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Nicolas Camps

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Probabilistic approaches for nonlinear Schrödinger equations

*Approches probabilistes pour des équations de Schrödinger
non-linéaires*

Thèse de doctorat de l'université Paris-Saclay

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Professeur, la co-direction de Frédéric Rousset, Professeur

Thèse soutenue à Paris-Saclay, le 21 septembre 2022, par

Nicolas CAMPS

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Titre : Approches probabilistes pour des équations de Schrödinger non-linéaires.....

Mots clés : équation de Schrödinger non-linéaire, problème de Cauchy, données initiales aléatoires, diffusion, instabilités

Résumé : Cette thèse a pour objet l'étude d'équations de Schrödinger non-linéaires avec des données initiales aléatoires, dans des régimes singuliers. Les équations de Schrödinger non-linéaires interviennent dans de nombreux modèles ondulatoires pour décrire l'enveloppe d'un paquet d'ondes qui oscillent rapidement. Quant à elle, l'étude statistique de ces ondes non-linéaires dans des régimes singuliers est motivée par la présence d'instabilités qui compromettent l'analyse des trajectoires individuelles.

Dans une première partie, on établit l'existence de deux types de données initiales : les données statistiques, qui donnent lieu à des solutions fortes, et les données pathologiques qui sont instantanément instables. Ces deux types de données initiales sont génériques, respectivement au sens de la me-

sure et de la topologie. Dans une seconde partie, des méthodes d'analyse sont combinées avec l'approche probabiliste pour décrire le comportement en temps long des données statistiques, ainsi que la stabilité de dynamiques non-linéaires sous l'effet de perturbations aléatoires. Enfin, une troisième partie est consacrée au développement de schémas itératifs quasi-linéaires dans un contexte probabiliste. Ces schémas exploitent l'information statistique des données initiales, et permettent l'analyse d'équations faiblement dispersives dans des régimes singuliers. La troisième partie se conclut par une prémisse au problème de l'invariance de la mesure de Gibbs pour l'équation de Schrödinger cubique sur la sphère. Une interaction singulière est isolée, ce qui permet la mise en place future d'un schéma quasi-linéaire adapté à ce problème.

Title : Probabilistic approaches for nonlinear Schrödinger equations.....

Keywords : nonlinear Schrödinger equation, Cauchy problem, random initial data, scattering, instabilities

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In a first part, we establish the existence of two types of initial data : statistical data, which give rise to strong solutions, and pathological data, which are instantaneously unstable. These two types of initial data are generic, respectively in the sense of measure theory and topology. In a se-

cond part, analytical methods are combined with the probabilistic approach to describe the long-time behavior of the statistical data, as well as the stability of nonlinear dynamics under random perturbations. Finally, a third part is devoted to the development of quasilinear iterative schemes in a probabilistic context. These schemes exploit the statistical information of the initial data and allow the analysis of weakly dispersive equations in singular regimes. The third part concludes with a premise for the problem of invariance of the Gibbs measure for the cubic Schrödinger equation on the sphere. A singular interaction is isolated, which allows the future implementation of a quasilinear scheme adapted to this problem.

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Résumé

Cette thèse a pour objet l'étude d'équations de Schrödinger non-linéaires avec des données initiales aléatoires, dans des régimes singuliers. Les équations de Schrödinger non-linéaires interviennent dans de nombreux modèles ondulatoires pour décrire l'enveloppe d'un paquet d'ondes qui oscillent rapidement. Quant à elle, l'étude statistique de ces ondes non-linéaires dans des régimes singuliers est motivée par la présence d'instabilités qui compromettent l'analyse des trajectoires individuelles.

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In a first part, we establish the existence of two types of initial data: statistical data, which give rise to strong solutions, and pathological data, which are instantaneously unstable. These two types of initial data are generic, respectively in the sense of measure theory and topology. In a second part, analytical methods are combined with the probabilistic approach to describe the long-time behavior of the statistical data, as well as the stability of nonlinear dynamics under random perturbations. Finally, a third part is devoted to the development of quasilinear iterative schemes in a probabilistic context. These schemes exploit the statistical information of the initial data and allow the analysis of weakly dispersive equations in singular regimes. The third part concludes with a premise for the problem of invariance of the Gibbs measure for the cubic Schrödinger equation on the sphere. A singular interaction is isolated, which allows the future implementation of a quasilinear scheme adapted to this problem.

The manuscript is made of five papers, two single authored and three collaborations.

- (Dec. 2020) “Asymptotic stability of small ground states for NLS under random perturbations”. published in *Ann. Inst. H. Poincaré Anal. Non Linéaire* [Cam22]
- (Oct. 2021) “Scattering for the cubic Schrödinger equation in 3D with randomized radial initial data” published in *Trans. Am. Soc.* [Cam23]
- (Mar. 2022) “Pathological set of initial data for scaling-supercritical nonlinear Schrödinger equations” with Louise Gassot, published in *Int. Math. Res. Not.* [CG22]
- (Oct 2022) “Refined probabilistic well-posedness for the Schrödinger-half wave equation” with Louise Gassot and Slim Ibrahim, *ArXiv preprint* [CGI22]
- (2023) “Lack of Regularization in the Second Picard Iteration for NLS on \mathbb{S}^2 ”. In preparation, with *Nicolas Burq, Mickaël Latocca, Chenmin Sun and Nikolay Tzvetkov*

Contents

0	Introduction	1
0.1	Nonlinear dispersive equations with random data	1
0.2	Presentation of the results in this manuscript	24
0.3	Synthèse en Français	53
I	Supercritical regimes: instabilities and well-posedness	67
1	Pathological set of initial data for NLS	69
1.1	Introduction	70
1.2	Pathological set of data where norm inflation occurs	77
1.3	Generic well-posedness	90
II	Long-time dynamics under random perturbations	97
2	Almost-sure scattering for the 3D NLS	99
2.1	Introduction	100
2.2	Preliminaries and probabilistic estimates	109
2.3	Cauchy theory for the forced NLS equation	117
2.4	Almost conservation laws	132
2.5	Almost-sure global well-posedness and scattering	144
3	Almost-sure asymptotic stability	149
3.1	Introduction	150
3.2	Spectral theory	159
3.3	Bilinear and Strichartz estimates in the perturbed case	166
3.4	Almost-sure scattering	176
3.5	Almost-sure asymptotic stability	187
3.6	Appendix	197
III	Quasilinear probabilistic Cauchy theory	199
4	Refined probabilistic LWP for NLS-HW	201

CONTENTS

4.1	Introduction	202
4.2	Set up and preliminaries	209
4.3	Iteration scheme	211
4.4	The adapted linear evolution	215
4.5	Estimates on the nonlinear term w_N	222
4.6	Conclusion	226
4.7	Paracontrolled trilinear estimates	229
4.8	Remarks on ill-posedness	239
5	Lack of regularisation for NLS on the sphere	245
5.1	Introduction	246
5.2	Notation and preliminaries	249
5.3	Concentration of spherical Harmonics	251
5.4	Proof of the main theorem in the case of \mathbb{S}^2	255
5.5	First iteration for the torus	261

Introduction

Chapter 0

Introduction

Contents

0.1	Nonlinear dispersive equations with random data	1
0.1.1	Generalities and main questions	1
0.1.2	A brief history of probabilistic Cauchy theory	6
0.1.3	Globalization of probabilistic flows and generic dynamics.	14
0.2	Presentation of the results in this manuscript	24
0.2.1	On the Cauchy theory in scaling-supercritical regimes	24
0.2.2	Long-time dynamics under stochastic perturbations	28
0.2.3	Probabilistic Cauchy theory for weakly dispersive equations	41
0.3	Synthèse en Français	53

0.1 Nonlinear dispersive equations with random data

0.1.1 Generalities and main questions

In classical physics, dispersion is a linear wave phenomenon that occurs in media where waves of different frequencies propagate differently. Dispersion is mathematically captured by a class of linear equations called *dispersive equations*. Specifically, these equations describe vibratory phenomena in dispersive media, without friction or dissipation. A typical feature shared by dispersive equations is the presence of conserved quantities, such as the mass or the energy of the waves. On the other hand, due to the dispersion, the amplitude of the solutions decreases, at least when the domain is unbounded so that the waves have room to spread.

The propagation of waves in weakly nonlinear dispersive media is described by *nonlinear dispersive equations* that are typically of the form

$$i\partial_t u = Lu + f(u),$$

where the solution $u = u(t, x)$ has values in \mathbb{R}^d or \mathbb{C}^d , L is a self-adjoint operator usually given by a Fourier multiplier, and $f(u)$ is the nonlinear term, which may contain derivatives of u . The theory of nonlinear dispersive equations has been greatly developed over the last 30 years, and we refer

to the books [Bou99; Caz03; Dod19a; Tao06], to the note [Tao08], and to the references therein for general presentation and discussion on many aspects of the rich mathematical study of these equations. Illustrious examples of nonlinear dispersive models are the Korteweg-de Vries equation (KdV), the nonlinear Schrödinger equation (NLS) or the wave equation (NLW), the Klein-Gordon equation (KG), the Benjamin-Ono equation (BO), and the Hartree equation.

In this dissertation, we are interested in the study of nonlinear dispersive equation in singular regimes where the solutions are highly oscillating, in the presence of rough and stochastic perturbations. Before discussing some motivations, we emphasize that unlike solutions to parabolic equations, such as the heat equation, the solutions to dispersive equations generally do not experience immediate smoothing effects. Thus, when working at a prescribed regularity, one has to isolate the different scales of oscillations and to finely exploit the dispersive mechanisms that govern wave propagation. In addition, at the non-linear level, it is necessary to understand how the different waves interact with each other. Dispersive mechanisms are expressed by means of functional norms linked together by a bunch of powerful inequalities. Some of these inequalities, state them in the following paragraph, are widely used in the work that constitutes this manuscript.

The motivations to study nonlinear dispersive equations in the presence of randomness are of very different natures. First, randomness can be imposed by the model to describe microscopic fluctuations and collective behavior of large number of particles. In this situation, and when the equation is Hamiltonian, the relevant objects that provide a statistical description of the system at equilibrium are the *Gibbs measures*. Second, nonlinear dispersive equations display instabilities or blow-up solutions in certain regimes that compromise the full description of the evolution for individual initial data. Nevertheless, by endowing the set of initial data with a relevant probability measure, we can inject randomness in the equation to grab a statistical ensemble of initial data leading to strong solutions that are limits of smooth regularized solutions. To motivate the study of dispersive equations with rough data, let us mention that models involving small parameters, in a semiclassical regime for instance, can be studied by considering a fixed equation with a sequence of oscillatory initial data, possibly with an unbounded energy.

We end this paragraph by pointing out that, generally speaking, probabilistic methods are used to exhibit mathematical objects with prescribed properties. In this perspective, the use of the *probabilistic method* is an old idea in harmonic analysis where the study of random Fourier series has been a subject of intense research (see for instance the book [Kah93] and the references therein). In the context of probabilistic Cauchy theory, however, the use of randomness is slightly different since, up to a certain extent, the relevant probability measures are prescribed. Indeed, we want the statistical ensemble to be representative of the functions in the regime we are interested in. On the top of that, the Gibbs measures, or other relevant invariant measures that are usually supported at very low regularities, are imposed by the equation and its Hamiltonian structure.

The model We will mostly discuss the nonlinear Schrödinger equation in the three-dimensional Euclidean space \mathbb{R}^3 , but also on the periodic case \mathbb{T}^d and the sphere \mathbb{S}^2 . Our motivation is to investigate the Cauchy theory and the long-time dynamics of NLS at *low* regularity under stochastic perturbations that may be rough. The general form of the nonlinear Schrödinger equation posed on a Riemannian manifold (\mathcal{M}, g) is

$$\begin{cases} i\partial_t u + \Delta_g u = \mu|u|^{p-1}u, & (t, x) \in \mathbb{R} \times \mathcal{M}. \\ u|_{t=0} = u_0, \end{cases} \quad (\text{NLS})$$

Here, Δ_g is the Laplace-Beltrami operator on (\mathcal{M}, g) and $p > 1$ measures the strength of the semilinear perturbation of power type, seen as a self-interacting potential. The sign of the parameter $\mu \in \mathbb{R}$ dictates the focusing or defocusing nature of the equation, leading to drastically different dynamics. Roughly speaking, in the defocusing case ($\mu > 0$), the potential is repulsive and we expect the waves to propagate linearly. In the focusing case ($\mu < 0$), however, the potential is attractive and nonlinear phenomena are expected. We recall that equation NLS has the conserved Hamiltonian

$$E[u](t) := \frac{1}{2} \int_{\mathcal{M}} |\nabla u(t, x)|^2 dx + \frac{\mu}{p+1} \int_{\mathcal{M}} |u(t, x)|^{p+1} dx.$$

In addition, the L^2 -mass is also conserved along the evolution

$$M[u](t) := \frac{1}{2} \int_{\mathcal{M}} |u(t, x)|^2 dx.$$

The nonlinear Schrödinger equation serves as a theoretical model that appears in a number of dispersive models to describes the amplitude of the envelope of a fast oscillating wave packet. We refer to [SS99] for physical perspectives on the nonlinear Schrödinger equation. In light of the properties of the linear flow $\{e^{it\Delta}\}_{t \in \mathbb{R}}$ and of the conservation laws, the natural phase spaces generally associated with dispersive equations are the Sobolev spaces $H^s(\mathcal{M})$ with $s \in \mathbb{R}$, endowed with the norm

$$\|u\|_{H^s(\mathcal{M})} := \left(\int_{\mathcal{M}} |\Delta_g^{\frac{s}{2}} u(x)|^2 dx \right)^{\frac{1}{2}}.$$

The parameter s measures the regularity and the oscillations of the function. Even if the mass of the solution to NLS is conserved, its $H^s(\mathcal{M})$ -norm can, on the other hand, inflate for some $s > 0$ as time evolves. In such a case, the mass distributes on the *high* frequencies as small-scale oscillations appear. Such a low-to-high frequency cascade is referred as *weak turbulence*.

In this dissertation, we only consider equations that are not known to be *integrable* and we follow a perturbative approach materialized by the Duhamel's integral formula

$$u(t, x) = e^{it\Delta_g} u_0(x) - i\mu \int_0^t e^{i(t-\tau)\Delta_g} |u(\tau, x)|^{p-1} u(\tau, x) d\tau.$$

Specifically, the nonlinear evolution is seen as a perturbation of the linear evolution. The Duhamel's integral formula is used to prove local existence and uniqueness by a contraction mapping argument or, alternatively, a Picard's iteration scheme in suitable space-time functional space. Notice that this strategy is more likely to be successful when the data and the time of existence are small. In order to understand the long-time behavior, however, we need to iterate this local argument thanks to conserved quantities that control the growth of Sobolev norms, possibly combined with other uniform space-time bounds obtained from monotonicity formulas.

In some supercritical regimes where the initial data oscillates at high frequencies, however, instabilities rule out the Picard's iteration scheme. We face two types of instabilities. On the one hand, we have the scaling-supercriticality that leads to the *norm-inflation mechanism* discussed in Chapter 1. Roughly speaking, because of some scaling-symmetries, studying small data in small-time amounts to study large data. On the other hand, some instabilities are due to the presence of particular stationary solutions that concentrate their mass, for instance when the dispersion is too

weak. In these regimes, illustrated in Chapters 4 and 5, the system is *quasilinear* in the sense that the linear evolution is not a good approximation of the nonlinear evolution.

Finally, we emphasize that when studying a nonlinear equation, one needs to control some L^p -norms to ensure that the nonlinearity makes sense. Such L^p -bounds measure the concentration of the solutions. Specifically, if a function in L^2 has a huge L^p -norm for some $p > 2$, then the function has its mass concentrated at a finite distance from the origin. Guided by physical intuition, one can imagine that dispersive effects induce bounds on the L^p -norms of the linear evolution, at least when averaging in time. In this perspective, we recall some fundamental estimates that capture dispersive features shared by solutions to the linear Schrödinger equation in the Euclidean space, where waves have room to propagate.

- *The dispersive estimate.*

$$\|e^{it\Delta} u_0\|_{L^\infty(\mathbb{R}^d)} \lesssim |t|^{-\frac{d}{2}} \|u_0\|_{L^1(\mathbb{R}^d)}, \quad \|e^{it\Delta} u_0\|_{L^2(\mathbb{R}^d)} = \|u_0\|_{L^2(\mathbb{R}^d)}. \quad (0.1.1)$$

- *Strichartz estimates.* Given $2 \leq p, q \leq \infty$, satisfying the admissibility condition $\frac{2}{p} + \frac{d}{q} = \frac{d}{2}$, with $(d, p, q) \neq (2, 2, \infty)$,

$$\|e^{it\Delta} u_0\|_{L_t^p(\mathbb{R}, L_x^q(\mathbb{R}^d))} \lesssim \|u_0\|_{L_x^2(\mathbb{R}^d)}. \quad (0.1.2)$$

- *Bourgain's bilinear estimate.*

$$\|e^{it\Delta} P_{\leq N} u_0 e^{it\Delta} P_M v_0\|_{L_{t,x}^2(\mathbb{R} \times \mathbb{R}^d)} \lesssim N^{\frac{1}{2}} \left(\frac{N}{M}\right)^{\frac{d-2}{2}} \|P_{\leq N} u_0\|_{L_x^2(\mathbb{R}^d)} \|P_M v_0\|_{L_x^2(\mathbb{R}^d)} \quad (0.1.3)$$

- *Local smoothing estimate.* Let u be solution of the forced Schrödinger equation

$$\begin{cases} (i\partial_t + \Delta)u = f, \\ u(0) = u_0. \end{cases}$$

Then, for every $\delta > 0$,

$$\begin{aligned} \iint_{\mathbb{R} \times \mathbb{R}^d} \langle x \rangle^{-1-\delta} |\langle \nabla \rangle u(t, x)|^2 dx dt &\lesssim \|\langle \nabla \rangle^{\frac{1}{2}} u_0\|_{L_x^2}^2 \\ &+ \iint_{\mathbb{R} \times \mathbb{R}^d} \langle \nabla \rangle^{1+\delta} |f(t, x)|^2 dx dt. \end{aligned} \quad (0.1.4)$$

When the dispersion is too weak and does not gain enough integrability, we have no other choice than trading *integrability* against *regularity*. To do so, we can use the Sobolev embedding estimates, or, in the Euclidean space, the Bernstein estimate. For randomized initial data, however, we can exploit probabilistic decoupling to gain integrability without trading regularity. We will see in this introduction that there are a number of randomization procedures, depending on the context. The general idea that ties them together is that a typical function has better L^p -properties than expected thanks to random averaging effects that cancel out the interference. This is reflected by the Paley Zygmund theorem discussed below. Before going further, let us give an informal statement of a *probabilistic well-posedness* type of results, which reads as follows.

Theorem 0.1.1 (Probabilistic local well-posedness). *Let s_d be a deterministic exponent, such as the equation is not uniformly well-posed when $s < s_d$. There exists s_0 such that for all $s_0 < s < s_d$, there exist a probability measure μ supported in H^s and a full measure set Σ , such that for all $u_0 \in \Sigma$, the sequence of regularized solutions $(u^\varepsilon)_{\varepsilon>0}$, with initial data $(u_0 * \rho_\varepsilon)_{\varepsilon>0}$, converges to a limiting object in $C([-T, T], H^s)$ solution of NLS, for some $T = T(u_0) > 0$.*

Often, the probability measure μ is induced by the random Fourier series of type

$$\omega \in \Omega \mapsto u^\omega = \sum_k g_n(\omega) \pi_n u$$

where the $(\pi_n)_n$'s spectral projectors are associated with a spectral resolution of the Laplace operator $(\lambda_n^2, \pi_n)_n$, and the $(g_n)_n$'s are, say, a sequence of independent normalized Gaussian variables. Convergence makes sense in $L^2(\Omega; H^s(\mathcal{M}))$, when $u \sim \sum_n \pi_n u$ is a fixed function in $H^s(\mathcal{M})$. This can be seen with the orthogonality of the elementary pieces $(\pi_n u)_n$'s, and the uniform bound on the variance of the $(g_n)_n$'s:

$$\begin{aligned} \left\| \sum_{n=N}^M g_n(\omega) \pi_n u \right\|_{L^2(\Omega; H^s(\mathcal{M}))}^2 &= \mathbb{E} \left[\left\| \sum_{n=N}^M g_n(\omega) \langle \Delta \rangle^{\frac{s}{2}} (\pi_n u) \right\|_{L^2(\mathcal{M})}^2 \right] \\ &= \mathbb{E} \left[\sum_{n=N}^M |g_n(\omega)|^2 \langle \lambda_n \rangle^{2s} \|\pi_n u\|_{L^2(\mathcal{M})}^2 \right] \\ &\leq \sup_n \mathbb{E} [|g_n(\omega)|^2] \sum_{n=N}^M \langle \lambda_n \rangle^{2s} \|\pi_n u\|_{L^2(\mathcal{M})}^2 \xrightarrow{N, M \rightarrow \infty} 0. \end{aligned}$$

Main questions. To conclude this paragraph, we list some questions that motivate the study of dispersive equations at low regularity under stochastic perturbations. Specifically, in the presence of non-trivial dynamics, well understood for smooth solutions, one can consider stochastic perturbations due, for example, to rough microscopic fluctuations, and ask the following questions:

- What is the generic behavior of the solutions in super-critical regimes?
- Does randomness prevent deterministic instabilities?
- How does randomness affect the long-time behavior of coherent solutions?

In this thesis we study different manifestations of these questions, and we develop technics combining deterministic and probabilistic tools, at low regularity. The main challenge we faced, both in short-time and in long-time, is the aforementioned lack of regularizing effect for dispersive flows. Beyond the fact that we are working at supercritical regularities where instabilities occur, we do not have direct access to the conserved coercive energy, which is fundamental to understand the long-time dynamics of the solutions. Actually, the underlying question we faced in this manuscript that goes beyond the probabilistic considerations, can be stated as follows:

- How to exploit dispersive effects instead of spending regularity, especially when there is little of it?

0.1.2 A brief history of probabilistic Cauchy theory

This general introduction to the study of dispersive equation with randomness is not intended to be exhaustive. Instead, we orient the discussion in light of the works presented in this manuscript. In particular, we focus on the long-time dynamics of probabilistic flows, in the context of the Wiener decomposition in the Euclidean space. Nevertheless, we aim to provide a motivating and illustrative background on the rich history of this active research area.

Random Fourier series The use of randomness in harmonic analysis dates back to the early 1930s, when Paley and Zygmund observed in a series of papers [PZ30] that a typical function in $H^s(\mathbb{T})$ enjoys better $L^p(\mathbb{T})$ estimates than what the Sobolev embedding would predict. More precisely, consider a square-summable trigonometric series

$$\sum_n c_n e^{inx}, \quad \text{with } \sum_n |c_n|^2 < \infty,$$

which does not belong to $L^p(\mathbb{T})$, for $p > 2$. However, Paley and Zygmund observed that it suffices to randomly change the sign of each coefficient, independently, for the function to belong to $L^p(\mathbb{T})$. This is how the story of *random Fourier series* started. In modern mathematics, the result in [PZ30] states as follows.

Theorem 0.1.2 (Paley-Zygmund). *Given a sequence of complex numbers $(c_n) \in \ell^2(\mathbb{Z})$ and a family of independent mean zero Bernoulli $(b_n(\omega))$ random variables, the random series defined by*

$$u^\omega(x) := \sum_{n=-\infty}^{+\infty} b_n(\omega) c_n e^{inx}, \quad x \in \mathbb{T},$$

is almost-surely in $L^p(\mathbb{T})$ for every $p \in [2, \infty)$.

This observation has initiated a series of major contributions in harmonic analysis, and we refer to the book [Kah93] for an overview of the subject. The modern proof of Theorem 0.1.2 is brief, and relies on a standard quantitative large-deviation estimate. Nevertheless, this proof is illustrative of the probabilistic arguments used in this manuscript. We encourage interested readers to consult the note [Bur11] for a precise discussion of the Paley-Zygmund theorem and its applications in the context of dispersive partial differential equations. Of course, the result is valid for more general random variables. We point out that a major difficulty at that time was to make sense of an infinite family of independent Bernoulli variables. The first explicit realizations of infinite families of independent Bernoulli variables are due to Rademacher [Rad22] and Kolmogorov [KS24] with a dyadic decomposition of a number on the line.

The invariance of the Gibbs measure. The historical reason for interest in dispersive equations in the presence of randomness and low regularity comes from statistical mechanics. In the study of microscopic systems containing a large number of particles at equilibrium, macroscopic observables are accessed by integrating over Gibbs probability distributions. In their works [LRS88; LRS89], Lebowitz, Rose and Speer proved the existence of the Gibbs measure for some Hamiltonian systems governed by the nonlinear focusing Schrödinger equation, and the Zakharov system. Given

a finite number of particles N in a finite volume Ω , the Gibbs probability measure on a set of microscopic states is written

$$\mu(dX_N) = Z_N^{-1} \exp(-\beta H(X_N)) dX_N,$$

where H is the Hamiltonian of the system, β is the reciprocal temperature and Z_n is a normalization factor. In order to understand Langmuir waves and plasma turbulence, they considered the focusing case, so that the Hamiltonian is not bounded from below. They overcome this difficulty by using a cutoff of the mass, which interprets as a system with a finite number of particles, without taking the hydrodynamics limit as N goes to infinity. The Gibbs measure is absolutely continuous with respect to a Gaussian measure, and the realization of a Gibbs measure consists in Wiener paths that are quite rough. Indeed, the Gaussian measures are of type

$$(\omega, x) \mapsto \sum_{n \in \mathbb{Z}^d} \frac{g_n(\omega)}{\langle n \rangle^{\frac{1}{2}}} e^{inx} \in L^2(\Omega; H^{1-\frac{d}{2}-}(\mathbb{T}^d)).$$

Hence, at that time, the existence of a well-defined equilibrium dynamics on the support of the Gibbs measure was hypothetical. For instance, in the case of the cubic Schrödinger equation in dimension $d = 2$, the Gibbs measure is supported in H^{0-} but the deterministic threshold for local well-posedness is $s_c = 0$. In a paragraph called “Open Questions” [LRS88], Lebowitz, Rose and Speer raised the following problem: how to prove the existence of a nonlinear dynamics on the support of the Gibbs measure, at least for small-time? In addition, they wrote a short argument proposed by E. Stein to show that the free linear evolution is not bounded in $L^\infty(\mathbb{T})$. The case $d = 3$ is still an important open problem.¹

The birth of probabilistic Cauchy theory The construction of invariant measures has become a classical problem since the 1990s. Following the early works of Lebowitz, Rose and Speer [LRS88; LRS89], Frilander [Fri85] and Zhidkov [Zhi94] proved the invariance of the Gibbs measure for the one-dimensional nonlinear wave equation, followed by Bourgain [Bou94], who proved the invariance of the Gibbs measure for the cubic nonlinear Schrödinger equation on \mathbb{T} . In these contributions, even if the initial data in the support of the Gibbs measure is by essence random, the Cauchy theory is deterministic. Indeed, in one dimension, the Gibbs measure is supported at regularity $s < \frac{1}{2}$, while the deterministic critical regularity is $s_c = \frac{1}{2} - \frac{2}{p-1}$. Nevertheless, Bourgain needed to introduce the *Fourier restriction* norm to establish the local well-posedness in $H^{\frac{1}{2}-}(\mathbb{T})$ [Bou94]. To prove global well-posedness, however, Bourgain developed in the same paper [Bou96] a new globalization argument discussed in Section 0.1.3. This argument is considered *probabilistic* since it exploits the formal invariance of the Gibbs measure that serves as a substitute for the conserved energy.

Let us now move to the two-dimensional case that, strictly speaking, was the birth of the probabilistic Cauchy theory. In another celebrated work [Bou96], Bourgain obtained both the global existence for data in the statistical ensemble, and the invariance property of the Gibbs measure for

¹We refer to [DNY19; DNY22], where a notion of *probabilistic* scaling is introduced. It corresponds to $s_p = -\frac{1}{p-1}$, which is independent of the dimension. The cubic Schrödinger equation in dimension three is therefore critical for the probabilistic scaling, since the Gibbs measure is supported in $H^{-\frac{1}{2}-}(\mathbb{T}^3)$.

the two-dimensional Wick-ordering² cubic Schrödinger equation

$$\begin{cases} (i\partial_t + \Delta)u =: |u|^2u :, & (t, x) \in \mathbb{R} \times \mathbb{T}^2, \\ u(0) = u_0^\omega. \end{cases}$$

Up to a gauge transformation to remove the conserved mass, the initial data can be represented as a Fourier series

$$u_0^\omega(x) = \sum_{n \in \mathbb{Z}^2} \frac{g_n(\omega)}{\langle n \rangle} e^{inx},$$

where $\langle n \rangle = \sqrt{1 + |n|^2}$ and $(g_n)_{n \in \mathbb{Z}^2}$ is a sequence of independent normalized Gaussian variables with complex values. In contrast to the one-dimensional case, the support of the Gibbs measure lies at super-critical regularities $H^{0-}(\mathbb{T}^2)$, the critical scaling exponent being $s_c = 0$. To overcome this difficulty, Bourgain searched the solution under the sum of a rough linear evolution, and a smoother nonlinear remainder

$$u = e^{it\Delta} u_0^\omega + v. \quad (0.1.5)$$

Specifically, the nonlinear Duhamel term v turns out to be at regularity $H^{\frac{1}{2}-}(\mathbb{T}^2)$. Bourgain performed a fixed point argument for v at regularity H^ε , for some $\varepsilon > 0$, represented by the Duhamel integral written

$$\mathcal{I} \left(: |e^{it\Delta} u_0^\omega + v|^2 (e^{it\Delta} u_0^\omega + v) : \right).$$

To run the contraction mapping argument, Bourgain first observed a gain of $\frac{1}{2}$ -derivatives in the second Picard iteration. Then, Bourgain exploited multilinear cancellations to analyze the interactions between $e^{it\Delta} u_0^\omega$ and v^2 , by developing a TT^* -probabilistic argument. In this approach, v is treated deterministically, since one does not exploit any particular random structure of the remainder. The *linear-nonlinear decomposition* (0.1.5) in the context of random initial data is also called the Da Prato-Debussche method [DD02], which successfully used a similar strategy in the context of fluid mechanics. Subsequently, Burq and Tzvetkov extended Bourgain's strategy to nonlinear wave equations in the case of a general Riemannian compact manifolds (\mathcal{M}, g) without boundaries. In this way, they provided a general framework that made possible further developments of the theory. They proceeded as follows. Take $s < s_c$ so that the Cauchy problem in $\mathcal{H}^s(\mathcal{M}) := H^s(\mathcal{M}) \times H^{s-1}(\mathcal{M})$

$$\begin{cases} (\partial_{tt} - \Delta_g)u = -|u|^3, & (t, x) \in \mathbb{R} \times \mathcal{M} \\ (u(0), \partial_t u(0)) = (f_0, f_1), \end{cases} \quad (0.1.6)$$

is ill-posed. Then, consider randomized initial data defined by taking (f_0, f_1) in \mathcal{H}^s , expanding them in an orthonormal basis of $L^2(\mathcal{M})$ constructed from the eigenfunctions of the Laplace operator $-\Delta_g$ and decoupling each mode by independent standard Gaussian variables (h_n, l_n) :

$$\omega \mapsto (f_0^\omega, f_1^\omega) = \left(\sum_n h_n(\omega)(f_0 | e_n)e_n, \sum_n l_n(\omega)(f_1 | e_n)e_n \right).$$

This defines a $\mathcal{H}^s(\mathcal{M})$ -valued random variable, and we have almost-sure local existence according to the induced measure in the following sense:

²For the cubic case, the Wick-ordering reads $: |u|^2u := |u|^2u - 2\|u\|_{L^2}^2 u$. We use a gauge transformation and the conservation of the L^2 -mass to pass from the cubic nonlinearity, to the Wick-ordering nonlinearity. See, e.g., [OT18] for discussions on Wick-ordering renormalization in the context of dispersive equations.

Theorem 0.1.3 (Burq-Tzvetkov [BT08a]). *For $T > 0$, there exists a set Σ_T such that for all $\omega \in \Sigma_T$, the initial value problem (0.1.6) with initial data $(u(0), \partial_t u(0)) = (f_0^\omega, f_1^\omega)$ admits a unique solution in $C([-T, T], \mathcal{H}^s)$ under the form*

$$u(t) = \cos(t|\nabla|)f_0^\omega + \frac{\sin(t|\nabla|)}{|\nabla|}f_1^\omega + v.$$

Uniqueness holds for the nonlinear term v in a subspace of $C([-T, T], \mathcal{H}^\sigma)$, for some $s_c < \sigma$. Moreover, there exist $C, c, \delta > 0$ such that for all $T > 0$,

$$\mathbb{P}({}^c\Sigma_T) \leq C \exp(-cT^{-\delta} \|(f_0, f_1)\|_{\mathcal{H}^s}^{-2}).$$

The proof follows the strategy of [Bou96], performing a fixed point on v . Here, the authors exploit the L^p -regularization of the linear evolution under the effect of the randomization, in the spirit of Paley Zygmund's observation, and the gain of one derivative in the Duhamel formula for the wave equation. For Schrödinger-type equations, however, there is no comparable general result. As we point out in Chapter 5, the behavior of solutions to Schrödinger-type equations on compact manifolds, even in short time, is much more complex because of the infinite speed of propagation. In this perspective, another ensuing series of works on compact manifolds concerns the almost-sure global well-posedness of the Schrödinger equation on two and three-dimensional balls [Tzv08; BB14c; BB14a; BB14b].

Randomization procedure in the Euclidean space. In the Euclidean space, where the long-time dynamics of Schrödinger equations are completely different, the probabilistic framework is not the same since we do not have a discrete spectral resolution for the Laplace operator. For this reason, there is no canonical randomization procedure in the Euclidean space. One way to mimic the randomization procedures presented above in a compact setting is to use a unit-scale frequency decomposition, leading to the so-called *Wiener randomization*. The procedure, introduced by [LM14b] and [OP16] independently, consists in superposing unit-scale frequency blocks decoupled by independent Gaussian variables (see Figure 0.1.2).

Definition 0.1.4 (Wiener randomization). Take a unit-scale partition of unity in the frequency space

$$\sum_{k \in \mathbb{Z}^d} \psi(\xi - k) \equiv 1, \quad \xi \in \mathbb{R}^d,$$

for a suitable smooth cutoff function ψ supported in $[-1, 1]^d$. Then, take $u_0 \in H^s(\mathbb{R}^d)$, and write its *Wiener decomposition*

$$u_0 = \sum_{k \in \mathbb{Z}^d} P_{1;k} u_0, \quad P_{1;k} u_0 = \mathcal{F}^{-1}(\psi(\cdot - k) \mathcal{F} u_0).$$

Given a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and a family of independent complex valued standard Gaussian variables $(g_k)_{k \in \mathbb{Z}^d}$, the mapping

$$\omega \in \Omega \mapsto u_0^\omega = \sum_{k \in \mathbb{Z}^d} g_k(\omega) P_{1;k} u_0 \tag{0.1.7}$$

defines a random variable valued in $H^s(\mathbb{R}^d)$. Note that the regularization procedure does not regularize the initial data. For all $p \geq 2$ and $\sigma > 0$, if $u_0 \in W^{s,p}(\mathbb{R}^d) \setminus W^{\sigma,p}(\mathbb{R}^d)$ then so does u_0^ω , almost-surely as proved in Lemma B.1 from [BT08a].

This randomization procedure has for side effect to relax the admissibility condition in the Strichartz estimates, imposed by the scaling of the equation (see Figure 0.1.2). Given a function whose frequency support is of size one, we can use the Bernstein inequality to pass from L^q to L^2 without losing any derivative. Then, we glue the individual estimates together by the mean of probabilistic decoupling, in the spirit of the proof of Paley-Zygmund's theorem.

Proposition 0.1.5 (Probabilistic Strichartz estimates, Prop. 1.5 [BOP15b]). *Let (p, q) be a Strichartz-admissible pair and $s > 0$. For every $q < \tilde{q} < \infty$, there exist $C, c > 0$ such that for all $u_0 \in H^s(\mathbb{R}^d)$, and $\lambda > 0$, we have the large deviation bound*

$$\mathbb{P}(\{\omega \in \Omega : \|\langle \nabla \rangle^s e^{it\Delta} f_0^\omega\|_{L_t^p(\mathbb{R}; L_x^{\tilde{q}}(\mathbb{R}^3))} > \lambda\}) \leq C \exp(-c\lambda^2 \|u_0\|_{H^s}^{-2}).$$

Then, almost-sure local well-posedness at supercritical regularities follows from the Da Prato-Debussche trick, as described in the above paragraph. In this case, the nonlinear smoothing effect for the remainder v was evidenced in the analysis of [BOP15b], combining the probabilistic Strichartz estimates with bilinear estimates in Fourier restriction spaces, or more recently by exploiting the local-smoothing estimate in lateral spaces [DLM19]. In the first approach, used in this manuscript in dimension 3, the nonlinear remainder gains s derivatives:

$$v(t) = u(t) - e^{it\Delta} u_0^\omega = -i \underbrace{\int_0^t e^{i(t-t')\Delta} \left(|e^{it'\Delta} u_0^\omega|^2 e^{it'\Delta} u_0^\omega \right) dt'}_{\text{second Picard's iteration}} + \dots \in H^{2s-}(\mathbb{R}^3) \quad \text{a.s.}$$

Bényi, Oh and Pocovnicu same authors refined in [BOP15a] the analysis in U^2 and V^2 spaces for small data, to prove almost-sure global existence and scattering.

Let us conclude by mentioning another randomization procedure used to study the nonlinear Schrödinger equation on the line. It is based on the lens transform, that compactifies time and space by transforming the Laplacian into the harmonic oscillator. We refer to [Poi12; PRT14; BT20]. Notice, however, that in dimension greater than one this procedure gains derivatives in L^p -spaces, so we prefer not following this approach.

Quasilinear probabilistic decomposition: the paracontrolled approach. The linear-nonlinear decomposition we have discussed so far has shown its limits, and Da Prato-Debussche's trick does not always lead to the desired nonlinear probabilistic regularizing effect. Indeed, in some cases where the dispersion is too small or where the nonlinear interactions are prevalent, singular interactions may appear and limit the possible gain. In the worst cases, bad interactions can even exclude any possible nonlinear smoothing. In such cases, one cannot run the standard probabilistic semilinear resolution scheme à la Bourgain. An illustrative example is the *derivative nonlinear wave equation*, whose probabilistic study was carried out by Bringmann in [Bri21], who by the same time proposed the first probabilistic *quasilinear* resolution scheme.

The derivative wave equation considered in [Bri21] writes

$$\begin{cases} -\partial_{tt}^2 u + \Delta u = |\nabla u|^2, & (t, x) \in \mathbb{R} \times \mathbb{R}^3, \\ u(0) = u_0, \quad \partial_t u(0) = u_1. \end{cases}$$

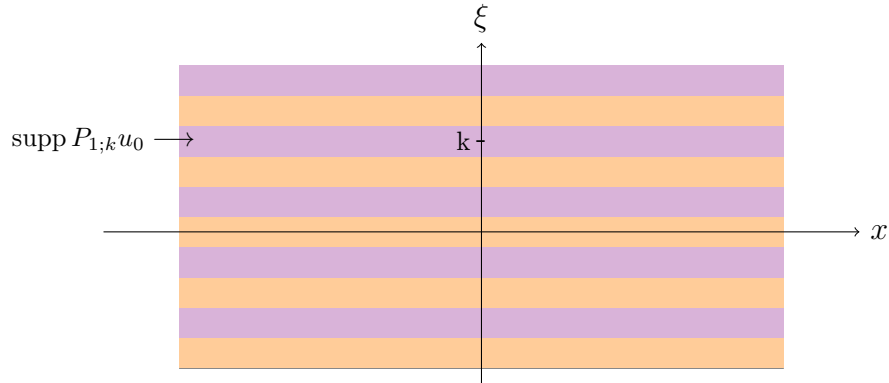


Figure 1: The colored stripes indicate the microlocal support of the elementary blocks $P_{1;k}$ in the Wiener randomization.

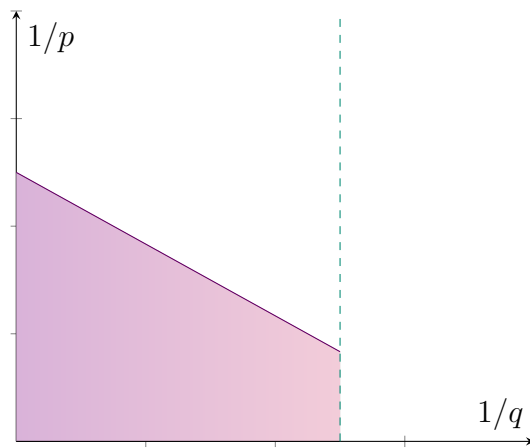


Figure 2: The purple region indicates the pairs of exponents for which the Strichartz estimate holds almost-surely. We observe that the Wiener randomization procedure relaxes the admissibility condition materialized by the purple line, and gains integrability without trading derivatives.

In addition to the scaling-critical exponent $s_c = \frac{3}{2}$, the Lorentz-critical critical regularity $s_d = 2$ dictates the well-posedness threshold: the equation is locally well-posed in $H^s \times H^{s-1}$ when $s > 2$, whereas ill-posedness occurs when $s < 2$ [Lin93], respectively. Still, Grünrock proved in [Grü11] that well-posed holds in Fourier-Lebesgue spaces, in scaling-subcritical regimes $\frac{3}{2} < s \leq 2$. At these regularities, Bringmann investigated in [Bri21] probabilistic well-posedness in the scale of the Sobolev spaces. Writing the linear ansatz

$$u = f^\omega + v, \quad f^\omega(t) = \cos(t|\nabla|)u_0^\omega + \frac{\sin(t|\nabla|)}{|\nabla|}u_1^\omega \in H^s,$$

and considering the *high-low* interactions $\nabla P_{\gg 1} f^\omega \nabla P_1 f^\omega$ in the second Picard's iteration, the author observed that the equation does not experience any smoothing. Indeed, given $\nu > s$ -derivatives, we have

$$|\nabla|^\nu \int_0^t \frac{\sin((t-\tau)|\nabla|)}{|\nabla|} \nabla P_{\gg 1} f^\omega(\tau) \nabla P_1 f^\omega(\tau) d\tau \simeq \int_0^t \sin(t|\nabla|) |\nabla|^\nu P_{\gg 1} f^\omega(\tau) \nabla P_1 f^\omega(\tau) d\tau.$$

It turns out that the ν -derivatives fall onto the linear term f^ω , which can only handle s derivatives. In contrast to the case of the Schrödinger equation, there is neither bilinear estimate nor local smoothing estimate to compensate for this surge of derivatives. To remove the *high-low* interactions, Bringmann successfully implemented the *paracontrolled calculus* framework, developed by Gubinelli, Imkeller and Perkowski [GIP15] in the context of parabolic equations with stochastic forcing. The key idea is to see the singular interactions, which consist in the paraproduct of a high-frequency function with a low-frequency potential, as part of the linear evolution. In this perspective, Bringmann used a refined probabilistic ansatz of type

$$u = f^\omega + C + w,$$

where³ roughly speaking, f^ω is the linear evolution of the randomized initial data, and the paracontrolled⁴ term C reads

$$C = \sum_N \mathcal{I}(t) (P_N f^\omega P_{\ll N} (u - f^\omega)), \quad (0.1.8)$$

where \mathcal{I} denotes the Duhamel operator. The nonlinear remainder w is smoother and lives above the deterministic threshold. In particular, the singular interactions are *paracontrolled* by the linear evolution [GIP15]. Notice that the proof in [Bri21], based on an induction on the size of frequencies of the initial data, is to be compared with quasilinear schemes. However, in contrast with the existing literature on paracontrolled calculus for parabolic equations, Bringmann further exploited the stochastic structure of the solution. Namely, when analyzing the singular interactions, Bringmann crucially exploited the independence between the high frequencies and the low frequencies of randomized initial data. In the spirit of [Bri21], we used in Chapter 4 a similar paracontrolled approach to prove almost-sure local well-posedness for a weakly dispersive equation, and we refer

³We denote the paracontrolled term by C referring to the common terminology used in [DNY19], which denotes the term C the *colored Gaussian* variables, alluding to the Gaussian structure of the linear evolution f^ω , altered by a low-frequency potential.

⁴We say that a function g is paracontrolled by f when there exists a function h such that $g = \Pi_{\text{hi}} f \Pi_{\text{lo}} h +$ a smoother remainder.

to the introduction of Chapter 4 and to paragraph 0.2.3 for a more detailed discussion on the paracontrolled probabilistic ansatz.

In the context of dispersive partial differential equations, paracontrolled calculus has also been developed in a spectacular way by Deng, Nahmod and Yue to prove the invariance of the Gibbs measure for the Wick-ordering Schrödinger equation on \mathbb{T}^2

$$\begin{cases} (i\partial_t + \Delta)u =: |u|^{p-1}u : , & (t, x) \in \mathbb{R} \times \mathbb{T}^2 , \\ u(0) = u_0 , \end{cases}$$

with an arbitrary odd power $p \geq 5$. This is an important breakthrough, and the use of *random averaging operators* has proven to be extremely powerful and promising in addressing other challenging problems open for many years (see e.g. paragraph 0.2.3 concerning the invariance of the Gibbs measure for NLS on \mathbb{S}^2). For Schrödinger equations, in contrast with the case of the nonlinear wave equation [OT20], the case $p \geq 5$ is much harder than the cubic case. The reason being that the *high-low-low . . . low*-type frequency interactions of the random data have regularity $\frac{1}{2}-$, whereas the deterministic critical threshold has regularity $s_c = 1 - \frac{2}{p-1}$, which is strictly larger than $\frac{1}{2}$ when p is larger than 5. To prove local existence on the support of the Gibbs measure, and to remove these bad interactions, they built upon tools not only from the paracontrolled calculus, but also from random matrix theory, and introduced the *random averaging operators*, that encapsulate the bad interactions. Subsequently, in other remarkable works, the random averaging operators method has been successfully implemented and refined to obtain strong solutions on the support of the Gibbs measure. For instance, [ST21] considered fractional Schrödinger equations posed on \mathbb{T} , with weak dispersion. Specifically, the linear operator is $(-\Delta)^\alpha$ with $\frac{1}{2} < \alpha \leq 1$ sufficiently close to $\frac{1}{2}$. They used random averaging operators in the physical space in place of the frequency space where [DNY19; DNY22] employed counting estimates. We also mention important results for wave-type equations. In [Bri20a; Bri20b], Bringmann proved invariance of the Gibbs measure for NLW with a Hartree nonlinearity. Recently, [Bri+22] proved the invariance of the Gibbs measure for the cubic wave equation in \mathbb{T}^3 . An important conjecture is the counterpart of this result for the Schrödinger equation.

We now give a very informal statement of the type of result one can hope to obtain from paracontrolled calculus, and we refer to the detailed introductions of [DNY19; ST21; Bri20a; Bri20b] for in-depth discussion on these technics.

Theorem 0.1.6 (Refined probabilistic ansatz (informal version)). *There exists a full measure set $\Sigma \subset H^s$ such that for all $\omega \in \Sigma$, the sequence of smooth local solutions $(u_n)_{n \geq 0}$ to*

$$\begin{cases} i\partial_t u_n + Lu_n = \mathcal{N}(u_n), & N = 2^n , \\ u_n(0) = P_{\leq N} u_0^\omega \end{cases}$$

converges to a unique limit in $u \in C([-T, T], H^s)$, for some $T = T(\omega) > 0$, which solves the equation with the full initial data. Moreover, u_n admits the decomposition

$$u_n = e^{itL} P_{\leq N} u_0^\omega + \mathcal{P}_n(u_0^\omega) + w_n .$$

- The sequence of smooth remainders (w_n) converges in a subcritical space H^σ .

- The random averaging operator \mathcal{P} isolates the singular part, and inherits the randomness structure from the initial data, captured by certain matrix norms. It roughly corresponds to a superposition of paracontrolled terms of type C written in the informal refined ansatz (0.1.8), which solve a linear equation with a low-frequency potential independent of the high frequency initial data.

In Part III, we present two situations in which a refined probabilistic ansatz is required. First, in Chapter 4, we prove probabilistic well-posedness for the Schrödinger-half wave equation, which is a weakly dispersive with rich long-time dynamics but for which the Cauchy theory is yet to be understood. Then, we evidence in Chapter 5 some bad interactions for NLS posed on the sphere \mathbb{S}^2 that compromise the regularization of the second Picard's iteration, and indicate therefore the need to use random averaging operators.

Recent progress and perspectives To end this historical journey in the study of dispersive equations with random initial data, let us mention the spectacular breakthroughs made recently in related problems. We mention the results of Collot and Germain [CG19; CG20] and of Deng and Hani [DH21b; DH21a] in the study of wave turbulence, concerning the propagation of statistical quantities (also called propagation of chaos) by wave kinetic equations. Another lines of research concern the study of geometric equations with Brownian paths [BLS21], or the statistical study of the formation of rogue waves, related with the derivation of large deviation principles for cubic Schrödinger equations with random initial data [Gar+21]. Very recently, [Bri+22] proved in an outstanding work the invariance of the Gibbs measure under the dynamics of the three-dimensional cubic wave equation, by introducing a new *caloric* decomposition of the initial data, and by combining the analytic framework of random tensors with the *combinatorial molecule estimates* developed in [DH21b; DH21a], respectively.

To conclude, we can say that modern techniques such as paracontrolled calculus and random tensors play an important role in these advances, and that they will continue to be explored and developed in order to solve other problems that were considered out of reach for a long time (such as the invariance of the Gibbs measure for the three-dimensional cubic Schrödinger equation). But these new and exciting stories go far beyond the scope of this manuscript...

0.1.3 Globalization of probabilistic flows and generic dynamics.

One of the main question addressed in this dissertation can be formulated as follows. Given a random initial data of the form

$$u(0) = u_0 + f_0^\omega, \quad u_0 \in H^\sigma, \quad f_0^\omega \in H^s, \quad \text{with } s < s_c \leq \sigma, \quad (0.1.9)$$

and the associated probabilistic local flow u^ω , can one prove that u^ω extends globally in time, and say something on its long-time dynamics? In this paragraph, we give a non-exhaustive presentation of the existing technics and results in this direction to provide a context for the contributions collected in this dissertation.

Qualitative properties on the support of a Gibbs measure The purpose of this paragraph is to briefly discuss how Gibbs measures can replace conserved energies to obtain global

well-posedness. We also give an insight into the qualitative properties of Hamiltonian dynamical systems in the presence of invariant measures, such as the *Poincaré recurrence* property. As explained in the next paragraph, these dynamics are in sharp contrast with scattering mechanisms that arise in the Euclidean space.

In order to fix the ideas, we consider the one-dimensional periodic Schrödinger equation, with a defocusing cubic nonlinearity:

$$(i\partial_t + \Delta)u(t, x) = |u|^2 u(t, x), \quad (t, x) \in \mathbb{R} \times \mathbb{T}. \quad (\text{NLS})$$

This is the historical model studied by Bourgain in [Bou94]. Besides, the method we outline here is referred to as the *Bourgain globalization argument*. We recall that the Gibbs measure associated with the one-dimensional (NLS) is supported in $H^s(\mathbb{T})$ for all $s < \frac{1}{2}$. Hence, (NLS) is locally well-posed on the support of the Gibbs measure, which is above the deterministic threshold $s = 0$ given by Galilean invariance. Therefore, to construct local solutions, there is no need for the probabilistic Cauchy theory initiated a little later by Bourgain in order to address the two-dimensional case [Bou96]. To understand the long-time dynamics, however, Bourgain used the formal invariance of the Gibbs measure as a substitute for the conservation of the energy, for the (finite dimensional) Galerkin approximated system

$$\begin{cases} (i\partial_t + \Delta)u_N = \Pi_N(|\Pi_N u_N|^2 \Pi_N u_N), \\ u(0) = \Pi_N u_0, \end{cases} \quad (\text{NLS}_N)$$

where, for $x \in \mathbb{T}$, $t \in \mathbb{R}$ and N , we write

$$\Pi_N u_0(x) = \sum_{|n| \leq N} (u_0 | e^{inx}) e^{inx}, \quad u_N(t, x) = \sum_{|n| \leq N} u_n(t) e^{inx}.$$

We briefly present the construction of the Gibbs measure associated with the Hamiltonian structure of NLS on \mathbb{T} , and the probabilistic globalization argument of Bourgain.

- *NLS as a Hamiltonian system.* Equation (NLS_N) can be seen as a finite-dimensional Hamiltonian system. Setting

$$p_n = \text{Re}(u_n), \quad q_n = \text{Im}(u_n),$$

it holds

$$\frac{d}{dt} \begin{pmatrix} p_n \\ q_n \end{pmatrix}_{n=1, \dots, N} = \begin{pmatrix} -\partial_{q_n} H_N(p, q) \\ \partial_{p_n} H_N(p, q) \end{pmatrix}_{n=1, \dots, N},$$

with Hamiltonian

$$H_N(p, q) = \frac{1}{2} \sum_{|n| \leq N} n^2 (p_n^2 + q_n^2) + \text{nonlinear terms}.$$

The Liouville theorem states that the volume measure $dp dq := \prod_{|n| \leq N} dp_n dq_n$ is invariant under the flow of (NLS_N). As a consequence of the conservation of the Hamiltonian under the flow, the (finite-dimensional) approximate Gibbs measure

$$d\mu_N(p, q) = Z_N^{-1} \exp(-\beta H_N(p, q)) dp dq$$

is also invariant under the flow. Namely, denoting the flow by $\Phi_N(t)$ and writing $(p(t), q(t)) = \Phi_N(t)(p_0, q_0)$, we have

$$\begin{aligned} \mu_N(\Phi_N(t)^{-1}(A)) &= \int_A \exp\left(-\beta H_N(p(t), q(t))\right) dp(t) dq(t) \\ &= \int_A \exp\left(-\beta H_N(p_0, q_0)\right) dp_0 dq_0 = \mu_N(A). \end{aligned}$$

Additionally, we observe from the Cauchy-Lipschitz theorem for ODE's and from the mass conservation that the solutions to (NLS_N) are global. This being said, the objective is now to remove the truncation and to construct the dynamics on the support of the (formal) infinite-dimensional Gibbs measure⁵ associated with (NLS)

$$“d\mu(u) = Z^{-1} \exp(-H(u) - M(u)) du”. \quad (0.1.10)$$

The Bourgain's globalization argument achieves these objectives. Since this method is now standard, we only sketch the contours of the argument, and we refer to the original article [Bou94] for further details.

- *Construction of the Gibbs measure.* Given a probability space (Ω, \mathbb{P}, A) and a sequence of independent normalized Gaussian variables, we consider the Gaussian free field ρ induced by the map

$$\omega \in \Omega \mapsto \sum_{n \in \mathbb{Z}} \frac{g_n(\omega)}{\langle n \rangle} e^{inx}. \quad (0.1.11)$$

Exploiting the independence of the Gaussian variables g_n , we see that the above series makes sense in $L^2(\Omega; H^s(\mathbb{T}))$ as the limit of a Cauchy sequence, for any $s < \frac{1}{2}$. From the Sobolev embedding, we see that $\|u\|_{L^4}$ is finite almost-surely with respect to the Gaussian measure. Then, the Gibbs measure (0.1.10) associated with (NLS) is the density measure with respect to $d\rho$

$$d\mu(u) = \exp(-V(u)) d\rho(u), \quad V(u) = \frac{1}{2} \int_{\mathbb{T}} |u(x)|^4 dx.$$

Furthermore, the sequence of finite-dimensional Gibbs measure (μ_N) associated with the truncated system weakly converges to $d\mu$. For every open set $U \subset H^s(\mathbb{T})$, with $s < \frac{1}{2}$, we have

$$\mu(U) = \lim_{N \rightarrow \infty} \mu_N(U \cap E_N), \quad \text{with } E_N = \text{span}\{e^{inx} \mid |n| \leq N\}.$$

This essentially follows from the almost-sure convergence of the series (0.1.11), and from the integrability of the density.

- *Global well-posedness on the support of the Gibbs measure.* We recall that (NLS) is locally well-posed on the support of μ , since it is well-posed on $H^{\frac{1}{2}-\varepsilon}(\mathbb{T})$. Next, to establish global well-posedness Bourgain combined local well-posedness for (NLS_N) together with the invariance

⁵We removed the mass $M(u) = \sum_n |u_n|^2$, which is a conserved quantity, in order to avoid the problem with zero Fourier mode, and to replace $|n|$ by $\langle n \rangle$ in the expression of the Gaussian measure $d\rho$.

of μ_N under the flow Φ_N . This leads to uniform bounds on $\Phi_N(t)$ on full-measure sets and on long-time intervals. The argument is as follows. Set

$$\Omega^{s,K} = \{u \in E_N \mid \|u\|_{H^s} \leq K\},$$

for fixed $s < \frac{1}{2}$ and $1 < K$. We have the large-deviation estimate, for some constant $c = c(s) > 0$:

$$\mu_N({}^c\Omega^{s,K}) \leq e^{-cK^2}.$$

Subsequently, we see from the local Cauchy theory for the approximated equation (NLS_N) that there exists $\tau \sim K^{-\theta}$, for some $\theta > 0$, such that

$$\|u_0\|_{H^s} \leq K \implies \sup_{t \in [-\tau, \tau]} \|\Phi_N(t)u_0\|_{H^s} \leq 2K.$$

Next, given $T > 1$, we set

$$\tilde{\Omega}^{s,K} = \bigcap_{j=0}^{\lceil \frac{T}{\tau} \rceil} \Phi(-\tau j)(\Omega^{s,K}).$$

By construction, for all $u_0^\omega \in \tilde{\Omega}^{s,K}$ we have

$$\sup_{|t| \leq T} \|\Phi_N(t)u_0\|_{H^s} \leq 2K.$$

Moreover, it follows from the sub-additivity of μ_N and its invariance under the flow Φ_N and the large-deviation estimate that

$$\mu_N({}^c\tilde{\Omega}^{s,K}) \leq TK^\theta e^{-cK^2}.$$

As a consequence, for every $\varepsilon > 0$ and $T > 1$, there exists a set $\tilde{\Omega}$ such that for all u_0 in this set,

$$\sup_{|t| \leq T} \|\Phi_N(t)u_0\|_{H^s} \leq C \log\left(\frac{T}{\varepsilon}\right)^{\frac{1}{2}}, \quad \mu_N({}^c\tilde{\Omega}^{s,K}) \leq \varepsilon.$$

To establish the long-time existence on the support of μ and the invariance of μ , it remains to combine the above uniform bounds with the weak convergence of the measures μ_N , together with a deterministic approximation lemma that yields the strong convergence of $\Phi_N(t)$ to $\Phi(t)$ in H^s , uniformly in time. The proof of this lemma consists in computing the difference between $\Phi(t)u_0$ and $\Phi_N(t)\Pi_N u_0$ by exploiting local well-posedness in $X^{s,b}$ -spaces, and iterating on a finite number of small spacetime slabs.

- *Long-time dynamics on the support of the Gibbs measure.*

As a byproduct of the existence and invariance of the Gibbs measure, the classical theory of dynamical systems yields asymptotic information about the flow, such as the Poincaré recurrence property. Specifically, by obtaining strong local solutions on the support of a Gibbs measure, and by proving its invariance under the flow-map, we are in a very good position to prove the following statement:

There exists a full μ -measure set Σ in H^s , such that for any $u_0 \in \Sigma$ the Cauchy problem with initial data u_0 has a strong global solution in $C(\mathbb{R}; H^s)$. Moreover, the flow map Φ satisfies the flow property

$$\Phi(t)(\Sigma) = \Sigma, \quad \Phi(t_1 + t_2) = \Phi(t_1) \circ \Phi(t_2),$$

and the Poincaré recurrence property: for all $t \in \mathbb{R}$ there is a subsequence of integers $(n_k)_k$ going to $+\infty$, such that

$$\lim_{k \rightarrow \infty} \|\Phi(tn_k)u_0 - u_0\|_{H^s} = 0.$$

We refer, for instance, to Theorem 6 and Corollary 1.2 in [ST20b] for a precise formulation.

The *Bourgain globalization argument* also works in the context of probabilistic Cauchy theory (below the deterministic threshold) provided that we can prove local well-posedness for solutions with random initial data according to the support of the Gibbs measure. An illustrative example is the subsequent series of works of Bourgain and Bulut [BB14c; BB14a; BB14b] who proved global existence for NLS and NLW on the ball with Dirichlet boundary conditions. Note, however, that their method does not provide the flow property and the invariance of the Gibbs measure. Furthermore, Bourgain's globalization argument adapts when probabilistic local well-posedness follows from the paracontrolled ansatz discussed in Section 0.1.2 (e.g. [DNY19; DNY22; ST21; Bri20a; Bri20b]). Finally, we mention that one can also systematically construct weak solutions from the existence of a Gibbs measure, by using Prokhorov compactness theorem and Skorokhod convergence theorem. We refer to [BTT18] for results in this direction, applied to a number of weakly dispersive models. We also refer to [Tzv10; Den15] for some important contributions to the invariance of the Gibbs measure for the Benjamin-Ono equation.

Before turning to other dynamical behavior of dispersive nonlinear equations in the presence of randomness, let us point out that the dynamics on the support of a Gibbs measure is not compatible with scattering mechanisms. We propose a proof of this statement, by showing that if a solution of a globally well-posed equation satisfies a certain time decay, materialized by uniform space-time bounds in Lebesgue spaces $L_t^q(\mathbb{R}; L_x^q(\mathbb{R}^d))$, with $q < \infty$, then the only invariant measure is the Dirac mass at zero.

Proposition 0.1.7 (Invariant measure and scattering are not compatible). *Let μ be a probability measure on H^s , invariant under a flow $\{\Phi(t)\}$ globally defined on a full μ -measure set Σ . Namely, for every Borelian set $A \in H^s$, and $t, \tau \in \mathbb{R}$,*

$$\mu(\Phi(t)(A)) = \mu(A), \quad \Phi(t)(\Sigma) = \Sigma, \quad \Phi(t + \tau) = \Phi(t) \circ \Phi(\tau).$$

If for some $2 < p, q < \infty$, we have that for all $K > 0$, there is a $C(K) \geq K$ such that for all $u_0 \in \Sigma$,

$$\|u_0\|_{H^s} \leq K \implies \|\Phi(\cdot)u_0\|_{L_t^\infty(\mathbb{R}; H^s) \cap L_t^p(\mathbb{R}; L_x^q)} \leq C(K),$$

then $\mu = \delta_0$ is the Dirac mass at zero.

Proof. For $\lambda > 0$, we set

$$A_\lambda = \{u_0 \in \Sigma : \|u_0\|_{L_x^q} > \lambda\},$$

and we prove that $\mu(A_\lambda) = 0$ for almost every λ . First, we observe from the global well-posedness assumption that for all $K > 0$,

$$B_K := \{u_0 \in \Sigma : \|u_0\|_{H^s} \leq K\} \subseteq \{u_0 \in \Sigma : \sup_{t \in \mathbb{R}} \|\Phi(t)u_0\|_{H^s} \leq C(K)\} =: \tilde{B}_K.$$

Then, given $\tau \in \mathbb{R}$, we use the invariance of μ and the flow property to deduce that

$$\begin{aligned} \mu(A_\lambda \cap B_K) &\leq \mu\left(\left\{u_0 \in \Sigma : \sup_{t \in \mathbb{R}} \|\Phi(t)u_0\|_{H^s} \leq C(K), \quad \|u_0\|_{L_x^q} > \lambda\right\}\right) \\ &= \mu\left(\left\{u_0 \in \Sigma : \sup_{t \in \mathbb{R}} \|\Phi(\tau) \circ \Phi(t)u_0\|_{H^s} \leq K, \quad \|\Phi(\tau)u_0\|_{L_x^q} > \lambda\right\}\right) \\ &= \mu(\tilde{B}_K \cap \Phi(-\tau)(A_\lambda)). \end{aligned}$$

Since the left-hand side is independent of time, integrating over $\lambda^{p-1}d\lambda$, and then over τ on an arbitrary long-time interval $[0, T)$, we obtain

$$\begin{aligned} T \int_0^1 \mu(A_\lambda \cap B_K) \lambda^{p-1} d\lambda &\leq \int_0^T \int_0^{+\infty} \mu(\{u_0 \in \tilde{B}_K : \Phi(\tau)u_0 \in A_\lambda\}) \lambda^{p-1} d\lambda d\tau \\ &\lesssim \int_0^{+\infty} \int_0^T \int_{\tilde{B}_K} \mathbf{1}_{\Phi(-\tau)(A_\lambda)}(u_0) d\mu(u_0) d\lambda d\tau. \end{aligned}$$

We denote the Lebesgue measure on the line by \mathcal{L} , and we deduce from the Fubini Tonelli's formula that

$$\begin{aligned} \int_0^{+\infty} \int_0^T \int_{\tilde{B}_K} \mathbf{1}_{\Phi(-\tau)(A_\lambda)}(u_0) d\mu(u_0) d\lambda d\tau &= \int_{\tilde{B}_K} \int_0^{+\infty} \int_0^T \mathbf{1}_{\Phi(-\tau)(A_\lambda)}(u_0) d\tau d\lambda d\mu(u_0) \\ &= \int_{\tilde{B}_K} \int_0^{+\infty} \mathcal{L}_{[0, T]}(\{\tau : \|\Phi(\tau)u_0\|_{L_x^q} > \lambda\}) \lambda^{p-1} d\lambda d\mu(u_0) \\ &= \frac{1}{p} \int_{\tilde{B}_K} \|\Phi(\cdot)u_0\|_{L_t^p L_x^q}^p d\mu(u_0) \leq \frac{1}{p} C(K)^p. \end{aligned}$$

Finally, we proved that for all $T > 0$,

$$T \int_0^1 \mu(A_{\lambda, K}) \lambda^{p-1} d\lambda \leq \frac{1}{p} C(K)^p.$$

By letting T go to $+\infty$, we obtain that $\mu(A_{K, \lambda}) = 0$ for almost every $0 < \lambda \leq 1$. Since K is arbitrary and Σ has full μ -measure, this proves that $\mu = \delta_0$. \square

Note however that scattering is not incompatible with the existence of quasi-invariant measures. Moreover, for the defocusing Schrödinger equation on the line with initial data supported just below $L^2(\mathbb{R})$, Burq and Thomann proved in [BT20] global existence when the nonlinearity $p > 1$, and scattering when $p > 3$. This result is a consequence of uniform space-time bounds on the density obtained from quasi-invariant measures, which are constructed through a compactifying procedure called the *Lens Transform*.

Quasi-invariant measures We briefly mention the use of quasi-invariant measures to obtain globalization of probabilistic flows, in the absence of invariant measures and conserved energies. This is a promising strategy, though beyond the scope of this dissertation. Given a Gaussian measure μ and a Hamiltonian flow $\{\Phi_t\}_t$, we say that μ is quasi-invariant under the flow if for every t ,

$$(\Phi_t)_\# \mu \ll \mu,$$

where $(\Phi_t)_\#$ denotes the push-forward of the measure by the flow. In such a case, by the Radon-Nikodym theorem, there exists a density $f_t \in L^1(d\mu)$ such that

$$(\Phi_t)_\# \mu = f_t \mu.$$

Then, one can deduce global well-posedness for the flow (Φ_t) from $L^p(d\mu)$ -bounds on the density, combined with the Bourgain globalization argument. This strategy, initiated by Tzvetkov in [Tzv15] generalizes the use of invariant Gibbs measure, which are also absolutely continuous measures with respect to a Gaussian measure. Yet, the use of quasi-invariant measure is much more flexible. We refer to [BT20; FT22; GLT21; OTT19] who have successfully implemented and developed this method, and to the references therein.

Deterministic methods In this section, we present some globalization methods that do not rely on the use of an invariant or quasi-invariant measure. In light of the previous discussion, these methods can be developed to study scattering mechanisms, and are employable at regularities not too far from the regularity required to access a conserved coercive quantity. They consist in obtaining a uniform a priori bound on the nonlinear remainder $v = u - u_L^\omega$, and in iterating the local existence result through the uniform bounds. They are called *deterministic methods*, as opposed to the probabilistic *Bourgain globalization argument* we described in the previous paragraph.

- *The high-low method.* This method, introduced by Bourgain in [Bou98a], consists in decomposing the solution into a low frequency part, and a high frequency part. The low frequency part being smooth, we can use the (almost) conservation of energy to control it. On the other hand, the high frequency part turns out to be well approximated by the linear flow, thanks to a certain nonlinear smoothing effect. Thus, each part of the solution is under control, and we obtain a priori bounds for the solution. In [CO12], Colliander and Oh implemented the high-low method in a probabilistic setting to prove the almost-sure global existence for the renormalized NLS on \mathbb{T} , at negative regularities. Note that in such regimes, conservation of mass, almost-surely defined for the nonlinear remainder $v = u - u_L^\omega$, has been exploited. We also mention the later work [PRT14], which applied the high-low method in a probabilistic framework to prove the almost-sure global existence and scattering for NLS in \mathbb{R}^d , $d \geq 3$, in a small data regime.
- *The use of modified energies* When the remainder $v = u - u_L^\omega$ is in H^1 one can try to obtain a priori bounds on v through a modified non-conserved energy, of type

$$\mathcal{E}[v](t) := \frac{1}{2} \int_{\mathcal{M}} (\nabla v(t, x))^2 dx + \frac{1}{2} \int_{\mathcal{M}} (\partial_t v(t, x))^2 dx + \frac{1}{p+1} \int_{\mathcal{M}} v^{p+1}(t, x) dx.$$

In [BT14], Burq and Tzvetkov proved almost-sure global existence for the cubic defocusing NLW in \mathbb{T}^3 , by the energy method for v together with a Cauchy-Schwarz estimate and a Gronwall argument. A refinement, proposed in [LM16], consists in putting the linear evolution in the potential energy in order to cancel certain terms. See also the works [OP16; OP17; LM14b; SX16] and the references therein, who proved almost-sure global well-posedness for NLW in different contexts, but in the same spirit.

However, in the case of NLS, the method is less favorable since the energy does not control $\partial_t v$. Instead, one has to replace it using the equation, and losing therefore Δv . We mention

the work of [OOP19] that develop from ideas used to treat NLW, but in higher dimensions $d \geq 5$. We also mention the conditional global well-posedness result [BOP15a].

Almost-sure scattering When a state evolves under nonlinear Schrödinger flow, with a defocusing self-interacting nonlinearity, its dynamics in long-time are well approximated by a free linear evolution. This *scattering* phenomenon is well understood in the energy space, when the equation is energy-subcritical, in or energy-critical as in [RV07; Vis12], for the four-dimensional case, [Col+08] for the three-dimensional, and [Dod16] for the mass-critical case, respectively. The defocusing cubic Schrödinger equation in \mathbb{R}^3 gives an illustrative example of scattering in energy space, and we sketch the proof as a prerequisite to presenting the general strategy leading to *probabilistic scattering* type of results. The cubic NLS is energy-subcritical, and hence it is locally well-posed in the energy space $H^1(\mathbb{R}^3)$. We see from the coercive conserved energy that the solution is global. Moreover, we deduce from Strichartz's estimates that a given solution scatters provided that we have a priori bound on the space-time norm

$$\iint_{\mathbb{R} \times \mathbb{R}^3} |u(t, x)|^p dx dt < \infty, \quad \text{for some } \frac{3}{10} < p < 10.$$

Such an a priori decay estimate follows from the so-called interaction Morawetz inequality,

$$\iint_{I \times \mathbb{R}^3} |u(t, x)|^4 dt dx \leq C \sup_{t \in I} \|u(t)\|_{L_x^2}^2 \|u(t)\|_{\dot{H}^{\frac{1}{2}}}^2. \quad (0.1.12)$$

combined with some conservation laws. Now that we have scattering in the energy space, a natural extension would be to prove that for any $\frac{1}{2} < s < 1$, the local solution in $H^s(\mathbb{R}^3)$ is global and scatters. Unfortunately, this simple question turns out to be a very challenging open problem that is still far from being solved in all generality. Partial results were obtained by employing the celebrated I-method [Col+04] and refinements based on *long-time Strichartz estimates* [Dod13; Dod19b]. We refer to paragraph 0.2.2 and to the introduction of Chapter 2 for a detailed discussion on this problem, as well as detailed presentation of the I-method and of the Morawetz estimate. Despite the difficulty of the scattering problem in the subcritical and critical regimes $\frac{1}{2} \leq s < 1$, it is interesting to achieve a probabilistic result. Besides, there exists a local-probabilistic flow even in supercritical regimes. The following question arises:

- Does a general supercritical initial data of type (0.1.9) scatter at infinity?

In recent years, the stability of the scattering mechanism under random perturbations have been extensively studied. In the small data regime, global well-posedness and scattering can be obtained as a by-product of almost-sure local well-posedness. See, for example, [Poi12; BOP15a]. Outside the small data regime, however, the problem is much more complicated and the first class of problems to consider are *energy-critical* problems, when the deterministic theory in the critical energy space is well understood. In such a context, the probabilistic flows are in H^s for s close to 1, but the nonlinear remainder is in H^1 . Therefore, the probabilistic ansatz (0.1.5) should allow analyzing the long-time behavior of the nonlinear term v in the energy space, solution to a nonlinear Schrödinger equation with stochastic forcing terms.

$$\begin{cases} (i\partial_t + \Delta)v = |v + u_L^\omega|^{p-1}(v + u_L^\omega), & (t, x) \in \mathbb{R} \times \mathbb{R}^d, \\ v(0) = 0, \end{cases} \quad (\text{NLS}_f)$$

where $u_L^\omega := e^{it\Delta} u_0^\omega$ is the linear evolution of the random initial data. The first step is to use the stability theory of the deterministic equation in H^1 to obtain a conditional scattering result. By doing so, we reduce the problem to obtaining a uniform control on the H^1 -norm of v . Next, in a second step, the strategy is to find appropriate modified energies and modified Morawetz inequalities for v , and to perform a double bootstrap argument on these two quantities.

For this strategy to work, another ingredient is often necessary to absorb the weights that come from the Lin Strauss and Morawetz type inequalities. It consists in concealing a radial symmetry assumption in the randomization procedure, by assuming that u_0 is radially symmetric in (0.1.7). Note that a Wiener randomization of a radially symmetric function need not be radially symmetric. Nevertheless, the solution retains some of the decay properties that radially symmetric functions have from the radial Sobolev embedding theorem. This strategy was first implemented by Dodson, Lührmann and Mendelson [DLM19] to prove almost-sure scattering for the energy critical wave equation in \mathbb{R}^4 . Then, Kilipp, Murphy and Visan obtained a similar almost-sure scattering result for the energy critical Schrödinger equation in \mathbb{R}^4 [KMV19], which was improved in [DLM19]. In the interest of considering the largest possible class of initial data, Bringmann implemented in [Bri18] tools from harmonic analysis, such as Bourgain bushes, and solved the scattering problem for the energy-critical wave equation, without any radial assumptions. This approach is less suitable for Schrödinger-type equations, since the wave packet propagates faster. Nevertheless, the transposition of this type of argument to the Schrödinger case is an interesting problem.

In the energy-subcritical framework (e.g., cubic NLS in \mathbb{R}^3), however, there was no such almost-sure scattering result outside the small data regime. We refer to 0.2.2 and Chapter 2 for the presentation of an almost-sure scattering result in this framework, with radially-symmetric randomized initial data.

Stability of coherent states under random perturbations The probabilistic Cauchy theory is mainly concerned with the perturbation of dispersive solutions, in the defocusing regime, or for small initial data. However, in the focusing regime where dispersive equations generally exhibit rich dynamical behaviors, much less is known about the effect of random perturbations. Nevertheless, stability results for special coherent solutions of dispersive partial differential equations under rough and random perturbations have been obtained recently. The first long-time result in this direction is due to Kenig and Mendelson, for an energy-critical model. In [KM19], they proved that when $s > \frac{5}{6}$, one can construct a global solution of the focusing quintic wave equation in \mathbb{R}^4 that scatters to a modulated soliton, for almost every initial data in $H^s \times H^{s-1}$. The considered solitons form a one-parameter family $(\phi_a)_{a>0}$ of stationary radially symmetric solutions to NLW satisfying the elliptic equation

$$\Delta\phi_a + \phi_a^5 = 0.$$

They have regularity \dot{H}^s , for any $s > \frac{1}{2}$, and are extremizers of the Sobolev embedding $\dot{H}^1(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$ given by the explicit formula

$$\phi_a(x) = \frac{(3a)^{\frac{1}{4}}}{(1 + a|x|^2)^{\frac{1}{2}}}.$$

The result states as follows.

Theorem 0.1.8 (Almost-sure asymptotic stability for energy critical NLW, [KM19]). *Let $s > \frac{5}{6}$, and (f_0^ω, f_1^ω) a suitably randomized initial data. There exists a set full-measure set $\Sigma \subset \mathcal{H}^s$ such that for all $f_0^\omega \in \Sigma$, there exist $\varepsilon > 0$, $h : \Sigma \rightarrow \mathbb{R}$ with $|h| \leq \varepsilon^2$ and a unique solution u to*

$$\begin{cases} -(\partial_{tt}^2 - \Delta)u = -u^5, \\ u(0) = \phi + \varepsilon f_0^\omega + h(\varepsilon f_0^\omega, \varepsilon f_1^\omega)Y, \quad \partial_t u(0) = \varepsilon f_1^\omega + h(\varepsilon f_0^\omega, \varepsilon f_1^\omega)\kappa Y, \end{cases} \quad (\text{NLW})$$

such that

$$u(t) = \phi_{a(t)} + v,$$

with $\dot{a} \in L_t^\infty \cap L_t^1$ is small, and v is a small radiation term in the sense that

$$\|v\|_{L_{t,x}^s} \lesssim \varepsilon.$$

The function h ensures an orthogonality condition for the pure point spectrum of the linearized operator around the ground state. Besides, the random perturbations are projected onto a Lipschitz manifold because of an unstable direction, and we refer to the article [KM19] for further details on the exact randomization procedure. To establish this type of result, the main challenge was to deal with a linear source term stemming from the soliton. To overcome this difficulty, the authors decided to adapt the randomization procedure to the linearized operator around the soliton. Instead of using standard Fourier multipliers in the randomization procedure (0.1.7), they used a distorted Fourier transform that diagonalizes the linearized operator around the ground state:

$$H_a = -\Delta - 5\phi_a^4.$$

Unfortunately, this operator has a resonance at zero, which obliges to use weighted spaces, and to derive technical kernel estimates. Note that a weighted Sobolev condition is also required on the initial data. The only existing counterpart of this almost-sure asymptotic stability result for the Schrödinger equation is the matter of Chapter 3. As for the Schrödinger case, we stress out that the Duhamel integral does not gain a derivative, so that a dyadic decomposition and bilinear estimates must be performed to establish the probabilistic local well-posedness. However, this becomes quite difficult in the presence of a potential, since the distorted Fourier transform does not share the multiplicative properties of the Fourier transform. We refer to paragraph 0.2 and to Chapter 3 for a detailed presentation of an *almost-sure asymptotic stability* result for NLS.

Inspired by the work of Kenig and Mendelson on the focusing energy-critical radial wave equation, Bringmann proved in the [Bri20c] the almost-sure stability of blow-up profile for the same wave equation below the energy space. The ODE blow-up profile is

$$u^{(T)}(t, x) := \kappa(T - t)^{-\frac{1}{2}}, \quad T > 0.$$

Thanks to finite propagation speed, it can be localized in the interior of the light cone

$$\mathcal{C}_T := \{(t, x) \in [0, T] \times \mathbb{R}^3, |x| \leq T - t\}.$$

The almost-sure stability result states as follows.

Theorem 0.1.9 (Stability of the blow-up profile under random perturbations, [Bri20c]). *For $0 < \delta < \delta_0$ small enough, there exist a set Σ_δ such that for all ω in this set, there is a $T(\omega) \in [1 - \delta, 1 + \delta]$ and a solution $u : \mathcal{C}_T \rightarrow \mathbb{R}$ to (NLW) with suitably randomized initial data (f_0^ω, f_1^ω) , satisfying*

$$\|(T - t)^{-\frac{3}{4}}(u - u_T)\|_{L_t^2 L_x^4(\mathcal{C}_T)} \lesssim \delta.$$

Moreover, there is $C, c > 0$ and $\theta > 0$ such that for all $0 < \delta < \delta_0$,

$$\mathbb{P}(\Omega \setminus \Sigma_\delta) \leq C \exp(-c\delta^\theta).$$

The proof revisits deterministic methods in the probabilistic setting, such as the Strichartz estimates for one-parameter semigroup $S(t)$ associated to the linearized operator around the blow-up profile in similarity coordinates. The challenge was to obtain probabilistic estimates for the difference operator $S(t) - S_0(t)$, where $S_0(t)$ is the free wave propagator. To achieve such a goal, Bringmann proved fine oscillatory integrals estimates. In a subsequent work [FM20], Fan and Mendelson proved the almost-sure stability of the log-log blow-up profile for the mass critical Schrödinger equation in \mathbb{R}^2 . However, the randomized data have a particular structure, since their Fourier transform is piecewise constant, with some decaying assumption.

To our knowledge, these aforementioned results are the only ones to date to prove the stability of coherent states under supercritical random perturbations.

0.2 Presentation of the results in this manuscript

In this section, we present the main contributions of the thesis put in perspective with the research themes discussed in the previous Sections 0.1.1, 0.1.2 and 0.1.3. We focus on the manifestations of the general questions raised at the end of Section 0.1.1. However, we merely state and motivate the theorems, and discuss the main ingredients of the proofs, avoiding technicalities. Thus, we defer the precise description of the arguments and the detailed list of references until the introduction of the individual chapters.

0.2.1 On the Cauchy theory in scaling-supercritical regimes

At scaling-supercritical regularities, the heuristics predict that instabilities will occur, thus undermining the possibility of establishing a general Cauchy theory. The matter of this section is to clarify the *genericity* of such instabilities, in the context of the random data Cauchy theory. Before going further, let us recall some terminology and basic facts about the local well-posedness for a general dispersive equation, which reads

$$i\partial_t u(t, x) - L(u)(t, x) = \mathcal{N}(u)(t, x), \quad (t, x) \in \mathbb{R} \times M, \quad u(0) \in H^s(M).$$

Here, (M, g) is a Riemannian manifold, L is a self adjoint operator on $H^k(M)$, for some k large enough, and \mathcal{N} denotes the nonlinearity, possibly depending on a finite number of space-derivatives of u .

Definition 0.2.1 (Local well-posedness, see Definition 1.1.1). We say that the above Cauchy problem is well-posed in $H^s(M)$ if for every bounded set $B \subset H^s(M)$, there exist $T = T(B) > 0$ and

a Banach space X_T continuously embedded in $C([t_0, T], H^s(M))$ such that for any $u_0 \in B$, there is a solution $u \in X_T$ satisfying the Duhamel integral representation formula

$$u(t) = e^{i(t-t_0)L} u_0 - i \int_{t_0}^t e^{i(t-\tau)L} \mathcal{N}(u(\tau)) d\tau. \quad (0.2.1)$$

Moreover, the following holds.

1. (Uniqueness) Let $u_1, u_2 \in X_T$ satisfying (0.2.1). If, for some $t_0 \leq t \leq T$, $u_1(t) = u_2(t)$, then $u_1 = u_2$ on $[t_0, T]$.
2. (Continuity) The flow map $u_0 \in B \mapsto u \in X_T$ is continuous.
3. (Persistence of regularity) If $u_0 \in H^\sigma(M)$ with $\sigma > s$, then $u \in C([t_0, T], H^\sigma(M))$.

When the flow-map is uniformly continuous, we say that the Cauchy problem is *semi-linearly* well-posed. This certifies the absence of instability for the high frequencies of the solution. Usually, we prove uniform well-posedness by performing a Picard's iteration argument, so that the flow-map is actually Lipschitz continuous. Furthermore, the flow-map is analytic when the nonlinearity is polynomial.

In the particular case of nonlinear Schrödinger and wave equations with power-type nonlinearity in the Euclidean space, or in the periodic case, the scaling invariance actually dictates the critical threshold for the uniform local well-posedness theory, at least when $s_c > 0$. Indeed, according to [CCT03b], we have local well-posedness in $H^s(\mathbb{R}^d)$ as soon as $s \geq \max(0, s_c)$, and norm-inflation occurs when $s < \max(0, s_c)$. For the Schrödinger equation on compact manifolds, however, the picture is less clear, since the behavior of the solution depends strongly on the geometry of the manifold, even in short time, as a consequence of the infinite speed of propagation for the solutions of NLS. For instance, the cubic NLS is not uniformly well-posed on the Euclidean sphere \mathbb{S}^2 when $s < \frac{1}{4}$, whereas local well-posedness holds up to the scaling threshold $0 < s$ on the torus \mathbb{T}^2 , respectively (see Subsection 0.2.3).

We briefly discuss the instability mechanism. The aforementioned work of Christ, Colliander and Tao [CCT03a] does not only prove that the flow map is not continuous at scaling-supercritical regularity $0 < s < s_c$, but it also provides explicit sequences of initial data that indicate the mechanism involved. Specifically, they constructed a sequence of concentrating blow-up profiles converging to zero in H^s , but for which the H^s -norm of the corresponding smooth local solution inflates, in arbitrarily small-time. Namely, by writing $\Phi(t)$ the local-in-time solution map at time t on H^σ , for σ large-enough, there is a sequence $(u_{0,n})_n$ in the Schwarz class and a sequence of times (t_n) going to zero, such that

$$\lim_{n \rightarrow \infty} \|u_{0,n}\|_{H^s} = 0, \quad \text{but} \quad \lim_{n \rightarrow \infty} \|\Phi(t_n)(u_{0,n})\|_{\dot{H}^s} = +\infty.$$

Such a *norm-inflation* mechanism indicates the existence of low-to-high frequency cascades in arbitrarily short-time, as mentioned in the general introduction 0.1.1. This is consistent with the scaling heuristic that when $s < s_c$, studying short-time existence for small data amounts to study large data. We refer to the pioneering work of [Leb01], and many other references cited in the introduction of Chapter 1, where we discuss norm-inflation in more details.

In contrast, the probabilistic Cauchy theory, as stated in Theorem 0.1.1, provides a full-measure set of initial data for which a strong solution exists, at least up to a certain time. Furthermore,

probabilistic solutions are natural in the sense that they are the limit of the regularized solutions by convolution. Notice, however, that by choosing a bad regularization procedure one can show norm-inflation for *any data*, as the result of [Xia21] indicates. Specifically, for any $u_0 \in H^s$, the author proved that there exist $(u_{0,n})$ in Schwartz-class, and (t_n) going to zero, such that

$$\lim_{n \rightarrow \infty} \|u_{0,n} - u_0\|_{H^s} = 0, \quad \text{but} \quad \lim_{n \rightarrow \infty} \|\Phi(t_n)u_{0,n}\|_{\dot{H}^s} = +\infty.$$

In light of this informal discussion, there are *bad* regularization procedure that lead to norm-inflation for any initial data, and *good* initial data for which the natural regularization by convolution converges to a strong solution. The natural question raised by Sun and Tzvetkov is about the genericity of *bad* initial data leading to norm-inflation even when regularized by convolution. Such bad initial data form a *pathological* set, defined as

$$\mathcal{P} = \left\{ u_0 \in H^s : \limsup_{\varepsilon, t \rightarrow 0} \|\Phi(t)(\rho_\varepsilon * u_0)\|_{H^s} = +\infty \right\}.$$

Here, (ρ_ε) is a fixed approximate identity.

Pathological set of initial data for scaling-supercritical NLS *joint work with L. Gassot* [CG22]. In the case of the wave equation, Sun and Tzvetkov proved in [ST20a] that the pathological set contains a G_δ dense set. Hence, two different types of genericity coexist:

- The *Baire category* genericity, in a complete metric space. It is a topological notion of genericity.
- The *probabilistic* genericity.

An important subsequent remark is that the full-measure set of data leading to local well-posedness cannot contain a dense G_δ set, since G_δ sets are stable under intersections.

In Chapter 1, we generalized the result of [ST20a] for Schrödinger-type equations, hence precisising the Cauchy theory at scaling supercritical regularities for NLS, whose study is the subject of Part II, and also for the Schrödinger half-wave equation, less studied, which is the matter of Chapter 4. The result can be formulated as follows.

Theorem 0.2.2 (Pathological set of scaling-supercritical data for NLS, [CG22]). *Let $0 < s < s_c$. The pathological set \mathcal{P} contains a dense G_δ set $S \subset H^s(\mathbb{R}^3)$: for every $u_0 \in S$, there is a sequence $\varepsilon_n \rightarrow 0$, and a sequence of times $t_n \rightarrow 0$, such that the local smooth solution $u_n^{\varepsilon_n} = \Phi(t)(u_0 * \rho_{\varepsilon_n})$ of NLS exists in $H^s(\mathbb{R}^3)$ up to the time t_n , but the family experiences norm-inflation*

$$\lim_{n \rightarrow \infty} \|u_n^{\varepsilon_n}(t_n)\|_{\dot{H}^s(\mathbb{R}^3)} = +\infty.$$

The result of [ST20a] does not only provide a dense G_δ set, but it also gives an explicit construction, which we reformulate here in the Schrödinger case.

- *The individual inflating bubbles.* The elementary building blocks are the inflating *bubbles*, widely used to prove norm-inflation in a variety of contexts, and introduced in [CCT03a]. Given a fixed, smooth, and radially-symmetric profile $\varphi \in C_c^\infty(\mathbb{R}^3)$, we define the bubble concentrating at scale n^{-1} by

$$v_n(0, x) = \log(n)^{-\gamma} n^{\frac{3}{2}-s} \varphi(nx), \quad v_n^\varepsilon(0) := \rho_\varepsilon * v_n(0),$$

for some $\gamma > 0$. The bubble is convoluted by an approximate identity in order to state the result in the context of probabilistic Cauchy theory (see Theorem 0.1.1). Then, the ODE profile is

$$v_n^\varepsilon(t, x) := v_n^\varepsilon(0, x)V(t|v_n^\varepsilon(0, x)|^{p-1}), \quad V(t) = e^{it\sigma}.$$

It solves the nonlinear ODE corresponding to NLS but without the Laplace operator (which is the dispersive term, that can be replaced by any other reasonable operator here)

$$\begin{cases} i\partial_t v_n^\varepsilon + \sigma|v_n^\varepsilon|^{p-1}v_n^\varepsilon = 0, \\ v_n^\varepsilon(0) = \rho_\varepsilon * v_n(0). \end{cases}$$

Uniformly in ε , we have the lower bound

$$\|v_n^\varepsilon(t)\|_{H^s} \geq c \log(n)^{-\gamma} (n^{3-2s}t)^s. \quad (0.2.2)$$

As a consequence, after a time $t_n \sim \log(n)^\beta n^{-(3-2s)}$ with $\beta > \gamma$, the H^s -norm of the profile starts to inflate, as n goes to infinity so that the initial data concentrate.

- *The small dispersion analysis.* The proof of [ST20a] consists in adapting the *small dispersion analysis* due to [CCT03b], in the presence of the convolution and with an infinite number of inflating bubbles, sufficiently distant from each other. First, let us outline the standard small dispersion analysis for one bubble. It consists in proving that in small-time, the nonlinear profile v_n dominates the effect of the dispersion, so that the solution to the dispersive equation NLS with data $v_n(0)$ stays close to the bubble $v_n(t)$. This is a small-time analysis. Still, one needs to give the inflating bubble time to burst (that is $t \geq t_n$). The control of the difference from the solutions and the bubble follows from energy considerations.

- *The “tanghuru” construction.* As in the preparation of tanghurus, candied fruit lollipops very popular in some Asian countries, the construction of the dense pathological set consists in putting side by side an infinity of bubbles, concentrated at arbitrarily small scales, and sufficiently distant from each other. In addition, the construction is made around an arbitrary function in $C_c^\infty(\mathbb{R}^3)$ in order to generate a dense set in $H^s(\mathbb{R}^3)$ that is, for some $k_0 \gg 1$,

$$S = \left\{ u_0 + \sum_{k=k_0}^{\infty} v_{n_k}(0, x - x_k), \quad n_k = e^{e^k}, \quad x_k = \frac{1}{k}, \quad u_0 \in C_c^\infty(\mathbb{R}^3) \right\}.$$

Then, the proof consists of two steps. In a first step, Sun and Tzvetkov isolate the n -th bubble at time t_n and scale ε_n , and they perform the standard small-dispersion analysis for \tilde{u} solution to NLS with initial data $v_n(0)$, which is just a piece of the initial data in S . Subsequently, they use the finite propagation of speed to ensure that the remaining part of the solution does not interfere with the inflating bubble v_n , and prove that the full solution u^{ε_n} stays close to $\tilde{u}^{\varepsilon_n}$ up to the time t_n , which is known from the first step to be close to the inflating profile $v_n^{\varepsilon_n}$. At the end of the day, the difference from u^{ε_n} to $v_n^{\varepsilon_n}$ in H^s is controlled up to time t_n , and the inflation of v_n , described by the lower bound (0.2.2) leads to the inflation of u^{ε_n} .

The second step of the proof strongly relies on the finite speed of propagation, which is a specific feature of wave-type equations. For this reason, it was not clear how to obtain an analogous result

in the case of Schrödinger-type equations, or for any other dispersive equation that does not satisfy finite speed of propagation. Our main observation is the following. Given a fixed regularization scale $\varepsilon > 0$, the convolution by the profile ρ_ε has for side effect to collapse the bubbles that concentrate at scale $\ll \varepsilon$. Therefore, at some appropriate time t_n , the bubble concentrated at scale $\sim \varepsilon_n$ is the only one to inflate. Consequently, when isolating the bubble that inflates at time t_n , there is no need to use finite speed propagation to make sure that the evolution of the other bubbles, more concentrated, does not interfere, and rules out the inflation. Let us now give some comments and consequences of this observation that is the key of our proof.

- In [CG22] we actually proved the result for any power-type nonlinearity $|u|^{p-1}u$, with $p \geq 3$ an odd integer, as stated in Chapter 4. This assumption is technical, and guarantees that the inflating bubbles are smooth. It should be possible to remove it.
- When $p \geq 7$, in the energy supercritical case, we only proved that \mathcal{P} contains a dense set. In order to get a dense G_δ -set, however, we need to uniformly control the time of existence of the smooth solution, say in H^2 , and the supercritical nature of the energy prevents us from doing so by basic energy considerations. Still, being able to treat the energy-supercritical case is an improvement of the method in [ST20a] that requires global existence of smooth solutions.
- A similar comment is about the focusing or defocusing nature of the nonlinearity. We prove that there is a G_δ dense subset when we have access to a conserved coercive energy, and at least a dense set of inflating data in the focusing case, which is also a novelty. Note however that ill-posedness results in the focusing case are of different nature, since there are special stationary solutions that may be used to evidence instabilities.
- Another consequence of not exploiting finite propagation speed is the freedom in the choice of the position of the bubbles in the construction. Indeed, the absence of interaction among them does not depend on their spacial localization. Hence, we can take bubbles centered at zero and obtain a pathological set that is radially symmetric.

We refer to the introduction of Chapter 1 for further remarks, detailed explanations and references. Let us just mention that a possible perspective would be to use the construction to describe a loss of regularity mechanism, as evidenced by [Leb01; AC09; CR09] for the energy-supercritical wave equation, and for Schrödinger equations, by the mean of nonlinear geometric optics.

0.2.2 Long-time dynamics under stochastic perturbations

In this part, we investigate the long-time behavior of the local probabilistic flow solution to the cubic Schrödinger equation in \mathbb{R}^3 , presented in Section 0.1.2. First, we address in Chapter 2 the scattering issue for the defocusing equation, outside the small data regime. In order to do so, we propose a way to revisit in a probabilistic setting the I-method with a Morawetz bootstrap. Then, we consider in Chapter 3 the stability of small ground states that emerge from the presence of an external short-range potential. By performing a “critical-weighted” strategy, we prove that these ground states are asymptotically stable even if the perturbations have infinite energy.

Almost-sure scattering for energy-subcritical Schrödinger equation We refer to the above Paragraph in Section 0.1.2 called “almost-sure scattering”, and to the introduction of the Chapter 2 for more detailed background and motivations. The model is the cubic Schrödinger equation with a defocusing nonlinearity, in the Euclidean space of dimension 3.

$$\begin{cases} i\partial_t u(t, x) + \Delta u(t, x) = |u(t, x)|^2 u(t, x), & (t, x) \in \mathbb{R} \times \mathbb{R}^3. \\ u(0, x) = u_0(x), \end{cases} \quad (\text{NLS})$$

This equation is energy subcritical in the sense that the critical scaling, which is $s_c = \frac{1}{2}$, lies between two conserved quantities, the L^2 -mass and the energy, at the level of H^1 . Hence, we have local solutions in $H^s(\mathbb{R}^3)$ for any $1 \leq s$. As explained in 0.2.2, the repulsive nature of the nonlinearity induces a linear behavior of the solution, and scattering is expected to occur. We do know that scattering holds in the energy space and above since the work of Ginibre and Velo [GV85]. However, at the level of H^s when $\frac{1}{2} \leq s < 1$ we have no conserved coercive quantity, and we cannot say for sure that every solution in the scaling-subcritical regime extend globally in time, and scatter at infinity.

Conjecture 0.2.3. Let $\frac{1}{2} \leq s < 1$. Local solutions to NLS initiated from initial data in $H^s(\mathbb{R}^3)$ extend globally in time, and scatter at infinity: for all $u_0 \in H^s(\mathbb{R}^3)$, there exist $u_{\pm} \in H^s(\mathbb{R}^3)$ such that

$$\lim_{t \rightarrow \pm\infty} \|u(t) - e^{it\Delta} u_{\pm}\|_{H^s} = 0.$$

This conjecture has attracted a lot of attention over the last two decades. For now on, the conjecture is proved up to $\frac{2}{3} < s$. The optimal result is due to [Su12], pursuing an important series of fruitful works. By important, we mean that these works have induced a number of major developments that are now essential in the study of dispersive differential equations. Let us briefly discuss some existing results below the energy space.

- The first global well-posedness and scattering results below the energy space is due to Bourgain [Bou98b] who developed by the way the *high-low* method in order to capture smoothing of the nonlinear term. Namely, Bourgain proved that $u(t) - e^{it\Delta} u_0 \in C(\mathbb{R}; H^1(\mathbb{R}^3))$ ⁶ for $s > \frac{11}{13}$. In addition, Bourgain proved scattering in the radial case when $s > \frac{5}{7}$ by employing Lin Strauss Morawetz estimate and radial Sobolev embedding.
- Then, the I-team [Col+02; Col+04] developed the I-method, which is roughly speaking a smooth version of the high-low method that exploits as well the conservation of the energy. More importantly, they obtained scattering for any data when $s > \frac{5}{6}$ by combining the I-method with a new Morawetz-type estimate, called the *interaction Morawetz estimate*, which yields a priori space-time $L^4_{t,x}$ -bounds.
- Subsequently, Dodson improved in [Dod13] the scattering result for any data when $s > \frac{5}{7}$, refining the I-method with a *double layer decomposition* (which we discuss below), and a linear-nonlinear decomposition.

⁶This reminds the probabilistic nonlinear smoothing. . . It is also a manifestation of the semi-linearity of the equation. For oscillating data, this means that the nonlinear solution is, in some sense, well-approximated by the linear evolution of the *same* initial data (in contrast with scattering results where the final state is different). We refer, e.g., to [BGT10] for a precise discussion on this aspect of semi-linear problems.

- The best result at this day is due to Su [Su12], proving scattering for $s > \frac{2}{3}$. The proof uses *long-time iteration* (meaning that the size of the interval increases at each set of the iteration), made possible by the use of second generation I-method (consisting in adding a correction to the energy function).
- Combining the long-time Strichartz estimates with local-smoothing and radial Sobolev embedding, Dodson [Dod19b] proved the conjecture in the radial case all the way down to $s > \frac{1}{2}$.
- In the critical case, Kenig and Merle [KM10] used a *rigidity* argument, based on the concentration- compactness method, to show that if the $\dot{H}^{\frac{1}{2}}$ -norm of the solution is uniformly bounded then the solution is global, and there is scattering.
- Recently, Dodson [Dod21] proved that there is scattering for initial data in a critical Besov space, with nice polynomial bounds.

