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THESE DE DOCTORAT DE

L'UNIVERSITE
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ECOLE DOCTORALE N° 601
*Mathématiques et Sciences et Technologies
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Spécialité : *Mathématiques*
Par

Hiba BIBI

Surfaces et Hypersurfaces Biharmoniques dans les Sphères et les Espaces Projectifs Complexes

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Chapter 1

Introduction

Throughout this thesis M is a connected smooth finite dimensional manifold. All tensor fields, maps, connections...etc, are assumed to be smooth. The connectedness is needed, for example, to conclude that a function is constant when its gradient vanishes on M , and also for reduction results.

1.1 Overview

In this chapter we introduce basic notions and results concerning biharmonic and biconservative submanifolds. In Sections [1.2](#) and [1.3](#) we recall some definitions and properties of Riemannian manifolds and Riemannian submanifolds which are foundation stones in our work. Then, in Section [1.4](#), after recovering some definitions and notions of biharmonic submanifolds in arbitrary target manifolds, we focus on the case when the target manifold is a Euclidean sphere, as our next chapter is concerned with spheres. The last section of the introduction mainly tackles biconservative submanifolds in complex space forms, where we present some classification results based on previous research works.

1.2 Riemannian manifolds

Let (M, g) be a Riemannian manifold. The Riemannian metric g defines the musical isomorphisms flat

$$\begin{aligned} \flat : T_p M &\rightarrow T_p^* M \\ X_p &\longmapsto X_p^\flat, \end{aligned}$$

by $X_p^\flat(Y_p) = g(X_p, Y_p)$, $\forall Y_p \in T_p M$, and its inverse sharp

$$\begin{aligned} \sharp = \flat^{-1} : T_p^* M &\rightarrow T_p M \\ \alpha_p &\longmapsto \alpha_p^\sharp, \end{aligned}$$

by $\alpha_p(Y_p) = g_p(\alpha_p^\sharp, Y_p)$, $\forall Y_p \in T_p M$.

We can approach these correspondences in a global way. The map

$$\begin{aligned} \flat : C(TM) &\rightarrow \Lambda^1(M) = C(T^*M) \\ X &\mapsto X^\flat, \quad X^\flat(Y) = g(X, Y), \quad \forall Y \in C(TM), \end{aligned}$$

is well-defined, i.e. $X^\flat \in \Lambda^1(M)$, C^∞ -linear and a bijection.

Denote

$$\begin{aligned} \sharp = \flat^{-1} : \Lambda^1(M) &\rightarrow C(TM) \\ \alpha &\mapsto \alpha^\sharp, \end{aligned}$$

and we have

$$\alpha(Y) = g(\alpha^\sharp, Y), \quad \forall Y \in C(TM).$$

The most important example is the gradient of a function

$$(df)^\sharp = \text{grad } f$$

i.e.

$$g(\text{grad } f, X) = df(X) = X(f), \quad \forall X \in C(TM).$$

If ∇ denotes the Levi-Civita connection of the metric g , then

$$\nabla_X \alpha^\sharp = (\nabla_X \alpha)^\sharp, \quad \forall \alpha \in \Lambda^1(M), \quad X \in C(TM).$$

The curvature tensor field of M , $R \in C(T_3^1(M))$, is defined with the convention

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z, \quad (1.2.1)$$

where $X, Y \in C(TM)$ and ∇ is the Levi-Civita connection on M .

The $(0, 4)$ -curvature tensor field of M , also denoted by R and, sometimes, simply called the curvature tensor field, is given by

$$R(X, Y, Z, W) = g(R(X, Y)W, Z) \quad \forall X, Y, Z, W \in C(TM). \quad (1.2.2)$$

The difference between the two tensors will be clear from the context.

The sectional curvature of a Riemannian manifold (M, g) for a 2-plane α spanned by two linearly independent vectors X and Y is given by

$$\text{Riem}^M(\alpha) = \frac{R(X, Y, X, Y)}{g(X, X)g(Y, Y) - g(X, Y)^2}. \quad (1.2.3)$$

The Ricci tensor field of M , either as a $(0, 2)$ or a $(1, 1)$ -tensor, is

$$g(\text{Ricci}(X), Y) = \text{Ricci}(X, Y) = \text{trace}\{Z \rightarrow R(Z, X)Y\}, \quad (1.2.4)$$

where $X, Y \in C(TM)$.

The scalar curvature of M , is given by

$$\text{Scal} = \text{trace Ricci}. \quad (1.2.5)$$

An m -dimensional Riemannian manifold of constant sectional curvature c , i.e. $\text{Riem}^M(\alpha) = c$ for any 2-plane α tangent to M , will be called a real space form and denoted by $M^m(c)$. Its curvature tensor field is

$$R(X, Y)Z = c\{g(Y, Z)X - g(X, Z)Y\}, \quad (1.2.6)$$

and its $(0, 4)$ curvature tensor field is

$$R(X, Y, Z, W) = c\{g(X, Z)g(Y, W) - g(X, W)g(Y, Z)\}. \quad (1.2.7)$$

Its Ricci tensor field is

$$\text{Ricci}(X, Y) = c(m-1)g(X, Y), \quad (1.2.8)$$

or as a $(1,1)$ -tensor

$$\text{Ricci}(X) = c(m-1)X,$$

so

$$\text{Scal} = cm(m-1). \quad (1.2.9)$$

The Riemannian product manifold $(M \times N, g_M + g_N)$ of two Riemannian manifolds (M, g_M) and (N, g_N) , where g_M and g_N are the Riemannian metrics on M and N respectively, is defined as the manifold $M \times N$ equipped with the metric

$$g_{M \times N}((X_1, Y_1), (X_2, Y_2)) = g_M(X_1, X_2) + g_N(Y_1, Y_2).$$

for all $X_1, X_2 \in C(TM)$ and $Y_1, Y_2 \in C(TN)$. Here we use the standard identifications

$$T_{(p,q)}(M \times N) = T_pM \oplus T_qN, \quad (p, q) \in M \times N,$$

and the pair $(X, Y) \in C(T(M \times N))$ is defined by

$$(X, Y)(p, q) = (X(p), Y(q)).$$

Also, $X \in C(TM)$ is identified with $X \in C(T(M \times N))$, $X(p, q) = (X(p), 0)$.

Lemma 1.2.1. *The Levi-Civita connection of the Riemannian product manifold $(M \times N, g_M + g_N)$ satisfies $\nabla_{(X_1, Y_1)}^{M \times N}(X_2, Y_2) = (\nabla_{X_1}^M X_2, \nabla_{Y_1}^N Y_2)$.*

Proof. Using Koszul formula (see for example [20]), we have

$$\begin{aligned}
2g_{M \times N} \left(\nabla_{(X_1, Y_1)}^{M \times N} (X_2, Y_2), (X_3, Y_3) \right) &= (X_1, Y_1) g_{M \times N} ((X_2, Y_2), (X_3, Y_3)) \\
&\quad + (X_2, Y_2) g_{M \times N} ((X_1, Y_1), (X_3, Y_3)) \\
&\quad - (X_3, Y_3) g_{M \times N} ((X_1, Y_1), (X_2, Y_2)) \\
&\quad + g_{M \times N} ([(X_1, Y_1), (X_2, Y_2)], (X_3, Y_3)) \\
&\quad - g_{M \times N} ([(X_1, Y_1), (X_3, Y_3)], (X_2, Y_2)) \\
&\quad + g_{M \times N} ([(X_2, Y_2), (X_3, Y_3)], (X_1, Y_1)),
\end{aligned}$$

where $X_1, X_2, X_3 \in C(TM)$ and $Y_1, Y_2, Y_3 \in C(TN)$. But

$$\begin{aligned}
(X_1, Y_1) g_{M \times N} ((X_2, Y_2), (X_3, Y_3)) &= (X_1, Y_1) (g_M(X_2, X_3) + g_N(Y_2, Y_3)) \\
&= X_1 g_M(X_2, X_3) + Y_1 g_N(Y_2, Y_3).
\end{aligned}$$

Similarly,

$$(X_2, Y_2) g_{M \times N} ((X_1, Y_1), (X_3, Y_3)) = X_2 g_M(X_1, X_3) + Y_2 g_N(Y_1, Y_3)$$

and

$$(X_3, Y_3) g_{M \times N} ((X_1, Y_1), (X_2, Y_2)) = X_3 g_M(X_1, X_2) + Y_3 g_N(Y_1, Y_2).$$

Also, we have

$$\begin{aligned}
&g_{M \times N} ([(X_1, Y_1), (X_2, Y_2)], (X_3, Y_3)) \\
&= g_{M \times N} ((X_1, Y_1)(X_2, Y_2) - (X_2, Y_2)(X_1, Y_1), (X_3, Y_3)) \\
&= g_{M \times N} ((X_1 X_2, Y_1 Y_2) - (X_2 X_1, Y_2 Y_1), (X_3, Y_3)) \\
&= g_{M \times N} ([[X_1, X_2], [Y_1, Y_2]], (X_3, Y_3)) \\
&= g_M([X_1, X_2], X_3) + g_N([Y_1, Y_2], Y_3).
\end{aligned}$$

Similarly,

$$g_{M \times N} ([(X_1, Y_1), (X_3, Y_3)], (X_2, Y_2)) = g_M([X_1, X_3], X_2) + g_N([Y_1, Y_3], Y_2)$$

and

$$g_{M \times N} ([(X_2, Y_2), (X_3, Y_3)], (X_1, Y_1)) = g_M([X_2, X_3], X_1) + g_N([Y_2, Y_3], Y_1).$$

Hence,

$$\begin{aligned}
2g_{M \times N} \left(\nabla_{(X_1, Y_1)}^{M \times N} (X_2, Y_2), (X_3, Y_3) \right) &= X_1 g_M (X_2, X_3) + Y_1 g_N (Y_2, Y_3) \\
&\quad + X_2 g_M (X_1, X_3) + Y_2 g_N (Y_1, Y_3) \\
&\quad - X_3 g_M (X_1, X_2) + Y_3 g_N (Y_1, Y_2) \\
&\quad + g_M ([X_1, X_2], X_3) + g_N ([Y_1, Y_2], Y_3) \\
&\quad - g_M ([X_1, X_3], X_2) + g_N ([Y_1, Y_3], Y_2) \\
&\quad - g_M ([X_2, X_3], X_1) + g_N ([Y_2, Y_3], Y_1) \\
&= 2g_{M \times N} \left((\nabla_{X_1}^M X_2, \nabla_{Y_1}^N Y_2), (X_3, Y_3) \right).
\end{aligned}$$

Therefore,

$$\nabla_{(X_1, Y_1)}^{M \times N} (X_2, Y_2) = (\nabla_{X_1}^M X_2, \nabla_{Y_1}^N Y_2).$$

□

Then the curvature tensor field of the product manifold is

$$R^{M \times N}((X_1, Y_1), (X_2, Y_2))(X_3, Y_3) = (R^M(X_1, X_2)X_3, R^N(Y_1, Y_2)Y_3), \quad (1.2.10)$$

where R^M and R^N are the curvature tensor fields of (M, g_M) and (N, g_N) respectively, $X_1, X_2, X_3 \in C(TM)$ and $Y_1, Y_2, Y_3 \in C(TN)$, and similarly, the $(0, 4)$ -curvature tensor field is

$$\begin{aligned}
R^{M \times N}((X_1, Y_1), (X_2, Y_2), (X_3, Y_3), (X_4, Y_4)) &= R^M(X_1, X_2, X_3, X_4) \\
&\quad + R^N(Y_1, Y_2, Y_3, Y_4), \quad (1.2.11)
\end{aligned}$$

where R^M and R^N are the $(0, 4)$ -curvature tensor fields of (M, g_M) and (N, g_N) respectively, $X_1, X_2, X_3, X_4 \in C(TM)$ and $Y_1, Y_2, Y_3, Y_4 \in C(TN)$.

The Ricci tensor field of $(M \times N, g_M + g_N)$ is

$$\text{Ricci}^{M \times N}((X_1, Y_1), (X_2, Y_2)) = \text{Ricci}^M(X_1, X_2) + \text{Ricci}^N(Y_1, Y_2), \quad (1.2.12)$$

where Ricci^M and Ricci^N are the Ricci tensor fields of (M, g_M) and (N, g_N) respectively, and its scalar curvature is

$$\text{Scal}^{M \times N} = \text{Scal}^M + \text{Scal}^N, \quad (1.2.13)$$

where Scal^M and Scal^N are the scalar curvatures of (M, g_M) and (N, g_N) respectively.

If $(M(c_1) \times N(c_2), g_M + g_N)$ is a Riemannian product of space forms, where c_1 and c_2 are the constant sectional curvatures of (M, g_M) and (N, g_N) respectively, then its curvature tensor is

$$\begin{aligned}
&R^{M \times N}((X_1, Y_1), (X_2, Y_2))(X_3, Y_3) \\
&= (c_1 \{g_M(X_2, X_3)X_1 - g_M(X_1, X_3)X_2\}, c_2 \{g_N(Y_2, Y_3)Y_1 - g_N(Y_1, Y_3)Y_2\}). \quad (1.2.14)
\end{aligned}$$

Recall the Divergence Theorem [6] relating the integral over a compact domain, with smooth boundary of the manifold M^m , of the divergence of a one-form (or vector field) to its values on the boundary. We will use this theorem repeatedly.

Proposition 1.2.2. (*Divergence Theorem*) *Let D be a compact domain with smooth boundary of a Riemannian manifold (M^m, g) . Let θ be a 1-form and X a vector field defined on an open neighborhood of D . Then*

$$\int_D \operatorname{div} \theta \, dv_g = \int_{\partial D} \theta(\eta) \, \partial v_g$$

and

$$\int_D \operatorname{div} X \, dv_g = \int_{\partial D} \langle X, \eta \rangle \, \partial v_g,$$

where ∂D denotes the boundary of D , ∂v_g denotes the induced volume form on the boundary and $\eta = \eta(x)$ denotes the outward pointing unit normal at a point $x \in \partial D$.

Corollary 1.2.3. *For any 1-form θ and vector field X with compact support included in D , we have*

$$\int_D \operatorname{div} \theta \, dv_g = 0 \quad \text{and} \quad \int_D \operatorname{div} X \, dv_g = 0.$$

We recall below the definition of a vector bundle.

Definition 1.2.4. Let E and M be two manifolds and $\pi : E \rightarrow M$ a map. Then E is called a vector bundle of rank k over M if the following two conditions are satisfied:

1. $\pi^{-1}(p)$ is a k -dimensional vector space for each $p \in M$.
2. There exists an open covering $\{U_i\}$ of M and diffeomorphisms $h_i : U_i \times \mathbb{R}^k \rightarrow \pi^{-1}(U_i)$ such that $\varphi_{i,p} : \mathbb{R}^k \rightarrow \pi^{-1}(p)$ are isomorphisms of vector spaces. In this case, $g_{ji}(p) = \varphi_{j,p}^{-1} \circ \varphi_{i,p} : \mathbb{R}^k \rightarrow \mathbb{R}^k$, $p \in U_j \cap U_i$, is an element of the general linear group $GL(k; \mathbb{R})$.

Prior to Bochner's work, Weitzenböck developed a formula very similar to the Bochner Formula (see for example [35]).

Theorem 1.2.5. (*Weitzenböck Formula*) [35] *Let $\pi : E \rightarrow M^m$ be a Riemannian vector bundle on a Riemannian manifold M^m . Then for any $r \geq 1$ and $\sigma \in C(\Lambda^r T^* M \otimes E)$, the space of smooth r -forms on M^m with values in the vector bundle E , we have*

$$\Delta^{\text{Hodge}} \sigma = -\operatorname{trace} \nabla^2 \sigma + S(\sigma),$$

where the operator $S \in C(\operatorname{hom}(\Lambda^r T^* M \otimes E, \Lambda^r T^* M \otimes E))$ is defined by $S = 0$ if $r = 0$, and

$$(S_p \sigma)(Y_1, \dots, Y_r) = \sum_{i,a} (-1)^{a+1} (R(X_i, Y_a) \sigma) (X_i, Y_1, \dots, \widehat{Y}_a, \dots, Y_r),$$

if $r \geq 1$, where $\{X_i\}_{i=1,\dots,m}$ is an orthonormal basis in T_pM . The curvature R is given by

$$(R(X, Y)\sigma)(X_1, \dots, X_r) = R^E(X, Y)(\sigma(X_1, \dots, X_r)) - \sum_{a=1}^r \sigma(X_1, \dots, R^M(X, Y)X_a, \dots, X_r).$$

The Bochner Formula follows from the Weitzenböck Formula.

Theorem 1.2.6. (Bochner Formula) [35] Let $\pi : E \rightarrow M^m$ be a Riemannian vector bundle on a Riemannian manifold M^m . Then for any $r \geq 1$ and $\sigma \in C(\Lambda^r T^*M \otimes E)$, the Bochner Formula is given by

$$\frac{1}{2}\Delta|\sigma|^2 = \langle \Delta^{\text{Hodge}}\sigma, \sigma \rangle - |\nabla\sigma|^2 - \langle S(\sigma), \sigma \rangle,$$

where $\Delta|\sigma|^2 = -\operatorname{div} \operatorname{grad} |\sigma|^2$.

The following lower bound was proved by Lichnerowicz [61], while Obata [76] considered the equality case.

Theorem 1.2.7. (Lichnerowicz-Obata) Let M be an m -dimensional compact manifold without boundary. Suppose that the Ricci curvature of M is bounded from below by

$$\operatorname{Ricci}^M(X, X) \geq (m-1)ag(X, X), \quad X \in C(TM),$$

for some constant $a > 0$, then the first non-zero eigenvalue of the Laplacian on M satisfies

$$\lambda_1 \geq ma.$$

Moreover, equality holds if and only if M is isometric to a standard sphere of radius $1/\sqrt{a}$.

1.3 Riemannian submanifolds

A submanifold of a given Riemannian manifold (N^n, h) is a pair (M^m, φ) , where M^m is a manifold and $\varphi : M \rightarrow N$ an immersion. We will always equip M with the induced metric $g = \varphi^*h$, so $\varphi : (M, g) \rightarrow (N, h)$ becomes an isometric immersion and, for simplicity, write $\varphi : M \rightarrow N$ without mentioning the metrics. We also write $\varphi : M \rightarrow N$, or even M , instead of (M, φ) .

To fix the notations, we recall the first-order fundamental equations of a submanifold in a Riemannian manifold, as these equations define the second fundamental form, the shape operator and the connection in the normal bundle. Let $\varphi : M^m \rightarrow N^n$ be an

isometric immersion. For each $p \in M$, $T_{\varphi(p)}N$ can be written as the orthogonal direct sum

$$T_{\varphi(p)}N = d\varphi(T_pM) \oplus d\varphi(T_pM)^\perp, \quad (1.3.1)$$

and $NM = \bigcup_{p \in M} d\varphi(T_pM)^\perp$ is referred to as the normal bundle of φ (or of M) in N .

Denote by ∇ and ∇^N the Levi-Civita connections on M and N , respectively, and by ∇^φ the induced connection in the pull-back bundle $\varphi^{-1}(TN) = \bigcup_{p \in M} T_{\varphi(p)}N$. Taking into account the decomposition (1.3.1), we obtain the *Gauss formula*

$$\nabla_X^\varphi d\varphi(Y) = d\varphi(\nabla_X Y) + B(X, Y), \quad X, Y \in C(TM).$$

The symmetric tensor field $B \in C(\odot^2 T^*M \otimes NM)$ is called the second fundamental form of M in N , T^*M being the cotangent bundle of M .

Definition 1.3.1. Let $\varphi : M^m \rightarrow N^n$ be an isometric immersion. If the second fundamental form B vanishes then M is called *totally geodesic*.

The mean curvature vector field of M in N is then defined as the section of the normal bundle NM

$$H = \frac{1}{m} \text{trace } B,$$

where the trace is taken with respect to the metric g .

We will often indicate metrics on various vector bundles by the same symbol $\langle \cdot, \cdot \rangle$.

If $\eta \in C(NM)$, the *Weingarten formula* is

$$\nabla_X^\perp \eta = -d\varphi(A_\eta(X)) + \nabla_X^\perp \eta, \quad X \in C(TM),$$

where $A_\eta \in C(T^*M \otimes TM)$ is called the shape operator of M in N in the direction η , and ∇^\perp is the induced connection in the normal bundle. The tensors B and A are related by

$$\langle B(X, Y), \eta \rangle = \langle A_\eta(X), Y \rangle,$$

for all $X, Y \in C(TM)$, $\eta \in C(NM)$.

For hypersurfaces, once we have chosen the unit normal vector field η , we can define the mean curvature function by

$$f = \frac{1}{m} \text{trace } A,$$

where $A = A_\eta$. The mean curvature vector field H is given by

$$H = f\eta.$$

Usually, when dealing with hypersurfaces, we assume them to be orientable (so there exists two unit normal vector fields globally defined), therefore η , A and f are globally defined. However, quantities like f^2 , $A(\text{grad } f)$, $|A|^2$, or expressions like $\Delta f = mf$ have a global meaning, even if M is not orientable.

When confusion is unlikely, since locally the immersion φ is an embedding, we identify M with its image by φ , X with $d\varphi(X)$ and $\nabla_X^\varphi d\varphi(Y)$ with $\nabla_X^N Y$. With this in mind, we can re-write the Gauss and the Weingarten formulas as

$$\nabla_X^N Y = \nabla_X Y + B(X, Y),$$

and

$$\nabla_X^N Y = -A_\eta(X) + \nabla_X^\perp \eta.$$

Next, we introduce some classes of submanifolds which are useful for our work.

Definition 1.3.2. Let $\varphi : M^m \rightarrow N^n$ be an isometric immersion. If the mean curvature vector field H vanishes then φ is called a *minimal immersion*.

Definition 1.3.3. Let M^m be a submanifold of a Riemannian manifold N^n . If the mean curvature vector field H of M^m is parallel in the normal bundle, i.e., $\nabla^\perp H = 0$, then M^m is called a *Parallel Mean Curvature (PMC) submanifold*.

Definition 1.3.4. Let M^m be a submanifold of a Riemannian manifold N^n . If the mean curvature vector field H has a constant norm, i.e. $|H|$ is constant, then it is called a *Constant Mean Curvature (CMC) submanifold*.

Now, we introduce two definitions for the case when the shape operator adopts a very simple form.

Definition 1.3.5. A submanifold $\varphi : M^m \rightarrow N^n$ is called *pseudo-umbilical* if $A_H = |H|^2 I$, where I is the identity tensor field of type $(1, 1)$.

Definition 1.3.6. A hypersurface $\varphi : M^m \rightarrow N^{m+1}$ is called *umbilical* if $A = fI$, where I is the identity tensor field of type $(1, 1)$ and f is the mean curvature function.

Remark 1.3.7. Any minimal submanifold is CMC, PMC and pseudo-umbilical. Of course, a minimal hypersurface is not necessarily umbilical; a minimal hypersurface is umbilical if and only if it is totally geodesic.

Consider $\varphi : M^m \rightarrow N^n$ a submanifold of N . Let $\{E_i\}_{i=1}^m$ be a local orthonormal frame field on M^m and tangent to M^m , and $\{E_a\}_{a=m+1}^n$ a local orthonormal frame field on M^m normal to M^m . Then (using Gram-Schmidt) the frame field $\{E_A\}_{A=1}^n$ along M^m can be extended to a local orthonormal frame field defined on an open subset of

N^n , tangent to N^n and denoted in the same way. We note that the pull-backs of the 1-forms $\{\omega^a\}$ on M vanish, where $\{\omega^A\}$ the dual basis of $\{E_A\}$, and we simply write

$$\omega^a = 0 \quad \text{on} \quad M.$$

Denote by $\{\omega_A^B\}$ the Levi-Civita connection 1-forms corresponding to $\{E_A\}$ and $\bar{\nabla}$, i.e. on N^n we have

$$\bar{\nabla}.E_A = \omega_A^B(\cdot)E_B,$$

and by $\{\omega^A\}$ the dual basis of $\{E_A\}$.

We recall that the $\{\omega_A^B\}$ are the unique 1-forms on N^n such that

$$d\omega^B = -\omega_A^B \wedge \omega^A$$

and

$$\omega_A^B + \omega_B^A = 0.$$

The curvature forms $\{\Omega_B^A\}$ associated to the orthonormal frame $\{\omega^A\}$ are defined by

$$d\omega_B^A = -\omega_C^A \wedge \omega_B^C + \Omega_B^A.$$

We have

$$\Omega_B^A + \Omega_A^B = 0$$

and

$$\Omega_B^A = \frac{1}{2}R_{BCD}^A \omega^C \wedge \omega^D,$$

where R_{BCD}^A is defined by

$$R(E_C, E_D)E_B = R_{BCD}^A E_A.$$

Remark 1.3.8. For the particular case $n = 2$, we have

$$d\omega_2^1 = K\omega^1 \wedge \omega^2,$$

where K is the Gaussian curvature of N^2 .

Remark 1.3.9. For the exterior product we choose the convention

$$\omega_1 \wedge \omega_2 = \omega_1 \otimes \omega_2 - \omega_2 \otimes \omega_1,$$

for 1-forms ω_1, ω_2 . Then, the exterior differential of ω is intrinsically defined by

$$d\omega(X, Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X, Y]).$$

Along M , we have

$$\bar{\nabla}_{E_i} E_j = \omega_j^A(E_i) E_A.$$

On the other hand,

$$\bar{\nabla}_{E_i} E_j = \nabla_{E_i} E_j + B(E_i, E_j),$$

thus, comparing the tangent and the normal parts, we obtain

$$\nabla_{E_i} E_j = \omega_j^k(E_i) E_k,$$

and

$$B(E_i, E_j) = \omega_j^a(E_i) E_a.$$

We note that the pull-backs of the 1-forms ω_j^k on M , also denoted by ω_j^k , coincide with the 1-forms defined by $\{E_i\}$ and ∇ .

Similarly, on M we have

$$\bar{\nabla}_{E_i} E_a = \omega_a^B(E_i) E_B,$$

and

$$\bar{\nabla}_{E_i} E_a = \nabla_{E_i}^\perp E_a - A_{E_a} E_i.$$

By comparing the tangent and the normal parts of both equations, we obtain

$$\nabla_{E_i}^\perp E_a = \omega_a^b(E_i) E_b, \quad (1.3.2)$$

and

$$-A_{E_a} E_i = \omega_a^j(E_i) E_j.$$

In the following we introduce the fundamental equations for submanifolds (see for example [20]).

Proposition 1.3.10. (*Gauss equation*). *Let $\varphi : M^m \rightarrow N^n$ be a submanifold. Then*

$$\langle R^N(X, Y)Z, W \rangle = \langle R(X, Y)Z, W \rangle - \langle B(X, W), B(Y, Z) \rangle + \langle B(Y, W), B(X, Z) \rangle, \quad (1.3.3)$$

where $X, Y, Z, W \in C(TM)$.

If M^m is a hypersurface in a space form $N^{m+1}(c)$, then the Gauss equation becomes

$$R(X, Y)Z = c(\langle Y, Z \rangle X - \langle X, Z \rangle Y) + \langle A(Y), Z \rangle A(X) - \langle A(X), Z \rangle A(Y)$$

for any $X, Y, Z \in C(TM)$.

Proposition 1.3.11. *Let $\varphi : M^m \rightarrow N^{m+1}$ be a hypersurface. Then we have*

$$\begin{aligned} \text{Ricci}^M(X, Y) &= \text{Ricci}^N(X, Y) - \langle AX, AY \rangle + mf \langle AX, Y \rangle \\ &\quad + \langle R^N(X, \eta)Y, \eta \rangle, \end{aligned}$$

and

$$\text{Scal}^M = \text{Scal}^N - |A|^2 + m^2 f^2 - 2 \text{Ricci}^N(\eta, \eta), \quad (1.3.4)$$

where A is the shape operator of M in N , X, Y are tangent vector fields on M and η is the unit normal vector field.

Corollary 1.3.12. *Let $\varphi : M^m \rightarrow N^{m+1}(c)$ be a hypersurface in a space form. Then we have*

$$\text{Ricci}^M(X, Y) = c(m-1) \langle X, Y \rangle + mf \langle AX, Y \rangle - \langle AX, AY \rangle, \quad (1.3.5)$$

and

$$\text{Scal}^M = cm(m-1) + m^2 f^2 - |A|^2. \quad (1.3.6)$$

where A is the shape operator of M in N , X, Y are tangent vector fields on M and η is the unit normal vector field.

Proof. As $\text{Scal}^N = cm(m+1)$ and $\text{Ricci}^N = cmI$, where I is the identity operator, replacing in (1.3.4), we obtain

$$|A|^2 = cm(m-1) + m^2 f^2 - \text{Scal}^M.$$

□

Proposition 1.3.13. *(Codazzi equation). Let $\varphi : M^m \rightarrow N^n$ be a submanifold. Then*

$$(\nabla_X A_\eta)(Y) - (\nabla_Y A_\eta)(X) = A_{\nabla_X^\perp \eta}(Y) - A_{\nabla_Y^\perp \eta}(X) - (R^N(X, Y)\eta)^\top,$$

where $X, Y \in C(TM)$ and $\eta \in C(NM)$, or equivalently,

$$\left(\nabla_X^\perp B \right)(Y, Z) - \left(\nabla_Y^\perp B \right)(X, Z) = (R^N(X, Y)Z)^\perp, \quad X, Y, Z \in C(TM), \quad (1.3.7)$$

where

$$\left(\nabla_X^\perp B \right)(Y, Z) = \nabla_X^\perp B(Y, Z) - B(\nabla_X Y, Z) - B(Y, \nabla_X Z).$$

In particular, if N is a space form the Codazzi equation becomes

$$(\nabla_X A_\eta)(Y) - (\nabla_Y A_\eta)(X) = A_{\nabla_X^\perp \eta}(Y) - A_{\nabla_Y^\perp \eta}(X),$$

or, equivalently,

$$\left(\nabla_X^\perp B \right)(Y, Z) = \left(\nabla_Y^\perp B \right)(X, Z).$$

If M^m is moreover a hypersurface in $N^{m+1}(c)$, then the Codazzi equation becomes

$$(\nabla_X A)(Y) - (\nabla_Y A)(X) = 0.$$

The following result holds for any submanifold in a Riemannian manifold, where by tracing the above Codazzi equation we obtain the following (see for example [24, 62]).

Proposition 1.3.14. *Let $\varphi : M^m \rightarrow N^n$ be a submanifold. Then*

$$\text{trace } \nabla A_H = \frac{m}{2} \text{grad} (|H|^2) + \text{trace } A_{\nabla^\perp H}(\cdot) + \text{trace} (R^N(\cdot, H)\cdot)^\top. \quad (1.3.8)$$

Corollary 1.3.15. *Let $\varphi : M^m \rightarrow N^n(c)$ be a submanifold, $c \in \mathbb{R}$. Then*

$$\text{trace } \nabla A_H = \frac{m}{2} \text{grad} (|H|^2) + \text{trace } A_{\nabla^\perp H}(\cdot). \quad (1.3.9)$$

Proposition 1.3.16. *(Ricci equation) Let $\varphi : M^m \rightarrow N^n$ be a submanifold. Then, for $X, Y \in C(TM)$ and $\eta \in C(NM)$*

$$(R^N(X, Y)\eta)^\perp = R^\perp(X, Y)\eta + B(A_\eta(X), Y) - B(X, A_\eta(Y)),$$

or, equivalently,

$$\langle R^N(X, Y)\eta, \xi \rangle = \langle R^\perp(X, Y)\eta, \xi \rangle - \langle [A_\eta, A_\xi]X, Y \rangle \quad (1.3.10)$$

where $\xi \in C(NM)$, and $[A_\eta, A_\xi] = A_\eta A_\xi - A_\xi A_\eta$.

Theorem 1.3.17. *An umbilical hypersurface in a space form is CMC.*

Next, we recall the fundamental theorem for submanifolds [19].

Theorem 1.3.18.

1. *Let M^m be a simply connected Riemannian manifold, $\pi : E \rightarrow M$ a Riemannian vector bundle of rank k with a compatible connection ∇ , and B a symmetric section of the homeomorphism bundle $\text{hom}(TM \times TM, E) \cong (TM \otimes TM)^* \otimes E = (TM)^* \otimes (TM)^* \otimes E$, i.e. $B : C(TM) \times C(TM) \rightarrow E$ is a $C^\infty(M)$ -bilinear and symmetric map. Define, for each local section η of E , a map $A_\eta : C(TM) \rightarrow C(TM)$ by*

$$\langle A_\eta(X), Y \rangle = \langle B(X, Y), \eta \rangle, \quad X, Y \in C(TM).$$

For submanifolds of space forms, if, for some $c \in \mathbb{R}$, B and ∇ satisfy the Gauss, Codazzi and Ricci equations, then there is an isometric immersion $\varphi : M^m \rightarrow N^{n=m+k}(c)$, and a vector bundle isomorphism $\tilde{\varphi} : C(E) \rightarrow C(NM)$ along φ , such that for every $X, Y \in C(TM)$ and any local sections η, ξ of E :

$$\langle \tilde{\varphi}(\eta), \tilde{\varphi}(\xi) \rangle = \langle \eta, \xi \rangle, \quad \tilde{\varphi}(B(X, Y)) = \tilde{B}(X, Y), \quad \tilde{\varphi}(\nabla_X \eta) = \nabla_X^\perp \tilde{\varphi}(\eta),$$

where \tilde{B} and ∇^\perp are the second fundamental form, and the normal connection of φ , respectively.

2. Suppose that φ and ψ are two isometric immersions of a connected manifold M^m into $N^{n=m+k}(c)$. Let NM_φ , B_φ and ∇^\perp_φ denote the normal bundle, the second fundamental form and the normal connection of φ , respectively; and let NM_ψ , B_ψ and ∇^\perp_ψ be the corresponding objects for ψ . If there exists a vector bundle isomorphism $\tilde{\varphi} : C(NM)_\varphi \rightarrow C(NM)_\psi$ such that, for every $X, Y \in C(TM)$ and every $\eta, \xi \in C(NM)_\varphi$:

$$\langle \tilde{\varphi}(\eta), \tilde{\varphi}(\xi) \rangle = \langle \eta, \xi \rangle, \quad \tilde{\varphi}(B_\varphi(X, Y)) = B_\psi(X, Y), \quad \tilde{\varphi}(\nabla_X^\perp \eta) = \nabla_X^\perp \tilde{\varphi}(\eta),$$

then there is an isometry $F : N^n(c) \rightarrow N^n(c)$ such that

$$\psi = F \circ \varphi \quad \text{and} \quad dF|_{NM_\varphi} = \tilde{\varphi}.$$

Theorem 1.3.19. (The fundamental theorem for hypersurfaces)

1. Let M^m be a simply connected Riemannian manifold and let $A : C(TM) \rightarrow C(TM)$ be a symmetric tensor field of type $(1, 1)$ satisfying the Gauss and Codazzi equations in the case of constant sectional curvature c . Then there is an isometric immersion $\varphi : M^m \rightarrow N^{m+1}(c)$ such that $A = A_\eta$, for some unit normal vector field $\eta \in C(NM)$, where A_η denotes the shape operator of the immersion φ .
2. Let $\varphi : M^m \rightarrow N^{m+1}(c)$ and $\psi : M^m \rightarrow N^{m+1}(c)$ be two connected hypersurfaces, and let $\tilde{\varphi} : C(NM)_\varphi \rightarrow C(NM)_\psi$ be one of the two vector bundle isomorphisms. Suppose that $B_\psi(X, Y) = \tilde{\varphi}(B_\varphi(X, Y))$ or $B_\psi(X, Y) = -\tilde{\varphi}(B_\varphi(X, Y))$, for every $X, Y \in C(TM)$, where B_ψ and B_φ denote, respectively, the second fundamental forms of ψ and φ . Then there exists an isometry $F : N^{m+1}(c) \rightarrow N^{m+1}(c)$ such that

$$\begin{cases} \psi = F \circ \varphi \\ dF|_{NM_\varphi} = \tilde{\varphi} \quad \text{or} \quad dF|_{NM_\varphi} = -\tilde{\varphi}. \end{cases}$$

1.4 Biharmonic submanifolds

Denote by $C^\infty(M, N)$ the space of smooth maps $\varphi : (M, g) \rightarrow (N, h)$ between two Riemannian manifolds. A map $\varphi \in C^\infty(M, N)$ is called *harmonic* if it is a critical point of the energy functional

$$E : C^\infty(M, N) \rightarrow \mathbb{R}, \quad E(\varphi) = \frac{1}{2} \int_M |d\varphi|^2 dv_g.$$

It is characterized by the vanishing of its tension field

$$\tau(\varphi) = \text{trace } \nabla d\varphi = 0.$$

The tension field is a smooth section of the pull-back bundle $\varphi^{-1}(TN)$.

