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Lucas Broux

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SORBONNE UNIVERSITÉ
École Doctorale de Sciences Mathématiques de Paris Centre
Laboratoire de Probabilités, Statistique et Modélisation

THÈSE DE DOCTORAT
Discipline : Mathématiques

présentée par

Lucas BROUX

Sewing, Reconstruction and Schauder in rough analysis and regularity structures

sous la direction de **Lorenzo ZAMBOTTI** et **Cyril LABBÉ**

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Couture, reconstruction et Schauder en analyse rugueuse et structures de régularité

Résumé

Dans cette thèse, nous obtenons des résultats analytiques liés aux théories des chemins rugueux et des structures de régularité, du point de vue des *germes*, c'est-à-dire des familles d'approximations locales de fonctions ou distributions.

D'abord, nous établissons un lemme de couture dans le régime $0 < \gamma \leq 1$, donnant une construction qui n'est pas unique ni canonique mais tout de même continue. En corollaire, nous exhibons une paramétrisation bicontinue de l'ensemble des chemins rugueux par un produit d'espaces de Hölder, généralisant à la fois le théorème d'extension de Lyons–Victoir et un résultat récent de Tapia–Zambotti.

Ensuite, nous proposons un théorème de reconstruction dans le contexte des espaces de Besov, généralisant des résultats de Hairer–Labbé et Caravenna–Zambotti. En corollaire, nous donnons une nouvelle preuve du théorème de multiplication dans les espaces de Besov, sans utiliser de paraproducts.

Enfin, nous étudions les propriétés régularisantes des noyaux singuliers contre les germes. Un premier résultat est la construction d'une application de convolution qui agit sur les germes cohérents et homogènes. Nous revisitons ensuite les estimées de Schauder multiniveaux de Hairer, donnant une présentation et une preuve qui font référence aussi peu que possible au formalisme des structures de régularité.

Mots-clés : chemins rugueux, structures de régularité, lemme de couture, théorème de reconstruction, estimées de Schauder.

Sewing, Reconstruction and Schauder in rough analysis and regularity structures

Abstract

In this thesis, we derive analytic results related to the theories of Rough Paths and Regularity Structures, with the point of view of *germs*, that is, families of local approximations of functions or distributions.

We first establish a Sewing Lemma in the regime $0 < \gamma \leq 1$, giving a construction which is non unique nor canonical but still continuous. As a corollary, we exhibit a bicontinuous parametrisation of the set of Rough Paths by a product of Hölder spaces, generalising both the Lyons–Victoir extension theorem and a recent result by Tapia–Zambotti.

Secondly, we propose a Reconstruction Theorem in the context of Besov spaces, generalising results of Hairer–Labbé and Caravenna–Zambotti. As a corollary, we provide a new proof of the multiplication theorem in Besov spaces without relying on paraproducts.

Finally, we study the regularising properties of singular kernels against germs. A first result is the construction of a convolution map which acts on general coherent and homogeneous germs. We also revisit Hairer’s multilevel Schauder estimates, providing a presentation and a proof which make only minimal references to the formalism of regularity structures.

Keywords: rough paths, regularity structures, sewing lemma, reconstruction theorem, Schauder estimates.

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Chapter 1

Introduction

Many physical systems can be modelised by mathematical equations in a satisfactory way. In some cases, it happens that the system is perturbed by some random noise, and that the physicist must take into account its inherent randomness and irregularity when writing the corresponding equation. It is then the role of the mathematician to elaborate tools in order to solve and discuss them.

In recent decades, many such tools have been developed. This thesis is concerned with some analytical aspects of two of them: the theory of Rough Paths and the theory of Regularity Structures.

While the theory of Rough Paths, introduced by Terry Lyons [Lyo98], is concerned with controlled differential equations of the form

$$dy_t = F(y_t)dx_t,$$

for some noise dx_t , the theory of Regularity Structures, introduced by Martin Hairer [Hai14], tackles stochastic partial differential equations (SPDEs) of the form

$$Lf = F(f, \xi),$$

where L is a differential operator, F is a non-linearity, and ξ is a forcing noise. In both cases, one must face the analytical difficulty that some of the objects in the equations are not properly defined in general: in the first case, the integral $\int F(y_t)dx_t$ cannot be canonically defined by Young–Kondurar integration as soon as the Hölder exponents of $F \circ y$ and x do not sum to a value strictly greater than 1; in the second case, the non-linearity F may formally introduce products of (Schwartz) distributions in the equation, which are notoriously ill-defined as soon as their corresponding Hölder exponents do not sum to a strictly positive value.

In order to overcome these difficulties, both theories adopt the point of view of not working with usual functions or distributions, but rather with families of local approximations, which will be called *germs* in both cases although their form and their role slightly differ in the two theories: on the one hand, in the theory of Rough Paths, germs take the form of two-parameter functions $A: [0, T]^2 \rightarrow \mathbb{R}$, where $A_{s,t}$ is intended to be a local description of the increments of some integral, i.e. typically $A_{s,t} \simeq \int_s^t f_u dg_u = I_t - I_s$ where one is interested in the integral $I_t = \int_0^t f_u dg_u$.

On the other hand, in the context of Regularity Structures, germs take the form of families of (Schwartz) distributions indexed by the space \mathbb{R}^d for some $d \in \mathbb{N}$, that is, families

$F = (F_x)_{x \in \mathbb{R}^d}$, where each $F_x \in \mathcal{D}'(\mathbb{R}^d)$ is a distribution: F_x is intended to be a local description of some distribution $f \in \mathcal{D}'(\mathbb{R}^d)$ when tested against test-functions which are concentrated around x .

Note that in some sense, the second point of view corresponds to a “differentiated” version of the first one: heuristically the theory of Rough Paths is concerned with defining ill-posed integrals $\int_s^t f_u dg_u$, while the theory of Regularity Structures would rather be concerned with the ill-posed integrand $f_u \dot{g}_u$ (where the derivative \dot{g} is understood in the distributional sense). As such, similarities will be noted between some of the results of the theories.

In both cases, the first natural general questions one can ask about germs are the following:

1. given a germ, can one construct a suitable global function or distribution which corresponds to the germ in the way hinted above?
2. what operations can one perform on germs in a way that is consistent with respect to the aforementioned construction?

In this thesis, we study some properties of germs in the light of those two questions.

The purpose of this introduction is threefold. First, we shall properly present (some) historical ideas which motivate the introduction of germs in the theories of Rough Paths and Regularity Structures. Second, we shall describe the pertaining tools and known results which will be useful in this thesis. Third, we shall introduce our contributions to this thesis.

1.1 Germs: motivations from Rough Paths

1.1.1 The Sewing problem

Let us start by discussing some ideas from the theory of Rough Paths, which is concerned with (controlled) differential equations of the form

$$dy_t = F(y_t)dx_t,$$

for functions $F, x, y: \mathbb{R} \rightarrow \mathbb{R}$, where y is the solution to find, and x is a given forcing noise which may be “rough”, by which we mean that it might satisfy Hölder conditions

$$|x_t - x_s| \lesssim |t - s|^\alpha,$$

for a possibly very small Hölder exponent $\alpha > 0$. As a typical example, sample paths of Brownian motion almost surely satisfy such bounds with $\alpha = \frac{1}{2} - \kappa$ for any $\kappa > 0$.

To give a meaning to the equation above, one can start by rewriting it in an integral form

$$y_t = y_0 + \int_0^t F(y_s)dx_s,$$

and now solving such a fixed-point problem requires to propose a strong meaning to integrals of the form

$$\int_0^t f_s dg_s, \tag{1.1.1}$$

where the functions f and g might admit very small Hölder exponents.

Of course, if $f \in C^0$ and $g \in C^1$, then (1.1.1) has a natural meaning via Riemann integration. On the other hand, when we can not make such assumptions, it is not clear how to define (1.1.1); however we can easily postulate a local approximation for the increments of the integral by simply expressing the fact that when $u \in [s, t]$, $f_u \simeq f_s$ and thus

$$\int_s^t f_u dg_u \simeq \int_s^t f_s dg_u = f_s(g_t - g_s). \tag{1.1.2}$$

Remarkably, when $f \in C^0$ and $g \in C^1$, the increments of the integral actually correspond to the best approximation of the two parameter function $(s, t) \mapsto f_s(g_t - g_s)$, in the following sense:

Observation 1.1.1. *Let $f, g: [0, T] \rightarrow \mathbb{R}$ be functions such that $f \in C^0$, $g \in C^1$, then the (Riemann) integral $I: t \mapsto \int_0^t f_s dg_s$ is the unique function $I: [0, T] \rightarrow \mathbb{R}$ such that*

$$I_0 = 0 \quad \text{and} \quad I_t - I_s - f_s(g_t - g_s) = o(|t - s|).$$

Proof. On one hand, $f_s(g_t - g_s) = \int_s^t f_s dg_u$, so that $I_t - I_s - f_s(g_t - g_s) = \int_s^t (f_u - f_s) dg_u$, which is indeed a $o(|t - s|)$. Furthermore, if there exists another such function \tilde{I} , then by subtraction, $(I - \tilde{I})_t - (I - \tilde{I})_s = o(|t - s|)$, so $I - \tilde{I}$ is differentiable with vanishing derivative, thus it is identically equal to its initial value $(I - \tilde{I})_0 = 0$: this establishes the uniqueness. \square

This assertion highlights the fact that two-parameter functions may play an important role in theories of integration; and replacing the function $(s, t) \mapsto f_s(g_t - g_s)$ by arbitrary functions $(s, t) \mapsto A_{s,t}$ motivates the following general question:

Given a two-parameter function $A: [0, T]^2 \rightarrow \mathbb{R}$, does there exist a (unique) one-parameter function $I: [0, T] \rightarrow \mathbb{R}$ such that $I_t - I_s - A_{s,t}$ is “small” as $s \rightarrow t$?

Such functions A shall be called *germs*, and the problem above of retrieving the integral I from the germ A the *sewing problem*. In a sense, the question is to recover the integral from a family of local approximations.

1.1.2 The classical Sewing Lemma

An elegant answer to this problem is provided by the following Sewing Lemma, due to Gubinelli [Gub04], see also Feyel–De la Pradelle [FLP06]

Theorem 1.1.2 (Sewing Lemma). *Let $\gamma > 1$ and $A: [0, T]^2 \rightarrow \mathbb{R}$ be a continuous function such that*

$$|A_{s,t} - A_{s,u} - A_{u,t}| \lesssim |t - s|^\gamma \quad \text{over } 0 \leq s \leq u \leq t \leq T, \quad (1.1.3)$$

then there exists a unique function $I: [0, T] \rightarrow \mathbb{R}$ such that $I_0 = 0$ and

$$|I_t - I_s - A_{s,t}| \lesssim |t - s|^\gamma \quad \text{over } 0 \leq s \leq t \leq T.$$

Furthermore, the “integral” I in this theorem is constructed as limit of Riemann-type sums:

$$I_t = \lim_{|\mathcal{P}| \rightarrow 0} \sum_{[t_k, t_{k+1}] \in \mathcal{P}} A_{t_k, t_{k+1}}, \quad (1.1.4)$$

where the limit runs over partitions $\mathcal{P} = \{0 = t_0 < t_1 < \dots < t_n = t\}$ of $[0, t]$ whose mesh $|\mathcal{P}| := \max_{k=0, \dots, n-1} |t_{k+1} - t_k|$ vanish.

Let us stress that the assumption $\gamma > 1$ is not purely cosmetic and in fact plays an essential part in the proof of this theorem¹.

Before proposing a few examples of applications of Theorem 1.1.2, we would like to mention that similar “sewing” results have been developed in a number of contexts, see for instance [FLPM08; DS13; Lê20; FS21; BL19; BL21; BL20].

¹Although we will later want to relax this constraint, see Section 1.1.5 below.

Example 1: Young–Kondurar integration

The Sewing Lemma is a tool which permits the construction of many objects in Rough Path theory. Let us shortly illustrate its power by recovering the classical Young–Kondurar theory [You36; Kon37] of integration for functions whose Hölder exponents sum to a value strictly greater than 1. Let $0 < \alpha, \beta < 1$ with $\alpha + \beta > 1$, and let $f \in C^\alpha([0, T])$, $g \in C^\beta([0, T])$, that is, f and g admit Hölder exponents α and β respectively. As suggested by the Observation 1.1.1, set

$$A_{s,t} := f_s(g_t - g_s),$$

then for $0 \leq s \leq u \leq t \leq T$,

$$A_{s,t} - A_{s,u} - A_{u,t} = (f_s - f_u)(g_t - g_u),$$

which is a $O(|t - s|^{\alpha+\beta})$ by the Hölder assumptions. Since $\alpha + \beta > 1$, the condition (1.1.3) is fulfilled and one can define

$$\int_0^t f_s dg_s := I_t = \lim_{\substack{\mathcal{P} \text{ partition of } [0,t] \\ |\mathcal{P}| \rightarrow 0}} \sum_{[t_k, t_{k+1}] \in \mathcal{P}} f_{t_k}(g_{t_{k+1}} - g_{t_k}),$$

where the convergence of the Riemann sum is guaranteed by the Sewing Lemma.

Example 2: Integration in lower regularity

In some cases however we want to consider functions whose Hölder exponents sum to a value smaller than 1. Let $0 < \alpha, \beta < 1$ with $\alpha + \beta \leq 1$, and let $f \in C^\alpha([0, T])$, $g \in C^\beta([0, T])$. From the calculation above, one observes that the germ $A_{s,t} := f_s(g_t - g_s)$ is not sufficient to define a canonical integral $\int f_s dg_s$ via an application of the Sewing Lemma. The solution proposed by the theory of Rough Paths is to assume that f has a further Taylor-type expansion with respect to g : more precisely, we assume that there exists a “Gubinelli derivative” $f' \in C^\alpha([0, T])$ such that

$$f_t = f_s + f'_s(g_t - g_s) + O(|t - s|^\gamma),$$

for some $\gamma > \alpha$. Then inserting this expansion in the integral as in (1.1.2) suggests that we should refine our germ by setting

$$A_{s,t} := f_s(g_t - g_s) + f'_s \int_s^t (g_u - g_s) dg_u,$$

where of course we need to ensure that the integral of g against itself indeed makes sense in some way. For this purpose, let us assume for the moment that g admits a Hölder exponent $\beta > \frac{1}{2}$ so that $\int_s^t (g_u - g_s) dg_u$ has a meaning as a Young–Kondurar integral. Now for $0 \leq s \leq u \leq t \leq T$,

$$A_{s,t} - A_{s,u} - A_{u,t} = (f_s - f_u - f'_u(g_s - g_u))(g_t - g_u) + (f'_s - f'_u) \int_u^t (g_r - g_s) dg_r,$$

which, by our assumptions, is a $O(|t - s|^{\min(\gamma+\beta, \alpha+2\beta)})$ where the exponent here is an actual improvement over $\alpha + \beta$ and thus may become strictly greater than 1, in which case the Sewing Lemma provides a natural and canonical notion of integral as²

$$\begin{aligned} \int_0^t f_s dg_s &:= I_t \\ &= \lim_{\substack{\mathcal{P} \text{ partition of } [0,t] \\ |\mathcal{P}| \rightarrow 0}} \sum_{[t_k, t_{k+1}] \in \mathcal{P}} \left(f_{t_k}(g_{t_{k+1}} - g_{t_k}) + f'_{t_k} \int_{t_k}^{t_{k+1}} (g_u - g_{t_k}) dg_u \right). \end{aligned}$$

²Notice however that the left-hand side depends not only on f , g , but also on the choice of f' (which may not be unique) and possibly of the integration theory used to define the iterated integral of g .

On the other hand, if the obtained exponent is still less or equal than 1, then the discussion above suggests that we should pursue the Taylor expansion of f against the *signature* of g , i.e. the sequence of its iterated integrals

$$\left(\int_{s \leq u_1 \leq \dots \leq u_n \leq t} dg_{u_1} \otimes \dots \otimes dg_{u_n} \right)_{n \in \mathbb{N}},$$

and that the signature itself should play an important role.

Of course, if g is not regular enough, in particular when the Hölder exponent of g is $\beta \leq \frac{1}{2}$ (which is the case for the important example of Brownian motion), the theory of Young–Kondurak integration alone is not sufficient in general to define the aforementioned sequence of iterated integrals.

The solution proposed by the theory of Rough Paths is to replace the signature here by a *rough path*, i.e. any sequence of two-parameter functions satisfying the nice algebraic and analytic properties which are enjoyed by the actual signature of smooth functions. This insight turns out to permit the construction of a strong integration theory.

1.1.3 Rough Paths and Hopf algebras

It has been known since the works of Chen [Che54] that the theory of signatures is algebraically rich. For instance, one can encode the signature of a smooth path $X: [0, T] \rightarrow \mathbb{R}^d$ as a biprocess \mathbb{X} acting on words over the alphabet $\{1, \dots, d\}$ via

$$\langle \mathbb{X}_{s,t}, i_1 i_2 \dots i_n \rangle := \int_{s \leq u_1 \leq \dots \leq u_n \leq t} dX_{u_1}^{i_1} \dots dX_{u_n}^{i_n},$$

which satisfies the following Chen’s relation for words w

$$\langle \mathbb{X}_{s,t}, w \rangle - \langle \mathbb{X}_{s,u}, w \rangle - \langle \mathbb{X}_{u,t}, w \rangle = \langle \mathbb{X}_{s,u} \otimes \mathbb{X}_{u,t}, \Delta' w \rangle,$$

(where Δ' denotes the reduced deconcatenation coproduct and is defined by $\Delta'(i_1 \dots i_n) = \sum_{k=1}^{n-1} i_1 \dots i_k \otimes i_{k+1} \dots i_n$), as well as the integration by parts formula, which takes the form for words w_1, w_2 ,

$$\langle \mathbb{X}_{s,t}, w_1 \sqcup w_2 \rangle = \langle \mathbb{X}_{s,t}, w_1 \rangle \langle \mathbb{X}_{s,t}, w_2 \rangle,$$

(where \sqcup denotes the shuffle product and is recursively defined for words w_1, w_2 and letter i_1, i_2 by $w_1 i_1 \sqcup w_2 i_2 = (w_1 \sqcup w_2 i_2) i_1 + (w_1 i_1 \sqcup w_2) i_2$). It turns out that these algebraic properties can be encoded in the more general framework of Hopf algebras, which we will not rigorously introduce here for the sake of simplicity (see for instance [Car07; CP21] for general treatments of Hopf algebras).

Thus, we will consider $H = \bigoplus_{n \in \mathbb{N}} H_n$ a graded Hopf algebra with product \cdot and reduced coproduct Δ' . Inspired by the above discussion, we say that a biprocess $\mathbb{X}: [0, T]^2 \rightarrow H^*$ is a (H, α) -*rough path* if

$$\begin{aligned} \langle \mathbb{X}_{s,t}, \sigma \cdot \tau \rangle &= \langle \mathbb{X}_{s,t}, \sigma \rangle \langle \mathbb{X}_{s,t}, \tau \rangle \quad \text{for } \sigma, \tau \in H, \\ \langle \mathbb{X}_{s,t}, \sigma \rangle - \langle \mathbb{X}_{s,u}, \sigma \rangle - \langle \mathbb{X}_{u,t}, \sigma \rangle &= \langle \mathbb{X}_{s,u} \otimes \mathbb{X}_{u,t}, \Delta' \sigma \rangle \quad \text{for } \sigma \in H, \\ |\langle \mathbb{X}_{s,t}, \sigma \rangle| &\lesssim |t - s|^{\alpha n} \quad \text{for } \sigma \in H_n. \end{aligned}$$

A few example of Hopf algebras of interest in this context are:

1. The shuffle algebra on the alphabet $\{1, \dots, d\}$, which gives rise to the theory of (weakly) geometric rough paths on \mathbb{R}^d [Lyo98]. In this case, H_n corresponds to (linear combinations of) words of length n .

2. The Butcher-Connes-Kreimer algebra of rooted forests with nodes decorated by $\{1, \dots, d\}$, which gives rise to the theory of branched rough paths on \mathbb{R}^d [Gub10].
3. Quasi-shuffle algebras, which gives rise to the theory of quasi-geometric rough paths [Bel20].
4. The Hopf algebra of Lie group integrators, which gives rise to the theory of planarly branched rough paths [CEFMMK20].

1.1.4 The question of extending rough paths

One important question is the following: given an α -rough path \mathbb{X} defined on $\oplus_{n=0}^N H_n$, can one extend it to $\oplus_{n=0}^{N+1} H_n$ while still preserving the three properties above? Or informally: “given a function $X: [0, T] \rightarrow \mathbb{R}^d$, does there exist a suitable sequence of iterated integrals of X ?”.

The positive answer was provided in 2007 by Lyons and Victoir [LV07] with an argument relying on the Lie structure of the characters on H .

Let us however suggest a (seemingly) more direct approach in terms of sewing: let $\sigma \in H_{N+1}$, we are looking for a biprocess $\langle \mathbb{X}_{s,t}, \sigma \rangle$ satisfying

$$\langle \mathbb{X}_{s,t}, \sigma \rangle - \langle \mathbb{X}_{s,u}, \sigma \rangle - \langle \mathbb{X}_{u,t}, \sigma \rangle = \langle \mathbb{X}_{s,u} \otimes \mathbb{X}_{u,t}, \Delta' \sigma \rangle, \quad |\langle \mathbb{X}_{s,t}, \sigma \rangle| \lesssim |t - s|^{\alpha(N+1)}. \quad (1.1.5)$$

It is not difficult to define a biprocess satisfying the first condition: setting $A_{s,t} := \langle \mathbb{X}_{0,s} \otimes \mathbb{X}_{s,t}, \Delta' \sigma \rangle$ (which is well-defined because $\Delta' \sigma$ only depends on $H_{\leq N}$), a straightforward computation gives

$$A_{s,t} - A_{s,u} - A_{u,t} = \langle \mathbb{X}_{s,u} \otimes \mathbb{X}_{u,t}, \Delta' \sigma \rangle,$$

but A will not in general satisfy $|A_{s,t}| \lesssim |t - s|^{\alpha(N+1)}$. On the other hand, A satisfies the assumption of the Sewing Lemma, Theorem 1.1.2, for $\gamma = \alpha(N+1)$, since it is straightforward to obtain the bound

$$|\langle \mathbb{X}_{s,u} \otimes \mathbb{X}_{u,t}, \Delta' \sigma \rangle| \lesssim |t - s|^{\alpha(N+1)},$$

and now $\langle \mathbb{X}, \sigma \rangle_{s,t} := A_{s,t} - (I_t - I_s)$, where I is the function provided by the Sewing Lemma, satisfies both constraints of (1.1.5), thus allowing to extend the rough path as wanted.

The catch here is that the Sewing Lemma as stated in Theorem 1.1.2 requires $\gamma > 1$, and thus the argument sketched above is valid only for the levels $N \geq \lfloor 1/\alpha \rfloor$: novel ideas are required for the lower levels.

1.1.5 Contribution I: the Sewing Lemma for $0 < \gamma \leq 1$

In this work written in collaboration with Lorenzo Zambotti, we prove that the Sewing Lemma, Theorem 1.1.2, can be extended to the regime $0 < \gamma \leq 1$:

Theorem 1.1.3 (L.B.–L.Zambotti, [BZ22, Theorem 2.2]). *Let $0 < \gamma \leq 1$ and $A: [0, T]^2 \rightarrow \mathbb{R}$ be a continuous function such that*

$$|A_{s,t} - A_{s,u} - A_{u,t}| \lesssim |t - s|^\gamma \text{ over } 0 \leq s \leq u \leq t \leq T,$$

then there exists a (non-unique) function $I: [0, T] \rightarrow \mathbb{R}$ such that $I_0 = 0$ and

$$|I_t - I_s - A_{s,t}| \lesssim \begin{cases} |t - s|^\gamma & \text{if } 0 < \gamma < 1, \\ |t - s|(1 + |\log |t - s||) & \text{if } \gamma = 1, \end{cases} \text{ over } 0 \leq s \leq t \leq T.$$

Let us comment briefly on some aspects of this result.

First, notice that the integral I here can not be unique anymore: let $J: [0, T] \rightarrow \mathbb{R}$ be *any* non-vanishing function such that $J_0 = 0$ and $|J_t - J_s| \lesssim |t - s|^\gamma$ if $0 < \gamma < 1$, $|J_t - J_s| \lesssim |t - s|(1 + |\log |t - s||)$ if $\gamma = 1$ (and such functions do exist), then $I + J \neq I$ is another solution to Theorem 1.1.3.

Second, notice that the integral I can not be constructed as a limit of Riemann sums as in (1.1.4) anymore. A simple counterexample is given by the function $A_{s,t} := \sqrt{t - s}$, which satisfies

$$|A_{s,t} - A_{s,u} - A_{u,t}| \lesssim |t - s|^{\frac{1}{2}},$$

yet if $\mathcal{P} = \{0 = t_0 < t_1 < \dots < t_n = t\}$ is a partition of $[0, t]$ of mesh $|\mathcal{P}| := \max_{k=0, \dots, n-1} |t_{k+1} - t_k|$, then the immediate estimate $\sqrt{t_{k+1} - t_k} \geq (t_{k+1} - t_k)/\sqrt{|\mathcal{P}|}$ yields

$$\sum_{[t_k, t_{k+1}] \in \mathcal{P}} A_{t_k, t_{k+1}} \geq \frac{t}{\sqrt{|\mathcal{P}|}},$$

which diverges as $|\mathcal{P}| \rightarrow 0$. Indeed, the construction of I in Theorem 1.1.3 relies on a different approach: we define I recursively on the dyadics, then extend I on $[0, T]$ by continuity.

Third, notice the great similarity between Theorem 1.1.3 and the Reconstruction Theorem (see Theorem 1.2.1 below), where the case $\gamma \leq 0$ provides a non-unique reconstruction which is built with different approximations as in the case $\gamma > 0$ (and where a logarithmic factor appears in the case $\gamma = 0$). Indeed, we prove that (a slightly weaker version) of Theorem 1.1.3 can be obtained as a corollary of the reconstruction theorem by considering the germ $F_t := \partial_2 A_{t, \cdot}$, where the differentiation in the second variable is understood in the distributional sense.

Finally, notice that – as motivated in Section 1.1.4 above – it is natural for Theorem 1.1.3 to admit applications in the context of the theory of Rough Paths, which is concerned with the study of (possibly non-canonical) iterated integrals. In particular, Theorem 1.1.3 permits to provide a simple proof of the Lyons–Victoir extension theorem [LV07], which states that given any α -Hölder path X for some $\alpha \in (0, 1)$, $\alpha^{-1} \notin \mathbb{N}$, there exists a (non-canonical) α -rough path lying above X . Our construction is furthermore continuous (with respect to the natural topologies).

In fact, we establish a slightly more general result of parametrisation of rough paths, stated in the general framework of Hopf algebras.

Theorem 1.1.4 (L.B.–L.Zambotti [BZ22, Theorem 4.7]). *Let H be a commutative graded connected locally finite Hopf algebra. Let $\alpha \in (0, 1)$, $\alpha^{-1} \notin \mathbb{N}$ and denote $\text{RP}^\alpha(H)$ the set of (H, α) -rough paths. Then there exists a finite subset B of H such that $\text{RP}^\alpha(H)$ is homeomorphic to the vector space*

$$\mathcal{C}_B^\alpha := \{(f^h)_{h \in B} : f^h \in \mathcal{C}^{\lfloor h \rfloor \alpha}, f^h(0) = 0\}.$$

More precisely, one such homeomorphism $\mathcal{P}: \text{RP}^\alpha(H) \rightarrow \mathcal{C}_B^\alpha$ is explicitly produced in terms of the map $A \mapsto I$ from Theorem 1.1.3. The interest of this parametrisation can be illustrated by the following applications: the map $f \mapsto \mathcal{P}^{-1}(f)$ yields a continuous Lyons–Victoir extension; and the map $(f, \mathbb{X}) \mapsto \mathcal{P}^{-1}(f + \mathcal{P}(\mathbb{X}))$ yields a continuous free transitive action of \mathcal{C}_B^α on $\text{RP}^\alpha(H)$, which retrieves a result from [TZ20].

1.2 Germs: motivations from Regularity Structures

1.2.1 Singular SPDEs

The theory of Regularity Structures is concerned with stochastic partial differential equations of the form

$$Lf = F(f, \xi),$$

where L is a differential operator, F is a non-linearity, ξ is a forcing noise, and f is the solution to find. Let us heuristically motivate the interest of germs, following a similar discussion from [Che21], by considering the example of the Φ_d^4 equation, for which the differential operator $L = \partial_t - \Delta$ is the heat operator and the non-linearity $F(f, \xi) = -f^3 + \xi$ involves the cube of f , that is, we consider

$$\partial_t f = \Delta f - f^3 + \xi, \quad \text{where } f = f(t, x), t \in \mathbb{R}, x \in \mathbb{R}^d.$$

This equation appears naturally in quantum field theory [PW81]. To solve it, one strategy is to start by rewriting it through a mild formulation

$$f = H * (-f^3 + \xi), \tag{1.2.1}$$

where H denotes the heat kernel, and now one wants to find such a fixed point f as limit of the Picard iteration

$$f_0 := 0, \quad f_{n+1} := H * (-f_n^3 + \xi), \tag{1.2.2}$$

that is, $f_1 := H * \xi$, $f_2 := H * (-(H * \xi)^3 + \xi)$, etc.

It turns out that a nice framework in general to set up such fixed-point problems is provided by the Hölder spaces C^α , which can be generalised for negative exponents as spaces of distributions.

1.2.2 Hölder and Zygmund spaces

In general, we will work in the space $\mathcal{D}'(\mathbb{R}^d)$ of (Schwartz) distributions, i.e. the topological dual of the space of test-functions $\mathcal{D}(\mathbb{R}^d) := C_c^\infty(\mathbb{R}^d)$. Within this framework one can generalise the notion of Hölder regularity to any real $\alpha \in \mathbb{R}$ by testing the behaviour of distributions $f \in \mathcal{D}'(\mathbb{R}^d)$ against suitable rescaled test-functions ψ_x^λ (recall that if $\psi \in \mathcal{D}(\mathbb{R}^d)$, we define $\psi_x^\lambda(\cdot) := \lambda^{-d} \psi(\lambda^{-1}(y - x))$): more precisely, we say that $f \in C^\alpha(\mathbb{R}^d)$ if

$$\sup_{x \in \mathbb{R}^d, \lambda \in (0, 1], \psi \in \mathcal{B}^r} |f(\psi_x)| < +\infty, \quad \text{and} \quad \sup_{x \in \mathbb{R}^d, \lambda \in (0, 1], \psi \in \mathcal{B}_\alpha^r} \frac{|f(\psi_x^\lambda)|}{\lambda^\alpha} < +\infty, \tag{1.2.3}$$

where r denotes any integer $r > -\alpha$; \mathcal{B}^r corresponds to the space of test-functions $\psi \in \mathcal{D}(\mathbb{R}^d)$ such that $\text{supp}(\psi) \subset B(0, 1)$, and $\|\psi\|_{C^r} := \max_{|k| \leq r} \|\partial^k \psi\|_\infty \leq 1$; and where \mathcal{B}_α^r denotes the space of test-functions $\psi \in \mathcal{B}^r$ such that $\int x^k \psi(x) dx = 0$ for $0 \leq |k| \leq \alpha$.

When $\alpha \in (0, 1)$, such distributions f can be shown to coincide (as the terminology ‘‘Hölder space’’ suggests) with actual functions satisfying $|f_y - f_x| \lesssim |y - x|^\alpha$.

At this point, one can propose some remarks. First, observe that the definition above subtly depends on the way we define the scaling ψ_x^λ of a test-function. In some cases it is convenient to slightly modify this notion: for instance when working with the heat operator $\partial_t - \Delta$ it is useful to encode the fact that time counts ‘‘twice as much’’ as space, by defining rather $\psi_x^\lambda(\cdot) := \lambda^{-(d+1)} \psi(\lambda^{-2}(y_1 - x_1), \lambda^{-1}(y_2 - x_2), \dots, \lambda^{-1}(y_d - x_d))$, where the first variable is the time-variable.

Another remark is that this definition gives rise to a slight conflict of notation at integer exponents, since for $n \in \mathbb{N}$ the space C^n as defined just above does *not* coincide with the space of n -times continuously differentiable functions (as an example, observe that $x \mapsto \log|x| \in C^0$ with the definition above). For this reason, the spaces as defined in (1.2.3) are also sometimes noted \mathcal{Z}^α and called the (Hölder)–Zygmund spaces, see [FH20, Section 14.3] for a presentation and Chapter 4 of this manuscript where we use this notation. For the sake of simplicity, we keep the notation C^α throughout this introduction.

The spaces C^α enjoy many nice properties, in particular:

1. (White noise) one can show that gaussian space-time white noise ξ in (space) dimension d almost surely belongs to $C^{-d/2+1-\kappa}(\mathbb{R}^{d+1})$ for any $\kappa > 0$ (where the Hölder regularity is measured here with the parabolic scaling), see [FM17];
2. (Schauder estimates) the heat kernel is 2-regularising i.e. the convolution with H is a continuous linear map from C^α to $C^{\alpha+2}$ for all $\alpha \in \mathbb{R}$ (this is an instance of the so-called Schauder estimates, see Chapter 4 below for a general discussion);
3. (Young multiplication) given $\alpha, \beta \in \mathbb{R}$, one can extend (in a canonical way) the usual pointwise multiplication of smooth functions $C^\infty \times C^\infty \rightarrow C^\infty$ to $C^\alpha \times C^\beta \rightarrow C^{\min(\alpha, \beta)}$, if and only if $\alpha + \beta > 0$ (see Section 1.2.4 below for a discussion and a sketch of proof using the reconstruction theorem).

1.2.3 The example of the Φ_d^4 equation and germs

Let us return to the fixed point problem (1.2.1), we heuristically discuss the Picard iteration (1.2.2) in (space) dimensions $d = 1, 2, 3$ successively.

When $d = 1$, the space-time dimension is 2 and so $\xi \in C^{-3/2^-}$, whence (by the Schauder estimates) $H * \xi \in C^{1/2^-}$, thus (by Young multiplication) its cube $(H * \xi)^3$ is properly defined and we can follow the Picard iteration (1.2.2) to solve (1.2.1) in the space $C^{1/2^-}$.

When $d = 2$, $\xi \in C^{-2^-}$ thus $H * \xi \in C^{0^-}$ which is unfortunately not sufficient for the cube $(H * \xi)^3$ to be properly defined, and so the Picard iteration (1.2.2) can not be performed. One idea going back to Da Prato and Debussche [DPD03] is to remark that in this Picard iteration, the term $\tilde{\xi} := H * \xi \in C^{0^-}$ should be the one with lowest regularity, suggesting that $g := f - \tilde{\xi}$ should have better regularity than C^{0^-} . Indeed, observe that the equation (1.2.1) implies (formally) on g :

$$g = -H * (g^3 + 3\tilde{\xi}g^2 + 3\tilde{\xi}^2g + \tilde{\xi}^3), \quad (1.2.4)$$

which can be solved via Picard iteration in C^{2^-} , as soon as one gives a sensible meaning to the powers $\tilde{\xi}^2 \in C^{0^-}$ and $\tilde{\xi}^3 \in C^{0^-}$, but this can be done by so-called Wick calculus techniques, exploiting probabilistic properties of the noise ξ .

When $d = 3$, $\xi \in C^{-3/2^-}$ thus $\tilde{\xi} := H * \xi \in C^{-1/2^-}$, whose cube is still ill-defined by Young multiplication. However, one can try the Da-Prato–Debusche trick above and it is still possible to define $\tilde{\xi}^2 \in C^{-1^-}$ and $\tilde{\xi}^3 \in C^{-3/2^-}$ in a sensible way via Wick calculus. A further problem arises though when performing the Picard iteration for (1.2.4), i.e.

$$g_0 := 0, \quad g_{n+1} := -H * (g_n^3 + 3\tilde{\xi}g_n^2 + 3\tilde{\xi}^2g_n + \tilde{\xi}^3).$$

Indeed,

$$g_1 = -H * \tilde{\xi}^3 \in C^{1/2^-},$$

whence the product $\tilde{\xi}^2g_1$ in g_2 is an ill-defined product of a C^{-1^-} distribution with a $C^{1/2^-}$ function. Here, one may still hope to pursue the Da Prato–Debusche technique and

iteratively remove the term of worst regularity. However, upon writing the calculations, one ends up in an infinite loop: the reason is that one can not expect to multiply a C^{-1^-} distribution with a $C^{1/2^-}$ function and obtain a distribution of regularity better than C^{-1^-} in general, as the regularity of a product is the minimum of the regularities.

An important remark here is that on the other hand, the regularity of the product of *local approximations around a point x* should be the *sum* of the local regularities around x : as an illustration of this principle, observe that if $|f_x(y)| \lesssim |y - x|^\alpha$ and $|g_x(y)| \lesssim |y - x|^\beta$ then $|f_x g_x(y)| \lesssim |y - x|^{\alpha+\beta}$.

This suggests that if, instead of working with distributions, we start to work with families of local approximations of distributions, then we could “improve” the regularities that appear in the Picard iterations, and successfully perform (iterated) Da Prato–Debussche type tricks for the equation Φ_d^4 in dimension $d = 3$ as discussed above (and also for many more similar singular SPDEs). This is one of the (many) ideas which were successfully implemented by Hairer in his theory of Regularity Structures [Hai14].

In general, we define a *germ* to be any such family $(F_x)_{x \in \mathbb{R}^d}$ of distributions $F_x \in \mathcal{D}'(\mathbb{R}^d)$. (For technical reasons we will also add the condition that for any test-function $\varphi \in \mathcal{D}(\mathbb{R}^d)$, the function $x \mapsto F_x(\varphi)$ is measurable). Of course, if we choose to work with germs rather than distributions, then we need to ensure that the germs we work with indeed correspond to local approximations of distributions of interest. In other words, the question of retrieving a distribution given a suitable family of local approximations needs to be addressed.

1.2.4 The Reconstruction Theorem

The discussion above thus motivates the following general question, called the *reconstruction problem*:

Given a germ $F = (F_x)_{x \in \mathbb{R}^d}$, does there exist a (unique) distribution $f \in \mathcal{D}'(\mathbb{R}^d)$ such that f is “sufficiently close to” F_x locally around x ?

An elegant answer to this problem is provided by the following Reconstruction Theorem, originally established by Hairer [Hai14] and later stated in the following formulation by Caravenna and Zambotti [CZ20].

Theorem 1.2.1 (Reconstruction Theorem). *Let $F = (F_x)_{x \in \mathbb{R}^d}$ be a germ such that there exist reals $\alpha \leq \gamma$ and a test-function $\varphi \in \mathcal{D}(\mathbb{R}^d)$ with $\int \varphi \neq 0$, such that the following estimate of “coherence”*

$$|(F_z - F_y)(\varphi_y^\lambda)| \lesssim \lambda^\alpha (|y - z| + \lambda)^{\gamma - \alpha}, \quad (1.2.5)$$

holds uniformly over z, y in compact sets, $\lambda \in (0, 1]$. Then there exists a distribution $\mathcal{R}(F) \in \mathcal{D}'(\mathbb{R}^d)$ such that

$$|(\mathcal{R}(F) - F_x)(\psi_x^\lambda)| \lesssim \lambda^\gamma, \quad (1.2.6)$$

uniformly over x in compact sets, $\lambda \in (0, 1]$, $\psi \in \mathcal{B}^r := \{\psi \in \mathcal{D}(\mathbb{R}^d), \text{supp}(\psi) \subset B(0, 1), \|\psi\|_{C^r} := \max_{|k| \leq r} \|\partial^k \psi\|_\infty \leq 1\}$, for any integer $r > -\alpha$.

Furthermore, such a distribution $\mathcal{R}(F)$ is unique if and only if $\gamma > 0$.

See also [HL17; MW20; RS21; BL22; ZK21] for similar “reconstruction” results in different contexts.

Note that the theorem as stated just above does not specify whether f admits Hölder regularity. However, one can establish that $\mathcal{R}(F) \in C^\beta$ for some $\beta \leq \gamma$, $\beta < 0$, as soon as the further condition of “homogeneity”

$$|F_y(\varphi_y^\lambda)| \lesssim \lambda^\beta, \quad (1.2.7)$$

holds uniformly over y in compacts and $\lambda \in (0, 1]$, where φ is the same test-function as in (1.2.5).

It is now convenient to introduce the notation $\mathcal{G}^{\beta;\alpha,\gamma}$ to denote the set of germs satisfying (1.2.5) and (1.2.7) (which can in fact be shown to be a vector space), as one can reformulate the Reconstruction Theorem as the existence (and uniqueness iff $\gamma > 0$) of a linear continuous operator

$$\mathcal{R}: \mathcal{G}^{\beta;\alpha,\gamma} \rightarrow C^\beta,$$

such that $\mathcal{R}(F) - F \in \mathcal{G}^{\gamma;\alpha,\gamma}$ for all $F \in \mathcal{G}^{\beta;\alpha,\gamma}$.

Let us shortly illustrate the power of the Reconstruction Theorem in two elementary situations: we refer to [CZ20] for more details.

Example 1: Taylor germs

Let $\alpha > 0$, $\alpha \notin \mathbb{N}$, and $f \in C^\alpha$ be a α -Hölder function. Then by the usual properties of Hölder functions, f admits derivatives up to degree α and we can introduce the Taylor germ

$$T_x^f(y) := \sum_{0 \leq |k| < \alpha} f^{(k)}(x) \frac{(y-x)^k}{k!}, \quad (1.2.8)$$

which furthermore satisfies $|f(y) - T_x^f(y)| \lesssim |y-x|^\alpha$ and as such provides a natural local approximation of f around x . Now an elementary calculation establishes

$$T^f \in \mathcal{G}^{0;0,\alpha}, \text{ and } \mathcal{R}(T^f) = f \text{ (as expected).}$$

Example 2: Young multiplication

Another elegant application of the Reconstruction Theorem permits to recover the celebrated result of multiplication of distributions whose Hölder exponents sum to a strictly positive value. Note that this can be seen as a “differentiated” version of the theory of Young–Kondurav integration, here constructing the integrand rather than the integral.

Let $\alpha > 0$, $\alpha \notin \mathbb{N}$, and $f \in C^\alpha$ be a α -Hölder function, so that we can consider its Taylor germ T^f as defined in (1.2.8). Now if $\beta \leq 0$ and $g \in C^\beta$, we can postulate that the (to-be-defined) product gf should be approximated around x by gT_x^f , i.e. the (well-defined) product of the distribution g with the C^∞ function T_x^f . This motivates the introduction of the germ

$$F_x := gT_x^f,$$

for which a straightforward calculation establishes

$$F \in \mathcal{G}^{\beta;\beta,\alpha+\beta},$$

and setting $gf := \mathcal{R}(F)$ continuously extends the product from $C^\beta \times C^\infty \rightarrow C^\beta$ to $C^\beta \times C^\alpha \rightarrow C^\beta$ in a canonical way as soon as $\alpha + \beta > 0$.

1.2.5 Contribution II: a Besov Reconstruction Theorem

In this work, written in collaboration with David Lee, we establish a reconstruction theorem with a Besov flavour.

Besov spaces are spaces of distributions which generalise the Hölder spaces, and the literature proposes (many) different equivalent ways of defining them. One possible way to measure Besov regularity of a distribution f is – similarly to the Hölder case – to consider the behaviour of $f(\psi_x^\lambda)/\lambda^\alpha$, but now instead of assuming this quantity to be bounded, we

rather impose assumptions of *integrability* in x and λ . More precisely, let $p, q \in [1, +\infty]$ and $\alpha \in \mathbb{R}$, we define the Besov space $\mathcal{B}_{p,q}^\alpha$ as the space of distributions $f \in \mathcal{D}'(\mathbb{R}^d)$ such that:

$$\left\| \sup_{\psi \in \mathcal{B}^r} |f(\psi_x)| \right\|_{L^p(x)} + \left\| \left\| \sup_{\psi \in \mathcal{B}_\alpha^r} \left| \frac{f(\psi_x^{2^{-n}})}{2^{-n\alpha}} \right| \right\|_{L^p(x)} \right\|_{\ell^q(n)} < +\infty,$$

where we denote \mathcal{B}_α^r the space of test-functions $\psi \in \mathcal{B}^r$ such that $\int x^k \psi(x) dx = 0$ for $0 \leq |k| \leq \alpha$. Note that taking $p = q = +\infty$ in this definition retrieves the Hölder–Zygmund space C^α .

Introducing the integrability exponents p and q permits greater flexibility in some applications. One example is the fact that for any $p \in [1, +\infty]$ the Dirac mass δ_0 at 0 is in $\mathcal{B}_{p,\infty}^{-d+d/p}$, and thus one can modulate the regularity exponent by “playing” with the integrability exponent. Another example is the fact that the spaces $\mathcal{B}_{2,2}^\alpha$ – also known as the (fractional) Sobolev spaces – enjoy a natural Hilbert structure and are well-studied in the context of Malliavin calculus. See [Lab19; GL20] for examples of applications in the context of SPDEs which were motivated by such ideas, and also [ST18; LPT21; FS21] for some works involving Besov-type spaces in the context of Rough Paths and Regularity Structures.

A reconstruction theorem in the framework of Besov spaces has been previously established by Hairer and Labbé in [HL17], where the reconstruction bound (1.2.6) is replaced by a version which takes into account integrability in x and λ . However, this theorem [HL17, Theorem 3.1] is presented in the formalism of regularity structures and proved using wavelet techniques.

We provide a generalisation (both of [HL17, Theorem 3.1] and of the “Hölder” reconstruction theorem of Caravenna–Zambotti [CZ20, Theorem 5.1]) which we state in the framework of distribution theory, and which we also prove using elementary techniques as in [CZ20]:

Theorem 1.2.2 (L.B.–D.Lee, [BL22, Theorem 3.2]). *Let $F = (F_x)_{x \in \mathbb{R}^d}$ be a germ such that there exist reals $p, q \in [1, +\infty]$, $\alpha, \beta \leq \gamma$, and a test-function $\varphi \in \mathcal{D}(\mathbb{R}^d)$ with $\int \varphi \neq 0$, such that for all $K \subset \mathbb{R}^d$,*

$$\left\| \left\| \left\| \frac{(F_{x+h} - F_x)(\varphi_x^{2^{-n}})}{2^{-n\alpha}(2^{-n} + |h|)^{\gamma-\alpha}} \right\|_{L^p(K, dx)} \right\|_{\ell^\infty(n \in \mathbb{N})} \right\|_{L^q(B(0,2), \frac{dh}{|h|^d})} < +\infty,$$

and

$$\left\| \left\| \frac{F_x(\varphi_x^{2^{-n}})}{2^{-n\beta}} \right\|_{L^p(K, dx)} \right\|_{\ell^\infty(n \in \mathbb{N})} < +\infty,$$

then there exists a distribution $\mathcal{R}(F) \in \mathcal{D}'(\mathbb{R}^d)$ such that for any integer $r > \max(-\alpha, -\beta)$ and any $K \subset \mathbb{R}^d$,

$$\left\| \left\| \sup_{\psi \in \mathcal{B}^r} \left| \frac{(\mathcal{R}(F) - F_x)(\psi_x^\lambda)}{k(\lambda)} \right| \right\|_{L^p(K, dx)} \right\|_{L^q(B(0,1), \frac{d\lambda}{\lambda})} < +\infty,$$

where $k(\lambda) = \lambda^\gamma$ if $\gamma \neq 0$; $k(\lambda) = 1 + |\log \lambda|$ if $\gamma = 0$, $q = +\infty$; $k(\lambda) = 1 + |\log \lambda|^{1+\kappa}$ for any $\kappa > 0$ if $\gamma = 0$, $q < +\infty$.

Furthermore, such a distribution $\mathcal{R}(F)$ is unique if and only if $\gamma > 0$.

Also, as soon as $\beta < 0$, $\mathcal{R}(F) \in \mathcal{B}_{p,\infty}^\beta$.

In fact, our main theorem [BL22, Theorem 4.5] is slightly stronger, but we refrain from presenting it here because it demands the introduction of heavier notations. At this point,

we also would like to mention the paper [ZK21] which provides a reconstruction result in the general case of quasi-normed spaces (including the case of the Besov spaces).

As in the case of the Hölder reconstruction theorem, see our discussion in Section 1.2.4, it is natural to expect Theorem 1.2.2 to retrieve a result of Young multiplication, this time in the more general Besov spaces. Indeed, the same approach as in Section 1.2.4 recovers the following result (which is already known in the literature, see e.g. [Mar18, Corollary 2.1.35] or [Zui20, Theorem 19.7] for alternative proofs using paraproduct techniques).

Theorem 1.2.3 (L.B.–D.Lee, [BL22, Theorem 3.12]). *Let $p_1, p_2, q_1, q_2 \in [1, +\infty]$ and let $p \in [1, +\infty]$ be defined by $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$. Let $\alpha, \beta \in \mathbb{R}$ be such that $\alpha < 0 < \beta$, $\alpha + \beta > 0$.*

Then there exists a bilinear continuous map $\mathcal{M}: \mathcal{B}_{p_1, q_1}^\alpha \times \mathcal{B}_{p_2, q_2}^\beta \rightarrow \mathcal{B}_{p, q_1}^\alpha$ which extends the usual (pointwise) product, i.e. when $g \in \mathcal{B}_{p_1, q_1}^\alpha$ and $f \in \mathcal{C}^\infty$, $\mathcal{M}(g, f) = gf$.

1.2.6 Contribution III: Schauder estimates for germs

In this work in preparation with Francesco Caravenna and Lorenzo Zambotti, we establish Schauder estimates in the context of coherent germs.

The celebrated “classical” Schauder estimates state that kernels with integrable singularities on the diagonal enjoy regularising properties. More precisely, let $\mathbf{K}: \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$ be such that for all multi-indices $k, l \in \mathbb{N}^d$,

$$|\partial_1^k \partial_2^l \mathbf{K}(x, y)| \lesssim |y - x|^{-d - |k| - |l| + \beta},$$

for some $\beta > 0$, then (assuming further technical assumptions, see Section 4.2.3 below for a more precise discussion), for all $\alpha \in \mathbb{R}$ the convolution with \mathbf{K} is a well-defined continuous map from C^α to $C^{\alpha+\beta}$: we say that \mathbf{K} is β -regularising.

This is a powerful tool in the context of partial differential equations as it allows to perform fixed-point arguments. Now in the context of stochastic partial differential equations, as suggested above one may want to argue at the level of germs rather than distributions, and it is natural to wonder whether one can “lift” the convolution operator in a meaningful way at the level of the spaces $\mathcal{G}^{\bar{\alpha}; \alpha, \gamma}$. More precisely, given $\bar{\alpha}, \alpha, \gamma \in \mathbb{R}$, does there exist a “nice” continuous linear map \mathcal{K} (and exponents $\bar{\alpha}', \alpha', \gamma' \in \mathbb{R}$) such that the following diagram commutes?

$$\begin{array}{ccc} \mathcal{G}^{\bar{\alpha}; \alpha, \gamma} & \xrightarrow{\mathcal{K}} & \mathcal{G}^{\bar{\alpha}'; \alpha', \gamma'} \\ \mathcal{R} \downarrow & & \downarrow \mathcal{R} \\ \mathcal{Z}^{\bar{\alpha}} & \xrightarrow{\mathbf{K}} & \mathcal{Z}^{\bar{\alpha} + \beta} \end{array} \quad (1.2.9)$$

In Chapter 4, we show that a natural answer to this question is provided by convolving the germ F with \mathbf{K} pointwise then subtracting a suitable “Taylor germ”:

Theorem 1.2.4 (L.B.–F.Caravenna–L.Zambotti). *Let $\bar{\alpha}, \alpha, \beta, \gamma \in \mathbb{R}$ satisfying $\bar{\alpha} \leq \min \gamma$, $\alpha \leq \gamma$, $\gamma \neq 0$, $\beta > 0$, $\bar{\alpha} + \beta \neq 0$, $\alpha + \beta \neq 0$, $\gamma + \beta \notin \mathbb{N}$, and let \mathbf{K} be a β -regularising kernel. Then the map*

$$\begin{aligned} \mathcal{K} : \mathcal{G}^{\bar{\alpha}; \alpha, \gamma} &\longrightarrow \mathcal{G}^{(\bar{\alpha} + \beta) \wedge 0; (\alpha + \beta) \wedge 0, \gamma + \beta} \\ F &\longmapsto \left(\mathbf{K} * F_x + \sum_{|k| < \gamma + \eta} (\mathbf{K} * (\mathcal{R}(F) - F_x))^{(k)}(x) \frac{(-x)^k}{k!} \right)_{x \in \mathbb{R}^d}, \end{aligned}$$

is well-defined and when $\gamma > 0$ the diagram in (1.2.9) commutes i.e. for all $F \in \mathcal{G}^{\bar{\alpha}; \alpha, \gamma}$,

$$\mathcal{R}(\mathcal{K}(F)) = \mathbf{K} * \mathcal{R}(F).$$

This statement is in some sense similar to that of Hairer’s multilevel Schauder estimates [Hai14, Theorem 5.12], which are however slightly more subtle. The reason is that in the context of SPDEs and Regularity Structures, we furthermore need to impose that the germs should admit further structure with respect to a given finite generating family of germs (which end up in practice containing the required “data” of the noise governing the equation). More precisely let Π^1, \dots, Π^n be a family of germs, germs of interest will be of the form

$$F_x = \sum_{i=1}^n f^i(x) \Pi_x^i,$$

where of course there is a priori no guarantee that such germs should satisfy the properties of coherence and homogeneity that are required for the reconstruction to exist. However, in practice the germs Π^i are stable by expansion and we have identities of the form

$$\Pi_y^i = \sum_{j=1}^n \Pi_x^j \Gamma_{x,y}^{i,j},$$

for some functions Γ . The couple (Π, Γ) is called a *model* in Hairer’s terminology. This allows to express

$$(F_y - F_x)(\varphi_x^\lambda) = \sum_{i=1}^n \left(\sum_{j=1}^n \Gamma_{x,y}^{i,j} f^j(y) - f^i(x) \right) \Pi_x^i(\varphi_x^\lambda),$$

and a straightforward calculation now shows that if Π^i satisfies the property of homogeneity (1.2.7) for some exponent $\bar{\alpha}_i$ and if there exists $\gamma > \max_i(\bar{\alpha}_i)$ such that

$$|f^i(x)| \lesssim 1, \quad \left| \sum_{j=1}^n \Gamma_{x,y}^{i,j} f^j(y) - f^i(x) \right| \lesssim |y - x|^{\gamma - \bar{\alpha}_i}, \quad (1.2.10)$$

then $F \in \mathcal{G}^{\alpha; \alpha, \gamma}$, where $\alpha = \min_i \bar{\alpha}_i$. Functions $f = (f^i)_i$ satisfying (1.2.10) are called “modelled distributions” in Hairer’s terminology and their space is denoted \mathcal{D}^γ .

Our calculations permit to recover Hairer’s multilevel Schauder estimates, the statement of which we now sketch (see the original theorem [Hai14, Theorem 5.12] or Theorem 4.4.10 below for a precise statement):

Theorem 1.2.5 (Multilevel Schauder estimates). *Under suitable (but not constraining) assumptions, the map \mathcal{K} defined in Theorem 1.2.4 induces a continuous linear map from \mathcal{D}^γ to $\tilde{\mathcal{D}}^{\gamma+\beta}$ where this last space is the space of modelled distributions with respect to a new³ (but explicit) model.*

Note that this kind of statement is stronger than Theorem 1.2.4 as the “condition of levels” (1.2.10) is stronger than requiring coherence and homogeneity.

³The fact that we change the model is not a problem in practice as we can generally choose to work with a (recursively constructed) model which is invariant by this transformation.

Chapter 2

The Sewing lemma for $0 < \gamma \leq 1$

Abstract

This chapter contains the results of [BZ22]. We establish a Sewing lemma in the regime $\gamma \in (0, 1]$, constructing a Sewing map which is neither unique nor canonical, but which is nonetheless continuous with respect to the standard norms. Two immediate corollaries follow, which hold on any commutative graded connected locally finite Hopf algebra: a simple constructive proof of the Lyons-Victoir extension theorem which associates to a Hölder path a rough path, with the additional result that this map can be made continuous; the bicontinuity of a transitive free action of a space of Hölder functions on the set of Rough Paths.

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2.1 Introduction

Rough paths were introduced by T. Lyons in [Lyo98] in order to give a robust theory for controlled ordinary differential equations: for $Y : [0, T] \rightarrow \mathbb{R}^k$ of class C^1 and $\sigma : \mathbb{R}^d \rightarrow \mathbb{R}^d \otimes \mathbb{R}^k$ smooth, one studies the equation

$$X_t = X_0 + \int_0^t \sigma(X_s) dY_s, \quad t \geq 0,$$

and the aim is to extend the map $Y \mapsto X$ to paths Y of class C^α with $\alpha \in (0, 1)$, in order to include the case of Brownian motion and therefore stochastic differential equations.

A few years later, a new analytical tool was introduced to study such rough differential equations: the Sewing lemma [Gub04; FLP06], which allows to define uniquely a notion of integral $I_t = \int_0^t X_s dY_s$ in situations where X and Y may be paths of low regularity. For example, in the so called Young regime, i.e. when X and Y have Hölder regularity α resp. β with $\alpha + \beta > 1$, it is a classical result due to Young [You36] and Kondurar [Kon37] that a canonical integration theory exists for $I_t = \int_0^t X_s dY_s$. The Sewing Lemma recovers this setting by showing that there exists *one and only one* $I : [0, T] \rightarrow \mathbb{R}$ such that

$$I_0 = 0, \quad |I_t - I_s - Y_s(X_t - X_s)| \lesssim |t - s|^{\alpha+\beta}.$$

This raises the following general question: given $A : [0, T]^2 \rightarrow \mathbb{R}$ and $\gamma > 0$, does there exist $I : [0, T] \rightarrow \mathbb{R}$ satisfying $|I_t - I_s - A_{s,t}| \lesssim |t - s|^\gamma$ uniformly over $s \leq t \in [0, T]$? The Sewing lemma gives a simple answer to this question when $\gamma > 1$: it asserts that such an I exists (uniquely if $I_0 = 0$) as soon as $|A_{s,t} - A_{s,u} - A_{u,t}| \lesssim |t - s|^\gamma$ uniformly over $s \leq u \leq t \in [0, T]$. This generalises Young-Kondurar's integration theory, and allows to build a well-posedness theory for rough differential equations.

Surprisingly, the Sewing lemma has not been extended yet to cover the case $\gamma \in (0, 1]$. It is clear that the situation is different, because this time the relation $|I_t - I_s - A_{s,t}| \lesssim |t - s|^\gamma$ does not characterise I anymore: indeed, any other \tilde{I} has the same property if and only if $I - \tilde{I}$ is γ -Hölder. However, existence of I under the hypothesis that $|A_{s,t} - A_{s,u} - A_{u,t}| \lesssim |t - s|^\gamma$ is not known. The aim of this paper is to fill this gap, including the case $\gamma = 1$ for which the result is slightly different.

As in the case of $\gamma > 1$, the Sewing lemma for $\gamma \leq 1$ is an analytic tool which can nicely interact with algebraic structures. The link between algebra, analysis and probability was already clear in Terry Lyons' seminal paper [Lyo98], where geometric rough paths were defined in terms of tensor algebras, following the work of Kuo-Tsai Chen on iterated integrals [Che54]. Later Massimiliano Gubinelli pushed this beautiful interaction further by defining branched rough paths in terms of the Butcher-Connes-Kreimer Hopf algebra [Gub10].

In this paper we show that the Sewing lemma, both for $\gamma > 1$ and $\gamma \leq 1$, allows to extend this framework to rough paths on a general (commutative graded connected locally finite) Hopf algebra. This includes geometric and branched rough paths and also other notions introduced recently, like quasi-geometric rough paths [Bel20], planarly branched rough paths [CEFMMK20] and the (so far un-named) Hopf algebra of [LOT21, Section 6].

In particular, the main application of the Sewing lemma for $\gamma \leq 1$ is the construction of rough paths over a Hölder path. The fact that one can always lift an α -Hölder path for $\alpha \in (0, 1)$ to a α -geometric rough path has been long known, at least since the paper by Lyons-Victoir [LV07]. However their construction was based on a (beautiful) geometric and algebraic construction, which famously mentioned the axiom of choice and was therefore considered as non-constructive.

In Section 2.4 we show that the Sewing lemma in the case $\gamma < 1$ allows to construct inductively in a simple way rough paths on a Hopf algebra over a α -Hölder path for $\alpha \in (0, 1)$,

up to level $N := \lfloor 1/\alpha \rfloor$ (higher levels are uniquely determined by the Sewing lemma for $\gamma > 1$). Even more, this construction is *continuous* with respect to the relevant metrics. This was already known for the second level of a geometric rough path, which can be reduced to an integral of the form $\int_0^t X_s dY_s$ in a case where the Young condition is not satisfied. Here we extend this to a general result.

Such a Sewing lemma in the regime $\gamma \in (0, 1]$ is proved in Theorem 2.2.8 below. Contrary to the case $\gamma > 1$, the constructed integral I is not unique and is not defined by Riemann-type sums. However, I can be chosen to be linear in A and continuous in an appropriate topology. Note that in the context of regularity structures this is very close to the Reconstruction Theorem [Hai14; CZ20; ZK21] in the negative exponent case, where uniqueness is lost and different approximations are used, see Section 2.5 below for a discussion.

Another application of the Sewing lemma for $\gamma \in (0, 1)$ is the bicontinuity of a natural bijection between the set of rough paths on a Hopf algebra and a linear space of Hölder functions. This also extends to a general result the content of [TZ20, Corollary 1.3], where a natural bijection between these spaces, in the context of branched rough paths, had been constructed using a constructive Lyons-Victoir extension technique, but no proof of the continuity of this map was available. We mention that the continuity of this action and of a Lyons-Victoir extension in the context of branched rough paths has been obtained using paraproducts in [BH21a, Theorem 21] and [BH21b, Corollary 3].

We note that other extensions results have been obtained since [LV07], see e.g. the renormalization method of [Unt10; Unt13], and [NT11], which uses probabilistic techniques in the case of the fractional Brownian motion. Finally, we mention that it should be possible to extend our method to the Besov setting [FS21]. Future extensions might involve the framework of the Stochastic Sewing Lemma [Lê20], the rough paths approach to non-commutative stochastic calculus [DS13; BG21] and the rough paths approach to McKean-Vlasov equations [DS21].

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2.2 Sewing lemmas

In the following we consider a time horizon $T > 0$. We note

$$\Delta_T^n := \{(x_1, \dots, x_n) \in \mathbb{R}^n : 0 \leq x_1 \leq \dots \leq x_n \leq T\}.$$

We denote $\mathcal{C}(\Delta_T^n)$ the space of \mathbb{R} -valued continuous functions on Δ_T^n ; and \mathcal{C}_n the space of \mathbb{R} -valued continuous functions on $[0, T]^n$.

Note well that here the subscript n corresponds to the dimension of the domain (and not the regularity of the function).

