes_{\mathcal{D}(B_q)} \mathcal{D}(L_q)$, where $\mathcal{E}(G_q)$ is the induction module built from the closed quantum subgroup B_q as defined in the previous chapter and where we recall that we defined $L_q = T \times A_q$.

Lemma 5.2.1. *The linear map $(\text{id} \otimes \pi_T) : \mathcal{D}(B_q) \rightarrow \mathcal{D}(L_q)$ is a *-Hopf homomorphism.*

Proof. We first show that $(\text{id} \otimes \hat{\pi}_T) : \mathcal{A}(L_q) \rightarrow M(\mathcal{A}(B_q))$ is a *-Hopf homomorphism, then we conclude with a duality argument. Before we start, we recall that $\hat{\pi}_T : \mathcal{A}(A_q) \rightarrow M(\mathcal{A}(\widehat{K}_q))$ is a Hopf *-morphism. We have seen in Section 1.3.3 that

$$\Delta_{B_q}(a \otimes f) = a_{(1)} \otimes \omega_{ii}^\sigma f_{(1)} \omega_{rr}^\nu \otimes \pi_T(u_{ii}^\sigma S(u_{rr}^\nu)) a_{(2)} \otimes f_{(2)},$$

for all $a \otimes f \in \mathcal{A}(B_q)$. Let $a \otimes h \in \mathcal{A}(L_q)$. We have on the one hand

$$((\text{id} \otimes \hat{\pi}_T) \otimes (\text{id} \otimes \hat{\pi}_T))(\Delta_{L_q}(a \otimes h)) = a_{(1)} \otimes \hat{\pi}_T(h_{(1)}) \otimes a_{(2)} \otimes \hat{\pi}_T(h_{(2)}).$$

And on the other hand

$$\Delta_{B_q}(a \otimes \hat{\pi}_T(h)) = a_{(1)} \otimes \omega_{ii}^\sigma \hat{\pi}_T(h_{(1)}) \omega_{rr}^\nu \otimes \pi_T(u_{ii}^\sigma S(u_{rr}^\nu)) a_{(2)} \otimes \hat{\pi}_T(h_{(2)}),$$

and since $\hat{\pi}_T$ maps $\mathcal{A}(A_q)$ into the set of diagonal elements of $\mathcal{A}(\widehat{K}_q)$, we obtain

$$\begin{aligned} \Delta_{B_q}(a \otimes \hat{\pi}_T(h)) &= a_{(1)} \otimes \omega_{ii}^\sigma \hat{\pi}_T(h_{(1)}) \otimes \pi_T(u_{ii}^\sigma S(u_{ii}^\sigma)) a_{(2)} \otimes \hat{\pi}_T(h_{(2)}) \\ &= a_{(1)} \otimes \hat{\pi}_T(h_{(1)}) \otimes a_{(2)} \otimes \hat{\pi}_T(h_{(2)}). \end{aligned}$$

Thus $(\text{id} \otimes \hat{\pi}_T)$ is compatible with the coproducts. The *-algebra structure of $\mathcal{A}(B_q)$ is not twisted so there is no difficulty to see that $(\text{id} \otimes \hat{\pi}_T)$ is a *-algebra homomorphism. To conclude we just notice that since the pairing between $\mathcal{D}(B_q)$ and $\mathcal{A}(B_q)$ is defined leg by leg it is clear that the dual morphism of $(\text{id} \otimes \hat{\pi}_T)$ is $(\text{id} \otimes \pi_T)$. \square

We now present the characters of B_q in a different way to that given in Section 1.3. First, for $(\mu, \lambda) \in \mathbf{P} \times \mathfrak{t}_q^*$, one can build the one dimensional representation of L_q on $\mathbb{C}_{\mu, \lambda} = \mathbb{C}_\mu \otimes \mathbb{C}_\lambda$ via