An isometric immersion, is a critical point of the energy functional if and only if it is a *minimal immersion*, i.e. a critical point of the volume functional [36], because

$$\tau(\varphi) = mH.$$

The study of *biharmonic maps* was introduced by G.-Y. Jiang in 1986 [54], as critical points of the bienergy functional

$$E_2 : C^\infty(M, N) \rightarrow \mathbb{R}, \quad E_2(\varphi) = \frac{1}{2} \int_M |\tau(\varphi)|^2 dv_g.$$

The associated Euler-Lagrange equation is given by the vanishing of the bitension field

$$\tau_2(\varphi) = -\Delta\tau(\varphi) - \text{trace } R^N(d\varphi(\cdot), \tau(\varphi))d\varphi(\cdot) = 0, \quad (1.4.1)$$

where, in this case, as $r = 0$ in Theorem [1.2.5],

$$\Delta = -\text{trace } \nabla^2.$$

This was first suggested by J. Eells and L. Lemaire in [35], as a variation on the theme of harmonic maps, they considered polyharmonic maps, i.e. critical points of the functional

$$F_k(\varphi) = \frac{1}{2} \int_M |(d + d^*)^k(\varphi)|^2 dv_g, \quad k \in \mathbb{N}^*,$$

so that $F_2(\varphi) = E_2(\varphi)$. Biharmonic maps generalize harmonic maps, and harmonic maps are clearly biharmonic. Using a simple Bochner formula, G.-Y. Jiang proved that biharmonic maps from a compact manifold to a non-positively curved space are harmonic, so the first interesting target manifold is the Euclidean sphere. Biharmonic submanifolds are defined as isometric immersions which are biharmonic maps.

Definition 1.4.1. Let $\varphi : (M, g) \rightarrow (N, h)$ be a smooth map between two Riemannian manifolds. If φ is biharmonic non-harmonic map then it is called *proper-biharmonic*.

In [24], B.-Y. Chen independently defined biharmonic submanifolds of the Euclidean space as isometric immersions with harmonic mean curvature vector field, that is, the components of the immersion are biharmonic functions. In [28, 52] biharmonic surfaces in \mathbb{E}^3 were proved to be minimal. This has led to:

Chen's Conjecture [23]: *Biharmonic submanifolds of Euclidean spaces are minimal.* Some particular subcases have been proved for example in [1, 34, 46].

An isometric immersion of a manifold M^m into a Riemannian manifold N^n is said to be biharmonic, if its mean curvature vector field H satisfies the following equation

$$\tau_2(\varphi) = -m(\Delta H + \text{trace } R^N(d\varphi(\cdot), H)d\varphi(\cdot)) = 0,$$

where Δ denotes the rough Laplacian on sections of the pull-back bundle $\varphi^{-1}(TN^n)$ and R^N is the curvature operator on N^n .

In the study of biharmonic submanifolds it is useful to split the bitension field with respect to its normal and tangent components.

When the ambient space is an arbitrary Riemannian manifold, the splitting of the biharmonic equation was obtained in [62].

Theorem 1.4.2. *A submanifold M^m in a Riemannian manifold N is biharmonic if and only if*

$$\begin{cases} \Delta^\perp H + \text{trace } B(\cdot, A_H \cdot) + \text{trace}(R^N(\cdot, H)\cdot)^\perp & = 0 \\ 4 \text{trace } A_{\nabla(\cdot)^\perp H}(\cdot) + m \text{grad}(|H|^2) + 4 \text{trace}(R^N(\cdot, H)\cdot)^\top & = 0, \end{cases} \quad (1.4.2)$$

where R^N is the curvature tensor of N . Moreover, the tangent part can also be written as

$$4 \text{trace}(\nabla A_H)(\cdot, \cdot) - m \text{grad}(|H|^2) = 0. \quad (1.4.3)$$

By direct application of formula (1.4.3) we have

Proposition 1.4.3. *Let M^m be a pseudo-umbilical biharmonic submanifold of a Riemannian manifold N^n with $m \neq 4$. Then M is CMC, and moreover*

$$|\nabla^\perp H|^2 + m|H|^2 + \langle \text{trace}(R^N(\cdot, H)\cdot), H \rangle = 0.$$

Corollary 1.4.4. *Let M^m be a pseudo-umbilical biharmonic submanifold of a Riemannian manifold N^n . If $m \neq 4$ and $\text{Riem}^N \leq 0$, then M^m is minimal.*

Formulas (1.4.2) and (1.4.3) were used to study biharmonic hypersurfaces in an arbitrary Riemannian manifold, for the first time in [79] (see Theorem 2.7 and Proposition 2.8). However, they were explicitly written for this case in [82].

Theorem 1.4.5. *Let M^m be a hypersurface in a Riemannian manifold N . Then M is biharmonic if and only if*

$$\begin{cases} \Delta f + (|A|^2 - \text{Ricci}^N(\eta, \eta)) f & = 0 \\ 2A(\text{grad } f) + m f \text{grad } f - 2f (\text{Ricci}^N(\eta))^\top & = 0, \end{cases}$$

where η is a unit normal vector field.

We present some further non-existence results for proper-biharmonic submanifolds, with the result of G.-Y. Jiang, the proof of which is a direct consequence of the Weitzenböck formula.

Theorem 1.4.6. [54] Let $\varphi : M \rightarrow N$ be a smooth map, where M is compact and $\text{Riem}^N \leq 0$. Then φ is biharmonic if and only if it is harmonic.

Replacing compactness with CMC, we can state the following proposition.

Proposition 1.4.7. [79] Let $\varphi : M^m \rightarrow N^n$ be an isometric immersion such that $|\tau(\varphi)|$ is constant and assume that $\text{Riem}^N \leq 0$. Then φ is biharmonic if and only if it is minimal.

For hypersurfaces, the next results hold.

Theorem 1.4.8. [79] Let $\varphi : M^m \rightarrow N^{m+1}$ be a compact hypersurface and assume that $\text{Ricci}^N \leq 0$. Then φ is biharmonic if and only if it is minimal.

Proposition 1.4.9. [79] Let $\varphi : M^m \rightarrow N^{m+1}$ be a hypersurface such that $|\tau(\varphi)|$ is constant and $\text{Ricci}^N \leq 0$. Then φ is biharmonic if and only if it is minimal.

Proposition 1.4.10. [79] Let $\varphi : M^m \rightarrow N^n$ be a smooth map such that $|\tau(\varphi)|$ is constant, and assume that there exists a point $p \in M$ where $\text{rank } \varphi(p) \geq 2$. If $\text{Riem}^N < 0$, then φ is biharmonic if and only if it is harmonic.

When the ambient space is the unit Euclidean sphere we have the following characterization.

Theorem 1.4.11. [24, 79] An immersion $\varphi : M^m \rightarrow \mathbb{S}^n$ is biharmonic if and only if

$$\begin{cases} \Delta^\perp H + \text{trace } B(\cdot, A_H \cdot) - mH & = 0 \\ 2 \text{trace } A_{\nabla^\perp(\cdot)} H + \frac{m}{2} \text{grad}(|H|^2) & = 0, \end{cases} \quad (1.4.4)$$

where A denotes the shape operator, B the second fundamental form, H the mean curvature vector field, ∇^\perp and Δ^\perp the connection and the Laplacian in the normal bundle of φ , respectively.

In the codimension one case, denoting by $A_\eta = A$ the shape operator with respect to a (local) unit section η in the normal bundle and putting $f = (\text{trace } A)/m$, so $f^2 = |H|^2$, the above result reduces to the following.

Corollary 1.4.12. [24, 32, 79] Let M^m be an orientable hypersurface in \mathbb{S}^{m+1} . Then φ is biharmonic if and only if

$$\begin{cases} \Delta f & = (m - |A|^2)f \\ A(\text{grad } f) & = -\frac{m}{2} f \text{grad } f. \end{cases} \quad (1.4.5)$$

Theorem 1.4.13. [53, 54] Let $\varphi : M^m \rightarrow \mathbb{S}^{m+1}$ be a non-minimal hypersurface of constant mean curvature. Then φ is proper-biharmonic if and only if $|A|^2 = m$.

Corollary 1.4.14. *Let $\varphi : M^m \rightarrow \mathbb{S}^{m+1}$ be a CMC proper-biharmonic hypersurface. Then the scalar curvature is a positive constant given by*

$$\text{Scal}^M = m^2(1 + |H|^2) - 2m.$$

A special class of immersions in \mathbb{S}^n consists of the parallel mean curvature immersions (PMC). For this class of immersions Theorem [1.4.11](#) reads as follows.

Corollary 1.4.15. [\[11\]](#) *Let $\varphi : M^m \rightarrow \mathbb{S}^n$ be a PMC immersion. Then φ is biharmonic if and only if*

$$\text{trace } B(A_H(\cdot), \cdot) = mH,$$

or equivalently,

$$\begin{cases} \langle A_H, A_\xi \rangle = 0, & \forall \xi \in C(NM) \text{ with } \xi \perp H, \\ |A_H|^2 = m |H|^2, \end{cases}$$

where NM denotes the normal bundle of M in \mathbb{S}^n .

The first example of a proper-biharmonic hypersurface [\[54\]](#) is the generalized Clifford torus $\mathbb{S}^{m_1}(\frac{1}{\sqrt{2}}) \times \mathbb{S}^{m_2}(\frac{1}{\sqrt{2}})$ in $\mathbb{S}^{m_1+m_2+1}$, with $m_1 \neq m_2$. Recall that when $m_1 = m_2$ this Clifford torus is minimal. In [\[15\]](#), using umbilicality the authors observed that the 45-th parallel $\mathbb{S}^m(\frac{1}{\sqrt{2}})$ in \mathbb{S}^{m+1} is a proper-biharmonic hypersurface. These two examples have constant mean curvature and motivate the following Conjecture:

C1 [\[10\]](#): *Any biharmonic submanifold in a Euclidean sphere is CMC.*

In the same paper, the authors conjectured for hypersurfaces in \mathbb{S}^{m+1} :

C2 [\[10\]](#): *Any proper-biharmonic hypersurface in \mathbb{S}^{m+1} is an open set of $\mathbb{S}^{m_1}(\frac{1}{\sqrt{2}}) \times \mathbb{S}^{m_2}(\frac{1}{\sqrt{2}})$, $m_1 + m_2 = m$ and $m_1 \neq m_2$, or $\mathbb{S}^m(\frac{1}{\sqrt{2}})$.*

Next, we construct examples considering first the hyperspheres of \mathbb{S}^{m+1} as follows;

Proposition 1.4.16. [\[15\]](#) *Let $M = \mathbb{S}^m(a) \times \{b\}$*

$$= \left\{ p = (x^1, \dots, x^{m+1}, b) \mid (x^1)^2 + \dots + (x^{m+1})^2 = a^2, a^2 + b^2 = 1, 0 < a < 1 \right\}$$

be a parallel hypersphere of \mathbb{S}^{m+1} . Then M is a biharmonic submanifold of \mathbb{S}^{m+1} if and only if $a = 1/\sqrt{2}$ and $b = \pm 1/\sqrt{2}$.

Then, an analogue for the product of spheres is given by the following proposition.

Proposition 1.4.17. [\[14, 53, 54\]](#) *Let m_1, m_2 be two positive integers such that $m = m_1 + m_2$, and let r_1, r_2 be two positive real numbers such that $r_1^2 + r_2^2 = 1$. Then we have two cases:*

1. $m_1 \neq m_2$, and $\mathbb{S}^{m_1}(r_1) \times \mathbb{S}^{m_2}(r_2)$ is a proper-biharmonic submanifold of \mathbb{S}^{m+1} if and only if $r_1 = r_2 = 1/\sqrt{2}$;

2. $m_1 = m_2 = q$, and the following statements are equivalent:

- $\mathbb{S}^q(r_1) \times \mathbb{S}^q(r_2)$ is a biharmonic submanifold of \mathbb{S}^{2q+1} .
- $\mathbb{S}^q(r_1) \times \mathbb{S}^q(r_2)$ is a harmonic submanifold of \mathbb{S}^{2q+1} .
- $r_1 = r_2 = 1/\sqrt{2}$.

Theorem 1.4.18. [14] Let $\psi : M \rightarrow \mathbb{S}^{n-1}(a)$ be a minimal submanifold in a small hypersphere $\mathbb{S}^{n-1}(a) \subset \mathbb{S}^n$, of radius $a \in (0, 1)$ and denote by $i : \mathbb{S}^{n-1}(a) \rightarrow \mathbb{S}^n$ the inclusion map. Then $\varphi = i \circ \psi : M \rightarrow \mathbb{S}^n$ is proper-biharmonic if and only if $a = 1/\sqrt{2}$.

Theorem 1.4.19. [14] Let $\psi_1 : M_1^{m_1} \rightarrow \mathbb{S}^{n_1}(a)$ and $\psi_2 : M_2^{m_2} \rightarrow \mathbb{S}^{n_2}(b)$ be two minimal submanifolds, where $n_1 + n_2 = n - 1$, $a^2 + b^2 = 1$, and denote by $i : \mathbb{S}^{n_1}(a) \times \mathbb{S}^{n_2}(b) \rightarrow \mathbb{S}^n$ the inclusion map. Then $\varphi = i \circ (\psi_1 \times \psi_2) : M_1 \times M_2 \rightarrow \mathbb{S}^n$ is proper-biharmonic if and only if $a = b = 1/\sqrt{2}$ and $m_1 \neq m_2$.

When a biharmonic immersion has constant mean curvature (CMC) the following bound for $|H|$ holds.

Theorem 1.4.20. [78] Let $\varphi : M \rightarrow \mathbb{S}^n$ be a CMC proper-biharmonic immersion. Then $|H| \in (0, 1]$, and $|H| = 1$ if and only if φ induces a minimal immersion of M into $\mathbb{S}^{n-1}(1/\sqrt{2}) \subset \mathbb{S}^n$.

Theorem 1.4.21. [8] Let $\varphi : M^m \rightarrow \mathbb{S}^{m+1}$ be a CMC proper-biharmonic hypersurface. Assume that $\text{Riem}^M \geq 0$. Then $\varphi(M)$ is either an open part of $\mathbb{S}^m(1/\sqrt{2})$, or an open part of $\mathbb{S}^{m_1}(1/\sqrt{2}) \times \mathbb{S}^{m_2}(1/\sqrt{2})$, $m_1 + m_2 = m$, $m_1 \neq m_2$.

When the hypersurface has at most two distinct principal curvatures everywhere, the following rigidity results can be obtained.

Theorem 1.4.22. [10] Let $\varphi : M^m \rightarrow \mathbb{S}^{m+1}$ be a hypersurface. Assume that φ is proper-biharmonic with at most two distinct principal curvatures everywhere. Then φ is CMC.

Theorem 1.4.23. [10] Let $\varphi : M^m \rightarrow \mathbb{S}^{m+1}$ be a hypersurface. Assume that φ is proper-biharmonic with at most two distinct principal curvatures everywhere. Then $\varphi(M)$ is either an open part of $\mathbb{S}^m(1/\sqrt{2})$, or an open part of $\mathbb{S}^{m_1}(1/\sqrt{2}) \times \mathbb{S}^{m_2}(1/\sqrt{2})$, $m_1 + m_2 = m$, $m_1 \neq m_2$. Moreover, if M is complete, then either $\varphi(M) = \mathbb{S}^m(1/\sqrt{2})$ and φ is an embedding, or $\varphi(M) = \mathbb{S}^{m_1}(1/\sqrt{2}) \times \mathbb{S}^{m_2}(1/\sqrt{2})$, $m_1 + m_2 = m$, $m_1 \neq m_2$ and φ is an embedding when $m_1 \geq 2$ and $m_2 \geq 2$.

Corollary 1.4.24. [15] Let $\varphi : M^2 \rightarrow \mathbb{S}^3$ be a proper-biharmonic surface. Then $\varphi(M)$ is an open part of $\mathbb{S}^2(1/\sqrt{2}) \subset \mathbb{S}^3$.

Theorem 1.4.25. [77] *If $\varphi : M^3 \rightarrow \mathbb{S}^4$ is a complete proper-biharmonic hypersurface, then $\varphi(M) = \mathbb{S}^3(1/\sqrt{2})$ or $\varphi(M) = \mathbb{S}^2(1/\sqrt{2}) \times \mathbb{S}^1(1/\sqrt{2})$.*

Proposition 1.4.26. [8] *Let $\varphi : M^m \rightarrow \mathbb{S}^{m+1}$ be a compact hypersurface. Assume that φ is proper-biharmonic and $|A|^2 \geq m$. Then φ is CMC and $|A|^2 = m$.*

In the following we present J.-H. Chen's lemma, with its proof (see also [77]), as it plays an essential role in Chapter 2 on unique continuation.

Lemma 1.4.27. (J.-H. Chen [32]) *Let $\varphi : M^m \rightarrow \mathbb{S}^{m+1}$ be a proper-biharmonic hypersurface. Then, at points where $\text{grad } f \neq 0$, we have*

$$|A|^2 \geq \frac{m^2(m+8)}{4(m-1)} f^2. \quad (1.4.6)$$

Proof. Let $p \in M$ such that $(\text{grad } f)(p) \neq 0$, then $e_1 = (\text{grad } f)(p)/|(\text{grad } f)(p)|$ is a principal direction for A with principal curvature $\lambda_1 = -\frac{m}{2}f(p)$. By considering $e_k \in T_p M$, $k = 2, \dots, m$, such that $\{e_i\}_{i=1}^m$ is an orthonormal basis in $T_p M$ and $A(e_k) = \lambda_k e_k$, we get at p

$$\begin{aligned} |A|^2 &= \sum_{i=1}^m |A(e_i)|^2 = |A(e_1)|^2 + \sum_{k=2}^m |A(e_k)|^2 = \frac{m^2}{4} f^2 + \sum_{k=2}^m \lambda_k^2 \\ &\geq \frac{m^2}{4} f^2 + \frac{1}{m-1} \left(\sum_{k=2}^m \lambda_k \right)^2 = \frac{m^2}{4} f^2 + \frac{1}{m-1} (\text{trace } A - \lambda_1)^2 \\ &\geq \frac{m^2}{4} f^2 + \frac{1}{m-1} \left(mf + \frac{m}{2} f \right)^2 = \frac{m^2}{4} f^2 + \frac{9m^2}{4(m-1)} f^2 = \frac{m^2(m+8)}{4(m-1)} f^2, \end{aligned}$$

thus Inequality (1.4.6) holds at p . \square

Remark 1.4.28. The equality case of Inequality (1.4.6), at a point $p \in M$ where $(\text{grad } f)(p) \neq 0$, holds if and only if the λ_i 's are equal for $i = 2, \dots, m$. Thus we end up having at most two distinct principal curvatures at p . Then, denoting by W the set of all points of M where $\text{grad } f$ does not vanish, due to Theorem 1.4.22 we have that Inequality (1.4.6) is strict on an open dense subset of W .

Theorem 1.4.29. [32] *Let $\varphi : M^m \rightarrow \mathbb{S}^{m+1}$ be a compact hypersurface in \mathbb{S}^{m+1} . If φ is proper-biharmonic and $|A|^2 \leq m$, then φ is CMC and $|A|^2 = m$.*

Proposition 1.4.30. [8] *Let $\varphi : M^m \rightarrow \mathbb{S}^{m+1}$ be a compact hypersurface. If φ is proper-biharmonic and $|H|^2 \geq 4(m-1)/(m(m+8))$, then φ is CMC. Moreover,*

1. if $m \in \{2, 3\}$, then $\varphi(M)$ is a small hypersphere $\mathbb{S}^m(1/\sqrt{2})$;
2. if $m = 4$, then $\varphi(M)$ is a small hypersphere $\mathbb{S}^4(1/\sqrt{2})$ or a standard product of spheres $\mathbb{S}^3(1/\sqrt{2}) \times \mathbb{S}^1(1/\sqrt{2})$.

For the non-compact case, we obtain the following.

Proposition 1.4.31. [8] *Let $\varphi : M^m \rightarrow \mathbb{S}^{m+1}$, $m > 2$, be a non-compact hypersurface. Assume that M is complete and has non-negative Ricci curvature. If φ is proper-biharmonic, $|A|^2$ is constant and $|A|^2 \geq m$, then φ is CMC and $|A|^2 = m$. In this case $|H|^2 \leq ((m-2)/m)^2$.*

Theorem 1.4.32. [32] *Let $\varphi : M^m \rightarrow \mathbb{S}^{m+1}$ be a compact hypersurface. If φ is proper-biharmonic, M has non-negative sectional curvature and $m \leq 10$, then φ is CMC and $\varphi(M)$ is either $\mathbb{S}^m(1/\sqrt{2})$, or $\mathbb{S}^{m_1}(1/\sqrt{2}) \times \mathbb{S}^{m_2}(1/\sqrt{2})$, $m_1 + m_2 = m$, $m_1 \neq m_2$.*

Proposition 1.4.33. *Let $\varphi : M^m \rightarrow \mathbb{S}^{m+1}$ be a compact proper-biharmonic hypersurface with constant scalar curvature. Then*

$$m(m-2) < \text{Scal}^M \leq 2m(m-1).$$

Moreover, $\text{Scal}^M = 2m(m-1)$ if and only if $\varphi(M) = \mathbb{S}^m(1/\sqrt{2})$.

1.5 Biconservative submanifolds

The notion of biconservative submanifolds is a generalisation of biharmonic submanifolds by only requiring the vanishing of the tangential part of the bitension field. Studying biconservative submanifolds will reveal the influence of the tangential part, how much we can rely on just one part of the biharmonic equation, and how many results remain valid with this half condition. The notion of *biconservative* hypersurfaces in an arbitrary manifold was first introduced in 2014 [16]. Since then, biconservative submanifolds were studied in [16, 42, 68–70, 74, 88, 89, 93, 94], where many classification results were obtained for such submanifolds in different ambient spaces. We mention that the idea of studying biconservative submanifolds can be tracked back as far as [46] which studied hypersurfaces in \mathbb{E}^4 with vanishing tangent part of the biharmonic equation.

Now fix a map φ and let the domain metric vary. We obtain a new functional on the set \mathcal{G} of all Riemannian metrics on M^m defined by

$$\mathcal{F}_2 : \mathcal{G} \rightarrow \mathbb{R}, \quad \mathcal{F}_2(g) = E_2(\varphi).$$

Critical points of this functional are characterized by the vanishing of the stress-energy tensor of the bienergy [62]. This tensor, denoted by S_2 , was introduced in [51] as

$$\begin{aligned} S_2(X, Y) &= \frac{1}{2} |\tau(\varphi)|^2 \langle X, Y \rangle + \langle d\varphi, \nabla \tau(\varphi) \rangle \langle X, Y \rangle - \langle d\varphi(X), \nabla_Y \tau(\varphi) \rangle \\ &\quad - \langle d\varphi(Y), \nabla_X \tau(\varphi) \rangle, \end{aligned}$$

and satisfies

$$\operatorname{div} S_2 = \langle \tau_2(\varphi), d\varphi \rangle.$$

We note that, for isometric immersions, $(\operatorname{div} S_2)^\sharp = -\tau_2(\varphi)^\top$, where $\tau_2(\varphi)^\top$ is the tangent part of the bitension field.

Definition 1.5.1. A submanifold $\varphi : M^m \rightarrow N^n$ of a Riemannian manifold N^n is called *biconservative* if $\operatorname{div} S_2 = 0$.

Note that we cannot expect properties of biharmonic submanifolds (i.e. derived from both the normal and tangent parts of the biharmonic equation) to hold for biconservative submanifolds. Conversely, properties satisfied by biconservative submanifolds can either simply fail for biharmonic submanifolds or, if they are satisfied, extra restrictions will be imposed by the normal part of the equation. For example:

Proposition 1.5.2. *Any biconservative surface M^2 which is pseudo-umbilical in \mathbb{E}^3 is either minimal or an open part of a Euclidean sphere.*

Proof. Using Proposition [1.5.10](#) we have that M^2 is CMC and then, it is either minimal or umbilical (with $H \neq 0$). But a non-minimal umbilical surface has to be an open part of the round sphere. \square

Now, for biharmonic submanifolds, Proposition [1.5.2](#) becomes:

Proposition 1.5.3. *Any biharmonic surface M^2 which is pseudo-umbilical in \mathbb{E}^3 is minimal.*

Proof. From the above result, we obtain that the surface M^2 is either minimal or an open part of a round sphere. But then, we can check that the second case cannot hold because of the normal part of the biharmonic equation. \square

Remark 1.5.4. In fact, it was proved in [\[28, 52\]](#) that any biharmonic surface in \mathbb{E}^3 is minimal. As there are many non-minimal biconservative surfaces in \mathbb{E}^3 and no proper-biharmonic surface in \mathbb{E}^3 , we can see that the normal part of the biharmonic equation is not compatible with its tangent part.

In the following, we list some general properties of biconservative submanifolds.

Proposition 1.5.5. ([\[62, 72\]](#)) *Let M^m be a submanifold of a Riemannian manifold N^n . Then the following properties are equivalent:*

1. M is biconservative;
2. $\operatorname{trace} A_{\nabla_{(\cdot)}^\perp H}(\cdot) + \operatorname{trace}(\nabla A_H)(\cdot, \cdot) + \operatorname{trace}(R^N(\cdot, H)\cdot)^\top = 0$;

3. $4 \operatorname{trace} A_{\nabla_{(\cdot)}^\perp H}(\cdot) + m \operatorname{grad}(|H|^2) + 4 \operatorname{trace}(R^N(\cdot, H)\cdot)^\top = 0;$
4. $4 \operatorname{trace}(\nabla A_H)(\cdot, \cdot) - m \operatorname{grad}(|H|^2) = 0.$

As an immediate consequence we get

Proposition 1.5.6. *Let M^m be a PMC submanifold of a Riemannian manifold N^n . Then M^m is biconservative if and only if*

$$\operatorname{trace}(R^N(\cdot, H)\cdot)^\top = 0.$$

When the ambient space is a space form, we have

Corollary 1.5.7. *([42]) Let M^m be a PMC submanifold of a real space form N^n . Then M^m is biconservative.*

Proposition 1.5.8. *([16]) A hypersurface M^m in a space form $N^{m+1}(c)$ is biconservative if and only if*

$$A(\operatorname{grad} f) = -\frac{m}{2} f \operatorname{grad} f.$$

Corollary 1.5.9. *Any CMC hypersurface in $N^{m+1}(c)$ is biconservative.*

Proposition 1.5.10. *([7]) Let M^m be a pseudo-umbilical submanifold of a Riemannian manifold N^n with $m \neq 4$. Then M is biconservative if and only if it is CMC.*

Remark 1.5.11. Compare with Proposition [1.4.3](#) where the normal part of the biharmonic equation imposes an additional constraint.

Remark 1.5.12. When $n = m + 1$ and N^n is a space form, the above result is trivial as any pseudo-umbilical hypersurface is either minimal or umbilical, and therefore, in a space form, it is CMC.

Corollary 1.5.13. *([73]) Let M^m be a biconservative hypersurface in $N^{m+1}(c)$. Then*

$$mf\Delta f - 3m|\operatorname{grad} f|^2 - 2\langle A, \operatorname{Hess} f \rangle = 0.$$

Theorem 1.5.14. *([73]) Let M^m be a biconservative hypersurface in $N^{m+1}(c)$ with $\operatorname{grad} f \neq 0$ at any point of M . Then the distribution D , orthogonal to that determined by $\operatorname{grad} f$, is completely integrable. Moreover, any integral manifold of maximal dimension of D has flat normal connection as a submanifold in $N^{m+1}(c)$.*

We see that in the proof of Lemma [1.4.27](#) we only used the tangent part of the biharmonic equation thus the result holds also for biconservative hypersurfaces, and for the sake of completeness we state it again.

Lemma 1.5.15. (*J.-H. Chen [32]*) Let $\varphi : M^m \rightarrow \mathbb{S}^{m+1}$ be a biconservative hypersurface. Then, at points where $\text{grad } f \neq 0$, we have

$$|A|^2 \geq \frac{m^2(m+8)}{4(m-1)} f^2.$$

In the special case of biconservative surfaces, we have some specific properties. First, we recall the following general definition.

Definition 1.5.16. A symmetric $(1, 1)$ -tensor field T is called a Codazzi tensor field if

$$(\nabla T)(X, Y) = (\nabla T)(Y, X),$$

for all $X, Y \in C(TM)$.

Theorem 1.5.17. [*64, 70, 72*] Let $\varphi : M^2 \rightarrow N^n$ be a CMC surface. Then the following properties are equivalent:

1. M is biconservative;
2. $\langle A_H(\partial_z), \partial_z \rangle$ is holomorphic, where $z = x + iy$, $\partial_z = \frac{1}{2} \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right)$ and $\partial_{\bar{z}} = \frac{1}{2} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right)$;
3. A_H is a Codazzi tensor field.

Note that the above theorem follows from general properties of divergence-free symmetric $(1, 1)$ -tensors defined on a Riemannian surface.

Remark 1.5.18. In the case of biharmonic surfaces, the fact that $\langle A_H(\partial_z), \partial_z \rangle$ is holomorphic if and only if M is CMC was proved in [64].

Remark 1.5.19. We can see $\langle A_H(\partial_z), \partial_z \rangle$ as a generalization of the Hopf function, as in 3-dimensional space forms we have the following: if M^2 is a topological sphere in $N^3(c)$, then it is CMC if and only if $\langle A_H(\partial_z), \partial_z \rangle$ is holomorphic [70]. For any surface in $N^3(c)$, $\langle A_H(\partial_z), \partial_z \rangle$ is holomorphic when M is CMC, but, in general, the converse does not hold. All non-CMC surfaces with holomorphic $\langle A_H(\partial_z), \partial_z \rangle$ must necessarily have curvature equal to c (though they are not umbilical). In [70], all surfaces in \mathbb{E}^3 with $\langle A_H(\partial_z), \partial_z \rangle$ holomorphic while not CMC were found. They are locally parametrized by

$$\varphi_\alpha(u, v) = (u \cos v, u \sin v, \alpha u), \quad (\alpha \in \mathbb{R}^+).$$

In \mathbb{S}^3 , these surfaces are parametrized by

$$\varphi_\alpha(u, v) = \left(\frac{\cos u}{\alpha} \cos v, \frac{\sin u}{\alpha} \cos v, \frac{\sqrt{\alpha^2 - 1}}{\alpha} \cos v, \sin v \right),$$

($\alpha > 1$) [72].

Remark 1.5.20. Let $\varphi : M^2 \rightarrow N^n$ be a CMC biconservative surface. If M^2 is not a pseudo-umbilical surface, then the set of pseudo-umbilical points has no accumulation point. Also, if M^2 is a CMC biconservative surface and a topological sphere, then it is pseudo-umbilical (see [70]); this should be compared with a classical result: a PMC surface M^2 of genus zero in a space form is pseudo-umbilical [47].

By using Theorem 1.5.17 and the CMC hypothesis, around non pseudo-umbilical points, one can obtain an explicit form of the metric on the surface and of the shape operator A_H . It follows that CMC biconservative surfaces without pseudo-umbilical points are globally conformally flat. Then, one can prove the following proposition.

Proposition 1.5.21. [72] *Let $\varphi : M^2 \rightarrow N^n$ be a CMC biconservative surface. Assume that M is compact and does not have pseudo-umbilical points. Then M is a topological torus and moreover, if $K \geq 0$ or $K \leq 0$, we have $\nabla A_H = 0$ and $K = 0$.*

Another important property of biconservative surfaces is given by the following result.

Theorem 1.5.22. [64, 72] *Let $\varphi : M^2 \rightarrow N^n$ be a compact CMC biconservative surface. If the Gaussian curvature K of the surface is non-negative, then $\nabla A_H = 0$ and M is flat or pseudo-umbilical.*

Since any PMC submanifold in a space form is biconservative, in codimension two we have the following rigidity result.

Theorem 1.5.23. [70] *Let $c \neq 0$ and $\varphi : M^2 \rightarrow N^4(c)$ a CMC biconservative surface. Then M^2 is PMC.*

For the special case of biconservative surfaces in \mathbb{E}^4 , the situation is a bit less rigid.

Proposition 1.5.24. [70] *Let $\varphi : M^2 \rightarrow \mathbb{E}^4$ be a biconservative surface with constant mean curvature different from zero, which is not PMC. Then, locally, the surface is given by*

$$\varphi(u, v) = (\gamma(u), v + a) = (\gamma^1(u), \gamma^2(u), \gamma^3(u), v + a) \quad a \in \mathbb{R},$$

where $\gamma : I \rightarrow \mathbb{E}^3$ is a curve in \mathbb{E}^3 parametrized by arc-length, with constant non-zero curvature and non-zero torsion.

Chapter 2

Unique Continuation Properties

The results of this chapter are extracted from the article:

H. Bibi, E. Loubeau, C. Oniciuc: Unique continuation property for biharmonic hypersurfaces in spheres. *Ann. Glob. Anal. Geom.* **60**, 807–827 (2021).

2.1 Introduction

This chapter is a contribution to Conjecture **C1** and gives rigidity results based on a new unique continuation theorem (UCT). In [13], a similar approach was used to prove that if a biharmonic map is harmonic on an open subset, then it must be harmonic everywhere.

Inspired by this work of V. Branding and C. Oniciuc, and relying on the UCT of N. Aronszajn [5], our objective in this chapter is not to show minimality, i.e. harmonicity, but rather to prove the weaker condition of CMC. Using a gradient inequality between the norm of A and the mean curvature we show that, for proper-biharmonic (i.e. non-minimal) hypersurfaces in a sphere, locally CMC implies globally CMC. As an application, we extend results of J.-H. Chen [32] and S. Maeta and Y.-L. Ou [67] to non-compact manifolds, with the hypothesis $\text{grad } f = 0$ on an open subset.

In Section 2.5, we exploit this UCT to prove new rigidity results and, in Theorem 2.5.1, use an integral condition involving both the scalar and mean curvatures to force biharmonic hypersurfaces to be CMC. This extends the main result of S. Maeta and Y.-L. Ou [67] to non-constant scalar curvature, while relying on a different technique of proof.

2.2 A mini survey on unique continuation

Proving unique continuation theorems is a central theme in the study of elliptic partial differential equations (PDEs). Here, the expression "unique continuation" refers to the property that the existence of a zero of infinite order at some point forces a function to vanish everywhere (strong version), and also to the property that the function which vanishes on some open subset has to vanish everywhere (weak version). In turns, these theorems have been used to prove uniqueness of solutions of systems of PDEs and are at the origin of breakthroughs in many different fields of research, for example solving inhomogeneous elliptic equations on non-compact sets or comparing eigenvalues of nested domains.

Historically, unique continuation properties began with T. Carleman [18] in 1933, and then with N. Aronszajn [5] in 1957. On Riemannian manifolds, they were exploited by J. H. Sampson [86] in 1978 to obtain a unique continuation theorem for harmonic maps. Some 40 years later V. Branding and C. Oniciuc [13] proved several unique continuation results for biharmonic maps.

One of the most important properties of analytic functions is the unique continuation property. It is well-known that harmonic functions on \mathbb{R}^2 are real analytic, so they possess the unique continuation property. However, some elliptic partial differential equations admit non-analytic solutions, even though they still satisfy the unique continuation property.

In [55] J. Kazdan says that "My personal belief was that unique continuation should hold for the solutions of all elliptic equations which arise "naturally" in geometry."

In his 1957 paper [5] N. Aronszajn was among the first to establish a unique continuation theorem for solutions of a large class of elliptic partial differential equations and inequalities of second-order.

Let D be an open connected subset of \mathbb{R}^m . Consider a linear elliptic second-order differential operator A which has its principal part $a^{ij} \frac{\partial^2}{\partial x^i \partial x^j}$ with coefficients a^{ij} of class $C^{(2,1)}$ (i.e. of class C^2 and with second derivatives Lipschitz), all other coefficients of A being uniformly bounded, and we suppose (because of ellipticity) that $\{a^{ij}\}$ is a positive definite matrix everywhere in D . Also, we consider the following regularity conditions on u , where it has locally strong L^2 -derivatives of first and second order. With these hypotheses, N. Aronszajn proved the following strong unique continuation theorem (Theorem 2.2.1).

Theorem 2.2.1. [5] *Assume that u is a solution of*

$$|Au(x)|^2 \leq C \left(\sum_{i=1}^m \left| \frac{\partial u(x)}{\partial x^i} \right|^2 + |u(x)|^2 \right), \quad \forall x \in D, \quad (2.2.1)$$

where C is a positive constant. If u has a zero of infinite order in the 1-mean at some point x_0 in D , then u vanishes identically.