It seems that this story is far from over, and that new techniques are needed to go further in understanding this problem. More modestly, we can try to show that scattering occurs for *almost all* initial data. This is all the more justified since a local probabilistic flow exists in H^s for $\frac{1}{4} < s < \frac{1}{2}$, but its long-time behavior is not understood outside the small-data regime. Indeed, *almost-sure* global well-posedness and scattering type of results outside the small-data regime only existed in the energy-critical case. The result presented in Chapter 2 proves almost sure scattering below the deterministic scaling. It reads as follows

Theorem 0.2.4 (Almost-sure scattering for energy-subcritical NLS [Cam23] and [SSW21]). *Take $f_0 \in H^s(\mathbb{R}^3)$ for $\frac{3}{7} < s$ be a radial function. We denote its corresponding Wiener randomization $\omega \mapsto f_0^\omega$, which is not radial. There exists a full-measure set Σ in $H^s(\mathbb{R}^3)$ such that for all $f_0^\omega \in \Sigma$, the corresponding probabilistic local strong solution extends globally in time and scatters at infinity: there exists $v_\pm \in H^\sigma(\mathbb{R}^3)$, such that*

$$\lim_{t \rightarrow \pm\infty} \|u(t) - e^{it\Delta}(f_0^\omega + v_\pm)\|_{H^\sigma} = 0, \quad \sigma = 2s - .$$

We give some comments on the result and on the main difficulties.

- The exact same result was proved independently by Shen, Soffer and Wu [SSW21]. The method is a bit different, and relies on the use of the *high-low* method. Surprisingly, they obtained the same lower bound $\frac{3}{7} < s$, but we think that there is nothing special with this threshold. By the way, we think that the method proposed in Chapter 2 is clear and robust, so that we can implement other deterministic refinements of the I-method and lower the threshold, hopefully reaching $\frac{1}{3} < s$, which corresponds to the best known result $\frac{2}{3} \sim \sigma = 2s -$.
- As usual, we do a linear-nonlinear decomposition $u = e^{it\Delta} f_0^\omega + v$. The key observation is that the nonlinear term is in H^σ , for $\sigma < 1$, but still close enough, so that the deterministic conjecture is known at this regularity. Yet, v solves a perturbed Schrödinger equation, with stochastic forcing terms. Subsequently, the challenge (but also the motivation) was to combine the probabilistic method with the I-method and a Morawetz bootstrap. The difficulty is that the stochastic forcing terms break the gauge-invariant structure of the nonlinearity. Hence, we have extra terms without frequency-cancellation to deal with. Instead, we can have lots of derivatives falling onto the rough, stochastic term.

- For this reason, and as in the energy-critical case, the concealed radial-symmetry assumption, or any other similar assumption, seems to be required. The idea is to exploit some mild radial Sobolev embedding in order to implement the local smoothing estimate, and overcome the loss of derivatives in the energy method. We recall that the randomized data f_0^ω , on the other hand, is not likely to be radially-symmetric. As mentioned in Section 0.2.2 the only work avoiding such an assumption is due to Bringmann, for the energy-critical wave equation in \mathbb{R}^4 [Bri18].
- The interaction Morawetz estimate needs at least $\frac{1}{2}$ derivatives, and we are tempted to only prove a Morawetz estimate for v . Though, such an estimate strongly relies on monotonicity formula that are sensitive to the structure of the nonlinearity of NLS. To overcome this difficulty, we proved a modified version of the interaction Morawetz estimate, where we consider Iu in place of u . This is possible since If_0^ω makes sense in $H^{\frac{1}{2}}$.
- As pointed out by N. Tzvetkov, it is remarkable that the convergence of the remainder holds in a space H^σ close to H^1 even if the initial is in H^s for some $s < \frac{1}{2}$. the fact that the scattering result for the nonlinear remainder holds in H^σ for some σ close to H^1 . This property might also be true for the deterministic results with $\frac{2}{3} < s < 1$, but it is less clear.

Let us now give an outline of the proof.

Blow up criterion. The first part of the analysis follows pre-existing lines. We consider the forced equation

$$\begin{cases} (i\partial_t + \Delta)v = |f^\omega + v|^2(f^\omega + v), & f^\omega = e^{it\Delta} f_0^\omega. \\ v(0) = v_0, & v_0 \in H^\sigma(\mathbb{R}^3), \end{cases} \quad (\text{NLS}_f)$$

Thanks to the refined probabilistic Strichartz estimates, we have access to space-time bounds for f^ω , almost-surely in the randomization. We proceed pathwise in ω , and fix ω in a full measure set such that

$$F^\omega(\mathbb{R}) := \|\langle \nabla \rangle^s f^\omega\|_{L_{t,x}^{10}} + \|\langle \nabla \rangle^s f^\omega\|_{L_{t,x}^4} + \|\langle \nabla \rangle^s f^\omega\|_{L_t^4 L_x^{12}} < +\infty.$$

In particular, this quantity controls the scaling-critical $L_{t,x}^5$ -norm, that requires loosing $\frac{1}{2}$ -derivatives to access the Strichartz-norm $L_t^5 L_x^{\frac{30}{11}}$. In the argument, we are frequently led to reduce the analysis to a finite number of space-time slabs where we impose a smallness condition on the forcing term:

$$F^\omega(J_j) \leq \varepsilon_0, \quad j \leq L, \quad L \lesssim_{\varepsilon_0, \omega} 1.$$

First, we set up a refined probabilistic Cauchy theory for (NLS_f) , with

$$v \in X^{\frac{1}{2}}(J) \subset C(J; H^{\frac{1}{2}}(\mathbb{R}^3)),$$

and where X is based on the U^2 and V^2 spaces. Since the probabilistic structure of the linear evolution is preserved under the evolution⁷, the hope is to iterate the local probabilistic Cauchy

⁷This in sharp contrast with the problems addressed in Part III, where the structure of the adapted linear evolution deteriorates as time evolves. Hence, it is not clear how to iterate the *refined* paracontrolled probabilistic Cauchy theory, except when there is an invariant measure, for which the globalization argument of Bourgain adapts. (See section 0.1.3)

theory, provided we are able to control the H^σ -norm of v . By doing so, we obtain the blow-up criterion, stated in Proposition 2.3.5: if v is a maximal lifespan solution to (NLS_f) on J^* , with $T^* = \sup J^*$, then

$$\|v\|_{L_{t,x}^5(J^* \times \mathbb{R}^3)} < \infty \implies T^* = +\infty, \quad \text{and } v \text{ scatters in } H^\sigma(\mathbb{R}^3).$$

The scattering result in H^σ , for $\frac{2}{3} < \sigma$, also come with the uniform space-time bound for the solution

$$\|u\|_{L_{t,x}^5(\mathbb{R} \times \mathbb{R}^3)} \leq C(\|u_0\|_{H^\sigma(\mathbb{R}^3)}),$$

that we use to develop a *stability* theory for (NLS_f) , in the functional space $X(J)$. Specifically, we reduce the blow-up criterion to

$$\sup_{t \in J^*} \|v(t)\|_{H^\sigma(\mathbb{R}^3)} < \infty \implies \|v\|_{L_{t,x}^5(J^* \times \mathbb{R}^3)} < \infty.$$

A priori bounds. As explained above, Theorem 0.2.4 reduces to obtain the a priori uniform bound for the maximal lifespan solution to (NLS_f) in $H^\sigma(\mathbb{R}^3)$, with $\frac{2}{3} < \sigma < 2s$:

$$\sup_{t \in J^*} \|v(t)\|_{H^\sigma(\mathbb{R}^3)} < \infty.$$

Since we have no conservation law at the level of $H^\sigma(\mathbb{R}^3)$, we need to use modified energies. Given a large integer N , the I-operator $I : H^\sigma(\mathbb{R}^3) \mapsto H^1(\mathbb{R}^3)$ is the Fourier multiplier given by the symbol

$$m_N(\xi) = \begin{cases} 1, & |\xi| \leq N, \\ \left(\frac{N}{|\xi|}\right)^{1-\sigma}, & |\xi| > 2N. \end{cases}$$

with

$$\|f\|_{H^\sigma} \lesssim \|If\|_{H^1} \lesssim N^{1-\sigma} \|f\|_{H^\sigma}.$$

It also maps $H^s(\mathbb{R}^3)$ to $H^{\frac{1}{2}}(\mathbb{R}^s)$, with

$$\|f\|_{H^s} \lesssim \|If\|_{H^{\frac{1}{2}}} \lesssim N^{\frac{1}{2}-s} \|f\|_{H^s},$$

where we used that $s < \frac{1}{2}$ and $\sigma < 2s$. We see from (0.2.2) that it suffices to control $\|Iv\|_{H^1}$. To recall the method, let us consider u the (deterministic) local solution to NLS, in H^σ for some $\frac{1}{2} < \sigma < 1$. Basically, since u has an infinite energy, one can fear that

$$\|Iu\|_{H^\sigma} \xrightarrow{N \rightarrow \infty} +\infty,$$

On the other hand, $E(Iu)$ is all the more close to the conserved energy since N is large. Hence, we expect that the time derivative of truncated energy decays. Basically, [Col+04] proved that when σ is close enough to 1, and if $J = [a, b]$ is a time interval with $E(u(a)) \lesssim 1$, such that we have a smallness condition on $\|u\|_{L_{t,x}^4(J \times \mathbb{R}^3)}$, then

$$\frac{d}{dt} E(Iu) \lesssim \frac{1}{N^{1-\sigma}}.$$

Yet, the overall goal is to prove that the conservation law eventually wins, so that uniformly in N ,

$$\sup_{t \leq T^*} E(Iu(t)) \lesssim 1$$

In our context, where the energy of v solution to (NLS_f) is not conserved due to the presence of the stochastic forcing terms, we need to control the time-increments of the modified energy

$$\mathcal{E}(v)(t) = \frac{1}{2} \int_{\mathbb{R}^3} |\nabla I v(t, x)|^2 dx + \frac{1}{4} \int_{\mathbb{R}^3} |Iu(t, x)|^4 dx.$$

Notice that we decided to place $u = v + f^\omega$ in the potential energy, in order to preserve the structure of the cubic Schrödinger equation as much as possible, and to exploit frequency cancellations. It holds

$$\begin{aligned} \frac{d}{dt} \mathcal{E}(v)(t) &= \text{Re} \int_{\mathbb{R}^3} \partial_t (Iv)(t, x) \underbrace{(|Iu(t, x)|^2 Iu(t, x) - I(|u(t, x)|^2 u(t, x)))}_{:=H} dx \\ &\quad - \text{Im} \int_{\mathbb{R}^3} \nabla I f(t, x) \nabla (|Iu(t, x)|^2 Iu(t, x)) dx. \end{aligned}$$

The first term in the right-hand side contains the commutator H , which displays some frequency cancellations and induces a gain of N^{-1+} , in suitable space-time norms. On the other hand, the second term is a remainder, coming from the stochastic perturbations. In order to be able to choose $s < \frac{1}{2}$, we have to gain derivatives. One way to do so is to use local smoothing effect for the linear evolution of randomized data. This is the place where we use the radial symmetry assumption, and we discuss the probabilistic estimate:

$$\|\langle \nabla \rangle I f^\omega\|_{L_\omega^2 L_t^2 L_x^\infty} \lesssim N^{\frac{1-\sigma}{2}} \quad (0.2.3)$$

in the next paragraph. We stress out that on the right-hand side of the energy estimate of the energy estimate, we keep track of sub-additive quantities (namely, L^2 -norm in time), following a nice idea from [Dod13; Dod19b]. In these two works, Dodson used the refined the energy estimate (under the same assumption as in [Col+04]):

$$\int_J \frac{d}{dt} E(Iu(\tau)) d\tau \lesssim \frac{\|P_{>cNu}\|_{L_t^2(J; L_x^6)}^2}{N^{1-}} + O\left(\frac{1}{N^2}\right).$$

The above quantity is controlled on J as soon as $E(Iu(t)) \lesssim 1$ for all $t \in J$ (this is the long-time Strichartz estimates). By doing so, we avoid losing too many powers of N when summing over the $L = L(N)$ intervals where the L^4 -norm is small.

To conclude, we perform a double bootstrap on the modified energy \mathcal{E} combined with the modified interaction Morawetz from Proposition 2.4.6, of type

$$\iint_{J \times \mathbb{R}^3} |Iu(t, x)|^4 dx dt \lesssim \|Iu\|_{L^\infty(J; \dot{H}^{\frac{1}{2}})}^2 \|u\|_{L^\infty(J; L^2)}^2 + R(\nabla Iu, H), \quad H := |Iu|^2 Iu - I(|u|^2 u),$$

where R is a remainder controlled thanks to the estimates on the commutator H and to the probabilistic estimates for f^ω . Then, we bootstrap the following estimates:⁸

$$\sup_{t \in J} \mathcal{E}(v)(t) \leq N^{2(1-\sigma)}, \quad \iint_{J \times \mathbb{R}^3} |Iu(t, x)|^4 dt dx \leq MN^{1-\sigma}.$$

Thanks to these a priori bounds, to the energy increment, to the modified interaction Morawetz and to the probabilistic estimate discussed below, we are able to prove that these bounds holds up to T^* .

Key probabilistic estimate. We discuss estimate (0.2.3). This is a consequence of a square function decay estimate that reads as follows: for all $\delta > 0$, and all $u_0 \in H^\delta$ radially-symmetric, there holds

$$\left\| \langle x \rangle \left(\sum_{k \in \mathbb{Z}^3} |P_{1;k} u_0(x)|^2 \right)^{\frac{1}{2}} \right\|_{L_x^\infty} \lesssim_\delta \|u_0\|_{H^\delta(\mathbb{R}^3)}.$$

This estimate is inspired from the stronger one used by Dodson, Lührmann and Mendelson [DLM19; DLM20] in dimension four, to address the energy-critical case. It drastically improves the radial Sobolev embedding,

$$\| |x|f \|_{L_x^\infty} \lesssim \|f\|_{H^1},$$

which losses one derivative. The strategy consists in study the following oscillatory integral. Up to changing the coordinates, we assume $x = |x|(1, 0, 0)$ and we use spherical coordinates for the frequency variable. It holds

$$P_{1;k} u_0(x) = \int_0^\infty \int_0^{2\pi} \underbrace{\left(\int_0^\pi e^{i|x|\rho \cos \theta} \psi(\xi(\rho, \theta, \varphi)) \sin(\theta) d\theta \right)}_{\text{oscillatory integral}} \widehat{u_0}(\rho) d\varphi \rho^2 d\rho.$$

Notice that the oscillatory integral does not depend on U_0 , thanks to the radial-symmetry assumption. Then, we use the frequency localization of the Fourier multiplier $P_{1;k}$ on the unit-scale cube centered around k , and we integrate by parts.

Next, as always when we estimate the moments of a random series, we have access to the square function thanks to probabilistic decoupling (see Section 0.1.2 for the discussion on the Paley-Zygmund's theorem): for all $2 \leq p < \infty$,

$$\mathbb{E} \left[\left| \sum_k g_k(\omega) P_{1;k} u_0(x) \right|^p \right]^{\frac{1}{p}} \lesssim \sqrt{p} \left(\sum_k |P_{1;k} u_0(x)|^2 \right)^{\frac{1}{2}}.$$

Then the probabilistic estimate (0.2.3) for the free Schrödinger evolution of Wiener randomization of radial functions follows from the local smoothing effect and probabilistic decoupling.

Almost-sure asymptotic stability for small ground states In Chapter 3 we study the asymptotic stability of small ground states of the Nonlinear Schrödinger equation, in the presence of a short range potential

$$(i\partial_t + \Delta)u(t, x) = \left(\underbrace{V(x)}_{\text{external potential}} + \underbrace{\mu |u(t, x)|^2}_{\text{nonlinear perturbation}} \right) u(t, x), \quad (t, x) \in \mathbb{R} \times \mathbb{R}^3.$$

⁸Notice that we have a power of N . Understanding how to use the subcritical-scaling in H^σ , would enable us to assume that $\mathcal{E} \lesssim 1$ and certainly lower the threshold for s .

Under some spectral assumptions on the Schrödinger operator that ensure the existence of linear bound states, the natural question is to study the persistence of these bound states under *small nonlinear* Hamiltonian perturbations. In a concrete situation, the understanding of such a model would allow to capture the stability of localized waves, emerging from the inhomogeneities of the medium in which they propagate under the effect of nonlinear repulsive interactions. This model has been studied by Soffer and Weinstein [SW90], where they described a *multichannel nonlinear scattering mechanism*, showing that the asymptotic behavior is given by a linear combination of a localized standing wave, which is periodic in time, and a dispersive part that is radiative, respectively. At that time, little was known about the stability of coherent solutions with nonlinear Hamiltonian perturbations, for equations that are not integrable. In this sense, the result was very innovative. In particular, they performed a rigorous and fruitful analysis of the so-called *modulation equations* discussed below.

The spectral assumptions on the Schrödinger operator $H := -\Delta + V$ are the following. First, for simplicity, we assume that V is in the Schwartz class, but the result would persist under mild decay and regularity assumptions. Then, we assume that $-\Delta + V$ has a unique simple eigenvalue $-e_0 < 0$, with eigenfunction ϕ_0 , and absolutely continuous spectrum on $[0, +\infty)$. In addition, we assume that there is no resonance at zero. This is a technical assumption that holds for a generic choice of V . Then, the one-complex parameter family of bound state solutions

$$z \mapsto Q(z, x) e^{-itE(z)}$$

are obtained by a bifurcation argument from the linear ground states $z e^{-ite_0} \phi_0$, parametrized by $z \in \mathbb{C}$. At fixed z , the nonlinear ground state Q satisfies the elliptic equation

$$(\Delta - V(x) + |Q(z, x)|^2)Q(z, x) = E(z)Q(z, x), \quad (z, x) \in \mathbb{C} \times \mathbb{R}^3,$$

where, at the vicinity of $z = 0$, there are some nonlinear corrections e and q obtained from inversion function theorem, such that

$$Q(z) = z\phi_0 + q(z), \quad E(z) = e_0 + e(z).$$

After contributions of Pillet and Wayne [PW97], revisiting the result of [SW90] with center stable manifold technics from dynamical systems, Gustafson, Nakanishi and Tsai proved in [GNT04] that *asymptotic stability* holds for any sufficiently small solution in the energy space $H^1(\mathbb{R}^3)$. Namely, small solutions uniquely resolve into the sum of a modulated bound state and a radiative term that scatters at infinity. Meanwhile, the modulation parameter converges to a fixed parameter. However, since the effect of the external potential is negligible in small-time, local well-posedness holds up to critical regularity $s_c = \frac{1}{2}$, but there is no reason for the bound state to remain stable under rough perturbations in H^s for $\frac{1}{2} < s < 1$. Similarly, what happens to the *probabilistic flow*, locally defined in H^s for some scaling-supercritical $\frac{1}{4} < s < \frac{1}{2}$? The concern of Chapter 3 is to address these two questions. Specifically, we consider random perturbations injected in the initial data, which are of type

$$u(0) = u_0 + f_0^\omega, \quad u_0 \in H^{\frac{1}{2}}, \quad f_0^\omega \in H^s \cap \text{Ran } P_c(H),$$

where $P_c(H)$ denotes the projection onto the continuous spectral subspace. The initial data are randomized according to the Wiener procedure (see Subsection 0.1.2), but they are adapted to the linear inhomogeneous evolution e^{itH} in the sense that the Fourier multipliers are replaced by distorted

Fourier multipliers. Under these perturbations, we prove asymptotic stability for these random perturbations, almost-surely with respect to the probability measure induced by the randomization. As a consequence, we also answer the first question by taking $f_0 = 0$. We give here an informal statement of the main theorem, and we refer to Chapter 3 for a more precise statement.

Theorem 0.2.5. *Let $\frac{1}{4} < s < \frac{1}{2}$. There exist $\delta_0 > 0$ and a full-measure set Σ in $H^s(\mathbb{R}^\omega)$ such that for all $u_0 \in H^{\frac{1}{2}}$ with norm less than δ_0 , and for all $f_0^\omega \in \Sigma$, there exist $\varepsilon = \varepsilon(\omega) > 0$, for which the Cauchy problem*

$$\begin{cases} (i\partial_t + \Delta)\psi(t, x) = \mu|\psi(t, x)|^2\psi(t, x) + V(x)\psi(t, x), & (t, x) \in \mathbb{R} \times \mathbb{R}^3, \\ \psi(0) = \varepsilon f_0^\omega + v_0, \end{cases}$$

admits a local solution under the form

$$\psi(t, x) = \varepsilon e^{-itH} f_0^\omega + v(t, x), \quad v \in C(\mathbb{R}, H^{\frac{1}{2}}(\mathbb{R}^3)).$$

Moreover, this solution extends globally in time, and resolves into the sum of a modulated ground state and a radiative term:

$$\psi(t, x) = Q(z(t)) + \eta(t, x).$$

As time goes to infinity we have asymptotic stability in the sense that there exist $z_\pm \in \mathbb{C}$ and final states $\eta_\pm \in H^{\frac{1}{2}}$ such that

$$\lim_{t \rightarrow \pm\infty} z(t) \exp\left(-i \int_0^t E(z(\tau)) d\tau\right) = z_\pm, \quad \lim_{t \rightarrow \pm\infty} \|\eta(t) - e^{-itH}(\varepsilon f_0^\omega + \eta_\pm)\|_{H^{\frac{1}{2}}} = 0.$$

- As mentioned above, when $f_0 = 0$, Theorem 0.2.5 extends the deterministic result of [GNT04] up to the critical regularity $s = \frac{1}{2}$.
- This work was inspired by the contribution of Kenig and Mendelson presented in Section 0.1.2 for the energy critical wave equation, contributing to the understanding of the persistence of coherent states under rough perturbations. As explained in Section 0.1.2, in the case of the Schrödinger equation, some challenging difficulties appear. The most serious difficulty is that the Duhamel integral term does not display any smoothing, and the linearized operator around the ground states is not self-adjoint. Hence, considering small ground states is a reasonable first step towards the goal of adapting the result of Kenig and Mendelson for Schrödinger equations.
- The main novelty is the *critical-weighted strategy*, used for the first time in the context of probabilistic well-posedness. We outline this strategy below.

Here are some of the key arguments in the proof.

Distorted Fourier transform and Wiener randomization. If one naively performs the standard Da Prato-Debussche's trick, and write $\psi = e^{it\Delta} f_0^\omega + v$, the nonlinear remainder solves

$$(i\partial_t + \Delta)v(t, x) = \underbrace{V(x)u(t, x)}_{\text{no better than } f_0^\omega} + |\psi(t, x)|^2\psi(t, x).$$

Instead, we pass the linear potential on the left-hand side of the equation, and we consider the recentered solution around the *inhomogeneous* linear evolution of the randomized initial data

$$u = e^{-itH} f_0^\omega + v .$$

The Wiener randomization, however, does not commute with e^{itH} , compromising therefore the probabilistic improvement of the Strichartz estimates. To overcome this issue, we follow [KM19] and we adapt the randomization procedure to the inhomogeneous case by employing the *distorted Fourier transform* in place of the free Fourier transform in order to define the unit-scale frequency localization that is at the basis of the procedure. This transformation is a partial isometry from L^2 onto the continuous spectral subspace $\text{Ran}(P_c)$, and it formally writes

$$\mathcal{F}_V f(\xi) \sim \int_{\mathbb{R}^3} \overline{e(x, \xi)} f(x) dx .$$

The generalized plane waves $e(x, \xi)$ formally solve the Helmholtz equation

$$(-\Delta + V) e(\xi, \cdot) = |\xi|^2 e(\xi, \cdot) ,$$

with the Sommerfeld radiation condition

$$e(\xi, x) - e^{i\xi \cdot x} = O(|x|^{-1}) .$$

Their construction is based upon the limiting absorption principle, and the study of such generalized plane waves and of the associated distorted Fourier date back to the early works of [Ike60; Agm75]. We refer to the introduction of Chapter 3 for detailed exposition and references. Roughly speaking, at the level of this present introduction, we should keep in mind that the linear theory of the Fourier transform adapts without any major modification to the distorted Fourier transform, at least when we assume no resonance at the bottom of the continuous spectrum. In our context, it is crucial to ensure that the Fourier multiplier theorem holds true:

$$\|\mathcal{F}_V^{-1}(m\mathcal{F}_V)f\|_{L^p} \lesssim_p |E|^{\frac{1}{2}-\frac{1}{p}} \|f\|_{L^2} , \quad p \geq 2, \quad \text{supp } m \subset E .$$

This is a key ingredient in the proof of the probabilistic Strichartz estimates from Proposition 0.1.5. In addition, we mention that we use the L^p -bound of the wave operator obtained by Yajima in a seminal paper [Yaj95]

$$\|W_\pm\|_{L^p \rightarrow L^p} \lesssim_p 1 .$$

We stress out that the use of the distorted Fourier transform in the randomization procedure is crucial when we study the modulation equations and perform the critical weighted argument. Basically, the adapted randomization procedure preserves the continuous spectral subspace and allows us to trace the structure of the different terms during the evolution. However, the *distorted plane waves* do not satisfy the group property

$$e^{i\xi_1 x} e^{i\xi_2 x} = e^{i(\xi_1 + \xi_2)x} .$$

For this reason, the nonlinear theory is more involved in the inhomogeneous case. For instance, the bilinear Bourgain estimates should not be taken for granted in this case. Nevertheless, in contrast to the study of the energy-critical wave equation addressed in [KM19], we do require such bilinear

estimates to overcome the lack of smoothing of the Schrödinger evolution, and to obtain the desired nonlinear probabilistic nonlinear smoothing effect.

Bilinear estimates for Schrödinger operators with a smooth potential. The subtle study of the quadratic interactions of distorted plane waves has attracted a lot of interest recently. Basically, the use of the deformed Fourier transform at the nonlinear level requires a fine analysis of certain bilinear operators with singular kernels. We refer to [PS20] and the references therein for important progresses in this direction. In Section 3.3.2 of Chapter 3, however, we consider the case when the potential is, say, in the Schwarz class. In this case, we propose a proof of the bilinear estimate that merely uses the L^p -bounds on the wave operators, proving that

$$\|(e^{it(\Delta-V)} \widetilde{P}_{\lesssim N} u_0)(e^{it(\Delta-V)} \widetilde{P}_M v_0)\|_{L^2_{t,x}(\mathbb{R} \times \mathbb{R}^3)} \lesssim NM^{-\frac{1}{2}} \|\widetilde{P}_{\lesssim N} u_0\|_{L^2_x(\mathbb{R}^3)} \|\widetilde{P}_M v_0\|_{L^2_x(\mathbb{R}^3)}, \quad (0.2.4)$$

where \widetilde{P}_K stands for the Littlewood-Paley multiplier associated with the functional calculus on H . The overall strategy consists in reducing the inhomogeneous case, to the homogeneous case by proceeding in two steps. First, we intertwine the Littlewood-Paley projectors with the distorted ones. Specifically, we deduce from some semi-classical functional calculus that for all K, L with $K = 2^k, L = 2^l, |k - l| \geq 3$, and all every $\alpha > 0$,

$$\|P_K \widetilde{P}_L\|_{L^2_x \rightarrow L^2_x} \lesssim_\alpha \max\{K, L\}^{-\alpha}.$$

Thus, roughly speaking, we can replace the inhomogeneous projector \widetilde{P}_N by the flat Littlewood-Paley projector P_N in (0.2.4). Then, in order to reduce the matter to the flat Bourgain's bilinear estimate (0.1.3), we need to replace the inhomogeneous linear evolution e^{-itH} by the free one $e^{it\Delta}$. To do so, we put back the potential term in its place of a source term, at the right-hand-side of the equation. Namely, given a dyadic number K , we write

$$P_K e^{it(\Delta-V)} u_0 = e^{it\Delta} P_K u_0 - i \underbrace{\int_0^t e^{i(t-\tau)\Delta} P_N (V e^{it(\Delta-V)} u_0) d\tau}_{:=F(t)}.$$

Similarly,

$$P_M e^{it(\Delta-V)} v_0 = e^{it\Delta} P_M v_0 - i \underbrace{\int_0^t e^{i(t-\tau)\Delta} P_M (V e^{it(\Delta-V)} v_0) d\tau}_{:=G(t)}.$$

Then, after developing the product $(P_K e^{it(\Delta-V)} u_0)(P_M e^{it(\Delta-V)} v_0)$, we are led to study the interaction of terms $e^{it\Delta} P_K u_0$ and $F(t)$, with the terms $e^{it\Delta} P_M v_0$ and $G(t)$. Hence, we are reduced in some sense to apply the bilinear estimate of Bourgain (0.1.3). This is enough to handle the free linear evolutions $e^{it\Delta} u_0, e^{it\Delta} v_0$. However, to control the source terms $F(t)$ and $G(t)$ we also need to use the local smoothing effect (0.1.4), and to exploit the localization of the external potential V . Overall the analysis is a bit technical, but in essence it combines the bilinear with the local smoothing effect in a suitable way.

Modulation equation. We adapt the framework developed in [GNT04] to settle the asymptotic stability for small data in the energy space. Their key idea is to use a time-dependent orthogonality

decomposition of the phase space, in order to cancel the terms of order one in the modulation equation. Basically, they decouple the local solution into

$$\psi(t, x) = Q(z(t), x) + \eta(z(t), x), \quad \eta(z) \in H_c(z),$$

where the space $H_c(z)$ associated with the modulation parameter z encodes the time-dependent orthogonality condition

$$H_c(z) = \left\{ \eta \in L^2 : \int_{\mathbb{R}^3} \langle i\eta(x), \partial_{z_1} Q(z, x) \rangle dx = \int_{\mathbb{R}^3} \langle i\eta(x), \partial_{z_2} Q(z, x) \rangle dx = 0 \right\}.$$

Thanks to this time-dependent orthogonality condition, they are able to control the projection onto the continuous spectrum of the nonlinear radiative interactions, by merely using the endpoint Strichartz estimate $L_t^2 L_x^6$. As a consequence, they removed the weights assumptions made in [SW90; PW97].

In our setting, on the other hand, we have to use the local smoothing estimate since we cannot obtain nonlinear probabilistic smoothing for the linear term $Q^2\psi$. The ansatz is as follows. From the continuous spectral comparison stated in Lemma 3.5.4 it suffices to control the continuous spectral component of the solution. To achieve such a goal, we decompose the local solution into

$$\psi(t, x) = Q(z(t), x) + \eta(t, x) = Q(z(t), x) + R(z(t))(\nu + e^{-itH} f_0^{\omega}),$$

where $R(z)$ is the isometry from $\text{Ran } P_c$ to $H_c(z)$, and

$$\nu(t, x) = P_c(H)\eta(t, x) - e^{-itH} f_0^{\omega} \in \text{Ran } P_c(H).$$

By doing so, we expect nonlinear probabilistic smoothing effect, so that ν is in $H^{\frac{1}{2}}$. It satisfies the dispersive equation

$$\begin{cases} (i\partial_t + \Delta - V)\nu = P_c(H) \left(\underbrace{|\eta|^2\eta - D_z Q\dot{m}}_{:=f, \text{ nonlinear, decay in time}} + \underbrace{Q^2\eta + Q\eta^2}_{:=g, \text{ localized}} \right) \\ \nu(0) = 0. \end{cases}$$

The main difficulty is to simultaneously control in critical spaces the higher order stochastic terms, collected in f , and the lower order localized terms, collected in g . To achieve this, we proposed a *critical-weighted* approach, inspired from the work of [KM12] for KdV equation which is, to the best of our knowledge, the only work in the spirit of Chapter 3 addressing asymptotic stability by employing critical spaces based upon U^2 and V^2 .

The critical weighted strategy. In order to encapsulate the probabilistic nonlinear smoothing effect and the linear local smoothing effect under the same norm, we do a continuity argument in the intersection of the two relevant spaces: given a time-interval J , possibly unbounded, we set

$$\mathcal{X}(J) := V_H^2(J; H^{\frac{1}{2}}(\mathbb{R}^3)) \cap L_t^2(J; H^{1, -\frac{1}{2}^-}(\mathbb{R}^3)).$$

The space V_H^2 is the space of bounded 2-variation, and is close to the critical space used in [BOP19a] to prove almost-sure global existence and scattering for scaling- supercritical NLS with small initial

data. We refer to Section 4 in Chapter 3 for some discussions on U^2 and V^2 spaces.⁹ Second, the weighed space captures the local-smoothing estimate. Given $\delta > 0$ arbitrarily small, the norm writes

$$\|u\|_{L_t^2(J; H^{1, -\frac{1}{2}-\delta})} := \left(\iint_{J \times \mathbb{R}^3} \langle x \rangle^{1-2\delta} |\langle \nabla \rangle u(t, x)|^2 dt \right)^{\frac{1}{2}}.$$

By suitably combining the transference principle in U^2 with the local smoothing estimate and the duality between U^2 and V^2 , we end up with the key estimate

$$\|\nu\|_{V_H^2(J; H^{\frac{1}{2}})} + \|\nu\|_{L_t^2(J; H^{1, -\frac{1}{2}-\delta})} \lesssim \|\nu_0\|_{H^{\frac{1}{2}}} + \sup_N N^{\frac{1}{2}} \|\mathcal{I}(t, \widetilde{P}_N f)\|_{DU_H^2} + \|g\|_{L_t^2(J; L_x^{2, \frac{1}{2}+\delta})}.$$

Here, \mathcal{I} denotes the Duhamel integral term. The rest of the proof consists in proving uniform a priori estimates on each term in the modulation equation, and to perform a continuity argument.

Over the long term, it would be of great interest to obtain a similar result outside the small data regime. In such a situation, the large ground states are not perturbations anymore. Instead, one would have to absorb them to the linear operator and revisit the analysis with some matrix-operators.

A perspective: almost-sure modified scattering Let us now consider the general defocusing Schrödinger equation, for some power $p \geq 1$

$$\begin{cases} i\partial_t u(t, x) + \Delta u(t, x) = |u(t, x)|^{p-1} u(t, x), & (t, x) \in \mathbb{R} \times \mathbb{R}^d. \\ u(0, x) = u_0(x), \end{cases} \quad (\text{NLS}_p)$$

If p is larger than a certain threshold p^* , depending on the dimension d , the nonlinear interaction is *short range* and there is scattering. However, when $p = p^*$, the solution keeps exhibiting *long range* effects, and we expect scattering but with a logarithmic-in-time phase correction. Such a *modified scattering* result holds in the small data regime, for solutions with decaying assumptions. It is a great challenge to widen our understanding of this phenomenon outside the small and decaying data regime. In this perspective, it would be meaningful to establish modified scattering on *generic* sets of large initial data. A good model for this is the nonlinear Schrödinger equation on the line, where $p^* = 3$. Indeed, for the cubic Schrödinger equation on the line, Burq and Thomann [BT20] recently evidenced the quasi-invariance of the nonlinear flow solution of (NLS_p) for any $p > 1$, with respect to some measures supported on a dense subset of $H^{-\varepsilon}(\mathbb{R})$, where $\varepsilon > 0$ is arbitrarily small. Moreover, they proved almost-sure scattering for short-range interactions, i.e. when $p > 3$. In the cubic case, when $p = p^* = 3$, the interaction is long-range, and NLS displays modified scattering for *any* small and decaying initial data in the energy space $H^1(\mathbb{R})$ [KP11; HN98]. In a future work with N. Burq we would like to extend this result *almost-surely* to the probabilistic flow constructed in [BT20]. The main difficulty is the presence of stochastic forcing terms that do not decay in time, and that rule out the strategy of [KP11; HN98]. Instead, one would have to absorb these terms in the effective dynamics, and to proceed as in [Han+13].

⁹In this argument, we use V^2 instead of U^2 for subtle technical reasons. From Proposition 3.4.12, the weighted space continuously embeds into U^2 , and therefore, by duality, it is the V^2 -norm of the Duhamel term that captures the local smoothing effect. However, it is not clear that doing things the other way around would work.

0.2.3 Probabilistic Cauchy theory for weakly dispersive equations

There has been a rich activity on the qualitative behavior of solutions to nonlinear evolution equations with weak dispersion. Examples are the half-wave equation [Poc11; Gér+18a], the Szegő equation [GG10; GG12b], the Schrödinger equation on compact manifold [BGT05a] or on the Heisenberg group [Gér06; Gas21a], where there is a lack of dispersion in one direction, the fractional Schrödinger equation [FL13; GPH15; ST20c], Kadomtsev-Petviashvili equations [HHK09] and many other models. In general, it is very challenging to solve the Cauchy problem for these equations at low regularity. The reason being that in such contexts, the lack of dispersion is materialized by some loss of derivatives in the Strichartz estimates (0.1.2). In some special cases where the Hamiltonian equation is completely integrable, which means that it has a Lax Pair structure providing infinite conservation laws, methods from integrable systems are substituted for the analytical perturbative approach to study Cauchy theory and long-time dynamics. Although beyond the scope of this thesis, let us mention the recent result of Bahouri and Perelman [BP20], proving the global well-posedness of the one-dimensional derivative NLS by combining analytical techniques, such as profile decomposition, with techniques from integrable systems. Recently [Har+22] proved global well-posedness in L^2 , where the flow-map is not uniformly continuous.

When the weakly dispersive equation is neither completely integrable nor semilinearly well-posed, it is relevant to develop a random data Cauchy theory inspired from quasilinear resolution schemes. Indeed, to run the probabilistic semilinear resolution scheme à la Bourgain one needs to combine the Paley-Zygmund observation from Theorem (0.1.2) with dispersive effects as in (0.1.3), (0.1.4), in order to exhibit probabilistic nonlinear smoothing that may not be sufficient in quasilinear regimes.¹⁰ The two articles presented in this section illustrate this point.

Refined probabilistic well-posedness for NLS half-wave equation *joint work with L. Gassot and S. Ibrahim* [CGI22]. In the perspective of studying weakly dispersive equations, we consider the Schrödinger half-wave equation with a power-type nonlinearity: for $\mu \in \mathbb{R}$, and $(t, x, y) \in \mathbb{R}^{1+2}$,

$$\begin{cases} i\partial_t u + (\partial_{xx} - |D_y|)u = \mu|u|^2 u, & \text{with } \widehat{|D_y|u}(\xi, \eta) = |\eta|\widehat{u}(\xi, \eta). \\ u_{t=0} = u_0, \end{cases} \quad (\text{NLS-HW})$$

In this anisotropic model¹¹, we have a Schrödinger equation in the x -direction, and a half-wave equation in the y -direction, respectively. Hence, in certain regimes with low frequency in the x -direction, and high frequencies in the y -direction, the transport dominates and there is no dispersion. As discussed at the end of this section, the long-time dynamics of smooth solutions to (NLS-HW) is quite rich (see [Xu17; BIK20; BIK21]), despite the fact that the Cauchy theory for this equation is poorly understood. To investigate the Cauchy theory for (NLS-HW) we measure the regularity in the scale of the anisotropic Sobolev spaces:

$$\mathcal{H}^s(\mathbb{R}^2) = L_x^2 H_y^s(\mathbb{R}^2) \cap H_x^{2s} L_y^2(\mathbb{R}^2).$$

¹⁰Another strategy, used by [GL21] for the Grushin Laplacian, would be to cook up a randomization procedure that penalizes some problematic features of the equation, and to remove therefore the singular interactions.

¹¹The study of such a model is rather driven by the mathematical analysis of weakly dispersive equations with anisotropy. Let us note however that (NLS-HW) can be seen as a special case of the fractional Ginzburg-Landau equation in media with fractal properties [Tar06].

The natural choice of these space is dictated by the Hamiltonian structure of the equation and by the invariance of the equation under the scaling symmetry

$$u \mapsto u_\lambda(t, x, y) = \lambda \psi(\lambda^2 t, \lambda x, \lambda^2 y).$$

In particular, the critical exponent associated with the natural scaling symmetry of the equation is $s_c = \frac{1}{4}$. The conserved energy is

$$H(u) = \frac{1}{2} \int_{\mathbb{R}^2} |D_x u(x, y)|^2 + ||D_y|^{\frac{1}{2}} u(x, y)|^2 + \frac{\nu}{4} \int_{\mathbb{R}^2} |u(x, y)|^4 dx dy,$$

Local well-posedness holds at regularities $s > \frac{1}{2}$. This was proved in [BIK20] from a fixed-point argument and Strichartz estimates with derivative loss. However, to the best of our knowledge, a satisfactory Cauchy theory in the energy space and below has not been established yet. The energy space corresponds to the endpoint $s = \frac{1}{2}$, and there are severe obstructions to settle a Cauchy theory in this space. These obstructions are raised in the introduction of Chapter 4, where we suggest adopting a rather sophisticated functional framework, used to tackle similar critical problems as in [BH16] for instance.

At scaling-supercritical regularities $0 < s < \frac{1}{4}$, equation (NLS-HW) is ill-posed. More specifically, we have norm-inflation (see [Kat21]), in a very strong sense. Indeed, the pathological set construction from Chapter 1, which is quite flexible, can be easily adapted here. Furthermore, we prove in Chapter 4 that the solution map is not of regularity \mathcal{C}^3 when $\frac{1}{4} < s < \frac{1}{2}$. As a consequence, the equation is semi-linearly ill-posed, and we have no hope to solve the Cauchy problem by performing a fixed point argument. We have to search for other types of solutions, obtained by a quasilinear scheme, a probabilistic method or in a Fourier-Lebesgue space (see [Grü11] for instance).

We prove almost-sure local well-posedness for (NLS-HW) by adapting the refined probabilistic ansatz of [Bri21], in the spirit of the informal Theorem 0.1.6. In doing so, we indicate that such a probabilistic method can overcome the lack of dispersion, and that it could be adapted to other weakly dispersive models in order to generate a statistical ensemble of initial data leading to strong solutions, when semi-linear well-posedness does not hold. For instance, [Oh11b] evidenced singular interactions for models such as Szegő or KdV, that we are very confident to overcome by the paracontrolled approach used in the present paper.

In addition, the results from Chapter 4 precise the Cauchy theory for (NLS-HW), summarized in figure 0.2.3.

- *Weak dispersion and lack of nonlinear smoothing.* Due to the absence of dispersion in the Half-wave direction, the linear evolution does not gain integrability, and we need to trade regularity against integrability. This is materialized by the Strichartz estimates with derivative loss, used in [BIK20] to prove local well-posedness strictly above the energy space. The proof follows from applying the Bernstein estimate in the y -direction, and the one-dimensional Strichartz estimate in the x -direction.

Proposition 0.2.6 (Strichartz estimate for Schrödinger half-wave equation). *Let (p, q) be a Schrödinger admissible pair in one dimension. For all $f_0 \in L^2_{x,y}(\mathbb{R}^2)$, if there exists a set E with finite Lebesgue measure such that*

$$\text{supp } \mathcal{F}_{y \rightarrow \eta} f_0 \subseteq E,$$



Figure 3: Cauchy theory for equation (NLS-HW) in the scale of Sobolev spaces.

then

$$\|e^{it(\partial_{xx}^2 - |D_y|)} f_0\|_{L_t^p L_{x,y}^q} \lesssim |E|^{\frac{1}{2} - \frac{1}{q}} \|f_0\|_{L_{x,y}^2}. \quad (0.2.5)$$

In particular, we deduce from a Littlewood Paley decomposition that

$$\|e^{it(\partial_{xx}^2 - |D_y|)} f_0\|_{L_t^4 L_{x,y}^\infty} \lesssim \| |D_y|^{\frac{1}{2}+} f_0 \|_{L_{x,y}^2}.$$

The probabilistic method can overcome this loss of derivative in the Strichartz estimates. Indeed, if we consider the Wiener randomization in the y -direction (see figure 4.2), which basically consists in superimposing unit-scale frequency localized initial data, decoupled by normalized independent Gaussian variables, one can easily show that the Strichartz estimates for the one-dimensional NLS hold almost-surely, without any derivative loss. Note, however, that we have no dispersive regularization effect, such as local smoothing or bilinear estimates in the spirit of (0.1.3) or (0.1.4), respectively. As a consequence, the second Picard's iteration has the same regularity as the initial data. To see this, we observe that the Duhamel integral of *high-low-low* type frequency interactions of a function localized at frequencies $\lesssim 1$ in the x -direction, is essentially the transport of the high frequency component in the y -direction. See (4.1) for a more precise statement. This phenomenon must be compared with Bringmann's observation for the derivative wave equation in [Bri21], discussed in Section 0.1.2. As a consequence, we follow [Bri21] and we use a refined probabilistic ansatz to prove almost-sure local well-posedness below the deterministic semi-linear threshold, for some $s < \frac{1}{2}$. The statement of our main theorem is as follows. Given a function $f_0 \in L^2(\mathbb{R}^2)$, we associate a random function obtained by the Wiener randomization

$$f_0^\omega = \sum_{k \in \mathbb{Z}} g_k(\omega) f_0(k), \quad f_0(k) = P_{1;k} f_0 = (\mathcal{F}_{y \rightarrow \eta})^{-1} (\varphi(\eta - k) \mathcal{F}_{y \rightarrow \eta}(f_0)) \quad (0.2.6)$$

where (g_k) is a sequence of independent mean zero complex standard Gaussian variables, $P_{1;k}$ is defined from a partition of unity $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}, [0, 1])$ supported in $[-\frac{1}{2}, \frac{1}{2}]$ as in Figure 4.2. Given $n \geq 0$ and $N = 2^n$, we define the projection $Q_N f_0^\omega$ onto the frequencies localized around N as

$$P_1 f_0^\omega = g_0(\omega) P_0 f(0), \quad P_N f_0^\omega = \sum_{\frac{N}{2} \leq |k| < N} g_k(\omega) f_0(k), \quad P_{\leq N} f^\omega = \sum_{M \leq N} Q_M f_0^\omega.$$

Theorem 0.2.7 (Probabilistic local well-posedness). *Let $s \in (13/28, 1/2]$, and $f_0 \in \mathcal{H}^s$. There exist $T_0 > 0$ and a sequence $(u_n)_{n \geq 1} \in \mathcal{C}([-T_0, T_0]; \mathcal{H}^\infty)^\mathbb{N}$ converging in expectation to a limiting object denoted u*

$$\lim_{n \rightarrow \infty} \mathbb{E} \left[\|u_n - u\|_{L^\infty([0, T_0]; \mathcal{H}^s)}^2 \right] = 0, \quad (0.2.7)$$

in such a way that, almost-surely in $\omega \in \Omega$, there exists $T^\omega > 0$ such that for all $n \in \mathbb{N}$, u_n and u exist in $\mathcal{C}([-T^\omega, T^\omega]; \mathcal{H}^s)$ and solve (NLS-HW) with initial data $P_{\leq n} f_0^\omega$ and f_0^ω , respectively.

A common feature of the derivative wave equation studied by Bringmann in [Bri21] shared by the Cauchy problem addressed here and many other weakly dispersive problems is that the second Picard's iteration of a randomized data does not become smoother.¹² Let us comment Theorem 0.2.7.

- In [Bri21], the presence of a derivative in the nonlinearity rules out the possibility of merely employing the recentering method of Bourgain $u = S(t)f_0^\omega + v$ to prove probabilistic well-posedness. Indeed, the derivative counteracts the gain of one derivative in the Duhamel formula for the linear wave equation. In the absence of bilinear estimates, this gain of one derivative is a key ingredient to show that v is smoother than f_0^ω , and to prove probabilistic well-posedness for wave-type equations. For equation (NLS-HW), however, the derivative loss comes from the lack of dispersion in the linear part of the equation. We are convinced that the refined probabilistic ansatz should be adapted to many weakly dispersive equations that are semi-linearly ill-posed, such as the Szegő equation [Oh11b].
- For wave-type equations (see for instance [DLM20; Bri21; Spi22]), the probabilistic Strichartz estimate for Wiener randomized initial data are refinements of the Klainerman-Tataru dispersive estimate from A.66 in [KT99]. For Schrödinger type equations, however, we just combine unit-scale Bernstein estimates with the Strichartz estimates.
- In contrast with the usual probabilistic Cauchy theory, the convergence does not hold *pathwise* in $\omega \in \Omega$, but in $L^2(\Omega)$. The reason being that we need to perform multiple contraction mapping arguments (one at each step n in the iteration scheme). Instead, we use the truncation method of De Bouard and Debussche discussed in the next paragraph.
- In the iteration scheme, we need to ensure frequency localization of the terms involved. This requires to make slight modifications in the framework of [Bri21]. Namely, in the absence of a convenient conserved energy, we use the conservation of the mass (which is not true for wave-type equations) to prove the almost frequency localization of the adapted linear at frequencies $\sim N$. Moreover, we replace the Gronwall arguments by using the TT^* Strichartz estimates.
- Since the standard Strichartz estimates for the one-dimensional free Schrödinger evolution provide control of the $L_x^\infty(\mathbb{R})$, we only need to consider randomized initial data in the y -direction, as in Figure 4.2. Hence, our result actually holds in $L_x^2 H_y^s$.
- Unfortunately, our result does not cover the whole range $\frac{1}{4} < s < \frac{1}{2}$, where the equation is quasi-linear. This would require to remove other bad frequency interactions from the second Picard's iteration, such as $F_n^\omega P_{>N^\gamma} u_{n-1} u_{n-1}$, but this would deteriorate the frequency structure of F_n^ω , ruling out the probabilistic Strichartz estimates. One possibility to lower the regularity threshold can be to adapt the *random-tensors* approach, developed in [DNY22].
- In Theorem 0.2.7, we give a different formulation of the result obtained from the paracontrolled approach by Bringmann in [Bri21]. Indeed, we emphasize that the solution is defined as the

¹²By *randomized initial data* we suggest that the associated measure in \mathcal{H}^s is not degenerate. Indeed, there exist randomization procedures that allow one to gain derivatives in L^p -spaces, and this is not what we want here.

limit of the smooth solutions with frequency-truncated data ¹³. This is to put in perspective with the informal Theorem 0.1.1, and with the situation described in Chapter 1. Besides, we explain how to easily adapt to (NLS-HW) the result of Chapter 1 on the existence of a *pathological set* of inflating data for NLS.

- *The iteration scheme and the paracontrolled approach.* Take $n \gg 1$, and set $N = 2^n$. We know from [BIK20] that (NLS-HW) is locally well-posed in $\mathcal{H}^{\frac{1}{2}+}$. In particular, the smooth solution initiated from the truncated initial data

$$\begin{cases} (i\partial_t + \partial_{xx} - |D_y|)u_n = \mu|u_n|^2u_n, \\ u_n(0) = Q_{\leq N}f_0^\omega, \end{cases}$$

exists and is unique in $C([-T, T], \mathcal{H}^{\frac{1}{2}+})$, given a certain time $T > 0$ depending on ω and on n . Our goal is to prove the convergence of (u_n) in $C([-T_0, T_0], \mathcal{H}^s)$, for $s_0 < s < \frac{1}{2}$ and a uniform time T_0 . To achieve this goal, we consider the increment $v_n = u_n - u_{n-1}$, solution of

$$\begin{cases} (i\partial_t + \partial_{xx} - |D_y|)v_n = \mu|v_n + u_{n-1}|^2(v_n + u_{n-1}) - \mu|u_{n-1}|^2u_{n-1}, \\ v_n(0) = Q_N f_0^\omega, \end{cases}$$

As explained above, the standard probabilistic method fails due to *high-low-low*-type interactions. Instead of merely employing the recentering method from Bourgain around the linear evolution, we introduce the adapted linear evolution F_n^ω . Dropping the complex conjugate from the notation, it solves

$$\begin{cases} (i\partial_t + \partial_{xx}^2 - |D_y|)F_n^\omega = \mu F_n^\omega (P_{\leq N^\gamma} u_{n-1})^2, \\ F_n^\omega(0) = Q_N f_0^\omega. \end{cases} \quad (0.2.8)$$

Here, $0 < \gamma < 1$ is a parameter to be adjusted to minimize the regularity s , and $P_{\leq N^\gamma}$ is the frequency cutoff in the y -direction. The adapted linear evolution F_n^ω encapsulates the bad interactions, and solves a linear equation with the high frequency initial data, multiplied by the low frequency components of the solution constructed at the previous steps of the iteration. Note that the potential $(P_{\leq N^\gamma} u_{n-1})^2$ is measurable in the sigma algebra generated by $\{g_k : |k| < \frac{1}{2}N\}$. As a consequence, they are independent of the sigma algebra generated by $\{g_k \mid \frac{1}{2}N \leq |k| < N\}$. Then, we use the linearity of (0.2.8) together with the random structure (0.2.6) of the initial data, to decompose

$$F_n^\omega = \sum_{\frac{N}{2} \leq |k| < N} g_k(\omega) F_{n,k},$$

where $F_{n,k}$ solves (0.2.8) with initial data $P_{1;k}f_0$. Using the frequency localization of the potential in (0.2.8) and the Strichartz estimate (0.2.5), we obtain a probabilistic Strichartz estimate for each $F_{n,k}$, losing $N^{\frac{\gamma}{2}}$ derivatives in y to control the $L_t^4 L_{x,y}^\infty$ instead of $N^{\frac{1}{2}}$. Next, we use the aforementioned probabilistic independence between the high-frequency initial data and the low-frequency potential

¹³Still, we only prove this convergence for dyadic truncation, and it is not easy to generalize this convergence for any truncation. Sun and Tzvetkov did it in [ST21].

in order to transfer the probabilistic Strichartz estimate from the individual $F_{n,k}$ to the adapted linear evolution F_n^ω . This is in the spirit of the proof of the Paley Zygmund theorem, but with a conditioning on the sigma algebra \mathcal{F}_{n-1}). With these probabilistic Strichartz estimates at hand, we decompose

$$v_n = F_n^\omega + w_n,$$

where w_n satisfies

$$(i\partial_t + \partial_{xx}^2 - |D_y|)w_n = \mu|F_n^\omega + w_n + u_{n-1}|^2(F_n^\omega + w_n + u_{n-1}) - \mu|u_{n-1}|^2u_{n-1} - \mu F_n^\omega (P_{\leq N^\gamma} u_{n-1})^2,$$

with zero initial condition. Since the worst nonlinear interactions are removed from the above equation, we expect some nonlinear smoothing to occur for w_n . Specifically, we run a contraction mapping argument on w_n at regularity \mathcal{H}^ν smoother than the deterministic threshold ($\nu > \frac{1}{2}$). There are two main ingredients.

First, we need to use the truncation method from De Bouard and Debussche [BD99]. The reason being that, in contrast with the standard probabilistic Cauchy theory, we need to do the contraction mapping argument at each step n , so that the time of existence might not be in uniform in n . Instead, we do the contraction-mapping argument for w_n in a $L_\omega^r L_t^p L_{x,y}^q$ -type space, on a fixed time-interval, not depending on n . However, since there is no smoothing in L_ω^r , one needs to truncate the nonlinearity. The truncation is subsequently removed, by implementing some a priori estimates for the relevant norms of (w_n) , on a uniform time-interval. We refer to Section 3 in [Bri21], to [BD99] and to Chapter 4 for a precise description of the truncation method.

Second, we prove some *paracontrolled* trilinear estimates on the nonlinear remainder w_n , in suitable functional spaces that encode the frequency localizations. Specifically, F_n^ω and w_n have to be localized at high frequencies of size $\sim N$. Roughly speaking, the general strategy consists in putting the stochastic terms F_n^ω in L^∞ in order to benefit from the refined probabilistic Strichartz estimates. Surprisingly, one must also proceed in this way in some cases when F_n^ω has the highest frequency. The reason being that the rough linear evolution F_n^ω has more derivatives in L^∞ than w_n . This is a consequence of the nice probabilistic structure of F_n^ω , and of the frequency localization of the unit blocks $F_{n,k}$. As explained below, however, this only holds in small-time, and it makes it even more difficult to iterate the local well-posedness result to extend the solutions globally in time.

- *Comments on the globalization of this refined probabilistic flow.* Unfortunately, we are not able to extend the solutions globally in time, say in the defocusing case ($\mu > 0$). Understanding the long-time behavior of the probabilistic solutions to (NLS-HW) obtained in Theorem 0.2.7, would be significant progress. In the presence of an invariant Gibbs measure, Deng, Nahmod and Yue [DNY19], and Sun and Tzvetkov [ST21] used the so-called Bourgain globalization argument to extend the local solution obtained by the paracontrolled strategy globally in time. Without any invariant, or quasi-invariant measure, however, we are not able yet to iterate the local well-posedness result. This is in sharp contrast with the standard linear-nonlinear probabilistic ansatz, as a discussion from paragraph 0.1.3 attests. Let us briefly discuss the main obstructions to iterate the local well-posedness result. First, at each step n of the recurrence, the adapted linear evolution F_n^ω has better L^∞ properties than the nonlinear term w_n (in the y -variable, the adapted linear evolution is in $W^{\sigma',\infty}(\mathbb{R}_y)$, whereas w_n is in $W^{\sigma,\infty}(\mathbb{R}_y)$ for some $\sigma < \sigma'$, respectively). In [ST20b], the authors

faced a similar issue where the nonlinear remainder is not in the same space as the rough linear evolution that is controlled in some Fourier-Lebesgue spaces. They used summed spaces to overcome this difficulty. In our case, however, the linear rough evolution has more derivatives in L^∞ than the nonlinear remainder term has, even if the latter is smoother in the scale of Sobolev spaces. More importantly, as time progresses, the frequency components intermingle so that the key frequency localization of the unit blocks $F_{n,k}$ and the random structure of F_n^ω have no reason to persist in long-time. Finally, let us point out that the deterministic Cauchy theory for (NLS-HW) is poorly understood, especially in long-time. For instance, we do not know yet if the smooth solutions extend globally in time.

- *Perspectives: long-time dynamics.* It is also of interest to study the long-time dynamics of (NLS-HW), at least for smooth solutions. In the defocusing case, we expect modified scattering as in [Xu17], but with different asymptotic effective dynamics. Meanwhile, for (NLS-HW) on the waveguide $\mathbb{R}_x \times \mathbb{T}_y$ with a focusing nonlinearity, there exists a family of ground state traveling waves. In [BIK21], the authors characterized the ground states $u = e^{it\omega} Q_\omega$ at low frequency ω , and proved that the traveling waves only depend on the x -variable, and correspond to the NLS line solitons. Then, they establish orbital stability properties, and prove transversal instability at high frequencies. Changing the geometry and considering the wave-guide $\mathbb{T}_x \times \mathbb{R}_y$, we expect to obtain similar results, but using the relevant family of half-wave traveling waves, that are characterized in [Gér+18b]. Specifically, at high frequencies we expect transverse instability, together with a possible bifurcation at the stability threshold.

Almost-sure well-posedness for NLS on \mathbb{S}^2 (*joint work with N.Burq, M. Latocca, C. Sun and N. Tzvetkov*)

In an effort to understand the effect of the background geometry on the dynamics of nonlinear Schrödinger equations, Burq Gérard and Tzvetkov proved in a series of works [BGT02; BGT04; BGT05b; BGT05a; BGT10] that the Cauchy problem for the cubic Schrödinger equation on the two-dimensional sphere \mathbb{S}^2 is semi-linearly well-posed in $H^s(\mathbb{S}^2)$ if and only if $s > \frac{1}{4}$ (see also [Ban04]). As a consequence, the gap to bridge between the deterministic threshold and the space where the corresponding Gibbs measure lives, which is $H^{0-}(\mathbb{S}^2)$, is significant. It is therefore extremely challenging to obtain the counterpart on the sphere \mathbb{S}^2 to the celebrated result of Bourgain [Bou96] on the two-dimensional torus \mathbb{T}^2 . Still, setting up a probabilistic Cauchy theory just below the deterministic threshold $s = \frac{1}{4}$ is yet to be done. Let us briefly discuss the difficulties. The most important one is the degeneracy of the spherical harmonics. Namely, given the energy level $\lambda_n^2 = n(n+1)$, the multiplicity of the corresponding eigenspace, generated by the spherical harmonics of degree n , is of dimension $2n + 1$. Moreover, some spherical harmonics exhibit concentration around the geodesics of the sphere, which is not the case for the plane waves $e^{in \cdot x}$ on the torus. Last but not least, and in sharp contrast with the case of the torus, *high-low*-type frequency interactions can generate low frequencies.

To initiate the study of NLS on \mathbb{S}^2 , we obtained a preliminary result by isolating the worst singular interaction, and proving that the Wick-ordering cubic Schrödinger equation posed on \mathbb{S}^2 with random data is still quasilinear. Namely, the first nontrivial Picard's iteration has no better regularity than the randomized initial data.

Before diving into the probabilistic problem, let us mention the result of [BGT10]. It describes

the nonlinear character of NLS on \mathbb{S}^2 by considering $H^s(\mathbb{S}^2)$ -normalized sequences (u_n) in $H^1(\mathbb{S}^2)$ that are *oscillatory* in the sense that

$$\lim_{n \rightarrow \infty} \|u_n\|_{L^2} = 0, \quad \|\nabla u_n\|_{L^2} \leq C \|u_n\|_{L^2}^{\frac{1-s}{s}}.$$

The oscillations are all the more important when n is large. When $s \leq \frac{1}{4}$, and in the asymptotic regime $n \rightarrow \infty$, [BGT10] proved that the nonlinear evolution of strongly oscillating data (u_n) is not well approximated by their linear evolution:

$$\lim_{n \rightarrow \infty} \|u^n(t) - e^{it\Delta} u_0^n\|_{H^s} = 0,$$

if and only if $s > \frac{1}{4}$. For the two-dimensional torus, however, the above convergence holds up to $s > 0$. This result indicates that the high oscillations are not linearly propagated, and it strongly suggests that a semilinear scheme is not appropriate. To exhibit the instabilities, Burq Gérard and Tzvetkov considered stationary solutions that are nonlinear versions of the spherical harmonics with the highest weight that concentrate on a great circle. In Chapter 5, we also use these spherical harmonics to prove that a probabilistic semilinear approach fails.

We need to introduce some notations and basic properties of the spherical harmonics, and we refer to the book [SW71] for interesting results and discussions on spherical harmonics through the perspective of harmonic analysis. We write $(Y_{n,k})_{1 \leq n, |k| \leq n}$ the canonical orthonormal basis of $L^2(\mathbb{S}^2)$ made of spherical harmonics, defined in spherical coordinates by

$$Y_{n,k}(\theta, \varphi) = e^{ik\varphi} L_{n,k}(\cos(\theta)),$$

where $L_{n,k}$ is the associated Legendre function of degree n and order k . They satisfy

$$-\Delta_{\mathbb{S}^2} Y_{n,k} = n(n+1) Y_{n,k}.$$

For $n \geq 1$, we write

$$\lambda_n^2 := n(n+1), \quad \text{the } n^{\text{th}} \text{ energy level, of multiplicity } [n] := 2n+1.$$

We keep in mind that for large n , we have $\lambda_n \sim [n] \sim n$ up to an insignificant multiplicative constant. The space of spherical harmonics of degree n writes

$$E_n = \text{span} \{Y_{n,k} : |k| \leq n\}, \quad \text{and } \pi_n := \ker(-\Delta_{\mathbb{S}^2} - \lambda_n^2 \text{Id}).$$

It is important to observe that E_n is invariant under the isometries of \mathbb{R}^3 . As a result,

$$\sum_{|k| \leq n} |Y_{n,k}(x)|^2 = [n], \quad \text{for all } x \in \mathbb{S}^2, \quad (0.2.9)$$

up to a normalization constant that does not depend on n . To see this, we observe that the integral kernel of the operator π_n is

$$K_n(x, y) = \sum_{|k| \leq n} Y_{n,k}(x) \overline{Y_{n,k}(y)}.$$

Since E_n is invariant under the isometries, we have that for all $R \in \mathcal{O}_3$, and for all $x, y \in \mathbb{S}^2$,

$$K_n(x, R^{-1}y) = K_n(x, y).$$

Since the rotations act transitively on \mathbb{S}^2 , we see that the quantity $K_n(x, x)$ is constant on \mathbb{S}^2 . Then (0.2.9) follows by integrating over \mathbb{S}^2 . To finish with the notations, and to give some meaning to the mass of an initial data on the support of the Gibbs measure, which is below $L^2(\mathbb{S}^2)$, we need to introduce a truncation at a large frequency N , and we set

$$\Pi_N = \sum_{n \leq N} \pi_n, \quad P_N = \Pi_N - \Pi_{\frac{N}{2}}.$$

Let us now set up the probabilistic framework. We fix $\alpha \in [\frac{1}{2}, \frac{3}{4})$, and a sequence of independent normalized Gaussian variables with complex values $(g_{n,k})_{1 \leq n, |k| \leq n}$. We set,

$$\phi^\omega = \sum_{1 \leq n} \frac{e_n^\omega}{\lambda_n^\alpha}, \quad e_n^\omega(x) = \sum_{|k| \leq n} \frac{g_{n,k}(\omega)}{[n]^{\frac{1}{2}}} Y_{n,k}(x). \quad (0.2.10)$$

The above series converges in $L^2(\Omega; H^{\alpha-\frac{1}{2}-}(\mathbb{S}^2))$, so that ϕ^ω lives in $H^{\alpha-\frac{1}{2}-}(\mathbb{S}^2)$ almost-surely. When $\alpha = \frac{1}{2}$, the initial data is distributed according to the Gibbs measure associated with the Hamiltonian equation NLS, as in the case of the torus (see Section 0.1.3 for discussion on Gibbs measures). However, the deterministic threshold is $H^{\frac{1}{4}}(\mathbb{S}^2)$. The *Wick-ordering* Schrödinger equation with a cubic nonlinearity reads

$$\begin{cases} i\partial_t u + \Delta u = (|u|^2 - 2\|u\|_{L^2(\mathbb{S}^2)}^2)u, & (t, x) \in \mathbb{R} \times \mathbb{S}^2. \\ u(0) = \phi^\omega, \end{cases} \quad (\text{NLS}_W)$$

Passing from (NLS) to (NLS_W) amounts to do the gauge transformation

$$v(t) = e^{2it\|u(t)\|_{L^2}^2} u(t),$$

and to use the mass conservation. Our result states as follows.

Theorem 0.2.8 (Lack of regularization of the second Picard's iteration, [Bur+22]). *Let $\alpha \geq \frac{1}{2}$, and $\phi^\omega \in H^{\alpha-\frac{1}{2}-}(\mathbb{S}^2)$ be the randomized initial data as in (0.3.2). There exist $\eta > 0$, $N_0 > 0$ such that for every $N \geq N_0$ and t ,*

$$|t|\eta \log(N) \leq \mathbb{E} \left[\left\| \int_0^t e^{i(t-\tau)\Delta} \Pi_N \mathcal{N}(\Pi_N e^{i\tau\Delta} \phi^\omega) d\tau \right\|_{H^{\alpha-\frac{1}{2}}(\mathbb{S}^2)}^2 \right],$$

where \mathcal{N} denotes the nonlinear term of the Wick-ordering cubic (NLS_W).

Let us briefly comment this statement.

- As N goes to infinity, we see that the quadratic moment of the $H^{\alpha-\frac{1}{2}}(\mathbb{S}^2)$ -norm of the second Picard's iteration of ϕ^ω has a $\log(N)$ divergence. Since the induced measure lives in $H^{\alpha-\frac{1}{2}-}(\mathbb{S}^2)$, this shows that there is no probabilistic regularizing effect.

- For technical reasons, we could not prove that the divergence occurs *almost-surely* in ω . It is not clear whether there exists an elementary proof of this fact that does not rely on sophisticated tools from stochastic analysis.
- In sharp contrast, the second Picard's iteration on \mathbb{T}^2 gains $\frac{1}{2}$ derivatives. This is also true in the case of irrational tori, and we give a self-contained proof of this result, which was previously proved in the analysis of [Fan+21].

We now outline the proof of Theorem 0.3.3, and see how to exhibit one singular interaction that causes the obstruction. For convenience, we do as if $[n] = \lambda_n = n$.

- *Exploiting some properties of the Gaussian spherical harmonics.* We shall essentially use two important properties of $(e_n^\omega)_{n \geq 1}$. The first one is that

$$\mathbb{E} [|e_n^\omega(x)|^2] = 1 \quad \text{for all } x \in \mathbb{S}^2.$$

This follows from (0.2.9):

$$\begin{aligned} \mathbb{E} [|e_n^\omega(x)|^2] &= \mathbb{E} \left[\frac{1}{[n]} \sum_{k, k' = -n}^n g_{n,k}(\omega) \overline{g_{n,k'}(\omega)} Y_{n,k}(x) \overline{Y_{n,k'}(x)} \right] \\ &= \frac{1}{[n]} \sum_{|k| \leq n} |Y_{n,k}(x)|^2 = 1, \end{aligned}$$

where we used that $(g_{n,k})_{|k| \leq n}$ are independent normalized Gaussian variables. Besides, as a consequence of the orthogonality of the spherical harmonics $(Y_{n,k})_{|k| \leq n}$, we have that

$$\mathbb{E} \left[\|e_n^\omega\|_{L^2(\mathbb{S}^2)}^2 \right] = 1.$$

Subsequently, the expectation of the renormalized potential in the nonlinearity (NLS_W) applied to the term e_n^ω vanishes pointwise in x :

$$\mathbb{E} \left[|e_n^\omega(x)|^2 - \|e_n^\omega\|_{L^2(\mathbb{S}^2)}^2 \right] = 0 \quad \text{for all } x \in \mathbb{S}^2. \quad (0.2.11)$$

In sharp contrast with the case of the torus, however, the above estimate does not hold pointwise in ω .¹⁴ By the way, we point out that a key feature is that the law induced by e_n^ω is invariant under the action of the rotations that act transitively on \mathbb{S}^2 . Then, in the spirit of Paley-Zygmund's theorem, Gaussian spherical harmonics have better integrability properties than expected by the Sobolev embedding. For $2 < q < +\infty$ arbitrarily large, there exists $C_q > 0$ such that for all n ,

$$\|e_n^\omega\|_{L^q(\mathbb{S}^2)} \leq C_q \quad \text{a.s.}$$

Once again, this follows from (0.2.9). We refer to [BL14] for general study of such a probabilistic gain of integrability on the sphere. To put this in perspective, we note that a given spherical harmonic

¹⁴This is a major obstruction in the probabilistic Cauchy theory, and we need to better understand the random structure of the colored terms in order to go all the way down to the support of the Gibbs measure.

can have its L^∞ -norm of size $\sim n$. To give a simple example, we can think of the *zonal* spherical harmonic (see e.g. [Ste70; BGT02]).

- *Dissociating the different regimes.* Let us now isolate a singular interaction. We formally decompose the nonlinearity $\mathcal{N}(u) := (|u|^2 - 2\|u\|_{L^2}^2)u$ into three different regimes:

$$\begin{aligned}\mathcal{N}_1(u) &= \sum_{\substack{n_1, n_2, n_3 \\ n_2 \neq n_1, n_3}} \pi_{n_1} u \overline{\pi_{n_2} u} \pi_{n_3} u, \\ \mathcal{N}_2(u) &= 2 \sum_{\substack{n_1, n_2 \\ n_2 \neq n_1}} (|\pi_{n_2} u|^2 - \|\pi_{n_2} u\|_{L^2(\mathbb{S}^2)}^2) \pi_{n_1} u, \\ \mathcal{N}_3(u) &= \sum_{n_1} |\pi_{n_1} u|^2 \pi_{n_1} u - 2 \sum_{n_1} \|\pi_{n_1} u\|_{L^2}^2 \pi_{n_1} u.\end{aligned}$$

Given a time $t > 0$ and a function $F \in C([0, t], L^2(\mathbb{S}^2))$, we denote the Duhamel integral term as

$$\mathcal{I}(t, F) = \int_0^t e^{i(t-\tau)\Delta_{\mathbb{S}^2}} F(\tau) d\tau,$$

so that given a large N , the truncated first nontrivial Picard's iteration is

$$u_N^{(1)}(t) := e^{it\Delta} \phi_N^\omega + \mathcal{I}(t, \Pi_N \mathcal{N}(e^{it\Delta} \phi_N^\omega)).$$

where

$$e^{it\Delta} \phi_N^\omega(t) = \Pi_N e^{it\Delta_{\mathbb{S}^2}} \phi^\omega = \sum_{n=1}^N e^{-itn^2} \frac{e_n^\omega}{n^\alpha}$$

is the truncated free Schrödinger evolution of ϕ^ω . With these notations at hand, we write

$$\text{I} := \mathcal{I}(t, \Pi_N \mathcal{N}_1(\phi_N^\omega)), \quad \text{II} := \mathcal{I}(t, \Pi_N \mathcal{N}_2(\phi_N^\omega)), \quad \text{III} := \mathcal{I}(t, \Pi_N \mathcal{N}_3(\phi_N^\omega)).$$

The singular interaction we study in Chapter 5 is II. Note, however, that the term I contains a singular interaction as well. Nevertheless, we prove from probabilistic independence of the Gaussian variable $(g_{n,k})$ that

$$\mathbb{E}[(\text{I} \mid \text{II})_{\mathbb{S}^2}] = \mathbb{E}[(\text{I} \mid \text{III})_{\mathbb{S}^2}] = 0,$$

so that the two singular interactions cannot cancel each other, and

$$\mathbb{E} \left[\|\text{II}\|_{L^2(\mathbb{S}^2)}^2 \right] - \mathbb{E} \left[\|\text{III}\|_{L^2(\mathbb{S}^2)}^2 \right] \leq \mathbb{E} \left[\|\mathcal{I}(t, \Pi_N \mathcal{N}(\phi_N^\omega))\|_{L^2(\mathbb{S}^2)}^2 \right].$$

As for the third interaction III, we easily prove that it gains one derivative. As a consequence, the contribution of this term is uniformly bounded with respect to N , so that it suffices to obtain a logarithmic divergence for the contribution of the singular interaction II.

- *Estimating from below the contribution of the singular interaction.* From probabilistic independence and (0.2.11), combined with the orthogonality of the spherical harmonics of different degrees, we obtain the lower bound

$$\mathbb{E} \left[\|\text{II}\|_{L^2(\mathbb{S}^2)}^2 \right] \gtrsim t \sum_{\substack{2 \leq n \leq N \\ |k| \leq n}} \frac{1}{n^2} \mathbb{E} \left[\left| \int_{\mathbb{S}^2} (|e_1^\omega(x)|^2 - \|e_1^\omega\|_{L^2(\mathbb{S}^2)}^2) |Y_{n,k}(x)|^2 d\sigma(x) \right|^2 \right].$$

Thus, we are reduced to prove that there is $\eta > 0$ such that for every n sufficiently large

$$\sum_{|k| \leq n} \mathbb{E} \left[\left| \int_{\mathbb{S}^2} (|e_1^\omega(x)|^2 - \|e_1^\omega\|_{L^2(\mathbb{S}^2)}^2) |Y_{n,k}(x)|^2 d\sigma(x) \right|^2 \right] \geq \eta n. \quad (0.2.12)$$

First, we observe that up to a normalization constant,

$$e_1^\omega(x) = g_{1,1}(\omega)\beta(x_1 + ix_2) + g_{1,1}(\omega)Y_{1,0} + g_{-1,1}(\omega)Y_{-1,1},$$

where $\beta > 1$ is a normalization constant. By taking ω in the set of positive measure

$$S_{\varepsilon,A} = \{\omega \in \Omega : |g_{1,1}(\omega)| > A, |g_{1,0}(\omega)| < \varepsilon, |g_{-1,1}(\omega)| < \varepsilon\},$$

for some suitable choice of $\varepsilon \ll 1$ and $A > 0$, we obtain

$$|e_1^\omega(x)|^2 - \|e_1^\omega\|_{L^2(\mathbb{S}^2)}^2 = |g_{1,1}(\omega)|^2(\beta^2(1 - x_3^2) - 1) + O(\varepsilon),$$

Next, we observe that in (0.2.12), the summands are all the larger as the spherical harmonics $Y_{n,k}$ are concentrated on the equator. Indeed, the function $\beta^2(1 - x_3^2) - 1$ concentrates its mass in the region $|x_3| \ll 1$. Subsequently, our key ingredient is a quantitative estimate that measures the number of spherical harmonics concentrated on the equator, in the semiclassical regime. Roughly speaking, we prove that $\propto n$ spherical harmonics with the highest weight $|k|$ concentrate on the equator. This yields the lower bound (0.2.12), which is sufficient to evidence the logarithmic divergence.

- *Obtaining quantitative concentration estimates for a number of spherical harmonics.* Take $0 < \delta \ll 1$, and n large enough. The key quantitative estimate reads

$$\|\mathbf{1}_{|x_3| > 2\delta} Y_{n,k}\|_{L^2(\mathbb{S}^2)} \lesssim_\delta \frac{1}{n} \quad \text{for all } k \text{ with } \sqrt{1 - \delta^2}n < |k| \leq n.$$

We sketch the proof, written in Section 5.3. Recall that in spherical coordinates the spherical harmonics write

$$Y_{n,k}(\theta, \varphi) = e^{ik\varphi} v_{n,k}(\theta).$$

Notice, however, that the spherical coordinates are singular at the poles. To overcome this issue, and to be able to use semiclassical functional calculus, we apply a pseudo-conformal transformation in order to send the singularities at infinity. Specifically, the map

$$y = \operatorname{arctanh}(\cos(\theta)).$$

is a C^∞ -diffeomorphism from $(0, \pi)$ to \mathbb{R} . We do the change of variable, and write $\tilde{v}_{n,k}(y) = v_{n,k}(\theta)$. It turns out that

$$P_h(y, hD_y, \alpha)\tilde{v}_{n,k}(y) = 0, \quad y \in \mathbb{R},$$

where $h \sim n^{-1}$ is the semiclassical parameter and $\alpha = |k|h$. The differential operator has a symbol of order 2

$$p_h(y, \xi, \alpha) := p_0(y, \xi, \alpha) + hp_1(y, \xi) + h^2, \quad p_0(y, \xi, \alpha) := \xi^2 + \alpha^2 - \cosh(y)^{-2}.$$

When $\alpha n < |k|$, this operator is elliptic near in the region $|x_3| \leq \delta$, in the sense that

$$\operatorname{Char} P_{h,\alpha} = \{(y, \xi) : p_0(y, \xi, \alpha) = 0\} \subseteq \{(y, \xi) : \tanh(y) \leq \delta\},$$

The quantitative estimate follows. To prove this standard result, we performed self-contained basic computations in the spirit of semiclassical analysis, in order to track the dependence on the parameter α .

Paracontrolled approach for NLS on the sphere Our result also come with some positive observations, and does not preclude any local well-posedness statement. Indeed, we have not only identified the obstruction for performing the Bourgain strategy, but we have also proved that other interactions display a smoothing effect.

In a long-term program N. Burq, C. Sun and N. Tzvetkov we develop a probabilistic Cauchy theory, employing paracontrolled calculus and the random averaging operators introduced in Section 0.1.2. Specifically, we overcome the lack of nonlinear smoothing by refining the probabilistic ansatz and exploiting the independence of high-frequency initial data and low-frequency remainders, employing the random averaging operators mentioned in Section 0.1.2. Our use of the random averaging operators is inspired by the works of Bringmann [Bri21], Deng, Nahmod, Yue [DNY19], and Sun, Tzvetkov [ST21], but it will come with novelties. The most important one is that we exploit the propagation by the flow map of the invariance of the Gaussian measures under the action of the isometries. This key feature allows us to significantly break the deterministic regularity threshold $s = \frac{1}{4}$, but it is not enough to go all the way down to the Gibbs measure. To achieve such a goal, we shall have a better understanding of the properties of the *colored Gaussian*, and their random structure. For now on, we merely use rough L^∞ estimates to handle these terms. To address the deterministic smoother remainders, we use the bilinear estimates form [BGT05b] and prove their convergence in $H^{\frac{1}{4}+}(\mathbb{S}^2)$.

A first step towards the goal of reaching the Gibbs measure would be to consider the equation with a Hartree nonlinearity instead. Indeed, for this equation, we prove deterministic local well-posedness in $H^{0+}(\mathbb{S}^2)$. Hence, the situation looks similar to the case of the two-dimensional torus, where we merely have to gain 2ε -derivatives to reach the support of a Gibbs measure. Nevertheless, it seems that singular *high-high-to-low*-type frequency interactions are ruling out the regularizing effect of the Hartree nonlinearity, and makes it hard to gain these 2ε derivatives. . .

0.3 Synthèse en Français

Cette thèse a pour objet l'étude d'équations de Schrödinger non-linéaires dans des régimes singuliers où les solutions sont fortement oscillantes, en présence de perturbations stochastiques. Avant de discuter de certaines motivations, on souligne que contrairement aux solutions d'équations paraboliques les solutions des équations dispersives ne bénéficient généralement pas d'effets régularisant. Ainsi, lorsqu'on travaille à une régularité donnée, il faut isoler les différentes échelles d'oscillations et exploiter finement les mécanismes dispersifs qui régissent leur propagation. En outre, au niveau non-linéaire, il est nécessaire de comprendre comment les différentes ondes interagissent entre elles.

Par ailleurs, les équations dispersives non-linéaires présentent des instabilités qui ont lieu dans certains régimes et qui compromettent la description de l'évolution de toutes les données initiales. Néanmoins, en dotant l'ensemble des données initiales d'une mesure de probabilité pertinente, on injecte naturellement un aléa dans l'étude. L'objectif est alors d'exhiber un *ensemble statistique* de données initiales qui donnent lieu à des solutions fortes, limites de solutions approchées qui sont elles régulières.

En présence d'une dynamique non triviale, bien comprise pour les solutions lisses, il est également pertinent de considérer des perturbations stochastiques dues par exemple à des fluctuations microscopiques irrégulières, et de se poser les questions suivantes :

- Quel est le comportement générique des solutions dans des régimes sur-critiques ?

- L'approche statistique permet-elle d'aller au delà des instabilités déterministes ?
- Comment l'aléatoire affecte-t-il la dynamique asymptotique de systèmes non-linéaires ?

Ce manuscrit se propose d'explorer différentes manifestations de ces questions, et d'y répondre en développant des techniques qui combinent outils déterministes et probabilistes, le tout à basse régularité. Le principal défi auquel ces travaux se sont confrontés, tant en temps court qu'en temps long, est le manque d'effet régularisant des systèmes dispersifs. Outre le fait que nous travaillons à des régularités sur-critiques où des instabilités se produisent, l'on ne dispose pas de lois de conservations qui nécessitent des solutions une certaines régularité. Hélas, l'utilisation de ces lois de conservation est indispensable si l'on veut comprendre la dynamique à long terme des solutions. Ainsi, la question sous-jacente aux travaux qui composent ce manuscrit, et qui va au-delà des considérations probabilistes, peut être énoncée comme suit :

- Comment exploiter les effets dispersifs lorsque les solutions sont très irrégulières ?

Le manuscrit de thèse comporte trois parties, dont les enjeux et les conclusions sont résumés dans les paragraphes qui suivent.

Les régimes sur-critiques: entre instabilités et ensemble statistique de données initiales

Dans l'étude de la théorie de Cauchy pour les équations dispersives, on rencontre généralement un seuil de régularité critique qui détermine si le problème de Cauchy est bien posé ou non. Étant donné un instant $T > 0$, rappelons qu'une courbe continue $u : t \in [-T, T] \mapsto u(t) \in H^s$ est une solution forte au problème de Cauchy avec la condition initiale $u(0) = u_0 \in H^s$

$$\begin{cases} i\partial_t u + \Delta u = \mu |u|^2 u, & (t, x) \in \mathbb{R} \times \mathcal{M}. \\ u|_{t=0} = u_0. \end{cases}$$

lorsqu'elle satisfait la formule de représentation intégrale de Duhamel

$$u(t) = e^{it\Delta} u_0 - i \int_0^t e^{i(t-\tau)\Delta} |u(\tau)|^2 u(\tau) d\tau. \quad (0.3.1)$$

Dans le contexte des équations aux dérivées partielles dispersives, la définition d'un problème de Cauchy *bien posé* est rappelée dans la Définition 1.1.1. Lorsque l'on sait montrer que le problème de Cauchy est bien posé pour des données initiales très régulières, disons dans H^σ pour un certain σ suffisamment grand, il est alors possible de définir un *flot* dans H^σ via l'application

$$u_0 \in H^\sigma \longmapsto u \in C([-T, T]; H^\sigma).$$

Pour montrer le caractère mal posé du problème de Cauchy dans un espace de faible régularité H^s , pour $s \ll \sigma$, il suffit alors de montrer que le flot, défini sur H^σ , ne s'étend pas à H^s . Cette stratégie est motivée par la persistance de la régularité du flot, qui permet de voir les solutions dans H^s , si elles existent, comme une limite de solutions régularisées, vivant dans l'espace H^σ .

Dans le cas des équations du type Schrödinger ou des ondes non-linéaires, la discontinuité du flot à des régularités sur-critiques se déduit de l'existence d'une suite de données initiales

régulières qui convergent vers 0 pour la norme H^s , mais telles que les normes H^s des solutions locales correspondantes, définies dans H^σ , vont à l'infini en des temps arbitrairement courts. Un tel mécanisme d'*inflation des normes* a été mis en évidence par [Leb01; Leb05; Lin93] pour des équation d'ondes nonlinéaires, puis étendu à l'équation de Schrödinger dans [CCT03a]. La stratégie générale est d'effectuer une *analyse de petite dispersion*. On génère dans un premier temps des données initiales lisses $(\psi_n)_n$ à partir d'une fonction test ré-échelonnée, qui convergent vers zéro dans H^s tout en se concentrant spatialement. Ensuite, on considère la solution lisse u_n dans H^σ associée à la donnée initiale ψ_n , et on étudie l'évolution du *profil* v_n , solution l'équation différentielle sans dispersion

$$\begin{cases} i\partial_t v_n = |v_n|^{p-1} v_n, \\ v_n(0) = \psi_n. \end{cases}$$

Le profil v_n encode les oscillations de u_n et commence à croître dans H^s après un certain temps t_n , qui tend vers zéro dans le régime où n est grand. Dans un second temps, on utilise des estimations d'énergie a priori jusqu'au temps t_n pour prouver que u_n reste proche de v_n dans l'espace H^s , au moins jusqu'au temps t_n :

$$\sup_{|t| \leq t_n} \|u_n(t) - v_n(t)\|_{H^s} \lesssim 1.$$

En d'autres termes, les oscillations dominent la dispersion de sorte que l'instabilité a lieu pour u_n . Ce phénomène de croissance de la norme H^s témoigne d'une cascade d'énergie des basses fréquences vers les hautes fréquences. Par ailleurs, Xia Bo a établi dans [Xia21] l'existence d'un tel mécanisme au voisinage de n'importe quelle donnée initiale dans H^s : pour tout $u_0 \in H^s$, il existe une suite ψ_n telle que

$$\lim_{n \rightarrow \infty} \|\psi_n - u_0\|_{H^s} = 0, \quad \lim_{n \rightarrow \infty} \|u_n\|_{H^s} = +\infty,$$

où u_n est la solution lisse associée à ψ_n . Mais que se passe-t-il lorsque la procédure de régularisation est choisie judicieusement ? En outre, la théorie de Cauchy probabiliste étudiée dans cette thèse permet d'établir l'existence d'un *ensemble statistique* qui fournit des données initiales donnant lieu à des solutions fortes. De plus, ces solutions fortes sont les limites des solutions lisses associées à la suite des données initiales régularisées par convolution avec une approximation de l'identité. Ainsi, pour cette procédure de régularisation, on dispose d'un ensemble de mesure pleine de données initiales pour lesquelles il n'y a pas d'inflation des normes.

A la lumière de cette discussion, il est pertinent de s'intéresser à l'ensemble *pathologique* des données initiales qui, lorsque régularisées par convolution, donnent quand même lieu à des solutions lisses approchées pour lesquelles l'inflation de la norme se produit. En d'autres termes, cet ensemble est contenu dans le complémentaire de l'ensemble statistique qui est dense et de mesure pleine. Néanmoins, Sun et Tzvetkov ont prouvé dans [ST20a] que pour les équations d'ondes non-linéaires l'ensemble pathologique est également générique (au sens de la topologie) puisqu'il contient un ensemble G_δ -dense. L'énoncé précis est écrit dans le Théorème 1.1.4.

Solutions fortes dans \mathbb{R}^d issues d'un ensemble statistique de données initiales A la suite des contributions pionnières de Bourgain [Bou94; Bou96] pour l'équation de Schrödinger sur le tore et des travaux de Burq, Tzvetkov [BT08a; BT08b] pour l'équations des ondes sur des variétés compactes sans bords, de nombreux auteurs ont adopté une approche statistique pour étudier le problème de Cauchy en régularité sur-critique. Plus précisément, étant donné un espace de Sobolev

H^s sur lequel le flot n'est pas prolongeable par continuité, il est possible de trouver un sous-ensemble *statistique* de H^s qui est de mesure pleine, pour une la mesure image d'une série de modes propres du Laplacien (dans le cas où il existe une résolution spectrale) découplés par des variables gaussiennes indépendentes. Les fonctions qui constituent l'ensemble statistique donnent lieu à des solutions fortes, . De plus, l'ensemble statistique est dense et la régularité qui n'est pas meilleure que H^s . Une telle approche est motivée par l'étude des mesures de Gibbs [Bou96], et a été largement développée dans d'autres contextes depuis. On renvoie à [BOP19a; BT08a] pour une introduction à la théorie de Cauchy dite probabiliste.

Dans l'espace euclidien où il n'y a pas de résolution spectrale pour l'opérateur de Laplace, il n'existe pas de procédure de randomisation canonique. Une façon de faire est d'utiliser une décomposition de l'espace des fréquences en des cubes unitaires (dits de Wiener). La randomisation qui en découle a été introduite par [LM14b] et [OP16] indépendamment (voir la figure 0.1.2). Cette procédure a pour effet de relaxer la condition d'admissibilité dans les estimations de Strichartz (voir Figure 0.1.2). En effet, pour une fonction dont le support en fréquence est unitaire, nous pouvons utiliser l'inégalité de Bernstein pour passer de L^q à L^2 sans perdre de dérivée. Ensuite, les estimations individuelles obtenues sur les blocs unitaires sont rassemblées au moyen d'un découplage probabiliste, dans l'esprit du théorème de Paley-Zygmund.

A l'aide d'inégalités de Strichartz probabilistes, Bényi, Oh et Pocovnicu [BOP15b] ont démontré l'existence d'un ensemble statistique Σ de données initiales à régularité sur-critique qui donnent lieu à des solutions locales fortes. Avec Louise Gassot, nous précisons ce résultat dans le Chapitre 1 en montrant qu'étant donnée une approximation de l'identité $(\rho_\varepsilon)_{\varepsilon>0}$ définie par

$$\rho_\varepsilon(x) := \varepsilon^{-3} \rho\left(\frac{x}{\varepsilon}\right), \quad \rho \in C_c^\infty(\mathbb{R}^d),$$

si $u_0 \in \Sigma$ alors la solution probabiliste issue de u_0 est la limite dans $C([-T, T]; H^s)$ des solutions approchées u_ε , associées aux données initiales régularisées $u_{0;\varepsilon} := u_0 * \rho_\varepsilon$.

Ensembles de données initiales *pathologiques* pour NLS. Le résultat principal du Chapitre 1 est la description de l'ensemble pathologique pour les équations de type Schrödinger. Pour cela, on utilise la même construction que [ST20a] qui consiste à mettre côte à côte une infinité de profils concentrés à des échelles arbitrairement petites. La preuve proposée dans [ST20a] revisite l'analyse de petite dispersion de [CCT03a] en régularisant les profils par convolution avec ρ_ε . Les auteurs utilisent ensuite le principe de propagation à vitesse finie de l'équation des ondes pour s'assurer que les différents profils n'interagissent pas avant le temps nécessaire à l'inflation des normes. Dans le cas des équations de type Schrödinger, on ne dispose pas d'un tel principe de propagation à vitesse finie. Pour remédier à cela, on exploite plutôt l'effet de la convolution sur les profils trop concentrés. Plus précisément, on observe qu'à une échelle de régularisation donnée $\varepsilon > 0$, la convolution par ρ_ε casse les profils qui se concentrent à l'échelle $\lambda^{-1} \ll \varepsilon$. Cette observation indique que le phénomène ne repose pas sur le principe de propagation à vitesse finie. Par conséquent, la même approche s'étend très certainement à de nombreuses autres équations non-linéaires dispersives.

Dynamiques en temps long pour des données initiales dans l'ensemble statistique

L'une des principales questions abordées dans cette thèse peut être formulée comme suit. Étant donné une donnée initiale *typique* (i.e. dans l'ensemble statistique) de la forme

$$u(0) = u_0 + f_0^\omega, \quad u_0 \in H^\sigma, \quad f_0^\omega \in H^s, \quad \text{avec } s < s_c \leq \sigma,$$

et la solution au problème de Cauchy qui lui est associée, que dire de son existence globale et de sa dynamique ?

Lorsque la mesure de probabilité est invariante ou quasi-invariante par le flot non-linéaire, l'argument de globalisation de Bourgain (présenté dans le paragraphe 0.1.3) permet de déduire l'existence globale des solutions probabilistes ainsi que des propriétés dynamiques en temps long (propriété de récurrence de Poincaré par exemple). En effet, la présence d'une mesure invariante ou quasi-invariante permet de transporter une information de nature probabiliste par le flot non-linéaire, nécessaire à la résolution du problème de Cauchy local. Au delà de cet argument de globalisation très général, et en l'absence de mesures invariantes, il s'agit de combiner des outils déterministes avec la théorie de Cauchy probabiliste afin de comprendre la stabilité de mécanisme dynamiques (scattering non-linéaire, stabilité asymptotique, croissance des normes de Sobolev,...) sous l'effet de perturbations aléatoires, très irrégulières. Les résultats qui constituent la partie II de ce manuscrit illustrent cette démarche.

Scattering Lorsqu'une donnée initiale évolue par une équation de Schrödinger non-linéaire dont la non-linéarité est *défocalisante* (c'est à dire répulsive), sa dynamique en temps long est parfois approchée par l'évolution linéaire d'une autre donnée initiale. Ce phénomène de *scattering non-linéaire* est bien compris dans l'espace d'énergie, lorsque l'équation est énergie sous-critique, ou énergie critique (voir par exemple [RV07; Vis12] en dimension 4, [Col+08] en dimension 3 et [Dod16] pour le cas masse-critique pour NLS cubique en dimension 2).

On considère dans le Chapitre 2 l'équation NLS avec une non-linéarité cubique défocalisante. Ce modèle est énergie sous-critique ce qui fait que le problème de Cauchy est bien posé dans l'espace d'énergie $H^1(\mathbb{R}^3)$. Il découle de la coercivité de l'énergie conservée que la solution est globale. De plus, on déduit des estimations de Strichartz qu'il suffit d'obtenir une estimation a priori du type

$$\iint_{\mathbb{R} \times \mathbb{R}^3} |u(t, x)|^p dx dt < \infty, \quad \text{pour certains } \frac{3}{10} < p < 10$$

pour prouver qu'il y a scattering dans $H^1(\mathbb{R}^3)$. Une telle estimation a priori de la décroissance découle d'inégalités de flux de type Morawetz

$$\iint_{I \times \mathbb{R}^3} |u(t, x)|^4 dt dx \leq C \sup_{t \in I} \|u(t)\|_{L_x^2}^2 \|u(t)\|_{\dot{H}^{\frac{1}{2}}}^2.$$

combinée à certaines lois de conservation. Une fois le scattering établi dans l'espace d'énergie, il s'agit de montrer que toutes les solutions locales (définies dans H^s avec $\frac{1}{2} < s < 1$) existent elles aussi globalement, et qu'il y a scattering. Malheureusement, cette question simple s'avère être un problème ouvert très difficile, qui est encore loin d'être résolu en toute généralité. Des résultats partiels ont été obtenus via la célèbre *I-method* [Col+04] et des raffinements basés sur *long-time Strichartz*

estimates [Dod13; Su12; Dod19b]. On réfère au paragraphe 0.2.2 et à l'introduction du Chapitre 2 pour une discussion à propos de cette conjecture. Compte tenu de sa difficulté, il est intéressant d'aborder ce problème avec une approche statistique. En outre, il existe des ensembles statistiques de données initiales même dans les régimes sur-critiques, mais leur comportement en temps long n'est compris que sous des hypothèses de petitesse. En effet, les résultats de type *scattering presque-sûr* en dehors du régime des petites données n'existaient que dans le cas d'équations énergie critique.

Dans ce chapitre, on démontre qu'il y a scattering presque sûrement dans H^s pour certains $s < 1/2$ en procédant comme suit. À partir de la décomposition de Da Prato et Debussche (0.1.5)

$$u = e^{it\Delta} f_0^\omega + v,$$

on analyse le comportement à long terme du terme non-linéaire v dans l'espace d'énergie H^1 qui est solution d'une équation de Schrödinger non-linéaire avec des termes de forçage stochastiques :

$$\begin{cases} (i\partial_t + \Delta)v = |v + e^{it\Delta} f_0^\omega|^{p-1}(v + e^{it\Delta} f_0^\omega), & (t, x) \in \mathbb{R} \times \mathbb{R}^d, \\ v(0) = 0, \end{cases} \quad (\text{NLS}_f)$$

La première étape consiste à utiliser la théorie de la stabilité de l'équation déterministe dans H^1 pour obtenir un résultat de scattering conditionnel. Ce faisant, on réduit le problème à l'obtention d'un contrôle uniforme sur la norme de H^1 de v . Ensuite, dans une deuxième étape, la stratégie consiste à trouver des énergies modifiées appropriées et des inégalités de Morawetz modifiées pour v puis à effectuer un double-argument de *bootstrap* à partir de ces deux quantités.

Le résultat présenté dans le Chapitre 2 prouve le scattering presque-sûr en pour des régularités sous le seuil critique.

Theorem 0.3.1 (Scattering presque-sûr pour NLS dans \mathbb{R}^3 [Cam23] et [SSW21]). *Pour $\frac{3}{7} < s$ on fixe $f_0 \in H^s(\mathbb{R}^3)$ une fonction radiale et nous notons $\omega \mapsto f_0^\omega$ la randomisation de Wiener qui lui est associée. Il existe un ensemble statistique de mesure pleine Σ dans $H^s(\mathbb{R}^3)$ tel que pour tout $f_0^\omega \in \Sigma$, la solution probabiliste correspondante existe globalement en temps, et il y a scattering : il existe $v_\pm \in H^\sigma(\mathbb{R}^3)$, tel que*

$$\lim_{t \rightarrow \pm\infty} \|u(t) - e^{it\Delta}(f_0^\omega + v_\pm)\|_{H^\sigma} = 0, \quad \sigma = 2s - .$$

Voici quelques commentaires sur le résultat et sur les principales difficultés.

- On effectue une décomposition linéaire-non-linéaire $u = e^{it\Delta} f_0^\omega + v$. L'observation clé est que le terme non-linéaire est dans H^σ , pour $\sigma < 1$, mais suffisamment proche de 1 pour que la conjecture déterministe soit connue à cette régularité (à savoir $2/3 < s$). Le défi (mais aussi la motivation) est de combiner la méthode probabiliste avec la *I-method* pour analyser le comportement asymptotique de v . La difficulté réside dans le fait que les termes de forçage stochastiques brisent la structure de la non-linéarité. Par conséquent, des termes supplémentaires apparaissent, pour lesquels il n'y a pas d'annulation de fréquence. Au lieu de cela, beaucoup de dérivés peuvent tomber sur le terme stochastique irrégulier.
- Pour cette raison, et comme dans le cas habituellement étudié des équations énergie critique, une hypothèse de symétrie radiale cachée, ou toute autre hypothèse similaire, semble être nécessaire. L'idée est d'exploiter une injection de Sobolev radiale afin de tirer profit de la

décroissance locale de l'énergie (0.1.4) et de surmonter la perte de dérivées inhérente aux méthodes d'énergie. On rappelle que les données aléatoires f_0^ω ne sont en général pas radiales. Comme mentionné dans la section 0.2.2, le seul travail évitant une telle hypothèse est dû à Bringmann, pour l'équation d'onde énergie critique dans \mathbb{R}^4 [Bri18].