2.2.1 Presentation of the result

Recall the usual Sewing lemma:

Theorem 2.2.1 (Sewing lemma for $\gamma > 1$ [Gub04, Proposition 1], [FLP06, Lemma 2.1]). *Let $\gamma > 1$ and $A: \Delta_T^2 \rightarrow \mathbb{R}$ be a continuous function such that*

$$|A_{s,t} - A_{s,u} - A_{u,t}| \lesssim |t - s|^\gamma,$$

uniformly over $0 \leq s \leq u \leq t \leq T$. Then there exists a unique function $I: [0, T] \rightarrow \mathbb{R}$ such that $I_0 = 0$ and:

$$|I_t - I_s - A_{s,t}| \lesssim |t - s|^\gamma,$$

uniformly over $0 \leq s \leq t \leq T$. Moreover I is the limit of Riemann-type sums

$$I_t = \lim_{|\mathcal{P}| \rightarrow 0} \sum_{i=0}^{\#\mathcal{P}-1} A_{t_i, t_{i+1}} \quad (2.2.1)$$

along arbitrary partitions \mathcal{P} of $[0, T]$ with vanishing mesh $|\mathcal{P}| \rightarrow 0$.

In (2.2.1), a *partition* of the interval $[a, b]$ is a finite sequence of ordered points $\mathcal{P} = \{a = t_0 < t_1 < \dots < t_k = b\}$; moreover we denote $\#\mathcal{P} = k$ and $|\mathcal{P}| := \max_{i=1, \dots, \#\mathcal{P}} |t_i - t_{i-1}|$.

In this section, we establish the following theorem, extending the scope of the Sewing lemma to the regime $0 < \gamma \leq 1$.

Theorem 2.2.2 (Sewing lemma for $0 < \gamma \leq 1$). *Let $0 < \gamma \leq 1$ and $A: \Delta_T^2 \rightarrow \mathbb{R}$ be a continuous function such that*

$$|A_{s,t} - A_{s,u} - A_{u,t}| \lesssim |t - s|^\gamma, \quad (2.2.2)$$

uniformly over $0 \leq s \leq u \leq t \leq T$. Then there exists a (non-unique) function $I: [0, T] \rightarrow \mathbb{R}$ such that $I_0 = 0$ and:

$$|I_t - I_s - A_{s,t}| \lesssim \begin{cases} |t - s|^\gamma & \text{if } 0 < \gamma < 1, \\ |t - s| (1 + |\log |t - s||) & \text{if } \gamma = 1, \end{cases} \quad (2.2.3)$$

uniformly over $0 \leq s \leq t \leq T$, and the map $A \mapsto I$ is linear.

In fact, we shall show a more general result in Theorem 2.2.8 below, where we replace the right-hand side of (2.2.2) with $V(t - s)$ for general control functions V , see Definition 2.2.6. Note that then, Theorem 2.2.2 is an immediate consequence of Theorem 2.2.8 and Example 2.2.7. Before turning to the statement and proof of Theorem 2.2.8, let us propose a few remarks.

Remark 2.2.3 (Non-uniqueness). *It is straightforward to observe by the triangle inequality that any I satisfying (2.2.3) for some $A: \Delta_T^2 \rightarrow \mathbb{R}$ and $\gamma \in (0, 1)$ is determined up to a γ -Hölder function, i.e. if I satisfies (2.2.3), then \tilde{I} also satisfies (2.2.3) if and only if $\tilde{I} - I$ is γ -Hölder.*

Remark 2.2.4. *In the usual Sewing lemma Theorem 2.2.1, I is constructed as limit of Riemann-type sums, see (2.2.1). In the case of Theorem 2.2.2, we define I with a different construction, namely I is defined explicitly (via a recursive formula) on the set of dyadic numbers and then extended to $[0, T]$ by density. Note that this approach is reminiscent of some constructive results of [LV07; TZ20].*

Remark 2.2.5. *The situation is similar to the setting of the Reconstruction Theorem [Hai14; CZ20], where different approximations are used depending on the sign of the exponent (named also γ), and where uniqueness is lost in the case of non-positive exponents. See Section 2.5 below for a discussion.*

2.2.2 The main technical result

Now we turn to the statement and proof of our main technical result, Theorem 2.2.8 below. As in [FLPM08], we introduce a (new) notion of control function for our purposes. We do not have a natural interpretation for this definition. Rather, it corresponds to the quantity that appears in our proof below.

Definition 2.2.6 (Control function). *If $V: [0, T] \rightarrow \mathbb{R}_+$, we set for $r, k_0 \in \mathbb{N}$:*

$$\begin{aligned} \bar{V}_{(k_0)}(T2^{-r}) &:= (k_0 + 1) \sum_{m=0}^{+\infty} 2^{m+1} V(T2^{-(r+mk_0)}) \\ &\quad + \sum_{m=0}^{+\infty} \sum_{k=0}^{k_0} \sum_{l=1}^{r+mk_0+k} 2^{m+1-r-mk_0-k+l} V(T2^{-(l-1)}). \end{aligned}$$

We say that V is a k_0 -control function if $V(0) = 0$, V is increasing and for each $r \in \mathbb{N}$, $\bar{V}_{(k_0)}(T2^{-r}) < +\infty$. We extend $\bar{V}_{(k_0)}: [0, T] \rightarrow \mathbb{R}_+$ as follows: set $\bar{V}_{(k_0)}(0) := 0$, and for $u \in (0, 1]$, set $\bar{V}_{(k_0)}(u) := \bar{V}_{(k_0)}(T2^{-r})$ where $r \in \mathbb{N}$ is uniquely defined by $T2^{-(r+1)} < u \leq T2^{-r}$.

Example 2.2.7. *Let $\gamma > 0$, $V(u) := u^\gamma$. Then for any integer $k_0 > \frac{1}{\gamma}$, V is a k_0 -control function and there exists a constant $C = C_{k_0, \gamma}$ such that:*

$$\bar{V}_{(k_0)}(u) \leq \begin{cases} Cu^\gamma & \text{if } 0 < \gamma < 1, \\ C(1 + |\log(T)|)(1 + |\log(u)|)u & \text{if } \gamma = 1, \\ CT^{\gamma-1}u & \text{if } \gamma > 1. \end{cases}$$

We need the operators δ defined as follows:

1. $\delta: \mathcal{C}_1 \rightarrow \mathcal{C}_2$: for $I: [0, T] \rightarrow \mathbb{R}$, we define

$$\delta I: [0, T]^2 \rightarrow \mathbb{R}, \quad \delta I_{s,t} := I_t - I_s.$$

2. $\delta: \mathcal{C}_2 \rightarrow \mathcal{C}_3$: for $A: [0, T]^2 \rightarrow \mathbb{R}$, we define

$$\delta A: [0, T]^3 \rightarrow \mathbb{R}, \quad \delta A_{s,u,t} := A_{s,t} - A_{s,u} - A_{u,t}.$$

It is easy to see that $\delta \circ \delta = 0$, so that these operators form a cochain complex, which is moreover *exact*: if $\delta Z = 0$ for $Z \in \mathcal{C}_2$, then $Z = \delta z$ for some $z \in \mathcal{C}_1$ (see [Gub04]).

We will work with dyadic numbers: for $m \in \mathbb{N}$, denote $D_m := \{k2^{-m}, k \in \llbracket 0, 2^m \rrbracket\}$, and $D := \bigcup_{m \in \mathbb{N}} D_m$. Our main technical result is the following.

Theorem 2.2.8. *Let $A: \Delta_T^2 \rightarrow \mathbb{R}$ be a function and V be a k_0 -control function for some $k_0 \in \mathbb{N}$. Assume that for all $0 \leq s \leq u \leq t \leq T$,*

$$|(\delta A)_{s,u,t}| \leq V(t-s).$$

Then:

1. (Sewing on dyadics) *There exists $I: TD \rightarrow \mathbb{R}$ such that $I_0 = 0$ and for all $0 \leq s \leq t \leq T$ with $s/T, t/T \in D$,*

$$|I_t - I_s - A_{s,t}| \leq \bar{V}_{(k_0)}(t-s). \quad (2.2.4)$$

2. (Sewing on $[0, T]$) *Assume furthermore that A is continuous and that*

$$\bar{V}_{(k_0)}(u) \xrightarrow{u \rightarrow 0} 0.$$

Let W be a continuous function such that $\bar{V}_{(k_0)} \leq W$. Then there exists $I: [0, T] \rightarrow \mathbb{R}$ such that $I_0 = 0$ and for all $0 \leq s \leq t \leq T$,

$$|I_t - I_s - A_{s,t}| \leq W(t-s).$$

Remark 2.2.9. Given a general control function V as in Definition 2.2.6, it is not clear in general whether the further condition $\tilde{V}_{(k_0)}(u) \rightarrow_{u \rightarrow 0} 0$ holds. However, note that the condition is satisfied when $V(u) = u^\gamma$ for some $\gamma > 0$, see Example 2.2.7.

Proof. We prove the items in the announced order. Without loss of generality we can suppose that $T = 1$: if $T \neq 1$, consider $\tilde{A}: \Delta_1^2 \rightarrow \mathbb{R}$ defined by $\tilde{A}_{s,t} := A_{sT,tT}$.

Proof of 1. First we introduce a sequence $u_{k,n}$ defined for $n \in \mathbb{N}$ and $k \in \llbracket 0, 2^n - 1 \rrbracket$ by the recurrence relations:

$$\begin{cases} u_{0,0} & := 0, \\ u_{2k,n+1} & := \frac{1}{2}u_{k,n}, \\ u_{2k+1,n+1} & := \frac{1}{2}u_{k,n} + (\delta A)_{k2^{-n}, (k+\frac{1}{2})2^{-n}, (k+1)2^{-n}}. \end{cases}$$

Now note that in order to define recursively I on D , it is enough for each $n \in \mathbb{N}$ to define I on elements of D_n of the form $k2^{-n}$ where $k = 2l + 1$ is odd. Indeed, when k is even, $k2^{-n}$ belongs to D_{n-1} . Thus, we set $I_0 := 0$ and then recursively:

$$I_{(2l+1)2^{-n}} := I_{2l2^{-n}} + A_{2l2^{-n}, (2l+1)2^{-n}} + u_{2l,n}. \quad (2.2.5)$$

(Notice that for each dyadic number $s \in D$, I_s is actually defined as a finite linear combination of the $A_{u,v}$ and there is no question of convergence here. Note also that this definition is linear in A .) We claim that $R_{s,t} := I_t - I_s - A_{s,t}$ can be expressed on consecutive elements of D_n by:

$$R_{k2^{-n}, (k+1)2^{-n}} = u_{k,n}. \quad (2.2.6)$$

This is not clear a priori as (2.2.5) only establishes this property when $k = 2l$ is even. However, when $k = 2l + 1$ is odd, we argue by recurrence on n , and this is where we exploit the definition of u . Indeed, by definition of R , it holds that $\delta R = -\delta A$ so that:

$$\begin{aligned} R_{(2l+1)2^{-n}, (2l+2)2^{-n}} &= R_{2l2^{-n}, (2l+2)2^{-n}} - R_{2l2^{-n}, (2l+1)2^{-n}} \\ &\quad + (\delta A)_{2l2^{-n}, (2l+1)2^{-n}, (2l+2)2^{-n}}. \end{aligned}$$

By recurrence on n , $R_{2l2^{-n}, (2l+2)2^{-n}} = u_{l,n-1}$. By (2.2.5), we have $R_{2l2^{-n}, (2l+1)2^{-n}} = u_{2l,n}$. By definition of u , $u_{2l,n} = \frac{1}{2}u_{l,n-1}$, thus:

$$R_{(2l+1)2^{-n}, (2l+2)2^{-n}} = \frac{1}{2}u_{l,n-1} + (\delta A)_{2l2^{-n}, (2l+1)2^{-n}, (2l+2)2^{-n}},$$

and by definition of u this in turn equals $u_{2l+1,n}$, which establishes (2.2.6).

Now let us turn to the Sewing bound (2.2.4). For $r, M \in \mathbb{N}$, set:

$$v_{r,M} := \max_{t,s \in D_M, 0 \leq t-s \leq 2^{-r}} |R_{s,t}|.$$

Note that when $r > M$, no distinct values $t, s \in D_M$ satisfy $0 \leq t-s \leq 2^{-r}$, so that $v_{r,M} = 0$. When $r = M$, the only elements $t, s \in D_r$ satisfying $0 \leq t-s \leq 2^{-r}$ are consecutive, i.e. of the form $s = k2^{-r}$, $t = (k+1)2^{-r}$ so that from (2.2.6),

$$v_{r,r} = \max_{k \in \llbracket 0, 2^r - 1 \rrbracket} |u_{k,r}|.$$

According to the recursive definition of u , and the hypothesis on A ,

$$v_{r,r} \leq \frac{1}{2}v_{r-1,r-1} + V\left(2^{-(r-1)}\right).$$

Iterating and since $v_{0,0} = 0$, we get:

$$v_{r,r} \leq \sum_{l=1}^r 2^{-(r-l)} V \left(2^{-(l-1)} \right). \quad (2.2.7)$$

Now we establish a first (recursive) estimate of $v_{r,r+k}$ for $r, k \in \mathbb{N}$. Let $t, s \in D_{r+k}$ be such that $0 \leq t - s \leq 2^{-r}$. Note that if $t = s$ then $R_{s,t} = 0$, so that now we assume $t - s > 0$. Remembering that $\delta R = -\delta A$, we decompose:

$$R_{s,t} = R_{s,s_1} + R_{s_1,t_1} + R_{t_1,t} - (\delta A)_{s_1,t_1,t} - (\delta A)_{s,s_1,t}. \quad (2.2.8)$$

where $s_1 := \min(u \in D_{r+k-1}, u \geq s)$, $t_1 := \max(u \in D_{r+k-1}, u \leq t)$. Note that s_1, t_1 are correctly defined, that $s_1, t_1 \in D_{r+k-1} \subset D_{r+k}$, $s \leq s_1$, $t_1 \leq t$, and that $s_1 - s \leq 2^{-(r+k)}$, $t - t_1 \leq 2^{-(r+k)}$. Furthermore, since $t - s > 0$ and $t, s \in D_{r+k}$, it holds that $t - s \geq 2^{-(r+k)}$ and thus $D_{r+k-1} \cap [s, t] \neq \emptyset$, whence $s_1 \leq t_1$. Thus from (2.2.8) and the definition of v :

$$v_{r,r+k} \leq 2v_{r+k,r+k} + v_{r,r+k-1} + 2V(2^{-r}).$$

Recalling (2.2.7), this yields:

$$v_{r,r+k} - v_{r,r+k-1} \leq 2 \left(V(2^{-r}) + \sum_{l=1}^{r+k} 2^{-(r+k-l)} V(2^{-(l-1)}) \right).$$

Summing from $k = 1$ to K and reusing (2.2.7), we obtain for $r, k \in \mathbb{N}$:

$$v_{r,r+K} \leq 2KV(2^{-r}) + 2 \sum_{k=0}^K \sum_{l=1}^{r+k} 2^{-(r+k-l)} V(2^{-(l-1)}). \quad (2.2.9)$$

Now fix $k_0 \in \mathbb{N}^*$, and let $r, M \in \mathbb{N}$ with $r \leq M$. Let $0 \leq t - s \leq 2^{-r}$ with $s, t \in D_M$. We consider several cases.

If $r + k_0 \geq M$, then using (2.2.9):

$$v_{r,M} \leq 2k_0 V(2^{-r}) + 2 \sum_{k=0}^{k_0} \sum_{l=1}^{r+k} 2^{-(r+k-l)} V(2^{-(l-1)}). \quad (2.2.10)$$

If $r + k_0 < M$ and $0 \leq t - s \leq 2^{-(r+k_0)}$ then by definition of v :

$$|R_{s,t}| \leq v_{r+k_0,M}. \quad (2.2.11)$$

If $r + k_0 < M$ and $2^{-(r+k_0)} < t - s \leq 2^{-r}$, then we consider

$$s_1 := \min(u \in D_{r+k_0}, u \geq s), \quad t_1 := \max(u \in D_{r+k_0}, u \leq t).$$

Observe that s_1, t_1 are correctly defined, $s \leq s_1$, $t_1 \leq t$, $s_1, t_1 \in D_{r+k_0} \subset D_M$. Also, since $t - s > 2^{-(r+k_0)}$, it holds that $D_{r+k_0} \cap [s, t] \neq \emptyset$, whence $s_1 \leq t_1$. Finally, the definition of s_1, t_1 implies $s_1 - s \leq 2^{-(r+k_0)}$, $t - t_1 \leq 2^{-(r+k_0)}$. Thus from (2.2.8):

$$|R_{s,t}| \leq 2v_{r+k_0,M} + v_{r,r+k_0} + 2V(2^{-r}). \quad (2.2.12)$$

We denote:

$$W_{(k_0)}(r) := 2(k_0 + 1)V(2^{-r}) + 2 \sum_{k=0}^{k_0} \sum_{l=1}^{r+k} 2^{-(r+k-l)} V(2^{-(l-1)}),$$

so that combining (2.2.10), (2.2.11), (2.2.12), and (2.2.9) gives for $r \leq M \in \mathbb{N}$:

$$v_{r,M} \leq \begin{cases} 2v_{r+k_0,M} + W_{(k_0)}(r) & \text{if } r + k_0 < M, \\ W_{(k_0)}(r) & \text{if } r + k_0 \geq M. \end{cases}$$

Iterating this recursive estimate, we obtain:

$$v_{r,M} \leq \sum_{m=0}^{+\infty} 2^m W_{(k_0)}(r + mk_0).$$

Note that from Definition 2.2.6, the right-hand term equals $\bar{V}_{(k_0)}(2^{-r})$, and thus this implies:

$$\sup_{t,s \in D, 2^{-(r+1)} < t-s \leq 2^{-r}} |R_{s,t}| \leq \bar{V}_{(k_0)}(2^{-r}).$$

Recall that by definition, $\bar{V}_{(k_0)}(2^{-r}) = \bar{V}_{(k_0)}(t-s)$ when $2^{-(r+1)} < t-s \leq 2^{-r}$. This is enough to conclude that for all $s, t \in D$ with $t \neq s$, $|R_{s,t}| \leq \bar{V}_{(k_0)}(t-s)$. When $t = s$, $|R_{t,t}| = |A_{t,t}| = |-(\delta A)_{t,t,t}| \leq V(0) = 0 \leq \bar{V}_{(k_0)}(0)$, whence the announced result. (Note that this is the only time in the proof where we use the assumption that $V(0) = 0$.)

Proof of 2. We extend I on $[0, T]$ by density, setting for $t \in [0, T]$:

$$I_t := \lim_{\substack{s \rightarrow t \\ s \in D}} I_s.$$

This is correctly defined, because for any choice of $(s_n)_{n \in \mathbb{N}} \in D^{\mathbb{N}}$ with $s_n \rightarrow t$, the Sewing estimate:

$$|I_{s_n} - I_{s_m}| \leq |A_{s_n \wedge s_m, s_n \vee s_m}| + \bar{V}_{(k_0)}(|s_n - s_m|),$$

implies that the sequence $(I_{s_n})_{n \in \mathbb{N}}$ is Cauchy. If W is a continuous function such that $\bar{V}_{(k_0)} \leq W$, then $|I_t - I_s - A_{s,t}| \leq W(t-s)$ for all $s, t \in [0, T]$. \square

Remark 2.2.10 (Non-locality). *Note that our construction of I in the theorem above is non-local, in the sense that $I_t - I_s$ may depend on the value of A outside of $[s, t]^2$. As an example, observe from the definition (2.2.5) of I that $I_{3/4} = \frac{1}{2}A_{0,1/2} + \frac{1}{2}A_{0,1} + A_{1/2,3/4} - \frac{1}{2}A_{1/2,1}$. Since $I_0 = 0$, it follows that $I_{3/4} - I_0$ depends on the value of $A_{0,1}$.*

Another sewing map I could possibly satisfy a locality property, but this would clearly require novel ideas. We point out that in the case of the Reconstruction Theorem it is also well known that in the regime $\gamma \leq 0$ (which corresponds to $\gamma \leq 1$ for the Sewing Lemma) the Reconstruction map is non-local, see e.g. [CZ20, Section 11].

Remark 2.2.11 (Lyons–Victoir). *The construction in the proof of Theorem 2.2.8 is reminiscent of the Lyons–Victoir extension technique [LV07], although it is not clear whether the extension we construct in Section 2.4 below does coincide with the Lyons–Victoir extension in general.*

Remark 2.2.12. *In the case of a smooth path or a Brownian motion, one can build a sewing map with the Riemann or (respectively) the Itô integral, but the results are bound to be different from that of our construction; the merit of our sewing map is of course its wider range of applicability.*

2.3 Continuity of the Sewing map

The Sewing Lemma for $\gamma > 1$ given in Theorem 2.2.1 finds its main applications in the theory of rough integration and rough differential equations [Gub04]. In this setting it is useful to introduce some function spaces and interpret the Sewing Lemma in terms of operators having nice properties on these spaces. Then we introduce for all $\gamma > 0$, $A \in \mathcal{C}(\Delta_T^2)$ and $B \in \mathcal{C}(\Delta_T^3)$

$$\|A\|_{\mathcal{C}_2^\gamma(\Delta_T^2)} := \sup_{0 \leq s < t \leq T} \frac{|A_{s,t}|}{|t-s|^\gamma}, \quad \|B\|_{\mathcal{C}_3^\gamma(\Delta_T^3)} := \sup_{0 \leq s < u < t \leq T} \frac{|B_{s,u,t}|}{|t-s|^\gamma},$$

with the associated normed spaces

$$\begin{aligned} \mathcal{C}_2^\gamma(\Delta_T^2) &:= \{A \in \mathcal{C}(\Delta_T^2) : \|A\|_{\mathcal{C}_2^\gamma(\Delta_T^2)} < +\infty\}, \\ \mathcal{C}_3^\gamma(\Delta_T^3) &:= \{B \in \mathcal{C}(\Delta_T^3) : \|B\|_{\mathcal{C}_3^\gamma(\Delta_T^3)} < +\infty\}. \end{aligned}$$

Then we can reformulate the Sewing Lemma for $\gamma > 1$ with a quantitative estimate as follows

Theorem 2.3.1 (Sewing map for $\gamma > 1$, see [Gub04; FLP06]). *Let $\gamma > 1$ and $A \in \mathcal{C}(\Delta_T^2)$ satisfy $\delta A \in \mathcal{C}_3^\gamma(\Delta_T^3)$. Then there exists a unique $R \in \mathcal{C}_2^\gamma(\Delta_T^2)$ such that $\delta R = \delta A$. Moreover we have the estimate*

$$\|R\|_{\mathcal{C}_2^\gamma(\Delta_T^2)} \leq C_\gamma \|\delta A\|_{\mathcal{C}_3^\gamma(\Delta_T^3)} \quad (2.3.1)$$

with $C_\gamma = (2^\gamma - 2)^{-1}$.

The two statements in Theorem 2.2.1 and Theorem 2.3.1 are essentially equivalent if we establish the correspondence $(A, R) \leftrightarrow (I, R)$ in such a way that

$$I : [0, T] \rightarrow \mathbb{R}, \quad I_0 = 0, \quad \delta I = A - R.$$

The uniqueness statement in Theorem 2.3.1 implies that if $A \in \mathcal{C}(\Delta_T^2)$ satisfies $\delta A = 0$, then the corresponding R is also equal to zero. We obtain that in Theorem 2.3.1 R is in fact a function of δA only. In other words, Theorem 2.3.1 allows to define the *Sewing map*

$$\Lambda : \delta(\mathcal{C}(\Delta_T^2)) \cap \mathcal{C}_3^\gamma(\Delta_T^3) \rightarrow \mathcal{C}_2^\gamma(\Delta_T^2), \quad B = \delta A \mapsto R = \Lambda B,$$

and the bound (2.3.1) yields the continuity property in the case $\gamma > 1$

$$\|\Lambda B\|_{\mathcal{C}_2^\gamma(\Delta_T^2)} \leq (2^\gamma - 2)^{-1} \|B\|_{\mathcal{C}_3^\gamma(\Delta_T^3)}, \quad \forall B \in \delta(\mathcal{C}(\Delta_T^2)) \cap \mathcal{C}_3^\gamma(\Delta_T^3).$$

Since $\delta R = \delta A$, or equivalently since $\delta \circ \delta I = 0$, we note that $\delta \Lambda = \text{Id}_{\delta(\mathcal{C}(\Delta_T^2)) \cap \mathcal{C}_3^\gamma(\Delta_T^3)}$, namely we can interpret Λ as a right inverse of δ .

We now state the Sewing Lemma for $0 < \gamma \leq 1$ with a quantitative estimate.

Theorem 2.3.2 (Sewing map for $0 < \gamma \leq 1$). *Let $0 < \gamma \leq 1$ and $A \in \mathcal{C}(\Delta_T^2)$ with $\delta A \in \mathcal{C}_3^\gamma(\Delta_T^3)$. Then there exists $R \in \mathcal{C}(\Delta_T^2)$ such that $\delta R = \delta A$, the map $\delta A \mapsto R =: \Lambda(\delta A)$ is linear and for some constant $C_\gamma \geq 0$ we have the estimate*

$$\|R\|_{\mathcal{C}_2^\gamma(\Delta_T^2)} \leq C_\gamma \|\delta A\|_{\mathcal{C}_3^\gamma(\Delta_T^3)} \quad \text{if } \gamma < 1, \quad (2.3.2)$$

$$\sup_{0 \leq s < t \leq T} \frac{|R_{s,t}|}{(1 + |\log |t-s||) |t-s|} \leq C_1 \|\delta A\|_{\mathcal{C}_3^1(\Delta_T^3)} \quad \text{if } \gamma = 1. \quad (2.3.3)$$

One can actually bound the constants in Example 2.2.7 and observe that one can take:

$$\begin{aligned} C_\gamma &= \frac{2^{\gamma+1}}{1 - 2^{1-\gamma}(\lfloor \frac{1}{\gamma} \rfloor + 1)} \left(2 + \left\lfloor \frac{1}{\gamma} \right\rfloor + \frac{2}{(2^{1-\gamma} - 1)(1 - 2^{-\gamma})} \right) && \text{if } 0 < \gamma < 1, \\ C_1 &= \frac{96}{\log 2} (1 + |\log T|) && \text{if } \gamma = 1. \end{aligned}$$

We have therefore *another* Sewing map for $\gamma < 1$

$$\Lambda : \delta(\mathcal{C}(\Delta_T^2)) \cap \mathcal{C}_3^\gamma(\Delta_T^3) \rightarrow \mathcal{C}_2^\gamma(\Delta_T^2), \quad B = \delta A \mapsto R = \Lambda B.$$

The fact that R is in fact a function of δA follows from the construction in Theorem 2.2.8 of the sequence $u_{k,n}$ which depends only on δA , see (2.2.6), (2.2.8). Linearity of the map $\delta A \mapsto R$ follows from the construction in the proof of Theorem 2.2.8.

The estimate (2.3.2) shows that also for $\gamma < 1$ the Sewing map can be constructed as a linear continuous operator $\Lambda : \delta(\mathcal{C}(\Delta_T^2)) \cap \mathcal{C}_3^\gamma(\Delta_T^3) \rightarrow \mathcal{C}_2^\gamma(\Delta_T^2)$. For $\gamma = 1$ this is also true, but the norm \mathcal{C}_2^γ has to be slightly modified in accordance with (2.3.3), but we refrain from introducing a notation for this.

We define the classical space of β -Hölder functions on $[0, T]$ for $\beta \in (0, 1]$:

$$\mathcal{C}_1^\beta := \{f \in C([0, T]) : \|\delta f\|_{\mathcal{C}_2^\beta} < +\infty\}.$$

Then the choice of R in Theorem 2.3.2 can not be unique for $\gamma \leq 1$, since $R + \delta f$ for any $f \in \mathcal{C}_1^\gamma$ also satisfies the desired properties.

Remark 2.3.3. *Note that there are two different Sewing maps: one for $\gamma > 1$ and one for $\gamma \leq 1$. We denote both by Λ , since it is always clear from the context which of the two is used.*

2.3.1 The integration map

We have seen above that the Sewing map allows to define an *integration map* $A \mapsto I$, for all $A \in \mathcal{C}(\Delta_T^2)$ such that $\delta A \in \mathcal{C}_3^\gamma(\Delta_T^3)$, where $I \in \mathcal{C}_1$ is defined by

$$I_0 = 0, \quad \delta I - A = -\Lambda(\delta A) \in \mathcal{C}_2^\gamma(\Delta_T^2). \quad (2.3.4)$$

We note this linear map by

$$\mathcal{I} : \{A \in \mathcal{C}(\Delta_T^2) : \delta A \in \mathcal{C}_3^\gamma(\Delta_T^3)\} \rightarrow \mathcal{C}_1, \quad \mathcal{I}(A) := I.$$

A natural question in this context is under which conditions I belongs to a space of β -Hölder functions.

Proposition 2.3.4. *Let $\beta \in (0, 1)$ and $A \in \mathcal{C}(\Delta_T^2)$ such that $\delta A \in \mathcal{C}_3^{\beta \vee \gamma}(\Delta_T^3)$. Then $I := \mathcal{I}(A) \in \mathcal{C}_1^\beta$ if and only if $A \in \mathcal{C}_2^\beta(\Delta_T^2)$, and in this case*

$$\|\delta I\|_{\mathcal{C}_2^\beta(\Delta_T^2)} \leq \|A\|_{\mathcal{C}_2^\beta(\Delta_T^2)} + C_{\beta \vee \gamma} \|\delta A\|_{\mathcal{C}_3^{\beta \vee \gamma}(\Delta_T^3)}.$$

Proof. Let $A \in \mathcal{C}(\Delta_T^2)$ with $\delta A \in \mathcal{C}_3^{\beta \vee \gamma}(\Delta_T^3)$. Since $\delta I = A - \Lambda(\delta A)$ we obtain for $\beta \in (0, 1)$:

$$\begin{aligned} \|\delta I\|_{\mathcal{C}_2^\beta(\Delta_T^2)} &\leq \|A\|_{\mathcal{C}_2^\beta(\Delta_T^2)} + \|\Lambda(\delta A)\|_{\mathcal{C}_2^\beta(\Delta_T^2)} \\ &\leq \|A\|_{\mathcal{C}_2^\beta(\Delta_T^2)} + \|\Lambda(\delta A)\|_{\mathcal{C}_2^{\beta \vee \gamma}(\Delta_T^2)} \\ &\leq \|A\|_{\mathcal{C}_2^\beta(\Delta_T^2)} + C_{\beta \vee \gamma} \|\delta A\|_{\mathcal{C}_3^{\beta \vee \gamma}(\Delta_T^3)}. \end{aligned}$$

On the other hand we have

$$\begin{aligned} \|A\|_{\mathcal{C}_2^\beta(\Delta_T^2)} &\leq \|\delta I\|_{\mathcal{C}_2^\beta(\Delta_T^2)} + \|\Lambda(\delta A)\|_{\mathcal{C}_2^\beta(\Delta_T^2)} \\ &\leq \|\delta I\|_{\mathcal{C}_2^\beta(\Delta_T^2)} + \|\Lambda(\delta A)\|_{\mathcal{C}_2^{\beta \vee \gamma}(\Delta_T^2)} \\ &\leq \|\delta I\|_{\mathcal{C}_2^\beta(\Delta_T^2)} + C_{\beta \vee \gamma} \|\delta A\|_{\mathcal{C}_3^{\beta \vee \gamma}(\Delta_T^3)}. \end{aligned}$$

The proof is complete. □

In the terminology of [CZ20]:

- $\delta A \in \mathcal{C}_3^{\beta\vee\gamma}(\Delta_T^3)$ is a “coherence” condition,
- $A \in \mathcal{C}_2^\beta(\Delta_T^2)$ is a “homogeneity” condition.

See Section 2.5 below for further discussions.

2.3.2 Optimality of the case $\gamma \in]0, 1[$.

Let us fix $\gamma \in]0, 1[$. The following example shows that the growth rate $|t - s|^\gamma$ from Theorem 2.2.2 in the case $\gamma < 1$ is optimal. Let us set $A_{s,t} := |t - s|^\gamma$. Then for $s \leq u \leq t$,

$$\delta A_{s,u,t} = |t - s|^\gamma - |u - s|^\gamma - |t - u|^\gamma \in [-|t - s|^\gamma, |t - s|^\gamma].$$

Therefore $\delta A \in \mathcal{C}_3^\gamma(\Delta_T^3)$. Let us suppose now that we can improve the bound (2.2.3) in the sense that there exists $I \in \mathcal{C}_1$ such that $I_0 = 0$ and

$$\lim_{\varepsilon \downarrow 0} \sup_{|t-s| \leq \varepsilon} \frac{|I_t - I_s - A_{s,t}|}{|t - s|^\gamma} = 0.$$

Then, denoting $R_{s,t} := I_t - I_s - A_{s,t}$ we must have $\delta R_{s,u,t} = o(|t - s|^\gamma)$ as $s \leq u \leq t, |t - s| \rightarrow 0$. However, taking $u = \frac{s+t}{2}$, observe that

$$\delta R_{s,u,t} = -\delta A_{s,u,t} = (2^{1-\gamma} - 1)|t - s|^\gamma,$$

which cannot be a $o(|t - s|^\gamma)$ and thus provides a contradiction.

2.3.3 Optimality of the case $\gamma = 1$.

The following example shows that the growth rate $|t - s| |\log |t - s||$ from Theorem 2.2.2 in the case $\gamma = 1$ is optimal. Let us set $A_{s,t} := |t - s| \log |t - s|$. Then for $s \leq u \leq t$,

$$\begin{aligned} \delta A_{s,u,t} &= |t - s| \log |t - s| - |u - s| \log |u - s| - |t - u| \log |t - u| \\ &= |t - s| \left(p \log \frac{1}{p} + (1 - p) \log \frac{1}{1 - p} \right) \in [0, (\log 2)|t - s|], \end{aligned}$$

where $p = \frac{|t-u|}{|t-s|} \in [0, 1]$. Therefore $\delta A \in \mathcal{C}_3^1(\Delta_T^3)$. Let us suppose now that we can improve the bound (2.2.3) in the sense that there exists $I \in \mathcal{C}_1$ such that $I_0 = 0$ and

$$\lim_{\varepsilon \downarrow 0} \sup_{|t-s| \leq \varepsilon} \frac{|I_t - I_s - A_{s,t}|}{|t - s| |\log |t - s||} = 0.$$

Then there exists $\delta_n \rightarrow 0$ such that

$$\sup_{|t-s| \leq \frac{1}{n}} |I_t - I_s - A_{s,t}| \leq \delta_n \frac{\log n}{n}.$$

Let us set $t = i/n$ and $s = (i - 1)/n$. Then $A_{s,t} = -\log n/n$ and

$$-(1 + \delta_n) \frac{\log n}{n} \leq I_{\frac{i}{n}} - I_{\frac{i-1}{n}} \leq -(1 - \delta_n) \frac{\log n}{n},$$

and summing over i

$$-(1 + \delta_n)(\log n)t \leq I_t \leq -(1 - \delta_n)(\log n)t, \quad t = \frac{i}{n}.$$

Since $t = \frac{im}{nm}$ for all $m \geq 1$, we obtain

$$-(1 + \delta_{nm})(\log n + \log m)t \leq I_t \leq -(1 - \delta_{nm})(\log n + \log m)t, \quad t = \frac{i}{n},$$

Letting $m \rightarrow +\infty$ we obtain $I_t = -\infty$, which is a contradiction.

2.3.4 Unordered times

For the applications to rough paths of Section 2.4, it is important to extend the Sewing Lemmas to functions $A: [0, T]^2 \rightarrow \mathbb{R}$ rather than $A: \Delta_T^2 \rightarrow \mathbb{R}$.

For this, we consider continuous functions $A: [0, T]^2 \rightarrow \mathbb{R}$ such that for some $\gamma > 0$

$$|A_{s,t} - A_{s,u} - A_{u,t}| \lesssim (|t - u| \vee |u - s|)^\gamma, \quad s, u, t \in [0, T].$$

This implies in particular

$$|A_{s,t} - A_{s,u} - A_{u,t}| \lesssim |t - s|^\gamma, \quad 0 \leq s \leq u \leq t \leq T.$$

Thus, $\delta A \in \mathcal{C}_3^\gamma(\Delta_T^3)$, and in particular we can apply Theorem 2.3.1 and Theorem 2.3.2, so that we can consider $\Lambda(\delta A) \in \mathcal{C}_2^\gamma(\Delta_T^2)$ and $\mathcal{I}(A) \in \mathcal{C}_1$. However, this gives us information on $\Lambda(\delta A)_{s,t}$ only when $s \leq t$. Still, we can recover information on $\Lambda(\delta A)_{s,t}$ when $s > t$ by writing from (2.3.4):

$$\Lambda(\delta A)_{s,t} + \Lambda(\delta A)_{t,s} = -\delta A_{s,t,s},$$

so that for $\gamma \neq 1$

$$\frac{|\Lambda(\delta A)_{s,t}|}{|t - s|^\gamma} \leq \|\Lambda(\delta A)\|_{\mathcal{C}_2^\gamma(\Delta_T^2)} + |\delta A_{s,t,s}| \leq (C_\gamma + 1)\|\delta A\|_{\mathcal{C}_3^\gamma(\Delta_T^3)},$$

with C_γ as in (2.3.1)-(2.3.2), respectively.

Thus, if we introduce for all $\gamma > 0$, $A \in \mathcal{C}_2$ and $B \in \mathcal{C}_3$

$$\|A\|_{\mathcal{C}_2^\gamma} := \sup_{s,t \in [0,T], s \neq t} \frac{|A_{s,t}|}{|t - s|^\gamma},$$

$$\|B\|_{\mathcal{C}_3^\gamma} := \sup_{s,u,t \in [0,T], s \neq t} \frac{|B_{s,u,t}|}{(|t - u| \vee |u - s|)^\gamma},$$

with the associated normed spaces

$$\mathcal{C}_2^\gamma := \{A \in \mathcal{C}_2 : \|A\|_{\mathcal{C}_2^\gamma} < +\infty\},$$

$$\mathcal{C}_3^\gamma := \{B \in \mathcal{C}_3 : \|B\|_{\mathcal{C}_3^\gamma} < +\infty\},$$

then the above argument combined with Theorem 2.3.1, Theorem 2.3.2, and Proposition 2.3.4, give the following sewing lemma for unordered times:

Theorem 2.3.5 (Sewing lemma for $\gamma > 0$ and unordered times). *Let $\gamma > 0$. There exist linear maps*

$$\Lambda: \delta(\mathcal{C}_2) \cap \mathcal{C}_3^\gamma \rightarrow \mathcal{C}_2,$$

$$\mathcal{I}: \{A \in \mathcal{C}_2 : \delta A \in \mathcal{C}_3^\gamma\} \rightarrow \mathcal{C}_1,$$

such that for $A \in \mathcal{C}_2$ with $\delta A \in \mathcal{C}_3^\gamma$:

$$\delta \Lambda(\delta A) = \delta A, \quad \mathcal{I}(A)_0 = 0, \quad \delta \mathcal{I}(A) - A = -\Lambda(\delta A).$$

Such maps are unique when $\gamma > 1$ but not when $0 < \gamma \leq 1$. Furthermore:

1. (Regularity of Λ) For $A \in \mathcal{C}_2$ with $\delta A \in \mathcal{C}_3^\gamma$

$$\|\Lambda(\delta A)\|_{\mathcal{C}_2^\gamma} \leq (C_\gamma + 1)\|\delta A\|_{\mathcal{C}_3^\gamma} \quad \text{if } \gamma \neq 1,$$

$$\sup_{s,t \in [0,T], s \neq t} \frac{|\Lambda(\delta A)_{s,t}|}{(1 + |\log |t - s||) |t - s|} \leq (C_1 + 1)\|\delta A\|_{\mathcal{C}_3^1} \quad \text{if } \gamma = 1.$$

2. (Regularity of \mathcal{I}) Let $0 < \beta < 1$ then:

$$\|\delta \mathcal{I}(A)\|_{\mathcal{C}_2^\beta} \leq \|A\|_{\mathcal{C}_2^\beta} + (C_{\beta \vee \gamma} + 1)\|\delta A\|_{\mathcal{C}_3^{\beta \vee \gamma}}.$$

2.4 A continuous Lyons-Victoir extension

One of the most important ideas of rough paths theory is that one can build strong integration theories involving non-smooth paths $X: [0, T] \mapsto \mathbb{R}^d$, under the condition that one “enriches” X with a collection of “iterated integrals of X against itself”. When X is not smooth, such iterated integrals can not be defined classically: therefore one enriches X rather with a collection of functions which retain some of the algebraic and analytic properties valid in the smooth case. The algebraic properties of the collection of its iterated integrals (also called its signature) were first discovered by Chen [Che54], and the theory of rough paths builds on this algebraic formalism.

In this section we show that the Sewing Lemma (Theorem 2.3.5) for $\gamma \in (0, 1)$ allows to construct rough paths over Hölder paths $X: [0, T] \mapsto \mathbb{R}^d$ in a continuous way, extending Lyons-Victoir’s result [LV07] to the setting of a general commutative graded conneted locally finite Hopf algebra (see below).

Let us discuss for example the case of (weakly) geometric rough paths. In this context, we have

1. a parameter $\alpha \in (0, 1)$, and the associated integer $N := \lfloor 1/\alpha \rfloor$,
2. the finite set $S := \cup_{n=1}^N \{1, \dots, d\}^n$ endowed with the *degree* function

$$S \ni (i_1, \dots, i_n) \mapsto |(i_1, \dots, i_n)| := n \in \{1, \dots, N\},$$

3. a family $(\langle X, \tau \rangle)_{\tau \in S} \subset \mathcal{C}_2$ such that for all $(i_1, \dots, i_n) \in S$ and $s, u, t \in [0, T]$, one has

$$|\langle X_{s,t}, (i_1, \dots, i_n) \rangle| \lesssim |t - s|^{\alpha n}, \quad (2.4.1)$$

$$\delta \langle X, (i_1, \dots, i_n) \rangle_{s,u,t} = \sum_{k=1}^{n-1} \langle X_{s,u}, (i_1, \dots, i_k) \rangle \langle X_{u,t}, (i_{k+1}, \dots, i_n) \rangle. \quad (2.4.2)$$

Then by (2.4.1)-(2.4.2) we have the estimate for all $\tau \in S$ and $s, u, t \in [0, T]$

$$|(\delta \langle X, \tau \rangle)_{s,u,t}| \lesssim \sum_{k=1}^{|\tau|-1} |u - s|^{\alpha k} |t - u|^{\alpha(|\tau|-k)} \lesssim (|s - u| \vee |t - u|)^{\alpha|\tau|}. \quad (2.4.3)$$

By definition, $\alpha|\tau| \leq \alpha N \leq 1$. The *extension problem* is the following: suppose that

1. $k \in \{1, \dots, N - 1\}$,
2. we have $(\langle X, \tau \rangle)_{\tau \in S, |\tau| \leq k} \subset \mathcal{C}_2$ satisfying (2.4.1)-(2.4.2),
3. we have $\tau_0 \in S$ with $|\tau_0| = k + 1$.

Is it then possible to find $\langle X, \tau_0 \rangle \in \mathcal{C}_2$ which satisfies (2.4.1)-(2.4.2) as well?

It is now clear that the Sewing Lemma (Theorem 2.3.5) for $\gamma < 1$ yields a positive answer to this problem for all $k \in \{1, \dots, N - 1\}$ as long as $\alpha N < 1$, by applying the Sewing map Λ to the right-hand side of (2.4.2) for $\tau = \tau_0$ and calling the result $\langle X, \tau_0 \rangle$. If $\alpha N = 1$ then for $k + 1 = N$ we can construct $\langle X, \tau_0 \rangle$ similarly but in the right-hand side of (2.4.1) we have $|t - s| \log |t - s|$ instead of $|t - s|$.

We note that for $|\tau| = 1$ formula (2.4.2) implies $\delta \langle X, \tau \rangle = 0$, namely $\langle X, \tau \rangle_{st} = f_t^\tau - f_s^\tau$ for some α -Hölder function f . Therefore one starts with a family of α -Hölder functions $(f^\tau)_{\tau \in S, |\tau|=1}$ on $[0, T]$, and applying recursively Theorem 2.3.5 one can construct a family $(\langle X, \tau \rangle)_{\tau \in S}$ satisfying (2.4.1)-(2.4.2). Moreover $(\langle X, \tau \rangle)_{\tau \in B}$ can be seen to depend in a continuous way on $(f^\tau)_{\tau \in B, |\tau|=1}$. This is the basis of the *extension theorem* that we prove in Theorem 2.4.4 below.

2.4.1 Rough paths and Hopf algebras

In what follows, we consider $H = (\bigoplus_{n \in \mathbb{N}} H_n, m, \mathbb{1}, \Delta, \epsilon, S)$, a graded, connected ($H_0 = \text{span}(\mathbb{1})$), locally finite ($0 < \dim(H_n) < +\infty$), commutative Hopf algebra on \mathbb{R} with antipode S . The degree $|\tau|$ of $\tau \in \bigcup_n H_n$ is defined by $|\tau| = n$ if $\tau \in H_n$.

The structure of Hopf algebra relies on many properties of compatibility between the operations (see e.g. [Car07; CP21] for more details on Hopf algebras). For instance, we will use the compatibility of the product m and the coproduct Δ , which reads (where we note $\tau_{2,3}$ to be the operator defined by $\tau_{2,3}(a \otimes b \otimes c \otimes d) := a \otimes c \otimes b \otimes d$):

$$\Delta \circ m = (m \otimes m) \circ \tau_{2,3}(\Delta \otimes \Delta). \quad (2.4.4)$$

We will also work with the reduced coproduct $\Delta': H \rightarrow H \otimes H$ defined as

$$\Delta' \tau := \Delta \tau - \tau \otimes \mathbb{1} - \mathbb{1} \otimes \tau.$$

Recall that for $n \geq 1$, $\Delta': H_n \rightarrow \bigoplus_{p,q \geq 1, p+q=n} H_p \otimes H_q$, and that Δ' satisfies the coassociativity property:

$$(\text{Id} \otimes \Delta') \Delta' = (\Delta' \otimes \text{Id}) \Delta'. \quad (2.4.5)$$

All those constraints on H are somehow quite restrictive. In fact, the following result, due to Milnor and Moore, asserts that such an H necessarily has a polynomial structure.

Theorem 2.4.1 (Milnor-Moore, see [MM65], [Car07, Theorem 3.8.3]). *Let H be a graded connected commutative Hopf algebra. Then H is a free polynomial algebra, whose indeterminates can be chosen to be homogeneous elements of H .*

This polynomial structure will be useful for the construction of linear maps on H that are also multiplicative, which we will call characters. More precisely:

Definition 2.4.2 (Characters). *Let H be a real algebra on \mathbb{R} .*

1. *Let $n \geq 1$. We say that a nonzero linear map $X \in L(H_{\leq n}, \mathbb{R})$ is a truncated character of order n on H if for all $\sigma, \tau \in H_{\leq n}$ with $\sigma\tau \in H_{\leq n}$, it holds that $\langle X, \sigma\tau \rangle = \langle X, \sigma \rangle \langle X, \tau \rangle$. We note G_n the set of truncated characters on H of order n .*
2. *We say that a nonzero linear map $X \in L(H, \mathbb{R})$ is a character on H if for all $\sigma, \tau \in H$, $\langle X, \sigma\tau \rangle = \langle X, \sigma \rangle \langle X, \tau \rangle$. We note G the set of characters on H .*

We will be interested in rough paths, which we define now. Heuristically, a rough path is a collection of biprocesses having the nice analytical and algebraic properties of iterated integrals.

Definition 2.4.3 (Rough paths). *Let H be a graded connected locally finite Hopf algebra on \mathbb{R} . Let $\alpha \in (0, 1)$ and $N := \lfloor 1/\alpha \rfloor$. A (H, α) -rough path is a map $X: [0, T]^2 \rightarrow G_N$ such that:*

1. *for all $\tau \in \bigcup_{n \leq N} H_n$ we have $\langle X, \tau \rangle \in \mathcal{C}_2^{\alpha|\tau|}$, i.e.*

$$\sup_{s \neq t \in [0, T]} \frac{|\langle X_{s,t}, \tau \rangle|}{|t - s|^{\alpha|\tau|}} < +\infty. \quad (2.4.6)$$

2. *(Chen's relation) for all $\tau \in H_{\leq N}$ and $s, u, t \in [0, T]$, one has*

$$\langle X_{s,t}, \tau \rangle = \langle X_{s,u} \otimes X_{u,t}, \Delta \tau \rangle. \quad (2.4.7)$$

We note $\text{RP}^\alpha(H)$ the set of all (H, α) -rough paths.

Then our main result will be the following generalisation of the Lyons-Victoir extension theorem [LV07]:

Theorem 2.4.4 (A continuous extension). *Let $\alpha \in (0, 1)$ with $\alpha^{-1} \notin \mathbb{N}$. Let us set*

$$\mathcal{C}_{H_1^*}^\alpha := \{f : [0, T] \rightarrow H_1^* : f_0 = 0, \|f_t - f_s\| \lesssim |t - s|^\alpha\}.$$

There exists a continuous “extension map” $\mathcal{E} : \mathcal{C}_{H_1^}^\alpha \rightarrow \text{RP}^\alpha(H)$ such that for all $h \in H_1$ and $f \in \mathcal{C}_{H_1^*}^\alpha$, $\langle \mathcal{E}(f), h \rangle = \langle f, h \rangle$.*

In other words, every α -Hölder path with values in the finite dimensional space H_1^* can be lifted to a α -rough path on H , and this extension can be made in a continuous way. Theorem 2.4.4 will be proved as a corollary of Theorem 2.4.7 below.

2.4.2 Extension to levels higher than N

By the definition of Δ' , the Chen relation (2.4.7) is equivalent to

$$(\delta \langle X, \tau \rangle)_{s,u,t} = \langle X_{s,u} \otimes X_{u,t}, \Delta' \tau \rangle. \quad (2.4.8)$$

Note that (2.4.1)-(2.4.2) are particular cases of (2.4.6)-(2.4.8), respectively. Then, a simple application of the Sewing Lemma (Theorem 2.3.5) for $\gamma > 1$ yields that

Proposition 2.4.5. *Every (H, α) -rough path $X : [0, T]^2 \rightarrow G_N$ has a unique extension $X : [0, T]^2 \rightarrow G$ satisfying (2.4.6)-(2.4.8) for all $\tau \in H$.*

Proof. Fix $\tau \in H_{N+1}$. Arguing as in (2.4.3), by (2.4.6)-(2.4.8) we have

$$|\langle X_{s,u} \otimes X_{u,t}, \Delta' \tau \rangle| \lesssim (|s - u| \vee |t - u|)^{\alpha(N+1)},$$

where by definition $\alpha(N+1) > 1$. In order to apply Theorem 2.3.5 to $(s, u, t) \mapsto \langle X_{s,u} \otimes X_{u,t}, \Delta' \tau \rangle$, we still have to check that this function belongs to $\delta(\mathcal{C}_2)$. This is true since, if we consider the function $F \in \mathcal{C}_2$, defined by $F_{s,t} := \langle X_{0,s} \otimes X_{s,t}, \Delta' \tau \rangle$, then

$$\begin{aligned} (\delta F)_{s,u,t} &= \langle X_{0,s} \otimes (\delta X)_{s,u,t} - (\delta X)_{0,s,u} \otimes X_{u,t} - X_{s,u} \otimes X_{u,t}, \Delta' \tau \rangle \\ &= \langle X_{0,s} \otimes X_{s,u} \otimes X_{u,t}, (\text{Id} \otimes \Delta') \Delta' \tau - (\Delta' \otimes \text{Id}) \Delta' \tau \\ &\quad - \langle X_{s,u} \otimes X_{u,t}, \Delta' \tau \rangle \\ &= -\langle X_{s,u} \otimes X_{u,t}, \Delta' \tau \rangle, \end{aligned}$$

by coassociativity of Δ' , recall (2.4.5). Therefore by Theorem 2.3.5 for $\gamma = \alpha(N+1) > 1$ there is a unique $\langle X, \tau \rangle \in \mathcal{C}_2^\gamma$ with the desired properties (2.4.6)-(2.4.8), given by $\langle X, \tau \rangle = \Lambda((s, u, t) \mapsto \langle X_{s,u} \otimes X_{u,t}, \Delta' \tau \rangle)$. By recurrence, one constructs in the same way $\langle X, \tau \rangle$ for all $\tau \in \bigoplus_{n \geq N+1} H_n$.

However we still have to prove that this construction gives an element of G , namely that $\langle X, \sigma\tau \rangle = \langle X, \tau \rangle \langle X, \sigma \rangle$ for all $\sigma, \tau \in H$. It is enough to assume $\sigma\tau \in \bigoplus_{n \geq N+1} H_n$. By (2.4.4) we have

$$\Delta'(\sigma\tau) = (m \otimes m) \circ \tau_{2,3} (\Delta\sigma \otimes \Delta\tau - 1 \otimes \sigma \otimes 1 \otimes \tau - \sigma \otimes 1 \otimes \tau \otimes 1).$$

By (2.4.8)

$$(\delta \langle X, \sigma\tau \rangle)_{s,u,t} = \langle X_{s,u} \otimes X_{u,t}, \Delta'(\sigma\tau) \rangle.$$

On the other hand

$$\begin{aligned} \delta(\langle X, \sigma \rangle \langle X, \tau \rangle)_{s,u,t} &= \\ &= \langle X_{s,t}, \sigma \rangle \langle X_{s,t}, \tau \rangle - \langle X_{s,u}, \sigma \rangle \langle X_{s,u}, \tau \rangle - \langle X_{u,t}, \sigma \rangle \langle X_{u,t}, \tau \rangle \end{aligned}$$

and by the Chen relation (2.4.7)

$$\begin{aligned} \langle X_{s,t}, \sigma \rangle \langle X_{s,t}, \tau \rangle &= \langle X_{s,u} \otimes X_{u,t}, (m \otimes m) \circ \tau_{2,3}(\Delta\sigma \otimes \Delta\tau) \rangle, \\ \langle X_{s,u}, \sigma \rangle \langle X_{s,u}, \tau \rangle &= \langle X_{s,u} \otimes X_{u,t}, 1 \otimes \sigma \otimes 1 \otimes \tau \rangle, \\ \langle X_{u,t}, \sigma \rangle \langle X_{u,t}, \tau \rangle &= \langle X_{s,u} \otimes X_{u,t}, \sigma \otimes 1 \otimes \tau \otimes 1 \rangle. \end{aligned}$$

We obtain $\delta \langle X, \sigma\tau \rangle = \delta(\langle X, \sigma \rangle \langle X, \tau \rangle) \in \mathcal{C}_3^{\alpha(N+1)}$. Since $\langle X, \sigma\tau \rangle$ and $\langle X, \sigma \rangle \langle X, \tau \rangle$ are both in \mathcal{C}_2^γ , we conclude by Theorem 2.3.5. \square

Since H is locally finite, the same can be said of the corresponding set of indeterminates. Hence from Theorem 2.4.1, for each $n \geq 1$, there exists a finite set $B_n \subset H_n$ such that:

$$H = \mathbb{R} \left[\mathbf{1} \cup \bigcup_{n \geq 1} B_n \right].$$

We denote

$$H_{\leq n} := \bigoplus_{k=0}^n H_k, \quad B_{\leq n} := \bigcup_{k=1}^n B_k.$$

We also note $B := \bigcup_{n \geq 1} B_n$, to be the set of generating monomials, so that $H = \mathbb{R}[\mathbf{1} \cup B]$. The Milnor-Moore theorem asserts the existence of such a basis of generating monomials, but B is neither unique nor canonical in general, see Example 2.4.6 below for some examples in usual cases. We note that B_1 is always a linear basis for H_1 .

Example 2.4.6 (Examples of (H, α) -rough paths). *Here are some examples of Hopf algebras to which the above applies:*

1. *The shuffle algebra on the alphabet $\{1, \dots, d\}$, and the theory of (weakly) geometric rough paths on \mathbb{R}^d ; in this case, one can take as a basis B the set of Lyndon words [Reu03].*
2. *The Butcher-Connes-Kreimer algebra of rooted forests with $\{1, \dots, d\}$ -decorated nodes, and the theory of branched rough paths on \mathbb{R}^d ; in this case, one can take as basis B the set of trees [Gub10].*

See [HK15], [TZ20, Section 4] and [BC19] for more details on geometric and branched rough paths.

3. *Quasi-shuffle algebras and quasi-geometric rough paths, see [Bel20].*
4. *The Hopf algebra of Lie group integrators and planarly branched rough paths, see [CEFMMK20].*
5. *The recently introduced (and so far un-named) Hopf algebra of [LOT21, Section 6].*

2.4.3 An isomorphism

What characterizes a (H, α) -rough path? Since B is a set of generators of H as an algebra, by linearity and multiplicativity, $X \in \text{RP}^\alpha(H)$ is uniquely determined by the values of $\langle X, h \rangle$ for $h \in B$. In fact, by Proposition 2.4.5, the values of $\langle X, h \rangle$ for $h \in \cup_{n>N} H_n$ are uniquely determined by the values for $h \in \cup_{n \leq N} B_n$.

It remains to characterize the family $(\langle X, h \rangle)_{h \in B_{\leq N}}$. For all $\tau \in B_{\leq N}$, by the Chen relation (2.4.8), $\delta \langle X, \tau \rangle$ is characterized by $(\langle X, h \rangle)_{h \in B_{\leq |\tau|-1}}$. However, since $\alpha|\tau| \leq 1$, the function $\langle X, \tau \rangle$ is determined by $\delta \langle X, \tau \rangle \in \mathcal{C}_3^{\alpha|\tau|}$ only up to a $(\alpha|\tau|)$ -Hölder function $f^\tau : [0, T] \rightarrow \mathbb{R}$ such that $f_0^\tau = 0$, see Remark 2.2.3.

It was indeed shown in [TZ20] that, in the case of branched rough paths, $\text{RP}^\alpha(H)$ is in a bijective correspondence with

$$\mathcal{C}_B^\alpha := \left\{ (f^h)_{h \in B_{\leq N}} : \text{for all } h \in B_{\leq N}, f^h \in \mathcal{C}_1^{|h|\alpha} \text{ and } f_0^h = 0 \right\}.$$

While the construction of [TZ20], restricted to the Butcher-Connes-Kreimer Hopf algebra, used the Hairer-Kelly map [HK15] and a Lyons-Victoir extension technique, the approach of this paper, based on the Sewing Lemma (Theorem 2.3.5) for $\gamma < 1$, is more elementary and more general at the same time. For a general (graded, connected, locally finite, commutative) Hopf algebra H we introduce the map

$$\begin{aligned} \mathcal{P} : \text{RP}^\alpha(H) &\longrightarrow \mathcal{C}_B^\alpha \\ X &\longmapsto \mathcal{P}(X) := (\mathcal{I}(\langle X, h \rangle))_{h \in B_{\leq N}}, \end{aligned} \quad (2.4.9)$$

where \mathcal{I} is the integration map defined in Theorem 2.3.5 and $N = \lfloor 1/\alpha \rfloor$. We prove that \mathcal{P} is bijective and furthermore we show that \mathcal{P} is bicontinuous with respect to the distances defined for $X, Y \in \text{RP}^\alpha(H)$ resp. $f, g \in \mathcal{C}_B^\alpha$ by:

$$\begin{cases} d_{\text{RP}^\alpha(H)}(X, Y) &:= \sum_{h \in B_{\leq N}} \|\langle X - Y, h \rangle\|_{\mathcal{C}_2^{|h|\alpha}}, \\ d_{\mathcal{C}_B^\alpha}(f, g) &:= \sum_{h \in B_{\leq N}} \|g^h - f^h\|_{\mathcal{C}_1^{|h|\alpha}}. \end{cases}$$

The main result of this section is the following.

Theorem 2.4.7. *Let $\alpha \in (0, 1)$ with $\alpha^{-1} \notin \mathbb{N}$. The map \mathcal{P} in (2.4.9) is a locally bi-Lipschitz homeomorphism between $(\text{RP}^\alpha(H), d_{\text{RP}^\alpha(H)})$ and $(\mathcal{C}_B^\alpha, d_{\mathcal{C}_B^\alpha})$.*

Before proving this result, let us discuss some corollaries. First we prove Theorem 2.4.4.

Proof of Theorem 2.4.4. It suffices to identify $\mathcal{C}_{H_1^*}^\alpha$ with $\mathcal{C}_{B_1}^\alpha$, where

$$\mathcal{C}_{B_1}^\alpha := \left\{ (f^h)_{h \in B_1} : \text{for all } h \in B_1, f^h \in \mathcal{C}_1^\alpha \text{ and } f_0^h = 0 \right\}$$

and then take $\mathcal{E} = \mathcal{P}^{-1} \circ \iota$, where $\iota : \mathcal{C}_{B_1}^\alpha \rightarrow \mathcal{C}_B^\alpha$ is the canonical injection obtained by defining $f^h := 0$ for all $h \in B_{\leq N} \setminus B_1$. \square

Corollary 2.4.8 (An action on Rough Paths, see [TZ20]). *Let $\alpha \in (0, 1)$ with $\alpha^{-1} \notin \mathbb{N}$ and define:*

$$\begin{aligned} T : \mathcal{C}_B^\alpha \times \text{RP}^\alpha(H) &\longrightarrow \text{RP}^\alpha(H) \\ (g, X) &\longmapsto gX := \mathcal{P}^{-1}(g + \mathcal{P}(X)). \end{aligned} \quad (2.4.10)$$

Then:

1. T is an action: for each $g, g' \in \mathcal{C}_B^\alpha$ and $X \in \text{RP}^\alpha(H)$, $g'(gX) = (g + g')X$.
2. T is free and transitive: for each $X, X' \in \text{RP}^\alpha(H)$, there exists a unique $g \in \mathcal{C}_B^\alpha$ such that $X' = gX$.
3. T is continuous.
4. Let $g \in \mathcal{C}_B^\alpha$ be such that there exists a unique $h \in B$ such that $g^h \neq 0$. Then $\langle gX, h \rangle = \langle X, h \rangle + \delta g^h$. Furthermore, let $\tilde{B} \subset B \setminus \{h\}$ be a set of monomials such that $\Delta'(\mathbb{R}[\tilde{B}]) \subset \mathbb{R}[\tilde{B}] \otimes \mathbb{R}[\tilde{B}]$. Then for any $\tau \in \mathbb{R}[\tilde{B}]$, $\langle gX, \tau \rangle = \langle X, \tau \rangle$.