Here, infinite order in the 1-mean is given by an integral condition (see [5]).

Remark 2.2.2.

1. If u vanishes on an open subset of D , then u has a zero of infinite order at any point of that open subset. Thus, the conclusion in Theorem 2.2.1 holds, and so we get the weak unique continuation version.
2. For the two dimension case, i.e. the case of a plane, Theorem 2.2.1 was proved by T. Carleman [18], thus N. Aronszajn generalizes T. Carleman's unique continuation theorem to higher dimensions.
3. By [5], Theorem 2.2.1 allows to establish the uniqueness of elliptic solutions of the Cauchy problem for general linear, quasi-linear, and for certain types of non-linear partial differential equations of second order.
4. The validity of Theorem 2.2.1 for domains of \mathbb{R}^m implies its validity for functions on a Riemannian manifold.

Theorem 2.2.1 can be further extended in the sense that u is vector-valued. Befitting our objectives, we will assume for simplicity's sake that all coefficients are smooth and the statement will be given in the weaker form, of smooth vector-valued functions u vanishing on an open subset of D .

Theorem 2.2.3. [5] *Let A be a linear elliptic second-order differential operator defined on an open subset D of \mathbb{R}^m . Let $u = (u^1, \dots, u^q)$ be an \mathbb{R}^q -valued function on D satisfying the inequality*

$$|Au^a| \leq C \left(\sum_{b,i} \left| \frac{\partial u^b}{\partial x^i} \right| + \sum_b |u^b| \right), \quad (2.2.2)$$

for some constant $C > 0$. If $u = 0$ in an open subset of D , then $u = 0$ on the whole set D .

Later, Aronszajn-Carleman's unique continuation theorems for elliptic equations were used by J. H. Sampson [86] to obtain a unique continuation theorem for harmonic maps between Riemannian manifolds. More precisely, the following result was proved.

Theorem 2.2.4. [86] *Let $\varphi_1, \varphi_2 : M \rightarrow N$ be two harmonic maps. If they agree on an open subset, then they are identical; and indeed the conclusion holds if φ_1 and φ_2 agree to infinitely high order at some point. In particular, a harmonic map which is constant on an open subset is a constant map.*

Remark 2.2.5. J. H. Sampson [86] observed that

1. If a harmonic map φ has rank 0 in an open set, i.e. φ is constant in an open set, it must have rank 0 everywhere.
2. In the case of a harmonic map between analytic Riemannian manifolds, the map is also analytic [36], and it follows that, if it has rank r in an open set, then it has rank r in an open, dense set.

Also, J. H. Sampson established the following geometric unique continuation property for harmonic maps.

Theorem 2.2.6. [86] *Let $\varphi : M \rightarrow N$ be a harmonic map and let P be a regular, closed, totally geodesic submanifold of N . If an open set of M is mapped into P , then all of M is mapped into P .*

But not all solutions of an elliptic partial differential equation possess the unique continuation property. One can construct explicit counterexamples of solutions of fourth-order elliptic partial differential equations where the unique continuation property does not hold [55].

Let P_1 and P_2 be two linear second-order elliptic operators having the same principal part, which is of real coefficients. Let

$$P = P_1 \circ P_2 + Q,$$

where Q is any differential operator of order ≤ 3 .

Theorem 2.2.7. [3] *The operator P has the strong unique continuation property.*

Let P_1, P_2, \dots, P_r the linear second-order elliptic operators having the same principal part which we assume to have real coefficients, and let

$$P = P_1 \circ P_2 \circ P_3 \circ \dots \circ P_r + Q,$$

where Q is any differential operator of order $\leq 3r/2$.

As a generalisation of Theorem [2.2.7] it was proved that we have

Theorem 2.2.8. *The operator P has the strong unique continuation property.*

Remark 2.2.9. Let (M, g) be a Riemannian manifold. One of the interesting applications of the above theorem is the following:

$$P = \Delta^r, \quad r \geq 1.$$

Thus, the Laplace operator has the strong unique continuation property regardless of its power.

Next, in [13], V. Branding and C. Oniciuc proved several unique continuation results for biharmonic maps between Riemannian manifolds.

When dealing with maps between Riemannian manifolds some difficulties appear, compared to the case of functions or vector-valued functions. In general, as we aim for a global result, we define a certain closed subset A and show that the boundary of its interior is the empty set. For this, we assume that the boundary contains at least a point p_0 , work around p_0 on a certain domain D included in the domain of a local chart of p_0 and reach a contradiction. Working on D , we must estimate some terms that contain geometric objects on the target manifold (like curvature terms) which are valued at the map (maps) that is (are) involved, according to the problem. For this, the mean value theorem and some specific tricks are required.

Theorem 2.2.10. [13, 14] *Let $\varphi : M \rightarrow N$ be a biharmonic map. If φ is harmonic on an open subset, then it is harmonic everywhere.*

Since there is a large interest in biharmonic maps into spheres, V. Branding and C. Oniciuc first established a unique continuation result for spherical targets.

Theorem 2.2.11. [13] *Let $\varphi_1, \varphi_2 : M \rightarrow \mathbb{S}^n$ be two biharmonic maps. If they agree on an open subset, then they are identical.*

Afterwards, V. Branding and C. Oniciuc proved a unique continuation theorem for biharmonic maps to an arbitrary target manifold, which is a biharmonic version of J. H. Sampson's unique continuation theorem (see Theorem 2.2.4) to obtain

Theorem 2.2.12. [13] *Let $\varphi_1, \varphi_2 : M \rightarrow N$ be two biharmonic maps. If they agree on an open subset, then they are identical.*

Finally, V. Branding and C. Oniciuc gave a geometric unique continuation property for biharmonic maps generalizing Sampson's result for harmonic maps (Theorem 2.2.6). First, they proved the following version for a spherical target.

Theorem 2.2.13. [13] *Let $\varphi : M \rightarrow \mathbb{S}^n$ be a biharmonic map. If an open subset of M is mapped into the equator \mathbb{S}^{n-1} , then all of M is mapped into \mathbb{S}^{n-1} .*

In addition, V. Branding and C. Oniciuc also showed a corresponding version of the above theorem for an arbitrary target.

Theorem 2.2.14. [13] *Let $\varphi : M \rightarrow N$ be a biharmonic map and let P be a regular, closed, totally geodesic submanifold of N . If an open subset of M is mapped into P , then all of M is mapped into P .*

- Remark 2.2.15.** 1. Theorems 2.2.11 and 2.2.14 were proved in 63 for the particular case of CMC biharmonic immersions into spheres. In the given reference, the main idea was: A CMC biharmonic immersion into \mathbb{S}^n composed with the canonical inclusion of the sphere in the ambient Euclidean space \mathbb{E}^{n+1} gives an immersion that can be written as a sum of two \mathbb{E}^{n+1} -valued eigenmaps of the Laplace operator on (M^m, g) . These maps induce harmonic maps into \mathbb{S}^n of appropriate radius. The results in 63 simply follow from Sampson's unique continuation for harmonic maps 86 and thus both Theorems 2.2.11 and 2.2.14 are more general and proved with different techniques.
2. Theorem 2.2.10 was first proved in 14 by simply applying 92, Proposition 1.2.3], but here, the proof is clearer and based on the classical result of Aronszajn 5.
3. In 4, Theorem 5.3], a unique continuation result for extrinsic biharmonic maps from $\Omega \subset \mathbb{E}^4$ to \mathbb{S}^4 was proved.

All the results presented in Theorems 2.2.10, 2.2.14 are on the unique continuation properties for biharmonic maps. Applying these theorems to biharmonic submanifolds the conclusion is that they must be minimal. However, we know some non-minimal CMC examples. Thus a natural step will be, in Section 2.4, to introduce a unique continuation theorem for proper-biharmonic hypersurfaces in spheres, proving CMC rather than minimality.

2.3 Proper-biharmonic hypersurfaces in spheres

It is known that 13, 14, for a proper-biharmonic map $\varphi : M \rightarrow N$, the subset $\{p \in M : \tau(\varphi)(p) \neq 0\}$ is open and dense in M . Thus,

$$\Omega = \{p \in M : f(p) \neq 0\}$$

is open and dense in M . Note that this subset can have several connected components.

We recall in this Chapter the following result of J.-H. Chen

Lemma 2.3.1. (*J.-H. Chen 32*) *Let $\varphi : M^m \rightarrow \mathbb{S}^{m+1}$ be a proper-biharmonic hypersurface. Then, at points where $\text{grad } f \neq 0$, we have*

$$|A|^2 \geq \frac{m^2(m+8)}{4(m-1)} f^2.$$

In low dimensions, since Conjecture C1 is settled for \mathbb{S}^3 and \mathbb{S}^4 , Lemma 2.3.1 has the following direct consequence.

Proposition 2.3.2. *Let $\varphi : M^m \rightarrow \mathbb{S}^{m+1}$ be a proper-biharmonic hypersurface. Assume that $m = 4$ and $\text{Scal}^M > m(m-1)$, then M has constant mean curvature.*

Proof. Assume that M does not have constant mean curvature, then there exists $p_0 \in M$ such that $(\text{grad } f)(p_0) \neq 0$. Thus, Lemma 2.3.1 allows us to infer that

$$|A(p_0)|^2 \geq \frac{m^2(m+8)}{4(m-1)} f^2(p_0).$$

On the other hand, using Equation (1.3.6), for $c = 1$, we have on M :

$$|A|^2 = m(m-1) + m^2 f^2 - \text{Scal}^M,$$

as $\text{Scal}^M > m(m-1)$ we obtain, at p_0 ,

$$\frac{m^2(m+8)}{4(m-1)} f^2(p_0) \leq |A(p_0)|^2 < m^2 f^2(p_0)$$

which forces m to be at least 5, which is a contradiction. Therefore, $\text{grad } f = 0$ on M . \square

The conjecture states that any proper-biharmonic hypersurface in \mathbb{S}^{m+1} has constant mean curvature. When M is compact, this conjecture was proved in several cases, under additional hypotheses. When M is not compact, and the additional hypotheses are still satisfied, we can only conclude that points where $\text{grad } f \neq 0$, if they exist, cannot form a set with a simple structure. We present here only one result of this type.

Proposition 2.3.3. *Let $\varphi : M^m \rightarrow \mathbb{S}^{m+1}$ be a proper-biharmonic hypersurface. Assume that M does not have constant mean curvature. If $|A|^2 \geq m$, or $|A|^2 \leq m$, then $W = \{p \in M : (\text{grad } f)(p) \neq 0\}$ cannot have a connected component W_0 with the following properties:*

1. $\overline{W_0}^M$ is compact;
2. the boundary of W_0 in M is a regular (not necessarily connected) hypersurface of M ;
3. there exists an open subset U of M such that $\overline{W_0}^M \subset U$ and $\text{grad } f = 0$ on $U \setminus W_0$.

Proof. Assume that W has a connected component W_0 with the above properties and we argue by contradiction.

Since ∂W_0 is a regular hypersurface of M , we have

$$\text{int}(U \setminus W_0) = U \setminus \overline{W_0}^U = U \setminus \overline{W_0}^M = U \setminus (W_0 \cup \partial W_0) \neq \emptyset,$$

otherwise $U = W_0 \cup \partial W_0$ is closed in M , so $U = M$, i.e. $M = W_0 \cup \partial W_0$ is a manifold with boundary; and

$$\overline{\text{int}(U \setminus W_0)}^U = U \setminus W_0.$$

On $\text{int}(U \setminus W_0)$, that may have several connected components, we have $\text{grad } f = 0$, but f cannot be zero on any of these connected components, so since $\Delta f = (m - |A|^2)f$, we get $|A|^2 = m$.

Assume now that $|A|^2 \leq m$. It was proved in [32], see also [77], Inequality (1.74)] that on M we have

$$\frac{1}{2}\Delta(|\text{grad } f|^2 + \frac{m^2}{8}f^4 + f^2) + \frac{1}{2}\text{div}(|A|^2 \text{grad } f^2) \leq \frac{8(m-1)}{m(m+8)}(|A|^2 - m)|A|^2 f^2.$$

Equivalently,

$$-\text{div } Z \leq \frac{8(m-1)}{m(m+8)}(|A|^2 - m)|A|^2 f^2 \leq 0, \quad (2.3.1)$$

where

$$Z = \frac{1}{2}\text{grad}(|\text{grad } f|^2 + \frac{m^2}{8}f^4 + f^2) - \frac{1}{2}|A|^2 \text{grad } f^2.$$

Since $Z = 0$ on $\text{int}(U \setminus W_0)$, it follows that $Z = 0$ on $U \setminus W_0$ and so on ∂W_0 . Integrating Inequality (2.3.1) on $\overline{W_0}^M$ and using the Divergence Theorem, as $Z = 0$ on ∂W_0 , we obtain $(|A|^2 - m)|A|^2 f^2 = 0$ on $\overline{W_0}^M$.

As in [77], we obtain $|A|^2 = m$ on W_0 , and so on $\overline{W_0}^M$. Indeed, assume that there exists a point $p \in W_0$ such that $|A(p)|^2 \neq m$. Therefore, there exists an open subset W'_0 containing p , $W'_0 \subset W_0$, such that $|A|^2 \neq m$ at any point of W'_0 . Now, it is easy to see that $f = 0$ at any point of W'_0 . Thus, $\text{grad } f = 0$ on W'_0 which is a contradiction.

By the first equation of (1.4.5), $|A|^2 = m$, implies $\Delta f = 0$ on $\overline{W_0}^M$.

Furthermore, we integrate Δf^2 on $\overline{W_0}^M$, and since $\text{grad } f^2 = 0$ on ∂W_0 , we obtain $\text{grad } f = 0$ on $\overline{W_0}^M$ which is impossible.

The case $|A|^2 \geq m$ is easy to prove as

$$\frac{1}{2}\Delta f^2 = (m - |A|^2)f^2 - |\text{grad } f|^2 \leq 0$$

on M , and integrating on $\overline{W_0}^M$ we obtain again $\text{grad } f = 0$ on $\overline{W_0}^M$. \square

In the next section (see Corollary 2.4.4) we will see that under a stronger hypothesis, i.e. $|A|^2$ is constant, the points of a non-CMC proper-biharmonic hypersurface where $\text{grad } f \neq 0$ form an open dense subset of M .

Before stating the last result of this section, we need to recall some well-known facts about the smoothness of principal curvatures.

Let $\varphi : M^m \rightarrow \mathbb{S}^{m+1}$ be a hypersurface with $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_m$ its principal curvatures, i.e. the eigenvalue functions of the shape operator A . The functions λ_i are continuous on M for all $i = 1, \dots, m$. The set of points where the numbers of distinct principal curvatures is locally constant is a set M_A that is open and dense in M . On a non-empty connected component of M_A , which is open in M_A , and so in M , the number of distinct principal curvatures is constant. Thus, the multiplicities of distinct principal curvatures are constant, and so, on that connected component, the λ_i 's are smooth and A is (smoothly) locally diagonalizable (see [12, 75, 84, 85]).

Proposition 2.3.4. *Let $\varphi : M^m \rightarrow \mathbb{S}^{m+1}$ be a proper-biharmonic hypersurface. Assume that at any point of M the multiplicity of each distinct principal curvature is at least 2. Then M has constant mean curvature.*

Proof. Assume that M is not CMC and denote

$$W := \{p \in M : (\text{grad } f)(p) \neq 0\}.$$

Clearly, W is a non-empty open subset of M . Since M_A is dense, $W \cap M_A \neq \emptyset$, and so W intersects a connected component of M_A . On that intersection, the principal curvatures λ_i are smooth, $i = 1, \dots, m$, and A is smoothly diagonalizable, i.e. $A(E_i) = \lambda_i E_i$, $i = 1, \dots, m$, where $\{E_i\}_{i=1}^m$ is a local orthonormal frame field on M and tangent to M .

Recall that, by the biharmonic equation (1.4.5), on W , $-(m/2)f$ is an eigenvalue of the shape operator A . From the hypothesis on the multiplicity of eigenvalues, we can assume for simplicity that $\lambda_1 = \lambda_2 = -\frac{m}{2}f$ and $E_1 = \frac{\text{grad } f}{|\text{grad } f|}$. Since $\langle E_a, E_1 \rangle = 0$, we have

$$E_a f = \langle \text{grad } f, E_a \rangle = |\text{grad } f| \langle E_1, E_a \rangle = 0, \quad a = 2, \dots, m. \quad (2.3.2)$$

Now, we use the connection equations with respect to the frame field $\{E_i\}_{i=1}^m$,

$$\nabla_{E_i} E_j = \omega_j^k(E_i) E_k,$$

and we rewrite the Codazzi equation

$$(\nabla_{E_i} A)(E_j) = (\nabla_{E_j} A)(E_i)$$

as

$$(E_i \lambda_j) E_j + \sum_{k=1}^m (\lambda_j - \lambda_k) \omega_j^k(E_i) E_k = (E_j \lambda_i) E_i + \sum_{k=1}^m (\lambda_i - \lambda_k) \omega_i^k(E_j) E_k. \quad (2.3.3)$$

For $i = 1$ and $j = 2$ we obtain

$$\begin{aligned} (E_1 \lambda_2) E_2 + \sum_{k=1}^m (\lambda_2 - \lambda_k) \omega_2^k(E_1) E_k &= (E_2 \lambda_1) E_1 + \sum_{k=1}^m (\lambda_1 - \lambda_k) \omega_1^k(E_2) E_k \\ &= \sum_{k=1}^m (\lambda_1 - \lambda_k) \omega_1^k(E_2) E_k. \end{aligned} \quad (2.3.4)$$

Furthermore, we take the scalar product of the above relation with E_2 , and obtain

$$E_1\lambda_2 = E_1\lambda_1 = 0,$$

i.e. $E_1f = 0$. Thus, from Equation (2.3.2) we conclude that $\text{grad } f = 0$ which is impossible. \square

2.4 A unique continuation theorem

Analytical properties of biharmonic maps have been studied by several researchers (see for example [95]). For the particular case of biharmonic submanifolds in Euclidean spheres, not too many of such properties are known.

An essential tool in the analysis of PDE's are unique continuation properties, which we establish in Theorem 2.4.1 under a global condition on the gradients of the norm of the shape operator and mean curvature.

The objective departs from [13] as the conclusion here is that the manifold has constant mean curvature, instead of the stronger condition of minimality, but the method is similar and based on Aronszajn's unique continuation theorem of 1957 [5].

In Corollaries 2.4.4 and 2.4.5, the main hypothesis of Theorem 2.4.1 is replaced by more geometrical constraints and allows known results to extend from compact to non-compact cases.

Theorem 2.4.1. *Let $\varphi : M^m \rightarrow \mathbb{S}^{m+1}$ be a proper-biharmonic hypersurface. Assume that there exists a non-negative function h on M such that $|\text{grad } |A|^2| \leq h |\text{grad } f|$ on M . If $\text{grad } f$ vanishes on a non-empty open connected subset of M , then M has constant mean curvature.*

Proof. Denote by V the non-empty open connected subset of M where $\text{grad } f = 0$. Consider the subset

$$A_0 := \{p \in M : (\text{grad } f)(p) = 0\}.$$

It is clear that A_0 is closed, $\text{int } A_0 \neq \emptyset$ and $\text{int } A_0$ may have several connected components. Indeed, $A_0 = (\text{grad } f)^{-1}(\{0\})$ where $\{0\}$ is closed in TM as the zero section, and $\text{grad } f : M \rightarrow TM$ is continuous, thus A_0 is closed, and since V is a non-empty open subset of A_0 , $\text{int } A_0$ is also non-empty.

Assume that $\partial(\text{int } A_0) = \emptyset$. Then

$$\begin{aligned} \emptyset &= \partial(\text{int } A_0) \\ &= \overline{(\text{int } A_0)}^M \cap \overline{(M \setminus \text{int } A_0)}^M \\ &= \overline{(\text{int } A_0)}^M \cap (M \setminus \text{int } A_0). \end{aligned}$$

Now, as $\emptyset = \overline{(\text{int } A_0)^M} \cap (M \setminus \text{int } A_0)$, we obtain $\overline{(\text{int } A_0)^M} \subset \text{int } A_0$ which implies that $\overline{(\text{int } A_0)^M} = \text{int } A_0$, thus $\text{int } A_0$ is closed in M . But $\text{int } A_0$ is non-empty open and M is connected, we conclude that $\text{int } A_0 = M$, so $\text{int } A_0 = A_0 = M$ and $\text{grad } f = 0$ on M .

Assume now that $\partial(\text{int } A_0) \neq \emptyset$, we will obtain a contradiction. Let $p_0 \in \partial(\text{int } A_0)$, $\partial(\text{int } A_0) = \overline{(\text{int } A_0)^M} \setminus \text{int } A_0$, necessarily $p_0 \notin \text{int } A_0$. Let U be an open subset containing p_0 , then $U \cap \text{int } A_0 \neq \emptyset$.

On the other hand, we have

$$p_0 \in \partial(\text{int } A_0) \subset \partial A_0,$$

so

$$p_0 \in \partial A_0 = \partial(M \setminus A_0).$$

Since A_0 is closed in M , then $M \setminus A_0$ is non-empty open in M , and so $p_0 \notin M \setminus A_0$. Of course, $p_0 \in \overline{(M \setminus A_0)^M}$ implies that $U \cap (M \setminus A_0) \neq \emptyset$.

In conclusion:

1. $U \cap \text{int } A_0$ is a non-empty open subset of $\text{int } A_0$ that does not contain p_0 , so there exists a non-empty open subset on which $\text{grad } f = 0$.
2. $U \cap (M \setminus A_0)$ is a non-empty open subset that does not contain p_0 , and is included in $M \setminus A_0$, so there exists a non-empty open subset on which $\text{grad } f \neq 0$ at any point.

Let $(U, x^i)_{i=1, \dots, m}$ be a local chart on M around $p_0 \in \partial(\text{int } A_0)$. Consider an open connected subset D in M containing p_0 , such that $\overline{D^M}$ is compact and $\overline{D^M} \subset U$. Note that D also contains a non-empty open subset where $\text{grad } f = 0$ everywhere, and a non-empty open subset where $\text{grad } f \neq 0$ at any point.

As usual, we identify $\text{grad } f \in C(TM)$ with $d\varphi(\text{grad } f) \in C(\varphi^{-1}T\mathbb{S}^{m+1})$, or

$$d(i \circ \varphi)(\text{grad } f) \in C((i \circ \varphi)^{-1}T\mathbb{E}^{m+2}),$$

where $i : \mathbb{S}^{m+1} \rightarrow \mathbb{E}^{m+2}$ is the canonical inclusion. Let us write $\text{grad } f = u^\alpha e_\alpha$, where $u^\alpha \in C^\infty(M)$, $\forall \alpha = 1, \dots, m+2$, and $\{e_\alpha\}_{\alpha=1}^{m+2}$ is the canonical basis in \mathbb{E}^{m+2} . For all $\alpha = 1, \dots, m+2$, the function u^α vanishes on V .

As φ is biharmonic, we have

$$\Delta f = (m - |A|^2)f,$$

and taking its differential we obtain

$$d\Delta f = (m - |A|^2)df - f d(|A|^2), \tag{2.4.1}$$

hence, by the musical isomorphism:

$$(d\Delta f)^\sharp = [(m - |A|^2)df - f d(|A|^2)]^\sharp.$$

Since $(df)^\sharp = \text{grad } f$ and $d\Delta^{\text{Hodge}} = \Delta^{\text{Hodge}}d$, we can rewrite Equation (2.4.1) as

$$\left(\Delta^{\text{Hodge}}(df)\right)^\sharp = (m - |A|^2) \text{grad } f - f \text{grad } |A|^2.$$

On the other hand, by the Weitzenböck formula (see Theorem 1.2.5), we have

$$\left(\Delta^{\text{Hodge}}(df)\right)^\sharp = -\text{trace } \nabla^2 \text{grad } f + \text{Ricci}^M(\text{grad } f),$$

thus

$$-\text{trace } \nabla^2 \text{grad } f = -\text{Ricci}^M(\text{grad } f) + (m - |A|^2) \text{grad } f - f \text{grad } |A|^2. \quad (2.4.2)$$

As $\text{Ricci}^M(\text{grad } f) = \text{Ricci}^M(u^\alpha e_\alpha)$ and

$$\text{grad } f = u^\alpha e_\alpha = u^\alpha (e_\alpha^\perp + e_\alpha^\top) = u^\alpha e_\alpha^\top,$$

where e_α^\perp and e_α^\top are the normal and the tangential components (to M) of e_α in \mathbb{E}^{m+2} .

Note that $|e_\alpha^\top|$ may vanish at some points. Since $\text{grad } f$ is tangent to M , we have $u^\alpha e_\alpha^\perp = 0$, i.e.

$$u^\alpha (e_\alpha - e_\alpha^\top) = 0 \quad (2.4.3)$$

and taking the covariant derivative of (2.4.3) we get

$$\begin{aligned} 0 &= \nabla_{\frac{\partial}{\partial x^i}}^{\mathbb{E}^{m+2}} u^\alpha (e_\alpha - e_\alpha^\top) \\ &= \frac{\partial u^\alpha}{\partial x^i} e_\alpha^\perp - u^\alpha \nabla_{\frac{\partial}{\partial x^i}}^{\mathbb{E}^{m+2}} e_\alpha^\top \\ &= \frac{\partial u^\alpha}{\partial x^i} e_\alpha^\perp - u^\alpha \nabla_{\frac{\partial}{\partial x^i}}^{S^{m+1}} e_\alpha^\top + u^\alpha \left\langle \frac{\partial}{\partial x^i}, e_\alpha^\top \right\rangle r \\ &= \frac{\partial u^\alpha}{\partial x^i} e_\alpha^\perp - u^\alpha \nabla_{\frac{\partial}{\partial x^i}}^M e_\alpha^\top - u^\alpha B \left(\frac{\partial}{\partial x^i}, e_\alpha^\top \right) + u^\alpha \left\langle \frac{\partial}{\partial x^i}, e_\alpha^\top \right\rangle r. \end{aligned}$$

It follows that

$$u^\alpha \nabla_{\frac{\partial}{\partial x^i}}^M e_\alpha^\top = 0$$

and

$$\frac{\partial u^\alpha}{\partial x^i} e_\alpha^\perp - u^\alpha B \left(\frac{\partial}{\partial x^i}, e_\alpha^\top \right) + u^\alpha \left\langle \frac{\partial}{\partial x^i}, e_\alpha^\top \right\rangle r = 0,$$

where r is the position vector field on \mathbb{E}^{m+2} . Equivalently, the last relation can be rewritten as

$$\frac{\partial u^\alpha}{\partial x^i} e_\alpha = \frac{\partial u^\alpha}{\partial x^i} e_\alpha^\top + u^\alpha B \left(\frac{\partial}{\partial x^i}, e_\alpha^\top \right) - u^\alpha \left\langle \frac{\partial}{\partial x^i}, e_\alpha^\top \right\rangle r = 0.$$

We have

$$\text{Ricci}^M(\text{grad } f) = \text{Ricci}^M(u^\alpha e_\alpha^\top) = u^\alpha \text{Ricci}^M(e_\alpha^\top). \quad (2.4.4)$$

On U , we combine the second fundamental forms of M in \mathbb{S}^{m+1} and of \mathbb{S}^{m+1} in \mathbb{E}^{m+2} to compute

$$\begin{aligned} \nabla_{\frac{\partial}{\partial x^i}}^M \text{grad } f &= \nabla_{\frac{\partial}{\partial x^i}}^{\mathbb{S}^{m+1}} \text{grad } f - B\left(\frac{\partial}{\partial x^i}, \text{grad } f\right) \\ &= \nabla_{\frac{\partial}{\partial x^i}}^{\mathbb{E}^{m+2}} \text{grad } f + \left\langle \frac{\partial}{\partial x^i}, u^\alpha e_\alpha^\top \right\rangle r - B\left(\frac{\partial}{\partial x^i}, \text{grad } f\right) \\ &= \nabla_{\frac{\partial}{\partial x^i}}^{\mathbb{E}^{m+2}} (u^\alpha e_\alpha) + \left\langle \frac{\partial}{\partial x^i}, u^\alpha e_\alpha^\top \right\rangle r - B\left(\frac{\partial}{\partial x^i}, u^\alpha e_\alpha^\top\right) \\ &= \frac{\partial u^\alpha}{\partial x^i} e_\alpha + u^\alpha \nabla_{\frac{\partial}{\partial x^i}}^{\mathbb{E}^{m+2}} e_\alpha + u^\alpha \left\langle \frac{\partial}{\partial x^i}, e_\alpha^\top \right\rangle r - u^\alpha B\left(\frac{\partial}{\partial x^i}, e_\alpha^\top\right) \\ &= \frac{\partial u^\alpha}{\partial x^i} e_\alpha + u^\alpha \left\langle \frac{\partial}{\partial x^i}, e_\alpha^\top \right\rangle r - u^\alpha B\left(\frac{\partial}{\partial x^i}, e_\alpha^\top\right). \end{aligned}$$

On the other hand,

$$\begin{aligned} \nabla_{\frac{\partial}{\partial x^i}}^M \text{grad } f &= \frac{\partial u^\alpha}{\partial x^i} e_\alpha^\top + u^\alpha \nabla_{\frac{\partial}{\partial x^i}}^M e_\alpha^\top \\ &= \frac{\partial u^\alpha}{\partial x^i} e_\alpha^\top. \end{aligned}$$

Put

$$Y_i = \nabla_{\frac{\partial}{\partial x^i}}^M \text{grad } f,$$

thus, for Y_i we have two equivalent expressions

$$\begin{aligned} Y_i &= \frac{\partial u^\alpha}{\partial x^i} e_\alpha + u^\alpha \left\langle \frac{\partial}{\partial x^i}, e_\alpha^\top \right\rangle r - u^\alpha B\left(\frac{\partial}{\partial x^i}, e_\alpha^\top\right) \\ &= \frac{\partial u^\alpha}{\partial x^i} e_\alpha^\top. \end{aligned}$$

For simplicity of notation, it is convenient to write Y_i as

$$Y_i = \frac{\partial u^\alpha}{\partial x^i} e_\alpha + u^\alpha Z_{\alpha,i},$$

where

$$Z_{\alpha,i} = \left\langle \frac{\partial}{\partial x^i}, e_\alpha^\top \right\rangle r - B\left(\frac{\partial}{\partial x^i}, e_\alpha^\top\right)$$

is a vector field normal to M in \mathbb{E}^{m+2} .

We repeat this process to obtain, on U , the second derivatives of $\text{grad } f$,

$$\begin{aligned}
\nabla_{\frac{\partial}{\partial x^i}}^M \nabla_{\frac{\partial}{\partial x^j}}^M \text{grad } f &= \nabla_{\frac{\partial}{\partial x^i}}^M Y_j \\
&= \nabla_{\frac{\partial}{\partial x^i}}^{\mathbb{E}^{m+2}} Y_j + \left\langle \frac{\partial}{\partial x^i}, Y_j \right\rangle r - B \left(\frac{\partial}{\partial x^i}, Y_j \right) \\
&= \nabla_{\frac{\partial}{\partial x^i}}^{\mathbb{E}^{m+2}} \left\{ \frac{\partial u^\alpha}{\partial x^j} e_\alpha + u^\alpha Z_{\alpha,j} \right\} + \left\langle \frac{\partial}{\partial x^i}, Y_j \right\rangle r - B \left(\frac{\partial}{\partial x^i}, Y_j \right) \\
&= \frac{\partial^2 u^\alpha}{\partial x^i \partial x^j} e_\alpha + \frac{\partial u^\alpha}{\partial x^i} Z_{\alpha,j} + u^\alpha \nabla_{\frac{\partial}{\partial x^i}}^{\mathbb{E}^{m+2}} Z_{\alpha,j} + \left\langle \frac{\partial}{\partial x^i}, Y_j \right\rangle r \\
&\quad - B \left(\frac{\partial}{\partial x^i}, Y_j \right). \tag{2.4.5}
\end{aligned}$$

To compute $\nabla_{\nabla_{\frac{\partial}{\partial x^i}}^M \frac{\partial}{\partial x^j}}^M \text{grad } f$, on U , we have

$$\begin{aligned}
\nabla_{\nabla_{\frac{\partial}{\partial x^i}}^M \frac{\partial}{\partial x^j}}^M \text{grad } f &= \nabla_{\nabla_{\frac{\partial}{\partial x^i}}^M \frac{\partial}{\partial x^j}}^{\mathbb{E}^{m+2}} \text{grad } f + \left\langle \nabla_{\frac{\partial}{\partial x^i}}^M \frac{\partial}{\partial x^j}, \text{grad } f \right\rangle r - B \left(\nabla_{\frac{\partial}{\partial x^i}}^M \frac{\partial}{\partial x^j}, \text{grad } f \right) \\
&= \left[\left(\nabla_{\frac{\partial}{\partial x^i}}^M \frac{\partial}{\partial x^j} \right) u^\alpha \right] e_\alpha + u^\alpha \left\langle \nabla_{\frac{\partial}{\partial x^i}}^M \frac{\partial}{\partial x^j}, e_\alpha^\top \right\rangle r - u^\alpha B \left(\nabla_{\frac{\partial}{\partial x^i}}^M \frac{\partial}{\partial x^j}, e_\alpha^\top \right). \tag{2.4.6}
\end{aligned}$$

Replacing (2.4.5) and (2.4.6) in (2.4.2), and using (2.4.4), we obtain

$$\begin{aligned}
(\Delta u^\alpha) e_\alpha - g^{ij} \frac{\partial u^\alpha}{\partial x^i} Z_{\alpha,j} - g^{ij} \left\langle \frac{\partial}{\partial x^i}, Y_j \right\rangle r + g^{ij} B \left(\frac{\partial}{\partial x^i}, Y_j \right) - g^{ij} u^\alpha \nabla_{\frac{\partial}{\partial x^i}}^{\mathbb{E}^{m+2}} Z_{\alpha,j} \\
+ g^{ij} u^\alpha \left\langle \nabla_{\frac{\partial}{\partial x^i}}^M \frac{\partial}{\partial x^j}, e_\alpha^\top \right\rangle r - g^{ij} u^\alpha B \left(\nabla_{\frac{\partial}{\partial x^i}}^M \frac{\partial}{\partial x^j}, e_\alpha^\top \right) \\
= -u^\alpha \text{Ricci}^M(e_\alpha^\top) + (m - |A|^2) u^\alpha e_\alpha^\top - f \text{grad } |A|^2,
\end{aligned}$$

so

$$\begin{aligned}
(\Delta u^\alpha) e_\alpha &= g^{ij} \frac{\partial u^\alpha}{\partial x^i} Z_{\alpha,j} + g^{ij} \left\langle \frac{\partial}{\partial x^i}, Y_j \right\rangle r - g^{ij} B \left(\frac{\partial}{\partial x^i}, Y_j \right) \\
&\quad + g^{ij} u^\alpha \nabla_{\frac{\partial}{\partial x^i}}^{\mathbb{E}^{m+2}} Z_{\alpha,j} - g^{ij} u^\alpha \left\langle \nabla_{\frac{\partial}{\partial x^i}}^M \frac{\partial}{\partial x^j}, e_\alpha^\top \right\rangle r \\
&\quad + g^{ij} u^\alpha B \left(\nabla_{\frac{\partial}{\partial x^i}}^M \frac{\partial}{\partial x^j}, e_\alpha^\top \right) - u^\alpha \text{Ricci}^M(e_\alpha^\top) + (m - |A|^2) u^\alpha e_\alpha^\top \\
&\quad - f \text{grad } |A|^2. \tag{2.4.7}
\end{aligned}$$

But

$$g^{ij} \left\langle \frac{\partial}{\partial x^i}, Y_j \right\rangle r = g^{ij} \frac{\partial u^\alpha}{\partial x^j} \left\langle \frac{\partial}{\partial x^i}, e_\alpha^\top \right\rangle r, \tag{2.4.8}$$

and

$$g^{ij} B \left(\frac{\partial}{\partial x^i}, Y_j \right) = g^{ij} \frac{\partial u^\alpha}{\partial x^j} B \left(\frac{\partial}{\partial x^i}, e_\alpha^\top \right), \tag{2.4.9}$$

so replacing (2.4.8) and (2.4.9) in (2.4.7) we obtain

$$\begin{aligned} (\Delta u^\alpha) e_\alpha &= g^{ij} \frac{\partial u^\alpha}{\partial x^i} Z_{\alpha,j} + g^{ij} \frac{\partial u^\alpha}{\partial x^j} \left\langle \frac{\partial}{\partial x^i}, e_\alpha^\top \right\rangle r - g^{ij} \frac{\partial u^\alpha}{\partial x^j} B \left(\frac{\partial}{\partial x^i}, e_\alpha^\top \right) \\ &\quad + g^{ij} u^\alpha \nabla_{\frac{\partial}{\partial x^i}}^{\mathbb{E}^{m+2}} Z_{\alpha,j} - g^{ij} u^\alpha \left\langle \nabla_{\frac{\partial}{\partial x^i}}^M \frac{\partial}{\partial x^j}, e_\alpha^\top \right\rangle r + g^{ij} u^\alpha B \left(\nabla_{\frac{\partial}{\partial x^i}}^M \frac{\partial}{\partial x^j}, e_\alpha^\top \right) \\ &\quad - u^\alpha \text{Ricci}^M(e_\alpha^\top) + (m - |A|^2) u^\alpha e_\alpha^\top - f \text{grad } |A|^2. \end{aligned} \quad (2.4.10)$$

Thus each term on the right-hand side of Equation (2.4.10), except for the very last one, contains either $\frac{\partial u^\alpha}{\partial x^i}$ or u^α .