$$(\tau \otimes h) \cdot 1 = \phi_T(e^{-\mu} \tau) \phi_{A_q}(K_{-\lambda} h),$$

for all $h \in \mathcal{D}(A_q)$, $\tau \in \mathcal{D}(T)$. Since $\mathcal{D}(L_q)$ is essential, we have $\mathcal{D}(L_q) \otimes_{\mathcal{D}(L_q)} \mathbb{C}_{\mu, \lambda} \cong \mathbb{C}_{\mu, \lambda}$. Furthermore since $\mathcal{D}(L_q)$ is a $\mathcal{D}(B_q)$ left module, one can consider the action of $\mathcal{D}(B_q)$ on $\mathcal{D}(L_q) \otimes_{\mathcal{D}(L_q)} \mathbb{C}_{\mu, \lambda}$, which happens to be exactly the character of B_q associated to (μ, λ) , according to the previous lemma. In particular this shows that such character can be factorized through the morphism $(\text{id} \otimes \pi_T) : \mathcal{D}(B_q) \rightarrow \mathcal{D}(L_q)$.

Recall that we defined in Section 1.3 the representation $\text{Ind}_{B_q}^{G_q} \mathbb{C}_{\mu, \lambda}$ in an ad hoc way. We now confirm that this agrees with the general induction method we developed in Chapter 4.

Lemma 5.2.2. *Let $a \otimes f \in \mathcal{A}(G_q)$. We have*

$$(a \otimes f)_{(1)} \otimes (\pi_T \otimes \pi_T)((a \otimes f)_{(2)}) = a_{(1)} \otimes f_{(1)} \otimes (\pi_T \otimes \pi_T)(a_{(2)} \otimes f_{(2)}),$$

where $(a \otimes f)_{(1)} \otimes (a \otimes f)_{(2)}$ refers to the coproduct of $\mathcal{A}(G_q)$.

Proof. Let $a \otimes f \in \mathcal{A}(G_q)$. Using the formula given at Section 1.3.2 we have

$$\begin{aligned}
(a \otimes f)_{(1)} \otimes (\pi_T \otimes \pi_T)((a \otimes f)_{(2)}) &= a_{(1)} \otimes \omega_{ij}^\sigma f_{(1)} \omega_{rs}^\nu \otimes \pi_T(S(u_{ij}^\sigma) a_{(2)} u_{rs}^\nu) \otimes \pi_T(f_{(2)}) \\
&= a_{(1)} \otimes \omega_{ii}^\sigma f_{(1)} \omega_{rr}^\nu \otimes \pi_T(u_{ii}^\sigma S(u_{rr}^\nu)) \pi_T(a_{(2)}) \otimes \pi_T(f_{(2)}) \\
&\stackrel{(*)}{=} a_{(1)} \otimes \omega_{ii}^\sigma \hat{\pi}_T(\pi_T(f_{(1)})) \omega_{rr}^\nu \otimes \pi_T(u_{ii}^\sigma S(u_{rr}^\nu)) \pi_T(a_{(2)}) \otimes \pi_T(f_{(2)}) \\
&= a_{(1)} \otimes f_{(1)} \otimes \pi_T(a_{(2)}) \otimes \pi_T(f_{(2)}),
\end{aligned}$$

where at equality (*) we used the Lemma 5.1.2. \square

We now consider the $\mathcal{D}(B_q)$ -inner product on $\mathcal{D}(G_q)$, given by Proposition 4.2.3. According to [VY20, Lemma 4.17] we have $\delta_{B_q} = 1 \otimes K_{-4\rho}$.

Lemma 5.2.3. *Let $a \otimes f, b \otimes g \in \mathcal{D}(G_q)$. We have*

$$(\text{id} \otimes \pi_T)(\langle a \otimes f, b \otimes g \rangle_{\mathcal{D}(B_q)}) = \pi_T(a^* * b) \otimes \pi_T(f^* * g) K_{-2\rho}.$$

Proof. $a \otimes f, b \otimes g \in \mathcal{D}(G_q)$. Using Remark 4.2.4 we obtain

$$\begin{aligned}
(\text{id} \otimes \pi_T)(\langle a \otimes f, b \otimes g \rangle_{\mathcal{D}(B_q)}) &= (\text{id} \otimes \pi_T)(\phi_{G_q}(\overline{(a \otimes f)}(b \otimes g)_{(1)})(\pi_T \otimes \text{id})((b \otimes g)_{(2)})(1 \otimes K_{-2\rho})) \\
&= \phi_{G_q}(\overline{(a \otimes f)}(b \otimes g)_{(1)})(\pi_T \otimes \pi_T)((b \otimes g)_{(2)})(1 \otimes K_{-2\rho}) \\
&\stackrel{(*)}{=} \phi_{G_q}((\bar{a} \otimes \bar{f})(b_{(1)} \otimes g_{(1)})) \pi_T(b_{(2)}) \otimes \pi_T(g_{(2)})(1 \otimes K_{-2\rho}) \\
&= \pi_T(a^* * b) \otimes \pi_T(f^* * g) K_{-2\rho},
\end{aligned}$$