Proof. Items 1., 2., 3. are straightforward from (2.4.10) and Theorem 2.4.7. Also, one obtains 4. recursively by using the explicit definition of T in (2.4.10) and the recursive construction of \mathcal{P}^{-1} from the proof of Theorem 2.4.7. \square

Remark 2.4.9 (Link with [TZ20]). *When H is the Butcher-Connes-Kreimer algebra, $\text{RP}^\alpha(H)$ corresponds to the set of branched rough paths, also denoted BRP^α in [TZ20]. In that case, let $h \in H$ and \tilde{B} be the set of trees not containing h . By definition of the coproduct from admissible cuts, $\Delta'(\mathbb{R}[\tilde{B}]) \subset \mathbb{R}[\tilde{B}] \otimes \mathbb{R}[\tilde{B}]$ and Corollary 2.4.8 generalises [TZ20, Theorem 1.2].*

2.4.4 Proof

Now we turn to the proof of Theorem 2.4.7.

Proof of Theorem 2.4.7. We organise this proof in five different steps.

Step 1: \mathcal{P} is well-defined. Indeed, if $X \in \text{RP}^\alpha(H)$ and $h \in B_{\leq N}$, by (2.4.6)-(2.4.8) $\langle X, h \rangle \in \mathcal{C}_2^{[h]\alpha}$ and $\delta \langle X, h \rangle \in \mathcal{C}_3^{[h]\alpha}$, so that by Theorem 2.3.5 we obtain $\mathcal{I}(\langle X, h \rangle) \in \mathcal{C}_1^{[h]\alpha}$.

Step 2: continuity of \mathcal{P} . Let $X, Y \in \text{RP}^\alpha(H)$, and $h \in B_{\leq N}$. By linearity of \mathcal{I} , $(\mathcal{P}(X))^h - (\mathcal{P}(Y))^h = \mathcal{I}(\langle X - Y, h \rangle)$. Now by continuity of \mathcal{I} (Theorem 2.3.5),

$$\|\mathcal{I}(\langle X - Y, h \rangle)\|_{\mathcal{C}_1^{[h]\alpha}} \lesssim \|\langle X - Y, h \rangle\|_{\mathcal{C}_2^{[h]\alpha}} + \|\delta \langle X - Y, h \rangle\|_{\mathcal{C}_3^{[h]\alpha}}.$$

Observe that $\|\delta \langle X - Y, h \rangle\|_{\mathcal{C}_3^{[h]\alpha}} \leq 3 \|\langle X - Y, h \rangle\|_{\mathcal{C}_2^{[h]\alpha}}$. Also, by definition, one has $\|\langle X - Y, h \rangle\|_{\mathcal{C}_2^{[h]\alpha}} \leq d_{\text{RP}^\alpha(H)}(X, Y)$, thus we obtain $d_{\mathcal{C}_B^\alpha}(\mathcal{P}(X), \mathcal{P}(Y)) \lesssim d_{\text{RP}^\alpha(H)}(X, Y)$, which establishes the Lipschitz continuity of \mathcal{P} .

Step 3: injectivity of \mathcal{P} . In this step, we use the following general fact: if $A, \tilde{A} \in \mathcal{C}_2$ are such that $\delta A = \delta \tilde{A}$ and $\mathcal{I}(A) = \mathcal{I}(\tilde{A})$, then $A = \tilde{A}$. Indeed, this follows immediately from the defining property (2.3.4) of \mathcal{I} :

$$\delta \circ \mathcal{I} = \text{id} - \Lambda \circ \delta.$$

Let $X, Y \in \text{RP}^\alpha$ and assume $\mathcal{P}(X) = \mathcal{P}(Y)$. That is, for all $h \in B$, $\mathcal{I}(\langle X, h \rangle) = \mathcal{I}(\langle Y, h \rangle)$. We prove by recurrence that for all $h \in B$, $\langle X, h \rangle = \langle Y, h \rangle$. If $h \in B_1$, then by definition of the reduced coproduct Δ' and Chen's relation, it holds that $\delta \langle X, h \rangle = \delta \langle Y, h \rangle = 0$ and thus from the remark just above, $\langle X, h \rangle = \langle Y, h \rangle$, whence the initialisation of the recurrence. Now let $n \in \mathbb{N}$ and assume that X and Y coincide on $B_{\leq n}$. Let $h \in B_{n+1}$. Once again using Chen's relation, the definition of the reduced coproduct, and the fact that X and Y are characters, one gets by the recurrence hypothesis that $\delta \langle X, h \rangle = \delta \langle Y, h \rangle$. Applying again the remark just above, it follows that $\langle X, h \rangle = \langle Y, h \rangle$. By recurrence, this yields the injectivity of \mathcal{P} .

Step 4: surjectivity of \mathcal{P} . In this step, we construct a right-inverse \mathcal{P}^{-1} of \mathcal{P} (which is also an actual inverse thanks to the previous step). For this purpose, we proceed recursively. Fix $f \in \mathcal{C}_B^\alpha$. We first define $X := \mathcal{P}^{-1}(f)$ on H_1 by setting for $h \in B_1$: $\langle X, h \rangle := f^h$, then extending X to H_1 by linearity. Now let $n \geq 1$ and assume that X is defined on $H_{\leq n}$. We start by defining X on the monomials i.e. on B_{n+1} . Fix $h \in B_{n+1}$ and let us show that there exists $\langle X, h \rangle : [0, T]^2 \rightarrow \mathbb{R}$ such that:

1. $|\langle X_{s,t}, h \rangle| \lesssim |t - s|^{\alpha(n+1)}$,
2. $(\delta \langle X, h \rangle)_{s,u,t} = \langle X_{s,u} \otimes X_{u,t}, \Delta' h \rangle$.

For this purpose, arguing as in the proof of Proposition 2.4.5 we consider the function $F \in \mathcal{C}_2$ defined for $s, t \in [0, T]$ by:

$$F_{s,t} := \langle X_{0,s} \otimes X_{s,t}, \Delta' h \rangle.$$

Then by the coassociativity of Δ' , the Chen relation and the recurrence hypothesis, for $s, u, t \in [0, T]$ we have $(\delta F)_{s,u,t} = -\langle X_{s,u} \otimes X_{u,t}, \Delta' h \rangle$. Also, from the definition of Δ' and the analytic constraint (2.4.6) on $(\langle X, \tau \rangle)_{\tau \in H_n}$, one observes that $\delta F \in \mathcal{C}_3^{(n+1)\alpha}$. Then, from the properties of the Sewing map, it suffices to set:

$$\langle X, h \rangle := -\Lambda(\delta F) + f^h. \quad (2.4.11)$$

Note that (2.4.11) can be rewritten as $\mathcal{I}(\langle X, h \rangle) = f^h$, i.e. $\mathcal{P}(X)^h = f^h$, as wanted for the construction of the inverse. Now it remains to suitably extend X to the whole of $H_{\leq n+1}$, which we do ‘‘polynomially’’: if P is a polynomial, we set

$$\langle X, P((h)_{h \in B_{\leq n+1}}) \rangle := P(\langle X, h \rangle)_{h \in B_{\leq n+1}}.$$

It is straightforward to observe that this correctly defines an element of G_{n+1} and enforces the estimate $|\langle X_{s,t}, \tau \rangle| \lesssim |t - s|^{\alpha(n+1)}$ for all $\tau \in H_{n+1}$. To conclude the recursive step, it suffices to establish Chen’s relation on H_{n+1} , knowing that it is satisfied on H_n and on B_{n+1} (by the construction just above). Observe that if Chen’s relation is satisfied on τ and σ , then it is also satisfied on $\tau + \sigma$. Now because we define X polynomially, it suffices to prove that if Chen’s relation is satisfied for $\sigma \in H_l$ and $\tau \in H_k$ where $k + l \leq n + 1$, $1 \leq k, l \leq n$, then it is also satisfied for $\sigma\tau$. For this, it is enough to prove that $\delta \langle X, \sigma\tau \rangle = \delta(\langle X, \sigma \rangle \langle X, \tau \rangle)$. This can be done arguing as in Proposition 2.4.5. (While in Proposition 2.4.5 we had to verify that the extension was multiplicative, here this property is enforced by the ‘‘polynomial’’ definition). This concludes the recursive step, so that we have constructed $X =: \mathcal{P}^{-1}(f) \in \text{RP}^\alpha(H)$.

Step 5: continuity of \mathcal{P}^{-1} . We first show that \mathcal{P}^{-1} maps bounded sets to bounded sets. For this purpose, let $C > 0$, we shall show that there exists $C' > 0$ such that for all $f \in \mathcal{C}_B^\alpha$ with $\sum_{h \in B} \|f^h\|_{\mathcal{C}_1^{|h|\alpha}} \leq C$, one has $\sum_{h \in B} \|\langle \mathcal{P}^{-1}(f), h \rangle\|_{\mathcal{C}_2^{|h|\alpha}} \leq C'$. We proceed recursively: when $h \in B_1$, by construction of \mathcal{P}^{-1} , it holds that $\langle \mathcal{P}^{-1}(f), h \rangle = f^h$, and thus $\|\langle \mathcal{P}^{-1}(f), h \rangle\|_{\mathcal{C}_2^{|h|\alpha}} = \|f\|_{\mathcal{C}_1^{|h|\alpha}} \leq C$. Now fix $n \geq 1$ and assume that for all $b \in B_{\leq n}$, $\|\langle \mathcal{P}^{-1}(f), b \rangle\|_{\mathcal{C}_2^{|b|\alpha}} \leq C'$ for some constant C' . Let $h \in B_{n+1}$, so that by the construction above:

$$\langle \mathcal{P}^{-1}(f), h \rangle = \Lambda(\delta \langle \mathcal{P}^{-1}(f), h \rangle) + f^h. \quad (2.4.12)$$

Using Chen’s relation, we obtain a decomposition:

$$\delta \langle \mathcal{P}^{-1}(f), h \rangle_{s,u,t} = \sum_{\substack{\sigma \in H_k, \tau \in H_l \\ k, l \geq 1, k+l=|h|}} \langle \mathcal{P}^{-1}(f)_{s,u}, \sigma \rangle \langle \mathcal{P}^{-1}(f)_{u,t}, \tau \rangle.$$

The definition of B implies that for all σ , there exists a polynomial $P = P_\sigma$ such that $\sigma = P_\sigma(B_{\leq|\sigma|})$. As a consequence, note that:

$$\langle \mathcal{P}^{-1}(f)_{s,u}, \sigma \rangle = |s - u|^{\sigma|\alpha|} P_\sigma \left(\left(\frac{\langle \mathcal{P}^{-1}(f)_{s,u}, b \rangle}{|s - u|^{|b|\alpha}} \right)_{b \in B_{\leq|\sigma|}} \right).$$

This yields:

$$\|\delta \langle \mathcal{P}^{-1}(f), h \rangle\|_{\mathcal{C}_3^{|h|\alpha}} \leq \sum_{\sigma, \tau} P_\sigma P_\tau \left(\left(\|\langle \mathcal{P}^{-1}(f), b \rangle\|_{\mathcal{C}_2^{|b|\alpha}} \right)_{b \in B_{\leq n}} \right),$$

which is bounded by the recurrence hypothesis. We conclude the recurrence step using (2.4.12) and the continuity of the Sewing map Λ , so that \mathcal{P}^{-1} does indeed map bounded sets to bounded sets.

Now let us tackle the continuity of \mathcal{P}^{-1} . We reason as above: we fix $f, g \in \mathcal{C}_B^\alpha$ and set $\zeta := \mathcal{P}^{-1}(f) - \mathcal{P}^{-1}(g)$. We estimate recursively $\|\langle \zeta, h \rangle\|_{\mathcal{C}_2^{|h|\alpha}}$ for monomials $h \in B_{\leq N}$. When $h \in B_1$, by construction of \mathcal{P}^{-1} , it holds that $\langle \zeta, h \rangle = f^h - g^h$, thus we have

$$\|\langle \zeta, h \rangle\|_{\mathcal{C}_2^{|h|\alpha}} \leq d_{\mathcal{C}_B^\alpha}(f, g).$$

Fix $n \geq 1$, and $h \in B_{n+1}$. By the construction above, $\langle \zeta, h \rangle = \Lambda(\delta \langle \zeta, h \rangle) + f^h - g^h$. Using Chen's relation and the notations above, one obtains a decomposition of the form, denoting $X := \mathcal{P}^{-1}(f)$, $Y := \mathcal{P}^{-1}(g)$:

$$\begin{aligned} \left| \frac{\delta \langle X - Y, h \rangle_{s,u,t}}{(|t - u| + |u - s|)^\alpha} \right| &\leq \sum_{\sigma, \tau} \left| P_\sigma \left(\left(\frac{\langle X_{s,u}, b \rangle}{|s - u|^{|b|\alpha}} \right)_b \right) P_\tau \left(\left(\frac{\langle X_{u,t}, b \rangle}{|t - u|^{|b|\alpha}} \right)_b \right) - \right. \\ &\quad \left. - P_\sigma \left(\left(\frac{\langle Y_{s,u}, b \rangle}{|s - u|^{|b|\alpha}} \right)_b \right) P_\tau \left(\left(\frac{\langle Y_{u,t}, b \rangle}{|t - u|^{|b|\alpha}} \right)_b \right) \right|. \end{aligned}$$

Using the fact that polynomials are locally Lipschitz, the fact established above that \mathcal{P}^{-1} maps bounded sets to bounded sets, and the continuity of Λ , one propagates the locally Lipschitz bound over $h \in B_{n+1}$, so that the theorem is proved by recurrence. \square

2.5 Sewing versus Reconstruction

In this section, we discuss the link between the Sewing Lemma (Theorem 2.2.1, Theorem 2.2.2) and the Reconstruction Theorem [Hai14, Theorem 3.10]-[CZ20, Theorem 5.1] in Regularity Structures. It is often claimed that the latter is a generalisation of the former. Here we want to test this claim in details, and show that it is correct only up to a point. We are going to see that the Sewing Lemma is actually slightly stronger than the 1-dimensional version of the Reconstruction Theorem, see Remark 2.5.6 below.

The take-home message is the following: if one wants to prove the Sewing Lemma (for any $\gamma > 0$) via the Reconstruction Theorem, then the "coherence" condition $\delta A \in \mathcal{C}_3^\gamma(\Delta_T^3)$ is not enough, and one needs the "homogeneity" condition $A \in \mathcal{C}_2^\beta(\Delta_T^2)$ for some $\beta > 0$. The original Sewing Lemma on the other hand holds with the coherence condition only.

The reason for that is the following: given $A_{s,t}$, one defines the distribution $F_s := \partial_t A_s$, depending on the parameter s ; the Reconstruction Theorem gives the existence of a distribution f with a desired property; in order to obtain the integral $I = \mathcal{I}(A)$, one need now to find a "primitive" of f , and this is the point where one need the homogeneity

condition $A \in \mathcal{C}_2^\beta(\Delta_T^2)$. In other words, by Proposition 2.3.4, the Reconstruction Theorem yields the Sewing Lemma only in the case where the primitive $I = \mathcal{I}(A)$ belongs to a Hölder space \mathcal{C}_1^β . The Sewing Lemma on the other hand does not need to differentiate A and the final integration step with the associated homogeneity condition is therefore unnecessary, see Remark 2.5.6.

2.5.1 Reconstruction

The Reconstruction Theorem is a result in analysis which was first established in the context of the theory of regularity structures [Hai14]. It has later been revisited in [CZ20], where it was stated and proved in a more elementary fashion, using only the language of distribution theory.

The Reconstruction Theorem in Hölder spaces is stated in the following way, see [CZ20] for a proof and a discussion of this result. For $r \in \mathbb{N}$ define $\mathcal{B}^r := \{\varphi \in \mathcal{D}(B(0,1)), \|\varphi\|_{\mathcal{C}^r} \leq 1\}$. For $\varphi \in \mathcal{D}(\mathbb{R}^d)$, $x \in \mathbb{R}^d$, $\lambda > 0$, we denote φ_x^λ the scaled and recentered version of φ , defined as follows: $\varphi_x^\lambda(\cdot) := \lambda^{-d} \varphi(\lambda^{-1}(\cdot - x))$.

Theorem 2.5.1 (Reconstruction Theorem [CZ20, Theorem 5.1]). *Let $(F_x)_{x \in \mathbb{R}^d}$ be a germ in the sense of [CZ20] i.e. for all x , $F_x \in \mathcal{D}'(\mathbb{R}^d)$ and for all $\varphi \in \mathcal{D}(\mathbb{R}^d)$, $x \mapsto F_x(\varphi)$ is measurable.*

Let $a, c \in \mathbb{R}$ with $a \leq 0 \wedge c$. Assume there exists a test-function $\varphi \in \mathcal{D}(\mathbb{R}^d)$ with $\int \varphi \neq 0$ such that for all compact $K \subset \mathbb{R}^d$:

$$\left| (F_{x+h} - F_x) \left(\varphi_x^\lambda \right) \right| \lesssim \lambda^a (|h| + \lambda)^{c-a}, \quad (2.5.1)$$

uniformly over $x \in K$, $h \in B(0,1)$, $\lambda \in (0,1]$. Then there exists a distribution $\mathcal{R}(F) \in \mathcal{D}'(\mathbb{R}^d)$ such that for all compact $K \subset \mathbb{R}^d$ there exists an integer r_K such that:

$$\left| (\mathcal{R}(F) - F_x) \left(\psi_x^\lambda \right) \right| \lesssim \begin{cases} \lambda^c & \text{if } c \neq 0, \\ 1 + |\log |\lambda|| & \text{if } c = 0, \end{cases} \quad (2.5.2)$$

uniformly over $x \in K$, $\lambda \in (0,1]$, $\psi \in \mathcal{B}^{r_K}$.

Furthermore, assume that there exists $b < 0 \wedge c$ and a test-function $\varphi \in \mathcal{D}(\mathbb{R}^d)$ with $\int \varphi \neq 0$ such that for all compact $K \subset \mathbb{R}^d$:

$$\left| F_x \left(\varphi_x^\lambda \right) \right| \lesssim \lambda^b, \quad (2.5.3)$$

uniformly over $x \in K$, $\lambda \in (0,1]$. Then one can take $r_K = \max(-a, -b)$ in (2.5.2), and it holds that $\mathcal{R}(F) \in \mathcal{C}_1^b$, in the sense that for all integer $s > -b$ and $K \subset \mathbb{R}^d$:

$$\left| \mathcal{R}(F) \left(\psi_x^\lambda \right) \right| \lesssim \lambda^b,$$

uniformly over $x \in K$, $\lambda \in (0,1]$, $\psi \in \mathcal{B}^s$.

Let us briefly comment on the relevance of this result. The Reconstruction Theorem states that one can retrieve a distribution $\mathcal{R}(F)$ from a collection $(F_x)_{x \in \mathbb{R}^d}$ of “local approximations”, under a suitable assumptions of “coherence” (2.5.1) on F . If we furthermore assume a condition of “homogeneity” (2.5.3) on F , then we have information on the regularity of $\mathcal{R}(F)$ as a distribution.

The similarity with the Sewing Lemma is evident, and we will indeed show that these two results are intimately related to each other. Let $0 < \gamma < 1$ and $A: [0, T] \rightarrow \mathbb{R}$ be a continuous function satisfying

$$|A_{s,t} - A_{s,u} - A_{u,t}| \lesssim (|t - u| \vee |u - s|)^\gamma,$$

uniformly over $s, u, t \in [0, T]$. Let us try to construct a function I satisfying the sewing bound

$$|I_t - I_s - A_{s,t}| \lesssim |t - s|^\gamma, \quad (2.5.4)$$

uniformly over $s, t \in [0, T]$, by invoking the Reconstruction Theorem. We first define a germ F by differentiating A in the second variable. More precisely, we extend A to \mathbb{R}^2 by setting for $s, t \in \mathbb{R}^2$, $A_{s,t} := A_{p(s),p(t)}$, where $p: s \mapsto \max(0, \min(s, T))$; then we consider the germ defined for $s \in \mathbb{R}$ by $F_s := \partial_t A_{s,\cdot}$, where the partial derivative is understood in the sense of distributions. That is, for test-functions $\varphi \in \mathcal{D}(\mathbb{R})$:

$$F_s(\varphi) = - \int_{\mathbb{R}} A_{s,t} \varphi'(t) dt.$$

Observe that F satisfies the coherence condition (2.5.1) with parameters $a = -1, c = \gamma - 1$ because if $\varphi \in \mathcal{D}(\mathbb{R})$ is *any* test-function, $s, u, t \in \mathbb{R}, \lambda \in (0, 1]$, then (note that p is 1-Lipschitz):

$$\begin{aligned} |(F_t - F_s)(\varphi_s^\lambda)| &= \lambda^{-1} \left| \int (\delta A_{p(t),p(s),p(s+\lambda v)}) \varphi'(v) dv \right| \\ &\lesssim \lambda^{-1} (\lambda + |t - s|)^\gamma. \end{aligned}$$

Thus, we can apply Theorem 2.5.1, so that there exists $\mathcal{R}(F) \in \mathcal{D}'(\mathbb{R})$ and integers r_K , such that for all compact $K \subset \mathbb{R}$:

$$\sup_{x \in K} \sup_{\lambda \in (0,1]} \sup_{\psi \in \mathcal{B}^{r_K}} \frac{|(F_s - \mathcal{R}(F))(\psi_s^\lambda)|}{\lambda^{\gamma-1}} < +\infty. \quad (2.5.5)$$

Now in order to obtain the sewing bound (2.5.4) on A , we want to “integrate” the reconstruction bound (2.5.5) just above.

Heuristically, this “integration” can be performed by testing the reconstruction bound against indicator functions: notice that $(\mathbb{1}_{[0,1]})_s^\lambda = \lambda^{-1} \mathbb{1}_{[s,s+\lambda]}$, and if I denotes a primitive of $\mathcal{R}(F)$, one should have $(\mathcal{R}(F) - F_s)(\mathbb{1}_{[s,s+\lambda]}) = -(I - A)_{s,s+\lambda}$. Taking $\psi := \mathbb{1}_{(0,1)}$ in the reconstruction bound above would then yield $|(I - A)_{s,s+\lambda}| \lesssim \lambda^\gamma$, which is the expected sewing bound because $(I - A)_{s,t} = I_t - I_s - A_{s,t}$.

However, this heuristic argument fails for two reasons:

1. $\mathbb{1}_{(0,1)}$ is not a test-function (because it is not smooth) and thus cannot be plugged into the reconstruction bound,
2. without further assumptions it is not clear in general whether $\mathcal{R}(F)$ admits a primitive I which is a function.

2.5.2 Approximating an indicator function

We will solve the first point above by suitably approximating $\mathbb{1}_{(0,1)}$. Specifically, we shall exploit the following decomposition, see [FH20, Exercise 13.10] for a similar statement.

Lemma 2.5.2 (Dyadic approximation of indicator functions). *There exist smooth functions $\varphi_n, \psi_n, n \in \mathbb{N}$ such that:*

1. for all $n \in \mathbb{N}$, $\text{supp}(\varphi_n) \subset \left[\frac{1}{16} 2^{-n}, \frac{15}{16} 2^{-n} \right]$,
2. for all $n \in \mathbb{N}$, $\text{supp}(\psi_n) \subset \left[1 - \frac{15}{16} 2^{-n}, 1 - \frac{1}{16} 2^{-n} \right]$,

$$3. \text{ for all } r \in \mathbb{N}, \sup_{n \in \mathbb{N}} \sup_{k \in [0, r]} \frac{\|D^k \varphi_n\|_\infty + \|D^k \psi_n\|_\infty}{2^{kn}} < +\infty,$$

$$4. \mathbf{1}_{(0,1)} = \sum_{n \geq 0} (\varphi_n + \psi_n).$$

Proof. It is well known [BCD11, Proposition 2.10] that there exists a function $\varphi \in \mathcal{C}_c^\infty([1/2, 2])$ such that for all $x \in \mathbb{R}$, $\mathbf{1}_{\mathbb{R}_+^*}(x) = \sum_{n \in \mathbb{Z}} \varphi(2^n x)$. For $x \in \mathbb{R}$, set

$$\eta(x) := \mathbf{1}_{(0,1)}(x) - \sum_{n \geq 3} (\varphi(2^n x) + \varphi(2^n(1-x))).$$

Observe that η is a test-function supported in $[1/16, 15/16]$, whence the announced decomposition. \square

As a consequence:

Proposition 2.5.3 (Approximation argument). *In the setting above, assume that there exists a continuous function I such that $I_0 = 0$ and $I' = \mathcal{R}(F)$ in the sense of distributions. Set, for $n \in \mathbb{N}$, $s \in [0, T]$, $\lambda > 0$:*

$$\Delta_{s,\lambda}^N := \sum_{n=0}^N (\mathcal{R}(F) - F_s) \left(\lambda (\varphi_n + \psi_n)_s^\lambda \right).$$

Then:

$$1. \text{ for all } s \in \mathbb{R}, \lambda > 0, \lim_{N \rightarrow \infty} \Delta_{s,\lambda}^N = (I - A)_{s,s+\lambda},$$

$$2. \text{ For all compact } K \subset \mathbb{R}, |\Delta_{s,\lambda}^N| \lesssim \lambda^\gamma \text{ uniformly over } s \in K, N \in \mathbb{N}, \lambda \in (0, 1].$$

Proof. Let us establish those points separately. On the one hand, since for $s \in \mathbb{R}$, $\mathcal{R}(F) - F_s = (I - A_s)'$, we have for $N \in \mathbb{N}$:

$$\Delta_{s,\lambda}^N = - \int (I_u - A_{s,u}) \sum_{n=0}^N \left((\varphi_n + \psi_n)' \right)_s^\lambda (u) du.$$

By construction of φ_n and ψ_n , for all $x \in [2^{-N}, 1 - 2^{-N}]$ one has $\sum_{n=0}^N (\varphi_n(x) + \psi_n(x)) = 1$. Hence, setting for $N \in \mathbb{N}$ and $u \in \mathbb{R}$:

$$\begin{cases} \eta_N(u) & := 2^{-N} \sum_{n=0}^N (\varphi_n + \psi_n)'(2^{-N}u) \mathbf{1}_{[0,T]}(u), \\ \tilde{\eta}_N(u) & := -2^{-N} \sum_{n=0}^N (\varphi_n + \psi_n)'(1 + 2^{-N}u) \mathbf{1}_{[-1,0]}(u), \end{cases}$$

then it follows that $\eta_N, \tilde{\eta}_N \in \mathcal{D}(B(0, 1))$, and $\sum_{n=0}^N (\varphi_n + \psi_n)' = (\eta_N)_0^{2^{-N}} - (\tilde{\eta}_N)_1^{2^{-N}}$. Note that $\|\tilde{\eta}_N\|_{L^\infty}, \|\eta_N\|_{L^\infty} \lesssim 1$ and $\int_{\mathbb{R}} \eta_N(u) du = \int_{\mathbb{R}} \tilde{\eta}_N(u) du = 1$. Hence:

$$\begin{aligned} \Delta_{s,\lambda}^N &= - \int I_{s+\lambda 2^{-N}u} \eta_N(u) du + \int A_{s,s+\lambda 2^{-N}u} \eta_N(u) du \\ &\quad + \int I_{s+\lambda+\lambda 2^{-N}u} \tilde{\eta}_N(u) du - \int A_{s,s+\lambda+\lambda 2^{-N}u} \tilde{\eta}_N(u) du. \end{aligned}$$

Let us treat the first term. We write:

$$\int I_{s+\lambda 2^{-N}u} \eta_N(u) du = I_s + \int (I_{s+\lambda 2^{-N}u} - I_s) \eta_N(u) du. \quad (2.5.6)$$

Because I is continuous, the integrand in the right-hand side converges pointwise to 0 and is bounded by a constant. By the dominated convergence theorem, it follows that the sequence of integrals in (2.5.6) converges to 0. Reasoning similarly for the other terms, one obtains as announced that $\Delta_{s,\lambda}^N = I_{s+\lambda} - I_s - A_{s,s+\lambda} + o_{N \rightarrow +\infty}(1)$.

Now let us bound $|\Delta_{s,\lambda}^N|$. For $n \in \mathbb{N}$ and $x \in \mathbb{R}$, set:

$$\eta_n(x) := \varphi_n(2^{-n}x), \quad \tilde{\eta}_n(x) := \psi_n(2^{-n}x + 1).$$

Then $(\varphi_n)_s^\lambda = 2^{-n}(\eta_n)_s^{2^{-n}\lambda}$, and $(\psi_n)_s^\lambda = 2^{-n}(\tilde{\eta}_n)_{s+\lambda}^{2^{-n}\lambda}$. Also, note that the quantity $C_K := \sup_{n \in \mathbb{N}} (\|\eta_n\|_{C^{r_K}} + \|\tilde{\eta}_n\|_{C^{r_K}})$ is finite, so that $\eta_n/C_K, \tilde{\eta}_n/C_K \in \mathcal{B}^2$. Then we have the decomposition:

$$\begin{aligned} \Delta_{s,\lambda}^N &= \underbrace{\sum_{n=0}^N (\mathcal{R}(F) - F_s) \left(\lambda 2^{-n} (\eta_n)_s^{\lambda 2^{-n}} \right)}_{=: \Delta_{s,\lambda}^{N;1}} \\ &\quad + \underbrace{\sum_{n=0}^N (\mathcal{R}(F) - F_{s+\lambda}) \left(\lambda 2^{-n} (\tilde{\eta}_n)_{s+\lambda}^{\lambda 2^{-n}} \right)}_{=: \Delta_{s,\lambda}^{N;2}} \\ &\quad + \underbrace{\sum_{n=0}^N (F_{s+\lambda} - F_s) \left(\lambda 2^{-n} (\tilde{\eta}_n)_{s+\lambda}^{\lambda 2^{-n}} \right)}_{=: \Delta_{s,\lambda}^{N;3}}. \end{aligned}$$

Using the reconstruction bound (2.5.5) on the 1-enlargement of K , one has $\Delta_{s,\lambda}^{N;1} \lesssim \sum_{n=0}^N \lambda 2^{-n} (\lambda 2^{-n})^{\gamma-1} \lesssim \lambda^\gamma$. Similarly, $\Delta_{s,\lambda}^{N;2} \lesssim \lambda^\gamma$, so that it only remains to treat $\Delta_{s,\lambda}^{N;3}$. For this purpose, we rewrite:

$$\Delta_{s,\lambda}^{N;3} = -\lambda \int_{\mathbb{R}} \delta A_{s+\lambda,s,u} \sum_{n=0}^N 2^{-n} \left((\tilde{\eta}_n)_{s+\lambda}^{\lambda 2^{-n}} \right)'(u) du.$$

Note that $\text{supp} \left((\tilde{\eta}_n)_{s+\lambda}^{\lambda 2^{-n}} \right) \subset [s + \lambda, s + \lambda + 2^{-n}\lambda]$, and thus:

$$\left| \Delta_{s,\lambda}^{N;3} \right| \leq \left(\sup_{u \in [s+\lambda, s+2\lambda]} |\delta A_{s+\lambda,s,u}| \right) \int_{\mathbb{R}} \left| \sum_{n=0}^N \lambda 2^{-n} \left((\tilde{\eta}_n)_{s+\lambda}^{\lambda 2^{-n}} \right)'(u) \right| du.$$

Now, the hypothesis on A implies that for $u \in [s + \lambda, s + 2\lambda]$, we have $|\delta A_{s+\lambda,s,u}| \lesssim (|u - s| + |s - (s + \lambda)|)^\gamma \lesssim \lambda^\gamma$. Now it remains to establish that the sequence of integrals above is bounded. We write:

$$\int_{\mathbb{R}} \left| \sum_{n=0}^N \lambda 2^{-n} \left((\tilde{\eta}_n)_{s+\lambda}^{\lambda 2^{-n}} \right)'(u) \right| du = \int_{\mathbb{R}} |f'_N(v)| dv.$$

where we set $f_N: v \mapsto \sum_{n=0}^N \psi_n(1 - v)$. Note by definition of ψ that $\text{supp}(f_N) \subset [0, T]$, and that f_N converges pointwise to $f := \sum_{n=0}^{+\infty} \psi_n(1 - v)$. From the definition of φ_n, ψ_n , one has $f = \mathbf{1}_{(0,1)} - \sum_{n=0}^{+\infty} \varphi_n(1 - \cdot)$ so that from the properties of the supports of φ_n, ψ_n , one has $f \mathbf{1}_{(0, \frac{1}{16})} = \mathbf{1}_{(0, \frac{1}{16})}$, and thus $\|f'\|_{L^\infty(0,1)} < +\infty$. Also, by construction, $\text{supp}(\psi_n(1 - \cdot)) \subset$

$\left[\frac{1}{16}2^{-n}, \frac{15}{16}2^{-n}\right]$. Thus:

$$\begin{aligned} \int_{\mathbb{R}} |f'_N(v)| dv &= \int_0^{2^{-N}} \sum_{n=N-4}^N |\psi'_n(1-v)| dv + \int_{2^{-N}}^1 |f'(v)| dv \\ &\leq 5 \sup_{n \in \mathbb{N}} (2^{-n} \|\psi_n\|_{C^1}) + \|f'\|_{L^\infty(0,1)} \lesssim 1. \end{aligned}$$

Thus we have established that $|\Delta_{s,\lambda}^N| \lesssim \lambda^\gamma$. This concludes the proof. \square

Recall that the calculations above assume the existence of a continuous primitive I of $\mathcal{R}(F)$, which is far from clear in general. However, assume now that there exists $0 < \beta < 1 \wedge \gamma$ such that A furthermore satisfies:

$$|A_{s,t}| \lesssim |t-s|^\beta,$$

uniformly over $s, t \in [0, T]$. Then, note that F satisfies the homogeneity condition (2.5.7) with parameter $b = \beta - 1$, because if $\varphi \in \mathcal{D}(\mathbb{R})$ is any test-function, $s, t \in \mathbb{R}$, $\lambda \in (0, 1]$, then:

$$\left| F_s(\varphi_s^\lambda) \right| = \lambda^{-1} \left| A_{p(s), p(s+\lambda v)} \varphi'(v) dv \right| \lesssim \lambda^{\beta-1}.$$

Thus, the Reconstruction Theorem 2.5.1 asserts that $\mathcal{R}(F) \in \mathcal{C}_1^{\beta-1}$, and it is then a well-known fact that there exists a function $I \in \mathcal{C}_1^\beta$ such that $I_0 = 0$ and $I' = \mathcal{R}(F)$.

Remark 2.5.4. *A rigorous proof of the latter fact can be found for example in [Bra19, Lemma 3.10] where the author uses wavelets, but let us briefly and informally present an alternative approach in the spirit of the calculations above. Indeed, one wishes to set $I_t := \mathcal{R}(F)(\mathbf{1}_{(0 \wedge t, 0 \vee t]})$, which is not possible because the indicator function is not a test-function. However, recalling Lemma 2.5.2, one can define for $t \geq 0$:*

$$I_t := \sum_{n \geq 0} \mathcal{R}(F) \left(t \left((\varphi_n)^{(t)} + (\psi_n)^{(t)} \right) \right),$$

where the sum is absolutely convergent because the fact that $\mathcal{R}(F) \in \mathcal{C}_1^{\beta-1}$ and the properties of φ_n and ψ_n imply that the absolute value of the summand is bounded by a constant times $2^{-n\beta}$, which is summable because $\beta > 0$. Now one can verify that indeed $I \in \mathcal{C}_1^\beta$ and that $I' = \mathcal{R}(F)$.

Hence, one retrieves the following weaker version of the Sewing Lemma (Theorem 2.2.1, Theorem 2.2.2) above. Note that the same arguments can be adapted to the case $\gamma = 1$.

Corollary 2.5.5 (Sewing via the Reconstruction Theorem 2.5.1). *Let $\beta, \gamma > 0$ with $\beta < 1$, and let $A: [0, T]^2 \rightarrow \mathbb{R}$ be a function satisfying:*

$$\begin{aligned} |A_{s,t} - A_{s,u} - A_{u,t}| &\lesssim (|t-u| \vee |u-s|)^\gamma, \\ |A_{s,t}| &\lesssim |t-s|^\beta, \end{aligned} \tag{2.5.7}$$

uniformly over $s, u, t \in [0, T]$. Then there exists a function $I \in \mathcal{C}_1^\beta$ such that:

$$|I_t - I_s - A_{s,t}| \lesssim \begin{cases} |t-s|^\gamma & \text{if } \gamma \neq 1, \\ |t-s|(1 + |\log|t-s||) & \text{if } \gamma = 1, \end{cases}$$

uniformly over $s, t \in [0, T]$

Remark 2.5.6. *Note that Corollary 2.5.5 is weaker than Theorem 2.2.1-Theorem 2.2.2 above because it further requires to assume the condition of homogeneity (2.5.7). Indeed our proof requires the assumption (2.5.7) above in order to retrieve this result as an application of the Reconstruction Theorem. Still, it is not clear to us whether the Reconstruction Theorem can yield the more general Theorem 2.2.1, Theorem 2.2.2.*

Chapter 3

Besov Reconstruction

Abstract

This chapter contains the results of [BL22]. The reconstruction theorem tackles the problem of building a global distribution, on \mathbb{R}^d or on a manifold, for a given family of sufficiently coherent local approximations. This theorem is a critical tool within Hairer’s theory of Regularity Structures. In this paper, we establish a reconstruction theorem in the Besov setting, extending recent results of Caravenna and Zambotti. A Besov reconstruction theorem was first formulated by Hairer and Labbé in the context of regularity structures, exploiting nontrivial results from wavelet analysis. Our calculations follow the more elementary approach of coherent germs due to Caravenna and Zambotti. With this formulation our results are both stated and proved with tools from the theory of distributions without the need of the theory of Regularity Structures. As an application, we present an alternative proof of a (Besov) Young multiplication theorem which does not require the use of para-differential calculus.

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3.1 Motivation and Background

Multiplying two distributions in a general setting is a notoriously difficult problem in many situations in PDEs and mathematical analysis. Recently this problem has motivated an intensive activity in stochastic analysis. Inspired by the theory of Rough Paths [Lyo98], two approaches to singular SPDEs have been developed in the last decade: regularity structures [Hai14; BHZ19; CH18; BCCH21] and paracontrolled distributions [GIP15]. These new theories allow to give a meaning and also a well-posedness result for PDEs with stochastic forcing terms which would be ill-defined using classical tools; indeed, due to the wild oscillations of the noise, the solutions are expected to be distributions; if the equations contain polynomial or analytical non-linearities, then such terms are ill-defined and require new ideas.

It is not our aim to enter into the details of regularity structures. On the contrary, we want to present one of its main results, the *reconstruction theorem* in Besov spaces, in a more elementary way.

One of the main ideas of the theory is to lift the equation to a space of “local approximations” of the solutions, rather than to work directly in the space of Schwartz distributions. The reconstruction theorem allows to retrieve a genuine distribution from such a family of local approximations. Note that although the theory permits deep results in stochastic analysis, this theorem is purely deterministic.

More precisely, we consider the following “reconstruction problem”:

Given the data, for all $x \in \mathbb{R}^d$, of a distribution $F_x \in \mathcal{D}'(\mathbb{R}^d)$, is there a distribution f on \mathbb{R}^d which is well approximated by F_x around each point $x \in \mathbb{R}^d$?

Of course, if for all $x \in \mathbb{R}^d$, F_x is an actual continuous function, then the function $f: x \mapsto F_x(x)$ is a natural answer to this question.

For instance, if $F_x(\cdot)$ is the Taylor polynomial of order $r \in \mathbb{N}$, at the point $x \in \mathbb{R}^d$, of some smooth function $f: \mathbb{R}^d \rightarrow \mathbb{R}$, then it indeed holds that $f(x) = F_x(x)$; furthermore f is well approximated by F_x around x , with the following estimate: $|f(y) - F_x(y)| \lesssim |y - x|^{r+1}$ for y close to x . Note that in this situation one also has $|F_z(w) - F_y(w)| \lesssim \sum_{|l| < r} |w - y|^{|l|} |y - z|^{r-|l|}$, which asserts that the family $(F_x)_{x \in \mathbb{R}^d}$ is sufficiently “coherent”.

The situation becomes more subtle when the objects at play are actual distributions, which is the case in the previously mentioned context of stochastic PDEs, where distributional terms arise from the white noise governing the equation.

The reconstruction theorem [Hai14, Theorem 3.10], which solves the above problem in the context of Hölder spaces, was originally stated in the formalism of regularity structures, and proved using wavelet analysis. Alternative proofs, within the context of regularity structures, were later established in [GIP15; OW19; FH20].

More recently, the theorem was revisited in [CZ20, Theorem 5.1], where it was stated and proven in an elementary and more general setting. In particular, [CZ20] exhibits a sufficient condition on $(F_x)_{x \in \mathbb{R}^d}$, dubbed *coherence*, under which such a reconstruction f exists. The reconstruction f is then built from a custom-made dyadic decomposition inspired by mollification, which replaces the wavelet decomposition of the original theorem. See also [RS21] for a version of this result over smooth manifolds (still in the Hölder setting).

Unfortunately, it turns out that working in Hölder spaces may not be enough for many purposes, and it is desirable to consider the more general Besov spaces $\mathcal{B}_{p,q}^\alpha(\mathbb{R}^d)$ with $p, q \in [1, \infty]$, as they allow for finer analysis of distributions. For instance, it is natural to consider the Dirac mass δ_0 on \mathbb{R}^d as an initial condition for some stochastic PDEs. In such cases, one may benefit from the fact that not only does $\delta_0 \in \mathcal{C}^{-d}$, it also holds that $\delta_0 \in \mathcal{B}_{p,\infty}^{-d+d/p}$ for any $p \in [1, +\infty]$, thus allowing to work with improved regularity (at the expense of integrability); see [HL18] for an application of this idea.

Another useful property of Besov spaces is that when $p = q = 2$, they match with (fractional) Sobolev spaces, which are the natural framework in many situations. For instance, this allows to construct random differential operators and study their spectral properties [Lab19], or to apply Malliavin calculus to solutions of stochastic PDEs [GL20].

This motivates the need for a reconstruction theorem in the more general context of Besov spaces, which, as it turns out, has already been established in the formalism of regularity structures in [HL17, Theorem 3.1], using once again wavelets in its proof, and later in [ST18]. See also [LPT21], where a similar reconstruction result is proposed and applied to the problem of lifting Sobolev paths to Sobolev rough paths. Note that a Besov sewing lemma – an analogous result in the context of Rough Paths theory – has recently been established in [FS21] and applied to rough differential equations.

In this article, we provide a version of this Besov reconstruction theorem, in the spirit of [CZ20]. In particular, our result generalises both [CZ20] and [HL17], in the sense that plugging $p = q = +\infty$ in our condition retrieves [CZ20, Theorem 5.1], and applying our condition to germs appearing in regularity structures retrieves [HL17, Theorem 3.1], see Example 3.3.8 and Example 3.3.9 for a more precise comparison. Furthermore, our result is independent of the theory of regularity structures and can be formulated in the language of distributions: note that we will not talk about regularity structures in this paper, except in comparing our results to [HL17].

We present our main results, Theorem 3.3.2 and Theorem 3.4.5, as sufficient conditions on the family $(F_x)_{x \in \mathbb{R}^d}$ for a reconstruction to exist in a prescribed Besov sense. Interestingly, in Theorem 3.3.2, we exhibit simple notions of “coherence” and “homogeneity”, which generalise the results of [HL17, Theorem 3.1] and [CZ20, Theorem 5.1]. It actually turns out that the conditions of Theorem 3.3.2 can be refined, and we propose a more general reconstruction result as Theorem 3.4.5, which we discuss in a later section because its statement is more technical. Remarkably, Theorem 3.4.5 allows us to tackle the classical problem of constructing a product between suitable Besov spaces, as stated in Theorem 3.3.12. Contrary to usual approaches, Theorem 3.3.12 does not require para-products, and to the best of our understanding our conditions on the parameters of the spaces are quite optimal, see Section 3.6 for a short review of the literature. See also [CZ20, Section 14] for a similar application in the Hölder case.

We would also like to mention the paper [ZK21], which has appeared after the first version of this manuscript was written, and proposes a reconstruction result in the general context of quasi-normed spaces (of which the Besov spaces are a particular case), with the slight difference that the condition of [ZK21] relies on integral averages and bypasses the need for homogeneity.

Outline

This paper is organised as follows. In Section 3.2, we set the main notations that will be used in the remainder of the paper.

In Section 3.3, we discuss the problem of reconstruction that we consider, state an important reconstruction result as Theorem 3.3.2, and compare it to the results of [CZ20; HL17]. We also discuss the problem of building a Young multiplication in Besov spaces, that is, a continuous bilinear map extending the usual pointwise product between smooth functions. We construct such a product in Theorem 3.3.12.

Section 3.4 is devoted to the proof of a general reconstruction result, Theorem 3.4.5, following an approach similar to [CZ20].

In Section 3.5, we prove Theorem 3.3.2 and Theorem 3.3.7 as a corollary of Theorem 3.4.5. Finally, in Section 3.6, we prove Theorem 3.3.12 as a corollary of Theorem 3.4.5.

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3.2 Notations

In this paper, we consider an integer $d \in \mathbb{N}$ and work in the space \mathbb{R}^d , equipped with its canonical Euclidean norm $|\cdot|$. If $x \in \mathbb{R}^d$ and $r \in \mathbb{R}_+$, we will denote $B(x, r) := \{z \in \mathbb{R}^d, |z - x| \leq r\}$ to be the closed ball of center x and radius r . When $K \subset \mathbb{R}^d$, and $R > 0$ we will denote $\bar{K}_R := K + B(0, R)$ its R -enlargement.

We shall work extensively in the following Lebesgue spaces, where the variables and domain of integration will usually be clear from context:

1. The variable $x \in \mathbb{R}^d$ will usually correspond to a space variable and we denote:

$$L^p = L^p(x) := L^p(\mathbb{R}^d, dx).$$

2. The variable $h \in \mathbb{R}^d$ will usually correspond to a space variable and we denote:

$$L_h^q = L_h^q(h) := L^q\left(B(0, 1), \frac{dh}{|h|^d}\right).$$

We might also integrate on a domain K rather than $B(0, 1)$. In this case, we will denote:

$$L_h^q(h \in K) := L^q\left(K, \frac{dh}{|h|^d}\right).$$

3. The variable $\lambda \in (0, 1]$ will usually correspond to a scaling variable and we denote:

$$L_\lambda^q = L_\lambda^q(\lambda) := L^q\left((0, 1], \frac{d\lambda}{\lambda}\right)$$

4. The variable $n \in \mathbb{N}$ will usually correspond to a scaling variable and we denote:

$$\ell^q = \ell^q(n) := \ell^q(n \in \mathbb{N}).$$

We might also sum on $n \geq n_0$ rather than \mathbb{N} . For $n_0 \in \mathbb{Z}$, we will denote $\ell^q(n \geq n_0)$ the corresponding space.

In the case where p or q are equal to ∞ then the associated norm corresponds to the usual supremum norm.

The space of test-functions is denoted $\mathcal{D} = \mathcal{D}(\mathbb{R}^d)$ and is defined as the space of $\mathcal{C}^\infty(\mathbb{R}^d)$ functions with compact support. More generally, if K is a compact set of \mathbb{R}^d , $\mathcal{D}(K)$ will denote the space of test-functions supported in K .

If $\varphi: \mathbb{R}^d \rightarrow \mathbb{R}$ is a sufficiently differentiable function, we will denote its partial derivatives by, for a multi-index $k = (k_1, \dots, k_d) \in \mathbb{N}^d$: $\partial^k \varphi := \partial_1^{k_1} \dots \partial_d^{k_d} \varphi$.

Let $r \in \mathbb{N}$, the \mathcal{C}^r norm of a sufficiently differentiable function φ is defined by:

$$\|\varphi\|_{\mathcal{C}^r} := \max_{|k| \leq r} \left\| \partial^k \varphi \right\|_\infty.$$

Let $\varphi \in \mathcal{D}(\mathbb{R}^d)$, $x \in \mathbb{R}^d$, $\lambda > 0$, we denote φ_x^λ the scaled and recentered version of φ , defined as follows: $\varphi_x^\lambda(\cdot) := \lambda^{-d} \varphi(\lambda^{-1}(\cdot - x))$.

We will denote the multinomial $x^k = \prod_i x_i^{k_i}$. Often we will use the notation $|k| := k_1 + \dots + k_d$.

For $r, s \in \mathbb{N}$ and K a compact set of \mathbb{R}^d we define:

$$\begin{cases} \mathcal{B}^r(K) & := \{\varphi \in \mathcal{D}(K), \|\varphi\|_{C^r} \leq 1\}, \\ \mathcal{B}_s^r(K) & := \left\{ \varphi \in \mathcal{D}(K), \|\varphi\|_{C^r} \leq 1 \text{ and } \int x^k \varphi(x) dx = 0, 0 \leq |k| \leq s \right\}. \end{cases}$$

Most of the time, K will be $B(0, 1)$, so for simplicity of notation, we will denote:

$$\begin{cases} \mathcal{B}^r & := \mathcal{B}^r(B(0, 1)), \\ \mathcal{B}_s^r & := \mathcal{B}_s^r(B(0, 1)). \end{cases}$$

Recall that a (Schwartz) distribution is a linear functional $f: \mathcal{D}(\mathbb{R}^d) \rightarrow \mathbb{R}$ such that for all compact $K \subset \mathbb{R}^d$, there exists $r = r_K \in \mathbb{N}$ and $C = C_K < +\infty$ such that for all $\varphi \in \mathcal{D}(\mathbb{R}^d)$ supported in K , $|f(\varphi)| \leq C \|\varphi\|_{C^r}$. When r does not depend on K , we say that f is a distribution of order r . We denote $\mathcal{D}'(\mathbb{R}^d)$ the space of distributions. When $f: \mathcal{D}(\mathbb{R}^d) \rightarrow \mathbb{R}$ is a linear functional satisfying the above condition, for a given compact set K , we say that f is a distribution on K and we note the corresponding space $\mathcal{D}'(K)$.

We will denote $\mathcal{B}_{p,q}^\alpha = \mathcal{B}_{p,q}^\alpha(\mathbb{R}^d)$ the (nonhomogeneous) Besov spaces of exponent $\alpha \in \mathbb{R}$ and integrability parameters $p, q \in [1, +\infty]$, on the whole space \mathbb{R}^d . We will denote $\mathcal{B}_{p,q,\text{loc}}^\alpha$ the corresponding local Besov space. See Section 3.A for the definition and properties of Besov spaces.

The reader might be surprised that we work in \mathcal{D}' rather than in the space of tempered distributions \mathcal{S}' , which is more natural in the context of Besov spaces. However, this is not really problematic, see Remark 3.A.3 below.

3.3 Main Results

3.3.1 The problem of reconstruction

To begin, let us properly define the notion of a *germ*.

Definition 3.3.1 (Germ). *A germ is a family of distributions $(F_x)_{x \in \mathbb{R}^d}$, i.e. for all $x \in \mathbb{R}^d$, $F_x \in \mathcal{D}'(\mathbb{R}^d)$. We also assume that for all test-functions $\varphi \in \mathcal{D}(\mathbb{R}^d)$, the map $x \mapsto F_x(\varphi)$ is measurable.*

This technical assumption is due to the fact that all the objects required for the reconstruction will be defined by integration over x ; for this reason it will also be sufficient to know F_x only for almost every $x \in \mathbb{R}^d$.

For simplicity, we will denote $(F_x)_{x \in \mathbb{R}^d}$ as F .

We think of a germ as a family of local approximations for a global distribution $\mathcal{R}(F)$ that is to be reconstructed in a suitable Besov sense. For this purpose, we shall consider in our main result the following scaling functions. Fix $\epsilon > 0$ arbitrary. For $\gamma \in \mathbb{R}$, $q \in [0, +\infty]$, and $\lambda \in (0, 1]$, set:

$$k(\lambda) := k_{\gamma,q,\epsilon}(\lambda) := \begin{cases} \lambda^\gamma & \text{if } \gamma \neq 0, \\ 1 + |\log(\lambda)| & \text{if } \gamma = 0, q = +\infty, \\ 1 + |\log(\lambda)|^{1+\epsilon} & \text{if } \gamma = 0, q < +\infty. \end{cases} \quad (3.3.1)$$

One important result that we prove in this paper (but not our most general, see Remark 3.3.3 below) is the following.

Theorem 3.3.2 (Besov reconstruction). *Let F be a germ, $p, q \in [1, +\infty]$, and $\alpha, \beta, \gamma \in \mathbb{R}$ be such that $\alpha \leq \gamma$. Assume that there exists a test-function φ such that $\int \varphi \neq 0$ and that for all $K \subset \mathbb{R}^d$, the following “homogeneity” property is satisfied:*

$$\|F\|_{p,\beta,K,\varphi}^{\text{hom}} := \left\| \left\| \frac{F_x(\varphi_x^{2^{-n}})}{2^{-n\beta}} \right\|_{L^p(K,dx)} \right\|_{\ell^\infty(n \in \mathbb{N})} < +\infty. \quad (3.3.2)$$

Assume also that the following “coherence” property is satisfied:

$$\|F\|_{p,q,\alpha,\gamma,K,\varphi}^{\text{coh}} := \left\| \left\| \left\| \frac{(F_{x+h} - F_x)(\varphi_x^{2^{-n}})}{2^{-n\alpha} (2^{-n} + |h|)^{\gamma-\alpha}} \right\|_{L^p(K,dx)} \right\|_{\ell^\infty(n \in \mathbb{N})} \right\|_{L^q\left(B(0,2), \frac{dh}{|h|^d}\right)} < +\infty. \quad (3.3.3)$$

Then there exists $\mathcal{R}(F) \in \mathcal{D}'(\mathbb{R}^d)$ satisfying the following reconstruction bound for any integer $r > \max(-\alpha, -\beta)$ and any $K \subset \mathbb{R}^d$ (recall that k is defined in (3.3.1)):

$$\left\| \left\| \sup_{\psi \in \mathcal{B}^r} \left| \frac{(\mathcal{R}(F) - F_x)(\psi_x^\lambda)}{k(\lambda)} \right| \right\|_{L^p(K,dx)} \right\|_{L^q(B(0,1), \frac{d\lambda}{\lambda})} < +\infty. \quad (3.3.4)$$

In fact, the quantity on the left-hand side of (3.3.4) can be bounded by a constant times $\|F\|_{p,q,\alpha,\gamma,\bar{K}_2,\varphi}^{\text{coh}}$. Furthermore, such an $\mathcal{R}(F)$ is unique when $\gamma > 0$ but not when $\gamma \leq 0$.

Let us propose a few remarks before giving more precise results on the reconstruction map.

Remark 3.3.3. *Theorem 3.3.2 is interesting because, as we will establish in Example 3.3.8 and Example 3.3.9, it is a generalisation of [HL17, Theorem 3.1] and [CZ20, Theorem 5.1] (in the case of global exponents). However, note that it is not clear whether the condition (3.3.3) is canonical, in particular regarding the order of integration in the Lebesgue norms. This suggests that (3.3.3) is not the “optimal” condition. Indeed, we actually establish a more general result of reconstruction, which we state in Theorem 3.4.5. However, the conditions of Theorem 3.4.5 require heavier notations, which is why we postpone its statement and proof to Section 3.4. In the remainder of this paper, we shall first prove Theorem 3.4.5, in Section 3.4. Then, in Section 3.5, we shall establish Theorem 3.3.2 as a corollary of Theorem 3.4.5.*

Remark 3.3.4. *The case $\gamma = 0$ appears as a critical case. Recall that this is the case also in [CZ20, cf. Theorem 5.1].*

Remark 3.3.5. *Note that in condition (3.3.3), we require integration over $h \in B(0,2)$, while in the reconstruction bound (3.3.4) one integrates over $\lambda \in B(0,1)$. It would be more natural to impose the constraint of integrability over $h \in B(0,1)$, as one would expect the estimates to propagate from $B(0,1)$ to $B(0,2)$, similarly to [CZ20]. However, in our context, the asymmetry between the roles of the variables x and h , and the fact that the variables x, h, n are “linked” by the integration, prevent the same argument as in [CZ20] to be applied. Of course, in practical situations it is usually equivalent to check the conditions for $B(0,1)$ or for $B(0,2)$.*

Remark 3.3.6. Note also that contrary to [CZ20], here we do not consider the converse problem of whether the existence of a reconstruction implies any coherence condition such as (3.3.3).

We shall also show that the reconstruction map \mathcal{R} admits the following properties (and see Theorem 3.4.5 for a more general version).

Theorem 3.3.7 (Properties of the reconstruction map). *In the context of Theorem 3.3.2:*

1. (Global version) if (3.3.2), (3.3.3) are satisfied for $K = \mathbb{R}^d$, then:

- (a) If $\beta \wedge \gamma > 0$, then $\mathcal{R}(F) = 0$.
- (b) If $\beta \wedge \gamma = 0$, then for all $\kappa > 0$, $\mathcal{R}(F) \in \mathcal{B}_{p,1}^{-\kappa}$.
- (c) If $\beta \wedge \gamma < 0$, then $\mathcal{R}(F) \in \mathcal{B}_{p,\infty}^{\beta \wedge \gamma}$.

2. (Local version) if (3.3.2), (3.3.3) are satisfied for all $K \subset \mathbb{R}^d$, then:

- (a) If $\beta \wedge \gamma > 0$, then $\mathcal{R}(F) = 0$.
- (b) If $\beta \wedge \gamma = 0$, then for all $\kappa > 0$, $\mathcal{R}(F) \in \mathcal{B}_{p,1,\text{loc}}^{-\kappa}$.
- (c) If $\beta \wedge \gamma < 0$, then $\mathcal{R}(F) \in \mathcal{B}_{p,\infty,\text{loc}}^{\beta \wedge \gamma}$.

The reconstruction map is continuous in the following sense: let \mathcal{B} denote the Besov space $\mathcal{B}_{p,1,K}^{-\kappa}$ if $\beta \wedge \gamma = 0$ and $\mathcal{B}_{p,\infty,K}^{\beta \wedge \gamma}$ if $\beta \wedge \gamma < 0$, then:

$$\|\mathcal{R}(F)\|_{\mathcal{B}} \lesssim \|F\|_{p,\beta,\bar{K}_2,\varphi}^{\text{hom}} + \|F\|_{p,q,\alpha,\gamma,\bar{K}_4,\varphi}^{\text{coh}}.$$

3.3.2 A comparison with the literature

Now let us compare Theorem 3.3.2 and Theorem 3.3.7 above with the existing literature.

Example 3.3.8 (Caravenna-Zambotti). Taking $p = q = +\infty$ in the previous theorem (in its local version) retrieves [CZ20, Theorem 5.1] in the situation where the coherence and homogeneity exponents α and β do not depend on the compact K . Recall also that in the context of [CZ20], the property of (local) homogeneity is implied by the property of (local) coherence, cf. [CZ20, Lemma 4.12]. However, it is not clear whether this generalises to the case of global exponents i.e. when α, β, γ do not depend on the choice of K , which is why we assume both homogeneity and coherence.

Example 3.3.9 (Hairer-Labbé). Let us shortly discuss how Theorem 3.3.2 (in its global version) generalises [HL17, Theorem 3.1], in the case of the canonical scaling $\mathfrak{s} = (1, \dots, 1)$. Here we assume that the reader is familiar with the framework and notations of [HL17]. Let $(\mathcal{A}, \mathcal{T}, \mathcal{G})$ be a regularity structure over \mathbb{R}^d , endowed with a model (Π, Γ) . Let $\gamma > 0$, $p, q \in [1, +\infty]$ and let $f \in \mathcal{D}_{p,q}^\gamma$ be a Besov modelled distribution. We define a germ F by setting for $x \in \mathbb{R}^d$, $F_x := \Pi_x(f(x))$. Then we claim that F satisfies the homogeneity property (3.3.2) and the coherence property (3.3.3), so that $\mathcal{R}(F)$ coincides with $\mathcal{R}^{\text{HL}}(f)$, the Hairer-Labbé reconstruction of f . Let us only discuss the coherence, as the homogeneity is obtained with a similar argument. One can observe that for any test-function $\varphi \in \mathcal{D}(\mathbb{R}^d)$,

$$\begin{aligned} (F_{x+h} - F_x)(\varphi_x^{2^{-n}}) &= (\Pi_{x+h}f(x+h) - \Pi_x f(x))(\varphi_x^{2^{-n}}) \\ &= \sum_{a \in \mathcal{A}_\gamma} \Pi_{x+h} \left((f(x+h) - \Gamma_{x+h,x} f(x))_a \right) (\varphi_x^{2^{-n}}) \\ &= \sum_{a \in \mathcal{A}_\gamma} \Pi_x \Gamma_{x,x+h} \left((f(x+h) - \Gamma_{x+h,x} f(x))_a \right) (\varphi_x^{2^{-n}}). \end{aligned}$$

Using the analytic bounds on Π and Γ , it holds:

$$|(F_{x+h} - F_x)(\varphi_x^{2^{-n}})| \lesssim \sum_{a \in \mathcal{A}_\gamma} 2^{-na} |f(x+h) - \Gamma_{x+h,x} f(x)|_a,$$

which implies that for $\alpha := \inf \mathcal{A}_\gamma \leq a < \gamma$ and $n \in \mathbb{N}$:

$$\frac{|(F_{x+h} - F_x)(\varphi_x^{2^{-n}})|}{2^{-n\alpha}(2^{-n} + |h|)^{\gamma-\alpha}} \lesssim \sum_{a \in \mathcal{A}_\gamma} 2^{-n(a-\alpha)} \frac{|f(x+h) - \Gamma_{x+h,x} f(x)|_a}{|h|^{\gamma-a}}.$$

Recall that the $\mathcal{D}_{p,q}^\gamma$ -norm of the modelled distribution f is given by:

$$\|f\|_{\mathcal{D}_{p,q}^\gamma} = \sum_{a \in \mathcal{A}_\gamma} \left(\| |f(x)|_a \|_{L^p(x)} + \left\| \left\| \frac{|f(x+h) - \Gamma_{x+h,x} f(x)|_a}{|h|^{\gamma-a}} \right\|_{L^p(x)} \right\|_{L_h^q(h)} \right).$$

Thus after integration over $h \in B(0,1)$:

$$\begin{aligned} & \left\| \left\| \left\| \frac{(F_{x+h} - F_x)(\varphi_x^{2^{-n}})}{2^{-n\alpha}(2^{-n} + |h|)^{\gamma-\alpha}} \right\|_{L^p(x)} \right\|_{\ell^\infty(n)} \right\|_{L_h^q(h \in B(0,1))} \\ & \lesssim \sum_{a \in \mathcal{A}_\gamma} \left\| \left\| \frac{|f(x+h) - \Gamma_{x+h,x} f(x)|_a}{|h|^{\gamma-a}} \right\|_{L^p(x)} \right\|_{L_h^q(h)} \lesssim \|f\|_{\mathcal{D}_{p,q}^\gamma} < +\infty, \end{aligned}$$

Also, when $h \in B(1,2)$, using the analytic bound on Γ it is straightforward to establish:

$$\left\| \left\| \frac{(F_{x+h} - F_x)(\varphi_x^{2^{-n}})}{2^{-n\alpha}(2^{-n} + |h|)^{\gamma-\alpha}} \right\|_{L^p(x)} \right\|_{L_h^q(h)} \lesssim \sum_{a \in \mathcal{A}_\gamma} \| |f(x)|_a \|_{L^p(x)} \leq \|f\|_{\mathcal{D}_{p,q}^\gamma},$$

whence the property of coherence (3.3.3). Recall that in [HL17] it is announced when $q < +\infty$ that for any $\kappa > 0$, $\mathcal{R}^{\text{HL}}(f) \in \mathcal{B}_{p,q}^{\alpha-\kappa}$, while our result yields the seemingly different $\mathcal{R}(F) \in \mathcal{B}_{p,\infty}^\alpha$. Our result is actually stronger than those presented in [HL17] since $\mathcal{B}_{p,\infty}^\alpha \subset \mathcal{B}_{p,1}^{\alpha-\kappa} \subset \mathcal{B}_{p,q}^{\alpha-\kappa}$.