Stabilité asymptotique de petits états fondamentaux La théorie probabiliste de Cauchy s'intéresse principalement à la perturbation des solutions dispersives, dans le régime défocalisant ou pour de petites données initiales. Cependant, dans le régime focalisant où les équations dispersives présentent des dynamiques très riches, l'effet de perturbations aléatoires est bien moins compris. Le premier résultat dans cette direction est dû à Kenig et Mendelson, pour l'équation d'ondes énergie critique. Dans [KM19], les auteurs ont prouvé que l'on peut construire une solution globale de l'équation d'onde quintique focalisante dans \mathbb{R}^4 qui se décompose comme un soliton modulé et un terme dispersif, pour un ensemble statistique de données initiales dans $H^s \times H^{s-1}$ lorsque $s > \frac{5}{6}$. Au lieu d'utiliser des multiplicateurs de Fourier classique, ils ont utilisé une transformée de Fourier dite *distordue* dans la procédure de randomisation (0.1.7), qui diagonalise l'opérateur linéarisé autour du soliton $-\Delta + V$.

Le seul équivalent existant de ce résultat de stabilité asymptotique presque sûr pour l'équation de Schrödinger fait l'objet du Chapitre 3. Dans le cas des équations de type Schrödinger, soulignons que l'intégrale de Duhamel ne gagne pas de dérivée de sorte qu'une décomposition dyadique et des estimations bilinéaires doivent être effectuées pour établir un schéma de point fixe. Cependant, cela devient assez difficile en présence d'un potentiel, puisque la transformée de Fourier déformée ne partage pas les propriétés multiplicatives de la transformée de Fourier classique.

Dans le Chapitre 3, on étudie la stabilité asymptotique des petits états fondamentaux de l'équation de Schrödinger non-linéaire, en présence d'un potentiel de courte portée $V \in \mathcal{S}(\mathbb{R}^3)$

$$(i\partial_t + \Delta)u(t, x) = \left(\underbrace{V(x)}_{\text{potentiel externe}} + \underbrace{\mu|u(t, x)|^2}_{\text{perturbation non-linéaire}} \right) u(t, x), \quad (t, x) \in \mathbb{R} \times \mathbb{R}^3,$$

où l'on suppose que l'opérateur de Schrödinger $-\Delta + V$ a une valeur propre simple négative. Ce modèle a d'abord été étudié par Soffer et Weinstein [SW90], qui montrent que le comportement asymptotique des petites solutions, sous des hypothèses de décroissance, est donné par la superposition d'une onde stationnaire localisée, qui est périodique dans le temps, et d'une partie dispersive. Gustafson, Nakanishi et Tsai ont ensuite prouvé dans [GNT04] que la *stabilité asymptotique* a lieu pour toute solution suffisamment petite dans l'espace d'énergie $H^1(\mathbb{R}^3)$, sans aucune hypothèse de décroissance.

Qu'advient-il des états fondamentaux sous l'effet de perturbations plus irrégulières, qui ont une énergie infinie ? Plus précisément, étant donnée une petite donnée initiale u_0 dans l'espace critique $H^{\frac{1}{2}}$, on considère dans le Chapitre 3 des perturbations aléatoires de la forme

$$u(0) = u_0 + f_0^\omega, \quad u_0 \in H^{\frac{1}{2}}, \quad f_0^\omega \in H^s \cap \text{Ran } P_c(H),$$

où $P_c(H)$ désigne la projection sur le sous-espace spectral continu. Les données initiales sont randomisées selon la procédure de Wiener (voir la sous-section 0.1.2), mais elles sont adaptées à l'évolution linéaire inhomogène e^{itH} dans le sens où les multiplicateurs de Fourier sont remplacés par des multiplicateurs de Fourier distordus. On démontre la stabilité asymptotique des états fondamentaux sous l'effet de ces perturbations.

Theorem 0.3.2. *Soit $\frac{1}{4} < s < \frac{1}{2}$. Il existe $\delta_0 > 0$ et un ensemble de mesure pleine Σ dans $H^s(\mathbb{R}^\omega)$ tel que pour tout u_0 dans $H^{\frac{1}{2}}$ dont la norme inférieure à δ_0 , et pour tout f_0^ω dans Σ , il existe $\varepsilon = \varepsilon(\omega) > 0$ pour lequel le problème de Cauchy*

$$\left\{ (i\partial_t + \Delta)\psi(t, x) = \mu|\psi(t, x)|^2\psi(t, x) + V(x)\psi(t, x), \quad (t, x) \in \mathbb{R} \times \mathbb{R}^3, \quad \psi(0) = \varepsilon f_0^\omega + v_0, \right.$$

admet une solution locale sous la forme

$$\psi(t, x) = \varepsilon e^{-itH} f_0^\omega + v, \quad v \in C(\mathbb{R}, H^{\frac{1}{2}}(\mathbb{R}^3)).$$

De plus, cette solution s'étend globalement dans le temps, et se décompose en la somme d'un état fondamental modulé et d'un terme radiatif :

$$\psi(t, x) = Q(z(t)) + \eta(t, x).$$

En temps long, les états fondamentaux sont asymptotiquement stables, dans le sens où il existe $z_\pm \in \mathbb{C}$ et des états finaux $\eta_\pm \in H^{\frac{1}{2}}$ tels que

$$\lim_{t \rightarrow \pm\infty} z(t) \exp\left(-i \int_0^t E(z(\tau)) d\tau\right) = z_\pm, \quad \lim_{t \rightarrow \pm\infty} \|\eta(t) - e^{-itH}(\varepsilon f_0^\omega + \eta_\pm)\|_{H^{\frac{1}{2}}} = 0.$$

- On réfère au Chapitre 3 pour un énoncé plus précis.
- Comme mentionné ci-dessus, lorsque $f_0 = 0$, le théorème 0.2.5 étend le résultat déterministe de [GNT04] jusqu'à la régularité critique $s = \frac{1}{2}$ en prenant $f_0 = 0$.
- Ce travail a été inspiré par la contribution de Kenig et Mendelson présentée dans la section 0.1.2 pour l'équation d'onde énergie critique. Comme expliqué dans la Section 0.1.2, dans le cas de l'équation de Schrödinger, certaines difficultés apparaissent. La difficulté la plus sérieuse est que le terme intégral de Duhamel ne présente aucun gain de dérivée, et que l'opérateur linéarisé autour des états fondamentaux n'est pas auto-adjoint. Par conséquent, la prise en compte de petits états fondamentaux est un premier pas raisonnable vers l'objectif d'adapter le résultat de Kenig et Mendelson aux équations de Schrödinger.
- La principale nouveauté est le fait d'avoir combiné l'utilisation d'espace fonctionnels critiques avec le principe de décroissance locale de l'énergie, pour la première fois dans le contexte de la théorie de Cauchy probabiliste.

Equations faiblement dispersives : vers une théorie de Cauchy probabiliste quasilinéaire

Dans certains cas où la dispersion est trop faible ou lorsque les interactions non-linéaires sont prépondérantes, des interactions singulières peuvent apparaître et limiter le gain possible de régularité des itérations de Picard. Dans le pire des cas, les mauvaises interactions peuvent même exclure tout gain de régularité. Dans de tels cas, on ne peut pas utiliser le schéma de résolution probabiliste semi-linéaire de Bourgain. Bringmann dans [Bri21] a étudié une équation d'onde quasi-linéaire, et a proposé le premier schéma de résolution probabiliste *quasi-linéaire*. Il s'agit d'un schéma itératif sur

la taille des fréquences de la donnée initiale, qui s'inspire du calcul para-contrôlé, dans un contexte dispersif. L'idée clé est d'utiliser l'indépendance probabiliste entre les hautes fréquences de la donnée initiale et la solution approchée, qui est construite à partir des basses fréquences de la donnée initiale.

Ce schéma itératif quasi-linéaire a ensuite été développé de manière spectaculaire par Deng, Nahmod et Yue pour prouver l'invariance de la mesure de Gibbs associée à l'équation de Schrödinger semi-linéaire (avec une renormalisation de Wick) sur \mathbb{T}^2 ,

$$\begin{cases} (i\partial_t + \Delta)u =: |u|^{2p}u : & (t, x) \in \mathbb{R} \times \mathbb{T}^2, \\ u(0) = u_0, \end{cases}$$

où la puissance $p \geq 2$ de la non linéarité est arbitraire. Il s'agit d'une percée importante, dans laquelle les auteurs ont introduits des *opérateurs de moyennisation aléatoires* dont l'utilisation s'est avérée extrêmement puissante et prometteuse, dans la perspective de résoudre d'autres problèmes difficiles ouverts depuis de nombreuses années. On renvoie à l'introduction du Chapitre 4 et au paragraphe 0.2.3 pour une discussion plus détaillée à propos du schéma de résolution quasi-linéaire.

Théorie de Cauchy probabiliste raffinée pour des équations faiblement dispersives Dans l'esprit de [Bri21], on utilise dans le Chapitre 4 un schéma itératif quasi-linéaire similaire pour résoudre le problème de Cauchy local associé à une équation faiblement dispersive qui est par ailleurs anisotrope.

Le comportement qualitatif des solutions aux équations d'évolution non-linéaires à faible dispersion fait l'objet d'une activité intense. Des exemples éloquentes sont l'équation de demi-ondes (HW) [Poc11; Gér+18a], l'équation de Szegő [GG10; GG12b] (Sz), l'équation de Schrödinger sur une variété compacte [BGT05a] ou sur le groupe de Heisenberg [Gér06; Gas21a], les équations de Schrödinger fractionnaire [FL13; GPH15; ST20c] (FNLS), les équations de Kadomtsev-Petviashvili [HHK09] (KPI et KPII) ainsi que de nombreux autres modèles. En général, il est très difficile de résoudre le problème de Cauchy pour ces équations à faible régularité, la raison étant que dans de tels contextes le manque de dispersion se traduit par une perte de dérivées dans les estimations de Strichartz (0.1.2). Lorsque l'équation faiblement dispersive n'est ni complètement intégrable ni semi-linéairement bien posée, il est pertinent de développer une théorie de Cauchy à données aléatoires inspirée des schémas de résolution quasi-linéaires. En effet, pour faire fonctionner le schéma de résolution probabiliste semi-linéaire de Bourgain, il faut combiner l'observation de Paley-Zygmund du théorème (0.1.2) avec des effets dispersifs comme dans (0.1.3), (0.1.4).

Dans la perspective d'étudier les équations faiblement dispersives à faible régularité, on considère dans le Chapitre 4 l'équation de Schrödinger demi-onde (NLS-HW) avec une non-linéarité de type puissance : pour $\mu \in \mathbb{R}$, et $(t, x, y) \in \mathbb{R}^{1+2}$, cette équation s'écrit

$$\begin{cases} i\partial_t u + (\partial_{xx} - |D_y|)u = \mu |u|^2 u, & \text{où } \widehat{|D_y|u}(\xi, \eta) = |\eta| \widehat{u}(\xi, \eta). \\ u_{t=0} = u_0, \end{cases}$$

Il s'agit là d'un modèle anisotrope, où on a une équation de Schrödinger dans la direction x et une équation de demi-onde dans la direction y , respectivement. Par conséquent, dans les régimes à basse fréquence dans la direction x et à haute fréquence dans la direction y , le transport domine et il n'y a pas de dispersion. Comme discuté à la fin de cette section, la dynamique asymptotique des solutions lisses de cette équations est très riche (voir [Xu17; BIK20; BIK21]), malgré le fait que la théorie de

Cauchy pour cette équation est mal comprise. Pour étudier la théorie de Cauchy pour (NLS-HW) on mesure la régularité dans l'échelle des espaces de Sobolev anisotropes :

$$\mathcal{H}^s(\mathbb{R}^2) = L_x^2 H_y^s(\mathbb{R}^2) \cap H_x^{2s} L_y^2(\mathbb{R}^2).$$

Le choix naturel de ces espaces est dicté par la structure hamiltonienne de l'équation et par l'invariance de l'équation par changement d'échelle :

$$u \mapsto u_\lambda(t, x, y) = \lambda \psi(\lambda^2 t, \lambda x, \lambda^2 y).$$

En particulier, l'exposant critique associé cette symétrie est $s_c = \frac{1}{4}$. L'énergie conservée s'écrit

$$H(u) = \frac{1}{2} \int_{\mathbb{R}^2} |D_x u|^2 + ||D_y|^{\frac{1}{2}} u|^2 dx dy + \frac{\mu}{4} \int_{\mathbb{R}^2} |u|^4 dx dy.$$

Le problème de Cauchy est bien posé lorsque $s > \frac{1}{2}$. Ceci a été prouvé dans [BIK20] à partir d'un point fixe et d'estimations de Strichartz avec perte de dérivée. Cependant une théorie de Cauchy satisfaisante dans l'espace d'énergie et en dessous n'a pas encore été établie. L'espace d'énergie correspond au point limite $s = \frac{1}{2}$, et il existe de sévères obstructions pour établir une théorie de Cauchy dans cet espace. Ces obstructions sont soulevées dans l'introduction du Chapitre 4.

On démontre que le problème de Cauchy est presque-sûrement bien posé dans \mathcal{H}^s pour des $1/4 < s < 1/2$ en adaptant le schéma itératif de [Bri21]. On montre à cette occasion qu'une telle méthode probabiliste peut surmonter le manque de dispersion, et qu'elle pourrait être adaptée à d'autres modèles faiblement dispersifs afin de générer des grands ensembles de données initiales qui donnent lieu à des solutions fortes. Par exemple, [Oh11b] a mis en évidence des interactions singulières pour des modèles tels que Szegő ou KdV, qui seront surmontées par l'approche para-contrôlée utilisée dans le présent article. On précise par ailleurs dans le Chapitre 4 la théorie de Cauchy pour (NLS-HW), résumée dans la figure 0.2.3.

Invariance de la mesure de Gibbs pour NLS sur la sphère \mathbb{S}^2 ? Afin de comprendre l'effet de la géométrie ambiante sur la dynamique des équations de Schrödinger non-linéaires, Burq, Gérard et Tzvetkov ont prouvé dans une série de travaux [BGT02; BGT04; BGT05b; BGT05a; BGT10] que le problème de Cauchy pour l'équation de Schrödinger cubique sur la sphère \mathbb{S}^2 est semi-linéairement bien posé dans $H^s(\mathbb{S}^2)$ si et seulement si $s > \frac{1}{4}$. Il y a par conséquent un écart très important à combler entre le seuil déterministe et l'espace où vit la mesure de Gibbs qui est $H^{0-}(\mathbb{S}^2)$. Il est donc extrêmement difficile d'obtenir l'analogie sur la sphère \mathbb{S}^2 du célèbre résultat de Bourgain [Bou96] sur le tore \mathbb{T}^2 . En outre, la théorie de Cauchy probabiliste juste en dessous du seuil déterministe $s = \frac{1}{4}$ reste à établir. Discutons brièvement des difficultés. La plus importante est la dégénérescence des harmoniques sphériques. En effet, étant donné le niveau d'énergie $\lambda_n^2 = n(n+1)$, la multiplicité de l'espace propre correspondant, généré par les harmoniques sphériques de degré n , est de dimension $2n+1$. De plus, certaines harmoniques sphériques se concentrent autour des géodésiques de la sphère (ce qui n'est pas le cas pour les ondes planes $e^{in \cdot x}$ sur le tore), et cela engendre des instabilités non-linéaires.

Pour amorcer l'étude probabiliste de l'équation NLS sur la sphère, nous avons obtenu un résultat préliminaire où l'on isole une interaction singulière dans la première itération de Picard, et on démontre que l'équation de Schrödinger cubique posée sur \mathbb{S}^2 avec des données aléatoires est quasi-linéaire (malgré la renormalisation de Wick et le découplage probabiliste). En particulier, la seconde itération de Picard n'est pas plus régulière (en espérance) que les données initiales aléatoires.

Avant de discuter du résultat principal et de ses conséquences, présentons le cadre probabiliste. On fixe $\alpha \in [\frac{1}{2}, \frac{3}{4})$ ainsi qu'une suite de variables gaussiennes normalisées indépendantes à valeurs complexes $(g_{n,k})_{1 \leq n, |k| \leq n}$. On définit ensuite

$$\phi^\omega = \sum_{1 \leq n} \frac{e_n^\omega}{\lambda_n^\alpha}, \quad e_n^\omega(x) = \sum_{|k| \leq n} \frac{g_{n,k}(\omega)}{[n]^{\frac{1}{2}}} Y_{n,k}(x). \quad (0.3.2)$$

La série ci-dessus converge dans $L^2(\Omega; H^{\alpha-\frac{1}{2}^-}(\mathbb{S}^2))$, de sorte que ϕ^ω vit dans $H^{\alpha-\frac{1}{2}^-}(\mathbb{S}^2)$ presque sûrement. Lorsque $\alpha = \frac{1}{2}$, les données initiales sont distribuées selon la mesure de Gibbs associée à NLS vu comme un système hamiltonien (on renvoie à la section 0.1.3 pour une discussion sur les mesures de Gibbs). Cependant, le seuil déterministe pour que l'équation soit bien posée est $H^{\frac{1}{4}}(\mathbb{S}^2)$. Le résultat est le suivant.

Theorem 0.3.3 (Absence de régularisation de la deuxième itération de Picard, [Bur+22]). *Soit $\alpha \geq \frac{1}{2}$, et $\phi^\omega \in H^{\alpha-\frac{1}{2}^-}(\mathbb{S}^2)$ les données initiales randomisées comme dans (0.3.2). Il existe $\eta > 0$, $N_0 > 0$ tels que pour chaque $N \geq N_0$ et t ,*

$$|t|\eta \log(N) \leq \mathbb{E} \left[\left\| \int_0^t e^{i(t-\tau)\Delta} \Pi_N \mathcal{N}(\Pi_N e^{i\tau\Delta} \phi^\omega) d\tau \right\|_{H^{\alpha-\frac{1}{2}}(\mathbb{S}^2)}^2 \right],$$

où \mathcal{N} désigne le terme non-linéaire de l'équation cubique renormalisée (NLS_W).

Commentons brièvement ce résultat.

- Lorsque N tend vers l'infini, le moment quadratique de la norme $H^{\alpha-\frac{1}{2}}(\mathbb{S}^2)$ de la seconde itération de Picard de ϕ^ω a une divergence logarithmique. Puisque la mesure induite vit dans $H^{\alpha-\frac{1}{2}^-}(\mathbb{S}^2)$, cela montre qu'il n'y a pas d'effet de régularisation probabiliste.
- Pour des raisons techniques, nous n'avons pas pu prouver que la divergence se produit *presque-sûrement* en ω . Il n'est pas clair s'il existe une preuve élémentaire de ceci, qui ne repose pas sur des outils sophistiqués d'analyse stochastique. On montre simplement une divergence logarithmique en espérance.
- A l'opposé, la seconde itération de Picard sur \mathbb{T}^2 gagne $\frac{1}{2}$ dérivées. On démontre que ceci reste vrai dans le cas des tores irrationnels, et on donne une preuve auto-contenue de ce résultat, déjà évoqué dans l'article [Fan+21].

Le résultat s'accompagne également de quelques observations positives, et n'exclut pas la résolution du problème de Cauchy en dessous de l'espace critique $H^{\frac{1}{4}}(\mathbb{S}^2)$. En effet, on a non seulement identifié l'obstacle à l'exécution d'un schéma semi-linéaire, mais on a également prouvé que d'autres interactions ont un effet régularisant.

Dans un programme à long terme avec N. Burq, C. Sun et N. Tzvetkov, nous développons une théorie probabiliste de Cauchy à l'aide du schéma itératif quasi-linéaire décrit précédemment, en employant le calcul paracontrôlé. Plus précisément, nous surmontons l'absence d'effet régularisant en affinant l'ansatz probabiliste et en exploitant l'indépendance des données initiales à haute fréquence et des solutions approchées construites à partir des basses fréquences, en utilisant les opérateurs de moyenne aléatoire mentionnés dans la section 0.1.2. L'analyse reposera ainsi sur l'utilisation

des opérateurs de moyenne aléatoire, s’inspirant des travaux de Bringmann [Bri21], Deng, Nahmod, Yue [DNY19], et Sun, Tzvetkov [ST21], et elle devra comporter des nouveautés importantes. Il nous faudra en outre exploiter la propagation par le flot de l’invariance des mesures gaussiennes sous l’action des isométries de la sphère. Cette caractéristique clé nous permet de briser significativement le seuil de régularité déterministe $s = \frac{1}{4}$. Afin de descendre jusqu’à la mesure de Gibbs, il nous faudra être capable de traiter un terme complètement résonnant, qui est de nature critique.

Bilan Dans le régime sur-critique, nous avons deux comportements différents pour le flot non-linéaire approché issu de données initiales régularisées par convolution. D’une part, le théorème 1.1.4 met en évidence un ensemble *pathologique* de données initiales où se produit l’inflation des normes, qui contient un sous-ensemble G_δ -dense. D’autre part, l’approche statistique initiée par Bourgain puis Bruq, Tzvetkov fournit un ensemble *statistique* de données initiales de mesure pleine décrit dans le théorème 1.1.3, pour lequel les solutions régularisées convergent vers un objet limite qui est une solution forte de l’équation. Il y a ainsi deux notions différentes de généricité dans les régimes sur-critiques. Alors que l’ensemble pathologique décrit dans le Théorème 1.1.4 est générique pour la topologie, il est négligeable pour la mesure. De plus, nous déduisons du théorème de Baire que l’ensemble statistique donné par le théorème 1.1.3 ne peut pas contenir d’ensemble G_δ -dense.

L’étude du comportement en temps long de ces solutions est illustrée dans la partie II, où nous avons obtenus deux résultats de natures différentes. Rappelons que les solutions probabilistes issues de données initiales dans l’ensemble statistique s’obtiennent à partir de la décomposition de Da Prato et Debussche

$$u(t) = e^{it\Delta} f_0^\omega + v, \quad (0.3.3)$$

où la partie non-linéaire v est suffisamment régulière pour être obtenue par un schéma itératif de Picard. D’une part, nous établissons *le scattering presque sûr* pour l’équation NLS défocalisante dans un régime où l’espace d’énergie est sous-critique. D’autre part, on démontre la stabilité asymptotique de petits états fondamentaux sous l’effet de perturbations irrégulières. Cela montre qu’il est possible d’étudier l’évolution non-linéaire en temps long d’une donnée initiale dans l’ensemble statistique, en adoptant une approche perturbative qui combine des outils probabilistes et des méthodes d’analyse des équations dispersives non-linéaires. Une perspective est l’étude de l’évolution de la mesure elle-même par le flot non-linéaire, afin de comprendre finement la propagation de l’information probabiliste des données initiales (invariance, quasi-invariance)...

L’on a ensuite étudié des modèles faiblement non-linéaires, pour lesquels l’effet régularisant des itérations de Picard n’est pas suffisant pour établir la convergence de ce schéma, bien que l’on soit dans un cadre probabiliste. Il n’est alors pas possible d’obtenir une solution de la forme (0.3.3). Pour obtenir des solutions locales, il faut plutôt utiliser un schéma itératif *quasi-linéaire* qui exploite la structure probabiliste de la solution, dans la continuité des développements récents Bringmann puis Deng, Nahmod, Yue et Sun, Tzvetkov. Se pose alors la question du comportement en temps long de ces solutions. Pour espérer y répondre, il faut comprendre comment se propage l’information probabiliste des données initiales. Ainsi, les techniques perturbatives déterministes développées dans la partie II ne sont pas suffisantes pour pouvoir itérer en temps le schéma quasi-linéaire. Il faudrait démontrer une forme d’invariance ou de quasi-invariance de la mesure de probabilité puis adapter l’argument de globalisation de Bourgain. Comme l’attestent les chapitres qui constituent la partie III, soulignons pour conclure que l’étude des mesures invariantes (e.g. Gibbs) ou quasi-invariantes dans de tels régimes non-linéaires et faiblement dispersifs, et à basse régularité, constitue un domaine de

recherche difficile et actif, dont les défis nécessiteront le développement de nouvelles approches aussi bien stochastiques que déterministes.

Part I

Supercritical regimes: instabilities and well-posedness

Chapter 1

Pathological set of initial data for scaling-supercritical nonlinear Schrödinger equations

joint work with Louise Gassot [CG22].

Résumé. Ce travail vise à mettre en évidence un ensemble pathologique de données initiales, pour lesquelles la norme de Sobolev des solutions régularisées par convolution croît en un temps arbitrairement court. Le résultat est dans l'esprit de la construction de Sun et Tzvetkov [ST20a], où l'ensemble pathologique contient une superposition de profils qui se concentrent en différents points. Leur argument repose sur le fait que, grâce au principe de propagation à vitesse finie de l'équation d'onde, au plus un profil présente une croissance significative en un temps t suffisamment petit. Cependant, pour les équations de type Schrödinger, nous ne pouvons pas exclure que les profils interagissent entre eux. La méthode que nous proposons exploite l'effet régularisant de l'approximation de l'identité par laquelle les données initiales sont lissées, qui permet d'éviter la croissance de la norme des profils qui sont concentrés à des échelles plus petites.

Abstract. The purpose of this work is to evidence a pathological set of initial data for which the regularized solutions by convolution experience a norm-inflation mechanism, in arbitrarily short time. The result is in the spirit of the construction from Sun and Tzvetkov [ST20a], where the pathological set contains superposition of profiles that concentrate at different points. Thanks to finite propagation speed of the wave equation, and given a certain time, at most one profile exhibits significant growth. However, for Schrödinger-type equations, we cannot preclude the profiles from interacting between each other. Instead, we propose a method that exploits the regularizing effect of the approximate identity which, at a given scale, rules out the norm inflation of the profiles that are concentrated at smaller scales.

Contents

1.1	Introduction	70
1.2	Pathological set of data where norm inflation occurs	77
1.2.1	Unstable profile	78
1.2.2	Pathological set	82
1.2.3	Perturbative analysis	84
1.3	Generic well-posedness	90
1.3.1	Preliminaries	91
1.3.2	Probabilistic local well-posedness	92
1.3.3	Approximate solutions by convolution and convergence	95

1.1 Introduction

Our model will be the semi-linear Schrödinger equation on \mathbb{R}^3

$$i\partial_t u + \Delta u + \sigma|u|^{p-1}u = 0, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^3, \quad p > 1, \quad (\text{NLS})$$

where $\sigma = 1$ in the focusing case and $\sigma = -1$ in the defocusing case, respectively. Our main result is the description of a *pathological set* for equation NLS, based on the approach of Sun and Tzvetkov [ST20a] for the nonlinear wave equation. We prove that if we regularize the initial data using a given convolution profile, there exists a dense set of initial data in $H^s(\mathbb{R}^3)$ such that the regularized solutions exhibit norm inflation. The precise statement can be found in Theorem 1.1.4 below. Such a result aims at precisizing the probabilistic well-posedness theory, summarized in our context in Theorem 1.1.3. Let us first recall the motivation for this result. In the study of the Cauchy theory for dispersive equations, one generally encounters a critical regularity threshold that dictates whether the equation is well-posed or not, in the Hadamard sense. We recall that a general dispersive equation on a Riemannian manifold (M, g) can be written under the form

$$i\partial_t u + Lu = F(u), \quad u(0) \in H^s(M), \quad (1.1.1)$$

where L is a well-defined operator on $H^s(M)$ such that the linear propagator $\{\exp(itL)\}_{t \in \mathbb{R}}$ is a group of isometries of $H^s(M)$, whereas the nonlinear interaction, denoted $F(u)$, has to make sense in $H^s(M)$. Given a time interval $I \subset \mathbb{R}$ containing t_0 , the curve $t \in I \subset \mathbb{R} \mapsto u(t) \in C([t_0, T], H^s(M))$ is a strong solution to (1.1.1), with initial condition $u(t_0) = u_0 \in H^s(M)$, if it satisfies the following Duhamel integral representation formula, in the distributional sense

$$u(t) = e^{i(t-t_0)L} u_0 - i \int_{t_0}^t e^{i(t-\tau)L} F(u(\tau)) d\tau. \quad (1.1.2)$$

For dispersive equations, a suitable notion of well-posedness, that we recall here, can be found in [Tzv07].

Definition 1.1.1 (Local well-posedness). Given a general dispersive equation as (1.1.1), we say that the Cauchy problem is well-posed in $H^s(M)$ if for every bounded set $B \subset H^s(M)$, there exist $T = T(B) > 0$ and a Banach space X_T continuously embedded in $C([t_0, T], H^s(M))$ such that for any $u_0 \in H^s(M)$, there exists a solution $u \in X_T$ satisfying the Duhamel integral representation formula (1.1.2). Moreover, the following holds.

1. (Uniqueness) ¹ Let $u_1, u_2 \in X_T$ satisfying (1.1.2). If, for some $t_0 \leq t \leq T$, $u_1(t) = u_2(t)$, then $u_1 = u_2$.
2. (Continuity) The flow map $u_0 \in B \mapsto u \in X_T$ is continuous.
3. (Persistence of regularity) If $u_0 \in H^\sigma(M)$ with $\sigma > s$, then $u \in C([t_0, T], H^\sigma(M))$.

In addition, when the flow-map is uniformly continuous, we say that the Cauchy problem is semi-linearly well-posed.

This definition deserves some comments, especially in the case of critical regimes where the time of existence does not only depend on the bounded set B but also on the profile of the initial data. In Definition 1.1.1, we ask for some persistence of regularity in order to see the solutions in $H^s(M)$ as limits of regularized smooth solutions, and this motivates the study of the Cauchy problem at low regularity. A natural question that arises is whether the flow map is always uniformly continuous, as soon as it is continuous. In other words, is the critical threshold for the well-posedness equal to the critical threshold for the semi-linear well-posedness? In general, the answer is no, especially in the context of KdV type equations, see [Bou93a; TT04]. We refer to [Tzv07] for discussions on this question, and for further comments on Definition 1.1.1.

Sometimes, an invariance property of the equation under the action of a group of symmetries indicates the aforementioned well-posedness threshold. The first example, that we exploit in this work, is the *scaling* symmetry. If an equation is invariant under a scaling symmetry

$$u(t, x) \mapsto u_\lambda(t, x) := \lambda^\alpha u(\lambda^\beta t, \lambda^\gamma x), \quad \lambda > 0,$$

then, in the scale of homogeneous Sobolev spaces, its critical regularity s_c corresponds to the Sobolev norm left invariant under the action of the scaling

$$\|u_\lambda(0, x)\|_{\dot{H}^{s_c}} = \|u(0, \cdot)\|_{\dot{H}^{s_c}}.$$

We expect well-posedness at *scaling-subcritical* regularities $s \geq s_c$, and ill-posedness at *scaling-supercritical* regularities $s < s_c$. Note, however, that this heuristic is quite informal and in general, there might exist other properties of the equation that actually dictate the well-posedness threshold.

For instance, in the case of integrable equations, the scaling, semi-linear and well-posedness thresholds often differ. A classic example is the KdV equation on \mathbb{T} , which is continuously well-posed in $H^s(\mathbb{T})$ if $s \geq -1$, but the solution map is locally uniformly continuous only when $s \geq -\frac{1}{2}$ (in which case it is real analytic). Besides, the next equation in the KdV hierarchy is continuously well-posed for $s \geq -\frac{1}{2}$, but the solution map is nowhere locally uniformly continuous for this range of exponents [KM18], whereas these equations are invariant under the scaling transformation with critical Sobolev index $s_c = -\frac{3}{2}$. Similarly, the Benjamin-Ono equation on \mathbb{T} is continuously well-posed in $H^s(\mathbb{T})$ for $s > -\frac{1}{2}$ [GKT20], the flow map is locally uniformly continuous only when $s \geq 0$ (in which case it is real analytic) [GKT21]. The next equation in the Benjamin-Ono hierarchy is continuously well-posed for $s \geq 0$ [Gas21b], but an adaptation of [KM18] implies that the flow map is nowhere uniformly continuous. For all the equations in the Benjamin-Ono hierarchy, the scaling-critical Sobolev exponent is $s_c = -\frac{1}{2}$. Geometry also influences the dispersion property of

¹This statement is sometimes referred as *conditional uniqueness*, whereas unconditional uniqueness in $H^s(M)$ holds when it is possible to prove uniqueness in $C([t_0, T], H^s)$ and not just in X_T .

the equation, for instance the NLS equation on \mathbb{T} , with scaling-critical Sobolev exponent $s_c = -\frac{1}{2}$, is ill-posed for $s < 0$ [CCT03c] but continuously well-posed when $s \geq 0$ with a smooth solution map [Bou93b]. In dimension 2, NLS is well-posed on \mathbb{T}^2 for $s_c = 0 < s$, whereas the flow map of NLS on the sphere \mathbb{S}^2 is not uniformly locally well-posed as soon as $s < \frac{1}{4}$, see [Ban04; BGT02]. We also point out that the flow map for the NLS equation on the Heisenberg group cannot be \mathcal{C}^3 in $H^s(\mathbb{H}^1)$ for $s < 2$, whereas the scaling-critical exponent is $s_c = 1$, see for instance [GG10].

For the nonlinear Schrödinger equation NLS, as well as for the nonlinear wave equation in \mathbb{R}^3 or \mathbb{T}^3 , the scaling symmetry

$$u \longmapsto u_\lambda(t, x) = \lambda^{\frac{2}{p-1}} u(\lambda^2 t, \lambda x)$$

actually dictates the local well-posedness threshold, which is independent of the focusing or defocusing nature of the equation, see [CCT03a].

Theorem 1.1.2 (Well-posedness [CW90; CCT03a]). *Let $p > 1$.* ²

- *The Cauchy problem associated to NLS is well-posed in $H^s(\mathbb{R}^3)$ in the sense of Definition 1.1.1 when $\max(0, s_c) < s$.*
- *When $0 \leq s_c$, the Cauchy problem is well-posed in $H^{s_c}(\mathbb{R}^3)$ but the time of existence does depend on the initial data and not just on its H^{s_c} -norm.*
- *When $s < s_c$ or when $s < 0$, the Cauchy problem is ill-posed.*

We comment on the ill-posedness result due to [CCT03a] in the next paragraph. Before that, we recall that the mass and the energy are formally conserved under the flow, we denote them

$$M(u) = \frac{1}{2} \int_{\mathbb{R}^3} |u(x)|^2 dx, \quad H(u) = \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u(x)|^2 dx - \frac{\sigma}{p+1} \int_{\mathbb{R}^3} |u(x)|^{p+1} dx.$$

Note that these conservation laws ensure the existence of smooth global solutions in the energy-subcritical regime $p \leq 5$, when the energy is coercive.

Ill-posedness issues. Assume that the Cauchy problem is known to be well-posed for smooth initial data in the sense of Definition 1.1.1, say in H^σ for some σ large enough. Then, one can try to prove ill-posedness in a low-regularity space H^s by showing that the flow-map, defined on H^σ for some $\sigma > s$, does not extend continuously to $C([0, T], H^s)$, with T being arbitrarily small. This strategy is motivated by the persistence of regularity of the flow, that enables one to see the solutions in H^s , if they exist, as a limit of smoother solutions in H^σ .

Before going further, let us point out that it is easier to prove ill-posedness in the presence of special nonlinear solutions. For instance, for focusing Schrödinger equations where nonlinear solutions are accessible, we have the following [SS99]. If $s_c < 0$, the Schrödinger equation is mass-subcritical and there exist blow-up solutions that, combined with scaling arguments and virial identities, lead to solutions blowing up in arbitrarily short time in H^s when $s < s_c$. In the mass-critical case, a blow-up solution is obtained by applying a pseudo-conformal transformation to the ground state solution, and for the mass-supercritical case [KPV01] proved that the flow map is not uniformly continuous, respectively. Meanwhile, in the defocusing case, we do not have explicit nonlinear profiles

²If p is not an odd integer we further assume that $\lfloor s \rfloor + 1 < p$, so that the nonlinearity makes sense in H^s .

generating blow-up solutions, except in the Schrödinger case with $p = 3, d = 1$ where one can make use of modified scattering solutions [CCT03d]. The norm inflation instability we describe in the next paragraph is by no mean the only way to indicate that a dispersive PDE is ill-posed, and we refer to [KPV01; Tzv07] for the description of ill-posedness results in a broader context.

Norm inflation In the case of the Schrödinger or wave type equations, the discontinuity of the flow-map at scaling-supercritical regularities is stronger. Specifically, there exists sequences of smooth initial data going to zero in H^s , such that the H^s -norms of the corresponding local solutions go to infinity, for arbitrarily short times. Such a norm inflation mechanism was evidenced by [Leb01; Leb05; Lin93] for the wave equation³, and extended to the Schrödinger equation in [CCT03a].

The general strategy to prove that norm inflation occurs in the scaling-supercritical regime is to perform a *small dispersion analysis*. Namely, by rescaling an arbitrary bump function, one generates a sequence of smooth initial data $(\psi_n)_n$ that go to zero in H^s , while spatially concentrating around a point. Then, one considers the smooth solution u_n with initial data ψ_n , and studies the *bubble* solution v_n to the dispersionless ODE

$$\begin{cases} i\partial_t v_n = \sigma |v_n|^{p-1} v_n, \\ v_n(0) = \psi_n. \end{cases}$$

The ODE profile v_n , that captures the oscillations of u_n , grows in H^s after a time t_n going to zero as n goes to infinity. Then, when $0 < s_c$ and $s < s_c$, one uses a priori energy estimates up to time t_n and establishes that uniformly in n ,

$$\|u_n(t_n) - v_n(t_n)\|_{H^s} \lesssim 1.$$

Therefore, the oscillations dominate the dispersion so that the instability stems from a frequency cascade from low to high Fourier modes.

We note that such a mechanism is independent of the focusing or defocusing nature of the equation. Specifically, in the case of Schrödinger equation, the results from [CCT03a] state as follows. Norm inflation occurs when $s < -\frac{d}{2}$, and also in the mass-supercritical case when $0 < s < s_c$. Furthermore, by using the frequency modulation method, Christ Colliander and Tao evidenced in [CCT03a] the *high to low* frequency cascade when $s < -\frac{d}{2}$, and the *low to high* frequency cascade when $0 < s$. They also proved that the flow map fails to be uniformly continuous for $s < 0$ by using the Galilean symmetries of NLS. Note that a similar small dispersion analysis has been performed earlier, in [Kuk95]. We refer to [BT08a; BGT05b] for some subsequent norm inflation results on compact manifolds, with similar proofs that rely on concentrating profiles in a point. We also mention the norm inflation results in negative Sobolev spaces initiated in [BT06], then further developed in [CDS12; CK17], see also [Kis18; Oh21].

Some variations The norm inflation does not only occur around the zero solution, but also around any solution, as shown in [Xia21] for the wave equation. The proof does not rely on finite propagation speed, so that it can be adapted to NLS, see also [Xia22] establishing the norm inflation at any point for the fourth-order Schrödinger equation. The general idea is that if the regularizing sequence is

³In [Lin93], the norm inflation is due to another concentration phenomenon related with the Lorentz invariance of the wave equation.

arbitrary, norm inflation is very likely to occur for a dense set of initial data. However, as presented in the next paragraph, the picture is more intricate when one uses the convolution by an approximate identity, which is the most natural regularization procedure. It has the property to commute with the linear flow and to be uniformly bounded on L^p when $p < \infty$.

To push this result further, a key construction consists in gluing together blow-up profiles that concentrate at different points. Such a construction is inspired from the work of Lebeau [Leb01], and was used in [BT08a; Xia21] for wave equations to generate initial data from which any solution u satisfying the finite propagation speed inflates instantaneously

$$\limsup_{t \rightarrow 0^+} \|u(t)\|_{H^s} = +\infty.$$

To give a precise meaning of the above inflating solution associated with such a pathological initial data, [ST20a] adapted the construction in the presence of a regularizing convolution. As described in the next paragraph, this is also motivated by the probabilistic Cauchy theory at supercritical regularities.

Let us conclude this paragraph by mentioning the result of [Leb01], that evidenced a loss of regularity mechanism for the energy-supercritical wave equation by using some nonlinear geometric optics. For Schrödinger equation we refer to [AC09; CR09], that indicate as well loss of regularity.

Pathological set in the context of probabilistic well-posedness After the pioneering contributions from Bourgain [Bou94; Bou96] for the cubic Schrödinger equation on \mathbb{T}^2 , followed by the works from Burq and Tzvetkov for the nonlinear wave equation on compact Riemannian manifolds without boundary [BT08a; BT08b], many authors have investigated the well-posedness issue at scaling-supercritical regularities by using a statistical approach. Specifically, given a Sobolev space $H^s(M)$ where the equation is known to be ill-posed, or semi-linearly ill-posed, one can search for a non-degenerate probability measure supported on a dense subset of $H^s(M)$, and prove that the equation is well-posed for initial data on the support of this measure. Such an approach stems from the study of Gibbs measures [Bou96; BTT18], and has been extensively developed in other contexts since then. We refer to [BOP19a] for a comprehensive survey on the so-called probabilistic Cauchy theory. Concerning the cubic Schrödinger equation in the Euclidean space \mathbb{R}^d , with $d \geq 3$, Bényi, Oh, and Pocovnicu [BOP15b] proved generic well-posedness by using a Wiener randomization procedure, based on a unit-scale decomposition of the frequency space that leads to refined Strichartz estimates. To state the result in the context of the present work, we fix an approximate identity $(\rho_\varepsilon)_{\varepsilon > 0}$ defined as

$$\rho_\varepsilon(x) := \frac{1}{\varepsilon^3} \rho\left(\frac{x}{\varepsilon}\right), \quad (1.1.3)$$

where $\rho \in C_c^\infty(\mathbb{R}^3)$ is valued in $[0, 1]$ and satisfies $\int_{\mathbb{R}^3} \rho(x) dx = 1$. Let us consider the cubic NLS equation, with $p = 3$, at a scaling-supercritical regularity $s < s_c = \frac{1}{2}$.

Theorem 1.1.3 (Local well-posedness on a full-measure set for cubic NLS). *Let $\frac{1}{4} < s < \frac{1}{2}$. There exist a probability measure μ supported on $H^s(\mathbb{R}^3)$ and a dense set $\Sigma \subset H^s(\mathbb{R}^3)$ with full μ -measure, such that the following holds. For all f_0 in Σ , the smooth local solutions (u^ε) to the cubic NLS with initial data $(f_0 * \rho_\varepsilon)$ are well-defined up to a time $T = T(f_0) > 0$, and they converge to a limiting object u*

$$\lim_{\varepsilon \rightarrow 0} \|u^\varepsilon - u\|_{L^\infty([0, T]; H^s(\mathbb{R}^3))} = 0, \quad u \in C([0, T]; H^s(\mathbb{R}^3)). \quad (1.1.4)$$

Moreover, u solves NLS in the distributional sense (1.1.2) on $[0, T]$, with initial data f_0 .

The local well-posedness result is due to [BOP15b] in dimension 4, and [BOP15a] in dimension $d \geq 3$ for cubic NLS, written in the functional framework of U^p, V^p spaces in order to prove scattering for small data. However, the convergence of the approximate solutions (1.1.4) is not written in [BOP15a], and we prove it in Section 1.3 by adapting the analysis performed in [BOP15b; BOP15a]. Note that the regularization by convolution is essential since, as pointed out in [Xia21] for the wave equation, norm inflation occurs over a dense set of initial data when the regularization procedure is arbitrary. The convergence of the regularized solutions is in the spirit of Theorems 2.6 and 2.7 from [Tzv17] for the wave equation. Let us mention that the large-time behavior of the probabilistic solution has been investigated by many authors, by combining the probabilistic local well-posedness result of Theorem 1.1.3 with deterministic methods. Concerning the cubic Schrödinger equation in \mathbb{R}^3 , [BOP15a; PRT14] established almost-sure scattering for small data, and [Cam22] proved almost-sure asymptotic stability for small ground states. Outside the small data regime, [KMV19; DLM20; Spi21] proved almost-sure scattering for the cubic NLS in \mathbb{R}^4 , and [Cam23; SSW21] extended the scattering result in \mathbb{R}^3 , where the cubic NLS is energy-subcritical.

Outside this full-measure set where the statistical approach provides a strong notion of well-posedness, there exist initial data that concentrate in a point so that a low to high frequency cascade occurs. From the result of [Xia21], we know that if the regularization procedure is arbitrary, such an instability phenomenon may appear in the vicinity of any initial data. In addition, Sun and Tzvetkov proved in [ST20a] that for nonlinear wave equations, when the initial data are regularized by convolution, norm inflation occurs on a *pathological set* containing a dense G_δ set, even when a probabilistic well-posedness result in the spirit of Theorem 1.1.3 holds.

Main result Following the approach of Sun and Tzvetkov [ST20a] for the nonlinear wave equation, we describe the pathological set for equation NLS, which is the counterpart of the probabilistic well-posedness theory presented in Theorem 1.1.3. We show that even if we naturally regularize the initial by convolution, there still exist inflating solutions arising from a dense set of initial data in H^s .

Theorem 1.1.4 (Pathological set of scaling-supercritical data for NLS). *Let $p \geq 3$ be an odd integer⁴ and $0 < s < s_c = \frac{3}{2} - \frac{2}{p-1}$. There exists a dense set $S \subset H^s(\mathbb{R}^3)$ such that for every $f_0 \in S$, the family of local solutions (u^ε) of NLS with initial data $(f_0 * \rho_\varepsilon)$ does not converge because of norm inflation. More precisely, there exist $\varepsilon_n \rightarrow 0$ and $t_n \rightarrow 0$ such that u^{ε_n} exists in $C([0, t_n], H^s(\mathbb{R}^3))$, and*

$$\lim_{n \rightarrow \infty} \|u^{\varepsilon_n}(t_n)\|_{H^s(\mathbb{R}^3)} = \infty. \quad (1.1.5)$$

Moreover, in the defocusing energy-subcritical and critical cases $p \in \{3, 5\}$, where NLS is known to be globally⁵ well-posed in H^1 , the pathological set \mathcal{P} contains a dense G_δ set.

To exhibit the pathological set, we use in part 1.2.2 the same construction as in [ST20a] that consists in putting side by side an infinity of inflating bubbles concentrated at arbitrarily small scales.

⁴This assumption is technical, and might be removed. It guarantees that the bubbles constructed in paragraph 1.2.1 are C^∞ .

⁵We refer to [Col+08] for the proof of global well-posedness and scattering in the energy space of the quintic ($p = 5$) defocusing Schrödinger equation.

As discussed above, this construction is contained in [BT08a; CCT03a], and was inspired from [Leb01]. The proof proposed in [ST20a] revisits the small dispersion analysis from [CCT03a] with an additional convolution. Indeed, since their motivation is to make precise the probabilistic well-posedness result, the smooth initial data need to be obtained by convolution with an approximate identity. Then, they use finite propagation speed to indicate that the different bubbles do not interact in short time. Moreover, in order to generate a dense set in $H^s(\mathbb{R}^3)$ and not only around zero, [ST20a] implement the idea from [Xia21] that consists in performing the standard small dispersion analysis around an arbitrary function in $C^\infty(\mathbb{R}^3)$ with compact support.

In the case of Schrödinger type equations, we do not have such a finite propagation speed principle. To overcome this, we play with the regularizing convolutions. The observation is that given a fixed scale of regularization $\varepsilon > 0$, the convolution by ρ_ε breaks the bubbles that concentrate at scale $\lambda^{-1} \ll \varepsilon$. Hence, at a certain suitable time $t_\varepsilon \ll 1$, the bubble concentrated at scale $\lambda^{-1} \sim \varepsilon$ is the only one to inflate. This observation indicates that there is no need to use the finite propagation speed in the proof of Theorem 1.1.4, even in the case of the nonlinear wave equation analyzed in [ST20a]. Therefore, the same approach probably extends to any equation of Schrödinger type, provided there exists a reasonable Cauchy theory for smooth initial data. Indeed, we do not directly use the particular dispersion of NLS in the analysis. Let us mention some features of the approach developed in this work.

Finite propagation speed and localization of bubbles Since we do not use any finite propagation speed, we have access to Schrödinger type equations. Another consequence is that we can superpose the bubbles in the construction. Indeed, we do not exploit the absence of interaction between the different bubbles, that was crucial in the short time analysis from [ST20a] about the wave equation. Hence, we can take the center of the bubbles to be $x = 0$, so that the pathological set is left invariant under the action of the rotations and the construction persists in the radial case. Note, however, that we are not able to prove that the solution inflates along the whole sequence as ε goes to zero, but only along the subsequence ε_n (see remark 1.1 in [ST20a], and remark 1.2.5 below).

Focusing and defocusing equations and range of nonlinearities We show that the a priori control on the growth of Sobolev norms is sufficient to establish a lower bound on the time of existence for the smooth solutions obtained by convolution. Hence, we can bypass the use of the coercive energy, and we treat the focusing and defocusing case without distinction. Also, we have access to the energy-supercritical case $p > 5$. In such a case, we point out that the norm inflation of the $H^1(\mathbb{R}^3)$ -norm does not contradict the conservation of the energy, since when $p > 5$ the potential energy of an initial data in $H^1(\mathbb{R}^3)$ is infinite. However, to prove that \mathcal{P} contains a dense G_δ set we need to assume that the smooth solutions are global, say in $H^{\frac{3}{2}^+}(\mathbb{R}^3)$, see Proposition 1.2.10. Hence, the reason for the technical constraint on $p \in \{3, 5\}$ in the statement of the second part of Theorem 1.1.4.

Lower bound on s Sun and Tzvetkov observed in [ST20a; ST20c] that a lower bound on s might be required. However, in the statement of Theorem 1.1.4, we do not need such a lower bound, and we propose an explanation of this difference in remark 1.2.9. Yet, we do not know if the limitation in the case of the wave equation is whether technical or not. Note also that we do not have $s = 0$ because of mass conservation. When $s < 0$ other mechanisms operate, and the possible norm inflation is due

to a *high to low* frequency cascade.

Other geometries and dimensions Since the construction of the pathological set is local, up to working in local coordinates, our proof works as well in the periodic case, or in the case of a compact Riemannian manifold. Nevertheless, we decided to state the result in the Euclidean case, since there is no general probabilistic Cauchy theory in the spirit of [BT08a] for NLS on a compact manifold of dimension 3. Our proof can also be adapted to higher dimension, but we write it in the context of [ST20a] for simplicity, in the sense that we only need to compute two derivatives to control the L^∞ -norm of a function in \mathbb{R}^3 . In higher dimensions, we would have to use a semiclassical energy with higher order derivatives.

Conclusion In the scaling-supercritical regime, and at least in the cubic case, we have two different behaviors for the smooth solutions to NLS initiated from regularized initial data obtained by convolution with an approximate identity. On the one hand, Theorem 1.1.4 states that the *pathological* set of initial data where norm inflation occurs contains a dense G_δ set. On the other hand, the statistical approach from Theorem 1.1.3 provides of dense full-measure set of initial data for which the regularized solutions converge to a limiting object that is a strong solution to NLS. This is the counterpart for Schrödinger-type equations to the result from Sun and Tzvetkov [ST20a] for wave equations, and the same comments apply. Namely, there are two different notions of genericity. While the norm inflation set from Theorem 1.1.4 is generic for the topology, it is negligible for the measure. Meanwhile, we know from the Baire category Theorem that the well-posedness set from Theorem 1.1.3, which has full measure, cannot be a dense G_δ set.

Outline of the paper We first establish the generic ill-posedness Theorem 1.1.4 in section 1.2. To achieve such a goal, we set up in part 1.2.1 the fundamental bubbles v_n that encode the oscillations, and we construct the pathological set in part 1.2.2 by superposing these bubbles. In paragraph 1.2.3 we fix a given scale ε_n , its corresponding inflation time t_n , and we compare in $H^s(\mathbb{R}^3)$ the solution to NLS initiated from an initial data in the pathological set, with the bubble concentrated at scale ε_n , supposed to inflate after the time t_n . In a second section 1.3, we prove the generic well-posedness result. After introducing preliminaries for the probabilistic method in part 4.2 and reviewing relevant probabilistic Cauchy theory results in part 1.3.2, we prove in part 1.3.3 the convergence in $H^s(\mathbb{R}^3)$ of the regularized solution by convolution to the solution generated from a randomized initial data.

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1.2 Pathological set of data where norm inflation occurs

The purpose of this section is to prove Theorem 1.1.4. First, we follow the strategy from [ST20a] and we study the unstable profile, or the *bubble*, regularized by an approximate identity. Our key observation can be stated as follows. When convoluted with an approximated identity (ρ_ε) , a

blow-up profile concentrated at scale n^{-1} , and supposed to inflate at time t_n , actually gets smoother and remains bounded at time t_n , as soon as $n^{-1} \ll \varepsilon$. We draw the attention to Lemmas 1.2.1 and 1.2.4 that indicate such a mechanism. Owing to this observation, we show in Section 1.2.3 how to generalize the small-dispersion analysis from [ST20a] without using finite propagation speed.

1.2.1 Unstable profile

We construct unstable profiles by following the approach in [BGT05b]. The construction of the bubbles, that have growing H^s -norm, is based on the solution $V(t) = e^{i\sigma t}$ to the dispersionless ODE

$$\begin{cases} iV' + \sigma|V|^{p-1}V = 0 \\ V(0) = 1. \end{cases}$$

Note that V is a bounded periodic function. Set a smooth and radial profile $\varphi \in \mathcal{C}_c^\infty(\{\|x\| \leq 1\})$, such that $0 \leq \varphi \leq 1$, from which we define a bubble concentrated at scale n

$$v_n(0, x) := \lambda_n^{\frac{2}{p-1}} \varphi(nx) = \kappa_n n^{\frac{3}{2}-s} \varphi(nx).$$

Given $\gamma > 0$ to be determined later in the analysis, the parameters of the above bubble are

$$\kappa_n := (\log(n))^{-\gamma}, \quad \lambda_n := \kappa_n^{\frac{p-1}{2}} n^{(\frac{3}{2}-s) \cdot \frac{p-1}{2}}.$$

We recall that we have fixed an approximate identity defined in (1.1.3), and consider the bubble after convolution by the approximate identity

$$v_n^\varepsilon(0) := \rho_\varepsilon * v_n(0).$$

Given a time t , we consider the profile

$$v_n^\varepsilon(t, x) := v_n^\varepsilon(0, x) V(t|v_n^\varepsilon(0, x)|^{p-1}),$$

solution to the dispersionless ODE

$$\begin{cases} i\partial_t v_n^\varepsilon + \sigma|v_n^\varepsilon|^{p-1}v_n^\varepsilon = 0. \\ v_n^\varepsilon(0) = \rho_\varepsilon * v_n(0), \end{cases}$$

We now fix the parameters,

$$\varepsilon_n := \frac{1}{100n}, \quad t_n := \log(n)^{\beta(p-1)} n^{(s-\frac{3}{2})(p-1)} = \lambda_n^{-2} \log(n)^{(\beta-\gamma)(p-1)}, \quad (1.2.1)$$

for suitable $0 < \gamma < \beta < 1$ adjusted in the analysis, and we follow [ST20a] to adapt the bounds from [BGT05b] in the presence of the convolution parameter.

Lemma 1.2.1 (Growth of the profile). *Let $0 \leq s < \frac{3}{2}$.*⁶

⁶Note that when $s = 0$ the lower bound goes to zero as n goes to infinity. This is in agreement with the mass conservation.

1. There exists $c > 0$ independent of n such that for all $\varepsilon \leq \varepsilon_n$, we have the lower bound

$$\|v_n^\varepsilon(t_n)\|_{H^s} \geq c\kappa_n(\lambda_n^2 t_n)^s.$$

2. For all $m \in \mathbb{N}$, there exists $C > 0$ such that for all $n \in \mathbb{N}$, $t \in \mathbb{R}$ and $\varepsilon > 0$,

$$\| |\nabla|^m v_n^\varepsilon(t) \|_{L^2} \leq C\kappa_n n^{m-s} (1 + (t\lambda_n^2)^m) \min\left(1, \frac{1}{\varepsilon n}\right)^{m+\frac{3}{2}}.$$

In addition,

$$\| |\nabla|^m v_n^\varepsilon(t) \|_{L^\infty} \leq C\kappa_n n^{m+\frac{3}{2}-s} (1 + (t\lambda_n^2)^m) \min\left(1, \frac{1}{\varepsilon n}\right)^{m+\frac{3}{2}}.$$

The factor $\min\left(1, \frac{1}{\varepsilon n}\right)^{m+\frac{3}{2}}$ obtained on top of Lemma 2.1 in [ST20a] comes from the fact that we get better estimates when estimating the derivatives and the L_x^∞ norm of ρ_ε instead of those of v_n when ε is large compared to $\frac{1}{n}$.

Proof. We first establish the upper bounds.

Upper bounds Let $T_n(f) := f(n \cdot)$ be the scaling operator. We have

$$\begin{aligned} v_n^\varepsilon(0, x) &= \kappa_n n^{\frac{3}{2}-s} \int \varphi(n(x-x')) \frac{1}{\varepsilon^3} \rho\left(\frac{x'}{\varepsilon}\right) dx' \\ &= \frac{\kappa_n n^{\frac{3}{2}-s}}{\varepsilon^3} (T_n \varphi * T_{\frac{1}{\varepsilon}} \rho)(0, x), \end{aligned}$$

so that when $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{N}^3$, denoting $m = |\alpha| = \alpha_1 + \alpha_2 + \alpha_3$,

$$\begin{aligned} \partial^\alpha v_n^\varepsilon(0, x) &= \kappa_n n^{\frac{3}{2}-s} n^m \int T_n(\partial^\alpha \varphi)(x-x') \frac{1}{\varepsilon^3} \rho\left(\frac{x'}{\varepsilon}\right) dx' \\ &= \frac{\kappa_n n^{\frac{3}{2}-s}}{\varepsilon^3} n^m (T_n(\partial^\alpha \varphi) * T_{\frac{1}{\varepsilon}} \rho)(0, x). \end{aligned}$$

The Young's convolution inequality implies

$$\begin{aligned} \|\partial^\alpha v_n^\varepsilon(0)\|_{L^\infty} &\lesssim \kappa_n n^{\frac{3}{2}-s} n^m \|T_n(\partial^\alpha \varphi)\|_{L^\infty} \|\rho_\varepsilon\|_{L^1} \lesssim \lambda_n^{\frac{2}{p-1}} n^m, \\ \|\partial^\alpha v_n^\varepsilon(0)\|_{L^2} &\lesssim \kappa_n n^{\frac{3}{2}-s} n^m \|T_n(\partial^\alpha \varphi)\|_{L_x^2} \|\rho_\varepsilon\|_{L^1} \lesssim \kappa_n n^{m-s}. \end{aligned}$$

But one can also derivate the ρ part

$$\begin{aligned} \partial^\alpha v_n^\varepsilon(0, x) &= \kappa_n n^{\frac{3}{2}-s} \left(\frac{1}{\varepsilon}\right)^m \int T_n(\varphi)(x') \frac{1}{\varepsilon^3} (\partial^\alpha \rho)\left(\frac{x-x'}{\varepsilon}\right) dx' \\ &= \frac{\kappa_n n^{\frac{3}{2}-s}}{\varepsilon^3} \left(\frac{1}{\varepsilon}\right)^m (T_n(\varphi) * T_{\frac{1}{\varepsilon}}(\partial^\alpha \rho))(0, x), \end{aligned}$$

so that there also holds

$$\begin{aligned} \|\partial^\alpha v_n^\varepsilon(0)\|_{L^\infty} &\lesssim \kappa_n n^{\frac{3}{2}-s} \left(\frac{1}{\varepsilon}\right)^m \|T_n(\varphi)\|_{L^1} \frac{1}{\varepsilon^3} \|T_{\frac{1}{\varepsilon}}(\partial^\alpha \rho)\|_{L^\infty} \lesssim \lambda_n^{\frac{2}{p-1}} \left(\frac{1}{\varepsilon}\right)^m \frac{1}{(\varepsilon n)^3}, \\ \|\partial^\alpha v_n^\varepsilon(0)\|_{L^2} &\lesssim \kappa_n n^{\frac{3}{2}-s} \left(\frac{1}{\varepsilon}\right)^m \|T_n(\varphi)\|_{L^1} \frac{1}{\varepsilon^3} \|T_{\frac{1}{\varepsilon}}(\partial^\alpha \rho)\|_{L^2} \\ &\lesssim \kappa_n n^{\frac{3}{2}-s} \left(\frac{1}{\varepsilon}\right)^{m-\frac{3}{2}} \frac{1}{(\varepsilon n)^3} \\ &= \kappa_n n^{m-s} \frac{1}{(\varepsilon n)^{m+\frac{3}{2}}}. \end{aligned}$$

Note that when $m \leq 1$, we can still improve this estimate using that

$$\|\partial^\alpha v_n^\varepsilon(0)\|_{L^2} \lesssim \kappa_n n^{\frac{3}{2}-s} \left(\frac{1}{\varepsilon}\right)^m \|T_n(\varphi)\|_{L^2} \frac{1}{\varepsilon^3} \|T_{\frac{1}{\varepsilon}}(\partial^\alpha \rho)\|_{L^1} \lesssim \kappa_n n^{m-s} \frac{1}{(\varepsilon n)^3}.$$

These two observations sum up as

$$\begin{aligned} \|\partial^\alpha v_n^\varepsilon(0)\|_{L^\infty} &\lesssim \lambda_n^{\frac{2}{p-1}} n^m \min\left(1, \frac{1}{\varepsilon n}\right)^{m+3}, \\ \|\partial^\alpha v_n^\varepsilon(0)\|_{L^2} &\lesssim \kappa_n n^{m-s} \min\left(1, \frac{1}{\varepsilon n}\right)^{m+\frac{3}{2}}. \end{aligned}$$

Next, we deduce the estimates at time t . Using V being bounded by 1, we have

$$\begin{aligned} \|v_n^\varepsilon(t)\|_{L^\infty} &\lesssim \lambda_n^{\frac{2}{p-1}} \min\left(1, \frac{1}{\varepsilon n}\right)^3, \\ \|v_n^\varepsilon(t)\|_{L^2} &\lesssim \kappa_n n^{-s} \min\left(1, \frac{1}{\varepsilon n}\right)^{\frac{3}{2}}. \end{aligned}$$

Finally, we compute

$$|\nabla v_n^\varepsilon(t)| \lesssim t |v_n^\varepsilon(0)|^{p-1} |\nabla v_n^\varepsilon(0) V'(t|v_n^\varepsilon(0)|^{p-1})| + |\nabla v_n^\varepsilon(0) V(t|v_n^\varepsilon(0)|^{p-1})|, \quad (1.2.2)$$

and

$$\begin{aligned} |\Delta v_n^\varepsilon(t)| &\lesssim t^2 |v_n^\varepsilon(0)|^{2p-3} |\nabla v_n^\varepsilon(0)|^2 V''(t|v_n^\varepsilon(0)|^{p-1})| \\ &\quad + 2t |v_n^\varepsilon(0)|^{p-2} |\nabla v_n^\varepsilon(0)|^2 V'(t|v_n^\varepsilon(0)|^{p-1})| \\ &\quad + t |v_n^\varepsilon(0)|^{p-1} |\Delta v_n^\varepsilon(0) V'(t|v_n^\varepsilon(0)|^{p-1})| \\ &\quad + |\Delta v_n^\varepsilon(0) V(t|v_n^\varepsilon(0)|^{p-1})|. \end{aligned}$$

Iterating, we obtain a similar formula when $\alpha \in \mathbb{N}^m$ is arbitrary, the leading term being the one where all the derivatives fall on the ODE profile V

$$|\partial^\alpha v_n^\varepsilon(t)| \lesssim |v_n^\varepsilon(0)|^{(p-1)m} |\nabla v_n^\varepsilon(0)|^m |v_n^\varepsilon(0)| t^m |V^{(m)}(t|v_n^\varepsilon(0)|^{p-1})|.$$

Since $|V|$ and its derivatives are bounded by 1, we deduce that

$$\begin{aligned}\|\nabla v_n^\varepsilon(t)\|_{L^\infty} &\lesssim \lambda_n^{\frac{2}{p-1}} n(1+t\lambda_n^2) \min\left(1, \frac{1}{\varepsilon n}\right)^{1+3}, \\ \|\Delta v_n^\varepsilon(t)\|_{L^\infty} &\lesssim \lambda_n^{\frac{2}{p-1}} n^2(1+t\lambda_n^2+(t\lambda_n^2)^2) \min\left(1, \frac{1}{\varepsilon n}\right)^{2+3}.\end{aligned}$$

On the L^2 -based spaces, we also get

$$\begin{aligned}\|\nabla v_n^{\varepsilon_n}(t)\|_{L^2} &\lesssim \kappa_n n^{1-s}(1+t\lambda_n^2) \min\left(1, \frac{1}{\varepsilon n}\right)^{1+\frac{3}{2}}, \\ \|\Delta v_n^{\varepsilon_n}(t)\|_{L^2} &\lesssim \kappa_n n^{2-s}(1+(t\lambda_n^2)^2) \min\left(1, \frac{1}{\varepsilon n}\right)^{2+\frac{3}{2}}.\end{aligned}$$

The other estimates follow similarly.

Lower bound We now establish the lower bound $\|v_n^\varepsilon(t_n)\|_{H^s} \geq c\kappa_n(t_n\lambda_n^2)^s$ for every $\varepsilon \leq \varepsilon_n$, where we recall that $t_n\lambda_n^2 \geq 1$ when $n \gg 1$. Fix $\varepsilon \leq \varepsilon_n$. When $0 \leq s < 1$, we have from interpolation that

$$\|v_n^\varepsilon(t)\|_{H^1}^{2-s} \lesssim \|v_n^\varepsilon(t)\|_{H^s} \|v_n^\varepsilon(t)\|_{H^2}^{1-s}.$$

Using the upper bound

$$\|v_n^\varepsilon(t)\|_{H^2} \lesssim \kappa_n n^{2-s}(1+(t\lambda_n^2)^2),$$

it is enough to establish the lower bound

$$\|v_n^\varepsilon(t_n)\|_{H^1} \gtrsim \kappa_n(t_n\lambda_n^2)n^{1-s}.$$

Since the second term on the right-hand side in (1.2.2) has the upper bound

$$\|\nabla v_n^\varepsilon(0)V(t_n|v_n^\varepsilon(0)|^{p-1})\|_{L^2} \lesssim \kappa_n n^{1-s},$$

it is sufficient to establish the lower bound for the leading term

$$\|t_n|v_n^\varepsilon(0)|^{p-1}\nabla v_n^\varepsilon(0)V'(t_n|v_n^\varepsilon(0)|^{p-1})\|_{L^2}.$$

Since V' is of modulus one,

$$\begin{aligned}\|t_n|v_n^\varepsilon(0)|^{p-1}\nabla v_n^\varepsilon(0)V'(t_n|v_n^\varepsilon(0)|^{p-1})\|_{L^2} &= t_n\||v_n^\varepsilon(0)|^{p-1}\nabla v_n^\varepsilon(0)\|_{L^2} \\ &= t_n\lambda_n^{2+\frac{2}{p-1}}n\| |T_n(\varphi) * \rho_\varepsilon|^{p-1}(T_n(\nabla\varphi) * \rho_\varepsilon)\|_{L^2} \\ &= t_n\lambda_n^{2+\frac{2}{p-1}}n^{1-\frac{3}{2}}\| |\varphi * \rho_{n\varepsilon}|^{p-1}(\nabla\varphi * \rho_{n\varepsilon})\|_{L^2},\end{aligned}$$

the latter equality coming from the change of variable

$$T_n f * \rho_\varepsilon(x) = T_n(f * \rho_{n\varepsilon}).$$

Thanks to the choice of ε such that $n\varepsilon \leq n\varepsilon_n = \frac{1}{100}$, the L^2 norm of $\varphi * \rho_{n\varepsilon}$ admits a lower bound that does not depend on n . Moreover, it is bounded from below by $c_0 > 0$ since $\varphi * \rho_{n\varepsilon}$ and its derivative tends to φ and its derivative in the L^p spaces when $n\varepsilon$ goes to 0. To conclude, it only remains to observe that

$$t_n \lambda_n^{2+\frac{2}{p-1}} n^{1-\frac{3}{2}} = \kappa_n (t_n \lambda_n^2) n^{1-s}.$$

When $1 \leq s < \frac{3}{2}$, the idea is the same except that we need to interpolate between higher order Sobolev spaces

$$\|v_n^\varepsilon(t)\|_{H^2}^{3-s} \lesssim \|v_n^\varepsilon(t)\|_{H^s} \|v_n^\varepsilon(t)\|_{H^3}^{2-s}.$$

The leading term in the expression for $\|v_n^\varepsilon(t)\|_{H^2}$ is

$$\begin{aligned} \|t_n^2 |v_n^\varepsilon(0)|^{2p-3} |\nabla v_n^\varepsilon(0)|^2 V''(t_n |v_n^\varepsilon(0)|^{p-1})\|_{L^2} &= t_n^2 \| |v_n^\varepsilon(0)|^{2p-3} (\nabla v_n^\varepsilon(0))^2 \|_{L^2} \\ &= t_n^2 \lambda_n^{\frac{2(2p-1)}{p-1}} n^2 \| |T_n(\varphi) * \rho_\varepsilon|^{2p-3} (T_n(\nabla\varphi) * \rho_\varepsilon)^2 \|_{L^2} \\ &= t_n^2 \lambda_n^{4+\frac{2}{p-1}} n^{2-\frac{3}{2}} \| |\varphi * \rho_{n\varepsilon}|^{2p-3} (\nabla\varphi * \rho_{n\varepsilon})^2 \|_{L^2}, \end{aligned}$$

where $t_n^2 \lambda_n^{4+\frac{2}{p-1}} n^{2-\frac{3}{2}} = \kappa_n (t_n \lambda_n^2)^2 n^{2-s}$. To conclude, it only remains to use the upper bound

$$\|v_n^\varepsilon(t_n)\|_{H^3} \lesssim \kappa_n n^{3-s} (t_n \lambda_n^2)^3. \quad \square$$

1.2.2 Pathological set

The pathological set is defined as

$$\mathcal{P} := \{f \in H^s(\mathbb{R}^3) \mid \limsup_{\varepsilon, t \rightarrow 0} \|\Phi(t)(\rho_\varepsilon * f)\|_{H^s} \rightarrow \infty\}. \quad (1.2.3)$$

In this part, we prove that the pathological contains a dense G_δ set, that stems from the “*tanghuru*” construction [ST20a] consisting in the superposition of inflating bubbles. We fix $a \gg 1$. For arbitrary $k \in \mathbb{N}$, $x_k \in \mathbb{R}^3$, we set $n_k := e^{a^k}$, and define the k -th bubble centered at $x_k \in \mathbb{R}^3$, and concentrated at scale $n_k^{-1} v_{0,k}$, writes

$$v_{0,k}(x) := v_{n_k}(0, x - x_k) = \log(n_k)^{-\gamma} n_k^{\frac{3}{2}-s} \varphi(n_k(x - x_k)).$$

Remark 1.2.2 (Position of the bubbles). In the construction from [ST20a], the k -th bubble concentrates in the point $x_k = \frac{1}{k}$, so that the distance between two consecutive bubbles is much larger than the scale of the bubble. Namely, $|x_k - x_{k-1}| \sim \frac{1}{k^2} \gg e^{-a^k}$. Hence, the different bubbles are far enough and do not interact in short time, thanks to the finite propagation speed. Since we do not make use of this, we take arbitrary x_k in the construction. Note that when $x_k = 0$ for all k , the construction is radial and provides a pathological set even in the radial case.

Definition 1.2.3 (Dense subset of the pathological set). We denote by S the set of initial data f_0 that can be decomposed under the form

$$f_0 = u_0 + \sum_{k=k_0}^{\infty} v_{0,k}, \quad k_0 \geq 1, \quad u_0 \in \mathcal{C}_c^\infty(\mathbb{R}^3).$$

Our aim is to show that the pathological set contains S , which is dense in $H^s(\mathbb{R}^3)$, and a G_δ subset as soon as every smooth solution to NLS is global. For this, we first establish upper bounds for the series in the definition of S , after regularization by convolution. Recall that we have set up the parameters in (1.2.1), and, given $k \in \mathbb{N}$, we may denote for simplicity

$$n = n_k = e^{a^k}, \quad \varepsilon_k = \varepsilon_{n_k}, \quad t_k = t_{n_k},$$

with $a \gg 1$ to be chosen later in the analysis. At time t_{n_k} and at scale ε_{n_k} we expect the k -th bubble to be responsible for the norm inflation, while the other bubbles concentrated at the larger scales $l^{-1} > k^{-1}$ and at smaller scales $l^{-1} < k^{-1}$ do not inflate.

Lemma 1.2.4 (H^m -bounds for initial data in the pathological set). *Let $m \in \mathbb{N}$, $1 \leq k_0 \leq k$, and denote $\varepsilon_k = (100n_k)^{-1}$, with $n_k = e^{a^k}$. If $m < s$, there holds*

$$\sum_{l=k_0}^{k-1} \|\rho_{\varepsilon_k} * v_{0,l}\|_{H^m(\mathbb{R}^3)} \lesssim 1, \quad (1.2.4)$$

$$\sum_{l=k+1}^{\infty} \|\rho_{\varepsilon_k} * v_{0,l}\|_{H^m(\mathbb{R}^3)} \lesssim n_{k+1}^{m-s} \left(\frac{n_k}{n_{k+1}} \right)^{\frac{3}{2}}. \quad (1.2.5)$$

Moreover, for every $m > s$, we have

$$\sum_{l=k_0}^{k-1} \|\rho_{\varepsilon_k} * v_{0,l}\|_{H^m(\mathbb{R}^3)} \lesssim n_{k-1}^{m-s}, \quad (1.2.6)$$

$$\sum_{l=k+1}^{\infty} \|\rho_{\varepsilon_k} * v_{0,l}\|_{H^m(\mathbb{R}^3)} \lesssim n_k^m n_{k+1}^{-s} \left(\frac{n_k}{n_{k+1}} \right)^{\frac{3}{2}}. \quad (1.2.7)$$

Remark 1.2.5. Note that the estimate for the sum of the profiles concentrated at scale smaller than n_k is only valid when $\varepsilon \geq \varepsilon_k$. As a consequence, we cannot evidence the norm inflation along the whole sequence of regularized initial data $(\rho_\varepsilon * f_0)_{\varepsilon>0}$, but only along the subsequence $(\rho_{\varepsilon_k} * f_0)_{k \geq k_0}$, with $\varepsilon_k = (100n_k)^{-1}$. For this reason, we only prove (1.1.5) instead of

$$\lim_{T, \varepsilon \rightarrow 0} \|u^\varepsilon\|_{L_t^\infty([0,T]; H^s(\mathbb{R}^3))} = \infty. \quad (1.2.8)$$

The norm inflation mechanism was also exhibited along a subsequence in [BT08a; BGT05b; Xia21]. However, thanks to the finite propagation speed, [ST20a] proved that the norm inflation for energy-subcritical nonlinear wave equations occurs along the whole sequence of regularized data. Nevertheless, defining the pathological set with the convergence type (1.1.5) is enough for our purpose. Indeed, in the context of probabilistic well-posedness, the convergence of the regularized strong solutions to the local solution from a randomized initial data holds for any sequence $\varepsilon \rightarrow 0$. Therefore, the pathological set and the good set of randomized initial data are disjoint.

Proof. When $m < s$, the large bubbles (i.e. with small index l compared to k) contribute the most to the mass of the initial data.

$$\sum_{l=k_0}^{k-1} \|\rho_{\varepsilon_k} * v_{0,l}\|_{H^m} \lesssim \sum_{l=k_0}^{k-1} \kappa_{n_l} n_l^{m-s} \lesssim n_{k_0}^{m-s},$$

whereas

$$\sum_{l=k+1}^{\infty} \|\rho_{\varepsilon_k} * v_{0,l}\|_{H^m} \lesssim \sum_{l=k+1}^{\infty} \kappa_{n_l} n_l^{m-s} \left(\frac{n_k}{n_l}\right)^{\frac{3}{2}} \lesssim n_{k+1}^{m-s} \left(\frac{n_k}{n_{k+1}}\right)^{\frac{3}{2}}.$$

This proves (1.2.4) and (1.2.5), respectively. Next, we assume that $m > s$ and we establish (1.2.6) and (1.2.7). It follows from Lemma 1.2.1 that

$$\sum_{l=k_0}^{k-1} \|\rho_{\varepsilon_k} * v_{0,l}\|_{H^m} \lesssim \sum_{l=k_0}^{k-1} \kappa_{n_l} n_l^{m-s} \min\left(1, \frac{1}{\varepsilon_k n_l}\right)^{m+\frac{3}{2}} \lesssim n_k^{m-s},$$

Similarly,

$$\sum_{l=k+1}^{\infty} \kappa_{n_l} n_l^{m-s} \min\left(1, \frac{1}{\varepsilon_k n_l}\right)^{m+\frac{3}{2}} \lesssim \left(\frac{1}{\varepsilon_k}\right)^m \sum_{l=k+1}^{\infty} \kappa_{n_l} n_l^{-s} (\varepsilon_k n_l)^{-\frac{3}{2}} \lesssim n_k^m n_{k+1}^{-s} \left(\frac{n_k}{n_{k+1}}\right)^{\frac{3}{2}}.$$

□

1.2.3 Perturbative analysis

In this part, we fix $f_0 \in S$ and consider the solution u^{ε_k} to NLS with initial data $f_0 * \rho_{\varepsilon_k}$. We prove that in the scaling-supercritical regime $s < s_c$, and up to the time t_{n_k} , the nonlinear part of u^{ε_k} dominates the dispersion. Specifically, we prove that u^{ε_k} behaves like the ODE profile $v_k^{\varepsilon_k}$, which is at scale n_k . Therefore, u^{ε_k} experiences a norm inflation at time t_{n_k} as well. To do so, we perform a perturbative analysis by proving some a priori energy estimates on the difference between the ODE profile and u_k^{ε} .

As a byproduct of the analysis, we can deduce from the a priori energy estimates combined with a standard continuity argument that the regularized solution u^{ε_k} exists in $C([0, t_{n_k}]; H^2(\mathbb{R}^3))$.

Proposition 1.2.6 (Perturbative analysis). *Let $k \geq 1$, $n_k = e^{a^k}$, $\varepsilon_k := (100n_k)^{-1}$. When $s < s_c$, the local solution u^{ε_k} to NLS with initial data $f_0 * \rho_{\varepsilon_k}$ can be extended in $H^2(\mathbb{R}^3)$ up to time t_{n_k} . Moreover, there exists $C > 0$ such that for k large enough,*

$$\|u^{\varepsilon_k}(t_{n_k}) - v_n^{\varepsilon_k}(t_{n_k})\|_{H^s(\mathbb{R}^3)} \leq C. \quad (1.2.9)$$

Corollary 1.2.7 (Norm inflation). *For all $0 < s < s_c$ there exist $0 < c \ll 1$ and a suitable choice of the parameters β, γ , such that for any k large enough we have*

$$\|u^{\varepsilon_k}(t_{n_k})\|_{H^s} \geq c \log(n_k)^c.$$

Corollary 1.2.7, from which we deduce Theorem 1.1.4, follows from a priori estimate (1.2.9) in the small-dispersion analysis, combined with the lower bound on $v_n^{\varepsilon_k}$ from Lemma 1.2.1

$$\|u^{\varepsilon_k}(t_{n_k})\|_{H^s} \geq c \log(n)^{s(\beta-\gamma)(p-1)-\gamma} - C.$$

To prove Proposition 1.2.6, we perform a standard perturbative analysis inspired from [BGT05b] and [ST20a], using semi-classical energy estimates. The main difference is the presence of a linear correction,

$$u_L^{\varepsilon_k} := e^{it\Delta} u_0^{\varepsilon_k} + \sum_{l=k_0}^{k-1} e^{it\Delta} v_{0,l}^{\varepsilon_k}.$$

This correction ensures the smallness of the L^2 -norm of $w(0)$. The correction is quite harmless up to time t_{n_k} , since $u_L^{\varepsilon_k}$ concentrates at scale at most $(n_{k-1})^{-1}$, which is much larger than the scale $(n_k)^{-1}$ of the inflating bubble provided a is chosen large enough in the construction.

Remark 1.2.8. We remove the linear evolution of the large bubbles in $u_L^{\varepsilon_k}$, in order to have an initial upper bound for the mass of order at most n_k^{-s} initially. Indeed, as a consequence of Lemma 1.2.4 we may have

$$\left\| \sum_{l=k_0}^{k-1} v_{0,l}^{\varepsilon_k} \right\|_{L^2} \sim 1.$$

Removing this contribution would deteriorate a bit the estimates on the Sobolev norms H^m of $u_L^{\varepsilon_k}$, that will be of size n_{k-1}^m instead of C . Nevertheless, since we run the argument up to time $t_{n_k} \ll n_{k-1}^m$ we shall be able to close the estimates as explained above.

Remark 1.2.9. In the case of the wave equation [ST20a], the norm inflation occurs much later, at time $t_{n_k}(NLW) \sim \lambda_{n_k}^{-1}$, since it occurs at time $t_{n_k}(NLS) \sim \lambda_{n_k}^{-2}$ for NLS, respectively. Consequently, a lower bound on s is needed in [ST20a] to control the growth of some source terms (containing $u_L^{\varepsilon_k}$) up to time $t_{n_k}(NLW) \gg t_{n_k}(NLS)$. Nevertheless, it should be possible to remove this condition by using a refined nonlinear correction instead of $u_L^{\varepsilon_k}$ in the perturbative analysis.

Proof of Proposition 1.2.6. By using the local Cauchy theory for NLS in H^σ for $\sigma > \frac{3}{2}$, we see that a lower bound on the time of existence of u_k^ε in say H^2 follows from an a priori bound on its H^2 -norm. We deduce such an a priori bound from energy estimates. To do so, we write

$$w := u^{\varepsilon_k} - u_L^{\varepsilon_k} - v_n^{\varepsilon_k}.$$

Using the definition of the linear correction $u_L^{\varepsilon_k}$, we see that initially,

$$w(0) = \sum_{l=k+1}^{\infty} v_{0,l} * \rho_{\varepsilon_k}. \quad (1.2.10)$$

We shorten the notation by denoting $n := n_k$, and we consider the following semi-classical energy

$$E_n(t) = \left(n^{2s} \|w\|_{L^2}^2 + n^{2(s-2)} \|w\|_{H^2}^2 \right)^{\frac{1}{2}}.$$

From the Gagliardo-Nirenberg inequality and the interpolation, we have that for $\sigma \in [0, 2]$,

$$\|w(t)\|_{L^\infty} \lesssim n^{\frac{3}{2}-s} E_n(t), \quad \|w(t)\|_{H^\sigma} \lesssim n^{\sigma-s} E_n(t). \quad (1.2.11)$$

In particular, $\|w(t)\|_{H^s} \lesssim E_n(t)$. Moreover, the H^s -norm of the linear correction is bounded uniformly in n since

$$\|u_L^{\varepsilon_k}\|_{L_t^\infty H_x^s} \leq \|u_0^{\varepsilon_k}\|_{H^s} + \sum_{l=k_0}^{k-1} \|v_{0,l}^{\varepsilon_k}\|_{L^2}^{1-s} \|v_{0,l}^{\varepsilon_k}\|_{H^1}^s \lesssim \|u_0\|_{H^s} + \sum_{k=k_0}^{k-1} \kappa_{n_l} \lesssim 1. \quad (1.2.12)$$

For these reasons, inequality (1.2.9) can be reduced to prove the a priori energy bound

$$\sup_{0 \leq t \leq t_n} E_n(t) \leq 1.$$

Initially, we have from (1.2.5) and (1.2.7) that

$$\|w(0)\|_{L^2} \leq C n_{k+1}^{-s} \left(\frac{n}{n_{k+1}}\right)^{\frac{3}{2}}, \quad \|w(0)\|_{H^2} \leq C n^2 n_{k+1}^{-s} \left(\frac{n}{n_{k+1}}\right)^{\frac{3}{2}},$$

so that

$$E_n(0)^2 = n^{2s} \|w(0)\|_{L^2}^2 + n^{-2(s-2)} \|\Delta w(0)\|_{L^2}^2 \leq 2C \left(\frac{n}{n_{k+1}}\right)^{3+2s} = C n^{-c_0}, \quad (1.2.13)$$

with $c_0 := (a-1)(3+2s)$. Now, we run a bootstrap argument to prove that $E_n(t) < 1$ for all $t \leq t_n$. For shortness of notation, we drop the complex conjugation sign. By construction, w is solution to the perturbed Schrödinger equation

$$(i\partial_t + \Delta)w = |w + u_L^{\varepsilon_k} + v_n^{\varepsilon_k}|^{p-1} (w + u_L^{\varepsilon_k} + v_n^{\varepsilon_k}) - |v_n^{\varepsilon_k}|^{p-1} v_n^{\varepsilon_k} + \Delta v_n^{\varepsilon_k} =: F + \Delta v_n^{\varepsilon_k},$$

with initial condition $w(0)$ given by (1.2.10), and where the forcing term is

$$F = (w + u_L^{\varepsilon_k}) O(|w|^{p-1} + |u_L^{\varepsilon_k}|^{p-1} + |v_n^{\varepsilon_k}|^{p-1}).$$

Denoting $\Lambda := \Delta F$, we have

$$\begin{aligned} |\Lambda| &\lesssim (|\Delta w| + |\Delta u_L^{\varepsilon_k}|) (|w|^{p-1} + |u_L^{\varepsilon_k}|^{p-1} + |v_n^{\varepsilon_k}|^{p-1}) \\ &\quad + (|\nabla w| + |\nabla u_L^{\varepsilon_k}|)^2 (|w|^{p-2} + |u_L^{\varepsilon_k}|^{p-2} + |v_n^{\varepsilon_k}|^{p-2}) \\ &\quad + (|\nabla w| + |\nabla u_L^{\varepsilon_k}|) |\nabla v_n^{\varepsilon_k}| (|w|^{p-2} + |u_L^{\varepsilon_k}|^{p-2} + |v_n^{\varepsilon_k}|^{p-2}) \\ &\quad + (|w| + |u_L^{\varepsilon_k}|) |\nabla^2 v_n^{\varepsilon_k}| (|w|^{p-2} + |u_L^{\varepsilon_k}|^{p-2} + |v_n^{\varepsilon_k}|^{p-2}) \\ &\quad + (|w| + |u_L^{\varepsilon_k}|) |\nabla v_n^{\varepsilon_k}|^2 (|w|^{p-3} + |u_L^{\varepsilon_k}|^{p-3} + |v_n^{\varepsilon_k}|^{p-3}). \end{aligned}$$

First, we control the source term coming from derivatives of $\Delta v_n^{\varepsilon_k}$. We observe that $2 < (\frac{3}{2} - s)(p-1)$ when $s < s_c$.⁷ Hence, we deduce from Lemma 1.2.1 that

$$\begin{aligned} n^s \|\Delta v_n^{\varepsilon_k}\|_{L_{t_n}^\infty L_x^2} + n^{s-2} \|\Delta^2 v_n^{\varepsilon_k}\|_{L_{t_n}^\infty L_x^2} &\leq C (t\lambda_n^2)^4 n^2 \leq C \log(n)^{4(\beta-\gamma)(p-1)} n^2 \\ &\leq C n^{(\frac{3}{2}-s)(p-1)-\delta_0}, \end{aligned} \quad (1.2.14)$$

with, say,

$$\delta_0 := \frac{1}{2} \left(\left(\frac{3}{2} - s \right) (p-1) - 2 \right) > 0.$$

Now, we control the source terms coming from $u_L^{\varepsilon_k}$ and $v_n^{\varepsilon_k}$ in the expression of F and ΔF . First, notice from (1.2.6) and from the H^s -bound (1.2.12) that for any $m \geq s$, there exists $C_m > 0$ such that

$$\|u_L^{\varepsilon_k}\|_{L_t^\infty H_x^m} \leq C_m n_{k-1}^{m-s}, \quad (1.2.15)$$

whereas when $m < s$ we can only use that thanks to (1.2.12), we have $\|u_L^{\varepsilon_k}\|_{L_t^\infty H_x^m} \leq C$. Using the Gagliardo-Nirenberg inequality with $s < \frac{3}{2}$, we deduce that

$$\|u_L^{\varepsilon_k}\|_{L_t^\infty L_x^\infty} \lesssim (\|u_L^{\varepsilon_k}\|_{L_t^\infty H_x^s})^{\frac{1}{2(2-s)}} (\|u_L^{\varepsilon_k}\|_{L_t^\infty H_x^2})^{\frac{3-2s}{2(2-s)}} \lesssim n_{k-1}^{\frac{3-s}{2}}.$$

⁷This seems to be the only place where we see the condition $s < s_c$.

In order to prevent another unnecessary loss later on, we also observe from the Sobolev embedding $\dot{H}^{\frac{3}{4}} \hookrightarrow L^4$ that

$$\|\nabla|u_L^{\varepsilon_k}\|_{L_t^\infty L_x^4} \lesssim \|u_L^{\varepsilon_k}\|_{L_t^\infty H_x^{1+\frac{3}{4}}},$$

and since $1 + \frac{3}{4} > \frac{3}{2} > s_c$, we deduce

$$\|\nabla|u_L^{\varepsilon_k}\|_{L_t^\infty L_x^4} \lesssim n_{k-1}^{1+\frac{3}{4}-s}.$$

Using the estimates on the profile v from Lemma 1.2.1, we can therefore control the source terms $u_L^{\varepsilon_k}$ and v up to time $t_n \sim \lambda_n^{-2}$, uniformly in n and estimate

$$\begin{aligned} n^s \|F(t)\|_{L^2} &\lesssim n^s \|w\|_{L^2} (\|w\|_{L^\infty}^{p-1} + \|u_L^{\varepsilon_k}\|_{L^\infty}^{p-1} + \|v_n^{\varepsilon_k}\|_{L^\infty}^{p-1}) + \\ &\quad n^s \|u_L^{\varepsilon_k}\|_{L^\infty} (\|w\|_{L^2} \|w\|_{L^\infty}^{p-2} + \|u_L^{\varepsilon_k}\|_{L^2} \|u_L^{\varepsilon_k}\|_{L^\infty}^{p-2} + \|v_n^{\varepsilon_k}\|_{L^2} \|v_n^{\varepsilon_k}\|_{L^\infty}^{p-2}). \end{aligned}$$

As a consequence,

$$\begin{aligned} n^s \|F(t)\|_{L^2} &\lesssim E_n(t) (E_n(t)^{p-1} n^{(\frac{3}{2}-s)(p-1)} + n_{k-1}^{(\frac{3}{2}-s)(p-1)} + n^{(\frac{3}{2}-s)(p-1)}) \\ &\quad + n^s n_{k-1}^{\frac{3}{2}-s} (n^{-s} E_n(t) n^{(\frac{3}{2}-s)(p-2)} + n_{k-1}^{(\frac{3}{2}-s)(p-2)} + n^{-s} n^{(\frac{3}{2}-s)(p-2)}). \end{aligned}$$

Supposing that $E_n(t) \ll 1$, which is the case at time $t = 0$, this leads to the estimate

$$n^s \|F(t)\|_{L^2} \lesssim E_n(t) n^{(\frac{3}{2}-s)(p-1)} + n^{(\frac{3}{2}-s)(p-1)} \left(\left(\frac{n_{k-1}}{n} \right)^{\frac{3}{2}-s} + n^s \left(\frac{n_{k-1}}{n} \right)^{(\frac{3}{2}-s)(p-1)} \right).$$

We recall that we choose $n = n_k = e^{a^k}$ for some $a \gg 1$, so that $\frac{n_{k-1}}{n} = n^{-\theta}$ with $\theta = 1 - \frac{1}{a}$. Therefore, there holds

$$n^s \|F(t)\|_{L^2} \lesssim E_n(t) n^{(\frac{3}{2}-s)(p-1)} + n^{(\frac{3}{2}-s)(p-1)} (n^{-\theta(\frac{3}{2}-s)} + n^{s-\theta(\frac{3}{2}-s)(p-1)}).$$

We want to ensure that

$$s < \theta \left(\frac{3}{2} - s \right) (p-1), \quad (1.2.16)$$

so that there exist $C > 0$ and $c > 0$ such that

$$n^s \|F(t)\|_{L^2} \leq C E_n(t) n^{(\frac{3}{2}-s)(p-1)} + n^{(\frac{3}{2}-s)(p-1)-c}. \quad (1.2.17)$$

Since $s < s_c = \frac{3}{2} - \frac{2}{p-1}$, the condition (1.2.16) reduces to

$$s_c < \theta \left(\frac{3}{2} - s_c \right) (p-1),$$

which is satisfied as soon as θ is sufficiently close to 1, that is when a is chosen sufficiently large so that

$$\frac{3}{4} < 1 - \frac{1}{a}.$$

Similarly, we estimate the increment of the H^2 -norm as in [BGT05b], controlling the terms involving $v_n^{\varepsilon_k}, \nabla v_n^{\varepsilon_k}, \Delta v_n^{\varepsilon_k}$ or $u_L^{\varepsilon_k}, \nabla u_L^{\varepsilon_k}, \Delta u_L^{\varepsilon_k}$ in L^∞ :

$$\begin{aligned}
\|\Lambda(t)\|_{L^2} &\lesssim \|\Delta w\|_{L^2} (\|u_L^{\varepsilon_k}\|_{L^\infty}^{p-1} + \|w\|_{L^\infty}^{p-1} + \|v_n^{\varepsilon_k}\|_{L^\infty}^{p-1}) \\
&\quad + \|\Delta u_L^{\varepsilon_k}\|_{L^\infty} (\|w\|_{L^2} \|w\|_{L^\infty}^{p-2} + \|v_n^{\varepsilon_k}\|_{L^2} \|v_n^{\varepsilon_k}\|_{L^\infty}^{p-2}) + \|\Delta u_L^{\varepsilon_k}\|_{L^2} \|u_L^{\varepsilon_k}\|_{L^\infty}^{p-1} \\
&\quad + \|\nabla|w|\|_{L^4}^2 (\|u_L^{\varepsilon_k}\|_{L^\infty}^{p-2} + \|w\|_{L^\infty}^{p-2} + \|v_n^{\varepsilon_k}\|_{L^\infty}^{p-2}) \\
&\quad + \|\nabla|u_L^{\varepsilon_k}|\|_{L^\infty}^2 (\|w\|_{L^2} \|w\|_{L^\infty}^{p-3} + \|v_n^{\varepsilon_k}\|_{L^2} \|v_n^{\varepsilon_k}\|_{L^\infty}^{p-3}) + \|\nabla|u_L^{\varepsilon_k}|\|_{L^4}^2 \|u_L^{\varepsilon_k}\|_{L^\infty}^{p-2} \\
&\quad + (\|\nabla|w|\|_{L^2} \|\nabla|v_n^{\varepsilon_k}|\|_{L^\infty} + \|\nabla|u_L^{\varepsilon_k}|\|_{L^\infty} \|\nabla|v_n^{\varepsilon_k}|\|_{L^2}) (\|u_L^{\varepsilon_k}\|_{L^\infty}^{p-2} + \|w\|_{L^\infty}^{p-2} + \|v_n^{\varepsilon_k}\|_{L^\infty}^{p-2}) \\
&\quad + (\|w\|_{L^2} \|\Delta v_n^{\varepsilon_k}\|_{L^\infty} + \|u_L^{\varepsilon_k}\|_{L^\infty} \|\Delta v_n^{\varepsilon_k}\|_{L^2}) (\|u_L^{\varepsilon_k}\|_{L^\infty}^{p-2} + \|w\|_{L^\infty}^{p-2} + \|v_n^{\varepsilon_k}\|_{L^\infty}^{p-2}) \\
&\quad + (\|w\|_{L^2} \|\nabla|v_n^{\varepsilon_k}|\|_{L^\infty}^2 + \|u_L^{\varepsilon_k}\|_{L^\infty} \|\nabla|v_n^{\varepsilon_k}|\|_{L^2} \|\nabla|v_n^{\varepsilon_k}|\|_{L^\infty}) (\|u_L^{\varepsilon_k}\|_{L^\infty}^{p-3} + \|w\|_{L^\infty}^{p-3} + \|v_n^{\varepsilon_k}\|_{L^\infty}^{p-3}).
\end{aligned}$$

In addition to inequality (1.2.11), we shall use the Sobolev embedding $\dot{H}^{\frac{3}{4}} \hookrightarrow L^4$ and interpolation to get

$$\|\nabla|w(t)|\|_{L^4} \leq n^{1+\frac{3}{4}-s} E_n(t).$$

Using the bounds on $v_n^{\varepsilon_k}$ and on $u_L^{\varepsilon_k}$, we obtain

$$\begin{aligned}
n^{s-2} \|\Lambda(t)\|_{L^2} &\lesssim E_n(t) (n_{k-1}^{(\frac{3}{2}-s)(p-1)} + n^{(\frac{3}{2}-s)(p-1)} (E_n(t)^{p-1} + \kappa_n^{p-1})) \\
&\quad + n_{k-1}^{2+\frac{3}{2}-s} n^{-2+(\frac{3}{2}-s)(p-2)} (E_n(t)^{p-1} + \kappa_n^{p-1}) + n^{s-2} n_{k-1}^{2-s+(\frac{3}{2}-s)(p-1)} \\
&\quad + n_{k-1}^{\frac{3}{2}-s} E_n(t)^2 (n_{k-1}^{(\frac{3}{2}-s)(p-2)} + n^{(\frac{3}{2}-s)(p-2)} (E_n(t)^{p-2} + \kappa_n^{p-2})) \\
&\quad + n_{k-1}^{2+3-2s} n^{-2+(\frac{3}{2}-s)(p-3)} (E_n(t)^{p-3} + \kappa_n^{p-3}) + n^{s-2} n_{k-1}^{2+\frac{3}{2}-2s+(\frac{3}{2}-s)(p-2)} \\
&\quad + \kappa_n \left(n_{k-1}^{\frac{3}{2}-s} E_n(t) + n_{k-1}^{1+\frac{3}{2}-s} n^{-1} \right) (n_{k-1}^{(\frac{3}{2}-s)(p-2)} + n^{(\frac{3}{2}-s)(p-2)} (E_n(t)^{p-2} + \kappa_n^{p-2})) \\
&\quad + \kappa_n \left(n_{k-1}^{\frac{3}{2}-s} E_n(t) + n_{k-1}^{\frac{3}{2}-s} \right) (n_{k-1}^{(\frac{3}{2}-s)(p-2)} + n^{(\frac{3}{2}-s)(p-2)} (E_n(t)^{p-2} + \kappa_n^{p-2})) \\
&\quad + \kappa_n^2 \left(n_{k-1}^{2(\frac{3}{2}-s)} E_n(t) + n_{k-1}^{\frac{3}{2}-s} n_{k-1}^{\frac{3}{2}-s} \right) (n_{k-1}^{(\frac{3}{2}-s)(p-3)} + n^{(\frac{3}{2}-s)(p-3)} (E_n(t)^{p-3} + \kappa_n^{p-3})).
\end{aligned}$$

Once again, we assume that $E_n(t) \ll 1$. Noting that $\kappa_n \leq 1$ and that $n_{k-1} \leq n$, we use the bounds

$$n_{k-1}^{(\frac{3}{2}-s)(p-i)} + n^{(\frac{3}{2}-s)(p-i)} (E_n(t)^{p-i} + \kappa_n^{p-i}) \lesssim \max(1, n^{(\frac{3}{2}-s)(p-i)}), \quad i \in \{1, 2, 3, \dots\}.$$

We then simplify the above expression by writing

$$\begin{aligned}
n^{s-2} \|\Lambda(t)\|_{L^2} &\lesssim E_n(t) n^{(\frac{3}{2}-s)(p-1)} \\
&\quad + n_{k-1}^{2+\frac{3}{2}-s} n^{-2+(\frac{3}{2}-s)(p-2)} + n^{s-2} n_{k-1}^{2-s+(\frac{3}{2}-s)(p-1)} \\
&\quad + n_{k-1}^{2+3-2s} n^{-2+(\frac{3}{2}-s)(p-3)} + n^{s-2} n_{k-1}^{2-s+(\frac{3}{2}-s)(p-1)} \\
&\quad + n_{k-1}^{1+\frac{3}{2}-s} n^{-1+(\frac{3}{2}-s)(p-2)} \\
&\quad + n_{k-1}^{\frac{3}{2}-s} n^{(\frac{3}{2}-s)(p-2)}.
\end{aligned}$$

The cross-terms between n_{k-1} and n are equal to

$$n^{(\frac{3}{2}-s)(p-1)} \left(\left(\frac{n_{k-1}}{n} \right)^{2+\frac{3}{2}-s} + \left(\frac{n_{k-1}}{n} \right)^{2-s} + \left(\frac{n_{k-1}}{n} \right)^{2+3-2s} + \left(\frac{n_{k-1}}{n} \right)^{1+\frac{3}{2}-s} + \left(\frac{n_{k-1}}{n} \right)^{\frac{3}{2}-s} \right).$$

Since $\frac{n_{k-1}}{n} = n^{-\theta}$, there exist $C, c > 0$ such that

$$n^{s-2} \|\Lambda(t)\|_{L^2} \leq C E_n(t) n^{(\frac{3}{2}-s)(p-1)} + n^{(\frac{3}{2}-s)(p-1)-c\theta}. \quad (1.2.18)$$

Combining (1.2.14), (1.2.17) and (1.2.18) with the energy estimate

$$\frac{d}{dt} (E_n(t)^2) \leq 2E_n(t) (n^s \|F(t)\|_{L^2} + n^{s-2} \|\Lambda(t)\|_{L^2}) + n^s \|\Delta v_n^{\varepsilon_k}\|_{L_{t_n}^\infty L_x^2} + n^{s-2} \|\Delta^2 v\|_{L_{t_n}^\infty L_x^2},$$

we get that there exist $C, c > 0$ such that

$$\frac{d}{dt} E_n(t)^2 \leq C E_n(t)^2 n^{(\frac{3}{2}-s)(p-1)} + C n^{(\frac{3}{2}-s)(p-1)-c} = C \lambda_n^2 E_n^2 + C \lambda_n^2 n^{-2c}.$$

It follows from the Gronwall estimate that

$$E_n(t)^2 \leq (E_n(0)^2 + C n^{-2c} t_n \lambda_n^2) \exp(C t_n \lambda_n^2) \leq (E_n(0)^2 + C n^{-c}) \exp(C \log(n)^{(\beta-\gamma)(p-1)}).$$

We recall that from (1.2.13), we have $E_n(0)^2 \leq C n^{-c_0}$ for some $c_0 > 0$. Finally, we fix parameters β, γ small enough such that $(p-1)\beta < \frac{1}{2}$ and $\gamma < \frac{1}{2}\beta$. Then, we fix $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, for some constant $C > 0$ encountered in the analysis, we have

$$C \log(n)^{(\beta-\gamma)(p-1)} \leq \frac{\delta}{2} \log(n) + C(\delta), \quad \delta := \min(c_0, c) > 0,$$

so that

$$E_n(t)^2 \lesssim_\delta n^{-\delta} n^{\frac{\delta}{2}} \lesssim_\delta n^{-\frac{\delta}{2}} \ll 1.$$

We conclude by a standard bootstrap argument. \square

We proved that $S \subset \mathcal{P}$. Finally, we show that S is dense in H^s . Then, by using this inclusion, we deduce that \mathcal{P} contains a dense G_δ set as soon as NLS is globally well-posed in H^2 .