By the well-known inequalities

$$|\Delta u^{\alpha_0}| \leq |(\Delta u^\alpha) e_\alpha|,$$

and since all functions and vector fields are smooth on U , they are bounded on \overline{D}^M , and so, on D . Using the hypothesis and the triangle inequality, we obtain

$$|\Delta u^{\alpha_0}| \leq C \left(\sum_{\alpha,i} \left| \frac{\partial u^\alpha}{\partial x^i} \right| + \sum_{\alpha} |u^\alpha| \right)$$

on D . Since u^α is zero on a non-empty open subset of D , by Aronszajn's unique continuation principle we deduce that u^α is equal to zero on D , and thus $\text{grad } f$ vanishes on D . This is impossible, hence the assumption $\partial(\text{int } A_0) \neq \emptyset$ is false. In conclusion, $\partial(\text{int } A_0) = \emptyset$, and so $\text{grad } f$ vanishes on the whole of M . \square

Remark 2.4.2. Since $\nabla_X^M \text{grad } f = (Xu^\alpha) e_\alpha^\top$, for any X tangent to M , from (2.4.2) we could directly obtain $(\Delta u^\alpha) e_\alpha^\top$, but this would not be enough to estimate Δu^α .

Theorem 2.4.1 can be rephrased as follows:

Corollary 2.4.3. *Let $\varphi : M^m \rightarrow \mathbb{S}^{m+1}$ be a proper-biharmonic hypersurface. Assume that there exists a non-negative function h on M such that $|\text{grad } |A|^2| \leq h |\text{grad } f|$ on M . Then, either M has constant mean curvature or the set of points where $\text{grad } f \neq 0$ is an open dense subset of M .*

Proof. Assume that M is not CMC. Let

$$W := \{p \in M : (\text{grad } f)(p) \neq 0\},$$

be a non-empty open subset in M . Assume that $\overline{W}^M \subsetneq M$, then $V = M \setminus \overline{W}^M$ is a non-empty, open subset of M and $\text{grad } f|_V = 0$, therefore f is constant on a connected component V_1 of V . As f is constant on V_1 and $|\text{grad } |A|^2| \leq h |\text{grad } f|$ over M , by Theorem 2.4.1 we deduce that f is constant on M , which is a contradiction, therefore

$$\overline{W}^M = M.$$

\square

The hypothesis on the existence of the function h in Theorem [2.4.1](#) can be obtained under natural conditions on $|A|^2$ or the scalar curvature of M .

Corollary 2.4.4. *Let $\varphi : M^m \rightarrow \mathbb{S}^{m+1}$ be a proper-biharmonic hypersurface with $|A|^2$ constant. Then, either M has constant mean curvature, or the set of points where $\text{grad } f \neq 0$ is an open dense subset of M .*

Proof. As $|A|^2$ is constant, the condition $|\text{grad } |A|^2| \leq h |\text{grad } f|$ on M is automatically satisfied, thus, by Corollary [2.4.3](#) we conclude. \square

Corollary 2.4.5. *Let $\varphi : M^m \rightarrow \mathbb{S}^{m+1}$ be a proper-biharmonic hypersurface with constant scalar curvature. Then, either M has constant mean curvature, or the set of points where $\text{grad } f \neq 0$ is an open dense subset of M .*

Proof. Using Equation [\(1.3.4\)](#) we have

$$|A|^2 = m(m-1) + m^2 f^2 - \text{Scal}^M,$$

which implies

$$\begin{aligned} \text{grad } |A|^2 &= m^2 \text{grad } f^2 \\ &= 2m^2 f \text{grad } f, \end{aligned}$$

thus

$$|\text{grad } |A|^2| = 2m^2 |f| |\text{grad } f|.$$

Therefore, the condition

$$|\text{grad } |A|^2| \leq h |\text{grad } f|$$

holds on M and we apply Corollary [2.4.3](#) to conclude. \square

Remark 2.4.6.

1. Corollaries [2.4.4](#) and [2.4.5](#) are meaningful because M is not assumed to be compact:
 - (a) A direct consequence of J.-H. Chen's result (see lemma [2.3.1](#)) is that if M is compact and $|A|^2$ is constant, then $\text{grad } f$ vanishes on the whole manifold M (see [\[77\]](#)).
 - (b) If M is compact and its scalar curvature is constant, Maeta and Ou show in [\[67\]](#) that f is constant.

Therefore, Corollaries [2.4.4](#) and [2.4.5](#) can be seen as extensions of results in [\[67\]](#) and [\[77\]](#), where it is shown that if f is constant on a non-empty open subset of M then f is constant on M .

2. Theorem [2.4.1](#) is meaningful even in the compact case.
3. Consider $\varphi : M^m \rightarrow N^{m+1}(c)$, ($c \leq 0$) a proper-biharmonic hypersurface. Assume that $\text{grad } f$ vanishes on an open subset. Then, it follows that f is constant on an open (connected) subset. But, as $c \leq 0$, the constant has to be zero (see [79](#) for a more general statement), so φ is harmonic on an open subset, therefore on the whole manifold M .

As a direct application of Corollary [2.4.5](#) we can give the following result.

Proposition 2.4.7. *Let $\varphi : M^m \rightarrow \mathbb{S}^{m+1}$ be a proper-biharmonic hypersurface with constant scalar curvature. Assume that there exists a connected component of M_A where the number of distinct principal curvatures is at most six. Then M has constant mean curvature.*

Proof. Let U be a connected component of M_A . The number of distinct principal curvatures is constant and at most 6.

As Scal^M is constant, by Theorem 1.1 of [44](#) we obtain that f is constant on U . On the other hand, by Corollary [2.4.5](#), we deduce that f is constant on M . \square

Corollary 2.4.8. *Let $\varphi : M^m \rightarrow \mathbb{S}^{m+1}$ be a proper-biharmonic hypersurface. Assume that there exists a non-negative function h such that $|\text{grad } |A|^2| \leq h |\text{grad } f|$, and M is not CMC. Denote by U a connected component of M_A . Then, on U , we have:*

1. $-\frac{m}{2}f$ is a principal curvature with multiplicity equal to 1;
2. $\frac{\text{grad } f}{|\text{grad } f|}$ is a vector field defined on an open dense subset of U and its integral curves are geodesics;
3. the number of distinct principal curvatures is at least 3 and $|A|^2 > \frac{m^2(m+8)}{4(m-1)}f^2$ on an open dense subset of U (see [10](#)).

Proof. Since M does not have constant mean curvature, by Corollary [2.4.3](#) we deduce that the points of U where $\text{grad } f \neq 0$ form an open dense subset of U . Now, by continuity, we obtain $-\frac{m}{2}f = \lambda_{i_0}$, for some i_0 , on U , and by Proposition [2.3.4](#) we obtain that the multiplicity of λ_{i_0} is 1.

Furthermore, for simplicity, we consider $i_0 = 1$, and work on an open connected subset of U where $\text{grad } f \neq 0$ at any point.

Let $E_1 = \frac{\text{grad } f}{|\text{grad } f|}$. Consider Equation [\(2.3.3\)](#) for $i = 1$ and $j = a$, $a = 2, \dots, m$, and taking the inner product with E_1 we get

$$\langle (E_1 \lambda_a) E_a + \sum_{k=1}^m (\lambda_a - \lambda_k) \omega_a^k(E_1) E_k, E_1 \rangle = \langle (E_a \lambda_1) E_1 + \sum_{k=1}^m (\lambda_1 - \lambda_k) \omega_1^k(E_a) E_k, E_1 \rangle.$$

Therefore,

$$(\lambda_a - \lambda_1)\omega_a^1(E_1) = 0.$$

As λ_1 has multiplicity equal to 1, i.e. $\lambda_a \neq \lambda_1$, we get

$$\omega_1^a(E_1) = 0,$$

and thus $\nabla_{E_1} E_1 = 0$, so the integral curves of E_1 are geodesics.

If the number of distinct principal curvatures is at most 2 then U is CMC (see [10]). As J.-H. Chen's Inequality is based on the Cauchy-Schwarz Inequality applied to the principal curvatures, we have a strict inequality. Indeed, since the number of distinct principal curvatures on U must be at least 3, and since λ_1 can only have multiplicity equal to 1, there exists $a \neq b$ such that $\lambda_a \neq \lambda_b$, $a, b = 2, \dots, m$. Therefore,

$$|A|^2 = \lambda_1^2 + \sum_{k=2}^m \lambda_k^2 > \frac{m^2}{4} f^2 + \frac{1}{(m-1)} \left(\sum_{k=2}^m \lambda_k \right)^2.$$

Then, the proof follows from Lemma [1.4.27]. \square

Remark 2.4.9. We note that the distribution orthogonal to that determined by $\frac{\text{grad } f}{|\text{grad } f|}$ is completely integrable. The level hypersurfaces of the mean curvature f have flat normal connection as submanifolds in S^{m+1} of codimension 2 (see [73], Theorem 1.40).

Corollary [2.4.3] allows the re-writing of some known results replacing their global hypothesis with local variants.

Corollary 2.4.10. *Let $\varphi : M^m \rightarrow \mathbb{S}^{m+1}$ be a proper-biharmonic hypersurface. Assume that $|\text{grad } |A|^2| \leq h |\text{grad } f|$ on M , where h is a non-negative function on M . If M is not CMC, then J.-H. Chen's Inequality*

$$|A|^2 \geq \frac{m^2(m+8)}{4(m-1)} f^2 \quad (2.4.11)$$

is valid everywhere on M .

Proof. Inequality [2.4.11] holds on W and, as W is dense, we conclude by continuity. \square

J.-H. Chen's Inequality enables us to obtain a more geometric version of Theorem [2.4.1].

Theorem 2.4.11. *Let $\varphi : M^m \rightarrow \mathbb{S}^{m+1}$ be a proper-biharmonic hypersurface. Assume that $f^2 > \frac{4(m-1)}{m(m+8)}$. If $\text{grad } f$ vanishes on a non-empty open connected subset of M , then M has constant mean curvature. Moreover, if $m \in \{2, 3, 4\}$, then $\varphi(M)$ is an open subset of the small hypersphere $\mathbb{S}^m \left(\frac{1}{\sqrt{2}} \right)$.*

Proof. Let us denote

$$A_0 := \{p \in M : (\text{grad } f)(p) = 0\}.$$

In the proof of Theorem 2.4.1 we have shown that A_0 is a closed subset of M , $\text{int } A_0 \neq \emptyset$, and if $\partial(\text{int } A_0) = \emptyset$ then $\text{grad } f$ vanishes on M .

As in the proof of Theorem 2.4.1, assume that $\partial(\text{int } A_0) \neq \emptyset$, to reach a contradiction. Let $p_0 \in \partial(\text{int } A_0)$, it follows that there exists a sequence of points $\{p_n^1\}_{n \in \mathbb{N}^*}$ converging to p_0 , $p_n^1 \neq p_0$ and $p_n^1 \in \text{int } A_0$ for any $n \in \mathbb{N}^*$, and there exists a sequence of points $\{p_n^2\}_{n \in \mathbb{N}^*}$, that converges to p_0 , $p_n^2 \neq p_0$ and $(\text{grad } f)(p_n^2) \neq 0$ for any $n \in \mathbb{N}^*$.

From Lemma 2.3.1 we have

$$|A|^2(p_n^2) \geq \frac{m^2(m+8)}{4(m-1)} f^2(p_n^2), \quad \forall n \in \mathbb{N}^*. \quad (2.4.12)$$

Now, each connected component of $\text{int } A_0$ is open in $\text{int } A_0$ and so in M . Thus, on each connected component of $\text{int } A_0$ the function f is constant. But the constant cannot be zero as φ is not harmonic and so $|A|^2 = m$. In conclusion, we have $|A|^2 = m$ on $\text{int } A_0$ and

$$|A|^2(p_n^1) = m, \quad \forall n \in \mathbb{N}^*. \quad (2.4.13)$$

Passing to the limit in 2.4.12 and 2.4.13 we obtain

$$m = |A|^2(p_0) \geq \frac{m^2(m+8)}{4(m-1)} f^2(p_0),$$

thus

$$f^2 \leq \frac{4(m-1)}{m(m+8)}$$

which is impossible.

The last part of the theorem follows directly from 11. □

Remark 2.4.12. Compare the above result with [77, Proposition 1.38 and Corollary 1.40], where the inequality involving f^2 is not strict, and compactness or completeness assumptions are needed.

2.5 Rigidity results for biharmonic hypersurfaces

The unique continuation properties of Section 2 can be exploited to obtain new rigidity results. Theorem 2.5.1 relies essentially on the combination of the Bochner formula applied to the vector field $\text{grad } f$ and J.-H. Chen's Inequality, made possible thanks to Corollary 2.4.10. Theorem 2.5.2 is a more technical alternative which puts together a bound on the Ricci curvature and an averaged version (i.e. an integral version) of the condition which appears as a strict inequality in Theorem 2.4.11.

Theorem 2.5.1. *Let $\varphi : M^m \rightarrow \mathbb{S}^{m+1}$ be a compact proper-biharmonic hypersurface. Assume that $|\operatorname{grad} |A|^2| \leq h |\operatorname{grad} f|$ on M , where h is a non-negative function on M , $\operatorname{Scal}^M \geq 0$ and*

$$\int_M [m(m+8)f^2 - 4(m-1)] \operatorname{Scal}^M f^2 dv_g \geq 0. \quad (2.5.1)$$

Then M has constant mean curvature.

Proof. Assume that M does not have constant mean curvature, we will use Corollary 2.4.10 and we will argue by contradiction.

Starting with the Bochner Formula (see for example [83]), we have

$$-\frac{1}{2}\Delta |\operatorname{grad} f|^2 = |\nabla df|^2 - \langle \operatorname{grad} \Delta f, \operatorname{grad} f \rangle + \operatorname{Ricci}^M(\operatorname{grad} f, \operatorname{grad} f).$$

Now, using Equation (1.3.5), with $c = 1$, we get

$$\begin{aligned} \operatorname{Ricci}^M(\operatorname{grad} f, \operatorname{grad} f) &= (m-1)|\operatorname{grad} f|^2 - |A(\operatorname{grad} f)|^2 \\ &\quad + mf \langle A(\operatorname{grad} f), \operatorname{grad} f \rangle. \end{aligned} \quad (2.5.2)$$

Now, since M is a biharmonic submanifold of \mathbb{S}^{m+1} ,

$$A(\operatorname{grad} f) = -\frac{m}{2}f \operatorname{grad} f,$$

and

$$|A(\operatorname{grad} f)|^2 = \frac{m^2}{4}f^2 |\operatorname{grad} f|^2.$$

Furthermore,

$$-mf \langle A(\operatorname{grad} f), \operatorname{grad} f \rangle = \frac{m^2}{2}f^2 |\operatorname{grad} f|^2,$$

and

$$\begin{aligned} R^{\mathbb{S}^{m+1}}(\operatorname{grad} f, \eta, \operatorname{grad} f, \eta) &= \langle \operatorname{grad} f, \operatorname{grad} f \rangle \langle \eta, \eta \rangle - \langle \operatorname{grad} f, \eta \rangle \langle \operatorname{grad} f, \eta \rangle \\ &= |\operatorname{grad} f|^2. \end{aligned}$$

Then, replacing in Equation (2.5.2), we obtain

$$m|\operatorname{grad} f|^2 = \operatorname{Ricci}^M(\operatorname{grad} f, \operatorname{grad} f) + \frac{3}{4}m^2f^2 |\operatorname{grad} f|^2 + |\operatorname{grad} f|^2,$$

thus

$$\operatorname{Ricci}^M(\operatorname{grad} f, \operatorname{grad} f) = \left(m-1 - \frac{3m^2}{4}f^2 \right) |\operatorname{grad} f|^2.$$

For simplicity, we denote the scalar curvature Scal^M by s . Recall that using (1.3.6), with $c = 1$, we have

$$|A|^2 = m(m-1) + m^2f^2 - s.$$

As φ is biharmonic, we have

$$\Delta f = (m - |A|^2)f,$$

thus

$$\begin{aligned} \text{grad } \Delta f &= \text{grad}[(m - |A|^2)f] \\ &= m \text{grad } f - f \text{grad } |A|^2 - |A|^2 \text{grad } f \\ &= m \text{grad } f - f \text{grad}(m(m-1) + m^2 f^2 - s) - |A|^2 \text{grad } f \\ &= m \text{grad } f - 2m^2 f^2 \text{grad } f + f \text{grad } s - |A|^2 \text{grad } f \\ &= (m - 2m^2 f^2 - |A|^2) \text{grad } f + f \text{grad } s. \end{aligned} \tag{2.5.3}$$

On the other hand, for a local orthonormal frame field $\{E_i\}_{i=1}^m$,

$$\begin{aligned} |\nabla df|^2 &= \sum_{i,j=1}^m (\nabla df(E_i, E_j))^2 \\ &\geq \sum_{i=1}^m (\nabla df(E_i, E_i))^2 \\ &\geq \frac{1}{m} \left(\sum_{i=1}^m \nabla df(E_i, E_i) \right)^2 \\ &\geq \frac{1}{m} (\Delta f)^2. \end{aligned}$$

Now

$$\begin{aligned} -\frac{1}{2} \Delta |\text{grad } f|^2 &\geq \frac{1}{m} (\Delta f)^2 - \langle (m - 2m^2 f^2 - |A|^2) \text{grad } f + f \text{grad } s, \text{grad } f \rangle \\ &\quad + \left(m - 1 - \frac{3m^2}{4} f^2 \right) |\text{grad } f|^2 \\ &\geq \frac{1}{m} (\Delta f)^2 + \left(|A|^2 + \frac{5m^2}{4} f^2 - 1 \right) |\text{grad } f|^2 - f \langle \text{grad } s, \text{grad } f \rangle. \end{aligned}$$

We have

$$\begin{aligned} \frac{1}{m} \int_M (\Delta f)^2 dv_g &= \frac{1}{m} \int_M (\Delta f)(\Delta f) dv_g \\ &= -\frac{1}{m} \int_M (\Delta f) \text{div}(\text{grad } f) dv_g, \end{aligned}$$

and using the Divergence Theorem and [\(2.5.3\)](#) we obtain

$$\begin{aligned} \frac{1}{m} \int_M (\Delta f)^2 dv_g &= \frac{1}{m} \int_M \langle \text{grad } \Delta f, \text{grad } f \rangle dv_g \\ &= -\frac{1}{m} \int_M (|A|^2 + 2m^2 f^2 - m) |\text{grad } f|^2 dv_g \\ &\quad + \frac{1}{m} \int_M f \langle \text{grad } s, \text{grad } f \rangle dv_g. \end{aligned}$$

Integrating the Bochner Formula over M and using the Divergence Theorem, we get

$$\begin{aligned}
0 &\geq -\frac{1}{m} \int_M (|A|^2 + 2m^2 f^2 - m) |\text{grad } f|^2 dv_g + \frac{1}{m} \int_M f \langle \text{grad } s, \text{grad } f \rangle dv_g \\
&\quad + \int_M (|A|^2 + \frac{5m^2}{4} f^2 - 1) |\text{grad } f|^2 dv_g - \int_M f \langle \text{grad } s, \text{grad } f \rangle dv_g \\
&\geq \int_M \left[\left(1 - \frac{1}{m}\right) |A|^2 + \left(-\frac{2}{m} + \frac{5}{4}\right) m^2 f^2 \right] |\text{grad } f|^2 dv_g \\
&\quad + \left(\frac{1-m}{m}\right) \int_M f \langle \text{grad } s, \text{grad } f \rangle dv_g. \tag{2.5.4}
\end{aligned}$$

To obtain a lower bound of the first term, we need Corollary [2.4.10](#) and apply it to Equation [\(2.5.4\)](#) to obtain

$$\begin{aligned}
0 &\geq \int_M \left[\left(\frac{m-1}{m}\right) \left(\frac{m^2(m+8)}{4(m-1)}\right) f^2 + \left(\frac{5m-8}{4m}\right) m^2 f^2 \right] |\text{grad } f|^2 dv_g \\
&\quad + \left(\frac{1-m}{2m}\right) \int_M \langle \text{grad } s, \text{grad } f^2 \rangle dv_g \\
&\geq \int_M \left[\frac{m(m+8)}{4} f^2 + \frac{m(5m-8)}{4} f^2 \right] |\text{grad } f|^2 dv_g + \left(\frac{1-m}{2m}\right) \int_M s \Delta f^2 dv_g \\
&\geq \frac{3m^2}{2} \int_M f^2 |\text{grad } f|^2 dv_g + \left(\frac{1-m}{2m}\right) \int_M s \Delta f^2 dv_g.
\end{aligned}$$

Now, we have

$$\Delta f^2 = 2((m - |A|^2)f^2 - |\text{grad } f|^2),$$

thus

$$\begin{aligned}
&\frac{3m^2}{2} \int_M f^2 |\text{grad } f|^2 dv_g + \left(\frac{1-m}{2m}\right) \int_M 2s [(m - |A|^2)f^2 - |\text{grad } f|^2] dv_g \\
&= \frac{3m^2}{2} \int_M f^2 |\text{grad } f|^2 dv_g + \left(\frac{1-m}{m}\right) \int_M s(m - |A|^2)f^2 dv_g \\
&\quad + \left(\frac{m-1}{m}\right) \int_M s |\text{grad } f|^2 dv_g.
\end{aligned}$$

Using Corollary [2.4.10](#) and the fact that $s \geq 0$, we obtain

$$\begin{aligned}
0 &\geq \frac{3m^2}{2} \int_M f^2 |\text{grad } f|^2 dv_g + (1-m) \int_M s f^2 dv_g \\
&\quad + \frac{m(m+8)}{4} \int_M s f^4 dv_g + \left(\frac{m-1}{m}\right) \int_M s |\text{grad } f|^2 dv_g. \tag{2.5.5}
\end{aligned}$$

Now as $s \geq 0$, we obtain

$$\int_M s |\text{grad } f|^2 dv_g \geq 0 \tag{2.5.6}$$

From Inequality (2.5.5) we have

$$0 \geq \frac{3m^2}{2} \int_M f^2 |\text{grad } f|^2 dv_g + (1-m) \int_M s f^2 dv_g + \frac{m(m+8)}{4} \int_M s f^4 dv_g. \quad (2.5.7)$$

Multiplying Inequality (2.5.7) by 4, we obtain

$$0 \geq 6m^2 \int_M f^2 |\text{grad } f|^2 dv_g + \int_M [4(1-m) + m(m+8)f^2] s f^2 dv_g.$$

Now as $\int_M [4(1-m) + m(m+8)f^2] s f^2 dv_g \geq 0$ we obtain

$$0 \geq 6m^2 \int_M f^2 |\text{grad } f|^2 dv_g + \int_M [4(1-m) + m(m+8)f^2] s f^2 dv_g \geq 0,$$

by the sandwich rule we conclude that

$$6m^2 \int_M f^2 |\text{grad } f|^2 dv_g + \int_M [4(1-m) + m(m+8)f^2] s f^2 dv_g = 0.$$

Hence, at every point of M , $f^2 |\text{grad } f|^2 = 0$ which implies that, at each point of M , $f = 0$ or $\text{grad } f = 0$.

Let $p \in M$ be an arbitrary fixed point. If $(\text{grad } f)(p) \neq 0$, then $f = 0$ around p , thus $\text{grad } f = 0$ at p , which is a contradiction. Therefore, $\text{grad } f = 0$ at each point p , which implies that f is constant (on M), and contradicts our assumption. \square

A weaker version of Theorem 2.5.1 can be formulated, replacing the condition on the scalar curvature by a combination of two lower bounds on the Ricci and scalar curvature, and an inequality involving the average of the mean curvature.

Theorem 2.5.2. *Let $\varphi : M^m \rightarrow \mathbb{S}^{m+1}$ be a compact proper-biharmonic hypersurface. Assume that there exist a non-negative function h on M such that $|\text{grad } |A|^2| \leq h |\text{grad } f|$ on M , and a real number $a > 0$ such that:*

1. $\text{Ricci}^M(X, X) \geq a > 0$, for all $X \in T_p M$, $|X| = 1$ and for all $p \in M$;
2. $\int_M [m^2(m+8)af^2 - 4(m-1)\text{Scal}^M] f^2 dv_g \geq 0$.

Then M has constant mean curvature.

Proof. We assume that M does not have constant mean curvature, and will argue by contradiction, as in the proof of Theorem 2.5.1.

First, we consider the Bochner Formula

$$-\frac{1}{2} \Delta |\text{grad } f|^2 = |\nabla df|^2 - \langle \text{grad } \Delta f, \text{grad } f \rangle + \text{Ricci}^M(\text{grad } f, \text{grad } f).$$

Second, take Equation (1.3.5), with $c = 1$, together with $A(\text{grad } f) = -\frac{m}{2}f \text{ grad } f$, which comes from the biharmonicity of M in \mathbb{S}^{m+1} , to obtain an equation for the Ricci curvature of M

$$\text{Ricci}^M(\text{grad } f, \text{grad } f) = \left(m - 1 - \frac{3m^2}{4}f^2\right) |\text{grad } f|^2.$$

Third, from Equation (1.3.6), with $c = 1$, we deduce a relationship between the norm of the shape operator A and $s = \text{Scal}^M$, the scalar curvature of M

$$|A|^2 = m(m - 1) + m^2f^2 - s.$$

Then, we compute $\text{grad } \Delta f$ and find a lower bound for $|\nabla df|^2$, and obtain

$$\text{grad } \Delta f = (m - 2m^2f^2 - |A|^2) \text{grad } f + f \text{ grad } s \quad \text{and} \quad |\nabla df|^2 \geq \frac{1}{m}(\Delta f)^2.$$

Furthermore, we rewrite the Bochner Formula, integrate it and use the Divergence Theorem to obtain Inequality (2.5.4)

$$\begin{aligned} 0 \geq & \int_M \left[\left(1 - \frac{1}{m}\right) |A|^2 + \left(-\frac{2}{m} + \frac{5}{4}\right) m^2f^2 \right] |\text{grad } f|^2 dv_g \\ & + \left(\frac{1-m}{m}\right) \int_M f \langle \text{grad } s, \text{grad } f \rangle dv_g. \end{aligned}$$

Finally, we use Corollary 2.4.10 and the fact that $s > 0$, which comes from the assumption that $\text{Ricci}^M(X, X) \geq a > 0$, to obtain

$$\begin{aligned} 0 \geq & \frac{3m^2}{2} \int_M f^2 |\text{grad } f|^2 dv_g + (1-m) \int_M s f^2 dv_g \\ & + \frac{m(m+8)}{4} \int_M s f^4 dv_g + \left(\frac{m-1}{m}\right) \int_M s |\text{grad } f|^2 dv_g. \end{aligned} \quad (2.5.8)$$

Now, instead of using $\int_M s |\text{grad } f|^2 dv_g$ as in Theorem 2.5.1, we will use Obata Inequality on the first eigenvalue of the Laplacian operator to obtain a more precise lower bound of $\int_M s |\text{grad } f|^2 dv_g$.

To obtain a lower bound of the L^2 -norm of $\text{grad } f$ in terms of f , we use the spectrum properties of the Laplacian operator. Consider an orthonormal basis $\{f_i\}_{i=0}^\infty$ of $C^\infty(M)$ -eigenfunctions of the Laplacian, i.e. $\Delta f_i = \lambda_i f_i$, where $\lambda_0 = 0 < \lambda_1 \leq \lambda_2 \leq \dots$, and $\int_M f_i f_j dv_g = \delta_{ij}$.

Let $f \in C^\infty(M)$, then $f = \sum_{i=0}^\infty \mu_i f_i$, where $f_0 = \frac{1}{\sqrt{\text{Vol}(M)}}$ and $\mu_0 = \frac{1}{\sqrt{\text{Vol}(M)}} \int_M f dv_g$. Then, by Parseval's Identity (see for example [33])

$$\int_M f^2 dv_g = \sum_{i=0}^\infty \mu_i^2.$$

Also,

$$\Delta f = \Delta \left(\sum_{i=0}^{\infty} \mu_i f_i \right) = \sum_{i=0}^{\infty} \mu_i \Delta f_i = \sum_{i=0}^{\infty} \mu_i \lambda_i f_i,$$

but as $\lambda_0 = 0$, we obtain

$$\Delta f = \sum_{i=1}^{\infty} \mu_i \lambda_i f_i,$$

so again, using Parseval's Identity [33], we have

$$\begin{aligned} \int_M f \Delta f \, dv_g &= \int_M \left(\sum_{i=1}^{\infty} \mu_i f_i \right) \left(\sum_{j=1}^{\infty} \mu_j \lambda_j f_j \right) \, dv_g = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \mu_i \mu_j \lambda_j \int_M f_i f_j \, dv_g = \sum_{i=1}^{\infty} \lambda_i \mu_i^2 \\ &\geq \lambda_1 \sum_{i=1}^{\infty} \mu_i^2 = \lambda_1 \left(\int_M f^2 \, dv_g - \mu_0^2 \right). \end{aligned}$$

But

$$\int_M f \Delta f \, dv_g = \int_M |\text{grad } f|^2 \, dv_g,$$

so

$$\int_M |\text{grad } f|^2 \, dv_g \geq \lambda_1 \left[\int_M f^2 \, dv_g - \frac{1}{\text{Vol}(M)} \left(\int_M f \, dv_g \right)^2 \right].$$

Now, by Obata [76], $\text{Ricci}^M(X, X) \geq a|X|^2 > 0$ implies that $\lambda_1 \geq \frac{ma}{m-1}$. Since $s \geq ma$, we have

$$\begin{aligned} \int_M s |\text{grad } f|^2 \, dv_g &\geq ma \int_M |\text{grad } f|^2 \, dv_g \\ &\geq \frac{m^2 a^2}{m-1} \left[\int_M f^2 \, dv_g - \frac{1}{\text{Vol}(M)} \left(\int_M f \, dv_g \right)^2 \right]. \end{aligned} \quad (2.5.9)$$

Then from Inequality (2.5.8)

$$\begin{aligned} 0 &\geq \frac{3m^2}{2} \int_M f^2 |\text{grad } f|^2 \, dv_g + (1-m) \int_M s f^2 \, dv_g + \frac{m(m+8)}{4} \int_M s f^4 \, dv_g \\ &\quad + ma^2 \int_M f^2 \, dv_g - \frac{ma^2}{\text{Vol}(M)} \left(\int_M f \, dv_g \right)^2, \end{aligned} \quad (2.5.10)$$

and multiplying Inequality (2.5.10) by 4, and using $s \geq ma$ we obtain

$$\begin{aligned} 0 &\geq 6m^2 \int_M f^2 |\text{grad } f|^2 \, dv_g + 4(1-m) \int_M s f^2 \, dv_g + m^2(m+8)a \int_M f^4 \, dv_g \\ &\quad + 4ma^2 \int_M f^2 \, dv_g - \frac{4ma^2}{\text{Vol}(M)} \left(\int_M f \, dv_g \right)^2 \\ &\geq 6m^2 \int_M f^2 |\text{grad } f|^2 \, dv_g + 4 \int_M [(1-m)s + ma^2] f^2 \, dv_g \\ &\quad + m^2(m+8)a \int_M f^4 \, dv_g - \frac{4ma^2}{\text{Vol}(M)} \left(\int_M f \, dv_g \right)^2. \end{aligned}$$

Now, we recall the Cauchy-Schwarz Inequality

$$\int_M fg \, dv_g \leq \left(\int_M f^2 \, dv_g \right)^{\frac{1}{2}} \left(\int_M g^2 \, dv_g \right)^{\frac{1}{2}}.$$

Taking $g = 1$, we get

$$\left(\int_M f \, dv_g \right)^2 \leq \text{Vol}(M) \int_M f^2 \, dv_g.$$

As a result

$$\begin{aligned} 0 &\geq 6m^2 \int_M f^2 |\text{grad } f|^2 \, dv_g \\ &\quad + \int_M [4(1-m)s + 4ma^2 + m^2(m+8)af^2 - 4ma^2] f^2 \, dv_g. \end{aligned}$$

By Condition [2](#) we conclude that

$$6m^2 \int_M f^2 |\text{grad } f|^2 \, dv_g + \int_M [4(1-m)s + m^2(m+8)af^2] f^2 \, dv_g = 0.$$

Hence, at every point of M , $f^2 |\text{grad } f|^2 = 0$ which implies that, at each point of M , $f = 0$ or $\text{grad } f = 0$.

Let $p \in M$ be an arbitrary fixed point. If $(\text{grad } f)(p) \neq 0$, then $f = 0$ around p , thus $\text{grad } f = 0$ at p , which is a contradiction. Therefore, $\text{grad } f = 0$ at each point p , which implies that f is constant (on M), and contradicts our assumption. \square

Example 2.5.3. The hypotheses in Theorems [2.5.1](#) and [2.5.2](#) are satisfied by the 45th-parallel $\mathbb{S}^m \left(\frac{1}{\sqrt{2}} \right) \rightarrow \mathbb{S}^{m+1}$, for $m \geq 2$, as $\text{Ricci}^{\mathbb{S}^m \left(\frac{1}{\sqrt{2}} \right)} = 2(m-1)g$, $\text{Scal}^{\mathbb{S}^m \left(\frac{1}{\sqrt{2}} \right)} = 2m(m-1)$, $|A|^2 = m$ and $f^2 = 1$ (we choose a equal to $2(m-1)$).

Indeed, by Proposition [\(1.4.16\)](#) we know that the 45th-parallel is proper-biharmonic, has $|A|^2 = m$ and $f^2 = 1$. Also, it has $\text{Ricci}^{\mathbb{S}^m \left(\frac{1}{\sqrt{2}} \right)} = 2(m-1)g$ and $\text{Scal}^{\mathbb{S}^m \left(\frac{1}{\sqrt{2}} \right)} = 2m(m-1)$.

By direct computations, we can show that Inequality [\(2.5.1\)](#) of Theorem [2.5.1](#) is verified for $m \geq 2$.

As for Theorem [2.5.2](#) Condition [1](#) holds, and Condition [2](#) is satisfied if and only if $2m^2(m+8)(m-1) - 8m(m-1)^2 \geq 0$. which is true for all $m \geq 2$.

Example 2.5.4. The generalized Clifford torus $\mathbb{S}^{m_1} \left(\frac{1}{\sqrt{2}} \right) \times \mathbb{S}^{m_2} \left(\frac{1}{\sqrt{2}} \right) \rightarrow \mathbb{S}^{m+1}$, where $1 \leq m_1 < m_2$, and $m \geq 3$, has $\text{Ricci}^{\mathbb{S}^{m_1} \left(\frac{1}{\sqrt{2}} \right) \times \mathbb{S}^{m_2} \left(\frac{1}{\sqrt{2}} \right)} = 2(m_1-1)g_1 + 2(m_2-1)g_2$, $\text{Scal}^{\mathbb{S}^{m_1} \left(\frac{1}{\sqrt{2}} \right) \times \mathbb{S}^{m_2} \left(\frac{1}{\sqrt{2}} \right)} = 2m_1(m_1-1) + 2m_2(m_2-1)$, $|A|^2 = m_1 + m_2 = m$ and $f^2 = \left(\frac{m_1 - m_2}{m_1 + m_2} \right)^2$. Here g_1 is the canonical metric on $\mathbb{S}^{m_1} \left(\frac{1}{\sqrt{2}} \right)$ and g_2 is the canonical metric on $\mathbb{S}^{m_2} \left(\frac{1}{\sqrt{2}} \right)$.