where for the equality (*) we use the previous lemma and that the involution on $\mathcal{A}(G_q)$ is leg-wise. For the last line we simply use that $\phi_{G_q} = \phi_{K_q} \otimes \psi_{\widehat{K}_q}$ and identify convolutions on each legs. \square

Proposition 5.2.4. *The unitary representations $\mathcal{D}(G_q) \otimes_{\mathcal{D}(B_q)} \mathbb{C}_{\mu, \lambda}$ and $\text{Ind}_{B_q}^{G_q} \mathbb{C}_{\mu, \lambda}$ of $\mathcal{D}(G_q)$ are isomorphic.*

Proof. We consider the map Ψ such that

$$\begin{aligned}
\Psi : \mathcal{D}(G_q) &\longrightarrow \text{Ind}_{B_q}^{G_q} \mathbb{C}_{\mu, \lambda} \\
(a \otimes f) &\longmapsto a * \hat{\pi}_T(e^\mu) \otimes \phi_{\widehat{K}_q}(f K_{-\lambda-2\rho}) K_{\lambda+2\rho}.
\end{aligned}$$

We will show that this map is surjective, intertwines the $\mathcal{A}(G_q)$ coactions and descends to the balanced tensor product $\mathcal{D}(G_q) \otimes_{\mathcal{D}(B_q)} \mathbb{C}_{\mu, \lambda}$. Let $a \otimes f \in \mathcal{D}(G_q)$. We first show that $a * \hat{\pi}_T(e^\mu) \otimes \phi_{\widehat{K}_q}(f K_{-\lambda+2\rho}) K_{\lambda+2\rho}$ belongs to $\text{Ind}_{B_q}^{G_q} \mathbb{C}_{\mu, \lambda}$. It is enough to show that $(\text{id} \otimes \pi_T)(\Delta_{K_q}(a * \hat{\pi}_T(e^\mu))) = (a * \hat{\pi}_T(e^\mu)) \otimes e^\mu$. For this, since e^μ is group-like we have

$$\begin{aligned}
(\text{id} \otimes \pi_T)(\Delta_{K_q}(a * \hat{\pi}_T(e^\mu))) &= \phi_T(e^{-\mu} \pi_T(a_{(3)})) a_{(1)} \otimes \pi_T(a_{(2)}) \\
&= \phi_T(e^{-\mu} \pi_T(a_{(2)})) a_{(1)} \otimes e^\mu.
\end{aligned}$$

Next, let $a \otimes f, b \otimes g$ be in $\mathcal{D}(G_q)$ and consider the elements $[a \otimes f \otimes 1], [b \otimes g \otimes 1]$ of $\mathcal{D}(G_q) \otimes_{\mathcal{D}(B_q)} \mathbb{C}_{\mu, \lambda}$. We have

$$\begin{aligned}
\langle [(a \otimes f) \otimes 1], [(b \otimes g) \otimes 1] \rangle &= \langle a \otimes f, b \otimes g \rangle_{\mathcal{D}(B_q)} \cdot 1 \\
&= (\text{id} \otimes \pi_T)(\langle a \otimes f, b \otimes g \rangle_{\mathcal{D}(B_q)}) \cdot 1 \\
&= (\pi_T(a^* * b) \otimes \pi_T(f^* * g)K_{-2\rho}) \cdot 1 \\
&= \phi_T(\pi_T(a^* * b)e^{-\mu})\phi_{\widehat{K}_q}((f^* * g)K_{-2\rho-\lambda}) \\
&= \phi_T(\pi_T(a^* * b)e^{-\mu})\phi_{\widehat{K}_q}(f^*K_{-2\rho-\lambda})\phi_{\widehat{K}_q}(gK_{-2\rho-\lambda})
\end{aligned}$$