Proposition 1.2.10 (Density). *For $g \in H^2(\mathbb{R}^3)$, we denote $\Phi(t)g$ the maximal solution to NLS in $C([0, T^*]; H^2(\mathbb{R}^3))$ with initial data g . The pathological set \mathcal{P} defined in (1.2.3) is dense in $H^s(\mathbb{R}^3)$. Moreover, in the defocusing energy-subcritical and critical cases $p \in \{3, 5\}$, where the Cauchy problem associated with NLS is globally well-posed in $H^2(\mathbb{R}^3)$, \mathcal{P} contains a dense G_δ subset of $H^s(\mathbb{R}^3)$.*

Proof. By construction, we know that

$$\sum_{k=k_0}^{\infty} \|v_{0,k}\|_{H^s} \lesssim \sum_{k=k_0}^{\infty} \kappa_{n_k} \xrightarrow{k_0 \rightarrow \infty} 0,$$

therefore S is dense in $H^s(\mathbb{R}^3)$.

We now assume that global well-posedness in $H^2(\mathbb{R}^3)$ holds, and we follow the proof of Corollary 1.2 in [ST20a]. Specifically, we prove that \mathcal{P} contains a countable intersection of nonempty open sets. Let us consider the sequences (ε_k) and (t_k) going to zero as in (1.2.1), and define

$$\mathcal{O}_{N,k} := \{f \in H^s(\mathbb{R}^3) \mid \|\Phi(t_{n_k})(\rho_{\varepsilon_k} * f)\|_{H^s} > N\}.$$

By definition of the lim sup, we have

$$\bigcap_{N=1}^{\infty} \bigcap_{k=k_1}^{\infty} \bigcup_{l=k_1}^{\infty} \mathcal{O}_{N,k} \subset S.$$

We already know that $\bigcup_{k=k_1}^{\infty} \mathcal{O}_{N,k}$ is nonempty since it contains for instance the lollipop $\sum_{l=k_1}^{\infty} v_{0,l}$. Therefore, it remains to show that for every N and k , $\mathcal{O}_{N,k}$ is an open set.

Let $f_0 \in \mathcal{O}_{N,k}$, $t \leq t_{n_k}$ and $\delta > 0$ to be chosen later. Using the Duhamel formula and the Cauchy theory in H^2 , we know that there exists $C_0 > 0$ such that for every $f \in H^s$,

$$\|\Phi(t)(\rho_{\varepsilon_k} * f) - \Phi(t)(\rho_{\varepsilon_k} * f_0)\|_{H^2} \leq C_0(\|\rho_{\varepsilon_k} * f\|_{H^2}^{p-1} + \|\rho_{\varepsilon_k} * f_0\|_{H^2}^{p-1})\|\rho_{\varepsilon_k} * (f - f_0)\|_{H^2}.$$

Using the inequality

$$\|\rho_{\varepsilon} * f\|_{H^2} \lesssim \varepsilon^{-(2-s)}\|f\|_{H^s},$$

we deduce that for any $f \in H^s$,

$$\|\Phi(t)(\rho_{\varepsilon_k} * f) - \Phi(t)(\rho_{\varepsilon_k} * f_0)\|_{H^2} \leq C_0 \varepsilon_k^{-(2-s)p}(\|f\|_{H^s}^{p-1} + \|f_0\|_{H^s}^{p-1})\|f - f_0\|_{H^s}.$$

Since $\|\Phi(t)(\rho_{\varepsilon_k} * f_0)\|_{H^s} > N$, we deduce that when f is at distance less than δ from f_0 in H^s , then $\|\Phi(t)(\rho_{\varepsilon_k} * f)\|_{H^s} > N$ for some suitable $\delta > 0$, so that $\mathcal{O}_{N,k}$ is open. \square

1.3 Generic well-posedness

This section is devoted to the proof of Theorem 1.1.3 for the cubic Schrödinger equation, at scaling-supercritical regularity $H^s(\mathbb{R}^3)$, i.e. with $\frac{1}{4} < s < \frac{1}{2}$.⁸ The technics shall be more specific to the Schrödinger semigroup and its dispersive properties. Namely, we make use of Strichartz and bilinear estimates in the framework of Fourier restriction spaces, that we recall in parts 4.2 and 1.3.2. We then adapt the proof of the almost sure local well-posedness result from [BOP15b] to establish the convergence of the regularized solutions to the solution obtained from the random initial data regularized by convolution, as claimed in Theorem 1.1.4.

In what follows, we fix s a scaling-supercritical exponent such that $\frac{1}{4} < s < \frac{1}{2}$, and a scaling-subcritical (or critical) Sobolev space H^σ , with σ satisfying

$$s_c = \frac{1}{2} < \sigma < 2s.$$

⁸The lower bound on s stems from the contributions of the *high-high-low* and of the *high-low-low* interactions in the first Duhamel iteration. Besides, by refining the recentering method in [BOP19a], the regularity threshold was lowered to $\frac{1}{5} < s$. In [DLM19], a different approach based on local smoothing in lateral spaces improves the regularity threshold in dimension 4, and can certainly be adapted in dimension 3. Note also that s needs to be strictly greater than the threshold because of the summation over dyadic blocks.

1.3.1 Preliminaries

We first recall the basic dispersive estimate for the free Schrödinger evolution.

Lemma 1.3.1 (Strichartz estimates). *Given $2 \leq p, q \leq \infty$ satisfying the admissibility condition*

$$\frac{2}{p} + \frac{3}{q} = \frac{3}{2}, \quad (1.3.1)$$

there exists $C > 0$ such that for all $u_0 \in L^2(\mathbb{R}^3)$, we have

$$\|e^{it\Delta} u_0\|_{L_t^p(\mathbb{R}; L_x^q(\mathbb{R}^3))} \leq C \|u_0\|_{L_x^2(\mathbb{R}^3)}.$$

We emphasize that the admissibility condition is imposed by the scaling invariance. A key feature of the randomization procedure is to rule out the scaling invariance, and to relax the admissibility condition, as written in Lemma 1.3.4. The following bilinear estimate due to Bourgain indicates a nonlinear smoothing effect when high and low frequencies interact. Combined with the probabilistic Strichartz estimates, it is the main mechanism responsible for the smoothing effect observed in the Picard iteration from a randomized initial data.

Lemma 1.3.2 (Bilinear estimate [Bou98a]). *Given $N \ll M$ two dyadic integers and $u_0, v_0 \in L^2(\mathbb{R}^3)$ such that*

$$\text{supp } \mathcal{F}(u_0) \subset \{\xi \in \mathbb{R}^3 \mid |\xi| \lesssim N\}, \quad \text{supp } \mathcal{F}(v_0) \subset \{\xi \in \mathbb{R}^3 \mid \frac{1}{2}M \leq |\xi| \leq 2M\}.$$

We have

$$\|(e^{it\Delta} u_0)(e^{it\Delta} v_0)\|_{L_{t,x}^2(\mathbb{R} \times \mathbb{R}^3)} \lesssim NM^{-\frac{1}{2}} \|u_0\|_{L_x^2(\mathbb{R}^3)} \|v_0\|_{L_x^2(\mathbb{R}^3)}.$$

Let us now set up the relevant randomization procedure in the Euclidean space, and recall the key probabilistic Strichartz estimates obtained in this context.

Wiener randomization The *Wiener decomposition* of a function in $f_0 \in L^2$ is defined from a unit-scale partition of unity of the frequency space given by a function $\psi \in C_c^\infty(\mathbb{R}^3)$ supported in $[-2, 2]^3$, such that

$$\sum_{k \in \mathbb{Z}^3} \psi(\xi - k) = 1.$$

Defining the corresponding unit-scale Fourier multiplier $P_{1,k} = \mathcal{F}^{-1}(\psi(\cdot - k)\mathcal{F})$ on the unit cube centered around k , we decompose any function in $L^2(\mathbb{R}^3)$ into a sum of unit-blocks

$$f_0 = \sum_{k \in \mathbb{Z}^3} P_{1,k} f_0.$$

The randomization procedure consists in decoupling these individual blocks by some independent Gaussian variables. Specifically, given a function⁹ $f_0 \in H^s(\mathbb{R}^3)$, a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and

⁹In order to have a reasonable measure, we ask for some non-degeneracy conditions on f_0 . Namely, we want to ensure that $\text{supp } \mu$ is dense in H^s , and that $\mu(H^\sigma \setminus H^s) = 0$ for all $\sigma > s$. To do so, it is sufficient to assume that $\|P_{1,k} f_0\|_{L^2} > 0$ for all $k \in \mathbb{R}^3$. We refer to Lemma B.1 in [BT08a] and [BOP19a] for further details.

some independent complex Gaussian variables $(g_k)_{k \in \mathbb{Z}^3}$, we define a probability measure μ supported on $H^s(\mathbb{R}^3)$ induced by the random variable

$$\omega \in \Omega \longmapsto f_0^\omega := \sum_{k \in \mathbb{Z}^3} g_k(\omega) P_{1,k} f_0 \in H^s(\mathbb{R}^3) \quad \text{a.s.}$$

This randomization procedure is called the *Wiener randomization*.

Strichartz estimates The main idea is to combine the unit scale Bernstein estimate with probabilistic decoupling in order to improve the integrability bounds and to relax the Strichartz admissibility condition, for a large measure set of initial data.

Definition 1.3.3. Let $M \gg 1$ be a parameter. We say that an event $A = A(M)$ occurs M -certainly when there exist some positive constants $C, c, \theta > 0$ such that

$$\mathbb{P}(A(M)) \geq 1 - C \exp(-cM^\theta).$$

Lemma 1.3.4 (Probabilistic Strichartz estimates [BOP15b], Proposition 1.4). *Fix a Strichartz admissible pair (q, r) , with $2 \leq q, r < \infty$, satisfying the Strichartz admissibility condition (1.3.1). For any \tilde{r} such that $r \leq \tilde{r} < \infty$, there exist $C, c > 0$ such that for all $\lambda > 0$,*

$$\mathbb{P}(\{\omega \mid \|\langle \nabla \rangle^s e^{it\Delta} f_0^\omega\|_{L_t^q(\mathbb{R}; L_x^{\tilde{r}}(\mathbb{R}^3))} > \lambda\}) \leq C \exp(-c\lambda^2 \|f_0\|_{H^s}^{-2}). \quad (1.3.2)$$

As a consequence, for any $\lambda > 0$, the following Strichartz estimate holds λ -certainly

$$\|\langle \nabla \rangle^s e^{it\Delta} f_0^\omega\|_{L_t^q(\mathbb{R}; L_x^{\tilde{r}}(\mathbb{R}^3))} \leq \lambda \|f_0\|_{H^s}, \quad (1.3.3)$$

so that almost surely,

$$\|\langle \nabla \rangle^s e^{it\Delta} f_0^\omega\|_{L_t^q(\mathbb{R}; L_x^{\tilde{r}}(\mathbb{R}^3))} < \infty.$$

1.3.2 Probabilistic local well-posedness

With the above refined probabilistic Strichartz estimates at hand, one can observe some nonlinear smoothing effect. Specifically, starting from an initial data $f_0^\omega \in H^s$ that satisfies (1.3.3), and by performing a *linear-nonlinear* decomposition through the ansatz

$$u(t) = e^{it\Delta} f_0^\omega + v,$$

we run a contraction mapping argument for v in H^σ with $\frac{1}{2} < \sigma < 2s$, where v is solution to a cubic Schrödinger equation with stochastic forcing terms depending on $f(t) = e^{it\Delta} f_0^\omega$, and zero initial condition:

$$\begin{cases} (i\partial_t + \Delta)v = |v + f|^2(v + f), \\ v(0) = 0. \end{cases} \quad (\text{NLS}_f)$$

We use the following *refined Strichartz* norm for the linear evolution of the initial data

$$\|f\|_{\tilde{S}^s} := \|\langle \nabla \rangle^s f\|_{L_{t,x}^4(\mathbb{R} \times \mathbb{R}^3)} + \|\langle \nabla \rangle^s f\|_{L_{t,x}^5(\mathbb{R} \times \mathbb{R}^3)}.$$

It follows from Lemma 1.3.4 that for all $\lambda > 0$, we have λ -certainly that

$$\|e^{it\Delta} f_0^\omega\|_{\tilde{S}^s} \leq \lambda \|f_0\|_{H^s},$$

As a consequence, the set

$$\Sigma := \{f_0^\omega \mid \|e^{it\Delta} f_0^\omega\|_{\tilde{S}^s} < \infty\}$$

has full μ -measure. To control the source term in (NLS_f) we set

$$Y^s := L_t^\infty(\mathbb{R}; H^s) \cap \tilde{S}^s.$$

The local existence result due to Bényi, Oh and Pocovnicu for NLS with initial condition $f_0^\omega \in \Sigma$, that we now recall, states as follows.

Theorem 1.3.5 (Almost sure local well-posedness, see [BOP15b; BOP15a]). *For all $f_0^\omega \in \Sigma$, there exists $T = T(\|e^{it\Delta} f_0^\omega\|_{Y^s}) > 0$ such that the Cauchy problem NLS with initial data f_0^ω has a unique solution*

$$u \in e^{it\Delta} f_0^\omega + X_T^{\sigma,b} \hookrightarrow L_t^\infty([0, T]; H^s),$$

for some $b = \frac{1}{2}^+$. Moreover, $u \in C_t([0, T]; H^s)$.

Before sketching the proof, let us comment on the *Bourgain recentering method* used to obtain such a result. The idea is to reduce the Cauchy problem NLS with initial data f_0^ω to the Cauchy problem (NLS_f), with a forcing term $f = e^{it\Delta} f_0^\omega$ and with zero initial condition, at scaling-subcritical regularity $H^\sigma(\mathbb{R}^3)$. This is the reason why we have uniqueness for u in an affine subspace of $L_t^\infty H_x^s(\mathbb{R}^3)$. This strategy generates strong solutions for initial data in Σ , in the sense that the flow map is continuous from the time interval of existence to a functional space embedded in H^s . Then, the contraction mapping argument follows from trilinear estimates in Fourier restriction spaces, required to handle the bilinear estimates from Lemma 1.3.2. They are associated to the norm

$$\|f\|_{X^{\sigma,b}} = \|\langle \xi \rangle^\sigma \langle \tau + |\xi|^2 \rangle^b \widehat{f}(\tau, \xi)\|_{L_{\tau, \xi}^2(\mathbb{R} \times \mathbb{R}^3)}, \quad \|f\|_{X_T^{\sigma,b}} = \inf \left\{ \|F\|_{X^{\sigma,b}} \mid F \in X^{\sigma,b}, F|_{[0, T]} = f \right\}.$$

Combining the bilinear estimate from Lemma 1.3.2 with the probabilistic Strichartz estimate from lemma 1.3.4, we obtain trilinear estimates based on a Littlewood-Paley analysis. Before proceeding further, let us simplify the notation. We write the cubic nonlinearity as a trilinear operator, acting on functions $\{u_i\}_{1 \leq i \leq 3}$ of type either $e^{it\Delta} f_0^\omega$ or v . In this purpose, we define

$$\mathcal{N}(u_1, u_2, u_3) := u_1 \overline{u_2} u_3, \quad \mathcal{N}(u) := \mathcal{N}(u, u, u) = |u|^2 u.$$

Once again, we drop the complex conjugation sign since it plays no role in our short time perturbative analysis, so that the above notation is invariant under permutation of the terms u_1, u_2, u_3 . For convenience, we write the Duhamel integral term as $(i\partial_t - \Delta)^{-1}(F) := \int_0^t e^{i(t-\tau)\Delta} F(\tau) d\tau$.

Lemma 1.3.6 (Trilinear estimates, see Proposition 3.1 in [BOP15b; BOP15a]). *Let $\frac{1}{4} < s < \frac{1}{2}$ and $\frac{1}{2} < \sigma < 2s$. There exist $C > 0$, $b = \frac{1}{2}^+$ and $\theta = 0^+$ such that for all $0 < T \lesssim 1$, for all $f_1, f_2, f_3 \in Y^s$ and $v, v_1, v_2, v_3 \in X^{\sigma,b}([0, T])$, we have*

$$\begin{aligned}
\|(i\partial_t - \Delta)^{-1}\mathcal{N}(v_1, v_2, v_3)\|_{X_T^{\sigma,b}} &\leq CT^\theta \|v_1\|_{X_T^{\sigma,b}} \|v_2\|_{X_T^{\sigma,b}} \|v_3\|_{X_T^{\sigma,b}}, \\
\|(i\partial_t - \Delta)^{-1}\mathcal{N}(f_1, f_2, f_3)\|_{X_T^{\sigma,b}} &\leq CT^\theta \|f_1\|_{Y^s} \|f_2\|_{Y^s} \|f_3\|_{Y^s}, \\
\|(i\partial_t - \Delta)^{-1}\mathcal{N}(v_1, v_2, f_3)\|_{X_T^{\sigma,b}} &\leq CT^\theta \|v_1\|_{X_T^{\sigma,b}} \|v_2\|_{X_T^{\sigma,b}} \|f_3\|_{Y^s}, \\
\|(i\partial_t - \Delta)^{-1}\mathcal{N}(v_1, f_2, f_3)\|_{X_T^{\sigma,b}} &\leq CT^\theta \|v_1\|_{X_T^{\sigma,b}} \|f_2\|_{Y^s} \|f_3\|_{Y^s}.
\end{aligned}$$

In addition, we have the Lipschitz estimate

$$\|(i\partial_t - \Delta)^{-1}(\mathcal{N}(v_1 + f) - \mathcal{N}(v_2 + f))\|_{X_T^{\sigma,b}} \leq CT^\theta \|v_1 - v_2\|_{X_T^{\sigma,b}} (\|v\|_{X_T^{\sigma,b}}^2 + \|f\|_{Y^s}^2).$$

As a consequence of the above trilinear estimates, we have a good Cauchy theory for (NLS_f) .

Lemma 1.3.7 (Local well-posedness for the perturbed equation). *Let $R > 0$ and assume that*

$$\|f\|_{Y^s} \leq R.$$

For any $\frac{1}{2} < \sigma < 2s$ and $b = \frac{1}{2}^+$, there exist $T = T(R) > 0$ and a unique solution v to (NLS_f) in $X_T^{\sigma,b}$. Moreover,

$$v \in C([0, T]; H^\sigma), \quad \text{and } \|v\|_{X_T^{\sigma,b}} \leq R.$$

Theorem 1.3.5 follows from Lemma 1.3.7 applied with $f := e^{it\Delta} f_0^\omega$, for some initial data $f_0^\omega \in \Sigma$.

Proof of Lemma 1.3.7. We show that for T small enough with respect to R , the map

$$\Gamma : v \in X_T^{\sigma,b} \mapsto (i\partial_t - \Delta)^{-1}(|v + f|^2(v + f)) \in X_T^{\sigma,b}$$

is a contraction on the ball

$$B_R = \{v \in X_T^{\sigma,b} \mid \|v\|_{X_T^{\sigma,b}} \leq R\}.$$

It follows from the trilinear estimates of Lemma 1.3.6 that for all $v, v_1, v_2 \in X_T^{\sigma,b}$,

$$\begin{aligned}
\|\Gamma v\|_{X_T^{\sigma,b}} &\leq CT^\theta (\|v\|_{X_T^{\sigma,b}}^3 + \|f\|_{Y^s}^3), \\
\|\Gamma(v_2) - \Gamma(v_1)\|_{X_T^{\sigma,b}} &\leq CT^\theta \|v_2 - v_1\|_{X_T^{\sigma,b}} (\|v_1\|_{X_T^{\sigma,b}}^2 + \|v_2\|_{X_T^{\sigma,b}}^2 + \|f\|_{Y^s}^2),
\end{aligned}$$

so that

$$\begin{aligned}
\|\Gamma v\|_{X_T^{\sigma,b}} &\leq 2CT^\theta R^3, \\
\|\Gamma(v_2) - \Gamma(v_1)\|_{X_T^{\sigma,b}} &\leq 2CT^\theta R^2 \|v_2 - v_1\|_{X_T^{\sigma,b}}.
\end{aligned}$$

We conclude by choosing $T = T(R) > 0$ such that $2CT^\theta R^2 \leq \frac{1}{2}$. □

1.3.3 Approximate solutions by convolution and convergence

Let $f_0^\omega \in \Sigma$, and $u = e^{it\Delta} f_0^\omega + v$ be the local solution to NLS obtained in Theorem 1.3.5, with $v \in X^{\sigma,b}([0, T]) \subset C([0, T], H^\sigma)$. Let $(\varepsilon_k)_{k \geq 0}$ be a sequence of positive numbers that go to zero. Using the approximate identity (1.1.3), we define a sequence of regularized initial data

$$f_{0,k}^\omega := f_0^\omega * \rho_{\varepsilon_k} \in H^\infty.$$

Then, we denote u_k the maximal solution in H^1 to NLS. Using the conservation of the coercive energy (in the defocusing case), u_k is global. The purpose of this section is to prove the convergence claimed in Theorem 1.1.3. We set $R > 0$ be such that

$$\|e^{it\Delta} f_0^\omega\|_{Y^s} \leq R,$$

and $T = T(R)$ is as in the proof of Lemma 1.3.7, applied with $f := e^{it\Delta} f_0^\omega$.

Proposition 1.3.8. *We have the convergence*

$$\lim_{k \rightarrow \infty} \|u_k - u\|_{L^\infty([0, T], H^s)} = 0.$$

Proof. We do not have a good Cauchy theory in $H^s(\mathbb{R}^3)$. However, Theorem 1.3.5 provides a Cauchy theory for the perturbed equation (NLS_f), with a forcing term $f = e^{it\Delta} f_0^\omega$ in Y^s . Hence, it is relevant to run a linear-nonlinear decomposition of u_k as well

$$u_k = e^{it\Delta} f_{0,k}^\omega + v_k =: f_k + v_k, \quad f_k = e^{it\Delta} f_{0,k}^\omega,$$

We define

$$\delta_k := f_k - f = e^{it\Delta} (f_0^\omega * \rho_{\varepsilon_k} - f_0^\omega),$$

and note that the convergence

$$\lim_{k \rightarrow \infty} \|\delta_k\|_{L_t^\infty H_x^s} = \lim_{k \rightarrow \infty} \|f_0 - f_0 * \rho_{\varepsilon_k}\|_{H_x^s} = 0.$$

follows from the unitary property of the linear evolution together with the fact that the convolution commutes with the linear evolution. Then, we use the embedding $X^{\sigma,b}([0, T]) \hookrightarrow L^\infty([0, T], H^\sigma)$ to reduce the proof of Theorem 1.1.3 to the convergence of the nonlinear term v_k in the Fourier restriction space.

$$\lim_{k \rightarrow \infty} \|v - v_k\|_{X_T^{\sigma,b}} = 0. \tag{1.3.4}$$

To achieve such a goal, we start by proving that

$$\lim_{k \rightarrow \infty} \|\delta_k\|_{\tilde{S}^s} = 0. \tag{1.3.5}$$

We fix $2 \leq q, r < \infty$ and $\omega \in \Omega$ such that

$$\|e^{it\Delta} f_0^\omega\|_{L_t^q L_x^r} < \infty.$$

In particular, we observe that

$$\|e^{it\Delta} f_0^\omega\|_{L_x^r} < \infty, \quad \text{Lebesgue a.e. } t \in \mathbb{R}. \tag{1.3.6}$$

In addition, when t is such that (1.3.6) holds we

$$\|\delta_k(t)\|_{L_x^r} \leq (1 + \|\rho\|_{L_x^1}) \|e^{it\Delta} f_0^\omega\|_{L_x^r},$$

so that the sequence of measurable functions $t \rightarrow \|\delta_k(t)\|_{L_x^r}$ is uniformly bounded in k . From the assumption on f_0^ω , we also have the uniform bound

$$\|\delta_k\|_{L_t^q L_x^r} \leq (1 + \|\rho\|_{L_x^1}) \|e^{it\Delta} f_0^\omega\|_{L_t^q L_x^r}. \quad (1.3.7)$$

Next, we use the fact that the approximate identity converge in L_x^r , and commutes with the linear evolution $e^{it\Delta}$, to obtain

$$\lim_{k \rightarrow \infty} \|\delta_k(t)\|_{L_x^r} = \lim_{k \rightarrow \infty} \|e^{it\Delta}(f_0^\omega - f_0^\omega * \rho_{\varepsilon_k})\|_{L_x^r} = \lim_{k \rightarrow \infty} \|e^{it\Delta} f_0^\omega - (e^{it\Delta} f_0^\omega) * \rho_{\varepsilon_k}\|_{L_x^r} = 0.$$

This convergence, combined with the uniform bound (1.3.7), allows us to apply the dominated convergence theorem in L_t^q and to conclude that the convergence (1.3.5) holds.¹⁰ Next, we fix $\eta > 0$ arbitrarily small, and $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$, $\|\delta_k\|_{Y^s} \leq \eta$. We define

$$w_k := v_k - v \in X^{\sigma,b}([0, T]).$$

By using the equations for v and v_k , we see that w_k is solution to the following forced Schrödinger equation with zero initial condition

$$\begin{cases} (i\partial_t + \Delta)w_k = \mathcal{N}(f_k + v_k) - \mathcal{N}(f + v) = \mathcal{N}(w_k + \delta_k + f + v) - \mathcal{N}(f + v) =: F. \\ w_k(0) = 0, \end{cases}$$

We perform a bootstrap argument and propagate the smallness of the $X^{\sigma,b}$ -norm of w_k to prove that for all $k \geq k_0$,

$$\|w_k\|_{X^{\sigma,b}([0,T])} \leq 2\eta.$$

Up to some multiplicative universal constants, the forcing term writes

$$F \sim \mathcal{N}(w_k + \delta_k) + \mathcal{N}(w_k + \delta_k, w_k + \delta_k, f + v) + \mathcal{N}(w_k + \delta_k, f + v, f + v).$$

Using the trilinear estimates from Lemma 1.3.6, we get the following a priori estimate

$$\begin{aligned} \|w_k\|_{X_T^{\sigma,b}} &\leq CT^\theta \|w_k\|_{X_T^{\sigma,b}} (\|w_k\|_{X_T^{\sigma,b}}^2 + \|\delta_k\|_{Y^s}^2 + \|v\|_{X_T^{\sigma,b}}^2 + \|f\|_{Y^s}^2) \\ &\quad + CT^\theta \|\delta_k\|_{Y^s} (\|w_k\|_{X_T^{\sigma,b}}^2 + \|\delta_k\|_{Y^s}^2 + \|v\|_{X_T^{\sigma,b}}^2 + \|f\|_{Y^s}^2). \end{aligned}$$

We finally exploit the smallness of the Y^s -norm of δ_k when $k \geq k_0$, the estimates on $\|v\|_{X_T^{\sigma,b}}$ from Theorem 1.3.5 and the a priori estimate on $\|w_k\|_{X_T^{\sigma,b}}$ from the bootstrap hypothesis to obtain

$$\|w_k\|_{X_T^{\sigma,b}} \leq 2CT^\theta \|w_k\|_{X_T^{\sigma,b}} (\eta^2 + R^2) + 2CT^\theta \eta (\eta^2 + R^2).$$

Since we chose η arbitrarily small, and T is chosen such that $2CT^\theta(1 + R^2) \leq \frac{1}{2}$ we conclude that,

$$\|w_k\|_{X_T^{\sigma,b}} \leq \frac{1}{2} \|w_k\|_{X_T^{\sigma,b}} + \frac{1}{2} \eta.$$

This finishes the proof of the bootstrap. \square

¹⁰Note that the only features of this specific approximate identity by convolution we used is that it commutes with $e^{it\Delta}$, and that it is uniformly bounded in L^p -spaces.

Part II

Long-time dynamics for NLS under random perturbations

Chapter 2

Scattering for the cubic Schrödinger equation in 3D with randomized radial initial data

From the publication [Cam23].

Résumé. Nous démontrons le scattering presque-sûr pour l'équation de Schrödinger cubique défocalisante dans l'espace euclidien \mathbb{R}^3 , avec des données initiales aléatoires surcritiques qui sont obtenues à partir d'une fonction radiale. Notre résultat généralise le travail de Bényi, Oh et Pocovnicu [BOP15a] dans la mesure où nous ne faisons pas d'hypothèse de petitesse. Il étend également les résultats de Dodson, Lührmann et Mendelson [DLM19] pour l'équation énergétique-critique dans \mathbb{R}^4 , à l'équation énergie-sous-critique dans \mathbb{R}^3 . Dans un tel cadre, bien que le terme Duhamel bénéficie d'un effet régularisant qui le rend sous-critique, son énergie est infinie. Pour démontrer notre résultat, nous commençons par développer une théorie de stabilité à partir des résultats de scattering déterministe en-dessous de l'espace d'énergie, qui sont dus à Colliander, Keel, Staffilani, Takaoka et Tao. Ensuite, nous proposons un argument de globalisation dans lequel nous revisitons la “*I*-method” dans un cadre stochastique. À notre connaissance, il s'agit du premier résultat de scattering presque-sûr à grandes données pour une équation de Schrödinger énergie-sous-critique.

Abstract. We obtain almost-sure scattering for the cubic defocusing Schrödinger equation in the Euclidean space \mathbb{R}^3 , with randomized radially-symmetric initial data at some supercritical regularity scales. Since we make no smallness assumption, our result generalizes the work of Bényi, Oh and Pocovnicu [BOP15a]. It also extends the results of Dodson, Lührmann and Mendelson [DLM19] on the energy-critical equation in \mathbb{R}^4 , to the energy-subcritical equation in \mathbb{R}^3 . In this latter setting, even if the nonlinear Duhamel term enjoys a stochastic smoothing effect that makes it subcritical, it still has infinite energy. In the present work, we first develop a stability theory from the deterministic scattering results below the energy space, due to Colliander, Keel, Staffilani, Takaoka and Tao. Then, we propose a globalization argument in which we set up the *I*-method with a Morawetz bootstrap in a stochastic setting. To our knowledge, this is the first almost-sure scattering result for an energy-subcritical Schrödinger equation outside the small data regime.

Contents

2.1	Introduction	100
2.1.1	Known results	101
2.1.2	Main result and outline of the proof	106
2.2	Preliminaries and probabilistic estimates	109
2.2.1	The free Schrödinger evolution and critical spaces	109
2.2.2	Large deviation estimates for the free Schrödinger evolution	111
2.2.3	Additional estimate for randomized radial initial data	113
2.3	Cauchy theory for the forced NLS equation	117
2.3.1	Nonlinear estimates	118
2.3.2	Local well-posedness	126
2.3.3	Stability theory	129
2.4	Almost conservation laws	132
2.4.1	Setting up the I -method	133
2.4.2	Modified energy increment	137
2.4.3	Modified interaction Morawetz	141
2.5	Almost-sure global well-posedness and scattering	144
2.5.1	Double bootstrap argument	144
2.5.2	Proof of Theorem 2.1.4	148

2.1 Introduction

We consider the initial value problem for the cubic defocusing Schrödinger equation

$$\begin{cases} i\partial_t u(t, x) + \Delta u(t, x) = |u(t, x)|^2 u(t, x), & (t, x) \in \mathbb{R} \times \mathbb{R}^3. \\ u(0, x) = u_0(x), \end{cases} \quad (\text{NLS})$$

The equation has a Hamiltonian structure associated with the energy

$$E(u) = \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u(\cdot, x)|^2 dx + \frac{1}{4} \int_{\mathbb{R}^3} |u(\cdot, x)|^4 dx. \quad (2.1.1)$$

For smooth solutions, the above energy is conserved as time evolves. The mass

$$M(u) = \int_{\mathbb{R}^3} |u(\cdot, x)|^2 dx$$

is also formally conserved. In addition, the equation enjoys the scaling symmetry. Namely, given $\lambda > 0$ and u a solution to the equation,

$$u_\lambda(t, x) = \lambda^{-1} u(\lambda^{-2} t, \lambda^{-1} x), \quad (t, x) \in \mathbb{R} \times \mathbb{R}^3,$$

is also a solution. Since the scaling symmetry leaves invariant the homogeneous Sobolev space $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$, the scale $s_c = \frac{1}{2}$ is *critical*. In the *scaling-subcritical* regime $s > \frac{1}{2}$, it follows from a contraction mapping argument that the initial value problem is *locally well-posed* in the following sense :

- (Uniqueness) Given two solutions u_1, u_2 in $H^s(\mathbb{R}^3)$ and a time t , if $u_1(t) = u_2(t)$ then $u_1 = u_2$.
- (Existence) For all $u_0 \in H^s(\mathbb{R}^3)$, there exists $T(\|u_0\|_{H^s}) > 0$ and a solution to (NLS)

$$u(t) = e^{it\Delta} u_0 - i \int_0^t e^{i(t-\tau)\Delta} |u(\tau)|^2 u(\tau) d\tau, \quad u \in C([-T, T]; H^s(\mathbb{R}^3)).$$

- (Continuity) The solution map $u_0 \in H^s(\mathbb{R}^3) \mapsto u \in C([-T, T], H^s(\mathbb{R}^3))$ is continuous.

2.1.1 Known results

Note that in the *scaling-subcritical* regime, the maximal lifespan of the solution is bounded from below by a quantity that only depends on the $H^s(\mathbb{R}^3)$ norm of the initial data. Therefore, one can use a continuity argument and deduce global existence from a priori uniform bound on the $H^s(\mathbb{R}^3)$ norm of the solution.

In the present case, $s_c < 1$ and the equation is *energy-subcritical*. Note that an initial data in $H^1(\mathbb{R}^3)$ has a finite energy. Besides, it follows from the defocusing nature of the equation that the energy is coercive, and yields a uniform control on the $H^1(\mathbb{R}^3)$ norm of the solution so that (NLS) is globally well-posed in the energy space $H^1(\mathbb{R}^3)$. In addition, the solution display *linear scattering* in large time. Namely, for all $u_0 \in H^1(\mathbb{R}^3)$, there exist some unique final states $u_{\pm} \in H^1(\mathbb{R}^3)$ such that

$$\lim_{t \rightarrow \pm\infty} \|e^{-it\Delta} u(t) - u_{\pm}\|_{H^1} = 0.$$

It is conjectured that (NLS) is globally well-posed in $H^s(\mathbb{R}^3)$ for $\frac{1}{2} \leq s < 1$, and that the solution scatters at infinity. Still, we do not have a coercive conservation law at these regularities and, in general, we cannot always preclude the $H^s(\mathbb{R}^3)$ norm from blowing up in finite time. In the *scaling-critical* case, the time of existence of the solution does not only depend on its $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$ norm, but also on its profile. Nevertheless, by using a concentration-compactness method, Kenig and Merle proved in [KM10] some *conditional* scattering.

It is well known from [CCT03b] that at *scaling-supercritical* scales $s < \frac{1}{2}$, the equation is ill-posed because of *norm inflation*. Furthermore, [ST20a; CG22] have recently evidenced that there exists in $H^s(\mathbb{R}^3)$ a *pathological set* containing a dense G_{δ} -set of initial data for which the regularized solutions inflate in arbitrarily short time. Nevertheless, following the pioneering work of Bourgain, a *probabilistic Cauchy theory* has emerged. It displays the existence of a local flow defined on *generic* sets of initial data, in some regimes where a deterministic Cauchy theory does not hold. In the present work, we address the globalization of such a probabilistic flow for (NLS) in the Euclidean space, with randomized initial data at supercritical regularities $\frac{3}{7} < s$.

Global well-posedness & Scattering below the energy space Let us present the general framework used to study (NLS) at scaling-subcritical regularity, below the energy space, where we still have a local flow. At regularities $\frac{1}{2} \leq \sigma < 1$, there is no coercive conservation law that prevents the Sobolev norm to grow up. Hence, to obtain global a priori estimates for a general initial data ¹, one has to make use of some modified energies. The idea is to consider a frequency truncated

¹ Note that one can easily prove scattering under smallness assumption on the initial data at regularity $\frac{1}{2} \leq \sigma < 1$, by directly performing a contraction argument and obtaining global spacetime bounds.

version of the solution, and to control the increments of its energy along the evolution. Bourgain implemented this idea in [Bou98a] to prove global existence for the cubic Schrödinger equation in $H^\sigma(\mathbb{R}^2)$, with $\frac{2}{3} < \sigma$, by using the *high-low* argument together with some bilinear estimates. The observation is that the nonlinear evolution of the high frequencies is well approximated by the linear evolution. Namely, if we evolve on a short time step the low-frequency modes by the nonlinear flow, and the high-frequency modes by the linear flow, the error we make turns out to be in the energy space $H^1(\mathbb{R}^3)$. Furthermore, the method displays a *nonlinear smoothing effect* : for all $t \in \mathbb{R}$,

$$u(t) - e^{it\Delta} u_0 \in H^1(\mathbb{R}^3). \quad (2.1.2)$$

This globalization method has become standard, and was used in many contexts (see, e.g., [KPV00] for the wave equation). Bourgain extended the method to the 3D case in [Bou98b], where he proved global existence when $\frac{11}{13} < \sigma$. Then, Colliander, Keel, Staffilani, Takaoka and Tao refined this frequency-cutoff in [Col+04] and introduced the *I*-operator, which goes from $H^\sigma(\mathbb{R}^3)$ to $H^1(\mathbb{R}^3)$. By exploiting the structure of the nonlinearity, the authors evidenced frequency cancellations that come with a nonlinear smoothing effect, loosening therefore the regularity threshold at which the local flow extends globally.

Definition 2.1.1 (*I*-operator). Given $\frac{1}{2} < \sigma < 1$ and $N \gg 1$, the *I*-operator is the Fourier multiplier

$$I = \mathcal{F}^{-1}(m\mathcal{F}), \quad m(\xi) = \begin{cases} 1 & \text{if } |\xi| \leq N, \\ \left(\frac{N}{|\xi|}\right)^{1-\sigma} & \text{if } |\xi| \geq 2N. \end{cases} \quad (2.1.3)$$

It holds that

$$\|\nabla Iu\|_{L^2} \leq \|u\|_{H^\sigma}, \quad \|u\|_{H^\sigma} \leq N^{1-\sigma} \|Iu\|_{H^1}. \quad (2.1.4)$$

As N becomes larger, the energy is supposed to grow to infinity, while we expect it's time derivative to decrease. Specifically, if we assume that $J \times \mathbb{R}^3$ is a spacetime slab where we have a smallness assumption on the $L^4_{t,x}$ norm of u , then

$$\int_J \left| \frac{d}{dt} E(Iu)(\tau) \right| d\tau \lesssim \frac{1}{N} + O\left(\frac{1}{N^2}\right). \quad (2.1.5)$$

The method usually comes with a rescaling argument, that allows one to assume that the energy of Iu is less than 1 initially. In addition, the *interaction Morawetz inequality* gives a bound on the $L^4_{t,x}$ norm, depending on the energy of Iu . Hence, we have a control on the number of spacetime slabs where the increment of $\mathcal{E}(v)$ satisfies (2.1.5). By finely tuning the parameters N and s , we would be able to sum (2.1.5) over finitely many spacetime slabs, and to obtain a uniform bound on both the energy of Iu and on the $L^4_{t,x}$ norm of u , from which scattering follows. The interaction Morawetz estimate reads

$$\int_J \int_{\mathbb{R}^3} |u(t, x)|^4 dx dt \lesssim \|u_0\|_{L^2}^2 \left(\sup_{t \in J} \|u(t)\|_{\dot{H}^{\frac{1}{2}}} \right)^2.$$

It is due to [Col+04], and its proof consists in averaging over y the recentered Morawetz action against the mass density, and in tracking some monotonicity. The *I*-method with a Morawetz bootstrap has been refined, and adapted in different contexts. Let us cite some works that are relevant to our problem. By the use of linear-nonlinear decomposition of the initial data, [Dod13]

reached the exponent $\frac{5}{7} < s$, further improved to $\frac{2}{3} < s$ in [Su12] by combining the arguments of [Dod13] with a second generation I -method. In the present work, we implement the following key idea. To improve the results of [Col+04], Dodson provided a refined version of the energy increment estimate (2.1.5), by keeping some sub-additive quantities on the right-hand-side : for some constant c independent of N ,

$$\int_J \left| \frac{d}{dt} E(Iu)(\tau) \right| d\tau \lesssim \frac{1}{N^{1-}} \|\nabla P_{>cN} Iu\|_{L_t^2(J; L_x^6(\mathbb{R}^3))}^2 + O\left(\frac{1}{N^2}\right). \quad (2.1.6)$$

For radial initial data, this comes together with the so-called *long-time Strichartz estimate*, that yields a uniform a priori estimate on $\|\nabla P_{>cN} Iu\|_{L_t^2 L_x^6}^2$. This strategy avoids losing too many powers of N when brutally multiplying the energy increments by the number of spacetime slabs where we have a control. Dodson implemented this idea in [Dod13; Dod19b] to prove scattering for $s > \frac{5}{7}$, and for $s > \frac{1}{2}$ in the radial case. However, the conjecture is still open for general data. Before ending this paragraph, we formulate the optimal result known at this date. We will use it throughout the analysis. ²

Theorem 2.1.2 (Global well-posedness below the energy space, [Col+04; Dod13; Su12]). *Let $\frac{2}{3} < \sigma < 1$ and $u_0 \in H_x^\sigma(\mathbb{R}^3)$. The Cauchy problem (NLS) with $u(t_0) = u_0$ is globally well-posed and the solution scatters at infinity in $H_x^\sigma(\mathbb{R}^3)$. Moreover, there exists a constant $C(\|u_0\|_{H_x^\sigma}) > 0$ such that*

$$\|u\|_{L_{t,x}^5(\mathbb{R} \times \mathbb{R}^3)} \leq C(\|u_0\|_{H_x^\sigma}). \quad (2.1.7)$$

Probabilistic Cauchy theory Given a randomized initial data f_0^ω , which is basically the superposition of modes decoupled by multiplying each of them with independent Gaussian variables, the probabilistic approach consists in performing a linear-nonlinear decomposition in the Duhamel integral representation, and in studying the nonlinear term

$$v(t) = u(t) - e^{it\Delta} f_0^\omega = \int_0^t e^{i(t-t')\Delta} (|e^{it'\Delta} f_0^\omega|^2 e^{it'\Delta} f_0^\omega) dt' + \dots$$

The remaining terms correspond to higher order Picard iterates. It turns out that the probabilistic decoupling enhances the nonlinear smoothing effect (2.1.2), even in scaling-supercritical regimes and, in the favorable cases, we expect v to be at critical, or subcritical regularity, where we have some local existence results. Besides, v is solution to (NLS) with a stochastic forcing term.

$$\begin{cases} i\partial_t v + \Delta v = |v|^2 v + (|v + f|^2 (v + f) - |v|^2 v), & (t, x) \in \mathbb{R} \times \mathbb{R}^3. \\ v(t_0) = v_0 \in H^\sigma(\mathbb{R}^3), \end{cases} \quad (\text{NLS}_f)$$

At time $t = 0$, we have $v(0) = 0$. The strategy is to extend the deterministic Cauchy theory at scaling-subcritical regularities to the perturbed equation (NLS_f), at least locally in time. Specifically, we expect to prove that for almost every ω , (NLS_f) is locally well-posed by some perturbative

² Some stronger restrictions will appear in the analysis, and we shall only use (2.1.7) when $\sigma > \frac{6}{7}$. The high-low method from Bourgain [Bou98b] already settle global existence at these range of regularities, and the original version of the I -method from [Col+04] yields the global spacetime estimates (2.1.7) and scattering.

arguments, with $f^\omega = e^{it\Delta} f_0^\omega$. This strategy corresponds to the Da Prato Debussche trick [DD02]. In the context of dispersive equations, it was first implemented by Bourgain for the 2D Schrödinger equation on the torus [Bou94], and then by Burq and Tzvetkov for the wave equation on compact manifolds without boundaries [BT08a; BT08b]. In such contexts, the randomization procedure is based on the spectral resolution of the Laplacian. However, on the Euclidean space, we do not have such natural decompositions for the initial data. It turns out that the so-called *Wiener randomization* proved to be suitable, in the sense that the nonlinear Duhamel term benefits from a nonlinear probabilistic smoothing that makes it subcritical, thanks to some improved spacetime integrability for almost every f^ω .

Definition 2.1.3 (Wiener randomization). Let $f_0 \in L_x^2(\mathbb{R}^3)$ be radially-symmetric. We define the unit-scale Wiener decomposition of f_0 by

$$f_0 \sim \sum_{k \in \mathbb{Z}^3} Q_k f_0,$$

where Q_k is the Fourier projector associated with a smooth multiplier localized on the unit cube centered around $k \in \mathbb{Z}^3$. Next, given a sequence of real valued mean zero and independent complex Gaussian variables $\{g_k\}_{k \in \mathbb{Z}^3}$, we define a random variable

$$\omega \in \Omega \mapsto f_0^\omega = \sum_{k \in \mathbb{Z}^3} g_k(\omega) Q_k f_0. \quad (2.1.8)$$

Let us emphasize that the procedure does not gain regularity in the sense that if $f_0 \in H^s(\mathbb{R}^3) \setminus \bigcap_{\delta > s} H^\delta(\mathbb{R}^3)$, then $f_0^\omega \in H^s(\mathbb{R}^3) \setminus \bigcap_{\delta > s} H^\delta(\mathbb{R}^3)$ for almost every ω . We refer to the works [LM14a; BOP15b; BOP15a; BOP19b; BOP19a; DLM20; DLM19] and the references therein for further details on the Wiener randomization.

Globalization arguments in the stochastic setting Besides the construction of local solutions in supercritical regimes, one may ask about the global existence and asymptotic behavior of these solutions. This question is out of reach in general, and the probabilistic method is mainly concerned with perturbations of the zero solution. Still, there are different globalization procedures in such a stochastic context outside the small data regime that make use, in a mild sense, of the conservation laws. In the compact setting, we can use invariant or quasi-invariant measures as a substitute for the conservation of energy. This approach in the context of dispersive PDE's was initiated by Bourgain, who settled in [Bou94; Bou96] the invariance property of the Gibbs measure for (NLS) on the torus, in dimension 1 and 2. In [BT08b], Burq and Tzvetkov proved almost sure global well-posedness by using the energy increment for v and a Gronwall's argument.

In the Euclidean space, we do not have invariant measures and the globalization arguments are mostly based on deterministic methods. Still, some works concern the harmonic oscillator. Thanks to the so-called *lens transform* that intertwine the harmonic oscillator and (NLS), a randomization procedure emerges from the decomposition on the orthonormal basis of the Hermite functions, that are the eigenfunctions of the harmonic oscillator. On the support of the induced measure, that lies in a space just below L^2 , we have global existence and scattering under smallness assumptions (see [Poi12; PRT14] in dimension $d \geq 2$). Recently, Burq and Thomann evidenced in [BT20] that this randomization procedure provides quasi-invariant measures for Schrödinger equations in 1D,

and proved almost-sure scattering. However, in higher dimensions $d \geq 2$, the randomization on the Hermite functions induces a gain of regularity in L^p for $p \neq 2$, as well as a gain of decay in weighted L^2 -spaces. To address energy-critical equations, for which the conservation law is not enough, the natural strategy in the random data setting consists in using a stability theory for (NLS_f) in the energy space, and to infer a priori spacetime bounds from which we deduce scattering. This method was used by [DLM20] to prove almost sure scattering for the 4D energy critical wave equation. Note that the case of the wave equation is more favorable, since the energy controls $\partial_t v$. Nevertheless, Killip, Murphy and Visan proved the counterpart of [DLM20] for the energy critical Schrödinger equation in 4D via a double bootstrap argument, that combines a modified energy and a Morawetz like estimate. In addition, the result of [KMV19] was improved in [DLM19], where the authors finely exploited the local smoothing effect. Namely, they refined the probabilistic local well-posedness theory in a new functional framework, inspired by the *lateral* spaces used to study Schrödinger maps, together with an improved Sobolev embedding for randomized radial initial data. We stress out that the stability-perturbation strategy is more likely to work under some radial assumption on the data, in order to use local energy decay estimate and to gain regularity when estimating the modified energy increments, with rough data. Specifically, one needs to use a radial Sobolev embedding at some point in the analysis, in order to absorb a weight that comes from the need to use local energy decay, or from the Lin-Strauss estimate. However, Oh, Okamoto and Pocovnicu [OOP19] proved almost sure global existence for the energy critical Schrödinger in higher dimension $d \geq 5$, where the potential energy controls more terms. To our knowledge, stability-perturbation arguments for NLS outside the small data regime were only used in the energy-critical case. In the energy-subcritical case, one may be led to consider modified energy arguments as well. Colliander and Oh already used the high-low method in a stochastic context, to prove almost-sure global well-posedness for the cubic periodic Schrödinger equation below $L^2(\mathbb{T})$ in [CO12], using the mass as a conservation law. Then, Lührmann and Mendelson [LM14a] applied the high-low method to prove global well-posedness for the NLW. Poiret Robert and Thomann also used this method to prove scattering for supercritical small initial data for the harmonic oscillator in [PRT14]. More recently, the authors of [Gub+21] used the I -method in the context of the stochastic nonlinear wave equation. In [FM20], the I -method was used to address the log-log blow up profile for the mass critical NLS in $L^2(\mathbb{R}^2)$. Let us also mention the scattering result from [Spi21] in \mathbb{R}^4 , where the author uses an extra randomization of the angular variable as a substitute for the radial assumption. Note, however, that this randomization yields better estimates on the linear evolution than we use under the radial assumption.

In the present work, we consider the 3D case, where the equation is energy-subcritical. Hence, there is a gap to bridge between the scaling-critical space $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$ and the energy space $\dot{H}^1(\mathbb{R}^3)$. The known results for this equation with supercritical initial data are the followings. On the one hand, we have the almost-sure scattering result for small-data that comes from [Poi12] by the use of the lens transform. On the other hand, the works of [BOP19a] based on the wiener randomization address scattering for small data, and conditional global existence for general data. Let us also mention the work [BOP19a], where the authors used higher order expansion in the Duhamel formula. The idea is to refine Da Prato Debussche's trick and include higher order terms in the Ansatz. By doing so, they prove that the nonlinear term lies in $H^1(\mathbb{R}^3)$ almost surely, and has finite energy. Yet, it is not clear how to deduce global well-posedness from this observation.

2.1.2 Main result and outline of the proof

Our approach is a bit different. Since v is not in the energy space but just below, we use the scattering theory in $H^\sigma(\mathbb{R}^3)$ for $\frac{1}{2} < \sigma < 1$ detailed in paragraph 2.1.1, and we establish a stability theory for the perturbed equation (NLS $_f$) to obtain a conditional scattering result. Then, we obtain global a priori estimates and we prove the following result.

Theorem 2.1.4. *Let s, σ be such that $\frac{3}{7} < s \leq 1$, and $\frac{6}{7} < \sigma < 2s$. Given a radially-symmetric function f_0 in $H^s(\mathbb{R}^3)$, we define the randomized initial data f_0^ω as in (2.1.8). Then, for almost every ω , there exists a global solution u to the defocusing cubic Schrödinger equation (NLS), with*

$$u - e^{it\Delta} f_0^\omega \in C(\mathbb{R}; H_x^\sigma(\mathbb{R}^3)).$$

Moreover, there exist $u_\pm \in H_x^\sigma(\mathbb{R}^3)$ such that

$$\lim_{t \rightarrow \pm\infty} \|u(t) - e^{it\Delta} (f_0^\omega + u_\pm)\|_{H_x^\sigma(\mathbb{R}^3)} = 0.$$

Uniqueness holds for $v = u - e^{it\Delta} f_0^\omega \in X^\sigma(\mathbb{R}) \subset C(\mathbb{R}; H_x^\sigma(\mathbb{R}^3))$, which is a strong solution to the nonlinear Schrödinger equation with some stochastic forcing terms (NLS $_f$), in the critical space X^σ defined in (2.2.2) and with zero initial data. Note that we have scattering at regularity close to H^{2s} , which has to be compared to Theorem 2.1.2, where we do not directly have scattering in $H^1(\mathbb{R}^3)$. The threshold $\frac{3}{7} < s$ is certainly not optimal. Even if the local probabilistic flow exists up to $\frac{1}{4} < s$, it would be more realistic to reach $\frac{1}{3} < s$, since the deterministic theory is known for $\frac{2}{3} < \sigma$. One way of loosening the threshold on s would be to have a local-smoothing estimate as (2.2.20) independent of N , and then to perform a scaling argument to have an energy smaller than 1. In addition, one can also hope to prove a long-time Strichartz estimate in a probabilistic setting, and to refine the local existence theory by the use of lateral spaces as in [DLM19].

We stress out that randomized initial data f_0^ω are not radial.³ Nevertheless, we follow [DLM20; DLM19] and we use the radial assumption on each frequency localized pieces $Q_k f_0$, for $k \in \mathbb{Z}^3$. Then, we deduce from probabilistic decoupling an estimate on the linear Schrödinger evolution of the whole function $f_0^\omega \sim \sum g_n(\omega) Q_k f_0$, which is not radially-symmetric. Hence, the radial-symmetry assumption is concealed by the randomization procedure, but it still gives access to the energy decay estimates. The radial-symmetry assumption might be loosened, but probably not entirely removed. Indeed, when we estimate the modified energy increment, a lot of derivatives fall on the rough term f , and we need at least to gain $\frac{1}{2}$ derivative when taking the L^2 -in-time spacetime norms of f . For this purpose, we use the local-energy decay. Subsequently, since we avoid using some decay assumptions, we need to use a radial Sobolev embedding to absorb the weights that comes from the local smoothing estimates. This is also the reason why Dodson settled in [Dod19b] the deterministic subcritical theory for radial data only, since the long time Strichartz estimates used in [Dod19b] exploit the local energy decay effect.

Remark 2.1.5. A few days after the submission of this paper on arXiv, a preprint [SSW21] also appeared on arXiv, proving a similar result independently, with the same threshold $s > \frac{3}{7}$ and radial assumption. Their proof is based on the high-low method.

³One can also perform an extra microlocal randomization to decouple the orientation of each block that appears in (2.1.8). For instance, one can multiply in the physical space each block by some independent Gaussian vectors on the sphere, and make therefore the initial data as generic as possible.

Outline of the proof First, we fix $\alpha > 0$, and $N_0 = N_0(\alpha, s, \sigma, \|f_0\|_{H^s}) \gg 1$, to be chosen at the end of the analysis. From the large deviation bounds presented in section 2.2, we can find a set $\tilde{\Omega}_\alpha$, with $\mathbb{P}(\Omega \setminus \tilde{\Omega}_\alpha) \leq 2\alpha$, such that for every ω , the linear evolution of the random initial data f_0^ω satisfies refined global spacetime estimates. Specifically, for every $\omega \in \tilde{\Omega}_\alpha$ and every $N \geq N_0$, we have the finiteness of the quantities $F^\omega(\mathbb{R})$, $F_\infty^\omega(\mathbb{R})$ and $F_2^\omega(\mathbb{R})$ defined in (2.1.11). The large deviation estimate (2.2.21) for this latter quantity $F_2^\omega(\mathbb{R})$ follows from an improved radial Sobolev embedding estimate, which is a weakened version in dimension 3 of the estimate (1.10) from [DLM20]. In some sense, it reflects the local energy decay for the linear Schrödinger evolution. Then, we first prove in section 2.3.3 a *conditional* scattering result, written in Proposition 2.3.8. It states that the nonlinear Duhamel term v scatters in $H^\sigma(\mathbb{R}^3)$ provided that we have a uniform bound for its $H^s(\mathbb{R}^3)$ norm, on its maximal lifespan. This follows from the stability theory for the perturbed equation (NLS_f), at the subcritical energy scales $H^s(\mathbb{R}^3)$ for $\frac{2}{3} < s < 1$ where one can use the global spacetime bounds provided by the recent progresses collected in Theorem 2.1.2. We avoid the use of (2.2.21) and of the radial assumption in this section, and we restrict the analysis to some $\omega \in \Omega_\alpha$, where we only assume (2.2.12).

The rest of the paper is devoted to the proof of such an a priori bound for v . For this purpose, we fix $\frac{6}{7} < \sigma < 1$ and $v \in C(J^*; H^\sigma(\mathbb{R}^3))$, the maximal lifespan solution of (NLS_f). Then, we perform a double bootstrap argument that involves a modified energy and a modified interaction Morawetz inequality, together with large deviation estimates (2.2.21) on F_2^ω , that comes from the local energy decay. In section 2.4, we set up the I -method in a stochastic setting. First, we define the modified energy, and compute its increment on some intervals where we assume a smallness condition for the $L_{t,x}^4$ norm of v . In addition, we establish a modified interaction Morawetz estimate for the frequency truncated solution Iu , where $I : H^\sigma(\mathbb{R}^3) \rightarrow H^1(\mathbb{R}^3)$ is the I -operator at regularity $H^\sigma(\mathbb{R}^3)$. Indeed, the usual Morawetz cannot hold for the whole solution, since it requires $H^{\frac{1}{2}}(\mathbb{R}^3)$ regularity. Yet, the general strategy of the globalization argument is to consider Iu instead, solution to the cubic Schrödinger equation with the forcing term $H = IN(u) - \mathcal{N}(Iu)$. Namely, H is the commutator between the nonlinearity and I . By doing so, we can exploit the nonlinear smoothing effect yielded by the frequency cancellations in the commutator H . Basically, we can extract a power N^{-1} from spacetime quantities where H appears, and we write a general key estimate in Lemma 2.4.3 that encapsulate this smoothing effect. Then, we specify this estimate to handle the modified energy increment and the modified Morawetz interaction. Let us now comment a bit on the modified energy we use. It contains two terms, and writes

$$\mathcal{E}(v)(t) := \frac{1}{2} \int_{\mathbb{R}^3} |\nabla I v(t, x)|^2 dx + \frac{1}{4} \int_{\mathbb{R}^3} |I v + I f|^4 dx.$$

The kinetic energy only depends on v , truncated by the I -operator. However, the potential energy depends on f , as in [SX16] and [OOP19]. The motivation for this is to preserve the structure of the unperturbed equation (NLS), and to benefit from the key frequency cancellations. Formally, we have

$$\frac{d}{dt} \mathcal{E}(v) = \operatorname{Re} \int_{\mathbb{R}^3} \partial_t(Iv)(\mathcal{N}(Iu) - IN(u)) dx - \operatorname{Re} i \int_{\mathbb{R}^3} \nabla(I f) \mathcal{N}(\nabla I u, I u, I u) dx.$$

To control the first term, we exploit the frequency cancellations in the commutator. However, these cancellations do not appear in the second term, since this latter term is purely perturbative. We

handle it thanks to local smoothing (2.2.27). Similarly, we write the interaction Morawetz estimate for Iu , and not only Iv , in order to have some perturbation terms that contain H . Note that in the modified energy increment (2.4.11), we use the aforementioned idea from Dodson [Dod13; Dod19b], and we keep track of some sub-additive quantities on the right-hand side as in (2.1.6), not to lose too many powers of N when summing over the spacetime slabs. Finally, we perform the double bootstrap argument in section 2.4 as follows. From the almost conservation of energy, we gain one negative power of N , multiplied by some polynomial powers of N on intervals where the $L_{t,x}^4$ norm of Iv is small. However, the interaction Morawetz estimate yields a control on such a spacetime norm, that depends on the energy of Iv . Hence, by doing a continuity argument, and after tuning up the parameter σ , we are able to sum the energy increments over a partition of J^* made of spacetime slabs where the $L_{t,x}^4$ norm of Iv is small, and to prove that the energy cannot exceed $N^{2(1-\sigma)}$ on the lifespan of v . At the difference of the standard framework, we cannot use the scaling argument in this perturbed setting. Indeed, some perturbation terms coming from $\|\nabla I f^\omega\|_{L_t^2 L_x^6}$ of size $N^{\frac{1-\sigma}{2}}$ appear on the right-hand side of the energy increment estimate, so that the energy cannot remain less than 1 as time evolves. Furthermore, we use the almost-conserved mass, which is harmless, although it is supercritical regarding the scaling.

Organization of the paper The analysis is divided into two parts. In the first one, we prove the conditional scattering result for v , solution to the perturbed equation (NLS_f). In section 2.2, we recall standard large deviation estimates, as well as an improved bound for radially-symmetric randomized initial data. Then, we address the stability theory in section 2.3.3. In the second part of the analysis, we prove the global a priori estimate for v . Namely, we obtain almost conservation laws in section 2.4, used in section 2.5.1 to perform the double bootstrap argument.

Notations

σ is a scaling-subcritical exponent, and s is a scaling-supercritical one. They are linked together by the relation

$$s < \frac{1}{2} < \sigma < 2s.$$

We shall keep in mind that the nonlinear Duhamel term is in $H^\sigma(\mathbb{R}^3)$, whereas the initial data is in $H^s(\mathbb{R}^3)$. The I -operator is defined in (2.1.3), and N is a large dyadic integer. Throughout the analysis, one can keep in mind that the energy is a priori of order

$$\mathcal{E}(v) \lesssim N^{2(1-s)}.$$

We denote the Duhamel integral by

$$\mathcal{I}([t_0, t], F) := -i \int_{t_0}^t e^{i(t-t')\Delta} F(t') dt'. \quad (2.1.9)$$

The nonlinearity is seen as a trilinear operator, and writes

$$\mathcal{N}(u^{(1)}, u^{(2)}, u^{(3)}) := u^{(1)} \overline{u^{(2)}} u^{(3)}, \quad \mathcal{N}(u) := \mathcal{N}(u, u, u). \quad (2.1.10)$$

Given a time-interval $J \subseteq \mathbb{R}$, a randomized initial data f_0^ω corresponding to some $\omega \in \Omega$, and its linear evolution at time t denoted $f^\omega(t) = e^{it\Delta} f_0^\omega$, we define

$$\begin{aligned} F^\omega(J) &:= \|\langle \nabla \rangle^s f^\omega\|_{L_{t,x}^{10}(J)} + \|\langle \nabla \rangle^s f^\omega\|_{L_{t,x}^4(J)} + \|\langle \nabla \rangle^s f^\omega\|_{L_t^4 L_x^{12}(J)}, \\ F_\infty^\omega(J) &:= \|f^\omega\|_{L_t^\infty L_x^4(J)} + \|f^\omega\|_{L_t^\infty L_x^6(J)}, \\ F_2^\omega(J) &:= \|\langle \nabla \rangle I f^\omega\|_{L_t^2 L_x^\infty(J)} + \|\langle \nabla \rangle I f^\omega\|_{L_t^2 L_x^6(J)}. \end{aligned} \tag{2.1.11}$$

Note that the quantity $F^\omega(J)$ controls the critical norm $L_{t,x}^5$, by interpolation between $L_{t,x}^4$ and $L_{t,x}^{10}$. We will often drop the dependence on ω from the notations. Some large deviation estimates on these terms are proved in section 2.2.⁴ In addition, C is a constant that is irrelevant, and that may change from line to line. We may also write \lesssim . Finally, we denote by P_K the Littlewood Paley multiplier around the dyadic frequency of size $K = 2^k$:

$$\text{supp } \widehat{P_K f} \subset \{\xi \in \mathbb{R}^3 \mid 2^{k-1} \leq |\xi| \leq 2^{k+1}\}.$$

The counterpart in the physical space is χ_j :

$$\text{supp } \chi_j f \subset \{x \in \mathbb{R}^3 \mid 2^{j-1} \leq |x| \leq 2^{j+1}\}.$$

The Wiener multiplier around $k \in \mathbb{Z}^3$ is denoted by Q_k , and $\text{supp } \widehat{Q_k f} \subset k + [-1, 1]^3$.

2.2 Preliminaries and probabilistic estimates

2.2.1 The free Schrödinger evolution and critical spaces

Proposition 2.2.1 (Strichartz estimates in dimension $d = 3$). *Let $u_0 \in L_x^2(\mathbb{R}^3)$. We have*

$$\|e^{it\Delta} u_0\|_{L_t^q(\mathbb{R}; L_x^r(\mathbb{R}^3))} \lesssim \|u_0\|_{L_x^2(\mathbb{R}^3)},$$

for any Strichartz admissible pair

$$(q, r) \in [2, +\infty], \quad s(q, r) = \frac{2}{q} + d\left(\frac{1}{r} - \frac{1}{2}\right) = 0.$$

Definition 2.2.2 (Functions of bounded 2-variation V^2 and atomic space U^2). Let $J \subseteq \mathbb{R}$ be a time-interval, and \mathcal{Z} be the collection of every finite partitions of J . The set of functions with bounded 2-variation $V^2(J)$ is the set of functions $v : J \rightarrow L_x^2(\mathbb{R}^3)$, endowed with the norm

$$\|v\|_{V^2(J)} := \sup_{\{t_k\}_{k=0}^{K-1} \in \mathcal{Z}} \left(\sum_{k=1}^K \|v(t_k) - v(t_{k-1})\|_{L_x^2}^2 \right)^{\frac{1}{2}}.$$

⁴We might also incorporate the $L_{t,x}^{\frac{10}{3}}$ norm in the definition of $F^\omega(J)$, which corresponds to the Strichartz admissible pair $(\frac{10}{3}, \frac{10}{3})$. Indeed, at some point in the analysis, we will need to consider some spacetime slabs $J \times \mathbb{R}^3$ for which this quantity is small.

A function $a : J \rightarrow L_x^2(\mathbb{R}^3)$ is an atom if there exists a partition $\{t_k\}_{k=0\dots K}$ in \mathcal{Z} and $\{\phi_k\}_{k=0,\dots,K-1}$ some elements in L_x^2 such that

$$a(t) = \sum_{k=1}^K \mathbf{1}_{[t_{k-1}, t_k)}(t) \phi_{k-1}, \quad \sum_{k=0}^{K-1} \|\phi_k\|_{L_x^2}^2 \leq 1.$$

The atomic space $U^2(J)$ is the set of functions $u : J \rightarrow L_x^2(\mathbb{R}^3)$ endowed with the norm

$$\|u\|_{U^2(J)} := \inf \{ \|(\lambda_j)\|_{\ell^1} \mid u = \sum_{j \geq 1} \lambda_j a_j \text{ for some } U^2\text{-atoms } (a_j) \}. \quad (2.2.1)$$

Definition 2.2.3 (Function spaces adapted to the free Schrödinger evolution). Spaces adapted to the linear propagator $e^{it\Delta}$ are the Banach spaces endowed with the norms

$$\|u\|_{U_\Delta^2(J)} := \|e^{it\Delta} u\|_{U^2(J)}, \quad \|v\|_{V_\Delta^2(J)} := \|e^{it\Delta} v\|_{V^2(J)}.$$

Then, given $\sigma \in \mathbb{R}$, the critical spaces $X^\sigma(J)$ and $Y^\sigma(J)$ are the Banach spaces endowed with the norms

$$\|u\|_{X^\sigma(J)}^2 = \sum_{K \in 2^{\mathbb{N}}} K^{2\sigma} \|P_K u\|_{U_\Delta^2(J)}^2, \quad \|v\|_{Y^\sigma(J)}^2 = \sum_{K \in 2^{\mathbb{N}}} K^{2\sigma} \|P_K v\|_{V_\Delta^2(J)}^2. \quad (2.2.2)$$

Proposition 2.2.4 (Embeddings, [HHK09] Proposition 2.2 and Corollary 2.6⁵).

$$U_\Delta^2 H_x^\sigma \hookrightarrow X^\sigma(J) \hookrightarrow Y^\sigma(J) \hookrightarrow V_\Delta^2 H_x^\sigma \hookrightarrow L_t^\infty(J, H_x^\sigma). \quad (2.2.3)$$

Proposition 2.2.5 (Duality (see [HHK09] and [HTT11])). *There exists a unique bilinear map $B, : U^2 \times V^2 \rightarrow \mathbb{C}$ such that*

$$v \in V^2 \mapsto B(\cdot, v) \in (U^2)^*$$

is a surjective isometry, and

$$\|v\|_{V^2} = \sup_{\|u\|_{U^2} \leq 1} |B(u, v)|, \quad \|u\|_{U^2} = \sup_{\|v\|_{V^2} \leq 1} |B(u, v)|.$$

If $\partial_t u \in L_t^1(J; L_x^2(\mathbb{R}^3))$ we have the explicit formula

$$B(u, v) = \int_I (\partial_t u \mid v)_{L_x^2(\mathbb{R}^3)} dt.$$

In particular, for the Duhamel term (2.1.9) and when $F \in L_t^1(J; L_x^2(\mathbb{R}^3))$, we have

$$\|\mathcal{I}(J, F)\|_{U_\Delta^2} \lesssim \sup_{\|v\|_{V_\Delta^2} \leq 1} \left| \iint_{J \times \mathbb{R}} F \bar{v} dx dt \right|. \quad (2.2.4)$$

⁵ The spaces $U_\Delta^2 H^\sigma$ and $V_\Delta^2 H^\sigma$ are generalizations of the spaces U_Δ^2 and $V_\Delta^2 H^\sigma$, with functions $u, v : J \rightarrow H^\sigma(\mathbb{R}^3)$ instead of $L_x^2(\mathbb{R}^3)$.

Proposition 2.2.6 (Transferred linear and bilinear estimates, Lemma 3.3 in [BOP15a]). *Let (q, r) be an admissible pair as in Proposition 2.2.1. We have*

$$\|u\|_{L_t^q(\mathbb{R}; L_x^r(\mathbb{R}^3))} \lesssim \|u\|_{U_\Delta^2}. \quad (2.2.5)$$

Let K, M be two dyadic integers. The bilinear estimate from Bourgain

$$\|(e^{it\Delta} P_K u_0)(e^{it\Delta} P_M v_0)\|_{L_{t,x}^2(\mathbb{R} \times \mathbb{R}^3)} \lesssim KM^{-\frac{1}{2}} \|P_{\leq K} u_0\|_{L_x^2} \|P_M v_0\|_{L_x^2} \quad (2.2.6)$$

has transferred versions to the context of the spaces U^2 and V^2 , that read

$$\begin{aligned} \|P_K u P_M v\|_{L_{t,x}^2(\mathbb{R} \times \mathbb{R}^3)} &\lesssim KM^{-\frac{1}{2}} \|P_K v\|_{U_\Delta^2} \|P_M v\|_{U_\Delta^2}, \\ \|P_K u P_M v\|_{L_{t,x}^2(\mathbb{R} \times \mathbb{R}^3)} &\lesssim KM^{-\frac{1}{2}+} \|P_K v\|_{V_\Delta^2} \|P_M v\|_{V_\Delta^2}. \end{aligned} \quad (2.2.7)$$

Lemma 2.2.7 (Time continuity and sub-additivity, see Lemma A.4 and A.8 in [BOP15a]). *Let $J = [a, b)$ and $u \in X^\sigma(J) \cap C(J; H_x^\sigma(\mathbb{R}^3))$. The mapping*

$$t \in J \rightarrow \|u\|_{X^\sigma[a,t]}$$

is continuous. Moreover, given a partition $J = \bigcup_{k=1}^L J_k$, we have

$$\|u\|_{X^\sigma(J)} \leq \sum_{k=1}^L \|u\|_{X^\sigma(J_k)}.$$

2.2.2 Large deviation estimates for the free Schrödinger evolution

First, we recall the following probabilistic decoupling estimate. It expresses the gain of integrability that emerges from probabilistic averaging effects, due to the cancelation of interferences.

Lemma 2.2.8 (Lemma 3.1 in [BT08a]). *Let $\{g_n\}$ be a sequence of real valued, zero-mean and independent random variables with distribution $\{\mu_n\}$ on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$. Assume that there exists $c > 0$ such that for any $\gamma \in \mathbb{R}$ and any $n \geq 0$ we have*

$$\left| \int_{\mathbb{R}} e^{\gamma x} d\mu_n(x) \right| \leq e^{c\gamma^2}.$$

Then, there exists $C > 0$ such that for every $2 \leq p \leq \infty$ and for every sequence of complex numbers $\{c_n\}$ in $\ell^2(\mathbb{N}; \mathbb{C})$ it holds that

$$\left\| \sum_{n \in \mathbb{N}} c_n g_n(\omega) \right\|_{L_\omega^p(\Omega)} \leq C \sqrt{p} \left(\sum_{n \in \mathbb{N}} |c_n|^2 \right)^{\frac{1}{2}}. \quad (2.2.8)$$

From this, we obtain the general large deviation estimate, on which rely all the estimates stated in this section.

Lemma 2.2.9 (Large deviation estimate). *Let F be a real valued measurable function on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$. Assume that there exist some constants $C_0 > 0$, $M > 0$ and $p_0 \geq 1$ such that for all $p \geq p_0$ we have*

$$\|F\|_{L^\omega_p(\Omega)} \leq C_0 \sqrt{p} M. \quad (2.2.9)$$

Then, there exist $C = C(C_0)$ and $c = c(C_0) > 0$ independent of M , such that for all $\lambda > 0$,

$$\mathbb{P}(\{\omega \in \Omega \mid |F(\omega)| > \lambda\}) \leq C \exp(-c\lambda^2 M^{-2}). \quad (2.2.10)$$

In the following lemma, we recall the *probabilistic* Strichartz estimates, that provide improved global spacetime bounds by loosening the Strichartz admissibility condition.

Lemma 2.2.10 (Improved Strichartz estimates). *Let (q, r) be a Strichartz admissible pair, i.e $s(q, r) = 0$. For all $\tilde{r} \geq r$, there exist $C = C(q, \tilde{r}) > 0$ and $c = c(q, r) > 0$ such that for all $f_0 \in H^s(\mathbb{R}^3)$ and $\lambda > 0$, we have*

$$\mathbb{P}(\{\omega \in \Omega \mid \|\langle \nabla \rangle^s e^{it\Delta} f_0^\omega\|_{L_t^q(\mathbb{R}; L_x^{\tilde{r}}(\mathbb{R}^3))} > \lambda\}) \leq C \exp(-c\lambda^2 \|f_0\|_{H^s}^{-2}). \quad (2.2.11)$$

Proof. The proof is standard, and we refer to [BOP15a; BOP15b]. To prove (2.2.9) for $p > \max(q, \tilde{r})$, we apply Minkowski inequality and the decoupling estimate (2.2.8). Then, we apply the unit-scale Bernstein estimate on each frequency bloc, that yields a bound on the Fourier multiplier Q_k from L_x^r to $L_x^{\tilde{r}}$, uniformly in k . The Strichartz estimate applied with (q, \tilde{r}) concludes. \square

Next, we state a uniform in time large deviation estimate for the linear evolution of randomized initial data.

Lemma 2.2.11. *Let f_0 in $H^s(\mathbb{R}^3)$ for some $s > 0$. For all $q > 2$, there exist $C, c > 0$ such that for all $\lambda > 0$,*

$$\mathbb{P}(\{\omega \in \Omega \mid \|e^{it\Delta} f_0^\omega\|_{L_t^\infty(\mathbb{R}; L_x^q(\mathbb{R}^3))} > \lambda\}) \leq C \exp(-c\lambda^2 \|f_0\|_{H^s}^{-2}).$$

Proof. See Lemma 5.15 in [DLM19]. This follows from a Sobolev embedding in time, and from the improved Strichartz estimate stated in the previous Lemma. \square

We deduce from these large deviation bounds, together with lemma 2.2.9, the following spacetime estimates on f^ω , for some ω in a large measure set.

Lemma 2.2.12. *For every $\alpha > 0$ there exist a set Ω_α and a constant C_α depending on $\|f_0\|_{H^s}$ such that $\mathbb{P}(\Omega \setminus \Omega_\alpha) \leq \alpha$, and for every $\omega \in \Omega_\alpha$ we have*

$$F^\omega(\mathbb{R}) + F_\infty^\omega(\mathbb{R}) \leq C_\alpha. \quad (2.2.12)$$

In what follows, $\omega \in \Omega_\alpha$ is fixed such that (2.2.12) holds. At this stage, we do not need the radial assumption on f_0 .

2.2.3 Additional estimate for randomized radial initial data

The following improved Sobolev embedding for radial functions was introduced in [DLM20], and used in [DLM19] in order to prove almost-sure scattering for the energy critical Schrödinger and wave equation in \mathbb{R}^4 . Here, we present an analog of this lemma in dimension 3, whose proof needs some slight modifications.

Lemma 2.2.13 (Improved Sobolev embedding for radial functions in \mathbb{R}^3). *For all $\delta > 0$, there exist $C_\delta > 0$ such that for all radially-symmetric function f_0 in $H^\delta(\mathbb{R}^3)$ we have*

$$\left\| \langle x \rangle \left(\sum_{k \in \mathbb{Z}^3} |Q_k f_0|^2 \right)^{\frac{1}{2}} \right\|_{L_x^\infty(\mathbb{R}^3)} \leq C_\delta \|f_0\|_{H^\delta(\mathbb{R}^3)}. \quad (2.2.13)$$

Proof. Up to changing the coordinates we assume that $x = |x|e_3$, where $e_3 = (0, 0, 1)$ and $|x| \gg 1$. We denote by (ρ, θ, φ) the spherical coordinates in the Fourier space \mathbb{R}_ξ^3 :

$$\xi = \xi(\rho, \theta, \varphi) = (\rho \cos(\varphi) \sin(\theta), \rho \sin(\varphi) \sin(\theta), \rho \sin(\varphi) \cos(\theta)),$$

with $\rho > 0$, $\theta \in (0, \pi)$ and $\varphi \in [0, 2\pi]$. Given $k \in \mathbb{Z}^3$ such that $1 \ll |k|$, we write

$$\begin{aligned} Q_k f(x) &= \int_{\mathbb{R}^3} e^{i\xi \cdot x} \psi_k(\xi) \widehat{f}(\xi) d\xi \\ &= \int_0^{+\infty} \int_0^{2\pi} \left(\int_0^\pi e^{i|x|\rho \cos \theta} \psi_k(\xi(\rho, \theta, \varphi)) \sin \theta d\theta \right) d\varphi \rho^2 \widehat{f}_0(\rho) d\rho. \end{aligned} \quad (2.2.14)$$

Hence, we need to study the following oscillatory integral

$$I(x, \rho, \varphi) = \int_0^\pi e^{i|x|\rho \cos \theta} \psi_k(\xi(\rho, \theta, \varphi)) \sin \theta d\theta. \quad (2.2.15)$$

The function ψ_k and its derivatives provide a localization on the unit cube centered around k . More precisely, this localization reads in spherical coordinates as follows : if $\xi(\rho, \theta, \varphi)$ is in $\text{supp}(\psi_k)$, then

- ρ is in an interval of size ~ 1 around $|k|$.
- θ in an interval of size $\sim \frac{1}{|k|}$ around the angle θ_k , where $\sin(\theta_k) \sim \sqrt{1 - \left(\frac{k}{|k|} e_3\right)^2}$.
- φ is in an interval J_k of size $\sim \min(1, \frac{1}{|k| \sin(\theta_k)})$.

To gain decay with respect to x , we integrate by parts in the variable θ . It holds

$$\begin{aligned} I(x, \rho, \varphi) &= -\frac{1}{i|x|\rho} \int_0^\pi \partial_\theta (e^{i\rho \cos \theta |x|}) \psi_k(\xi(\rho, \theta, \varphi)) d\theta \\ &= -\frac{1}{i|x|\rho} \left[e^{i\rho \cos \theta |x|} \psi_k(\xi(\rho, \theta, \varphi)) \right]_0^\pi + \frac{1}{i|x|\rho} \int_0^\pi e^{i\rho \cos \theta |x|} \partial_\theta \varphi_k(\xi(\rho, \theta, \varphi)) d\theta. \end{aligned} \quad (2.2.16)$$

Moreover, we have for all $\xi(\rho, \theta, \varphi) \in \mathbb{R}^3$, all $x \in \mathbb{R}^3$ and $k \in \mathbb{Z}^3$ that

$$\left| \frac{\partial}{\partial \theta} \psi_k(\xi) \right| \leq \rho |\nabla_\xi \psi_k(\xi)|. \quad (2.2.17)$$

Keeping in mind the localization in the variables ρ, θ yielded by the amplitude ψ_k and its derivatives, we deduce from (2.2.16) and (2.2.17) that

$$|I(x, \rho, \varphi)| \lesssim \frac{1}{|x|\rho} \mathbf{1}_{(|k|-1, |k|+1)}(\rho) \mathbf{1}_{J_k}(\varphi). \quad (2.2.18)$$

Using that φ is localized on the interval J_k of size $\min(1, \frac{1}{|k|\sin\theta_k})$, integrating over the domain $[|k|-1, |k|+1] \times J_k$ and using Cauchy-Schwarz in ρ , we get

$$\begin{aligned} |Q_k(f_0)(x)| &\lesssim \frac{1}{|k||x|} \min(1, \frac{1}{|k|\sin\theta_k}) \|\mathbf{1}_{[|k|-1, |k|+1]}(\rho) \widehat{f_0}(\rho) \rho^2\|_{L^1_\rho(\mathbb{R})} \\ &\lesssim \min(1, \frac{1}{|k|\sin\theta_k}) \|\mathbf{1}_{[|k|-1, |k|+1]}(\rho) \widehat{f_0}(\rho) \rho\|_{L^2_\rho(\mathbb{R})}. \end{aligned}$$

Summing the above estimate over \mathbb{Z}^3 gives

$$\begin{aligned} |x|^2 \sum_{k \in \mathbb{Z}^3} |Q_k(f_0)(x)|^2 &\lesssim \sum_{k \in \mathbb{Z}^3} \min(1, \frac{1}{|k|\sin\theta_k})^2 \|\mathbf{1}_{[|k|-1, |k|+1]}(\rho) \widehat{f_0}(\rho) \rho\|_{L^2_\rho(\mathbb{R})}^2 \\ &\lesssim \sum_{j \in \mathbb{N}} \sum_{l=0}^j \frac{\#\{k \in \mathbb{Z}^3 \mid |k|=j, |j\sin(\theta_k) - l| \leq 1\}}{(1+l)^2} \|\mathbf{1}_{[j-1, j+1]}(\rho) \widehat{f_0}(\rho) \rho\|_{L^2_\rho(\mathbb{R})}^2 \\ &\lesssim \sum_{j \in \mathbb{N}} \sum_{l=0}^j \frac{1}{1+l} \|\mathbf{1}_{[j-1, j+1]}(\rho) \widehat{f_0}(\rho) \rho\|_{L^2_\rho(\mathbb{R})}^2 \\ &\lesssim \sum_{j \geq 1} \log(1+j) \|\mathbf{1}_{[j-1, j+1]}(\rho) \widehat{f_0}(\rho) \rho\|_{L^2_\rho(\mathbb{R})}^2 \lesssim_\delta \|f_0\|_{H^\delta(\mathbb{R}^3)}^2. \end{aligned}$$

This ends the proof of Lemma 2.2.13. \square

Corollary 2.2.14. *Let $2 \leq r \leq \infty$, $K \in 2^{\mathbb{N}}$ and $\delta > 0$. There exists a constant $C_\delta > 0$ such that for any radial function $f_0 \in L^2(\mathbb{R}^3)$ we have*

$$\left\| \langle x \rangle^{1-\frac{2}{r}} \left(\sum_{k \in \mathbb{Z}^3} |Q_k P_K f_0|^2 \right)^{\frac{1}{2}} \right\|_{L^r_x(\mathbb{R}^3)} \leq C_\delta K^\delta \|P_K f_0\|_{L^2_x(\mathbb{R}^3)}. \quad (2.2.19)$$

Proof. The estimate (2.2.19) is obtained by interpolation between (2.2.13) and the trivial estimate

$$\left\| \left(\sum_{k \in \mathbb{Z}^3} |Q_k P_K f_0|^2 \right)^{\frac{1}{2}} \right\|_{L^2_x(\mathbb{R}^3)} \leq \|P_K f_0\|_{L^2_x(\mathbb{R}^3)}.$$

It holds

$$\begin{aligned} \left\| \langle x \rangle^{1-\frac{2}{r}} \left(\sum_{k \in \mathbb{Z}^3} |Q_k P_K f_0|^2 \right)^{\frac{1}{2}} \right\|_{L^r_x} &\leq \left\| \langle x \rangle \left(\sum_{k \in \mathbb{Z}^3} |Q_k P_K f_0|^2 \right)^{\frac{1}{2}} \right\|_{L^\infty_x}^{1-\frac{2}{r}} \left\| \left(\sum_{k \in \mathbb{Z}^3} |Q_k P_K f_0|^2 \right)^{\frac{1}{2}} \right\|_{L^2_x}^{\frac{2}{r}} \\ &\leq C_\delta \|P_K f_0\|_{H^\delta}^{1-\frac{2}{r}} \|P_K f_0\|_{L^2_x}^{\frac{2}{r}} \\ &\lesssim C_\delta K^\delta \|P_K f_0\|_{L^2_x}. \end{aligned}$$

\square

We will use the local smoothing estimate under the following form.

Lemma 2.2.15 (Local smoothing, [CS88]). *For any $d \geq 3$ and $\alpha > 0$, there exists a constant C such that for all $f_0 \in L^2(\mathbb{R}^d)$,*

$$\sup_{R>0} \|e^{it\Delta} f_0\|_{L^2_{t,x}(\mathbb{R} \times \{|x|<R\})} \leq CR^{\frac{1}{2}} \|\nabla|^{-\frac{1}{2}} f_0\|_{L^2_x(\mathbb{R}^d)}. \quad (2.2.20)$$

Recall that the Fourier multiplier $I : H_x^\sigma \rightarrow H_x^1$ was defined in (2.1.3)

Proposition 2.2.16. *Let $s > \frac{1}{4}$ and $\sigma > \frac{1}{2}$ such that $\sigma < 2s$. Then, for all $r \in (4, +\infty]$ and $\delta > 0$, there exist $N_0, C > 0$ and $c > 0$ depending on r, δ, s, σ such that for all $\lambda > 0$ and $N \geq N_0$, we have the large deviation estimates*

$$\mathbb{P}(\{\omega \in \Omega \mid \|\nabla I e^{it\Delta} f_0^\omega\|_{L^2_t(\mathbb{R}; L^r_x(\mathbb{R}^3))} > \lambda\}) \leq C \exp(-c\lambda^2(N^{-\delta} N^{\frac{1-\sigma}{2}} \|f_0\|_{H_x^s})^{-2}). \quad (2.2.21)$$

Proof. It suffices to prove that for any $\delta > 0$ small enough (depending on s and σ), there exists a constant C_δ such that for any $p > r$ one has

$$\|\nabla I e^{it\Delta} f_0^\omega\|_{L^p_\omega L^2_t L^r_x} \leq C_\delta \sqrt{p} N^{-\delta} N^{\frac{1-\sigma}{2}} \|f_0\|_{H_x^s(\mathbb{R}^3)}. \quad (2.2.22)$$

Subsequently, (2.2.21) follows from Lemma 2.2.9, with $F := \|\nabla I e^{it\Delta} f_0^\omega\|_{L^2_t(\mathbb{R}; L^r_x(\mathbb{R}^3))}$ and $M = N^{0-\frac{1-\sigma}{2}} \|f_0\|_{H_x^s}$. To prove (2.2.22), we dyadically decompose f_0 in the frequency space. We reduce the case when $r = \infty$ to the case where r is finite by applying the Bernstein estimate. For any $3 < r < \infty$, we get

$$\|\nabla e^{it\Delta} I f_0^\omega\|_{L^p_\omega L^2_t L^r_x} \lesssim \|\nabla P_{\leq 1} e^{it\Delta} f_0^\omega\|_{L^p_\omega L^2_t L^r_x} + \sum_{K \in 2^{\mathbb{N}^*}} K^{\frac{3}{r}} \|\nabla P_K I e^{it\Delta} f_0^\omega\|_{L^p_\omega L^2_t L^r_x}. \quad (2.2.23)$$

Estimating the low-frequency term is standard, and we refer to [BOP15b]. However, to handle the high-frequency terms, we used the radial Sobolev embedding (2.2.19) and the local smoothing estimate. Let us fix $K \geq 2$ a dyadic integer. Then, we dyadically decompose the term on right-hand side of (2.2.23) in the physical space, with some cutoff χ_j introduced in the notations paragraph. By the triangle inequality, we have

$$\|\nabla P_K I e^{it\Delta} f_0^\omega\|_{L^p_\omega L^2_t L^r_x} \leq \sum_{j \geq 0} \|\chi_j \nabla P_K I e^{it\Delta} f_0^\omega\|_{L^p_\omega L^2_t L^r_x}.$$

Next, for each $j \geq 0$, and $p \geq r$, we use Minkowski's inequality and the decoupling estimate (2.2.8) to get

$$\|\chi_j \nabla P_K I e^{it\Delta} f_0^\omega\|_{L^p_\omega L^2_t L^r_x} \lesssim \sqrt{p} \left\| \left(\sum_{\substack{k \in \mathbb{Z}^3 \\ |k| \sim K}} |\chi_j Q_k \nabla P_K I e^{it\Delta} f_0| \right)^2 \right\|_{L^2_t L^r_x}^{\frac{1}{2}}.$$

We split the sum into a localized term, and a remainder.

$$\begin{aligned} \|\chi_j \nabla P_K I e^{it\Delta} f_0^\omega\|_{L^p_\omega L^2_t L^r_x} &\lesssim \sqrt{p} \left\| \left(\sum_{\substack{k \in \mathbb{Z}^3 \\ |k| \sim K}} |\chi_j Q_k \chi_{\leq j+5} \nabla P_K I e^{it\Delta} f_0| \right)^2 \right\|_{L^2_t L^r_x}^{\frac{1}{2}} \\ &\quad + \sqrt{p} \left\| \left(\sum_{\substack{k \in \mathbb{Z}^3 \\ |k| \sim K}} |\chi_j Q_k \chi_{> j+5} \nabla P_K I e^{it\Delta} f_0| \right)^2 \right\|_{L^2_t L^r_x}^{\frac{1}{2}}. \end{aligned}$$

To estimate the remainder, we refer to [DLM19]. The proof is a bit technical but only uses Young inequality the local smoothing estimate (2.2.20) and some operator bounds collected in the following lemma that holds for dimension 3 without changing the proof.

Lemma 2.2.17 (Lemma 5.10 and Lemma 5.11 from [DLM19]). *Let $2 \leq r \leq \infty$. For any $k \in \mathbb{Z}^3$, any dyadic $J, L \in 2^{\mathbb{N}}$ with $2^5 J < L$, and any $\nu > 0$, there exists $C_\nu > 0$ such that*

$$\|\chi_j Q_k \chi_l\|_{L_x^2(\mathbb{R}^3) \rightarrow L_x^r(\mathbb{R}^3)} \leq C_\nu L^{-\nu}. \quad (2.2.24)$$

Similarly, given $L \in 2^{\mathbb{N}}$ and $k, m \in \mathbb{Z}^3$ such that $|k - m| \geq 100$, it holds that

$$\|Q_k \chi_l Q_m\|_{L_x^2(\mathbb{R}^3) \rightarrow L_x^r(\mathbb{R}^3)} \leq C_\nu L^{-\alpha} |k - m|^{-\nu}. \quad (2.2.25)$$

However, we detail how to handle the main term, since this requires to combine the local smoothing estimate (2.2.20) with the improved radial Sobolev embedding (2.2.13).⁶ Given $j \geq 0$, we have

$$\begin{aligned} & \left\| \left(\sum_{\substack{k \in \mathbb{Z}^3 \\ |k| \sim K}} |\chi_j Q_k \chi_{\leq j+5} \nabla P_K I e^{it\Delta} f_0|^2 \right)^{\frac{1}{2}} \right\|_{L_t^2 L_x^r} \\ & \lesssim 2^{-j(1-\frac{2}{r})} \left\| \langle x \rangle^{1-\frac{2}{r}} \left(\sum_{\substack{k \in \mathbb{Z}^3 \\ |k| \sim K}} |Q_k \chi_{\leq j+5} \nabla P_K I e^{it\Delta} f_0|^2 \right)^{\frac{1}{2}} \right\|_{L_t^2 L_x^r}. \end{aligned}$$

By the improved radial Sobolev embedding (2.2.19), we have

$$\left\| \langle x \rangle^{1-\frac{2}{r}} \left(\sum_{\substack{k \in \mathbb{Z}^3 \\ |k| \sim K}} |Q_k \chi_{\leq j+5} \nabla P_K I e^{it\Delta} f_0|^2 \right)^{\frac{1}{2}} \right\|_{L_t^2 L_x^r} \lesssim_\delta K^\delta \|\chi_{\leq j+5} |\nabla| P_K I e^{it\Delta} f_0\|_{L_t^2 L_x^2}.$$

Next, we apply the local smoothing estimate (2.2.20) to see that the above line is controlled by

$$\left\| \left(\sum_{\substack{k \in \mathbb{Z}^3 \\ |k| \sim K}} |\chi_j Q_k \chi_{\leq j+5} \nabla P_K I e^{it\Delta} f_0|^2 \right)^{\frac{1}{2}} \right\|_{L_t^2 L_x^r} \lesssim_\delta 2^{-j(1-\frac{2}{r}-\frac{1}{2})} K^{\frac{1}{2}+\delta} \|P_K I f_0\|_{L_x^2}.$$

The above expression can be summed over j provided that $r > 4$. In addition, we observe that if $K > 2N$, there exists $\gamma > 0$ such that

$$K^{\frac{1}{2}+\delta} \|P_K I f_0\|_{L_x^2} \leq N^{1-\sigma} K^{\sigma-\frac{1}{2}-s+\delta} \|P_K f_0\|_{H_x^s}.$$

Indeed, Plancherel estimate and the definition of the I -multiplier (2.1.3) yield

$$\begin{aligned} K^{1+2\delta} \|P_K I f_0\|_{L_x^2}^2 &= K^{1+2\delta} \int_{2^{-1}K < |\xi| < 2K} \left(\frac{N}{|\xi|} \right)^{2(1-\sigma)} |\widehat{P_K f_0}(\xi)|^2 d\xi \\ &\lesssim N^{2(1-\sigma)} K^{2(\sigma-1)+1-2s+2\delta} \|P_K f_0\|_{H_x^s}^2, \end{aligned}$$

⁶ The case when $j = 0$ is actually easier and does not require to use the improved radial Sobolev embedding. We refer to the original proof in [DLM20] where this term is handled separately.

When $K \leq 2N$, we directly get that

$$\|\nabla|\cdot|^{\frac{1}{2}+\delta}P_K I f_0\|_{L_x^2} \leq N^{\frac{1}{2}-s+\delta}\|P_K f_0\|_{H_x^s}.$$

We now sum over K in estimate (2.2.23) and get that for all $r > 4$ and $\delta > 0$,⁷

$$\begin{aligned} \|\nabla e^{it\Delta} I F\|_{L_t^p L_t^2 L_x^\infty} &\lesssim_{\delta,r} \sqrt{p}\|P_{\leq 1} f_0\|_{L_x^2} + \sqrt{p} \sum_{\substack{K \in 2^{\mathbb{N}^*} \\ K \leq 2N}} K^{\frac{1}{2}-s+\delta}\|P_K f_0\|_{H_x^s} \\ &\quad + \sqrt{p} N^{1-\sigma} \sum_{\substack{K \in 2^{\mathbb{N}^*} \\ K > 2N}} K^{\sigma-\frac{1}{2}-s+\delta}\|P_K f_0\|_{H_x^s}. \end{aligned}$$

Define $\gamma_0(s, \sigma) = s - \frac{\sigma}{2} > 0$ such that $\frac{1}{2} - s = \frac{1-\sigma}{2} - \gamma_0$.⁸ Next, we chose δ and γ_0 such that

$$\sigma - \frac{1}{2} - s + \delta < 0, \quad \text{and} \quad \frac{1}{2} - s \leq \frac{1-\sigma}{2} - \gamma_0. \quad (2.2.26)$$

By using Cauchy Schwarz, we conclude that

$$\|\nabla I e^{it\Delta} f_0^\omega\|_{L_t^p L_t^2 L_x^\infty} \lesssim_{\delta,r} \sqrt{p} N^{\frac{1}{2}-s+\delta}\|P_K f_0\|_{H_x^s}.$$

This ends the proof of Proposition 2.2.16. \square

We deduce from the above large deviation estimates the following bounds on the linear evolution of the randomized initial data.

Proposition 2.2.18. *For all $\alpha > 0$, there exist a set $\tilde{\Omega}_\alpha \subset \Omega_\alpha$, with $\mathbb{P}(\Omega_\alpha \setminus \tilde{\Omega}_\alpha) \leq \alpha$, a constant $C_\alpha > 0$ and $N_0(\alpha)$ such that for all $\omega \in \Omega_\alpha$, and all $N \geq N_0$ we have*

$$F_2^\omega(\mathbb{R}) \leq C_\alpha N^{\frac{1-\sigma}{2}}. \quad (2.2.27)$$

Proof. Estimate (2.2.27) follows from the large deviation estimate (2.2.21) by choosing

$$\lambda = C_\alpha N^{0-} N^{\frac{1-\sigma}{2}} \|f_0\|_{H_x^s}, \quad C_\alpha^2 \geq c^{-1} \ln(C_\alpha^{-1}). \quad \square$$

2.3 Cauchy theory for the forced NLS equation

The main step of this section is to come with a conditional scattering result for the solutions v to the perturbed Schrödinger equation with a random forcing term (NLS_f). More precisely, we prove that an a priori uniform estimate for the $H^\sigma(\mathbb{R}^3)$ norm of v on its maximal lifespan yields global existence and scattering, when $\frac{2}{3} < \sigma \leq 1$. To prove such a result, we essentially follow the same lines as in the proof of Proposition 3.1 from [KMV19]. However, since v lies below the energy space,

⁷In the case when we look at the $L_t^2 L_x^\infty$ norm, we need to add a K to the power $\frac{3}{r}$, for r arbitrarily large, that comes from the Sobolev embedding from $W^{\frac{3}{r}, r}(\mathbb{R}^3)$ to $L^\infty(\mathbb{R}^3)$. This term is harmless, and can be absorbed in the residual power K^δ .