Concerning Inequality [2.5.1](#) in Theorem [2.5.1](#), by straightforward computations, we can see that it is satisfied if and only if $m_1 \in \left[1, \frac{m}{2} - \sqrt{\frac{m(m-1)}{m+8}}\right]$, for $m \geq 4$. Note that this is not a very restrictive condition, as for m large enough, we have

$$\frac{m}{3} < \frac{m}{2} - \sqrt{\frac{m(m-1)}{m+8}}.$$

Indeed, by Proposition [\(1.4.17\)](#) we know that the Clifford torus has $|A|^2 = m$ and $f^2 = \left(\frac{m_1 - m_2}{m_1 + m_2}\right)^2$. It also has $\text{Ricci}^{\mathbb{S}^{m_1}(\frac{1}{\sqrt{2}}) \times \mathbb{S}^{m_2}(\frac{1}{\sqrt{2}})} = 2(m_1 - 1)g_1 + 2(m_2 - 1)g_2$ and $\text{Scal}^{\mathbb{S}^{m_1}(\frac{1}{\sqrt{2}}) \times \mathbb{S}^{m_2}(\frac{1}{\sqrt{2}})} = 2m_1(m_1 - 1) + 2m_2(m_2 - 1)$. Obviously, the first two conditions in Theorem [2.5.1](#) are satisfied. Now, we verify Inequality [2.5.1](#) and it is enough to show that

$$(m + 8)(m_1 - m_2)^2 - 4m(m - 1) \geq 0.$$

We have

$$\begin{aligned} (m + 8)(m_1 - m_2)^2 - 4m(m - 1) &= (m + 8)(m_1 - m + m_1)^2 - 4m(m - 1) \\ &= 4(m + 8)m_1^2 - 4m(m + 8)m_1 + m(m + 2)^2. \end{aligned}$$

We consider the inequality

$$4(m + 8)m_1^2 - 4m(m + 8)m_1 + m(m + 2)^2 \geq 0$$

as a second order inequality in m_1 . Clearly, we obtain $m_1 \in (0, x_1] \cup [x_2, m)$, where x_1 and x_2 are the roots of this quadratic equation,

$$x_1 = \frac{m}{2} - \sqrt{\frac{m(m-1)}{m+8}} \quad \text{and} \quad x_2 = \frac{m}{2} + \sqrt{\frac{m(m-1)}{m+8}}.$$

It is clear that $x_1 \in (0, \frac{m}{2})$ and $x_2 \in (\frac{m}{2}, m)$. Since $m_1 < m_2$, we get $m_1 < \frac{m}{2}$ and therefore $m_1 \in (0, x_1]$. Thus, to have a solution m_1 , which must be a positive natural number, we need to impose $x_1 \geq 1$, that is

$$\frac{m}{2} - \sqrt{\frac{m(m-1)}{m+8}} \geq 1,$$

which is satisfied for any $m \geq 4$.

As to the conditions of Theorem [2.5.2](#), they are more restrictive compared to Theorem [2.5.1](#), but we still can construct many examples that satisfy them.

Indeed, condition $|\text{grad}|A|^2| \leq h|\text{grad}f|$ in Theorem [2.5.2](#) is automatically satisfied. Then, Condition [1](#) in Theorem [2.5.2](#) is satisfied since $\text{Ricci}^{\mathbb{S}^{m_1}(\frac{1}{\sqrt{2}}) \times \mathbb{S}^{m_2}(\frac{1}{\sqrt{2}})} = 2(m_1 - 1)g_1 + 2(m_2 - 1)g_2$ and we can choose $a = 2(m_1 - 1)$.

We have

$$\begin{aligned} \text{Scal}^{\mathbb{S}^{m_1}(\frac{1}{\sqrt{2}}) \times \mathbb{S}^{m_2}(\frac{1}{\sqrt{2}})} &= 2m_1(m_1 - 1) + 2m_2(m_2 - 1) \\ &= 2m_1(m_1 - 1) + 2(m - m_1)(m - m_1 - 1) \\ &= 2(2m_1^2 - 2mm_1 + m^2 - m). \end{aligned}$$

Substituting $\text{Scal}^{\mathbb{S}^{m_1}(\frac{1}{\sqrt{2}}) \times \mathbb{S}^{m_2}(\frac{1}{\sqrt{2}})}$, f^2 and a in Condition [2](#) we obtain

$$\int_M \left[2m^2(m+8)(m_1-1) \left(\frac{m_1-m_2}{m_1+m_2} \right)^2 - 8(m-1)(2m_1^2 - 2mm_1 + m^2 - m) \right] f^2 dv_g \geq 0$$

which is true if and only if

$$2m^2(m+8)(m_1-1) \left(\frac{m_1-m_2}{m_1+m_2} \right)^2 - 8(m-1)(2m_1^2 - 2mm_1 + m^2 - m) \geq 0.$$

We have

$$\begin{aligned} &2m^2(m+8)(m_1-1) \left(\frac{m_1-m_2}{m_1+m_2} \right)^2 - 8(m-1)(2m_1^2 - 2mm_1 + m^2 - m) \\ &= 2m^2(m+8)(m_1-1) \left(\frac{2m_1-m}{m} \right)^2 - 8(m-1)(2m_1^2 - 2mm_1 + m^2 - m) \\ &= 8(m+8)m_1^3 - 8(m^2 + 11m + 6)m_1^2 + 2(m^3 + 20m^2 + 24m)m_1 - 2(5m^3 + 4m). \end{aligned} \tag{2.5.11}$$

The first case to consider is $m_1 = 2$, so Condition [2](#) is equivalent to

$$64(m+8) - 32(m^2 + 11m + 6) + 4(m^3 + 20m^2 + 24m) - 2(5m^3 + 4m) \geq 0. \tag{2.5.12}$$

It is not difficult to check that Inequality [\(2.5.12\)](#) is never satisfied for $m \geq 5$.

To obtain examples, it is convenient to think of m_1 as a parameter and m as a variable. So we define the function $f : \mathbb{R} \rightarrow \mathbb{R}$ by

$$f(x) = (2m_1 - 10)x^3 + (40m_1 - 8m_1^2)x^2 + (8m_1^3 - 88m_1^2 + 48m_1 - 8)x + 64m_1^3 - 48m_1^2.$$

For any $m_1 \in \{3, 4, 5\}$, one can see that $f(x)$ is always negative after a certain value of the variable x (depending on m_1).

However, for $m_1 \geq 6$, beyond a certain x , an infinite number of Clifford tori with $f(x) \geq 0$, are obtained.

For instance, for $m_1 = 6$, we have

$$f(x) = 2x^3 - 48x^2 - 1160x + 12096$$

which is non-negative for $x \geq 36$.

When $m_1 = 7$

$$f(x) = 4x^3 - 112x^2 - 1240x + 19600$$

is non-negative for $x \geq 33$.

Further, for $m_1 = 8$, we have

$$f(x) = 6x^3 - 192x^2 - 1160x + 29696$$

which is non-negative for $x \geq 34$.

Also, for $m_1 = 9$, we have

$$f(x) = 8x^3 - 288x^2 - 872x + 42768$$

which is non-negative for $x \geq 35$.

Chapter 3

PMC biconservative surfaces

The results of this chapter are extracted from the article:

H. Bibi, B.-Y. Chen, D. Fetcu, C. Oniciuc: PMC Biconservative Surfaces in Complex Space Forms, 2021. [arXiv: 2107.11558v1](https://arxiv.org/abs/2107.11558v1)

3.1 Introduction

This chapter starts with a proof that parallel mean curvature (PMC) surfaces in a non-flat complex space form are biconservative if and only if totally real. Then we develop a Simons type formula for a well-chosen vector field constructed from the mean curvature vector field, to deduce that a complete PMC totally real surface with non-negative Gaussian curvature in a complex space form must have parallel shape operator. This surface must be either flat or pseudo-umbilical. Next, restricting ourselves to complex space forms of complex dimension 2, we find optimal conditions so that a CMC biconservative surface must then be PMC. Then, we use codimension-reduction techniques used in [2, 37, 38] to improve results of [39, 43], and show that a non pseudo-umbilical PMC biconservative surface in a non-flat complex space form $N^n(c)$ must lie in some $N^4(c) \subset N^n(c)$. Then we find a particular case where we further reduce the real codimension to 2. We conclude using the Segre embedding to construct examples of CMC biconservative submanifolds M^{1+2q} of the complex projective space $\mathbb{C}P^{1+2q}(4)$, which are neither PMC nor totally real. Moreover, we discuss their biharmonicity. This illustrates how CMC biconservative submanifolds are less rigid in higher dimensions.

A complex space form of complex dimension n and constant holomorphic sectional curvature c will be denoted by $N^n(c)$. Since a minimal submanifold is trivially biconservative, we will always assume that the mean curvature vector field H does not vanish.

When dealing with a submanifold M of a complex space form N , i.e. we have an isometric immersion $\varphi : M \rightarrow N$, we will indicate the objects on the target manifold N

by $\overline{(\cdot)}$.

3.2 PMC and biconservative submanifolds

We will briefly recall some notions regarding Kähler manifolds and complex space forms.

Definition 3.2.1. A Hermitian manifold is a complex manifold N endowed with a Riemannian metric $\langle \cdot, \cdot \rangle$, such that

$$\langle J\overline{X}, J\overline{Y} \rangle = \langle \overline{X}, \overline{Y} \rangle, \quad \forall \overline{X}, \overline{Y} \in C(TN),$$

where J is the complex structure associated to the complex manifold N .

It is well-known that $J \in C(T_1^1(N))$ and $J^2 = -1$.

Definition 3.2.2. A Kähler manifold is a Hermitian manifold $(N, J, \langle \cdot, \cdot \rangle)$ such that $d\Phi = 0$, where Φ is the 2-form defined by

$$\Phi(\overline{X}, \overline{Y}) = \langle \overline{X}, J\overline{Y} \rangle, \quad \forall \overline{X}, \overline{Y} \in C(TN).$$

Theorem 3.2.3. A Hermitian manifold $(N, J, \langle \cdot, \cdot \rangle)$ is Kähler if and only if $\overline{\nabla}J = 0$.

Definition 3.2.4. A complex space form $N(c)$ is a Kähler manifold $(N, J, \langle \cdot, \cdot \rangle)$ with constant holomorphic sectional curvature c , i.e. $c = \text{Riem}^N(\alpha)$, where α is spanned by \overline{X} and $J\overline{X}$, \overline{X} being any non-zero tangent vector to N .

We recall that the curvature tensor field \overline{R} of a complex space form $N^n(c)$ of complex dimension n is given by

$$\begin{aligned} \overline{R}(\overline{X}, \overline{Y})\overline{Z} &= \frac{c}{4} \left\{ \langle \overline{Y}, \overline{Z} \rangle \overline{X} - \langle \overline{X}, \overline{Z} \rangle \overline{Y} + \langle J\overline{Y}, \overline{Z} \rangle J\overline{X} - \langle J\overline{X}, \overline{Z} \rangle J\overline{Y} \right. \\ &\quad \left. + 2\langle J\overline{Y}, \overline{X} \rangle J\overline{Z} \right\}, \end{aligned} \quad (3.2.1)$$

where \overline{X} , \overline{Y} and \overline{Z} are vector fields tangent to N .

Definition 3.2.5. A submanifold M^m of the complex manifold N equipped with the complex structure J is said to be *complex* if JTM^m lies in the tangent bundle of M^m .

Definition 3.2.6. A submanifold M^m of the complex manifold N equipped with the complex structure J is said to be *totally real* if JTM^m lies in the normal bundle of M^m .

Proposition 3.2.7. [40] Let M^m be a real submanifold of a complex space form $N^n(c)$ of dimension m such that JH is tangent to M^m . Then M^m is biharmonic if and only if

$$\begin{cases} \Delta^\perp H + \text{trace } B(\cdot, A_H \cdot) - \frac{c}{4}(m+3)H &= 0 \\ 2 \text{trace } A_{\nabla_{(\cdot)}^\perp H}(\cdot) + m \text{grad}(|H|^2) &= 0, \end{cases}$$

where A denotes the shape operator, B the second fundamental form, H the mean curvature vector field, $|H|$ the mean curvature function, ∇^\perp and Δ^\perp the connection and the Laplacian in the normal bundle of M^m in $N^n(c)$.

We denote $JH = T + N$, T being the tangential part of JH and N the normal part of JH , i.e., $T = (JH)^\top$ and $N = (JH)^\perp$.

Distinguishing between N the normal part of JH and by the same N the target manifold should be clear from the context.

Theorem 3.2.8. *Let M^m be a PMC submanifold of a complex space form $N^n(c)$. If $c = 0$, then M^m is biconservative, and if $c \neq 0$, then M^m is biconservative if and only if $JT \in C(NM^m)$.*

Proof. Clearly, from Equation (3.2.1) we have

$$\begin{aligned} \sum_{i=1}^m \bar{R}(E_i, H)E_i &= \frac{c}{4} \sum_{i=1}^m \left\{ \langle H, E_i \rangle E_i - \langle E_i, E_i \rangle H + \langle JH, E_i \rangle JE_i \right. \\ &\quad \left. - \langle JE_i, E_i \rangle JH + 2\langle JH, E_i \rangle JE_i \right\} \\ &= \frac{c}{4} \sum_{i=1}^m \left\{ -H + 3\langle JH, E_i \rangle JE_i \right\}, \\ &= \frac{c}{4} \sum_{i=1}^m \left\{ -H + 3J(\langle JH, E_i \rangle E_i) \right\}, \end{aligned}$$

where $\{E_i\}_{i=1}^m$ is a local orthonormal frame field on M^m and tangent to M^m , thus

$$\text{trace}(\bar{R}(\cdot, H)\cdot)^T = \frac{3}{4}c(JT)^\top = 0.$$

One can see that for a PMC submanifold M^m the biconservativity condition (see Proposition 1.5.6) holds if and only if either $c = 0$ or $JT \in C(NM^m)$. \square

Corollary 3.2.9. *Let M^m be a PMC totally real submanifold of a complex space form $N^n(c)$. Then M^m is biconservative.*

Proof. As M^m is totally real, $JT \in C(NM^m)$, so we obtain

$$\text{trace}(\bar{R}(\cdot, H)\cdot)^T = \frac{3}{4}c(JT)^\top = 0.$$

Thus the biconservativity condition (see Proposition 1.5.6) holds. \square

Corollary 3.2.10. *Any PMC real hypersurface M^{2n-1} of a complex space form $N^n(c)$ is biconservative.*

Proof. In view of Theorem 3.2.8, we may assume that $c \neq 0$. Since the mean curvature vector H is normal to M^{2n-1} in $N^n(c)$, and the codimension is one, we have $JH = T$, hence $JT = -H$ is a normal vector field. Therefore, $JT \in C(NM^{2n-1})$. Consequently, the real hypersurface M^{2n-1} is always biconservative. \square

Alternative conditions for biconservativity are given by the following results.

Theorem 3.2.11. *Let M^m be a PMC submanifold of a complex space form $N^n(c)$ with $c \neq 0$. If $JH \in C(NM^m)$, then M^m is biconservative.*

Proof. If $JH \in C(NM^m)$, then $T = JT = 0$. Furthermore, as M^m is a PMC submanifold, the biconservativity condition (see Proposition 1.5.6)

$$\text{trace}(\bar{R}(\cdot, H)\cdot)^T = \frac{3}{4}c(JT)^\top = 0$$

holds. Therefore, M^m is biconservative. \square

Theorem 3.2.12. *Let M^m be a PMC submanifold of a complex space form $N^n(c)$ with $c \neq 0$. If $JH \in C(TM^m)$, then M^m is biconservative.*

Proof. If $JH \in C(TM^m)$, then $T = JH$ and so $JT = -H$ is normal. Furthermore, as M^m is a PMC submanifold, the biconservativity condition (see Proposition 1.5.6)

$$\text{trace}(\bar{R}(\cdot, H)\cdot)^T = \frac{3}{4}c(JT)^\top = 0$$

holds. Therefore, M^m is biconservative. \square

3.3 PMC biconservative surfaces in $N^n(c)$

In this section we study PMC biconservative surfaces in the complex space form $N^n(c)$ of complex dimension n . The first result shows that, when $c \neq 0$, such surfaces are totally real. More precisely, we have the following.

Theorem 3.3.1. *Let M^2 be a PMC surface in a complex space form $N^n(c)$. If $c = 0$, then M^2 is biconservative. If $c \neq 0$, then M^2 is biconservative if and only if M^2 is totally real.*

Proof. Since the case $c = 0$ was already proved in Theorem 3.2.8, we will only consider the case $c \neq 0$.

First, we prove that a PMC biconservative surface M^2 in $N^n(c)$ is totally real. Since M^2 is PMC and biconservative, from Theorem 3.2.8 we have $JT \in C(NM^m)$, thus

$$(JT)^\top = 0. \tag{3.3.1}$$

Taking $\eta = H$ and $\xi = V$ in the Ricci Equation (1.3.10), with $V \in C(NM)$, since M^2 is PMC, we obtain

$$\langle [A_H, A_V]X, Y \rangle = -\langle \bar{R}(X, Y)H, V \rangle.$$

Now, using (3.2.1)

$$\begin{aligned} \langle \bar{R}(X, Y)H, JT \rangle &= \frac{c}{4} \left\{ \langle Y, H \rangle \langle X, JT \rangle - \langle X, H \rangle \langle Y, JT \rangle + \langle JY, H \rangle \langle JX, JT \rangle \right. \\ &\quad \left. - \langle JX, H \rangle \langle JY, JT \rangle + 2\langle JY, X \rangle \langle JH, JT \rangle \right\} \\ &= \frac{c}{4} \left\{ \langle JY, H \rangle \langle JX, JT \rangle - \langle JX, H \rangle \langle JY, JT \rangle \right. \\ &\quad \left. + 2\langle JY, X \rangle \langle JH, JT \rangle \right\} \\ &= \frac{c}{4} \left\{ -\langle JH, Y \rangle \langle X, T \rangle + \langle JH, X \rangle \langle Y, T \rangle \right. \\ &\quad \left. - 2\langle JX, Y \rangle \langle H, T \rangle \right\} \\ &= \frac{c}{4} \left\{ -\langle T, Y \rangle \langle X, T \rangle + \langle T, X \rangle \langle Y, T \rangle \right\} \\ &= 0, \end{aligned}$$

so

$$[A_H, A_{JT}] = 0. \quad (3.3.2)$$

From Equation (3.3.2) it follows that, at each point of M^2 , there exists a (positive) orthonormal basis $\{e_1, e_2\}$ tangent to M^2 that diagonalizes both A_H and A_{JT} at that point.

Moreover, the following equality holds on M^2

$$\begin{aligned} \text{trace } A_{JT} &= \sum_{i=1}^2 \langle A_{JT}e_i, e_i \rangle = \sum_{i=1}^2 \langle B(e_i, e_i), JT \rangle = \langle \text{trace } B, JT \rangle \\ &= 2\langle H, JT \rangle = -2\langle JH, T \rangle = -2\langle T, T \rangle \\ &= -2|T|^2. \end{aligned}$$

Further, as $\nabla^\perp H = 0$, we have

$$\begin{aligned} \bar{\nabla}_X JH &= J\bar{\nabla}_X H = J(\nabla_X^\perp H - A_H X) \\ &= -JA_H X, \end{aligned}$$

and

$$\begin{aligned} \bar{\nabla}_X JH &= \bar{\nabla}_X (T + N) \\ &= \nabla_X T + B(X, T) - A_N X + \nabla_X^\perp N, \end{aligned}$$

thus

$$-JA_HX = \nabla_X T + B(X, T) - A_N X + \nabla_X^\perp N. \quad (3.3.3)$$

We fix a point p and then, for $X = e_i$, take the inner product of (3.3.3) with e_j , $i \neq j$, and, at p , we obtain

$$\begin{aligned} -\langle JA_H e_i, e_j \rangle &= \langle \nabla_{e_i} T, e_j \rangle - \langle A_N e_i, e_j \rangle + \langle B(e_i, T), e_j \rangle + \langle \nabla_{e_i}^\perp N, e_j \rangle \\ &= \langle \nabla_{e_i} T, e_j \rangle - \langle A_N e_i, e_j \rangle. \end{aligned}$$

With respect to the basis $\{e_1, e_2\}$, we have

$$A_H = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \quad A_{JT} = \begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{pmatrix}.$$

Thus, at p ,

$$-\lambda_i \langle J e_i, e_j \rangle = \langle \nabla_{e_i} T, e_j \rangle - \langle A_N e_i, e_j \rangle. \quad (3.3.4)$$

One can see that

$$\lambda_1 + \lambda_2 = \text{trace } A_H = 2|H|^2$$

and

$$\mu_1 + \mu_2 = \text{trace } A_{JT} = -2|T|^2.$$

On the other hand, on M^2 ,

$$\bar{\nabla}_X J T = -A_{JT} X + \nabla_X^\perp J T,$$

and

$$\bar{\nabla}_X J T = J \bar{\nabla}_X T = J \nabla_X T + J B(X, T).$$

For $X = e_i$ in the above relations, taking the inner product with e_i , at the point p , we get

$$\begin{aligned} -\langle A_{JT} e_i, e_i \rangle &= \langle J \nabla_{e_i} T, e_i \rangle + \langle J B(e_i, T), e_i \rangle \\ &= -\langle \nabla_{e_i} T, J e_i \rangle - \langle B(e_i, T), J e_i \rangle, \end{aligned}$$

and therefore,

$$-\langle \mu_i e_i, e_i \rangle = -\langle \nabla_{e_i} T, J e_i \rangle - \langle B(e_i, T), J e_i \rangle,$$

which implies

$$\begin{aligned} \mu_i &= \sum_{j=1}^2 \langle \nabla_{e_i} T, e_j \rangle \langle J e_i, e_j \rangle + \langle B(e_i, T), J e_i \rangle. \\ &= \langle \nabla_{e_i} T, e_j \rangle \langle J e_i, e_j \rangle + \langle B(e_i, T), J e_i \rangle, \quad i \neq j. \end{aligned} \quad (3.3.5)$$

Now, multiply Equation (3.3.4) by $\langle Je_i, e_j \rangle$, $i \neq j$, to obtain

$$-\lambda_i \langle Je_i, e_j \rangle^2 = \langle \nabla_{e_i} T, e_j \rangle \langle Je_i, e_j \rangle - \langle A_N e_i, e_j \rangle \langle Je_i, e_j \rangle. \quad (3.3.6)$$

From Equations (3.3.5) and (3.3.6), we get

$$-\lambda_i \langle Je_i, e_j \rangle^2 = \mu_i - \langle B(e_i, T), Je_i \rangle - \langle A_N e_i, Je_i \rangle.$$

Thus, summing up, we have

$$\begin{aligned} \mu_1 + \mu_2 &= -(\lambda_1 + \lambda_2) \langle Je_1, e_2 \rangle^2 + \sum_{i=1}^2 \langle B(e_i, T), Je_i \rangle \\ &\quad + \sum_{i=1}^2 \langle A_N e_i, Je_i \rangle. \end{aligned} \quad (3.3.7)$$

Since

$$\begin{aligned} \langle A_N e_1, Je_1 \rangle &= \langle A_N e_1, e_2 \rangle \langle e_2, Je_1 \rangle = -\langle e_1, A_N e_2 \rangle \langle Je_2, e_1 \rangle \\ &= -\langle A_N e_2, Je_2 \rangle, \end{aligned}$$

then

$$\mu_1 + \mu_2 = -(\lambda_1 + \lambda_2) \langle Je_1, e_2 \rangle^2 + \sum_{i=1}^2 \langle B(e_i, T), Je_i \rangle,$$

equivalently,

$$2|T|^2 = 2|H|^2 \langle Je_1, e_2 \rangle^2 + \sum_{i=1}^2 \langle JB(e_i, T), e_i \rangle,$$

i.e.

$$2|T|^2 = 2|H|^2 \langle Je_1, e_2 \rangle^2 + \text{trace} \langle JB(\cdot, T), \cdot \rangle, \quad (3.3.8)$$

which holds at any point $p \in M^2$.

We note that Equation (3.3.8) has a geometrical meaning, i.e. it does not depend on the choice of the orthonormal basis $\{e_1, e_2\}$. Indeed, let $\{\tilde{e}_1, \tilde{e}_2\}$ be an orthogonal basis such that

$$\begin{pmatrix} \tilde{e}_1 \\ \tilde{e}_2 \end{pmatrix} = \begin{pmatrix} \cos \varphi & -\sin \varphi \\ \sin \varphi & \cos \varphi \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}, \quad \text{or} \quad \begin{pmatrix} \tilde{e}_1 \\ \tilde{e}_2 \end{pmatrix} = \begin{pmatrix} \cos \varphi & \sin \varphi \\ \sin \varphi & -\cos \varphi \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}.$$

For the first case, we have

$$\begin{aligned} \langle J\tilde{e}_1, \tilde{e}_2 \rangle &= \langle \cos \varphi Je_1 - \sin \varphi Je_2, \sin \varphi e_1 + \cos \varphi e_2 \rangle \\ &= \langle Je_1, e_2 \rangle. \end{aligned}$$

Similarly, for the second case, where $\tilde{e}_1 = \cos \varphi e_1 + \sin \varphi e_2$ and $\tilde{e}_2 = \sin \varphi e_1 - \cos \varphi e_2$, we get

$$\langle J\tilde{e}_1, \tilde{e}_2 \rangle = -\langle Je_1, e_2 \rangle.$$

Now, let $p \in M^2$ be an arbitrary point. If $T_p \neq 0$, we can consider the orthonormal basis $\{X_1, X_2\}$, where $X_1 = T_p/|T_p|$, tangent to M^2 . Then, since JT is normal, we have $\langle JX_2, X_1 \rangle = 0$, and it is easy to see that

$$J(T_p M^2) \subset N_p M^2.$$

Now assume that $T_p = 0$. From (3.3.8), it follows that

$$2|H|^2 \langle Je_1, e_2 \rangle^2 = 0,$$

that is $\langle Je_1, e_2 \rangle = 0$, which shows that

$$J(T_p M^2) \subset N_p M^2.$$

Therefore, we conclude that M^2 is totally real.

Conversely, if M^2 is totally real, then $(JT)^\top = 0$ and therefore, $\text{trace}(\overline{R}(\cdot, H), \cdot)^\top = 0$. Since M^2 is also PMC, from Proposition 1.5.6, it follows that M^2 is biconservative. \square

We recall that

Theorem 3.3.2. [90] *Let M^m be an m -dimensional pseudo-umbilical PMC submanifold of a non-flat complex space form $N^n(c)$ of complex dimension n , with $H \neq 0$. Then $n > m$ and M^m is a totally real submanifold of $N^n(c)$.*

For the particular case $m = 2$ and $n > 2$, we can give an alternative proof of the above result. Indeed, as a pseudo-umbilical PMC surface is biconservative, by Theorem 3.3.1 it follows that it is totally real.

For the sake of completeness, we present here a sketch of the proof of Theorem 3.3.2 which, in particular, says that in $\mathbb{C}P^2$ there exists no pseudo-umbilical PMC surface. Indeed, let M^m be a pseudo-umbilical PMC submanifold in $N^n(c)$, $c \neq 0$. We directly obtain $\overline{R}(X, Y)H = 0$ for any X, Y vector fields tangent to M^m . Taking the inner product of $\overline{R}(X, Y)H$ with JH in the above relation and using (3.2.1) we get that M^m is totally real. Moreover, taking the inner product of $\overline{R}(X, Y)H$, but now with JX , in the same relation we get $\langle JY, H \rangle = 0$, i.e., at any $p \in M^m$, $H(p) \perp J(T_p M^m)$. Since

$$T_p M^m \oplus J(T_p M^m) \oplus \text{span}\{H(p)\} \subset T_p N,$$

We get $2m + 1 < 2n$ and so $m < n$.

As an application of Theorem 3.3.1, we have

Corollary 3.3.3. *Let M^2 be a PMC surface in a complex space form $N^n(c)$. If $JH \in C(NM^2)$, then M^2 is totally real and $n > 2$.*

Proof. Since $T = 0$, formula (3.3.1) holds and therefore (3.3.8) becomes $\langle Je_1, e_2 \rangle = 0$, thus M^2 is totally real.

We can prove that M^2 is totally real in a straightforward way as follows. Consider $\{e_1, e_2\}$ an orthonormal basis that diagonalizes A_H at $p \in M^2$, then

$$A_H = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}.$$

Now, since $\langle H, JH \rangle = 0$, then

$$\text{trace } A_{JH} = \sum_{i=1}^2 \langle A_{JH}e_i, e_i \rangle = \sum_{i=1}^2 \langle B(e_i, e_i), JH \rangle = 2\langle H, JH \rangle = 0.$$

Thus

$$A_{JH} = \begin{pmatrix} a & b \\ b & -a \end{pmatrix}.$$

Let $X \in C(TM^2)$, on M^2 , we have

$$\bar{\nabla}_X JH = J\bar{\nabla}_X H,$$

and as M^2 is PMC we obtain

$$\bar{\nabla}_X JH = -JA_H X.$$

On the other hand,

$$\bar{\nabla}_X JH = -A_{JH} X + \nabla_X^\perp JH$$

then, on M^2 , we have

$$JA_H X = A_{JH} X - \nabla_X^\perp JH. \tag{3.3.9}$$

Taking the inner product of (3.3.9) with $Y \in C(TM^2)$, we obtain

$$\langle JA_H X, Y \rangle = \langle A_{JH} X, Y \rangle.$$

Thus, at p , we have the following

1. if $X = e_1, Y = e_2$, we get

$$\lambda_1 \langle Je_1, e_2 \rangle = \langle ae_1 + be_2, e_2 \rangle = b, \tag{3.3.10}$$

and

2. if $X = e_2, Y = e_1$, we get

$$\lambda_2 \langle Je_2, e_1 \rangle = \langle be_1 - ae_2, e_1 \rangle = b. \quad (3.3.11)$$

From Equations (3.3.10), and (3.3.11) we obtain

$$\lambda_1 \langle Je_1, e_2 \rangle = \lambda_2 \langle Je_2, e_1 \rangle,$$

hence $\langle Je_1, e_2 \rangle = 0$ or $\lambda_1 = -\lambda_2$ which is impossible as $|H| \neq 0$. Therefore, $\langle Je_1, e_2 \rangle = 0$, and so M^2 is totally real. \square

Remark 3.3.4. For $c = 0$, every PMC submanifold of a complex n -dimensional Euclidean space \mathbb{C}^n is biconservative, but not necessarily totally real. For instance, $\mathbb{S}^2(1) \subset \mathbb{E}^3 \subset \mathbb{C}^2$ is PMC and biconservative in \mathbb{C}^2 but not totally real, where \mathbb{E}^3 is the real 3-dimensional Euclidean space, as we will see in the following example.

Example 3.3.5. Consider the following embeddings $\mathbb{S}^2(1) \subset \mathbb{E}^3 \subset \mathbb{C}^2$, where $\mathbb{S}^2(1)$ is a Euclidean sphere of radius 1 and \mathbb{E}^3 is the real 3-dimensional Euclidean space. The sphere $\mathbb{S}^2(1)$ is PMC and biconservative in \mathbb{C}^2 but not totally real.

Indeed, first we need to show that $\mathbb{S}^2(1)$ is a PMC submanifold in \mathbb{C}^2 , i.e. we need to prove that $\langle \nabla_X^{\mathbb{C}^2} H, \eta \rangle = 0$, for all η , where η is a normal vector field of $\mathbb{S}^2(1)$ in \mathbb{C}^2 . As \mathbb{E}^3 is totally geodesic in \mathbb{E}^4 and $\mathbb{S}^2(1)$ is CMC in \mathbb{E}^3 , we have

$$\begin{aligned} \langle \nabla_X^{\mathbb{C}^2} H, \eta \rangle &= \langle \nabla_X^{\mathbb{E}^4} H, \eta \rangle \\ &= \langle \nabla_X^{\mathbb{E}^3} H, \eta \rangle \\ &= -\langle A_H X, \eta \rangle \\ &= 0. \end{aligned}$$

Now to show that $\mathbb{S}^2(1)$ is not totally real in \mathbb{C}^2 , we proceed as follows. Let $(U_N, (u, v))$ be the stereographic projection from the north pole which is defined by

$$S(u, v) := \left(\frac{2u}{1+u^2+v^2}, \frac{2v}{1+u^2+v^2}, \frac{-1+u^2+v^2}{1+u^2+v^2} \right).$$

Let X and Y be two tangent vector fields to $\mathbb{S}^2(1)$ such that

$$\begin{aligned} X = X(u, v) &= \left(\frac{\partial}{\partial u} \right)_{(u,v)} \\ &= \frac{2}{(u^2+v^2+1)^2} (v^2 - u^2 + 1, -2uv, 2u), \end{aligned}$$

and

$$\begin{aligned} Y = Y(u, v) &= \left(\frac{\partial}{\partial v} \right)_{(u,v)} \\ &= \frac{2}{(u^2+v^2+1)^2} (-2uv, u^2 - v^2 + 1, 2v). \end{aligned}$$

Clearly X and Y are orthogonal to each other, and we can write X and Y in \mathbb{E}^4 as

$$X = \alpha(v^2 - u^2 + 1, -2uv, 2u, 0) \quad \text{and} \quad Y = \alpha(-2uv, u^2 - v^2 + 1, 2v, 0),$$

where $\alpha = 2/(u^2 + v^2 + 1)^2$ is positive. We have

$$JX = \alpha(2uv, v^2 - u^2 + 1, 0, 2u),$$

thus

$$\begin{aligned} \langle JX, Y \rangle &= \alpha^2(-4u^2v^2 + (u^2 - v^2 + 1)(-u^2 + v^2 + 1)) \\ &= \alpha^2(-2u^2v^2 - u^4 - v^4 + 1) \\ &= \alpha^2(1 - (u^2 + v^2)^2). \end{aligned}$$

Since $\langle JX, Y \rangle \neq 0$, except when $1 = u^2 + v^2$, for all u and v we conclude that $\mathbb{S}^2(1)$ is not totally real in \mathbb{C}^2 .

Concerning slant surfaces (see [21]), we have the following non-existence result, a direct application of Theorem 3.3.1.

Corollary 3.3.6. *A PMC proper slant surface in a non-flat complex space form $N^n(c)$ cannot be biconservative.*

Proof. Recall that a proper slant surface is neither complex nor totally real. Now, as $c \neq 0$, and our surface is PMC by Theorem 3.3.1 we obtain that the surface is not biconservative. \square

The proof of Theorem 3.3.1 also yields the following general result.

Theorem 3.3.7. *Let M^2 be a PMC surface in a complex space form $N^n(c)$. Then $JT \in C(NM^2)$ if and only if M^2 is totally real.*

Proof. We have seen in the proof of Theorem 3.3.1 that the relation (3.3.1) implies (3.3.8) and further, it implies that M^2 is totally real regardless of whether $c = 0$ or not. \square

Now we recall the following result for surfaces.

Theorem 3.3.8. [72] *Let M^2 be a complete CMC biconservative surface in a Riemannian manifold N^n , with Gaussian curvature K . Assume that $K \geq 0$ and $\text{Riem}^N \leq K_0$, where K_0 is a constant. Then $\nabla A_H = 0$ and M^2 is either flat or pseudo-umbilical.*

By Theorem 3.3.1, a PMC totally real surface in a complex space form $N^n(c)$ is biconservative. Moreover, a complex space form has sectional curvature bounded by $c/4$ and c :

Indeed, let \bar{X}_1 , and \bar{X}_2 be two unit vectors tangent to $N^n(c)$ such that $\bar{X}_1 \perp \bar{X}_2$. We have

$$\begin{aligned} \text{Riem}^N(\bar{X}_1, \bar{X}_2) = \langle \bar{R}(\bar{X}_1, \bar{X}_2)\bar{X}_2, \bar{X}_1 \rangle &= \frac{c}{4} \left\{ \langle \bar{X}_2, \bar{X}_2 \rangle \langle \bar{X}_1, \bar{X}_1 \rangle - \langle \bar{X}_1, \bar{X}_2 \rangle \langle \bar{X}_2, \bar{X}_1 \rangle \right. \\ &\quad \left. + \langle J\bar{X}_2, \bar{X}_2 \rangle \langle J\bar{X}_1, \bar{X}_1 \rangle - \langle J\bar{X}_1, \bar{X}_2 \rangle \langle J\bar{X}_2, \bar{X}_1 \rangle \right. \\ &\quad \left. + 2\langle J\bar{X}_2, \bar{X}_1 \rangle \langle J\bar{X}_2, \bar{X}_1 \rangle \right\} \\ &= \frac{c}{4} \left\{ 1 + 3\langle J\bar{X}_2, \bar{X}_1 \rangle^2 \right\}. \end{aligned}$$

Using Cauchy-Schwartz Inequality we have

$$\langle J\bar{X}_2, \bar{X}_1 \rangle^2 \leq |J\bar{X}_2|^2 |\bar{X}_1|^2 = 1,$$

and conclude.