where at the last line we used that $\phi_{\widehat{K}_q}(x * y) = \phi_{\widehat{K}_q}(x)\phi_{\widehat{K}_q}(y)$, $\forall x, y \in \mathcal{D}(\widehat{K}_q)$. Note also that $\phi_{\widehat{K}_q}(f^*K_{-2\rho-\lambda}) = (K_{-2\rho-\lambda}^*f^*)$ since $K_{-2\rho-\lambda}$ is self-adjoint and $\sigma_{\widehat{K}_q}(K_{-2\rho-\lambda}) = K_{-2\rho-\lambda}$. For the calculation on the right hand side we will use that $(e^\mu)^* = e^\mu$ and that $e^\mu * e^\mu = e^\mu$. We also use Lemma 2.3.7. Using the definition of the inner product on $\text{Ind}_{B_q}^{G_q} \mathbb{C}_{\mu, \lambda}$ from Section 1.3.4 we have

$$\begin{aligned}
\langle a * \widehat{\pi}_T(e^\mu) \otimes K_{\lambda+2\rho}, b * \widehat{\pi}_T(e^\mu) \otimes K_{\lambda+2\rho} \rangle &= \langle a * \widehat{\pi}_T(e^\mu), b * \widehat{\pi}_T(e^\mu) \rangle \\
&= \epsilon_{K_q}((a * \widehat{\pi}_T(e^\mu))^* * b * \widehat{\pi}_T(e^\mu)) \\
&= \epsilon_T(\pi_T(\widehat{\pi}_T(e^\mu)^* * a^*)) * (b * \widehat{\pi}_T(e^\mu)) \\
&= \epsilon_T(e^\mu * \pi_T(a^* * b) * e^\mu) \\
&= \epsilon_T(\pi_T(a^* * b) * (e^\mu)^*) \\
&= \phi_T(\pi_T(a^* * b)e^{\overline{\mu}}) \\
&= \phi_T(\pi_T(a^* * b)e^{-\mu}).
\end{aligned}$$

This then shows that Ψ descends to an unitary map on the balanced tensor product $\mathcal{D}(G_q) \otimes_{\mathcal{D}(B_q)} \mathbb{C}_{\mu, \lambda} \rightarrow \text{Ind}_{B_q}^{G_q} \mathbb{C}_{\mu, \lambda}$.

To conclude, we show that Ψ is surjective. To this end we first notice that $\text{Ind}_{B_q}^{G_q} \mathbb{C}_{\mu, \lambda}$ is spanned by elements of type $a \otimes K_{\lambda+2\rho}$ for $a \in \Gamma(\mathcal{E}_{\mu, \lambda})$. This follows from the fact that the map $\text{ext} : \Gamma(\mathcal{E}_{\mu, \lambda}) \rightarrow \text{Ind}_{B_q}^{G_q} \mathbb{C}_{\mu, \lambda}$ from [VY20, Lemma 6.18] is an isomorphism, where $\Gamma(\mathcal{E}_{\mu, \lambda})$ is defined in Section 1.3.4 and we have $\text{ext}(a) = a \otimes K_{\lambda+2\rho}$ for all $a \in \Gamma(\mathcal{E}_{\mu, \lambda})$. Let then $a \in \Gamma(\mathcal{E}_{\mu, \lambda})$. We have that $a * \widehat{\pi}_T(e^\mu) = a$; thus the element $a \otimes \widehat{1_{K_q}} \otimes 1$ of $\mathcal{D}(G_q) \otimes_{\mathcal{D}(B_q)} \mathbb{C}_{\mu, \lambda}$ is an antecedent of $a \otimes K_{\lambda+2\rho}$, where $\widehat{1_{K_q}}$ is defined in Section 1.3.1. \square

One can now consider the $\mathcal{D}(L_q)$ -inner product module $\mathcal{D}(G_q) \otimes_{\mathcal{D}(B_q)} \mathcal{D}(L_q)$ and we have

$$\mathcal{D}(G_q) \otimes_{\mathcal{D}(B_q)} \mathcal{D}(L_q) \otimes_{\mathcal{D}(L_q)} \mathbb{C}_{\mu, \lambda} \cong \mathcal{D}(G_q) \otimes_{\mathcal{D}(B_q)} \mathbb{C}_{\mu, \lambda}.$$

As a consequence, $\mathcal{D}(G_q) \otimes_{\mathcal{D}(B_q)} \mathcal{D}(L_q)$ is the parabolic induction module.