⁸ Recall the assumption that $s \leq \frac{1}{2}$ and $\sigma < 2s$.

we need to develop a stability theory in $H^\sigma(\mathbb{R}^3)$, and not in the energy space. To achieve such a goal, we appeal to the global Cauchy theory in such subcritical regimes provided by [Col+04; Dod13], that provides the spacetime global estimate

$$\|u\|_{L_{t,x}^5} \leq C(\|u(t_0)\|_{H^\sigma(\mathbb{R}^3)}).$$

for solutions u to (NLS) with $u(t_0) \in H^\sigma(\mathbb{R}^3)$. With this global spacetime bound at hand, we can use sub-additivity and reduce the analysis to intervals J where we have a smallness assumption on the $L_{t,x}^5(J)$ norm of u , which is a scaling-critical quantity, and hence on v which is expected to stay close to u in $H^\sigma(\mathbb{R}^3)$. For this reason, we are led to refine the trilinear stochastic estimates from [BOP19a], and to estimate the Duhamel nonlinear term not only by the critical norm $X^\sigma(J)$, which is not small, but also by some powers of the $L_{t,x}^5(J)$ norm of v to gain smallness. This is the matter of the refined nonlinear estimates from Propositions 2.3.2 and 2.3.3. Throughout this section, we fix $\alpha > 0$ and $\omega \in \Omega_\alpha$ such that

$$F^\omega(\mathbb{R}) \leq C_\alpha.$$

In particular, we do not use the quantity $F_2^\omega(\mathbb{R})$, so that we can release the radial assumption on f_0 .

2.3.1 Nonlinear estimates

Before diving into the nonlinear analysis, let us detail how to address the Littlewood-Paley summation of L^q norms of dyadic blocs u_N , for a given function u that comes with a gain of regularity materialized by the presence of a negative power of N in front of each paraproduct u_N .

Observation 2.3.1. *Let $q > 2$ and u in $L_t^q(J; W_x^{\gamma,q}(\mathbb{R}^3))$ for some $\gamma \in \mathbb{R}$. We denote by $u \sim \sum_{N \in 2^{\mathbb{N}}} u_N$ the Littlewood-Paley decomposition of u . It holds*

$$\sum_{N \in 2^{\mathbb{N}}} N^{0-} N^\gamma \|u_N\|_{L_{t,x}^q(J \times \mathbb{R}^3)} \lesssim \|u\|_{L_t^q(J; W_x^{\gamma,q}(\mathbb{R}^3))}. \quad (2.3.1)$$

Proof. From Hölder, we have

$$\begin{aligned} \sum_{N \in 2^{\mathbb{N}}} N^{0-} N^\gamma \|u_N\|_{L_{t,x}^q(J \times \mathbb{R}^3)} &\leq \left(\sum_{N \in 2^{\mathbb{N}}} N^{0-} \right)^{\frac{1}{q'}} \left(\sum_{N \in 2^{\mathbb{N}}} (N^\gamma \|u_N\|_{L_{t,x}^q(J \times \mathbb{R}^3)})^q \right)^{\frac{1}{q}} \\ &\lesssim \|N^\gamma u_N\|_{\ell_N^q(2^{\mathbb{N}}; L_{t,x}^q(J \times \mathbb{R}^3))} = \|N^\gamma u_N\|_{L_{t,x}^q(J \times \mathbb{R}^3; \ell_N^q(2^{\mathbb{N}}))}. \end{aligned}$$

Since $q \geq 2$, we obtain

$$\|N^\gamma u_N\|_{L_{t,x}^q(J \times \mathbb{R}^3; \ell_N^q(2^{\mathbb{N}}))} \leq \|N^\gamma u_N\|_{L_{t,x}^q(J \times \mathbb{R}^3; \ell_N^2(2^{\mathbb{N}}))}.$$

We conclude the proof of (2.3.1) by applying the Littlewood-Paley square function theorem

$$\|u_N\|_{L_{t,x}^q(J \times \mathbb{R}^3; \ell_N^2(2^{\mathbb{N}}))} = \left\| \left(\sum_{N \in 2^{\mathbb{N}}} |u_N|^2 \right)^{\frac{1}{2}} \right\|_{L_{t,x}^q} \lesssim \|u\|_{L_{t,x}^q(J \times \mathbb{R}^3)}. \quad \square$$

Now, we establish quadrilinear estimates that involve three types of terms. First, we have the stochastic forcing terms of type $f = e^{it\Delta} f_0^\omega$, for which we obtained in Proposition 2.2.10 improved Strichartz estimates. More precisely, we have that $F^\omega(\mathbb{R}) < +\infty$, and using sub-additivity, we will reduce the analysis to a finite number of intervals where $F^\omega(J)$ is small. Then, we have the term v solution to the forced Schrödinger equation (NLS_f) at the subcritical regularity $H^\sigma(\mathbb{R}^3)$, and that corresponds to the nonlinear Duhamel term for the solution $u = e^{it\Delta} f_0^\omega + v$. We want to obtain a priori estimates for v in the spacetime spaces X^σ and $L_{t,x}^5$, at least, on some intervals where there holds a smallness assumption on the forcing term. Finally, the terms $w \in Y^0$ that appear in the analysis come from duality.

Proposition 2.3.2 (Trilinear estimates with random terms). *Let $\frac{1}{4} < s \leq \frac{1}{2}$ and $\frac{1}{2} < \sigma < 2s$. Denote by $f^\omega = e^{it\Delta} f_0^\omega$ the linear evolution of the randomized initial data. There exists a constant $C(\|f_0\|_{H^\sigma}) > 0$ such that for all interval $J \subseteq \mathbb{R}$ where $F^\omega(J) \leq 1$, and all $v \in X^\sigma(J)$ with $\|v\|_{L_{t,x}^5(J)} \leq 1$, it holds*

$$\|\mathcal{I}(\cdot, \mathcal{N}(f, f, f))\|_{X^\sigma(J)} \leq C(\|f_0\|_{H^\sigma}) F^\omega(J), \quad (2.3.2)$$

$$\|\mathcal{I}(\cdot, \mathcal{N}(f, f, v))\|_{X^\sigma(J)} \leq C(\|f_0\|_{H^\sigma}) F^\omega(J) \|v\|_{X^\sigma(J)}, \quad (2.3.3)$$

$$\|\mathcal{I}(\cdot, \mathcal{N}(f, v, v))\|_{X^\sigma(J)} \leq C(\|f_0\|_{H^\sigma}) F^\omega(J)^{\frac{1}{2}} \|v\|_{L_{t,x}^5(J)} \|v\|_{X^\sigma(J)}. \quad (2.3.4)$$

In particular, the above trilinear estimates yield the following estimate for the forcing term of (NLS_f) at the regularity H_x^σ : there exists $C = C(\|f_0\|_{H^s}) > 0$ such that

$$\|\mathcal{I}(\cdot, \mathcal{N}(v+f) - \mathcal{N}(v))\|_{X^\sigma(J)} \lesssim C(\|f_0\|_{H^s}) F^\omega(J) + C\{F^\omega(J) + F^\omega(J)^{\frac{1}{2}} \|v\|_{L_{t,x}^5(J)}\} \|v\|_{X^\sigma(J)}. \quad (2.3.5)$$

Proof. In the following, we take all the spacetime norms over $J \times \mathbb{R}^3$. Using the duality between U^2 and V^2 (see Proposition 2.2.5), we have ⁹

$$\|v\|_{X^\sigma(J)}^2 \leq \sum_N \mathcal{I}_N^2,$$

where, denoting $w^{(i)} \in \{v, f\}$ for $i \in \{1, 2, 3\}$, we have that for all $v \in X^\sigma(J)$

$$\mathcal{I}_N := N^\sigma \|P_N v\|_{U_\Delta^2(J)} \leq CN^\sigma \sup_{\|w\|_{V_\Delta^2(J)} \leq 1} \left| \iint_{J \times \mathbb{R}^3} P_N(w^{(1)} \overline{w^{(2)}} w^{(3)}) \overline{w} dx dt \right|.$$

Given a fixed function w with $\|w\|_{V_\Delta^2(J)} \leq 1$, we perform a Littlewood-Paley decomposition of each term ¹⁰ and we are left to estimate quadrilinear spacetime integrals of the form

$$\mathcal{I}_N \leq \sum_{L=(N_1, N_2, N_3)} N_{(1)}^\sigma \left| \iint_{J \times \mathbb{R}^3} w_{N_1}^{(1)} \overline{w_{N_2}^{(2)}} w_{N_3}^{(3)} \overline{w_N} dx dt \right| =: \sum_{L=(N_1, N_2, N_3)} N_{(1)}^\sigma \mathcal{I}_N(L). \quad (2.3.6)$$

⁹ To apply Proposition 2.2.5 and to be in position to apply formula (2.2.4), we need to make sure that $\mathcal{N}(u)$ is in $L_t^1(J; H_x^\sigma)$. We omit this short verification which is essentially contained in the analysis, and we refer to [BOP15a] where the authors prove that $P_{\leq N} \mathcal{N}(u) \in L_t^1(J; H_x^\sigma)$ for any $N \in 2^{\mathbb{N}}$.

¹⁰ Note that the sums over N_1, N_2, N_3 under the integral are absolutely convergent in \mathbb{C} . Hence, we can intertwine the sum and the integral, and then apply the triangle inequality to obtain, (2.3.6).

For dyadic integers N_1, N_2, N_3 , we denote by $N_{(1)} \geq N_{(2)} \geq N_{(3)}$ the non-increasing ordering among them, and we use the shorthand notation $w_{N_i} := P_{N_i} w^{(i)}$, $i \in \{1, 2, 3\}$. Note that it suffices to consider the nontrivial cases, where $N \lesssim N_{(1)}$. To be able to sum the terms \mathcal{I}_N over N , we need to gain a negative power of N after summing \mathcal{I}_N^L over L . Hence, we shall bound the above quadrilinear integrals by some negative power of the highest frequency $N_{(1)}$, and by some appropriate norms of each w_i . Then, we proceed as in observation 2.3.1.

Proof of (2.3.2) Here we address the first Duhamel iteration, that contains products with three random terms

$$\mathcal{I}_N(N_1, N_2, N_3) := -i \int_0^t e^{i(t-t')\Delta} \mathcal{N}(f_{N_1}, f_{N_2}, f_{N_3})(t') dt'.$$

By symmetry, we can assume that $N_1 \geq N_2 \geq N_3$ without loss of generality.

- **Case 1 : High-high-high** $N_1^{\frac{1}{2}} \lesssim N_3$. Applying Hölder with $1 = \frac{3}{10} + \frac{7}{10}$ yields

$$\begin{aligned} N_1^\sigma |\mathcal{I}_N(N_1, N_2, N_3)| &= N_1^\sigma \left| \iint_{J \times \mathbb{R}^3} f_{N_1} \overline{f_{N_2}} f_{N_3} w_N dx dt \right| \\ &\lesssim N_1^{\sigma-s} N_2^{-s} N_3^{-s} \prod_{1 \leq j \leq 3} \|\langle \nabla \rangle^s f_{N_j}\|_{L_{t,x}^{\frac{30}{7}}} \|w_N\|_{L_{t,x}^{\frac{10}{3}}} \\ &\lesssim N_1^{\sigma-2s} \prod_{1 \leq j \leq 3} \|\langle \nabla \rangle^s f_{N_j}\|_{L_{t,x}^{\frac{10}{3}}} \|w_N\|_{V_\Delta^2}. \end{aligned}$$

Recall that $\sigma < 2s$. Hence, we have a negative power of the highest frequency, and we can proceed as in observation 2.3.1 to obtain

$$\sum_{N_1, N_2, N_3, N} N_1^\sigma |\mathcal{I}_N(N_1, N_2, N_3)| \lesssim \sum_N N^{0-} \|\langle \nabla \rangle^s f\|_{L_{t,x}^{\frac{10}{3}}}^3 \|w_N\|_{V_\Delta^2}.$$

When we eventually sum over N , we apply Cauchy-Schwarz and use that $\|v\|_{Y_0} \leq 1$ to conclude that

$$\sum_N N^{0-} \|w_N\|_{V_\Delta^2} \lesssim \left(\sum_N \|w_N\|_{V_\Delta^2}^2 \right)^{\frac{1}{2}} \lesssim 1.$$

We conclude similarly in the other cases.

- **Case 2 : High-low-low** $N_2 \leq N_1^{\frac{1}{2}}$ and $N \sim N_1$. We apply Cauchy-Schwarz, we use interpolation and the bilinear estimate (2.2.6) in its transferred version (2.2.7) to obtain

$$\begin{aligned} N_1^\sigma |\mathcal{I}_N(N_1, N_2, N_3)| &\lesssim N_1^\sigma \|f_{N_1} f_{N_2}\|_{L_{t,x}^2}^{\frac{1}{2}} \|f_{N_1}\|_{L_{t,x}^4}^{\frac{1}{2}} \|f_{N_2}\|_{L_{t,x}^4}^{\frac{1}{2}} \|f_{N_3} w_N\|_{L_{t,x}^2} \\ &\lesssim N_1^{\sigma-\frac{1}{4}} N_2^{\frac{1}{2}} N^{-\frac{1}{2}+0} N_3 \|f_{N_1}(0)\|_{L_x^2}^{\frac{1}{2}} \|f_{N_2}(0)\|_{L_x^2}^{\frac{1}{2}} \|f_{N_1}\|_{L_{t,x}^4}^{\frac{1}{2}} \|f_{N_2}\|_{L_{t,x}^4}^{\frac{1}{2}} \|f_{N_3}(0)\|_{L_x^2} \|w_N\|_{V_\Delta^2} \\ &\lesssim N_1^{\sigma-\frac{1}{4}-s} N_2^{\frac{1}{2}-s} N^{-\frac{1}{2}+0} N_3^{1-s} \\ &\quad \|f_{N_1}(0)\|_{H_x^s}^{\frac{1}{2}} \|f_{N_2}(0)\|_{H_x^s}^{\frac{1}{2}} \|\langle \nabla \rangle^s f_{N_1}\|_{L_{t,x}^4}^{\frac{1}{2}} \|\langle \nabla \rangle^s f_{N_2}\|_{L_{t,x}^4}^{\frac{1}{2}} \|f_{N_3}(0)\|_{H_x^s} \|w_N\|_{V_\Delta^2}. \end{aligned}$$

Using that $s < \frac{1}{2}$, $N_2 \leq N_1^{\frac{1}{2}}$ and $N_3 \leq N_1^{\frac{1}{2}} \sim N^{\frac{1}{2}}$, we have

$$N_1^{\sigma - \frac{1}{4} - s} N_2^{\frac{1}{2} - s} N^{-\frac{1}{2} + 0} N_3^{1 - s} \lesssim N_1^{\sigma - 2s + 0}.$$

Then, as in the first case, we then use that $\sigma < 2s$ and we sum over the N_j 's to get

$$\sum_{N_1, N_2, N_3, N} N_1^\sigma |\mathcal{I}_N(N_1, N_2, N_3)| \lesssim \|\langle \nabla \rangle^s f\|_{L_{t,x}^4} \|f_0\|_{H_x^s}^2.$$

- **Case 3 : High-high-low** $N_3 \leq N_1^{\frac{1}{2}} \leq N_2$. Similarly, it follows from Hölder's inequality, from the bilinear estimate (2.2.6) and from the Strichartz embedding (2.2.5) that

$$\begin{aligned} N_1^\sigma |\mathcal{I}_N(N_1, N_2, N_3)| &\lesssim N_1^\sigma \|f_{N_1} f_{N_3}\|_{L_{t,x}^2} \|f_{N_2}\|_{L_{t,x}^5} \|w_N\|_{L_{t,x}^{\frac{10}{3}}} \\ &\lesssim N_1^{\sigma - \frac{1}{4} - s} N_3^{\frac{1}{2} - s} \|f_{N_1}(0)\|_{H_x^s}^{\frac{1}{2}} \|f_{N_3}(0)\|_{H_x^s}^{\frac{1}{2}} \|\langle \nabla \rangle^s f_{N_1}\|_{L_{t,x}^4}^{\frac{1}{2}} \|\langle \nabla \rangle^s f_{N_3}\|_{L_{t,x}^4}^{\frac{1}{2}} N_2^{-s} \|\langle \nabla \rangle^s f_{N_2}\|_{L_{t,x}^5} \|w_N\|_{V_\Delta^2} \\ &\lesssim N_1^{\sigma - 2s} \|f_{N_1}(0)\|_{H_x^s}^{\frac{1}{2}} \|f_{N_3}(0)\|_{H_x^s}^{\frac{1}{2}} \|\langle \nabla \rangle^s f_{N_1}\|_{L_{t,x}^4}^{\frac{1}{2}} \|\langle \nabla \rangle^s f_{N_3}\|_{L_{t,x}^4}^{\frac{1}{2}} N_2^{-s} \|\langle \nabla \rangle^s f_{N_2}\|_{L_{t,x}^5} \|w_N\|_{V_\Delta^2}. \end{aligned}$$

Summing over the different dyadic integers yields

$$\sum_{N_1, N_2, N_3, N} N_1^\sigma |\mathcal{I}_N(N_1, N_2, N_3)| \lesssim \|\langle \nabla \rangle^s f\|_{L_{t,x}^4} \|\langle \nabla \rangle^s f\|_{L_{t,x}^5} \|f_0\|_{H_x^s}.$$

Proof of estimates (2.3.3), (2.3.4) with mixed terms Here we consider the case when at least one term is random, say $w^{(1)} = f$, and at least another one is deterministic, say $w^{(2)} = v$. As for the last term $w^{(3)} \in \{f, v\}$, we always place it in $L_{t,x}^5(J \times \mathbb{R}^3)$.

- **Case 1 : One deterministic term comes with the highest frequency** Without loss of generality, we assume that $N_{(1)} = N_2$. By applying Hölder's inequality, we get

$$N_2^\sigma |\mathcal{I}_N(N_1, N_2, N_3)| = N_2^\sigma \left| \iint_{J \times \mathbb{R}^3} f_{N_1} \bar{v}_{N_2} w_{N_3}^{(3)} \bar{w}_N dx dt \right| \lesssim N_2^\sigma \|f_{N_1} v_{N_2}\|_{L_{t,x}^2} \|w_{N_3}^{(3)}\|_{L_{t,x}^5} \|w_N\|_{L_{t,x}^{\frac{10}{3}}}.$$

The transferred bilinear estimate and Strichartz embedding yield

$$\begin{aligned} N_2^\sigma |\mathcal{I}_N(N_1, N_2, N_3)| &\lesssim N_2^\sigma \|f_{N_1} v_{N_2}\|_{L_{t,x}^2}^{\frac{1}{2}} \|f_{N_1}\|_{L_{t,x}^{\frac{5}{2}}}^{\frac{1}{2}} \|v_{N_2}\|_{L_{t,x}^{\frac{10}{3}}}^{\frac{1}{2}} \|w_{N_3}^{(3)}\|_{L_{t,x}^5} \|w_N\|_{V_\Delta^2} \\ &\lesssim N_2^{-\frac{1}{4}} N_1^{\frac{1}{2} - s} (N_2^\sigma \|v_{N_2}\|_{U_\Delta^2}) \|f_{N_1}(0)\|_{H_x^s}^{\frac{1}{2}} \|\langle \nabla \rangle^s f_{N_1}\|_{L_{t,x}^{\frac{5}{2}}}^{\frac{1}{2}} \|w_{N_3}^{(3)}\|_{L_{t,x}^5} \|w_N\|_{V_\Delta^2}. \end{aligned}$$

Finally, we sum over the N_i 's to obtain that the contribution of this term to (2.3.6) is less than

$$\begin{aligned} &\sum_{N_1, N_2, N_3, N} N_2^{-\frac{1}{4}} N_1^{\frac{1}{2} - s} N_2^\sigma \|v_{N_2}\|_{U_\Delta^2} \|P_{N_1} f_0\|_{H_x^s}^{\frac{1}{2}} \|f_{N_1}\|_{L_{t,x}^{\frac{5}{2}}}^{\frac{1}{2}} \|w_{N_3}^{(3)}\|_{L_{t,x}^5} \|w_N\|_{V_\Delta^2} \\ &\lesssim \sum_{N_1, N_2, N_3, N} N_2^{\frac{1}{4} - s} \|P_{N_1} f_0\|_{H_x^s}^{\frac{1}{2}} \|f_{N_1}\|_{L_{t,x}^{\frac{5}{2}}}^{\frac{1}{2}} \|w_{N_3}^{(3)}\|_{L_{t,x}^5} \|w_N\|_{V_\Delta^2} \\ &\lesssim N^{0-} \|f_0\|_{H_x^{\frac{1}{2}}}^{\frac{1}{2}} \|\langle \nabla \rangle^s f\|_{L_{t,x}^{\frac{5}{2}}}^{\frac{1}{2}} \|w^{(3)}\|_{L_{t,x}^5} \|v\|_{X^\sigma} \|w_N\|_{V_\Delta^2}. \end{aligned}$$

Note that we used the assumption that $\frac{1}{4} < s$.

- **Case 2 : One random term comes with the highest frequency:** $N_{(1)} = N_1$. We distinguish between two cases.

- **Case 2a : High-high interactions** $N_1^{\frac{1}{2}} \leq N_2$. We apply Hölder's inequality and the Strichartz embedding (2.2.5), to get

$$\begin{aligned} N_1^\sigma |\mathcal{I}_N(N_1, N_2, N_3)| &\lesssim N_1^{\sigma-s} N_2^{-\sigma} \|\langle \nabla \rangle^s f_{N_1}\|_{L_{t,x}^5} \|\langle \nabla \rangle^\sigma v_{N_2}\|_{L_{t,x}^{\frac{10}{3}}} \|w_{N_3}^{(3)}\|_{L_{t,x}^5} \|w_N\|_{L_{t,x}^{\frac{10}{3}}} \\ &\lesssim N_1^{\frac{\sigma}{2}-s} \|\langle \nabla \rangle^s f_{N_1}\|_{L_{t,x}^5} \|\langle \nabla \rangle^\sigma v_{N_2}\|_{U_\Delta^2} \|w_{N_3}^{(3)}\|_{L_{t,x}^5} \|w_N\|_{V_\Delta^2}. \end{aligned}$$

Since $\sigma < 2s$, we can sum over the dyadic frequencies and obtain

$$\sum_{N_1, N_2, N_3, N} N_1^\sigma |\mathcal{I}_N(N_1, N_2, N_3)| \lesssim \|\langle \nabla \rangle^s f\|_{L_{t,x}^5} \|v\|_{L_{t,x}^5} \|v\|_{X^\sigma} \|w_N\|_{V_\Delta^2}.$$

- **Case 2b : High-low-low interactions** $N_2, N_3 \ll N_1^{\frac{1}{2}}$, and $N_1 \sim N$. We make a different analysis depending on the type of $w^{(3)}$.

- * **Case 2b(i) :** $w^{(3)} = f$. The idea is to use Cauchy Schwarz in order to apply the bilinear estimate twice to gain derivatives, and to use Hölder's inequality in order to access the $L_{t,x}^5(J)$ norm of f , which is small.

$$\begin{aligned} N_1^\sigma |\mathcal{I}_N(N_1, N_2, N_3)| &\leq N_1^\sigma \|f_{N_1} v_{N_2}\|_{L_{t,x}^2} \|f_{N_3} w_N\|_{L_{t,x}^2} \\ &\lesssim N_1^\sigma \|f_{N_1} v_{N_2}\|_{L_{t,x}^2}^{\frac{1}{2}} \|f_{N_1}\|_{L_{t,x}^5}^{\frac{1}{2}} \|v_{N_2}\|_{L_{t,x}^{\frac{10}{3}}}^{\frac{1}{2}} \|f_{N_3} w_N\|_{L_{t,x}^2}^{\frac{1}{2}} \|f_{N_3}\|_{L_{t,x}^5}^{\frac{1}{2}} \|w_N\|_{L_{t,x}^{\frac{10}{3}}}^{\frac{1}{2}}. \end{aligned}$$

Applying bilinear Strichartz estimate yields

$$\begin{aligned} N_1^\sigma |\mathcal{I}_N(N_1, N_2, N_3)| &\lesssim N_1^{\sigma-s-\frac{1}{4}} N_2^{\frac{1}{2}-\sigma} N_3^{\frac{1}{2}-\frac{s}{2}} N^{-\frac{1}{4}+0} \|\langle \nabla \rangle^s f_{N_1}\|_{L_{t,x}^{\frac{5}{2}}} \|f_{N_1}(0)\|_{H_x^s} \\ &\quad \|\langle \nabla \rangle^\sigma v_{N_2}\|_{U_\Delta^2}^{\frac{1}{2}} \|f_{N_3}(0)\|_{H_x^s}^{\frac{1}{2}} \|\langle \nabla \rangle^\sigma v_{N_2}\|_{L_{t,x}^{\frac{10}{3}}}^{\frac{1}{2}} \|\langle \nabla \rangle^s f_{N_3}\|_{L_{t,x}^5}^{\frac{1}{2}} \|w_N\|_{V_\Delta^2} \\ &\lesssim N_1^{\sigma-s-\frac{1}{2}+0} N_2^{\frac{1}{2}-\sigma} N_3^{\frac{1}{2}-\frac{s}{2}} \|\langle \nabla \rangle^\sigma v_{N_2}\|_{U_\Delta^2} \|w_N\|_{V_\Delta^2} \\ &\quad (\|f_{N_1}(0)\|_{H_x^s} \|f_{N_3}(0)\|_{H_x^s})^{\frac{1}{2}} (\|\langle \nabla \rangle^s f_{N_1}\|_{L_{t,x}^5} \|\langle \nabla \rangle^s f_{N_3}\|_{L_{t,x}^5})^{\frac{1}{2}}. \end{aligned}$$

Finally, we observe that

$$N_1^{\sigma-s-\frac{1}{2}+0} N_2^{\frac{1}{2}-\sigma} N_3^{\frac{1}{2}-\frac{s}{2}} \leq N_1^{\sigma-s-\frac{1}{2}+0} N_3^{1-\sigma-\frac{s}{2}} \leq N_1^{\frac{\sigma}{2}-s},$$

and we can sum over the N_i 's to see that the contribution for this term is less than

$$\sum_{N_i} N_1^\sigma \left| \iint_{J \times \mathbb{R}^3} f_{N_1} \bar{v}_{N_2} f_{N_3} \bar{w}_N dx dt \right| \lesssim N^{0-} \|f_0\|_{H_x^s} \|\langle \nabla \rangle^s f\|_{L_{t,x}^5} \|v\|_{X^\sigma}.$$

* **Case 2b(ii)** : $w^{(3)} = v$. We use Hölder's inequality and interpolation to get

$$N_1^\sigma |\mathcal{I}_N(N_1, N_2, N_3)| \lesssim N_1^\sigma \left(\|f_{N_1}\|_{L_{t,x}^5} \|v_{N_2}\|_{L_{t,x}^{\frac{10}{3}}} \|f_{N_1} v_{N_2}\|_{L_{t,x}^2} \right)^{\frac{1}{2}} \\ \left(\|v_{N_3}\|_{L_{t,x}^5} \|w_N\|_{L_{t,x}^{\frac{10}{3}}} \|v_{N_3} w_N\|_{L_{t,x}^2} \right)^{\frac{1}{2}}.$$

As a consequence of the bilinear estimate (2.2.7) and Strichartz estimates,

$$N_1^\sigma |\mathcal{I}_N(N_1, N_2, N_3)| \lesssim N_1^{\sigma - \frac{1}{4} - s} N_2^{\frac{1}{2} - \sigma} N^{-\frac{1}{4} + 0} N_3^{\frac{3}{4} - \sigma} \\ \left(\|f_{N_1}(0)\|_{H^s} \|f_{N_1}\|_{L_{t,x}^5} \right)^{\frac{1}{2}} \|\langle \nabla \rangle^\sigma v_{N_2}\|_{U_\Delta^2} \|\langle \nabla \rangle^\sigma v_{N_3}\|_{U_\Delta^2} \|w_N\|_{V_\Delta^2}.$$

We get a negative power of $N_1 \sim N$ as soon as $\sigma > \frac{3}{4}$. If $\sigma < \frac{3}{4}$, it follows from $N_1 \sim N$ and $N_3 \leq N_1^{\frac{1}{2}}$ that

$$N_1^{\sigma - \frac{1}{4} - s} N_2^{\frac{1}{2} - \sigma} N^{-\frac{1}{4} + 0} N_3^{\frac{3}{4} - \sigma} \lesssim N_1^{\sigma - \frac{1}{2} - s + 0} N_2^{\frac{1}{2} - \sigma} N_1^{\frac{3}{8} - \frac{\sigma}{2}} \lesssim N_1^{\frac{\sigma}{2} - s - \frac{1}{2} + \frac{3}{8} + 0}.$$

Next, we use the assumption that $\sigma < 2s$ to conclude that

$$\sum_{N_i} N_1^\sigma \left| \iint_{J \times \mathbb{R}^3} f_{N_1} \bar{v}_{N_2} v_{N_3} \bar{w}_N dx dt \right| \lesssim N^{0-} \|f_0\|_{H_x^s}^{\frac{1}{2}} \|\langle \nabla \rangle^s f\|_{L_{t,x}^{\frac{10}{3}}}^{\frac{1}{2}} \|v\|_{L_{t,x}^5} \|v\|_{X^\sigma} \|w_N\|_{V_\Delta^2}.$$

This finishes the proofs of estimates (2.3.3) and (2.3.4), and of Proposition 2.3.2. \square

In order to perform a double bootstrap argument with $\|v\|_{L_{t,x}^5(J)}$ and $\|v\|_{X^\sigma(J)}$ to prove local well-posedness, we need to estimate the spacetime critical $L_{t,x}^5$ norm of v .

Proposition 2.3.3 (Additional trilinear estimates).

$$\|\mathcal{I}(\cdot, \mathcal{N}(f, f, v))\|_{L_{t,x}^5(J)} \lesssim \|f_0\|_{H_x^s}^{\frac{1}{2}} F^\omega(J)^{\frac{3}{2}} \|v\|_{X^{\frac{1}{2}}(J)}, \quad (2.3.7)$$

$$\|\mathcal{I}(\cdot, \mathcal{N}(f, v, v))\|_{L_{t,x}^5(J)} \lesssim \|f_0\|_{H_x^s}^{\frac{1}{2}} F^\omega(J)^{\frac{1}{2}} \|v\|_{L_{t,x}^5(J)} \|v\|_{X^{\frac{1}{2}}(J)}. \quad (2.3.8)$$

Proof. By the TT* Strichartz estimate and the Sobolev embedding, we have

$$\|\mathcal{I}(\cdot, \mathcal{N})\|_{L_{t,x}^5(J)} \lesssim \|\langle \nabla \rangle^{\frac{1}{2}} \mathcal{I}(\cdot, \mathcal{N})\|_{L_t^5 L_x^{\frac{30}{11}}} \lesssim \|\langle \nabla \rangle^{\frac{1}{2}} \mathcal{N}\|_{L_{t,x}^{\frac{10}{7}}}.$$

Once again we perform a Littlewood-Paley decomposition of each term and conduct a case by case analysis. We write $N_{(1)}, N_{(2)}, N_{(3)}$ the non-increasing ordering among them.

Proof of (2.3.7) We need to estimate

$$N_{(1)}^{\frac{1}{2}} \|f_{N_1} f_{N_2} v_{N_3}\|_{L_{t,x}^{\frac{10}{7}}}.$$

Without loss of generality, we may assume that $N_1 \geq N_2$.

- **Case 1 :** $N_{(1)} = N_1$. We use Hölder's inequality with $\frac{7}{10} = \frac{1}{5} + \frac{1}{2}$, interpolation, the bilinear estimate and the Strichartz embedding of $L_{t,x}^{\frac{10}{3}} \hookrightarrow U_{\Delta}^2$ to get

$$\begin{aligned} N_1^{\frac{1}{2}} \|f_{N_1} f_{N_2} v_{N_3}\|_{L_{t,x}^{\frac{10}{7}}} &\leq N_1^{\frac{1}{2}} \|f_{N_1} v_{N_3}\|_{L_{t,x}^2}^{\frac{1}{2}} \|f_{N_1}\|_{L_{t,x}^5}^{\frac{1}{2}} \|v_{N_3}\|_{L_{t,x}^{\frac{10}{3}}}^{\frac{1}{2}} \|f_{N_2}\|_{L_{t,x}^5} \\ &\lesssim N_1^{\frac{1}{2}-s-\frac{1}{4}} N_3^{\frac{1}{2}-\sigma} \|f_{N_1}(0)\|_{H^s}^{\frac{1}{2}} \|\langle \nabla \rangle^s f_{N_1}\|_{L_{t,x}^5}^{\frac{1}{2}} N_3^{\sigma} \|v_{N_3}\|_{U_{\Delta}^2} \|f_{N_2}\|_{L_{t,x}^5}. \end{aligned}$$

Under the assumption that $\frac{1}{4} < s$ and $\frac{1}{2} < \sigma$, we can use Observation 2.3.1 and see that in this case,

$$\sum_{N_i} N_1^{\frac{1}{2}} \|f_{N_1} f_{N_2} v_{N_3}\|_{L_{t,x}^{\frac{10}{7}}} \lesssim \|f_0\|_{H^s}^{\frac{1}{2}} \|\langle \nabla \rangle^s f\|_{L_{t,x}^5}^{\frac{3}{2}} \|v\|_{X^0}.$$

- **Case 2 :** $N_{(1)} = N_3$.

– **Case 2a :** $N_1 < N_3^{\frac{1}{2}}$. In this case we shall apply the bilinear estimate once.

$$\begin{aligned} N_3^{\frac{1}{2}} \|f_{N_1} f_{N_2} v_{N_3}\|_{L_{t,x}^{\frac{10}{7}}} &\leq N_3^{\frac{1}{2}} \|f_{N_2} v_{N_3}\|_{L_{t,x}^2} \|f_{N_1}\|_{L_{t,x}^5} \\ &\lesssim N_3^{-\frac{1}{4}} N_2^{\frac{1}{2}-s} N_1^{-s} \|f_{N_2}(0)\|_{H_x^s}^{\frac{1}{2}} \|\langle \nabla \rangle^s f_{N_2}\|_{L_{t,x}^5}^{\frac{1}{2}} (N_3 \|v_{N_3}\|_{U_{\Delta}^2} \|v_{N_3}\|_{L_{t,x}^{\frac{10}{3}}})^{\frac{1}{2}} \|\langle \nabla \rangle^s f_{N_1}\|_{L_{t,x}^5} \\ &\lesssim N_3^{-s} \|f_{N_2}(0)\|_{H_x^s}^{\frac{1}{2}} \|\langle \nabla \rangle^s f_{N_2}\|_{L_{t,x}^5}^{\frac{1}{2}} (N_3^{\frac{1}{2}} \|v_{N_3}\|_{U_{\Delta}^2}) \|\langle \nabla \rangle^s f_{N_1}\|_{L_{t,x}^5}. \end{aligned}$$

Summing over the N_i 's yields

$$\sum_{N_i} N_3^{\frac{1}{2}} \|f_{N_1} f_{N_2} v_{N_3}\|_{L_{t,x}^{\frac{10}{7}}} \lesssim \|f_0\|_{H_x^s}^{\frac{1}{2}} \|\langle \nabla \rangle^s f\|_{L_{t,x}^5}^{\frac{3}{2}} \|v\|_{X^{\frac{1}{2}}}.$$

– **Case 2b :** $N_3^{\frac{1}{2}} \leq N_1$. In this case there is no need to apply the bilinear estimate and we only use Hölder. We have

$$\begin{aligned} N_3^{\frac{1}{2}} \|f_{N_1} f_{N_2} v_{N_3}\|_{L_{t,x}^{\frac{10}{7}}} &\leq N_3^{\frac{1}{2}} N_1^{-s} \|\langle \nabla \rangle^s f_{N_1}\|_{L_{t,x}^5} \|f_{N_2}\|_{L_{t,x}^5} \|v_{N_3}\|_{L_{t,x}^{\frac{10}{3}}} \\ &\leq N_3^{-\frac{s}{2}} \|\langle \nabla \rangle^s f_{N_1}\|_{L_{t,x}^5} \|f_{N_2}\|_{L_{t,x}^5} (N_3^{\frac{1}{2}} \|v_{N_3}\|_{U_{\Delta}^2}). \end{aligned}$$

Then we sum over the N_i 's and get

$$\sum_{N_i} N_3^{\frac{1}{2}} \|f_{N_1} f_{N_2} v_{N_3}\|_{L_{t,x}^{\frac{10}{7}}} \lesssim \|\langle \nabla \rangle^s f\|_{L_{t,x}^5} \|f\|_{L_{t,x}^5} \|v\|_{X^{\frac{1}{2}}}.$$

This concludes the proof of (2.3.7).

Proof of (2.3.8) We need to estimate

$$N_{(1)}^{\frac{1}{2}} \|f_{N_1} v_{N_2} v_{N_3}\|_{L_{t,x}^{\frac{10}{7}}}.$$

Without loss of generality, we assume that $N_2 \geq N_3$.

- **Case 1 :** $N_{(1)} = N_1$. We use Hölder with $\frac{7}{10} = \frac{1}{2} + \frac{1}{5}$ to apply the bilinear estimate once.

$$\begin{aligned} N_1^{\frac{1}{2}} \|f_{N_1} v_{N_2} v_{N_3}\|_{L_{t,x}^{\frac{10}{7}}} &\leq \|f_{N_1} v_{N_3}\|_{L_{t,x}^2} \|v_{N_2}\|_{L_{t,x}^5} \\ &\lesssim N_1^{\frac{1}{4}-s} N_3^{\frac{1}{2}} \|\langle \nabla \rangle^s f_{N_1}\|_{L_{t,x}^5} \|f_{N_1}(0)\|_{H_x^s}^{\frac{1}{2}} \|v_{N_3}\|_{U_{\Delta}^2}^{\frac{1}{2}} \|v_{N_3}\|_{L_{t,x}^{\frac{10}{3}}}^{\frac{1}{2}} \|v_{N_2}\|_{L_{t,x}^5} \\ &\lesssim N_1^{\frac{1}{4}-s} \|\langle \nabla \rangle^s f_{N_1}\|_{L_{t,x}^5} \|f_{N_1}(0)\|_{H_x^s}^{\frac{1}{2}} \|v_{N_3}\|_{X^{\frac{1}{2}}} \|v_{N_2}\|_{L_{t,x}^5}. \end{aligned}$$

Under the assumption that $\frac{1}{4} < s$ we can use Observation 2.3.1 and see that in this case,

$$\sum_{N_i} N_1^{\frac{1}{2}} \|f_{N_1} v_{N_2} v_{N_3}\|_{L_{t,x}^{\frac{10}{7}}} \lesssim \|f_0\|_{H_x^s}^{\frac{1}{2}} \|\langle \nabla \rangle^s f\|_{L_{t,x}^5}^{\frac{1}{2}} \|v\|_{L_{t,x}^5} \|v\|_{X^{\frac{1}{2}}}.$$

- **Case 2 :** $N_{(1)} = N_2$.

- **Case 2a :** $N_1 < N_2^{\frac{1}{2}}$. We proceed similarly, and get

$$\begin{aligned} N_2^{\frac{1}{2}} \|f_{N_1} v_{N_2} v_{N_3}\|_{L_{t,x}^{\frac{10}{7}}} &\leq \|f_{N_1} v_{N_2}\|_{L_{t,x}^2} \|v_{N_3}\|_{L_{t,x}^5} \\ &\lesssim N_2^{\frac{1}{4}} N_1^{\frac{1}{2}-s} \|\langle \nabla \rangle^s f_{N_1}\|_{L_{t,x}^5} \|f_{N_1}(0)\|_{H_x^s}^{\frac{1}{2}} \|v_{N_2}\|_{U_{\Delta}^2}^{\frac{1}{2}} \|v_{N_2}\|_{L_{t,x}^{\frac{10}{3}}}^{\frac{1}{2}} \|v_{N_3}\|_{L_{t,x}^5} \\ &\lesssim N_2^{-\frac{s}{2}} \|\langle \nabla \rangle^s f_{N_1}\|_{L_{t,x}^5} \|f_{N_1}(0)\|_{H_x^s}^{\frac{1}{2}} \|v_{N_2}\|_{X^{\frac{1}{2}}} \|v_{N_3}\|_{L_{t,x}^5}. \end{aligned}$$

Hence,

$$\sum_{N_i} N_2^{\frac{1}{2}} \|f_{N_1} v_{N_2} v_{N_3}\|_{L_{t,x}^{\frac{10}{7}}} \lesssim \|f_0\|_{H_x^s}^{\frac{1}{2}} \|\langle \nabla \rangle^s f\|_{L_{t,x}^5}^{\frac{1}{2}} \|v\|_{L_{t,x}^5} \|v\|_{X^{\frac{1}{2}}}.$$

- **Case 2b :** $N_2^{\frac{1}{2}} \leq N_1$. We apply Hölder and get

$$\begin{aligned} N_2^{\frac{1}{2}} \|f_{N_1} v_{N_2} v_{N_3}\|_{L_{t,x}^{\frac{10}{7}}} &\leq N_2^{\frac{1}{2}} \|f_{N_1}\|_{L_{t,x}^5} \|v_{N_2}\|_{L_{t,x}^{\frac{10}{3}}} \|v_{N_3}\|_{L_{t,x}^5} \\ &\leq N_2^{-\frac{s}{2}} \|\langle \nabla \rangle^s f_{N_1}\|_{L_{t,x}^5} (N_2^{\frac{1}{2}} \|v_{N_2}\|_{U_{\Delta}^2}) \|v_{N_3}\|_{L_{t,x}^5}. \end{aligned}$$

We deduce from this and from Observation 2.3.1 that in this case,

$$\sum_{N_i} N_2^{\frac{1}{2}} \|f_{N_1} v_{N_2} v_{N_3}\|_{L_{t,x}^{\frac{10}{7}}} \lesssim \|\langle \nabla \rangle^s f\|_{L_{t,x}^5} \|v\|_{L_{t,x}^5} \|v\|_{X^{\frac{1}{2}}}.$$

□

2.3.2 Local well-posedness

In the following local well-posedness statement, the smallness assumption comes from the $L_{t,x}^5$ norm of the free evolution of the initial data. Since the problem is at a subcritical regularity scale $\frac{1}{2} < \sigma$, we could avoid using such a critical norm. Nevertheless, the $L_{t,x}^5$ norm provides a blow-up and scattering criterion. Then, we prove in the next section that this blow-up criterion can be exploited as soon as we are able to obtain an a priori uniform bound on the H^σ norm of the solution.

Proposition 2.3.4 (Local well-posedness). *Let $\frac{1}{4} < s \leq \frac{1}{2}$ and $\omega \in \Omega_\alpha$. Then, for every $\frac{1}{2} < \sigma < 2s$, the Cauchy problem (NLS_f) with data $v(t_0) \in H^\sigma$ is locally well-posed. More precisely, for every $E > 0$, there exists $\varepsilon_0(E)$ such that for every $\varepsilon \leq \varepsilon_0$, every interval J that contains t_0 on which*

$$F^\omega(J) \leq \varepsilon, \quad \|e^{it\Delta} v_0\|_{L_{t,x}^5(J)} \leq \varepsilon, \quad (2.3.9)$$

and for every $v_0 \in H^\sigma$ with $\|v_0\|_{H^\sigma} \leq E$, there exists a unique solution v to (NLS_f) in $X^\sigma(J)$, with data $v(t_0) = v_0$. Moreover, there exists $C = C(\|f_0\|_{H^s}) > 0$ such that

$$\|v\|_{X^\sigma(J)} \leq 2E, \quad \|v\|_{L_{t,x}^5(J)} \leq C\varepsilon, \quad \text{and } v \in C(J; H^\sigma).$$

Proof. We perform a contraction mapping argument in the Banach space,

$$\mathcal{B}_{E,\varepsilon} = \{v \in X^\sigma(J) \mid \|v\|_{X^\sigma(J)} \leq 2E, \quad \|v\|_{L_{t,x}^5(J)} \leq C\varepsilon\},$$

where $C = C(\|f_0\|_{H^s}) > 0$ is an irrelevant constant that comes from the multilinear estimates of Proposition 2.3.3. We endow this space with the norm $X^\sigma(J)$, and we define the mapping

$$\Phi : v \in \mathcal{B}_{E,\varepsilon} \mapsto e^{it\Delta} v_0 - i \int_{t_0}^t e^{i(t-t')\Delta} \mathcal{N}(v+f)(t') dt'.$$

From the Duhamel integral representation formula, we see that v is solution to (NLS_f) with $v(t_0) = v_0$ if and only if $\Phi(v) = v$. Let us prove that

$$\Phi(\mathcal{B}_{E,\varepsilon}) \subseteq \mathcal{B}_{E,\varepsilon}$$

when $\varepsilon \ll E^{-1}$. We get from Strichartz and from the trilinear estimate (2.3.5) that for all $v \in \mathcal{B}_{E,\varepsilon}$,

$$\begin{aligned} \|\Phi(v)\|_{X^\sigma(J)} &\leq \|v_0\|_{H_x^\sigma} + \|\mathcal{I}(\cdot, \mathcal{N}(v))\|_{X^\sigma(J)} + \|\mathcal{I}(\cdot, \mathcal{N}(v+f) - \mathcal{N}(v))\|_{X^\sigma(J)} \\ &\leq E + C \|\langle \nabla \rangle^\sigma (\mathcal{N}(v))\|_{L_{t,x}^{\frac{10}{7}}} + C(\|f_0\|_{H^s}) F^\omega(J) \\ &\quad + C(\|f_0\|_{H^s}) \{F^\omega(J) + \|v\|_{L_{t,x}^5(J)}\} \|v\|_{X^\sigma(J)}. \end{aligned}$$

Applying the Fractional Leibniz rule with $\frac{7}{10} = \frac{2}{5} + \frac{3}{10}$ yields

$$\|\langle \nabla \rangle^\sigma (\mathcal{N}(v))\|_{L_{t,x}^{\frac{10}{7}}} \leq \|v\|_{L_{t,x}^5(J)}^2 \|\langle \nabla \rangle^\sigma v\|_{L_{t,x}^{\frac{10}{3}}} \leq \|v\|_{L_{t,x}^5(J)}^2 \|v\|_{X^\sigma(J)} \leq C^2 \varepsilon^2 E.$$

Hence, for all $v \in \mathcal{B}_{E,\varepsilon}$ and under the smallness assumption (2.3.9) we get

$$\|\Phi(v)\|_{X^\sigma(J)} \leq E + C\varepsilon^2 E + C(1+C)\varepsilon E. \quad (2.3.10)$$

Similarly, we have

$$\begin{aligned} \|\Phi(v)\|_{L_{t,x}^5} &\leq \|e^{it\Delta} v_0\|_{L_{t,x}^5(J)} + \|\mathcal{I}(\cdot, \mathcal{N}(v))\|_{L_{t,x}^5(J)} + \|\mathcal{I}(\cdot, \mathcal{N}(f))\|_{L_{t,x}^5(J)} \\ &\quad + \|\mathcal{I}(\cdot, \mathcal{N}(f, f, v) + \mathcal{N}(f, v, v))\|_{L_{t,x}^5(J)}. \end{aligned} \quad (2.3.11)$$

First, observe from the Sobolev embedding and from (2.3.2) that

$$\|\mathcal{I}(\cdot, \mathcal{N}(f))\|_{L_{t,x}^5(J)} \leq C\|\mathcal{I}(\cdot, \mathcal{N}(f))\|_{X^{\frac{1}{2}}(J)} \leq C(\|f_0\|_{H^\sigma})F^\omega(J).$$

By Hölder's inequality, Sobolev embedding and Strichartz embedding we have

$$\|\mathcal{I}(\cdot, \mathcal{N}(v))\|_{L_{t,x}^5(J)} \leq \|v\|_{X^{\frac{1}{2}}(J)} \|v\|_{L_{t,x}^5(J)}^2.$$

Then, using the trilinear estimates presented in Proposition 2.3.3 we get that for any $v \in \mathcal{B}_{E,\varepsilon}$,

$$\|\mathcal{I}(\cdot, \mathcal{N}(f, f, v) + \mathcal{N}(f, v, v))\|_{L_{t,x}^5(J)} \leq C(F^\omega(J)^{\frac{3}{2}} + F^\omega(J)^{\frac{1}{2}}\|v\|_{L_{t,x}^5(J)})\|v\|_{X^\sigma(J)}.$$

Collecting the above estimates, we deduce from (2.3.11) that under the smallness condition (2.3.9), we have for $v \in \mathcal{B}_{E,\varepsilon}$

$$\|\Phi(v)\|_{L_{t,x}^5} \leq \varepsilon + C\varepsilon + 2C^3\varepsilon^2E + 2CE(\varepsilon^{\frac{3}{2}} + C\varepsilon^{\frac{3}{2}}). \quad (2.3.12)$$

Combining (2.3.10), (2.3.12) and choosing $\eta = 3C\varepsilon$, $R = 2CE$, for $\varepsilon \leq \varepsilon_0(E)$ with, say $\varepsilon_0(E) \sim E^{-1}$ when $E \gg 1$, we obtain

$$\|\Phi(v)\|_{X^\sigma(J)} \leq 2CE, \quad \|\Phi(v)\|_{L_{t,x}^5(J)} \leq 3C\varepsilon.$$

for any $v \in \mathcal{B}_{E,\varepsilon}$. Thus, $\Phi : \mathcal{B}_E \mapsto \mathcal{B}_E$. Similarly, we prove that

$$\|\Phi(v_1) - \Phi(v_2)\|_{X^\sigma(J)} \leq C(\|v_1\|_{L_{t,x}^5(J)}\|v_1\|_{X^\sigma(J)} + \|v_2\|_{L_{t,x}^5(J)}\|v_2\|_{X^\sigma(J)})\|v_1 - v_2\|_{X^\sigma(J)}.$$

Choosing $\varepsilon_0(E) \ll E^{-1}$, we see that Φ is a contraction mapping, and it admits a unique fixed point v , solution to (NLS_f). Finally, we deduce the continuity of $t \rightarrow v(t, \cdot)$ from the embedding $X^\sigma(J) \hookrightarrow C(J; H_x^\sigma(\mathbb{R}^3))$. \square

Proposition 2.3.5 (Blow-up criterion). *Let v be the maximal lifespan solution to (NLS_f) on $J^* \times \mathbb{R}^3$ given by the local well-posedness theory. If we have*

$$\|v\|_{L_{t,x}^5(J^*)} < +\infty, \quad (2.3.13)$$

then $\sup J^* = +\infty$ and the solution scatters as t goes to $+\infty$.

Proof. Denote $E = \|v(0)\|_{H_x^\sigma}$, and $\varepsilon > 0$ to be chosen later on. We proceed by contradiction and assume that $\|v\|_{L_{t,x}^5} < +\infty$, but $T^* = \sup J^* < +\infty$. We will raise the contradiction by extending the solution up to T^* . By the local Cauchy theory from Proposition 4.2.2, it is enough to prove that there exist a constant $C(E, f_0)$, $t_0 \in J^*$ and $\delta > 0$ such that

$$\|v(t)\|_{L_t^\infty H_x^\sigma(J^*)} \leq C(E, f_0), \quad (2.3.14)$$

$$\|e^{i(t-t_0)\Delta} v(t_0)\|_{L_{t,x}^5[t_0-\delta, T^*+\delta]} \leq \varepsilon_0, \quad (2.3.15)$$

where $\varepsilon_0 = \varepsilon_0(E)$ is as in the local well-posedness Proposition 4.2.2. To prove the above estimates, we proceed as follows. First, we use the global assumption (2.3.13) to decompose J^* into a finite number $L(\varepsilon)$ of intervals $\{I_l\}_{1 \leq l \leq L}$ such that for $j \in \{1, \dots, L\}$,

$$\|v\|_{L_{t,x}^5(I_j)} \leq \varepsilon.$$

Using the assumption (2.2.12) that $\|f\|_{L_{t,x}^5(\mathbb{R})} < +\infty$, and up to an extra decomposition of J^* into $O(L)$ intervals, we may assume that for each $l \in \{1, \dots, L\}$,

$$F(J_l) \leq \varepsilon.$$

Next, we take $\tilde{J} \Subset J^*$ a compact sub-interval of J^* , and denoting $\tilde{J} \cap J_l$ by \tilde{J}_l , we prove by induction on $l \leq L$ that there exists $C(\|f_0\|_{H^s})$ such that

$$\|v\|_{X^\sigma(\tilde{J}_l)} \leq C2^l E. \quad (2.3.16)$$

For this purpose, we apply the trilinear estimates (2.3.5) to get

$$\|v\|_{X^\sigma(\tilde{J}_1)} \leq \|v(0)\|_{H^\sigma} + C\varepsilon + 2C\varepsilon\|v\|_{X^\sigma(\tilde{J}_1)}.$$

Hence, choosing ε small enough we see that

$$\|v\|_{X^\sigma(\tilde{J}_1)} \leq 2E.$$

Iterating this on $\tilde{J}_2, \dots, \tilde{J}_L$ with the same ε , we obtain that for $l \in \{1, \dots, L\}$,

$$\|v\|_{X^\sigma(\tilde{J}_l)} \leq 2^l E.$$

Since ε does not depend on $\tilde{J} \Subset J^*$, we have from sub-additivity of the norm X^σ (see Lemma 3.4.4) that

$$\|v\|_{L_t^\infty H^\sigma(J^*)} \leq \sup_{\tilde{J} \Subset J^*} \|v\|_{X^\sigma(\tilde{J})} \leq \left(\sum_{l=1}^L 2^l \right) E \leq C(L, E). \quad (2.3.17)$$

Similarly, the multilinear estimates from Proposition 2.3.3 yield for all $t_0 \in J^*$,

$$\|\mathcal{I}([t_0, \cdot], \mathcal{N}(f+v))\|_{L_{t,x}^5(\tilde{J})} \leq \tilde{C}(E, f_0).$$

Therefore, taking the sup over all the possible \tilde{J} we obtain a uniform bound with respect to t_0 on the $L_{t,x}^5(J^*)$ norm of the Duhamel integral. Consequently, we have from Duhamel's formula that

$$\begin{aligned} \|e^{i(t-t_0)\Delta} v(t_0)\|_{L_{t,x}^5[t_0, T^*]} &= \|v - \mathcal{I}([t_0, \cdot], \mathcal{N}(f+v))\|_{L_{t,x}^5[t_0, T^*]} \\ &\leq \|v\|_{L_{t,x}^5(J^*)} + \|\mathcal{I}([t_0, \cdot], \mathcal{N}(f+v))\|_{L_{t,x}^5(t_0, T^*)}, \end{aligned}$$

and we conclude the proof of (2.3.15) by monotone convergence

$$\lim_{t_0 \rightarrow T^*} \|e^{i(t-t_0)\Delta} v(t_0)\|_{L_{t,x}^5[t_0, T^*]} = 0.$$

As explained above, this finishes the proof of the global existence. At that point, it is standard to deduce scattering from these global bounds. Indeed, when $J^* = [0, +\infty)$ and $\|v\|_{L_{t,x}^5(J^*)} < +\infty$, we use the uniform bound (2.3.17), the nonlinear estimates from Proposition 2.3.2 and monotone convergence to deduce from the Duhamel integral formulation that $\{e^{-it_n\Delta} v\}_n$ is a Cauchy sequence in H_x^σ as t_n goes to $+\infty$. This ends the proof of Proposition 2.3.5. \square

2.3.3 Stability theory

In this subsection, we prove that global existence and scattering for v can be deduced from a priori uniform estimate on the H^σ norm of $v(t)$. Note that this is true when v is a solution to (NLS) without forcing terms, and when $\sigma > \frac{2}{3}$ (see Theorem 2.1.2). Hence, since v is actually solution to the forced equation (NLS), we will settle a stability theory at regularity H^σ . In particular, on some spacetime slabs where the perturbation satisfies a smallness condition, we shall be able to approach v by a solution to (NLS) and to infer local spacetime bound on v . Then, provided we have a global a priori estimate for the H^σ norm of v , we perform a bootstrap argument to extend the spacetime bound to the whole maximal lifespan of v , and to deduce conditional scattering. For now on, we fix $\sigma > \frac{2}{3}$, which corresponds to the lowest regularity where the global Cauchy theory for (NLS) is known.

Lemma 2.3.6 (Short-time stability in H_x^σ). *Let $t_0 \in \mathbb{R}$, $E \geq 0$ and $v_0 \in H_x^\sigma$ with $\|v_0\|_{H_x^\sigma} \leq E$. Let v be the local solution to (NLS_f) as in Proposition 4.2.2 associated with $v(t_0) = v_0$. Then, take $u_0 \in H_x^\sigma$ and denote by u the global solution to (NLS) in $C^\infty(\mathbb{R}; H_x^\sigma)$, with data $u(t_0) = u_0$. There exist $\varepsilon_1 = \varepsilon_1(E)$ and $C_0(E)$ such that if*

$$\|v_0 - u_0\|_{H_x^\sigma} \leq \varepsilon, \quad F^\omega(J) \leq \varepsilon, \quad \|u\|_{L_{t,x}^5(J)} \leq \varepsilon, \quad (2.3.18)$$

for some $\varepsilon \leq \varepsilon_1$, then v stays close to u on any compact interval $J \Subset J^*$ that contains t_0

$$\|v - u\|_{X^\sigma(J)} \leq C_0(E)\varepsilon. \quad (2.3.19)$$

Proof. Let us consider the difference $w = v - u$ ¹¹, solution to the equation

$$\begin{cases} i\partial_t w + \Delta w = \mathcal{N}(v + f) - \mathcal{N}(v) + \mathcal{N}(v) - \mathcal{N}(u), & (t, x) \in \mathbb{R} \times \mathbb{R}^3. \\ w_{t=0} = v_0 - u_0, \end{cases} \quad (2.3.20)$$

We introduce the function $g : t \in J \mapsto \|w\|_{X^\sigma(t_0, t)}$, and we perform a continuity argument. Note that as a consequence of the smallness condition $\|w(0)\|_{H_x^\sigma} \leq \varepsilon$ and from Lemma 3.4.4, we have that g is continuous and $\limsup_{t \rightarrow t_0} g(t) \lesssim \varepsilon$. Moreover, it follows from Duhamel's formulation that

$$\|w\|_{X^\sigma(J)} \leq C\|v_0 - u_0\|_{H_x^\sigma} + \|\mathcal{I}(\cdot, \mathcal{N}(v + f) - \mathcal{N}(v))\|_{X^\sigma(J)} + \|\mathcal{I}(\cdot, \mathcal{N}(u) - \mathcal{N}(v))\|_{X^\sigma(J)}.$$

From the trilinear estimate (2.3.5) of Proposition 2.3.2 and the smallness assumptions, we get

$$\begin{aligned} \|\mathcal{I}(\cdot, \mathcal{N}(v + f) - \mathcal{N}(v))\|_{X^\sigma[t_0, t]} &\leq CF^\omega(J) + C\|v\|_{X^\sigma[t_0, t]}(F^\omega(J) + \|v\|_{L_{t,x}^5([t_0, t])}F^\omega(J)^{\frac{1}{2}}) \\ &\leq C\varepsilon + C(\|u\|_{X^\sigma[t_0, t]} + \|w\|_{X^\sigma([t_0, t])})(\varepsilon + \varepsilon^{\frac{1}{2}}\|w\|_{L_{t,x}^5[t_0, t]} + \varepsilon^{\frac{1}{2}}\|u\|_{L_{t,x}^5(J)}) \\ &\leq C\varepsilon + C\varepsilon^{\frac{1}{2}}(CE + g(t))(2\varepsilon + Cg(t)). \end{aligned}$$

Moreover,

$$\begin{aligned} \|\mathcal{I}(\cdot, \mathcal{N}(u) - \mathcal{N}(v))\|_{X^\sigma[t_0, t]} &\leq C\|w\|_{X^\sigma([t_0, t])}(\|v\|_{L_{t,x}^5([t_0, t])}^2 + \|u\|_{L_{t,x}^5(J)}^2) \\ &\leq C\|w\|_{X^\sigma[t_0, t]}(\|w\|_{L_{t,x}^5[t_0, t]}^2 + \|u\|_{L_{t,x}^5(J)}^2) \leq Cg(t)(g(t)^2 + \varepsilon^2). \end{aligned}$$

¹¹ Note that u , hence w , is in $X^\sigma(J)$. This follows from the uniqueness part of Theorem 2.1.2 combined with a contraction mapping argument in $X^\sigma(J)$. Moreover, $\|u\|_{X^\sigma(J)} \leq CE$.

Hence, by combining the above estimates, we prove that for all $t \in J$ with $t_0 \leq t$,

$$g(t) \leq 2C\varepsilon + 2C\varepsilon^{\frac{3}{2}}E + g(t)(\varepsilon^{\frac{1}{2}}CE + \varepsilon^{\frac{3}{2}} + g(t)^2 + \varepsilon^2),$$

and the result follows from a continuity argument, by choosing $\varepsilon_1 \ll E^{-\frac{1}{2}}$. \square

Next, we combine the local well-posedness result with the blow-up criterion to turn the short-time stability result into a long time stability statement.

Lemma 2.3.7 (Long-time stability). *Let $\Lambda > 0$, $t_0 \in \mathbb{R}$, $J = [t_0, T]$, $v_0 \in H_x^\sigma$ with $\|v_0\|_{H_x^\sigma} \leq E$. Let u be the global-in-time solution to (NLS) in H_x^σ with data $u(t_0) = v_0$, and assume that*

$$\|u\|_{L_{t,x}^5(J)} \leq \Lambda.$$

There exists $\varepsilon_2 = \varepsilon_2(E, \Lambda) > 0$ such that under the smallness condition

$$F^\omega(J) \leq \varepsilon_2,$$

there exist a unique solution v to (NLS_f) and a constant $C(E, \Lambda) > 0$ such that for every compact interval $\tilde{J} \Subset J$, we have $v \in X^\sigma(\tilde{J})$, and

$$\|v\|_{L_{t,x}^5(J)} \leq C(E, \Lambda). \quad (2.3.21)$$

Proof. Let v be the maximal lifespan solution to (NLS_f) with $v(t_0) = v_0$, on an interval $J^* \times \mathbb{R}^3$. The whole statement reduces to the estimate (2.3.21). Indeed, we would be in position to apply the blow-up criterion, and to deduce that $J \subset J^*$. The strategy to prove such a global estimate is to break J into a finite number $L(E, \Lambda)$ of intervals where the $L_{t,x}^5$ norm of u is small, so that we can apply the short-time stability Lemma 2.3.6. Then, we sum over the different spacetime slabs. Since their number does only depend on E and Λ , we obtain (2.3.21).

We proceed as follows. First, we use the deterministic global well-posedness Theorem 2.1.2 and we get a constant $C_0(E)$, such that for any $\tilde{J} \Subset J^*$, we have

$$\|u\|_{X^\sigma(\tilde{J})} \leq C_0(E).$$

In addition, we take $\varepsilon_1 = \varepsilon_1(2C_0(E))$ as in Lemma 2.3.6, and $\varepsilon_2 < \varepsilon_1$, to be determined shortly. Next, we decompose J into $L(\varepsilon, \Lambda, E)$ intervals $\{J_i\}_{1 \leq i \leq L}$, with $J_i = [t_i, t_{i+1})$, such that for $i \in \{0, \dots, L-1\}$ and for some $\varepsilon < \varepsilon_2$,

$$\|u\|_{L_{t,x}^5(J_i)} \leq \varepsilon.$$

Now, we write $w = v - u$, we fix $\tilde{J}^* \Subset J^*$ and we denote $\tilde{J}_i = \tilde{J}^* \cap J_i$. We would like to apply Lemma 2.3.6 on each \tilde{J}_i . To this end, we need to make sure that for all $i \in \{1, \dots, L-1\}$, we have

$$\|v(t_i)\|_{H_x^\sigma} \leq 2C_0(E), \quad \|w(t_i)\|_{H^\sigma} \leq \varepsilon_1(2C_0(E)).$$

Indeed, we already know by assumption and by construction that

$$F^\omega(\tilde{J}_i) \leq \varepsilon_1, \quad \|u\|_{L_{t,x}^5(\tilde{J}_i)} \leq \varepsilon_1.$$

Hence, we shall prove by induction on i that for some $\varepsilon_2 < \varepsilon_1$, for every $\varepsilon < \varepsilon_2$ and for every $i \in \{0, \dots, L_1\}$, there exists $C_i = C_i(E, \Lambda)$ such that

$$\|w\|_{X^\sigma(\tilde{J}_i)} \leq C_i \varepsilon, \quad \left(\sup_{1 \leq i \leq L} C_i \right) \varepsilon_2 \leq \varepsilon_1, \quad \|v(t_i)\|_{H_x^\sigma} \leq 2C_0(E). \quad (2.3.22)$$

In the case when $i = 0$, we have $w(t_0) = 0$ and it follows from the short-time stability Lemma 2.3.6 applied on J_1 , where v and f satisfy the smallness condition (2.3.18), that there exists $C_1(E)$ with

$$\|w\|_{X^\sigma(\tilde{J}_1)} \leq C_1(E) \varepsilon.$$

Hence, we chose ε_2 such that $C_1(E) \varepsilon_2 < \varepsilon_1(2C_0(E))$. Next, we assume that (2.3.22) holds up to time t_i . By Duhamel's integral formulation, and since $w(t_0) = 0$, we have

$$\|w(t_i)\|_{H_x^\sigma} \leq \|\mathcal{I}(t_0, t_i, \mathcal{N}(v+f) - \mathcal{N}(u))\|_{H_x^\sigma}.$$

Then, we proceed as in the proof of the short-time stability Lemma 2.3.6, and we use the multilinear estimates from Propositions 2.3.2, 2.3.3, as well as the induction assumption, to see that

$$\|\mathcal{I}(t_0, t_i, \mathcal{N}(v+f) - \mathcal{N}(u))\|_{H_x^\sigma} \leq C \sum_{l=0}^{i-1} \|\mathcal{I}(\cdot, \mathcal{N}(v+f) - \mathcal{N}(u))\|_{X^\sigma(\tilde{J}_l)} \leq C \sum_{l=0}^{i-1} C_l(E) \varepsilon.$$

We define $C_i = C(E)C \sum_{1 \leq l \leq i-1} C_l$ and choose ε_2 such that $C_i \varepsilon_2 \leq \varepsilon_1(2C_0(E))$. Subsequently, the smallness condition (2.3.18) is satisfied on the interval $J_i = [t_i, t_{i+1})$, so that we can apply Lemma 2.3.6 and obtain

$$\|w\|_{X^\sigma(\tilde{J}_i)} \leq C(E) \sum_{0 \leq l \leq i-1} C_l \varepsilon.$$

In addition, we get from the triangle inequality that

$$\|v(t_{i+1})\|_{H_x^\sigma} \leq \|u\|_{L^\infty(J_{i+1}; H_x^\sigma)} + \|w\|_{L^\infty(J_{i+1}; H_x^\sigma)} \leq C_0(E) + C_{i+1} \varepsilon \leq 2C_0(E).$$

This finishes the proof of the induction result (2.3.22). Moreover, it follows from the sub-additivity of the X^σ norm (see Lemma 3.4.4) that

$$\|v\|_{X^\sigma(\tilde{J})} \leq \|u\|_{X^\sigma(J)} + \sum_{j=1}^L \|w\|_{X^\sigma(J_j)} \leq C(E, \Xi).$$

The proof of (2.3.21) follows by using the embedding $X^\sigma(\tilde{J}) \hookrightarrow L_{t,x}^5(\tilde{J})$, and the monotone convergence theorem which implies that $\|v\|_{L_{t,x}^5(J)} = \sup_{\tilde{J} \in J} \|v\|_{L_{t,x}^5(\tilde{J})}$. \square

Proposition 2.3.8 (Uniform bound in H_x^σ implies scattering). *Let $v_0 \in H_x^\sigma$, and let v be the maximal-lifespan solution in H_x^σ to (NLS_f) from initial data $v(0) = v_0$. Suppose that we have the uniform a priori bound*

$$\sup_{t \in J^*} \|v(t)\|_{H_x^\sigma} \leq E.$$

Then, the solution v is global and scatters as t goes to ∞ .

Proof. We get from the blow-up criterion of Lemma 2.3.5 that it suffices to prove the global spacetime bound

$$\|v\|_{L_{t,x}^5(J^*)} < +\infty. \quad (2.3.23)$$

Recall that the *deterministic* Cauchy theory in H^σ , for $\frac{2}{3} < \sigma \leq 1$, provides such a bound. Indeed, Theorem 2.1.2 claims that there exists $\Lambda(E)$ such that for any solution u to (NLS) starting from a data $u(t_0)$ with $\|u(t_0)\|_{H_x^\sigma} \leq E$, we have

$$\|u\|_{L_{t,x}^5(\mathbb{R})} \leq \Lambda(E).$$

Subsequently, we fix $\varepsilon_2(E, \Lambda(E)) = \varepsilon_2(E)$ as in Lemma 2.3.7, and we divide J^* into a finite number $L(E)$ of intervals $J_i = [t_i, t_{i+1})$, with $t_0 = 0$, such that

$$F^\omega(J_i) \leq \varepsilon_1, \quad j \in \{1, \dots, L\}.$$

Next, for each $i \in \{0, \dots, L-1\}$, we consider the global solution u to (NLS) with initial data $u(t_i) = v(t_i)$. Since $\|v(t_i)\| \leq E$ by assumption, we have

$$\|v\| \leq \Lambda(E).$$

Hence, we can apply Lemma 2.3.7 on J_i , and obtain

$$\|v\|_{L_{t,x}^5(J_i)} \leq C_i(E).$$

By summing over the spacetime slabs, we conclude that

$$\|v\|_{L_{t,x}^5(J^*)} \leq C(E, \Lambda(E)) \leq C(E).$$

This finishes the proof of Proposition 2.3.8. □

2.4 Almost conservation laws

In this section, we prove some almost conservation laws in order to obtain the uniform bound on the $H^\sigma(\mathbb{R}^3)$ norm of v . Recall that throughout this section, $\alpha > 0$ and $\omega \in \tilde{\Omega}_\alpha$ are fixed, such that for N large enough,

$$F^\omega(\mathbb{R}) + F_\infty^\omega(\mathbb{R}) \leq C_\alpha, \quad F_2^\omega(\mathbb{R}) \leq C_\alpha N^{\frac{1-\sigma}{2}-\gamma}.$$

Since there is no coercive conservation law at the level of H^σ for (NLS), we need to use a modified energy. More precisely, we consider the energy of Iv , which is finite but not preserved, in order to damp the frequencies of size larger than N . However, we expect the time-derivative of the modified energy to be small when N is large, at least on a spacetime slab where there hold some smallness conditions for the $L_{t,x}^4$ norm of v and spacetime bounds for the stochastic forcing term.

2.4.1 Setting up the I -method

Let $v \in H_x^\sigma(\mathbb{R}^3)$ be the maximal lifespan solution to (NLS_f) on $[0, T^*)$. There is no difficulty to estimate the low frequencies of v since the L^2 norm is almost-conserved, and the forcing term is uniformly in L^2 .

Lemma 2.4.1 (Almost conservation of the mass (see Lemma 7.1 in [OOP19])). *Let v be solution to (NLS_f) on a time interval J . Then,*

$$\int_{\mathbb{R}^3} |v(t, x)|^2 dx \lesssim \int_{\mathbb{R}^3} |v_0(x)|^2 dx + \int_{\mathbb{R}^3} |f_0(x)|^2 dx.$$

Proof. The proof is a straightforward consequence of the conservation of mass for (NLS) and the unitarity of the linear Schrödinger propagator $e^{it\Delta}$, from which we deduce that

$$\int_{\mathbb{R}^3} |v(t)|^2 dx = -2\text{Re}\left(\int v(t)\overline{f^\omega(t)} dx\right) \leq \frac{1}{2} \int |v(t)|^2 dx + 2 \int |f^\omega(t)|^2 dx.$$

We refer to Lemma 7.1 in [OOP19] for further details. \square

To obtain some estimates on the high frequencies of v , which has infinite energy, we need to smooth it by applying the I -operator, and by considering Iv , which has finite energy, or sometimes $Iu = I(v + f)$. The aim is to prove the uniform bound

$$\sup_{t \in [0, T^*)} \|Iv(t)\|_{H^1(\mathbb{R}^3)} < +\infty. \quad (2.4.1)$$

Indeed, we deduce from this that

$$\sup_{t \in [0, T^*)} \|v(t)\|_{H_x^\sigma(\mathbb{R}^3)} < +\infty. \quad (2.4.2)$$

With this uniform estimate (2.4.2) on the $H^\sigma(\mathbb{R}^3)$ norm of v at hand, we are in position to apply Proposition 2.3.8 to show that the solution v to (NLS_f) is global-in-time and scatters. Now, observe that the truncated solution Iu , as well as Iv , satisfies the perturbed cubic Schrödinger equation

$$i\partial_t Iu + \Delta Iu = \mathcal{N}(Iu) + H, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^3, \quad (I\text{-NLS})$$

where we denoted by H the commutator

$$H := IN(u) - \mathcal{N}(Iu). \quad (2.4.3)$$

This perturbation H can be seen as a trilinear operator acting on functions $u^{(j)}$ of type v or f . In some appropriate spacetime norms, H enjoys some decay with respect to N thanks to frequency cancellations stemming from the gauge-invariant structure of the equation, and from the definition of the I -operator.

Definition 2.4.2. Given a time-interval $J = [t_0, b] \subset [0, T^*)$, we define the Strichartz norm of a frequency truncated function v by

$$Z_I(J) := \|Iv\|_{S^1(J)} = \sup_{s(q,r)=0} \|\langle \nabla \rangle Iv\|_{L_t^q(J; L_x^r(\mathbb{R}^3))},$$

where we recall that $s(q, r) = \frac{2}{q} - d(\frac{1}{2} - \frac{1}{r})$ encodes the admissibility condition for the Strichartz estimate to hold.

We shall prove that $Z_I(J)$ is of size $\sim N^{1-\sigma}$ provided $E(Iv(t_0)) \leq N^{2(1-\sigma)}$ and $\|Iv\|_{L_{t,x}^4(J)}^4 \ll 1$. This is the matter of Lemma 2.4.4. Before this, let us state and prove the main result of this section, where we estimate the spacetime Lebesgue norm of wH for a given function w that lies uniformly in L_x^2 .

Lemma 2.4.3. *For all $\delta > 0$, there exists a constant C such that for every $w \in L_t^\infty(J; L_x^2(\mathbb{R}^3))$ and $N \in 2^{\mathbb{N}}$, if $J \subset [0, T^*)$ is such that $Z_I(J) < +\infty$, then*

$$\int_J \left| \int_{\mathbb{R}^3} w(t, x) \overline{H}(t, x) dx \right| dt \leq CN^{-1+\delta} \sup_{K \in 2^{\mathbb{N}}} (K^{-1} \|P_K w\|_{L_t^\infty L_x^2(J)}) (Z_I(J)^2 + F_2^\omega(J)^2) (Z_I(J) + F_\infty^\omega(J)). \quad (2.4.4)$$

The idea is to use the frequency cancellations from the I operator in the error term H to gain a negative power of N . As we will see, we shall often come across terms as in the left-hand-side of (2.4.4), especially when we estimate the modified energy and the modified Morawetz interaction. Hence, obtaining estimates like (2.4.4) turns out to be the crucial part of our analysis.

Proof. The strategy follows the same lines as the one of Proposition 3 from [Col+04], but with some differences. At a fixed time $t \in J$, we use Plancherel's formula, and we develop the nonlinearity in Fourier to obtain

$$\int_{\mathbb{R}^3} w(t, x) \overline{H}(t, x) dx = \int_{\Sigma_{\xi_i=0}} \left(1 - \frac{m(\xi_2 + \xi_3 + \xi_4)}{m(\xi_2)m(\xi_3)m(\xi_4)} \right) \widehat{w}(t, \xi_1) \widehat{Iu}(t, \xi_2) \widehat{Iu}(t, \xi_3) \widehat{Iu}(t, \xi_4). \quad (2.4.5)$$

First, we perform a Littlewood-Paley decomposition of each term appearing in the above integral, localized around a dyadic frequency N_i . We denote by B the pointwise bound of the Fourier multiplier

$$\sup_{\substack{\xi_1 + \xi_2 + \xi_3 + \xi_4 = 0 \\ \xi_i \sim N_i}} \left| 1 - \frac{m(\xi_2 + \xi_3 + \xi_4)}{m(\xi_2)m(\xi_3)m(\xi_4)} \right| \leq B(N_1, N_2, N_3, N_4).$$

As observed in Proposition 3.1 in [Col+04], a case by case analysis leads to the following decay estimate for B : for all $\delta > 0$, there exists $C > 0$ such that for all $N_1, N_2, N_3, N_4, N \in 2^{\mathbb{N}}$, we have

$$B(N_1, N_2, N_3, N_4) \frac{N_1}{N_2 N_3} \leq CN^{-1+\delta} \max(N_1, N_2, N_3, N_4)^{-\delta}. \quad (2.4.6)$$

The above estimate stems from frequency cancellations and indicates an *almost* conservation of the energy. Next, we factorize (2.4.5) by the pointwise bound B on the Fourier multiplier. We are left to control

$$B(N_1, N_2, N_3, N_4) \left| \int_{\mathbb{R}^3} \Lambda(\widehat{w}, \widehat{Iu}, \widehat{Iu})(\xi) \widehat{Iu}(\xi) d\xi \right|,$$

where Λ is a multilinear operator obtained from (2.4.5) (see also (3.22) from [Col+04]). To estimate such a multilinear integral, we apply Plancherel, Hölder's inequality and the Coifman-Meyer estimate (see Page 179 in [MC78]). We also use Hölder's inequality in time, to see that the left-hand-side of (2.4.4) is bounded from above by

$$\sum_{N_i} B(N_1, N_2, N_3, N_4) \|P_{N_1} w\|_{L_t^\infty L_x^2(J)} \|P_{N_2} Iu\|_{L_t^2 L_x^6(J)} \|P_{N_3} Iu\|_{L_t^2 L_x^6(J)} \|P_{N_4} Iu\|_{L_t^\infty L_x^6(J)}.$$

Using the decay estimate (2.4.6), we can sum over the N_i 's and gain one power of N

$$\begin{aligned} & \int_J \left| \int_{\mathbb{R}^3} w(t, x) \overline{H}(t, x) dx \right| dt \\ & \leq CN^{-1+\delta} \left(\sup_{N_1} N_1^{-1} \|P_{N_1} w\|_{L_t^\infty L_x^2} \right) \|\nabla Iu\|_{L_t^2 L_x^6(J)} \|\nabla Iu\|_{L_t^2 L_x^6(J)} \|Iu\|_{L_t^\infty L_x^6(J)}. \end{aligned} \quad (2.4.7)$$

It remains to estimate the contributions of the terms on the right-hand side in (2.4.7). Each term Iu is decomposed into $Iu = If + Iv$. To estimate the terms Iv in $L_t^2 L_x^6$, we use the endpoint Strichartz estimate, while we need Sobolev embedding to estimate Iv in $L_t^\infty L_x^6$, respectively. As for the terms If , we only use the definitions from Section 2.2. This concludes the proof of Lemma 2.4.3. \square

Let us now address the a priori estimate for $Z_I(J)$ on a spacetime slab, where we assume a bound for the modified energy of v , as well as a smallness condition on spacetime norms of Iv and f . More precisely, provided that the energy of Iv is initially of size $N^{2(1-\sigma)}$, we prove that $Z_I(J)$ is of size $\sim N^{1-\sigma}$ on an interval J where there holds (2.4.8) and (2.4.9).

Lemma 2.4.4 (Modified local existence theory). *Let $t_0 \in J^*$ be such that*

$$\mathcal{E}(v)(t_0) \leq N^{2(1-\sigma)}.$$

There exist $C > 0$ and $\eta > 0$ that do not depend on N , $\varepsilon_N = N^{-2(1-\sigma)-\delta}$ for some arbitrarily small δ , and $N_0 = N_0(C_\alpha, f_0)$ such that for any interval $J = [t_0, b) \subset [0, T^)$ on which the following smallness conditions¹² are satisfied*

$$\|Iv\|_{L_{t,x}^4(J)}^4 \leq \varepsilon_N, \quad (2.4.8)$$

$$\|f\|_{L_t^4 L_x^3(J)} \leq \eta, \quad (2.4.9)$$

we have that for all $N \geq N_0$,

$$Z_I(J) \leq CN^{1-\sigma}. \quad (2.4.10)$$

For now on, we assume that $N > N_0$, such that (2.4.10) holds whenever the assumptions of Lemma 2.4.4 are satisfied.

Proof. Recall that Iv is solution to (I–NLS). By the Duhamel's integral formulation for equation (I–NLS), we have that

$$\begin{aligned} Iv(t) &= e^{i(t-t_0)\Delta} Iv(t_0) - \int_{t_0}^t e^{i(t-\tau)\Delta} (\mathcal{N}(Iu) + H) d\tau \\ &= e^{i(t-t_0)\Delta} Iv(t_0) + \mathcal{I}([t_0, t]; \mathcal{N}_1) + \mathcal{I}([t_0, t]; \mathcal{N}_1) + \mathcal{I}([t_0, t]; H), \end{aligned}$$

where¹³

$$\mathcal{N}_1 := \mathcal{N}(v, v, v) + \mathcal{N}(f, f, f) + \mathcal{N}(f, v, v), \quad \mathcal{N}_2 = \mathcal{N}(f, f, v).$$

¹²Note that (4, 3) is an admissible pair. Hence, $\|f\|_{L_t^4 L_x^3(\mathbb{R})} \leq C\|f_0\|_{L_x^2}$ for every ω .

¹³We also have similar terms, but the analysis does not depend on the complex conjugates at this stage.

It follows from the dual Strichartz estimate of Proposition 2.2.1 that

$$Z_I(J) \leq C \left(\|Iv(t_0)\|_{H^1} + \underbrace{\|\langle \nabla \rangle I \mathcal{N}_1\|_{L_t^{\frac{6}{5}} L_x^{\frac{18}{11}}(J)}}_{I_1} + \underbrace{\|\langle \nabla \rangle I \mathcal{N}_2\|_{L_t^1 L_x^2(J)}}_{I_2} + \underbrace{\|\mathcal{I}([t_0, t], \langle \nabla \rangle I H)\|_{U_{\Delta}^2(J)}}_{I_3} \right).$$

Term I_1 : We deduce from the Leibniz rule that

$$\|\langle \nabla \rangle I(\mathcal{N}(Iv))\|_{L_t^{\frac{6}{5}} L_x^{\frac{18}{11}}} \leq \|\langle \nabla \rangle Iv\|_{L_t^2 L_x^6} \|Iv\|_{L_t^6 L_x^{\frac{9}{2}}}^2.$$

By interpolation and Sobolev embedding, we have

$$\|Iv\|_{L_t^6 L_x^{\frac{9}{2}}} \leq \|Iv\|_{L_{t,x}^4}^{\frac{2}{3}} \|Iv\|_{L_t^\infty L_x^6}^{\frac{1}{3}} \leq C \varepsilon_N^{\frac{1}{6}} Z_I(J)^{\frac{1}{3}}.$$

Consequently, the contribution of this term is less than

$$\|\langle \nabla \rangle I(\mathcal{N}(Iv))\|_{L_t^{\frac{6}{5}} L_x^{\frac{18}{11}}} \leq C \varepsilon_N^{\frac{1}{3}} Z_I(J)^{1+\frac{2}{3}}.$$

Similarly, we have

$$\begin{aligned} \|\langle \nabla \rangle I(\mathcal{N}(Iv, Iv, If))\|_{L_t^{\frac{6}{5}} L_x^{\frac{18}{11}}} &\leq \|\langle \nabla \rangle If\|_{L_t^2 L_x^6} \|Iv\|_{L_t^6 L_x^{\frac{9}{2}}}^2 + \|\langle \nabla \rangle Iv\|_{L_t^2 L_x^6} \|Iv\|_{L_t^6 L_x^{\frac{9}{2}}} \|If\|_{L_t^6 L_x^{\frac{9}{2}}} \\ &\leq C Z_I(J) (F_2^\omega(J) \varepsilon_N^{\frac{1}{3}} Z_I(J)^{-\frac{1}{3}} + \varepsilon_N^{\frac{1}{6}} Z_I(J)^{\frac{1}{3}} \|If\|_{L_{t,x}^4}^{\frac{2}{3}} \|If\|_{L_t^\infty L_x^6}^{\frac{1}{3}}) \\ &\leq C Z_I(J) (F_2^\omega(J) \varepsilon_N^{\frac{1}{3}} Z_I(J)^{-\frac{1}{3}} + \varepsilon_N^{\frac{1}{6}} Z_I(J)^{\frac{1}{3}} F_\infty^\omega(J)^{\frac{1}{3}} F^\omega(J)^{\frac{2}{3}}). \end{aligned}$$

Additionally,

$$\|\langle \nabla \rangle (If)(If)^2\|_{L_t^{\frac{6}{5}} L_x^{\frac{18}{11}}} \leq \|\langle \nabla \rangle If\|_{L_t^2 L_x^6} \|If\|_{L_t^6 L_x^{\frac{9}{2}}}^2 \leq C \|f_0\|_{L_x^2}^2 F_2^\omega(J).$$

Hence, the contribution of the first term is less than

$$I_1 \leq C Z_I(J) (F_2^\omega(J) Z_I(J)^{-1} + \varepsilon_N^{\frac{1}{3}} Z_I(J)^{\frac{2}{3}} + \varepsilon_N^{\frac{1}{3}} N^{\frac{1-\sigma}{2}} Z_I(J)^{-\frac{1}{3}} C_\alpha + \varepsilon_N^{\frac{1}{6}} Z_I(J)^{\frac{1}{3}} C_\alpha).$$

Term I_2 : Once again, when a derivative falls on the term If , we need to place it inside L_t^2 , and use the improved Sobolev embedding for randomized radial initial data. It follows from the Leibniz rule that

$$\|\langle \nabla \rangle I \mathcal{N}_2\|_{L_t^1 L_x^2} \leq C \|\langle \nabla \rangle If\|_{L_t^1 L_x^2} \|Iv\|_{L_t^1 L_x^2} + \|\langle \nabla \rangle Iv\|_{L_t^1 L_x^2} \|If\|_{L_t^1 L_x^2}^2.$$

We use Hölder's inequality, Sobolev embedding and the assumption that $s \geq \frac{3}{7}$ to get

$$\|\langle \nabla \rangle If\|_{L_t^1 L_x^2} \|Iv\|_{L_t^1 L_x^2} \leq C \|\langle \nabla \rangle If\|_{L_t^2 L_x^6} \|If\|_{L_t^4 L_x^{12}} \|Iv\|_{L_{t,x}^4} \leq C C_\alpha^2 \varepsilon_N^{\frac{1}{4}} N^{\frac{1-\sigma}{2}}.$$

Finally,

$$\|\langle \nabla \rangle Iv(If)^2\|_{L_t^1 L_x^2} \lesssim \|\langle \nabla \rangle Iv\|_{L_t^2 L_x^6} \|If\|_{L_t^4 L_x^3}^2 \lesssim \eta^2 Z_I(J).$$

Collecting the above estimates, we see that the contribution of this term is less than

$$I_2 \leq CZ_I(J)(C_\alpha^3 N^{\frac{1-\sigma}{2}} Z_I(J)^{-1} + C_\alpha^2 \varepsilon_N^{\frac{1}{4}} N^{-\frac{1-\sigma}{2}} + \eta^2).$$

Term I_3 : It follows from the duality estimate in U_Δ^2 as stated in Proposition 2.2.5 and from Lemma 2.4.4 that

$$\begin{aligned} \|\mathcal{I}((t_0, t), \langle \nabla \rangle IH)\|_{U_\Delta^2} &\leq C \sup_{\|w\|_{V_\Delta^2} \leq 1} \left| \iint_{J \times \mathbb{R}^3} \bar{w}(t, x) \langle \nabla \rangle IH dx dt \right| \\ &\leq CN^{-1+\delta} \sup_{\|w\|_{V_\Delta^2} \leq 1} \sup_{K \in 2^{\mathbb{N}}} (K^{-1} \|P_K(\langle \nabla \rangle Iw)\|_{L_t^\infty L_x^2(J)}) (Z_I(J)^2 + F_2^\omega(J)^2) (Z_I(J) + F_\infty^\omega(J)) \\ &\leq CN^{-1+\delta} (Z_I(J)^2 + F_2^\omega(J)^2) (Z_I(J) + F_\infty^\omega(J)). \end{aligned}$$

Next, we take $\gamma > 0$ arbitrarily small, and we fix

$$\varepsilon_N = N^{-2(1-\sigma)-\gamma}.$$

We have

$$I_1 \leq CZ_I(J) \left(N^{-\frac{2(1-\sigma)}{3}-\frac{\gamma}{3}} Z_I(J)^{\frac{2}{3}} + C_\alpha N^{-\frac{2(1-\sigma)}{3}-\frac{\gamma}{3}} N^{\frac{1-\sigma}{2}} Z_I(J)^{-\frac{1}{3}} + C_\alpha N^{-\frac{1-\sigma}{3}-\frac{\gamma}{3}} Z_I(J)^{\frac{1}{3}} \right).$$

If $Z_I(J) \sim N^{1-\sigma}$, then the term in the parenthesis on the right-hand-side of the above estimate is smaller than a negative power of N . Moreover, we have

$$\lim_{|J| \rightarrow 0} Z_I(J) = 0.$$

Hence, we apply a continuity argument in this case to prove that $Z_I(J)$ is indeed of size $\sim N^{1-\sigma}$. Similarly, we have

$$I_2 \leq CZ_I(J)(C_\alpha^3 N^{\frac{1-\sigma}{2}} Z_I(J)^{-1} + C_\alpha^2 Z_I(J)^{-1} + \eta^2).$$

Here again, we shall be able use the continuity method. We emphasize that η does not have to depend on N . The last contribution is estimated as follows :

$$I_3 \leq CN^{-1+\delta} (Z_I(J)^2 + N^{1-\sigma}) (Z_I(J) + C_\alpha).$$

In this case, we observe that when $Z_I(J) \sim N^{1-\sigma}$, the parenthesis in the right-hand-side of the above estimate is bounded by a negative power of N provided $\frac{1}{2} < \sigma$. Finally, we conclude by collecting the above contributions and by using the continuity method for $\frac{1}{2} < \sigma$, $N > N_0$ large enough and η small enough. We stress out that η and N_0 only depend on some universal constants and on C_α , which is fixed throughout the analysis. This concludes the proof of Lemma 2.4.4. \square

2.4.2 Modified energy increment

In this section, we revisit in a probabilistic setting the analysis of modified-energy increment performed when using the I -method (see for instance [Dod13; Dod19b; Col+04]). A similar strategy was developed in [FM20]. Here, we propose a different approach that strongly relies on the structure

of the nonlinearity. Subsequently, we follow [OOP19] and we include the probabilistic forcing term in the potential part of the energy

$$\mathcal{E}(v(t)) := \frac{1}{2} \int_{\mathbb{R}^3} |\nabla I v(t, x)|^2 dx + \frac{1}{4} \int_{\mathbb{R}^3} |I v + I f|^4 dx.$$

The idea is to benefit from the frequency cancellations appearing in H . More precisely, when we handle the increment of the energy, we aim to come up with terms under the form wH , for which we already obtained some spacetime bounds in Lemma 2.4.3. In the next Proposition, we estimate the energy increments on some intervals where the $L_{t,x}^4$ norm of Iv satisfies a smallness condition.

Proposition 2.4.5. *Let $t_0 \in J^*$, such that*

$$\mathcal{E}(v)(t_0) \leq N^{2(1-\sigma)}.$$

Then, for all interval $J = [t_0, t] \subset [0, T^)$ where Iv satisfies the smallness conditions (2.4.8) and (2.4.9), that lead to the modified local well-posedness statement of Lemma 2.4.4, there exists $C(\alpha)$ such that*

$$\int_J \left| \frac{d}{dt} \mathcal{E}(v) \right| dt \leq CN^{0+} N^{6(1-\sigma)-1} + CN^{1-\sigma} F_2^\omega(J)^2 + CN^{1-\sigma} F_2^\omega(J) \|Iv\|_{L_{t,x}^4(J)}^2. \quad (2.4.11)$$

Observe that we gain a negative power of N from the perturbative term that contains H . However, as explained in the introduction, we keep track of some sub-additive quantities when we estimate the other terms. This will be useful when we will sum over the intervals where we have the modified local well-posedness result of Lemma 2.4.4. Still, we are forced to lose some powers of N when we estimate the terms where the gradient hits $I f$ by F_2 . This indicates that we have no hope to prevent the energy from extending $O(1)$, even if we rescale the initial data at the beginning of the analysis.

Proof. We deduce from (I -NLS) that

$$\begin{aligned} \frac{d}{dt} E(Iv(\tau)) &= \operatorname{Re} \int_{\mathbb{R}^3} \partial_t(Iv) (\mathcal{N}(Iu) - IN(u)) dx + \operatorname{Re} \int_{\mathbb{R}^3} \partial_t(I f) \mathcal{N}(Iu) dx \\ &= \operatorname{Re} \int_{\mathbb{R}^3} \partial_t(Iv) H dx + \operatorname{Re} \int_{\mathbb{R}^3} i \Delta(I f) \mathcal{N}(Iu) dx. \end{aligned}$$

Consequently,

$$\int_J \left| \frac{d}{dt} E(Iv(\tau)) \right| d\tau \leq C \underbrace{\int_J \left| \int_{\mathbb{R}^3} \partial_t(Iv) \bar{H} dx \right| dt}_I + \underbrace{\|\Delta(I f) \mathcal{N}(Iu)\|_{L_{t,x}^1(J)}}_{II}. \quad (2.4.12)$$

Estimate of term I In light of Lemma 2.4.3, it suffices to prove that

$$\sup_{K \in 2^{\mathbb{N}}} (K^{-1} \|(\partial_t I v)\|_{L_t^\infty L_x^2}) \leq N^{3(1-\sigma)}. \quad (2.4.13)$$

Indeed, we have

$$Z_I(J) \leq CN^{1-\sigma}, \quad F_\infty^\omega(\mathbb{R}) \leq C_\alpha, \quad F_2^\omega(J) \leq C_\alpha N^{\frac{1-\sigma}{2}-\gamma}.$$

The first estimate on $Z_I(J)$ follows from the assumptions (2.4.8), (2.4.9), and Lemma 2.4.4. Subsequently, we use Lemma 2.4.3 to get that

$$\begin{aligned} & \int_J \left| \int_{\mathbb{R}^3} \partial_t(Iu) \overline{H} dx \right| dt \\ & \leq CN^{-1+\delta} \sup_{K \in 2^{\mathbb{N}}} (K^{-1} \|P_K(\partial_t(Iu))\|_{L_t^\infty L_x^2(J)}) (Z_I(J)^2 + F_2^\omega(J)^2) (Z_I(J) + F_\infty^\omega(J)) \\ & \leq C(C_\alpha) N^{-1+\delta} \sup_{K \in 2^{\mathbb{N}}} (K^{-1} \|P_K(\partial_t(Iu))\|_{L_t^\infty L_x^2(J)}) N^{3(1-\sigma)}. \end{aligned}$$

Let us turn to the proof of (2.4.13), and fix $K \in 2^{\mathbb{N}}$. Recall that Iv solves

$$\partial_t Iv = i\Delta Iv - iN(u).$$

First, we have from the definition of $Z_I(J)$ and from (2.4.10) that

$$\|P_K \Delta Iv\|_{L_t^\infty L_x^2(J)} \leq CK Z_I(J) \leq CK N^{1-\sigma}.$$

It remains to handle the nonlinear term $IN(u) \sim IN(u^{(1)}, u^{(2)}, u^{(3)})$ with $u^{(i)} \in \{v, f\}$. For this purpose, we shall lose some derivatives and apply the fractional Leibniz rule with the operator ∇I , especially when all the three terms are of type v since we only have some controls on norms of ∇Iv , and not on ∇v . One way of losing some derivatives is to use Sobolev embedding. We eventually prove that

$$\|P_K IN(u)\|_{L_t^\infty L_x^2(J)} \leq CK (Z_I(J)^3 + F_\infty^\omega(J)^3). \quad (2.4.14)$$

In what follows, all the spacetime norms will be taken over $J \times \mathbb{R}^3$. First, we observe that

$$\|v\|_{L_t^\infty L_x^3} \leq C \|\langle \nabla \rangle^{\frac{1}{2}} v\|_{L_t^\infty L_x^2} \leq C \|\langle \nabla \rangle Iv\|_{L_t^\infty L_x^2} \leq CZ_I(J).$$

This follows from the Sobolev embedding and from the definition of the I -operator (see (2.1.3)). Next, we perform a case by case analysis depending on the type of terms that occur in the trilinear form.

Case 1 : Three random terms In this case, we use Hölder inequality and the probabilistic estimates (2.2.12) to get

$$\|P_K I(f^3)\|_{L_t^\infty L_x^2(J)} \leq \|f\|_{L_t^\infty L_x^6(J)}^3 \leq F_\infty^\omega(J)^3 \leq C_\alpha^3.$$

Case 2 : Two random terms In this case, we use Sobolev embedding, Hölder's inequality and the almost conservation of the mass to obtain

$$\|P_K I(f^2 v)\|_{L_t^\infty L_x^2} \leq C \|\langle \nabla \rangle P_K I(f^2 v)\|_{L_t^\infty L_x^{\frac{6}{5}}} \leq CK \|f\|_{L_t^\infty L_x^6}^2 \|v\|_{L_t^\infty L_x^2} \leq CK F_\infty^\omega(J)^2 \|f_0\|_{L_x^2}.$$

Case 3 : One random term Similarly,

$$\|P_K I(f v^2)\|_{L_t^\infty L_x^2} \leq C \|\langle \nabla \rangle P_K I(f v^2)\|_{L_t^\infty L_x^{\frac{6}{5}}} \leq CK \|f\|_{L_t^\infty L_x^6} \|v\|_{L_t^\infty L_x^3} \leq CK F_\infty^\omega(J) Z_I(J)^2.$$

Case 4 : Three deterministic terms Here, we need to use the Leibniz rule for $I\langle\nabla\rangle$. We obtain

$$\begin{aligned} \|P_K I(v^3)\|_{L_t^\infty L_x^2} &\leq C\|\langle\nabla\rangle^{\frac{5}{4}} P_K I(v^3)\|_{L_t^\infty L_x^{\frac{12}{11}}} \leq CK\|\langle\nabla\rangle^{\frac{1}{4}} P_K I(v^3)\|_{L_t^\infty L_x^{\frac{12}{11}}} \\ &\leq CK\|\langle\nabla\rangle^{\frac{1}{4}} I v\|_{L_t^\infty L_x^4} \|v\|_{L_t^\infty L_x^3}^2 \leq CK\|\langle\nabla\rangle I v\|_{L_t^\infty L_x^2} \|v\|_{L_t^\infty L_x^3}^2 \leq CK Z_I(J)^3. \end{aligned}$$

This addresses the contribution for the term I in the energy increment.

Estimate of term II After integrating by parts and applying the Leibniz rule, we write

$$\|\Delta(I f)\mathcal{N}(I u)\|_{L_{t,x}^1(J)} \leq C \iint_{J \times \mathbb{R}^3} |\nabla(I f)\mathcal{N}(\nabla I u^{(1)}, I u^{(2)}, I u^{(3)})| dt dx \quad (2.4.15)$$

with $u^{(i)} \in \{f, v\}$ for $i \in \{1, 2, 3\}$. Next, we perform a case by case analysis to estimate the contribution of the different terms on the right-hand side of (2.4.15).