Example 3.3.10 (Taylor germ). Let us add one more pedagogical example of germs related to classical Taylor expansions, which have been discussed in [CZ20, Examples 4.11 & 5.4] in the Hölder case. Fix $\gamma > 0$ with $\gamma \notin \mathbb{N}$, $p, q \in [1, +\infty]$, and $f \in \mathcal{B}_{p,q}^\gamma$. Proposition 3.A.5 implies that for $0 \leq |k| < \gamma$, $\partial^k f$ coincides with an L^p function (up to a set of Lebesgue measure 0). Thus we define, for $x, z \in \mathbb{R}^d$ the following germ, called the Taylor germ of f :

$$F_x(z) := \sum_{0 \leq |k| < \gamma} \partial^k f(x) \frac{(z-x)^k}{k!}.$$

Then, F satisfies the properties of coherence and homogeneity in the sense that for all test-functions φ with $\int \varphi \neq 0$:

$$\|F\|_{p,0,\mathbb{R}^d,\varphi}^{\text{hom}} + \|F\|_{p,q,0,\gamma,\mathbb{R}^d,\varphi}^{\text{coh}} < +\infty.$$

Indeed, this corresponds to a particular case of the calculations in Section 3.6 (taking $g \equiv 1$). Using Proposition 3.A.5, Item (2) and the uniqueness part in Theorem 3.3.2, it is straightforward to observe that $\mathcal{R}(F) = f$.

Remark 3.3.11 (Sewing Lemma). *The sewing lemma [Gub04; FLP06] is a result of Rough Path theory which plays a role similar to that of the reconstruction theorem in the theory of Regularity Structures. In fact, it is known that the (Hölder) reconstruction theorem can be understood as a “generalisation” of the (Hölder) sewing lemma, see for instance [BZ22, Section 4] for a rigorous discussion. It is thus interesting to wonder if this is still the case in the Besov setting, where a sewing lemma was recently obtained in [FS21].*

For a result in this direction, see [ZK21, Section 4] where Zorin-Kranich’s general reconstruction theorem in quasi-normed spaces is shown to imply the Besov sewing lemma of [FS21], though only in the particular case of smooth functions.

However, it is still an open question whether our results Theorem 3.3.2– Theorem 3.4.5 can be shown to generalise the Besov sewing lemma of [FS21].

3.3.3 The problem of Young multiplication in Besov spaces

Now let us discuss the classical problem of multiplying two distributions, provided they belong to suitable Besov spaces, mirroring a similar discussion from [CZ20, Section 14] in the Hölder case. We will construct such a multiplication as a consequence of our general result Theorem 3.4.5.

The question can be formulated as follows: given $\alpha, \beta, \gamma \in \mathbb{R}$, $p_1, p_2, p_3, q_1, q_2, q_3 \in [1, +\infty]$, does there exist a continuous bilinear application $\mathcal{M}: \mathcal{B}_{p_1, q_1}^\alpha \times \mathcal{B}_{p_2, q_2}^\beta \rightarrow \mathcal{B}_{p_3, q_3}^\gamma$ that extends the canonical pointwise multiplication between smooth functions?

Usually, such multiplication maps are constructed with tools from the theory of para-products, and it is sometimes claimed in the literature that it is enough to assume:

$$\alpha < 0 < \beta, \quad \alpha + \beta > 0, \quad \gamma = \alpha, \quad \frac{1}{p_3} = \frac{1}{p_1} + \frac{1}{p_2}, \quad \frac{1}{q_3} = \frac{1}{q_1} + \frac{1}{q_2}.$$

However, it turns out that only the “resonant” term of the paraproduct decomposition is well-defined under these conditions. In fact, the last condition on q_3 is incorrect as [Joh95, Theorem 4.2] exhibits sequences of smooth functions f_n, g_n such that for any such $\alpha, \beta, p_1, p_2, p_3, q_1, q_2$:

$$\begin{cases} \|g_n\|_{\mathcal{B}_{p_1, q_1}^\alpha}, \|f_n\|_{\mathcal{B}_{p_2, q_2}^\beta} &= 1, \\ \|g_n \cdot f_n\|_{\mathcal{B}_{p_3, q_3}^\gamma} &\xrightarrow{n \rightarrow +\infty} +\infty \text{ when } q < q_1. \end{cases}$$

In this article, we construct a suitable multiplication map \mathcal{M} under the conditions:

$$\alpha < 0 < \beta, \quad \alpha + \beta > 0, \quad \gamma = \alpha, \quad \frac{1}{p_3} = \frac{1}{p_1} + \frac{1}{p_2}, \quad q_3 = q_1.$$

That is, we build a multiplication:

$$\mathcal{M}: \mathcal{B}_{p_1, q_1}^\alpha \times \mathcal{B}_{p_2, q_2}^\beta \rightarrow \mathcal{B}_{(p_1^{-1} + p_2^{-1})^{-1}, q_1}^\alpha.$$

Note that it is known that such a map \mathcal{M} can be built with paraproducts, see [Joh95, Theorem 6.6], [Mar18, Corollary 2.1.35] or [Zui20, Theorem 19.7]. However, our construction does not require paraproducts and relies instead on Theorem 3.4.5 below. We will provide a proof in Section 3.6, but let us outline the strategy here.

Recalling the embedding $\mathcal{B}_{p_2, q_2}^\beta \subset \mathcal{B}_{p_2, q_2}^{\beta - \epsilon}$ for any $\epsilon > 0$, we can assume without loss of generality that $\beta \notin \mathbb{N}$.

Thus, we shall fix $\alpha < 0, \beta > 0$ with $\alpha + \beta > 0$ and $\beta \notin \mathbb{N}$, as well as $p_1, p_2, q_1, q_2, p, q \in [1, +\infty]$ with $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}, \frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$. Fix distributions $g \in \mathcal{B}_{p_1, q_1}^\alpha, f \in \mathcal{B}_{p_2, q_2}^\beta$. Note that our conventions for the sign of α and β are interverted with those of [CZ20].

Since $f \in \mathcal{B}_{p_2, q_2}^\beta$ with $\beta > 0$ and $\beta \notin \mathbb{N}$, we know from Proposition 3.A.5 that for $0 \leq |k| < \beta$, $\partial^k f$ coincides with an L^{p_2} function (up to a set of Lebesgue measure 0), so that we can define, for $x, z \in \mathbb{R}^d$:

$$F_x(z) := \sum_{0 \leq |k| < \beta} \partial^k f(x) \frac{(z-x)^k}{k!}. \quad (3.3.5)$$

This is the Taylor germ of f . Now we define a germ P by setting for $x \in \mathbb{R}^d$ and $\varphi \in \mathcal{D}(\mathbb{R}^d)$:

$$P_x(\varphi) := g(\varphi F_x). \quad (3.3.6)$$

Recall that this is the same germ as considered in [CZ20] in the case of Young multiplication for Hölder distributions. Note also that here, P_x is only correctly defined for x away from a null set, which is not a problem since all the objects required for the reconstruction are defined by integration over x , recall Definition 3.3.1.

In Section 3.6 we will prove that the germ P satisfies the hypotheses of the more general Reconstruction Theorem 3.4.5 below, and that P admits a unique reconstruction $\mathcal{R}(P)$. Therefore the following result will follow by setting $\mathcal{M}(g, f) := \mathcal{R}(P)$:

Theorem 3.3.12 (Young multiplication in Besov spaces). *Let $p_1, p_2, q_1, q_2 \in [1, +\infty]$ and let $p, q \in [1, +\infty]$ be defined by $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$. Let $\alpha, \beta \in \mathbb{R}$ be such that $\alpha < 0 < \beta$, $\alpha + \beta > 0$.*

Then there exists a bilinear continuous map $\mathcal{M}: \mathcal{B}_{p_1, q_1}^\alpha \times \mathcal{B}_{p_2, q_2}^\beta \rightarrow \mathcal{B}_{p, q}^{\alpha+\beta}$ that extends the usual product, i.e. when $g \in \mathcal{B}_{p_1, q_1}^\alpha$ and $f \in \mathcal{C}^\infty$, $\mathcal{M}(g, f) = g \cdot f$, where the product in the right-hand side is understood as the product of a distribution against a smooth function.

Furthermore, when $\beta \notin \mathbb{N}$, our map \mathcal{M} is characterised by the following property: for any $r \in \mathbb{N}$ with $r > -\alpha$:

$$\left\| \left\| \sup_{\psi \in \mathcal{B}^r} \left| \frac{(\mathcal{M}(g, f) - g \cdot F_x)(\psi_x^\lambda)}{\lambda^{\alpha+\beta}} \right| \right\|_{L^p(x)} \right\|_{L_\lambda^q(\lambda)} < +\infty. \quad (3.3.7)$$

Remark 3.3.13. *A similar bound as (3.3.7) is established in [LPT21, Equation (3.1)] in the case $p_1 = p_2 = q_1 = q_2$.*

3.4 A general Besov reconstruction theorem

Now let us turn to the statement and proof of our most general reconstruction result, Theorem 3.4.5 below.

3.4.1 Statement of the result

Let us introduce the following notations.

Definition 3.4.1 (Besov reconstruction of a Germ). *Let F be a germ, $p, q \in [1, +\infty]$, and $r \in \mathbb{N}$. Let $k: (0, 1] \rightarrow \mathbb{R}_+$ be a function (which we will call a scaling function). Let $K \subset \mathbb{R}^d$. We say that a distribution $\mathcal{R}(F) \in \mathcal{D}'(\bar{K}_1)$ is a k, p, q -reconstruction of F on K if the following estimate, called the reconstruction bound, holds for any test-function $\psi \in \mathcal{D}(B(0, 1))$:*

$$\left\| \left\| \frac{(\mathcal{R}(F) - F_x)(\psi_x^\lambda)}{k(\lambda)} \right\|_{L^p(x \in K)} \right\|_{L_\lambda^q(\lambda)} < +\infty.$$

We say that the reconstruction $\mathcal{R}(F)$ is r -uniform if the reconstruction bound is uniform in ψ in the following sense:

$$\left\| \left\| \sup_{\psi \in \mathcal{B}^r} \left| \frac{(\mathcal{R}(F) - F_x)(\psi_x^\lambda)}{k(\lambda)} \right| \right\|_{L^p(x \in K)} \right\|_{L_\lambda^q(\lambda)} < +\infty.$$

Finally, if $\gamma \in \mathbb{R}$, we say that a distribution is a γ, p, q -reconstruction of F on K if it is a $k_{\gamma, q, \epsilon, p, q}$ -reconstruction of F on K , where k is defined by (3.3.1).

The reconstruction bound quantifies how close $\mathcal{R}(F)$ is to F_x locally in space (the x variable) and scale (the λ variable), in a way similar to the definition of Besov spaces, see Section 3.A.

It is interesting to note that this definition already guarantees some properties of the reconstruction:

Proposition 3.4.2. *In the context of the previous definition, let $k_1, k_2: (0, 1] \rightarrow \mathbb{R}_+$ be two scaling functions.*

1. *Assume there exists $C > 0$ such that $k_1 \leq Ck_2$ pointwise, then a k_1, p, q -reconstruction of F is also a k_2, p, q -reconstruction of F .*
2. *Assume there exists $C > 0$ such that for all $\lambda \in (0, 1]$, $C^{-1}k(\lambda) \leq k(2^{\lfloor \log_2(\lambda) \rfloor}) \leq Ck(\lambda)$. Then a distribution $\mathcal{R}(F) \in \mathcal{D}'(\bar{K}_1)$ is a r -uniform k, p, q -reconstruction of F on K if and only if:*

$$\left\| \left\| \sup_{\psi \in \mathcal{B}^r} \left| \frac{(\mathcal{R}(F) - F_x)(\psi_x^{2^{-n}})}{k(2^{-n})} \right| \right\|_{L^p(x \in K)} \right\|_{\ell^q(n)} < +\infty. \quad (3.4.1)$$

3. *Assume that k_1, k_2 are two scaling functions such that Item (2) just above applies, and $k_1, k_2(\lambda) \rightarrow_{\lambda \rightarrow 0} 0$. Then any k_1, p, q -reconstruction of a germ F on all $K \subset \mathbb{R}^d$ must coincide with any k_2, p, q -reconstruction of F on all $K \subset \mathbb{R}^d$. In particular, when $k(\lambda) \rightarrow_{\lambda \rightarrow 0} 0$, a germ F can have at most one k, p, q -reconstruction.*

Proof. (1) and (2) are elementary to check. Now let us tackle (3). Denote T the difference of the two reconstructions. The embedding $\ell^q \subset \ell^\infty$, Hölder's inequality, and the triangle inequality, yield $\lim_{n \rightarrow +\infty} \int_K |T(\varphi_x^{\epsilon_n})| dx = 0$ for any compact $K \subset \mathbb{R}^d$ and any test-function φ . Now we fix a test-function φ such that $\int \varphi = 1$. Let $\eta \in \mathcal{D}(\mathbb{R}^d)$ and let us show that $T(\eta) = 0$. By mollification, $T(\eta) = \lim_{n \rightarrow +\infty} T_n(\eta)$ where $T_n(\eta) := \int_{\text{supp}(\eta)} T(\varphi_x^{\epsilon_n}) \eta(x) dx$. Now $|T_n(\eta)| \leq \|\eta\|_\infty \int_{\text{supp}(\eta)} |T(\varphi_x^{\epsilon_n})| dx = o_{n \rightarrow +\infty}(1)$, hence $T(\eta) = 0$ as announced, so that the two reconstructions coincide. \square

Notation 3.4.3. *If F is a germ, $p \in [1, +\infty]$, $\alpha, \beta, \gamma \in \mathbb{R}$, $K \subset \mathbb{R}^d$, and $\varphi \in \mathcal{D}(\mathbb{R}^d)$ is a test-function, we denote:*

$$\begin{cases} f_K(n, h) := f_{F, \varphi, \alpha, \gamma, p, K}(n, h) & := \left\| \frac{(F_{x+h} - F_x)(\varphi_x^{2^{-n}})}{2^{-n\alpha} (2^{-n} + |h|)^{\gamma - \alpha}} \right\|_{L^p(x \in \bar{K}_2)}, \\ g_K(n) := g_{F, \varphi, \beta, p, K}(n) & := \left\| \frac{F_x(\varphi_x^{2^{-n}})}{2^{-n\beta}} \right\|_{L^p(x \in \bar{K}_2)}. \end{cases}$$

We will usually drop the subscripts when the dependence in $F, \varphi, \alpha, \beta, \gamma, p$ is clear from the context.

The following sequences, corresponding to *averaged* versions of f , will play an important role in our calculations. Note that these quantities will appear naturally in our calculations, but we do not really have an interpretation of what they represent.

Notation 3.4.4. For any function $f: \mathbb{N} \times \mathbb{R}^d \rightarrow \mathbb{R}_+$ and real $c \in \mathbb{R}$, set for $n \in \mathbb{N}$:

$$\begin{cases} m_f^{(1)}(n) & := \int_{h \in B(0, 2^{-n+1})} 2^{nd} f(n, h) dh, \\ m_{c,f}^{(2)}(n) & := \sum_{k=n}^{+\infty} \int_{h \in B(0, 2^{-k})} 2^{-(k-n)c+kd} f(k, h) dh, \\ m_{c,f}^{(3)}(n) & := \sum_{k=n}^{+\infty} \int_{h \in B(0, 2^{-n+1})} 2^{-(k-n)c+nd} f(k, h) dh, \\ m_{c,f}^{(4)}(n) & := \sum_{k=0}^{n-1} \int_{h \in B(0, 2^{-k+1})} 2^{-(k-n)c+kd} f(k, h) dh. \end{cases} \quad (3.4.2)$$

Once again, we will drop the subscripts when the dependence in the parameters is clear from context.

We will establish the following general version of the Besov reconstruction theorem.

Theorem 3.4.5 (Besov reconstruction, general case). *Let F be a germ, $p, q \in [1, +\infty]$, and $\alpha, \beta, \gamma \in \mathbb{R}$ be such that $\alpha \leq \gamma$. Let $r \in \mathbb{N}$ be an integer such that $r > -\beta$. Let $K \subset \mathbb{R}^d$ and $\hat{\varphi} \in \mathcal{D}(B(0, 1/2))$ be a test-function such that $\int \hat{\varphi} = 1$ and $\int x^k \hat{\varphi}(x) dx = 0$ for $1 \leq |k| \leq r-1$. For simplicity of notation, denote:*

$$\begin{cases} f_K & = f_{F, \hat{\varphi}, \alpha, \gamma, p, K}, & m_K^{(2)} & = m_{\gamma, f_K}^{(2)}, \\ g_K & = g_{F, \hat{\varphi}, \beta, p, K}, & m_K^{(3)} & = m_{\alpha+r, f_K}^{(3)}, \\ m_K^{(1)} & = m_{f_K}^{(1)}, & m_K^{(4)} & = m_{\gamma, f_K}^{(4)}. \end{cases} \quad (3.4.3)$$

For any $q_1 \in [1, +\infty]$, we denote:

$$\|F\|_{\mathcal{G}_{p, q, q_1, K, \varphi}^{\alpha, \beta, \gamma}} := \begin{cases} \|g_K\|_{\ell^{q_1}} + \|m_K^{(1)}\|_{\ell^q} + \|m_K^{(2)}\|_{\ell^q} + \|m_K^{(3)}\|_{\ell^q} & \text{if } \gamma > 0, \\ \|g_K\|_{\ell^{q_1}} + \|m_K^{(3)}\|_{\ell^q} + \|m_K^{(4)}\|_{\ell^q} & \text{if } \gamma < 0, \\ \|g_K\|_{\ell^{q_1}} + \|\tilde{m}_K^{(3)}\|_{\ell^q} + \|\tilde{m}_K^{(4)}\|_{\ell^q} & \text{if } \gamma = 0, \end{cases}$$

where in the case $\gamma = 0$, we define, for $n \in \mathbb{N}$ and $i \in \{3, 4\}$, $\tilde{m}_K^{(i)}(n) := m^{(i)}(n) / k(2^{-n})$, where k is any scaling function such that $(1/k(2^{-n}))_{n \in \mathbb{N}} \in \ell^q$.

Assume that:

$$\|F\|_{\mathcal{G}_{p, q, q_1, K, \varphi}^{\alpha, \beta, \gamma}} < +\infty. \quad (3.4.4)$$

Then there exists a k, p, q -reconstruction of F on K , noted $\mathcal{R}(F)$ or $\mathcal{R}_K(F)$, that is also r -uniform.

Furthermore:

1. (Global version) if (3.4.4) holds for $K = \mathbb{R}^d$, then:

- (a) Such an $\mathcal{R}(F)$ is unique when $\gamma > 0$ but not when $\gamma \leq 0$.
- (b) If $\beta \wedge \gamma > 0$, then $\mathcal{R}(F) = 0$.
- (c) If $\beta \wedge \gamma = 0$, then for all $\kappa > 0$, $\mathcal{R}(F) \in \mathcal{B}_{p, 1}^{-\kappa}$.

(d) If $\beta \wedge \gamma < 0$, then $\mathcal{R}(F) \in \mathcal{B}_{p,q_1 \vee q}^{\beta \wedge \gamma}$.

2. (Local version) if (3.4.4) holds for all $K \subset \mathbb{R}^d$, then there exists a global distribution $\mathcal{R}(F) \in \mathcal{D}'(\mathbb{R}^d)$ that is a r -uniform k, p, q -reconstruction of F on all $K \subset \mathbb{R}^d$ and:

(a) Such an $\mathcal{R}(F)$ is unique when $\gamma > 0$ but not when $\gamma \leq 0$.

(b) If $\beta \wedge \gamma > 0$, then $\mathcal{R}(F) = 0$.

(c) If $\beta \wedge \gamma = 0$, then for all $\kappa > 0$, $\mathcal{R}(F) \in \mathcal{B}_{p,1,\text{loc}}^{-\kappa}$.

(d) If $\beta \wedge \gamma < 0$, then $\mathcal{R}(F) \in \mathcal{B}_{p,q_1 \vee q,\text{loc}}^{\beta \wedge \gamma}$.

The reconstruction map is continuous in the following sense: let \mathcal{B} denote $\mathcal{B}_{p,1,K}^{-\kappa}$ if $\beta \wedge \gamma = 0$ and $\mathcal{B}_{p,q_1 \vee q,K}^{\beta \wedge \gamma}$ if $\beta \wedge \gamma < 0$, then:

$$\|\mathcal{R}(F)\|_{\mathcal{B}} \lesssim \|F\|_{\mathcal{G}_{p,q,q_1,\bar{K}_2,\varphi}^{\alpha,\beta,\gamma}}.$$

Remark 3.4.6. The cases $\gamma \neq 0$ could also be slightly modified to consider general scaling functions k as in the case of $\gamma = 0$.

Let us turn to the proof of Theorem 3.4.5. In the remainder of this section, we consider a germ F , reals $p, q \in [1, +\infty]$, $\alpha, \beta, \gamma \in \mathbb{R}$ such that $\alpha \leq \gamma$, an integer $r \in \mathbb{N}$, $K \subset \mathbb{R}^d$, and a single test-function $\hat{\varphi} \in \mathcal{D}(B(0, 1/2))$ such that $\int \hat{\varphi} = 1$ and $\int x^k \hat{\varphi}(x) dx = 0$ for $1 \leq |k| \leq r-1$. We denote $f, g, m^{(1)}, m^{(2)}, m^{(3)}, m^{(4)}$ to be the functions defined in (3.4.3).

We break up the proof into several sections.

3.4.2 Uniqueness of reconstruction

Recall from Proposition 3.4.2, Item (3), that when $\gamma > 0$, the reconstruction, if it exists, is unique. Nevertheless, when $\gamma \leq 0$, the reconstruction is not unique in general. We now focus on the existence of $\mathcal{R}(F)$.

3.4.3 Existence for $\gamma > 0$

We now construct a reconstruction in the case $\gamma > 0$. First, let us recall the strategy of [CZ20].

In order to establish the existence of the reconstruction, we proceed by mollification. Recall that if $\rho \in \mathcal{D}(\mathbb{R}^d)$ is any test-function such that $\int \rho = 1$, and if $\xi \in \mathcal{D}'(\mathbb{R}^d)$ is any distribution, then we have an approximation of ξ provided for $\psi \in \mathcal{D}(\mathbb{R}^d)$ by:

$$\xi(\psi) = \lim_{n \rightarrow \infty} \xi(\rho^{2^{-n}} * \psi) = \lim_{n \rightarrow \infty} \int_{\mathbb{R}^d} \xi(\rho_z^{2^{-n}}) \psi(z) dz.$$

This yields a natural candidate for the reconstruction of the germ F , as we would like to set by analogy:

$$\mathcal{R}(F)(\psi) \stackrel{?}{=} \lim_{n \rightarrow \infty} \int_{\mathbb{R}^d} F_z(\rho_z^{2^{-n}}) \psi(z) dz.$$

Of course, the convergence of such a sequence is a priori far from obvious. However, as our goal is an existence result for $\mathcal{R}(F)$, it is enough for us to exhibit just one choice of ρ such that this sequence converges. For this purpose, we follow the strategy of [CZ20] i.e. construct a specific mollifier ρ from the single test-function $\hat{\varphi}$ provided by the assumption of the theorem.

Explicitly, define as in [CZ20]:

$$\rho := \hat{\varphi}^2 * \hat{\varphi}. \tag{3.4.5}$$

The motivation behind this choice is that it allows us to rewrite the difference $\rho^{1/2} - \rho$ as a convolution, paving the way for a nice dyadic decomposition. Explicitly, set $\check{\varphi} := \hat{\varphi}^{1/2} - \hat{\varphi}^2$, then:

$$\rho^{1/2} - \rho = \hat{\varphi} * \check{\varphi}.$$

This directly implies that for $n \in \mathbb{N}$,

$$\rho^{2^{-(n+1)}} - \rho^{2^{-n}} = \hat{\varphi}^{2^{-n}} * \check{\varphi}^{2^{-n}}.$$

Notice that by assumption on $\hat{\varphi}$, it holds that $\text{supp}(\check{\varphi}) \subset B(0, 1)$, and that $\check{\varphi}$ cancels all polynomials of degree less than $r - 1$: $\int x^k \hat{\varphi}(x) dx = 0$ for $0 \leq |k| \leq r - 1$.

Now we can exploit the dyadic structure of our mollifier: for all $n_0, n \in \mathbb{N}$,

$$\rho^{2^{-n}} = \rho^{2^{-n_0}} + \sum_{k=n_0}^{n-1} \hat{\varphi}^{2^{-k}} * \check{\varphi}^{2^{-k}}.$$

This gives us a natural definition of approximating our reconstruction.

Definition 3.4.7. Let $F = (F_x)_{x \in \mathbb{R}^d}$ be a germ. For simplicity of notation, denote $\epsilon_n := 2^{-n}$. We define a sequence of approximating distributions $\mathcal{R}_n(F) \in \mathcal{D}'(\mathbb{R}^d)$ by setting, for $n \in \mathbb{N}$, $\psi \in \mathcal{D}(\mathbb{R}^d)$, and any $n_0 \in \mathbb{N}$:

$$\begin{aligned} \mathcal{R}_n(F)(\psi) &:= \int_{\mathbb{R}^d} F_z(\rho_z^{\epsilon_n}) \psi(z) dz \\ &= \int_{\mathbb{R}^d} F_z(\hat{\varphi}^{\epsilon_{n_0}} * \hat{\varphi}_z^{2\epsilon_{n_0}}) \psi(z) dz + \sum_{k=n_0}^{n-1} \int_{\mathbb{R}^d} F_z(\hat{\varphi}^{\epsilon_k} * \check{\varphi}_z^{\epsilon_k}) \psi(z) dz. \end{aligned}$$

where ρ is defined as in (3.4.5). Note that as explained above, this definition does not depend on the choice of $n_0 \in \mathbb{N}$. If the sequence converges, then we denote for $\psi \in \mathcal{D}(\mathbb{R}^d)$:

$$\mathcal{R}(F)(\psi) := \lim_{n \rightarrow \infty} \mathcal{R}_n(F)(\psi).$$

Now, we want to establish whether this limit $\lim_{n \rightarrow \infty} \mathcal{R}_n(F)$ exists. For this, we shall pursue even further the decomposition of $\mathcal{R}_n(F)$. Recall that for any distribution $\xi \in \mathcal{D}'(\mathbb{R}^d)$ and any two test-functions $\eta, \tilde{\eta} \in \mathcal{D}(\mathbb{R}^d)$,

$$\xi(\eta * \tilde{\eta}) = \int_{\mathbb{R}^d} \xi(\eta_x) \tilde{\eta}(x) dx. \quad (3.4.6)$$

From Definition 3.4.7, it follows that the existence of $\mathcal{R}(F)$ is implied by the absolute convergence of the series $\sum_k u_k$, where we set for $k \in \mathbb{N}$:

$$\begin{aligned} u_k &:= \int_{\mathbb{R}^d} F_z(\hat{\varphi}^{\epsilon_k} * \check{\varphi}_z^{\epsilon_k}) \psi(z) dz \\ &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} F_z(\hat{\varphi}_x^{\epsilon_k}) \check{\varphi}^{\epsilon_k}(x - z) \psi(z) dx dz. \end{aligned}$$

Writing $F_z = F_x + (F_z - F_x)$, we decompose $u_k = u'_k + u''_k$, where:

$$\begin{aligned} u'_k &:= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} F_x(\hat{\varphi}_x^{\epsilon_k}) \check{\varphi}^{\epsilon_k}(x - z) \psi(z) dx dz \\ &= \int_{\mathbb{R}^d} F_x(\hat{\varphi}_x^{\epsilon_k}) (\check{\varphi}^{\epsilon_k} * \psi)(x) dx, \end{aligned}$$

and:

$$\begin{aligned} u_k'' &:= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (F_z - F_x) (\hat{\varphi}_x^{\varepsilon_k}) \check{\varphi}^{\varepsilon_k}(x-z) \psi(z) dx dz \\ &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (F_{x+h} - F_x) (\hat{\varphi}_x^{\varepsilon_k}) \check{\varphi}^{\varepsilon_k}(-h) \psi(x+h) dx dh. \end{aligned}$$

Hence, the existence of the reconstruction $\mathcal{R}(F)$ can be determined by the absolute convergence of

$$\sum_k u_k', \quad \sum_k u_k''.$$

The following lemma from [CZ20] will also be useful for us here. For completeness, we concisely recall its proof. Note that [CZ20, Lemma 9.2] states the bound in the L^1 case but its proof actually treats the L^∞ case.

Lemma 3.4.8. [CZ20, Lemma 9.2] *Assume that $\check{\varphi} \in \mathcal{D}(\mathbb{R}^d)$ is a test-function that cancels polynomials of degree $r-1 \in \mathbb{N}$ i.e. for all $0 \leq k \leq r-1$, $\int_{\mathbb{R}^d} x^k \check{\varphi}(x) dx = 0$. Then for any test-function $\eta \in \mathcal{D}(\mathbb{R}^d)$ and $\lambda > 0$:*

$$\|\check{\varphi}^\lambda * \eta\|_{L^\infty} \leq \|\eta\|_{C^r} \|\check{\varphi}\|_{L^1} \lambda^r.$$

Proof. We fix $y \in \mathbb{R}^d$ and denote $p_y(\cdot) := \sum_{|k| \leq r-1} \frac{\partial^k \eta(y)}{k!} (\cdot - y)^k$ to be the Taylor polynomial of η of order $r-1$ based at y , so that $|\eta(z) - p_y(z)| \leq \|\eta\|_{C^r} |z - y|^r$. As noted above, $\check{\varphi}$ cancels polynomials of degree $r-1$, so that $\int_{\mathbb{R}^d} \check{\varphi}^\lambda(y-z) p_y(z) dz = 0$, thus:

$$(\check{\varphi}^\lambda * \eta)(y) = \int_{\mathbb{R}^d} \check{\varphi}^\lambda(y-z) \{\eta(z) - p_y(z)\} dz.$$

Hence,

$$\begin{aligned} |(\check{\varphi}^\lambda * \eta)(y)| &\leq \int_{\mathbb{R}^d} |\check{\varphi}^\lambda(y-z)| |\eta(z) - p_y(z)| dz \\ &\leq \|\eta\|_{C^r} \int_{\mathbb{R}^d} |\check{\varphi}^\lambda(y-z)| |z - y|^r dz \\ &= \lambda^r \|\eta\|_{C^r} \|\check{\varphi}\|_{L^1}, \end{aligned}$$

which gives us our result. \square

Proposition 3.4.9 (Convergence of approximating distributions). *In the setting of this section:*

1. *Suppose that $\beta + r > 0$ and that $g_K \in \ell^\infty$. Then $\sum_k u_k'$ is absolutely convergent as soon as $\text{supp}(\psi) \subset \bar{K}_1$.*
2. *Suppose that $\gamma > 0$, $\gamma \geq \alpha$, and $m_K^{(1)} \in \ell^q$. Then $\sum_k u_k''$ is absolutely convergent as soon as $\text{supp}(\psi) \subset \bar{K}_1$.*
3. *Suppose that assumptions (1) and (2) just above apply. Then $\mathcal{R}(F) \in \mathcal{D}'(\bar{K}_1)$ is a distribution of order r .*

Proof. We start with (1). Denote \tilde{p} to be the Hölder conjugate of p . By Hölder's inequality, and since $\text{supp}(\check{\varphi}^{\varepsilon_k} * \psi) \subset \bar{K}_2$

$$\begin{aligned} |u_k'| &\leq \int_{\bar{K}_2} |F_x(\hat{\varphi}_x^{\varepsilon_k}) (\check{\varphi}^{\varepsilon_k} * \psi)(x)| dx \\ &\leq \|F_x(\hat{\varphi}_x^{\varepsilon_k})\|_{L^p(x \in \bar{K}_2)} \|\check{\varphi}^{\varepsilon_k} * \psi\|_{L^{\tilde{p}}}. \end{aligned}$$

Applying Lemma 3.4.8, we obtain:

$$\|\check{\varphi}^{\epsilon_k} * \psi\|_{L^{\bar{p}}} \lesssim \|\psi\|_{C^r} \|\check{\varphi}\|_{L^1} 2^{-kr}.$$

Thus, recalling the definition of g_K , and since $\beta + r > 0$,

$$\begin{aligned} \sum_k |u'_k| &\lesssim \|\psi\|_{C^r} \|\check{\varphi}\|_{L^1} \sum_k \left(\frac{\|F_x(\hat{\varphi}_x^{\epsilon_k})\|_{L^p(x \in \bar{K}_2)}}{2^{-k\beta}} \right) 2^{-k(\beta+r)} \\ &\lesssim \|\psi\|_{C^r} \|\check{\varphi}\|_{L^1} \|g_K\|_{\ell^\infty} \sum_k 2^{-k(\beta+r)} < \infty. \end{aligned}$$

This yields the announced result.

Now we prove (2). By definition:

$$|u''_k| \leq \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |(F_{x+h} - F_x)(\hat{\varphi}_x^{\epsilon_k}) \check{\varphi}^{\epsilon_k}(-h) \psi(x+h)| dx dh$$

Because of the supports of $\check{\varphi}$ and ψ , we have that h runs over $B(0, \epsilon_k)$ and x runs over \bar{K}_2 . If we denote \tilde{p} to be the Hölder conjugate of p , then by Hölder's inequality, with respect to the x variable,

$$|u''_k| \leq \int_{B(0, \epsilon_k)} \|(F_{x+h} - F_x)(\hat{\varphi}_x^{\epsilon_k})\|_{L^p(x \in \bar{K}_2)} \|\psi(x+h)\|_{L^{\tilde{p}}(x)} |\check{\varphi}^{\epsilon_k}(-h)| dh.$$

By substitution, $\|\psi(x+h)\|_{L^{\tilde{p}}(x)} = \|\psi\|_{L^{\tilde{p}}}$. Also, $|\check{\varphi}^{\epsilon_k}(-h)| \leq 2^{dk} \|\check{\varphi}\|_{L^\infty}$, so that:

$$|u''_k| \leq 2^{dk} \|\check{\varphi}\|_{L^\infty} \|\psi\|_{L^{\tilde{p}}} \int_{B(0, \epsilon_k)} \|(F_{x+h} - F_x)(\hat{\varphi}_x^{\epsilon_k})\|_{L^p(x)} dh.$$

Recall that by definition of $f_K(k, h)$:

$$\|(F_{x+h} - F_x)(\hat{\varphi}_x^{\epsilon_k})\|_{L^p(x \in \bar{K}_2)} = f_K(k, h) 2^{-k\alpha} (2^{-k} + |h|)^{\gamma-\alpha},$$

and since here $|h| \leq 2^{-k}$ and $\gamma \geq \alpha$, we bound $2^{-k\alpha} (2^{-k} + |h|)^{\gamma-\alpha} \leq 2^{-k\gamma}$. Thus:

$$|u''_k| \leq \|\psi\|_{L^{\tilde{p}}} \|\check{\varphi}\|_{L^\infty} 2^{-k\gamma} \int_{B(0, \epsilon_k)} f_K(k, h) 2^{kd} dh \leq \|\psi\|_{L^{\tilde{p}}} \|\check{\varphi}\|_{L^\infty} 2^{-k\gamma} m_K^{(1)}(k).$$

Denoting \tilde{q} to be the Hölder conjugate of q , we have by Hölder's inequality

$$\sum_k |u''_k| \leq \|\psi\|_{L^{\tilde{p}}} \|\check{\varphi}\|_{L^\infty} \left\| \left(2^{-k\gamma} \right)_{k \in \mathbb{N}} \right\|_{\ell^{\tilde{q}}} \left\| m_K^{(1)} \right\|_{\ell^q},$$

which is finite because $\gamma > 0$ and $m_K^{(1)} \in \ell^q$.

Finally, (3) follows immediately from the established estimates. \square

3.4.4 Reconstruction bound for $\gamma > 0$

In Proposition 3.4.9 just above, we have established the existence of a distribution $\mathcal{R}(F) \in \mathcal{D}'(\bar{K}_1)$ that is a natural candidate for the reconstruction of the germ F . In this section we focus on establishing that $\mathcal{R}(F)$ does indeed satisfy the following reconstruction bound (recall the discussion of Definition 3.4.1 and Proposition 3.4.2):

$$\left\| \left\| \sup_{\psi \in \mathcal{B}^r} \left| \frac{(\mathcal{R}(F) - F_x)(\psi_x^{2^{-n}})}{2^{-n\gamma}} \right| \right\|_{L^p(x \in K)} \right\|_{\ell^q(n)} < +\infty. \quad (3.4.7)$$

In fact, we shall show that the left-hand term of (3.4.7) is bounded by a constant times $\|F\|_{\mathcal{G}_{p,q,q_1,\bar{K}_2,\varphi}^{\alpha,\beta,\gamma}}$. For simplicity of notation, we denote:

$$G_w(\psi) := (\mathcal{R}(F) - F_w)(\psi) \quad \text{for } w \in \mathbb{R}^d \text{ and } \psi \in \mathcal{D}(\bar{K}_1).$$

As in the previous section, we shall be able to discuss G thanks to the dyadic decomposition provided by the mollifier (3.4.5). Remember that we defined $\mathcal{R}(F)$ as:

$$\mathcal{R}(F)(\psi) := \int_{\mathbb{R}^d} F_z \left(\hat{\varphi}^{\epsilon_{n_0}} * \hat{\varphi}_z^{2\epsilon_{n_0}} \right) \psi(z) dz + \sum_{k=n_0}^{+\infty} \int_{\mathbb{R}^d} F_z \left(\hat{\varphi}^{\epsilon_k} * \check{\varphi}_z^{\epsilon_k} \right) \psi(z) dz,$$

where the right-hand term does not depend on the choice of $n_0 \in \mathbb{Z}$. Now for any fixed $w \in \mathbb{R}^d$, it holds by classical mollification of the distribution F_w (with the mollifier ρ defined as (3.4.5)) that for any $n_0 \in \mathbb{Z}$:

$$F_w(\psi) = \int_{\mathbb{R}^d} F_w \left(\hat{\varphi}^{\epsilon_{n_0}} * \hat{\varphi}_z^{2\epsilon_{n_0}} \right) \psi(z) dz + \sum_{k=n_0}^{+\infty} \int_{\mathbb{R}^d} F_w \left(\hat{\varphi}^{\epsilon_k} * \check{\varphi}_z^{\epsilon_k} \right) \psi(z) dz.$$

Thus, we can decompose G as:

$$\begin{aligned} G_w(\psi) &= \int_{\mathbb{R}^d} (F_z - F_w) \left(\hat{\varphi}^{\epsilon_{n_0}} * \hat{\varphi}_z^{2\epsilon_{n_0}} \right) \psi(z) dz \\ &\quad + \sum_{k=n_0}^{+\infty} \int_{\mathbb{R}^d} (F_z - F_w) \left(\hat{\varphi}^{\epsilon_k} * \check{\varphi}_z^{\epsilon_k} \right) \psi(z) dz \end{aligned}$$

Recalling (3.4.6), writing $F_z - F_w = (F_z - F_x) + (F_x - F_w)$ and taking into consideration the support of $\check{\varphi}_z^{\epsilon_n}$ and ψ_w^λ (recall that $\text{supp}(\hat{\varphi}) \subset B(0, 1/2)$ and $\text{supp}(\check{\varphi}) \subset B(0, 1)$), we obtain, for $\psi \in \mathcal{B}^r$, $\lambda \in (0, 1]$ and any $n \in \mathbb{N}$, the more refined decomposition:

$$\begin{aligned} G_w(\psi_w^\lambda) &= \int_{B(w,\lambda)} \int_{B(z,\epsilon_n)} (F_z - F_x) \left(\hat{\varphi}_x^{\epsilon_n} \hat{\varphi}_z^{\epsilon_{n-1}}(x) \right) \psi_w^\lambda(z) dx dz \\ &\quad + \int_{B(w,\lambda)} \int_{B(z,\epsilon_n)} (F_x - F_w) \left(\hat{\varphi}_x^{\epsilon_n} \hat{\varphi}_z^{\epsilon_{n-1}}(x) \right) \psi_w^\lambda(z) dx dz \\ &\quad + \sum_{k=n}^{\infty} \int_{B(w,\lambda)} \int_{B(z,\epsilon_k)} (F_z - F_x) \left(\hat{\varphi}_x^{\epsilon_k} \check{\varphi}_z^{\epsilon_k}(x) \right) \psi_w^\lambda(z) dx dz \\ &\quad + \sum_{k=n}^{\infty} \int_{B(w,\lambda)} \int_{B(z,\epsilon_k)} (F_x - F_w) \left(\hat{\varphi}_x^{\epsilon_k} \check{\varphi}_z^{\epsilon_k}(x) \right) \psi_w^\lambda(z) dx dz. \end{aligned}$$

Choosing $\lambda = 2^{-n} = \epsilon_n$ gives us four terms which we define in the following way for $n \in \mathbb{N}$ and $w \in \mathbb{R}^d$:

$$\begin{aligned} a_n(w) &:= \sup_{\psi \in \mathcal{B}^r} \left| \int_{B(w,\epsilon_n)} \int_{B(z,\epsilon_n)} (F_z - F_x) \left(\hat{\varphi}_x^{\epsilon_n} \hat{\varphi}_z^{\epsilon_{n-1}}(x) \right) \psi_w^{\epsilon_n}(z) dx dz \right|, \\ b_n(w) &:= \sup_{\psi \in \mathcal{B}^r} \left| \int_{B(w,\epsilon_n)} \int_{B(z,\epsilon_n)} (F_x - F_w) \left(\hat{\varphi}_x^{\epsilon_n} \hat{\varphi}_z^{\epsilon_{n-1}}(x) \right) \psi_w^{\epsilon_n}(z) dx dz \right|, \\ c_n(w) &:= \sup_{\psi \in \mathcal{B}^r} \left| \sum_{k=n}^{\infty} \int_{B(w,\epsilon_n)} \int_{B(z,\epsilon_k)} (F_z - F_x) \left(\hat{\varphi}_x^{\epsilon_k} \check{\varphi}_z^{\epsilon_k}(x) \right) \psi_w^{\epsilon_n}(z) dx dz \right|, \\ d_n(w) &:= \sup_{\psi \in \mathcal{B}^r} \left| \sum_{k=n}^{\infty} \int_{B(w,\epsilon_n)} \int_{B(z,\epsilon_k)} (F_x - F_w) \left(\hat{\varphi}_x^{\epsilon_k} \check{\varphi}_z^{\epsilon_k}(x) \right) \psi_w^{\epsilon_n}(z) dx dz \right|, \end{aligned}$$

Now, the reconstruction bound is established in the following proposition.

Proposition 3.4.10. *Assume that $m_K^{(1)}, m_K^{(2)}, m_K^{(3)} \in \ell^q$ and that $\gamma \geq \alpha$. Then*

$$\left(\frac{\|a_n(w)\|_{L^p(w \in K)}}{2^{-n\gamma}} \right)_{n \in \mathbb{N}}, \left(\frac{\|b_n(w)\|_{L^p(w \in K)}}{2^{-n\gamma}} \right)_{n \in \mathbb{N}}, \\ \left(\frac{\|c_n(w)\|_{L^p(w \in K)}}{2^{-n\gamma}} \right)_{n \in \mathbb{N}}, \left(\frac{\|d_n(w)\|_{L^p(w \in k)}}{2^{-n\gamma}} \right)_{n \in \mathbb{N}} \in \ell^q.$$

As an immediate consequence, (3.4.7) holds, i.e. $\mathcal{R}(F)$ is a r -uniform γ, p, q -reconstruction of F on K .

Proof. To begin, we focus on $\left(\frac{\|a_n(w)\|_{L^p(w \in K)}}{2^{-n\gamma}} \right)_{n \in \mathbb{N}}$.

By definition of a_n ,

$$a_n(w) \leq \sup_{\psi \in \mathcal{B}^r} \int_{B(w, \epsilon_n)} \int_{B(z, \epsilon_n)} |(F_z - F_x)(\hat{\varphi}_x^{\epsilon_n})| |\hat{\varphi}_z^{\epsilon_{n-1}}(x)| |\psi_w^{\epsilon_n}(z)| dx dz.$$

Using the estimates $|\hat{\varphi}_z^{\epsilon_{n-1}}(x)| \leq 2^{nd} \|\hat{\varphi}\|_{L^\infty}$ and $|\psi_w^{\epsilon_n}(z)| \leq 2^{nd} \|\psi\|_{C^r}$, we get:

$$a_n(w) \lesssim 2^{2nd} \int_{B(w, \epsilon_n)} \int_{B(z, \epsilon_n)} |(F_z - F_x)(\hat{\varphi}_x^{\epsilon_n})| dx dz.$$

We apply the substitution $\tilde{x} = -(x - z)$ then $\tilde{z} = z - w$ in this integral, which yield:

$$a_n(w) \lesssim 2^{2nd} \int_{B(0, \epsilon_n)} \int_{B(0, \epsilon_n)} |(F_{\tilde{z}+w} - F_{-\tilde{x}+\tilde{z}+w})(\hat{\varphi}_{-\tilde{x}+\tilde{z}+w}^{\epsilon_n})| d\tilde{x} d\tilde{z}.$$

Now Minkowski's inequality implies:

$$\|a_n(w)\|_{L^p(w \in K)} \\ \lesssim 2^{2nd} \int_{B(0, \epsilon_n)} \int_{B(0, \epsilon_n)} \|(F_{\tilde{z}+w} - F_{-\tilde{x}+\tilde{z}+w})(\hat{\varphi}_{-\tilde{x}+\tilde{z}+w}^{\epsilon_n})\|_{L^p(w \in K)} d\tilde{x} d\tilde{z}.$$

Applying the substitution $\tilde{w} = w + \tilde{z} - \tilde{x}$ in the L^p norm yields:

$$\|a_n(w)\|_{L^p(w \in K)} \lesssim 2^{2nd} \int_{B(0, \epsilon_n)} \int_{B(0, \epsilon_n)} \|(F_{\tilde{w}+\tilde{x}} - F_{\tilde{w}})(\hat{\varphi}_{\tilde{w}}^{\epsilon_n})\|_{L^p(\tilde{w} \in \bar{K}_2)} d\tilde{x} d\tilde{z} \\ = 2^{2nd} \int_{B(0, \epsilon_n)} \|(F_{\tilde{w}+\tilde{x}} - F_{\tilde{w}})(\hat{\varphi}_{\tilde{w}}^{\epsilon_n})\|_{L^p(\tilde{w} \in \bar{K}_2)} d\tilde{x}.$$

By definition of f ,

$$\|(F_{\tilde{w}+\tilde{x}} - F_{\tilde{w}})(\hat{\varphi}_{\tilde{w}}^{\epsilon_n})\|_{L^p(\tilde{w} \in \bar{K}_2)} = f_K(n, \tilde{x}) 2^{-n\alpha} (2^{-n} + |\tilde{x}|)^{\gamma-\alpha}.$$

Since $|\tilde{x}| \leq 2^{-n}$ in the integral and $\gamma \geq \alpha$, this implies:

$$\|a_n(w)\|_{L^p(w \in K)} \lesssim 2^{2nd} \int_{B(0, \epsilon_n)} f_K(n, \tilde{x}) 2^{-n\gamma} d\tilde{x} \leq 2^{-n\gamma} m_K^{(1)}(n).$$

The assertion on a_n follows.

We now focus on $\left(\frac{\|b_n(w)\|_{L^p(w \in K)}}{2^{-n\gamma}} \right)_{n \in \mathbb{N}}$. By definition of b_n :

$$b_n(w) \leq \sup_{\psi \in \mathcal{B}^r} \int_{B(w, \epsilon_n)} \int_{B(z, \epsilon_n)} |(F_x - F_w)(\hat{\varphi}_x^{\epsilon_n})| |\hat{\varphi}_z^{\epsilon_{n-1}}(x)| |\psi_w^{\epsilon_n}(z)| dx dz.$$

Once again, since $|\hat{\varphi}_z^{\epsilon_{n-1}}(x)| \leq 2^{nd} \|\hat{\varphi}\|_{L^\infty}$ and $|\psi_w^{\epsilon_n}(z)| \leq 2^{nd} \|\psi\|_{C^r}$, we obtain:

$$b_n(w) \lesssim 2^{2nd} \int_{B(w, \epsilon_n)} \int_{B(z, \epsilon_n)} |(F_x - F_w)(\hat{\varphi}_x^{\epsilon_n})| dx dz.$$

Observe that for every $z \in B(w, \epsilon_n)$, we have $B(z, \epsilon_n) \subset B(w, 2\epsilon_n)$. In turn we have

$$b_n(w) \lesssim 2^{nd} \int_{B(w, \epsilon_{n-1})} |(F_x - F_w)(\hat{\varphi}_x^{\epsilon_n})| dx.$$

Substituting $\tilde{x} = -(x - w)$ then applying Minkowski's inequality yields:

$$\|b_n(w)\|_{L^p(w \in K)} \lesssim 2^{nd} \int_{B(0, \epsilon_{n-1})} \|(F_{w-\tilde{x}} - F_w)(\hat{\varphi}_{w-\tilde{x}}^{\epsilon_n})\|_{L^p(w \in K)} d\tilde{x}.$$

Substituting $\tilde{w} = w - \tilde{x}$ in the L^p norm yields:

$$\|b_n(w)\|_{L^p(w \in K)} \lesssim 2^{nd} \int_{B(0, \epsilon_{n-1})} \|(F_{\tilde{w}+\tilde{x}} - F_{\tilde{w}})(\hat{\varphi}_{\tilde{w}}^{\epsilon_n})\|_{L^p(\tilde{w} \in \tilde{K}_2)} d\tilde{x}.$$

Recalling the definition of f_K and using the facts that $|\tilde{x}| \leq \epsilon_{n-1}$ and $\gamma \geq \alpha$:

$$\|b_n(w)\|_{L^p(w \in K)} \lesssim 2^{-n\gamma} \int_{B(0, \epsilon_{n-1})} 2^{nd} f_K(n, \tilde{x}) d\tilde{x} = 2^{-n\gamma} m_K^{(1)}(n).$$

The assertion on b_n follows.

Let us now consider $\left(\frac{\|c_n(w)\|_{L^p(w \in K)}}{2^{-n\gamma}}\right)_{n \in \mathbb{N}}$. By definition of c_n :

$$c_n(w) \leq \sup_{\psi \in \mathcal{B}^r} \sum_{k=n}^{\infty} \int_{B(w, \epsilon_n)} \int_{B(z, \epsilon_k)} |(F_z - F_x)(\hat{\varphi}_x^{\epsilon_k})| |\check{\varphi}_z^{\epsilon_k}(x)| |\psi_w^{\epsilon_n}(z)| dx dz.$$

Once again, since $|\check{\varphi}_z^{\epsilon_k}(x)| \leq 2^{kd} \|\check{\varphi}\|_{L^\infty}$ and $|\psi_w^{\epsilon_n}(z)| \leq 2^{nd} \|\psi\|_{C^r}$, we obtain:

$$c_n(w) \lesssim 2^{nd} \sum_{k=n}^{\infty} 2^{kd} \int_{B(w, \epsilon_n)} \int_{B(z, \epsilon_k)} |(F_z - F_x)(\hat{\varphi}_x^{\epsilon_k})| dx dz.$$

Reasoning as for a_n we obtain:

$$\|c_n(w)\|_{L^p(w \in K)} \lesssim \sum_{k=n}^{\infty} 2^{kd} \int_{B(0, \epsilon_k)} 2^{-k\gamma} f_K(k, \tilde{x}) d\tilde{x}.$$

Thus,

$$\|c_n(w)\|_{L^p(w \in K)} \lesssim 2^{-n\gamma} m_K^{(2)}(n).$$

The assertion on c_n follows.

Let us now consider $\left(\frac{\|d_n(w)\|_{L^p(w)}}{2^{-n\gamma}}\right)_{n \in \mathbb{N}}$. Applying the substitution $\tilde{z} = z - w$ then $\tilde{x} = -(x - w)$, and remarking that the obtained integrand is supported in $\tilde{z} \in B(0, \epsilon_n)$, $\tilde{x} \in B(-\tilde{z}, \epsilon_k)$, we can integrate over $\tilde{z} \in B(0, \epsilon_n)$, $\tilde{x} \in B(0, \epsilon_{n-1})$ without changing the value of the integral so that:

$$\begin{aligned} & d_n(w) \\ &= \sup_{\psi \in \mathcal{B}^r} \left| \sum_{k=n}^{\infty} \int_{B(0, \epsilon_n)} \int_{B(0, \epsilon_{n-1})} (F_{-\tilde{x}+w} - F_w)(\hat{\varphi}_{-\tilde{x}+w}^{\epsilon_k}) \check{\varphi}_{\tilde{z}}^{\epsilon_k}(-\tilde{x}) \psi^{\epsilon_n}(\tilde{z}) d\tilde{x} d\tilde{z} \right| \\ &= \sup_{\psi \in \mathcal{B}^r} \left| \sum_{k=n}^{\infty} \int_{B(0, \epsilon_{n-1})} ((F_{-\tilde{x}+w} - F_w)(\hat{\varphi}_{-\tilde{x}+w}^{\epsilon_k})) (\check{\varphi}^{\epsilon_k} * \psi^{\epsilon_n}(-\tilde{x})) d\tilde{x} \right| \\ &\leq \sup_{\psi \in \mathcal{B}^r} \sum_{k=n}^{\infty} \int_{B(0, \epsilon_{n-1})} |(F_{-\tilde{x}+w} - F_w)(\hat{\varphi}_{-\tilde{x}+w}^{\epsilon_k})| |\check{\varphi}^{\epsilon_k} * \psi^{\epsilon_n}(-\tilde{x})| d\tilde{x}. \end{aligned}$$

Now from Lemma 3.4.8, $|\check{\varphi}^{\epsilon_k} * \psi^{\epsilon_n}(-\tilde{x})| \leq 2^{n(r+d)-kr} \|\psi\|_{C^r} \|\check{\varphi}\|_{L^1}$ so that:

$$d_n(w) \lesssim \sum_{k=n}^{\infty} 2^{n(r+d)-kr} \int_{B(0, \epsilon_{n-1})} |(F_{-\tilde{x}+w} - F_w)(\hat{\varphi}_{-\tilde{x}+w}^{\epsilon_k})| d\tilde{x}.$$

Taking the $L^p(w \in K)$ norm, applying Minkowski's inequality and substituting $\tilde{w} = w - \tilde{x}$ yields:

$$\|d_n(w)\|_{L^p(w \in K)} \lesssim \sum_{k=n}^{\infty} 2^{n(r+d)-kr} \int_{B(0, \epsilon_{n-1})} \|(F_{\tilde{w}+\tilde{x}} - F_{\tilde{w}})(\hat{\varphi}_{\tilde{w}}^{\epsilon_k})\|_{L^p(\tilde{w} \in \bar{K}_2)} d\tilde{x}.$$

By definition of f_K :

$$\begin{aligned} \|d_n(w)\|_{L^p(w \in K)} &\lesssim \sum_{k=n}^{\infty} 2^{n(r+d)-kr} \int_{B(0, \epsilon_{n-1})} f_K(k, \tilde{x}) 2^{-k\alpha} 2^{-n(\gamma-\alpha)} d\tilde{x} \\ &= 2^{-n\gamma} m_K^{(3)}(n). \end{aligned}$$

This is enough to conclude. \square

Remark 3.4.11. Note that we did not use the assumption $\gamma > 0$ in establishing the reconstruction bound (Proposition 3.4.10) above.

3.4.5 The reconstruction for $\gamma \leq 0$

Now that we have treated the case $\gamma > 0$, let us discuss the problem of reconstruction when $\gamma \leq 0$. It is very natural a priori to consider the same sequence of approximating distributions as in Definition 3.4.7. However, note from Proposition 3.4.9, Item (2), that the convergence of those approximating distributions fundamentally requires $\gamma > 0$. Namely, in this case, we cannot control the series $\sum_k u_k''$.

The idea, as in [CZ20], is to simply remove the term u_k'' from the approximating sequence of Definition 3.4.7. In particular, from Proposition 3.4.9 we can still define:

$$\mathcal{R}(F)(\psi) := \int_{\mathbb{R}^d} F_z(\hat{\varphi} * \hat{\varphi}_z^2) \psi(z) dz + \sum_{k=0}^{\infty} u_k', \quad (3.4.8)$$

where we recall that u_k' is defined as:

$$u_k' := \int_{\mathbb{R}^d} F_x(\hat{\varphi}_x^{\epsilon_k})(\check{\varphi}^{\epsilon_k} * \psi)(x) dx, \quad \text{for } k \in \mathbb{N}.$$

Note that without the term u_k'' , there is no simplification allowing to start the decomposition at a scale 2^{-n_0} for any n_0 . Hence we need to take into account the fact that the sum starts at index $k = 0$ in (3.4.8).

It remains to establish the reconstruction bound. As in the previous section, define for $w \in \mathbb{R}^d$ and $\psi \in \mathcal{D}(\mathbb{R}^d)$, $G_w(\psi) := (\mathcal{R}(F) - F_w)(\psi)$. Then, in a similar way to the previous section, it is straightforward to obtain the following decomposition:

$$\begin{aligned} G_w(\psi_w^{\epsilon_n}) &= \int_{B(w, \epsilon_n)} \int_{B(z, 1)} (F_z - F_x)(\hat{\varphi}_x) \hat{\varphi}_z^2(x) \psi_w^{\epsilon_n}(z) dx dz \\ &+ \int_{B(w, \epsilon_n)} \int_{B(z, 1)} (F_x - F_w)(\hat{\varphi}_x) \hat{\varphi}_z^2(x) \psi_w^{\epsilon_n}(z) dx dz \\ &+ \sum_{k=0}^{n-1} \int_{B(w, \epsilon_n)} \int_{B(z, \epsilon_k)} (F_x - F_w)(\hat{\varphi}_x^{\epsilon_k}) \check{\varphi}_z^{\epsilon_k}(x) \psi_w^{\epsilon_n}(z) dx dz \\ &+ \sum_{k=n}^{+\infty} \int_{B(w, \epsilon_n)} \int_{B(z, \epsilon_k)} (F_x - F_w)(\hat{\varphi}_x^{\epsilon_k}) \check{\varphi}_z^{\epsilon_k}(x) \psi_w^{\epsilon_n}(z) dx dz, \end{aligned}$$

so that checking the reconstruction bound follows from estimating the following quantities:

$$\begin{aligned} a_n(w) &:= \sup_{\psi \in \mathcal{B}^r} \left| \int_{B(w, \epsilon_n)} \int_{B(z, 1)} (F_z - F_x) (\hat{\varphi}_x) \hat{\varphi}_z^2(x) \psi_w^{\epsilon_n}(z) dx dz \right|, \\ b_n(w) &:= \sup_{\psi \in \mathcal{B}^r} \left| \int_{B(w, \epsilon_n)} \int_{B(z, 1)} (F_x - F_w) (\hat{\varphi}_x) \hat{\varphi}_z^2(x) \psi_w^{\epsilon_n}(z) dx dz \right|, \\ c_n(w) &:= \sup_{\psi \in \mathcal{B}^r} \left| \sum_{k=0}^{n-1} \int_{B(w, \epsilon_n)} \int_{B(z, \epsilon_k)} (F_x - F_w) (\hat{\varphi}_x^{\epsilon_k}) \check{\varphi}_z^{\epsilon_k}(x) \psi_w^{\epsilon_n}(z) dx dz \right|, \\ d_n(w) &:= \sup_{\psi \in \mathcal{B}^r} \left| \sum_{k=n}^{+\infty} \int_{B(w, \epsilon_n)} \int_{B(z, \epsilon_k)} (F_x - F_w) (\hat{\varphi}_x^{\epsilon_k}) \check{\varphi}_z^{\epsilon_k}(x) \psi_w^{\epsilon_n}(z) dx dz \right|. \end{aligned}$$

Note that these quantities are different from those named with the same letters in the section corresponding to $\gamma > 0$. The reconstruction bound is established in the following proposition:

Proposition 3.4.12. *In the setting of this section, assume that $\tilde{m}_K^{(3)}, \tilde{m}_K^{(4)} \in \ell^q$ and that $\gamma \geq \alpha, \gamma \leq 0$. Let k be the scaling function defined in (3.3.1). Then:*

$$\begin{aligned} &\left(\frac{\|a_n(w)\|_{L^p(w \in K)}}{k(2^{-n})} \right)_{n \in \mathbb{N}}, \left(\frac{\|b_n(w)\|_{L^p(w \in K)}}{k(2^{-n})} \right)_{n \in \mathbb{N}}, \\ &\left(\frac{\|c_n(w)\|_{L^p(w \in K)}}{k(2^{-n})} \right)_{n \in \mathbb{N}}, \left(\frac{\|d_n(w)\|_{L^p(w \in K)}}{k(2^{-n})} \right)_{n \in \mathbb{N}} \in \ell^q. \end{aligned}$$

As an immediate consequence, $\mathcal{R}(F)$ is a r -uniform γ, p, q -reconstruction of F on K .