5.3 Geometric presentation of the induction module

We consider the linear space

$$\mathcal{A}(G_q/N_q) = \mathcal{A}(K_q) \otimes \mathcal{A}(A_q),$$

equipped with its natural structure of untwisted $*$ -algebra. We endow $\mathcal{A}(G_q/N_q)$ with a left $\mathcal{A}(G_q)$ -coaction given, for $a \otimes h \in \mathcal{A}(G_q/N_q)$, by

$$\Delta_{G_q/N_q}(a \otimes h) = W_{32}^{-1}(\Delta_{K_q}(a) \otimes \alpha_{A_q}(h))W_{32} \in M(\mathcal{A}(G_q)) \otimes \mathcal{A}(G_q/N_q),$$

where the coaction α_{A_q} is defined after Lemma 5.1.2. Let $f \in \mathcal{A}(\widehat{K}_q)$. We have

$$\Delta_{G_q/N_q}(a \otimes \pi_T(f)) = W_{32}^{-1}(a_{(1)} \otimes f_{(1)} \otimes a_{(2)} \otimes \pi_T(f_{(2)}))W_{32}.$$

From this we see that $\Delta_{G_q/N_q}(a \otimes \pi_T(f)) = (\text{id} \otimes \text{id} \otimes \text{id} \otimes \pi_T)(\Delta_{G_q}(a \otimes f))$ and it directly follows that the map Δ_{G_q/N_q} is coassociative. This remark also implies the next proposition.

Proposition 5.3.1. *The map $\text{id} \otimes \pi_T : \mathcal{A}(G_q) \rightarrow \mathcal{A}(G_q/N_q)$ intertwines the left- $\mathcal{A}(G_q)$ -coactions where $\mathcal{A}(G_q)$ is considered with its natural comodule structure given by the co-product.*

We now define a right $\mathcal{A}(L_q)$ -coaction on $\mathcal{A}(G_q/N_q)$, denoted Δ'_{G_q/N_q} . For all $a \otimes h \in \mathcal{A}(G_q/N_q)$ we set

$$\Delta'_{G_q/N_q}(a \otimes h) = a_{(1)} \otimes h_{(1)} \otimes \pi_T(a_{(2)}) \otimes h_{(2)} \in \mathcal{A}(G_q/N_q) \otimes \mathcal{A}(L_q).$$

Proposition 5.3.2. *The coactions Δ'_{G_q/N_q} and Δ_{G_q/N_q} commute.*

Proof. We first claim that we have

$$(\text{id} \otimes \hat{\pi}_T \otimes \text{id} \otimes \hat{\pi}_T)(\Delta'_{G_q/N_q}(a \otimes h)) = (\text{id} \otimes \text{id} \otimes \pi_T \otimes \text{id})[\Delta_{G_q}(a \otimes \hat{\pi}_T(h))].$$

We calculate

$$\begin{aligned} & (\text{id} \otimes \text{id} \otimes \pi_T \otimes \text{id})[\Delta_{G_q}(a \otimes \hat{\pi}_T(h))] \\ &= a_{(1)} \otimes \omega_{ij}^\sigma \hat{\pi}_T(h_{(1)}) \omega_{rs}^\nu \otimes \pi_T(S(u_{ij}^\sigma) a_{(2)} u_{rs}^\nu) \otimes \hat{\pi}_T(h_{(2)}) \\ &= a_{(1)} \otimes \hat{\pi}_T(h_{(1)}) \otimes \pi_T(a_{(2)}) \otimes \hat{\pi}_T(h_{(2)}) \\ &= (\text{id} \otimes \hat{\pi}_T \otimes \text{id} \otimes \hat{\pi}_T)(\Delta'_{G_q/N_q}(a \otimes h)), \end{aligned}$$

We have also that

$$(\text{id} \otimes \text{id} \otimes \text{id} \otimes \hat{\pi})(\Delta_{G_q/N_q}(a \otimes h)) = \Delta_{G_q}(a \otimes \hat{\pi}(h)).$$