No random term. In this case, we have $u^{(i)} = v$ for $i \in \{1, 2, 3\}$. We apply Cauchy-Schwarz inequality to obtain

$$(2.4.15) \leq C\|\nabla I f\|_{L_t^2 L_x^\infty(J)} \|\nabla I v\|_{L_t^\infty L_x^2(J)} \|I v\|_{L_{t,x}^4}^2 \leq C F_2^\omega(J) Z_I(J) \|I v\|_{L_{t,x}^4}^2.$$

To handle the other cases, we apply Cauchy-Schwarz and we use the estimate (2.2.27).

One random term.

- **Case 1 :** $u^{(1)} = f, u^{(2)} = u^{(3)} = v$. It follows from the almost conservation from Lemma (2.4.1) that

$$(2.4.15) \leq C\|\nabla I f\|_{L_t^2 L_x^\infty(J)}^2 \|I v\|_{L_t^\infty L_x^2}^2 \leq C F_2^\omega(J)^2.$$

- **Case 2 :** $u^{(1)} = u^{(2)} = v, u^{(3)} = f$. Similarly,

$$(2.4.15) \leq C\|\nabla I f\|_{L_t^2 L_x^\infty(J)} \|I f\|_{L_t^2 L_x^\infty(J)} \|\nabla I v\|_{L_t^\infty L_x^2(J)} \|I v\|_{L_t^\infty L_x^2(J)} \leq C F_2^\omega(J)^2 Z_I(J).$$

Two random terms.

- **Case 1 :** $u^{(1)} = u^{(2)} = f, u^{(3)} = v$. We deduce from Cauchy-Schwarz and from the almost conservation of the mass that 2.4.1 that

$$(2.4.15) \leq \|\nabla I f\|_{L_t^2 L_x^\infty(J)}^2 \|I f\|_{L_t^\infty L_x^2(J)} \|I v\|_{L_t^\infty L_x^2(J)} \leq C F_2^\omega(J)^2.$$

- **Case 2 :** $u^{(1)} = v, u^{(2)} = u^{(3)} = f, u^{(2)} = u^{(3)} = f$. In this case, we have

$$\begin{aligned} (2.4.15) &\leq C\|\nabla I f\|_{L_t^2 L_x^\infty(J)} \|\nabla I v\|_{L_t^\infty L_x^2(J)} \|I f\|_{L_t^2 L_x^\infty(J)} \|I v\|_{L_t^\infty L_x^2(J)} \\ &\leq C F_2^\omega(J) \|I f\|_{L_t^\infty L_x^2(J)} Z_I(J) \leq C F_2^\omega(J) Z_I(J). \end{aligned}$$

Three random terms. Here, we have $u^{(i)} = f$ for $i \in \{1, 2, 3\}$, so that

$$(2.4.15) \leq C \|\nabla I f\|_{L_t^2 L_x^\infty(J)}^2 \|I f\|_{L_t^\infty L_x^2(J)}^2 \leq C F_2^\omega(J)^2.$$

Collecting all the contributions, we obtain the desired estimate on the modified energy increment. \square

2.4.3 Modified interaction Morawetz

We revisit the standard interaction Morawetz inequality in a perturbed setting, for the equation (*I-NLS*) satisfied by Iu . First, we recall the formal computations that leads to the so-called Lin-Strauss inequality. Given a convex weight a , we define the action ¹⁴

$$M_0(t) = 2\text{Im} \int_{\mathbb{R}^3} \partial_j a(x) (\partial_j Iu)(x) \overline{Iu}(x) dx = 2\text{Im} \int_{\mathbb{R}^3} \partial_j a T_{0,j} dx.$$

It corresponds to the contraction of the current density $T_{0,j} = T_{j,0} = 2\text{Im}(\partial_j(Iu)\overline{Iu})$ against the vector field ∇a . Thanks to the convexity of a , we hope to get some monotonicity for M_0 . The general virial identity reads

$$\frac{d}{dt} M_y(t) = \int_{\mathbb{R}^3} -\Delta^2 a |Iu|^2 + 4\partial_{k_j}^2 a \text{Re}(\partial_k(\overline{Iu})\partial_j(Iu)) + \Delta a |Iu|^4 + 2\text{Re}(2\partial_j a \partial_j(Iu)H + \Delta a(Iu)H) dx.$$

Given a point $y \in \mathbb{R}^3$, the recentered Morawetz action, denoted M_y , corresponds to the particular weight $a_y(x) = |x - y|$, that satisfies

$$\partial_j a = \frac{x_j - y_j}{|x - y|}, \quad \Delta a = \frac{2}{|x - y|}, \quad \Delta^2 a = -2\pi\delta_y, \quad \partial_{j_k}^2 a = \text{I} - \frac{x - y}{|x - y|} \otimes \frac{x - y}{|x - y|} =: \text{P}.$$

Hence,

$$\begin{aligned} -\frac{d}{dt} M_y(t) &= \pi |Iu(t, y)|^2 + 2 \int_{\mathbb{R}^3} |\text{P} \nabla Iu(x)|^2 dx + \int_{\mathbb{R}^3} \frac{1}{|x - y|} |Iu(x)|^4 dx \\ &\quad + 4\text{Re} \int \frac{x_j - y_j}{|x - y|} \partial_j(Iu) \overline{H} dx + 4\text{Re} \int_{\mathbb{R}^3} \frac{1}{|x - y|} Iu \overline{H} dx. \end{aligned} \quad (2.4.16)$$

Observe the presence of two extra terms, that come from the forcing term H . Next, we average the recentered Morawetz action against the mass density $|Iu(y)|^2$ to obtain the Morawetz interaction,

$$\int_{\mathbb{R}^3} M_y(t) |Iu(y)|^2 dy.$$

Note that this quantity is bounded by the conserved L^2 norm times the critical $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$ norm of Iu .

$$\left| \int_{\mathbb{R}^3} M_y(t) |Iu(y)|^2 dy \right| \leq C \|Iu(t)\|_{L_x^2}^2 \|Iu(t)\|_{\dot{H}^{\frac{1}{2}}}^2. \quad (2.4.17)$$

We can now state and prove the modified Morawetz interaction, that will provide a spacetime control on Iu .

¹⁴ we sum over the indices, and $\partial_j f = \sum_{j=1}^3 \partial_{x_j} f$.

Proposition 2.4.6 (Modified interaction Morawetz). *Assume that $J \subset J^*$ is an interval such that $Z_I(J_k) < \infty$. Then, for any $\delta > 0$, there exists $C > 0$ such that for any partition $J = \bigcup_{k=0}^{L-1} J_k$, there holds*

$$\begin{aligned} \iint_{J \times \mathbb{R}^3} |Iu|^4 dt dx &\leq C \|Iu\|_{L^\infty \dot{H}^{\frac{1}{2}}(J)}^2 \|Iu\|_{L^\infty L^2(J)}^2 \\ &+ CN^{-1+\delta} \|Iu\|_{L_t^\infty L_x^2(J)} \|Iu\|_{L_t^\infty H^{\frac{1}{2}}(J)}^2 \left(\sum_{k=0}^{L-1} (Z_I(J_k)^2 + F_2(J_k)^2) (Z_I(J_k) + F_\infty(J_k)) \right). \end{aligned} \quad (2.4.18)$$

Proof. By using the Leibniz rule, and then integrating over time, we obtain

$$\begin{aligned} \left[\int_{\mathbb{R}^3} M_y(t) |Iu(y)|^2 dy \right]_{t=t_0}^{t=t_1} &= \underbrace{\iint_{J \times \mathbb{R}^3} |Iu(t, y)|^2 \left(\frac{d}{dt} \int_{\mathbb{R}^3} \frac{x_j - y_j}{|x - y|} T_{0,j}(x) dx \right) dy dt}_I \\ &+ \underbrace{\iiint_{J \times \mathbb{R}^3 \times \mathbb{R}^3} \frac{x_j - y_j}{|x - y|} T_{0,j}(x) 2\text{Re}(\partial_t Iu(y) \overline{Iu}(y)) dx dy dt}_{II}. \end{aligned}$$

Next, we plug (2.4.16) into I and II. We have

$$\begin{aligned} I &= \pi \iint_{J \times \mathbb{R}^3} |Iu(t, y)|^4 dy dt \\ &+ 2 \iiint_{J \times \mathbb{R}^3 \times \mathbb{R}^3} |P_{x,y} \nabla u(t, x)|^2 |Iu(y)|^2 dx dy dt + \iiint_{J \times \mathbb{R}^3 \times \mathbb{R}^3} \frac{2}{|x - y|} |Iu(x)|^4 |Iu(y)|^2 dx dy dt \\ &+ 4\text{Re} \underbrace{\iiint_{\mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^3} |Iu(y)|^2 \frac{x_j - y_j}{|x - y|} \partial_j(Iu)(x) \overline{H}(x) dx dy dt}_{I_1} \\ &+ 4\text{Re} \underbrace{\iint_{J \times \mathbb{R}^3 \times \mathbb{R}^3} \frac{|Iu(y)|^2}{|x - y|} Iu(x) \overline{H}(x) dx dy dt}_{I_2}. \end{aligned} \quad (2.4.19)$$

To expand the term II we use equation (*I-NLS*) and we write the time derivative of Iu in terms of the perturbation H .

$$\begin{aligned} II &= - \iiint_{J \times \mathbb{R}^3 \times \mathbb{R}^3} \frac{x_j - y_j}{|x - y|} T_{0,j}(x) 2\text{Re}(i\Delta Iu(y) \overline{Iu}(y)) dx dy dt \\ &+ \underbrace{\iiint_{J \times \mathbb{R}^3 \times \mathbb{R}^3} \frac{x_j - y_j}{|x - y|} I_{0,j}(x) 2\text{Re}(iIu(y) \overline{H}(y)) dx dy dt}_{II_1}. \end{aligned} \quad (2.4.20)$$

Next, we observe that the terms which does not come from the perturbation have a sign. Hence, we can use the bound (2.4.17) and see that

$$\pi \iint_{J \times \mathbb{R}^3} |Iu(t, y)|^4 dy dt \leq 2C \|Iu\|_{L_t^\infty L_x^2(J)}^2 \|Iu\|_{L_t^\infty \dot{H}^{\frac{1}{2}}(J)}^2 + |I_1 + I_2 + II_1|.$$

We are left to estimate three terms I_1, I_2 and II_1 , that contain a spacetime norm of the commutator H . Hence, we will use Lemma 2.4.3 to address these three terms. Up to a partition, and using sub-additivity, we may assume that $L = 1$.

Term I_1 : We use the triangular inequality and Lemma 2.4.3 to get

$$\begin{aligned} |I_1| &\leq \|Iu\|_{L_t^\infty L_y^2(J)}^2 \int_J \sup_y \left| \int_{\mathbb{R}^3} \frac{x_j - y_j}{|x - y|} \partial_j(Iu)(x) \overline{H}(x) dx \right| dt \\ &\leq CN^{-1+\delta} \|Iu\|_{L_t^\infty L_y^2(J)}^2 \sup_{y \in \mathbb{R}^3} \sup_{K \in 2^{\mathbb{N}}} (K^{-1} \|P_K(\frac{x_j - y_j}{|x - y|} \partial_j(Iu)(x))\|_{L_t^\infty L_x^2(J)} \\ &\quad (Z_I(J)^2 + F_2^\omega(J)^2)(Z_I(J) + F_\infty^\omega(J)). \end{aligned}$$

We fix $y \in \mathbb{R}^3$, $t \in J$ and $K \in 2^{\mathbb{N}}$, and we proceed as follows.

$$\|P_K(\frac{x_j - y_j}{|x - y|} \partial_j(Iu)(x))\|_{L_x^2} = \sup_{\|g\|_{L_2} \leq 1} \left| \int_{\mathbb{R}^3} (\frac{x_j - y_j}{|x - y|} \partial_j(Iu)(t, x)) \overline{P_K g}(x) dx \right|$$

We deduce from the Leibniz rule that for each $x \in \mathbb{R}^3$,

$$\frac{x - y}{|x - y|} \cdot \nabla \overline{Iu}(x) = \operatorname{div}(\frac{x - y}{|x - y|} \overline{Iu}(x)) - \frac{2}{|x - y|} \overline{Iu}(x).$$

Next, we apply Hardy's inequality to handle the singular term with $|x - y|^{-1}$, and we obtain

$$\begin{aligned} \|P_K(\frac{x_j - y_j}{|x - y|} \partial_j(Iu)(x))\|_{L_x^2} &\leq \sup_{\|g\|_{L_2} \leq 1} \left| \int_{\mathbb{R}^3} \frac{x - y}{|x - y|} \cdot Iu(t, x) \overline{\nabla P_K g}(x) + Iu(t, x) \frac{2}{|x - y|} \overline{P_K g}(x) dx \right| \\ &\leq C \|Iu\|_{L_x^2} \sup_{\|g\|_{L_2} \leq 1} (K \|g\|_{L_x^2} + \|\frac{1}{|x - y|} P_K g(x)\|_{L_x^2}) \\ &\leq CK \|Iu\|_{L_x^2}. \end{aligned}$$

This addresses the estimate for the term I_1 :

$$|I_1| \leq CN^{-1+\delta} \|Iu\|_{L_t^\infty L_x^2(J)}^3 (Z_I(J)^2 + F_2^\omega(J)^2)(Z_I(J) + F_\infty^\omega(J)).$$

Term I_2 : We proceed similarly. In this case, we are left to estimate

$$\begin{aligned} \|P_K(\frac{Iu(x)}{|x - y|})\|_{L_x^2(J)} &= \sup_{\|g\|_{L_2} \leq 1} \left| \int_{\mathbb{R}^3} Iu(t, x) \frac{1}{|x - y|} \overline{P_K g}(x) dx \right| \\ &\leq \|Iu\|_{L_x^2} \|\frac{1}{|x - y|} g\|_{L_x^2} \\ &\leq CK \|Iu\|_{L_x^2} \|g\|_{L_x^2}, \end{aligned}$$

where we used Hardy's inequality in the last line. Hence,

$$|I_2| \leq CN^{-1+\delta} \|Iu\|_{L_t^\infty L_x^2(J)}^3 (Z_I(J)^2 + F_2^\omega(J)^2)(Z_I(J) + F_\infty^\omega(J)).$$

Term Π_1 :

$$\begin{aligned} |\Pi_1| &\leq \sup_{t,y} \left| \int_{\mathbb{R}^3} \frac{x_j - y_j}{|x - y|} I_{0,j}(x) dx \right| \left| \int_J \left| \int_{\mathbb{R}^3} Iu(y) H(y) dy \right| dt \right| \\ &\leq CN^{-1+\delta} \|Iu\|_{L_t^\infty \dot{H}^{\frac{1}{2}}(J)}^2 \|Iu\|_{L_t^\infty L_x^2(J)} (Z_I(J)^2 + F_2^\omega(J)^2) (Z_I(J) + F_\infty^\omega(J)). \end{aligned}$$

This concludes the proof of Proposition 2.4.6. \square

2.5 Almost-sure global well-posedness and scattering

In this section, we glue together the pieces collected in the previous sections to prove the main theorem. First, given $\alpha > 0$ we take $\omega \in \Omega_\alpha$ such that

$$F^\omega(\mathbb{R}) + F_\infty^\omega(\mathbb{R}) \leq C_\alpha.$$

Then, we apply the local probabilistic Cauchy theory developed in Section 2.3.2. For such an ω , we have a local solution $u = e^{it\Delta} f_0^\omega + v$ to (NLS) with initial data $u|_{t=0} = f_0^\omega$, where v satisfies (NLS_f) in $H^\sigma(\mathbb{R}^3)$ with zero initial condition, up to a maximal lifetime T^* . In addition, v extends globally in time scatters in $H^\sigma(\mathbb{R}^3)$ as soon as its $H^\sigma(\mathbb{R}^3)$ norm is uniformly bounded up to the time T^* . To obtain such an a priori uniform bound on v , we perform a double bootstrap argument with a modified interaction Morawetz and the modified energy $\mathcal{E}(v)$. Recall that we obtained in this section a priori estimates on the time increments of these two quantities, under the extra assumption that f_0 is radially-symmetric, and that $\omega \in \tilde{\Omega}_\alpha$, yielding a control the $L_t^2(\mathbb{R}; L_x^\infty(\mathbb{R}^3))$ norm of $\langle \nabla \rangle I e^{it\Delta} f_0^\omega$.

2.5.1 Double bootstrap argument

Recall that we fixed $\alpha > 0$, $N_0(\alpha)$ large enough and $\tilde{\Omega}_\alpha$ with $\mathbb{P}(\Omega_\alpha \setminus \tilde{\Omega}_\alpha) \leq \alpha$, such that for every $\omega \in \Omega_\alpha$ and $N \geq N_0(\alpha)$, there holds (2.2.27). Furthermore, we can assume from the large deviation estimate on the L_x^p norm of the randomized initial data (see [BOP15b], Lemma 2.3), and from the fact that the I -operator commutes with the randomization and that it is bounded on L^2 , that for all $\omega \in \tilde{\Omega}_\alpha$ and all N ,

$$\|I f_0^\omega\|_{L_x^4} \leq C_\alpha. \quad (2.5.1)$$

Then, we write C the constant that appears in the modified Morawetz inequality (2.4.18), and we define

$$\mathcal{M}_0 := \|u\|_{L_t^\infty L_x^2([0, T^*])}, \quad M = 4C\mathcal{M}_0^3. \quad (2.5.2)$$

It follows from the almost conservation of the mass from Lemma 2.4.1 that the bound on \mathcal{M}_0 does only depend on $\|f_0\|_{L_x^2}$. Afterwards, we consider the set

$$\Theta_N = \left\{ T \in [0, T^*) \mid \sup_{t \in [0, T]} \mathcal{E}(v(t)) \leq N^{2(1-\delta)}, \|Iv\|_{L_{t,x}^4[0, T]}^4 \leq MN^{1-\sigma} \right\}.$$

We claim that for every $\frac{6}{7} < \sigma < 2s \leq 1$, there exists $N'_0(\varepsilon, \|f_0\|_{L_x^2(\mathbb{R}^3)}, s, \sigma)$ to be defined later such that for all $N \geq \max(N_0, N'_0)$, we have

$$\Theta_N = [0, T^*). \quad (2.5.3)$$

Before diving into the proof, we note that Θ_N is nonempty. Indeed, it follows from (2.5.1) and from $v(0) = 0$ that $\mathcal{E}(v)(0) = \frac{1}{4}\|If_0^\omega\|_{L_x^4}^4 \leq C_\alpha \ll N^{2(1-\sigma)}$, and hence $T = 0 \in \Theta_N$. Moreover, we see from a continuity argument that θ_N is also closed. Hence, it remains to prove that Θ_N is an open interval. Let us fix $T \in \Theta_N$, with $T < T^*$. By the local well-posedness theory for (NLS_f) presented in Proposition 4.2.2, there exists a small $T_2 > T$ such that

$$\|Iv\|_{L_{t,x}^4[0,T_2]}^4 \leq 2MN^{1-\sigma}. \quad (2.5.4)$$

The double bootstrap argument is the following. First, by using the a priori estimate (2.5.4), by tuning the parameter and by summing over the spacetime slabs where

$$\|Iv\|_{L_{t,x}^4[T_1,T]}^4 \leq \varepsilon_N,$$

we observe from the local-in-time energy increment of Proposition 2.4.5 that if a time T_1 is such that $\mathcal{E}(v(T_1)) \leq N^{2(1-\sigma)}$, where $\varepsilon_N \sim N^{-2(1-\sigma)}$ is as in Lemma 2.4.4, then the energy remains less than $N^{2(1-\sigma)}$ as long as we control the $L_{t,x}^4$ norm of Iv . Next, we deduce from the modified interaction Morawetz (2.4.18) that we eventually have (2.5.4) with $MN^{1-\sigma}$ on the right-hand side, instead of $2MN^{1-\sigma}$.

Proposition 2.5.1 (Double bootstrap argument). *Let $\frac{6}{7} < \sigma < 2s$. There exists $\tilde{N}_0(\varepsilon, \sigma, s, \|f_0\|_{L^2})$ such that for all $N \geq \max(N_0, \tilde{N}_0)$, if T_1 is in Θ_N and $T_1 < T_2 < T^*$ is such that*

$$\|Iv\|_{L_{t,x}^4(0,T_2)}^4 \leq 2MN^{1-\sigma},$$

then T_2 is also in Θ_N .

Proof. Given a fixed $N \geq N_0$, we consider $\varepsilon_N = N^{-2(1-\sigma)-\delta}$ as in Lemma 2.4.4, where $\delta > 0$ is arbitrarily small. As a consequence of the assumption $\|Iv\|_{L_{t,x}^4[0,T_2]}^4 \leq 2MN^{1-\sigma}$, there are at most $L = 2MN^{1-\sigma}\varepsilon_N^{-1}$ spacetime-slabs on which the smallness condition $\|Iv\|_{L_{t,x}^4}^4 \leq \varepsilon_N$ is satisfied. We decompose therefore $[0, T_2)$ into L intervals $J_k = [t_k, t_{k+1})$ such that for $k \in \{0, \dots, L-1\}$

$$\|Iv\|_{L_{t,x}^4(J_k)}^4 \leq \varepsilon_N,$$

with

$$0 = t_0 < t_1 < \dots < t_L = T_2, \quad \text{and } L = \frac{2MN^{1-\sigma}}{\varepsilon_N} = 2MN^{3(1-\sigma)+\delta}.$$

In order to apply Lemma 2.4.4 on each spacetime slab J_k , we need the $L_t^4 L_x^3$ norm of f on J_k to be less than η , which is an universal constant. This is possible up to some extra decomposition of the intervals J_k into a finite number of sub-intervals, that only depends on $\|f_0\|_{H^s}$ and on η . Hence, the total number of spacetime-slabs on which we perform the analysis is bounded by

$$L \leq C(f_0, \eta) \frac{2MN^{1-\sigma}}{\varepsilon_N} \leq C(f_0, \eta) 2MN^{3(1-\sigma)+\delta}.$$

Now, by iteration on k , we deduce from Proposition 2.4.5 that

$$\sup_{t \in [0, t_k]} \mathcal{E}(v(t)) \leq N^{2(1-\sigma)}. \quad (2.5.5)$$

First, we note from the definition of Θ_N that (2.5.5) is already known when $t_k \leq T_1$, since we assumed that $T_1 \in \Theta_N$. Afterwards, we consider k such that (2.5.5) holds for t_k . We prove that it still holds for any $t \in J_k$. To do so, we fix $t \in J_k$ and we use sub-additivity :

$$\mathcal{E}(v(t)) \leq \mathcal{E}(v(0)) + \int_0^{t_{k+1}} \left| \frac{d}{dt} \mathcal{E}(Iv(\tau)) \right| d\tau = \mathcal{E}(v(0)) + \sum_{0 \leq i \leq k-1} \int_{J_i} \left| \frac{d}{dt} \mathcal{E}(v(\tau)) \right| d\tau.$$

Initially, we know from (2.5.1) that $\mathcal{E}(v(0)) \leq C_\alpha$. Moreover, we have by construction that $\mathcal{E}(Iv(t_i)) \leq N^{2(1-\sigma)}$, and $\|Iv\|_{L_{t,x}^4(J_i)}^4 \leq \varepsilon_N$ for each $i \in \{0, \dots, k\}$. Hence, we can apply the modified energy increment (2.4.11) on each interval J_i , and get

$$\mathcal{E}(v(t)) \leq C_\alpha + C_0 \sum_{0 \leq i \leq k-1} (N^{6(1-\sigma)-1} + N^{1-\sigma} F_2(J_i)^2 + N^{1-\sigma} F_2(J_i) \|Iv\|_{L_{t,x}^4(J_i)}^2),$$

where $C_0 = C_0(f_0, s, \sigma)$ is a universal constant that arises in (2.4.11). Next, we brutally multiply the first term by the number of spacetime-slabs¹⁵, and we use sub-additivity for the second term and the third term. When summing over the spacetime slabs, we also apply Cauchy-Schwarz on the third term, and we obtain

$$\begin{aligned} \mathcal{E}(v(t)) &\lesssim C_\alpha + LN^{6(1-\sigma)-1} \\ &\quad + N^{1-\sigma} \left\{ \sum_{0 \leq i \leq k-1} F_2(J_i)^2 + \left(\sum_{0 \leq i \leq k-1} F_2(J_i)^2 \right)^{\frac{1}{2}} \left(\sum_{0 \leq i \leq k-1} \|Iv\|_{L_{t,x}^4(J_i)}^4 \right)^{\frac{1}{2}} \right\}. \end{aligned}$$

We conclude by using the upper bound on L and the estimate (2.2.27) on F_2 that

$$\begin{aligned} \mathcal{E}(v(t)) &\leq C_\alpha + C_0 (LN^{6(1-\sigma)-1} + N^{1-\sigma} (F_2^\omega(\mathbb{R})^2 + F_2^\omega(\mathbb{R})(2MN^{1-\sigma})^{\frac{1}{2}})) \\ &\leq C_\alpha + C_0 (2MN^{9(1-\sigma)-1+\delta} + N^{1-\sigma} (N^{-\gamma} N^{1-\sigma} + N^0 N^{\frac{1-\sigma}{2}} N^{\frac{1-\sigma}{2}})) \\ &\leq N^{2(1-\sigma)} (C_\alpha N^{-2(1-\sigma)} + 2C_0 MN^{7(1-\sigma)-1+\delta} + N^{-2\gamma} + N^{-\gamma}) \\ &\leq N^{2(1-\sigma)}, \end{aligned}$$

provided that $\frac{6}{7} < \sigma$, and $N \geq N'_0 = N'_0(C_0, \gamma, M)$. This finishes the first part of the proof of Proposition 2.5.1. To show that $T_2 \in \Theta_N$, it remains to prove that

$$\iint_{[0, T_2] \times \mathbb{R}^3} |Iv|^4 dt dx \leq MN^{1-\sigma}.$$

For this purpose, we perform the same analysis as in the first part of the proof to handle the remainders that contain the commutator H . Namely, we decompose $[0, T_2]$ into L intervals $\{J_k\}_{0 \leq k \leq L-1}$ such that (2.4.8) and (2.4.9) hold on each J_k . Since we already know from Proposition (2.5.1) that

$$\sup_{t \leq T_2} \mathcal{E}(v(t)) \leq N^{2(1-\sigma)},$$

¹⁵ If we prove a long time Strichartz estimate as in [Dod19b], we might be able to use also some sub-additivity to estimate the first term, and to loosen therefore our restriction on $s > \frac{3}{7}$.

we can deduce from Lemma 2.4.4 that for each k

$$Z_{N,J_k}(v) \leq CN^{1-\sigma}.$$

Therefore, it follows from the modified interaction Morawetz that (2.4.18)

$$\begin{aligned} \iint_{[0,T_2] \times \mathbb{R}^3} |Iu|^4 dt dx &\leq C \left(\|Iv\|_{L_t^\infty([0,T_2]; \dot{H}^{\frac{1}{2}})}^2 + \|If\|_{L_t^\infty([0,T_2]; \dot{H}^{\frac{1}{2}})}^2 \right) \|Iu\|_{L_t^\infty([0,T_2]; L_x^2)}^2 \\ &\quad + CN^{-1+\delta} L \|Iu\|_{L_t^\infty([0,T_2]; L_x^2)} \|Iu\|_{L_t^\infty([0,T_2]; \dot{H}^{\frac{1}{2}})}^2 \left(N^{2(1-\sigma)} + C_\alpha^2 N^{1-\sigma} \right) (N^{1-\sigma} + C_\alpha). \end{aligned}$$

As a consequence of the global energy estimate on $[0, T_2]$ shown in the first part of the proof, we deduce from interpolation and from the coercivity of $\mathcal{E}(v)$ that

$$\|Iv\|_{L_t^\infty([0,T_2]; \dot{H}^{\frac{1}{2}})}^2 \leq \|Iv\|_{L_t^\infty([0,T_2]; \dot{H}^1)} \|Iv\|_{L_t^\infty([0,T_2]; L_x^2)} \leq \sqrt{2} \mathcal{E}(v)^{\frac{1}{2}} \mathcal{M}_0 \leq \sqrt{2} \mathcal{M}_0 N^{1-\sigma}. \quad (2.5.6)$$

Moreover,

$$\|If\|_{L^\infty \dot{H}^{\frac{1}{2}}}^2 \leq CN^{1-2s} \|f_0\|_{H^s}^2 \leq C(f_0) N^{0-} N^{1-\sigma}, \quad (2.5.7)$$

for some irrelevant constant $C(f_0) = C(\|f_0\|_{H^s}, s, \sigma)$. Consequently, the almost conservation of the mas and estimates (2.5.6), (2.5.7) yield

$$\begin{aligned} C \left(\|Iv\|_{L_t^\infty([0,T_2]; \dot{H}^{\frac{1}{2}})}^2 + \|If\|_{L_t^\infty([0,T_2]; \dot{H}^{\frac{1}{2}})}^2 \right) \|Iu\|_{L_t^\infty([0,T_2]; L_x^2)}^2 \\ \leq C \mathcal{M}_0^3 (\sqrt{2} N^{1-\sigma} + C(f_0)^{1-\sigma-\delta}) \leq 2C \mathcal{M}_0^3 N^{1-\sigma} \leq \frac{M}{2} \mathcal{M}_0^3 N^{1-\sigma}, \end{aligned} \quad (2.5.8)$$

provided that $\frac{6}{7} < \sigma$, and that δ is taken sufficiently small. We conclude that

$$\iint_{[0,T_2] \times \mathbb{R}^3} |Iu|^4 dt dx \leq \frac{M}{2} N^{1-\sigma} + N^{1-\sigma} N^{0-} (C_0 + C'_0).$$

Moreover, using the bound ¹⁶ on F^ω written in (2.1.11),

$$\begin{aligned} \iint_{[0,T_2] \times \mathbb{R}^3} |Iu|^4 dt dx &\lesssim \iint_{[0,T_2] \times \mathbb{R}^3} |If|^4 dt dx + \iint_{[0,T_2] \times \mathbb{R}^3} |Iu|^4 dt dx \\ &\leq 4F_\infty^\omega([0, T_2])^4 + \frac{M}{2} N^{1-\sigma} + 4N^{0-} (C_0 + C'_0) N^{1-\sigma} \leq MN^{1-\sigma}. \end{aligned}$$

In the last estimate, we took $N \geq N_0'' = N_0''(f_0)$ large enough. This concludes the proof of Proposition 2.5.1 with $\tilde{N}_0 = \max(N_0', N_0'')$. \square

¹⁶We actually control F^ω defined with f instead of If . But we can easily replace f by If in the proof, if we replace f_0 by If_0 using the fact I is a Fourier multiplier and commutes therefore with the randomization. Moreover, the operator norm of I acting on L^2 is less than 1. Hence, we do not lose any power of N .

2.5.2 Proof of Theorem 2.1.4

We proved that for every $\alpha > 0$ and $\omega \in \Omega_\alpha$ as in Lemma 2.2.27, the Cauchy problem (NLS_f) admits a unique local solution v , with a maximal lifespan $[0, T^*)$. Moreover, we find a set $\tilde{\Omega}_\alpha$ such that $\mathbb{P}(\Omega_\alpha \setminus \tilde{\Omega}_\alpha) \leq \alpha$, where we proved in Proposition 2.5.1 by a double bootstrap argument on $\mathcal{E}(v)$ and a modified Morawetz estimate that if $\omega \in \tilde{\Omega}_\alpha$, there exists $N_0 = N_0(\varepsilon, \|f_0\|_{H^s}, s, \sigma)$ such that for all $N \geq N_0$, we have $\Theta_N = [0, T^*)$. In particular,

$$\sup_{0 \leq t \leq T^*} \mathcal{E}(v(t)) \leq N^{2(1-\sigma)}.$$

Hence, we deduce from the above estimate on the coercive modified energy, from the operator bound (2.1.4) for the I -operator and from the almost conservation of the mass, that there exists N large enough and a constant C_N such that

$$\sup_{0 \leq t \leq T^*} \|v\|_{H_x^\sigma} \leq \|v\|_{L^\infty([0, T^*]; L^2)} + CN^{1-\sigma} \|Iv\|_{L^\infty([0, T^*]; \dot{H}^1)} \leq CN^{3(1-\sigma)} =: C_N.$$

In what follow we drop the dependence on N . Since we have a global a priori estimate on the H^σ norm of v , we can apply the stability theory presented in Section 2.3.2, and deduce from Proposition 2.3.8 that $T^* = +\infty$, and that v scatters at infinity. Finally, we take

$$\Sigma = \bigcap_{\alpha > 0} \tilde{\Omega}_\alpha.$$

Since $\mathbb{P}(\Omega \setminus \tilde{\Omega}_\alpha) \leq 2\alpha$ for every $\alpha > 0$, we have that $\mathbb{P}(\Sigma) = 1$. By construction, a local solution to (NLS) initiated from an initial data $u(0) = f_0^\omega \in H^s$ with $\omega \in \Sigma$ extends globally in time and scatters.

Chapter 3

Asymptotic stability of small ground states for NLS under random perturbations

From the publication [Cam22].

Résumé. Nous étudions l'équation de Schrödinger cubique dans l'espace euclidien, perturbée par un potentiel de courte portée V . La présence d'une valeur propre négative pour $-\Delta + V$ génère une famille à un paramètre d'états fondamentaux non linéaires, qui sont localisés en espace et qui donnent lieu à de petites solutions périodiques, asymptotiquement stables dans l'espace d'énergie. Nous étudions la persistance de ces états cohérents sous l'effet de perturbations irrégulières et aléatoires. Plus précisément, nous construisons un ensemble de mesure pleine constitué de petites solutions sur-critiques, qui ont en outre une énergie infinie, mais qui laissent la famille d'états fondamentaux asymptotiquement stable. La principale difficulté réside dans l'analyse des interactions de termes localisés en espace avec des termes dispersifs qui ont une faible régularité.

Abstract. We consider the cubic Schrödinger equation on the Euclidean space perturbed by a short-range potential V . The presence of a negative simple eigenvalue for $-\Delta + V$ gives rise to a curve of small and localized nonlinear ground states that yield some time-periodic solutions known to be asymptotically stable in the energy space. We study the persistence of these coherent states under rough perturbations. We shall construct a large measure set of small scaling-supercritical solutions below the energy space that display some asymptotic stability behavior. The main difficulty is the need to handle the interactions of localized and dispersive terms in the modulation equations. To do so, we use a *critical-weighted strategy* to combine probabilistic nonlinear estimates in critical spaces based on U^p, V^q (controlling higher order terms) with some local energy decay estimates (controlling lower order terms). We also revisit in the perturbed setting the analysis of [BOP15a] on the probabilistic global well-posedness and scattering for small supercritical initial data. We use a distorted Fourier transform and semiclassical functional calculus to generalize probabilistic and bilinear Strichartz estimates.

Contents

3.1	Introduction	150
3.1.1	Main result	151
3.1.2	Background	152
3.1.3	Organization of the paper	155
3.2	Spectral theory	159
3.2.1	Wave operators and the intertwining property	159
3.2.2	Distorted Fourier multipliers	160
3.2.3	Application to the distorted Littlewood-Paley theory	164
3.3	Bilinear and Strichartz estimates in the perturbed case	166
3.3.1	Wiener randomization with respect to the distorted Fourier transform	166
3.3.2	Bilinear estimate	168
3.4	Almost-sure scattering	176
3.4.1	Critical spaces	177
3.4.2	Probabilistic nonlinear estimates	181
3.4.3	Proof of Theorem 3.1.5	183
3.4.4	Local smoothing and the critical weighted strategy	184
3.5	Almost-sure asymptotic stability	187
3.5.1	Local existence	187
3.5.2	Nonlinear ground states	190
3.5.3	Global a priori estimates	191
3.5.4	Proof of Theorem 3.1.1	196
3.6	Appendix	197

3.1 Introduction

In this paper we consider the cubic Schrödinger equation perturbed by a short-range potential in the Euclidean space of dimension $d \geq 3$:

$$\begin{cases} i\partial_t \psi + \Delta \psi = \mu |\psi|^2 \psi + V \psi, \\ \psi|_{t=0} = \psi_0. \end{cases} \quad (\text{NLS})$$

The constant μ , whose sign dictates whether the nonlinearity is focusing or defocusing, will play no role since we look at small initial data. Hence, we shall fix $\mu = 1$. The short-range potential V , whose properties are detailed below, is assumed to be in the Schwartz class $\mathcal{S}(\mathbb{R}^d)$. Our main result concerns the case when $d = 3$ and $\sigma(H) = \{e_0\} \cup \sigma_c(H)$ with no resonance at zero, and where $e_0 < 0$ is a simple negative eigenvalue with positive and normalized eigenfunction ϕ_0 . Then, the 1-dimensional eigenspace spanned by ϕ_0 bifurcates around zero to a family of small and localized nonlinear ground states. These ground states satisfy the elliptic equation

$$(\Delta - V + |Q|^2)Q = EQ, \quad (3.1.1)$$

and are written under the form

$$Q(z) = z\phi_0 + q(z), \quad E(z) = e_0 + e(z), \quad (3.1.2)$$

where z is the complex modulation parameter. They give rise to periodic solutions to (NLS) of the form $u(t, x) = e^{-itE} Q(x)$. Soffer and Weinstein [SW90], followed by Pillet and Wayne [PW97] and Gustafson & al. [GNT04] addressed the problem of asymptotic stability of these small ground states in the energy space. More precisely, any small local solution ψ to (NLS) in $\mathcal{H}^1(\mathbb{R}^3)$ is global and can be decomposed into

$$\psi(t) = Q(z(t)) + \eta(t, z(t)), \quad (3.1.3)$$

where the radiation term η satisfies a time dependent orthogonality condition (see (3.1.7)). In large time, asymptotic stability holds in the sense that $z(t)$ has a limit as t goes to infinity, and $\eta(t)$ scatters in $\mathcal{H}^1(\mathbb{R}^3)$. This result is a particular instance of the so-called *soliton resolution conjecture* (see [Tao08] for a general presentation). Our aim is to prove that the asymptotic stability property of these coherent states still holds at very low regularity, in a supercritical regime where we have a local probabilistic flow.

Before we present our main result, we recall that the critical exponent s_c for the homogeneous (NLS) equation (with $V=0$) is the regularity exponent for which the homogeneous Sobolev norm is invariant under the scaling

$$u_\lambda(t, x) = \lambda^{-1} u(\lambda^{-2}t, \lambda^{-1}x), \quad u_{0,\lambda}(x) = \lambda^{-1} u_0(\lambda^{-1}x).$$

For the cubic Schrödinger equation in \mathbb{R}^d , we have $s_c = \frac{d-2}{2}$. In light of the conservation laws of mass and energy

$$\mathcal{M}(\psi)(t) = \int_{\mathbb{R}^d} |\psi(t, x)|^2 dx, \quad \mathcal{E}(\psi)(t) = \int_{\mathbb{R}^d} \frac{1}{2} |\nabla \psi(t, x)|^2 + \frac{\mu}{4} |\psi(t, x)|^4 + V |\psi(t, x)|^2 dx,$$

we say that the problem is mass-critical when $s_c = 0$ and energy-critical when $s_c = 1$. There exists a vast literature on the Cauchy theory for (NLS) and we refer to the books [BKK07; Caz03; Dod19a; Tao06] or to the survey [GV79a; GV79b] and the references therein for a general presentation. Note that the equation is energy-subcritical for $d = 3$, and we have a local flow for $s \in [1/2, 1]$, where $s_c = 1/2$. On the other hand, in the scaling-*supercritical* regime where $s < s_c$, the local Cauchy problem for (NLS) is known to be ill-posed (see [CCT03c]). Nevertheless, the *probabilistic Cauchy theory* initiated by Bourgain [Bou96] and developed by many authors since then provides some large measure sets made of scaling-supercritical initial data u_0^ω in $\mathcal{H}^s(\mathbb{R}^d)$ for $s < s_c$ that give rise a local solutions to (NLS) of the form $\psi = u^\omega + v$. Here, $u^\omega = e^{-itH} u_0^\omega$ is the propagation of u_0^ω under the linear flow and v is solution to the cubic Schrödinger equation with a random forcing term, but with $v(0) = 0$.

3.1.1 Main result

We consider (NLS) in dimension $d = 3$ with small randomized initial data that lie in $\mathcal{H}^s(\mathbb{R}^3) \cap \text{Ran } P_c(H)$ for some $s \in (1/4, 1/2]$, where $P_c(H)$ is the projection onto the continuous spectral subspace for H , and we assume that $\sigma(H) = \{e_0\} \cup \sigma_c(H)$ with no resonance at zero, where $e_0 < 0$ is a simple negative eigenvalue. We generalize the result of [BOP15a] in the presence of such of potential by first constructing a local probabilistic flow, and extending to a global flow. Then, we prove that at each time of the evolution, the solution decouples into a sum of a ground state and a radiation term that scatters at infinity.

More precisely, we fix $s \in (1/4, 1/2]$, a function u_0 in \mathcal{H}^s and a small parameter ε . Then, we perform the *Wiener randomization procedure* on u_0 as detailed in section 3.3.1, and we get a large measure set $\tilde{\Omega}_\varepsilon$ made of rough and small initial data $u_0^\omega \in \mathcal{H}^s(\mathbb{R}^3)$ with improved Strichartz estimates. These initial data give rise to global solutions to (NLS) under the form (3.1.3) that display an asymptotic stability dynamic.

Theorem 3.1.1 (Probabilistic asymptotic stability of small ground states). *Assume that H has no resonance at zero and that $\sigma(H) = \{e_0\} \cup \sigma_c(H)$ with $e_0 < 0$ a simple eigenvalue. There exist a set $\tilde{\Omega}_\varepsilon$ and δ_0 such that for all ψ_0 with $\|\psi_0\|_{\mathcal{H}^{1/2}} < \delta_0$ and all $\omega \in \tilde{\Omega}_\varepsilon$, the initial value problem*

$$\begin{cases} i\partial_t \psi + \Delta \psi = |\psi|^2 \psi + V\psi, & (t, x) \in \mathbb{R} \times \mathbb{R}^3, \\ \psi|_{t=0} = \varepsilon u_0^\omega + \psi_0, \end{cases} \quad (3.1.4)$$

admits a unique global-in-time solution ψ of the form

$$\psi(t) = \varepsilon e^{it(\Delta-V)} u_0^\omega + v(t), \text{ where } v \in C(\mathbb{R}, \mathcal{H}^{1/2}(\mathbb{R}^3)).$$

Moreover, the solution uniquely resolves into $\psi(t) = Q(z(t)) + \eta(t)$, and there exist $z_+ \in \mathbb{C}$ and a final state $\eta_+ \in \mathcal{H}^{1/2}(\mathbb{R}^3) \cap \text{Ran}(P_c)$ such that

$$\lim_{t \rightarrow +\infty} z(t) \exp(i \int_0^t E(z) d\tau) = z_+, \quad \lim_{t \rightarrow +\infty} \|\eta - e^{-itH}(\varepsilon u_0^\omega + \eta_+)\|_{\mathcal{H}^{1/2}(\mathbb{R}^3)} = 0. \quad (3.1.5)$$

Remark 3.1.2. The measure of the set $\tilde{\Omega}_\varepsilon$ of initial with improved Strichartz estimates is all the more large since $\varepsilon \|u_0\|_{\mathcal{H}^s}$ is small. There exist $C, c > 0$ such that for all u_0 and $\varepsilon > 0$,

$$\mathbf{P}(\Omega \setminus \tilde{\Omega}_\varepsilon) \leq C \exp(-c\varepsilon^{-2} \|u_0\|_{\mathcal{H}^s(\mathbb{R}^3)}^{-2}).$$

Remark 3.1.3. We have conditional uniqueness for v in the critical space defined in (3.4.13), and embedded into $L^\infty(\mathbb{R}, \mathcal{H}^{1/2})$. Hence, v gains some regularity and lies in a space where there exists a deterministic Cauchy theory.

By taking $u_0 = 0$ in the statement of Theorem 3.1.1 and using persistence of regularity, we may extend the deterministic result of [GNT04] to the intercritical regime, where $1/2 \leq s \leq 1$.

Corollary 3.1.4 (Deterministic asymptotic stability). *For $1/2 \leq s \leq 1$, there exists $\delta_0 > 0$ such that for all $\|\psi_0\|_{\mathcal{H}^s} \leq \delta_0$, the Cauchy problem (NLS) with initial data ψ_0 has a unique global solution $\psi \in C(\mathbb{R}; \mathcal{H}^s)$ that resolves into $\psi(t) = Q(z(t)) + \eta(t)$, and asymptotic stability (3.1.5) holds in $\mathcal{H}^{1/2}$.*

3.1.2 Background

Asymptotic stability for small solitons in the energy space Previous results on the asymptotic stability of small ground states hold for initial data ψ_0 in the energy space $\mathcal{H}^1(\mathbb{R}^3)$, where (NLS) is known to be well-posed. Soffer and Weinstein proved in [SW90] that the equation displays some multichannel scattering for small and localized data in the energy space. More precisely, any initial data small in $\mathcal{H}^1(\mathbb{R}^3) \cap L_x^1(\mathbb{R}^3)$ gives rise to a global solution that resolves into a fixed ground state

and a radiation term whose $L_x^6(\mathbb{R}^3)$ -norm decays to zero, as well as some of its $L_x^2(\mathbb{R}^3)$ -weighted norm (see Theorem 4.1 in [SW90]). By the use of the *center stable manifold method*, Pillet and Wayne then extended this result in [PW97] to the case where the initial data are localized in some $L_x^2(\mathbb{R}^3)$ -weighted spaces rather than in $L_x^1(\mathbb{R}^3)$. While these works impose some fixed orthogonality conditions to the modulation parameters, Gustafson, Nakanishi and Tsai introduced a time-dependent orthogonality condition that leads to an asymptotic stability result in $\mathcal{H}^1(\mathbb{R}^3)$ (Theorem 1.7 in [GNT04]) without any assumption on the decay of the initial data, and the radiation term scatters in $\mathcal{H}^1(\mathbb{R}^3)$. A natural question that arises is whether the asymptotic stability holds true below the energy space. Indeed, we still have a local flow for $1/2 \leq s \leq 1$, and a local probabilistic flow for $1/4 < s < 1/2$. We note that our proof does not use any decay assumption on the initial data. Besides, the randomized initial are not likely to decay for a general u_0 (see the discussion in paragraph 3.3.1).

Probabilistic well-posedness theory Even though the Cauchy problem (NLS) is in general ill-posed for scaling-supercritical initial data below $\mathcal{H}^{1/2}(\mathbb{R}^3)$, the probabilistic method can provide some large measure sets of initial data that give rise to global-in-time solutions. Many works were done in this direction for different dispersive PDE's after the pioneer work of Jean Bourgain [Bou96] for the NLS equation on \mathbb{T}^2 , followed by the work of Burq and Tzvetkov on the nonlinear wave equation (NLW) on a 3D compact Riemannian manifold without a boundary [BT08a; BT08b]. The idea is to find a randomization procedure which improves some Strichartz estimates on the free evolution of the randomized data, but does not improve their regularity.

To extend local well-posedness below the scaling-critical Sobolev space, the strategy consists in decomposing the solution into $\psi = u^\omega + v$ where $u^\omega := e^{-itH} u_0^\omega$ is the linear evolution of the randomized data. The remaining term v is smoother and satisfies (NLS) with a stochastic forcing term:

$$\begin{cases} i\partial_t \psi - H\psi = \mathcal{N}(\psi), \\ \psi|_{t=0} = \psi_0 + u_0^\omega. \end{cases} \iff \begin{cases} i\partial_t v - Hv = \mathcal{N}(v + u^\omega), \\ v|_{t=0} = \psi_0. \end{cases} \quad (3.1.6)$$

Besides, we need to solve equation (3.1.6) for v in a well-chosen subspace of $C(I, \mathcal{H}^{sc})$ by a contraction mapping argument. Thanks to its random structure, the linear evolution of the randomized data u^ω displays some enhanced integrability properties that make it possible to gain regularity on the stochastic Duhamel term. After that, we globalize the local solutions by using the *Bourgain invariant measure argument* in the case of \mathbb{T}^d , or by using a priori estimates on some critical norm of v .

The general randomization procedure consists in finding a well-chosen decomposition of the initial data and in decoupling the terms of this decomposition by multiplying each of them by some independent random variables $\{g_n(\omega)\}_n$ centered around zero. Then, taking averages cancels interference and improves therefore the integrability of the data. There exists many versions of the randomization procedure, and we refer the interested reader to the survey [BOP19b] and the references therein for a detailed presentation. One has to distinguish between two cases. In the confining case where the physical space is a compact manifold, or when the equation contains a confining potential, we consider the decomposition $u_0^\omega = \sum_n g_n(\omega) u_n e_n(x)$ where $\{e_n\}$ is an orthonormal basis made of eigenfunctions of the Schrödinger operator. In the Euclidean case, the Schrödinger operator does not provide such a natural decomposition, and we use instead unit scale frequency decomposition on Wiener cubes (see [BOP15a; BOP15b; DLM19; DLM20]), or microlocal decomposition (see [Bri18; BK21; Mur19]). In some specific cases, we can also use a compactifying

transformation such as the Lens transform for NLS (see [BT20]) or the Penrose transform for NLW (see [De 13]) to apply in the Euclidean setting variants of the Bourgain invariant measure argument. Nevertheless, it must be emphasized that the probabilistic method used in the aforementioned works concerns perturbations of the zero solution and ends up with asymptotic results like global well-posedness and scattering for small data or in the defocusing case.

Still, Kenig and Mendelson recently addressed in [KM19] the problem of asymptotic stability of large solitons for the quintic focusing wave equation with randomized radial initial data. They introduced a randomization procedure based on the distorted Fourier transform adapted to the linearized operator around a soliton, and proved some intricate kernel estimates due to the presence of a resonance at zero for this operator. Then, Bringmann showed the stability of the ODE blow-up for the radial energy critical NLW in 4D under random perturbations below the energy space [Bri20c]. However, for nonlinear focusing Schrödinger equations the asymptotic stability around solitons is more subtle. Indeed, the solitons for (NLS) with $V = 0$ are stable when the nonlinearity is small and the critical exponent is negative. Hence, we cannot run the probabilistic strategy since the smoother term v , which is expected to lie in the critical space $C(\mathbb{R}; \mathcal{H}^{s_c}(\mathbb{R}^d))$, cannot serve as a substitute for the conservation laws. In addition, the linearized operator around the soliton is not self-adjoint. We refer to [CM21; Tao08] and the references therein for a general insight on the results and techniques about stability of solitons for NLS.

Schrödinger equation with a short-range potential The study of dispersive PDE's with a potential is a very general problem. For instance, a potential can arise in the modulation equations obtained to address stability problems around soliton solutions by the analytic method (see [Tao08]). There exists a vast literature about Schrödinger equations perturbed by a localized potential (see for instance [GHW14; GPR18; PS20]). These works rely on the use of a *distorted Fourier transform* \mathcal{F}_V , which is the analogue adapted to H of the Fourier transform. When it exists, the distorted Fourier transform defines a partial isometric map onto $L^2_{ac}(\mathbb{R}^d)$ that conjugates H with the operator of multiplication by $|\xi|^2$. Additionally, the distorted Fourier transform is related with the wave operators W_{\pm} . We refer the interested reader to the seminal work of Agmon [Agm75], grounded on the previous works [AS71; Ike60]. In the probabilistic context, we shall follow the strategy used by Kenig and Mendelson for the quintic focusing NLW in [KM19] to provide a randomization procedure based on a *distorted frequency* decomposition. This randomization procedure commutes with the flow e^{-itH} and is therefore suited to the underlying linear dynamic of (NLS). In the present work we consider the Schrödinger equation perturbed by a potential in order to generate a curve of small nonlinear ground states, and our analysis is easier than in [KM19] since we can assume no resonance at zero. Let us now precise the assumptions made on V .

Assumptions on the potential: V is a real-valued potential in the Schwartz class $\mathcal{S}(\mathbb{R}^d; \mathbb{R})$. We write

$$H := -\Delta + V, \quad H_0 = -\Delta.$$

Note that V is *short-range* in the sense of Agmon: there exists $\varepsilon > 0$ such that the operator

$$u \in \mathcal{H}^2(\mathbb{R}^d) \mapsto (1 + |x|)^{1+\varepsilon} V u \in L^2(\mathbb{R}^d)$$

is a compact operator. Namely, H is H_0 -compact. Hence, H admits a unique self-adjoint realization on $L^2(\mathbb{R}^d)$ with domain $\mathcal{D}(H) = \mathcal{H}^2(\mathbb{R}^d)$ and has the same essential spectrum as H_0 , that is the half

line $[0, +\infty)$. In addition, Agmon proved in [Agm75] that if V is short-range then the spectrum of H is given by

$$\sigma(H) = \sigma_p(H) \cup \sigma_{ess}(H) = \sigma_p(H) \cup [0, +\infty),$$

where $\sigma_p(H)$ (the discrete spectrum of H) is a countable set made of eigenvalues with finite multiplicity. We have the spectral decomposition:

$$L^2(\mathbb{R}^d) = L_{ac}^2(\mathbb{R}^d) \oplus L_p^2(\mathbb{R}^d),$$

where $L_p^2(\mathbb{R}^d)$ is the space spanned by the associated eigenvectors and $L_{ac}^2(\mathbb{R}^d)$ is the absolute continuous spectral subspace for H . In what follows, we assume that V is *generic*, in the sense that

- (i) $\sigma_p(H) \subset (-\infty, 0)$,
- (ii) 0 is not a resonance for H .

We refer to [GHW14] and the references therein for precise discussions about optimal assumptions on V . In the present work, we assume extra regularity on V to get the global-in-time local smoothing estimates of Proposition 1.33 in [RT15]. We also chose to take V smooth in order to derive a bilinear estimate for perturbed linear Schrödinger evolutions based on *semiclassical functional calculus*, and that does not rely on an explicit structure theory for wave operators (see [Bec14; Hon13]). The assumption that V is in $\mathcal{S}(\mathbb{R}^d)$ is far from being optimal and is made for simplicity.

3.1.3 Organization of the paper

Spectral theory We present in section 3.2 the basic properties of the *distorted Fourier transform*, and its connections with wave operators and functional calculus. Under our assumptions on $-\Delta + V$, the distorted Fourier transform is a unitary operator onto the continuous spectral subspace $L_c^2(\mathbb{R}^d)$ that provides very useful frequency decompositions adapted to the perturbed framework. Indeed, the distorted Fourier multipliers involved in these decompositions commute with the perturbed linear flow e^{-itH} . In particular, they preserve the continuous spectral subspace $\text{Ran}(P_c)$. We state an analogue of the Fourier multiplier theorem for this transformation that follows from the L^p -boundedness of wave operators, and we also stress out that distorted Fourier multipliers by radial functions coincide with spectral multipliers defined by the functional calculus on self-adjoint operators. Then, we shall briefly revisit some nonlinear tools such as the Littlewood-Paley inequality in order to perform standard harmonic analysis techniques for nonlinear dispersive PDE's in the perturbed setting.

Probabilistic and Bilinear Strichartz estimates for the inhomogeneous Schrödinger evolution First, we generalize the refined global-in-time Strichartz estimates for randomized data from [BOP15a], Lemma 2.3. To do so, we introduce a *distorted Wiener decomposition* by using distorted Fourier multipliers localized on unit cubes (whereas standard Fourier multipliers are used in the flat case). As mentioned above, the reason for this is to come with a randomization that commutes with e^{-itH} and that preserves $\text{Ran} P_c(H)$. Proposition (3.3.2) yields a large measure set $\tilde{\Omega}$ of initial data in $\text{Ran} P_c(H)$ that display refined Strichartz estimates. In order to revisit the standard proof written in [BOP15a], we use the variant of a Bernstein's estimate for distorted Fourier multipliers (3.2.12) presented in section 3.2. Furthermore, we stress out that since u_0^ω is

in $\text{Ran}(P_c)$ we can apply the local smoothing property (3.2.7) for e^{-itH} . For all these reasons, randomization adapted with a distorted frequency decomposition turns out to be the best suited to the dynamic of (NLS). Nevertheless, the distorted Fourier transform does not change a product into a product of convolution. Therefore, the nonlinear analysis is more intricate in this setting, and we cannot generalize bilinear estimates so easily. Still, we prove an extended version of the bilinear Strichartz estimate from J. Bourgain to the inhomogeneous case with V in the Schwartz class. Our proof does not rely on a structure formula for the wave operator, and we use instead semiclassical functional analysis to intertwine Fourier and distorted Fourier multipliers. More precisely, we quantify the interactions between functions of the form $\varphi(N^{-1}H)$ and $\varphi(M^{-1}H_0)$ in order to intertwine localization with respect to H and H_0 . Then, we decompose the perturbed evolution e^{-itH} into a superposition of free evolution $e^{it\Delta}$ of flat-Fourier localized data, and we apply the original Bourgain's estimate for the free evolution $e^{it\Delta}$, as well as local energy decay.

Probabilistic global existence and scattering on the continuous spectral subspace The purpose of section 3.4 is to lay the foundations for the analysis conducted in section 3.5 about the stability of small nonlinear ground states under rough and randomized perturbations. First, we recall the definitions and key properties of critical spaces of functions from an interval I to $\mathcal{H}^{\frac{d-2}{2}}(\mathbb{R}^d)$, written $X^{\frac{d-2}{2}}(I)$. This space is built upon the space of functions of finite q -variation V^q and its predual, the atomic space U^p . Next, we generalize Theorem 1.2 in [BOP15a] to the perturbed framework in order to understand the dynamic for solutions to (NLS) projected on the continuous spectral subspace for H . To do so, we shall specify and develop a bit on the random nonlinear estimates derived in [BOP15a]. We obtain the following result.

Theorem 3.1.5. *Assume that H has no resonance nor eigenvalue at zero. Let $s_d = \frac{d-1}{d+1} \cdot s_c$ and $s \in (s_d, s_c]$. For all $\omega \in \tilde{\Omega}_\varepsilon$ and $\psi_0 \in \mathcal{H}^{\frac{d-2}{2}}(\mathbb{R}^d) \cap \text{Ran}(P_c)$ small enough, the Cauchy problem*

$$\begin{cases} i\partial_t \psi + (\Delta - V)\psi = P_c(|\psi|^2 \psi), \\ \psi_{t=0} = \varepsilon u_0^\omega + \psi_0, \end{cases} \quad (\text{NLS}_c)$$

admits a unique global-in-time solution ψ in the class

$$\varepsilon e^{-itH} u_0^\omega + X^{\frac{d-2}{2}}[0, \infty) \subseteq C([0, \infty), \mathcal{H}^s(\mathbb{R}^d) \cap \text{Ran}(P_c)).$$

In addition, there exists $v_+ \in \mathcal{H}^{\frac{d-2}{2}}(\mathbb{R}^d) \cap \text{Ran}(P_c)$ such that

$$\lim_{t \rightarrow +\infty} \|\psi - e^{-itH} \varepsilon u_0^\omega + v_+\|_{\mathcal{H}^{\frac{d-2}{2}}(\mathbb{R}^d)} = 0.$$

Remark 3.1.6. If the pure point spectrum of $-\Delta + V$ is empty then $P_c = \text{Id}$ and (NLS_c) is (NLS). By time reversibility of (NLS) the same results hold true for negative times, and conditional uniqueness holds in the following sense: let v_1, v_2 be two solutions in $X^{\frac{d-2}{2}}(0, \infty)$. If there exists $t \in [0, \infty)$ such that $v_1(t) = v_2(t)$ then $v_1 \equiv v_2$.

In section 3.5, we consider perturbed solutions of (NLS) around a ground state under the form $Q + \eta$. In particular, the radiation term η satisfies a nonlinear Schrödinger equation with a stochastic forcing term that contains localized and linear terms. The main difficulty is that the

higher order terms are handled in critical spaces, while the localized lower order ones can only be controlled in weighted Sobolev spaces by the use of some *local smoothing estimates*. Therefore, we shall present in Proposition 3.4.15 a *critical-weighted strategy* that gives a way to perform nonlinear analysis in critical spaces and in weighted spaces simultaneously. In dimension $d = 3$, our analysis shows that this technique also gives an alternative version of Theorem 3.1.5 where global existence and uniqueness for the nonlinear part of the solution hold in V^2 intersected with the weighted Sobolev space $L_t^2(\mathbb{R}; \mathcal{H}^{1, -1/2-}(\mathbb{R}^3))$. A similar approach for Korteweg-de Vries equation can be found in [KM12].

Outline of the proof of Theorem 3.1.1 Let us now present the framework and some notations used in [GNT04]. In what follows, we identify $\mathbb{C} \simeq \mathbb{R}^2$, with $z = z_1 + iz_2 = (z_1, z_2)$. For convenience, we denote the operator $D_z f(z) := (\partial_{z_1} f(z), \partial_{z_2} f(z))$ acting on \mathbb{C} seen as a real vector space of dimension 2 endowed with the real scalar product $\langle (z_1, z_2), (z'_1, z'_2) \rangle = z_1 z'_1 + z_2 z'_2$. Then, we construct a local probabilistic flow for (NLS), and we decompose the solution at each time under the form (3.1.3) with the following time-dependent orthogonality condition imposed for the radiation term $\eta(z)$:

$$\eta(z) \in \mathcal{H}_c(z) := \left\{ \eta \in L^2(\mathbb{R}^3) \mid \int_{\mathbb{R}^3} \langle i\eta, \partial_{z_1} Q(z) \rangle dx = \int_{\mathbb{R}^3} \langle i\eta, \partial_{z_2} Q(z) \rangle dx = 0 \right\}. \quad (3.1.7)$$

In particular, observe that $\mathcal{H}_c(0) = \text{Ran}(P_c) = L_c^2(\mathbb{R}^3)$. In broad outline, we ask for η to be orthogonal to the center manifold of ground states at each time, and this has for effect to cancel some linear terms in the modulation equations (3.5.8). Note that these orthogonality conditions are all the more natural since any small function ϕ in $L^2(\mathbb{R}^3)$ can actually be decomposed into

$$\phi = Q(z) + \eta, \quad (3.1.8)$$

where $z \in \mathbb{C}$ and $\eta \in \mathcal{H}_c(z)$ satisfies (3.1.7) (see Lemma 2.3 in [GNT04]). Furthermore, the decomposition (3.1.8) is explicit and $P_p(\eta)$, the discrete part of η , can be expressed as a function of $P_c(\eta)$ and z . Indeed, for each z small enough there exists a bijective operator $R(z) : \mathcal{H}_c(0) \rightarrow \mathcal{H}_c(z)$ written in Lemma 3.5.4 such that

$$\eta = R(z) P_c(\eta).$$

We introduce $\nu = P_c(\eta) - \varepsilon u^\omega$. Since u^ω is in $\text{Ran}(P_c)$, we have that $\nu \in \text{Ran } P_c$, and

$$P_p(\eta) = (R(z) - I)(\varepsilon u^\omega + \nu).$$

Therefore, the evolution reduces to a system with two degrees of freedom z and ν

$$\psi = Q(z) + \eta(z), \quad \eta(z) = R(z)(\varepsilon u^\omega + \nu(z)). \quad (3.1.9)$$

The aim is to obtain some global a priori estimates for $(\nu, m(z))$, solution to the coupled system of modulation equations (3.5.8), (3.5.10), where $m(z)$ denotes bijective gauge defined in (3.5.7). By injecting the ansatz (3.1.9) into the equation (NLS), we see that ν satisfies the perturbed Schrödinger equation with a stochastic forcing term (3.5.10). It has the form

$$i\partial_t \nu + (\Delta - V)\nu = F(\nu, Q(z), z, \varepsilon u^\omega),$$

and stress out that the forcing term contains some linear terms with respect to η and its complex conjugate $\bar{\eta}$. The main difficulty is to prove scattering with these linear terms. Indeed, they are

time dependent, and they are not self-adjoint perturbations of H . Hence, we can not absorb them in the left-hand side in order to get a perturbed operator as studied in our framework. We rather need to consider linear terms as source terms. Fortunately, these terms come with powers of Q or ϕ_0 which are small in \mathcal{H}^2 and localized. Hence, we shall use local energy decay to control them. On the other hand, we need the critical spaces based on U^2 and V^2 to control the term $\mathcal{N}(u^\omega + \nu)$. Consequently, we shall exploit the aforementioned *critical-weighted strategy* (see Proposition 3.4.15) to simultaneously cope with the localized and critical terms and to get the desired a priori estimate on ν .

The dynamic of $m(z)$ is governed by an ODE that arises as when differentiating the orthogonality conditions (3.1.7) with respect to time. As explained above, orthogonality conditions (3.1.7) cancel terms which are linear with respect to η . Hence, the obtained ODE contains terms which are at least quadratic in η . Since the local smoothing estimate controls some global-in-time and weighted quadratic norms of η , a global-in-time control of $\dot{m}(z)$ in $L_t^1(\mathbb{R})$ follows from the analysis of the ODE and yields the convergence result written in (3.1.5).

Eventually, the analysis leads to the global a priori estimates on ν and $m(z)$ stated in Proposition 3.5.9, from which we deduce global existence by the use of a continuity argument. The desired asymptotic dynamic also follows from these a priori estimates.

Notations Let $s, \sigma \in \mathbb{R}$. We denote the Sobolev space by $\mathcal{H}^s(\mathbb{R}^d)$, and we define the weighted Sobolev space by

$$L^{2,\sigma}(\mathbb{R}^d) = \{u \mid \langle x \rangle^\sigma u \in L^2(\mathbb{R}^d)\}, \quad \mathcal{H}^{1,\sigma}(\mathbb{R}^d) = \{u \mid \langle x \rangle^\sigma u, \langle x \rangle^\sigma \nabla u \in L^2(\mathbb{R}^d)\}.$$

The projection onto the continuous spectral subspace $L_c^2(\mathbb{R}^d)$ for H is written P_c , and $P_p = I - P_c = (\cdot \mid \phi_0)\phi_0$ is the projection onto the pure point subspace for H . Note that

$$\text{Ran}(P_c) = L_c^2(\mathbb{R}) = \mathcal{H}_c(0),$$

where spaces $\mathcal{H}_c(z)$ encode the orthogonality conditions (3.1.7) to the ground state manifold. We write $e^{-itH} = e^{-itH} P_c$ the perturbed Schrödinger evolution projected on the continuous spectral subspace, whereas by $e^{it\Delta} = e^{-itH_0}$ is the free Schrödinger evolution. For $I \subseteq \mathbb{R}$, the space-time Lebesgue space $L_t^q(I; L_x^p(\mathbb{R}^d))$ is written $L_t^q L_x^p(I)$, or $L_t^q L_x^p$ when the dependence on I is clear. Given a Borel function $f : \mathbb{R}_+ \rightarrow \mathbb{C}$, we denote the operators defined by the usual functional calculus by $f(H_0)$ and $f(H) = f(H) P_c(H)$. In particular, we define the Littlewood-Paley multipliers (see (3.2.16)) around the dyadic frequency $N \in 2^{\mathbb{N}}$ by

$$\Delta_N u = \varphi(N^{-1}H_0), \quad \Pi_N = \sum_{K \leq N} \Delta_K u, \quad \tilde{\Delta}_N = \varphi(N^{-1}H), \quad \tilde{\Pi}_N = \sum_{K \leq N} \tilde{\Delta}_K u.$$

Given a function $m : \mathbb{R}^d \rightarrow \mathbb{C}$ we denote the distorted Fourier multiplier $\mathcal{F}_V^* P_c(H) m \mathcal{F}_V$ by $M_m(H)$ (see Definition 3.2.7) where \mathcal{F}_V is the distorted Fourier transform (see Proposition 3.2.6). Given a function u_0 in $L_c^2(\mathbb{R}^d)$, we define its Wiener randomization by u_0^ω in Definition 3.3.1, and $u^\omega = e^{-itH} u_0^\omega$ is the linear propagation of the randomized data by the under perturbed flow.

3.2 Spectral theory for generic short-range Schrödinger operators

In this section, we present well known facts about functional calculus on perturbed Schrödinger operators that gives rise to a natural framework to work with $-\Delta + V$ and $-\Delta$ all together. First, we collect some definitions and basic properties of *wave operators*, such as the intertwining property as well as their L^p -boundedness. As a consequence, we will see how dispersive estimates known for free Schrödinger evolution carry over to the perturbed setting. Additionally, we recall the construction of a *distorted Fourier transform* based on the *limiting absorption principle*. We shall see how the wave operators connect this transformation to the usual - or flat - Fourier transform, and lead to a useful generalization of the Fourier multiplier theorem. Some of the results we state here may not hold true in low dimensions $d < 3$.

3.2.1 Wave operators and the intertwining property

The wave operators are defined by the strong limit

$$W_{\pm} = s - \lim_{t \rightarrow \pm\infty} e^{itH} e^{-itH_0} \quad \text{in } \mathcal{B}(L^2(\mathbb{R}^d)).$$

They aim at understanding scattering of Schrödinger operators $-\Delta + V$ by comparing the asymptotic behavior of e^{-itH} with the asymptotic behavior of e^{-itH_0} . We say that the wave operators are asymptotically complete if

- (i) W_{\pm} is bounded and surjective from $L^2(\mathbb{R}^d)$ onto $L_{ac}^2(\mathbb{R}^d)$
- (ii) $\sigma_{sc}(H) = \emptyset$, and hence $L_{ac}^2(\mathbb{R}^d) = L_c^2(\mathbb{R}^d)$.

Agmon proved in [Agm75] that the wave operators W_{\pm} exist and are asymptotically complete under the decaying assumption $\langle x \rangle^{1+\varepsilon} V \in L^{\infty}(\mathbb{R}^d)$. Consequently, wave operators are partial isometries onto L_c^2 in this case, that is

$$W_{\pm}^* W_{\pm} = I, \quad W_{\pm} W_{\pm}^* = P_c.$$

It follows from the definition that

$$e^{-itH} W_{\pm} = W_{\pm} e^{-itH_0}.$$

This leads to the so-called intertwining property: for any Borel function $f : \mathbb{R} \rightarrow \mathbb{C}$

$$H P_c = W_{\pm} H_0 W_{\pm}^*, \quad f(H) P_c = W_{\pm} f(H_0) W_{\pm}^*, \quad (3.2.1)$$

where $f(H)$ and $f(H_0)$ are defined by the functional calculus on self-adjoint operators. In a seminal paper [Yaj95], Yajima proved the $W^{k,p}(\mathbb{R}^d)$ boundedness for wave operators when some extra regularity and decay on V and $\mathcal{F}(V)$ are assumed. We state the result in $L^p(\mathbb{R}^d)$, since we do not need more in our study.

Proposition 3.2.1 (*L^p -bound of wave operators, Theorem 1.1 in [Yaj95]*). *Let V be a generic potential with some regularity and decaying assumptions detailed in [Yaj95], and $1 \leq p \leq +\infty$. The wave operators W_{\pm} can be extended to bounded operators in $L^p(\mathbb{R}^d)$: there exists C_p such that for all f in $L^2(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$,*

$$\|W_{\pm} f\|_{L^p(\mathbb{R}^d)} \leq C_p \|f\|_{L^p(\mathbb{R}^d)}. \quad (3.2.2)$$

Given $1 \leq p, q \leq +\infty$, it follows from the intertwining property (3.2.1) and from the L^p bound (3.2.2) that $f(H)$ and $f(H_0)$ have equivalent norms on $\mathcal{B}(L^p(\mathbb{R}^d), L^q(\mathbb{R}^d))$. There exists $C_{p,q}$ such that for any Borel function $f : \mathbb{R} \rightarrow \mathbb{C}$,

$$C_{p,q}^{-1} \|f(H)\|_{\mathcal{B}(L^p, L^q)} \leq \|f(H_0)\|_{\mathcal{B}(L^p, L^q)} \leq C_{p,q} \|f(H)\|_{\mathcal{B}(L^p, L^q)}.$$

One striking but straightforward consequence of Proposition 3.2.1 is the generalization of the dispersive estimate to perturbed linear Schrödinger evolutions.

Proposition 3.2.2 (Dispersive estimate for perturbed Schrödinger evolution). *Let V be a potential as in Proposition 3.2.1, $2 \leq p \leq +\infty$ and q such that $1/p + 1/q = 1$. There exists a constant C_p such that for all $f \in L^2(\mathbb{R}^d) \cap L^q(\mathbb{R}^d)$ and $t \in \mathbb{R} \setminus \{0\}$*

$$\|e^{-itH} P_c f\|_{L^p(\mathbb{R}^d)} \leq C^p |t|^{-d(1/2-1/p)} \|f\|_{L^q(\mathbb{R}^d)}. \quad (3.2.3)$$

By the TT^* argument, we deduce from the above dispersive estimate the global-in-time Strichartz estimate for the perturbed linear evolution: for all Schrödinger admissible pairs $2 \leq q, r \leq \infty$, with $3 \leq d$ and $\frac{2}{q} + \frac{d}{r} = \frac{d}{2}$

$$\|e^{-itH} P_c f\|_{L_t^q(\mathbb{R}; L_x^r(\mathbb{R}^d))} \leq C_{q,r,d} \|f\|_{L^2(\mathbb{R}^d)}. \quad (3.2.4)$$

This result was originally proved in [JSS91], under the assumption that H has no eigenvalue nor resonance at zero, and that V satisfies the decay condition

$$|V(x)| \lesssim \langle x \rangle^{-2-\varepsilon} \quad \text{for some } 0 < \varepsilon.$$

3.2.2 Distorted Fourier multipliers

In order to refine the perturbed Strichartz estimates of Proposition 3.2.2 for randomized initial data as in [BOP15a], we have to use unit-scale decomposition of the frequency space adapted to the operator $-\Delta + V$. More precisely, we need to set out a *distorted Fourier transform* that conjugates the operator H with the multiplication by $|\xi|^2$, and to generalize the Fourier multiplier theorem. To construct such a transformation, the usual strategy is to determine some *generalized plane waves* $e(x, \xi)$ that are perturbations of the plane waves $e^{ix \cdot \xi}$ and that formally satisfy the Helmholtz equation

$$(-\Delta + V) e(\cdot, \xi) = |\xi|^2 e(\cdot, \xi), \quad (3.2.5)$$

with the asymptotic condition

$$v(x, \xi) := e(x, \xi) - e^{ix \cdot \xi} = O_{|x| \rightarrow \infty}(|x|^{-1}),$$

as well as the Sommerfeld radiation condition

$$r(\partial_r - i|\xi|)v(x, \xi) \xrightarrow{|x| \rightarrow 0} 0.$$

Equation (3.2.5) cannot be solved in $L^2(\mathbb{R}^d)$ since $|\xi|^2$ is in the essential spectrum of H . However, this equation can be solved in weighted spaces by a procedure called the *limiting absorption principle*. This principle states that the resolvent $R(z) := (H - z)^{-1}$, which is bounded from $L^2(\mathbb{R}^d)$ to $\mathcal{H}^2(\mathbb{R}^d)$ and that defines an analytic operator valued function on $\mathbb{C} \setminus [0, +\infty)$, can be extended to the boundary of its domain of definition in the following sense.

Proposition 3.2.3 (Limiting absorption principle, Theorem 4.2 in [Agm75]). *Let $\lambda > 0$ and $\delta > 1/2$. The sequence*

$$\lim_{\substack{z \rightarrow \lambda \\ \text{Im}(z) > 0}} R(z) =: R^+(\lambda)$$

converges in $\mathcal{B}(\langle x \rangle^{-\delta} L^2(\mathbb{R}^d), \langle x \rangle^\delta \mathcal{H}^2(\mathbb{R}^d))$ endowed with the uniform operator norm topology.

Then, the existence of a family of *distorted plane waves* follows from the above limiting absorption principle. Indeed, the solutions to equation (3.2.5) satisfy the *Lippman-Schwinger* equation

$$e(x, \xi) = e^{ix \cdot \xi} - R^+(|\xi|^2)(V e^{ix \cdot \xi}). \quad (3.2.6)$$

Proposition 3.2.4 (Generalized plane wave, Theorem 5.1 in [Agm75]). *There exists a measurable function e in $L^2_{loc}(\mathbb{R}^d \times (\mathbb{R}^d \setminus \{0\}))$ such that for every fixed ξ in $\mathbb{R}^d \setminus \{0\}$, the function $e(\cdot, \xi)$ belongs to $H^2_{loc}(\mathbb{R}^d) \cap C(\mathbb{R}^d)$ and is a solution to equation (3.2.5) in $H^2_{loc}(\mathbb{R}^d)$.*

See [Ike60] for further properties of the generalized plane waves when V is regular. Another important consequence of the limiting absorption principle is the *local smoothing estimate*, or *local energy decay*, for the perturbed linear Schrödinger evolution. We state a version taken from [RT15] (Proposition 1.33) where some extra regularity and decay assumptions on V are required. This local smoothing effect and its transferred version into the space U^2 is a key ingredient in the proof of Theorem 3.1.1, when we study the perturbations around the ground state.

Proposition 3.2.5 (Global-in-time local smoothing estimate for H , Proposition 1.33 in [RT15]). *Under our assumptions made on V , and for ψ solution to the forced Schrödinger equation*

$$i\partial_t \psi + (\Delta - V)\psi = F, \quad \psi|_{t=0} = \psi_0,$$

we have the following global-in-time control of the local energy of ψ

$$\int_{\mathbb{R}} \|\mathbb{P}_c(\psi)\|_{\mathcal{H}^{1, -1/2-}}^2 dt \lesssim_V \|\psi_0\|_{\mathcal{H}^{1/2}(\mathbb{R}^d)}^2 + \int_{\mathbb{R}} \|F(t)\|_{L_x^{2, 1/2+}}^2 dt. \quad (3.2.7)$$

The next step is to define a *distorted Fourier transform* from the generalized plane waves constructed in the above paragraph. Since H may have some pure point spectrum in our assumptions, the operator we get is only a partial isometry with range $L^2_c(\mathbb{R}^d)$.

Proposition 3.2.6 (Distorted Fourier transform, Theorem 5.1 in [Agm75]). *There exists a partial isometry \mathcal{F}_V from $L^2(\mathbb{R}^d)$ to $L^2_c(\mathbb{R}^d)$ which diagonalizes H on $\mathcal{H}^2(\mathbb{R}^d)$. For all $f \in \mathcal{H}^2(\mathbb{R}^d)$,*

$$(-\Delta + V)f = (\mathcal{F}_V^* M_{|\cdot|^2} \mathcal{F}_V)f.$$

Moreover, for any f in $L^2(\mathbb{R}^d)$ the following representation formula holds in $L^2(\mathbb{R}^d)$:

$$\mathcal{F}_V f(\xi) = (2\pi)^{-d/2} \lim_{n \rightarrow \infty} \int_{|x| < n} \overline{e(x, \xi)} f(x) dx, \quad (3.2.8)$$

$$\mathcal{F}_V^* f(x) = (2\pi)^{-d/2} \lim_{n \rightarrow \infty} \int_{|x| < n} e(x, \xi) \mathbb{P}_c(H)f(\xi) d\xi. \quad (3.2.9)$$

Furthermore, we have

$$W_+ = \mathcal{F}_V^* \mathcal{F}. \quad (3.2.10)$$

Thanks to the distorted Fourier transform we are now able to define some analogues of the Fourier multipliers, that commute with H .

Definition 3.2.7. Let $m : \mathbb{R}^d \rightarrow \mathbb{C}$ be a function in L^∞ . The distorted Fourier multiplier

$$M_m(H) := \mathcal{F}_V^* m(\xi) \mathcal{F}_V \quad (3.2.11)$$

is a bounded operator on $L^2(\mathbb{R}^d)$, where $m(\xi)$ denotes the operator $u \in L_\xi^2 \mapsto mu$. We have

$$\|M_m(H)\|_{\mathcal{B}(L^2(\mathbb{R}^d))} \leq \|m\|_{L^\infty(\mathbb{R}^d)}.$$

It follows from equation (3.2.10) that

$$W_+^* M_m(H) W_+ = M_m(H_0),$$

where $M_m(H_0)$ is the usual Fourier multiplier by m . Hence, we deduce from the L^p boundedness of the wave operator W_\pm mentioned in Proposition 3.2.1 that $M_m(H)$ is bounded on $L^p(\mathbb{R}^d)$ if and only if $M_m(H_0)$ is bounded on $L^p(\mathbb{R}^d)$. As a consequence of this principle, we generalize the Fourier multiplier theorem and its variations to the perturbed linear Schrödinger evolution.

Lemma 3.2.8. *Let $m : \mathbb{R}^d \rightarrow \mathbb{C}$ be a function in $L^\infty(\mathbb{R}^d)$ supported on a compact set E . Given q in $[2, +\infty]$, there exists a constant $C = C(q, V, \|m\|_{L^\infty})$ such that for all f in $L^2(\mathbb{R}^d)$,*

$$\|M_m(H)f\|_{L^q} \leq C|E|^{\frac{1}{2}-\frac{1}{q}}\|f\|_{L^2}. \quad (3.2.12)$$

Proof. Let f be a function in $L^2(\mathbb{R}^d)$. We have

$$|(\mathcal{F}^* m \mathcal{F})f(x)| = \left| \int_E e^{i\xi \cdot x} m(\xi) \mathcal{F}f(\xi) d\xi \right| \leq \|m\|_{L^\infty(\mathbb{R}^d)} \|e(\cdot, \xi)\|_{L^\infty} |E|^{\frac{1}{2}} \|f\|_{L^2}. \quad (3.2.13)$$

Then (3.2.12) for $M_m(H_0)$ (i.e. when $V = 0$) follows from complex interpolation, and we deduce the estimate for $M_m(H)$ from the intertwining property (3.2.1) and from the L^p -bound (3.2.2) on W_\pm . \square

Similarly, one can generalize the Mihlin multiplier theorem (see Theorem 3.2 in the book [Ste70]) to the perturbed setting.

Lemma 3.2.9 (Mihlin multiplier theorem). *Let $m : \mathbb{R}^d \setminus \{0\} \rightarrow \mathbb{C}$ be a multiplier and $1 < p < +\infty$. Assume that for all α in \mathbb{N}^d such that $0 \leq |\alpha| \leq \frac{d-1}{2} + 1$,*

$$|D^\alpha m(\xi)| \leq C_\alpha |\xi|^{-\alpha}. \quad (3.2.14)$$

Then,

$$\|m(H)f\|_{L^p(\mathbb{R}^d)} \lesssim C_\alpha \|f\|_{L^p(\mathbb{R}^d)}.$$

As a consequence of the Mihlin multiplier theorem, we show that the Sobolev spaces defined with H or H_0 are equivalent.

Lemma 3.2.10. *Let $1 < p < +\infty$ and $0 \leq s$. There exists $C = C(s, p, d)$ such that for all f in $\mathcal{S}(\mathbb{R}^d)$*

$$C^{-1} \|\langle \sqrt{H_0} \rangle^s f\|_{L^p} \leq \|\langle \sqrt{H} \rangle^s f\|_{L^p} \leq C \|\langle \sqrt{H_0} \rangle^s f\|_{L^p}$$

Proof. Let q be the conjugated exponent of p . Let us show that $\langle \sqrt{H_0} \rangle^s \langle \sqrt{H} \rangle^{-s}$ can be extended to a bounded operator in $\mathcal{B}(L^p)$.¹ We proceed by using complex interpolation and duality in L^p . Take f, g in $\mathcal{S}(\mathbb{R}^d)$ such that the support of \widehat{f} is compact, and note that such functions are dense in L^p . Next, we consider the map

$$F : z \mapsto \left(f \mid e^{z^2} \langle \sqrt{H_0} \rangle^z \langle \sqrt{H} \rangle^{-z} g \right)$$

with domain $D = \{z \in \mathbb{C} \mid 0 < \operatorname{Re}(z) < 2n\}$ for a given $n \in \mathbb{N}$. We see that F is a function that is analytic on D and continuous on the closure of D . Moreover, it follows from the Mihlin multiplier theorem and its distorted version (3.2.14) that for f in L^p with $1 < p < \infty$ and δ in \mathbb{R} we have

$$|F(e^{i\delta})| \lesssim \|f\|_{L^p} \|g\|_{L^q}.$$

Indeed,

$$\|\langle \sqrt{H_0} \rangle^{i\delta} f\|_{L^p} \lesssim (1 + |\delta|)^N \|f\|_{L^p}, \quad \|\langle \sqrt{H} \rangle^{i\delta} f\|_{L^p} \lesssim (1 + |\delta|)^N \|f\|_{L^p}.$$

Note that the constant can be made independent of δ thanks to the term $e^{-\delta^2}$ that arises from the function e^{z^2} . In addition, the distorted version of the Mihlin multiplier theorem and the fact that V is in \mathcal{S} yield

$$\|\langle \sqrt{H_0} \rangle^{2n} \langle \sqrt{H} \rangle^{-2n}\|_{\mathcal{B}(L^p)} \lesssim \sum_{i+j+k=n} \|V^i H^k V^j \langle \sqrt{H} \rangle^{-2n}\|_{\mathcal{B}(L^p)} = C_{n,p} < +\infty,$$

and we deduce from this that

$$|F(e^{2n+i\delta})| \lesssim \|f\|_{L^p} \|g\|_{L^q}.$$

It follows from the *three lines lemma*, we prove that $|F(z)| \lesssim \|f\|_{L^p} \|g\|_{L^q}$ on D_n uniformly in n , and the density of functions like f and g in L^p and L^q yields the desired result. \square

A straightforward consequence of Lemma 3.2.10 is the generalization to the perturbed setting of the fractional Leibniz rule.

Lemma 3.2.11 (Fractional Leibniz rule). *Let $1 < p, p_1, p_2, q_1, q_2 < +\infty$, $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{q_1} + \frac{1}{q_2}$, and let $0 \leq s$. Assume that f, g are in $\mathcal{S}(\mathbb{R}^d)$. Then,*

$$\|\langle \sqrt{H} \rangle^s (fg)\|_{L^p} \lesssim \|\langle \sqrt{H} \rangle^s f\|_{L^{p_1}} \|g\|_{L^{p_2}} + \|\langle \sqrt{H} \rangle^s g\|_{L^{q_1}} \|f\|_{L^{q_2}}.$$

We prove in the next lemma that if m is a radial function, the Fourier multiplier by m is precisely the spectral multiplier defined by the functional calculus on self-adjoint operators.

Lemma 3.2.12. *If there exists a Borel function $f : \mathbb{R} \rightarrow \mathbb{C}$ such that $m(\xi) = f(|\xi|^2)$ for all ξ in \mathbb{R}^d , then*

$$M_m(H) = f(H). \tag{3.2.15}$$

Proof. Let us first write the result for the free Laplacian H_0 , that is $\mathcal{F}^* m \mathcal{F} = f(H_0)$. By an approximation argument, the proof reduces to the cases where f is a Schwartz function. In this setting, $f(H_0)$ is given by the formula

$$f(H_0) = \int_{\mathbb{R}} e^{itH_0} \widehat{f}(t) dt, \quad \text{where} \quad e^{itH_0} = \mathcal{F}^* e^{it|\cdot|^2} \mathcal{F}.$$

¹ The proof provides the same result for the operator $\langle \sqrt{H} \rangle^s \langle \sqrt{H_0} \rangle^{-s}$.

By applying the Fourier inversion formula for f , we have that

$$f(H_0) = \mathcal{F}^* \left(\int_{\mathbb{R}} e^{it|\cdot|^2} \widehat{f}(t) dt \right) \mathcal{F} = \mathcal{F}^* f(|\cdot|^2) \mathcal{F} = M_m(H_0).$$

Hence,

$$\mathcal{F} f(H_0) \mathcal{F}^* = M_m(H_0).$$

Also, the intertwining property (3.2.1) and the equation (3.2.10) give

$$f(H) = W_+ f(H_0) W_+^* = \mathcal{F}_V^* \mathcal{F} f(H_0) \mathcal{F}^* \mathcal{F}_V = \mathcal{F}_V^* M_m \mathcal{F}_V = M_m(H). \quad \square$$

3.2.3 Application to the distorted Littlewood-Paley theory

There exists a nonnegative radial function ψ in $\mathcal{C}^\infty(\mathbb{R}^d)$ such that $\varphi = \psi - \psi(2\cdot)$ is supported on $\{x \in \mathbb{R}^d \mid \frac{1}{2} \leq |x| \leq 2\}$ and satisfies for all x in $\mathbb{R}^d \setminus \{0\}$

$$\sum_{N \in 2^{\mathbb{Z}}} \varphi(N^{-1}x) = 1.$$

This partition of unity provides frequency decomposition on dyadic annuli for functions f in $L^2(\mathbb{R}^d)$ called the Littlewood-Paley decomposition. We consider such a decomposition for both the flat Fourier transform and the distorted one:

$$f = \sum_{N \in 2^{\mathbb{Z}}} \Delta_N f = \Delta_{\leq 1} f + \sum_{N \in 2^{\mathbb{N}}} \Delta_N f, \quad (3.2.16)$$

where $\Delta_N := \varphi(N^{-1}H_0)$ is the spectral multiplier around frequencies of size N for H_0 as well as

$$f = \sum_{N \in 2^{\mathbb{Z}}} \widetilde{\Delta}_N f = \widetilde{\Delta}_{\leq 1} f + \sum_{N \in 2^{\mathbb{N}}} \widetilde{\Delta}_N f, \quad (3.2.17)$$

where $\widetilde{\Delta}_N := \varphi(N^{-1}H)$. Note that the Bernstein's estimates are still true in the perturbed setting, as a consequence of the intertwining property, and of the L^p bound on the wave operators. For $1 \leq p \leq q \leq \infty$, $s \geq 0$ there exist $C_{p,q}$ and $C_{p,s}$ and such that for all $N \geq 1$,

$$\begin{aligned} \|\widetilde{\Delta}_N f\|_{L^q(\mathbb{R}^d)} &\leq C_{p,q} N^{d(\frac{1}{p}-\frac{1}{q})} \|\widetilde{\Delta}_N f\|_{L^p(\mathbb{R}^d)}, \\ \|\widetilde{\Delta}_N f\|_{W^{s,p}(\mathbb{R}^d)} &\leq C_{p,s} N^s \|\widetilde{\Delta}_N f\|_{L^p(\mathbb{R}^d)}. \end{aligned} \quad (3.2.18)$$

Given f in $L^2(\mathbb{R}^d)$, the associated distorted Littlewood-Paley square function is defined by

$$\Lambda f(x) = \left(\sum_{N \in 2^{\mathbb{Z}}} |\widetilde{\Delta}_N f(x)|^2 \right)^{1/2}.$$

One can see from Plancherel's theorem that the $L^2(\mathbb{R}^d)$ -norm of the Littlewood-Paley square function is equivalent to the $L^2(\mathbb{R}^d)$ -norm of the function itself. The so-called *Littlewood-Paley square function* theorem extends this result to the $L^p(\mathbb{R}^d)$ -norms for $1 < p < \infty$. We state a generalized version of this theorem in the perturbed framework.

Proposition 3.2.13. *For $1 < p < +\infty$ and f in $L^p(\mathbb{R}^d)$, the Littlewood-Paley square function Λf is in $L^p(\mathbb{R}^d)$, and*

$$\|\Lambda f\|_{L^p(\mathbb{R}^d)} \sim \|f\|_{L^p(\mathbb{R}^d)}. \quad (3.2.19)$$

The constant involved in the above equivalence only depends on p and on the cutoff function ψ used in the Littlewood-Paley decomposition.

We outline the classical proof (see Theorem 5 in the book [Ste70]) that relies on the Mihlin multiplier theorem from Lemma 3.2.9. The strategy is to randomize the sum Δf with independent Rademacher variables, and to use Khinchin's estimates to reduce the proof to the L^2 -case. As mentioned above, this case is a straightforward consequence of Plancherel theorem.

Proof. Let us fix $1 < p < +\infty$ and f in $L^2(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$. Take $(\varepsilon_N)_{N \in 2^{\mathbb{Z}}}$ a sequence of independent Rademacher variables on a probability space $(\Omega, \mathcal{A}, \mathbf{P})$, i.e. $\varepsilon_N \sim \mathcal{U}(\{-1, 1\})$. Next, we define the random Fourier multiplier for ω in Ω and ξ in \mathbb{R}^d by

$$m^\omega(\xi) = \sum_{N \in 2^{\mathbb{Z}}} \varepsilon_N(\omega) \varphi(N^{-1}\xi).$$

Since $\varphi(N^{-1}\cdot)$ is supported on frequencies of size $\sim N$, the random multiplier m^ω satisfies the Mihlin's assumption: for all ξ in \mathbb{R}^d and α a multi-index,

$$|D^\alpha m^\omega(\xi)| \leq \sum_{N \in 2^{\mathbb{Z}}} N^{-|\alpha|} |(D^\alpha \varphi)(N^{-1}\xi)| \leq C_\alpha |\xi|^{-|\alpha|}.$$

Besides, m^ω is a radial multiplier, and it follows from Lemma 3.2.12 and from Lemma 3.2.9 that

$$\|m^\omega(H)f\|_{L^p(\mathbb{R}^d)} \leq \|f\|_{L^p(\mathbb{R}^d)}. \quad (3.2.20)$$

That being said, the Khinchin's estimate claims that

$$\left(1 - \frac{1}{p}\right)^{1/2} \left(\sum_n |c_n|^2\right)^{1/2} \lesssim \left(\mathbf{E} \left| \sum_n \varepsilon_N(\omega) c_n \right|^p\right)^{1/p}, \quad (3.2.21)$$

and yields

$$\Lambda f = \left(\sum_{N \in 2^{\mathbb{Z}}} |\tilde{\Delta}_N f|^2\right)^{1/2} \lesssim \left(\mathbf{E} \sum_{N \in 2^{\mathbb{Z}}} \varepsilon_N(\omega) |\tilde{\Delta}_N f|^p\right)^{1/p} \lesssim (\mathbf{E} |m^\omega(H)f|^p)^{1/p}.$$

By taking the $L^p(\mathbb{R}^d)$ -norm on both sides and by using (3.2.20), we have

$$\|\Lambda f\|_{L^p(\mathbb{R}^d)} \lesssim \mathbf{E} \|m^\omega(H)f\|_{L^p(\mathbb{R}^d)} \lesssim \|f\|_{L^p(\mathbb{R}^d)}.$$

We prove the reversed inequality by duality. □

3.3 Probabilistic and bilinear Strichartz estimates in the perturbed setting

3.3.1 Wiener randomization with respect to the distorted Fourier transform

Let us now detail how to adapt the Wiener randomization for the operator $-\Delta + V$ and to generalize probabilistic Strichartz estimates. This will be made possible thanks to the distorted Fourier transform presented in the above section. We shall follow the framework detailed by [BOP15a] which can be adapted with no difficulty to the perturbed case.

Wiener decomposition on unit cubes We denote the unit cube $[-1, 1]^d$ by Q_0 and the translated cube centered around $n \in \mathbb{Z}^d$ by $Q_n = Q_0 + n$. Then, we take a well-chosen bump function ψ supported on Q_0 that provides a partition of unity on the frequency space, that is $\sum_{n \in \mathbb{Z}^d} \psi(\cdot - n) \equiv 1$, and we define the Wiener's decomposition of a function in $L^2(\mathbb{R}^d)$ as

$$f = \sum_{n \in \mathbb{Z}^d} M_{\psi_n}(H)f, \quad (3.3.1)$$

where we defined the distorted Fourier multiplier $M_{\psi_n}(H)$ in (3.2.11). After that, we consider a probability space $(\Omega, \mathcal{A}, \mathbf{P})$ and a sequence of mean zero complex valued random variables $(g_n)_{n \in \mathbb{Z}^d}$ of laws $(\mu_n)_{n \in \mathbb{Z}^d}$, with uniform bound:

$$\left| \int_{\mathbb{R}} e^{\gamma x} d\mu_n(x) \right| \leq e^{c\gamma^2} \quad \text{for all } n \text{ in } \mathbb{Z}^d \text{ and } \gamma \text{ in } \mathbb{R}. \quad (3.3.2)$$

Such a bound is satisfied by Gaussian random variables, Bernoulli variables or any random variables with compactly supported distributions. We also assume that $(\operatorname{Re}(g_n), \operatorname{Im}(g_n))_{n \in \mathbb{Z}^d}$ are independent variables.

Definition 3.3.1. The Wiener randomization of a function u_0 in $L^2(\mathbb{R}^d)$ is

$$u_0^\omega := \sum_{n \in \mathbb{Z}^d} g_n(\omega) M_{\psi_n}(H)u_0. \quad (3.3.3)$$

The Khinchin's inequality asserts that

$$\left(\mathbf{E} \left| \sum_n g_n(\omega) c_n \right|^p \right)^{1/p} \lesssim \sqrt{p} \left(\sum_n |c_n|^2 \right)^{1/2} \quad \text{for } (c_n) \in \ell^2, \quad 1 \leq p < +\infty, \quad (3.3.4)$$

and we deduce from it that the randomization procedure improves integrability and preserves regularity. Namely, for all $2 \leq p < +\infty$ and s in \mathbb{R} there exist some constants $0 < c, 0 < C$ such that for all $0 < \alpha, u$ in $L^2(\mathbb{R}^d)$ and v in $\mathcal{H}^s(\mathbb{R}^d)$ we have

$$\mathbf{P}(\|u^\omega\|_{L^p(\mathbb{R}^d)} > \lambda) \leq C e^{-c\lambda^2 \|u\|_{L^2(\mathbb{R}^d)}^{-2}}, \quad \mathbf{P}(\|v^\omega\|_{\mathcal{H}^s(\mathbb{R}^d)} > \lambda) \leq C e^{-c\lambda^2 \|u\|_{\mathcal{H}^s(\mathbb{R}^d)}^{-2}}.$$

We emphasize that the procedure does not improve regularity provided that the distribution μ_n does not concentrate around zero when n goes to ∞ , i.e. $u_0^\omega \in \mathcal{H}^s(\mathbb{R}^d) \setminus \mathcal{H}^{s+}(\mathbb{R}^d)$ almost surely when $u_0 \in \mathcal{H}^s(\mathbb{R}^d) \setminus \mathcal{H}^{s+}(\mathbb{R}^d)$ (see Lemma B.1 in [BT08a]). As for the possible gain of decay, we use the independence of the $\{g_n\}_{n \in \mathbb{Z}^d}$'s, and we observe that for $\varepsilon > 0$,

$$\begin{aligned} \mathbf{E}(\| |x|^\varepsilon u_0^\omega \|_{L_x^2}^2) &= \sum_{(n,m) \in \mathbb{Z}^{2d}} \mathbf{E}(g_n \overline{g_m}) \int |x|^{2\varepsilon} M_{\psi_n}(H) u_0 \overline{M_{\psi_m}(H) f} dx \\ &= \sum_{n \in \mathbb{Z}^d} \| |x|^\varepsilon M_{\psi_n}(H) u_0 \|_{L_x^2(\mathbb{R}^d)}^2 \end{aligned}$$

diverges if we assume that

$$\sum_{n \in \mathbb{Z}^d} \| |x|^\varepsilon M_{\psi_n}(H) u_0 \|_{L_x^2(\mathbb{R}^d)}^2 = +\infty. \quad (3.3.5)$$

Hence, $\| |x|^\varepsilon u_0^\omega \|_{L^2(\mathbb{R}^d)}$ diverges in $L^2(\Omega)$ when u_0 is chosen such that (3.3.5). This indicates that the randomization does not gain decay in general.² Still, the initial data $\psi_0 = u_0^\omega + v_0$ in the Cauchy problem (3.1.4) does not decay for a general v_0 in $\mathcal{H}^{1/2}$. Besides, we don't use any decay of the data in our arguments to prove Theorem 3.1.1.

Improved probabilistic Strichartz estimates Let us now recall how to improve the Strichartz estimates (3.2.3) for data randomized according to the distorted Wiener procedure (3.3.3) (see Lemma 2.3 in [BOP15a]).

Proposition 3.3.2 (Improved global-in-time Strichartz estimate). *Given a Schrödinger admissible pair (q, r) as in Proposition (3.2.2), a real number \tilde{r} with $r \leq \tilde{r} < +\infty$ and a function $f \in \mathcal{H}^s(\mathbb{R}^d)$ for some $s \geq 0$, there exist constants $0 < C, 0 < c$ such that*

$$\mathbf{P}(\| \langle \sqrt{H} \rangle^s e^{-itH} u_0^\omega \|_{L_t^q(\mathbb{R}; L_x^{\tilde{r}}(\mathbb{R}^d))} > \lambda) \leq C \exp(-c\lambda^2 \|f\|_{\mathcal{H}_x^s(\mathbb{R}^d)}^{-2}). \quad (3.3.6)$$

Consequently, $\| \langle \sqrt{H} \rangle^s e^{-itH} u_0^\omega \|_{L_t^q(\mathbb{R}; L_x^{\tilde{r}}(\mathbb{R}^d))} < +\infty$ almost-surely when f is in \mathcal{H}^s .