Remark 3.4.13. *The proof below actually works for any scaling function k such that $\left(\frac{1}{k(2^{-n})}\right)_{n \in \mathbb{N}} \in \ell^q$.*

Proof. The proof follows similarly as in Proposition 3.4.10. The same calculations as in Proposition 3.4.10 allow to establish:

$$\begin{cases} \|a_n(w)\|_{L^p(w \in K)} &\lesssim m_K^{(1)}(0), \\ \|b_n(w)\|_{L^p(w \in K)} &\lesssim m_K^{(1)}(0), \\ \|d_n(w)\|_{L^p(w \in K)} &\lesssim 2^{-n\gamma} m_K^{(3)}(n). \end{cases}$$

Let us now focus on c_n . By definition of c_n :

$$c_n(w) \leq \sup_{\psi \in \mathcal{B}^r} \sum_{k=0}^{n-1} \int_{B(w, \epsilon_n)} \int_{B(z, \epsilon_k)} |(F_x - F_w) (\hat{\varphi}_x^{\epsilon_k})| |\check{\varphi}_z^{\epsilon_k}(x)| |\psi_w^{\epsilon_n}(z)| dx dz.$$

Using the estimates $|\check{\varphi}_z^{\epsilon_k}(x)| \leq 2^{kd} \|\check{\varphi}\|_{L^\infty}$ and $|\psi_w^{\epsilon_n}(z)| \leq 2^{nd} \|\psi\|_{C^r}$, we get:

$$c_n(w) \lesssim \sum_{k=0}^{n-1} 2^{nd} 2^{kd} \int_{B(w, 2^{-n})} \int_{B(z, 2^{-k})} |(F_x - F_w) (\hat{\varphi}_x^{2^{-k}})| dx dz.$$

In this integral, $B(z, 2^{-k}) \subset B(w, 2^{-k+1})$ so that:

$$c_n(w) \lesssim \sum_{k=0}^{n-1} 2^{kd} \int_{B(w, 2^{-k+1})} |(F_x - F_w) (\hat{\varphi}_x^{2^{-k}})| dx.$$

Substituting $\tilde{x} = -(x - w)$ in this integral:

$$c_n(w) \lesssim \sum_{k=0}^{n-1} 2^{kd} \int_{B(0, 2^{-k+1})} |(F_{w-\tilde{x}} - F_w) (\hat{\phi}_{w-\tilde{x}}^{2^{-k}})| d\tilde{x}.$$

Now we take the L^p norm in w , apply Minkowski's inequality and change variable $\tilde{w} = w - \tilde{x}$ in the L^p norm:

$$\|c_n(w)\|_{L^p(w \in K)} \lesssim \sum_{k=0}^{n-1} 2^{kd} \int_{B(0, 2^{-k+1})} \|(F_{\tilde{w}+\tilde{x}} - F_{\tilde{w}}) (\hat{\phi}_{\tilde{w}}^{2^{-k}})\|_{L^p(\tilde{w} \in \tilde{K}_2)} d\tilde{x}.$$

Finally, remembering the definition of f_K and using the fact that $|\tilde{x}| \leq 2^{-k+1}$ in this integral:

$$\|c_n(w)\|_{L^p(w \in K)} \lesssim 2^{-n\gamma} m_K^{(4)}(n).$$

This is enough to conclude, even for the case $\gamma = 0$. Indeed, for the terms a and b , we use the fact that $\left(\frac{1}{k(2^{-n})}\right)_{n \in \mathbb{N}} \in \ell^q$. And for the terms c and d , we have just established that:

$$\begin{cases} \frac{\|c_n(w)\|_{L^p(w \in K)}}{k(2^{-n})} \lesssim \tilde{m}_K^{(4)}(n), \\ \frac{\|d_n(w)\|_{L^p(w \in K)}}{k(2^{-n})} \lesssim \tilde{m}_K^{(3)}(n). \end{cases}$$

□

3.4.6 The reconstruction is Besov

We now show that $\mathcal{R}(F)$ lies in a suitable Besov space.

Proposition 3.4.14. *In the setting of this section:*

1. (Global version) Assume that $\|F\|_{\mathcal{G}_{p,q,q_1,\mathbb{R}^d,\varphi}^{\alpha,\beta,\gamma}} < +\infty$. Then:

- (a) If $\beta \wedge \gamma > 0$, then $\mathcal{R}(F) = 0$.
- (b) If $\beta \wedge \gamma = 0$, then for all $\kappa > 0$, $\mathcal{R}(F) \in \mathcal{B}_{p,1}^{-\kappa}$.
- (c) If $\beta \wedge \gamma < 0$, then $\mathcal{R}(F) \in \mathcal{B}_{p,q_1 \vee q}^{\beta \wedge \gamma}$.

2. (Local version) Assume that for all $K \subset \mathbb{R}^d$, $\|F\|_{\mathcal{G}_{p,q,q_1,K,\varphi}^{\alpha,\beta,\gamma}} < +\infty$. Then there exists a global distribution $\mathcal{R}(F) \in \mathcal{D}'(\mathbb{R}^d)$ satisfying (3.4.1) over all $K \subset \mathbb{R}^d$ and:

- (a) If $\beta \wedge \gamma > 0$, then $\mathcal{R}(F) = 0$.
- (b) If $\beta \wedge \gamma = 0$, then for all $\kappa > 0$, $\mathcal{R}(F) \in \mathcal{B}_{p,1,\text{loc}}^{-\kappa}$.
- (c) If $\beta \wedge \gamma < 0$, then $\mathcal{R}(F) \in \mathcal{B}_{p,q_1 \vee q,\text{loc}}^{\beta \wedge \gamma}$.

Furthermore, the reconstruction map is continuous in the following sense: let \mathcal{B} denote $\mathcal{B}_{p,1,K}^{-\kappa}$ if $\beta \wedge \gamma = 0$ and $\mathcal{B}_{p,q_1 \vee q,K}^{\beta \wedge \gamma}$ if $\beta \wedge \gamma < 0$, then:

$$\|\mathcal{R}(F)\|_{\mathcal{B}} \lesssim \|F\|_{\mathcal{G}_{p,q,q_1,\tilde{K}_2,\varphi}^{\alpha,\beta,\gamma}}. \quad (3.4.9)$$

Proof. Let us first prove the global version of the result. The item (a) follows from Proposition 3.4.2, Item (3).

Now we turn to (c). Recall, from the equivalent definition of a Besov space Proposition 3.A.5, Item (1), that it is sufficient to show, denoting $r := q_1 \vee q$:

$$\left\| \left\| \frac{\mathcal{R}(F)(\hat{\varphi}_x^{2^{-n}})}{2^{-n(\beta \wedge \gamma)}} \right\|_{L^p(x \in \mathbb{R}^d)} \right\|_{\ell^r(n)} < +\infty.$$

By the assumption and the reconstruction bound obtained in the previous sections, we know that:

$$\left\| \left\| \frac{F_x(\hat{\varphi}_x^{\epsilon_n})}{2^{-n\beta}} \right\|_{L^p(x)} \right\|_{\ell^{q_1}(n)} + \left\| \left\| \frac{(\mathcal{R}(F) - F_x)(\hat{\varphi}_x^{\epsilon_n})}{2^{-n\gamma}} \right\|_{L^p(x)} \right\|_{\ell^q(n)} \lesssim \|F\|_{\mathcal{G}_{p,q,q_1,\mathbb{R}^d,\varphi}^{\alpha,\beta,\gamma}}.$$

Using the fact that $\beta, \gamma \geq \beta \wedge \gamma$ and the embeddings $\ell^{q_1} \subset \ell^r$, $\ell^q \subset \ell^r$:

$$\left\| \left\| \frac{F_x(\hat{\varphi}_x^{\epsilon_n})}{2^{-n(\beta \wedge \gamma)}} \right\|_{L^p(x)} \right\|_{\ell^r(n)} + \left\| \left\| \frac{(\mathcal{R}(F) - F_x)(\hat{\varphi}_x^{\epsilon_n})}{2^{-n(\beta \wedge \gamma)}} \right\|_{L^p(x)} \right\|_{\ell^r(n)} \lesssim \|F\|_{\mathcal{G}_{p,q,q_1,\mathbb{R}^d,\varphi}^{\alpha,\beta,\gamma}}.$$

The triangle inequality yields the announced result.

Finally, (b) immediately follows from (c) after noticing that the condition of homogeneity $g \in \ell^{q_1}$ for some $\beta > 0$ implies $g \in \ell^1$ for any $\beta' < \beta$; that the reconstruction bound for γ implies the reconstruction bound for any $\gamma' < \gamma$, see Proposition 3.4.2, Item (1); and that $\mathcal{B}_{p,\infty}^{-\kappa/2} \subset \mathcal{B}_{p,1}^{-\kappa}$ for any $\kappa > 0$.

Now let us discuss the local version of the result. A global reconstruction $\mathcal{R}(F)$ can be built by localization, as in [CZ20, Section 11]. Then, properties (a), (b), (c), and (3.4.9) follow from the same arguments as in the global case, using the local version of Proposition 3.A.5, Item (1). \square

3.5 Proof of Theorems 3.3.2 and 3.3.7 from Theorem 3.4.5

In this section, we prove Theorem 3.3.2 and Theorem 3.3.7 from Theorem 3.4.5.

The case $\gamma \leq 0$ is treated similarly to the case $\gamma > 0$ so we only treat the case $\gamma > 0$ for concision.

Thus, we consider a germ F , and reals $p, q \in [1, +\infty]$, $\alpha, \beta, \gamma \in \mathbb{R}$ with $\alpha \leq \gamma$ and $\gamma > 0$. We let $K \subset \mathbb{R}^d$ and we assume that there is a test-function $\varphi \in \mathcal{D}(\mathbb{R}^d)$ with $\int \varphi \neq 0$ such that (3.3.2) and (3.3.3) hold. Let $r \in \mathbb{N}$ be an integer such that $r > \max(-\alpha, -\beta)$.

We shall show that the hypotheses of Theorem 3.4.5 are satisfied. First, notice that this requires us to exhibit a test-function $\hat{\varphi} \in \mathcal{D}(B(0, 1/2))$ such that $\int \hat{\varphi} = 1$ and $\int x^k \hat{\varphi}(x) dx = 0$ for $1 \leq |k| \leq r - 1$.

For this purpose, we *tweak* the test-function φ as presented in [CZ20].

Lemma 3.5.1. *[Tweaking a test-function, [CZ20, Lemma 8.1]] Fix $r \in \mathbb{N}$, distinct reals $\lambda_0, \lambda_1, \dots, \lambda_{r-1} \in (0, \infty)$ and a test-function $\varphi \in \mathcal{D}(\mathbb{R}^d)$ such that $\int \varphi \neq 0$ and $\text{supp } \varphi \subset B(0, R_\varphi)$.*

Then there exists constants $c_0, c_1, \dots, c_{r-1} \in \mathbb{R}$ such that the tweaked test-function $\hat{\varphi}$, defined by

$$\hat{\varphi} := \frac{1}{\int \varphi} \sum_{i=0}^{r-1} c_i \varphi^{\lambda_i},$$

has the following properties:

1. $\int_{\mathbb{R}^d} \hat{\varphi} = 1$,
2. $\hat{\varphi}$ annihilates monomials of degree 1 to $r - 1$, specifically

$$\int_{\mathbb{R}^d} y^k \hat{\varphi}(y) dy = 0, \quad \text{for all } k \in \mathbb{N}_0^d : 1 \leq |k| \leq r - 1,$$

3. moreover, if $0 < \lambda_i < \frac{1}{2R_\varphi}$ for $i = 0, 1, \dots, r - 1$, then:

$$\text{supp } \hat{\varphi} \subset B(0, 1/2).$$

Remark 3.5.2. Whilst we direct the reader to [CZ20, Lemma 8.1] for the proof we outline the approach here. The main idea is to consider an arbitrary $\hat{\varphi} := (1/\int \varphi) \sum_{i=0}^{r-1} c_i \varphi^{\lambda_i}$ and write a system of linear equations in c_i from the relations given by Item 1 and Item 2 of Lemma 3.5.1. The obtained system of equations involves the Vandermonde matrix of $(\lambda_i)_{0 \leq i \leq r-1}$. Since the λ_i are distinct, this system is invertible, and one can even provide explicit expressions for c_i [CZ20, see Equation (8.1)].

Remark 3.5.3. In [CZ20], the authors choose the specific values $\lambda_i := \frac{2^{-i-1}}{1+R_\varphi}$ for $i = 0, 1, \dots, r - 1$. The reason for this choice is that it allows for explicit quantitative bounds throughout their calculations. In this paper however, we do not track the precise constants that appear in the estimates, which is why we do not pick explicit values for λ_i .

Now we show that the properties of homogeneity and coherence are stable by tweaking.

Proposition 3.5.4. Assume that there exists $\varphi \in \mathcal{D}(\mathbb{R}^d)$ such that $\int \varphi \neq 0$ and (3.3.2) resp. (3.3.3) is satisfied for φ . Then, for any $r \in \mathbb{N}$, there exists $\hat{\varphi} \in \mathcal{D}(B(0, 1/2))$ such that $\int \hat{\varphi} = 1$, $\int x^k \hat{\varphi}(x) dx = 0$ for $1 \leq |k| \leq r - 1$ and (3.3.2) resp. (3.3.3) is satisfied for $\hat{\varphi}$.

Proof. Let us only present the proof for the coherence condition (3.3.3) as the other case is similar to treat. Thus, assume that $\varphi \in \mathcal{D}(\mathbb{R}^d)$ is such that $\int \varphi \neq 0$ and:

$$\left\| \left\| \left\| \frac{(F_{x+h} - F_x) (\varphi_x^{2^{-n}})}{2^{-n\alpha} (2^{-n} + |h|)^{\gamma-\alpha}} \right\|_{L^p(x \in \bar{K}_2)} \right\|_{\ell^\infty(n)} \right\|_{L_h^q(h \in B(0,2))} < +\infty.$$

As in Lemma 3.5.1, let $R_\varphi > 0$ be such that $\text{supp}(\varphi) \subset B(0, R_\varphi)$. Let $r \in \mathbb{N}$ and fix distinct $m_0, \dots, m_{r-1} \in \mathbb{N}$ such that for $0 \leq |k| \leq r - 1$, $2^{-m_k} < \frac{1}{2R_\varphi}$. We apply Lemma 3.5.1 to $\lambda_k := 2^{-m_k}$. Let us denote

$$\hat{\varphi} := \frac{1}{\int \varphi} \sum_{i=0}^{r-1} c_i \varphi^{2^{-m_i}},$$

the obtained test-function, then $\hat{\varphi} \in \mathcal{D}(B(0, 1/2))$ and $\int \hat{\varphi} = 1$, $\int x^k \hat{\varphi}(x) dx = 0$ for $1 \leq |k| \leq r - 1$. To conclude, it is enough by triangle inequality to show that for any $m \in \mathbb{N}$:

$$\left\| \left\| \left\| \frac{(F_{x+h} - F_x) (\varphi_x^{2^{-n-m}})}{2^{-n\alpha} (2^{-n} + |h|)^{\gamma-\alpha}} \right\|_{L^p(x \in \bar{K}_2)} \right\|_{\ell^\infty(n)} \right\|_{L_h^q(h \in B(0,2))} < +\infty.$$

But this follows immediately from the estimate:

$$2^{-n\alpha} (2^{-n} + |h|)^{\gamma-\alpha} \geq 2^{m\alpha} 2^{-(n+m)\alpha} (2^{-(n+m)} + |h|)^{\gamma-\alpha}.$$

(Remember that $\gamma \geq \alpha$ from our hypotheses). □

We still have to translate the conditions of Theorem 3.3.2 into the conditions of Theorem 3.4.5. This is possible thanks to the Lemma 3.B.1, which implies the following result. Note that Lemma 3.B.1 is a purely elementary result. However, as its proof is a bit technical, we state it and prove it in the appendix.

Corollary 3.5.5. *Let $f: \mathbb{N} \times \mathbb{R}^d \rightarrow \mathbb{R}_+$ be a positive function, and $c > 0$. Define $m_f^{(1)}$, $m_{c,f}^{(2)}$, $m_{c,f}^{(3)}$ as in (3.4.2). Assume that:*

$$\left\| \|f(k, x)\|_{\ell^\infty(k)} \right\|_{L_x^q(x \in B(0,2))} < +\infty. \quad (3.5.1)$$

Then $m_f^{(1)}$, $m_{c,f}^{(2)}$, $m_{c,f}^{(3)}$ are in ℓ^q .

Proof. Applying Lemma 3.B.1 to $a_{k,n} := \delta_{k,n}$ resp. $a_{k,n} := 2^{-(k-n)c} \mathbf{1}_{\{k \geq n\}}$ immediately gives the result for $m_f^{(1)}$ resp. $m_{c,f}^{(2)}$.

Now let us treat $m_{c,f}^{(3)}$. For $n \in \mathbb{N}$, $h \in \mathbb{R}^d$, set:

$$\tilde{f}(n, h) := \sum_{k=0}^{+\infty} 2^{-(k-n)c} \mathbf{1}_{\{k \geq n\}} f(k, h),$$

so that $m_{c,f}^{(3)} = m_{\tilde{f}}^{(1)}$. It is straightforward to see that if (3.5.1) is satisfied for f , then (3.5.1) is also satisfied for \tilde{f} . We conclude by applying lemma Lemma 3.B.1 to $a_{k,n} := \delta_{k,n}$ and the function \tilde{f} . \square

Combining Proposition 3.5.4, Corollary 3.5.5 and Theorem 3.4.5 yields Theorem 3.3.2 and Theorem 3.3.7 (in the case $\gamma > 0$).

3.6 Proof of Theorem 3.3.12 from Theorem 3.4.5

Let us now prove Theorem 3.3.12 from Theorem 3.4.5. In the remainder of this section, we consider the setting of Section 3.3.3. That is, we let $\alpha < 0, \beta > 0$ with $\alpha + \beta > 0$ and $\beta \notin \mathbb{N}$, as well as $p_1, p_2, q_1, q_2, p, q \in [1, +\infty]$ with $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$; we fix distributions $g \in \mathcal{B}_{p_1, q_1}^\alpha$, $f \in \mathcal{B}_{p_2, q_2}^\beta$; and we define germs F and P as in (3.3.5) resp. (3.3.6).

Fix any test-function $\varphi \in \mathcal{D}(\mathbb{R}^d)$. Recalling the statement of Theorem 3.4.5, Theorem 3.3.12 holds as soon as the following quantities are finite (recall that $\epsilon_n := 2^{-n}$):

$$\left\{ \begin{array}{l} v_1 := \left\| \left\| \frac{P_x(\varphi_x^{\epsilon_n})}{2^{-n\alpha}} \right\|_{L^p(x)} \right\|_{\ell^{q_1}(n)}, \\ v_2 := \left\| \int_{h \in B(0, \epsilon_{n-1})} 2^{nd} \left\| \frac{(P_{x+h} - P_x)(\varphi_x^{\epsilon_n})}{2^{-n\alpha} (2^{-n} + |h|)^\beta} \right\|_{L^p(x)} dh \right\|_{\ell^q(n)}, \\ v_3 := \left\| \sum_{k=n}^{+\infty} \int_{h \in B(0, \epsilon_k)} 2^{-(k-n)(\alpha+\beta)+kd} \left\| \frac{(P_{x+h} - P_x)(\varphi_x^{\epsilon_k})}{2^{-k\alpha} (2^{-k} + |h|)^\beta} \right\|_{L^p(x)} dh \right\|_{\ell^q(n)}, \\ v_4 := \left\| \sum_{k=n}^{+\infty} \int_{h \in B(0, \epsilon_{n-1})} 2^{-(k-n)(\alpha+r)+nd} \left\| \frac{(P_{x+h} - P_x)(\varphi_x^{\epsilon_k})}{2^{-k\alpha} (2^{-k} + |h|)^\beta} \right\|_{L^p(x)} dh \right\|_{\ell^q(n)}. \end{array} \right.$$

This is the content of the following proposition.

Proposition 3.6.1. $v_1, v_2, v_3, v_4 < +\infty$. As a consequence, Theorem 3.3.12 holds.

Proof. We start with v_1 . For $x \in \mathbb{R}^d, \varphi \in \mathcal{D}(\mathbb{R}^d)$ we have:

$$P_x(\varphi) = \sum_{0 \leq |k| < \beta} \frac{\partial^k f(x)}{k!} g\left((\cdot - x)^k \varphi(\cdot)\right).$$

Thus by triangle inequality and Hölder's inequality:

$$\left\| \frac{P_x(\varphi_x^{\epsilon_n})}{2^{-n\alpha}} \right\|_{L^p(x)} \leq \sum_{0 \leq |k| < \beta} \frac{\|\partial^k f\|_{L^{p_2}}}{k!} \frac{\|g\left((\cdot - x)^k \varphi_x^{\epsilon_n}(\cdot)\right)\|_{L^{p_1}(x)}}{2^{-n\alpha}}.$$

Now summing in n :

$$v_1 \leq \sum_{0 \leq |k| < \beta} \frac{\|\partial^k f\|_{L^{p_2}}}{k!} \left\| \frac{\|g\left((\cdot - x)^k \varphi_x^{\epsilon_n}(\cdot)\right)\|_{L^{p_1}(x)}}{2^{-\alpha n}} \right\|_{\ell^{q_1}(n)}.$$

Considering the test-function $\psi: z \mapsto z^k \phi(z)$ and recalling the definition of Besov spaces, Definition 3.A.1, one obtains:

$$\left\| \frac{\|g\left((\cdot - x)^k \varphi_x^{\epsilon_n}(\cdot)\right)\|_{L^{p_1}(x)}}{2^{-(\alpha+|k|)n}} \right\|_{\ell^{q_1}(n)} \lesssim \|g\|_{\mathcal{B}_{p_1, q_1}^\alpha}.$$

Recall also that since $\beta > |k|$ in the sum above, one has $\|\partial_k f\|_{L^{p_2}} \lesssim \|\partial_k f\|_{\mathcal{B}_{p_2, q_2}^{\beta-|k|}} \lesssim \|f\|_{\mathcal{B}_{p_2, q_2}^\beta}$, and thus:

$$v_1 \lesssim \|g\|_{\mathcal{B}_{p_1, q_1}^\alpha} \|f\|_{\mathcal{B}_{p_2, q_2}^\beta} < +\infty.$$

Now we consider the quantities v_2, v_3, v_4 . Let us use the following notation for the Taylor expansions: if f is a sufficiently regular function, $\alpha \in \mathbb{R}$ and $x, h \in \mathbb{R}^d$, set $T_f^\alpha(x, h) := f(x+h) - \sum_{0 \leq |l| < \alpha} \frac{1}{l!} \partial^l f(x) h^l$.

Let $x, h, z \in \mathbb{R}^d$ then a straightforward calculation establishes (recall that F is defined in (3.3.5)):

$$(F_{x+h} - F_x)(z) = - \sum_{0 \leq |k| < \beta} \frac{(z-x)^k}{k!} T_{\partial^k f}^{\beta-|k|}(x, h).$$

For simplicity of notations, denote $T_k(x, h) := T_{\partial^k f}^{\beta-|k|}(x, h)$ for the remainder of this proof. Let $\varphi \in \mathcal{D}(\mathbb{R}^d)$ be any test-function. We deduce that for $x, h \in \mathbb{R}^d$:

$$\begin{aligned} (P_{x+h} - P_x)(\varphi) &= g(\varphi(\cdot)(F_{x+h} - F_x)(\cdot)) \\ &= - \sum_{0 \leq |k| < \beta} \frac{1}{k!} T_k(x, h) g\left((\cdot - x)^k \varphi(\cdot)\right). \end{aligned}$$

Applying the triangle inequality then Hölder's inequality:

$$\|(P_{x+h} - P_x)(\varphi)\|_{L^p(x)} \leq \sum_{0 \leq |k| < \beta} \frac{\|g\left((\cdot - x)^k \varphi(\cdot)\right)\|_{L^{p_1}(x)} \|T_k(x, h)\|_{L^{p_2}(x)}}{k!}.$$

For $0 \leq |k| < \beta$, it holds that $\lambda^\alpha (\lambda + |h|)^\beta \gtrsim \lambda^{\alpha+|k|} |h|^{\beta-|k|}$, so that:

$$\begin{aligned} & \left\| \frac{(P_{x+h} - P_x) (\varphi_x^\lambda)}{\lambda^\alpha (\lambda + |h|)^\beta} \right\|_{L^p(x)} \\ & \lesssim \sum_{0 \leq |k| < \beta} \frac{1}{k!} \frac{\left\| g \left((\cdot - x)^k \varphi_x^\lambda(\cdot) \right) \right\|_{L^{p_1}(x)}}{\lambda^{\alpha+|k|}} \frac{\|T_k(x, h)\|_{L^{p_2}(x)}}{|h|^{\beta-|k|}}. \end{aligned}$$

For $0 \leq |k| < \beta$, set :

$$\begin{cases} \mu_k(n) & := \frac{1}{k!} \frac{\left\| g \left((\cdot - x)^k \varphi_x^\lambda(\cdot) \right) \right\|_{L^{p_1}(x)}}{\lambda^{\alpha+|k|}}, \\ \nu_k(h) & := \frac{\|T_k(x, h)\|_{L^{p_2}(x)}}{|h|^{\beta-|k|}} \end{cases}$$

Reasoning as above, it holds that $\mu_k \in \ell^{q_1}$ ($n \in \mathbb{N}$). Also, recalling Proposition 3.A.5, one observes that $\nu_k \in L_h^{q_2}$ ($h \in B(0, 2)$). We conclude by applying Lemma 3.B.1, Item (ii) to the quantities ν_2, ν_3, ν_4 in the same way as in the proof of Corollary 3.5.5.

We obtain Theorem 3.3.12 by setting $\mathcal{M}(g, f) := \mathcal{R}(P)$. Note that by collecting all the inequalities, we even obtain the following continuity estimate:

$$\|\mathcal{M}(g, f)\|_{\mathcal{B}_{p, q_1}^\alpha} \lesssim \|g\|_{\mathcal{B}_{p_1, q_1}^\alpha} \|f\|_{\mathcal{B}_{p_2, q_2}^\beta}.$$

□

Appendices to Chapter 3

3.A Besov spaces

In this section, we define Besov spaces, and recall some of their properties. There are many different equivalent norms used in the literature to define and study Besov spaces. In our context, the following definition “by local means” from [HL17] will be the most useful.

Definition 3.A.1 (Besov spaces). *Let $\alpha \in \mathbb{R}$, $p, q \in [1, +\infty]$. Let $r \in \mathbb{N}$ be such that $r > -\alpha$, and let $n_0 \in \mathbb{Z}$. We define $\mathcal{B}_{p,q}^\alpha = \mathcal{B}_{p,q}^\alpha(\mathbb{R}^d)$ to be the space of distributions $f \in \mathcal{D}'(\mathbb{R}^d)$ such that:*

$$\left\{ \begin{array}{l} \left\| \left\| \sup_{\psi \in \mathcal{B}^r} \left| \frac{f(\psi_x^{2^{-n}})}{2^{-n\alpha}} \right| \right\|_{L^p(x)} \right\|_{\ell^q(n \geq n_0)} < +\infty & \text{if } \alpha < 0, \\ \left\| \sup_{\psi \in \mathcal{B}^r} |f(\psi_x)| \right\|_{L^p(x)} + \left\| \left\| \sup_{\psi \in \mathcal{B}_{[\alpha]}^r} \left| \frac{f(\psi_x^{2^{-n}})}{2^{-n\alpha}} \right| \right\|_{L^p(x)} \right\|_{\ell^q(n \geq n_0)} < +\infty & \text{if } \alpha \geq 0. \end{array} \right.$$

Here, recall that $\mathcal{B}_{[\alpha]}^r$ denotes the space of test-functions $\psi \in \mathcal{B}^r$ such that $\int x^k \psi(x) dx = 0$ for $0 \leq |k| \leq [\alpha]$.

Remark 3.A.2. In [HL17, Proposition 2.4], it is established (in the case $n_0 = 0$) that this definition does not depend on the choice of $r > -\alpha$ and that it is equivalent to the usual definition “by wavelets”. It is also straightforward to establish that the definition does not depend on the choice of $n_0 \in \mathbb{Z}$, so that unless specified, n_0 is taken to be 0.

Remark 3.A.3. From Definition 3.A.1, we a priori only have $\mathcal{B}_{p,q}^\alpha \subset \mathcal{D}'$ (the space of Schwartz distributions), while usual definitions of Besov spaces impose $\mathcal{B}_{p,q}^\alpha \subset \mathcal{S}'$ (the space of tempered distributions). However, the latter inclusion is actually a consequence of our definition, which can be seen for instance from the wavelet characterisation [HL17, Proposition 2.4].

In some situations, it is useful to have local versions of the spaces $\mathcal{B}_{p,q}^\alpha(\mathbb{R}^d)$. In this case, the bounds of Definition 3.A.1 are required to hold on $L^p(x \in K)$ for all compact K , rather than on $L^p(x \in \mathbb{R}^d)$.

Definition 3.A.4 (Local Besov spaces). *Let $\alpha \in \mathbb{R}$, $p, q \in [1, +\infty]$. Let $r \in \mathbb{N}$ be such that $r > -\alpha$. We define $\mathcal{B}_{p,q,\text{loc}}^\alpha = \mathcal{B}_{p,q,\text{loc}}^\alpha(\mathbb{R}^d)$ to be the space of distributions $f \in \mathcal{D}'(\mathbb{R}^d)$ such*

that for all compact $K \subset \mathbb{R}^d$:

$$\begin{cases} \left\| \left\| \sup_{\psi \in \mathcal{B}^r} \left| \frac{f(\psi_x^{2^{-n}})}{2^{-n\alpha}} \right| \right\|_{L^p(x \in K)} \right\|_{\ell^q(n)} < \infty & \text{if } \alpha < 0, \\ \left\| \left\| \sup_{\psi \in \mathcal{B}^r} |f(\psi_x)| \right\|_{L^p(x \in K)} + \left\| \left\| \sup_{\psi \in \mathcal{B}^r_{[\alpha]}} \left| \frac{f(\psi_x^{2^{-n}})}{2^{-n\alpha}} \right| \right\|_{L^p(x \in K)} \right\|_{\ell^q(n)} < \infty & \text{if } \alpha \geq 0. \end{cases}$$

We note $\|f\|_{\mathcal{B}_{p,q,K}^\alpha}$ the norm provided by the quantity just above. Using the same argument as in Remark 3.A.2, this definition does not depend on the choice of $r > -\alpha$.

The following equivalent norms will be useful for us.

Proposition 3.A.5. *Let $\alpha \in \mathbb{R}$, $p, q \in [1, +\infty]$, and $f \in \mathcal{D}'(\mathbb{R}^d)$.*

1. *If $\alpha < 0$, then $f \in \mathcal{B}_{p,q}^\alpha(\mathbb{R}^d)$ if and only if there exists a test-function $\varphi \in \mathcal{D}(\mathbb{R}^d)$ such that $\int \varphi \neq 0$ and:*

$$\left\| \left\| \frac{f(\varphi_x^{2^{-n}})}{2^{-n\alpha}} \right\|_{L^p(x \in \mathbb{R}^d)} \right\|_{\ell^q(n)} < +\infty.$$

The same statement holds for the local Besov spaces $\mathcal{B}_{p,q,\text{loc}}^\alpha(\mathbb{R}^d)$, when one replaces $L^p(x \in \mathbb{R}^d)$ by $L^p(x \in K)$ for all compact $K \subset \mathbb{R}^d$ in the condition above.

2. *If $\alpha > 0$ and $\alpha \notin \mathbb{N}$, then $f \in \mathcal{B}_{p,q}^\alpha(\mathbb{R}^d)$ if and only if for all $0 \leq |k| < \alpha$, $\partial^k f \in L^p$ and for any $h_0 > 0$,*

$$\left\| \left\| \frac{\partial^k f(x+h) - \sum_{0 \leq |l| < \alpha - |k|} \frac{1}{l!} \partial^{k+l} f(x) h^l}{h^{\alpha - |k|}} \right\|_{L^p(x)} \right\|_{L_h^q(h \in B(0, h_0))} < +\infty. \quad (3.A.1)$$

The same statement holds for the local Besov spaces $\mathcal{B}_{p,q,\text{loc}}^\alpha(\mathbb{R}^d)$, when one replaces $L^p(x \in \mathbb{R}^d)$ by $L^p(x \in K)$ for all compact $K \subset \mathbb{R}^d$ in the condition above.

We choose to provide a proof of this proposition for the sake of completeness, although we believe that these properties are well-known in the literature of Besov spaces. For instance, Item (1) is proven to be equivalent to the usual ‘‘Littlewood-Paley’’ definition of Besov spaces in [Tri06, Corollary 1.12]. Also, see [JW78; MP20] for examples of papers using a definition of Besov spaces similar to Item (2).

The techniques used in the proof below are very reminiscent of those used in the remainder of this paper, which is another reason for us to include it.

Proof. We prove the assertions separately. The local versions of the results are established with similar calculations so we only prove the global versions.

- (1) The direct implication is straightforward. Now let us concentrate on the converse. Let $f \in \mathcal{D}'(\mathbb{R}^d)$ and $\varphi \in \mathcal{D}(\mathbb{R}^d)$ be a test-function as in the statement, we shall show that $f \in \mathcal{B}_{p,q}^\alpha$. We ‘‘tweak’’ the test-function φ as in Lemma 3.5.1. Let $r \in \mathbb{N}$ be such that

$r > -\alpha$, and fix distinct $\lambda_0, \dots, \lambda_{r-1}$ small enough so that we can define $\hat{\varphi} \in \mathcal{B}_{[\alpha]}^r$ as in Lemma 3.5.1. Note that also:

$$\left\| \left\| \frac{f(\hat{\varphi}_x^{2^{-n}})}{2^{-n\alpha}} \right\|_{L^p(x)} \right\|_{\ell^q(n)} < +\infty.$$

As above, set $\check{\varphi} := \hat{\varphi}^{1/2} - \hat{\varphi}^2$ so that by mollification we have the following decomposition for all $\psi \in \mathcal{D}(\mathbb{R}^d)$ (see Definition 3.4.7):

$$\begin{aligned} f(\psi_x^{2^{-n}}) &= \int_{\mathbb{R}^d} f(\hat{\varphi}_z^{2^{-n}}) (\hat{\varphi}^{2^{-n+1}} * \psi^{2^{-n}})(z-x) dz \\ &\quad + \sum_{m \geq n} \int_{\mathbb{R}^d} f(\hat{\varphi}_z^{2^{-m}}) (\check{\varphi}^{2^{-m}} * \psi^{2^{-n}})(z-x) dz. \end{aligned}$$

Substituting $\tilde{z} := z - x$ and integrating only on the support of the integrand:

$$\begin{aligned} f(\psi_x^{2^{-n}}) &= \int_{\tilde{z} \in B(0, 2^{-n+1})} f(\hat{\varphi}_{\tilde{z}+x}^{2^{-n}}) (\hat{\varphi}^{2^{-n+1}} * \psi^{2^{-n}})(\tilde{z}) d\tilde{z} \\ &\quad + \sum_{m \geq n} \int_{\tilde{z} \in B(0, 2^{-n+1})} f(\hat{\varphi}_{\tilde{z}+x}^{2^{-m}}) (\check{\varphi}^{2^{-m}} * \psi^{2^{-n}})(\tilde{z}) d\tilde{z}. \end{aligned}$$

Now we use the estimates (see Lemma 3.4.8):

$$\begin{cases} \|\hat{\varphi}^{2^{-n+1}} * \psi^{2^{-n}}\|_{L^\infty} &\leq 2^{nd} \|\psi\|_{L^\infty} \|\hat{\varphi}\|_{L^1} \leq 2^{nd} \|\psi\|_{C^r} \|\hat{\varphi}\|_{L^1}, \\ \|\check{\varphi}^{2^{-m}} * \psi^{2^{-n}}\|_{L^\infty} &\leq 2^{n(r+d)-mr} \|\psi\|_{C^r} \|\check{\varphi}\|_{L^1}. \end{cases}$$

This yields:

$$\begin{aligned} \sup_{\psi \in \mathcal{B}^r} |f(\psi_x^{2^{-n}})| &\leq 2^{nd} \|\hat{\varphi}\|_{L^1} \int_{\tilde{z} \in B(0, 2^{-n+1})} f(\hat{\varphi}_{\tilde{z}+x}^{2^{-n}}) d\tilde{z} \\ &\quad + \sum_{m \geq n} 2^{n(r+d)-mr} \|\check{\varphi}\|_{L^1} \int_{\tilde{z} \in B(0, 2^{-n+1})} f(\hat{\varphi}_{\tilde{z}+x}^{2^{-m}}) d\tilde{z}. \end{aligned}$$

In order to simplify notations, denote $B_n := B(0, 2^{-n})$. Thus, integrating over x :

$$\begin{aligned} \left\| \sup_{\psi \in \mathcal{B}^r} |f(\psi_x^{2^{-n}})| \right\|_{L^p(x)} &\leq 2^{nd} \|\hat{\varphi}\|_{L^1} \int_{\tilde{z} \in B_{n-1}} \|f(\hat{\varphi}_{\tilde{z}+x}^{2^{-n}})\|_{L^p(x)} d\tilde{z} \\ &\quad + \sum_{m \geq n} 2^{n(r+d)-mr} \|\check{\varphi}\|_{L^1} \int_{\tilde{z} \in B_{n-1}} \|f(\hat{\varphi}_{\tilde{z}+x}^{2^{-m}})\|_{L^p(x)} d\tilde{z}. \end{aligned}$$

In those integrals in \tilde{z} , the integrand is actually constant so that after integration, we obtain:

$$\left\| \sup_{\psi \in \mathcal{B}^r} |f(\psi_x^{2^{-n}})| \right\|_{L^p(x)} \lesssim \|f(\hat{\varphi}_x^{2^{-n}})\|_{L^p(x)} + \sum_{m \geq n} 2^{(n-m)r} \|f(\hat{\varphi}_x^{2^{-m}})\|_{L^p(x)}.$$

Then:

$$\|f\|_{\mathcal{B}_{p,q}^\alpha} \lesssim \left\| \left\| \frac{f(\hat{\varphi}_x^{2^{-n}})}{2^{-n\alpha}} \right\|_{L^p(x)} \right\|_{\ell^q(n)} + \left\| \sum_{m \geq n} 2^{(n-m)(r+\alpha)} \left\| \frac{f(\hat{\varphi}_x^{2^{-m}})}{2^{-m\alpha}} \right\|_{L^p(x)} \right\|_{\ell^q(n)}.$$

Since we chose $r + \alpha > 0$, applying Jensen's inequality then interverting the sums in m and n in the second term of the right-hand side yields as announced $\|f\|_{\mathcal{B}_{p,q}^\alpha} < +\infty$.

(2) For simplicity, we reason with $h_0 = 1$ (but the same arguments generalise to any $h_0 > 0$). Let us first concentrate on the direct statement. Let $f \in \mathcal{B}_{p,q}^\alpha$. By definition of the Besov space and the distributional definition of $\partial^k f$, it holds that for $0 \leq |k| < \alpha$, $\partial^k f \in \mathcal{B}_{p,q}^{\alpha-|k|}(\mathbb{R}^d)$, so it suffices to prove the claim for $|k| = 0$. First, let us show that $f \in L^p$, in the sense that there exists $\tilde{f} \in L^p$ such that $f = \tilde{f}$ as distributions. We reason by mollification. Fix $\varphi \in \mathcal{B}^r$ a single test-function such that $\int \varphi = 1$ and $\int x^l \varphi(x) dx = 0$ for $1 \leq |l| < \alpha$. For $m, n \in \mathbb{N}$, define $\tilde{f}_{m,n}(x) := f(\varphi_x^{2^{-m-n}})$. Using the embedding $\ell^q \subset \ell^\infty$, it holds that:

$$\begin{aligned} \left\| \left\| \frac{\tilde{f}_{m,n}(x) - \tilde{f}_{m,n+1}(x)}{2^{-n\alpha}} \right\|_{L^p(x)} \right\|_{\ell^\infty(n)} &\leq \left\| \left\| \frac{\tilde{f}_{m,n}(x) - \tilde{f}_{m,n+1}(x)}{2^{-n\alpha}} \right\|_{L^p(x)} \right\|_{\ell^q(n)} \\ &= \left\| \left\| \frac{f(\varphi_x^{2^{-m-n}} - \varphi_x^{2^{-m-n-1}})}{2^{-n\alpha}} \right\|_{L^p(x)} \right\|_{\ell^q(n)}. \end{aligned}$$

Let $\psi := \varphi - \varphi^{\frac{1}{2}}$. Note that $\frac{1}{C}\psi \in \mathcal{B}_{[\alpha]}^r$ for a suitable $C > 0$. In particular, we have:

$$\begin{aligned} \left\| \left\| \frac{f(\varphi_x^{2^{-m-n}} - \varphi_x^{2^{-m-n-1}})}{2^{-n\alpha}} \right\|_{L^p(x)} \right\|_{\ell^q(n)} &= \left\| \left\| \frac{f(\psi_x^{2^{-m-n}})}{2^{-n\alpha}} \right\|_{L^p(x)} \right\|_{\ell^q(n)} \\ &\leq 2^{-m\alpha} \|f\|_{\mathcal{B}_{p,q}^\alpha}. \end{aligned}$$

We deduce that $\|\tilde{f}_{m,n}(x) - \tilde{f}_{m,n+1}(x)\|_{L^p(x)} \leq 2^{-(m+n)\alpha} \|f\|_{\mathcal{B}_{p,q}^\alpha}$. This implies that for each $m \in \mathbb{N}$, the sequence $(\tilde{f}_{m,n})_{n \in \mathbb{N}}$ is Cauchy in L^p . Hence it has a limit, which we call $\tilde{f}_m \in L^p$, satisfying:

$$\|\tilde{f}_m - \tilde{f}_{m,n}\|_{L^p} \xrightarrow{n \rightarrow +\infty} 0.$$

By summation of a geometric series, we even have $\|\tilde{f}_m - \tilde{f}_{m,n}\|_{L^p} \lesssim 2^{-(m+n)\alpha}$. And for any $n_0 \in \mathbb{N}$, the following series converges in L^p :

$$\tilde{f}_m = \tilde{f}_{m,n_0} + \sum_{n=n_0}^{+\infty} (\tilde{f}_{m,n+1} - \tilde{f}_{m,n}).$$

Also, since for any $m, n \in \mathbb{N}$, $\tilde{f}_{m,n} = \tilde{f}_{m+1,n-1}$, we deduce by triangle inequality that:

$$\begin{aligned} \|\tilde{f}_{m+1} - \tilde{f}_m\|_{L^p} &\leq \|\tilde{f}_{m+1} - \tilde{f}_{m+1,n-1}\|_{L^p} + \|\tilde{f}_m - \tilde{f}_{m,n}\|_{L^p} \\ &\xrightarrow{n \rightarrow +\infty} 0. \end{aligned}$$

Thus for all $m \in \mathbb{N}$, $\tilde{f}_m = \tilde{f}_{m+1} =: \tilde{f}$, where this equality holds in L^p (hence also in \mathcal{D}'). Now let us show that $\tilde{f} = f$ in \mathcal{D}' . Let $\psi \in \mathcal{D}(\mathbb{R}^d)$. By mollification,

$$(\tilde{f} - f)(\psi) = \lim_{n \rightarrow +\infty} \int_{\mathbb{R}^d} (\tilde{f}(x) - f(\varphi_x^{2^{-n}})) \psi(x) dx.$$

But for $n \in \mathbb{N}$, Hölder's inequality:

$$\begin{aligned} \left| \int_{\mathbb{R}^d} (\tilde{f}(x) - f(\varphi_x^{2^{-n}})) \psi(x) dx \right| &\lesssim \|\tilde{f}(x) - f(\varphi_x^{2^{-n}})\|_{L^p(x)} \\ &= \|\tilde{f} - \tilde{f}_{0,n}\|_{L^p} \\ &\xrightarrow{n \rightarrow +\infty} 0. \end{aligned}$$

This establishes the announced equality. Now we establish (3.A.1) for $|k| = 0$, which, according to our previous remark, suffices to establish (3.A.1). For $x, h \in \mathbb{R}^d$, set $T_f^\alpha(x, h) := f(x+h) - \sum_{0 \leq |l| < \alpha} \frac{1}{l!} \partial^l f(x) h^l$, so that we shall show:

$$\left\| \left\| \frac{T_f^\alpha(x, h)}{|h|^\alpha} \right\|_{L^p(x)} \right\|_{L_h^q(h)} < +\infty.$$

For each $h \in \mathbb{R}^d$ we consider $m_h \in \mathbb{Z}$ defined to be explicated later. We write with the notations of the previous item:

$$\begin{aligned} T_f^\alpha(x, h) &= T_{f_{m_h}}^\alpha(x, h) \\ &= T_{f_{m_h, 0}}^\alpha(x, h) + \left(T_{f_{m_h}}^\alpha(x, h) - T_{f_{m_h, 0}}^\alpha(x, h) \right). \end{aligned}$$

More explicitly:

$$\begin{aligned} T_f^\alpha(x, h) &= f \left(\varphi_{x+h}^{2^{-m_h}} - \sum_{0 \leq |l| < \alpha} \frac{(-1)^{|l|} h^l}{l!} \partial^l \left(\varphi_x^{2^{-m_h}} \right) \right) \\ &\quad + \sum_{n=0}^{+\infty} f \left(\varphi_{x+h}^{2^{-m_h-n-1}} - \varphi_{x+h}^{2^{-m_h-n}} \right) \\ &\quad - \sum_{0 \leq |l| < \alpha} \sum_{n=0}^{+\infty} \frac{h^l}{l!} \partial^l f \left(\varphi_x^{2^{-m_h-n-1}} - \varphi_x^{2^{-m_h-n}} \right). \end{aligned}$$

Now we bound each of these terms using our definition of Besov spaces. For $h \in B(0, 1)$ and $z \in \mathbb{R}^d$, define:

$$\begin{cases} \psi(z) &:= \varphi(z - 2^{m_h} h) - \sum_{0 \leq |l| < \alpha} \frac{(-2^{m_h} h)^l}{l!} \partial^l \varphi(z), \\ \tilde{\psi}(z) &:= \varphi^{\frac{1}{2}}(z) - \varphi(z). \end{cases}$$

Then:

$$\begin{aligned} \left\| \left\| \frac{T_f^\alpha(x, h)}{|h|^\alpha} \right\|_{L^p(x)} \right\|_{L_h^q(h)} &\leq \left\| \left\| \frac{f \left(\psi_x^{2^{-m_h}} \right)}{|h|^\alpha} \right\|_{L^p(x)} \right\|_{L_h^q(h)} \\ &\quad + \sum_{n=0}^{+\infty} \left\| \left\| \frac{f \left(\tilde{\psi}_{x+h}^{2^{-m_h-n}} \right)}{|h|^\alpha} \right\|_{L^p(x)} \right\|_{L_h^q(h)} \\ &\quad + \sum_{0 \leq |l| < \alpha} \frac{1}{l!} \sum_{n=0}^{+\infty} \left\| \left\| \frac{\partial^l f \left(\tilde{\psi}_x^{2^{-m_h-n}} \right)}{|h|^{\alpha-|l|}} \right\|_{L^p(x)} \right\|_{L_h^q(h)}. \end{aligned}$$

Changing variable $\tilde{x} = x + h$ in the second term and noting that there exists a constant $C > 0$ such that $\frac{1}{C}\psi, \frac{1}{C}\tilde{\psi} \in \mathcal{B}_{[\alpha]}^r$ (note that actually $\text{supp}(\psi) \subset B(0, 2)$ rather than $B(0, 1)$, but this is not a problem after invoking the definition of Besov

spaces Definition 3.A.1 for $n_0 = -1$; note also that this is where we require α to be non-integer), we obtain:

$$\begin{aligned} & \left\| \left\| \frac{T_f^\alpha(x, h)}{|h|^\alpha} \right\|_{L^p(x)} \right\|_{L_h^q(h)} \\ & \leq C \left\| \left\| \sup_{\psi \in \mathcal{B}_{[\alpha]}^r} \left| \frac{f(\psi_x^{2^{-m_h}})}{|h|^\alpha} \right| \right\|_{L^p(x)} \right\|_{L_h^q(h)} \\ & \quad + \sum_{0 \leq |l| < \alpha} \frac{2C}{l!} \sum_{n=0}^{+\infty} \left\| \left\| \sup_{\tilde{\psi} \in \mathcal{B}_{[\alpha]}^r} \left| \frac{\partial^l f(\tilde{\psi}_x^{2^{-m_h-n}})}{|h|^{\alpha-|l|}} \right| \right\|_{L^p(x)} \right\|_{L_h^q(h)}. \end{aligned}$$

To conclude, it is enough to prove that if $f \in \mathcal{B}_{p,q}^\alpha$, then:

$$v(f) := \left\| \left\| \sup_{\psi \in \mathcal{B}_{[\alpha]}^r} \left| \frac{f(\psi_x^{2^{-m_h}})}{|h|^\alpha} \right| \right\|_{L^p(x)} \right\|_{L_h^q(h)} < +\infty.$$

We cut the integral in h along the annuli: for $n \in \mathbb{N}$, set $B_n := B(2^{-(n+1)}, 2^{-n})$ then:

$$v(f) = \left\| \left\| \left\| \sup_{\psi \in \mathcal{B}_{[\alpha]}^r} \left| \frac{f(\psi_x^{2^{-m_h}})}{|h|^\alpha} \right| \right\|_{L^p(x)} \right\|_{L_h^q(h \in B_n)} \right\|_{\ell^q(n)}.$$

Now we choose m_h so that for $h \in B_n$, $m_h = n$. Using the fact that $\left\| \frac{1}{|h|^\alpha} \right\|_{L_h^q(h \in B_n)} \lesssim 2^{n\alpha}$ uniformly in $n \in \mathbb{N}$, we get $v(f) \lesssim \|f\|_{\mathcal{B}_{p,q}^\alpha} < +\infty$, which concludes the direct statement of the proposition.

Now let us turn to the converse. Assume that for all $0 \leq |k| < \alpha$, $\partial^k f \in L^p$ and that (3.A.1) holds, we shall prove that $f \in \mathcal{B}_{p,q}^\alpha$. On the one hand, when $\psi \in \mathcal{B}^r$ it holds that $f(\psi_x) = \int_{\text{supp}(\psi)} f(y-x) \psi(y) dy$, so that:

$$\begin{aligned} \left\| \sup_{\psi \in \mathcal{B}^r} |f(\psi_x)| \right\|_{L^p(x)} & \leq \left\| \sup_{\psi \in \mathcal{B}^r} \|\psi\|_{\mathcal{B}^r} \int_{B(0,1)} |f(y-x)| dy \right\|_{L^p(x)} \\ & \leq \int_{B(0,1)} \|f\|_{L^p} dy \\ & < +\infty. \end{aligned}$$

On the other hand, when $\psi \in \mathcal{B}_{[\alpha]}^r$, by subtracting a suitable Taylor polynomial it

holds that $f\left(\psi_x^{2^{-n}}\right) = \int_{\mathbb{R}^d} T_f^\alpha(x, y-x) \psi\left(\frac{y-x}{2^{-n}}\right) 2^{nd} dy$, so that:

$$\begin{aligned} & \left\| \left\| \sup_{\psi \in \mathcal{B}_{[\alpha]}^r} \left| \frac{f\left(\psi_x^{2^{-n}}\right)}{2^{-n\alpha}} \right| \right\|_{L^p(x)} \right\|_{\ell^q(n)} \\ & \leq \left\| \left\| \sup_{\psi \in \mathcal{B}_{[\alpha]}^r} \left| \frac{\int_{B(x, 2^{-n})} |T_f^\alpha(x, y-x)| \|\psi\|_{\mathcal{B}_{[\alpha]}^r} dy}{2^{-n(\alpha+d)}} \right| \right\|_{L^p(x)} \right\|_{\ell^q(n)} \\ & \leq \left\| \int_{B(0, 2^{-n})} \left(\frac{\|T_f^\alpha(x, h)\|_{L^p(x)}}{|h|^\alpha} \right) \left(\frac{|h|^\alpha}{2^{-n(\alpha+d)}} \right) dh \right\|_{\ell^q(n)}. \end{aligned}$$

Since $\sup_{n \in \mathbb{N}} \int_{B(0, 2^{-n})} \frac{|h|^\alpha}{2^{-n(\alpha+d)}} dh < +\infty$, we have by applying Jensen's inequality as well as switching summations,

$$\begin{aligned} & \left\| \int_{B(0, 2^{-n})} \left(\frac{\|T_f^\alpha(x, h)\|_{L^p(x)}}{|h|^\alpha} \right) \left(\frac{|h|^\alpha}{2^{-n(\alpha+d)}} \right) dh \right\|_{\ell^q(n)} \\ & \lesssim \left(\sum_{n=0}^{+\infty} \int_{B(0, 2^{-n})} \left(\frac{\|T_f^\alpha(x, h)\|_{L^p(x)}}{|h|^\alpha} \right)^q \left(\frac{|h|^\alpha}{2^{-n(\alpha+d)}} \right) dh \right)^{\frac{1}{q}} \\ & = \left(\int_{B(0, 1)} \sum_{n=0}^{\lfloor -\log_2(|h|) \rfloor} \left(\frac{\|T_f^\alpha(x, h)\|_{L^p(x)}}{|h|^\alpha} \right)^q \left(\frac{|h|}{2^{-n}} \right)^{\alpha+d} \frac{dh}{|h|^d} \right)^{\frac{1}{q}}. \end{aligned}$$

Since $\sup_{h \in B(0, 1)} \sum_{n=0}^{\lfloor -\log_2(|h|) \rfloor} \left(\frac{|h|}{2^{-n}} \right)^{\alpha+d} < +\infty$, we deduce:

$$\begin{aligned} & \left\| \left\| \sup_{\psi \in \mathcal{B}_{[\alpha]}^r} \left| \frac{f\left(\psi_x^{2^{-n}}\right)}{2^{-n\alpha}} \right| \right\|_{L^p(x)} \right\|_{\ell^q(n)} \lesssim \left\| \left\| \frac{T_f^\alpha(x, h)}{|h|^\alpha} \right\|_{L^p(x)} \right\|_{L_h^q(h)} \\ & < +\infty. \end{aligned}$$

Hence, $f \in \mathcal{B}_{p, q}^\alpha$. □

3.B A technical lemma on series

In this section, we establish the following technical result, used in Section 3.5 for the proof of Theorem 3.3.2, and in Section 3.6 for the proof of Theorem 3.3.12.

Lemma 3.B.1. *Let $(f_k : \mathbb{R}^d \rightarrow \mathbb{R}_+)_{k \in \mathbb{N}}$ be a family of positive functions, and $(a_{k, n})_{k, n \in \mathbb{N}} \in \mathbb{R}_+^{\mathbb{N}^2}$ be a sequence of positive reals. Consider the sequence defined by:*

$$u_n := \sum_{k=0}^{+\infty} a_{k, n} \int_{|x| \leq 2^{-k+1}} 2^{kd} f_k(x) dx \quad \text{for } n \in \mathbb{N}.$$

Assume that there exists $A > 0$ such that:

$$\left\{ \begin{array}{l} \text{for all } n \in \mathbb{N} : \sum_{k=0}^{+\infty} a_{k,n} \leq A, \\ \text{for all } k \in \mathbb{N} : \sum_{n=0}^{+\infty} a_{k,n} \leq A. \end{array} \right. \quad (3.B.1)$$

Fix $q \in \mathbb{N}$ and assume also that either of the following conditions is satisfied:

(i) $\left\| \|f_k(x)\|_{\ell^\infty(k)} \right\|_{L^q_x(x \in B(0,2))} < +\infty$, or

(ii) There exists $\mu \in \ell^{q_1}$ ($n \in \mathbb{N}$), $\nu \in L^{q_2}_h$ ($h \in B(0,2)$) for some $q_1, q_2 \in [1, +\infty]$ satisfying $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$, and such that $f_k(x) \leq \mu(k) \nu(x)$.

Then $(u_n)_{n \in \mathbb{N}} \in \ell^q$.

Proof. Let us first prove the result under the assumption (i). For simplicity, we assume that $q < +\infty$ since the case for $q = \infty$ is straightforward. For a fixed $n \in \mathbb{N}$, we apply Jensen's inequality on u_n and we obtain:

$$\left(\frac{|u_n|}{\sum_{k=0}^{+\infty} a_{k,n}} \right)^q \leq \frac{1}{\sum_{k=0}^{+\infty} a_{k,n}} \sum_{k=0}^{+\infty} a_{k,n} \left(\int_{|x| \leq 2^{-k+1}} 2^{kd} f_k(x) dx \right)^q.$$

Hence, we have

$$\begin{aligned} |u_n|^q &\leq \left(\sum_{k=0}^{+\infty} a_{k,n} \right)^{q-1} \sum_{k=0}^{+\infty} a_{k,n} \left(\int_{|x| \leq 2^{-k+1}} 2^{kd} f_k(x) dx \right)^q \\ &\leq A^{q-1} \sum_{k=0}^{+\infty} a_{k,n} \left(\int_{|x| \leq 2^{-k+1}} 2^{kd} f_k(x) dx \right)^q \end{aligned}$$

from (3.B.1).

Applying Jensen's inequality, on the integral, with the probability measure:

$$c_d 2^{kd} \mathbf{1}_{|x| \leq 2^{-k+1}} dx, \quad \text{where } c_d = \frac{1}{\text{Vol}(B(0,2))},$$

we obtain

$$\left(\int_{|x| \leq 2^{-k+1}} 2^{kd} f_k(x) dx \right)^q \leq c_d^{1-q} \int_{|x| \leq 2^{-k+1}} 2^{kd} |f_k(x)|^q dx.$$

which yields:

$$\|(u_n)_{n \in \mathbb{N}}\|_{\ell^q}^q \leq c_d^{1-q} A^{q-1} \sum_{n=0}^{+\infty} \sum_{k=0}^{+\infty} a_{k,n} \int_{|x| \leq 2^{-k+1}} 2^{kd} |f_k(x)|^q dx.$$

Note that the integral can be decomposed over annuli:

$$\int_{|x| \leq 2^{-k+1}} 2^{kd} |f_k(x)|^q dx = \sum_{l=k-1}^{\infty} \int_{2^{-(l+1)} \leq |x| \leq 2^{-l}} 2^{kd} |f_k(x)|^q dx.$$

By applying Tonelli's theorem and rearranging sequences we obtain:

$$\begin{aligned}
\|u\|_{\ell^q}^q &\leq c_d^{1-q} A^{q-1} \sum_{n=0}^{+\infty} \sum_{k=0}^{+\infty} \sum_{l=k-1}^{+\infty} a_{k,n} \int_{2^{-(l+1)} \leq |x| \leq 2^{-l}} 2^{kd} |f_k(x)|^q dx \\
&= c_d^{1-q} A^{q-1} \sum_{n=0}^{+\infty} \sum_{0 \leq k \leq l+1 < +\infty} a_{k,n} \int_{2^{-(l+1)} \leq |x| \leq 2^{-l}} 2^{kd} |f_k(x)|^q dx \\
&= c_d^{1-q} A^{q-1} \sum_{l=-1}^{+\infty} \sum_{k=0}^{l+1} \sum_{n=0}^{+\infty} a_{k,n} \int_{2^{-(l+1)} \leq |x| \leq 2^{-l}} 2^{kd} |f_k(x)|^q dx \\
&= c_d^{1-q} A^{q-1} \sum_{l=-1}^{+\infty} \int_{2^{-(l+1)} \leq |x| \leq 2^{-l}} \sum_{k=0}^{l+1} 2^{kd} |f_k(x)|^q \sum_{n=0}^{+\infty} a_{k,n} dx.
\end{aligned}$$

Applying assumption (3.B.1) we obtain

$$\|u\|_{\ell^q}^q \leq c_d^{1-q} A^q \sum_{l=-1}^{+\infty} \int_{2^{-(l+1)} \leq |x| \leq 2^{-l}} \sum_{k=0}^{l+1} 2^{kd} |f_k(x)|^q dx.$$

For l fixed, we can obtain the following

$$\sum_{k=0}^{l+1} 2^{kd} |f_k(x)|^q \leq \frac{2^{dl+2d} - 1}{2^d - 1} \|f_k(x)\|_{l^\infty}^q \leq \frac{2^{2d}}{2^d - 1} 2^{ld} \|(f_k(x))_{k \in \mathbb{N}}\|_{l^\infty}^q.$$

Hence:

$$\begin{aligned}
\|u\|_{\ell^q}^q &\leq c_d^{1-q} A^q \frac{2^{2d}}{2^d - 1} \sum_{l=-1}^{+\infty} \int_{2^{-(l+1)} \leq |x| \leq 2^{-l}} 2^{ld} \|f_k(x)\|_{\ell^\infty(k)}^q dx \\
&\leq c_d^{1-q} A^q \frac{2^{2d}}{2^d - 1} \sum_{l=-1}^{+\infty} \int_{2^{-(l+1)} \leq |x| \leq 2^{-l}} \|f_k(x)\|_{\ell^\infty(k)}^q \frac{dx}{|x|^d} \\
&= c_d^{1-q} A^q \frac{2^{2d}}{2^d - 1} \int_{0 \leq |x| \leq 2} \|f_k(x)\|_{\ell^\infty(k)}^q \frac{dx}{|x|^d} \\
&= c_d^{1-q} A^q \frac{2^{2d}}{2^d - 1} \left\| \|f_k(x)\|_{\ell^\infty(k)} \right\|_{L_x^q(x \in B(0,2))}^q \\
&< +\infty,
\end{aligned}$$

from (i).

Now let us prove the result under the assumption (ii). By Jensen's inequality:

$$|u_n|^q \lesssim \sum_{k=0}^{+\infty} a_{k,n} |\mu(k)|^q \int_{|x| \leq 2^{-k+1}} 2^{kd} |\nu(x)|^q dx.$$

We sum over $n \in \mathbb{N}$ and intervert summations in k and n by Fubini:

$$\sum_{n \in \mathbb{N}} |u_n|^q \lesssim \sum_{k=0}^{+\infty} \left(\sum_{n=0}^{+\infty} a_{k,n} \right) |\mu(k)|^q \int_{|x| \leq 2^{-k+1}} 2^{kd} |\nu(x)|^q dx.$$

By assumption, the sum of $a_{k,n}$ is bounded, hence:

$$\sum_{n \in \mathbb{N}} |u_n|^q \lesssim \sum_{k=0}^{+\infty} |\mu(k)|^q \int_{|x| \leq 2^{-k+1}} 2^{kd} |\nu(x)|^q dx.$$

We now apply Hölder's inequality with the conjugate exponents $\frac{q_1}{q}$, $\frac{q_2}{q}$:

$$\sum_{n \in \mathbb{N}} |u_n|^q \lesssim \left(\sum_{k=0}^{+\infty} |\mu(k)|^{q \frac{q_1}{q}} \right)^{\frac{q}{q_1}} \left(\sum_{k=0}^{+\infty} \left(\int_{|x| \leq 2^{-k+1}} 2^{kd} |\nu(x)|^q dx \right)^{\frac{q_2}{q}} \right)^{\frac{q}{q_2}}.$$

Applying Jensen's in the integral:

$$\|u\|_{\ell^q}^q \lesssim \|\mu\|_{\ell^{q_1}}^q \left(\sum_{k=0}^{+\infty} \int_{|x| \leq 2^{-k+1}} 2^{kd} |\nu(x)|^{q_2} dx \right)^{\frac{q}{q_2}}.$$

We decompose the domain of the integral as an union of dyadic annuli:

$$\|u\|_{\ell^q}^q \lesssim \|\mu\|_{\ell^{q_1}}^q \left(\sum_{k=0}^{+\infty} \sum_{l=k-1}^{+\infty} \int_{2^{-(l+1)} \leq |x| \leq 2^{-l}} 2^{kd} |\nu(x)|^{q_2} dx \right)^{\frac{q}{q_2}}.$$

Interverting the sums:

$$\|u\|_{\ell^q}^q \lesssim \|\mu\|_{\ell^{q_1}}^q \left(\sum_{l=-1}^{+\infty} \int_{2^{-(l+1)} \leq |x| \leq 2^{-l}} \left(\sum_{k=0}^{l+1} 2^{kd} \right) |\nu(x)|^{q_2} dx \right)^{\frac{q}{q_2}}.$$

Now in this integral $\sum_{k=0}^{l+1} 2^{kd} \lesssim \frac{1}{|x|^d}$ and thus:

$$\|u\|_{\ell^q}^q \lesssim \|\mu\|_{\ell^{q_1}}^q \|\nu\|_{L_x^{q_2}(x \in B(0,2))}^q.$$

By assumption, this is finite, and thus our assertion is proved. \square

Chapter 4

Hairer's multilevel Schauder estimates without Regularity Structures

Abstract

This chapter is based on joint work with Francesco Caravenna and Lorenzo Zambotti. We investigate the regularising properties of singular kernels at the level of germs, i.e. families of distributions indexed by points in \mathbb{R}^d . First we construct a suitable convolution map which acts on general coherent and homogeneous germs. Then we focus on germs that can be decomposed along a basis (corresponding to the so-called modelled distributions in Regularity Structures) and we prove a version of Hairer's multi-level Schauder estimates in this setting, with minimal assumptions. Our results are strongly inspired by the theory of Regularity Structures, but their formulation makes no explicit reference to this theory.