Now we can prove the proposition. First we rewrite above equalities using the leg notation (we write π and $\hat{\pi}$ instead of π_T and $\hat{\pi}_T$):

$$\begin{aligned} (\hat{\pi} \otimes \hat{\pi})_{24} \circ \Delta'_{G_q/N_q} &= \pi_3 \circ \Delta_{G_q} \circ \hat{\pi}_2 \\ \hat{\pi}_4 \circ \Delta_{G_q/N_q} &= \Delta_{G_q} \circ \hat{\pi}_2 \end{aligned}$$

Now observe that we have on the one hand

$$\begin{aligned} (\hat{\pi} \otimes \hat{\pi})_{46} \circ (\Delta'_{G_q/N_q})_{34} \circ \Delta_{G_q/N_q} & \\ &= ((\hat{\pi} \otimes \hat{\pi})_{24} \circ \Delta'_{G_q/N_q})_{34} \circ \Delta_{G_q/N_q} \\ &= (\pi_3 \circ \Delta_{G_q} \circ \hat{\pi}_2)_{34} \circ \Delta_{G_q/N_q} \\ &= \pi_5 \circ (\Delta_{G_q})_{34} \circ \Delta_{G_q} \circ \hat{\pi}_2 \end{aligned}$$

and on the other hand

$$\begin{aligned} (\hat{\pi} \otimes \hat{\pi})_{46} \circ (\Delta_{G_q/N_q})_{12} \circ \Delta'_{G_q/N_q} & \\ &= (\Delta_{G_q})_{12} \circ (\hat{\pi} \otimes \hat{\pi})_{24} \circ \Delta'_{G_q/N_q} \\ &= (\Delta_{G_q})_{12} \circ \pi_3 \circ \Delta_{G_q} \circ \hat{\pi}_2 \\ &= \pi_5 \circ (\Delta_{G_q})_{12} \circ \Delta_{G_q} \circ \hat{\pi}_2 \end{aligned}$$

and we conclude the proof using the coassociativity of Δ_{G_q} and injectivity of $\hat{\pi}_T$. \square

Observe now that $\mathcal{A}(G_q/N_q) = \mathcal{D}(K_q) \otimes \mathcal{D}(A_q)$ as linear space. On the one hand $\mathcal{D}(K_q)$ can be considered as a $\mathcal{D}(T)$ -inner product module, in the sense of Chapter 4, since T is a closed quantum subgroup of K_q . On the other hand $K_{2\rho}$ is a self-adjoint and group-like element of $M(\mathcal{A}(A_q))$; thus $\mathcal{D}(A_q)$ has a structure of $\mathcal{D}(A_q)$ -inner product module with right action

$$h \cdot l = h * (lK_{2\rho}),$$

and the sesquilinear map defined by

$$\langle h, k \rangle_{\mathcal{D}(A_q)} = (h^* * k)K_{-2\rho},$$

for all $h, k, l \in \mathcal{D}(A_q)$. One can thus endow $\mathcal{A}(G_q/N_q) = \mathcal{D}(K_q) \otimes \mathcal{D}(A_q)$ with the structure of a $(\mathcal{D}(T) \otimes \mathcal{D}(A_q))$ -inner product module induced by the tensor product. Let $a \otimes h, b \otimes k \in \mathcal{A}(G_q/N_q)$ and $\tau \otimes l \in \mathcal{D}(L_q)$. We have

$$\begin{aligned} \langle a \otimes h, b \otimes k \rangle_{\mathcal{D}(L_q)} &= \pi_T(a^* * b) \otimes (h^* * k)K_{-2\rho}, \\ (a \otimes k) \cdot (\tau \otimes l) &= a * \hat{\pi}_T(\tau) \otimes k * (lK_{2\rho}). \end{aligned}$$

Lemma 5.3.3. *The left action of $\mathcal{D}(K_q)$ on $\mathcal{A}(G_q/N_q)$ induced by Δ_{G_q/N_q} commutes with the right $\mathcal{D}(L_q)$ action.*