The strategy of the proof is to exploit the enhanced space integrability of Wiener randomized data. More precisely, the unit scale frequency components $\psi_n(H)f$ benefit from better integrability (uniformly in n) thanks to the distorted Fourier multiplier theorem, while the randomization procedure cancels interference between these different pieces $\psi_n(H)f$. Hence, the space integrability of $e^{-iH} u_0^\omega$ is improved and one can apply the deterministic Strichartz estimate for the admissible pair (q, r) . The main point of our revisited proof is that the distorted Fourier transform commutes with the perturbed linear flow e^{-itH} , and satisfies the unit-scale Bernstein estimate of Lemma 3.2.8.

Proof. We write the proof for $s = 0$. The large deviation bound (3.3.6) will be deduced from an estimate on the moments of the random variable $\| e^{-itH} u_0^\omega \|_{L_t^q(\mathbb{R}; L_x^{\tilde{r}}(\mathbb{R}^d))}$. To get such an estimate, we take p with $\max(q, \tilde{r}) < p$ and we use Minkowski inequality to get

$$\mathbf{E}(\| e^{-itH} u_0^\omega \|_{L_t^q(\mathbb{R}; L_x^{\tilde{r}}(\mathbb{R}^d))}^p)^{1/p} \leq \| e^{-itH} u_0^\omega \|_{L_t^q(\mathbb{R}; L_x^{\tilde{r}}(\mathbb{R}^d); L_\omega^p(\Omega))}.$$

² Since $\| |x|^\varepsilon \sum_n M_{\psi_n} u_0 \|_{L^2}^2 \neq \sum_n \| |x|^\varepsilon M_{\psi_n} u_0 \|_{L^2}^2$, we are not able to reproduce the proof of Lemma B.1 in [BT08a], and to preclude any gain of decay by our randomization procedure almost-surely.

At fixed $(t, x) \in \mathbb{R} \times \mathbb{R}^d$, it follows from Khinchin inequality (3.3.4) that

$$\|e^{-itH} u_0^\omega\|_{L_\omega^p(\Omega)} = \left\| \sum_{n \in \mathbb{Z}^d} g_n(\omega) e^{-itH} M_{\psi_n}(H) f \right\|_{L_\omega^p(\Omega)} \lesssim \sqrt{p} \|M_{\psi_n}(H) e^{-itH} f\|_{\ell_n^2(\mathbb{Z}^d)}.$$

Hence, we use Minkowski inequality once again with $2 \leq \min(q, \tilde{r})$, and get

$$\mathbf{E}(\|e^{-itH} u_0^\omega\|_{L_t^q(\mathbb{R}; L_x^{\tilde{r}}(\mathbb{R}^d))}^p)^{1/p} \lesssim \sqrt{p} \|M_{\psi_n}(H) e^{-itH} f\|_{\ell_n^2(\mathbb{Z}^d; L_t^q(\mathbb{R}; L_x^{\tilde{r}}(\mathbb{R}^d)))}. \quad (3.3.7)$$

Next, we use the distorted Fourier multiplier Lemma 3.2.8 at fixed t with \tilde{r}, r and the unit-scale multiplier $\xi \mapsto e^{-it|\xi|^2} \psi(\xi - n)$ (in particular the volume of the support does not depend on n):

$$\|e^{-itH} M_{\psi_n}(H) f\|_{L_x^{\tilde{r}}(\mathbb{R}^d)} \lesssim_{r, \tilde{r}} \|e^{-itH} M_{\psi_n}(H) f\|_{L_x^r(\mathbb{R}^d)}.$$

Now, we apply the deterministic global-in-time Strichartz estimate for the perturbed Schrödinger evolution (3.2.3) with the admissible pair (q, r) and we get

$$\|e^{-itH} M_{\psi_n}(H) f\|_{L_t^q(\mathbb{R}; L_x^r(\mathbb{R}^d))} \lesssim \|M_{\psi_n}(H) f\|_{L_x^2(\mathbb{R}^d)}.$$

Hence, there exists $0 < C$ such that

$$\mathbf{E}(\|e^{-itH} u_0^\omega\|_{L_t^q(\mathbb{R}; L_x^{\tilde{r}}(\mathbb{R}^d))}^p)^{1/p} \lesssim \sqrt{p} \|M_{\psi_n}(H) f\|_{\ell_n^2 L_x^2(\mathbb{Z}^d \times \mathbb{R}^d)} \leq \sqrt{p} C \|f\|_{L_x^2(\mathbb{R}^d)}.$$

To conclude, we apply Markov inequality to get

$$\mathbf{P}(\|e^{-itH} u_0^\omega\|_{L_t^q(\mathbb{R}; L_x^{\tilde{r}}(\mathbb{R}^d))} > \lambda) \leq (\sqrt{p} C \lambda^{-1} \|f\|_{L_x^2(\mathbb{R}^d)})^p.$$

After that we chose p such that $\sqrt{p} := e^{-1} (C \lambda^{-1} \|f\|_{L_x^2})^{-1}$ and we distinguish the cases when $\max(q, \tilde{r}) < p$ or not. \square

Given $0 < \varepsilon, R$ we define the set

$$\tilde{\Omega}_{\varepsilon, R} = \bigcup_{(q, \tilde{r})} \{\omega \in \Omega \mid \|\varepsilon u^\omega\|_{L_t^q(\mathbb{R}; L_x^{\tilde{r}}(\mathbb{R} \times \mathbb{R}^d))} \leq R\}, \quad (3.3.8)$$

where the union is taken over a finite number of pairs (q, \tilde{r}) as in Proposition 3.3.2 that occur in the nonlinear analysis presented in the section 3.4. It follows from (3.3.6) that there exist $0 < c, C$ such that for all $0 < \varepsilon, R$

$$\mathbf{P}(\Omega \setminus \tilde{\Omega}_{\varepsilon, R}) \leq C \exp(-R^2 \varepsilon^{-2} \|u_0\|_{\mathcal{H}^s}^{-2}).$$

When R is an irrelevant universal constant we denote this set by $\tilde{\Omega}_\varepsilon$.

3.3.2 Bilinear estimate for the perturbed Schrödinger evolution

Bourgain's bilinear estimate states that given N, M two dyadic numbers with $N \leq M$ and u_0, v_0 in $L^2(\mathbb{R}^d)$ localized in the Fourier space, say $\text{supp } \hat{u}_0 \subseteq \{|\xi| \lesssim N\}$ and $\text{supp } \hat{v}_0 \subseteq \{|\xi| \sim M\}$, we have

$$\|(e^{it\Delta} u_0)(e^{it\Delta} v_0)\|_{L_{t,x}^2(\mathbb{R} \times \mathbb{R}^d)} \lesssim N^{\frac{d-1}{2}} M^{-\frac{1}{2}} \|u_0\|_{L_x^2(\mathbb{R}^d)} \|v_0\|_{L_x^2(\mathbb{R}^d)}. \quad (3.3.9)$$

We refer to Lemma 5 in [Bou98a] for the original proof of this result, and to [Col+08] for a more detailed proof. Basically, the proof relies on Plancherel's formula, on the explicit form of the free Schrödinger flow and on the multiplicative property of the flat Fourier transform. As mentioned in the first section, the distorted Fourier transform does not have such a property. That being said, we propose to generalize (3.3.9) to the perturbed case, and we consider the interaction of two initial data localized by perturbed Littlewood-Paley projectors $\tilde{\Delta}_N, \tilde{\Delta}_M$ introduced in (3.2.17) that evolve under the linear flow e^{-itH} .

Proposition 3.3.3 (Bilinear estimate for perturbed linear Schrödinger evolutions). *Given N, M two dyadic integers, we have that for all $u_0, v_0 \in L^2(\mathbb{R}^d)$,*

$$\|(e^{-itH} \tilde{\Delta}_N u_0)(e^{-itH} \tilde{\Delta}_M v_0)\|_{L^2_{t,x}(\mathbb{R} \times \mathbb{R}^d)} \lesssim N^{\frac{d-1}{2}} M^{-\frac{1}{2}} \|\tilde{\Delta}_N u_0\|_{L^2_x(\mathbb{R}^d)} \|\tilde{\Delta}_M v_0\|_{L^2_x(\mathbb{R}^d)}. \quad (3.3.10)$$

The general strategy of our proof is to reduce (3.3.10) to (3.3.9). To do so, we use a perturbative argument that relies on the Duhamel formula that connects the free evolution to the perturbed one. More precisely, given N, K two dyadic numbers and $u = e^{-itH} \tilde{\Delta}_N u_0$ a solution to the perturbed linear Schrödinger equation with initial data localized around a distorted Fourier frequencies of size N , we have

$$\Delta_K u(t) = e^{it\Delta} \Delta_K \tilde{\Delta}_N u_0 - i \int_0^t e^{i(t-\tau)\Delta} \Delta_K (Vu(\tau)) d\tau. \quad (3.3.11)$$

Hence, we will need to estimate some space-time Lebesgue norms of terms like $\Delta_K (V \tilde{\Delta}_N e^{-itH} u_0)$.

Semiclassical functional calculus This section is devoted to the statement of some lemmas used in the proof of Proposition 3.3.3. When studying the interactions of a solution localized around a high frequency $N \gg 1$ with other solutions, we introduce the small parameter $h = N^{-1}$ to place ourselves in the semiclassical analysis framework that provides precise asymptotic expansions with respect to h . This strategy, that can essentially be used when V is smooth, provides an easy but efficient way to intertwine Fourier and distorted Fourier localization. For the convenience of the reader, we detail some notations and results from semiclassical analysis (we refer to [Zwo12] for a general presentation). The Weyl quantization of a given symbol $a \in C^\infty(\mathbb{R}^{2d})$ is

$$\text{Op}_h(a)u(x) := (2\pi h)^{-2d} \iint_{\mathbb{R}^d \times \mathbb{R}^d} e^{i\frac{(x-y)\cdot\xi}{h}} a\left(\frac{x+y}{2}, \xi\right) u(y) dy d\xi. \quad (3.3.12)$$

It defines an operator acting on $\mathcal{S}(\mathbb{R}^d)$ and for instance $h^2 H_0 = \text{Op}_h(|\xi|^2)$, $h^2 H = \text{Op}_h(|\xi|^2 + h^2 V)$. Moreover, if the symbol a is in the class

$$S(1) := \left\{ a(h) \in C^\infty(\mathbb{R}^{2d}) \mid \forall \alpha, \beta \in \mathbb{N}^d \sup_{h \in [0,1]} \sup_{\rho \in \mathbb{R}^{2d}} |\partial_x^\alpha \partial_\xi^\beta a(\rho)| < \infty \right\},$$

we can extend $\text{Op}_h(a)$ to a bounded operator on $L^2(\mathbb{R}^d)$ thanks to the Calderon-Vaillancourt theorem that yields:

$$\|\text{Op}_h(a)\|_{\mathcal{B}(L^2(\mathbb{R}^d))} \lesssim \sum_{|\alpha| \leq 6d+2} h^{|\alpha|/2} \|\partial_{x,\xi}^\alpha a\|_{L^\infty(\mathbb{R}^{2d})}. \quad (3.3.13)$$

Moreover, given a and b two symbols in the class $S(1)$ there exists a symbol c also in $S(1)$ such that $\text{Op}_h(a) \circ \text{Op}_h(b) = \text{Op}_h(c)$. In addition, c has an explicit semiclassical asymptotic expansion of the form

$$c(\rho) = a(\rho)b(\rho) - h \frac{i}{2} \{a, b\}(\rho) + \cdots + \frac{1}{(n-1)!} \left(\frac{ih}{2}\right)^{n-1} a(\omega(\overleftarrow{D}, \overrightarrow{D}))^{n-1} b(\rho) + O(h^n)_{S(\mathbb{R}^d)}, \quad (3.3.14)$$

where $a\omega(\overleftarrow{D}, \overrightarrow{D})b(\rho) := (\partial_{\xi_0} a(\rho_0) \partial_{x_1} b(\rho_1) - \partial_{\xi_1} b(\rho_1) \partial_{x_0} a(\rho_0))|_{\rho_0=\rho_1=\rho}$. The symbol c is written $c =: (a \#_h b)$. In particular, we observe from (3.3.13) and (3.3.14) that whenever a and b are in $S(1)$ with disjoint supports then

$$\text{Op}_h(a) \circ \text{Op}_h(b) = O(h^\infty)_{\mathcal{B}(L^2(\mathbb{R}^d))}. \quad (3.3.15)$$

We can now state an approximation result of spectral multipliers by pseudo-differential operators. This result comes from [Rob87], and was extended to more general multipliers in [Sog17]. We chose to write a version presented in the context of compact manifolds in [BGT04], Proposition 2.1. The proof relies on the so-called Helffer-Sjöstrand formula and can be rewritten in our setting with no difficulty.

Lemma 3.3.4. *Let η in $C_c^\infty(\mathbb{R})$ be a smooth cutoff function. There exists a sequence of symbols $(c_j)_{j \geq 0}$ in $S(1)$ such that for every $h \in (0, 1]$, $n \in \mathbb{N}$ and $0 \leq \sigma \leq n$,*

$$\|\eta(h^2 H) - \sum_{0 \leq j \leq n-1} h^j \text{Op}_h(c_j)\|_{\mathcal{B}(L^2(\mathbb{R}^d), \mathcal{H}^\sigma(\mathbb{R}^d))} \lesssim_{n, \sigma} h^n. \quad (3.3.16)$$

Besides, for $(x, \xi) \in \mathbb{R}^{2d}$

$$c_0(x, \xi) = \eta(|\xi|^2), \quad c_j(x, \xi) = \sum_{k \geq 2} \frac{1}{(k-1)!} \eta^{(k-1)}(|\xi|^2) q_{j,k}(x, \xi),$$

where the finite sum representing c_j is made of functions $q_{j,k}$ which are polynomials of degree less than $2(k-1)$ in the frequency variable ξ , and they are Schwartz functions with respect to the space variable x .

We emphasize that $\text{supp } c_j \subseteq \{(x, \xi) \in \mathbb{R}^{2d} \mid |\xi|^2 \in \text{supp } \eta\}$. Hence, we can approximate $\tilde{\Delta}_N := \varphi(N^{-1}H)$ by pseudo-differential operators supported on frequencies of size N . Since the Fourier multiplier $\Delta_K := \varphi(K^{-1}H_0)$ is already a pseudo-differential operator localized on frequencies of size K , we can use the semiclassical asymptotic expansion (3.3.14) to see that $\Delta_K \tilde{\Delta}_N$ is negligible when $1 \ll |K - N|$. Namely, for all $s, \sigma \in \mathbb{R}$ and $0 < \alpha$ there exists $C_{s, \sigma, \alpha}$ such that for all $N = 2^n$ and $K = 2^k$ with $3 \leq |k - n|$,

$$\|\Delta_K \tilde{\Delta}_N\|_{\mathcal{B}(\mathcal{H}^{-s}, \mathcal{H}^\sigma)} \leq C_{s, \sigma, \alpha} 2^{-\alpha \max(n, k)}. \quad (3.3.17)$$

Therefore one can intertwine localization with respect to spectral Littlewood-Paley multiplier for H and H_0 up to a negligible term. The next lemma, written in light of equation (3.3.11), encapsulate the above discussion.

Lemma 3.3.5. *For any $\alpha \in \mathbb{N}$, there exists $C = C(\alpha, V) > 0$ depending on some weighted norms of V and its derivatives³ such that for any dyadic integer $K = 2^k$ and $f \in L_x^2(\mathbb{R}^d)$, we have*

$$\left(\sum_{|k-l| \geq 3} \|\Delta_K(V\Delta_L f)\|_{L_x^2}^2 \right)^{1/2} \leq CK^{-\alpha} \|\langle x \rangle^{-2} f\|_{L_x^2}. \quad (3.3.18)$$

Note that we decided to exploit not only the localization in frequency, but also the decay of V in order to use the local smoothing effect later on.

Proof. We prove that there exists $C_\alpha > 0$ depending on weighted norms of the potential V and its derivatives such that for all $L = 2^{-l}$ with $3 \leq |k-l|$, we have

$$\|\Delta_K(V\Delta_L f)\|_{L_x^2} \leq C2^{-2\alpha \max(k,l)} \|\langle x \rangle^{-2} f\|_{L_x^2}. \quad (3.3.19)$$

Let us first consider the case when $k+3 \leq l$. Plancherel's formula yields

$$\|\Delta_K(V\Delta_L \langle x \rangle^2 f)\|_{L^2(\mathbb{R}^d)} = \sup_{\|g\|_{L^2(\mathbb{R}^d)} \leq 1} \left| \iint_{\mathbb{R}^d \times \mathbb{R}^d} \varphi(K^{-1}\xi) \varphi(L^{-1}\eta) \widehat{V}(\xi - \eta) \langle \nabla_\eta \rangle^2 \widehat{f}(\eta) \overline{\widehat{g}(\xi)} d\xi d\eta \right|,$$

where we used that $\mathcal{F}_{x \rightarrow \eta} \langle x \rangle^2 f \sim \langle \nabla_\eta \rangle^2 \mathcal{F}_{x \rightarrow \eta} f$. Integrating by parts⁴ with respect to η gives

$$\|\Delta_K(V\Delta_L \langle x \rangle^2 f)\|_{L^2(\mathbb{R}^d)} = \sup_{\|g\|_{L^2(\mathbb{R}^d)} \leq 1} \left| \iint_{\mathbb{R}^d \times \mathbb{R}^d} \varphi(K^{-1}\xi) \langle \nabla_\eta \rangle^2 (\varphi(L^{-1}\eta) \widehat{V}(\xi - \eta)) \widehat{f}(\eta) \overline{\widehat{g}(\xi)} d\xi d\eta \right|.$$

Since φ_K (resp. φ_L) is supported on $2^{k-1} \leq |\xi| \leq 2^{k+1}$ (resp. $2^{l-1} \leq |\eta| \leq 2^{l+1}$), we have that $2^{l-2} \leq |\xi - \eta|$ on the support of the integrand. Therefore, for all $0 \leq \delta$ we have that $1 \leq 2^{-\delta(l-2)} |\xi - \eta|^\delta$, and the right-hand-side of the above estimate is less than

$$\begin{aligned} &\lesssim 2^{-\delta(l-2)} \iint_{\mathbb{R}^d \times \mathbb{R}^d} |\varphi(K^{-1}\xi)| |\xi - \eta|^\delta \langle \nabla_\eta \rangle^2 (\widehat{\varphi}(L^{-1}\eta) \widehat{V}(\xi - \eta)) \widehat{f}(\eta) \overline{\widehat{g}(\xi)} |d\xi d\eta \\ &\lesssim 2^{-\delta(l-2)} \|\langle \nabla \rangle^\delta (\langle x \rangle^2 V)\|_{L_x^2} \|f\|_{L_x^2} \|g\|_{L_x^2} \|\varphi(K^{-1}\cdot)\|_{L_\xi^2} \\ &\leq C(V) 2^{-\delta(l-2) + \frac{d}{2}k} \|f\|_{L_x^2} \leq C(V) 2^{-\alpha l} \|f\|_{L_x^2}, \end{aligned}$$

by choosing $\delta \geq \frac{d}{2} + 3\alpha$, and by using Cauchy-Schwarz and Plancherel. In the case when $l+3 \leq k$, the same computations yield

$$\|\Delta_K(V\Delta_L \langle x \rangle^2 f)\|_{L^2(\mathbb{R}^d)} \leq C(V) 2^{-\delta(k-2) + \frac{d}{2}k} \|f\|_{L_x^2} \leq C(V) 2^{-\alpha k} \|f\|_{L_x^2},$$

provided that $\delta \geq \frac{3d}{2} + 3\alpha$. We finish the proof of estimate (3.3.18) by summing over L . \square

Proof of bilinear estimate for perturbed Schrödinger evolutions We now come to the proof of Proposition 3.3.3 itself. For now on, we use the notation

$$u(t) := e^{-itH} \widetilde{\Delta}_N u_0, \quad v(t) := e^{-itH} \widetilde{\Delta}_M v_0.$$

³ More precisely, our brutal computations give $C \lesssim \|\langle \nabla \rangle^{\frac{3d}{2} + 3\alpha} (\langle x \rangle^2 V)\|_{L_x^2}$, but they are far from being optimal.

⁴ We stress out that the operator $\langle \nabla_\eta \rangle^2 \sim 1 + \sum_{i=1}^d \partial_{\eta_i}^2$ is a local operator and has no effect on the frequency localization $\eta \sim L$.

Step 1: Reduction to the case where initial data are flat Fourier localized. First, we reduce Proposition 3.3.3 to the following proposition where the initial data are also localized by flat Fourier multipliers. In what follows, Δ_K is the projector around a frequency K , $\Delta_{\lesssim N}$ is the projector on frequencies below $2^5 N$ and $\Delta_{\sim M}$ is a fattened projector around frequency M . Given N, M two dyadic numbers, we write

$$u(t) := e^{-itH} \tilde{\Delta}_N u_0, \quad v(t) := e^{-itH} \tilde{\Delta}_M v_0,$$

where $\tilde{\Delta}_N, \tilde{\Delta}_M$ are the distorted Fourier multipliers around frequencies N, M as defined in (3.2.17).

Proposition 3.3.6. *There exists $C > 0$ such that for any time interval $I \subseteq \mathbb{R}$ and any dyadic integers N, M with $N \leq M$, we have*

$$\|uv\|_{L^2_{t,x}(I)} \leq \|\Delta_{\lesssim N} u \Delta_{\sim M} v\|_{L^2_{t,x}(I)} + \left(\sum_{K > 2^5 N} \|\Delta_K u \Delta_{\sim M} v\|_{L^2_{t,x}(I)}^2 \right)^{1/2} + N^{\frac{d-1}{2}} M^{-\frac{1}{2}} \gamma(I). \quad (3.3.20)$$

For any interval I and for any partition $\mathbb{R} = \bigcup_{\ell} I_{\ell}$, we have

$$\gamma(I) \leq C \|\tilde{\Delta}_N u_0\|_{L^2_x} \|\tilde{\Delta}_M v_0\|_{L^2_x}, \quad \left(\sum_{\ell} \gamma(I_{\ell})^2 \right)^{1/2} \leq C \|\tilde{\Delta}_N u_0\|_{L^2_x} \|\tilde{\Delta}_M v_0\|_{L^2_x}. \quad (3.3.21)$$

Proof. We introduce the fattened projector $\Delta_{\sim M}$, a flat Fourier multiplier $\tilde{\varphi}(M^{-1}H_0)$ by a smooth cutoff function $\tilde{\varphi}$ chosen such that $(1 - \tilde{\varphi})\varphi \equiv 0$. Under this assumption, we deduce from Lemma 3.3.4 that for all $\sigma \in \mathbb{R}$ and for all $\alpha \in \mathbb{N}$,

$$\|(1 - \Delta_{\sim M})\tilde{\Delta}_M\|_{\mathcal{B}(L^2(\mathbb{R}^d), \mathcal{H}^{\sigma}(\mathbb{R}^d))} \lesssim_{\alpha, \sigma} M^{-\alpha}. \quad (3.3.22)$$

Then, v is decomposed into $v = \Delta_{\sim M} v + (1 - \Delta_{\sim M})v$, and

$$uv = u\Delta_{\sim M} v + u(1 - \Delta_{\sim M})v. \quad (3.3.23)$$

We deduce from Cauchy-Schwarz, Sobolev embedding, Strichartz estimate and (3.3.22) that the second term on the right-hand side of (3.3.23) is negligible:

$$\begin{aligned} \|u(1 - \Delta_{\sim M})v\|_{L^2_{t,x}(I)} &\leq \|u\|_{L^4_{t,x}(I)} \|(1 - \Delta_{\sim M})v\|_{L^4_{t,x}(I)} \\ &\lesssim N^{\frac{d}{12}} \|u\|_{L^4_t L^3_x(I)} \|\langle \nabla \rangle^{\frac{d}{12}} (1 - \Delta_{\sim M})v\|_{L^4_t L^3_x(I)} \\ &\leq C_{\alpha} M^{-\alpha + \frac{d}{12}} N^{\frac{d}{12}} \|\tilde{\Delta}_N u_0\|_{L^2_x} \|\tilde{\Delta}_M v_0\|_{L^2_x}. \end{aligned}$$

Choosing $\alpha = \frac{d}{12} + \frac{1}{2}$, writing

$$\gamma(I) = (N^{\frac{d-1}{2}} M^{-\frac{1}{2}})^{-1} \|u(1 - \Delta_{\sim M})v\|_{L^2_{t,x}(I)},$$

and, by using sub-additivity, we easily obtain (3.3.21). Finally, (3.3.20) follows from a Littlewood-Paley decomposition of u in the first term on the right-hand side of (3.3.23). \square

Step 2: Replacing the perturbed evolution e^{-itH} by the free one $e^{it\Delta}$. To prove Proposition 3.3.6, it remains to estimate the first two terms on the right-hand side of (3.3.20). The idea is to use Bourgain's estimate (3.3.9) in the free case. To do so, we shall understand how to describe the perturbed evolution e^{-itH} in terms of $e^{it\Delta}$. For this purpose, we see V as a forcing term, and we follow the proof of the inhomogeneous bilinear estimate from [Col+08]. Before that, we prove the following lemma that we will use to estimate the Duhamel term that comes from the potential, seen as the forcing term in what follows.

Lemma 3.3.7. *For all $\alpha \in \mathbb{N}$, there exists $C = C(\alpha, V)$ such that for all dyadic integer K, N and all $u_0 \in L_x^2$ we have*

$$\left(\int_{\mathbb{R}} \|\Delta_K(V\tilde{\Delta}_N e^{-itH} u_0)\|_{L_x^2}^2 d\tau \right)^{1/2} \leq C \begin{cases} K^{-\alpha} \|\tilde{\Delta}_N u_0\|_{L_x^2} & \text{if } K \geq 2^5 N, \\ \|\tilde{\Delta}_N u_0\|_{L_x^2} & \text{if } K \lesssim N. \end{cases} \quad (3.3.24)$$

Proof. In the case when $K \lesssim N$, we just apply the local smoothing (3.2.7). Otherwise, we need to gain a negative power of K . To do so, we fix $K > 2^5 N$ and we do a Littlewood-Paley decomposition of $\tilde{\Delta}_N e^{-itH} u_0$:

$$\begin{aligned} \|\Delta_K(V\tilde{\Delta}_N e^{-itH} u_0)\|_{L_x^2}^2 &\sim \sum_{|l-k| \geq 3} \|\Delta_K(V\Delta_L \tilde{\Delta}_N e^{-itH} u_0)\|_{L_x^2}^2 \\ &\quad + \sum_{|l-k| \leq 2} \|\Delta_K(V\Delta_L \tilde{\Delta}_N e^{-itH} u_0)\|_{L_x^2}^2 = A + B. \end{aligned}$$

Estimate of term A: We use Lemma 3.3.5 applied with $f = \tilde{\Delta}_N e^{-itH} u_0$ to get

$$\sum_{|l-k| \geq 3} \|\Delta_K(V\Delta_L e^{-itH} \tilde{\Delta}_N e^{-itH} u_0)\|_{L_x^2}^2 \leq C_\delta K^{-\alpha} \|\langle x \rangle^{-2} e^{-itH} \tilde{\Delta}_N u_0\|_{L_x^2}^2.$$

Then, we conclude by the local smoothing estimate (3.2.7) for e^{-itH} that

$$\begin{aligned} \int_{\mathbb{R}} \sum_{|l-k| \geq 3} \|\Delta_K(V\Delta_L e^{-itH} \tilde{\Delta}_N e^{-itH} u_0)\|_{L_x^2}^2 dt &\leq C_\alpha K^{-\alpha} \int_{\mathbb{R}} \|\langle x \rangle^{-2} e^{-itH} \tilde{\Delta}_N u_0\|_{L_x^2}^2 dt \\ &\leq C_\alpha K^{-\alpha} \|\tilde{\Delta}_N u_0\|_{L_x^2}^2. \end{aligned}$$

Estimate of term B: We use (3.3.17) and the endpoint Strichartz estimate for e^{-itH} , that is the Strichartz estimate 3.2.4 with the admissible pair $(2, \frac{2d}{d-2})$. Let us fix l such that $|k-l| \leq 2$ and write

$$\begin{aligned} \|\Delta_K(V\Delta_L \tilde{\Delta}_N e^{-itH} u_0)\|_{L_x^2} &= \sup_{\|g\|_{L_x^2} \leq 1} |(\Delta_K(V\Delta_L \tilde{\Delta}_N e^{-itH} u_0) | g)_{L_x^2}| \\ &= |(e^{-itH} \tilde{\Delta}_N u_0 | \tilde{\Delta}_N \Delta_L(V\Delta_K g))_{L_x^2}|. \end{aligned}$$

By Hölder and Sobolev embedding, we have

$$\begin{aligned} \|\Delta_K(V\Delta_L \tilde{\Delta}_N e^{-itH} u_0)\|_{L_x^2} &\leq \|e^{-itH} \tilde{\Delta}_N u_0\|_{L_x^{\frac{2d}{d-2}}} \|(\tilde{\Delta}_N \Delta_L)V\Delta_K g\|_{L_x^d} \\ &\leq \|e^{-itH} \tilde{\Delta}_N u_0\|_{L_x^{\frac{2d}{d-2}}} \|(\tilde{\Delta}_N \Delta_L)V\Delta_K g\|_{\mathcal{H}_x^{\frac{d-2}{2}}}. \end{aligned} \quad (3.3.25)$$

Since $|k - n| \geq 5$ and $|l - k| \leq 2$, we have $|n - l| \geq 3$. Therefore, we can apply (3.3.17), and get

$$\|(\tilde{\Delta}_N \Delta_L) V \Delta_K g\|_{\mathcal{H}^{\frac{d-2}{2}}} \lesssim_\alpha K^{-\alpha} \|V \Delta_K g\|_{L_x^2} \lesssim_\alpha K^{-\alpha} \|g\|_{L_x^2}.$$

Hence, we conclude from (3.3.25) that

$$\|\Delta_K (V \Delta_L \tilde{\Delta}_N e^{-itH} \tilde{\Delta}_N u_0)\|_{L_x^2} \lesssim_\alpha K^{-\alpha} \|g\|_{L_x^2} \|e^{-itH} \tilde{\Delta}_N u_0\|_{L_x^{\frac{2d}{d-2}}} \lesssim_\alpha K^{-\alpha} \|e^{-itH} \tilde{\Delta}_N u_0\|_{L_x^{\frac{2d}{d-2}}},$$

and the endpoint Strichartz estimate (3.2.4) for e^{-itH} yields

$$\int_{\mathbb{R}} \sum_{|k-l| \leq 2} \|\Delta_K (V \Delta_L \tilde{\Delta}_N e^{-itH} \tilde{\Delta}_N u_0)\|_{L_x^2}^2 \lesssim_\alpha K^{-\alpha} \int_{\mathbb{R}} \|e^{-itH} \tilde{\Delta}_N u_0\|_{L_x^{\frac{2d}{d-2}}}^2 dt \lesssim_\alpha K^{-\alpha} \|\tilde{\Delta}_N u_0\|_{L_x^2}^2.$$

This proves the estimate for term B , and finishes the proof of Lemma 3.3.7. \square

Let us fix an interval $I \subseteq \mathbb{R}$. Without loss of generality, we assume that $0 \in I$ and, given a dyadic integer K , we write Duhamel's formula as follows:

$$\begin{aligned} \Delta_K u(t) &= e^{it\Delta} \Delta_K \tilde{\Delta}_N u_0 - i \int_0^t e^{i(t-\tau)\Delta} \Delta_K (V u(\tau)) d\tau, \\ \Delta_{\sim M} v(t) &= e^{it\Delta} \Delta_{\sim M} \tilde{\Delta}_M v_0 - i \int_0^t e^{i(t-\tau)\Delta} \Delta_{\sim M} (V v(\tau)) d\tau. \end{aligned} \tag{3.3.26}$$

Then, we write

$$F(t) = \int_0^t e^{-i\tau\Delta} \Delta_K (V u(\tau)) d\tau, \quad G(t) = \int_0^t e^{-i\tau\Delta} \Delta_{\sim M} (V v(\tau)) d\tau.$$

Using decomposition (3.3.26) and developing the product, we obtain

$$\begin{aligned} \|\Delta_K u \Delta_{\sim M} v\|_{L_{t,x}^2(I)} &\leq \|e^{it\Delta} (\Delta_K \tilde{\Delta}_N u_0) e^{it\Delta} (\Delta_{\sim M} \tilde{\Delta}_M v_0)\|_{L_{t,x}^2(I)} \\ &\quad + \|e^{it\Delta} F(t) e^{it\Delta} (\Delta_{\sim M} \tilde{\Delta}_M v_0)\|_{L_{t,x}^2(I)} \\ &\quad + \|e^{it\Delta} (\Delta_K \tilde{\Delta}_N u_0) e^{it\Delta} G(t)\|_{L_{t,x}^2(I)} \\ &\quad + \|e^{it\Delta} F(t) e^{it\Delta} G(t)\|_{L_{t,x}^2(I)} \\ &= I_1 + I_2 + I_3 + I_4. \end{aligned} \tag{3.3.27}$$

Proof of Proposition 3.3.3. It remains to estimate each term in the above decomposition.

Estimate of term I_1 : We can directly use the bilinear estimate (3.3.9) for the free evolution to estimate I_1 , where we have the interaction of two free Schrödinger evolutions of frequency-localized data. For any interval I

$$\begin{aligned} &\|e^{it\Delta} (\Delta_K \tilde{\Delta}_N u_0) e^{it\Delta} (\Delta_{\sim M} \tilde{\Delta}_M v_0)\|_{L_{t,x}^2(I)} \\ &\leq \|e^{it\Delta} (\Delta_K \tilde{\Delta}_N u_0) e^{it\Delta} (\Delta_{\sim M} \tilde{\Delta}_M v_0)\|_{L_{t,x}^2(\mathbb{R})} \\ &\lesssim N^{\frac{d-1}{2}} M^{-\frac{1}{2}} \|\Delta_K \tilde{\Delta}_N u_0\|_{L_x^2} \|\tilde{\Delta}_M v_0\|_{L_x^2} \\ &\lesssim C(K, N) N^{\frac{d-1}{2}} M^{-\frac{1}{2}} \|\tilde{\Delta}_N u_0\|_{L_x^2} \|\tilde{\Delta}_M v_0\|_{L_x^2}, \end{aligned}$$

where we have from (3.3.17) that for any α , there exists C_α such that for all N, K

$$C(K, N) = \begin{cases} 1 & \text{if } K \lesssim N, \\ C_\alpha K^{-\alpha} & \text{if } K \geq 2^5 N. \end{cases} \quad (3.3.28)$$

Moreover, we can use sub-additivity to deduce that for any partition $\bigcup_\ell I_\ell = \mathbb{R}$,

$$\left(\sum_\ell \| e^{it\Delta} (\Delta_K \tilde{\Delta}_N u_0) e^{it\Delta} (\Delta_{\sim M} \tilde{\Delta}_M v_0) \|_{L^2_{t,x}(I_\ell)}^2 \right)^{1/2} \lesssim C(K, N) \| \tilde{\Delta}_N u_0 \|_{L^2_x} \| \tilde{\Delta}_M v_0 \|_{L^2_x}$$

as well. To estimate the others terms in (3.3.27), we shall use Lemma 3.3.5 and exploit the local smoothing effect to get some global in time estimates. For now on we fix an interval I of size $|I| \sim 1$, and see how to handle terms I_2, I_3 and I_4 .

Estimate of term I_2 : We use Minkowski and the bilinear estimate (3.3.9) for the free Schrödinger equation to get

$$\begin{aligned} \| e^{it\Delta} F(t) e^{it\Delta} (\Delta_{\sim M} \tilde{\Delta}_M v_0) \|_{L^2_{t,x}(I)} &= \| e^{it\Delta} \left(\int_0^t e^{-i\tau\Delta} \Delta_K (Vu(\tau)) d\tau \right) e^{it\Delta} (\Delta_{\sim M} \tilde{\Delta}_M v_0) \|_{L^2_{t,x}(I)} \\ &\leq \int_I \| e^{it\Delta} (e^{-i\tau\Delta} \Delta_K (Vu(\tau))) e^{it\Delta} (\Delta_{\sim M} \tilde{\Delta}_M v_0) \|_{L^2_{t,x}(I)} d\tau \\ &\lesssim K^{\frac{d-1}{2}} M^{-\frac{1}{2}} \int_I \| e^{-i\tau\Delta} \Delta_K (Vu(\tau)) \|_{L^2_x} \| \tilde{\Delta}_M v_0 \|_{L^2_x} d\tau \\ &\lesssim K^{\frac{d-1}{2}} M^{-\frac{1}{2}} \left(\int_I \| \Delta_K (V \tilde{\Delta}_N e^{-i\tau H} u_0) \|_{L^2_x}^2 d\tau \right)^{1/2} \| \tilde{\Delta}_M v_0 \|_{L^2_x}, \end{aligned}$$

where we used the assumption that $|I| \sim 1$. If $K \geq 2^5 N$, we apply Lemma 3.3.7 to have

$$\begin{aligned} K^{\frac{d-1}{2}} M^{-\frac{1}{2}} \left(\int_I \| \Delta_K (V \tilde{\Delta}_N e^{-i\tau H} u_0) \|_{L^2_x}^2 d\tau \right)^{1/2} &\lesssim K^{\frac{d-1}{2} - \alpha} M^{-\frac{1}{2}} \| \tilde{\Delta}_N u_0 \|_{L^2_x} \| \tilde{\Delta}_M v_0 \|_{L^2_x} \\ &\lesssim K^{-1} M^{-\frac{1}{2}} \| \tilde{\Delta}_N u_0 \|_{L^2_x} \| \tilde{\Delta}_M v_0 \|_{L^2_x}, \end{aligned}$$

by choosing $\alpha = \frac{d-1}{2} - 1$. Otherwise, $K \leq 2^5$ and we use Minkowski, we apply the bilinear estimate (3.3.9) and the local smoothing estimate (3.2.7) to get

$$\begin{aligned} \| e^{it\Delta} F(t) e^{it\Delta} (\Delta_{\sim M} \tilde{\Delta}_M v_0) \|_{L^2_{t,x}(I)} &\lesssim \int_I \| e^{it\Delta} (e^{-i\tau\Delta} \Delta_{\lesssim N} (Vu(\tau))) e^{it\Delta} (\Delta_{\sim M} \tilde{\Delta}_M v_0) \|_{L^2_{t,x}(I)} d\tau \\ &\lesssim N^{\frac{d-1}{2}} M^{-\frac{1}{2}} \left(\int_I \| V e^{-i\tau H} \tilde{\Delta}_N u_0 \|_{L^2_x}^2 d\tau \right)^{1/2} \| \tilde{\Delta}_M v_0 \|_{L^2_x} \\ &\lesssim N^{\frac{d-1}{2}} M^{-\frac{1}{2}} \| \tilde{\Delta}_N u_0 \|_{L^2_x} \| \tilde{\Delta}_M v_0 \|_{L^2_x}. \end{aligned}$$

Estimate of term I_3 : Similarly, we apply the bilinear estimate (3.3.9) and Minkowski to get

$$\begin{aligned} \| e^{it\Delta} (\Delta_K \tilde{\Delta}_N u_0) e^{it\Delta} G(t) \|_{L^2_{t,x}(I)} &= \| e^{it\Delta} (\Delta_K \tilde{\Delta}_N u_0) e^{it\Delta} \left(\int_0^t e^{-i\tau\Delta} \Delta_{\sim M} (Vv(\tau)) d\tau \right) \|_{L^2_{t,x}(I)} \\ &\lesssim K^{\frac{d-1}{2}} M^{-\frac{1}{2}} \| \Delta_K \tilde{\Delta}_N u_0 \|_{L^2_x} \int_I \| \Delta_{\sim M} (V e^{-i\tau H} \tilde{\Delta}_M v_0) \|_{L^2_x} d\tau. \end{aligned}$$

It follows from (3.3.17) that

$$K^{\frac{d-1}{2}} \|\Delta_K \tilde{\Delta}_N u_0\|_{L_x^2} \lesssim K^{-1} \|\tilde{\Delta}_N u_0\|_{L_x^2}.$$

Moreover, we deduce from the fact that $|I| \sim 1$ and from the local smoothing estimate (3.2.7) that

$$\int_I \|\Delta_{\sim M}(V e^{-i\tau H} \tilde{\Delta}_M v_0)\|_{L_x^2} d\tau \lesssim \left(\int_I \|V e^{-i\tau H} \tilde{\Delta}_M v_0\|_{L_x^2}^2 d\tau \right)^{1/2} \lesssim \|\tilde{\Delta}_M v_0\|_{L_x^2}.$$

Estimate of term I_4 : We use Minkowski and (3.3.9) to get

$$\begin{aligned} & \|e^{it\Delta} F(t) e^{it\Delta} G(t)\|_{L_{t,x}^2(I)} = \|e^{it\Delta} \left(\int_0^t e^{-i\tau\Delta} \Delta_K(Vu(\tau)) d\tau \right) e^{it\Delta} \left(\int_0^t e^{-i\tau\Delta} \Delta_{\sim M}(Vv(\tau)) d\tau \right)\|_{L^2(I)} \\ & \leq \iint_{I \times I} \|e^{it\Delta} (e^{-i\tau_1\Delta} \Delta_K(Vu(\tau_1))) e^{it\Delta} (e^{-i\tau_2\Delta} \Delta_{\sim M}(Vv(\tau_2)))\|_{L_{t,x}^2} d\tau_1 d\tau_2 \\ & \lesssim K^{\frac{d-1}{2}} M^{-\frac{1}{2}} \left(\int_I \|\Delta_K(V \tilde{\Delta}_N e^{-i\tau H} u_0)\|_{L_x^2} d\tau \right) \left(\int_I \|\Delta_{\sim M}(V \tilde{\Delta}_M e^{-i\tau H} v_0)\|_{L_x^2} d\tau \right). \end{aligned}$$

Then, we conclude as in case I_2 and I_3 , by using Lemma 3.3.5 and the local smoothing estimate (3.2.7). We complete the proof of Proposition 3.3.6 in the case when $|I| \sim 1$ by summing over K . To prove the global estimate when $I = \mathbb{R}$, we consider a partition $\bigcup_\ell I_\ell$ made of intervals of size $|I_\ell| \sim 1$. Then, we exploit the sub-additivity property of the L^2 norm, that is

$$\|f\|_{L_t^2(\mathbb{R})} \sim \left(\sum_\ell \|f\|_{L_t^2(I_\ell)}^2 \right)^{1/2}$$

for a given function $f \in L^2(\mathbb{R})$. Moreover, when estimating terms I_2, I_3 and I_4 , we use that the estimate (3.3.24) from Lemma 3.3.7 is global in time. This is a consequence of the local smoothing effect. \square

3.4 Probabilistic scattering on the continuous spectral subspace

The probabilistic nonlinear a priori estimates and other partial results, such as the critical-weighted strategy presented in this section will be reused in the next when proving Theorem 3.1.1. Here, we chose to state and to prove Theorem 3.1.5 from a pedagogical perspective before addressing the proof of the main theorem. We recall that we search for a solution of (NLS_c), which is (NLS) projected on the continuous spectral subspace, under the form $\psi = \varepsilon u^\omega + v$ where $u^\omega = e^{-itH} u_0^\omega$ and where v , that lies in a critical space embedded into $L^\infty(\mathbb{R}; \mathcal{H}^{s_c}(\mathbb{R}^d))$, is solution to the cubic NLS equation with a stochastic forcing term

$$\begin{cases} i\partial_t v - H v = P_c \mathcal{N}(\varepsilon u^\omega + v), \\ v|_{t=0} = \psi_0 \in \mathcal{H}^{s_c}(\mathbb{R}^d). \end{cases} \quad (3.4.1)$$

For the sake of completeness, we first recall definitions and essential properties of *critical spaces* introduced by [HHK09; HTT11]. See also ([Gri13], p. 49-67) for an expository presentation. These

spaces of functions from an interval $I \subseteq \mathbb{R}$ to a Hilbert space \mathcal{H} are constructed upon V^q, U^p spaces. Roughly speaking, they can be seen as the extensions of *Bourgain spaces* $X^{s,b}$ for $b = 1/2$, they embed into $L^\infty(\mathbb{R}, \mathcal{H}^s)$ and are well-behaved with respect to sharp cutoff functions in time. Furthermore, they are well suited for global-in-time a priori estimates thanks to the *duality argument* detailed in section 3.4.1, while Bourgain spaces are rather used for local-in-time estimates.

3.4.1 Critical spaces

For now on, the Hilbert space \mathcal{H} is $L^2(\mathbb{R}^d)$ unless otherwise specified. We fix a real number $1 \leq q < +\infty$ and an interval $I = (a, b)$ with $-\infty \leq a < b \leq +\infty$ and we denote the collection of finite partitions of I by \mathcal{Z} :

$$\mathcal{Z} := \left\{ \{t_k\}_{k=0 \dots K} \mid a = t_0 < t_1 < \dots < t_K = b \right\}.$$

Definitions and embeddings

Definition 3.4.1 (Functions of bounded q -variation). $V^q(I)$ is the set of functions $v : I \rightarrow \mathcal{H}$ endowed with the norm

$$\|v\|_{V^q(I)} := \sup_{\{t_k\}_{k=0}^{K-1} \in \mathcal{Z}} \left(\sum_{k=1}^K \|v(t_k) - v(t_{k-1})\|_{\mathcal{H}}^q \right)^{1/q}.$$

Functions in V^q have one-sided limits everywhere, and they may have at most countably many discontinuities. In what follows, we consider the closed subspace V_{rc}^q made of right-continuous functions v in V^q with $\lim_{t \rightarrow -\infty} v(t) = 0$. We still write them $V^q = V_{rc}^q$. Next we introduce the predual space of V^q (in a sense detailed in paragraph 3.4.1), namely the atomic space U^p with $\frac{1}{p} + \frac{1}{q} = 1$.

Definition 3.4.2 (Atomic space U^p). A function $a : I \rightarrow \mathcal{H}$ is a p -atom if there exists a partition $\{t_k\}_{k=0 \dots K}$ in \mathcal{Z} and $\{\phi_k\}_{k=0, \dots, K-1}$ some elements in \mathcal{H} such that

$$a(t) = \sum_{k=1}^K \mathbf{1}_{[t_{k-1}, t_k)}(t) \phi_{k-1}, \quad \sum_{k=0}^{K-1} \|\phi_k\|_{\mathcal{H}}^p \leq 1.$$

The atomic space $U^p(I)$ is the set of functions $u : I \rightarrow \mathcal{H}$ endowed with the norm

$$\|u\|_{U^p(I)} := \inf \left\{ \|(\lambda_j)\|_{\ell^1} \mid u = \sum_{j \geq 1} \lambda_j a_j \quad \text{for some } U^p\text{-atoms } (a_j) \right\}. \quad (3.4.2)$$

Functions in U^p are right-continuous, they admit left limits everywhere, and they may have at most countably many discontinuities. As we shall see in the next paragraph, U^p is the predual space of V^q when $\frac{1}{p} + \frac{1}{q} = 1$.

Proposition 3.4.3 (Embeddings, [HHK09] Proposition 2.2 and Corollary 2.6).

For $1 \leq p < q < \infty$,

$$U^p \hookrightarrow V^p \hookrightarrow U^q \hookrightarrow L_t^\infty(I, \mathcal{H}). \quad (3.4.3)$$

Let us take from [BOP15a] the continuity property of these norms that is particularly crucial when one wants to use some bootstrap argument.

Lemma 3.4.4 (Time continuity, see Lemma A.6 in [BOP15a]). *Let $J = [a, b)$, $u \in U^p(J) \cap C(J; H_x^s(\mathbb{R}^3))$ and $v \in V^q(J) \cap C(J; H_x^s(\mathbb{R}^3))$. The following mappings are continuous*

$$t \in J \rightarrow \|u\|_{U^p[a,t]}, \quad t \in J \rightarrow \|v\|_{V^q[a,t]}.$$

Duality argument There exists a unique bilinear map $B : U^p \times V^q \rightarrow \mathbb{C}$ such that

$$B(u, v) = \sum_{i=1}^K (u(t_i) - u(t_{i-1}) \mid v(t_i))$$

when u is a right-continuous step function with associated partition $\{t_k\}_{k=0}^K$ such that $u(t_0) := u(a) = 0$ and when v is a function in V^q . We have

$$|B(u, v)| \lesssim \|u\|_{U^p} \|v\|_{V^q}.$$

Moreover,

$$v \in V^q \mapsto B(\cdot, v) \in (U^p)^*$$

is a surjective isometry, and

$$\|v\|_{V^q} = \sup_{u \in U^p, \|u\|_{U^p} \leq 1} |B(u, v)|, \quad \|u\|_{U^p} = \sup_{v \in V^q, \|v\|_{V^q} \leq 1} |B(u, v)|. \quad (3.4.4)$$

The second estimate follows from Hahn-Banach theorem. Furthermore, if $\partial_t u \in L^1(I)$ we have the explicit formula

$$B(u, v) = \int_I (\partial_t u \mid v) dt. \quad (3.4.5)$$

We refer to [HHK09] and [HTT11] for proofs and details.

Definition 3.4.5 (Function spaces adapted to linear propagators). (i) We define by

$$U_H^p(I) := e^{-itH} (U^p(I) \cap L^\infty(I; \text{Ran } P_c)), \quad V_H^q(I) := e^{-itH} (V^q(I) \cap L^\infty(I; \text{Ran } P_c))$$

the critical spaces adapted to the perturbed linear propagator. They are Banach spaces when endowed with norms

$$\|u\|_{U_H^p} := \|e^{itH} u\|_{U^p}, \quad \|v\|_{V_H^q} := \|e^{itH} v\|_{V^q}.$$

We have similar definitions for the space U_Δ^p and V_Δ^q adapted to the free evolution $e^{it\Delta}$.

(ii) The spaces DU^p and DU_H^p are defined by

$$DU^p = \{f = \partial_t u \mid u \in U^p, f(a^+) = 0\}, \quad DU_H^p = e^{-itH} \{f = e^{-itH} \partial_t u \mid u \in U^p, f(a^+) = 0\}.$$

These spaces are endowed with norms

$$\|f\|_{DU^p} = \|u\|_{U^p}, \quad \|f\|_{DU_H^p} = \|e^{itH} u\|_{U^p}.$$

Spaces DV^q and DV_H^q are defined in the same way.

Remark 3.4.6. Here, ∂_t has to be understood in the sense of distributions on I . Let us recall that for distributions in one dimension, $\partial_t u = 0$ implies that u is constant, and thus we note that there is no ambiguity in the above definition, since we imposed the condition $\lim_{t \rightarrow a} f = 0$.

The definition of DU_H^p and DV_H^q are motivated by the need to control the Duhamel integral representation of a solution u to a forced Schrödinger equation. Namely, given f in $L_t^1(I, \mathcal{H})$ and u such that

$$\begin{cases} i\partial_t u - Hu = f, \\ u|_{t=0} = u_0, \end{cases}$$

the Duhamel's formulation reads

$$u(t) = e^{-itH} u_0 - i \int_0^t e^{-i(t-\tau)H} f(\tau) d\tau.$$

Consequently, we get

$$\|u\|_{U_H^p} \leq \|u_0\|_{\mathcal{H}} + \|f\|_{DU_H^p} = \|u_0\|_{\mathcal{H}} + \left\| \int_0^t e^{i\tau H} f(\tau) d\tau \right\|_{U^p}.$$

Furthermore, it follows from the duality argument (3.4.4) that

$$\|f\|_{DU_H^p} = \left\| e^{-itH} \int_0^t e^{i\tau H} f(\tau) d\tau \right\|_{U_H^2} = \sup_{\|v\|_{V^q} \leq 1} \left| \mathbb{B} \left(\int_0^t e^{i\tau H} f(\tau) d\tau, v \right) \right|.$$

Since f is in $L_t^1(I, \mathcal{H})$ we apply the explicit formula for \mathbb{B} (3.4.5) to get that

$$\|f\|_{DU_H^p} = \sup_{\|v\|_{V_H^q} \leq 1} \left| \int_I (f | v) d\tau \right|. \quad (3.4.6)$$

The corresponding formula for DV_H^q can also be deduced from (3.4.4):

$$\|f\|_{DV_H^q} = \sup_{\|u\|_{U_H^p} \leq 1} \left| \int_I (f | u) d\tau \right|. \quad (3.4.7)$$

Transferred linear and bilinear estimates

Proposition 3.4.7. *For $q > 2$ and r with $\frac{2}{q} + \frac{d}{r} = \frac{d}{2}$ and $2(d+2) \leq dp$ we have*

(i) *Linear Strichartz estimates:*

$$\|v\|_{L_t^q(\mathbb{R}; L_x^r(\mathbb{R}^d))} \lesssim \|v\|_{V_H^2}, \quad (3.4.8)$$

$$\|v\|_{L_{t,x}^p(\mathbb{R} \times \mathbb{R}^d)} \lesssim \|\langle \sqrt{H} \rangle^{\frac{d}{2} - \frac{d+2}{p}} v\|_{V_H^2}. \quad (3.4.9)$$

(ii) *Bilinear Strichartz estimate: Let N, M be two dyadic numbers,*

$$\|\tilde{\Delta}_N u \tilde{\Delta}_M v\|_{L_{t,x}^2(\mathbb{R} \times \mathbb{R}^d)} \lesssim N^{\frac{d-2}{2}} \left(\frac{N}{M} \right)^{\frac{1}{2}-} \|\tilde{\Delta}_N u\|_{V_H^2} \|\tilde{\Delta}_M v\|_{V_H^2}. \quad (3.4.10)$$

Proof. To prove the embedding (3.4.8), we first deduce from the global-in-time Strichartz estimate (3.2.4) that for any Schrödinger admissible pair (p, q) the mixed Lebesgue space $L_t^q L_x^r$ embeds into U^q , which embeds into V^2 provided that $2 < q$ (see Proposition 3.4.3). Estimate (3.4.9) follows from (3.4.8) and from the Bernstein estimate (3.2.18). We refer to Lemma 3.3 [BOP15a] and the references therein to see how to transfer the bilinear estimate (3.3.10) into the space V_H^2 . \square

In order to derive probabilistic nonlinear estimates, we consider the Duhamel integral representation of the solution to (NLS_c). Provided that the nonlinear forcing term $P_c \mathcal{N}(v + \varepsilon u^\omega)$ lies in $L^1(\mathbb{R}, \mathcal{H}^{\frac{d-2}{2}})$, it follows from the *duality argument* (3.4.6) that the U_H^2 -norm of the solution u to (3.4.1) can be estimated by

$$\|v\|_{U_H^2(\mathcal{H}^{\frac{d-2}{2}})} \leq \|\psi_0\|_{\mathcal{H}^{1/2}} \sup_{\|w\|_{V_H^2} \leq 1} \iint_{\mathbb{R} \times \mathbb{R}^d} \langle \sqrt{H} \rangle^{\frac{d-2}{2}} (P_c \mathcal{N}(v + \varepsilon u^\omega)) w dt dx.$$

If we proceed to a Littlewood-Paley decomposition of each term u_j that occurs in the forcing term, we are reduced to estimate sums of multilinear integrals of the form

$$I_{N_1, N_2, N_3, N_4} = \iint_{\mathbb{R} \times \mathbb{R}^d} \langle \sqrt{H} \rangle^{\frac{d-2}{2}} (\tilde{\Delta}_{N_1} u_1 \tilde{\Delta}_{N_2} u_2 \tilde{\Delta}_{N_3} u_3) \tilde{\Delta}_{N_4} w dx dt, \quad (3.4.11)$$

where each term u_j can be either u^ω, v or their complex conjugate for $j \in \{1, 2, 3\}$, and w is in V_H^2 with norm less than one. Although $P_c \mathcal{N}(v + \varepsilon u^\omega)$ does not a priori lie in $L^1(\mathbb{R}, \mathcal{H}^{\frac{d-2}{2}})$, it follows from Bernstein and Strichartz estimates that we have

$$\tilde{\Pi}_N P_c \mathcal{N}(v + \varepsilon u^\omega) \in L^1(I, \mathcal{H}^{\frac{d-2}{2}}), \quad (3.4.12)$$

for any dyadic number N as observed in [BOP15a]. It turns out that the estimates on multilinear integrals obtained by the authors are uniform in N . Hence, we have to estimate $\tilde{\Pi}_N v$ rather than v , and to use a space slightly smaller than U_H^2 which keeps track of the frequency cut-off.

Definition 3.4.8. The critical space $X^{\frac{d-2}{2}}(I)$ is the space made of tempered distributions $u : I \rightarrow \mathcal{H}^{\frac{d-2}{2}}(\mathbb{R}^d) \cap \text{Ran}(P_c)$ such that $\|u\|_{X^{\frac{d-2}{2}}(I)}$ is finite, where

$$\|u\|_{X^{\frac{d-2}{2}}(I)} := \left(\sum_{N \in 2^{\mathbb{N}}} N^{d-2} \|\tilde{\Delta}_N u\|_{U_H^2(I)}^2 \right)^{1/2}. \quad (3.4.13)$$

We define as well $DX^{\frac{d-2}{2}}(I)$ by replacing in (3.5.11) the $U_H^2(I)$ -norm by the $DU_H^2(I)$ one.

Remark 3.4.9. In the case of V_H^2 it follows from Definition 3.4.1 and from the almost orthogonality property of Littlewood-Paley decomposition that

$$\|\langle \sqrt{H} \rangle^{\frac{d-2}{2}} v\|_{V_H^2(I)} \sim \left(\sum_{N \in 2^{\mathbb{N}}} N^{d-2} \|\tilde{\Delta}_N v\|_{V_H^2(I)}^2 \right)^{1/2}. \quad (3.4.14)$$

3.4.2 Probabilistic nonlinear estimates

Thanks to the probabilistic and bilinear improved Strichartz estimates for the perturbed operator obtained in section 3.3 and their transferred versions collected in Proposition 3.4.7 we are now able to reproduce in the perturbed framework the same analysis conducted by [BOP15a]. We get the following probabilistic nonlinear estimates.

Proposition 3.4.10. *Let $0 < \varepsilon$, $0 < R$ and $I = (a, b)$ with $-\infty \leq a < b \leq +\infty$. For all $v \in X^{\frac{d-2}{2}}(I)$ and $\omega \in \widetilde{\Omega}_{\varepsilon, R}$ (see (3.3.8)) we have*

$$\sup_{2^{\mathbb{N}}} N^{\frac{d-2}{2}} \|\widetilde{\Pi}_N(\mathcal{N}(v + \varepsilon u^\omega))\|_{DU_H^2(I)} \lesssim \|\psi_0\|_{\mathcal{H}^{1/2}} + \|v\|_{V_H^2(I)}^3 + R^3, \quad (3.4.15)$$

as well as the Lipschitz estimate: for all $v, v' \in V_H^2(I)$,

$$\sup_{2^{\mathbb{N}}} N^{\frac{d-2}{2}} \|\widetilde{\Pi}_N(\mathcal{N}(v' + \varepsilon u^\omega) - \mathcal{N}(v + \varepsilon u^\omega))\|_{DU_H^2(I)} \lesssim (\|v\|_{V_H^2(I)}^2 + \|v'\|_{V_H^2(I)}^2 + R^2) \|v' - v\|_{V_H^2(I)}.$$

Contribution of terms like $v^2 u^\omega$ with high-low interactions: We illustrate the case by case analysis performed in [BOP15a] for the flat Laplacian, and we transpose it to the perturbed setting. In order to be as brief as possible, we chose to detail one enlightening case. We consider the high-high-low regime, where

$$v_1 = \widetilde{\Delta}_{N_1} v, \quad v_2 = \widetilde{\Delta}_{N_2} v, \quad u_3^\omega = \widetilde{\Delta}_{N_3} u^\omega, \quad w_4 = \widetilde{\Delta}_{N_4} w, \quad \text{and } N_1 \leq N_3^{\frac{1}{d-1}} \leq N_2 \leq N_3, N_4 \sim N. \quad ^5$$

We refer to ([BOP15a], Proposition 4.1) for the other subcases. As we explained in a previous paragraph, the proof of Proposition 3.4.10 consists in getting some a priori estimates on multilinear integrals of the form (3.4.11). Under the assumption made on the frequencies, the main term is the one where the derivatives fall onto $\widetilde{\Delta}_{N_3} v_3$. Hence, it follows from Hölder inequality that ⁶

$$\begin{aligned} \left| \iint_{\mathbb{R} \times \mathbb{R}^d} \langle \sqrt{H} \rangle^{\frac{d-2}{2}} (v_1 v_2 \varepsilon u_3^\omega) \widetilde{\Delta}_{N_4} w dx dt \right| &\lesssim N_3^{\frac{d-2}{2}} \|v_2\|_{L_{t,x}^{\frac{2(d+2)}{d}}} \|\varepsilon u_3^\omega\|_{L_{t,x}^{d+2}} \|v_1 w_4\|_{L_{t,x}^2} \\ &\lesssim N_3^{\frac{d-2}{2}} N_1^{\frac{d-1}{2}-} N_4^{-\frac{1}{2}+} \|v_2\|_{V_H^2} \|\varepsilon u_3^\omega\|_{L_{t,x}^{d+2}} \|v_1\|_{V_H^2} \|w_4\|_{V_H^2} \\ &\lesssim C(N_1, N_2, N_3, N_4) (N_2^{\frac{d-2}{2}} \|v_2\|_{V_H^2}) (N_3^s \|\varepsilon u_3^\omega\|_{L_{t,x}^{d+2}}) (N_1^{\frac{d-2}{2}} \|v_1\|_{V_H^2}) \|w_4\|_{V_H^2}, \end{aligned}$$

with

$$C(N_1, N_2, N_3, N_4) = N_3^{\frac{d-2}{2}-s} N_1^{\frac{d-1}{2}-0-\frac{d-2}{2}} N_2^{-\frac{d-2}{2}} N_4^{-\frac{1}{2}+0} \lesssim N_3^{\frac{d-2}{2}-s+\frac{1}{2(d-1)}-\frac{d-2}{2(d-1)}-\frac{1}{2}+0} =: N_3^{\delta-s}.$$

We used the transferred Strichartz estimate (3.4.8) to control the term v_1 and the transferred bilinear estimate (3.4.10) to control the term with $v_2 w_4$. Note that under the assumption that $\frac{d-1}{d+1} \cdot \frac{d-2}{2} < s$, we have

$$\delta = \frac{d-3}{d-1} \cdot \frac{d-2}{2} + 0 < \frac{d-1}{d+1} \cdot \frac{d-2}{2} < s.$$

⁵For convenience, we define $\Lambda_N = \{(N_1, N_2, N_3, N_4) \mid N_1 \leq N_3^{\frac{1}{d-1}} \leq N_2 \leq N_3, N_4 \sim N\}$.

⁶For a dyadic number N we shall denote small positive (resp. negative) power of N by N^{+0} (resp. N^{-0}).

Therefore, since the highest frequency comes with a negative power we are able to apply *Schur's test* and to sum over N_1, N_2, N_4 . To sum over the dyadic numbers N_3 , we use first Hölder

$$\begin{aligned} \sum_{N_3 \in 2^{\mathbb{N}}} N_3^{\delta-s} N_3^s \|\tilde{\Delta}_{N_3} \varepsilon u^\omega\|_{L_{t,x}^{d+2}} &\leq N^{\delta-s+0} \left(\sum_{N_3 \in 2^{\mathbb{N}}} N_3^{-0} \right)^{\frac{d+1}{d+2}} \|N_3^s \tilde{\Delta}_{N_3} \varepsilon u^\omega\|_{\ell_{N_3}^{d+2} L_{t,x}^{d+2}} \\ &\lesssim N^{\delta-s+0} \|N_3^s \tilde{\Delta}_{N_3} \varepsilon u^\omega\|_{L_t^{d+2} \ell_{N_3}^{d+2} L_x^{d+2}} \lesssim N^{\delta-s+0} \|N_3^s \tilde{\Delta}_{N_3} \varepsilon u^\omega\|_{L_t^{d+2} \ell_{N_3}^2 L_x^{d+2}}, \end{aligned}$$

and we deduce from Littlewood-Paley's inequality (3.2.19) and probabilistic Strichartz estimate (3.3.6) that for $\omega \in \tilde{\Omega}_{\varepsilon,R}$,

$$\sum_{N_3 \in 2^{\mathbb{N}}} N_3^{\delta-s} N_3^s \|\tilde{\Delta}_{N_3} \varepsilon u^\omega\|_{L_{t,x}^{d+2}} \lesssim N^{\delta-s+0} \|\langle \sqrt{H} \rangle^{\delta+} \varepsilon u^\omega\|_{L_{t,x}^{d+2}} \lesssim N^{\delta-s+0} R. \quad (3.4.16)$$

Hence,

$$\sum_{N_3} \iint_{\mathbb{R} \times \mathbb{R}^d} |\langle \sqrt{H} \rangle^{\frac{d-2}{2}} (v_1 v_2 \varepsilon u_3^\omega) w_4| dx dt \lesssim N^{\delta-s+0} R (N_1^{\frac{d-2}{2}} \|v_1\|_{V_H^2}) (N_2^{\frac{d-2}{2}} \|v_2\|_{V_H^2}) \|w_4\|_{V_H^4}.$$

It follows from the *Schur's test* (see [BOP15a], Lemma 3.6) that

$$\sum_{\Lambda_N} \iint_{\mathbb{R} \times \mathbb{R}^d} |\langle \sqrt{H} \rangle^{\frac{d-2}{2}} (v_1 v_2 \varepsilon u_3^\omega) w_4| dx dt \lesssim R \|N_1^{\frac{d-2}{2}} v_1\|_{\ell_{N_1}^2 V_H^2} \|N_2^{\frac{d-2}{2}} v_2\|_{\ell_{N_2}^2 V_H^2} \|w_4\|_{\ell_{N_4}^2 V_H^2},$$

and the contribution of the integral (3.4.11) in this case is less than

$$\sup_{\|w\|_{V_H^2} \leq 1} R \|\langle \sqrt{H} \rangle^{\frac{d-2}{2}} v\|_{V_H^2}^2 \|w\|_{V_H^2} \lesssim R \|\langle \sqrt{H} \rangle^{\frac{d-2}{2}} v\|_{V_H^2}^2.$$

Remark 3.4.11. As observed in [BOP15a], there is at least a portion of $\|\langle \sqrt{H} \rangle^s \varepsilon u^\omega\|_{L_{t,x}^p}$ that bounds from above the multilinear integrals where u^ω occurs, with $p \in \{4, d+2, 6\frac{d+2}{d+4}\}$ (see for instance (3.4.16) where $p = d+2$). Hence, it follows from the probabilistic Strichartz estimate (3.3.6) and from the monotone convergence theorem that for all $\omega \in \tilde{\Omega}$,

$$\lim_{|I| \rightarrow 0} \|\varepsilon u^\omega\|_{L_{t,x}^p(I \times \mathbb{R}^d)} = 0.$$

In the $vv\bar{v}$ -case, we also have

$$\sup_{\|w\|_{V_H^2} \leq 1} \left| \int_I \int_{\mathbb{R}^d} \mathbf{1}_I(t) \langle \sqrt{H} \rangle^{\frac{d-2}{2}} v v^2 w dx dt \right| \leq \|\langle \sqrt{H} \rangle^{\frac{d-2}{2}} v\|_{L_{t,x}^{\frac{2(d+2)}{d}}(I \times \mathbb{R}^d)} \|v\|_{L_{t,x}^{d+2}(I \times \mathbb{R}^d)}^2, \quad (3.4.17)$$

where the right-hand side of (3.4.17) is uniformly bounded by the global-in-time critical $X^{\frac{d-2}{2}}(\mathbb{R})$ -norm. Hence, the left-hand side of (3.4.17) also goes to zero when $|I| \rightarrow 0$ (in order to prove continuity or local well-posedness) or when $I \subset [t, \infty)$ with $t \rightarrow +\infty$ (in order to prove scattering). Consequently, we have in case that

$$\lim_I \|\mathcal{N}(v + \varepsilon u^\omega)\|_{DU_H^2(I)} = 0 \quad \text{whenever } |I| \rightarrow 0 \text{ or } I \subset [t, \infty) \text{ with } t \rightarrow +\infty. \quad (3.4.18)$$

3.4.3 Proof of Theorem 3.1.5

We are now ready to extend the probabilistic global Cauchy theory for (NLS) to the case where we have a short-range potential, and when the equation is projected onto the continuous spectral subspace.

Global existence We denote the interval $[0, \infty)$ by I and we introduce the Banach space

$$B_R = \left\{ v \in X^{\frac{d-2}{2}}(I) \cap C(I, \mathcal{H}^{\frac{d-2}{2}}) \mid \|v\|_{X^{\frac{d-2}{2}}(0, \infty)} \leq R \right\}$$

for a fixed small $R > 0$, as well as the operator

$$\mathcal{T} : v \in B_R \mapsto \left(t \mapsto e^{-itH} \psi_0 - i \int_0^t e^{-i(t-\tau)H} \mathbf{P}_c(\mathcal{N}(v + \varepsilon u^\omega)) d\tau \right).$$

It follows from the a priori estimates collected in Proposition 3.4.10 that for all v in B_R ,

$$\|\mathcal{T}v\|_{X^{\frac{d-2}{2}}(0, \infty)} \leq \|e^{-itH} \psi_0\|_{X^{\frac{d-2}{2}}(0, \infty)} + \|\mathbf{P}_c(\mathcal{N}(v + \varepsilon u^\omega))\|_{DX^{\frac{d-2}{2}}(0, \infty)} \leq \|\psi_0\|_{\mathcal{H}^{\frac{d-2}{2}}} + 2C_1 R^3,$$

and that for all v, v' in B_R ,

$$\|\mathcal{T}(v') - \mathcal{T}(v)\|_{X^{\frac{d-2}{2}}(0, \infty)} \leq \|\mathbf{P}_c(\mathcal{N}(v' + \varepsilon u^\omega) - \mathcal{N}(v + \varepsilon u^\omega))\|_{DX^{\frac{d-2}{2}}(0, \infty)} \leq 3C_2 R^2 \|v' - v\|_{X^{\frac{d-2}{2}}(0, \infty)}.$$

Hence, \mathcal{T} is a contraction mapping on B_R provided that

$$2C_1 R \leq 1/2, \quad 3C_2 R \leq 1/2 \quad \text{and} \quad \|\psi_0\|_{\mathcal{H}^{\frac{d-2}{2}}} \leq R/2.$$

With this fixed $R = R_0$, we write $\tilde{\Omega}_\varepsilon = \tilde{\Omega}_{\varepsilon, R_0}$ the set defined in (3.3.8). It satisfies

$$\mathbf{P}(\Omega \setminus \tilde{\Omega}_\varepsilon) \leq C \exp(-\tilde{c}\varepsilon^{-2} \|u_0\|_{\mathcal{H}^s}^{-2})$$

where $\tilde{c} = cR_0^{-2}$ is a universal constant.

Continuity and scattering : Let v be a solution to (NLS_c) with random initial data associated with an $\omega \in \tilde{\Omega}_\varepsilon$. We need to show that the limit of $\mathcal{I}(t)$ exists in $\mathcal{H}^{\frac{d-2}{2}}$ when t goes to infinity, where

$$\mathcal{I}(t) := \int_0^t e^{i\tau H} \mathbf{P}_c(\mathcal{N}(v + \varepsilon u^\omega)) d\tau.$$

We use the Cauchy criterion, and chose $t_1, t_2 \in (0, \infty)$ with, say, $t_1 < t_2$. We denote the interval (t_1, t_2) by I . It follows from (3.4.18) in remark 3.4.11 that

$$\|\mathcal{I}(t_2) - \mathcal{I}(t_1)\|_{\mathcal{H}^{\frac{d-2}{2}}} \leq \|\mathbf{P}_c(\mathcal{N}(v + \varepsilon u^\omega))\|_{X^{\frac{d-2}{2}}(I)} \xrightarrow{t_1, t_2 \rightarrow +\infty} 0.$$

This yields scattering with

$$v_+ := \int_0^\infty e^{i\tau H} \mathbf{P}_c(\mathcal{N}(v + \varepsilon u^\omega)) d\tau \in \mathcal{H}^{\frac{d-2}{2}}(\mathbb{R}^d),$$

and the same argument applied when $|I| \rightarrow 0$ proves that $\mathcal{T}v$ is in $C(I, \mathcal{H}^{\frac{d-2}{2}})$.

Uniqueness : We use a connectedness argument. Let v_1, v_2 be two solutions to (NLS_c) in the space $X^{\frac{d-2}{2}}[0, \infty)$. Define $A = \{t \in [0, \infty) \mid v_1(t) = v_2(t)\}$. By continuity in time, A is closed. Let $t \in A$ and, for small τ , we define $I = (t - \tau, t + \tau)$ if $t > 0$, or $I = [0, \tau)$ otherwise. It follows from remark 3.4.11 that

$$\|v_2 - v_1\|_{X^{\frac{d-2}{2}}(I)} = o(\tau) \|v_2 - v_1\|_{X^{\frac{d-2}{2}}(I)}.$$

Hence, $v_2 = v_1$ on I if τ is small enough, which proves that A is open. We conclude that $A = [0, \infty)$ whenever A is nonempty, and this finishes the proof of Theorem 3.1.5.

3.4.4 Local smoothing and the critical weighted strategy

First, we transfer in U_H^2 the local smoothing effect stated in Proposition 3.2.5

Proposition 3.4.12 (Local smoothing in U^2). *The local smoothing estimate reads in $U_H^2(\mathcal{H}^{1/2})$:*

$$\int_{\mathbb{R}} \|P_c(u)\|_{\mathcal{H}^{1,-1/2-}}^2 dt \lesssim_V \|u\|_{U_H^2(\mathcal{H}^{1/2})}^2. \quad (3.4.19)$$

The proof is straightforward and follows from the transference principle and from Proposition 3.2.5 with $F = 0$. Note in particular that the result is true when $V = 0$ and when the space U_H^2 is replaced by U_{Δ}^2 .

Remark 3.4.13. Estimate (3.4.19) means that $U_H^2(\mathcal{H}^{1/2})$ embeds into $L_t^2(\mathbb{R}; \mathcal{H}^{1,-1/2-}(\mathbb{R}^d))$, and we can deduce from this and from the duality argument (3.4.6) that

$$L_t^2(\mathbb{R}; L_x^{2,1/2+}) \hookrightarrow DV_H^2(\mathcal{H}^{1/2}). \quad (3.4.20)$$

Indeed, provided that $f \in L^1(\mathbb{R}, L_x^2)$, we have

$$\begin{aligned} \|\langle \sqrt{H} \rangle^{1/2} f\|_{DV_H^2} &= \sup_{\|u\|_{U_H^2} \leq 1} \left| \int_{\mathbb{R}} (\langle \sqrt{H} \rangle^{1/2} f \mid u) dt \right| = \sup_{\|u\|_{U_H^2} \leq 1} \left| \int_{\mathbb{R}} (f \mid \langle \sqrt{H} \rangle^{1/2} u) dt \right| \\ &\leq \sup_{\|u\|_{U_H^2} \leq 1} \|\langle x \rangle^{1/2+} f\|_{L_{t,x}^2} \|\langle x \rangle^{-1/2-} \langle \sqrt{H} \rangle (\langle \sqrt{H} \rangle^{-1/2} u)\|_{L_{t,x}^2} \\ &\leq \sup_{\|u\|_{U_H^2} \leq 1} \|\langle x \rangle^{1/2+} f\|_{L_{t,x}^2} \|u\|_{U_H^2} \lesssim \|f\|_{L_t^2(\mathbb{R}; L_x^{2,1/2+})}. \end{aligned}$$

The general case follows from the density of step functions in V^p (see the proof in [Gri13], Lemma 4 p.56). Furthermore, embedding (3.4.3) directly yields the other dual embedding

$$DU_H^2(\mathcal{H}^{1/2}) \hookrightarrow DV_H^2(\mathcal{H}^{1/2}). \quad (3.4.21)$$

Remark 3.4.14. In light of the nonlinear analysis performed in section 3.4.2, we shall use the slightly better estimate

$$\int_{\mathbb{R}} \|P_c u\|_{\mathcal{H}^{1,-1/2-}} dt \lesssim \sup_{2^{\mathbb{N}}} N^{1/2} \|\tilde{\Pi}_N u\|_{U_H^2}.$$

Let us denote the Schwartz class by $\mathcal{S}(\mathbb{R}^d)$ and the space of tempered distributions $\mathcal{S}'(\mathbb{R}^d)$, and deduce the above refined estimate from (3.4.19) and duality. Given $u \in \mathcal{H}^{1,-1/2-}$, we have

$$\|P_c u\|_{\mathcal{H}^{1,-1/2-}} = \sup_{\varphi \in \mathcal{S}, \|\varphi\|_{L^2} \leq 1} |(P_c u | \langle x \rangle^{-1/2-} \langle \sqrt{H} \rangle \varphi)|.$$

Since $\langle x \rangle^{-1/2-} \langle \sqrt{H} \rangle \varphi \in \mathcal{S}(\mathbb{R}^d)$ and $u = \lim_{N \rightarrow \infty} \tilde{\Pi}_N u$ in $\mathcal{S}'(\mathbb{R}^d)$, we have

$$\|P_c u\|_{\mathcal{H}^{1,-1/2-}} = \sup_{\varphi \in \mathcal{S}, \|\varphi\|_{L^2} \leq 1} \left| \left(\lim_{N \rightarrow \infty} \tilde{\Pi}_N u | \langle x \rangle^{-1/2-} \langle \sqrt{H} \rangle \varphi \right) \right| \leq \sup_{2^{\mathbb{N}}} \|\tilde{\Pi}_N u\|_{\mathcal{H}^{1,-1/2-}}.$$

Critical-weighted strategy The next Proposition details the *critical-weighted strategy*. As explained in the introduction, this strategy makes it possible to handle both linear and nonlinear terms that arise in the stability equation around a ground state. We state two versions of this strategy. The first one is suited to the operator $-\Delta$ and has an interest in itself. One can use it when Vu is not absorbed by the linear operator, but when it is seen instead as a source term. The second version is suited to the dynamic around the nonlinear ground states, when V is absorbed by the linear operator so that e^{-itH} preserves $\text{Ran } P_c$ and the decomposition of the phase space from Lemma 3.5.4. This second version plays a key role in the proof of Theorem 3.1.1. See also [KM12] for a similar approach in the context of Korteweg–de Vries equation.

Proposition 3.4.15. *Let u be a solution to the forced ⁷ Schrödinger equation*

$$\begin{cases} i\partial_t u + (\Delta - V)u &= f + g, \\ u|_{t=0} &= u_0. \end{cases} \quad (3.4.22)$$

There exists $C = C(V)$ depending on weighted Sobolev norms of V such that for all $u \in \text{Ran } P_c(H)$,

$$\|u\|_{V_{\Delta}^2(\mathcal{H}^{1/2})} + \|u\|_{L_t^2(\mathbb{R}; \mathcal{H}^{1,-1/2-})} \leq C(\|u_0\|_{\mathcal{H}^{1/2}} + \sup_{2^{\mathbb{N}}} N^{1/2} \|\Pi_N f\|_{DU_{\Delta}^2} + \|g\|_{L_t^2(\mathbb{R}; L_x^{2,1/2+})}), \quad (3.4.23)$$

$$\|u\|_{V_H^2(\mathcal{H}^{1/2})} + \|u\|_{L_t^2(\mathbb{R}; \mathcal{H}^{1,-1/2-})} \leq C(\|u_0\|_{\mathcal{H}^{1/2}} + \sup_{2^{\mathbb{N}}} N^{1/2} \|\tilde{\Pi}_N f\|_{DU_H^2} + \|g\|_{L_t^2(\mathbb{R}; L_x^{2,1/2+}))). \quad (3.4.24)$$

Remark 3.4.16. We prove the estimate with $\sup_{2^{\mathbb{N}}} \|\tilde{\Pi}_N f\|_{DU^2}$ instead of $\|f\|_{DU^2}$ in order to use (3.4.15).

Proof. Estimate (3.4.24) is slightly easier to prove since we don't have to control the term Vu which is absorbed by the left-hand side in (3.4.22). Also, the local smoothing estimates we use are the same for $-\Delta$ and $-\Delta + V$ provided that u in $\text{Ran}(P_c)$ so we only write the proof of (3.4.23). To that end, we pass the term Vu on the right-hand side of equation (3.4.22)

$$\begin{cases} i\partial_t u + \Delta u &= f + g + Vu, \\ u|_{t=0} &= u_0. \end{cases}$$

⁷ Here, f (resp. g) has to be thought of as a nonlinear term controlled in the critical space (resp. a linear and localized term controlled in a weighted L^2 space).

By definition of the space DV_{Δ}^2 , Duhamel's formulation gives

$$\begin{aligned} \|u\|_{V_{\Delta}^2(\mathcal{H}^{1/2})} &\lesssim \sup_{2^{\mathbb{N}}} \|\Pi_N f\|_{DV_{\Delta}^2(\mathcal{H}^{1/2})} + \|g\|_{DV_{\Delta}^2(\mathcal{H}^{1/2})} + \|Vu\|_{DV_{\Delta}^2(\mathcal{H}^{1/2})}, \\ &\lesssim \sup_{2^{\mathbb{N}}} N^{1/2} \|\Pi_N f\|_{DU_{\Delta}^2} + \|g\|_{L_t^2(\mathbb{R}; L_x^{2,1/2+})} + \|Vu\|_{L_t^2(\mathbb{R}; L_x^{2,1/2+})}. \end{aligned} \quad (3.4.25)$$

Note that we used the dual embedding (3.4.21) to control $\Pi_N f$ in DU_{Δ}^2 and the local smoothing dual embedding (3.4.20) to control g and Vu . As for the weighted norm of u , we decompose it into $u = v + w$ where $v, w \in \text{Ran } P_c(H)$ and are solution to

$$\begin{cases} i\partial_t v + \Delta v = f, & i\partial_t w + (\Delta - V)w = g - Vv, \\ v|_{t=0} = 0, & w|_{t=0} = u_0. \end{cases}$$

To control u in $L_t^2(\mathbb{R}; \mathcal{H}^{1,-1/2-}(\mathbb{R}^3))$ we first control the weighted norm of v and then the weighted norm of w . To do so, we apply the transferred local smoothing estimate (3.4.19) with $V = 0$.

$$\|v\|_{L_t^2(\mathbb{R}; \mathcal{H}^{1,-1/2-})} \leq \sup_{2^{\mathbb{N}}} \|\Pi_N v\|_{L_t^2(\mathbb{R}; \mathcal{H}^{1,-1/2-})} \lesssim \sup_{2^{\mathbb{N}}} N^{1/2} \|\Pi_N v\|_{U_{\Delta}^2} = \sup_{2^{\mathbb{N}}} N^{1/2} \|\Pi_N f\|_{DU_{\Delta}^2}. \quad (3.4.26)$$

Afterwards, we use the fact that $w \in P_c(H)$ and we apply the local smoothing estimate for H :

$$\|w\|_{L_t^2(\mathbb{R}; \mathcal{H}^{1,-1/2-})} \lesssim \|u_0\|_{\mathcal{H}^{1/2}} + \|g\|_{L_t^2(\mathbb{R}; L_x^{2,1/2+})} + \|Vv\|_{L_t^2(\mathbb{R}; L_x^{2,1/2+})}. \quad (3.4.27)$$

To estimate the second term in the right-hand side, we use the fact that V is localized.

$$\begin{aligned} \|Vv\|_{L_t^2(\mathbb{R}; L_x^{2,1/2+})} &\lesssim \|\langle x \rangle^{1+} V\|_{L_x^{\infty}(\mathbb{R}^d)} \|v\|_{L_t^2(\mathbb{R}; L_x^{2,-1/2-})} \\ &\leq C(V) \|v\|_{L_t^2(\mathbb{R}; \mathcal{H}^{1,-1/2-})} \leq C(V) \sup_{2^{\mathbb{N}}} N^{1/2} \|\Pi_N f\|_{DU_{\Delta}^2}, \end{aligned} \quad (3.4.28)$$

where we used (3.4.26) to get the last inequality. By collecting estimates (3.4.27) and (3.4.28) we obtain

$$\|w\|_{L_t^2(\mathbb{R}; \mathcal{H}^{1,-1/2-})} \lesssim_V \|u_0\|_{L_x^2} + \sup_{2^{\mathbb{N}}} 2^{\mathbb{N}} \|\Pi_N f\|_{DU_{\Delta}^2} + \|g\|_{L_t^2(\mathbb{R}; L_x^{2,1/2+})}. \quad (3.4.29)$$

Finally, we deduce from (3.4.26) and (3.4.29) the estimate for u in the weighted space.

$$\|u\|_{L_t^2(\mathbb{R}; \mathcal{H}^{1,-1/2-})} \lesssim \|u_0\|_{\mathcal{H}^{1/2}} + \sup_{2^{\mathbb{N}}} 2^{\mathbb{N}} \|\Pi_N f\|_{DU_{\Delta}^2} + \|g\|_{L_t^2(\mathbb{R}; L_x^{2,1/2+})}. \quad (3.4.30)$$

It remains to control Vu in $V_{\Delta}^2(\mathcal{H}^{1/2})$ to close the estimate (3.4.25). This is again a consequence of the fact that V is localized, and we proceed as in (3.4.28) to get

$$\|Vu\|_{L_t^2(\mathbb{R}; L_x^{2,1/2+})} \lesssim_V \|u\|_{L_t^2(\mathbb{R}; \mathcal{H}^{1,-1/2-})}. \quad (3.4.31)$$

We conclude by injecting (3.4.31) and (3.4.30) into (3.4.25) to control u in $V_{\Delta}^2(\mathcal{H}^{1/2})$. \square

3.5 Probabilistic asymptotic stability for small ground states

For now on, we fix the dimension $d = 3$ and we make the extra assumption that $\sigma(H) = \{e_0\} \cup \sigma_c(H)$ where $e_0 < 0$ is a simple negative eigenvalue with positive and normalized eigenfunction ϕ_0 . We recall that $\sigma_c(H) = [0, +\infty)$, with no resonance nor eigenvalue at zero.

Since complex conjugate and numerical constants play no role in what follows, we might sometimes drop them from the notation for the sake of clarity. We will also drop the dependence on time in the notation and write $\|D_z^\alpha Q\| := \sup_t \|D_z^\alpha Q(z(t))\|$, as well as $O(z) := O(\sup_t |z(t)|)$.

3.5.1 Local existence

First, let us briefly transpose the probabilistic Cauchy theory, at least locally in time, to our setting where the Schrödinger operator $-\Delta + V$ has some discrete spectrum. More precisely, we prove that under smallness assumptions on the $\mathcal{H}^{1/2}$ -norm of the initial data ψ_0 , (NLS) still admits a unique local solution under the form

$$\psi(t) = \varepsilon u^\omega(t) + v(t), \quad v \in \mathcal{H}^{1/2}.$$

Indeed, the discrete part of the solution does not contribute in short time, and we shall be able to reproduce the scheme developed in section 3.4, with the same gain of regularity for the nonlinear part of the solution. Note that by time reversibility, we only consider forward-in-time solutions in what follows.

Proposition 3.5.1 (Local existence). *There exist $\delta_0 > 0$ and $c > 0$ such that for all $T \lesssim 1$, all $\alpha_0 \in \mathbb{C}$, $\nu_0 \in \text{Ran } P_c \cap \mathcal{H}^{1/2}$ with*

$$\|v_0\|_{\mathcal{H}^{1/2}} + |\alpha_0| < \delta_0,$$

and all $\omega \in \tilde{\Omega}_{\varepsilon, R}$ with $R = c\delta_0$, the Cauchy problem (NLS) with data

$$\psi(t_0) = \varepsilon u_0^\omega + \nu_0 + \alpha_0 \phi_0$$

admits a unique solution ψ on $[t_0, t_0 + T)$ under the form

$$\psi(t) = \varepsilon u^\omega(t) + v(t), \quad v \in C([t_0, t_0 + T); \mathcal{H}^{1/2}).$$

Uniqueness holds for $P_c v \in X^{1/2}(0, T)$ and $(v | \phi_0) \in L_t^\infty(0, T)$.

We recall that the set of random initial data with improved Strichartz estimates defined in (3.3.8). In light of the above Proposition, we fix $R = c\delta_0$ for now on, and take an initial value u_0^ω that corresponds to some ω in $\tilde{\Omega}_{\varepsilon, c\delta_0}$. We also recall that ε is much smaller than R .

Proof. For simplicity, we assume that $t_0 = 0$. At each time t , we decompose $v(t)$ into $v(t) = \nu(t) + \alpha(t)\phi_0$, with $\nu(t) = P_c v(t)$, and $\alpha(t) = (v(t) | \phi_0)$. Then, ψ is solution to (NLS) if and only if (ν, α) are solution to the coupled system

$$\begin{cases} i\partial_t \nu + (\Delta - V)\nu = P_c \mathcal{N}(\varepsilon u^\omega + \nu + \alpha \phi_0), \\ \nu(0) = \nu_0, \end{cases} \quad \text{and} \quad \begin{cases} i\dot{\alpha} + e_0 \alpha = (\mathcal{N}(\varepsilon u^\omega + \nu + \alpha \phi_0) | \phi_0), \\ \alpha(0) = \alpha_0. \end{cases}$$

It follows from the Duhamel's formulation that solving the above system reduces to find a fixed point for the map $\Gamma = (\Gamma_1, \Gamma_2)$, where

$$\begin{aligned}\Gamma_1(\nu)(t) &= e^{-itH} \nu_0 - i \int_0^t e^{-i(t-\tau)H} \mathbb{P}_c \mathcal{N}(\varepsilon u^\omega(\tau) + \nu(\tau) + \alpha(\tau)\phi_0) d\tau, \\ \Gamma_2(\alpha)(t) &= e^{ie_0 t} \alpha_0 - i \int_0^t e^{i(t-\tau)e_0} (\mathcal{N}(\varepsilon u^\omega(\tau) + \nu(\tau) + \alpha(\tau)\phi_0) | \phi_0) d\tau.\end{aligned}$$

Given $T > 0$ and A to be chosen later on, we search a fixed point for Γ in the Banach space

$$\mathcal{E}_{T,A} = \left\{ (\nu, \alpha) \in X^{1/2}(0, T) \times L^\infty((0, T); \mathbb{C}) \mid \|\nu\|_{X^{1/2}(0, T)} + \sup_{0 \leq t \leq T} |\alpha(t)| \leq A \right\}.$$

In this setting, we have the following a priori estimates: given $(\nu, \alpha) \in \mathcal{E}_{T,A}$,

$$\|\Gamma_1(\nu)\|_{X^{1/2}(0, T)} \leq C \|\nu_0\|_{\mathcal{H}^{1/2}} + C(1 + T^{1/5})(A^3 + R^3), \quad (3.5.1)$$

and

$$\sup_{0 \leq t \leq T} |\Gamma_2(\alpha)(t)| \leq \alpha_0 + C(1 + T)(A^3 + R^3). \quad (3.5.2)$$

Under our assumptions, we now fix $A = R = 2C_0\delta_0 =: c\delta_0$ and the above estimates yield

$$\|\Gamma_1(\nu)\|_{X^{1/2}(0, T)} + \sup_{0 \leq t \leq T} |\Gamma_2(\alpha)(t)| \leq C_0(\delta_0 + A^3 + T^{1/2}) \leq 2C_0\delta_0$$

provided that $\delta_0 C_0 < 1/2$, and where $C_0 = C_0(\|\phi_0\|_{\mathcal{H}^1}, \| |x|^{-1/2} \phi \|_{L_x^\infty})$. This proves that Γ preserves the space $\mathcal{E}_{T,A}$. Let us briefly explain how we obtained estimates (3.5.1) and (3.5.2). To prove (3.5.1), we first note that the terms without the discrete part $\alpha\phi_0$ are handled by the analysis conducted in section 3.4.1, globally in time. Therefore, by dropping the complex conjugate from the notations, it remains to control some terms of the form

$$(\alpha\phi_0)^3, \quad (\alpha\phi_0)^2(\varepsilon u^\omega + \nu), \quad \alpha\phi_0(\varepsilon u^\omega + \nu)^2.$$

To handle the terms without any power of the linear part u^ω , we use the Leibniz rule and we obtain

$$\begin{aligned}\|(\alpha\phi_0)^3\|_{DX_H^{1/2}(0, T)} + \|(\alpha\phi_0)^2\nu\|_{DX_H^{1/2}(0, T)} + \|(\alpha\phi_0)\nu^2\|_{DX_H^{1/2}(0, T)} \\ \lesssim T \left(\sup_{0 \leq t \leq T} |\alpha(t)|^3 \|\phi_0\|_{\mathcal{H}^{1/2}}^3 + \|\nu\|_{X_H^{1/2}(0, T)} \right) \lesssim TA^3.\end{aligned}$$

As for the terms with some power of u^ω , we can use the Leibniz rule and the local smoothing estimate (3.2.7) as follows. We do a Littlewood-Paley decomposition of each term in the definition of Writing the definition of the $X^{1/2}$ -norm: given a fixed $N \in 2^{\mathbb{N}}$, we denote by N_1, N_2, N_3 the frequencies at which $\phi_0, u^\omega, \nu^\omega$ are localized.

Case $\alpha\phi_0(u^\omega)^2$: We consider the worst case where ϕ_0 , which is the smoothest term, comes with the lowest frequency $N_1 \leq N_2 \leq N_3$. In the high-low-low case, when we have $N_2 \ll N_3$, $\tilde{\Delta}_{N_3} u^\omega$

cannot absorb all the derivatives, and we need to use the local smoothing effect. To that hand, we apply Hölder with $1 = \frac{1}{2} + \frac{1}{5} + \frac{3}{10}$ and get

$$\begin{aligned} N^{1/2} \|P_N(\alpha\phi_0(\varepsilon u^\omega)^2)\|_{U_H^2(0,T)} &\leq \varepsilon^2 N_3^{1/2} \sup_{\|w\|_{V_H^2(0,T)} \leq 1} \left| \int_0^T \int_{\mathbb{R}^3} \alpha(t) \tilde{\Delta}_{N_1} \phi_0 \tilde{\Delta}_{N_2} u^\omega \tilde{\Delta}_{N_3} u^\omega w dx dt \right| \\ &\lesssim \sup_{0 \leq t \leq T} |\alpha(t)| N_3^{1/2} \|\tilde{\Delta}_{N_1} \phi_0 \tilde{\Delta}_{N_3} \varepsilon u^\omega\|_{L_{t,x}^2(0,T)} \|\varepsilon u^\omega\|_{L_{t,x}^5(0,T)} \|w\|_{V_H^2} \\ &\lesssim A \|\langle x \rangle^{1/2+} \phi_0\|_{L_x^\infty} N_3^{1/2} \|\langle x \rangle^{-1/2-} \tilde{\Delta}_{N_3} \varepsilon u^\omega\|_{L_{t,x}^2(0,T)} \|\varepsilon u^\omega\|_{L_{t,x}^5(0,T)}. \end{aligned}$$

Then, we use the probabilistic Strichartz estimate (3.5.14) for the $L_{t,x}^5$ norm and the local smoothing effect (3.2.7) to get

$$N^{1/2} \|P_N(\alpha\phi_0(\varepsilon u^\omega)^2)\|_{U_H^2(0,T)} \lesssim A \|\tilde{\Delta}_{N_3} \varepsilon u_0^\omega\|_{L_x^2} R \lesssim A \varepsilon R \lesssim A^3 + R^3,$$

since ε has to be chosen much smaller than R in (3.3.8) for the probability of the bad set of initial data to be small.⁸

Case $(\alpha\phi_0)^2 u^\omega$: Similarly, we assume that $N_1 \leq N_2 \leq N_3 \sim N$ and do the same computations with $\alpha(t)\phi_0$ instead of εu^ω . The only difference is that

$$\|\alpha\phi_0\|_{L_{t,x}^5(0,T)} \leq AT^{1/5} \|\phi_0\|_{L_x^5}.$$

This yields

$$N^{1/2} \|P_N(\alpha\phi_0)^2 \varepsilon u^\omega\|_{U_H^2(0,T)} \lesssim T^{1/5} A^2 \varepsilon \lesssim T^{1/5} (A^3 + R^3).$$

Then, we obtain (3.5.1) by summing over N_1, N_2, N_3 and N as in section 3.4.1. To get estimate (3.5.2), we just use Hölder, the endpoint Strichartz estimate and the embedding $L_t^3((0, T); L_x^4) \hookrightarrow X_H^0(0, T)$

$$\begin{aligned} &\sup_{0 < t < T} \left| \int_0^t e^{i(t-\tau)e_0} (\mathcal{N}(\varepsilon u^\omega(\tau) + \nu(\tau) + \alpha(\tau)\phi_0) | \phi_0) d\tau \right| \\ &\lesssim \int_0^T \int_{\mathbb{R}^3} (|\nu(\tau, x) + \varepsilon u^\omega(\tau, x)|^3 + |\alpha(\tau)\phi_0(x)|^3) |\phi_0(x)| dx d\tau \\ &\lesssim \|\phi_0\|_{L_x^2} \left(\int_0^T \|\nu(\tau) + \varepsilon u^\omega\|_{L_x^6}^3 d\tau \right) + T \sup_{0 < t < T} |\alpha(t)|^3 \|\phi_0\|_{L_x^4}^4 \\ &\lesssim C(\phi_0) (\|\langle \nabla \rangle^{1/4} \nu\|_{L_t^3 L_x^4}^3 + \|\varepsilon \langle \nabla \rangle^{1/4} u^\omega\|_{L_t^3 L_x^4}^3 + TA^3). \end{aligned}$$

By the embedding $L_t^3((0, T); L_x^4) \hookrightarrow X_H^0(0, T)$, the right-hand side is less than

$$\begin{aligned} &\lesssim C(\phi_0) (\|\nu\|_{X_H^{1/4}(0,T)}^3 + \varepsilon^3 \|u_0\|_{\mathcal{H}^{1/4}}^3 + TA^3) \\ &\lesssim C(\phi_0) (1 + T) (A^3 + R^3). \end{aligned}$$

This finishes the proof of estimates (3.5.1) and (3.5.2). The proof of the Lipschitz estimates for Γ follows similarly, and establishes the contraction mapping property for Γ . \square

⁸ In this case, note that the estimates do not depend on T .

3.5.2 Nonlinear ground states

Before we dive into the proof of asymptotic stability we establish the main global estimates on the local flow constructed in the above paragraph, we collect from [GNT04] some properties of the curve of *nonlinear ground states* that bifurcates from the eigenspace spanned by ϕ_0 . In particular, we will be able to decompose the solution into a ground state plus a radiation term at each time, where the radiation term will be shown to scatter in the last paragraph. First, we recall how the ground states are constructed.

Lemma 3.5.2 (Nonlinear ground states (Lemma 2.1 in [GNT04])). *There exists $\delta > 0$ small enough such that for all $z \in \mathbb{C}$ with $|z| \leq \delta$, there exists a nonlinear ground state Q and E solution to*

$$(\Delta - V + |Q|^2)Q = EQ \quad (3.5.3)$$

under the form

$$Q(z) = z\phi_0 + q(z), \quad E(z) = e_0 + e(z).$$

We have uniqueness for (q, e) in the class

$$\{(q, e) \in (\mathcal{H}^2 \cap \text{Ran}(P_c)) \times \mathbb{R} \mid \|q\|_{\mathcal{H}^2} \leq \delta, |e| \leq \delta\}.$$

Thanks to the gauge invariance of the nonlinear part,

$$Q(z e^{i\alpha}) = Q(z) e^{i\alpha}, \quad E(z) = E(|z|).$$

In addition, q and its derivatives are small in $\mathcal{H}^2(\mathbb{R}^3)$:

$$q = O(z^3), \quad D_z Q = O(z^2), \quad D_z^2 q = O(z). \quad (3.5.4)$$

The first two derivatives of e are also small and satisfy

$$|D_z e| = O(z), \quad |D_z^2 e| = O(1). \quad (3.5.5)$$

The following elliptic estimates on some Sobolev weighted norms of the ground states and its derivatives with respect to the parameter z are crucial, and our arguments heavily rely on the smallness of these quantities.