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4.1 Introduction

It is well-known that the convolution of a (Schwartz) distribution against a kernel admitting an integrable singularity on the diagonal will result in a distribution with improved Hölder-Besov regularity: this is the content of the celebrated Schauder estimates, and is a fundamental result in analysis, since examples of such regularising kernels include the Heat kernel and the Green’s function of many differential operators.

One of the key insights of Hairer’s theory of regularity structures [Hai14; BHZ19; CH18; BCCH21] is that this phenomenon of regularisation still occurs when one rather works at the level of *families* of distributions, which one should think of as local approximations to a distribution of interest. In the framework of regularity structures, this is formalised by the notion of “modelled distribution”, and the resulting *multilevel Schauder estimates* [Hai14, Theorem 5.12] admit powerful consequences, as they are one of the crucial steps to set up fixed point results at the level of modelled distributions in order to study singular stochastic partial differential equations. See [FH20; BH20; Ber22] for expository works containing sections on the multilevel Schauder estimates. We would also like to mention the works [OSSW18; OW19; MW20; OSSW21], where similar (though kernel-free) Schauder estimates are established at the level of families of distributions for the analysis of SPDEs.

On the other hand, there has recently been an effort, see e.g. [OSSW21; MW20; CZ20; ZK21], to isolate the other key analytic result of regularity structures – the *reconstruction theorem* [Hai14, Theorem 3.10] – as a standalone result in distribution theory, without any reference to the formalism of regularity structures. More precisely, consider a *germ*, i.e. a family $F = (F_x)_{x \in \mathbb{R}^d}$ of distributions indexed by \mathbb{R}^d , then the reconstruction theorem as presented in [CZ20, Theorem 5.1] states that under simple conditions on F named *coherence* and *homogeneity*, there exists a distribution $\mathcal{R}(F)$ which is best-approximated by the germ F in a suitable sense. See also [HL17; RS21; BL22; ZK21] for similar results of reconstruction in this direction.

The purpose of the present paper is to revisit Hairer’s multilevel Schauder estimates in the light of [CZ20], with minimal references to the formalism of regularity structures. Our first main result concerns integration of coherent and homogeneous germs with respect to a regularizing kernel, and can be loosely stated as follows.

Theorem 4.1.1. *Let F be an (α, γ) -coherent germ. Let K be a β -regularising kernel. Then the germ*

$$\mathcal{K}^{\gamma+\beta}(F)_x := \mathsf{K} * F_x - \sum_{|k| < \gamma+\beta} \frac{(\mathsf{K} * \{F_x - \mathcal{R}(F)\})^{(k)}(x)}{k!} (\cdot - x)^k,$$

*is well-defined, $((\alpha + \beta) \wedge 0, \gamma + \beta)$ -coherent, and $\mathcal{R}(\mathcal{K}^{\gamma+\beta}(F)) = \mathsf{K} * \mathcal{R}(F)$.*

If furthermore F is $\bar{\alpha}$ -homogeneous, then $\mathcal{K}^{\gamma+\beta}(F)$ is $(\bar{\alpha} + \beta) \wedge 0$ -homogeneous.

The convolution map $\mathcal{K}^{\gamma+\beta}$ is furthermore continuous for the natural topologies, see Theorem 4.3.12 for a precise formulation of the result.

However, this result in itself does not recover the multilevel Schauder estimates of regularity structures. The reason is that modelled distributions encode more structural constraints than coherent and homogeneous germs: notably, they arise as (finite) linear combinations of basis germs $(\Pi^i)_{i \in I}$ which can be “reexpanded” by a certain operator Γ . The data of $M = (\Pi, \Gamma)$ is called a model in the terminology of regularity structures and naturally induce spaces \mathcal{D}_M^γ of functions f such that the germ $\langle f, M \rangle_x := \sum_{i \in I} f^i(x) \Pi_x^i$ is α -homogeneous and (α, γ) -coherent, see section 4.4.1 below for more precise definitions.

As the second main result of the present paper, we propose a proof of Hairer’s multilevel Schauder estimates [Hai14, Theorem 5.12], which can loosely be stated as follows.

Theorem 4.1.2 ([Hai14, Theorem 5.12]). *Let M be a model and $f \in \mathcal{D}_M^\gamma$ be a modelled distribution. Let K be a β -regularising kernel. Then there exists a new (explicit) model \hat{M} and a new (explicit) modelled distribution $\hat{f} \in \mathcal{D}_{\hat{M}}^{\gamma+\beta}$ such that $\mathcal{R}(\hat{f}) = K * \mathcal{R}(f)$.*

Furthermore, the maps $M \mapsto \hat{M}$, $f \mapsto \hat{f}$ are continuous in natural topologies, and $\langle \hat{f}, \hat{M} \rangle = \mathcal{K}^{\gamma+\beta}(\langle f, M \rangle)$, see Theorem 4.4.10 below for a precise formulation of the result. In fact, it turns out that there are some small differences between [Hai14, Theorem 5.12] and our Theorem 4.4.10 below, which we highlight now:

1. [Hai14, Theorem 5.12] is established under the assumption that $K = \sum_n K_n$ with $\int K_n(x, y) y^k dy = 0$ for all $|k| \leq \gamma$. We replace this constraint by the slightly more general assumption that $\int K_n(x, y) y^k dy$ is polynomial of degree $\leq |k|$ for all $|k| \leq \gamma$, see assumption 4.2.6 below. This is an improvement as this new assumption is automatically satisfied if K is translation invariant i.e. $K(x, y) = K(y - x)$.
2. Our definition of model slightly differs from the one in [Hai14], as we impose neither a group property, a Hölder bound, nor a triangular structure, on the reexpansion operator Γ , see remark 4.4.2. However, we prove that those properties are preserved by the operation $M \mapsto \hat{M}$, see theorem 4.4.13.
3. We define the pointwise evaluations $(K * \{F_x - \mathcal{R}(F)\})^{(k)}(x)$ appearing in the convolution map in a different, more canonical way. Indeed, while in [Hai14] this quantity is defined as $\sum_n \{F_x - \mathcal{R}(F)\}(\partial_1^k K_n(x, \cdot))$, seemingly relying on the choice of the decomposition $K = \sum_n K_n$, we show that it can be defined in a more general way as the limit $\lim_{\lambda \rightarrow 0} (K * \{F_x - \mathcal{R}(F)\})^{(k)}(\eta_x^\lambda)$ where η denotes any (suitable) mollifier, see Lemma 4.3.11.

This paper is organised as follows. In section 4.2, we recall notations and classical results. In section 4.3, we present our first main result of convolution of coherent and homogeneous germs, Theorem 4.3.12. In section 4.4, we present our second main result of convolution of modelled distributions, Theorem 4.4.10. In section 4.5, we prove Theorem 4.3.12. In section 4.6, we prove Theorem 4.4.10.

4.2 Classical results (revisited)

We will work in \mathbb{R}^d where $d \geq 1$ is a fixed integer.

4.2.1 Test functions

For a test-function $\varphi \in \mathcal{D}(\mathbb{R}^d)$, $x \in \mathbb{R}^d$, $\lambda > 0$ we denote by φ_x^λ its scaled and centered version: $\varphi_x^\lambda(\cdot) := \lambda^{-d} \varphi(\lambda^{-1}(\cdot - x))$. Note that $\int \varphi_x^\lambda = \int \varphi$.

Given $r \in \mathbb{N}_0$, we denote by C^r the space of functions $\varphi : \mathbb{R}^d \rightarrow \mathbb{R}$ which admit partial derivatives of order k for all $|k| \leq r$. The corresponding norm is

$$\|\varphi\|_{C^r} := \max_{|k| \leq r} \|\partial^k \varphi\|_\infty,$$

where for a multi-index $k \in \mathbb{N}_0^d$ we set $|k| = k_1 + \dots + k_d$. Similarly, given $r, m \in \mathbb{N}_0$, we denote by $C^{m,r}$ the space of functions $\psi : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$ which admit partial derivatives of order (k_1, k_2) for all multi-indices $|k_1| \leq m, |k_2| \leq r$.

4.2.2 Hölder-Zygmund spaces

For $\gamma \in \mathbb{R}$ we denote by $\mathcal{Z}^\gamma := \mathcal{B}_{\infty, \infty, \text{loc}}^\gamma$ the (local) Hölder-Zygmund spaces, see [FH20, Section 14.3], which coincide with the usual (local) Hölder-Besov spaces \mathcal{C}^γ when γ is not an integer. To recall their definition, we first introduce for $r \in \mathbb{N}_0$ and $\gamma \in \mathbb{R}$ the families of test-functions

$$\begin{aligned} \mathcal{B}^r &:= \left\{ \varphi \in \mathcal{D}(B(0,1)) : \|\partial^k \varphi\|_\infty \leq 1 \text{ for all } 0 \leq |k| \leq r \text{ (i.e. } \|\varphi\|_{C^r} \leq 1) \right\}, \\ \mathcal{B}_\gamma &:= \left\{ \varphi \in \mathcal{D}(B(0,1)) : \int_{\mathbb{R}^d} \varphi(z) z^k dz = 0 \text{ for all } 0 \leq |k| \leq \gamma \right\}, \end{aligned} \quad (4.2.1)$$

and we denote their intersection by

$$\mathcal{B}_\gamma^r := \mathcal{B}^r \cap \mathcal{B}_\gamma. \quad (4.2.2)$$

Note that we have $\mathcal{B}_\gamma^r = \mathcal{B}_m^r$ where $m = \lfloor \gamma \rfloor$ is the largest integer $m \leq \gamma$. Also note that for $\gamma < 0$ the constraint $0 \leq |k| \leq \gamma$ is empty and we have $\mathcal{B}_\gamma^r = \mathcal{B}^r$.

We can now define the spaces \mathcal{Z}^γ . Note that for $\gamma < 0$ we denote by $r = \lfloor -\gamma + 1 \rfloor$ the smallest positive integer $r > -\gamma$.

Definition 4.2.1 (Hölder-Zygmund spaces \mathcal{Z}^γ). *Let $\gamma \in \mathbb{R}$, we define \mathcal{Z}^γ as the set of distributions $f \in \mathcal{D}'(\mathbb{R}^d)$ such that*

$$\|f\|_{\mathcal{Z}_{K, \bar{\lambda}}^\gamma} < +\infty$$

for all compacts $K \subset \mathbb{R}^d$ and for some (hence any) $\bar{\lambda} \in (0, \infty)$, where

$$\|f\|_{\mathcal{Z}_{K, \bar{\lambda}}^\gamma} = \begin{cases} \sup_{\substack{x \in K, \lambda \in (0, \bar{\lambda}], \\ \varphi \in \mathcal{B}^r \text{ with } r = \lfloor -\gamma + 1 \rfloor}} \frac{|f(\varphi_x^\lambda)|}{\lambda^\gamma} & \text{if } \gamma < 0, \\ \sup_{\substack{x \in K \\ \varphi \in \mathcal{B}^0}} |f(\varphi_x)| + \sup_{\substack{x \in K, \lambda \in (0, \bar{\lambda}], \\ \varphi \in \mathcal{B}_\gamma^0}} \frac{|f(\varphi_x^\lambda)|}{\lambda^\gamma} & \text{if } \gamma \geq 0. \end{cases} \quad (4.2.3)$$

We often set $\bar{\lambda} = 1$ and omit it from notation.

4.2.3 Singular kernels

Intuitively, we consider kernels $\mathbf{K}(x, y)$ such that

$$|\mathbf{K}(x, y)| \lesssim \frac{1}{|y - x|^{d-\beta}} \mathbb{1}_{\{|y-x| \leq c\}} \quad \text{for some } \beta, c > 0 \quad (4.2.4)$$

We call such kernels β -regularising, for reasons that will soon be clear. In practice, it is convenient to assume a suitable decomposition for $\mathbf{K}(x, y)$ inspired by (4.2.4), as in [Hai14, Assumption 5.1]. Then we discuss the precise relation with (4.2.4).

Definition 4.2.2 (Regularising kernel). *Let $\beta > 0$, $m, r \in \mathbb{N}_0$. A function $\mathsf{K}: \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$ is called β -regularising kernel of order (m, r) if one can write*

$$\mathsf{K}(x, y) = \sum_{n=0}^{+\infty} \mathsf{K}_n(x, y) \quad \text{for a.e. } x, y \in \mathbb{R}^d, \quad (4.2.5)$$

where K_n are measurable functions satisfying the following properties: there are constants c and c_K , depending on compact sets $K \subset \mathbb{R}^d$, such that, for all $n \in \mathbb{N}_0$:

1. $\text{supp}(\mathsf{K}_n) \subset \{(x, y) : c^{-1} 2^{-n} \leq |y - x| \leq c 2^{-n}\}$;
2. for $k, l \in \mathbb{N}_0^d$ with $|k| \leq m$, $|l| \leq r$ we have, for $x, y \in K$,

$$\left| \partial_1^k \partial_2^l \mathsf{K}_n(x, y) \right| \leq c_K 2^{(d-\beta+|l|+|k|)n}; \quad (4.2.6)$$

3. for $k, l \in \mathbb{N}_0^d$ with $|k| \leq m$, $|l| \leq r$ we have, for $x, y \in K$,

$$\left| \int_{\mathbb{R}^d} (y - x)^l \partial_2^k \mathsf{K}_n(x, y) dx \right| \leq c_K 2^{-\beta n}. \quad (4.2.7)$$

Let us show that these assumptions are closely related to (4.2.4) and they are less restrictive than they might appear.

Remark 4.2.3 (Singular kernels). *Let K be a kernel which satisfies (4.2.4), and an analogous estimate for derivatives of order $k, l \in \mathbb{N}_0^d$ with $|k| \leq m$, $|l| \leq r$:*

$$\left| \partial_1^k \partial_2^l \mathsf{K}(x, y) \right| \lesssim \frac{1}{|y - x|^{d-\beta+|l|+|k|}} \mathbb{1}_{\{|y-x| \leq c\}},$$

uniformly for x, y in compact sets. Then K satisfies assumptions (1) and (2) above: it suffices to multiply K by a dyadic partition of unity.

Remark 4.2.4 (Translation invariance, I). *If the kernel K_n satisfies (1) and (2) above, assumption (3) is easily seen to hold for $|l| \geq |k|$. The issue is whether (3) is satisfied for $|l| < |k|$. This holds automatically in the translation invariant case:*

$$\forall n \in \mathbb{N}_0 : \quad \mathsf{K}_n(x, y) = \mathsf{K}_n(y - x) \quad \forall x, y \in \mathbb{R}^d, \quad (4.2.8)$$

because the integral in the l.h.s. of (4.2.7) vanishes for $|l| < |k|$, as one sees through integration by parts, since $\partial_2^k \mathsf{K}_n = (-1)^{|k|} \partial_1^k \mathsf{K}$ and $\partial_x^k (y - x)^l = 0$.

Remark 4.2.5. *If $\mathsf{K}: \mathbb{R}^d \setminus \{0\} \rightarrow \mathbb{R}$ is a function admitting the scaling property $\mathsf{K}(x/\lambda) = \lambda^{d-\beta} \mathsf{K}(x)$ for all $\lambda > 0$, then – modulo a smooth function – the kernel $(x, y) \mapsto \mathsf{K}(y - x)$ is β -regularising for any (m, r) , see [Hai14, Lemma 5.5] for a precise statement.*

Examples of kernels falling in this situation include the Heat kernel, the Green's function of usual differential operators with constant coefficients, the Green's function of the fractional Laplacian [BK17], etc.

In some cases, we will require a last assumption on the kernel.

Assumption 4.2.6 (Preserving polynomials). *Let $\mathsf{K}: \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$ admit a decomposition $\mathsf{K} = \sum_{n=0}^{+\infty} \mathsf{K}_n$ as in (4.2.5). For $\gamma \in \mathbb{R}$, we say that K preserves polynomials at level γ if, for every $n \in \mathbb{N}_0$ and for all $k \in \mathbb{N}_0^d$ with $0 \leq |k| \leq \gamma$,*

$$x \mapsto \int_{\mathbb{R}^d} \mathsf{K}_n(x, y) y^k dy \quad \text{is a polynomial of degree } \leq |k|. \quad (4.2.9)$$

This condition is automatically satisfied for $\gamma < 0$. For $\gamma \geq 0$, it is not so restrictive.

Remark 4.2.7 (Translation invariance, II). *A sufficient condition for (4.2.9) is that*

$$\int_{\mathbb{R}^d} \mathbf{K}_n(x, y) (y - x)^k \, dy \quad \text{does not depend on } x.$$

This condition clearly holds if the kernels \mathbf{K}_n are translation invariant, see (4.2.8), in which case Assumption 4.2.6 is satisfied at any level γ .

Remark 4.2.8. *In [Hai14, Assumption 5.4] a much stronger assumption is made, namely that for all multi-indices k with $|k| \leq \gamma$ and any $n \in \mathbb{N}_0$,*

$$\forall x \in \mathbb{R}^d : \quad \int_{\mathbb{R}^d} \mathbf{K}_n(x, y) y^k \, dy = 0.$$

4.2.4 Singular convolution and classical Schauder estimate

The convolution of a distribution $f \in \mathcal{D}'(\mathbb{R}^d)$ with a kernel \mathbf{K} is formally defined by

$$(\mathbf{K} * f)(x) := f(\mathbf{K}(x, \cdot)) = \int_{\mathbb{R}^d} \mathbf{K}(x, y) f(dy),$$

which makes sense when $\mathbf{K}(x, \cdot)$ is regular enough. If, on the other hand, \mathbf{K} is singular, then one expects $\mathbf{K} * f$ to be a distribution, defined for a test function $\psi \in \mathcal{D}(\mathbb{R}^d)$ by

$$(\mathbf{K} * f)(\psi) := f(\mathbf{K}^* \psi) \quad \text{where} \quad (\mathbf{K}^* \psi)(y) := \int_{\mathbb{R}^d} \psi(x) \mathbf{K}(x, y) \, dx,$$

provided $\mathbf{K}^* \psi$ is regular enough, so that $f(\mathbf{K}^* \psi)$ makes sense.

Let us now assume that \mathbf{K} satisfies the conditions in Definition 4.2.2. From (4.2.6) and Fubini's theorem we can formally write

$$(\mathbf{K}^* \psi)(y) = \sum_{n \in \mathbb{N}_0} (\mathbf{K}_n^* \psi)(y) \quad \text{where} \quad (\mathbf{K}_n^* \psi)(y) := \int_{\mathbb{R}^d} \psi(x) \mathbf{K}_n(x, y) \, dx,$$

and note that $\mathbf{K}_n^* \psi \in \mathcal{D}$ is a genuine test function, for any $n \in \mathbb{N}_0$. We can thus define the convolution of a distribution $f \in \mathcal{D}'(\mathbb{R}^d)$ with \mathbf{K} by setting, for $\psi \in \mathcal{D}(\mathbb{R}^d)$,

$$(\mathbf{K} * f)(\psi) := \sum_{n \in \mathbb{N}_0} f(\mathbf{K}_n^* \psi), \tag{4.2.10}$$

as soon as this sum converges. A sufficient condition for this convolution to be well-posed is given by the following result.

Proposition 4.2.9 (Singular convolution). *Let $\beta > 0$ and $r \in \mathbb{N}_0$. If \mathbf{K} is a β -regularising kernel of order $(0, r)$, and f is any distribution of order at most r , the convolution $\mathbf{K} * f$ is well-defined by (4.2.10) as a distribution of order at most r .*

We can now show that the convolution by a β -regularising kernel \mathbf{K} improves the Hölder regularity of a distribution by β : this result is known as the classical Schauder estimate and can be stated as follows (see also [FH20, Theorem 14.17]).

Theorem 4.2.10 (Classical Schauder estimate). *Let $\gamma \in \mathbb{R}$. Let \mathbf{K} be a β -regularising kernel of order (m, r) , where $\beta > 0$ and $m, r \in \mathbb{N}_0$ satisfy:*

$$m > \gamma + \beta, \quad r > -\gamma.$$

Also assume (if $\gamma \geq 0$) that \mathbf{K} preserves polynomials at level γ , see Assumption 4.2.6. Then, the convolution by \mathbf{K} defines a continuous linear map from \mathcal{Z}^γ to $\mathcal{Z}^{\gamma+\beta}$.

We will prove Proposition 4.2.9 and Theorem 4.2.10 in Section 4.5.3 below.

4.3 Main result I: Schauder estimate for germs

4.3.1 Germs and reconstruction

Our first goal is to extend Theorem 4.2.10 in the context of germs, that is, families of distributions indexed by \mathbb{R}^d .

Definition 4.3.1 (Germs). *A germ is a family $F = (F_x)_{x \in \mathbb{R}^d}$ of distributions $F_x \in \mathcal{D}'(\mathbb{R}^d)$, such that for all $\varphi \in \mathcal{D}(\mathbb{R}^d)$, the map $x \mapsto F_x(\varphi)$ is measurable.*

We will denote \mathcal{G} the vector space of germs. In general, we will see a germ $F \in \mathcal{G}$ as a family of local approximations of a global distribution f . The reconstruction problem, i.e. the problem of constructing a suitable f from F , has been previously considered in a number of different contexts, see [Hai14; CZ20]. In [CZ20], it was established that this reconstruction can be performed under the assumption that F satisfies properties named coherence and homogeneity, which we recall now.

Definition 4.3.2 (Coherence and homogeneity [CZ20]). *Let $\bar{\alpha}, \alpha, \gamma \in \mathbb{R}$ with $\bar{\alpha}, \alpha \leq \gamma$. Let $r_{\bar{\alpha}, \alpha} \in \mathbb{N}_0$ be the smallest non-negative integer $r > \max\{-\bar{\alpha}, -\alpha\}$.*

We say that a germ F is (α, γ) -coherent with homogeneity $\bar{\alpha}$ if it satisfies the following two conditions, for any given compact $K \subset \mathbb{R}^d$, with $r = r_{\bar{\alpha}, \alpha}$:

1. (homogeneity) uniformly over $x \in K$, $\lambda \in (0, 1]$, $\varphi \in \mathcal{B}^r$:

$$|F_x(\varphi_x^\lambda)| \lesssim \lambda^{\bar{\alpha}}; \quad (4.3.1)$$

2. (coherence) uniformly over $x, y \in K$, $\lambda \in (0, 1]$, $\varphi \in \mathcal{B}^r$:

$$|(F_y - F_x)(\varphi_x^\lambda)| \lesssim \lambda^\alpha (|y - x| + \lambda)^{\gamma - \alpha}. \quad (4.3.2)$$

The vector space of such germs will be denoted by $\mathcal{G}^{\bar{\alpha}; \alpha, \gamma}$, or simply by \mathcal{G}^γ .

Remark 4.3.3 (Monotonicity). *For $\bar{\alpha}' \geq \bar{\alpha}$, $\alpha' \geq \alpha$, $\gamma' \geq \gamma$ we have $\mathcal{G}^{\bar{\alpha}'; \alpha', \gamma'} \subseteq \mathcal{G}^{\bar{\alpha}; \alpha, \gamma}$.*

Remark 4.3.4 (Uniformity). *The uniformity of (4.3.1) and (4.3.2) over $\varphi \in \mathcal{B}^r$ needs not be required beforehand. Indeed, leaving aside the case $\gamma = 0$ for simplicity, it was shown in [CZ20, Propositions 13.1 and 13.2] that if (4.3.1) and (4.3.2) hold for a single (arbitrary) test function $\varphi \in \mathcal{D}$ with $\int \varphi \neq 0$, then they actually hold uniformly over $\varphi \in \mathcal{B}^r$, for an arbitrary choice¹ of $r > \max\{-\bar{\alpha}, -\alpha\}$.*

Remark 4.3.5 (General scales). *For coherent and homogeneous germs $F \in \mathcal{G}^{\bar{\alpha}; \alpha, \gamma}$, relations (4.3.1) and (4.3.2) hold uniformly for $\lambda \in (0, \bar{\lambda}]$, for any given $\bar{\lambda} \in (1, \infty)$. To this goal, it is enough to show that $|F_y(\varphi_x^\lambda)| \lesssim 1$ uniformly in $\lambda \in [1, \bar{\lambda}]$, $x, y \in K$, $\varphi \in \mathcal{B}^r$ (because for $\lambda \in [1, \bar{\lambda}]$ we have $1 \lesssim \lambda^\alpha$ and also $1 \lesssim \lambda^\alpha (|y - x| + \lambda)^{\gamma - \alpha}$).*

By (4.3.1) and (4.3.2) we can bound, uniformly over $x, y \in K$, $\lambda \in (0, 1]$ and $\varphi \in \mathcal{B}^r$,

$$|F_y(\varphi_x^\lambda)| \leq |(F_y - F_x)(\varphi_x^\lambda)| + |F_x(\varphi_x^\lambda)| \lesssim \lambda^\alpha + \lambda^{\bar{\alpha}} \lesssim \lambda^{\min\{\bar{\alpha}, \alpha\}}. \quad (4.3.3)$$

If $\lambda \in [1, \bar{\lambda}]$, then φ_x^λ may not be as in (4.3.3), however (using for instance a partition of unity) one can perform a decomposition $\varphi_x^\lambda = \sum_{k=1}^n (\hat{\varphi}_k)_x^1$ where $x_k \in K$, $\hat{\varphi}_k \in \mathcal{B}^r$, and n is bounded by a value depending only on $\bar{\lambda}$ and of the compact K . It follows that relation (4.3.3) holds uniformly for $\lambda \in (0, \bar{\lambda}]$, in particular $|F_y(\varphi_x^\lambda)| \lesssim 1$ as claimed.

¹The proof in [CZ20, Propositions 13.1] requires $\alpha \leq 0$, however when $\alpha > 0$ one can show that a (α, γ) -coherent germ must be constant, hence the conclusion still holds.

Remark 4.3.6. Let us introduce the semi-norms corresponding to (4.3.1) and (4.3.2): given a compact set $K \subset \mathbb{R}^d$, $r \in \mathbb{N}$ and $\bar{\lambda} \in (0, \infty)$, we set

$$\|F\|_{\mathcal{G}_{\text{hom};K,\bar{\lambda},r}^{\bar{\alpha}}} := \sup_{\substack{x \in K, \lambda \in (0, \bar{\lambda}] \\ \varphi \in \mathcal{B}^r}} \frac{|F_x(\varphi_x^\lambda)|}{\lambda^{\bar{\alpha}}}, \quad (4.3.4)$$

$$\|F\|_{\mathcal{G}_{\text{coh};K,\bar{\lambda},r}^{\alpha,\gamma}} := \sup_{\substack{x,y \in K, \lambda \in (0, \bar{\lambda}] \\ \varphi \in \mathcal{B}^r}} \frac{|(F_y - F_x)(\varphi_x^\lambda)|}{\lambda^\alpha (|y - x| + \lambda)^{\gamma - \alpha}}, \quad (4.3.5)$$

and define the joint semi-norm, with $r = r_{\bar{\alpha}, \alpha}$, as in Definition 4.3.2,

$$\|F\|_{\mathcal{G}_{K,\bar{\lambda}}^{\bar{\alpha};\alpha,\gamma}} := \|F\|_{\mathcal{G}_{\text{hom};K,\bar{\lambda},r_{\bar{\alpha},\alpha}}^{\bar{\alpha}}} + \|F\|_{\mathcal{G}_{\text{coh};K,\bar{\lambda},r_{\bar{\alpha},\alpha}}^{\alpha,\gamma}}. \quad (4.3.6)$$

We usually fix $\bar{\lambda} = 1$ and omit it from notation. In particular, a germ F is $\mathcal{G}^{\bar{\alpha};\alpha,\gamma}$ if and only if $\|F\|_{\mathcal{G}_K^{\bar{\alpha};\alpha,\gamma}} < \infty$ for any compact set $K \subset \mathbb{R}^d$.

Now we can state the reconstruction theorem. The following result is a reformulation of [CZ20, Theorem 5.1] in the case of a nonzero exponent γ (we exclude the case $\gamma = 0$ for simplicity, as this case requires to introduce a logarithmic term), see also [ZK21].

Theorem 4.3.7 (Reconstruction for $\gamma \neq 0$). *Let $\alpha, \gamma \in \mathbb{R}$ with $\alpha \leq \gamma$ and assume that $\gamma \neq 0$. For any germ F which is (α, γ) -coherent, there exists a distribution $\mathcal{R}(F) \in \mathcal{D}'(\mathbb{R}^d)$ which is “locally approximated by F ” in the following sense: for any integer $r \in \mathbb{N}_0$ with $r > -\alpha$ and any compact $K \subset \mathbb{R}^d$ we have*

$$|(F_x - \mathcal{R}(F))(\varphi_x^\lambda)| \lesssim \lambda^\gamma \quad \text{uniformly over } \lambda \in (0, 1], x \in K, \varphi \in \mathcal{B}^r. \quad (4.3.7)$$

Such a distribution $\mathcal{R}(F)$ is unique if and only if $\gamma > 0$; and for any γ one can construct it in such a way that $F \mapsto \mathcal{R}(F)$ is linear.

If furthermore, F has homogeneity $\bar{\alpha} \leq \gamma$, then $\mathcal{R}(F) \in \mathcal{Z}^{\bar{\alpha}}$. That is, \mathcal{R} defines a linear map

$$\mathcal{R}: \mathcal{G}^{\bar{\alpha};\alpha,\gamma} \rightarrow \mathcal{Z}^{\bar{\alpha}}.$$

Remark 4.3.8 (Reconstruction bounds). *Note that since F is (α, γ) -coherent, the bound (4.3.7) shows that $F - \mathcal{R}(F) = (F_x - \mathcal{R}(F))_{x \in \mathbb{R}^d} \in \mathcal{G}^{\gamma;\alpha,\gamma}$, more precisely*

$$\|F - \mathcal{R}(F)\|_{\mathcal{G}_{K,\bar{\lambda}}^{\gamma;\alpha,\gamma}} \lesssim \|F\|_{\mathcal{G}_{\text{coh};K',1}^{\alpha,\gamma}},$$

for the enlarged compact $K' = K \oplus B(0, \bar{\lambda})$.

If furthermore F has homogeneity $\bar{\alpha}$, then

$$\|\mathcal{R}(F)\|_{\mathcal{Z}_{K,\bar{\lambda}}^{\bar{\alpha}}} \lesssim \|F\|_{\mathcal{G}_{K',1}^{\bar{\alpha};\alpha,\gamma}}.$$

Remark 4.3.9. *If $\bar{\alpha} > 0$ then $\mathcal{R} = 0$. Indeed, it follows by (4.3.1) and (4.3.7) that we have $|\mathcal{R}(F)(\varphi_x^\lambda)| \lesssim \lambda^{\bar{\alpha}} + \lambda^\gamma \lesssim \lambda^{\bar{\alpha}}$, since $\bar{\alpha} \leq \gamma$. If $\bar{\alpha} > 0$, this implies that $\mathcal{R}(F) = 0$.*

4.3.2 Schauder estimate for coherent germs

A natural and interesting problem is to find a “nice” continuous linear map \mathcal{K} which “lifts the convolution with K on the space of coherent and homogeneous germs”. To be more

precise, given $\bar{\alpha}, \alpha, \gamma \in \mathbb{R}$, we are looking for a continuous linear map \mathcal{K} such that the following diagram commutes, for suitable $\bar{\alpha}', \alpha', \gamma' \in \mathbb{R}$:

$$\begin{array}{ccc} \mathcal{G}^{\bar{\alpha}; \alpha, \gamma} & \xrightarrow{\mathcal{K}} & \mathcal{G}^{\bar{\alpha}'; \alpha', \gamma'} \\ \mathcal{R} \downarrow & & \downarrow \mathcal{R} \\ \mathcal{Z}^{\bar{\alpha}} & \xrightarrow{\mathcal{K} * \cdot} & \mathcal{Z}^{\bar{\alpha} + \beta} \end{array} \quad (4.3.8)$$

A naive guess would be to define $\mathcal{K}(F)_x$ as

$$\mathcal{K} * F_x \quad (4.3.9)$$

but this choice of germ is typically neither coherent nor homogeneous. However, it turns out that we can nicely modify (4.3.9) by subtracting a suitable polynomial term.

Remark 4.3.10. One “trivial” solution would be to define $\mathcal{K}(F)_x$ as

$$\mathcal{K} * \mathcal{R}(F).$$

However, such a germ is independent of x and does not contain F_x . This is not useful for applications, e.g. to stochastic equations, where one needs modelled distributions which do depend on x (see below), to reflect the local fluctuations of the noise.

As a first ingredient, we canonically define the “pointwise derivatives” of a germ, provided it is locally homogeneous on test functions which annihilate polynomials.

Lemma 4.3.11 (Pointwise derivatives). *Let $f \in \mathcal{D}'$ be a distribution. Assume that for some $x \in \mathbb{R}^d$ and $\delta > 0$ the following holds, uniformly for $\lambda \in (0, 1]$:*

$$|f(\varphi_x^\lambda)| \lesssim \lambda^\delta \quad \text{for any } \varphi \in \mathcal{B}_\delta \quad (4.3.10)$$

(we recall that functions in \mathcal{B}_δ annihilate polynomials up to degree δ , see (4.2.1)). Then, for any multi-index k with $0 \leq |k| < \delta$, we can define the pointwise derivative

$$f^{(k)}(x) := \lim_{\lambda \rightarrow 0} f^{(k)}(\eta_x^\lambda) \in \mathbb{R},$$

where $\eta \in \mathcal{D}$ is any test function with $\int \eta = 1$ and $\int \eta(x) x^l dx = 0$ for all $1 \leq |l| < \delta$ (the limit does not depend on the choice of η).

Consider now a coherent and homogeneous germ $F \in \mathcal{G}^{\bar{\alpha}; \alpha, \gamma}$ with $\gamma > 0$, so that the reconstruction $\mathcal{R}(F)$ is unique and the germ $F - \mathcal{R}(F) = (F_x - \mathcal{R}(F))_{x \in \mathbb{R}^d}$ has homogeneity and coherence exponents both equal to γ , see (4.3.7):

$$F - \mathcal{R}(F) = (F_x - \mathcal{R}(F))_{x \in \mathbb{R}^d} \in \mathcal{G}^{\gamma; \alpha, \gamma}.$$

Given a β -regularizing kernel \mathcal{K} , we will show that $\mathcal{K} * \{F_x - \mathcal{R}(F)\}$ satisfies (4.3.10) with $\delta = \gamma + \beta$, hence we can define $\mathcal{K} = \mathcal{K}^{\gamma + \beta}$ as follows:

$$\mathcal{K}^{\gamma + \beta}(F)_x := \mathcal{K} * F_x - \sum_{|k| < \gamma + \beta} \frac{(\mathcal{K} * \{F_x - \mathcal{R}(F)\})^{(k)}(x)}{k!} (\cdot - x)^k, \quad (4.3.11)$$

that is, we subtract a “Taylor-like polynomial” of $\mathcal{K} * \{F_x - \mathcal{R}(F)\}$ of order $\gamma + \beta$.

We can now state our first main result.

Theorem 4.3.12 (Schauder estimate for coherent germs). *Consider:*

- an (α, γ) -coherent germ F with $\gamma > 0$;
- a β -regularising kernel K of order (m, r) large enough:

$$m > \gamma + \beta, \quad r > -\alpha, \quad (4.3.12)$$

where $\beta > 0$ is such that

$$\alpha + \beta \neq 0, \quad \gamma + \beta \notin \mathbb{N},$$

Then, if we define $\mathcal{K}^{\gamma+\beta}(F)$ as in (4.3.11), we have that:

1. the germ $\mathcal{K}^{\gamma+\beta}(F)$ is well-defined and is $((\alpha + \beta) \wedge 0, \gamma + \beta)$ -coherent,
2. $\mathcal{R}(\mathcal{K}^{\gamma+\beta}(F)) = K * \mathcal{R}(F)$.

If furthermore F has homogeneity $\bar{\alpha}$ with $\bar{\alpha} + \beta \neq 0$, then $\mathcal{K}^{\gamma+\beta}(F)$ has homogeneity $(\bar{\alpha} + \beta) \wedge 0$.

In conclusion, if $F \in \mathcal{G}^{\bar{\alpha}; \alpha, \gamma}$, then $\mathcal{K}^{\gamma+\beta}(F) \in \mathcal{G}^{\bar{\alpha}'; \alpha', \gamma'}$, i.e. the diagram (4.3.8) commutes, with $(\bar{\alpha}', \alpha', \gamma') = ((\bar{\alpha} + \beta) \wedge 0, (\alpha + \beta) \wedge 0, \gamma + \beta)$; and furthermore the map $\mathcal{K}^{\gamma+\beta}$ is continuous.

4.4 Main result II: multi-level Schauder estimate

In some applications, the space $\mathcal{G}^{\bar{\alpha}; \alpha, \gamma}$ of all coherent and homogeneous germs is “too big”. This happens for instance when one wants to define singular operations on germs, such as the product with a non smooth function, or even a distribution: one can sometimes make sense of such a product for a restricted family of germs Π^i and the best one can hope is to extend it to those germs that locally look like the Π^i 's, in a suitable sense. This leads to the notion of *models* and *modelled distributions*, which are cornerstones of the theory of regularity structures [Hai14].

4.4.1 Models and modelled distributions

We fix a “basis” $\Pi = (\Pi^i)_{i \in I}$ of germs $\Pi^i = (\Pi_x^i)_{x \in \mathbb{R}^d}$ on \mathbb{R}^d , indexed by a finite set I . We look at germs $F = \langle f, \Pi \rangle$ given by linear combinations of the germs in Π with real coefficients $f^i(x)$, that is

$$F_x = \langle f, \Pi \rangle_x := \sum_{i \in I} f^i(x) \Pi_x^i. \quad (4.4.1)$$

We will call the basis Π a *model* and the family of coefficients f a *modelled distribution*, provided they satisfy assumptions that we now discuss.

To define a model $\Pi = (\Pi^i)_{i \in I}$, we require that each germ Π^i is *homogeneous* (i.e. it satisfies condition (4.3.1) for some exponent $\alpha_i \in \mathbb{R}$) and furthermore that the vector space $\text{Span}\{\Pi_x^i : i \in I\} \subset \mathcal{D}'(\mathbb{R}^d)$ does not depend on x (i.e. each distribution Π_x^i is a linear combination of $(\Pi_y^j)_{j \in I}$, for any $x, y \in \mathbb{R}^d$ and $i \in I$). This leads to:

Definition 4.4.1 (Model). *Fix a finite set I and a family $\alpha = (\alpha_i)_{i \in I}$ of real numbers. A pair $M = (\Pi, \Gamma)$ is called a model on \mathbb{R}^d with homogeneities α if there exists $r = r_\Pi \in \mathbb{N}_0$ such that:*

1. $\Pi = (\Pi^i)_{i \in I}$ is a family of germs on \mathbb{R}^d such that $\|\Pi^i\|_{\mathcal{G}_{\text{hom}; K, 1, r}^{\alpha_i}} < \infty$ for any $i \in I$ and $K \subset \mathbb{R}^d$ compact, that is

$$|\Pi_x^i(\varphi_x^\lambda)| \lesssim \lambda^{\alpha_i} \quad (4.4.2)$$

uniformly over $x \in K$, $\lambda \in (0, 1]$, $\varphi \in \mathcal{B}^r$;

2. $\Gamma = (\Gamma_{xy}^{ji})_{j,i \in I}$ for $x, y \in \mathbb{R}^d$ are real numbers such that, for all $i \in I$,

$$\Pi_y^i = \sum_{j \in I} \Pi_x^j \Gamma_{xy}^{ji}. \quad (4.4.3)$$

With an abuse of notation, we also call Π alone a model (as it often determines Γ). We denote by \mathcal{M}^α the class of models with homogeneities α and we set, see (4.3.4),

$$\|\Pi\|_{\mathcal{M}_{K,\bar{\lambda}}^\alpha} := \sup_{i \in I} \|\Pi^i\|_{\mathcal{G}_{\text{hom};K,\bar{\lambda},r_\Pi}^{\alpha_i}}. \quad (4.4.4)$$

Remark 4.4.2 (Models in Regularity Structures). *In our definition of a model we do not enforce the following requirements, which are present in Hairer's original definition [Hai14, Definition 2.17]:*

1. Group Property: $\Gamma_{xy} \Gamma_{yz} = \Gamma_{xz}$ (that is $\sum_{k \in I} \Gamma_{xy}^{jk} \Gamma_{yz}^{ki} = \Gamma_{xz}^{ji}$);
2. Triangular Structure: $\Gamma_{xy}^{ii} = 1, \Gamma_{xy}^{ij} = 0$ if $\alpha_i > \alpha_j$;
3. Hölder Bound: $|\Gamma_{xy}^{ij}| \lesssim |y - x|^{\alpha_j - \alpha_i}$.

Property (1) is natural, in view of (4.4.3) (indeed, when the Π^i 's are linearly independent, the coefficients Γ_{xy}^{ji} are univocally determined by (4.4.3) and (1) holds automatically). The role of the other properties (2) and (3) is discussed below, see Remark 4.4.5.

Example 4.4.3 (Polynomial model). *The simplest choice of a model is obtained taking as basis of germs the usual (normalized) monomials $\mathbb{X}^k = (\mathbb{X}_x^k)_{x \in \mathbb{R}^d}$, where*

$$\mathbb{X}_x^k := \frac{(\cdot - x)^k}{k!}, \quad k \in \mathbb{N}_0^d. \quad (4.4.5)$$

More precisely, if we fix any $\ell \in \mathbb{N}_0$, the polynomial model at level ℓ is defined by

$$\Pi_{\leq \ell}^{\text{poly}} := \{\Pi^k = \mathbb{X}^k\}_{k \in \mathbb{N}_0^d: |k| \leq \ell}.$$

It is an exercise to check that $\Pi_{\leq \ell}^{\text{poly}}$ is indeed a model, as in Definition 4.4.1, with

$$\alpha_k := |k|, \quad (\Gamma^{\text{poly}})_{xy}^{lk} := \frac{(x - y)^{k-l}}{(k-l)!} \mathbf{1}_{\{l \leq k\}}, \quad (4.4.6)$$

where by $l \leq k$ we mean $l_1 \leq k_1, l_2 \leq k_2, \dots, l_d \leq k_d$.

We next define modelled distributions. Consider a germ $F = \langle f, \Pi \rangle$ as in (4.4.1), for some model $\Pi = (\Pi^i)_{i \in I}$. Applying (4.4.3), for any $x, y \in \mathbb{R}^d$ we can write

$$F_y - F_x = \sum_{i \in I} \left\{ \sum_{j \in I} \Gamma_{xy}^{ij} f^j(y) - f^i(x) \right\} \Pi_x^i. \quad (4.4.7)$$

In order to ensure that F is coherent, it is natural to require scaling properties of the quantities in brackets. This leads to the following definition:

Definition 4.4.4 (Modelled distribution). *Consider a model $M = (\Pi, \Gamma)$ with homogeneities $\alpha = (\alpha_i)_{i \in I}$ and fix a real number $\gamma > \max\{\alpha_i: i \in I\}$.*

A function $f = (f^i(x))_{i \in I}: \mathbb{R}^d \rightarrow \mathbb{R}^I$ is called modelled distribution of order γ if for any compact set $K \subset \mathbb{R}^d$ we have, for $i \in I$ and uniformly for $x, y \in K$,

$$|f^i(x)| \lesssim 1 \quad \text{and} \quad \left| \sum_{j \in I} \Gamma_{xy}^{ij} f^j(y) - f^i(x) \right| \lesssim |y - x|^{\gamma - \alpha_i}.$$

We denote by $\mathcal{D}^\gamma = \mathcal{D}_M^\gamma = \mathcal{D}_{\Gamma, \alpha}^\gamma$ the space of modelled distributions of order γ , relative to a model $M = (\Pi, \Gamma)$ with homogeneities α . This is a vector space with a Fréchet structure through the semi-norms

$$\|f\|_{\mathcal{D}_K^\gamma} = \|f\|_{\mathcal{D}_{\Gamma, \alpha; K}^\gamma} := \sup_{x \in K, i \in I} |f^i(x)| + \sup_{x, y \in K, i \in I} \frac{\left| \sum_{j \in I} \Gamma_{xy}^{ij} f^j(y) - f^i(x) \right|}{|y - x|^{\gamma - \alpha_i}}. \quad (4.4.8)$$

Remark 4.4.5 (Modelled distributions in Regularity Structures). *Our definition of modelled distributions mimics Hairer's original one [Hai14, Definition 3.1]. The additional properties of the models enforced in [Hai14], see Remark 4.4.2, ensure that the spaces \mathcal{D}^γ contain non-zero elements $f \neq 0$ and that they are ordered. More precisely, given a model $M = (\Pi^i, \Gamma^{ji})_{i, j \in I}$ with index set I and homogeneities $\alpha = (\alpha_i)_{i \in I}$, let us fix $\gamma > \max\{\alpha_i : i \in I\}$ and, for $\gamma' < \gamma$, denote by $I' := \{i \in I : \alpha_i < \gamma'\}$ the truncation of I at level γ' . Then:*

- property (2) (triangular structure) ensures that the truncation $M' := (\Pi^i, \Gamma^{ji})_{i, j \in I'}$ is also a model;
- property (3) (Hölder bound) ensures that if $f = (f^i)_{i \in I} \in \mathcal{D}^\gamma$ is a modelled distribution (of order γ relative to M), the truncation $f' = (f^i)_{i \in I'} \in \mathcal{D}^{\gamma'}$ is also a modelled distribution (of order γ' relative to M').

Given a model (Π, Γ) and a related modelled distribution f , we now check that the germ $F = \langle f, \Pi \rangle$ in (4.4.1) is coherent and homogeneous, see [CZ20, Example 4.10].

Proposition 4.4.6 (Modelled distribution yield coherent germs). *Let (Π, Γ) be a model with homogeneities $\alpha = (\alpha_i)_{i \in I}$ and set $\bar{\alpha} := \min_{i \in I} \alpha_i$.*

For any modelled distribution f of order γ , the germ $F = \langle f, \Pi \rangle$ in (4.4.1) is γ -coherent, more precisely $(\bar{\alpha}, \gamma)$ -coherent with homogeneity $\bar{\alpha}$:

$$f \in \mathcal{D}^\gamma \implies F = \langle f, \Pi \rangle \in \mathcal{G}^{\bar{\alpha}; \bar{\alpha}, \gamma},$$

and the map $f \mapsto F = \langle f, \Pi \rangle$ is continuous:

$$\|F\|_{\mathcal{G}_{K, \bar{\lambda}}^{\bar{\alpha}; \bar{\alpha}, \gamma}} \leq |I| \|\Pi\|_{\mathcal{M}_{K, \bar{\lambda}}^\alpha} \|f\|_{\mathcal{D}_K^\gamma}. \quad (4.4.9)$$

Proof. By (4.3.4), the homogeneity semi-norm of F can be bounded by

$$\|F\|_{\mathcal{G}_{\text{hom}; K, \bar{\lambda}, r}^{\bar{\alpha}}} \leq |I| \sup_{x \in K, i \in I} |f^i(x)| \|\Pi^i\|_{\mathcal{G}_{\text{hom}; K, \bar{\lambda}, r}^{\alpha_i}} \leq |I| c_1^f \|\Pi\|_{\mathcal{M}_{K, \bar{\lambda}}^\alpha}, \quad (4.4.10)$$

where c_1^f denotes the first term in the r.h.s. of (4.4.8), see (4.4.4). Turning to coherence, by (4.4.7) we can bound, arguing as in [CZ20, Example 4.10],

$$|(F_y - F_x)(\varphi_x^\lambda)| \leq |I| c_2^f \|\Pi\|_{\mathcal{M}_{K, \bar{\lambda}}^\alpha} (\lambda + |y - x|)^{\gamma - \bar{\alpha}} \lambda^{\bar{\alpha}},$$

where c_2^f denotes the second term in the r.h.s. of (4.4.8). Then, by (4.3.5), we obtain

$$\|F\|_{\mathcal{G}_{\text{coh}; K, \bar{\lambda}, r}^{\bar{\alpha}, \gamma}} \leq |I| c_2^f \|\Pi\|_{\mathcal{M}_{K, \bar{\lambda}}^\alpha},$$

which together with (4.4.10) yields (4.4.9). \square

Example 4.4.7 (Polynomial modelled distributions). Let $f : \mathbb{R}^d \rightarrow \mathbb{R}$ be a function of class C^ℓ , for some $\ell \in \mathbb{N}_0$. Its Taylor polynomial of order ℓ based at x is

$$F_x(\cdot) := \sum_{|k| \leq \ell} \partial^k f(x) \mathbb{X}_x^k(\cdot),$$

where \mathbb{X}_x^k are normalized monomials, see (4.4.5). The germ $F = (F_x)_{x \in \mathbb{R}^d}$ of Taylor polynomials can be expressed as $F = \langle \mathbf{f}, \Pi_{\leq \ell}^{\text{poly}} \rangle$, see (4.4.1), where $\Pi_{\leq \ell}^{\text{poly}}$ is the polynomial model in Example 4.4.3 and $\mathbf{f} = (\mathbf{f}^k(x))_{|k| \leq \ell, x \in \mathbb{R}^d}$ is defined by $\mathbf{f}^k(x) := \partial^k f(x)$.

If the function f is Hölder continuous with exponent $\gamma \in (\ell, \ell + 1)$, it is an exercise to show that \mathbf{f} is a modelled distribution of order γ , see e.g. [CZ20, Example 4.11]. In particular, by Proposition 4.4.6, the germ $F = (F_x)_{x \in \mathbb{R}^d} = \langle \mathbf{f}, \Pi_{\leq \ell}^{\text{poly}} \rangle$ of Taylor polynomials of f is γ -coherent (more precisely: $(0, \gamma)$ -coherent with homogeneity 0).

4.4.2 Schauder estimate for modelled distributions

Given a model (Π, Γ) and a modelled distribution $f \in \mathcal{D}^\gamma$, by Proposition 4.4.6 we have that

$$F = \langle f, \Pi \rangle \text{ in (4.4.1) is } \gamma\text{-coherent.}$$

If $\gamma > 0$, we can apply our Schauder estimate in Theorem 4.3.12: given a β -regularising kernel K with $\beta > 0$, we have that

$$\mathcal{K}^{\gamma+\beta}(F) \text{ in (4.3.11) is } (\gamma + \beta)\text{-coherent} \quad \text{and} \quad \mathcal{R}(\mathcal{K}^{\gamma+\beta}(F)) = \mathsf{K} * \mathcal{R}(F).$$

Since the germ $F = \langle f, \Pi \rangle$ comes from a modelled distribution f , a natural question arises: do we have $\mathcal{K}^{\gamma+\beta}(F) = \langle \hat{f}, \hat{\Pi} \rangle$ for some model $\hat{\Pi}$ and modelled distribution \hat{f} ?

Our next main result shows that the answer is positive: see Theorem 4.4.10 below, which is a version of Hairer's *multi-level Schauder estimate* [Hai14, Theorem 5.12] in our context. We first need to define the extended model $(\hat{\Pi}, \hat{\Gamma})$ and the modelled distribution \hat{f} . A key role will be played by *polynomials*.

Remark 4.4.8 (Polynomials). Some germs in the model $\Pi = (\Pi^i)_{i \in I}$ may be monomials, see (4.4.5). It is then useful to write I as the disjoint union

$$I = I^{\text{sing}} \cup I^{\text{poly}}$$

where I^{poly} labels the monomial germs (possibly $I^{\text{poly}} = \emptyset$) and $I^{\text{sing}} := I \setminus I^{\text{poly}}$. Without loss of generality, we fix the natural parametrization

$$I^{\text{poly}} = \{k \in \mathbb{N}_0^d : |k| \leq \ell\} \quad \text{for some } \ell \in \mathbb{N}_0, \quad (4.4.11)$$

and we assume that $\Pi|_{I^{\text{poly}}} = \Pi_{\leq \ell}^{\text{poly}}$, see (4.4.6), that is

$$\text{for } k \in I^{\text{poly}} : \quad \Pi_x^k = \mathbb{X}_x^k, \quad \alpha_k = |k|, \quad \Gamma_{xy}^{lk} = (\Gamma^{\text{poly}})_{xy}^{lk} \mathbb{1}_{\{l \in I^{\text{poly}}\}}. \quad (4.4.12)$$

We make no assumptions on the germs Π^i for $i \in I^{\text{sing}}$. In concrete examples, they often consist of singular functions or distributions which play a special role in the problem under investigation, such as e.g. the noise in stochastic equations.

Extended model The extended model $\hat{\Pi} = (\hat{\Pi}_i)_{i \in \hat{I}}$ is labelled by a new set \hat{I} , obtained enlarging I^{poly} to include all multi-indexes of homogeneity up to $\gamma + \beta$:²

$$\hat{I} := \hat{I}^{\text{sing}} \cup \hat{I}^{\text{poly}} \quad \text{where} \quad \begin{cases} \hat{I}^{\text{sing}} := I^{\text{sing}}, \\ \hat{I}^{\text{poly}} := \{k \in \mathbb{N}_0^d : |k| < \gamma + \beta\} \supseteq I^{\text{poly}}. \end{cases} \quad (4.4.13)$$

The germs $\hat{\Pi}^i$ in the extended model are defined by

$$\hat{\Pi}_x^i := \begin{cases} \mathbb{K} * \Pi_x^a - \sum_{k \in \mathbb{N}_0^d : |k| < \alpha_a + \beta} (\mathbb{K} * \Pi_x^a)^{(k)}(x) \mathbb{X}_x^k & \text{if } i = a \in \hat{I}^{\text{sing}}, \\ \mathbb{X}_x^k & \text{if } i = k \in \hat{I}^{\text{poly}}, \end{cases} \quad (4.4.14)$$

with homogeneities $\hat{\alpha} = (\hat{\alpha}_i)_{i \in \hat{I}}$ given by

$$\hat{\alpha}_i := \begin{cases} \alpha_a + \beta & \text{if } i = a \in \hat{I}^{\text{sing}}, \\ |k| & \text{if } i = k \in \hat{I}^{\text{poly}}. \end{cases} \quad (4.4.15)$$

We will show that $\hat{\Pi}_i$ is well defined, thanks to Lemma 4.3.11, and it satisfies the homogeneity condition (4.4.2) with exponent $\hat{\alpha}_i$, provided we assume that $\hat{\alpha}_i \notin \mathbb{N}_0$.

We next define the coefficients $\hat{\Gamma} = (\hat{\Gamma}_{xy}^{ji})_{j,i \in \hat{I}}$. Using labels $a, b \in \hat{I}^{\text{sing}}$ and $k, l \in \hat{I}^{\text{poly}}$ for clarity, we have the triangular structure

$$\hat{\Gamma}_{xy}^{ji} := \begin{pmatrix} \Gamma_{xy}^{ba} & 0 \\ \dots & (\Gamma^{\text{poly}})_{xy}^{lk} \end{pmatrix} = \begin{cases} \Gamma_{xy}^{ba} & \text{if } (j, i) = (b, a) \in \hat{I}^{\text{sing}} \times \hat{I}^{\text{sing}}, \\ 0 & \text{if } (j, i) = (b, k) \in \hat{I}^{\text{sing}} \times \hat{I}^{\text{poly}}, \\ \dots & \text{if } (j, i) = (l, a) \in \hat{I}^{\text{poly}} \times \hat{I}^{\text{sing}}, \\ (\Gamma^{\text{poly}})_{xy}^{lk} & \text{if } (j, i) = (l, k) \in \hat{I}^{\text{poly}} \times \hat{I}^{\text{poly}}, \end{cases} \quad (4.4.16)$$

where Γ refers to the original model and Γ^{poly} refers to the polynomial model, see (4.4.6). It only remains to define $\dots = \hat{\Gamma}_{xy}^{la}$ for $l \in \hat{I}^{\text{poly}}$ and $a \in \hat{I}^{\text{sing}}$:

$$\hat{\Gamma}_{xy}^{la} := \sum_{\substack{j \in I: \\ \alpha_j + \beta > |l|}} (\mathbb{K} * \Pi_x^j)^{(l)}(x) \Gamma_{xy}^{ja} - \sum_{\substack{k \in \mathbb{N}_0^d: \\ k \geq l, |k| < \alpha_a + \beta}} (\mathbb{K} * \Pi_y^a)^{(k)}(y) (\Gamma^{\text{poly}})_{xy}^{lk}, \quad (4.4.17)$$

where we stress that the first sum runs over $j \in I = I^{\text{sing}} \cup I^{\text{poly}}$. (We also note that $\hat{\Gamma}_{xy}^{la} \neq 0$ only for $|l| \leq \max_{j \in I} \alpha_j + \beta$.)

We will check by direct computation that condition (4.4.3) in the definition of a model is satisfied by $\hat{\Pi}$ and $\hat{\Gamma}$, see Section 4.6.2.

Extended modelled distribution Given a modelled distribution $f = (f^i(x))_{i \in I}$ relative to the original model (Π, Γ) , we define for $i \in \hat{I} = \hat{I}^{\text{sing}} \cup \hat{I}^{\text{poly}}$

$$\hat{f}^i(x) := \begin{cases} f^a(x) & \text{if } i = a \in \hat{I}^{\text{sing}}, \\ \sum_{\substack{j \in I: \\ \alpha_j + \beta > |k|}} f^j(x) (\mathbb{K} * \Pi_x^j)^{(k)}(x) & \text{if } i = k \in \hat{I}^{\text{poly}}. \\ - (\mathbb{K} * \{\langle f, \Pi \rangle_x - \mathcal{R}\langle f, \Pi \rangle\})^{(k)}(x) & \end{cases} \quad (4.4.18)$$

We point out that the three lines in the r.h.s. of (4.4.18) correspond precisely to the three terms $\mathcal{I}, \mathcal{J}, \mathcal{N}$ in the setting of Regularity Structures, see [Hai14, (5.15)].

²One has $\hat{I}^{\text{poly}} \supseteq I^{\text{poly}}$ because $\gamma + \beta > \gamma > \max\{\alpha_i : i \in I\}$, see Definition 4.4.4.

Remark 4.4.9. For $t \in \mathbb{R}$, we define the restriction $\mathcal{Q}_{\leq t}f$ of a modelled distribution $f = (f^j(x))_{j \in I}$ where we only keep the components $f^j(x)$ with $\alpha_j \leq t$, that is

$$\mathcal{Q}_{\leq t}f(x) := (f^j(x) \mathbb{1}_{\{\alpha_j \leq t\}})_{j \in I}.$$

We can then rewrite (4.4.18) more compactly as follows:

$$\hat{f}^i(x) := \begin{cases} f^a(x) & \text{if } i = a \in \hat{I}^{\text{sing}}, \\ \left(\mathbb{K} * \left\{ \mathcal{R}\langle f, \Pi \rangle - \langle \mathcal{Q}_{\leq |k| - \beta} f, \Pi \rangle_x \right\} \right)^{(k)}(x) & \text{if } i = k \in \hat{I}^{\text{poly}}. \end{cases}$$

Multi-level Schauder estimate We can finally state our second main result.

Theorem 4.4.10 (Multi-level Schauder estimate). *Fix a model $M = (\Pi, \Gamma)$ with homogeneities $\alpha = (\alpha_i)_{i \in I}$ and a real number $\gamma > \max\{\alpha_i : i \in I\}$ with $\gamma > 0$. We decompose $I = I^{\text{sing}} \cup I^{\text{poly}}$, see (4.4.11)-(4.4.12).*

Let $f \in \mathcal{D}_M^\gamma$ be a modelled distribution of order γ relative to $M = (\Pi, \Gamma)$, hence

$$\langle f, \Pi \rangle := \left(\sum_{i \in I} f^i(x) \Pi_x^i \right)_{x \in \mathbb{R}^d} \quad \text{is a } \gamma\text{-coherent germ.}$$

Let \mathbb{K} be a β -regularising kernel of order (m, r) large enough:

$$m > \gamma + \beta, \quad r \geq r_\Pi,$$

with $\beta > 0$ such that

$$\alpha_i + \beta \notin \mathbb{N}_0 \quad \text{for } i \in I^{\text{sing}}, \quad \gamma + \beta \notin \mathbb{N}_0.$$

Assume that \mathbb{K} preserves polynomials at level ℓ , given in (4.4.11).

Then we can define:

- *a new model $\hat{M} = (\hat{\Pi}, \hat{\Gamma})$, see (4.4.14) and (4.4.16)-(4.4.17), indexed by \hat{I} in (4.4.13) with homogeneities $\hat{\alpha} = (\hat{\alpha}_i)_{i \in \hat{I}}$ in (4.4.15);*
- *a modelled distribution $\hat{f} \in \mathcal{D}_{\hat{M}}^{\gamma + \beta}$ of order $\gamma + \beta$ relative to $\hat{M} = (\hat{\Pi}, \hat{\Gamma})$, see (4.4.18), hence*

$$\langle \hat{f}, \hat{\Pi} \rangle := \left(\sum_{i \in \hat{I}} \hat{f}^i(x) \hat{\Pi}_x^i \right)_{x \in \mathbb{R}^d} \quad \text{is a } (\gamma + \beta)\text{-coherent germ};$$

such that the following equality holds, with $\mathcal{K}^{\gamma + \beta}$ as in (4.3.11):

$$\langle \hat{f}, \hat{\Pi} \rangle = \mathcal{K}^{\gamma + \beta} \langle f, \Pi \rangle. \quad (4.4.19)$$

In particular, by Theorem 4.3.12, we have

$$\mathcal{R} \langle \hat{f}, \hat{\Pi} \rangle = \mathbb{K} * \mathcal{R} \langle f, \Pi \rangle. \quad (4.4.20)$$

The proof of Theorem 4.4.10 is given in Section 4.6.2, and we proceed as follows.

- In Section 4.6.2, we prove that $\hat{M} = (\hat{\Pi}, \hat{\Gamma})$ is indeed a model: we first check the condition of reexpansion (4.4.3) for $\hat{\Pi}$ and $\hat{\Gamma}$ by a direct computation; then we show that each $\hat{\Pi}_x^i$ satisfies the homogeneity relation (4.4.2) with exponent $\hat{\alpha}_i$.
- In Section 4.6.2, we prove (4.4.19) by a simple calculation; then, as an immediate consequence of (4.4.19) and Theorem 4.3.12, relation (4.4.20) follows.

- In Section 4.6.2, we prove that \hat{f} is indeed a modelled distribution, and we deduce the continuity estimate (4.4.21).