Therefore, we get

Corollary 3.3.9. *Let M^2 be a complete PMC totally real surface with $K \geq 0$ in a complex space form $N^n(c)$. Then $\nabla A_H = 0$, and M^2 is either flat or pseudo-umbilical.*

We recall that

Theorem 3.3.10. [31, Theorem 5.4] *Let M^2 be a compact totally real surface immersed in $N^n(c)$. If*

1. *the Gauss curvature does not change sign,*
2. *there exists a parallel umbilic-free isoperimetric unit normal vector field,*

then M^2 is flat.

We can see that our Corollary 3.3.9 is similar to [31, Theorem 5.4], but here M^2 is only complete and not necessarily compact, and the proof of Theorem 3.3.8 relies on a different technique.

But, PMC totally real surfaces in complex space forms have even more specific properties.

Theorem 3.3.11. *Let M^2 be a PMC totally real surface in the complex space form $N^n(c)$ with Gaussian curvature K . Then $\nabla T = A_N$ and*

$$-\frac{1}{2}\Delta|T|^2 = K|T|^2 + |A_N|^2.$$

Proof. It is well-known that

$$-\frac{1}{2}\Delta|T|^2 = \langle \text{trace } \nabla^2 T, T \rangle + |\nabla T|^2. \quad (3.3.12)$$

Now, since M^2 is totally real, taking the inner product of (3.3.3) with Y , we obtain

$$-\langle JA_H X, Y \rangle = \langle \nabla_X T, Y \rangle + \langle B(X, T), Y \rangle - \langle A_N X, Y \rangle + \langle \nabla_X^\perp N, Y \rangle.$$

Thus, it follows that

$$\langle \nabla_X T, Y \rangle = \langle A_N X, Y \rangle$$

for any vector fields X, Y tangent to M^2 , and then

$$\nabla_X T = A_N X,$$

that is $\nabla T = A_N$, and therefore

$$|\nabla T| = |A_N|. \quad (3.3.13)$$

Now, we compute the first term on the right-hand side of Equation (3.3.12), and prove that

$$\langle \text{trace } \nabla^2 T, T \rangle = K|T|^2. \quad (3.3.14)$$

To begin, we note that, from the decomposition of JH , we have

$$0 = \langle H, JH \rangle = \langle H, T + N \rangle = \langle H, N \rangle,$$

thus $N \perp H$. Then

$$\begin{aligned} \text{trace } A_N &= \sum_{i=1}^2 \langle A_N E_i, E_i \rangle \\ &= \sum_{i=1}^2 \langle B(E_i, E_i), N \rangle \\ &= \langle \text{trace } B, N \rangle \\ &= 2\langle H, N \rangle = 0. \end{aligned} \quad (3.3.15)$$

Let $\{E_i\}_{i=1}^2$ be a local (positive) geodesic orthonormal frame field at $p \in M^2$. Then, at p , we have

$$\begin{aligned} \langle \text{trace } \nabla^2 T, T \rangle &= \langle \nabla_{E_i} \nabla_{E_i} T, T \rangle \\ &= \langle \nabla_{E_i} A_N E_i, T \rangle \\ &= \langle (\nabla_{E_i} A_N) E_i, T \rangle \\ &= \langle (\nabla_{E_i} A_N) T, E_i \rangle, \end{aligned} \quad (3.3.16)$$

where we used the fact that $\langle (\nabla_X A_N) \cdot, \cdot \rangle$ is symmetric. Indeed, $\langle (\nabla_X A_N) \cdot, \cdot \rangle$ is symmetric since

$$\begin{aligned}
\langle (\nabla_X A_N)Y, Z \rangle &= \langle \nabla_X A_N Y, Z \rangle - \langle A_N \nabla_X Y, Z \rangle \\
&= X \langle A_N Y, Z \rangle - \langle A_N Y, \nabla_X Z \rangle - \langle \nabla_X Y, A_N Z \rangle \\
&= X \langle Y, A_N Z \rangle - \langle A_N Y, \nabla_X Z \rangle - \langle \nabla_X Y, A_N Z \rangle \\
&= \langle Y, \nabla_X A_N Z \rangle - \langle A_N Y, \nabla_X Z \rangle \\
&= \langle Y, \nabla_X A_N Z \rangle - \langle Y, A_N \nabla_X Z \rangle \\
&= \langle Y, (\nabla_X A_N)Z \rangle.
\end{aligned}$$

The Codazzi Equation (1.3.7) becomes

$$(\nabla_X^\perp B)(Y, Z) - (\nabla_Y^\perp B)(X, Z) = (\bar{R}(X, Y)Z)^\perp = 0, \quad (3.3.17)$$

since M^2 is totally real.

In (3.3.17), we take $X = E_i$, $Y = E_j$ and $Z = E_k$. At p , we have

$$\begin{aligned}
\langle (\nabla_{E_i}^\perp B)(E_j, E_k), N \rangle &= \langle \nabla_{E_i}^\perp B(E_j, E_k), N \rangle \\
&= E_i \langle B(E_j, E_k), N \rangle - \langle B(E_j, E_k), \nabla_{E_i}^\perp N \rangle \\
&= E_i \langle A_N E_j, E_k \rangle - \langle B(E_j, E_k), \nabla_{E_i}^\perp N \rangle.
\end{aligned}$$

Then using Equation (3.3.3) we obtain

$$\begin{aligned}
\langle (\nabla_{E_i}^\perp B)(E_j, E_k), N \rangle &= \langle \nabla_{E_i}(A_N E_j), E_k \rangle \\
&\quad - \langle B(E_j, E_k), -J(A_H E_i) - \nabla_{E_i} T - B(E_i, T) + A_N E_i \rangle \\
&= \langle (\nabla_{E_i} A_N)E_j, E_k \rangle + \langle B(E_j, E_k), J(A_H E_i) \rangle \\
&\quad + \langle B(E_j, E_k), B(E_i, T) \rangle, \tag{3.3.18}
\end{aligned}$$

and similarly

$$\begin{aligned}
\langle (\nabla_{E_j}^\perp B)(E_i, E_k), N \rangle &= \langle (\nabla_{E_j} A_N)E_i, E_k \rangle + \langle B(E_i, E_k), J(A_H E_j) \rangle \\
&\quad + \langle B(E_i, E_k), B(E_j, T) \rangle. \tag{3.3.19}
\end{aligned}$$

From Equations (3.3.17)–(3.3.19) we have

$$\begin{aligned}
\langle (\nabla_{E_i} A_N)E_j, E_k \rangle - \langle (\nabla_{E_j} A_N)E_i, E_k \rangle &= \langle B(E_j, T), B(E_i, E_k) \rangle \\
&\quad - \langle B(E_i, T), B(E_j, E_k) \rangle \\
&\quad + \langle B(E_i, E_k), J(A_H E_j) \rangle \\
&\quad - \langle B(E_j, E_k), J(A_H E_i) \rangle. \tag{3.3.20}
\end{aligned}$$

Further, using the fact that M^2 is totally real, we deduce

$$\begin{aligned}
\langle B(E_i, E_k), J(A_H E_j) \rangle &= -\langle JB(E_i, E_k), A_H E_j \rangle \\
&= -\langle J(\bar{\nabla}_{E_i} E_k), A_H E_j \rangle \\
&= -\langle \bar{\nabla}_{E_i} J E_k, A_H E_j \rangle \\
&= \langle J E_k, \bar{\nabla}_{E_i} (A_H E_j) \rangle \\
&= -\langle J E_k, \bar{\nabla}_{E_i} \bar{\nabla}_{E_j} H \rangle \\
&= \langle E_k, J(\bar{\nabla}_{E_i} \bar{\nabla}_{E_j} H) \rangle,
\end{aligned}$$

where we also used $\nabla^\perp H = 0$. Similarly,

$$\langle B(E_j, E_k), J(A_H E_i) \rangle = \langle E_k, J(\bar{\nabla}_{E_j} \bar{\nabla}_{E_i} H) \rangle,$$

and, therefore,

(3.3.21)

$$\langle B(E_i, E_k), J(A_H E_j) \rangle - \langle B(E_j, E_k), J(A_H E_i) \rangle = \langle E_k, J(\bar{R}(E_i, E_j)H) \rangle.$$

The Gauss Equation (1.3.3) yields

$$\begin{aligned}
\langle B(E_i, E_k), B(E_j, T) \rangle - \langle B(E_j, E_k), B(E_i, T) \rangle & \quad (3.3.22) \\
= \langle \bar{R}(E_i, E_j)E_k, T \rangle - \langle R(E_i, E_j)E_k, T \rangle.
\end{aligned}$$

Now, we compute the first term on the right-hand side of (3.3.22) and the curvature term in (3.3.21). We have

$$\begin{aligned}
\langle \bar{R}(E_i, E_j)E_k, T \rangle &= \frac{c}{4} \left\{ \langle E_j, E_k \rangle \langle E_i, T \rangle - \langle E_i, E_k \rangle \langle E_j, T \rangle + \langle J E_j, E_k \rangle \langle J E_i, T \rangle \right. \\
&\quad \left. - \langle J E_i, E_k \rangle \langle J E_j, T \rangle + 2 \langle J E_j, E_i \rangle \langle J E_k, T \rangle \right\} \\
&= \frac{c}{4} \left\{ \langle E_j, E_k \rangle \langle E_i, T \rangle - \langle E_i, E_k \rangle \langle E_j, T \rangle \right\},
\end{aligned}$$

and

$$\bar{R}(E_i, E_j)H = \frac{c}{4} \left\{ \langle J E_j, H \rangle J E_i - \langle J E_i, H \rangle J E_j \right\},$$

which shows that

$$\begin{aligned}
\langle J(\bar{R}(E_i, E_j)H), E_k \rangle &= \frac{c}{4} \left\{ \langle J E_i, H \rangle \langle E_j, E_k \rangle - \langle J E_j, H \rangle \langle E_i, E_k \rangle \right\} \\
&= \frac{c}{4} \left\{ \langle E_j, J H \rangle \langle E_i, E_k \rangle - \langle E_i, J H \rangle \langle E_j, E_k \rangle \right\} \\
&= \frac{c}{4} \left\{ \langle E_j, T \rangle \langle E_i, E_k \rangle - \langle E_i, T \rangle \langle E_j, E_k \rangle \right\}.
\end{aligned}$$

We note that

$$\langle \bar{R}(E_i, E_j)E_k, T \rangle + \langle J(\bar{R}(E_i, E_j)H), E_k \rangle = 0. \quad (3.3.23)$$

It easily follows from Equations (3.3.20)–(3.3.23) that

$$\begin{aligned} \langle (\nabla_{E_i} A_N) E_j, E_k \rangle - \langle (\nabla_{E_j} A_N) E_i, E_k \rangle &= \langle \bar{R}(E_i, E_j) E_k, T \rangle - \langle R(E_i, E_j) E_k, T \rangle \\ &\quad + \langle E_k, J\bar{R}(E_i, E_j) H \rangle. \\ &= -\langle R(E_i, E_j) E_k, T \rangle. \end{aligned}$$

In the above relation, because of its tensorial character, we can consider $E_j = T$. Then taking $k = i$, summing up over i and using (3.3.15), we get

$$\begin{aligned} \sum_{i=1}^2 \langle (\nabla_{E_i} A_N) T, E_i \rangle &= \sum_{i=1}^2 \left\{ T \langle A_N E_i, E_i \rangle - \langle R(E_i, T) E_i, T \rangle \right\} \\ &= T(\text{trace } A_N) - \sum_{i=1}^2 \langle R(E_i, T) E_i, T \rangle \\ &= |T|^2 K. \end{aligned} \tag{3.3.24}$$

Thus, from Equations (3.3.16) and (3.3.24), we obtain Equation (3.3.14).

Finally, we use (3.3.13) and (3.3.14) to conclude. \square

Theorem 3.3.12. *Let M^2 be a complete PMC totally real surface with $K \geq 0$ in a complex space form $N^n(c)$. Then $\nabla T = A_N = 0$ and either $K = 0$ or $K > 0$ at some point and $T = 0$ on M^2 .*

Proof. As $|T|^2 \leq |JH|^2 = |H|^2$ and M^2 is CMC, we have that $|T|^2$ is a bounded function on M^2 . Further, since $\Delta|T|^2 \leq 0$, $|T|^2$ is a subharmonic function and it follows that $|T|^2$ is constant (48). Thus, $K|T|^2 + |A_N|^2 = 0$ on M^2 , and so $|A_N|^2 = 0$ and $K|T|^2 = 0$. Therefore $A_N = \nabla T = 0$, and either $K = 0$ (everywhere) or $K > 0$ at some point and $T = 0$. \square

From Corollary 3.3.9 and Theorem 3.3.12 we have the following direct result.

Corollary 3.3.13. *Let M^2 be a complete PMC totally real surface with $K \geq 0$ in a complex space form $N^n(c)$. Then*

$$\nabla A_H = \nabla T = A_N = 0,$$

and either M^2 is flat or pseudo-umbilical with $T = 0$. In the latter case, $n > 2$.

3.4 CMC biconservative surfaces in $N^2(c)$

Consider a CMC surface M^2 in a complex space form $N^2(c)$ of complex dimension 2, with $c \neq 0$. Let $\{E_3 = H/|H|, E_4\}$ be the global orthonormal frame field in the normal

bundle NM^2 , and $\{E_1, E_2\}$ a local positive orthonormal frame field tangent to M^2 . Then the frame field $\{E_1, E_2, E_3, E_4\}$ along M^2 can be extended to a local orthonormal frame field defined on an open subset of $N^2(c)$ and tangent to $N^2(c)$.

Recall the following. Denote by ω_A^B the connection 1-forms corresponding to $\{E_1, E_2, E_3, E_4\}$, i.e. on the open subset of $N^2(c)$ containing M^2 , we have

$$\bar{\nabla} \cdot E_A = \omega_A^B(\cdot) E_B,$$

and by $\{\omega^1, \omega^2, \omega^3, \omega^4\}$ the dual basis of $\{E_1, E_2, E_3, E_4\}$. It follows that on M^2 the following relations hold

$$\nabla_{E_1}^\perp E_3 = \omega_3^4(E_1) E_4, \quad \nabla_{E_2}^\perp E_3 = \omega_3^4(E_2) E_4. \quad (3.4.1)$$

Proposition 3.4.1. *There exists no pseudo-umbilical CMC surface M^2 in a complex space form $N^2(c)$.*

Proof. First, we will prove that a pseudo-umbilical CMC surface M^2 in a complex space form $N^2(c)$ has to be PMC. But then, as we saw before, according to N. Sato [90] such a surface cannot exist.

By definition of the curvature tensor field and the fact that M^2 is pseudo-umbilical, we get

$$\begin{aligned} \bar{R}(X, Y)H &= \bar{\nabla}_X \bar{\nabla}_Y H - \bar{\nabla}_Y \bar{\nabla}_X H - \bar{\nabla}_{[X, Y]} H \\ &= \bar{\nabla}_X (\nabla_Y^\perp H - A_H Y) - \bar{\nabla}_Y (\nabla_X^\perp H - A_H X) - \nabla_{[X, Y]}^\perp H + A_H [X, Y] \\ &= \bar{\nabla}_X \nabla_Y^\perp H - \bar{\nabla}_X (|H|^2 Y) - \bar{\nabla}_Y \nabla_X^\perp H + \bar{\nabla}_Y (|H|^2 X) - \nabla_{[X, Y]}^\perp H \\ &\quad + |H|^2 [X, Y] \\ &= |H|^2 (\bar{\nabla}_Y X - \bar{\nabla}_X Y + [X, Y]) + \bar{\nabla}_X \nabla_Y^\perp H - \bar{\nabla}_Y \nabla_X^\perp H - \nabla_{[X, Y]}^\perp H \\ &= \nabla_X^\perp \nabla_Y^\perp H - A_{\nabla_Y^\perp H} X - \nabla_Y^\perp \nabla_X^\perp H + A_{\nabla_X^\perp H} Y - \nabla_{[X, Y]}^\perp H \\ &= A_{\nabla_X^\perp H} Y - A_{\nabla_Y^\perp H} X + R^\perp(X, Y)H, \end{aligned} \quad (3.4.2)$$

for any X, Y tangent to M^2 .

Now, the Ricci Equation (1.3.10) and M^2 pseudo-umbilical imply

$$\begin{aligned} \langle R^\perp(X, Y)H, V \rangle &= \langle [A_H, A_V]X, Y \rangle + \langle \bar{R}(X, Y)H, V \rangle \\ &= \langle \bar{R}(X, Y)H, V \rangle, \end{aligned}$$

and, from (3.4.2), we have

$$A_{\nabla_X^\perp H} Y = A_{\nabla_Y^\perp H} X, \quad (3.4.3)$$

for any X and Y tangent to M^2 .

From (3.4.1) for $X = E_1$ and $Y = E_2$, we obtain

$$\begin{aligned} A_{|H|\nabla_{E_1}^\perp E_3} E_2 &= |H|A_{\omega_3^4(E_1)E_4} E_2 \\ &= |H|\omega_3^4(E_1)A_4 E_2 \end{aligned}$$

and

$$\begin{aligned} A_{|H|\nabla_{E_2}^\perp E_3} E_1 &= |H|A_{\omega_3^4(E_2)E_4} E_1 \\ &= |H|\omega_3^4(E_2)A_4 E_1. \end{aligned}$$

Then, by (3.4.3) we obtain

$$\omega_3^4(E_1)A_4 E_2 = \omega_3^4(E_2)A_4 E_1, \quad (3.4.4)$$

where $A_i = A_{E_i}$, $i \in \{3, 4\}$.

Assume that $\nabla^\perp H \neq 0$. Then there exists an open subset of M^2 where $\nabla^\perp H \neq 0$ at any point, i.e. $\omega_3^4 \neq 0$ at any point (since $\nabla^\perp H = |H|\nabla^\perp E_3 = |H|\omega_3^4(\cdot)E_4$), and we will work on that subset. For the sake of simplicity, we can assume that this subset is the whole manifold M^2 .

Let

$$A_4 = \begin{pmatrix} \mu_1 & \mu_0 \\ \mu_0 & \mu_2 \end{pmatrix},$$

with respect to $\{E_1, E_2\}$. Since

$$\text{trace } A_4 = \sum_{i=1}^2 \langle A_4 E_i, E_i \rangle = \sum_{i=1}^2 \langle B(E_i, E_i), E_4 \rangle = 2\langle H, E_4 \rangle = 0,$$

we obtain $\mu_2 = -\mu_1$, and therefore

$$A_4 = \begin{pmatrix} \mu_1 & \mu_0 \\ \mu_0 & -\mu_1 \end{pmatrix}.$$

By Equation (3.4.4), we obtain

$$\omega_3^4(E_1)\{\mu_0 E_1 - \mu_1 E_2\} = \omega_3^4(E_2)\{\mu_1 E_1 + \mu_0 E_2\},$$

that is

$$\begin{cases} \omega_3^4(E_1)\mu_0 - \omega_3^4(E_2)\mu_1 &= 0 \\ \omega_3^4(E_2)\mu_0 + \omega_3^4(E_1)\mu_1 &= 0. \end{cases}$$

Since $|\nabla^\perp H|^2 = |H|^2 \{(\omega_3^4(E_2))^2 + (\omega_3^4(E_1))^2\} = |H|^2 |\omega_3^4|^2 > 0$, we obtain $\mu_0 = \mu_1 = 0$, and then $\mu_2 = 0$. Thus, $A_4 = 0$ on M^2 .

Now, since M^2 is pseudo-umbilical, i.e. $A_H = |H|^2I$ then $A_3 = |H|I$. We have on M^2 :

$$B(E_1, E_1) = \langle A_3 E_1, E_1 \rangle E_3 + \langle A_4 E_1, E_1 \rangle E_4 = |H|E_3.$$

Similarly, for $B(E_2, E_2)$ and $B(E_1, E_2)$, we obtain

$$B(E_2, E_2) = |H|E_3 \quad \text{and} \quad B(E_1, E_2) = 0.$$

Since M^2 is CMC and biconservative,

$$\text{trace } A_{\nabla^\perp H}(\cdot) + \text{trace}(\overline{R}(\cdot, H)\cdot)^\top = 0.$$

On the other hand, as $A_4 = 0$, we get

$$\text{trace } A_{\nabla^\perp H}(\cdot) = |H|\omega_3^4(E_1)A_4E_1 + |H|\omega_3^4(E_2)A_4E_2 = 0.$$

Therefore, $\text{trace}(\overline{R}(\cdot, H)\cdot)^\top = 0$, which shows that, as $c \neq 0$, $(JT)^\top = 0$.

Next, we will again use the Codazzi Equation. From Equation (3.2.1) we have

$$\begin{aligned} (\overline{R}(X, Y)Z)^\perp &= \frac{c}{4} \left\{ \langle Y, Z \rangle X - \langle X, Z \rangle Y + \langle JY, Z \rangle JX - \langle JX, Z \rangle JY \right. \\ &\quad \left. + 2\langle JY, X \rangle JZ \right\}^\perp, \\ &= \frac{c}{4} \left\{ \langle JY, Z \rangle (JX)^\perp - \langle JX, Z \rangle (JY)^\perp + 2\langle JY, X \rangle (JZ)^\perp \right\}, \end{aligned}$$

and if

(i) $X = Z = E_1, Y = E_2$, we obtain

$$\begin{aligned} \langle (\overline{R}(E_1, E_2)E_1)^\perp, E_3 \rangle &= \frac{3c}{4} \langle JE_2, E_1 \rangle \langle JE_1, E_3 \rangle = \frac{3c}{4|H|} \langle JE_2, E_1 \rangle \langle JE_1, H \rangle \\ &= -\frac{3c}{4|H|} \langle JE_2, E_1 \rangle \langle E_1, JH \rangle = -\frac{3c}{4|H|} \langle JE_2, E_1 \rangle \langle E_1, T \rangle \\ &= -\frac{3c}{4|H|} \langle JE_2, T \rangle = -\frac{3c}{4|H|} \langle E_2, JT \rangle \\ &= 0 \end{aligned}$$

since JT is normal. The same way we obtain

$$\langle (\overline{R}(X, Y)Z)^\perp, V \rangle = 0 \tag{3.4.5}$$

in all the following cases:

(ii) $X = Z = E_1, Y = E_2, V = E_4$,

(iii) $X = E_1, Y = Z = E_2, V = E_3$,

(iv) $X = E_1, Y = Z = E_2, V = E_4$.

From the Codazzi Equation (1.3.7) and Equation (3.4.5), we get

$$(\nabla_X^\perp B)(Y, Z) = (\nabla_Y^\perp B)(X, Z).$$

If $X = Z = E_1$, $Y = E_2$, we obtain

$$(\nabla_{E_1}^\perp B)(E_2, E_1) = (\nabla_{E_2}^\perp B)(E_1, E_1)$$

thus

$$\begin{aligned} & \nabla_{E_1}^\perp B(E_2, E_1) - B(\nabla_{E_1} E_2, E_1) - B(E_2, \nabla_{E_1} E_1) = \\ & \nabla_{E_2}^\perp B(E_1, E_1) - B(\nabla_{E_2} E_1, E_1) - B(E_1, \nabla_{E_2} E_1) \end{aligned}$$

which implies

$$\begin{aligned} & -B(\omega_2^1(E_1)E_1, E_1) - B(E_2, \omega_1^2(E_1)E_2) = \\ & |H|\omega_3^4(E_1)E_4 - B(\omega_1^2(E_1)E_2, E_1) - B(E_1, \omega_1^2(E_2)E_2). \end{aligned}$$

Then $|H|\omega_3^4(E_1) = 0$.

Similarly, for X, Y, Z and V in any of the above cases, we get

$$|H|\omega_3^4(E_2) = 0 \quad \text{and} \quad |H|\omega_3^4(E_1) = 0,$$

that is $\omega_3^4 = 0$, which is a contradiction. \square

Proposition 3.4.2. *Let M^2 be a CMC biconservative surface with no pseudo-umbilical points in a complex space form $N^2(c)$, with $c \neq 0$. If JT is normal, then M^2 is PMC.*

Proof. Let $\{\lambda_1, \lambda_2\}$ be the smooth eigenvalue functions of A_3 on M^2 and consider $\{E_1, E_2\}$ such that

$$\lambda_1 < \lambda_2, \quad A_3 E_1 = \lambda_1 E_1 \quad \text{and} \quad A_3 E_2 = \lambda_2 E_2.$$

We note that $\text{trace } A_4 = 0$.

Assume that $\nabla^\perp H \neq 0$. We will work on an open subset of M^2 such that at any point $\nabla^\perp H \neq 0$. For the sake of simplicity, we assume this open subset to be the whole of M^2 .

As M^2 is CMC and JT is normal, the biconservativity condition

$$2 \text{trace } A_{\nabla^\perp H}(\cdot) + \text{grad}(|H|^2) + 2 \text{trace}(\overline{R}(\cdot, H)\cdot)^\top = 0$$

reduces to

$$\text{trace } A_{\nabla^\perp H}(\cdot) = 0. \tag{3.4.6}$$

Since

$$\begin{aligned} \text{trace } A_{\nabla^\perp E_3}(\cdot) &= A_{\nabla^\perp_{E_1} E_3} E_1 + A_{\nabla^\perp_{E_2} E_3} E_2 \\ &= A_{\omega_3^4(E_1)E_4} E_1 + A_{\omega_3^4(E_2)E_4} E_2 \\ &= \omega_3^4(E_1)A_4 E_1 + \omega_3^4(E_2)A_4 E_2, \end{aligned}$$

Equation (3.4.6) can be re-written as

$$\begin{cases} \omega_3^4(E_1)\langle A_4 E_1, E_1 \rangle + \omega_3^4(E_2)\langle A_4 E_2, E_1 \rangle = 0 \\ \omega_3^4(E_1)\langle A_4 E_1, E_2 \rangle + \omega_3^4(E_2)\langle A_4 E_2, E_2 \rangle = 0. \end{cases} \quad (3.4.7)$$

Since $\nabla^\perp H \neq 0$, i.e.

$$|\nabla^\perp E_3|^2 = (\omega_3^4(E_1))^2 + (\omega_3^4(E_2))^2 > 0,$$

System (3.4.7) admits a non-trivial solution. Therefore, its determinant is zero, i.e.

$$\begin{aligned} 0 &= \langle A_4 E_1, E_1 \rangle \langle A_4 E_2, E_2 \rangle - (\langle A_4 E_1, E_2 \rangle)^2 \\ &= -(\langle A_4 E_1, E_1 \rangle)^2 - (\langle A_4 E_1, E_2 \rangle)^2 \\ &= -|A_4 E_1|^2, \end{aligned} \quad (3.4.8)$$

where in the second equality we used $\text{trace } A_4 = 0$. Moreover, since

$$\begin{aligned} |A_4 E_1|^2 &= (\langle A_4 E_1, E_1 \rangle)^2 + (\langle A_4 E_1, E_2 \rangle)^2 \\ &= (\langle A_4 E_2, E_2 \rangle)^2 + (\langle E_1, A_4 E_2 \rangle)^2 \\ &= |A_4 E_2|^2, \end{aligned}$$

from (3.4.8) we get

$$|A_4|^2 = |A_4 E_1|^2 + |A_4 E_2|^2 = 2|A_4 E_1|^2 = 0,$$

that is $A_4 = 0$ on M^2 .

The second fundamental form B of M^2 is given by

$$B(E_1, E_1) = \lambda_1 E_3, \quad B(E_2, E_2) = \lambda_2 E_3 \quad \text{and} \quad B(E_1, E_2) = 0,$$

and therefore, $2H = (\lambda_1 + \lambda_2)E_3$ and $\lambda_1 + \lambda_2 = 2|H|$ is a non-zero constant.

Next, we will use again the Codazzi Equation. From Equation (3.2.1) we have

$$(\bar{R}(X, Y)Z)^\perp = \frac{c}{4} \{ \langle JY, Z \rangle (JX)^\perp - \langle JX, Z \rangle (JY)^\perp + 2\langle JY, X \rangle (JZ)^\perp \},$$

and if

(i) $X = Z = E_1, Y = E_2$, we obtain

$$\begin{aligned}
\langle (\overline{R}(E_1, E_2)E_1)^\perp, E_3 \rangle &= \frac{3c}{4} \langle JE_2, E_1 \rangle \langle JE_1, E_3 \rangle = \frac{3c}{4|H|} \langle JE_2, E_1 \rangle \langle JE_1, H \rangle \\
&= -\frac{3c}{4|H|} \langle JE_2, E_1 \rangle \langle E_1, JH \rangle = -\frac{3c}{4|H|} \langle JE_2, E_1 \rangle \langle E_1, T \rangle \\
&= -\frac{3c}{4|H|} \langle JE_2, T \rangle = -\frac{3c}{4|H|} \langle E_2, JT \rangle \\
&= 0
\end{aligned}$$

since JT is normal. Similarly,

$$\langle (\overline{R}(X, Y)Z)^\perp, V \rangle = 0 \quad (3.4.9)$$

for

(ii) $X = E_1, Y = Z = E_2, V = E_3$,

(iii) $X = Z = E_1, Y = E_2, V = E_4$,

(iv) $X = E_1, Y = Z = E_2, V = E_4$.

If $X = Z = E_1, Y = E_2$ and $V = E_3$ using the Codazzi Equation (1.3.7) we have

$$\langle (\nabla_{E_1}^\perp B)(E_2, E_1), E_3 \rangle = \langle (\nabla_{E_2}^\perp B)(E_1, E_1), E_3 \rangle$$

thus

$$\begin{aligned}
&\langle \nabla_{E_1}^\perp B(E_2, E_1), E_3 \rangle - \langle B(\nabla_{E_1} E_2, E_1), E_3 \rangle - \langle B(E_2, \nabla_{E_1} E_1), E_3 \rangle = \\
&\langle \nabla_{E_2}^\perp B(E_1, E_1), E_3 \rangle - \langle B(\nabla_{E_2} E_1, E_1), E_3 \rangle - \langle B(E_1, \nabla_{E_2} E_1), E_3 \rangle
\end{aligned}$$

which implies that

$$\begin{aligned}
&E_1 \langle B(E_2, E_1), E_3 \rangle - \langle B(E_2, E_1), \nabla_{E_1}^\perp E_3 \rangle - \langle B(\nabla_{E_1} E_2, E_1), E_3 \rangle \\
&- \langle B(E_2, \nabla_{E_1} E_1), E_3 \rangle = \\
&E_2 \langle B(E_1, E_1), E_3 \rangle - \langle B(E_1, E_1), \nabla_{E_2}^\perp E_3 \rangle - \langle B(\nabla_{E_2} E_1, E_1), E_3 \rangle \\
&- \langle B(E_1, \nabla_{E_2} E_1), E_3 \rangle.
\end{aligned}$$

Then

$$\begin{aligned}
-\langle B(\nabla_{E_1} E_2, E_1), E_3 \rangle - \langle B(E_2, \nabla_{E_1} E_1), E_3 \rangle &= E_2 \langle B(E_1, E_1), E_3 \rangle \\
&- \langle B(E_1, E_1), \nabla_{E_2}^\perp E_3 \rangle \\
&- 2 \langle B(\nabla_{E_2} E_1, E_1), E_3 \rangle \\
-\langle A_3 E_1, \nabla_{E_1} E_2 \rangle - \langle A_3 E_2, \nabla_{E_1} E_1 \rangle &= E_2 \langle \lambda_1 E_3, E_3 \rangle \\
&- \langle \lambda_1 E_3, \nabla_{E_2}^\perp E_3 \rangle \\
&- 2 \langle A_3 E_1, \nabla_{E_2} E_1 \rangle \\
-\langle \lambda_1 E_1, \omega_2^1(E_1) E_1 \rangle - \langle \lambda_2 E_2, \omega_1^2(E_1) E_2 \rangle &= E_2 (\lambda_1) \\
(\lambda_1 - \lambda_2) \omega_1^2(E_1) &= E_2 (\lambda_1) \quad (3.4.10)
\end{aligned}$$

Hence, similarly, for X, Y, Z and V as above in the previous cases (ii), (iii) and (iv), the Codazzi Equation (1.3.7) reads as follows:

- (2) $E_1(\lambda_2) = (\lambda_1 - \lambda_2)\omega_1^2(E_2)$,
- (3) $\lambda_1\omega_3^4(E_2) = 0$,
- (4) $\lambda_2\omega_3^4(E_1) = 0$.

Assume that $\omega_3^4(E_1) \neq 0$ at some point $p \in M^2$, then $\omega_3^4(E_1) \neq 0$ on a neighborhood of p . On this neighborhood, by (3.4) we have $\lambda_2 = 0$ and so $\lambda_1 = 2|H|$, which is a contradiction since $\lambda_1 < \lambda_2 = 0$.

If $\omega_3^4(E_2) \neq 0$, we get $\lambda_1 = 0$ on an open subset and so $\lambda_2 = 2|H|$. From (3.4.10) and (3.4), we obtain $\omega_1^2(E_1) = 0$ and $\omega_1^2(E_2) = 0$. We will use the same notation for ω_1^2 and its pull-back on M^2 . Therefore, on M^2 , $\omega_1^2 = 0$ and $\nabla_{E_i}E_j = 0$, for any $i, j \in \{1, 2\}$. Since the curvature of M^2 is given by $d\omega_1^2 = -K\omega^1 \wedge \omega^2$, we conclude that M^2 is flat.

Furthermore, from the Gauss Equation (1.3.3) with $X = W = E_1, Y = Z = E_2$ and the fact that M^2 is flat, we obtain

$$\frac{c}{4}\{1 + 3\langle JE_1, E_2 \rangle^2\} = -\lambda_1\lambda_2 = 0,$$

which is a contradiction, as $c \neq 0$.

Therefore $\nabla^\perp H = 0$. □

From Propositions 3.4.1 and 3.4.2, we get the following theorem.

Theorem 3.4.3. *Let M^2 be a CMC biconservative surface in a complex space form $N^2(c)$, with $c \neq 0$. If JT is normal, then M^2 is PMC.*

Proof. We know that, with standard notations,

$$\langle A_H \partial_z, \partial_z \rangle$$

is holomorphic [72]. Since M^2 cannot be pseudo-umbilical in $N^2(c)$, the set W of non pseudo-umbilical points is a dense open subset of M^2 .

From Proposition 3.4.2, we get that W is PMC in $N^2(c)$, and then, by continuity, M^2 is PMC. □

Remark 3.4.4. When the ambient space is a real space form of dimension 4, a similar result to Theorem 3.4.3 was obtained in [70, Theorem 5.1], where it was shown that if M^2 is a CMC biconservative surface in a space form of dimension 4 with non-zero constant sectional curvature, then M^2 is PMC. Thus, Theorem 3.4.3 is almost the same as [70, Theorem 5.1] except that in the case of complex space forms an extra condition on JT was added to conclude that M^2 is PMC.

In the following, we want to check whether the above result extends to the case $c = 0$, i.e. whether a CMC biconservative surface in $\mathbb{C}^2 = \mathbb{E}^4$ with JT normal is PMC. Equivalently, we investigate a CMC biconservative surface which is not PMC can have (JT) not normal, i.e. $(JT)^\top \neq 0$.

The parametric equations for CMC biconservative surfaces which are not PMC are already known [70], but, for the sake of completeness, we include the proof.

Proposition 3.4.5. [70] *Let M^2 be a non-PMC biconservative surface with constant mean curvature in \mathbb{E}^4 . Then, locally, the surface is given by*

$$X(u, v) = (\gamma^1(u), \gamma^2(u), \gamma^3(u), v), \quad (3.4.11)$$

where $\gamma : I \rightarrow \mathbb{E}^3$ is a curve in \mathbb{E}^3 parametrized by arc-length, with constant non-zero curvature, and non-zero torsion.

Proof. We use the local frame introduced at the beginning of this section. Since M^2 is not pseudo-umbilical and $\langle A_H(\partial_z), \partial_z \rangle$ is holomorphic, pseudo-umbilical points are isolated. Without loss of generality, we can assume that M^2 has no pseudo-umbilical points. Further, since $\nabla^\perp H \neq 0$, for simplicity, we can assume that $\nabla^\perp H \neq 0$ at any point of M^2 . As in the proof of Proposition 3.4.2 (since the ambient manifold is flat, Equation (3.4.6) still holds) we get $A_4 = 0$ and

$$B(E_1, E_1) = \lambda_1 E_3, \quad B(E_2, E_2) = \lambda_2 E_3 \quad \text{and} \quad B(E_1, E_2) = 0.$$

We have $\omega_3^4(E_2) \neq 0$, $\omega_3^4(E_1) = 0$, $\lambda_1 = 0$, $\lambda_2 = 2|H|$ which is a non-zero constant and $\omega_1^2 = 0$.