Proof. This is almost equivalent to Proposition 5.3.2. Observe that if one precomposes the right $\mathcal{D}(L_q)$ action by the $*$ -algebra homomorphism of $\mathcal{D}(L_q)$ given by $x \mapsto x(1 \otimes K_{2\rho})$ we obtain exactly the action induced by the coaction Δ'_{G_q/N_q} . \square

Proposition 5.3.4. *The map defined by*

$$\begin{aligned} \Phi : \mathcal{D}(G_q) \otimes \mathcal{D}(L_q) &\longrightarrow \mathcal{A}(G_q/N_q) \\ (a \otimes f) \otimes (\tau \otimes h) &\longmapsto (a \otimes \pi_T(f)) \cdot (\tau \otimes h), \end{aligned}$$

is a $\mathcal{D}(L_q)$ -linear map which intertwines the left action of $\mathcal{D}(G_q)$ and descends to a unitary isomorphism on the balanced tensor product $\mathcal{D}(G_q) \otimes_{\mathcal{D}(B_q)} \mathcal{D}(L_q)$.

Proof. The $\mathcal{D}(L_q)$ -linearity of Φ is immediate from the definition since the right $\mathcal{D}(L_q)$ action on $\mathcal{A}(G_q/N_q)$ is associative. The intertwining property directly follows from Proposition 5.3.1 and the previous proposition. Let $(a \otimes f) \otimes (\tau \otimes h)$ and $(b \otimes g) \otimes (\zeta \otimes k)$ be in $\mathcal{D}(G_q) \otimes \mathcal{D}(L_q)$ and consider the elements $[(a \otimes f) \otimes (\tau \otimes h)]$ and $[(b \otimes g) \otimes (\zeta \otimes k)]$ of the balanced tensor product $\mathcal{D}(G_q) \otimes_{\mathcal{D}(B_q)} \mathcal{D}(L_q)$. We have

$$\begin{aligned} &\langle [(a \otimes f) \otimes (\tau \otimes h)], [(b \otimes g) \otimes (\zeta \otimes k)] \rangle_{\mathcal{D}(L_q)} \\ &= \left\langle (\tau \otimes h), \langle (a \otimes f, b \otimes g)_{\mathcal{D}(B_q)} \cdot (\zeta \otimes k) \right\rangle_{\mathcal{D}(L_q)} \\ &= \langle (\tau \otimes h), (\pi_T(a^* * b) \otimes \pi_T(f^* * g)K_{-2\rho}) * (\zeta \otimes k) \rangle_{\mathcal{D}(L_q)} \\ &= \langle (\tau \otimes h), (\pi_T(a^* * b) * \zeta) \otimes (\pi_T(f^* * g)K_{-2\rho}) * k \rangle_{\mathcal{D}(L_q)} \\ &= (\tau^* * \pi_T(a^* * b) * \zeta) \otimes (h^* * \pi_T((f^* * g)K_{-2\rho}) * k) \\ &= (\pi_T(a * \hat{\pi}_T(\tau))^* * b * \hat{\pi}_T(\zeta)) \otimes (h^* * \pi_T(f^* * g)K_{-2\rho} * k) \\ &= (\pi_T(a * \hat{\pi}_T(\tau))^* * b * \hat{\pi}_T(\zeta)) \otimes (h^* * \pi_T(f)^* K_{2\rho} * \pi_T(g)K_{2\rho} * k)K_{-2\rho} \\ &= \langle (a \otimes \pi_T(f)) \cdot (\tau \otimes h), (b \otimes \pi_T(g)) \cdot (\zeta \otimes k) \rangle_{\mathcal{D}(L_q)}. \end{aligned}$$

Thus the map Ψ descend to a unitary map on the balanced tensor product. With regard to the surjectivity it is enough to observe that the right $\mathcal{D}(L_q)$ action on $\mathcal{A}(G_q/N_q)$ is essential. \square

The following theorem is now immediate.

Theorem 5.3.5. *The pre-Hilbert $\mathcal{D}(L_q)$ -module $\mathcal{A}(G_q/N_q)$ can be completed into a Hilbert $C^*(L_q)$ -module $\mathcal{E}(G_q/N_q)$ and we have*

$$\mathcal{E}(G_q/N_q) \cong \mathcal{E}(G_q) \otimes_{C^*(B_q)} C^*(L_q),$$

as G_q -representations. The tensor product $\mathcal{E}(G_q/N_q) \otimes_{C^*(L_q)} -$ defines a functor from the category of unitary $C^*(L_q)$ -representations to the category of unitary $C_u^*(G_q)$ -representations which coincides with parabolic induction.