Remark 4.4.11 (On the properties of \hat{f}). *We can rephrase Theorem 4.4.10 by stating that the map $\mathcal{K}^{\gamma+\beta}$ acting on germs can be lifted to a map $\hat{\mathcal{K}}^{\gamma+\beta} f$ acting on modelled distributions, defined by (4.4.18), so that the following diagram commutes:*

$$\begin{array}{ccc} \mathcal{D}_M^\gamma & \xrightarrow{\hat{\mathcal{K}}^{\gamma+\beta}} & \mathcal{D}_{\hat{M}}^{\gamma+\beta} \\ \langle \cdot, \Pi \rangle \downarrow & & \downarrow \langle \cdot, \hat{\Pi} \rangle \\ \mathcal{G}^\gamma & \xrightarrow{\mathcal{K}^{\gamma+\beta}} & \mathcal{G}^{\gamma+\beta} \end{array}$$

where we set $\mathcal{G}^\gamma := \mathcal{G}^{\bar{\alpha}; \bar{\alpha}, \gamma}$ and $\mathcal{G}^{\gamma+\beta} := \mathcal{G}^{(\bar{\alpha}+\beta) \wedge 0; (\bar{\alpha}+\beta) \wedge 0, \gamma+\beta}$ for short.

The map $f \mapsto \hat{f} = \hat{\mathcal{K}}^{\gamma+\beta} f$ is linear, and we will prove that it is continuous:

$$\|\hat{f}\|_{\mathcal{D}_{K'}^{\gamma+\beta}} \lesssim \|\Pi\|_{\mathcal{M}_{K'}^\alpha} \|f\|_{\mathcal{D}_K^\gamma}, \quad (4.4.21)$$

see (4.4.8) and (4.4.4), where the compact $K' \supset K$ in the r.h.s. depends only on the compact K : e.g., we can take $K' := K \oplus B(0, 1)$ to be the 1-enlargement of K .

4.4.3 Further properties

We observe that the space \mathcal{M}^α of models is not a vector space, despite the semi-norm like notation $\|\cdot\|_{\mathcal{M}_{K, \bar{\lambda}}^\alpha}$, see (4.4.4), because the coefficients Γ appearing in (4.4.3) depend on the model Π . Nevertheless, given two models $M_1 = (\Pi_1, \Gamma_1)$ and $M_2 = (\Pi_2, \Gamma_2)$ (with the same homogeneities $\alpha = (\alpha_i)_{i \in I}$ and the same value of $r = r_{\Pi_1} = r_{\Pi_2}$), we can consider the distance

$$\|\Pi_1 - \Pi_2\|_{\mathcal{M}_K^\alpha}$$

which is well defined by (4.4.4) (even though $\Pi_1 - \Pi_2$ needs not be a model).

We next compare two modelled distributions $f_1 \in \mathcal{D}_{M_1}^\gamma$ and $f_2 \in \mathcal{D}_{M_2}^\gamma$ of the same order γ , but relative to different models $M_1 = (\Pi_1, \Gamma_1)$ and $M_2 = (\Pi_2, \Gamma_2)$ (with the same homogeneities $\alpha = (\alpha_i)_{i \in I}$ and $r = r_{\Pi_1} = r_{\Pi_2}$), as in [Hai14, Remark 3.6]. To this purpose, we define for compacts $K \subset \mathbb{R}^d$ the distance

$$\begin{aligned} \|f_1; f_2\|_{\mathcal{D}_K^\gamma} &= \|f_1; f_2\|_{\mathcal{D}_{M_1, M_2; K}^\gamma} = \|f_1; f_2\|_{\mathcal{D}_{\Gamma_1, \Gamma_2, \alpha; K}^\gamma} \\ &:= \sup_{x \in K, i \in I} \left| f_1^i(x) - f_2^i(x) \right| \\ &\quad + \sup_{x, y \in K, i \in I} \frac{\left| \sum_{j \in I} \{(\Gamma_1)_{xy}^{ij} f_1^j(y) - (\Gamma_2)_{xy}^{ij} f_2^j(y)\} - (f_1^i(x) - f_2^i(x)) \right|}{|x - y|^{\gamma - \alpha_i}}. \end{aligned}$$

We can then improve the estimate (4.4.21), showing that the distance between \hat{f}_1 and \hat{f}_2 can be controlled by the distances between f_1 and f_2 and between the models Π_1 and Π_2 (provided $\|\Pi_i\|_{\mathcal{M}_{K'}^\alpha}$ and $\|f_i\|_{\mathcal{D}_{K'}^\gamma}$ are uniformly bounded). More precisely, the following local Lipschitz estimate holds.

Proposition 4.4.12 (Enhanced continuity). *The following bound holds:*

$$\|\hat{f}_1; \hat{f}_2\|_{\mathcal{D}_{K'}^{\gamma+\beta}} \lesssim \|\Pi_1\|_{\mathcal{M}_{K'}^\alpha} \|f_1; f_2\|_{\mathcal{D}_{K'}^{\gamma+\beta}} + \|\Pi_1 - \Pi_2\|_{\mathcal{M}_{K'}^\alpha} \|f_2\|_{\mathcal{D}_{K'}^\gamma},$$

for some enlarged compact $K' \supset K$ (e.g. we can take $K' = K \oplus B(0, 1)$).

We finally come back to the additional properties (1), (2), (3) of the coefficients Γ that one may require in a model $M = (\Pi, \Gamma)$, see Remark 4.4.2. We show that these properties are preserved when one considers the new model $\hat{M} = (\hat{\Pi}, \hat{\Gamma})$.

Theorem 4.4.13. *Fix a model (Π, Γ) , a real number $\gamma > 0$ and a β -regularizing kernel K which satisfy the assumptions of Theorem 4.4.10. Consider the new model $\hat{M} = (\hat{\Pi}, \hat{\Gamma})$, see (4.4.14) and (4.4.16)-(4.4.17).*

If any of the properties (1), (2), (3) is satisfied by Γ , the same property is satisfied by $\hat{\Gamma}$ (with respect to the homogeneities $\hat{\alpha} = (\hat{\alpha}_i)_{i \in \hat{I}}$ in (4.4.15)).

4.5 Proof of our Main Result I

The purpose of this section is to establish the Schauder estimates on coherent germs, Theorem 4.3.12. Rather than establishing Theorem 4.3.12 by direct calculation, we prefer to divide our proof into two steps.

1. We first establish that the operation of convolution $F_x \mapsto K * F_x$ maps the space of coherent and homogeneous germs $\mathcal{G}^{\bar{\alpha}; \alpha, \gamma}$ into a new space of *weakly* coherent and homogeneous germs, denoted by $\check{\mathcal{G}}^{\bar{\alpha} + \beta; \alpha + \beta, \gamma + \beta}$, for which the coherence and homogeneity conditions (4.3.2) and (4.3.3) hold for test-functions which annihilate *suitable polynomials*, in a manner which is reminiscent of Hölder-Zygmund spaces \mathcal{Z} , see Definition 4.2.1. This result is a direct generalisation of the classical Schauder estimate Theorem 4.2.10.
2. Then we prove that subtracting a suitable Taylor polynomial, as in (4.3.11), maps the space $\check{\mathcal{G}}^{\bar{\alpha}; \alpha, \gamma}$ of *weakly* coherent and homogeneous germs, *in the special case* $\gamma = \bar{\alpha}$, into the usual space $\mathcal{G}^{\bar{\alpha}; \alpha, \gamma}$ of coherent and homogeneous germs.

4.5.1 Weakly coherent and homogeneous germs

Let us define precisely the space of *weakly coherent and homogeneous* germs, generalising Definition 4.3.2. We recall that \mathcal{B}_δ^r denotes, for $r \in \mathbb{N}_0$ and $\delta \in \mathbb{R}$, the space of test functions $\varphi \in \mathcal{B}^r$ which annihilate polynomials of degree $\leq \delta$, see (4.2.2).

Definition 4.5.1 (Weak coherence and homogeneity). *Let $\bar{\alpha}, \alpha, \gamma \in \mathbb{R}$ with $\bar{\alpha}, \alpha \leq \gamma$. Let $r_{\bar{\alpha}, \alpha} \in \mathbb{N}_0$ be the smallest non-negative integer $r > \max\{-\bar{\alpha}, -\alpha\}$.*

We say that a germ F is weakly (α, γ) -coherent with homogeneity $\bar{\alpha}$ if it satisfies the following three conditions, for any given compact $K \subset \mathbb{R}^d$, with $r = r_{\bar{\alpha}, \alpha}$:

1. (weak homogeneity) uniformly over $x \in K$, $\lambda \in (0, 1]$, $\varphi \in \mathcal{B}_{\bar{\alpha}}^r$:

$$|F_x(\varphi_x^\lambda)| \lesssim \lambda^{\bar{\alpha}}; \quad (4.5.1)$$

2. (weak coherence) uniformly over $x, y \in K$, $\lambda \in (0, 1]$, $\varphi \in \mathcal{B}_\gamma^r$:

$$|(F_y - F_x)(\varphi_x^\lambda)| \lesssim \lambda^\alpha (|y - x| + \lambda)^{\gamma - \alpha}. \quad (4.5.2)$$

3. (boundedness) uniformly over $x, y \in K$ and $\varphi \in \mathcal{B}^r$:

$$|F_y(\varphi_x)| \lesssim 1. \quad (4.5.3)$$

The vector space of such germs will be denoted by $\check{\mathcal{G}}^{\bar{\alpha}; \alpha, \gamma}$.

Remark 4.5.2. Each condition (4.5.1), (4.5.2), (4.5.3) involves a different class of test functions: $\mathcal{B}_{\bar{\alpha}}^r$ for (4.5.1), \mathcal{B}_{γ}^r for (4.5.2), \mathcal{B}^r for (4.5.3). In the special case when $\bar{\alpha} < 0$ and $\gamma < 0$, conditions (4.5.1), (4.5.2) and (4.5.3) reduce to the usual homogeneity and coherence conditions (4.3.1) and (4.3.2), because $\mathcal{B}_{\bar{\alpha}}^r = \mathcal{B}_{\gamma}^r = \mathcal{B}^r$.

Remark 4.5.3. For the usual space $\mathcal{G}^{\bar{\alpha};\alpha,\gamma}$ of coherent and homogeneous germs, see Definition 4.3.2, we do not impose the boundedness condition (4.5.3) because it follows by (4.3.1) and (4.3.2): it suffices to apply (4.3.3) for $\lambda = 1$. This shows that coherent and homogeneous germs are weakly coherent and homogeneous: more precisely $\mathcal{G}^{\bar{\alpha};\alpha,\gamma} \subseteq \check{\mathcal{G}}^{\bar{\alpha};\alpha,\gamma}$, and the inclusion is an equality when both $\bar{\alpha} < 0$ and $\gamma < 0$.

Remark 4.5.4 (General scales). As in the Remark 4.3.5, if we fix $\bar{\lambda} \in (1, \infty)$ and consider $\lambda \in [1, \bar{\lambda}]$, then we can decompose $\varphi_x^\lambda = \sum_{k=1}^n (\varphi_k)_{x_k}^1$ where $x_k \in K$, $\varphi_k \in \mathcal{B}^r$, and n is bounded by a value depending only on the choice of $\bar{\lambda}$ and K . Then, by applying (4.5.3), we see that for a germ $F \in \check{\mathcal{G}}^{\bar{\alpha};\alpha,\gamma}$, hence satisfying (4.5.1) and (4.5.2), the same estimates are actually satisfied uniformly over $\lambda \in (0, \bar{\lambda}]$.

4.5.2 Conditional proof of Theorem 4.3.12

We state two basic results on (weakly) coherent and homogeneous germs, which yield Theorem 4.3.12 as a corollary.

The first result describes how convolution with a kernel K acts on germs. This will be proved in Section 4.5.4 below.

Theorem 4.5.5 (Convolution of germs). *Let $\bar{\alpha}, \alpha, \gamma \in \mathbb{R}$ with $\bar{\alpha}, \alpha \leq \gamma$.*

If K is a β -regularising kernel with $\beta > 0$ of order (m, r) large enough:

$$r > \max\{-\bar{\alpha}, -\alpha\}, \quad m > \gamma + \beta,$$

convolution by K , see (4.3.9), is a continuous linear map from $\mathcal{G}^{\bar{\alpha};\alpha,\gamma}$ to $\check{\mathcal{G}}^{\bar{\alpha}+\beta;\alpha+\beta,\gamma+\beta}$.

If furthermore K preserves polynomials at level γ , see Assumption 4.2.6, then convolution by K is also a continuous linear map from $\check{\mathcal{G}}^{\bar{\alpha};\alpha,\gamma}$ to $\check{\mathcal{G}}^{\bar{\alpha}+\beta;\alpha+\beta,\gamma+\beta}$.

Remark 4.5.6 (Classical Schauder estimate). *Given any $\bar{\alpha} \in \mathbb{R}$ and any distribution $f \in \mathcal{Z}^{\bar{\alpha}}$, see Definition 4.2.1, we can consider the constant germ ($F_x = f$) which is clearly coherent for any exponents α, γ and weakly homogeneous with exponent $\bar{\alpha}$, that is $F \in \check{\mathcal{G}}^{\bar{\alpha};\alpha,\gamma}$. By Theorem 4.5.5, we see that $(K * F_x) \in \check{\mathcal{G}}^{\bar{\alpha}+\beta;\alpha+\beta,\gamma+\beta}$, which means that $K * f \in \mathcal{Z}^{\bar{\alpha}+\beta}$ (compare (4.5.1) and (4.5.3) with (4.2.3)). We thus obtain the classical Schauder Theorem 4.2.10 as a corollary of our approach.*

Our second basic result links weakly coherent and homogeneous germs with ordinary ones, in the special case when homogeneity and coherence exponents coincide $\bar{\alpha} = \gamma$. This will be proved in Section 4.5.6 below.

Theorem 4.5.7 (Positive renormalisation). *Let $\alpha, \gamma \in \mathbb{R}$ with $\alpha \leq \gamma$ and*

$$\alpha \neq 0, \quad \gamma \notin \mathbb{N}_0.$$

Then, for any germ $F \in \check{\mathcal{G}}^{\gamma;\alpha,\gamma}$ (i.e. with $\bar{\alpha} = \gamma$), the germ

$$G_x := F_x - \sum_{|k| < \gamma} \frac{F_x^{(k)}(x)}{k!} (\cdot - x)^k$$

is well defined and belongs to $\mathcal{G}^{\gamma;\alpha \wedge 0, \gamma}$. The map $F \mapsto G$ is linear and continuous.

The terminology “positive renormalisation” is inspired by [BHZ19], where this notion is related to an operator called Δ^+ which yields an algebraic description of the subtraction of Taylor polynomials, see [BHZ19, Lemma 6.10 and Remark 6.11].

We can now deduce Theorem 4.3.12 from Theorems 4.5.5 and 4.5.7.

Proof of Theorem 4.3.12. Let F be an (α, γ) -coherent germ for some $\gamma > 0$, $\alpha \leq \gamma$. Applying the reconstruction Theorem 4.3.7, there exists a unique $\mathcal{R}(F) \in \mathcal{D}'(\mathbb{R}^d)$ with:

$$F - \mathcal{R}(F) \in \mathcal{G}^{\gamma; \alpha, \gamma}.$$

Note that $F - \mathcal{R}(F)$ is both coherent and homogeneous, even though we do not assume a priori that F satisfies a homogeneity bound. By Theorem 4.5.5, using the assumptions (4.3.12), it follows that

$$(\mathbb{K} * \{F_x - \mathcal{R}(F)\})_{x \in \mathbb{R}^d} \in \check{\mathcal{G}}^{\gamma + \beta; \alpha + \beta, \gamma + \beta}.$$

Since $\alpha + \beta \neq 0$, $\gamma + \beta \notin \mathbb{N}_0$, we can apply Theorem 4.5.7 to $(\mathbb{K} * \{F_x - \mathcal{R}(F)\})_x$, which implies that $\mathcal{K}(F)$ as in (4.3.11) is well-defined and:

$$\mathcal{K}(F) - \mathbb{K} * \mathcal{R}(F) \in \mathcal{G}^{\gamma + \beta; (\alpha + \beta) \wedge 0, \gamma + \beta}. \quad (4.5.4)$$

It follows that $\mathcal{K}(F)$ is $((\alpha + \beta) \wedge 0, \gamma + \beta)$ -coherent and its reconstruction is unique, since $\gamma + \beta > 0$. In fact, by the property of homogeneity in (4.5.4), we recognize

$$\mathcal{R}(\mathcal{K}(F)) = \mathbb{K} * \mathcal{R}(F).$$

Now assume that F is also $\bar{\alpha}$ -homogeneous, and let us establish the homogeneity of $\mathcal{K}(F)$. By the reconstruction Theorem 4.3.7, we know that $\mathcal{R}(F) \in \mathcal{Z}^{\bar{\alpha}}$, hence $\mathbb{K} * \mathcal{R}(F) \in \mathcal{Z}^{\bar{\alpha} + \beta}$ by the classical Schauder estimate, see Theorem 4.2.10. If we view $\mathbb{K} * \mathcal{R}(F)$ as a constant germ, then it is trivially (α', γ') -coherent for arbitrary α', γ' , therefore

$$\mathbb{K} * \mathcal{R}(F) \in \mathcal{G}^{(\bar{\alpha} + \beta) \wedge 0; \alpha', \gamma'}. \quad (4.5.5)$$

Since $\bar{\alpha} \leq \gamma$, summing (4.5.4) and (4.5.5) we obtain

$$\mathcal{K}(F) \in \mathcal{G}^{(\bar{\alpha} + \beta) \wedge 0; (\alpha + \beta) \wedge 0, \gamma + \beta},$$

which completes the proof. \square

The rest of this section is devoted to the proof of Theorems 4.5.5 and 4.5.7. These will be preceded by some technical tools, which will also yield the the proof of Proposition 4.2.9 (convolution of \mathbb{K} with a sufficiently nice distribution f is well-defined) and Theorem 4.2.10 (classical Schauder estimate).

4.5.3 Preliminary tools (I)

A key ingredient of our proofs is the following lemma which is similar to [FH20, Proposition 14.11] (though this reference only considers the translation invariant case $\mathbb{K}_n(x, y) = \mathbb{K}_n(y - x)$), and which quantifies the behaviour of the convolution $\mathbb{K}_n^* \varphi_x^\lambda$.

Let us heuristically suggest the expected answer. We start by remarking that since \mathbb{K} is β -regularising, we expect \mathbb{K}_n to behave like $\mathbb{K}_n \simeq 2^{-\beta n} \psi^{2^{-n}}$ for some test-function $\psi \in \mathcal{D}$, and thus $\mathbb{K}_n^* \varphi_x^\lambda \simeq 2^{-\beta n} \psi^{2^{-n}} * \varphi_x^\lambda$. For this type of convolution, we heuristically approximate the test-function corresponding to the lower scale $\max(2^{-n}, \lambda)$ to be constant compared to the other one, so that we expect in general

$$\mathbb{K}_n^* \varphi_x^\lambda \simeq 2^{-\beta n} \eta_x^{\max(2^{-n}, \lambda)} \quad \text{for some test-function } \eta \in \mathcal{D}.$$

In fact, it turns out that we can improve this approximation when φ is assumed to annihilate polynomials of degree up to some integer $c \in \mathbb{N}_0$: by subtracting the Taylor expansion of ψ at order c in the integral defining the convolution, we obtain the improved approximation in the regime $\lambda \leq 2^{-n}$:

$$\mathbf{K}_n^* \varphi_x^\lambda \simeq (2^n \lambda)^{c+1} 2^{-\beta n} \zeta_x^{2^{-n}} \quad \text{for some test-function } \zeta \in \mathcal{D}.$$

We can now state the precise result. Its proof is given in Appendix 4.A and proceeds as in [FH20, Proposition 14.11].

Lemma 4.5.8 (Convolving \mathbf{K}_n against test-functions). *Let \mathbf{K} be a β -regularising kernel of order (m_0, r_0) for some $\beta > 0$, $m_0, r_0 \in \mathbb{N}_0$, and let $c \in \mathbb{N}_0 \cup \{-1\}$. Then there exists a constant $\text{cst} > 0$ and a real $\bar{\lambda} > 1$ depending only on the kernel \mathbf{K} (and d), such that for $n \in \mathbb{N}_0$, $x \in \mathbb{R}^d$, $\lambda \in (0, 1]$, $\varphi \in \mathcal{B}_c^{r_0}$, one can write*

$$\mathbf{K}_n^* \varphi_x^\lambda = \begin{cases} 2^{-\beta n} (\eta^{[n, \lambda, x, \varphi]})_x^{\bar{\lambda} \lambda} & \text{for } 2^{-n} \leq \lambda, \\ 2^{-\beta n} (2^n \lambda)^{m_0} (\zeta^{[n, \lambda, x, \varphi]})_x^{\bar{\lambda} 2^{-n}} & \text{for } 2^{-n} \geq \lambda, c \geq m_0 - 1, \end{cases} \quad (4.5.6)$$

for some test-functions

$$\begin{cases} \eta^{[n, \lambda, x, \varphi]} \in \text{cst} \mathcal{B}^{r_0} & \text{for } 2^{-n} \leq \lambda, \\ \zeta^{[n, \lambda, x, \varphi]} \in \text{cst} \mathcal{B}^{r_0} & \text{for } 2^{-n} \geq \lambda, c \geq m_0 - 1. \end{cases} \quad (4.5.7)$$

If furthermore \mathbf{K} satisfies Assumption 4.2.6 for some $c_0 \in \mathbb{N}_0$, then the same conclusion holds replacing (4.5.7) by

$$\eta^{[n, \lambda, x, \varphi]} \in \text{cst} \mathcal{B}_{\min(c, c_0)}^{r_0}, \quad \zeta^{[n, \lambda, x, \varphi]} \in \text{cst} \mathcal{B}_{\min(c, c_0)}^{r_0}. \quad (4.5.8)$$

Remark 4.5.9 (The new scale $\bar{\lambda}$). *It follows from the proof of this result that one can take $\bar{\lambda} := 1 + \text{cst}$ where $\text{cst} > 0$ is the constant that appears in item 1 of the Definition 4.2.2 of the regularising kernel \mathbf{K} .*

We can now prove Proposition 4.2.9 and Theorem 4.2.10.

Proof of Proposition 4.2.9. Let $r \in \mathbb{N}_0$, \mathbf{K} be a β -regularising kernel of order $(0, r)$ for some $\beta > 0$, and let f be a distribution of order less or equal to r . In particular, for any compact $K \subset \mathbb{R}^d$,

$$\sup_{\eta \in \mathcal{B}^r(K)} |f(\eta)| < +\infty.$$

Now let $\varphi \in \mathcal{D}(\mathbb{R}^d)$ be a test-function supported in some compact $K \subset \mathbb{R}^d$. Applying Lemma 4.5.8, there exists $\bar{\lambda} > 0$ and a constant $\text{cst} > 0$, which depend only on the kernel \mathbf{K} and the compact K , such that

$$\mathbf{K}_n^* \varphi = 2^{-\beta n} (\eta^{[n, \varphi]})^{\bar{\lambda}},$$

for some test-function $\eta^{[n, \varphi]} \in \text{cst} \mathcal{B}^r$. As a consequence, $|f(\mathbf{K}_n^* \varphi)| \lesssim 2^{-\beta n}$, where the multiplicative constant depends only on the compact K , the distribution f , the kernel \mathbf{K} and the dimension d . Thus, the sum in (4.2.10) converges so that the convolution is well-defined, and furthermore

$$\sup_{\eta \in \mathcal{B}^r(K)} |\mathbf{K} * f(\eta)| < +\infty,$$

where the compact K is arbitrary, so that $\mathbf{K} * f$ is of order less or equal than r . \square

Proof of Theorem 4.2.10. Using the estimate of Lemma 4.5.8 more precisely, we can now also give a simple proof of Theorem 4.2.10, in the same manner as in [FH20, Section 14]. We will also perform similar calculations when proving Theorem 4.5.5 below.

We fix $\gamma \in \mathbb{R}$, \mathbf{K} a β -regularising kernel of order (m, r) for some $\beta > 0$ and $m, r \in \mathbb{N}_0$ such that $m > \gamma + \beta$, $r > -\gamma$; and (when $\gamma \geq 0$), we also assume that \mathbf{K} preserves polynomials at level γ , see Assumption 4.2.6.

Let $f \in \mathcal{Z}^\gamma$. It is straightforward from definition 4.2.1 that f is a distribution of order r so that from Theorem 4.2.9 the convolution $\mathbf{K} * f$ is well-defined.

Fix $K \subset \mathbb{R}^d$ compact, $x \in K$, $\lambda \in (0, 1]$, and $\varphi \in \mathcal{B}_{\gamma+\beta}^r$. Set $N_\lambda := \min\{n \in \mathbb{N}, 2^{-n} \leq \lambda\}$, we cut the sum in two regimes and apply Lemma 4.5.8 to obtain for some $\bar{\lambda} > 1$ and test-functions $\eta^{[n, \lambda, x, \varphi]}, \zeta^{[n, \lambda, x, \varphi]} \in \text{cst}\mathcal{B}_\gamma^r$ (note that this is where we use the assumption $m > \gamma + \beta$):

$$\begin{aligned} \mathbf{K} * f(\varphi_x^\lambda) &= \sum_{n=0}^{N_\lambda-1} f(\mathbf{K}_n^* \varphi_x^\lambda) + \sum_{n=N_\lambda}^{+\infty} f(\mathbf{K}_n^* \varphi_x^\lambda) \\ &= \sum_{n=0}^{N_\lambda-1} 2^{-\beta n} (2^n \lambda)^m f((\zeta^{[n, \lambda, x, \varphi]})_x^{\bar{\lambda} 2^{-n}}) + \sum_{n=N_\lambda}^{+\infty} 2^{-\beta n} f((\eta^{[n, \lambda, x, \varphi]})_x^{\bar{\lambda} \lambda}). \end{aligned}$$

From (4.5.7), the assumption $f \in \mathcal{Z}^\gamma$ (this is where we use the assumption $r > -\gamma$), and summing the geometric series:

$$\left| \mathbf{K} * f(\varphi_x^\lambda) \right| \lesssim \sum_{n=0}^{N_\lambda-1} 2^{-\beta n} (2^n \lambda)^m 2^{-n\gamma} + \sum_{n=N_\lambda}^{+\infty} 2^{-\beta n} \lambda^\gamma \lesssim \lambda^{\gamma+\beta}, \quad (4.5.9)$$

where the multiplicative constant depends only on the kernel \mathbf{K} , the compact K , and the distribution f .

Now let $\varphi \in \mathcal{B}^r$, which may not annihilate polynomials. By definition, and from (4.5.6),

$$\mathbf{K} * f(\varphi_x) = \sum_{n \in \mathbb{N}} f(\mathbf{K}_n^* \varphi_x) = \sum_{n \in \mathbb{N}} 2^{-\beta n} f((\eta^{[n, \lambda, x, \varphi]})_x^{\bar{\lambda}}).$$

From (4.5.7) and the assumption $f \in \mathcal{Z}^\gamma$ this yields as announced:

$$|\mathbf{K} * f(\varphi_x)| \lesssim \sum_{n \in \mathbb{N}} 2^{-\beta n} \lesssim 1. \quad (4.5.10)$$

From the calculations above, the estimate (4.5.9)-(4.5.10) hold uniformly over $x \in K$, $\lambda \in (0, 1]$, $\varphi \in \mathcal{B}_{\gamma+\beta}^r$ for any $r > -\gamma$. In fact, from the results of Appendix 4.B, see Proposition 4.B.1, this remains true even for $r > -\gamma - \beta$, and thus $f \in \mathcal{Z}^{\gamma+\beta}$.

By tracking the constants in the estimates,

$$\|\mathbf{K} * f\|_{\mathcal{Z}^{\gamma+\beta}} \lesssim \|f\|_{\mathcal{Z}^\gamma},$$

where the compact $K' \supset K$ on the right-hand side depends only on K and the kernel \mathbf{K} , whence the continuity of the map $\mathcal{Z}^\gamma \rightarrow \mathcal{Z}^{\gamma+\beta}$, $f \mapsto \mathbf{K} * f$. \square

4.5.4 Proof of Theorem 4.5.5 (convolution of germs)

Before proceeding with the proof of Theorem 4.5.5, let us give more precise notations for weakly homogeneous and coherent germs. As in Remark 4.3.6, introduce the semi-norms

corresponding to (4.5.1), (4.5.2), (4.5.3):

$$\begin{aligned} \|F\|_{\check{\mathcal{G}}_{\text{hom};K,\bar{\lambda},r}^{\bar{\alpha}}} &:= \sup_{\substack{x \in K, \lambda \in (0, \bar{\lambda}] \\ \varphi \in \mathcal{B}_{\bar{\alpha}}^r}} \frac{|F_x(\varphi_x^\lambda)|}{\lambda^{\bar{\alpha}}} + \sup_{x \in K, \varphi \in \mathcal{B}^r} |F_x(\varphi_x)|, \\ \|F\|_{\check{\mathcal{G}}_{\text{coh};K,\bar{\lambda},r}^{\alpha,\gamma}} &:= \sup_{\substack{x,y \in K, \lambda \in (0, \bar{\lambda}] \\ \varphi \in \mathcal{B}_\gamma^r}} \frac{|(F_y - F_x)(\varphi_x^\lambda)|}{\lambda^\alpha (|y-x| + \lambda)^{\gamma-\alpha}} + \sup_{x,y \in K, \varphi \in \mathcal{B}^r} |(F_y - F_x)(\varphi_x)|, \end{aligned}$$

and define the joint semi-norm, with $r = r_{\bar{\alpha},\alpha}$, as in Definition 4.3.2,

$$\|F\|_{\check{\mathcal{G}}_{K,\bar{\lambda}}^{\bar{\alpha};\alpha,\gamma}} := \|F\|_{\check{\mathcal{G}}_{\text{hom};K,\bar{\lambda},r_{\bar{\alpha},\alpha}}^{\bar{\alpha}}} + \|F\|_{\check{\mathcal{G}}_{\text{coh};K,\bar{\lambda},r_{\bar{\alpha},\alpha}}^{\alpha,\gamma}}. \quad (4.5.11)$$

By Remark 4.5.4, we may fix $\bar{\lambda} = 1$ in (4.5.11) and omit it from the notation.

Proof of Theorem 4.5.5. We only prove the first statement, that is convolution by K is continuous from $\mathcal{G}^{\bar{\alpha};\alpha,\gamma}$ to $\check{\mathcal{G}}^{\bar{\alpha}+\beta;\alpha+\beta,\gamma+\beta}$. The second part, i.e. continuity on $\check{\mathcal{G}}^{\bar{\alpha};\alpha,\gamma}$, is proved essentially in the same way, with the only difference that in this case one uses the conclusion (4.5.8) rather than (4.5.7) of Lemma 4.5.8.

Let $F \in \mathcal{G}^{\bar{\alpha};\alpha,\gamma}$. We prove that $(K * F_x)_x$ satisfies (4.5.1)-(4.5.2)-(4.5.3) for $\bar{\alpha} + \beta, \alpha + \beta, \gamma + \beta$, and $r := r_{\bar{\alpha},\alpha}$. Note that by the Proposition 4.B.1 of Appendix 4.B, the same estimates will automatically hold also for $r_{\bar{\alpha}+\beta,\alpha+\beta}$, which completely establishes $(K * F_x)_x \in \check{\mathcal{G}}^{\bar{\alpha}+\beta;\alpha+\beta,\gamma+\beta}$.

In the remainder of this proof, we fix a compact $K \subset \mathbb{R}^d$, $x, y \in K$, $\bar{\lambda}_0 > 0$, $\lambda \in (0, \bar{\lambda}_0]$. The decomposition argument is the same as in the proof of Theorem 4.2.10. We prove the three estimates separately.

- (Weak homogeneity) Let $\varphi \in \mathcal{B}_{\bar{\alpha}+\beta}^r$. By the assumption $m_0 > \bar{\alpha} + \beta$, this implies $\varphi \in \mathcal{B}_{m_0-1}^r$. Set $N_\lambda := \min\{n \in \mathbb{N}, 2^{-n} \leq \lambda\}$. We cut the sum in two regimes:

$$K * F_x(\varphi_x^\lambda) = \sum_{n=0}^{N_\lambda-1} F_x(K_n^* \varphi_x^\lambda) + \sum_{n=N_\lambda}^{+\infty} F_x(K_n^* \varphi_x^\lambda).$$

From (4.5.6), one has for some $\bar{\lambda} > 1$,

$$K * F_x(\varphi_x^\lambda) = \sum_{n=0}^{N_\lambda-1} 2^{-\beta n} (2^n \lambda)^{m_0} F_x((\zeta^{[n,\lambda,x,\varphi]})_x^{\bar{\lambda} 2^{-n}}) + \sum_{n=N_\lambda}^{+\infty} 2^{-\beta n} F_x((\eta^{[n,\lambda,x,\varphi]})_x^{\bar{\lambda} \lambda}).$$

From (4.5.7) and the assumption of homogeneity on F ,

$$\left| K * F_x(\varphi_x^\lambda) \right| \lesssim \sum_{n=0}^{N_\lambda-1} 2^{-\beta n} (2^n \lambda)^{m_0} 2^{-n\bar{\alpha}} + \sum_{n=N_\lambda}^{+\infty} 2^{-\beta n} \lambda^{\bar{\alpha}},$$

where the multiplicative constant depends only on the kernel K , the compact K , and the germ F . Summing the geometric series yields as announced:

$$\left| K * F_x(\varphi_x^\lambda) \right| \lesssim \lambda^{\bar{\alpha}+\beta}.$$

- (Weak coherence) Let $\varphi \in \mathcal{B}_{\gamma+\beta}^r$. By the assumption $m_0 > \gamma + \beta$, this implies $\varphi \in \mathcal{B}_{m_0-1}^r$. By definition,

$$(K * F_y - K * F_x)(\varphi_x^\lambda) = \sum_{n \in \mathbb{N}} (F_y - F_x)(K_n^* \varphi_x^\lambda).$$

As previously we cut the sum using (4.5.6):

$$\begin{aligned} (\mathbf{K} * F_y - \mathbf{K} * F_x)(\varphi_x^\lambda) &= \sum_{n=0}^{N_\lambda-1} 2^{-\beta n} (2^n \lambda)^{m_0} (F_y - F_x)((\zeta^{[n,\lambda,x,\varphi]})_x^{\bar{\lambda}} 2^{-n}) \\ &\quad + \sum_{n=N_\lambda}^{+\infty} 2^{-\beta n} (F_y - F_x)((\eta^{[n,\lambda,x,\varphi]})_x^{\bar{\lambda}} \lambda). \end{aligned}$$

From (4.5.7) and the assumption of coherence on F ,

$$\begin{aligned} |(\mathbf{K} * F_y - \mathbf{K} * F_x)(\varphi_x^\lambda)| &\lesssim \sum_{n=0}^{N_\lambda-1} 2^{-\beta n} (2^n \lambda)^{m_0} 2^{-n\alpha} (|y-x| + 2^{-n})^{\gamma-\alpha} \\ &\quad + \sum_{n=N_\lambda}^{+\infty} 2^{-\beta n} \lambda^\alpha (|y-x| + \lambda)^{\gamma-\alpha}, \end{aligned}$$

where the multiplicative constant depends only on the kernel \mathbf{K} , the compact K , and the germ F . Summing the geometric series yields as announced

$$|(\mathbf{K} * F_y - \mathbf{K} * F_x)(\varphi_x^\lambda)| \lesssim \lambda^{\alpha+\beta} (|y-x| + \lambda)^{(\gamma+\beta)-(\alpha+\beta)}.$$

3. (Boundedness) Now let $\varphi \in \mathcal{B}^r$, which possibly does not annihilate polynomials. By definition, and from (4.5.6),

$$\mathbf{K} * F_y(\varphi_x) = \sum_{n \in \mathbb{N}} 2^{-\beta n} F_y((\eta^{[n,\lambda,x,\varphi]})_x^{\bar{\lambda}}) + 2^{-\beta n} (F_x - F_y)((\eta^{[n,\lambda,x,\varphi]})_x^{\bar{\lambda}}).$$

From (4.5.7) and the assumption of homogeneity and coherence on F , this yields as announced:

$$|\mathbf{K} * F_y(\varphi_x)| \lesssim \sum_{n \in \mathbb{N}} 2^{-\beta n} \lesssim 1.$$

In fact, by tracking the constants in the estimates, one obtains the continuity estimate:

$$\|\mathbf{K} * F\|_{\check{\mathcal{G}}_K^{\bar{\alpha}+\beta;\alpha+\beta,\gamma+\beta}} \lesssim \|F\|_{\mathcal{G}_{K'}^{\bar{\alpha};\alpha,\gamma}},$$

where the compact $K' \supset K$ on the right-hand side depends only on the compact K on the left-hand side, the kernel \mathbf{K} and the exponents $\bar{\alpha}, \alpha$. \square

Remark 4.5.10. *In the proof just above, the property of homogeneity on $\mathbf{K} * F$ did not require the assumption of coherence on F and inversely.*

Indeed, under the assumption that \mathbf{K} is β -regularising of order (m, r) for some $r \in \mathbb{N}_0$ and $m > \bar{\alpha} + \beta$ resp. $m > \gamma + \beta$, the same calculations as in the proof just above establish that convolution with \mathbf{K} defines a linear map $\mathcal{G}_{\text{hom};r}^{\bar{\alpha}} \rightarrow \check{\mathcal{G}}_{\text{hom};r}^{\bar{\alpha}+\beta}$ resp. $\mathcal{G}_{\text{coh};r}^{\alpha,\gamma} \rightarrow \mathcal{G}_{\text{coh};r}^{\alpha+\beta,\gamma+\beta}$, with continuity estimates

$$\|\mathbf{K} * F\|_{\check{\mathcal{G}}_{\text{hom};K,\bar{\lambda},r}^{\bar{\alpha}+\beta}} \lesssim \|F\|_{\mathcal{G}_{\text{hom};K,\bar{\lambda}',r}^{\bar{\alpha}}}, \quad \text{resp.} \quad \|\mathbf{K} * F\|_{\check{\mathcal{G}}_{\text{coh};K,\bar{\lambda},r}^{\alpha+\beta,\gamma+\beta}} \lesssim \|F\|_{\mathcal{G}_{\text{coh};K,\bar{\lambda}',r}^{\alpha,\gamma}}.$$

where $\bar{\lambda}' > \bar{\lambda}$ depends only on $\bar{\lambda}$ and the kernel \mathbf{K} .

4.5.5 Preliminary tools (II)

The following lemma will be useful to perform decompositions involving test-functions which annihilate polynomials.

Lemma 4.5.11 (Some test-functions which annihilate polynomials). *Let $r \in \mathbb{N}_0, c \in \mathbb{N}_0 \cup \{-1\}$, and $\eta \in \mathcal{D}(B(0, 1))$ with $\int \eta(x) dx = 1$ and $\int \eta(x) x^l dx = 0$ for $1 \leq |l| \leq c$.*

Then there exists a constant $\text{cst} > 0$ depending only on r, c, η, d , such that for all multi-indices k with $0 \leq |k| \leq c$ and all test-functions $\psi \in \mathcal{B}^r$, one has

$$\begin{cases} 2^{|k|}(\eta^{(k)})^{2^{-1}} - \eta^{(k)} & \in \text{cst} \mathcal{B}_{c+|k|}^r, \\ \psi - \sum_{|k| \leq c} (-1)^{|k|} \mathbb{X}_0^k(\psi) \eta^{(k)} & \in \text{cst} \mathcal{B}_c^r. \end{cases}$$

Proof. This is straightforward. □

As a consequence of Lemma 4.5.11, we provide a proof of the existence of pointwise derivatives for weakly homogeneous germs, as announced in Lemma 4.3.11.

Lemma 4.5.12 (Existence of pointwise evaluations). *Let $\bar{\alpha} \in \mathbb{R}, r \in \mathbb{N}_0$, and $F \in \check{\mathcal{G}}_{\text{hom}; r}^{\bar{\alpha}}$. Then for any multi-index $k \in \mathbb{N}_0^d$ with $|k| < \bar{\alpha}$, any $x \in \mathbb{R}^d$ and any test-function $\eta \in \mathcal{D}(\mathbb{R}^d)$ such that $\int \eta(x) dx = 1$ and $\int \eta(x) x^l dx = 0$ for all multi-indices l with $1 \leq |l| < \bar{\alpha}$, the limit*

$$F_x^{(k)}(x) := \lim_{\lambda \rightarrow 0} F_x^{(k)}(\eta_x^\lambda), \quad (4.5.12)$$

exists and does not depend on the choice of such an η .

Furthermore, one has the following estimate: there exists a constant $\text{cst} > 0$ depending only on $\bar{\alpha}, r, d$ such that for $k \in \mathbb{N}_0^d$ with $|k| < \bar{\alpha}$:

$$|F_x^{(k)}(x)| \leq \text{cst} \|F\|_{\check{\mathcal{G}}_{\text{hom}; K, \bar{\lambda}=1, r}^{\bar{\alpha}}}, \quad (4.5.13)$$

for any $x \in \mathbb{R}^d$ and $K \subset \mathbb{R}^d$ containing x .

Proof. Without loss of generality, let us assume that $\text{supp}(\eta) \subset B(0, 1)$. Observe first that the definition (4.5.12) does not depend on such an η because if $\tilde{\eta}$ is another such function, then $(\eta - \tilde{\eta})^{(k)}$ annihilates polynomials of degree less or equal than $\bar{\alpha}$, and thus by assumption of homogeneity and the distributional definition of the derivative,

$$F_x^{(k)}(\eta_x^\lambda) - F_x^{(k)}(\tilde{\eta}_x^\lambda) = (-\lambda)^{-|k|} F_x \left(((\eta - \tilde{\eta})^{(k)})_x^\lambda \right) = O_{\lambda \rightarrow 0}(\lambda^{\bar{\alpha}-|k|}) = o_{\lambda \rightarrow 0}(1).$$

Now let us establish the convergence of (4.5.12). For any $\lambda \in (0, 1]$ and $N \in \mathbb{N}_0$,

$$F_x^{(k)}(\eta_x^{\lambda 2^{-(N+1)}}) = F_x^{(k)}(\eta_x^\lambda) + \sum_{n=0}^N F_x^{(k)}(\eta_x^{\lambda 2^{-(n+1)}} - \eta_x^{\lambda 2^{-n}}). \quad (4.5.14)$$

By the distributional definition of the derivative,

$$\begin{aligned} F_x^{(k)}(\eta_x^{\lambda 2^{-(n+1)}} - \eta_x^{\lambda 2^{-n}}) &= (-\lambda^{-1} 2^{n+1})^{|k|} F_x \left((\eta^{(k)})_x^{2^{-(n+1)}} \right) - (-\lambda^{-1} 2^n)^{|k|} F_x \left((\eta^{(k)})_x^{2^{-n}} \right) \\ &= (-\lambda^{-1} 2^n)^{|k|} F_x \left((\varphi^{(k)})_x^{\lambda 2^{-n}} \right), \end{aligned}$$

where $\varphi^{(k)} := 2^{|k|}(\eta^{(k)})^{2^{-1}} - \eta^{(k)}$. From Lemma 4.5.11, there exists a constant $\text{cst} > 0$ depending only on $\bar{\alpha}, r, d, \eta$ such that for $|k| < \bar{\alpha}$,

$$\varphi^{(k)} \in \text{cst} \mathcal{B}_{\bar{\alpha}}^r.$$

Thus, by the assumption of homogeneity of the germ F ,

$$\left| (-\lambda^{-1}2^n)^{|k|} F_x((\varphi^{(k)})_x^{\lambda 2^{-n}}) \right| \lesssim \lambda^{\bar{\alpha}-|k|} 2^{-n(\bar{\alpha}-|k|)}. \quad (4.5.15)$$

By assumption, $|k| < \bar{\alpha}$, whence the sum in (4.5.14) converges, which justifies the well-posedness of $F_x^{(k)}(x)$. In fact, (4.5.14) yields the following useful decomposition: for any $\lambda \in (0, 1]$ and any such η ,

$$F_x^{(k)}(x) = (-\lambda^{-1})^{|k|} F_x((\eta^{(k)})_x^\lambda) + \sum_{n=0}^{+\infty} (-\lambda^{-1}2^n)^{|k|} F_x((\varphi^{(k)})_x^{\lambda 2^{-n}}). \quad (4.5.16)$$

Finally, the estimate (4.5.13) follows immediately from the above calculations (taking $\lambda = 1$). \square

Using a similar decomposition argument, one obtains also the following decomposition result.

Lemma 4.5.13 (Decomposition at lower scale). *Let $r, d \in \mathbb{N}_0$, $c \in \mathbb{N}_0 \cup \{-1\}$. There exists a constant $\text{cst} > 0$ depending only on r, c, d , such that for all $M \in \mathbb{N}_0$ and all $\psi \in \mathcal{B}^r$, there exist test-functions*

$$\tilde{\psi}^{[M]} \in \text{cst} \mathcal{B}^r, \quad \check{\psi}^{[n]} \in \text{cst} \mathcal{B}_c^r \quad \text{for } 0 \leq n \leq M,$$

such that one has the decomposition

$$\psi = (\tilde{\psi}^{[M]})^{2^M} + \sum_{n=0}^M (\check{\psi}^{[n]})^{2^n}.$$

Proof. We fix a test-function $\eta \in \mathcal{D}(B(0, 1))$ such that $\int \eta(x) dx = 1$ and $\int x^k \eta(x) dx = 0$ for multi-indices k with $|k| \leq c$.

We start by defining:

$$\check{\psi} := \psi - \sum_{|k| \leq c} (-1)^{|k|} \mathbb{X}_0^k(\psi) \eta^{(k)},$$

From Lemma 4.5.11, there exists a constant $\text{cst} > 0$ depending only on c (and d), such that

$$\check{\psi} \in \text{cst} \mathcal{B}_c^r.$$

Perform the following telescoping sum, for any $|k| \leq c$:

$$\begin{aligned} \eta^{(k)} &= 2^{-M|k|} (\eta^{(k)})^{2^M} + \sum_{n=0}^{M-1} \left(2^{-n|k|} (\eta^{(k)})^{2^n} - 2^{-(n+1)|k|} (\eta^{(k)})^{2^{n+1}} \right) \\ &= 2^{-M|k|} (\eta^{(k)})^{2^M} + \sum_{n=0}^{M-1} 2^{-(n+1)|k|} (\varphi^{(k)})^{2^{n+1}}. \end{aligned}$$

where $\varphi^{(k)} := 2^{|k|} (\eta^{(k)})^{2^{-1}} - \eta^{(k)}$. From Lemma 4.5.11, there exists a constant $\text{cst} > 0$ depending only on d, r, c such that for $|k| \leq c$,

$$\varphi^{(k)} \in \text{cst} \mathcal{B}_c^r.$$

Thus, we have decomposed:

$$\begin{aligned} \psi &= \check{\psi} + \sum_{|k| \leq c} (-1)^{|k|} \mathbb{X}_0^k(\psi) 2^{-M|k|} (\eta^{(k)})^{2^M} \\ &\quad + \sum_{n=0}^{M-1} \sum_{|k| \leq c} (-1)^{|k|} \mathbb{X}_0^k(\psi) 2^{-(n+1)|k|} (\varphi^{(k)})^{2^{n+1}}. \end{aligned}$$

Define:

$$\begin{cases} \tilde{\psi}^{[M]} & := \sum_{|k| \leq c} (-1)^{|k|} \mathbb{X}_0^k(\psi) 2^{-M|k|} \eta^{(k)}, \\ \check{\psi}^{[n]} & := \sum_{|k| \leq c} (-1)^{|k|} \mathbb{X}_0^k(\psi) 2^{-n|k|} \varphi^{(k)} \quad \text{for } n = 1, \dots, M, \\ \check{\psi}^{[0]} & := \check{\psi}. \end{cases}$$

Then there exists a constant $\text{cst} > 0$ depending only on d, r, c such that:

$$\tilde{\psi}^{[M]} \in \text{cst} \mathcal{B}^r, \quad \check{\xi}^{[n]} \in \text{cst} \mathcal{B}_c^r \text{ over } 1 \leq n \leq M.$$

Also:

$$\psi = (\tilde{\psi}^{[M]})^{2^M} + \sum_{n=0}^M (\check{\xi}^{[n]})^{2^n}.$$

This provides the announced decomposition. \square

4.5.6 Proof of Theorem 4.5.7 (positive renormalisation)

We first establish the following result which considers the case of germs which admit only the property of (weak) homogeneity.

Theorem 4.5.14. *Let $\bar{\alpha} \in \mathbb{R}_+ \setminus \mathbb{N}$, $r \in \mathbb{N}$, and F be a germ. The following assertions are equivalent:*

(i) $F \in \check{\mathcal{G}}_{\text{hom};r}^{\bar{\alpha}}$,

(ii) Let G be defined by $G_x := F_x - \sum_{|k| < \bar{\alpha}} F_x^{(k)}(x) \mathbb{X}_x^k$.

Then G is correctly defined and $G \in \mathcal{G}_{\text{hom};r}^{\bar{\alpha}}$. Furthermore, we have the following continuity estimate for compacts $K \subset \mathbb{R}^d$ and reals $\bar{\lambda} > 0$:

$$\|G\|_{\mathcal{G}_{\text{hom};K,\bar{\lambda},r}^{\bar{\alpha}}} \lesssim \|F\|_{\check{\mathcal{G}}_{\text{hom};K,\bar{\lambda},r}^{\bar{\alpha}}}. \quad (4.5.17)$$

Remark 4.5.15. *Compare theorem 4.5.14 with the well-known (see [BL22, Proposition A.5], or the similar [FH20, Proposition 14.15]) result from Hölder spaces that for a distribution $f \in \mathcal{D}'(\mathbb{R}^d)$ and an exponent $\alpha \in \mathbb{R}_+ \setminus \mathbb{N}$, there is equivalence between:*

(i) $f \in \mathcal{C}^\alpha$ i.e. $|f(\varphi_x^\lambda)| \lesssim \lambda^\alpha$ over x in compacts, $\lambda \in (0, 1]$, $\varphi \in \mathcal{B}_\alpha^0$.

(ii) f is a $C^{[\alpha]}$ function and $\left| f(y) - \sum_{|k| < \alpha} \frac{f^{(k)}(x)}{k!} (y-x)^k \right| \lesssim |y-x|^\alpha$.

In the case of integer exponents, only the implication (ii) \Rightarrow (i) holds.

Proof of Theorem 4.5.14. (ii) \Rightarrow (i) is straightforward.

Let us now prove (i) \Rightarrow (ii). Let $F \in \check{\mathcal{G}}_{\text{hom};r}^{\bar{\alpha}}$. Thanks to Lemma 4.5.12, we know that the germ G is well-defined. Now let us prove that $G \in \mathcal{G}_{\text{hom};r}^{\bar{\alpha}}$.

Let $K \subset \mathbb{R}^d$ be compact and $\psi \in \mathcal{B}^r$ be a test-function (which does not necessarily annihilate polynomials). Let $\lambda \in (0, 1]$, $x \in K$, we shall estimate $G_x(\psi_x^\lambda)$. By definition,

$$G_x(\psi_x^\lambda) = F_x(\psi_x^\lambda) - \sum_{|k| < \bar{\alpha}} F_x^{(k)}(x) \mathbb{X}_x^k(\psi_x^\lambda).$$

Note that $\mathbb{X}_x^k(\psi_x^\lambda) = \lambda^{|k|} \mathbb{X}_0^k(\psi)$. Now we fix $\eta \in \mathcal{D}(B(0,1))$ such that $\int \eta(x) dx = 1$ and $\int \eta(x) x^l dx = 0$ for all multi-indices l with $1 \leq |l| \leq \lfloor \bar{\alpha} \rfloor$. Thus, (4.5.16) yields the decomposition

$$G_x(\psi_x^\lambda) = \underbrace{F_x(\psi_x^\lambda) - \sum_{|k| < \bar{\alpha}} \lambda^{|k|} \mathbb{X}_0^k(\psi) (-\lambda^{-1})^{|k|} F_x((\eta^{(k)})_x^\lambda)}_{=:A} - \underbrace{\sum_{|k| < \bar{\alpha}} \lambda^{|k|} \mathbb{X}_0^k(\psi) \sum_{n=0}^{+\infty} (-\lambda^{-1} 2^n)^{|k|} F_x((\varphi^{(k)})_x^{\lambda 2^{-n}})}_{=:B}.$$

Recalling (4.5.15), and summing the geometric series, we obtain $|B| \lesssim \lambda^{\bar{\alpha}}$. Furthermore, denote

$$\check{\psi} := \psi - \sum_{|k| < \bar{\alpha}} (-1)^{|k|} \mathbb{X}_0^k(\psi) \eta^{(k)},$$

so that from Lemma 4.5.11, there exists a constant $\text{cst} > 0$ depending only on $\bar{\alpha}$ (and d), such that $\check{\psi} \in \text{cst} \mathcal{B}_\alpha^r$. As a consequence, using the homogeneity property of the germ F :

$$|A| = |F_x(\check{\psi}_x^\lambda)| \lesssim \lambda^{\bar{\alpha}}.$$

This concludes the proof. Finally, the estimate (4.5.17) follows from keeping track of the constants in the calculations just above. \square

Now we are in position to provide a proof for Theorem 4.5.7.

Proof of Theorem 4.5.7. Let $F \in \check{\mathcal{G}}^{\gamma; \alpha, \gamma}$. We know from Lemma 4.5.12 that G is well-defined and from Theorem 4.5.14 that $G \in \mathcal{G}_{\text{hom}; r}^\gamma$ where $r := r_{\gamma, \alpha}$ is fixed for the remainder of this proof, see Definitions 4.3.2-4.5.1. Thus, it remains to obtain the property of coherence on the germ G .

Let $K \subset \mathbb{R}^d$ be compact and $\psi \in \mathcal{B}^r$ be a test-function (which does not necessarily annihilate polynomials). Let $\lambda \in (0, 1]$, $x, y \in K$, we shall estimate $(G_y - G_x)(\psi_x^\lambda)$.

A first heuristic observation is that when $\lambda \sim |y - x|$, then $\psi_x^\lambda \simeq \check{\psi}_y^\lambda$ can also be seen as a test-function centered in y with scale λ , and thus using the fact established in Theorem 4.5.14 that $G \in \mathcal{G}_{\text{hom}; r}^\gamma$ is γ -homogeneous (even against test-functions which do not annihilate polynomials),

$$|(G_y - G_x)(\psi_x^\lambda)| \lesssim |G_y(\check{\psi}_y^\lambda)| + |G_x(\psi_x^\lambda)| \lesssim \lambda^\gamma \simeq \lambda^0 (|y - x| + \lambda)^{\gamma-0},$$

implying $(0, \gamma)$ -coherence on G .

However, in general λ and $|y - x|$ are not at comparable scales, which is why we resort to the result of decomposition at lower scale Lemma 4.5.13. More precisely, define:

$$M := M_{x, y, \lambda} := \min \left\{ n \in \mathbb{N}_0, 2^{-n} \leq \frac{\lambda}{\lambda + |y - x|} \right\},$$

and apply Lemma 4.5.13 to obtain the existence of test-functions

$$\check{\psi}^{[M]} \in \text{cst} \mathcal{B}^r, \quad \check{\psi}^{[n]} \in \text{cst} \mathcal{B}_\gamma^r \text{ for } 0 \leq n \leq M,$$

where the constant $\text{cst} > 0$ depends only on γ, r, d , and such that

$$\psi = (\check{\psi}^{[M]})^{2^M} + \sum_{n=0}^M (\check{\psi}^{[n]})^{2^n}.$$

Hence:

$$(G_y - G_x)(\psi_x^\lambda) = \underbrace{(G_y - G_x)((\tilde{\psi}^{[M]})_x^{\lambda 2^M})}_{=:A} + \underbrace{\sum_{n=0}^M (G_y - G_x)((\check{\psi}^{[n]})_x^{\lambda 2^n})}_{=:B}.$$

We now estimate A and B separately.

Estimate of A. Because the choice of M implies $\lambda 2^M \sim \lambda + |y - x|$, we are at the correct scale to apply the argument sketched above. Let us recenter $(\tilde{\psi}^{[M]})_x^{\lambda 2^M}$ at point y : observe that

$$(\tilde{\psi}^{[M]})_x^{\lambda 2^M} = \underbrace{\left((\tilde{\psi}^{[M]})^{\frac{1}{2}} \right)_{\frac{x-y}{\lambda 2^{M+1}}}}_{=: \tilde{\psi}^{[M,y,x,\lambda]}} \lambda 2^{M+1}.$$

By construction of M , $|(\lambda 2^{M+1})^{-1}(y - x)| \leq 1$ so that $\tilde{\psi}^{[M,y,x,\lambda]} \in \text{cst}\mathcal{B}^r$. As a consequence, using the fact that $G \in \mathcal{G}_{\text{hom};r}^\gamma$, and noting that $\lambda 2^{M+1} \leq 4(\lambda + \text{Diam}(K)) =: \bar{\lambda}'$ by definition of M :

$$\begin{aligned} |A| &= |G_y((\tilde{\psi}^{[M,y,x,\lambda]})_y^{\lambda 2^{M+1}}) - G_x((\tilde{\psi}^{[M]})_x^{\lambda 2^M})| \\ &\leq |G_y((\tilde{\psi}^{[M,y,x,\lambda]})_y^{\lambda 2^{M+1}})| + |G_x((\tilde{\psi}^{[M]})_x^{\lambda 2^M})| \\ &\leq \text{cst} \|G\|_{\mathcal{G}_{\text{hom};K,\bar{\lambda}',r}^\gamma} (\lambda 2^M)^\gamma \\ &\leq \text{cst} \|F\|_{\check{\mathcal{G}}_{\text{hom};K,\bar{\lambda}',r}^\gamma} \lambda^0 (\lambda + |y - x|)^{\gamma-0}, \end{aligned}$$

where in the last estimate we used (4.5.17) and the definition of M .

Estimate of B. By construction, $\check{\psi}^{[n]}$ annihilates polynomials up to degree γ , so $(G_y - G_x)((\check{\psi}^{[n]})_x^{\lambda 2^n}) = (F_y - F_x)((\check{\psi}^{[n]})_x^{\lambda 2^n})$. Using the coherence assumption on F :

$$\begin{aligned} |B| &\lesssim \sum_{n=0}^M (\lambda 2^n)^\alpha (\lambda 2^n + |y - x|)^{\gamma-\alpha} \\ &\lesssim \lambda^\gamma \sum_{n=0}^M 2^{n\gamma} + \lambda^\alpha |y - x|^{\gamma-\alpha} \sum_{n=0}^M 2^{n\alpha}. \end{aligned}$$

By assumption, $\gamma > 0$ so $\lambda^\gamma \sum_{n=0}^M 2^{n\gamma} \lesssim (\lambda 2^M)^\gamma \lesssim (\lambda + |y - x|)^\gamma$ by the definition of M . Furthermore, there are two possibilities for α . If $\alpha > 0$ then $\lambda^\alpha |y - x|^{\gamma-\alpha} \sum_{n=0}^M 2^{n\alpha} \lesssim (\lambda 2^M)^\alpha |y - x|^{\gamma-\alpha} \lesssim (\lambda + |y - x|)^\gamma$. If $\alpha < 0$ then $\lambda^\alpha |y - x|^{\gamma-\alpha} \sum_{n=0}^M 2^{n\alpha} \lesssim \lambda^\alpha |y - x|^{\gamma-\alpha} \lesssim \lambda^\alpha (|y - x| + \lambda)^{\gamma-\alpha}$. The case $\alpha = 0$ is not considered in the statement of the theorem. In all cases:

$$|B| \lesssim \lambda^{\alpha \wedge 0} (|y - x| + \lambda)^{\gamma - \alpha \wedge 0}.$$

This concludes the proof. Note finally that keeping track of the constants in the calculations just above yields the continuity estimate for $\bar{\lambda}' := 4(\bar{\lambda} + \text{Diam}(K))$:

$$\|G\|_{\mathcal{G}_{\text{hom};K,\bar{\lambda},r}^\gamma} \lesssim \|F\|_{\check{\mathcal{G}}_{\text{hom};K,\bar{\lambda},r}^\gamma}, \quad \|G\|_{\mathcal{G}_{\text{coh};K,\bar{\lambda},r}^{\alpha \wedge 0, \gamma}} \lesssim \|F\|_{\check{\mathcal{G}}_{K,\bar{\lambda}',r}^{\gamma; \alpha \wedge 0, \gamma}},$$

where the implicit multiplicative constant depends only on α, γ, d , whence the announced continuity of the map $F \mapsto G$. \square

4.6 Proofs for of our Main Result II

4.6.1 Preliminary lemma

The following lemma will be useful in order to establish the multi-level Schauder estimate of section 4.4.

Lemma 4.6.1 (Test-functions and Taylor expansions). *Let $r \in \mathbb{N}_0$, $c \in \mathbb{N}_0 \cup \{-1\}$. Then there exists a constant $\text{cst} > 0$ depending only on r, c, d , such that for all test-functions $\varphi \in \mathcal{B}_c^{r+c+1}$ and $x, y \in \mathbb{R}^d$, $n \in \mathbb{N}_0$ with $|y - x| \leq 2^{-n}$, there exists a test-function*

$$\psi = \psi^{[x,y,n]} \in \text{cst}\mathcal{B}_c^r,$$

such that for such x, y, n ,

$$\varphi_x^{2^{-n}} - \sum_{|k| \leq c} \frac{(x-y)^k}{k!} (-2^n)^{|k|} (\varphi^{(k)})_y^{2^{-n}} = (2^n(x-y))^{c+1} (\psi^{[x,y,n]})_y^{2^{-n+1}}.$$

Remark 4.6.2. *Note that the scale 2^{-n+1} may be greater than 1 for $n = 0$.*

Proof. The test-function $\psi^{[x,y,n]}$ is defined by:

$$\psi^{[x,y,n]}(\cdot) := 2^d (2^n(x-y))^{-c-1} \left(\varphi(2 \cdot + 2^n(y-x)) - \sum_{|k| \leq c} \frac{(2^n(y-x))^k}{k!} \varphi^{(k)}(2 \cdot) \right).$$

The required properties on ψ follow from this expression, in particular after applying Taylor-Lagrange's formula. \square

4.6.2 Proof of Theorem 4.4.10 (multi-level Schauder estimate)

In this section, we prove Theorem 4.4.10.

Proof that $(\hat{\Pi}, \hat{\Gamma})$ is a model

There are two properties to establish.