Then

$$\begin{aligned} \nabla_{E_1}^{\mathbb{E}^4} E_1 &= \nabla_{E_1} E_1 + B(E_1, E_1) = \omega_1^2(E_1) E_2 + \lambda_1 E_3 \\ &= 0, \end{aligned}$$

and

$$\begin{aligned} \nabla_{E_2}^{\mathbb{E}^4} E_1 &= \nabla_{E_2} E_1 + B(E_1, E_2) = \omega_1^2(E_2) E_2 \\ &= 0, \end{aligned}$$

thus

$$\nabla_{E_1}^{\mathbb{E}^4} E_1 = \nabla_{E_2}^{\mathbb{E}^4} E_1 = 0.$$

Hence, E_1 is the restriction to M^2 of a constant vector field of \mathbb{E}^4 , which up to an isometry of \mathbb{E}^4 , can be chosen as

$$E_1 = (0, 0, 0, 1).$$

Now, let σ be the integral curve of E_1 parametrized by arc-length that is

$$\sigma(v) = (b_1, b_2, b_3, b_4 + v), \quad b_1, b_2, b_3, b_4 \in \mathbb{R}.$$

Further, let $p_0 = (b_1, b_2, b_3, b_4) \in M^2$ be an arbitrary point, and let $\gamma = \gamma(u) = (\gamma^1(u), \gamma^2(u), \gamma^3(u), \gamma^4(u))$ be an integral curve of E_2 with $\gamma(0) = p_0$. Then let φ be the flow of E_1 and $(U; u, v)$ local coordinates around p_0 given by

$$X(u, v) = \varphi_{\gamma(u)}(v) = \varphi(\gamma(u), v) = (\gamma^1(u), \gamma^2(u), \gamma^3(u), \gamma^4(u) + v).$$

We have

$$X_u(u, 0) = \gamma'(u) = E_2(\gamma(u)),$$

and

$$X_v(u, v) = E_1(u, v) = (0, 0, 0, 1).$$

Since $X_u(u, 0)$ is orthogonal to $X_v(u, 0)$, we obtain $(\gamma^4)'(u) = 0$ for all u thus $\gamma^4 = b_4$, where b_4 is a constant. Therefore,

$$X(u, v) = (\gamma^1(u), \gamma^2(u), \gamma^3(u), b_4 + v).$$

We have

$$\begin{cases} \nabla_{E_2}^{\mathbb{R}^4} E_2 &= \nabla_{E_2} E_2 + B(E_2, E_2) = \omega_2^1(E_2)E_1 + \lambda_2 E_3 = \lambda_2 E_3 \\ \nabla_{E_2}^{\mathbb{R}^4} E_3 &= \nabla_{E_2}^\perp E_3 - A_3 E_2 = \omega_3^4(E_2)E_4 - A_3 E_2 = \omega_3^4(E_2)E_4 - \lambda_2 E_2 \\ \nabla_{E_2}^{\mathbb{R}^4} E_4 &= \nabla_{E_2}^\perp E_4 - A_4 E_2 = -\omega_3^4(E_2)E_3. \end{cases}$$

From the Ricci equation [\(1.3.10\)](#), we have

$$R^\perp(E_1, E_2)E_3 = 0,$$

which is equivalent to

$$E_1(\omega_3^4(E_2))E_4 - \omega_3^4(E_2)\omega_4^3(E_1)E_3 - E_2(\omega_3^4(E_1))E_4 - \omega_3^4(E_1)\omega_4^3(E_2)E_3 = 0.$$

Now, since $\omega_3^4(E_2) \neq 0$ and $\omega_3^4(E_1) = 0$ we obtain

$$E_1(\omega_3^4(E_2)) = 0,$$

that is $\omega_3^4(E_2)$ depends only on u . Thus, $\{E_2, E_3, E_4\}$ is the Frenet frame field along $\gamma(u)$, its curvature is given by $k = \lambda_2$ and the torsion is $\tau = \tau(u) = \omega_3^4(E_2)$. Moreover, since

$$\nabla_{E_2}^\perp H = |H| \nabla_{E_2}^\perp E_3 = \frac{k\tau}{2} E_4,$$

and the surface is not PMC, we conclude that $\tau \neq 0$ at any point.

Replacing the variable $b_4 + v$ by v , we conclude. \square

Now we will prove that any CMC biconservative surface in \mathbb{E}^4 which is not PMC has $(JT)^\top \neq 0$.

Proposition 3.4.6. *Let M^2 be a non-PMC biconservative surface with constant mean curvature in \mathbb{E}^4 . Then $(JT)^\top \neq 0$.*

Proof. Let $X(u, v) = (\gamma^1(u), \gamma^2(u), \gamma^3(u), v)$, where $\gamma : I \rightarrow \mathbb{E}^3 \equiv \mathbb{E}^3 \times \{0\} \subset \mathbb{E}^4$ is a curve parametrized by arc-length, i.e. $|\gamma'| = 1$, with a non-zero constant curvature κ , and $\tau \in C^\infty(I)$ is a non-zero function (we can assume that $\tau > 0$). We denote the Frenet frame field along γ by

$$\{\gamma'(u), \mathbf{n}(u), \mathbf{b}(u)\}, \quad u \in I.$$

We have

$$\begin{cases} X_u &= (\gamma', 0) = \gamma' \\ X_v &= e_4. \end{cases}$$

It is clear that \mathbf{n} and \mathbf{b} are orthogonal to γ' and e_4 , thus $\{\mathbf{n}, \mathbf{b}\}$ is an orthonormal frame field in the normal bundle of M^2 in \mathbb{E}^4 .

Also, we have $g_{11} = |X_u|^2 = 1$, $g_{12} = \langle X_u, X_v \rangle = 0$ and $g_{22} = |X_v|^2 = 1$.

$$\begin{cases} \nabla_{X_u}^{\mathbb{E}^4} X_u &= X_{uu} = \gamma'' = \kappa \mathbf{n} \\ \nabla_{X_u}^{\mathbb{E}^4} X_v &= \nabla_{X_v}^{\mathbb{E}^4} X_u = X_{uv} = X_{vu} = 0 \\ \nabla_{X_v}^{\mathbb{E}^4} X_v &= X_{vv} = 0, \end{cases}$$

then $B(X_u, X_u) = \kappa \mathbf{n}$, $B(X_u, X_v) = 0$ and $B(X_v, X_v) = 0$. We have

$$\begin{aligned} H &= \frac{1}{2} \text{trace } B = \frac{1}{2} \{B(X_u, X_u) + B(X_v, X_v)\} \\ &= \frac{1}{2} \kappa \mathbf{n} = \frac{1}{2} \gamma'', \end{aligned}$$

and, therefore,

$$JH = \frac{1}{2} J\gamma''.$$

Now, the tangential part of JH is given by

$$\begin{aligned} T &= (JH)^\top = \langle JH, X_u \rangle X_u + \langle JH, X_v \rangle X_v \\ &= \frac{1}{2} \{ \langle J\gamma'', \gamma' \rangle \gamma' + \langle J\gamma'', e_4 \rangle e_4 \}. \end{aligned}$$

We will prove that JT is not normal. For this purpose, we will assume that JT is normal and reach a contradiction.

Since JT is normal, we have

$$\langle JT, X_u \rangle = 0 \quad \text{and} \quad \langle JT, X_v \rangle = 0.$$

Thus

$$\begin{aligned}\langle JT, X_u \rangle = 0 &\Leftrightarrow \langle J\gamma'', \gamma' \rangle \langle J\gamma', \gamma' \rangle + \langle J\gamma'', e_4 \rangle \langle Je_4, \gamma' \rangle = 0 \\ &\Leftrightarrow \langle J\gamma'', e_4 \rangle \langle Je_4, \gamma' \rangle = 0,\end{aligned}$$

and

$$\begin{aligned}\langle JT, X_v \rangle = 0 &\Leftrightarrow \langle J\gamma'', \gamma' \rangle \langle J\gamma', e_4 \rangle + \langle J\gamma'', e_4 \rangle \langle Je_4, e_4 \rangle = 0 \\ &\Leftrightarrow \langle J\gamma'', \gamma' \rangle \langle J\gamma', e_4 \rangle = 0.\end{aligned}$$

Therefore, JT is normal if and only if

$$\langle J\gamma', e_4 \rangle = 0$$

on I , or

$$\langle J\gamma'', e_4 \rangle = 0 \quad \text{and} \quad \langle J\gamma'', \gamma' \rangle = 0.$$

We have, $J\gamma' = (-\gamma^2)', (\gamma^1)', 0, (\gamma^3)'$ and $J\gamma'' = (-\gamma^2)'', (\gamma^1)'', 0, (\gamma^3)''$, so

$$\langle J\gamma'', e_4 \rangle = 0 \Leftrightarrow (\gamma^3)'' = 0,$$

$$\langle J\gamma', e_4 \rangle = 0 \Leftrightarrow (\gamma^3)' = 0$$

and

$$\langle J\gamma'', \gamma' \rangle = 0 \Leftrightarrow -(\gamma^1)'(\gamma^2)'' + (\gamma^2)'(\gamma^1)'' = 0.$$

Assume that $\langle J\gamma', e_4 \rangle = 0$, i.e. $(\gamma^3)' = 0$. We obtain that γ^3 is constant, so γ is a plane curve, and therefore $\tau = 0$ which is a contradiction. Hence we are only left with the second case, i.e.

$$(\gamma^3)'' = 0 \quad \text{and} \quad -(\gamma^1)'(\gamma^2)'' + (\gamma^2)'(\gamma^1)'' = 0.$$

From $(\gamma^3)'' = 0$, we get $\gamma^3(u) = au + b$, for any $u \in I$ (or a smaller open interval), and from $|\gamma'| = 1$ we obtain $((\gamma^1)')^2 + ((\gamma^2)')^2 + a^2 = 1$. As $\tau \neq 0$, $a^2 \in (0, 1)$. Then, there exists a smooth function f such that

$$(\gamma^1)' = \sqrt{1 - a^2} \cos f \quad \text{and} \quad (\gamma^2)' = \sqrt{1 - a^2} \sin f$$

and

$$\begin{cases} (\gamma^1)'' &= -\sqrt{1 - a^2} f' \sin f \\ (\gamma^2)'' &= \sqrt{1 - a^2} f' \cos f. \end{cases}$$

Then, the condition

$$-(\gamma^1)'(\gamma^2)'' + (\gamma^2)'(\gamma^1)'' = 0$$

is equivalent to

$$-(1 - a^2)f' \cos^2 f - (1 - a^2)f' \sin^2 f = 0$$

which means

$$f = \text{constant.} \quad (3.4.12)$$

On the other hand,

$$\kappa^2 = |\gamma''|^2 = [(\gamma^1)']^2 + [(\gamma^2)']^2 = (1 - a^2)(f')^2.$$

As $\kappa > 0$ we obtain

$$(f')^2 > 0. \quad (3.4.13)$$

From (3.4.12) and (3.4.13) we get a contradiction. \square

In conclusion, Theorem 3.4.3 extends to the case $c = 0$ and we can state the following theorem.

Theorem 3.4.7. *Let M^2 be a CMC biconservative surface in a complex space form $N^2(c)$. If JT is normal, then M^2 is PMC and totally real.*

Proof. If $c \neq 0$ and JT is normal, by Theorem 3.4.3, we get that M^2 is PMC. Now, as M^2 is a PMC biconservative surface, by Theorem 3.3.1 we conclude that M^2 is totally real.

If $c = 0$ and JT is normal, by Proposition 3.4.6, we get that M^2 is PMC. Since M^2 is a PMC surface and JT is normal by Theorem 3.3.7 we conclude that M^2 is totally real. \square

Remark 3.4.8. The full classification of complete PMC surfaces in a complex space form $N^2(c)$ was achieved in [56-59] and, when $c > 0$, they are totally real flat tori.

3.5 Codimension reduction for biconservative surfaces

We recall that if M^2 is a PMC surface in a real Euclidean space \mathbb{E}^n , then it is biconservative, and it is either pseudo-umbilical (and lies as a minimal surface in a Euclidean hypersphere of \mathbb{E}^n), or lies as a CMC (possibly minimal) surface in a 3-dimensional sphere (and this sphere lies in \mathbb{E}^4), or lies as a CMC surface in \mathbb{E}^3 [26, 27, 96].

In this section we will assume that M^2 is a PMC totally real surface in a complex space form $N^n(c)$ of complex dimension n , n large enough, $c \neq 0$, with $H \neq 0$, to

obtain a reduction of the codimension. More precisely, we will reduce the complex dimension of the ambient space for non pseudo-umbilical PMC totally real surfaces to 4 (see Theorem 3.5.8). For this purpose, we will follow [39], where it was proved that a non pseudo-umbilical PMC surface in a complex space form $N^n(c)$, $c \neq 0$, lies in $N^5(c)$. The strategy for obtaining reduction results was initiated and developed in [2, 37, 38]. Our result extends [43] which required the stronger condition of biharmonicity. We mention that a reduction of codimension for totally real submanifolds of complex space forms, with parallel f-structure in the normal bundle, was obtained in [60].

We recall three results which play an essential role in the reduction of codimension in Theorem 3.5.8

Theorem 3.5.1. [37, Theorem 2] *Let N^n be a manifold with complete connection $\bar{\nabla}$ with parallel torsion T^N and curvature tensor R^N . Let $\varphi : M^m \rightarrow N^n$ be a smooth map such that the image of $d\varphi$ lies in a parallel subbundle $\tilde{L} \subset \varphi^{-1}(TN^n)$ which is invariant under T^N and R^N . Then there is a totally geodesic submanifold $N' \subset N^n$ with $\varphi(M^m) \subset N'$.*

Theorem 3.5.2. [38, Theorem 1] *Let $\varphi : M^2 \rightarrow N^n$ be an isometric immersion from a Riemannian surface into a Riemannian symmetric space. If φ has parallel mean curvature $H \neq 0$, one of the following conditions holds:*

1. φ is pseudo-umbilical;
2. the dimension d of $R_{\varphi(p)}^N(O_p^2)$ is independent of $p \in M^2$ and there exists an embedded totally geodesic submanifold $N' \subset N^n$, with dimension d , such that $\varphi(M^2) \subset N'$, where R^N is the curvature tensor field of N^n and $O_p^n(\varphi)$ denotes the n -th osculating space of φ at $p \in M^2$.

The most useful result for our work in this section is the following

Theorem 3.5.3. [38, Theorem 2] *Let $\varphi : M^m \rightarrow N^n$ be an isometric immersion from a Riemannian manifold into a Riemannian symmetric space. If there exists a parallel fiber bundle \tilde{L} over M^m such that $R^N(\tilde{L}) = \tilde{L}$ and $TM^m \subset \tilde{L}$, then there exists an embedded totally geodesic submanifold N' of N^n with $\varphi(M^m) \subset N'$ and $\tilde{L}_p = T_{\varphi(p)}N'$ for any $p \in M^m$.*

Now, we introduce our work concerning this section.

Lemma 3.5.4. *For any vector field V normal to M^2 and orthogonal to JTM^2 , we have $[A_H, A_V] = 0$, i.e. A_H commutes with A_V .*

Proof. Since M^2 is a PMC surface, from the Ricci Equation (1.3.10) we have

$$\langle R^\perp(X, Y)H, V \rangle = 0,$$

where X and Y are tangent to M^2 . As M^2 is totally real and V is orthogonal to JTM^2 , we obtain

$$\begin{aligned} \langle \bar{R}(X, Y)H, V \rangle &= \frac{c}{4} \left\{ \langle Y, H \rangle \langle X, V \rangle - \langle X, H \rangle \langle Y, V \rangle + \langle JY, H \rangle \langle JX, V \rangle \right. \\ &\quad \left. - \langle JX, H \rangle \langle JY, V \rangle + 2 \langle JY, X \rangle \langle JH, V \rangle \right\} \\ &= 0, \end{aligned}$$

therefore $[A_H, A_V] = 0$. □

Corollary 3.5.5. *At any point $p \in M^2$, either H is an umbilical direction, or there exists an orthonormal frame field $\{E_1, E_2\}$ around p that diagonalizes simultaneously A_H and A_V , for any vector field V normal to M^2 and orthogonal to JTM^2 .*

Proof. If at a point $p \in M^2$ the mean curvature vector field H is not an umbilical direction then, on an open subset containing p , H remains a non-umbilical direction, i.e. $A_H \neq |H|^2 I$ at any point of that open subset. Therefore, we have two distinct principal curvatures corresponding to A_H , each of multiplicity one, and there exists an orthonormal frame field $\{E_1, E_2\}$ around p that diagonalizes A_H . Working on that open subset, as $A_H A_V = A_V A_H$, we get that $\{E_1, E_2\}$ diagonalizes also A_V . □

Proposition 3.5.6. *Assume that H is nowhere an umbilical direction. Then there exists a parallel subbundle L of the normal bundle of real dimension less or equal to 6 that contains the image of the second fundamental form B .*

Proof. We define a subbundle of the normal bundle of M^2 in the complex space form $N^n(c)$ by

$$L = \text{span}\{\text{Im } B \cup (J \text{Im } B)^\perp \cup JTM^2\},$$

where $(J \text{Im } B)^\perp = \{(JB(X, Y))^\perp : X, Y \text{ tangent vector fields to } M^2\}$.

We prove that L is parallel by taking U a section in L and showing that, for any X , $\nabla_X^\perp U$ is also a section in L . This translates as $\langle \nabla_X^\perp U, V \rangle = 0$, for any V normal to M^2 and orthogonal to L ; equivalently, $\langle U, \nabla_X^\perp V \rangle = 0$.

Let V be a normal vector field orthogonal to L , i.e.

$$\langle V, B(X, Y) \rangle = \langle V, JB(X, Y) \rangle = \langle V, JX \rangle = 0,$$

for any X, Y tangent to M^2 .

Consider $\{E_1, E_2\}$ a local orthonormal frame field that diagonalizes simultaneously A_H and A_V (see Corollary 3.5.5). We want to prove that $\nabla_{E_k}^\perp V$ is orthogonal to $\text{Im } B$, $(J \text{Im } B)^\perp$ and JTM^2 .

To do this, we first prove that $\nabla_{E_k}^\perp V$ is orthogonal to JTM^2 :

$$\begin{aligned}
\langle JE_j, \nabla_{E_k}^\perp V \rangle &= -\langle \nabla_{E_k}^\perp JE_j, V \rangle \\
&= -\langle \bar{\nabla}_{E_k} JE_j, V \rangle - \langle A_{JE_j} E_k, V \rangle \\
&= -\langle J\bar{\nabla}_{E_k} E_j, V \rangle \\
&= -\langle J\nabla_{E_k} E_j, V \rangle - \langle JB(E_k, E_j), V \rangle \\
&= 0.
\end{aligned}$$

To prove that $\nabla_{E_k}^\perp V$ is orthogonal to $\text{Im } B$ we set

$$A_{ijk} = -\langle B(E_i, E_j), \nabla_{E_k}^\perp V \rangle = \langle \nabla_{E_k}^\perp B(E_i, E_j), V \rangle$$

and show that $A_{ijk} = 0$. As B is symmetric, $A_{ijk} = A_{jik}$.

Now, notice that

$$\begin{aligned}
\langle (\nabla_{E_k}^\perp B)(E_i, E_j), V \rangle &= \langle \nabla_{E_k}^\perp B(E_i, E_j), V \rangle - \langle B(\nabla_{E_k} E_i, E_j), V \rangle \\
&\quad - \langle B(E_i, \nabla_{E_k} E_j), V \rangle \\
&= \langle \nabla_{E_k}^\perp B(E_i, E_j), V \rangle \\
&= A_{ijk}.
\end{aligned}$$

Using Codazzi Equation (1.3.7) we have

$$\begin{aligned}
A_{ijk} &= \langle (\nabla_{E_k}^\perp B)(E_i, E_j), V \rangle \\
&= \langle (\nabla_{E_i}^\perp B)(E_k, E_j) + (\bar{R}(E_k, E_i)E_j)^\perp, V \rangle \\
&= \langle (\nabla_{E_i}^\perp B)(E_k, E_j), V \rangle \\
&= A_{kji},
\end{aligned}$$

hence $A_{ijk} = A_{kji} = A_{jki} = A_{ikj}$.

Next, since the normal vector field $\nabla_{E_k}^\perp V$ is orthogonal to JTM^2 , by Corollary 3.5.5, the basis $\{E_1, E_2\}$ also diagonalizes $A_{\nabla_{E_k}^\perp V}$, and for $i \neq j$

$$\begin{aligned}
A_{ijk} &= -\langle B(E_i, E_j), \nabla_{E_k}^\perp V \rangle = -\langle (A_{\nabla_{E_k}^\perp V} E_i), E_j \rangle = -\langle \lambda_i E_i, E_j \rangle \\
&= 0.
\end{aligned}$$

Hence, $A_{ijk} = 0$ whenever two indices differ.

Finally,

$$\begin{aligned}
A_{iii} &= -\langle B(E_i, E_i), \nabla_{E_i}^\perp V \rangle \\
&= -\langle 2H, \nabla_{E_i}^\perp V \rangle + \langle B(E_j, E_j), \nabla_{E_i}^\perp V \rangle \quad (j \neq i) \\
&= 2\langle \nabla_{E_i}^\perp H, V \rangle - A_{jji} \\
&= 0.
\end{aligned}$$

Thus $A_{ijk} = 0$ for any $i, j, k \in \{1, 2\}$.

Now, if V is normal to M^2 and orthogonal to L , JV must be normal and orthogonal to L . Further

$$\begin{aligned}
\langle (JB(E_i, E_j))^\perp, \nabla_{E_k}^\perp V \rangle &= -\langle \bar{\nabla}_{E_k}(JB(E_i, E_j))^\perp, V \rangle \\
&= -\langle \bar{\nabla}_{E_k} JB(E_i, E_j), V \rangle + \langle \bar{\nabla}_{E_k}(JB(E_i, E_j))^\top, V \rangle \\
&= \langle JA_{B(E_i, E_j)} E_k, V \rangle - \langle J\nabla_{E_k}^\perp B(E_i, E_j), V \rangle \\
&\quad + \langle B(E_k, (JB(E_i, E_j))^\top), V \rangle \\
&= \langle \nabla_{E_k}^\perp B(E_i, E_j), JV \rangle \\
&= 0.
\end{aligned}$$

Therefore, we conclude that L is parallel.

Finally, we need to prove that L has real dimension less or equal to 6. Indeed, since JE_1 and JE_2 are unit and orthogonal, we can consider a local orthonormal frame field

$$\{JE_1, JE_2, V_1, \dots, V_{2n-4}\},$$

in NM^2 . We have,

$$B(E_1, E_2) = \alpha JE_1 + \beta JE_2 + \gamma_1 V_1 + \dots + \gamma_{2n-4} V_{2n-4},$$

and then, for any $i = 1, \dots, 2n-4$,

$$\begin{aligned}
\gamma_i &= \langle B(E_1, E_2), V_i \rangle = \langle A_{V_i} E_1, E_2 \rangle \\
&= 0.
\end{aligned}$$

Therefore, $B(E_1, E_2) = \alpha JE_1 + \beta JE_2$ and $JB(E_1, E_2) = -\alpha E_1 - \beta E_2$.

Let $X, Y \in C(TM^2)$, we have

$$\begin{aligned}
B(X, Y) &= B(X^1 E_1 + X^2 E_2, Y^1 E_1 + Y^2 E_2) \\
&= X^1 Y^1 B(E_1, E_1) + (X^1 Y^2 + Y^1 X^2) B(E_1, E_2) + X^2 Y^2 B(E_2, E_2).
\end{aligned}$$

Then:

1. As J and $(\cdot)^\perp$ are linear operators, we have

$$\begin{aligned}
(JB(X, Y))^\perp &= X^1 Y^1 (JB(E_1, E_1))^\perp + (X^1 Y^2 + X^2 Y^1) (JB(E_1, E_2))^\perp \\
&\quad + X^2 Y^2 (JB(E_2, E_2))^\perp \\
&= X^1 Y^1 (JB(E_1, E_1))^\perp + (X^1 Y^2 + X^2 Y^1) (-\alpha E_1 - \beta E_2)^\perp \\
&\quad + X^2 Y^2 (JB(E_2, E_2))^\perp \\
&= X^1 Y^1 (JB(E_1, E_1))^\perp + X^2 Y^2 (JB(E_2, E_2))^\perp.
\end{aligned}$$

Thus, the real dimension of $(J \operatorname{Im} B)^\perp$ is at most 2.

2. Consider the normal vector $B(X, Y) + JZ$, we have

$$\begin{aligned}
B(X, Y) + JZ &= X^1 Y^1 B(E_1, E_1) + (X^1 Y^2 + Y^1 X^2) B(E_1, E_2) + X^2 Y^2 B(E_2, E_2) \\
&\quad + J(Z_1 E_1 + Z_2 E_2) \\
&= X^1 Y^1 B(E_1, E_1) + (X^1 Y^2 + Y^1 X^2)(\alpha J E_1 + \beta J E_2) \\
&\quad + X^2 Y^2 B(E_2, E_2) + Z_1 J E_1 + Z_2 J E_2 \\
&= X^1 Y^1 B(E_1, E_1) + (\alpha X^1 Y^2 + \alpha Y^1 X^2 + Z_1) J E_1 \\
&\quad + (\beta X^1 Y^2 + \beta Y^1 X^2 + Z_2) J E_2 + X^2 Y^2 B(E_2, E_2).
\end{aligned}$$

Then, $\text{span}\{\text{Im } B \cup JTM^2\}$ has real dimension at most 4.

Therefore, the subbundle L has real dimension less or equal to 6. \square

Lemma 3.5.7. *Assume that H is nowhere an umbilical direction. Let $\tilde{L} = L \oplus TM^2$, then \tilde{L} is parallel with respect to the Levi-Civita connection of the complex space form $N^n(c)$ and is invariant by the curvature tensor \bar{R} , i.e. $\bar{R}(\bar{u}, \bar{v})\bar{w} \in \tilde{L}$, for all $\bar{u}, \bar{v}, \bar{w} \in \tilde{L}$.*

Proof. From Proposition [3.5.6](#), it is easy to see that \tilde{L} is parallel with respect to the Levi-Civita connection $\bar{\nabla}$ of the complex space form $N^n(c)$. Indeed, if $\sigma \in C(L) \subset C(\tilde{L})$, we have

$$\bar{\nabla}_X \sigma = \nabla_X^\perp \sigma - A_\sigma X$$

as $\nabla_X^\perp \sigma \in C(L)$ and $A_\sigma X \in C(TM^2)$, we obtain $\bar{\nabla}_X \sigma \in C(\tilde{L})$.

If $Y \in C(TM^2) \subset C(\tilde{L})$, we have

$$\bar{\nabla}_X Y = \nabla_X Y + B(X, Y),$$

and since $\nabla_X Y \in C(TM^2)$ and $B(X, Y) \in C(L)$, we get $\bar{\nabla}_X Y \in C(\tilde{L})$.

In order to show that \tilde{L} is invariant by the curvature tensor \bar{R} , we first need to prove that $J\tilde{L} \subset \tilde{L}$, which implies, by reasons of dimension, $J\tilde{L} = \tilde{L}$:

1. Let $X \in TM^2 \subset \tilde{L}$. By the definition of L , we obtain $JX \in L \subset \tilde{L}$.
2. Let $B(X, Y) \in L \subset \tilde{L}$, then

$$JB(X, Y) = (JB(X, Y))^\top + (JB(X, Y))^\perp.$$

Since $(JB(X, Y))^\top \in TM^2 \subset \tilde{L}$ and $(JB(X, Y))^\perp \in L \subset \tilde{L}$, we get $JB(X, Y) \in \tilde{L}$, for all $X, Y \in TM^2$.

3. Let $(JB(X, Y))^\perp \in L \subset \tilde{L}$, then

$$J((JB(X, Y))^\perp) = J(JB(X, Y) - (JB(X, Y))^\top).$$

Take $Z = (JB(X, Y))^{\top}$, then

$$J((JB(X, Y))^{\perp}) = -B(X, Y) - JZ,$$

and since $B(X, Y) \in L \subset \tilde{L}$ and $JZ \in L \subset \tilde{L}$, we obtain that $J((JB(X, Y))^{\perp}) \in \tilde{L}$, for all $X, Y \in TM^2$.

4. Let $JX \in L \subset \tilde{L}$, then

$$J(JX) = -X,$$

thus $J(JX) \in \tilde{L}$, for all $X \in TM^2$.

Therefore, $J\tilde{L} = \tilde{L}$, and we conclude by inspecting the terms in (3.2.1), the formula of the curvature tensor field \bar{R} of $N^n(c)$. \square

Now we can state the main result of this section:

Theorem 3.5.8. *Let M^2 be a non pseudo-umbilical PMC totally real surface in a complex space form $N^n(c)$, $c \neq 0$, $n \geq 4$. Then there exists a totally geodesic complex submanifold $N^4(c) \subset N^n(c)$ such that $M^2 \subset N^4(c)$.*

Proof. In the first case, we assume that M^2 is nowhere pseudo-umbilical. We apply Theorem 3.5.3 to the subbundle \tilde{L} as defined in Lemma 3.5.7 and Proposition 3.5.6 to conclude that there exists a totally geodesic submanifold N' of $N^n(c)$ such that $M \subset N'$ and $\tilde{L}_p = T_p N'$ for all $p \in M^2$. Since $J\tilde{L} = \tilde{L}$, N' is a complex space form $N' = N^4(c)$ [29, 30].

For the second case, assume that M^2 admits some pseudo-umbilical points and denote by W the set of all non pseudo-umbilical points of M^2 . As $M^2 \setminus W$ admits no accumulation point, the subset W is open, dense and connected.

In order to prove that W is connected, we show that W is path-connected. Indeed, let $p, q \in W$, thus, $p, q \in M^2$ and there exists a path γ in M^2 that joins p and q . If the path is already in W , we directly conclude. As W is dense assume that γ passes through a finite number of pseudo-umbilical points, and denote by z one of these points. Since pseudo-umbilical points are isolated, choose a neighborhood of z containing no other pseudo-umbilical point. In this neighborhood, we smoothly modify the curve γ to avoid the point z . This way, we obtain a new path that joins p and q and lies entirely in W .

We apply the same argument as for the first case to W to conclude that $W \subset N^4(c)$. By a standard argument, we conclude that the whole of M^2 lies in that $N^4(c)$. More precisely, let $z \in M^2 \setminus W$. Since z is isolated, there exists a sequence $\{p_n\}_{n \in \mathbb{N}^*}$ that converges to z in M^2 , $p_n \neq z$, and p_n belongs to the neighborhood of z that isolates z from the other points of $M^2 \setminus W$. As $\{p_n\}_{n \in \mathbb{N}^*}$ converges to z in M^2 , it follows that

$\{p_n\}_{n \in \mathbb{N}^*}$ is a Cauchy sequence in M^2 , or in W , and from here it follows that $\{p_n\}_{n \in \mathbb{N}^*}$ is a Cauchy sequence also in $N^4(c)$. As $N^4(c)$ is complete, $\{p_n\}_{n \in \mathbb{N}^*}$ converges to some point $z' \in N^4(c)$ in $N^4(c)$. However, as $\{p_n\}_{n \in \mathbb{N}^*}$ converges to z in $N^n(c)$ and $\{p_n\}_{n \in \mathbb{N}^*}$ converges to z' in $N^n(c)$, we get $z = z'$, thus $z \in N^4(c)$. \square

Remark 3.5.9. When M^2 is pseudo-umbilical and a topological sphere, then the situation is different. First, we recall that if M^2 is a topological sphere, CMC and biconservative in an arbitrary Riemannian manifold, then it is pseudo-umbilical (see [70, Corollary 4.3]). Now, according to a result in [80], when M^2 is a PMC totally real surface in a complex space form $N^n(c)$, $c \neq 0$, and M^2 is a topological sphere, then there exists a totally geodesic totally real submanifold N' such that $M^2 \subset N'$. We note that the technique used in [80] is a completely different one.

Under a slightly stronger assumption, we can further reduce the codimension that takes us back to the case of surfaces in complex space forms of complex dimension 2, as in Section 3.4.

Theorem 3.5.10. *Let M^2 be a non pseudo-umbilical PMC totally real surface in a complex space form $N^n(c)$, $c \neq 0$. If $H \in C(JTM^2)$, then there exists a totally geodesic complex submanifold $N^2(c) \subset N^n(c)$ such that $M^2 \subset N^2(c)$.*

Proof. Assume M^2 is nowhere pseudo-umbilical, otherwise we follow the argument of the second part of the proof of Theorem 3.5.8. Consider $\{E_1, E_2\}$ a local orthonormal frame field that diagonalizes simultaneously A_H and A_V , where V is normal to M^2 and orthogonal to JTM^2 .

In the proof of Proposition 3.5.6, we have shown that $B(E_1, E_2) \in C(JTM^2)$. We will prove that furthermore

$$B(E_1, E_1), B(E_2, E_2) \in C(JTM^2),$$

implying

$$\text{Im } B \subset JTM^2, \quad J \text{Im } B \subset TM^2$$

and so $L = JTM^2$.

Indeed, let V be normal to M^2 and orthogonal to JTM^2 . Then $V \perp H$. We have $\langle V, JE_i \rangle = 0$, so $\langle JV, E_i \rangle = 0$ and therefore JV is normal to M^2 . And since $\langle JV, JE_i \rangle = \langle V, E_i \rangle = 0$, we also get $JV \perp JTM^2$.

We have

$$\bar{\nabla}_{E_i} JE_j = -A_{JE_j} E_i + \nabla_{E_i}^\perp JE_j, \quad (3.5.1)$$

while, on the other hand,

$$\bar{\nabla}_{E_i} JE_j = J\bar{\nabla}_{E_i} E_j = J\nabla_{E_i} E_j + JB(E_i, E_j). \quad (3.5.2)$$

Taking inner products of Equations (3.5.1) and (3.5.2) with V , we obtain

$$\langle JB(E_i, E_j), V \rangle = \langle \nabla_{E_i}^\perp JE_j, V \rangle, \quad \forall i, j = 1, 2. \quad (3.5.3)$$

As $H \in JTM^2$, we obtain

$$H = \langle H, JE_1 \rangle JE_1 + \langle H, JE_2 \rangle JE_2,$$

and, since M^2 is PMC, we get

$$\begin{aligned} 0 = \nabla_X^\perp H &= X(\langle H, JE_1 \rangle) JE_1 + \langle H, JE_1 \rangle \nabla_X^\perp JE_1 \\ &\quad + X(\langle H, JE_2 \rangle) JE_2 + \langle H, JE_2 \rangle \nabla_X^\perp JE_2. \end{aligned} \quad (3.5.4)$$

Taking the inner product of Equation (3.5.4) with V , we obtain

$$\langle H, JE_1 \rangle \langle \nabla_X^\perp JE_1, V \rangle + \langle H, JE_2 \rangle \langle \nabla_X^\perp JE_2, V \rangle = 0. \quad (3.5.5)$$

Using Equation (3.5.3) and taking $X = E_1$, as $JB(E_1, E_2)$ is tangent, we have

$$\begin{aligned} \langle H, JE_1 \rangle \langle JB(E_1, E_1), V \rangle &= -\langle H, JE_2 \rangle \langle \nabla_{E_1}^\perp JE_2, V \rangle \\ &= \langle H, JE_2 \rangle \langle JB(E_1, E_2), V \rangle \\ &= 0. \end{aligned}$$

Hence, $\langle H, JE_1 \rangle = 0$ or $\langle JB(E_1, E_1), V \rangle = 0$.

Let $p \in M^2$. If, at the point p , we have $\langle JB(E_1, E_1), V \rangle = 0$, then $\langle B(E_1, E_1), JV \rangle = 0$, so $\langle B(E_1, E_1), U \rangle = 0$ for all U normal to M^2 and $U \perp J(T_p M^2)$. Therefore, $B(E_1, E_1) \in J(T_p M^2)$ and, as $H \in JTM^2$, we also get $B(E_2, E_2) \in J(T_p M^2)$.