By the Fourier transform, we have

$$C^*(L_q) \cong C_0(\widehat{L_q}) = C_0(\mathbf{P} \times T)$$

such that the characters of $C^*(L_q)$ become the evaluation maps

$$\text{ev}_{(\mu,\lambda)} : C_0(\mathbf{P} \times T) \rightarrow \mathbb{C}_{\mu,\lambda}.$$

According to [VY20, Theorem 7.1] we have

$$C_r^*(G_q) \cong C_0(\mathbf{P} \times \mathfrak{t}_q^*, \mathcal{K}(H))^W,$$

where H is a countable dimensional Hilbert space, and the action of the Weyl group W is a lifting of its action by reflections on $\mathbf{P} \times \mathfrak{t}_q^*$ to an action on the bundle of C^* -algebras. More precisely, the Hilbert space H at the parameter $(\mu, \lambda) \in \mathbf{P} \times \mathfrak{t}_q^*$ is identified with the parabolically induced representation of G_q ,

$$\begin{aligned} H = H_{\mu,\lambda} &= \text{Ind}_{B_q}^{G_q} \mathbb{C}_{\mu,\lambda} \\ &\cong H_\mu = \text{Ind}_T^{K_q} \mathbb{C}_\mu \\ &= \overline{\{\xi \in \mathcal{A}(K_q) \mid \Delta(\xi) = \xi \otimes e^\mu\}}^{\|\cdot\|_{L^2(K_q)}}, \end{aligned}$$

which is a trivial Hilbert bundle on each connected component $\{\mu\} \times \mathfrak{t}_q^*$ of the parameter space. The action of W is via intertwiners of principal series representations. In this way, we have

$$C_r^*(G_q) = (\mathfrak{K}(\bigoplus_{\mu \in \mathbf{P}} C_0(\mathfrak{t}_q^*, H_\mu)))^W,$$

where \mathfrak{K} denotes compact operators on the right Hilbert $C_0(\mathbf{P} \times \mathfrak{t}_q^*)$ -module.

By theorem 5.3.5 we have

$$H_{\mu,\lambda} \cong \mathcal{E}(G_q) \otimes_{C^*(B_q)} \mathbb{C}_{\mu,\lambda}$$

as left $C_u^*(G_q)$ -module. Therefore

$$C_0(\mathfrak{t}_q^*, H_\mu) \cong \mathcal{E}(G_q/N_q) \otimes_{C^*(L_q)} C_0(\mathfrak{t}_q^*)_\mu,$$

as left $C_u^*(G_q)$ -module and right $C_0(\mathfrak{t}_q^*)$ -Hilbert module, where $C_0(\mathfrak{t}_q^*)_\mu$ denotes $C_0(\mathfrak{t}_q^*)$ equipped with the left action of $C^*(L_q) = C^*(T) \hat{\otimes} C^*(A_q) = C_0(\mathbf{P}) \hat{\otimes} C_0(\mathfrak{t}_q^*)$ such that

$C_0(\mathfrak{t}_q^*)$ acts by pointwise multiplication and $C_0(\mathbf{P})$ acts by evaluation at μ . We thus obtain

$$\begin{aligned} \bigoplus_{\mu \in \mathbf{P}} C_0(\mathfrak{t}_q^*, H_\mu) &= \mathcal{E}(G_q/N_q) \otimes_{C^*(L_q)} C_0(\mathbf{P} \times \mathfrak{t}_q^*) \\ &= \mathcal{E}(G_q/N_q). \end{aligned}$$

We have therefore proven the following result.

Corollary 5.3.6. *Let G_q be a complex semi-simple quantum group. Then*

$$C_r^*(G_q) \cong \mathfrak{K}(\mathcal{E}(G_q/N_q))^W,$$

where \mathfrak{K} indicates the algebra of compact operators in the sense of Hilbert modules.

In the classical case, this result has been first obtained in [Was87] and reformulated in [CCH16] with the Rieffel induction framework.

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