Property of reexpansion. Let us first check that condition (4.4.3) in the definition of a model is satisfied by $\hat{\Pi}$ and $\hat{\Gamma}$, defined in (4.4.13)-(4.4.17):

$$\hat{\Pi}_y^i = \sum_{j \in \hat{I}} \hat{\Pi}_x^j \hat{\Gamma}_{xy}^{ji}.$$

This condition is immediate when $i = k \in \hat{I}^{\text{poly}}$, because $\hat{\Pi}^i$ is a polynomial. We then fix $i = a \in \hat{I}^{\text{sing}} = I^{\text{sing}}$ and write

$$\hat{\Pi}_y^a = \mathbb{K} * \Pi_y^a - \sum_{k \in \mathbb{N}_0^d: |k| < \alpha_a + \beta} (\mathbb{K} * \Pi_y^a)^{(k)}(y) \mathbb{X}_y^k.$$

We can expand $\Pi_y^a = \sum_{j \in I} \Pi_x^j \Gamma_{xy}^{ja}$ and $\mathbb{X}_y^k = \sum_{l \in \hat{I}^{\text{poly}}: l \leq k} \mathbb{X}_x^l (\Gamma^{\text{poly}})_{xy}^{lk}$, then we split the sum over $j = b \in I^{\text{sing}}$ and $j = k \in I^{\text{poly}}$, to get

$$\begin{aligned} \hat{\Pi}_y^a &= \sum_{b \in I^{\text{sing}}} (\mathbb{K} * \Pi_x^b) \Gamma_{xy}^{ba} + \sum_{k \in I^{\text{poly}}} (\mathbb{K} * \mathbb{X}_x^k) \Gamma_{xy}^{ka} \\ &\quad - \sum_{k \in \mathbb{N}_0^d: |k| < \alpha_a + \beta} (\mathbb{K} * \Pi_y^a)^{(k)}(y) \sum_{l \in \hat{I}^{\text{poly}}: l \leq k} \mathbb{X}_x^l (\Gamma^{\text{poly}})_{xy}^{lk}. \end{aligned}$$

We next write $(\mathbf{K} * \Pi_x^b) = \hat{\Pi}_x^b - \sum_{|l| < \alpha_b + \beta} (\mathbf{K} * \Pi_x^b)^{(l)}(x) \mathbb{X}_x^l$, see (4.4.14). Moreover, since we require that \mathbf{K} preserves polynomials, see Assumption 4.2.6, we have that $\mathbf{K} * \mathbb{X}_x^k$ is a polynomial of degree at most $|k| < \gamma + \beta$, hence it coincides with its Taylor expansion and we can write $\mathbf{K} * \mathbb{X}_x^k = \sum_{l \in \hat{I}^{\text{poly}}} (\mathbf{K} * \mathbb{X}_x^k)^{(l)}(x) \mathbb{X}_x^l$. This yields

$$\begin{aligned} \hat{\Pi}_y^a &= \sum_{b \in I^{\text{sing}}} \hat{\Pi}_x^b \Gamma_{xy}^{ba} - \sum_{b \in I^{\text{sing}}} \left(\sum_{l \in \mathbb{N}_0^d: |l| < \alpha_b + \beta} (\mathbf{K} * \Pi_x^b)^{(l)}(x) \mathbb{X}_x^l \right) \Gamma_{xy}^{ba} \\ &\quad + \sum_{k \in I^{\text{poly}}} \left(\sum_{l \in \hat{I}^{\text{poly}}} (\mathbf{K} * \mathbb{X}_x^k)^{(l)}(x) \mathbb{X}_x^l \right) \Gamma_{xy}^{ka} \\ &\quad - \sum_{k \in \mathbb{N}_0^d: |k| < \alpha_a + \beta} (\mathbf{K} * \Pi_y^a)^{(k)}(y) \sum_{l \in \hat{I}^{\text{poly}}: l \leq k} \mathbb{X}_x^l (\Gamma^{\text{poly}})_{xy}^{lk}. \end{aligned}$$

Switching the sums, we can write $\hat{\Pi}_y^a = \sum_{b \in I^{\text{sing}}} \hat{\Pi}_x^b \Gamma_{xy}^{ba} + \sum_{l \in \hat{I}^{\text{poly}}} \mathbb{X}_x^l \hat{\Gamma}_{xy}^{la}$ where

$$\begin{aligned} \hat{\Gamma}_{xy}^{la} &= \sum_{b \in I^{\text{sing}}: \alpha_b + \beta > |l|} (\mathbf{K} * \Pi_x^b)^{(l)}(x) \Gamma_{xy}^{ba} + \sum_{k \in I^{\text{poly}}} (\mathbf{K} * \mathbb{X}_x^k)^{(l)}(x) \Gamma_{xy}^{ka} \\ &\quad - \sum_{k \in \mathbb{N}_0^d: k \geq l, |k| < \alpha_a + \beta} (\mathbf{K} * \Pi_y^a)^{(k)}(y) (\Gamma^{\text{poly}})_{xy}^{lk}. \end{aligned}$$

Finally, we insert the constraint $\alpha_k + \beta > \alpha_l$ in the second sum, because when this is violated we have $|l| \geq |k| + \beta > |k|$ and the derivative $(\mathbf{K} * \mathbb{X}_x^k)^{(l)}(x)$ vanishes, since $\mathbf{K} * \mathbb{X}_x^k$ is a polynomial of degree at most $|k|$. We can thus rewrite the first two sums as a single sum over $j \in I = I^{\text{sing}} \cup I^{\text{poly}}$, which leads exactly to (4.4.17).

Property of homogeneity. Now let us prove that each $\hat{\Pi}_x^i$ ($i \in \hat{I} = \hat{I}^{\text{sing}} \cup \hat{I}^{\text{poly}}$) satisfies the homogeneity relation (4.4.2) with exponent $\hat{\alpha}_i$. On the one hand, if $i \in \hat{I}^{\text{poly}}$, this is straightforward, recall Example 4.4.3.

On the other hand, if $i \in \hat{I}^{\text{sing}}$, then since \mathbf{K} is regularising of order (m, r) for some $m > \alpha_i + \beta$ and since $\Pi_x^i \in \mathcal{G}_{\text{hom};r}^{\alpha_i}$, applying the content of Remark 4.5.10 yields $\mathbf{K} * \Pi_x^i \in \check{\mathcal{G}}_{\text{hom};r}^{\hat{\alpha}_i}$. Now applying Theorem 4.5.14 to $\mathbf{K} * \Pi_x^i$ using the assumption $\hat{\alpha}_i \notin \mathbb{N}$, and recalling the definition (4.4.13) of $\hat{\Pi}_x^i$, one obtains as announced $\hat{\Pi}_x^i \in \mathcal{G}_{\text{hom};r}^{\hat{\alpha}_i}$.

Proof of relation (4.4.19)

We need to show that

$$\langle \hat{f}, \hat{\Pi} \rangle_x = \sum_{i \in \hat{I}} \hat{f}^i(x) \hat{\Pi}_x^i = \mathcal{K}^{\gamma+\beta}(\langle f, \Pi \rangle)_x, \quad (4.6.1)$$

where we recall that $\mathcal{K}^{\gamma+\beta}$ is defined in (4.3.11).

We recall that $\hat{I} = \hat{I}^{\text{sing}} \cup \hat{I}^{\text{poly}}$, see (4.4.13). The key observation is that

$$\sum_{a \in \hat{I}^{\text{sing}}} \hat{f}^a(x) \hat{\Pi}_x^a = \sum_{j \in I = I^{\text{sing}} \cup I^{\text{poly}}} f^j(x) \left(\mathbf{K} * \Pi_x^j - \sum_{k \in \mathbb{N}_0^d: |k| < \alpha_j + \beta} (\mathbf{K} * \Pi_x^j)^{(k)}(x) \mathbb{X}_x^k \right),$$

which follows by (4.4.14) and (4.4.18) for $i = a \in \hat{I}^{\text{sing}} = I^{\text{sing}}$, because *the parenthesis vanishes for $j \in I^{\text{poly}}$* : indeed, $\Pi_x^j = \mathbb{X}_x^k$ is a polynomial for $j = k \in I^{\text{poly}}$, see (4.4.12), hence $\mathbf{K} * \Pi_x^j$ is a polynomial of degree at most $\alpha_j = |k|$, by Assumption 4.2.6, which then coincides

with its Taylor development at level $\alpha_j + \beta$. We thus obtain

$$\begin{aligned} \sum_{i \in \hat{I}} \hat{f}^i(x) \hat{\Pi}_x^i &= \sum_{a \in \hat{I}^{\text{sing}}} \hat{f}^a(x) \hat{\Pi}_x^a + \sum_{k \in \hat{I}^{\text{poly}}} \hat{f}^k(x) \mathbb{X}_x^k \\ &= \sum_{j \in I} f^j(x) \mathbb{K} * \Pi_x^j - \sum_{k \in \mathbb{N}_0^d: |k| < \gamma + \beta} (\mathbb{K} * \{\langle f, \Pi \rangle_x - \mathcal{R}\langle f, \Pi \rangle\})^{(k)}(x) \mathbb{X}_x^k \\ &= \mathbb{K} * \langle f, \Pi \rangle_x - \sum_{k \in \mathbb{N}_0^d: |k| < \gamma + \beta} (\mathbb{K} * \{\langle f, \Pi \rangle_x - \mathcal{R}\langle f, \Pi \rangle\})^{(k)}(x) \mathbb{X}_x^k, \end{aligned}$$

which shows that (4.6.1) holds.

Proof that \hat{f} is a modelled distribution

We now prove that \hat{f} defined in (4.4.18) is a modelled distribution of order $\gamma + \beta$ relative to the model $(\hat{\Pi}, \hat{\Gamma})$. Recalling Definition 4.4.4, we have to bound

$$|\hat{f}^i(x)|, \quad |\sum_{j \in \hat{I}} \hat{\Gamma}_{x,y}^{i,j} \hat{f}^j(y) - \hat{f}^i(x)|, \quad \text{for } i \in \hat{I}, x, y \text{ in compacts } K \subset \mathbb{R}^d.$$

Estimate of $|\hat{f}^i(x)|$. On the one hand, if $i \in \hat{I}^{\text{sing}}$ then $|\hat{f}^i(x)| = |f^i(x)| \leq \|f\|_{\mathcal{D}_{\Gamma, \alpha; K}^\gamma}$. On the other hand, if $i = k \in \hat{I}^{\text{poly}}$, then by construction

$$\hat{f}^k(x) = \sum_{\substack{j \in I = I^{\text{sing}} \cup I^{\text{poly}}: \\ \alpha_j + \beta > |k|}} f^j(x) (\mathbb{K} * \Pi_x^j)^{(k)}(x) - (\mathbb{K} * \{\langle f, \Pi \rangle_x - \mathcal{R}\langle f, \Pi \rangle\})^{(k)}(x).$$

Now recall that for $j \in I$, since \mathbb{K} is regularising of order (m, r) for some $m > \alpha_j + \beta$ and since $\Pi_x^j \in \mathcal{G}_{\text{hom}; r}^{\alpha_j}$, applying the content of Remark 4.5.10 yields $\mathbb{K} * \Pi_x^j \in \check{\mathcal{G}}_{\text{hom}; r}^{\alpha_j}$, with the continuity estimate

$$\|\mathbb{K} * \Pi_x^j\|_{\check{\mathcal{G}}_{\text{hom}; K, \bar{\lambda}, r}^{\alpha_j + \beta}} \lesssim \|\Pi_x^j\|_{\mathcal{G}_{\text{hom}; K, \bar{\lambda}, r}^{\alpha_j}} \lesssim \|\Pi\|_{\mathcal{M}_{K, \bar{\lambda}}^\alpha}, \quad (4.6.2)$$

for some $\bar{\lambda}' \geq \bar{\lambda}$, so that from Lemma 4.5.12, the pointwise evaluations $(\mathbb{K} * \Pi_x^j)^{(k)}(x)$ are well-defined with the estimates

$$|(\mathbb{K} * \Pi_x^j)^{(k)}(x)| \lesssim \|\Pi\|_{\mathcal{M}_{K, 1}^\alpha}.$$

Similarly, exploiting Lemma 4.5.12, Remark 4.5.10, the reconstruction theorem and the estimate (4.4.9), one obtains

$$|(\mathbb{K} * \{\langle f, \Pi \rangle_x - \mathcal{R}\langle f, \Pi \rangle\})^{(k)}(x)| \lesssim \|\Pi\|_{\mathcal{M}_{K', 1}^\alpha} \|f\|_{\mathcal{D}_{\Gamma, \alpha; K'}^\gamma}, \quad (4.6.3)$$

for some $K' \supset K$. Thus collecting the estimates above we have established for all $i \in \hat{I}$, $x \in K$,

$$|\hat{f}^i(x)| \lesssim \|\Pi\|_{\mathcal{M}_{K', 1}^\alpha} \|f\|_{\mathcal{D}_{\Gamma, \alpha; K'}^\gamma}.$$

Estimate of $|\sum_{j \in \hat{I}} \hat{\Gamma}_{x,y}^{i,j} \hat{f}^j(y) - \hat{f}^i(x)|$. Once again we distinguish the cases $i \in \hat{I}^{\text{sing}}$ and $i \in \hat{I}^{\text{poly}}$. If $i \in \hat{I}^{\text{sing}}$ then by construction

$$|\sum_{j \in \hat{I}} \hat{\Gamma}_{x,y}^{i,j} \hat{f}^j(y) - \hat{f}^i(x)| = |\sum_{j \in I} \Gamma_{x,y}^{i,j} f^j(y) - f^i(x)| \leq \|f\|_{\mathcal{D}_{\Gamma, \alpha; K}^\gamma} |y - x|^{\gamma - \alpha_i},$$

whence the desired estimate because $\gamma - \alpha_i = (\gamma + \beta) - \hat{\alpha}_i$.

Now fix $i = k \in \hat{I}^{\text{poly}}$, i.e. k denotes some multi-index with $|k| < \gamma + \beta$. Replacing $\hat{\Gamma}$ and \hat{f} with their definition and simplifying the expression yields

$$\begin{aligned} & \sum_{j \in \hat{I}} \hat{\Gamma}_{x,y}^{i,j} \hat{f}^j(y) - \hat{f}^i(x) = \\ &= \sum_{j \in I, \alpha_j + \beta > |i|} (\mathbf{K} * \Pi_x^j)^{(i)}(x) \left(\sum_{k \in I} \Gamma_{x,y}^{j,k} f^k(y) - f^j(x) \right) \\ &+ (\mathbf{K} * \{\langle f, \Pi \rangle_x - \mathcal{R}\langle f, \Pi \rangle\})^{(i)}(x) \\ &- \sum_{j \in \mathbb{N}_0^d, i \leq j, |j| < \gamma + \beta} \frac{(x-y)^{j-i}}{(j-i)!} (\mathbf{K} * \{\langle f, \Pi \rangle_y - \mathcal{R}\langle f, \Pi \rangle\})^{(j)}(y). \end{aligned}$$

In order to simplify our expressions it is convenient to define $G_x := \mathbf{K} * \{\langle f, \Pi \rangle_x - \mathcal{R}\langle f, \Pi \rangle\}$. Recalling (4.6.2), (4.6.3), one has $\mathbf{K} * \Pi^j \in \check{\mathcal{G}}_{\text{hom};r}^{\hat{\alpha}_j}$, $G \in \check{\mathcal{G}}_{\text{hom};r}^{\check{\gamma} + \beta}$, with continuity estimates

$$\|\mathbf{K} * \Pi^j\|_{\check{\mathcal{G}}_{\text{hom};K,\bar{\lambda},r}^{\hat{\alpha}_j}} \lesssim \|\Pi\|_{\mathcal{M}_{K,\bar{\lambda}}^\alpha}, \quad \|G\|_{\check{\mathcal{G}}_{\text{hom};K,\lambda,r}^{\check{\gamma} + \beta}} \lesssim \|\Pi\|_{\mathcal{M}_{K',\bar{\lambda}'}^\alpha} \|f\|_{\mathcal{D}_{\Gamma,\alpha;K'}^\gamma}, \quad (4.6.4)$$

for some enlarged compact $K' \supset K$ and $\bar{\lambda}' \geq \bar{\lambda}$. We now replace the pointwise evaluations in the expression above by their corresponding multi-scale decompositions as in (4.5.16). More precisely, fix $\eta \in \mathcal{D}(B(0,1))$ such that $\int \eta(x) dx = 1$ and $\int \eta(x) x^l dx = 0$ for $1 \leq |l| < \gamma + \beta$, and let $\varphi := 2\eta^{2^{-1}} - \eta$ so that there exists a constant $\text{cst} > 0$ such that $\varphi^{(l)} \in \text{cst} \mathcal{B}_{\gamma+\beta}^r$ for all $|l| < \gamma + \beta$; then

$$\begin{aligned} (\mathbf{K} * \Pi_x^j)^{(l)}(x) &= (-1)^{|l|} \mathbf{K} * \Pi_x^j((\eta^{(l)})_x) + \sum_{n=0}^{+\infty} (-2^n)^{|l|} \mathbf{K} * \Pi_x^j((\varphi^{(l)})_x^{2^{-n}}) \quad \text{for } |l| < \hat{\alpha}_j, \\ G_x^{(l)}(x) &= (-1)^{|l|} G_x((\eta^{(l)})_x) + \sum_{n=0}^{+\infty} (-2^n)^{|l|} G_x((\varphi^{(l)})_x^{2^{-n}}) \quad \text{for } |l| < \gamma + \beta, \end{aligned}$$

This yields a decomposition:

$$\sum_{j \in \hat{I}} \hat{\Gamma}_{x,y}^{i,j} \hat{f}^j(y) - \hat{f}^i(x) = \Delta_{x,y}^{i;0}(\eta) + \sum_{n=0}^{+\infty} \Delta_{x,y}^{i;n}(\varphi),$$

where for a test-function $\psi \in \mathcal{D}$ and $n \in \mathbb{N}_0$ we set

$$\begin{aligned} \Delta_{x,y}^{i;n}(\psi) &:= \sum_{j \in I, \hat{\alpha}_i + \beta > |i|} (-2^n)^{|i|} \mathbf{K} * \Pi_x^j((\psi^{(i)})_x^{2^{-n}}) \left(\sum_{k \in I} \Gamma_{x,y}^{j,k} f^k(y) - f^j(x) \right) \\ &+ (-2^n)^{|i|} G_x((\psi^{(i)})_x^{2^{-n}}) - \sum_{\substack{j \in \mathbb{N}_0^d \\ i \leq j, |j| < \gamma + \beta}} \frac{(x-y)^{j-i}}{(j-i)!} (-2^n)^{|j|} G_y((\psi^{(j)})_y^{2^{-n}}). \end{aligned}$$

We bound this quantity differently depending on whether $|y-x| \leq 2^{-n}$. For this purpose, denote $N = N_{x,y} = \min\{n \in \mathbb{N}_0, 2^{-n} \leq |y-x|\}$. On the one hand, using the properties of homogeneity (4.6.4) and the fact that f is a modelled distribution in the expression above, one obtains

$$\begin{aligned} |\Delta_{x,y}^{i;n}(\varphi)| &\lesssim \left(\sum_{j \in I, \hat{\alpha}_i > |i|} 2^{-n(-|i| + \hat{\alpha}_i)} |y-x|^{\gamma - \alpha_i} \right. \\ &\quad \left. + \sum_{\substack{j \in \mathbb{N}_0^d \\ i \leq j, |j| < \gamma + \beta}} |x-y|^{j-i} 2^{-n(-|j| + \gamma + \beta)} \right) \|\Pi\|_{\mathcal{M}_{K'}^\alpha} \|f\|_{\mathcal{D}_{\Gamma,\alpha;K'}^\gamma}, \end{aligned}$$

so that summing the geometric series yields

$$\left| \sum_{n=N_{x,y}}^{+\infty} \Delta_{x,y}^{i;n}(\varphi) \right| \lesssim \|\Pi\|_{\mathcal{M}_{K'}^\alpha} \|f\|_{\mathcal{D}_{\Gamma,\alpha;K'}^\gamma} |y-x|^{\gamma+\beta-\hat{\alpha}_i}.$$

On the other hand, observe that by definition of the reexpansion operator Γ ,

$$\sum_{j \in I} \left(\sum_{k \in I} \Gamma_{x,y}^{j,k} f^k(y) - f^j(x) \right) \mathbb{K} * \Pi_x^j = F_y - F_x = G_y - G_x,$$

whence we can rewrite

$$\begin{aligned} \Delta_{x,y}^{i;n}(\varphi) = & - \sum_{j \in I, \alpha_j + \beta < |l|} (-2^n)^{|l|} \mathbb{K} * \Pi_x^j ((\varphi^{(i)})_x^{2^{-n}}) \left(\sum_{k \in I} \Gamma_{x,y}^{j,k} f^k(y) - f^j(x) \right) \\ & + (-2^n)^{|l|} G_y \left((\varphi^{(i)})_x^{2^{-n}} - \sum_{\substack{j \in \mathbb{N}_0^d \\ i \leq j, |j| < \gamma + \beta}} \frac{(x-y)^{j-i}}{(j-i)!} (-2^n)^{|j|-|i|} (\varphi^{(j)})_y^{2^{-n}} \right). \end{aligned}$$

Thus, using the properties of homogeneity (4.6.4), Lemma 4.6.1 and the fact that f is a modelled distribution in the expression above, one obtains

$$\begin{aligned} |\Delta_{x,y}^{i;n}(\varphi)| \lesssim & \left(\sum_{j \in I, \alpha_j + \beta < |l|} 2^{n(|i| - \hat{\alpha}_i)} |y-x|^{\gamma - \alpha_i} \right. \\ & \left. + 2^{n(|i| - \gamma - \beta)} (2^n |y-x|)^{\lfloor \gamma + \beta \rfloor + 1 - |i|} \right) \|\Pi\|_{\mathcal{M}_{K'}^\alpha} \|f\|_{\mathcal{D}_{\Gamma,\alpha;K'}^\gamma}, \end{aligned}$$

and similarly for $\Delta_{x,y}^{i;0}(\eta)$, so that summing the geometric series,

$$\left| \Delta_{x,y}^{i;0}(\eta) + \sum_{n=0}^{N_{x,y}} \Delta_{x,y}^{i;n}(\varphi) \right| \lesssim \|\Pi\|_{\mathcal{M}_{K'}^\alpha} \|f\|_{\mathcal{D}_{\Gamma,\alpha;K'}^\gamma} |y-x|^{\gamma+\beta-\hat{\alpha}_i},$$

which concludes the proof that \hat{f} is a modelled distribution with the continuity estimate (4.4.21).

4.6.3 Proof of Theorem 4.4.13 (properties of reexpansion)

In this section we prove Theorem 4.4.13.

The first two properties follow from direct calculations and the definition (4.4.16)-(4.4.17) of $\hat{\Gamma}$.

Let us prove the third property, i.e. the Hölder bound: assume we have estimates $|\Gamma_{x,y}^{i,j}| \lesssim |y-x|^{\alpha_j - \alpha_i}$. Inspecting the definition (4.4.16)-(4.4.17) of $\hat{\Gamma}$, the following bound

$$|\hat{\Gamma}_{x,y}^{i,j}| \lesssim |y-x|^{\hat{\alpha}_j - \hat{\alpha}_i}, \quad \text{over } x, y \in K, i, j \in \hat{I},$$

is straightforward except when $i \in \hat{I}^{\text{poly}}$, $j \in \hat{I}^{\text{sing}}$, which is the case we tackle now, arguing as in the proof that \hat{f} is a modelled distribution, see Section 4.6.2: recall from (4.6.2) that $K * \Pi^j \in \check{\mathcal{G}}_{\text{hom};r}^{\hat{\alpha}_j}$ with the continuity estimates (4.6.4). We then replace the pointwise evaluations in the expression (4.4.17) of $\hat{\Gamma}$ by their corresponding multi-scale decompositions as in (4.5.16). More precisely, fix $\eta \in \mathcal{D}(B(0,1))$ such that $\int \eta(x) dx = 1$ and $\int \eta(x) x^l dx = 0$

for $1 \leq |l| < \gamma + \beta$, and let $\varphi := 2\eta^{2^{-1}} - \eta$ so that there exists a constant $\text{cst} > 0$ such that $\varphi^{(l)} \in \text{cst} \mathcal{B}_{\gamma+\beta}^r$ for all $|l| < \gamma + \beta$; then we obtain a decomposition:

$$\hat{\Gamma}_{x,y}^{i,j} = \hat{\Gamma}_{x,y}^{i,j;0}(\eta) + \sum_{n=0}^{+\infty} \hat{\Gamma}_{x,y}^{i,j;n}(\varphi),$$

where for a test-function $\psi \in \mathcal{D}$ and $n \in \mathbb{N}_0$ we set

$$\begin{aligned} \hat{\Gamma}_{x,y}^{i,j;n}(\psi) := & \sum_{\substack{k \in I = I^{\text{sing}} \cup I^{\text{poly}}: \\ \alpha_k + \beta > |i|}} (-2^n)^{|i|} (\mathbb{K} * \Pi_x^k) ((\psi^{(i)})_x^{2^{-n}}) \Gamma_{x,y}^{k,j} \\ & - \sum_{\substack{k \in \mathbb{N}_0^d: \\ k \geq i, |k| < \alpha_j + \beta}} (-2^n)^{|k|} (\mathbb{K} * \Pi_y^j) ((\psi^{(k)})_y^{2^{-n}}) (\Gamma^{\text{poly}})_{x,y}^{i,k}. \end{aligned}$$

Again, we produce different bounds depending on whether $|y - x| \leq 2^{-n}$, so that we denote $N = N_{x,y} = \min\{n \in \mathbb{N}_0, 2^{-n} \leq |y - x|\}$. On the one hand, using the properties of homogeneity (4.6.4) and the Hölder assumption on Γ ,

$$|\hat{\Gamma}_{x,y}^{i,j;n}(\varphi)| \lesssim \sum_{\substack{k \in I: \\ \alpha_k + \beta > |i|}} 2^{-n(\hat{\alpha}_k - |i|)} |y - x|^{\alpha_j - \alpha_k} + \sum_{\substack{k \in \mathbb{N}_0^d: \\ k \geq i, |k| < \alpha_j + \beta}} 2^{-n(\alpha_j - |k|)} |y - x|^{|k| - |i|},$$

so that summing the geometric series yields

$$\left| \sum_{n=N_{x,y}}^{+\infty} \hat{\Gamma}_{x,y}^{i,j;n}(\varphi) \right| \lesssim |y - x|^{\hat{\alpha}_j - |i|}.$$

On the other hand, observe that by the reexpansion property (4.4.3),

$$\sum_{k \in I} (\mathbb{K} * \Pi_x^k) \Gamma_{x,y}^{k,j} = \mathbb{K} * \Pi_y^j,$$

so that we can rewrite

$$\begin{aligned} \hat{\Gamma}_{x,y}^{i,j;n}(\varphi) := & \sum_{\substack{k \in I: \\ \alpha_k + \beta < |i|}} (-2^n)^{|i|} (\mathbb{K} * \Pi_x^k) ((\varphi^{(i)})_x^{2^{-n}}) \Gamma_{x,y}^{k,j} \\ & + (-2^n)^{|i|} (\mathbb{K} * \Pi_y^j) \left((\varphi^{(i)})_x^{2^{-n}} - \sum_{\substack{k \in \mathbb{N}_0^d: \\ k \geq i, |k| < \alpha_j + \beta}} \frac{(x - y)^{k-i}}{(k - i)!} (-2^n)^{|k| - |i|} (\varphi^{(k)})_y^{2^{-n}} \right), \end{aligned}$$

hence using the properties of homogeneity (4.6.4), the Hölder assumption on Γ , and Lemma 4.6.1,

$$|\hat{\Gamma}_{x,y}^{i,j;n}(\varphi)| \lesssim \sum_{\substack{k \in I: \\ \alpha_k + \beta < |i|}} 2^{n(|i| - \hat{\alpha}_k)} |y - x|^{\alpha_j - \alpha_k} + 2^{n(|i| - \hat{\alpha}_j)} (2^n |y - x|)^{[\hat{\alpha}_j] + 1 - i},$$

and similarly for $\hat{\Gamma}_{x,y}^{i,j;0}(\eta)$, so that summing the geometric series,

$$\left| \hat{\Gamma}_{x,y}^{i,j;0}(\eta) + \sum_{n=0}^{N_{x,y}} \hat{\Gamma}_{x,y}^{i,j;n}(\varphi) \right| \lesssim |y - x|^{\hat{\alpha}_j - |i|},$$

which gives the announced Hölder bound. In fact, keeping track of the constants in the estimates above, if one defines

$$\|\Gamma\|_{\mathcal{M}_K^\alpha} := \sup_{x,y \in K, i,j \in I} \frac{|\Gamma_{x,y}^{i,j}|}{|x-y|^{\alpha_j - \alpha_i}},$$

one gets the continuity estimate

$$\|\hat{\Gamma}\|_{\mathcal{M}_K^{\hat{\alpha}}} \lesssim \|\Pi\|_{\mathcal{M}_{K'}^\alpha} \|\Gamma\|_{\mathcal{M}_K^\alpha},$$

for some enlarged compact $K' \supset K$.

Appendices to Chapter 4

4.A Proof of Lemma 4.5.8

We prove Lemma 4.5.8 under the more general assumption that K satisfies Assumption 4.2.6 for some $c_0 \in \mathbb{N}_0 \cup \{-1\}$. The functions η and ζ are defined for $y \in \mathbb{R}^d$ by

$$\begin{cases} \eta^{[n,\lambda,x,\varphi]}(y) & := 2^{\beta n} (\bar{\lambda}\lambda)^d \int_{\mathbb{R}^d} \varphi(z) K_n(x + \lambda z, x + \bar{\lambda}\lambda y) dz, \\ \zeta^{[n,\lambda,x,\varphi]}(y) & := 2^{\beta n} (2^n \lambda)^{-m_0} (\bar{\lambda} 2^{-n})^d \int_{\mathbb{R}^d} \varphi(z) K_n(x + \lambda z, x + \bar{\lambda} 2^{-n} y) dz, \end{cases} \quad (4.A.1)$$

so that (4.5.6) follows from (4.A.1). Now let us prove (4.5.7). From the Assumption 4.2.6, see (4.2.9), observe that it is straightforward that η and ζ annihilate polynomials of degree up to $\min(c, c_0)$. It remains to establish the support and the C^{r_0} norm of η resp. ζ .

Support of $\eta^{[n,\lambda,x,\varphi]}$. Let $y \in \mathbb{R}^d$ be such that $\eta^{[n,\lambda,x,\varphi]}(y) \neq 0$. Then by (4.A.1) there exists $z \in \text{supp}(\varphi) \subset B(0, 1)$ such that $K_n(x + \lambda z, x + \bar{\lambda}\lambda y) \neq 0$. By assumption on the kernel K , this implies that $\lambda|\bar{\lambda}y - z| \leq \text{cst} 2^{-n}$ where the constant $\text{cst} > 0$ is the one appearing in item 1 of the Definition 4.2.2 of the regularising kernel K . Since we consider only the regime $2^{-n} \leq \lambda$, this implies that $|\bar{\lambda}y - z| \leq \text{cst}$ and thus by triangle inequality $|\bar{\lambda}y| \leq \text{cst} + 1$. Choosing $\bar{\lambda} := \text{cst} + 1$ yields $|y| \leq 1$ i.e. $\text{supp}(\eta^{[n,\lambda,x,\varphi]}) \subset B(0, 1)$ as wanted.

Support of $\zeta^{[n,\lambda,x,\varphi]}$. Let $y \in \mathbb{R}^d$ be such that $\zeta^{[n,\lambda,x,\varphi]}(y) \neq 0$. Then by (4.A.1) there exists $z \in \text{supp}(\varphi) \subset B(0, 1)$ such that $K_n(x + \lambda z, x + \bar{\lambda} 2^{-n} y) \neq 0$. By assumption on the kernel K , this implies that $|\bar{\lambda} 2^{-n} y - \lambda z| \leq \text{cst} 2^{-n}$ where the constant $\text{cst} > 0$ is the one appearing in item 1 of the Definition 4.2.2 of the regularising kernel K . Thus by the triangle inequality, $|\bar{\lambda}y| \leq \text{cst} + \frac{\lambda}{2^{-n}}$. Since we consider only the regime $\lambda \leq 2^{-n}$, choosing $\bar{\lambda} := \text{cst} + 1$ yields once again $\text{supp}(\zeta^{[n,\lambda,x,\varphi]}) \subset B(0, 1)$. Now we fix $\bar{\lambda} := \text{cst} + 1$.

Bound on C^{r_0} norm of $\eta^{[n,\lambda,x,\varphi]}$. Let $k \in \mathbb{N}_0^d$ be a multi-index with $|k| \leq r_0$, and $y \in \mathbb{R}^d$, then by differentiation under the integral,

$$\partial^k \eta^{[n,\lambda,x,\varphi]}(y) = 2^{\beta n} (\bar{\lambda}\lambda)^{d+|k|} \int_{\mathbb{R}^d} \varphi(z) \partial_2^k K_n(x + \lambda z, x + \bar{\lambda}\lambda y) dz.$$

In this integral, we subtract and add the Taylor polynomial of φ at $\bar{\lambda}y$ of order $|k| - 1$:

$$\begin{aligned} & \partial^k \eta^{[n, \lambda, x, \varphi]}(y) \\ &= 2^{\beta n} (\bar{\lambda} \lambda)^{d+|k|} \int_{\mathbb{R}^d} \left(\varphi(z) - \sum_{|l| \leq |k|-1} \frac{\partial^l \varphi(\bar{\lambda} y)}{l!} (z - \bar{\lambda} y)^l \right) \partial_2^k K_n(x + \lambda z, x + \bar{\lambda} \lambda y) dz \\ & \quad + 2^{\beta n} (\bar{\lambda} \lambda)^{d+|k|} \sum_{|l| \leq |k|-1} \frac{\partial^l \varphi(\bar{\lambda} y)}{l!} \int_{\mathbb{R}^d} (z - \bar{\lambda} y)^l \partial_2^k K_n(x + \lambda z, x + \bar{\lambda} \lambda y) dz. \end{aligned}$$

Using Taylor-Lagrange's inequality in the first integral (and absorbing $\bar{\lambda}^{d+|k|}$ into the implicit constant):

$$\begin{aligned} & |\partial^k \eta^{[n, \lambda, x, \varphi]}(y)| \\ & \leq 2^{\beta n} \lambda^{d+|k|} \sum_{|l|=|k|} \frac{1}{l!} \|\varphi\|_{C^{|k|}} \int_{\mathbb{R}^d} |z - \bar{\lambda} y|^{|k|} |\partial_2^k K_n(x + \lambda z, x + \bar{\lambda} \lambda y)| dz \\ & \quad + 2^{\beta n} \lambda^{d+|k|} \sum_{|l| \leq |k|-1} \frac{\|\varphi\|_{C^{|k|}}}{l!} \left| \int_{\mathbb{R}^d} (z - \bar{\lambda} y)^l \partial_2^k K_n(x + \lambda z, x + \bar{\lambda} \lambda y) dz \right|. \end{aligned}$$

In the first resp. the second integral we use the property (4.2.6) resp. (4.2.7) of the kernel K , noting that $\int_{\mathbb{R}^d} (z - \bar{\lambda} y)^l \partial_2^k K_n(x + \lambda z, x + \bar{\lambda} \lambda y) dz = \lambda^{-|l|-d} \int_{\mathbb{R}^d} (\tilde{z} - \bar{\lambda} \lambda y)^l \partial_2^k K_n(x + \tilde{z}, x + \bar{\lambda} \lambda y) d\tilde{z}$:

$$\begin{aligned} & |\partial^k \eta^{[n, \lambda, x, \varphi]}(y)| \\ & \lesssim 2^{\beta n} \lambda^{d+|k|} \|\varphi\|_{C^{|k|}} 2^{(d-\beta+|k|)n} \int_{z \in B(\bar{\lambda} y, \text{cst} \frac{2^{-n}}{\lambda})} |z - \bar{\lambda} y|^{|k|} dz \\ & \quad + 2^{\beta n} \lambda^{d+|k|} \sum_{|l| \leq |k|-1} \|\varphi\|_{C^{|k|}} \lambda^{-d-|l|} 2^{-\beta n}. \end{aligned}$$

As a consequence:

$$\|\eta^{[n, \lambda, x, \varphi]}\|_{C^{|r_0|}} \lesssim \|\varphi\|_{C^{|r_0|}} \leq 1.$$

Bound on C^{r_0} norm of $\zeta^{[n, \lambda, x, \varphi]}$. Let $k \in \mathbb{N}_0^d$ be a multi-index with $|k| \leq r_0$, and $y \in \mathbb{R}^d$, then by differentiation under the integral,

$$\partial^k \zeta^{[n, \lambda, x, \varphi]}(y) = 2^{\beta n} (2^n \lambda)^{-m_0} (\bar{\lambda} 2^{-n})^{d+|k|} \int_{\mathbb{R}^d} \varphi(z) \partial_2^k K_n(x + \lambda z, x + 2^{-n} y) dz.$$

Recall that by assumption, φ annihilates polynomials of degree $c \geq m_0 - 1$ so in this integral we can subtract the Taylor polynomial of $\partial_2^k K_n(\cdot, x + \bar{\lambda} 2^{-n} y)$ at x of order $m_0 - 1$: let us denote:

$$R_{n, \lambda, x, y, k}^{[m_0-1]}(z) := \partial_2^k K_n(x + \lambda z, x + \bar{\lambda} 2^{-n} y) - \sum_{|l| \leq m_0-1} \partial_1^l \partial_2^k K_n(x, x + \bar{\lambda} 2^{-n} y) \frac{(\lambda z)^l}{l!},$$

then

$$\partial^k \zeta^{[n, \lambda, x, \varphi]}(y) = 2^{\beta n} (2^n \lambda)^{-m_0} (\bar{\lambda} 2^{-n})^{d+|k|} \int_{\mathbb{R}^d} \varphi(z) R_{n, \lambda, x, y, k}^{[m_0-1]}(z) dz.$$

By Taylor-Lagrange's formula,

$$\left| R_{n, \lambda, x, y, k}^{[m_0-1]}(z) \right| \leq \sum_{|l|=m_0} \frac{1}{l!} \left\| \partial_1^l \partial_2^k K_n \right\|_{\infty} |\lambda z|^{m_0}.$$

Thus, by the property (4.2.6) of the kernel K ,

$$\left| R_{n,\lambda,x,y,k}^{[m_0-1]}(z) \right| \lesssim 2^{(d-\beta+m_0+|k|)n} \lambda^{m_0} |z|^{m_0},$$

where the multiplicative constant depends only on the kernel K (and the dimension d of the underlying space). Consequently,

$$\left| \partial^k \zeta^{[n,\lambda,x,\varphi]}(y) \right| \lesssim \int_{\mathbb{R}^d} |\varphi(z)| |z|^{m_0} dz.$$

Recall that by assumption, $\varphi \in \mathcal{B}_{m_0-1}^{r_0}$, so this last integral can be bounded by 1. Thus, this establishes:

$$\|\zeta^{[n,\lambda,x,\varphi]}\|_{C^{r_0}} \lesssim 1,$$

where the multiplicative constant depends only on the kernel K (and the underlying dimension d). This concludes the proof. \square

4.B Spaces of germs and distributions are “independent in r ”

In this section we prove that the choice of the regularity r of test-functions in the different spaces of distributions and germs studied in this paper generally does not matter.

Proposition 4.B.1. *Let $\bar{\alpha}, \alpha, \gamma \in \mathbb{R}$ with $\bar{\alpha}, \alpha \leq \gamma$. Then:*

1. *The Definition 4.2.1 of Hölder-Zygmund spaces \mathcal{Z}^γ does not depend on the choice of $r \geq r_\gamma := \min\{r \in \mathbb{N}_0, r > -\gamma\}$.*
2. *The Definition 4.3.2 of homogeneous and coherent germs $\mathcal{G}^{\bar{\alpha};\alpha,\gamma}$ does not depend on the choice of $r \geq r_{\bar{\alpha},\alpha} := \min\{r \in \mathbb{N}_0, r > \max(-\bar{\alpha}, -\alpha)\}$.*
3. *The Definition 4.5.1 of weakly homogeneous and coherent germs $\check{\mathcal{G}}^{\bar{\alpha};\alpha,\gamma}$ does not depend on the choice of $r \geq r_{\bar{\alpha},\alpha} := \min\{r \in \mathbb{N}_0, r > \max(-\bar{\alpha}, -\alpha)\}$.*

A proof in the case of the Hölder-Zygmund spaces \mathcal{Z}^γ can be found for instance in [FH20, Lemma 14.13].

A proof in the case of the space of homogeneous and coherent germs $\mathcal{G}^{\bar{\alpha};\alpha,\gamma}$ when $\gamma \neq 0$ can be found in [CZ20, Propositions 13.1 and 13.2], see Remark 4.3.4. However the approach in this reference fails to cover the case $\gamma = 0$.

We prove Proposition 4.B.1 using the following result from wavelet theory:

Theorem 4.B.2 (Daubechies’ wavelets, see [Dau88; Dau92; Mey92]). *For any $r, d \in \mathbb{N}_0$, there exist a compactly supported function $\varphi \in C_c^r(\mathbb{R}^d)$ and a finite family Ψ of compactly supported functions $\psi \in C_c^r(\mathbb{R}^d)$ satisfying $\int_{\mathbb{R}^d} \psi(x) x^k dx = 0$ for all multi-indices $k \in \mathbb{N}_0^d$ with $|k| \leq r$, such that for all $n_0 \in \mathbb{Z}$, the family*

$$\left\{ 2^{\frac{-n_0 d}{2}} \varphi_k^{2^{-n_0}}, k \in 2^{-n_0} \mathbb{Z} \right\} \cup \left\{ 2^{\frac{-n d}{2}} \psi_k^{2^{-n}}, k \in 2^{-n} \mathbb{Z}, \psi \in \Psi \right\}, \quad (4.B.1)$$

is a Hilbert basis of $L^2(\mathbb{R}^d)$.

In fact, for C^r functions the convergence along the basis (4.B.1) holds in C^r norm. This allows us to prove Proposition 4.B.1.

Proof of Proposition 4.B.1. As a proof in the case of the spaces \mathcal{Z}^γ can be found in the literature, see [FH20, Lemma 14.13], we only consider the case of spaces of germs. We argue slightly differently in the case of \mathcal{G} and in the case of $\check{\mathcal{G}}$:

- in the case of \mathcal{G} , we exploit the decomposition (4.B.1) starting from $n_0 \in \mathbb{Z}$ with $2^{-n_0} \sim \lambda$,
- in the case of $\check{\mathcal{G}}$, we exploit the decomposition (4.B.1) starting from $n_0 = 0$.

Let $\alpha, \bar{\alpha}, \gamma \in \mathbb{R}$ be such that $\alpha \leq \gamma$, $\bar{\alpha} \leq \gamma$, and define

$$r_{\alpha, \bar{\alpha}} := \min\{r \in \mathbb{N}_0, r > \max\{-\alpha, -\bar{\alpha}\}\}.$$

For any $r \in \mathbb{N}_0$ arbitrary, we denote $\mathcal{G}_r^{\bar{\alpha}; \alpha, \gamma}$ the space of germs corresponding to the family of seminorms given by (4.3.6); and similarly $\check{\mathcal{G}}_r^{\bar{\alpha}; \alpha, \gamma}$ corresponding to the family of seminorms (4.5.11).

Let $r \in \mathbb{N}_0$ with $r \geq r_{\alpha, \bar{\alpha}}$, we shall show that

$$\mathcal{G}_r^{\bar{\alpha}; \alpha, \gamma} = \mathcal{G}_{r_{\alpha, \bar{\alpha}}}^{\bar{\alpha}; \alpha, \gamma}, \quad (4.B.2)$$

$$\check{\mathcal{G}}_r^{\bar{\alpha}; \alpha, \gamma} = \check{\mathcal{G}}_{r_{\alpha, \bar{\alpha}}}^{\bar{\alpha}; \alpha, \gamma}. \quad (4.B.3)$$

Proof of (4.B.2). It suffices to show the inclusion

$$\mathcal{G}_r^{\bar{\alpha}; \alpha, \gamma} \subset \mathcal{G}_{r_{\alpha, \bar{\alpha}}}^{\bar{\alpha}; \alpha, \gamma},$$

because the other one follows from the definitions. Let $F \in \mathcal{G}_r^{\bar{\alpha}; \alpha, \gamma}$, we start with the estimate of homogeneity. Let φ, Ψ be as in Theorem 4.B.2 applied to r . Let $K \subset \mathbb{R}^d$ be compact, $x \in K$, $\lambda \in (0, 1]$, $\eta \in \mathcal{B}^{r_{\alpha, \bar{\alpha}}}$, we want to estimate $F_x(\eta_x^\lambda)$. Set $N := N_\lambda := \min\{n \in \mathbb{N}, 2^{-n} \leq \lambda\}$. From the decomposition (4.B.1) starting at N_λ , we have:

$$\begin{aligned} F_x(\eta_x^\lambda) &= \sum_{k \in 2^{-N_\lambda} \mathbb{Z}} 2^{-N_\lambda d} \langle \eta_x^\lambda, \varphi_k^{2^{-N_\lambda}} \rangle F_x(\varphi_k^{2^{-N_\lambda}}) \\ &\quad + \sum_{n=N_\lambda}^{+\infty} \sum_{k \in 2^{-n} \mathbb{Z}} \sum_{\psi \in \Psi} 2^{-nd} \langle \eta_x^\lambda, \psi_k^{2^{-n}} \rangle F_x(\psi_k^{2^{-n}}). \end{aligned}$$

In the first line, for reasons of support one has $|x - k| \lesssim \lambda$ and only a finite number of k contribute to the sum. In the second line, for reasons of support one has $|x - k| \lesssim \lambda$ and $\sim 2^{(n-N_\lambda)d}$ values of k contribute to the sum. Thus, because of the coherence and homogeneity of F , one has $|F_x(\varphi_k^{2^{-N_\lambda}})| \lesssim \lambda^{\bar{\alpha}} + \lambda^\gamma \lesssim \lambda^{\bar{\alpha}}$ in the first line, and $|F_x(\psi_k^{2^{-n}})| \lesssim 2^{-n\bar{\alpha}} + 2^{-n\alpha} \lambda^{\gamma-\alpha}$ in the second line. Also, since the functions ψ cancel polynomials of degree up to $r \geq r_{\alpha, \bar{\alpha}}$, by subtracting a Taylor polynomial of degree $\tilde{r} := r_{\alpha, \bar{\alpha}} - 1$ in the integral one obtains $|\langle \eta_x^\lambda, \psi_k^{2^{-n}} \rangle| \lesssim \|\eta\|_{C^{\tilde{r}+1}} \lambda^{-d} \left(\frac{2^{-n}}{\lambda}\right)^{\tilde{r}+1}$. Thus, collecting these estimate:

$$\begin{aligned} |F_x(\eta_x^\lambda)| &\lesssim 2^{-N_\lambda d} \|\eta\|_\infty 2^{N_\lambda d} 2^{-N_\lambda \bar{\alpha}} \\ &\quad + \sum_{n=N_\lambda}^{+\infty} 2^{-nd} 2^{(n-N_\lambda)d} \|\eta\|_{C^{\tilde{r}+1}} \lambda^{-d} \left(\frac{2^{-n}}{\lambda}\right)^{\tilde{r}+1} (2^{-n\bar{\alpha}} + 2^{-n\alpha} \lambda^{\gamma-\alpha}). \end{aligned}$$

Recalling that by choice of \tilde{r} one has $\tilde{r} + 1 > -\alpha$ and $\tilde{r} + 1 > -\bar{\alpha}$, by summing the geometric series one obtains the wanted homogeneity estimate

$$|F_x(\eta_x^\lambda)| \lesssim \|\eta\|_{C^{r_{\alpha, \bar{\alpha}}}} \lambda^{\bar{\alpha}}.$$

We establish the estimate of coherence similarly. Let again φ, Ψ be as in Theorem 4.B.2 applied to r . Let $K \subset \mathbb{R}^d$ be compact, $x, y \in K$, $\lambda \in (0, 1]$, $\eta \in \mathcal{B}^{r_{\alpha, \bar{\alpha}}}$, we want to estimate

$(F_y - F_x)(\eta_x^\lambda)$. As above, set $N := N_\lambda := \min\{n \in \mathbb{N}, 2^{-n} \leq \lambda\}$, from the decomposition (4.B.1) starting at N_λ , we have:

$$\begin{aligned} (F_y - F_x)(\eta_x^\lambda) &= \sum_{k \in 2^{-N_\lambda} \mathbb{Z}} 2^{-N_\lambda d} \langle \eta_x^\lambda, \varphi_k^{2^{-N_\lambda}} \rangle (F_y - F_x)(\varphi_k^{2^{-N_\lambda}}) \\ &\quad + \sum_{n=N_\lambda}^{+\infty} \sum_{k \in 2^{-n} \mathbb{Z}} \sum_{\psi \in \Psi} 2^{-nd} \langle \eta_x^\lambda, \psi_k^{2^{-n}} \rangle (F_y - F_x)(\psi_k^{2^{-n}}). \end{aligned}$$

We perform the same estimate as above except this time from the assumption of coherence on the germ F (and the fact that $|x - k| \lesssim \lambda$ for reasons of support) one has $|(F_y - F_x)(\varphi_k^{2^{-N_\lambda}})| \lesssim \lambda^\alpha (|y - x| + \lambda)^{\gamma - \alpha}$ in the first line, and similarly in the second line $|(F_y - F_x)(\psi_k^{2^{-n}})| \lesssim 2^{-n\alpha} (|y - x| + \lambda)^{\gamma - \alpha}$. Thus, collecting all the estimate, one obtains for $\tilde{r} := r_{\alpha, \bar{\alpha}} - 1$:

$$\begin{aligned} |(F_y - F_x)(\eta_x^\lambda)| &\lesssim 2^{-N_\lambda d} \|\eta\|_\infty 2^{N_\lambda d} \lambda^\alpha (|y - x| + \lambda)^{\gamma - \alpha} \\ &\quad + \sum_{n=N_\lambda}^{+\infty} 2^{-nd} 2^{(n-N_\lambda)d} \|\eta\|_{C^{\tilde{r}+1}} \lambda^{-d} \left(\frac{2^{-n}}{\lambda} \right)^{\tilde{r}+1} 2^{-n\alpha} (|y - x| + \lambda)^{\gamma - \alpha}, \end{aligned}$$

so that using the fact that $\tilde{r} + 1 > -\alpha$ one obtains after summing the geometric series

$$|(F_y - F_x)(\eta_x^\lambda)| \lesssim \|\eta\|_{C^{r_{\alpha, \bar{\alpha}}}} \lambda^\alpha (|y - x| + \lambda)^{\gamma - \alpha}.$$

This concludes the proof of (4.B.2).

Proof of (4.B.3). It suffices to show the inclusion

$$\check{\mathcal{G}}_r^{\bar{\alpha}; \alpha, \gamma} \subset \check{\mathcal{G}}_{r_{\alpha, \bar{\alpha}}}^{\bar{\alpha}; \alpha, \gamma},$$

because the other one follows from the definitions. Let $F \in \check{\mathcal{G}}_r^{\bar{\alpha}; \alpha, \gamma}$, we start with the estimate of homogeneity. Let again φ, Ψ be as in Theorem 4.B.2 applied to r . Let $K \subset \mathbb{R}^d$ be compact, $x \in K$, $\lambda \in (0, 1]$, $\eta \in \mathcal{B}^{r_{\alpha, \bar{\alpha}}}$, $\check{\eta} \in \mathcal{B}_{\bar{\alpha}}^{r_{\alpha, \bar{\alpha}}}$. We want to estimate $F_x(\eta_x)$ and $F_x(\check{\eta}_x^\lambda)$. From the decomposition (4.B.1) starting at 0, we have:

$$\begin{aligned} F_x(\eta_x) &= \sum_{k \in \mathbb{Z}} \langle \eta_x, \varphi_k \rangle F_x(\varphi_k) \\ &\quad + \sum_{n=0}^{+\infty} \sum_{k \in 2^{-n} \mathbb{Z}} \sum_{\psi \in \Psi} 2^{-nd} \langle \eta_x, \psi_k^{2^{-n}} \rangle F_x(\psi_k^{2^{-n}}). \end{aligned}$$

In the first line, for reasons of support one has $|x - k| \lesssim 1$ and only a finite number of k contribute to the sum. In the second line, for reasons of support one has $|x - k| \lesssim 1$ and $\sim 2^{nd}$ values of k contribute to the sum. Thus, because of the coherence and homogeneity of the germ F , one has $|F_x(\varphi_k)| \lesssim 1$ in the first line, and $|F_x(\psi_k^{2^{-n}})| \lesssim 2^{-n\bar{\alpha}} + 2^{-n\alpha}$ in the second line. Also, since the functions ψ cancel polynomials of degree up to $r \geq r_{\alpha, \bar{\alpha}}$, by subtracting a Taylor polynomial of degree $\tilde{r} := r_{\alpha, \bar{\alpha}} - 1$ in the integral one obtains $|\langle \eta_x, \psi_k^{2^{-n}} \rangle| \lesssim \|\eta\|_{C^{\tilde{r}+1}} 2^{-n(\tilde{r}+1)}$. Thus, collecting these estimates:

$$|F_x(\eta_x)| \lesssim \|\eta\|_\infty + \sum_{n=0}^{+\infty} 2^{-nd} 2^{nd} \|\eta\|_{C^{\tilde{r}+1}} 2^{-n(\tilde{r}+1)} (2^{-n\bar{\alpha}} + 2^{-n\alpha}),$$

so that summing the geometric series and recalling that $\tilde{r} + 1 > -\alpha$, $\tilde{r} + 1 > -\bar{\alpha}$, one obtains $|F_x(\eta_x)| \lesssim \|\eta\|_{C^{r_{\alpha, \bar{\alpha}}}}$.

Similarly:

$$\begin{aligned}
F_x(\check{\eta}_x^\lambda) &= \sum_{k \in \mathbb{Z}} \langle \check{\eta}_x^\lambda, \varphi_k \rangle F_x(\varphi_k) \\
&+ \sum_{n=0}^{N_\lambda} \sum_{k \in 2^{-n}\mathbb{Z}} \sum_{\psi \in \Psi} 2^{-nd} \langle \check{\eta}_x^\lambda, \psi_k^{2^{-n}} \rangle F_x(\psi_k^{2^{-n}}) \\
&+ \sum_{n=N_\lambda+1}^{+\infty} \sum_{k \in 2^{-n}\mathbb{Z}} \sum_{\psi \in \Psi} 2^{-nd} \langle \check{\eta}_x^\lambda, \psi_k^{2^{-n}} \rangle F_x(\psi_k^{2^{-n}}).
\end{aligned}$$

In the first line, for reasons of support one has $|x - k| \lesssim 1$ and only a finite number of k contribute to the sum. Also, since $\check{\eta}$ annihilate polynomials of degree up to $[\bar{\alpha}]$, by subtracting a Taylor polynomial of φ of degree $[\bar{\alpha}]$ in the integral one obtains $|\langle \check{\eta}_x^\lambda, \varphi_k \rangle| \lesssim \|\check{\eta}\|_\infty \lambda^{[\bar{\alpha}]+1} \lesssim \|\check{\eta}\|_\infty \lambda^{\bar{\alpha}}$. Furthermore, because of the coherence and homogeneity of the germ F , $|F_x(\varphi_k)| \lesssim 1$.

In the second line, for reasons of support one has $|x - k| \lesssim 2^{-n}$ and only a finite number of k contribute to the sum. Also, since $\check{\eta}$ annihilate polynomials of degree up to $[\bar{\alpha}]$, by subtracting a Taylor polynomial of φ of degree $[\bar{\alpha}]$ in the integral one obtains $|\langle \check{\eta}_x^\lambda, \psi_k^{2^{-n}} \rangle| \lesssim \|\check{\eta}\|_\infty 2^{nd} \left(\frac{\lambda}{2^{-n}}\right)^{[\bar{\alpha}]+1}$. Furthermore, because of the coherence and homogeneity of the germ F , $|F_x(\psi_k^{2^{-n}})| \lesssim 2^{-n\gamma} + 2^{-n\bar{\alpha}} \lesssim 2^{-n\bar{\alpha}}$ (since we assume $\bar{\alpha} \leq \gamma$).

In the third line, for reasons of support one has $|x - k| \lesssim \lambda$ and $\sim 2^{(n-N_\lambda)d}$ values of k contribute to the sum. Also, since the functions ψ cancel polynomials of degree up to $r \geq r_{\alpha, \bar{\alpha}}$, by subtracting a Taylor polynomial of $\check{\eta}$ of degree $\tilde{r} := r_{\alpha, \bar{\alpha}} - 1$ in the integral one obtains $|\langle \check{\eta}_x^\lambda, \psi_k^{2^{-n}} \rangle| \lesssim \|\check{\eta}\|_{C^{\tilde{r}+1}} 2^{-n(\tilde{r}+1)}$. Furthermore, because of the coherence and homogeneity of the germ F , $|F_x(\psi_k^{2^{-n}})| \lesssim 2^{-n\alpha} \lambda^{\gamma-\alpha} + 2^{-n\bar{\alpha}}$.

Collecting these estimates yields:

$$\begin{aligned}
|F_x(\check{\eta}_x^\lambda)| &\lesssim \|\check{\eta}\|_\infty \lambda^{\bar{\alpha}} \\
&+ \sum_{n=0}^{N_\lambda} 2^{-nd} \|\check{\eta}\|_\infty 2^{nd} \left(\frac{\lambda}{2^{-n}}\right)^{[\bar{\alpha}]+1} 2^{-n\bar{\alpha}} \\
&+ \sum_{n=N_\lambda+1}^{+\infty} 2^{-nd} 2^{(n-N_\lambda)d} \|\check{\eta}\|_{C^{\tilde{r}+1}} 2^{-n(\tilde{r}+1)} \left(2^{-n\alpha} \lambda^{\gamma-\alpha} + 2^{-n\bar{\alpha}}\right),
\end{aligned}$$

so that summing the geometric series and recalling that $\tilde{r} + 1 > -\alpha$, $\tilde{r} + 1 > -\bar{\alpha}$, $[\bar{\alpha}] + 1 > \bar{\alpha}$, one obtains:

$$|F_x(\check{\eta}_x^\lambda)| \lesssim \|\check{\eta}\|_{C^{r_{\alpha, \bar{\alpha}}}} \lambda^{\bar{\alpha}}.$$

Once again, we establish the property of coherence similarly. Let φ, Ψ be as in Theorem 4.B.2 applied to r . Let $K \subset \mathbb{R}^d$ be compact, $x, y \in K$, $\lambda \in (0, 1]$, $\eta \in \mathcal{B}^{r_{\alpha, \bar{\alpha}}}$, $\check{\eta} \in \mathcal{B}^{r_{\alpha, \bar{\alpha}}}$, we want to estimate $(F_y - F_x)(\eta_x)$ and $(F_y - F_x)(\check{\eta}_x^\lambda)$. From the decomposition (4.B.1) starting at 0, we have:

$$\begin{aligned}
(F_y - F_x)(\eta_x) &= \sum_{k \in \mathbb{Z}} \langle \eta_x, \varphi_k \rangle (F_y - F_x)(\varphi_k) \\
&+ \sum_{n=0}^{+\infty} \sum_{k \in 2^{-n}\mathbb{Z}} \sum_{\psi \in \Psi} 2^{-nd} \langle \eta_x, \psi_k^{2^{-n}} \rangle (F_y - F_x)(\psi_k^{2^{-n}}).
\end{aligned}$$

We perform the same estimates as in the case of the homogeneity, except for the fact that in the first line $|(F_y - F_x)(\varphi_k)| \lesssim 1$ because of the assumption of coherence of F (and the fact that $|x - k| \lesssim 1$ for reasons of support); and the fact that in the second line,

$|(F_y - F_x)(\psi_k^{2^{-n}})| \lesssim 2^{-n\alpha}$ because of the assumption of coherence of F (and the facts that $x, y \in K$ for a compact K and $|x - k| \lesssim 1$ for reasons of support). Thus this yields for $\tilde{r} := r_{\alpha, \bar{\alpha}} - 1$:

$$\begin{aligned} |(F_y - F_x)(\eta_x)| &\lesssim \|\eta\|_\infty \\ &\quad + \sum_{n=0}^{+\infty} 2^{-nd} 2^{nd} \|\eta\|_{C^{\tilde{r}+1}} 2^{-n(\tilde{r}+1)} 2^{-n\alpha}, \end{aligned}$$

so that summing the geometric series yields $|(F_y - F_x)(\eta_x)| \lesssim 1$.

Similarly:

$$\begin{aligned} (F_y - F_x)(\check{\eta}_x^\lambda) &= \sum_{k \in \mathbb{Z}} \langle \check{\eta}_x^\lambda, \varphi_k \rangle (F_y - F_x)(\varphi_k) \\ &\quad + \sum_{n=0}^{N_\lambda} \sum_{k \in 2^{-n}\mathbb{Z}} \sum_{\psi \in \Psi} 2^{-nd} \langle \check{\eta}_x^\lambda, \psi_k^{2^{-n}} \rangle (F_y - F_x)(\psi_k^{2^{-n}}) \\ &\quad + \sum_{n=N_\lambda+1}^{+\infty} \sum_{k \in 2^{-n}\mathbb{Z}} \sum_{\psi \in \Psi} 2^{-nd} \langle \check{\eta}_x^\lambda, \psi_k^{2^{-n}} \rangle (F_y - F_x)(\psi_k^{2^{-n}}). \end{aligned}$$

We perform the same estimates as in the case of the homogeneity above except that here, in the first line $|(F_y - F_x)(\varphi_k)| \lesssim 1$; in the second line $|(F_y - F_x)(\psi_k^{2^{-n}})| \lesssim 2^{-n\alpha}(|y - x| + 2^{-n})^{\gamma-\alpha}$; and in the third line $|(F_y - F_x)(\psi_k^{2^{-n}})| \lesssim 2^{-n\alpha}(|y - x| + \lambda)^{\gamma-\alpha}$. Thus this yields for $\tilde{r} := r_{\alpha, \bar{\alpha}} - 1$:

$$\begin{aligned} |(F_y - F_x)(\check{\eta}_x^\lambda)| &\lesssim \|\check{\eta}\|_\infty \lambda^\gamma \\ &\quad + \sum_{n=0}^{N_\lambda} 2^{-nd} \|\check{\eta}\|_\infty 2^{nd} \left(\frac{\lambda}{2^{-n}}\right)^{[\gamma]+1} 2^{-n\alpha} (|y - x| + 2^{-n})^{\gamma-\alpha} \\ &\quad + \sum_{n=N_\lambda+1}^{+\infty} 2^{-nd} 2^{(n-N_\lambda)d} \|\check{\eta}\|_{C^{\tilde{r}+1}} 2^{-n(\tilde{r}+1)} 2^{-n\alpha} (|y - x| + \lambda)^{\gamma-\alpha}, \end{aligned}$$

so that after summing the geometric series and recalling that $\tilde{r}+1 > -\alpha$ and $[\gamma]+1 > \gamma \geq \alpha$, we obtain:

$$|(F_y - F_x)(\check{\eta}_x^\lambda)| \lesssim \|\check{\eta}\|_{C^{r_{\alpha, \bar{\alpha}}}} \lambda^\alpha (|y - x| + \lambda)^{\gamma-\alpha}.$$

This concludes the proof. □

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