Now, assume that, at p , $\langle JB(E_1, E_1), V \rangle \neq 0$. Thus, $\langle H, JE_1 \rangle = 0$ around p . Then $E_2 = JH/|H|$ and so $JE_2 = -H/|H|$, hence,

$$\bar{\nabla}_{E_2} JE_2 = -A_{JE_2} E_2 + \nabla_{E_2}^\perp \left(\frac{-H}{|H|} \right). \quad (3.5.6)$$

On the other hand,

$$\bar{\nabla}_{E_2} JE_2 = J\bar{\nabla}_{E_2} E_2 = J\nabla_{E_2} E_2 + JB(E_2, E_2). \quad (3.5.7)$$

Taking the inner product of Equations (3.5.6) and (3.5.7) with V , we obtain

$$\langle JB(E_2, E_2), V \rangle = 0,$$

for any V normal to M^2 and orthogonal to JTM^2 . Thus, $\langle B(E_2, E_2), JV \rangle = 0$, then $\langle B(E_2, E_2), U \rangle = 0$ for all U normal to M^2 and orthogonal to JTM^2 .

Therefore, $B(E_2, E_2) \in JTM^2$ and, as the mean curvature vector field $H \in JTM^2$, we conclude that $B(E_1, E_1) \in JTM^2$. \square

3.6 Higher dimensions

In this section, we construct examples with the Segre embedding (see for example [22, 91]) to show that non-PMC CMC biconservative submanifolds of complex projective spaces do exist. This shows that the existence of biconservative submanifolds of dimension greater than two is less rigid. One may find interesting new examples. Moreover, we compute when these examples are proper-biharmonic.

Recall that the Segre embedding (see [22])

$$S_{hq} : \mathbb{C}P^h(4) \times \mathbb{C}P^q(4) \rightarrow \mathbb{C}P^{h+q+hq}(4),$$

is defined by

$$S_{hq}([(z_0, \dots, z_h)], [(\omega_0, \dots, \omega_q)]) = \left[(z_j \omega_t)_{0 \leq j \leq h, 0 \leq t \leq q} \right],$$

where (z_0, \dots, z_h) and $(\omega_0, \dots, \omega_q)$ are the homogeneous coordinates of $\mathbb{C}P^h(4)$ and $\mathbb{C}P^q(4)$, respectively. This embedding was introduced by Segre in 1891 (see [91]).

Now, we introduce three properties of the Segre embedding that we will use later [22, 25, 71]. First, it is well-known that $\mathbb{C}P^h(4)$ and $\mathbb{C}P^q(4)$ are both totally geodesic in $\mathbb{C}P^{h+q+hq}(4)$. Since $\mathbb{C}P^h(4)$ and $\mathbb{C}P^q(4)$ are totally geodesic in the product manifold $\mathbb{C}P^h(4) \times \mathbb{C}P^q(4)$, we get

$$B(X_1, X_2) = 0,$$

for any vector fields X_1, X_2 on $\mathbb{C}P^h(4)$ tangent to $\mathbb{C}P^h(4)$ and similarly,

$$B(Y_1, Y_2) = 0,$$

for any vector fields Y_1, Y_2 on $\mathbb{C}P^q(4)$ tangent to $\mathbb{C}P^q(4)$, where B is the second fundamental form of S_{hq} in $\mathbb{C}P^{h+q+hq}$.

Also, we have

$$\begin{aligned} 0 &= (\nabla^\perp B)(X, Y, Z) \\ &= (\nabla_X^\perp B)(Y, Z) \\ &= \nabla_X^\perp (B(Y, Z)) - B(\nabla_X^{\mathbb{C}P^h \times \mathbb{C}P^q} Y, Z) - B(Y, \nabla_X^{\mathbb{C}P^h \times \mathbb{C}P^q} Z), \end{aligned}$$

for all X, Y and Z in $C(T(\mathbb{C}P^h \times \mathbb{C}P^q))$.

Now, let M_1 be a totally real submanifold of $\mathbb{C}P^h(4)$, of real dimension h . Consider $M = M_1 \times \mathbb{C}P^q(4)$ and we embed it in $\mathbb{C}P^{h+q+hq}(4)$ via S_{hq} . Then, M_1 is a totally real submanifold of $\mathbb{C}P^{h+q+hq}(4)$ and $\mathbb{C}P^q(4)$ is a totally geodesic submanifold of $\mathbb{C}P^{h+q+hq}(4)$.

Moreover, denoting again by B the second fundamental form of the embedding of M in $\mathbb{C}P^{h+q+hq}(4)$, we have that

$$B(X_i, Y_j), \quad i = 1, \dots, h \quad \text{and} \quad j = 1, \dots, 2q$$

are orthonormal vector fields, where $\{X_i\}_{i=1}^h$ is a local orthonormal frame field on M_1 and $\{Y_j\}_{j=1}^{2q}$ is a local orthonormal frame field on $\mathbb{C}P^q(4)$.

Now we state the following theorem.

Theorem 3.6.1. *Let γ be a curve of nowhere vanishing curvature κ in the complex projective space $\mathbb{C}P^1(4)$ of complex dimension 1. Then, we have:*

1. *via the Segre embedding of $\mathbb{C}P^1(4) \times \mathbb{C}P^q(4)$ into $\mathbb{C}P^{1+2q}(4)$, the product $M^{1+2q} = \gamma \times \mathbb{C}P^q(4)$ is a biconservative submanifold of $\mathbb{C}P^{1+2q}(4)$ if and only if κ is constant; in this case, M^{1+2q} is CMC non-PMC, and moreover, it is not totally real;*
2. *M^{1+2q} is a proper-biharmonic submanifold of $\mathbb{C}P^{1+2q}(4)$ if and only if $\kappa^2 = 4$, i.e., γ is proper-biharmonic in $\mathbb{C}P^1(4)$.*

Proof. Let

$$\gamma : I \rightarrow \mathbb{C}P^1(4),$$

where γ is parametrized by arc-length, with nowhere vanishing curvature κ , and identify $\mathbb{C}P^1(4)$ with the sphere \mathbb{S}^2 of curvature 4. The curve γ is a totally real submanifold of $\mathbb{C}P^1(4)$.

Further, consider the following embeddings

$$\mathbf{i} : M^{1+2q} = \gamma \times \mathbb{C}P^q(4) \rightarrow \mathbb{C}P^1(4) \times \mathbb{C}P^q(4)$$

and

$$\mathbf{j} : \mathbb{C}P^1(4) \times \mathbb{C}P^q(4) \rightarrow \mathbb{C}P^{1+2q}(4),$$

where, for simplicity, we denote by \mathbf{j} the Segre embedding. Let $\varphi = \mathbf{j} \circ \mathbf{i}$ be the composition map, we have

$$B^\varphi(X, Y) = B^{\mathbf{i}}(X, Y) + B^{\mathbf{j}}(X, Y),$$

for all $X, Y \in C(TM^{1+2q})$. Indeed,

$$\begin{aligned} B^\varphi(X, Y) &= \nabla_X^{\mathbb{C}P^{1+2q}} Y - \nabla_X^M Y \\ &= B^{\mathbf{j}}(X, Y) + \nabla_X^{\mathbb{C}P^1(4) \times \mathbb{C}P^q(4)} Y - \nabla_X^M Y \\ &= B^{\mathbf{j}}(X, Y) + B^{\mathbf{i}}(X, Y). \end{aligned}$$

Thus

$$H^\varphi = H^{\mathbf{i}} + \frac{1}{1+2q} \sum_{k=1}^{1+2q} B^{\mathbf{j}}(E_k, E_k),$$

where $E_1 = \gamma'$ and $\{E_2, \dots, E_{1+2q}\}$ is a local orthonormal frame field defined on $\mathbb{C}P^q(4)$ and tangent to $\mathbb{C}P^q(4)$.

Recall that, since \mathbf{j} is the Segre embedding, we have

$$B^{\mathbf{j}}(E_k, E_k) = 0, \quad \forall k = 1, \dots, 1+2q.$$

Since $B^{\mathbf{i}}(E_1, E_l) = 0$, for $l = 2, \dots, 1+2q$, we get

$$B^{\mathbf{j}}(E_1, E_l), \quad l = 2, \dots, 1+2q$$

are orthonormal vector fields. Moreover, as $B^{\mathbf{j}}(E_l, JE_1) = JB^{\mathbf{j}}(E_l, E_1)$, for $l = 2, \dots, 1+2q$, we also obtain that

$$B^{\mathbf{j}}(JE_1, E_l), \quad l = 2, \dots, 1+2q$$

are orthonormal vector fields.

In particular we obtain,

$$H^\varphi = H^{\mathbf{i}}.$$

Now, we have

$$\nabla_X^{\mathbb{C}P^{1+2q}} H^\varphi = \nabla_X^{\perp\varphi} H^\varphi - A_{H^\varphi}^\varphi X, \quad (3.6.1)$$

and, on the other hand,

$$\begin{aligned} \nabla_X^{\mathbb{C}P^{1+2q}} H^\varphi &= \nabla_X^{\mathbb{C}P^{1+2q}} H^{\mathbf{i}} \\ &= \nabla_X^{\mathbb{C}P^1 \times \mathbb{C}P^q} H^{\mathbf{i}} + B^{\mathbf{j}}(X, H^{\mathbf{i}}). \end{aligned} \quad (3.6.2)$$

In order to compute $H^{\mathbf{i}}$, we consider $\{E_1 = \gamma' = \mathbf{t}, \mathbf{n}\}$ the Frenet frame field along γ in $\mathbb{C}P^1(4)$. Since

$$\begin{aligned} B^{\mathbf{i}}(E_1, E_1) &= \nabla_{E_1}^{\mathbb{C}P^1 \times \mathbb{C}P^q} E_1 - \nabla_{E_1}^{M^{1+2q}} E_1 = \nabla_{E_1}^{\mathbb{C}P^1} E_1 \\ &= \kappa \mathbf{n} \end{aligned}$$

and

$$\begin{aligned} B^{\mathbf{i}}(E_l, E_l) &= \nabla_{E_l}^{\mathbb{C}P^1 \times \mathbb{C}P^q} E_l - \nabla_{E_l}^{M^{1+2q}} E_l \\ &= 0 \end{aligned}$$

for $l = 2, \dots, 1 + 2q$, we obtain

$$H^i = \frac{1}{1 + 2q} \kappa \mathbf{n}. \quad (3.6.3)$$

Replacing (3.6.3) in (3.6.2), we get

$$\nabla_X^{\mathbb{C}P^{1+2q}} H^\varphi = \frac{\kappa}{1 + 2q} \left\{ \nabla_X^{\mathbb{C}P^1 \times \mathbb{C}P^q} \mathbf{n} + B^j(X, \mathbf{n}) \right\} + \frac{X(\kappa)}{1 + 2q} \mathbf{n}. \quad (3.6.4)$$

To find the shape operator $A_{H^\varphi}^\varphi$ and to prove that the immersion φ is not PMC, we consider first $X = E_1$ in Equation (3.6.4), and we obtain

$$\begin{aligned} \nabla_{E_1}^{\mathbb{C}P^{1+2q}} H^\varphi &= \frac{\kappa}{1 + 2q} \left\{ \nabla_{E_1}^{\mathbb{C}P^1} \mathbf{n} + B^j(E_1, \mathbf{n}) \right\} + \frac{\kappa'}{1 + 2q} \mathbf{n} \\ &= \frac{\kappa}{1 + 2q} \{-\kappa \mathbf{t}\} + \frac{\kappa'}{1 + 2q} \mathbf{n} \\ &= -\frac{\kappa^2}{1 + 2q} \mathbf{t} + \frac{\kappa'}{1 + 2q} \mathbf{n} \\ &= -\frac{\kappa^2}{1 + 2q} E_1 + \frac{\kappa'}{1 + 2q} \mathbf{n}. \end{aligned} \quad (3.6.5)$$

From Equations (3.6.1) and (3.6.5), we conclude

$$\nabla_{E_1}^{\perp \varphi} H^\varphi = \frac{\kappa'}{1 + 2q} \mathbf{n} \quad \text{and} \quad A_{H^\varphi}^\varphi E_1 = \frac{\kappa^2}{1 + 2q} E_1. \quad (3.6.6)$$

Second, if $X = E_l$, $l = 2, \dots, 2q + 1$, from Equation (3.6.4), we get

$$\begin{aligned} \nabla_{E_l}^{\mathbb{C}P^{1+2q}} H^\varphi &= \frac{\kappa}{1 + 2q} \left\{ \nabla_{E_l}^{\mathbb{C}P^1 \times \mathbb{C}P^q} \mathbf{n} + B^j(E_l, \mathbf{n}) \right\} \\ &= \frac{\kappa}{1 + 2q} B^j(E_l, \mathbf{n}). \end{aligned} \quad (3.6.7)$$

Using Equations (3.6.1) and (3.6.7), we conclude

$$\nabla_{E_l}^{\perp \varphi} H^\varphi = \frac{\kappa}{1 + 2q} B^j(E_l, \mathbf{n}) \quad \text{and} \quad A_{H^\varphi}^\varphi E_l = 0. \quad (3.6.8)$$

Since $|B^j(E_l, E_l)| = 1$, for all $l = 2, \dots, 1 + 2q$, $B^j(E_l, \mathbf{n}) \neq 0$. Therefore, $\nabla_{E_l}^{\perp \varphi} H^\varphi \neq 0$, i.e. M^{1+2q} is a non-PMC submanifold of $\mathbb{C}P^{1+2q}(4)$.

In the following, in order to study the biconservativity of φ , we compute the curvature term in the biconservative equation. We have

$$\begin{aligned} R^{\mathbb{C}P^{1+2q}}(X, H^\varphi)X &= \langle H^\varphi, X \rangle X - \langle X, X \rangle H^\varphi + \langle JH^\varphi, X \rangle JX \\ &\quad - \langle JX, X \rangle JH^\varphi + 2\langle JH^\varphi, X \rangle JX \\ &= -H^\varphi + 3\langle JH^\varphi, X \rangle JX, \end{aligned}$$

for all $X \in C(TM^{1+2q})$. Then

$$\begin{aligned}
\text{trace}(R^{\mathbb{C}P^{1+2q}}(\cdot, H^\varphi)\cdot)^{\top\varphi} &= 3 \{ \text{trace}\langle T, \cdot \rangle J \cdot \}^{\top\varphi} \\
&= 3 \left\{ \sum_{i=2}^{1+2q} \langle T, E_i \rangle J E_i \right\}^{\top\varphi} = 3 \sum_{i=2}^{1+2q} \langle T, E_i \rangle J E_i \\
&= 3 \sum_{i=2}^{1+2q} \langle JT, J E_i \rangle J E_i + 3 \langle JT, E_1 \rangle E_1 \\
&= 3 (JT)^{\top\varphi},
\end{aligned}$$

where $JH^\varphi = T + N$ with respect to φ .

In our case,

$$JH^\varphi = JH^{\mathbf{i}} = \frac{\kappa}{1+2q}(\pm \mathbf{t}) \in C(TM^{1+2q}),$$

so $T = JH^\varphi$ and then $JT = -H^\varphi$ which implies $(JT)^{\top\varphi} = 0$. Therefore,

$$\text{trace}(R^{\mathbb{C}P^{1+2q}}(\cdot, H^\varphi)\cdot)^{\top\varphi} = 0.$$

Now, according to Proposition [1.5.5](#), to show that φ is biconservative, we must prove

$$4 \text{trace} A_{\nabla_{(\cdot)}^\perp H^\varphi}^\varphi(\cdot) + (1+2q) \text{grad}(|H^\varphi|^2) = 0. \quad (3.6.9)$$

The second term on the left-hand side of [\(3.6.9\)](#) can be written as

$$(1+2q) \text{grad}(|H^\varphi|^2) = \frac{2\kappa\kappa'}{1+2q} E_1. \quad (3.6.10)$$

For the first term, by [\(3.6.6\)](#) we have

$$\begin{aligned}
A_{\nabla_{E_1}^\perp H^\varphi}^\varphi E_1 &= A_{\frac{\kappa'}{1+2q} \mathbf{n}}^\varphi E_1 = \frac{\kappa'}{\kappa} A_{H^\varphi}^\varphi E_1 \\
&= \frac{\kappa\kappa'}{1+2q} E_1.
\end{aligned} \quad (3.6.11)$$

Next, if $l = 2, \dots, 1+2q$ and $k = 1, \dots, 1+2q$, we have

$$\begin{aligned}
\left\langle A_{\nabla_{E_l}^\perp H^\varphi}^\varphi E_l, E_k \right\rangle &= \frac{\kappa}{1+2q} \left\langle A_{B^{\mathbf{j}}(E_l, \mathbf{n})}^\varphi E_l, E_k \right\rangle \\
&= \frac{\kappa}{1+2q} \left\langle B^\varphi(E_l, E_k), B^{\mathbf{j}}(E_l, \mathbf{n}) \right\rangle \\
&= \frac{\kappa}{1+2q} \left\langle B^{\mathbf{i}}(E_l, E_k) + B^{\mathbf{j}}(E_l, E_k), B^{\mathbf{j}}(E_l, \mathbf{n}) \right\rangle \\
&= \frac{\kappa}{1+2q} \left\{ \left\langle B^{\mathbf{i}}(E_l, E_k), B^{\mathbf{j}}(E_l, \mathbf{n}) \right\rangle + \left\langle B^{\mathbf{j}}(E_l, E_k), B^{\mathbf{j}}(E_l, \mathbf{n}) \right\rangle \right\} \\
&= \frac{\kappa}{1+2q} \left\langle B^{\mathbf{j}}(E_l, E_k), B^{\mathbf{j}}(E_l, \mathbf{n}) \right\rangle.
\end{aligned}$$

Further, from the Gauss Equation (1.3.3) for the immersion \mathbf{j} we have

$$\begin{aligned} \langle B^{\mathbf{j}}(E_l, E_k), B^{\mathbf{j}}(E_l, \mathbf{n}) \rangle &= \langle B^{\mathbf{j}}(\mathbf{n}, E_k), B^{\mathbf{j}}(E_l, E_l) \rangle - \langle R^{\mathbb{C}P^{1+2q}}(E_l, \mathbf{n})E_l, E_k \rangle \\ &\quad + \langle R^{\mathbb{C}P^1 \times \mathbb{C}P^q}(E_l, \mathbf{n})E_l, E_k \rangle. \end{aligned}$$

But

$$B^{\mathbf{j}}(E_l, E_l) = 0$$

and

$$\begin{aligned} R^{\mathbb{C}P^{1+2q}}(E_l, \mathbf{n})E_l &= \langle \mathbf{n}, E_l \rangle E_l - \langle E_l, E_l \rangle \mathbf{n} + \langle J\mathbf{n}, E_l \rangle J E_l \\ &\quad - \langle J E_l, E_l \rangle J \mathbf{n} + 2 \langle J\mathbf{n}, E_l \rangle J E_l \\ &= -\mathbf{n}, \end{aligned}$$

it follows that

$$\langle R^{\mathbb{C}P^{1+2q}}(E_l, \mathbf{n})E_l, E_k \rangle = 0.$$

We also have

$$\begin{aligned} R^{\mathbb{C}P^1 \times \mathbb{C}P^q}(E_l, \mathbf{n})E_l &= \nabla_{E_l}^{\mathbb{C}P^1 \times \mathbb{C}P^q} \nabla_{\mathbf{n}}^{\mathbb{C}P^1 \times \mathbb{C}P^q} E_l - \nabla_{\mathbf{n}}^{\mathbb{C}P^1 \times \mathbb{C}P^q} \nabla_{E_l}^{\mathbb{C}P^1 \times \mathbb{C}P^q} E_l \\ &\quad - \nabla_{[E_l, \mathbf{n}]}^{\mathbb{C}P^1 \times \mathbb{C}P^q} E_l \\ &= 0, \end{aligned}$$

therefore,

$$\langle B^{\mathbf{j}}(E_l, E_k), B^{\mathbf{j}}(E_l, \mathbf{n}) \rangle = 0. \quad (3.6.12)$$

Thus $\langle A_{\nabla_{E_l}^{\perp \varphi} H^\varphi} E_l, E_k \rangle = 0$, i.e.

$$A_{\nabla_{E_l}^{\perp \varphi} H^\varphi} E_l = 0. \quad (3.6.13)$$

Replacing (3.6.10), (3.6.11) and (3.6.13) in (3.6.9) we obtain

$$\begin{aligned} 4 \operatorname{trace} A_{\nabla_{(\cdot)}^{\perp \varphi} H^\varphi}(\cdot) + (1 + 2q) \operatorname{grad}(|H^\varphi|^2) &= \frac{4\kappa\kappa'}{1+2q} E_1 + \frac{2\kappa\kappa'}{1+2q} E_1 \\ &= \frac{6\kappa\kappa'}{1+2q} E_1. \end{aligned}$$

In conclusion, φ is biconservative if and only if $\kappa' = 0$, i.e. κ is constant.

Now, we prove that M^{1+2q} is a proper-biharmonic submanifold of $\mathbb{C}P^{1+2q}(4)$ if and only if $\kappa^2 = 4$.

Assume κ to be constant, then the tangential part of the biharmonic equation (1.4.1) holds. Therefore, we only need to solve the normal part of the biharmonic equation. As

JH^φ is tangent to M^{1+2q} , using Proposition 3.2.7 the normal part of the biharmonic equation becomes

$$-\Delta^{\perp\varphi} H^\varphi - \text{trace } B^\varphi(\cdot, A_{H^\varphi}^\varphi(\cdot)) + (m+3)H^\varphi = 0 \quad (3.6.14)$$

(see also 40).

For the first term of Equation (3.6.14), we have

$$\begin{aligned} -\Delta^{\perp\varphi} H^\varphi &= \nabla_{E_1}^{\perp\varphi} \nabla_{E_1}^{\perp\varphi} H^\varphi - \nabla_{\nabla_{E_1}^M}^{\perp\varphi} H^\varphi + \sum_{l=2}^{1+2q} \left\{ \nabla_{E_l}^{\perp\varphi} \nabla_{E_l}^{\perp\varphi} H^\varphi - \nabla_{\nabla_{E_l}^M}^{\perp\varphi} H^\varphi \right\} \\ &= \sum_{l=2}^{1+2q} \left\{ \nabla_{E_l}^{\perp\varphi} \nabla_{E_l}^{\perp\varphi} H^\varphi - \nabla_{\nabla_{E_l}^{\mathbb{C}P^q}}^{\perp\varphi} H^\varphi \right\} \\ &= \frac{\kappa}{1+2q} \sum_{l=2}^{1+2q} \left\{ \nabla_{E_l}^{\perp\varphi} \nabla_{E_l}^{\perp\varphi} \mathbf{n} - \nabla_{\nabla_{E_l}^{\mathbb{C}P^q}}^{\perp\varphi} \mathbf{n} \right\}. \end{aligned} \quad (3.6.15)$$

Using (3.6.8), we obtain

$$\nabla_{E_l}^{\perp\varphi} \nabla_{E_l}^{\perp\varphi} \mathbf{n} = \nabla_{E_l}^{\perp\varphi} B^{\mathbf{j}}(E_l, \mathbf{n}).$$

Next,

$$\nabla_{E_l}^{\mathbb{C}P^{1+2q}} B^{\mathbf{j}}(E_l, \mathbf{n}) = \nabla_{E_l}^{\perp\varphi} B^{\mathbf{j}}(E_l, \mathbf{n}) - A_{B^{\mathbf{j}}(E_l, \mathbf{n})}^\varphi E_l. \quad (3.6.16)$$

On the other hand,

$$\nabla_{E_l}^{\mathbb{C}P^{1+2q}} B^{\mathbf{j}}(E_l, \mathbf{n}) = \nabla_{E_l}^{\perp\mathbf{j}} B^{\mathbf{j}}(E_l, \mathbf{n}) - A_{B^{\mathbf{j}}(E_l, \mathbf{n})}^{\mathbf{j}} E_l. \quad (3.6.17)$$

Using the Codazzi Equation (1.3.7), for the immersion \mathbf{j} , we have

$$(\nabla_{E_l}^{\perp\mathbf{j}} B^{\mathbf{j}})(n, E_l) - (\nabla_n^{\perp\mathbf{j}} B^{\mathbf{j}})(E_l, E_l) = (R^{\mathbb{C}P^{1+2q}}(E_l, n)E_l)^{\perp\mathbf{j}},$$

but

$$\begin{aligned} (\nabla_n^{\perp\mathbf{j}} B^{\mathbf{j}})(E_l, E_l) &= \nabla_n^{\perp\mathbf{j}} B^{\mathbf{j}}(E_l, E_l) - B^{\mathbf{j}}(\nabla_n^{\mathbb{C}P^1 \times \mathbb{C}P^q} E_l, E_l) \\ &\quad - B^{\mathbf{j}}(E_l, \nabla_n^{\mathbb{C}P^1 \times \mathbb{C}P^q} E_l) \\ &= 0. \end{aligned}$$

Also, we have

$$\begin{aligned} R^{\mathbb{C}P^{1+2q}}(E_l, n)E_l &= \langle n, E_l \rangle E_l - \langle E_l, E_l \rangle n + \langle Jn, E_l \rangle J E_l \\ &\quad - \langle J E_l, E_l \rangle J n + 2 \langle Jn, E_l \rangle J E_l \\ &= -\langle E_l, E_l \rangle n \\ &= -n, \end{aligned}$$

where $Jn = -E_1$, then $(R^{\mathbb{C}P^{1+2q}}(E_l, n)E_l)^\perp = 0$. Therefore, $(\nabla^\perp B^j)(E_l, n, E_l) = (\nabla_{E_l}^\perp B^j)(n, E_l) = 0$. Now, as $\nabla^\perp B^j = 0$, Equation (3.6.17) becomes

$$\begin{aligned} \nabla_{E_l}^{\mathbb{C}P^{1+2q}} B^j(E_l, \mathbf{n}) &= B^j(\nabla_{E_l}^{\mathbb{C}P^q} E_l, \mathbf{n}) + B^j(E_l, \nabla_{E_l}^{\mathbb{C}P^1 \times \mathbb{C}P^q} \mathbf{n}) \\ &\quad - A_{B^j(E_l, \mathbf{n})}^j E_l \\ &= B^j(\nabla_{E_l}^{\mathbb{C}P^q} E_l, \mathbf{n}) - A_{B^j(E_l, \mathbf{n})}^j E_l. \end{aligned} \quad (3.6.18)$$

Thus, from Equations (3.6.16) and (3.6.18), and using (3.6.12), we obtain

$$\nabla_{E_l}^\perp B^j(E_l, \mathbf{n}) = B^j(\nabla_{E_l}^{\mathbb{C}P^q} E_l, \mathbf{n}) - \langle A_{B^j(E_l, \mathbf{n})}^j E_l, \mathbf{n} \rangle \mathbf{n}.$$

Hence, using (3.6.8) and the above result we obtain

$$\begin{aligned} \nabla_{E_l}^\perp \nabla_{E_l}^\perp \mathbf{n} &= \nabla_{E_l}^\perp B^j(E_l, \mathbf{n}) \\ &= B^j(\nabla_{E_l}^{\mathbb{C}P^q} E_l, \mathbf{n}) - \langle B^j(E_l, \mathbf{n}), B^j(E_l, \mathbf{n}) \rangle \mathbf{n} \\ &= B^j(\nabla_{E_l}^{\mathbb{C}P^q} E_l, \mathbf{n}) - \mathbf{n}. \end{aligned} \quad (3.6.19)$$

Replacing (3.6.19) in (3.6.15), we get

$$\begin{aligned} -\Delta^\perp H^\varphi &= \frac{\kappa}{1+2q} \sum_{l=2}^{1+2q} \{ \nabla_{E_l}^\perp \nabla_{E_l}^\perp \mathbf{n} - \nabla_{\nabla_{E_l}^{\mathbb{C}P^q} E_l}^\perp \mathbf{n} \} \\ &= \frac{\kappa}{1+2q} \sum_{l=2}^{1+2q} \{ B^j(\nabla_{E_l}^{\mathbb{C}P^q} E_l, \mathbf{n}) - \mathbf{n} - B^j(\nabla_{E_l}^{\mathbb{C}P^q} E_l, \mathbf{n}) \} \\ &= \frac{\kappa}{1+2q} \sum_{l=2}^{1+2q} \{ -\mathbf{n} \} \\ &= \frac{-2q\kappa}{1+2q} \mathbf{n}. \end{aligned} \quad (3.6.20)$$

Now, we compute trace $B^\varphi(\cdot, A_{H^\varphi}^\varphi(\cdot))$. From Equations (3.6.6) and (3.6.8) we recall that

$$A_{H^\varphi}^\varphi E_1 = \frac{\kappa^2}{1+2q} E_1 \quad \text{and} \quad A_{H^\varphi}^\varphi E_l = 0.$$

It follows that

$$\begin{aligned} \text{trace } B^\varphi(\cdot, A_{H^\varphi}^\varphi(\cdot)) &= B^\varphi(E_1, A_{H^\varphi}^\varphi E_1) = \frac{\kappa^2}{1+2q} B^\varphi(E_1, E_1) \\ &= \frac{\kappa^2}{1+2q} \{ B^i(E_1, E_1) + B^j(E_1, E_1) \} \\ &= \frac{\kappa^3}{1+2q} \mathbf{n}. \end{aligned} \quad (3.6.21)$$

From (3.6.14), (3.6.20) and (3.6.21) we obtain that M^{1+2q} is biharmonic if and only if

$$-\frac{2q\kappa}{1+2q} \mathbf{n} - \frac{\kappa^3}{1+2q} \mathbf{n} + \frac{(m+3)\kappa}{1+2q} \mathbf{n} = 0,$$

that is, as $\kappa \neq 0$, $\kappa^2 = 4$.

Using the isometry of $\mathbb{C}P^1(4)$ with the sphere \mathbb{S}^2 of radius $1/2$ and by a standard argument, we get that a curve γ with constant curvature $\kappa = 2$ is a small circle of radius $(1/2)/\sqrt{2}$ of the above sphere \mathbb{S}^2 . Thus, it is proper-biharmonic in $\mathbb{C}P^1(4)$ [15, 17].

Indeed, first, in general, let us compute the second fundamental form of the sphere of radius r in \mathbb{E}^3 . In this particular example we place the radius in the brackets and not the curvature as before. We have

$$\mathbb{S}^2(r) = \{(x^1, x^2, x^3) \in \mathbb{E}^3 : (x^1)^2 + (x^2)^2 + (x^3)^2 = r^2\}.$$

Let $X \in C(T\mathbb{S}^2(r))$, i.e. $X(\bar{x}) \in \mathbb{E}^3$ and $\langle \bar{x}, X(\bar{x}) \rangle = 0$, for any $\bar{x} \in \mathbb{S}^2(r)$. We have

$$\begin{aligned} \nabla_X^{\mathbb{E}^3} Y &= \nabla_X^{\mathbb{S}^2(r)} Y + B(X, Y) \\ &= \nabla_X^{\mathbb{S}^2(r)} Y + b(X, Y) \frac{\bar{x}}{r}. \end{aligned} \quad (3.6.22)$$

Taking the inner product of (3.6.22) with \bar{x} we obtain

$$\begin{aligned} rb(X, Y) &= \langle \nabla_X^{\mathbb{E}^3} Y, \bar{x} \rangle = -\langle Y, \nabla_X^{\mathbb{E}^3} \bar{x} \rangle \\ &= -\langle Y, X \rangle \end{aligned}$$

Then $b(X, Y) = -\frac{1}{r} \langle X, Y \rangle$ and thus $B(X, Y) = -\frac{1}{r^2} \langle X, Y \rangle \bar{x}$.

Let γ be a curve with constant curvature $\kappa = 2$, we consider $\{\gamma' = \mathbf{t}, \mathbf{n}\}$ the Frenet frame field along γ in $\mathbb{S}^2(1/2)$. We have

$$\begin{cases} \nabla_{\gamma'}^{\mathbb{S}^2(1/2)} \gamma' &= \kappa \mathbf{n} = 2\mathbf{n} \\ \nabla_{\gamma'}^{\mathbb{S}^2(1/2)} \mathbf{n} &= -\kappa \mathbf{t} = -2\gamma'. \end{cases}$$

Looking at γ in \mathbb{E}^3 , we have

$$\begin{aligned} \nabla_{\gamma'}^{\mathbb{E}^3} \gamma' &= \nabla_{\gamma'}^{\mathbb{S}^2(1/2)} \gamma' + B(\gamma', \gamma') \\ &= \nabla_{\gamma'}^{\mathbb{S}^2(1/2)} \gamma' - 4\gamma' = 2\mathbf{n} - 4\gamma' \\ &= \overline{K} \overline{N}, \end{aligned}$$

where $\overline{K} = \|2\mathbf{n} - 4\gamma'\| = \sqrt{8}$ and $\overline{N} = (2\mathbf{n} - 4\gamma')/\overline{K}$. Also,

$$\begin{aligned} \nabla_{\gamma'}^{\mathbb{E}^3} \overline{N} &= \frac{1}{\overline{K}} \nabla_{\gamma'}^{\mathbb{E}^3} (2\mathbf{n} - 4\gamma') \\ &= \frac{1}{\overline{K}} \left\{ 2(\nabla_{\gamma'}^{\mathbb{S}^2(1/2)} \mathbf{n}) + B(\gamma', \overline{N}) - 4\gamma' \right\} \\ &= \frac{1}{\overline{K}} \left\{ 2(-2\gamma') - 4\gamma' \right\} \\ &= -\frac{8\gamma'}{\sqrt{8}}. \end{aligned}$$

Thus

$$\begin{cases} \nabla_{\gamma'}^{\mathbb{E}^3} \gamma' &= \sqrt{8} \bar{N} \\ \nabla_{\gamma'}^{\mathbb{E}^3} \bar{N} &= -\sqrt{8} \gamma'. \end{cases}$$

Therefore, γ is a circle in \mathbb{E}^3 with radius equal to $(1/2)/\sqrt{2}$. By [17, Remark 4.1], one sees that γ is a proper-biharmonic curve on the sphere of radius r if and only if it is a small circle of radius $r/\sqrt{2}$. Hence, we conclude that γ is proper-biharmonic in $\mathbb{C}P^1(4)$. \square

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Titre : Surfaces et Hypersurfaces Biharmoniques dans les Sphères et les Espaces Projectifs Complexes

Mots clés : Sous-variétés Biharmoniques, Courbure Moyenne Constante, Continuation Unique, Surfaces Biconservatrices, Surfaces PMC, Espaces de Formes Complexes.

Résumé : Dans cette thèse, nous commençons par démontrer un Théorème de Continuation Unique pour les hypersurfaces non-minimales biharmoniques dans les sphères. Sous les bonnes hypothèses, ce résultat montre que, pour ces immersions, CMC sur un sous-ensemble ouvert implique globalement CMC. Après, nous déduisons des nouveaux théorèmes de rigidité pour soutenir la conjecture que sous-variétés biharmoniques dans les sphères Euclidiennes doit être de la courbure moyenne constante. Nous considérons aussi les surfaces PMC dans les espaces de formes complexes, et nous étudions l'interaction entre les notions de PMC, totalement réel et biconservative. Nous considérons d'abord les surfaces PMC dans une espace de forme complexe non-plate et prouvons qu'elles sont biconservatrices si et seulement si totalement réelles.

Ensuite, nous trouvons une formule de type Simons pour un champ de vecteur bien choisi construit à partir du champ de vecteur de la courbure moyenne et l'utilisons pour prouver un résultat de rigidité pour les surfaces biconservatrices CMC dans les espaces de formes complexes de dimension 2. Nous prouvons ensuite un résultat d'une réduction de codimension pour les surfaces PMC biconservatrices dans les espaces de formes complexes non-plates. Enfin, nous concluons en construisant des exemples de sous-variétés biconservatrices CMC non-PMC à partir du plongement de Segre, et discutons quand elles sont proprement biharmoniques.

Title : Biharmonic Surfaces and Hypersurfaces in Spheres and Complex Projective Spaces

Keywords : Biharmonic Submanifolds, Constant Mean Curvature, Unique Continuation, Biconservative Surfaces, PMC Surfaces, Complex Space Forms.

Abstract : In this thesis, we start by proving a Unique Continuation Theorem for non-minimal biharmonic hypersurfaces of spheres. Under the right hypotheses, this result shows that, for these immersions, CMC on an open subset implies globally CMC. We then deduce new rigidity theorems to support the Conjecture that biharmonic submanifolds of Euclidean spheres must be of constant mean curvature. Also, we consider PMC surfaces in complex space forms, and study the interaction between the notions of PMC, totally real and biconservative. We first consider PMC surfaces in a non-flat complex space form and prove that they are biconservative if and only if totally real.

Then, we find a Simons type formula for a well-chosen vector field constructed from the mean curvature vector field and use it to prove a rigidity result for CMC biconservative surfaces in 2-dimensional complex space forms. We prove then a reduction codimension result for PMC biconservative surfaces in non-flat complex space forms. Finally, we conclude by constructing examples of CMC non-PMC biconservative submanifolds from the Segre embedding, and discuss when they are proper-biharmonic.