Lemma 3.5.3 (Weighted elliptic estimates for the derivatives of ground states, see [GNT04]). *There exists $\delta > 0$ such that for all $|z| < \delta$ and all $k \in \{1, 2\}$ we have*

$$\begin{aligned} \|\langle x \rangle^k q\|_{\mathcal{H}^2(\mathbb{R}^3)} &= O(z^3), \\ \|\langle x \rangle^k D_z Q\|_{\mathcal{H}^2(\mathbb{R}^3)} &= O(z^2). \end{aligned} \quad (3.5.6)$$

We recall that the generalized continuous spectral spaces $\mathcal{H}_c(z)$ parametrized by z were introduced to encode orthogonality conditions (3.1.7):

$$\mathcal{H}_c(z) := \left\{ \eta \in L^2(\mathbb{R}^3) \mid \int_{\mathbb{R}^3} \langle i\eta, \partial_{z_1} Q \rangle dx = \int_{\mathbb{R}^3} \langle i\eta, \partial_{z_2} Q \rangle dx = 0 \right\}.$$

In particular, note that $\text{Ran}(P_c)$ corresponds to $\mathcal{H}_c(0)$. The following result (Lemma 2.2 in [GNT04]) yields a bijection from $\mathcal{H}_c(0)$ to $\mathcal{H}_c(z)$. This useful correspondence reduces the dynamic of η in ansatz (3.1.9) to the dynamic of its continuous spectral part ν .

Lemma 3.5.4 (Continuous spectral subspace comparison). *There exists $\delta > 0$ such that any function $\psi \in L^2(\mathbb{R}^3)$ can be uniquely decomposed into*

$$\psi = Q(z) + \eta, \quad \eta \in \mathcal{H}_c(z).$$

If $\|\psi\|_{\mathcal{H}^s} \leq \delta$ for some $s \geq 0$ then $\eta \in \mathcal{H}^s$ and $\|\eta\|_{\mathcal{H}^s} + |z| \lesssim \|\psi\|_{\mathcal{H}^s}$. Moreover, for all $|z| \leq \delta$ there exists a bijective operator

$$\begin{aligned} \mathbf{R}(z) &: \mathcal{H}_c(0) &\rightarrow & \mathcal{H}_c(z) \\ &u &\mapsto & u + \alpha(z)(u)\phi_0. \end{aligned}$$

Here, $\alpha(z)u$ is solution to

$$\Lambda(z)\alpha(z)u = \langle iu, D_z Q \rangle,$$

with

$$\Lambda(z) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} \langle \phi_0, \text{Im}(\partial_{z_2} q) \rangle & -\langle \phi_0, \text{Re}(\partial_{z_2} q) \rangle \\ \langle \phi_0, \text{Im}(\partial_{z_1} q) \rangle & \langle \phi_0, \text{Re}(\partial_{z_1} q) \rangle \end{pmatrix} =: \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + \gamma(z).$$

Finally, $\mathbf{R}(z) - I$ is compact and continuous in the operator norm on any space Y that satisfies $\mathcal{H}^{-2} \subset Y \subset \mathcal{H}^2$.

3.5.3 Global a priori estimates

Now that we have a local in time solution ψ for small initial data, we need to obtain some global estimates on ϕ to extend it as a global solution and understand to its asymptotic behavior. First, we recall that in a small neighborhood of 0 the solution can be uniquely decomposed into

$$\psi = Q(z) + \eta,$$

where η satisfies the time-dependent orthogonality conditions $\eta(t) \in \mathcal{H}_c(z(t))$. In what follows, the continuous spectral part of the nonlinear part ⁹ of the solution is denoted by ν :

$$\nu = P_c(\eta) - e^{-itH} \varepsilon u_0^\omega \in \mathcal{H}_c(0).$$

Then, we deduce from Lemma 3.5.4 that $\eta = R(z)(P_c \eta)$ so that the solution decouples into

$$\psi = Q(z) + \eta = Q(z) + \mathbf{R}(z)(\varepsilon u^\omega + \nu).$$

In particular, we note that

$$P_c(\eta) = \varepsilon u^\omega + \nu, \quad P_p(\eta) = (\mathbf{R}(z) - I) P_c(\eta) = \alpha(z) P_c(\eta) \phi_0 = \alpha(z) (\varepsilon u^\omega + \nu) \phi_0.$$

In what follows, we fix $\delta > 0$ as in Lemmas 3.5.3, 3.5.4 and $\delta_0 > 0$ as in the local existence result from Proposition 3.5.1. Then, we take ε and $\delta' < \delta$ such that

$$\|\nu\|_{L^2} + |z| \leq \delta' \implies \|\psi\|_{\mathcal{H}^s} \leq \delta.$$

⁹ We emphasize that the so-called linear part of the solution $u^\omega(t) = e^{-itH} u_0^\omega$ lies in $\mathcal{H}_c(0)$ at each time t , since u_0 is in $\mathcal{H}_c(0)$ and the randomization preserves this space, as well as the flow e^{-itH} .

The goal is to obtain some global bounds on η in a critical space contained in $L_t^\infty \mathcal{H}_x^{1/2}$ and on z to prove that the solutions stay small. Then we bootstrap the local existence result to obtain global existence, and further exploit the global estimates to prove asymptotic stability. We denote by m the following gauge transformation of the parameter z

$$m(z) = z \exp\left(i \int_0^t E(z)(\tau) d\tau\right). \quad (3.5.7)$$

It turns out that m is the interesting evolution parameter. Indeed, if we assume orthogonality conditions (3.1.7) to hold then m solves the ODE

$$\begin{cases} \frac{d}{dt} m(z) = \dot{z} + iE(z)z = -A(z, \eta)^{-1} \langle F, D_z Q \rangle, \\ m(z)|_{t=0} = z_0, \end{cases} \quad (3.5.8)$$

where for $j, k \in \{1, 2\}$ we have

$$A(z, \eta)_{j,k} = \langle i D_j Q, D_k Q \rangle + \langle i\eta, D_j D_k Q \rangle = j - k + O(\delta^2). \quad (3.5.9)$$

The forcing term is

$$F = F(z, \eta) = \mathcal{N}(Q + \eta) - |Q|^2 \eta - Q^2 \bar{\eta}.$$

Remark 3.5.5. As a consequence of (3.5.4), the matrix A is invertible provided that δ is small enough.

Remark 3.5.6. If we drop numerical constants and complex conjugate in the notations, we can write

$$F = \eta^3 + Q\eta^2.$$

In particular, we observe that orthogonality conditions (3.1.7) cancelled the terms which are linear in η . Since we do not control the L^1 -in-time norm of the radiation term, we couldn't have handled these linear terms. That's why the time dependent orthogonality conditions are crucial.

Remark 3.5.7. Since $E(z) = E(|z|) = E(|m(z)|)$ we have that $\bar{z} = m(\overline{m(z)})$. Hence, m is a bijective operator, so we can recover z from $m(z)$.

To get the evolution equation for ν , we inject the ansatz (3.1.9) into (NLS) and we use the fact that Q is solution to (3.5.3). This yields that the radiation term η solves the equation

$$i\partial_t \eta + (\Delta - V)\eta = \mathcal{N}(Q + \eta) - \mathcal{N}(Q) - i D_z Q(\dot{z} + iEz).$$

By projecting the above equation on the continuous spectral subspace, and noting that $P_c(D_z Q) = D_z q$, we get

$$\begin{cases} i\partial_t \nu + (\Delta - V)\nu = P_c(\mathcal{N}(Q + \eta) - \mathcal{N}(Q)) - i D_z q \dot{m}(z) =: f + g, \\ \nu|_{t=0} = \nu_0, \end{cases} \quad (3.5.10)$$

where we decomposed the forcing term into the sum of a nonlinear term f and a localized term g . More precisely, we chose to collect in f the higher order nonlinear terms $\mathcal{N}(\nu + \varepsilon u^\omega)$ as well as the modulation term:

$$f = P_c(\mathcal{N}(\nu + \varepsilon u^\omega)) - i D_z q \dot{m}(z).$$

On the other hand, g contains the localized lower order terms which involve at least a power of Q or of $P_p(\eta)$:

$$g = P_c(Q^2\eta + Q\eta^2 + |P_p(\eta)|^2 P_p(\eta)).$$

To get some global in time bounds on ν , we then follow the *critical-weighted strategy* detailed in Proposition 3.4.15, and we are reduced to control f in $DU_H^2(\mathcal{H}^{1/2})$ and g in $L_t^2(\mathbb{R}; L_x^{2,1/2+})$. Hence, given an interval $I \subseteq \mathbb{R}$ where the solution is defined, we will control ν in the critical-weighted space

$$\mathcal{X}(I) = V_H^2(I; \mathcal{H}_x^{1/2}) \cap L_t^2(I; \mathcal{H}_x^{1,-1/2-}(\mathbb{R}^d)), \quad (3.5.11)$$

endowed with its natural norm, and \dot{m} in $L_t^1(I)$. The aim is to get bounds that are independent of I .

Global-in-time a priori estimates on ν and \dot{m} In the following Lemma, we prove some preliminary estimates on the radiation term η that will be needed in the analysis. We recall that $\tilde{\Omega}_{\varepsilon, c\delta_0}$ is the set defined in (3.3.8) made of randomized initial data which display some improved Strichartz estimates, with $R = c\delta_0$ (see Proposition 3.5.1).

Lemma 3.5.8 (Preliminary estimates). *Take $\omega \in \tilde{\Omega}_{\varepsilon, R}$ and let η, z, ν be as in ansatz (3.1.9). For any interval $I \subseteq [0, \infty)$,*

$$\|\eta\|_{L^2(I; L_x^{2,-1/2-})} \lesssim \|\nu\|_{\mathcal{X}(I)} + \varepsilon \|u_0\|_{L^2(\mathbb{R}^3)}, \quad (3.5.12)$$

$$\|\eta\|_{L_t^4 L_x^4(I \times \mathbb{R}^3)} \lesssim \|\nu\|_{\mathcal{X}(I)} + R. \quad (3.5.13)$$

Proof. We can estimate the discrete part of η by its continuous part. Indeed, we recall that

$$|P_p(\eta)| = |(R - I)(z)(\nu + u^\omega)| \lesssim |\phi_0| |(\nu + u^\omega | D_z Q)|. \quad (3.5.14)$$

Hence,

$$\begin{aligned} \|P_p(\eta)\|_{L_t^2(I; L_x^{2,-1/2-})} &\leq \|\langle x \rangle^{-1/2-} \phi_0\|_{L_x^2} \|\langle x \rangle^{1/2+} D_z Q\|_{L_x^\infty} \|\nu + \varepsilon u^\omega\|_{L_t^2(I; L_x^{2,-1/2-})} \\ &\lesssim \|\langle x \rangle D_z Q\|_{L_t^\infty(I; \mathcal{H}^2)} \|\nu + \varepsilon u^\omega\|_{L_t^2(I; L_x^{2,-1/2-})} \\ &\lesssim \|\nu\|_{\mathcal{X}} + \varepsilon \|u_0\|_{L_x^2}, \end{aligned}$$

where we used the local smoothing estimate (3.2.7) to control the weighted norm of the perturbed linear propagation u^ω in the last inequality, as well as the fact that $\|u_0\|_{L_x^2} \sim \|u_0^\omega\|_{L_x^2}$. To prove (3.5.13) we apply Hölder in (3.5.14) and Sobolev embedding for $\|\phi_0\|_{L^4(\mathbb{R}^3)}$. It comes

$$\|(R - I)(\nu + \varepsilon u^\omega)\|_{L_t^4 L_x^4} \lesssim \|\phi_0\|_{\mathcal{H}^{3/4}} \|\nu + \varepsilon u^\omega\|_{L_t^4 L_x^4} \|D_z Q\|_{L_x^{4/3}}.$$

Applying Hölder once again, we get

$$\|D_z Q\|_{L_x^{4/3}} \leq \left(\int_{\mathbb{R}^3} \langle x \rangle^{-3+dx} \right)^{1/3} \|\langle x \rangle^{3/4+} D_z Q\|_{L_x^2} \lesssim \|\langle x \rangle^2 D_z Q\|_{L_x^2}.$$

Then, it follows from (3.5.6) that

$$\|(R - I)(\nu + \varepsilon u^\omega)\|_{L_t^4 L_x^4} \lesssim \|\langle x \rangle^2 D_z Q\|_{L_t^\infty L_x^2} \|\nu + \varepsilon u^\omega\|_{L_t^4 L_x^4} \lesssim \|\nu\|_{L_t^4 L_x^4} + \varepsilon \|u^\omega\|_{L_t^4 L_x^4}.$$

Note that we used the Sobolev weighted estimate (3.5.6) on $D_z Q$. To control the deterministic term ν in $L_{t,x}^4$, we use Sobolev embedding, Strichartz estimate for the admissible pair (4, 3) and the transference principle in V_H^2 . Since we chose $\omega \in \tilde{\Omega}_{\varepsilon,R}$ we can use the improved global-in-time Strichartz estimate (3.3.6) to control the random term u^ω and obtain

$$\|\nu + \varepsilon u^\omega\|_{L_t^4 L_x^4} \lesssim \|\langle \sqrt{H} \rangle^{1/4} \nu\|_{L_t^4 L_x^3} + R \lesssim \|\langle \sqrt{H} \rangle^{1/4} \nu\|_{V_H^2} + R \lesssim \|\nu\|_{\mathcal{X}} + R.$$

This gives both the desired estimate for the discrete part of η and its continuous part $\nu + \varepsilon u^\omega$, and finishes the proof of Lemma 3.5.8. \square

We can now state and prove the main global-in-time a priori estimate.

Proposition 3.5.9 (Global a priori estimates). *There exists $C = C(\phi_0)$ such that for all interval $I \subseteq \mathbb{R}$, for all $\omega \in \tilde{\Omega}_{\varepsilon,R}$ and $\psi_0 \in \mathcal{H}^{1/2}$, if*

$$\psi = Q(z) + \eta = Q(z) + \mathbf{R}(z)(\varepsilon u^\omega + \nu), \quad \psi|_{t=t_0} = \varepsilon u_0^\omega + \psi_0,$$

is solution to (NLS) on I with $\|\nu_0\|_{\mathcal{H}^{1/2}} + |z_0| \leq \delta_0$, $\|\nu\|_{\mathcal{X}(I)} \leq \delta'$, $\sup_{t \in I} |z| \leq \delta'$, then ¹⁰

$$\|\nu\|_{\mathcal{X}(I)} \leq C(\|\nu\|_{\mathcal{X}(I)}^3 + R^3 + |z_0|^3 + \|\dot{m}(z)\|_{L^1(I)}^3 + \|\nu_0\|_{\mathcal{H}^{1/2}}), \quad (3.5.15)$$

$$\|\dot{m}(z)\|_{L^1(I)} \leq C(\|\nu\|_{\mathcal{X}(I)}^3 + R^3 + |z_0|^3 + \|\dot{m}(z)\|_{L^1(I)}^3). \quad (3.5.16)$$

Proof of (3.5.15). It follows from *weighted-critical* estimate (3.4.24) that

$$\|\nu\|_{\mathcal{X}} \lesssim \|\nu_0\|_{\mathcal{H}^{1/2}} + \sup_{2^{\mathbb{N}}} N^{1/2} \|\tilde{\Pi}_N f\|_{DU_H^2} + \|g\|_{L_t^2(\mathbb{R}; L_x^{2,1/2+})}.$$

We recall that $f = \mathcal{N}(\nu + u^\omega) + i D_z q \dot{m}(z)$ and we observe that $\tilde{\Pi}_N f \in L^1(I, \mathcal{H}^{1/2})$ as explained in (3.4.12). Hence, we apply the probabilistic nonlinear estimate (3.4.15) on the continuous spectral subspace stated in section 3.4 to get that for all $\omega \in \tilde{\Omega}_{\varepsilon,R}$

$$\sup_{2^{\mathbb{N}}} N^{1/2} \|\tilde{\Pi}_N \mathcal{N}(\nu + u^\omega)\|_{DU_H^2} \lesssim \|\nu\|_{\mathcal{X}}^3 + R^3.$$

To handle the modulation term, we observe that $P_c(D_z Q) = D_z Q$, and we deduce from the duality argument between U^2 and V^2 detailed in remark 3.4.13 that

$$\begin{aligned} \|D_z Q(z) \dot{m}(z)\|_{DU_H^2} &\leq \sup_{\|v\|_{V_H^2} \leq 1} \int_{\mathbb{R}} (D_z Q \dot{m}(z) | v) dt \\ &\leq \sup_{\|v\|_{V_H^2} \leq 1} \|v\|_{L_t^\infty L_x^2} \|D_z Q(z)\|_{L_t^\infty L_x^2} \|\dot{m}(z)\|_{L_t^1} \\ &= \|D_z Q(z)\|_{L_x^2} \|\dot{m}(z)\|_{L_t^1}, \end{aligned}$$

where we used the embedding $V^2 \hookrightarrow L_t^\infty(\mathbb{R}; L_x^2)$. Consequently,

$$\sup_{2^{\mathbb{N}}} N^{1/2} \|\tilde{\Pi}_N D_z Q(z) \dot{m}(z)\|_{DU_H^2} \lesssim \|D_z Q\|_{\mathcal{H}^{1/2}} \|\dot{m}(z)\|_{L_t^1} = O(z^2) \|\dot{m}(z)\|_{L_t^1}.$$

¹⁰ We recall that $\delta' < \delta$ is chosen such that $\|\psi\|_{\mathcal{H}^s} \leq \delta$. This forces the solution to remain small and allows us to apply Lemma 3.5.3 and Lemma 3.5.4.

Note that we used (3.5.4) to control $\|D_z Q\|_{\mathcal{H}^{1/2}}$. Now, we need to estimate the localized lower order terms collected in g , defined in (3.5.10).

$$\|g\|_{L_t^2(\mathbb{R}; L_x^{2,1/2+})} \leq \|P_c(Q^2\eta)\|_{L_t^2(\mathbb{R}; L_x^{2,1/2+})} + \|P_c(Q\eta^2)\|_{L_t^2(\mathbb{R}; L_x^{2,1/2+})} + \| |P_p(\eta)|^2 P_p(\eta) \|_{L_t^2(\mathbb{R}; L_x^{2,1/2+})}.$$

It follows from (3.5.6) and (3.5.12) that

$$\|P_c(Q^2\eta)\|_{L_t^2(\mathbb{R}; L_x^{2,1/2+})} \leq \|\langle x \rangle^{1/2+} Q\|_{L_x^\infty}^2 \|\eta\|_{L_t^2(\mathbb{R}; L_x^{2,-1/2-})} \lesssim \|z\|_{L_t^\infty}^2 (\|\nu\|_{\mathcal{X}} + \varepsilon \|u_0\|_{L_x^2}).$$

Similarly, (3.5.6) and (3.5.13) yield

$$\|P_c(Q\eta^2)\|_{L_t^2(\mathbb{R}; L_x^{2,1/2+})} \leq \|\langle x \rangle^{1/2+} Q\|_{L_x^\infty} \|\eta\|_{L_t^4 L_x^4}^2 \lesssim \|z\|_{L_t^\infty} (\|\nu\|_{\mathcal{X}} + R)^2 \lesssim \|z\|_{L_t^\infty}^3 + \|\nu\|_{\mathcal{X}}^3 + R^3.$$

To conclude we recall that

$$|P_p(\eta)| \lesssim |\phi_0| |\langle \nu + u^\omega, D_z Q \rangle|^3,$$

and therefore

$$\begin{aligned} \| |P_p(\eta)|^2 P_p(\eta) \|_{L_t^2(\mathbb{R}; L_x^{2,1/2+})} &\lesssim \|\phi_0\|_{L_x^{2,1/2+}} \left(\int_{\mathbb{R}} \langle \nu + u^\omega, D_z Q \rangle^6 \right)^{1/2} \\ &\lesssim \left(\int_{\mathbb{R}} \langle \nu + u^\omega, D_z Q \rangle^2 \right)^{1/2} \sup_{\mathbb{R}} |\langle \nu + u^\omega, D_z Q \rangle|^2 \\ &\lesssim \|\langle x \rangle D_z Q\|_{L_x^2} \|\nu + u^\omega\|_{L_t^2(\mathbb{R}; L_x^{2,-1/2-})} \|\nu + u^\omega\|_{L_t^\infty L_x^2} \|D_z Q\|_{L_x^2} \\ &\lesssim (\|\nu\|_{\mathcal{X}} + \varepsilon \|u_0\|_{L_x^2})^3 \lesssim \|\nu\|_{\mathcal{X}}^3 + \varepsilon^3 \|u_0\|_{L_x^2}^3. \quad \square \end{aligned}$$

Proof of (3.5.16). We recall that

$$|\dot{m}(z)| \lesssim |\langle \eta^3, D_z Q \rangle| + |\langle Q\eta^2, D_z Q \rangle|.$$

We apply Cauchy-Schwarz inequality firstly in space and then in time to get that

$$\begin{aligned} \int_{\mathbb{R}} |\langle \eta^3, D_z Q \rangle| dt &\leq \|\langle x \rangle^{1/2+} D_z Q\|_{L_x^\infty} \int_{\mathbb{R}} \|\langle x \rangle^{-1/2-} \eta\|_{L_x^2} \|\eta\|_{L_{t,x}^4}^2 dt \\ &\lesssim \|\langle x \rangle D_z Q\|_{\mathcal{H}^2} \|\eta\|_{L_t^2(\mathbb{R}; L_x^{2,-1/2-})} \|\eta\|_{L_{t,x}^4}^2. \end{aligned}$$

By using estimate (3.5.6) on the weighted norm of $D_z Q = J\phi_0 + D_z Q$ and estimates of Lemma 3.5.8 on η , we conclude that

$$\int_{\mathbb{R}} |\langle \eta^3, D_z Q \rangle| dt \lesssim (\|\nu\|_{\mathcal{X}} + \varepsilon \|u_0\|_{L^2}) (\|\nu\|_{\mathcal{X}} + R)^2 \lesssim \|\nu\|_{\mathcal{X}}^3 + \varepsilon^3 \|u_0\|_{L^2}^3 + R^3.$$

Similarly, we have

$$\begin{aligned} \int_{\mathbb{R}} |\langle Q\eta^2, D_z Q \rangle| dt &\leq \|\langle x \rangle^{1/2+} Q\|_{L^\infty} \|\langle x \rangle^{1/2+} D_z Q\|_{L^\infty} \|\langle x \rangle^{-1} \eta\|_{L_t^2 L_x^2}^2 \\ &\lesssim \|\langle x \rangle Q\|_{\mathcal{H}^2} \|\langle x \rangle D_z Q\|_{\mathcal{H}^2} \|\eta\|_{L_t^2(\mathbb{R}; L_x^{2,-1/2-})}^2 \\ &\lesssim \|z\|_{L_t^\infty} (\|\nu\|_{\mathcal{X}}^2 + \varepsilon^2 \|u_0\|_{L^2}^2) \lesssim \|z\|_{L_t^\infty}^3 + \|\nu\|_{\mathcal{X}}^3 + \varepsilon^3 \|u_0\|_{L^2}^3. \end{aligned}$$

This concludes the proof of Proposition 3.5.9. \square

3.5.4 Proof of Theorem 3.1.1

Now that we have a local solution with some global a priori estimates both on the radiation term η and on the modulation parameter z , we shall be able to prove global well-posedness, and then deduce from these global estimates asymptotic stability.

Global existence and uniqueness It remains to finely tune the parameters that appear in Proposition 3.4.10. First, we take δ_0 such that the local existence result of Proposition 3.5.1 holds true, $R = c\delta_0$. Then, if the solution exists up to a certain time T we prove that

$$\|\nu\|_{\mathcal{X}[0,T]} + \|\dot{m}(z)\|_{L^1[0,T]} \leq \delta' \implies \|\nu\|_{\mathcal{X}[0,T]} + \|\dot{m}(z)\|_{L^1[0,T]} \leq \frac{\delta'}{2}. \quad (3.5.17)$$

Indeed, we have from (3.5.15) and (3.5.16) that

$$\|\nu\|_{\mathcal{X}[0,T]} \leq C(2(\delta')^3 + (c\delta_0)^3 + \delta_0^3 + \delta_0), \quad \|\dot{m}(z)\|_{L^1[0,T]} \leq C(2(\delta')^3 + (c\delta_0)^3 + \delta_0^3).$$

The a priori estimate (3.5.17) follows if we chose δ_0 even smaller, say such that $\delta_0 \leq (4C)^{-1}\delta'$, and $\delta' \lesssim (c^3C)^{-1/2}$. Then, we use a continuity argument (see Lemma 3.4.4) and prove that the solution is global, with the bounds

$$\|\nu\|_{\mathcal{X}(\mathbb{R})} + \|\dot{m}(z)\|_{L^1(\mathbb{R})} \leq \delta'.$$

Since $\dot{m}(z)$ lies in $L^1[0, \infty)$ the convergence of the modulation parameter follows immediately. There exists $z_+ \in \mathbb{C}$ such that

$$\lim_{t \rightarrow +\infty} m(z)(t) = z_+.$$

Note that $|z|$ converges, and hence $E(z)$ is also convergent.

Scattering We turn to the proof of the so-called completeness property of the flow, that is the fact that the radiation part η scatters. To this end, we first prove scattering for $\nu = P_c(\eta) - \varepsilon u^\omega$ by following the proof of the scattering from Theorem 3.1.5 we made in section 3.4. We recall that ν is solution to (3.5.10), and we prove that the Cauchy criterion is satisfied:

$$\lim_{t_1, t_2 \rightarrow +\infty} \|e^{-it_1(\Delta-V)} \nu(t_1) - e^{-it_2(\Delta-V)} \nu(t_2)\|_{\mathcal{H}^{1/2}(\mathbb{R}^3)} = 0.$$

Indeed, by the Duhamel integral formulation,

$$\|e^{-it_1(\Delta-V)} \nu(t_1) - e^{-it_2(\Delta-V)} \nu(t_2)\|_{\mathcal{H}^{1/2}(\mathbb{R}^3)} \leq \|f + g\|_{DV^2(t_1, t_2)}.$$

As observed in remark 3.4.11, it follows from the proof of Proposition 3.5.9 that there exists a constant C depending on some norms of the solution such that

$$\begin{aligned} \|f + g\|_{DV_H^2(t_1, t_2)} &\leq \|f + g\|_{DU_H^2(t_1, t_2)} \\ &\leq C(\|\nu\|_{\mathcal{X}(t_1, +\infty)}, \|\dot{m}(z)\|_{L_t^1(t_1, +\infty)}, \|u^\omega\|_{L_t^q((t_1, +\infty); L_x^r)}), \end{aligned} \quad (3.5.18)$$

and C goes to zero when t_1 goes to $+\infty$. Hence, the Cauchy criterion is satisfied and there exists $\nu_+ \in \mathcal{H}^{1/2} \cap \text{Ran}(P_c)$ such that

$$\lim_{t \rightarrow +\infty} \|\nu(t) - e^{-itH}(\varepsilon u_0^\omega + \nu_+)\|_{\mathcal{H}^{1/2}(\mathbb{R}^3)} = 0.$$

It remains to prove that the discrete part of η goes to zero when t goes to infinity. First, the weak convergence of

$$e^{-itH}(\varepsilon u_0^\omega + \nu) \xrightarrow{t \rightarrow +\infty} 0 \quad \text{in } \mathcal{H}^{1/2}(\mathbb{R}^3)$$

follows from the Riemann-Lebesgue theorem and the Plancherel formula for the distorted Fourier transform. Then, we use the compactness of the operator $R - I$ from $\mathcal{H}^{1/2}(\mathbb{R}^3)$ to \mathbb{C} stated in Lemma 3.5.4 to get that

$$\lim_{t \rightarrow +\infty} P_p(\eta) = \lim_{t \rightarrow +\infty} (R - I) e^{-itH}(\varepsilon u_0^\omega + \nu) = 0 \quad \text{in } \mathcal{H}^{1/2}(\mathbb{R}^3).$$

This concludes the proof of Theorem 3.1.1 with $\eta_+ = \nu_+$.

3.6 Appendix. Weighted elliptic estimates for the ground states

We show how to prove estimates of Lemma 3.5.3 for the ground states $Q(z)$ and its derivative. For the convenience of the reader we recall the equations satisfied by $e, q, D_e, D_z q$ (see A.4 in [GNT04]), and we omit numerical constants as well as complex conjugate from the notations. The nonlinear ground state $Q(z) = z\phi_0 + q$ and $E(z) = e_0 + e$ are solution to (3.1.1), and they satisfy

$$\begin{aligned} (H - e_0)q &= -P_c \mathcal{N}(Q) + eq, \\ ez &= (\phi_0 | \mathcal{N}(Q)). \end{aligned} \quad (3.6.1)$$

We recall that $\mathcal{N}(u) = |u|^2 u$, $D_z = (\partial_{z_1}, \partial_{z_2})$ and we use the notation $J = D_z(z) = (1, i)$. Then, if we differentiate the above expressions with respect to z we obtain

$$\begin{aligned} (H - e_0)D_z Q &= -P_c D_z \mathcal{N}(Q) + q D_z e + e D_z Q, \\ z D_z e + J e &= (\phi_0 | D_z \mathcal{N}(Q)), \end{aligned} \quad (3.6.2)$$

with

$$D_z \mathcal{N}(Q) = Q^2(J\phi_0 + D_z Q) = O(z^2) \quad \text{in } \mathcal{H}^2.$$

Proof of Lemma 3.5.3. We first prove the estimate for q in \mathcal{H}^1 with weight $\langle x \rangle$. Given a Schwartz function φ , we have

$$(H - e_0)\langle x \rangle \varphi = [-\Delta, \langle x \rangle] \varphi + \langle x \rangle (H - e_0) \varphi = -\frac{2x \cdot \nabla \varphi}{\langle x \rangle} - \frac{3 + 2|x|^2}{\langle x \rangle^3} \varphi + \langle x \rangle (H - e_0) \varphi. \quad (3.6.3)$$

Then, we multiply (3.6.3) by $\langle x \rangle \bar{\varphi}$ and we integrate over \mathbb{R}^3 to obtain

$$\|\nabla(\langle x \rangle \varphi)\|_{L^2}^2 - e_0 \|\langle x \rangle \varphi\|_{L^2}^2 = \int_{\mathbb{R}^3} \left(\langle x \rangle^2 V(x) + 3 + \frac{3 + 2|x|^2}{\langle x \rangle^2} \right) |\varphi(x)|^2 dx + (\langle x \rangle^2 (H - e_0) \varphi | \varphi). \quad (3.6.4)$$

Since the first term on the right-hand side is equivalent to $\|\varphi\|_{L^2}^2$, it suffices to handle the second term. Formally taking $\varphi = q$ and using (3.6.1), we have

$$\begin{aligned} (\langle x \rangle (H - e_0)q | \langle x \rangle q) &= (-P_c \mathcal{N}(Q) + eq | \langle x \rangle^2 q) \\ &= -(\mathcal{N}(Q) | \langle x \rangle^2 q) + (\phi_0 | \mathcal{N}(Q))(\phi_0 | \langle x \rangle^2 q) + e \|\langle x \rangle q\|_{L^2}. \end{aligned}$$

Since $q = O(z^3)$ in \mathcal{H}^2 (see (3.5.4)), we have

$$|(\mathcal{N}(Q) | \langle x \rangle^2 q)| + |(\phi_0 | \mathcal{N}(Q))(\phi_0 | \langle x \rangle^2 q)| \lesssim |z|^6 (\|\langle x \rangle q\|_{L^2}^2 + C(\|\langle x \rangle^2 \phi_0\|_{L^\infty})).$$

Therefore, we deduce from (3.6.4) that

$$\|\nabla(\langle x \rangle q)\|_{L^2}^2 - (e_0 + e + C|z|^6)\|\langle x \rangle q\|_{L^2}^2 \lesssim \|q\|_{L^2}^2 + |z|^6 C(\|\langle x \rangle^2 \phi_0\|_{L^\infty}).$$

To conclude we observe that there exists $\gamma > 0$ and $c_\gamma > 0$ such that $E(z) = e_0 + e(z) < -c_\delta$ for all $|z| \leq \delta$. Next, we just mention that the estimate for the weight $\langle x \rangle^2$ follows similarly, using that

$$[-\Delta, \langle x \rangle^2] \varphi = 4x \cdot \nabla \varphi + 6\varphi.$$

Let us now derive the estimate with the weight $\langle x \rangle$ in \mathcal{H}^2 for $D_z q$. To this end, we first differentiate (3.6.3) with respect to x_k and we get

$$(H - e_0) \partial_k(\langle x \rangle \varphi) + (\partial_k V) \langle x \rangle \varphi = \partial_k([-\Delta, \langle x \rangle]) \varphi + \partial_k(\langle x \rangle (H - e_0) \varphi).$$

In addition, we multiply by $\partial_k(\langle x \rangle \bar{\varphi})$ on both sides, and we integrate over \mathbb{R}^3 to see that

$$\begin{aligned} \|\nabla(\partial_k \langle x \rangle \varphi)\|_{L^2}^2 - e_0 \|\partial_k \langle x \rangle \varphi\|_{L^2}^2 &= - \int_{\mathbb{R}^3} (V \partial_k(\langle x \rangle \varphi) + \partial_k([-\Delta, \langle x \rangle]) \varphi) \partial_k(\langle x \rangle \bar{\varphi}) dx \\ &\quad + (\partial_k(\langle x \rangle (H - e_0)) \varphi | \partial_k(\langle x \rangle \varphi)). \end{aligned} \quad (3.6.5)$$

Since

$$\partial_k([-\Delta, \langle x \rangle]) \varphi = -2 \left(\frac{\partial_k \varphi}{\langle x \rangle} + \frac{x \cdot \partial_k \nabla \varphi}{\langle x \rangle} - x_k \frac{x \cdot \nabla \varphi}{\langle x \rangle^3} \right) + (3 + 2|x|^2) \frac{3x_k \varphi}{\langle x \rangle^5} - \frac{4x_k \varphi}{\langle x \rangle^3} - (3 + 2|x|^2) \frac{\partial_k \varphi}{\langle x \rangle^3},$$

we have that

$$\left| \int_{\mathbb{R}^3} (V \partial_k(\langle x \rangle \varphi) + \partial_k([-\Delta, \langle x \rangle]) \varphi) \partial_k(\langle x \rangle \bar{\varphi}) dx \right| \lesssim \|\langle x \rangle \varphi\|_{\mathcal{H}^1}^2 + \|\varphi\|_{\mathcal{H}^2}^2.$$

Moreover, by using (3.5.4), (3.5.5), (3.6.2) and the estimates for q and $D_z q$ with weight $\langle x \rangle$ in \mathcal{H}^1 already proved in the first part of the proof, we show that

$$|(\partial_k(\langle x \rangle (H - e_0)) \varphi | \partial_k(\langle x \rangle \varphi))| \leq C(\phi_0) |z|^4.$$

Hence,

$$\|\nabla(\partial_k \langle x \rangle \varphi)\|_{L^2}^2 - (e_0 + e(z)) \|\partial_k \langle x \rangle \varphi\|_{L^2}^2 \lesssim C(\phi_0) |z|^4.$$

This concludes the proof since $e_0 + e(z) \leq -c_\gamma < 0$ for γ small enough and $|z| \leq \gamma$. \square

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Part III

Quasilinear probabilistic Cauchy theory

Chapter 4

Refined probabilistic well-posedness for the Schrödinger half-wave equation

joint work with Louise Gassot and Slim Ibrahim [CGI22].

Résumé. Nous montrons le caractère localement bien posé en dessous de l'espace d'énergie pour l'équation de Schrödinger-demi-onde avec une non-linéarité cubique pour presque toutes les données initiales. Pour cette équation, qui est énergie sous-critique, la théorie de Cauchy dans l'espace d'énergie n'a pas encore été établie. En outre, nous montrons que le flot ne peut pas être régulier en dessous de l'espace d'énergie. Afin d'établir une théorie de Cauchy locale probabiliste, nous devons utiliser le calcul paracontrôlé en l'absence d'effet régularisants non-linéaires la deuxième itération de Picard. La preuve est une adaptation de la méthode de Bringmann [Bri21] développée pour une équation d'ondes. Nous discutons par ailleurs du caractère mal-posé de cette équation à basse régularité.

Abstract. We obtain almost-sure local well-posedness below the energy space for the Schrödinger half-wave equation with a cubic nonlinearity. We need to use a refined probabilistic ansatz because of the lack of probabilistic smoothing in the second Picard's iteration, due to the high-low-low nonlinear interactions. The proof is an adaptation of the method of Bringmann [Bri21]. In addition, we discuss ill-posedness results for this equation.

Contents

4.1	Introduction	202
4.2	Set up and preliminaries	209
	4.2.1 Dispersive estimates with a derivative loss	210
	4.2.2 Local existence above the energy space	211
4.3	Iteration scheme	211
	4.3.1 The refined Ansatz	211
	4.3.2 Functional spaces	212
	4.3.3 The recurrence and the truncation method from de Bouard and Debussche	213
4.4	The adapted linear evolution	215

4.4.1	Almost frequency localization	215
4.4.2	Strichartz estimates for the adapted linear evolution	219
4.5	Estimates on the nonlinear term w_N	222
4.5.1	A priori trilinear estimates	223
4.5.2	Contraction mapping argument	225
4.6	Conclusion	226
4.6.1	Optimizing the constraints	226
4.6.2	Convergence of $(u_N)_{N \in \mathbb{2}^{\mathbb{N}}}$	226
4.6.3	The limit is a strong solution	228
4.6.4	The $H_x^{2s} L_y^2$ part	228
4.6.5	Limit of the whole sequence	229
4.7	Paracontrolled trilinear estimates	229
4.8	Remarks on ill-posedness	239
4.8.1	Semilinear ill-posedness	239
4.8.2	Pathological set	240

4.1 Introduction

We consider the Cauchy problem at low regularity for the cubic Schrödinger half-wave equation on \mathbb{R}^2

$$\begin{cases} i\partial_t u + (\partial_{xx} - |D_y|)u = \mu|u|^2 u, & (t, x, y) \in \mathbb{R}^{1+2}, \quad \widehat{|D_y|u}(\xi, \eta) = |\eta|\widehat{u}(\xi, \eta), \\ u_{t=0} = u_0, \end{cases} \quad (\text{NLS-HW})$$

with $\mu \in \mathbb{R}^*$. The study of this equation is motivated by mathematical interests, as a model for understanding weakly dispersive equations that are not completely integrable, but for which some global solutions have unbounded Sobolev norms.

Background on the Schrödinger half-wave equation This model was first introduced by Xu [Xu17] to evidence weak turbulence mechanisms. In the defocusing case ($\mu > 0$), global existence and modified scattering is obtained for a class of sufficiently smooth and decaying small initial data on the wave guide $\mathbb{R}_x \times \mathbb{T}_y$. In addition, the author shows that the limiting effective dynamics is governed by the Szegő equation on the torus. As a consequence of [GG12a; GG17], this analysis provides arbitrarily small initial data such that for every $s > \frac{1}{2}$ and $N \geq 1$, the solution u exhibits weak turbulence in the sense that

$$\limsup_{t \rightarrow \infty} \frac{\|u(t)\|_{L_x^2 H_y^s}}{\log(t)^N} = \infty, \quad \liminf_{t \rightarrow \infty} \|u(t)\|_{L_x^2 H_y^s} < \infty.$$

Subsequently, Bahri, Ibrahim and Kikuchi consider in [BIK20; BIK21] the focusing case $\mu < 0$ and study traveling wave solutions on the wave guide $\mathbb{R}_x \times \mathbb{T}_y$. They obtain orbital stability and transverse instability results, subject to the condition of a good Cauchy theory in the energy space. Unfortunately, such a Cauchy theory is yet to be addressed, and not much is known about the global existence of smooth solutions in Sobolev spaces.

The main concern in this paper is to address the local Cauchy problem at low regularity, for initial data in a *statistical* ensemble. Before presenting our results, let us recall that (NLS-HW) is a Hamiltonian system with a conserved energy

$$H(u) = \frac{1}{2} \int_{\mathbb{R}^2} |\partial_x u|^2 + \left| |D_y|^{\frac{1}{2}} u \right|^2 dx dy + \frac{\mu}{4} \int_{\mathbb{R}^2} |u|^4 dx dy.$$

The L^2 mass is also formally conserved by the flow, and the solutions are left invariant by the scaling symmetry

$$u \mapsto u_\lambda(t, x, y) = \lambda u(\lambda^2 t, \lambda x, \lambda^2 y). \quad (4.1.1)$$

In light of the conservation laws and of the scaling invariance, the relevant regularity spaces for this equation are the following anisotropic Sobolev spaces \mathcal{H}^s defined by

$$\mathcal{H}^s := L_x^2 H_y^s \cap H_x^{2s} L_y^2, \quad \dot{\mathcal{H}}^s := L_x^2 \dot{H}_y^s \cap \dot{H}_x^{2s} L_y^2.$$

The scaling (4.1.1) leaves the $\dot{\mathcal{H}}^{\frac{1}{4}}$ -norm invariant. As a consequence, when $0 < s < \frac{1}{4}$, a *low-to-high frequency cascade* occurs and there is a short-time inflation of the $\dot{\mathcal{H}}^s$ -norm for the regularized solutions [Kat21]. We make precise the type of ill-posedness by describing a *pathological set* in Appendix 4.8. When $\frac{1}{4} < s$, the equation is scaling-subcritical. Yet, semilinear local well-posedness is only known in \mathcal{H}^s when $\frac{1}{2} < s$. Specifically, semilinear well-posedness is obtained in [BIK20] from Strichartz estimates with a derivative loss. Furthermore, we prove in Appendix 4.8 that the flow map cannot be of class \mathcal{C}^3 when $\frac{1}{4} < s < \frac{1}{2}$. To do so, we consider a one-parameter family of traveling wave profiles for the one-dimensional Szegő equation to invalidate some relevant Strichartz estimates. It is therefore not possible to run a contraction mapping argument when $\frac{1}{4} < s < \frac{1}{2}$, since otherwise the flow-map would be analytical. The state-of-the-art Cauchy theory results for (NLS-HW) are summarized in the following diagram:

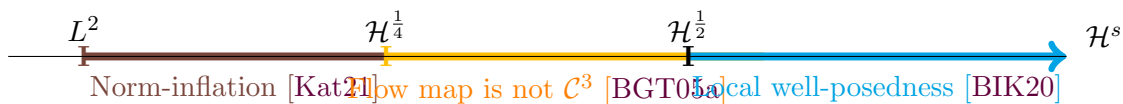


Figure 4.1: Cauchy theory for equation (NLS-HW) in the scale of Sobolev spaces.

Main result As mentioned above, it turns out to be quite challenging to prove semilinear local well-posedness in the energy space, or to run a quasi linear scheme in \mathcal{H}^s when $\frac{1}{4} < s < \frac{1}{2}$. Instead, our goal is to study the *probabilistic* local well-posedness of equation (NLS-HW) in this range of regularities. Given $f_0 \in \mathcal{H}^s(\mathbb{R}^2)$ and a sequence of independent normalized Gaussian variables $(g_k(\omega))_{k \in \mathbb{Z}}$ on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$, we define a random variable

$$\omega \in \Omega \mapsto f_0^\omega := \sum_{k \in \mathbb{Z}} g_k(\omega) P_{1,k} f_0,$$

where $P_{1,k}$ is the partial Fourier projector (in the y -variable) on an interval of unit length centered around $k \in \mathbb{Z}$. We refer to (4.2.1) for the precise definition of the relevant anisotropic Wiener

randomization procedure we employ, and to Figure 4.2. The considered probability measure on \mathcal{H}^s is the induced measure by the above random variable.

However, due to the lack of dispersion and to the absence of regularizing features (such as bilinear estimates or local energy decay), Picard's iterations have the same regularity as the initial data and the standard probabilistic method, which is discussed below, fails. Instead, we need to consider a refined probabilistic ansatz adapted from the probabilistic quasilinear scheme developed by Bringmann [Bri21] to prove probabilistic well-posedness for a derivative wave equation. For $T > 0$ and some s and σ in \mathbb{R} , we denote

$$X_{T_0}^{s,\sigma} := \mathcal{C}_t([-T_0, T_0]; \mathcal{H}^s(\mathbb{R}^2)) \cap L_t^8([-T_0, T_0]; L_x^4 W_y^{\sigma,\infty}(\mathbb{R}^2)). \quad (4.1.2)$$

We also denote the truncated initial data as

$$P_{\leq n} f_0^\omega := \sum_{|k| \leq n} g_k(\omega) P_{1,k} f_0.$$

Our main result reads as follows.

Theorem 4.1.1 (Probabilistic local well-posedness). *Let $s \in (13/28, 1/2]$, some suitable $0 < \sigma < s$ and $f_0 \in \mathcal{H}^s$. There exist $T_0 > 0$ and a sequence $(u_n)_{n \geq 1} \in \mathcal{C}([-T_0, T_0]; \mathcal{H}^\infty)^\mathbb{N}$ converging in expectation to a limiting object denoted u*

$$\lim_{n \rightarrow \infty} \mathbb{E} \left[\|u_n - u\|_{X_{T_0}^{s,\sigma}}^2 \right] = 0, \quad (4.1.3)$$

in such a way that, almost-surely in $\omega \in \Omega$, there exists $T^\omega > 0$ such that for all $n \in \mathbb{N}$, u_n and u exist in $\mathcal{C}([-T^\omega, T^\omega]; \mathcal{H}^s)$ and solve (NLS-HW) with initial data $P_{\leq n} f_0^\omega$ and f_0^ω , respectively.

Remark 4.1.2 (Time of existence). Theorem 1.1 provides a uniform time of existence $T^\omega > 0$ for the smooth solutions u_n initiated from regularized initial data $(P_{\leq n} f_0^\omega)_n$ in the statistical ensemble. This is not a consequence of the known local well-posedness result from [BIK20]. Then, the main claim of Theorem 1.1 is the convergence of $(u_n)_n$ to a strong solution u in \mathcal{H}^s , on the time interval $[-T^\omega, T^\omega]$.

Remark 4.1.3 (Regularizing sequence). Note that the regularizing sequence is general and not dyadic. Actually, we first prove the result for dyadic frequencies N and deduce the convergence for general approximations by adapting an argument from [ST21].

Remark 4.1.4 (Other models). Theorem 1.1 is an adaptation to Schrödinger-type equations of the quasilinear scheme from [Bri21], and allows addressing in the same way other weakly dispersive models. For instance, the case of the one-dimensional half-wave equation and of the Szegő equation are both contained in our analysis by simply forgetting about the variable x everywhere.

Remark 4.1.5 (Nonlinearity). We decided to only consider the cubic nonlinearity on \mathbb{R}^2 for simplicity. But up to some technicalities, other nonlinearities $p \geq 5$ and other geometries (e.g. on the wave guide $\mathbb{R} \times \mathbb{T}$) would work.

Remark 4.1.6 (Sobolev threshold). The threshold $s > s_0$ with $s_0 = 13/28$ is a convenient fractional approximation of the exponent s_0 we will actually get in (4.6.1). This is by no means optimal, and it would be interesting to go all the way down to $s_0 = 1/4$ to cover the whole quasilinear range of exponents.

Background This work is part of the study of nonlinear weakly dispersive equations, in the presence of randomness.

Weakly dispersive equations. There has been a rich activity on the qualitative behavior of solutions to nonlinear evolution equations with weak dispersion. Examples are the half-wave equation [Poc11; Gér+18a] and other fractional Schrödinger equations [FL13; GPH15; ST21], the Szegő equation [GG10; GG12b], the Schrödinger equation on compact manifold [BGT05a], the Schrödinger equation on the Heisenberg group [Gér06] for which there is a lack of dispersion in one direction, the Kadomtsev-Petviashvili equations [HHK09] and many other models. It is in general quite challenging to solve the Cauchy problem for these equations at low regularity. The reason being that in such contexts, dispersive estimates come with a derivative loss. In some special cases where the Hamiltonian equation is integrable in the sense that it has a Lax pair structure providing infinite conservation laws, methods from integrable systems are substituted for the analytical perturbative approach, to study both Cauchy theory and the long-time dynamics.

A first illustrative example one can think of is the Szegő equation, which is known to be globally well-posed in $H^{\frac{1}{2}}_+(\mathbb{T})$ since [GG10]. Using the Lax pair structure, the flow map was extended to BMO in [GK16], then the Cauchy problem was recently shown to be globally well-posed even on $L^2_+(\mathbb{T})$ in [GP22] also by using integrable techniques. Note that the cubic Szegő equation is invariant by a $L^4_+(\mathbb{T}) \hookrightarrow H^{\frac{1}{4}}_+(\mathbb{T})$ -critical scaling. Another model is the derivative Schrödinger equation on the line, which is L^2 -critical with respect to the scaling, but the flow map fails to be uniformly continuous in $H^s(\mathbb{R})$ when $s < \frac{1}{2}$ [BL01; Tak01]. Nevertheless, [Har+22] recently proved from integrable techniques that the equation is globally well-posed in $L^2(\mathbb{R})$. Previously, integrable techniques were combined with concentration-compactness argument in [BP20] to prove global well-posedness in $H^{\frac{1}{2}}(\mathbb{R})$. A third model is the Schrödinger equation on the Heisenberg group, for which local well-posedness is only known to hold in adapted Sobolev spaces $H^s(\mathbb{H}^1)$ for $s > 2$, but is $H^{\frac{1}{2}}$ scaling-invariant, moreover, the flow map is known not to be of class \mathcal{C}^3 for $\frac{1}{2} < s < 2$. The lack of dispersion comes from a special direction, the vertical direction, where the equation is a transport equation rather than a dispersive one.

For the Grushin Laplacian (a simplified version of the Heisenberg Laplacian), a probabilistic approach was attempted in [GL21], proving almost-sure local well-posedness according to a specific randomization procedure which penalizes the direction where there is no dispersion, and gain derivatives in L^p spaces. Finally, we stress out that it is sometimes possible to close the derivative gap (namely $1/4 < s < 1/2$ in our situation) *deterministically* by considering more general function spaces than the Sobolev spaces, such as Fourier-Lebesgue spaces, see e.g. [Grü05] for the derivative NLS equation, and [Grü11] for the wave equation.

Probabilistic Cauchy theory and its limitations. The so-called probabilistic Cauchy theory was initiated by Bourgain [Bou96] and Burq, Tzvetkov [BT08a; BT08b], and can be sketched as follows. When $s < s_c$ is below a critical threshold s_c so that instabilities are known to occur, a general initial data in \mathcal{H}^s may have better integrability properties than expected by the Sobolev embedding. In contexts when the dispersion is strong enough to give some local energy decay or bilinear estimates, or for wave-type equations where the Duhamel formula gains one derivative, one can exploit such an enhanced integrability property to prove that the Picard's iterations are smoother than the linear

evolution of the initial data. A rigorous analysis of this observation makes it possible to evidence strong solutions initiated from *statistical* initial data, according to a non-degenerate probability measure which charges any open set in \mathcal{H}^s .

A *randomized* initial data is typically defined from a given function $\phi \in \mathcal{H}^s$ and its unit-scale frequency decomposition $(\phi_n^\omega)_n$ in the frequency space (or according to a spectral resolution of the Laplace operator in compact settings), where each mode is decoupled by a sequence $(g_n(\omega))_n$ of normalized independent Gaussian variables, defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$:

$$\omega \in \Omega \mapsto \phi^\omega \sim \sum_n g_n(\omega) \phi_n, \quad \text{where } \phi \sim \sum_n \phi_n \in \mathcal{H}^s.$$

Then, for *many* initial data ϕ^ω in a statistical ensemble $\Sigma \subset \mathcal{H}^s$ which has full measure, one expects to observe a *nonlinear smoothing effect* for the recentered solution around the linear evolution thanks to the combination of space-time oscillations (dispersion) and probabilistic oscillations (randomization). Namely, the goal is to show that for some $\nu > s_c$,

$$v(t) := u(t) - e^{it\Delta} \phi^\omega \in \mathcal{C}([-T, T]; \mathcal{H}^\nu) \quad \text{for all } \phi^\omega \in \Sigma.$$

The above recentered solution v is obtained from a fixed point argument at subcritical regularities in \mathcal{H}^ν and solves the original equation perturbed by stochastic terms stemming from the linear evolution $e^{it\Delta} \phi^\omega$. We refer to [BT08a; BT08b] for an introduction to the probabilistic Cauchy theory for dispersive equations.

Nevertheless, to observe the aforementioned probabilistic smoothing effect, we need to exploit dispersive properties of the equations to gain decay without trading regularity. In equation (NLS-HW), however, there is no dispersion in the y -direction. Therefore, in the low \hat{x} -frequency regimes, the Strichartz estimates come with a derivative loss, so that we have neither usable bilinear estimate, nor local smoothing estimates at our disposal. Besides, the second Picard iteration of the randomized initial data does not have a better regularity than the initial data. To see this, we consider *high-low-low* type interactions in high y -frequencies $|\eta| \gg 1$, for an initial data ϕ^ω projected at low x -frequencies $|\xi| \lesssim 1$. For simplicity, we assume that there is no dependence in the variable x , then the *high-low-low* interactions take on the form

$$\int_0^t e^{-i(t-\tau)|D_y|} P_{|\eta| \gg 1} e^{i\tau|D_y|} \phi^\omega \overline{P_{|\eta| \leq 1} e^{i\tau|D_y|} \phi^\omega} P_{|\eta| \leq 1} e^{i\tau|D_y|} \phi^\omega d\tau.$$

Such interactions are basically transported by the half-wave linear flow, so that every derivative of the second Picard's iteration can fall onto the first linear term. Hence, one can only handle s derivatives instead of the desired ν derivatives. Similarly, Oh [Oh11a] considered the Szegő equation on the circle and proved that the first nontrivial Picard's iterate does not gain regularity compared to the initial data.

To undertake this type of issues, Bringmann [Bri21] developed a refined probabilistic ansatz in a quasilinear setting thanks to paracontrolled calculus. In the present work, we adapt the strategy from [Bri21] to prove almost-sure local well-posedness below the energy space, in the quasilinear regime. We conclude this paragraph by mentioning that the paracontrolled approach was further developed in a spectacular way in a series of papers of Deng, Nahmod and Yue [DNY19; DNY21] and Sun, Tzvetkov [ST21] bringing tools from random matrix theory and introducing powerful methods such as the random averaging operators and the random tensors. More recently, the paracontrolled

approach has taken a step forward, and were successfully implied in the resolution by Bringmann, Deng, Nahmod and Yue [Bri+22] of the ϕ_4^3 problem for NLW.

Consequences of Theorem 4.1.1 and perspectives

- From the perspective of probabilistic Cauchy theory, equation (NLS-HW) is the first dispersionless Schrödinger-type equation for which we prove probabilistic well-posedness in a quasilinear regime, where the second Picard's iteration does not gain any regularity. Indeed, we provide a measure induced by a *mild* unit-scale and one-directional randomization procedure in part 4.2.1. In contrast with [GL21] on the Schrödinger equation on the Heisenberg group, the randomization does not gain any regularity in L^p -spaces and does not penalize the dispersionless direction. We believe that the present approach would also be successful in this context.
- From the deterministic perspective, Theorem 4.1.1 yields a statistical ensemble Σ (a dense full-measure set) of initial data leading to strong solutions, in regimes where we show that the equation is semi-linearly ill-posed. The theorem also provides strong solutions in the energy space, which is a first modest step in the comprehension of the long-time Cauchy theory of (NLS-HW), motivated by the rich possible asymptotic behaviors of solutions studied in [Xu17] in the defocusing case and [BIK21; BIK20] in the focusing case. In contrast to the half-wave equation or the Szegő equation on the line, local existence in the energy space $\mathcal{H}^{\frac{1}{2}}$ does not follow from a Yudovich argument since the L^q -norms are not controlled by the energy when $q > 6$. We cannot use the conservation of the energy together with a Brezis-Gallouët estimate either in order to show that in the defocusing case, smooth solutions extend globally in time, since \mathcal{H}^s is not an algebra when $s < \frac{3}{4}$, and there is no conservation law that controls the $\mathcal{H}^{\frac{3}{4}}$ -norm.
- In addition, equation (NLS-HW) can serve as a model to study the long-time behavior of probabilistic solutions generated by the paracontrolled decomposition, in the absence of an invariant measure. Indeed, we construct a probabilistic solution in the presence of a conserved energy which is not enough to globalize the solutions since *probabilistic* information is crucial in the iteration scheme. However, to understand how this information is transported by the flow and to prove quasi-invariance of the measure, one needs dispersion ([OST18]). Hence, understanding the long-time behavior of the solutions provided by Theorem 4.1.1 requires the combination of energy methods with some measure-theory argument. This is a challenging but interesting problem.

Strategy of the proof The proof amounts to show the convergence of $(u_n)_{n \in \mathbb{N}}$ (whose existence is ensured by Theorem 1.6 in [BIK20] recalled in Proposition 4.2.2), on a fixed time interval. First, we establish the convergence along the subsequence $(u_N)_{N \in 2^{\mathbb{N}}}$ by adapting the iterative scheme from [Bri21], which is made up of an induction on frequencies. At every step N of the induction, we consider separately the problematic *high-low-low* frequency interactions, absorbed in the *adapted linear evolution* F_N studied in Section 4.4 and solution to

$$\begin{cases} i\partial_t F_N + (\partial_{xx}^2 - |D_y|)F_N = \mathcal{N}(F_N, P_{\leq N\gamma} u_{\frac{N}{2}}, P_{\leq N\gamma} u_{\frac{N}{2}}). \\ F_N(0) = P_N f_0^\omega, \end{cases}$$

We get *probabilistic Strichartz estimates* for small times, controlling the L^∞ norm of F_N with a loss $N^{\frac{\gamma}{2}-\sigma}$ instead of $N^{\frac{1}{2}}$. To achieve such a goal, we exploit the probabilistic structure of the initial data f_0^ω , which is the sum of terms frequency localized in unit-scale intervals and decoupled by independent Gaussian variables. A key ingredient is the independence between the approximate solution $u_{\frac{N}{2}}$, which is constructed from the low modes $P_{\leq \frac{N}{2}} f_0^\omega$ of the initial data, and $P_N f_0^\omega$. As for the remainder w_N , it solves the equation with zero initial data and stochastic forcing terms but without the singular interaction

$$\mathcal{N}_s(F_N, u_{\frac{N}{2}}, u_{\frac{N}{2}}) := \mathcal{N}(F_N, P_{\leq N^\gamma} u_{\frac{N}{2}}, P_{\leq N^\gamma} u_{\frac{N}{2}}).$$

In order to get convergence on a fixed time interval, not depending on the iteration step N , we follow [Bri21] by making use of the *truncation method* from De Bouard and Debussche [BD99] sketched in Section 4.3.3. We stress out that we only do this in the half-wave variable y , whereas a deterministic analysis is performed in the Schrödinger variable x . Namely, we exploit the dispersion materialized by the Strichartz estimates and use mixed Lebesgue spaces and TT^* -type argument instead of Gronwall inequalities and energy estimates obtained in [Bri21].

Outline of the paper The paper is organized as follows. We first introduce the unit-scale one-directional randomization of the initial in Section 4.2, and prove refined Strichartz estimates for frequency localized functions. Then, we define in Section 4.3 the dyadic subsequence of smooth approximate solutions u_N and set up the truncation method. The adapted linear evolution is handled in Section 4.4, and the a priori bounds for the nonlinear remainders are established in Section 4.5. These bounds rely on paracontrolled trilinear estimates that are detailed in Section 4.7. Since the nonlinearity is cubic, we have to control more various types of interactions than in [Bri21] where the nonlinearity is quadratic. Finally, in Section 4.6, we show convergence of the approximating sequence $(u_N)_N$ to a limit u on a fixed interval, and we prove that it solves the equation (NLS-HW) on a smaller time interval, which is almost-surely nonempty. Then we deduce the convergence of the whole sequence $(u_n)_n$. We conclude by detailing some ill-posedness results for equation (NLS-HW) in Appendix 4.8.

Notation We denote the linear operator $\mathcal{A} = \partial_{xx}^2 - |D_y|$, and the interactions stemming from the cubic nonlinear term by

$$\mathcal{N}(u) = |u|^2 u, \quad \mathcal{N}(u_1, u_2, u_3) = \overline{u_1} u_2 u_3 + u_1 \overline{u_2} u_3 + u_1 u_2 \overline{u_3}.$$

We write \mathcal{F} or $\mathcal{F}_{y \rightarrow \eta}$ the partial Fourier transform with respect to the half-wave direction y , by $\eta = \widehat{y}$ the corresponding Fourier frequency, and by \mathcal{F}^{-1} or $\mathcal{F}_{\eta \rightarrow y}^{-1}$ the inverse Fourier transform. For fixed $\alpha \in \mathbb{R}$, a parameter β satisfies the relation $\beta = \alpha - 0$ if there exists $\varepsilon > 0$ such that $\beta = \alpha - \varepsilon$, where $\varepsilon > 0$ can be chosen arbitrarily close to zero independently of the parameters.

Given two Banach spaces E, F , we denote $\mathcal{B}(E \rightarrow F)$ the set of bounded operators from E to F .

4.2 Set up and preliminaries

Since the standard Strichartz estimates for the one-dimensional free Schrödinger evolution provide control on the norm $L_x^\infty(\mathbb{R})$ without losing derivatives, we only need to consider randomized initial data in the y -direction as in Figure 4.2. We fix $\varphi \in C_c^\infty(\mathbb{R}, [0, 1])$ supported in $[-1, 1]$ and equal to 1 on $[-\frac{1}{2}, \frac{1}{2}]$ such that $\{\varphi(\cdot - k) : k \in \mathbb{Z}\}$ form a partition of unity, and set $\psi(x) = \varphi(x) - \varphi(\frac{x}{2})$. For dyadic M , the spectral projector $P_{M,k}$ around $k \in \mathbb{Z}$ of size M for y -frequencies is defined as follows (see Figure 4.2). For $f_0 \in L^2(\mathbb{R}^2)$ we set

$$\begin{aligned} P_{1,k}f_0 &= \mathcal{F}_{y \rightarrow \eta}^{-1}(\varphi(\eta - k)\mathcal{F}_{y \rightarrow \eta}f_0), \\ P_{M,k}f_0 &= \mathcal{F}_{y \rightarrow \eta}^{-1}(\psi(M^{-1}(\eta - k))\mathcal{F}_{y \rightarrow \eta}f_0). \end{aligned}$$

Given a dyadic integer $N \geq 1$, $P_N := P_{N,0}$ denotes the Littlewood-Paley projector at frequencies $\frac{N}{2} \leq \eta \leq 2N$ in the y -direction only, and \widetilde{P}_N denotes a similar fattened projector using a function $\widetilde{\psi}$ with a slightly larger support. Let $(g_k)_k$ be a sequence of complex-valued independent normalized Gaussian variables. Given $f_0 \in \mathcal{H}^s(\mathbb{R}^2)$, we associate a random function obtained by the Wiener randomization

$$f_0^\omega = \sum_{k \in \mathbb{Z}} g_k(\omega) P_{1,k}f_0. \quad (4.2.1)$$

This randomization induces a non-degenerate probability measure on $\mathcal{H}^s(\mathbb{R}^2)$, densely supported (see [BT08a] Lemma B.1). In addition, for and $N \in 2^{\mathbb{N}}$, we set

$$P_1f_0^\omega = g_0(\omega)P_1f_0, \quad P_Nf_0^\omega = \sum_{\frac{N}{2} \leq |k| < N} g_k(\omega)P_{1,k}f_0, \quad P_{\leq N}f_0^\omega = \sum_{M \leq N} P_Mf_0^\omega.$$

Finally, we emphasize that the family $\{P_Nf_0^\omega\}_{N \geq 1}$ is jointly independent.

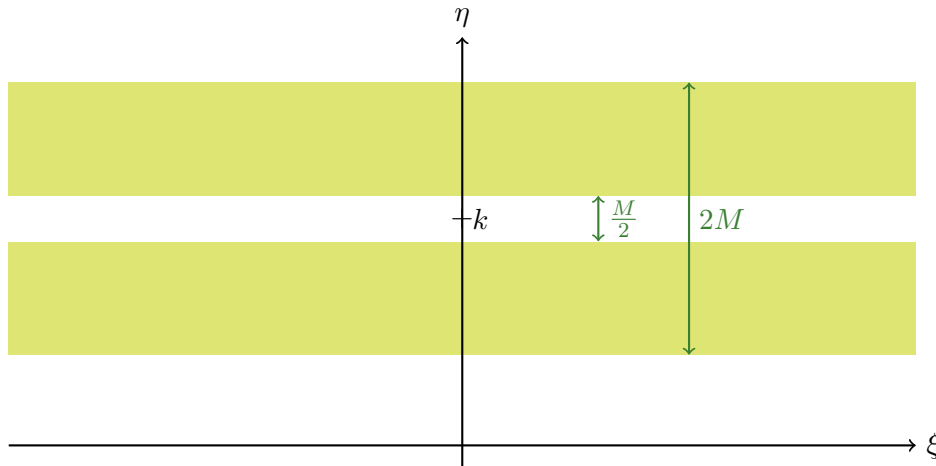


Figure 4.2: Support of the projection $P_{M,k}$ in the frequency space $(\mathbb{R}_\xi, \mathbb{R}_\eta)$.

4.2.1 Dispersive estimates with a derivative loss

We say that the pair (p, q) , with $2 \leq p, q \leq \infty$, is admissible if it is a Schrödinger-admissible pair in dimension 1, i.e.

$$\frac{2}{p} + \frac{1}{q} = \frac{1}{2}. \quad (4.2.2)$$

Lemma 4.2.1 (Bernstein - Strichartz estimates). *Given a Borel function $m : E \rightarrow \mathbb{C}$ with support in a bounded measurable set $E \subset \mathbb{R}$ of finite Lebesgue measure $|E| < \infty$, we define T_E to be the linear evolution frequency-localized in E :*

$$T_E = e^{itA} \mathcal{F}_{\eta \rightarrow y}^{-1}(m(\eta) \mathcal{F}_{y \rightarrow \eta}) \in \mathcal{B}(L_{x,y}^2(\mathbb{R}^2) \rightarrow L_{x,y}^2(\mathbb{R}^2)), \quad t \in \mathbb{R}.$$

Given (p, q) and (\tilde{p}, \tilde{q}) two admissible pairs as in (4.2.2), and $r \in [2, \infty]$, we have the following Strichartz estimates with a derivative loss:

- Bernstein-Strichartz estimate

$$\|T_E f_0\|_{L_t^p L_x^q L_y^r} \lesssim |E|^{\frac{1}{2} - \frac{1}{r}} \|f_0\|_{L_{x,y}^2}, \quad (4.2.3)$$

- T^* -estimate

$$\left\| \int_{\mathbb{R}} e^{-i\tau A} \mathcal{F}_{\eta \rightarrow y}^{-1}(m(\eta) \mathcal{F}_{y \rightarrow \eta}) f_0(\tau) d\tau \right\|_{L_{x,y}^2} \lesssim |E|^{\frac{1}{2} - \frac{1}{r}} \|f_0\|_{L_t^{\tilde{p}} L_x^{\tilde{q}} L_y^{\tilde{r}}},$$

- TT^* -estimate

$$\left\| \int_{\mathbb{R}} \mathbf{1}_{0 \leq \tau \leq t} e^{i(t-\tau)A} \mathcal{F}_{\eta \rightarrow y}^{-1}(m(\eta) \mathcal{F}_{y \rightarrow \eta}) f_0(\tau) d\tau \right\|_{L_t^p L_x^q L_y^r} \lesssim |E|^{\frac{1}{2} - \frac{1}{r}} \|f_0\|_{L_t^{\tilde{p}} L_x^{\tilde{q}} L_y^{\tilde{r}}}. \quad (4.2.4)$$

Notice that when $r = 2$ there is no loss in the Bernstein estimate and we can choose $E = \mathbb{R}$.

Proof. In the mixed Lebesgue norms, we first use the unit-scale Bernstein estimate in the y -direction to pass from L_y^r to L_y^2 , and then the unitarity of $e^{it|D_y|}$ on L_y^2 to reduce the matter to the free Schrödinger evolution on the line. There holds

$$\|T_E f_0\|_{L_t^p L_x^q L_y^r} \lesssim |E|^{\frac{1}{2} - \frac{1}{r}} \|e^{itA} f_0\|_{L_t^p L_x^q L_y^2} = |E|^{\frac{1}{2} - \frac{1}{r}} \| \|e^{it\partial_{xx}^2} f_0\|_{L_y^2} \|_{L_t^p L_x^q}.$$

Subsequently, we use the Minkowski inequality and the Strichartz estimate for $e^{it\partial_{xx}^2}$ in the x -direction to conclude that

$$|E|^{\frac{1}{2} - \frac{1}{r}} \| \|e^{it\partial_{xx}^2} f_0\|_{L_y^2} \|_{L_t^p L_x^q} \lesssim |E|^{\frac{1}{2} - \frac{1}{r}} \| \|e^{it\partial_{xx}^2} f_0\|_{L_t^p L_x^q} \|_{L_y^2} \lesssim |E|^{\frac{1}{2} - \frac{1}{r}} \|f_0\|_{L_{x,y}^2}.$$

The T^* -estimate follows from duality, and the TT^* -estimate is a consequence of the first two estimates, by using the T^* estimate with $r = 2$. We also need an application of the standard Christ-Kiselev argument to localize in time $0 \leq \tau \leq t$. \square

4.2.2 Local existence above the energy space

We recall the best known local well-posedness result for equation (NLS-HW).

Proposition 4.2.2 (Local well-posedness above the energy space, [BIK20] Theorem 1.6). *Let $s > \frac{1}{2}$. For every $f_0 \in \mathcal{H}^s(\mathbb{R}^2)$, there exists $T = T(\|f_0\|_{\mathcal{H}^s}) > 0$ such that equation (NLS-HW) admits a unique local solution in $\mathcal{C}((-T, T), \mathcal{H}^s)$.*

It is shown in [BIK20] that local well-posedness actually holds in $L_x^2 H_y^s$. The proof follows from a fixed point argument in $L_t^\infty([-T, T]; \mathcal{H}^s(\mathbb{R}^2)) \cap L_t^4([-T, T]; L_{x,y}^\infty(\mathbb{R}^2))$, for some $T > 0$ depending on the $\mathcal{H}^s(\mathbb{R}^2)$ -norm of the initial data. The use of the Strichartz space $L_t^4([-T, T]; L_{x,y}^\infty(\mathbb{R}^2))$ requires that $s > \frac{1}{2}$ because of the lack of dispersion in the y -direction. Consequently, we have no control on the growth of the \mathcal{H}^s -norm of the solution since we do not have access to the conserved energy, which is at the level of $\mathcal{H}^{\frac{1}{2}}$. As discussed in the introduction, local existence for smooth solutions to (NLS-HW) is an open problem.

4.3 Iteration scheme

4.3.1 The refined Ansatz

We construct the solution to (NLS-HW) by iteration on the frequencies, that we regroup as dyadic packets. Let $N \in 2^{\mathbb{N}}$, and let u_N be the maximal solution¹ to (NLS-HW) with initial data localized at frequencies at most N

$$\begin{cases} (i\partial_t + \mathcal{A})u_N = |u_N|^2 u_N, \\ u_N(0) = P_{\leq N} f_0^\omega. \end{cases}$$

We write $u_N = u_{\frac{N}{2}} + v_N$, so that $v_N(0)$ is localized at frequency $\sim N$, and v_N solves

$$\begin{cases} (i\partial_t + \mathcal{A})v_N = \mathcal{N}(v_N + u_{\frac{N}{2}}) - \mathcal{N}(u_{\frac{N}{2}}), \\ v_N(0) = P_N f_0^\omega. \end{cases}$$

Then, we consider at each step n the adapted *linear-nonlinear decomposition* $v_N = F_N + w_N$, defined as follows.

- The adapted linear evolution F_N is solution to a linear Schrödinger half-wave equation that encapsulates the bad nonlinear interactions, with initial data $P_N f_0^\omega$ localized at frequency $\sim N$:

$$\begin{cases} (i\partial_t + \mathcal{A})F_N = \mathcal{N}(F_N, P_{\leq N^\gamma} u_{\frac{N}{2}}, P_{\leq N^\gamma} u_{\frac{N}{2}}), \\ F_N(0) = P_N f_0^\omega. \end{cases}$$

The parameter $0 < \gamma < 1$, to be determined later in the analysis, separates two ranges of frequency scales $N \gg N^\gamma$.

¹Observe that the local well-posedness result from [BIK20], recalled in Proposition 4.2.2, implies existence and uniqueness for u_N in $\mathcal{H}^{\frac{1}{2}+}(\mathbb{R}^2)$ up to a time $T(n) > 0$. In Theorem 4.1.1 we claim that (u_N) converges almost-surely in $\mathcal{H}^s(\mathbb{R}^2)$ to a strong solution to (NLS-HW) up to a time $T^\omega > 0$ almost-surely in the randomization.

- The nonlinear remainder w_N is solution to (NLS-HW) with a stochastic forcing term and zero initial condition:

$$\begin{cases} (i\partial_t + \mathcal{A})w_N = \mathcal{N}(u_N) - \mathcal{N}(u_{\frac{N}{2}}) - \mathcal{N}(F_N, P_{\leq N^\gamma} u_{\frac{N}{2}}, P_{\leq N^\gamma} u_{\frac{N}{2}}), \\ w_N(0) = 0. \end{cases} \quad (4.3.1)$$

The key point is that the singular interactions between the *high* frequencies of the initial data and the *low* frequencies of the approximate solution $u_{N/2}$ are removed from the equation satisfied by w_N . Therefore, we can expect that a smoothing effect now occurs on w_N . Note that at each step we can decompose the solution as a series

$$u_N = u_{\frac{N}{2}} + F_N + w_N = u_{N_0} + \sum_{L=2N_0+1}^N (w_L + F_L^\omega). \quad (4.3.2)$$

The goal is to prove that the series with general term $(w_N)_{N \geq N_0}$ almost-surely converges in a *subcritical* space $\mathcal{C}([-T_0, T_0]; \mathcal{H}^\nu(\mathbb{R}^2))$, for some $\nu > \frac{1}{2}$ and some $0 < T_0 \ll 1$, and the series with general term $(F_N)_{N \geq N_0}$ converges almost-surely in $\mathcal{C}([-T_0, T_0]; \mathcal{H}^s(\mathbb{R}^2))$. In addition, we prove that there exists a time $T^\omega > 0$ such that the limit of (u_N) solves (NLS-HW) in $\mathcal{C}([-T^\omega, T^\omega]; \mathcal{H}^s(\mathbb{R}^2))$.

4.3.2 Functional spaces

In the nonlinear analysis, we need to keep track of the frequency localization of the adapted linear evolution F_N . Specifically, in order to avoid the *high-low-low* interactions in the nonlinear analysis that are responsible for the lack of nonlinear smoothing, we want to ensure that for a given N , the y -frequencies of the functions w_N and F_N are localized at size $|\eta| \sim N$. To capture such a frequency localization, rigorously stated in Lemmas 4.4.1 and 4.4.2, we fix a parameter $D > 0$ and some weights

$$C_{N,D}(M) := \max\left(\frac{N}{M}, \frac{M}{N}\right)^D, \quad C_{\leq N,D}(M) := \max\left(1, \frac{M}{N}\right)^D.$$

Then we set the weighted norms

$$\|u\|_{X_{N,D}[-T,T]} := \sum_{M \geq 1} C_{N,D}(M) \|P_M u\|_{L_t^\infty([-T,T], L_{x,y}^2(\mathbb{R}^2))}, \quad (4.3.3)$$

$$\|u\|_{X_{\leq N,D}[-T,T]} := \sum_{M \geq 1} C_{\leq N,D}(M) \|P_M u\|_{L_t^\infty([-T,T], L_{x,y}^2(\mathbb{R}^2))}.$$

For some $r \in [2, \infty]$, we define as well some frequency-localized mixed Strichartz norms

$$\begin{aligned} \|u\|_{S_{N,D}^r[-T,T]} &:= \sum_{M \geq 1} C_{N,D}(M) \|P_M u\|_{L_t^r([-T,T], L_x^4 L_y^r(\mathbb{R}^2))}, \\ \|u\|_{S_{\leq N,D}^r[-T,T]} &:= \sum_{M \geq 1} C_{\leq N,D}(M) \|P_M u\|_{L_t^r([-T,T], L_x^4 L_y^r(\mathbb{R}^2))}. \end{aligned}$$

When $r = \infty$ we simply write

$$S_{N,D} := S_{N,D}^\infty, \quad S_{\leq N,D} := S_{\leq N,D}^\infty.$$

The frequency localization of F_N^ω around frequencies N stems from the structure of F_N . Specifically, each F_N is a superposition of functions $(F_{N,k})_{\frac{N}{2} \leq |k| \leq N}$, decoupled by independent Gaussian variables. We observe that $F_{N,k}$ is solution to a linear Schrödinger equation with a potential truncated at frequencies of size $\lesssim N^\gamma$, with initial data $P_{1,k}f_0$ for some $|k| \sim N$. Hence, at least in short-time, we expect $F_{N,k}$ to be localized at a distance $\lesssim N^\gamma$ from the frequency k . In particular,

$$|\text{supp } \mathcal{F}(F_{N,k})| \lesssim N^\gamma \quad \text{a.s.} \quad (4.3.4)$$

This key frequency localization would reduce the derivative loss in the Strichartz estimates from Lemma 4.2.1 for $F_{N,k}$, and therefore for F_N by using probabilistic decoupling. To encapsulate (4.3.4), rigorously stated in Lemma 4.4.1, we follow [Bri21] and we consider the recentered Besov spaces

$$\|f\|_{B_{k,D}^{\rho,\gamma}[-T,T]} := \sum_{M \geq 1} c_{k,D}^{\rho,\gamma}(M) \|P_{M,k}f\|_{L_t^\infty([-T,T]; L_{x,y}^2(\mathbb{R}^2))}, \quad c_{k,D}^{\rho,\gamma}(M) := M^\rho \max\left(1, \frac{M}{N^\gamma}\right)^D. \quad (4.3.5)$$

The weight M^ρ accounts for an extra gain of derivative $\rho = \sigma - 0$ on $F_{N,k}$ that comes from the frequency localization of $P_{M,k}F_{N,k}$. In Section 4.6, we implement a fixed point argument on the nonlinear term w in the space associated with the norm

$$\|w\|_{Y_N^\nu} = \|\langle D_y \rangle^\nu w\|_{X_{N,\alpha}} + \|\langle D_y \rangle^\nu w\|_{X_{\leq N,D}} + \|\langle D_y \rangle^\sigma w\|_{S_{N,\alpha}} + \|\langle D_y \rangle^\sigma w\|_{S_{\leq N,D}},$$

where α, D are parameters, $\nu > \frac{1}{2}$ is the regularity of the remainder w_N determined by $0 < \sigma = \nu - \frac{1}{2} - 0$ from the Sobolev embedding $H_y^\nu \hookrightarrow W_y^{\sigma,\infty}$.

4.3.3 The recurrence and the truncation method from de Bouard and Debussche

In contrast with the standard probabilistic well-posedness argument implemented in [BT08a], which consists in a single fixed point argument for a fixed $\omega \in \Omega$, we cannot solve the equation pathwise. The reason is that we need to perform a fixed point argument at each step of the iteration scheme, leading to a sequence of times $(T_N(\omega))_N$ that would depend intricately on ω . In particular, we cannot ensure that this sequence is uniformly bounded from below in n . Instead, we do have to work in a space $L^\rho(\Omega, \mathcal{E})$ for some $\rho \geq 1$, where \mathcal{E} is the space-time functional space in which we establish local existence.

To handle a similar problem arising in the study of a Schrödinger equation with a multiplicative white noise [BD99], de Bouard and Debussche truncated the nonlinearity. This allows to perform a fixed-point argument in some space $L^\rho(\Omega, \mathcal{E})$ despite the absence of gain of integrability in the ω -variable. We refer to [Bri21] for some discussion on motivation for the truncation method of de Bouard and Debussche. The method consists in fixing a smooth function $\theta \in C_c^\infty(\mathbb{R}, [-1, 1])$ compactly supported in $(-2, 2)$ and such that $\theta \equiv 1$ on $[-1, 1]$, and to essentially reduce the matter to a linear problem by assuming most of the terms to be at most 1 in the nonlinearity. Let $F_{N,\theta}$ and

$w_{M,\theta}$ to be determined later as the solutions to some equation. The cutoff functions are denoted

$$\begin{aligned}\theta_{F;N}(\tau) &= \theta \left(\|\langle D_y \rangle^{\sigma'} F_{N,\theta}\|_{S_{\leq N, D'}[-\tau, \tau]} \right), \\ \theta_{w;N}(\tau) &= \theta \left(\|\langle D_y \rangle^{\sigma} w_{N,\theta}\|_{Y_N^{\nu}[-\tau, \tau]} \right), \\ \theta_{F,w;\leq \frac{N}{2}}(\tau) &= \theta \left(\sum_M^{N/2} \|\langle D_y \rangle^{\sigma'} F_{M,\theta}^{\omega}\|_{S_{\leq M, D'}[-\tau, \tau]} + \|\langle D_y \rangle^{\sigma} w_{M,\theta}\|_{S_{\leq M, D}[-\tau, \tau]} \right. \\ &\quad \left. + \|\langle D_y \rangle^{\nu} w_{M,\theta}\|_{X_{\leq M, D}[-\tau, \tau]} \right).\end{aligned}\tag{4.3.6}$$

We define by induction $F_{N,\theta}$ and $w_{N,\theta}$ as solutions to the following truncated equations. First,

$$F_{N,\theta}(t) = e^{itA} P_N f_0^{\omega} + i \int_0^t e^{i(t-\tau)A} \theta_{F,w;\leq \frac{N}{2}}(\tau)^2 \mathcal{N}(F_{N,\theta}, P_{\leq N\gamma} u_{\frac{N}{2},\theta}, P_{\leq N\gamma} u_{\frac{N}{2},\theta})(\tau) d\tau, \tag{4.3.7}$$

and similarly, the block $F_{N,k,\theta}$ is solution to the decomposed truncated equation

$$F_{N,k,\theta}(t) = e^{itA} P_{1,k} f_0 + i \int_0^t e^{i(t-\tau)A} \theta_{F,w;\leq \frac{N}{2}}(\tau)^2 \mathcal{N}(F_{N,k,\theta}, P_{\leq N\gamma} u_{\frac{N}{2},\theta}, P_{\leq N\gamma} u_{\frac{N}{2},\theta})(\tau) d\tau, \tag{4.3.8}$$

so that

$$F_{N,\theta} = \sum_{\frac{N}{2} \leq |k| < N} g_k(\omega) F_{N,k,\theta}. \tag{4.3.9}$$

Then, the truncated nonlinear remainder $w_{N,\theta}$ solves

$$w_{N,\theta}(t) = i \int_0^t e^{i(t-\tau)A} \tilde{\mathcal{N}}_{\theta}(w_{N,\theta})(\tau) d\tau, \tag{4.3.10}$$

where the nonlinearity $\tilde{\mathcal{N}}_{\theta}(w_{N,\theta})$ is given by

$$\begin{aligned}\tilde{\mathcal{N}}_{\theta}(w_{N,\theta}) &= \theta_{F;N}^2 \mathcal{N}(F_{N,\theta}) + \theta_{F;N}^2 \mathcal{N}(F_{N,\theta}, F_{N,\theta}, w_{N,\theta}) + \theta_{F;N} \theta_{w;N} \mathcal{N}(F_{N,\theta}, w_{N,\theta}, w_{N,\theta}) \\ &\quad + \theta_{w;N}^2 \mathcal{N}(w_{N,\theta}) + \theta_{F,w;\leq \frac{N}{2}}^2 \mathcal{N}(u_{\frac{N}{2},\theta}, u_{\frac{N}{2},\theta}, w_{N,\theta}) + \theta_{F,w;\leq \frac{N}{2}} \theta_{w;N} \mathcal{N}(u_{\frac{N}{2},\theta}, w_{N,\theta}, w_{N,\theta}) \\ &\quad + 2\theta_{F,w;\leq \frac{N}{2}} \theta_{F;N} \mathcal{N}(u_{\frac{N}{2},\theta}, w_{N,\theta}, F_{N,\theta}) + \theta_{F,w;\leq \frac{N}{2}} \theta_{F;N} \mathcal{N}(F_{N,\theta}, F_{N,\theta}, u_{\frac{N}{2},\theta}) \\ &\quad + \theta_{F,w;\leq \frac{N}{2}}^2 \left(\mathcal{N}(F_{N,\theta}, u_{\frac{N}{2},\theta}, u_{\frac{N}{2},\theta}) - \mathcal{N}(F_{N,\theta}, P_{\leq N\gamma} u_{\frac{N}{2},\theta}, P_{\leq N\gamma} u_{\frac{N}{2},\theta}) \right).\end{aligned}\tag{4.3.11}$$

Finally, we denote the truncated approximate solution

$$u_{N,\theta} = u_{\frac{N}{2},\theta} + F_{N,\theta} + w_{N,\theta}.$$

We prove the convergence of $(u_{N,\theta})_N$ on a time-interval $[-T_0, T_0]$. Subsequently, we use a priori estimates and a continuity argument to remove the truncation, and to get a solution to the original equation (4.3.1) on a sub-interval $[-T^{\omega}, T^{\omega}]$ with $0 < T^{\omega} \leq T_0$ almost-surely. This is the matter of Section 4.6, once we solved the fixed point for the truncated equations and proved the convergence of the approximate solutions $(u_{N,\theta})_N$ in $L^2(\Omega; \mathcal{E})$.

4.4 The adapted linear evolution

From now on, and until Section 4.6, we often drop the index θ for simplicity in the notation. At each step n , the adapted linear evolution is decomposed as (4.3.9)

$$F_N = \sum_{\frac{N}{2} \leq |k| < N} g_k(\omega) F_{N,k},$$

where each unit block $F_{N,k}$ is the solution to the linear equation

$$\begin{cases} (i\partial_t + \mathcal{A})F_{N,k} = \theta_{F,w;\leq \frac{N}{2}}^2 \mathcal{N}(F_{N,k}, P_{\leq N^\gamma u_{\frac{N}{2}}}, P_{\leq N^\gamma u_{\frac{N}{2}}}), \\ F_{N,k}(0) = P_{1,k} f_0. \end{cases}$$

We denote the *potential* by $\phi := \theta_{F,w;\leq \frac{N}{2}} P_{\leq N^\gamma u_{\frac{N}{2}}}$, and we use the following features of ϕ in this section:

- ϕ is localized at low frequencies $\lesssim N^\gamma$;
- ϕ satisfies the following a priori estimate, encoded in the truncation defined in (4.3.6)

$$\sum_M \|\langle D_y \rangle^\sigma \phi\|_{S_{M,D}[-\tau,\tau]} \leq 1, \quad \tau \in \mathbb{R};$$

- ϕ is measurable in the σ -algebra $\mathcal{B}_{< \frac{N}{2}}$ generated by the random variables $(g_k)_{|k| < \frac{N}{2}}$. Therefore, ϕ is independent of the Gaussian variables $(g_k)_{\frac{N}{2} \leq |k| < N}$.

4.4.1 Almost frequency localization

In what follows, we fix $n \geq 1$, $N = 2^n$, $k \in \mathbb{Z}$ with $\frac{N}{2} \leq |k| < N$, and the potential $\phi = \theta_{F,w;\leq \frac{N}{2}} P_{\leq N^\gamma u_{\frac{N}{2}}}$ is as above. First, we prove the key frequency localization of $F_{N,k}$ at distance $\lesssim N^\gamma$ from k , with a gain of $\rho = \sigma - 0$ derivatives captured by the recentered Besov spaces $B_{k,D}^{\rho,\gamma}$ defined in (4.3.5). This stems from the localization of the initial condition $P_{1,k} f_0$ in a unit-scale interval around k , together with the localization of the potential ϕ .

Lemma 4.4.1 (Frequency localization of the individual blocks). *Let $D'' \gg 1$. Then, for $0 \leq \rho < \sigma$ arbitrarily close to σ , there exist $0 < T_0$ and $0 < C$ such that for every N and $\frac{N}{2} \leq |k| < N$ the solution $F_{N,k}$ to (4.3.8) satisfies*

$$\|F_{N,k}\|_{B_{k,D''}^{\rho,\gamma}[-T_0,T_0]} \leq C \|P_{1,k} f_0\|_{L_{x,y}^2}. \quad (4.4.1)$$

As a consequence,

$$\|\mathbf{1}_{\frac{N}{2} \leq |k| < N} F_{N,k}\|_{\ell_k^2(\mathbb{Z}; B_{k,D''}^{\rho,\gamma}[-T_0,T_0])} \leq C \|\widetilde{P}_N f_0\|_{L_{x,y}^2}. \quad (4.4.2)$$

The proof follows the same lines as the proof of Proposition 4.1 in [Bri21]. Nevertheless, we use the Duhamel formula and the Strichartz estimate in place of a Gronwall argument in order to avoid using the quantity $L_{x,y}^\infty(\mathbb{R}^2)$ that we do not control.

Proof. The estimate (4.4.2) follows from (4.4.1) by using that

$$\|\mathbf{1}_{\frac{N}{2} \leq |k| < N} P_{1,k} f_0\|_{\ell_k^2(\mathbb{Z}; L_{x,y}^2(\mathbb{R}^2))} \lesssim \|\tilde{P}_N f_0\|_{L_{x,y}^2}.$$

We prove (4.4.1): given $0 < T_0 \leq 1$, we show that for all $D'' > 0$, there exists $C > 0$ such that for all $k \in \mathbb{Z}$ and $0 \leq T \leq T_0$,

$$\|F_{N,k}\|_{B_{k,D''}^{\rho,\gamma}[-T,T]} \leq \|P_{1,k} f_0\|_{L_{x,y}^2} + CT^{\frac{1}{2}} \|F_{N,k}\|_{B_{k,D''}^{\rho,\gamma}[-T,T]} \|\langle D_y \rangle^\sigma \phi\|_{L_t^8 L_x^4 L_y^\infty}^2. \quad (4.4.3)$$

We then use the truncation θ to see that the norms of the potential is less than one, and we chose $T_0 > 0$ (independent of N) such that $CT_0^{\frac{1}{2}} \leq \frac{1}{2}$. Let us establish (4.4.3) for fixed $k \in \mathbb{Z}$. First, observe that

$$\|P_{1,k} f_0\|_{B_{k,D''}^{\rho,\gamma}[-T,T]} \lesssim \|P_{1,k} f_0\|_{L_{x,y}^2}.$$

Next, it follows from the definition of the Besov norm (4.3.5) and from the Duhamel formula that we need to estimate

$$\sum_M c_{k,D''}^{\rho,\gamma}(M) \left\| \int_0^t e^{i(t-\tau)\mathcal{A}} P_{M,k}(\mathcal{N}(F_{N,k}, \phi, \phi))(\tau) d\tau \right\|_{L_{x,y}^2}.$$

We choose the Strichartz admissible pairs $(p, q) = (\infty, 2)$, $(\tilde{p}, \tilde{q}) = (4, \infty)$ and $r = 2$, then we apply the TT^* -estimate from Lemma 4.2.1 (recall that when $r = 2$ there is no need to apply the Bernstein estimate, and there is no derivative loss). For fixed M , we have

$$\left\| \int_0^t e^{i(t-\tau)\mathcal{A}} P_{M,k}(\mathcal{N}(F_{N,k}, \phi, \phi))(\tau) d\tau \right\|_{L_t^\infty L_{x,y}^2} \lesssim \|P_{M,k}(\mathcal{N}(F_{N,k}, \phi, \phi))\|_{L_t^{\frac{4}{3}} L_x^1 L_y^2}.$$

We do a Littlewood-Paley decomposition of $F_{N,k}(t)$ with frequencies K centered around k , and a standard Littlewood-Paley decomposition of ϕ with frequencies L_1, L_2 . Without loss of generality, we assume that $L_2 \leq L_1$. There holds

$$\|P_{M,k}(\mathcal{N}(F_{N,k}, \phi, \phi))\|_{L_t^{\frac{4}{3}} L_x^1 L_y^2} \lesssim \sum_K \sum_{L_2 \leq L_1 \lesssim N^\gamma} \|P_{M,k}(\mathcal{N}(P_{K,k} F_{N,k}, P_{L_1} \phi, P_{L_2} \phi))\|_{L_t^{\frac{4}{3}} L_x^1 L_y^2}.$$

Then, the interactions that contribute to the above sum are the following:

- $K \sim M$,
- $K \ll M$ and $L_1 \sim M$,
- $K \gg M$ and $L_1 \sim K$,

Hence,

$$\begin{aligned} & \|P_{M,k}(\mathcal{N}(F_{N,k}, \phi, \phi))\|_{L_t^{\frac{4}{3}} L_x^1 L_y^2} \lesssim T^{\frac{1}{2}} \left(\sup_L L^\sigma \|P_L \phi\|_{L_t^8 L_x^4 L_y^\infty} \right)^2 \\ & \left(\sum_{K \sim M} \|P_{K,k} F_{N,k}\|_{L_t^\infty L_{x,y}^2} + \sum_{K: K \ll M \sim L_1} L_1^{-\sigma} \|P_{K,k} F_{N,k}\|_{L_t^\infty L_{x,y}^2} + \sum_{K: K \sim L_1 \gg M} L_1^{-\sigma} \|P_{K,k} F_{N,k}\|_{L_t^\infty L_{x,y}^2} \right). \end{aligned}$$

It follows from the definition (4.3.5) of weights $c_{k,D''}^{\rho,\gamma}(K)$ and from the truncation (4.3.8) that

$$\sup_L L^\sigma \|P_L \phi\|_{L_t^s L_x^4 L_y^\infty} \lesssim 1.$$

Therefore,

$$\begin{aligned} & \|P_{M,k}(\mathcal{N}(F_{N,k}, \phi, \phi))\|_{L_t^{\frac{4}{3}} L_x^1 L_y^2} \lesssim T^{\frac{1}{2}} \sum_{K \sim M} \|P_{K,k} F_{N,k}(t)\|_{L_{x,y}^2} \\ & + T^{\frac{1}{2}} \left(\sum_{K \ll M \sim L_1} K^{-\rho} L_1^{-\sigma} + \sum_{L_1 \sim K \gg M} K^{-\rho} \max\left(1, \frac{K}{N^\gamma}\right)^{-D''} L_1^{-\sigma} \right) \|F_{N,k}(t)\|_{B_{k,D''}^{\rho,\gamma}[-T,T]}. \end{aligned}$$

Next, we multiply by $c_{k,D''}^{\rho,\gamma}(M) = M^\rho \max\left(1, \frac{M}{N^\gamma}\right)^D$ and we sum over M to obtain

$$\begin{aligned} \|F_{N,k}(t)\|_{B_{k,D''}^{\rho,\gamma}[-T,T]} & \lesssim \|P_{1,k} f_0\|_{L_{x,y}^2} + T^{\frac{1}{2}} \sum_{M \sim N} c_{k,D''}^{\rho,\gamma}(M) \|P_{M,k} F_{N,k}\|_{L_t^\infty L_{x,y}^2} \\ & + T^{\frac{1}{2}} \left(\sum_{M \lesssim N^\gamma} M^{\rho-\sigma} \right) \|F_{N,k}\|_{B_{k,D''}^{\rho,\gamma}[-T,T]}. \end{aligned}$$

The assumption that $\sigma > \rho$ allows us to conclude. \square

Now, we prove that F_N is essentially localized at frequency N . This turns out to be crucial in the estimates for the remainder w_N , in the situations when we want to avoid the *high-low-low* frequency interactions. We recall that such a frequency localization is captured by the weighted norm (4.3.3).

Lemma 4.4.2 (Frequency localization for the adapted evolution). *Let $D'' > 0$. There exists $T_0 > 0$ such that F_N satisfies*

$$\|\langle D_y \rangle^s F_N^\omega\|_{X_{N,D'}[-T_0, T_0]} \lesssim \|\widetilde{P}_N f_0^\omega\|_{\mathcal{H}^s(\mathbb{R}^2)}.$$

Proof. Given $M \gg 1$ and $0 \leq T \leq 1$, we need to estimate

$$M^s C_{N,D'}(M) \|P_M F_N\|_{L_t^\infty([-T, T]; L_{x,y}^2)}.$$

We recall that F_N is solution to the truncated equation (4.3.7), and that $\phi(t) = \theta_{F, w; \leq \frac{N}{2}} P_{\leq N^\gamma} u_{\frac{N}{2}}(t)$.

-Case when $M \sim N$. In contrast with [Bri21], we do not use an energy estimate. Instead, we proceed as in the proof of Lemma 4.4.1 in order to avoid using $L_{x,y}^\infty$. We start from the Duhamel formula, and we estimate

$$F_N(t) = P_N e^{it\mathcal{A}} f_0^\omega + \int_0^t e^{i(t-\tau)\mathcal{A}} \mathcal{N}(F_N, \phi, \phi)(\tau) d\tau.$$

From the Bernstein-Strichartz TT^* -estimate in Lemma 4.4.1, with $(p, q) = (\infty, 2)$, $(\widetilde{p}, \widetilde{q}) = (4, \infty)$ and $r = 2$, we obtain

$$\|F_N\|_{L_t^\infty L_{x,y}^2} \lesssim \|P_N f_0^\omega\|_{L_{x,y}^2} + \|\mathcal{N}(F_N, \phi, \phi)\|_{L_t^{\frac{4}{3}} L_x^1 L_y^2}.$$

Thanks to Hölder's inequality and to the truncation, we know that on $[-T, T]$,

$$\|\mathcal{N}(F_N, \phi, \phi)\|_{L_t^{\frac{4}{3}} L_x^1 L_y^2} \lesssim T^{\frac{1}{2}} \|F_N\|_{L_t^\infty L_{x,y}^2} \|\phi\|_{L_t^8 L_x^4 L_y^\infty}^2 \lesssim T^{\frac{1}{2}} \|F_N\|_{L_t^\infty L_{x,y}^2}.$$

Choosing T_0 small enough we conclude that if $T \leq T_0$,

$$\|F_N\|_{L_t^\infty L_{x,y}^2} \lesssim \|P_N f_0^\omega\|_{L_{x,y}^2}.$$

Therefore, when $M \sim N$, we deduce that

$$M^s \|P_M F_N\|_{L_t^\infty L_{x,y}^2} \lesssim N^s \|F_N\|_{L_t^\infty L_{x,y}^2} \lesssim \|\langle D_y \rangle^s \widetilde{P}_N f_0^\omega\|_{L_{x,y}^2}.$$

-*Case when $M \approx N$.* In this case, we first fix k with $\frac{N}{2} \leq |k| < N$ and we perform a dyadic decomposition in frequency around k for $F_{N,k}$. We see that when $M \ll N$,

$$\|P_M F_{N,k}\|_{L_t^\infty L_{x,y}^2} \lesssim \sum_{K \sim N} \|P_{K;k} F_{N,k}\|_{L_t^\infty L_{x,y}^2} \lesssim \left(\frac{N}{N^\gamma}\right)^{-D''} \|F_{N,k}\|_{B_{k,D''}^{\rho,\gamma}},$$

whereas when $N \ll M$, we have

$$\|P_M F_{N,k}\|_{L_t^\infty L_{x,y}^2} \lesssim \sum_{K \sim M} \|P_{K;k} F_{N,k}\|_{L_t^\infty L_{x,y}^2} \lesssim \left(\frac{M}{N^\gamma}\right)^{-D''} \|F_{N,k}\|_{B_{k,D''}^{\rho,\gamma}}.$$

Hence,

$$\|P_M F_{N,k}\|_{L_t^\infty L_{x,y}^2} \lesssim \left(\frac{\max(M, N)}{N^\gamma}\right)^{-D''} \|F_{N,k}\|_{B_{k,D''}^{\rho,\gamma}}.$$

We deduce that if D'' is large enough with respect to D' , then

$$C_{N,D'}(M) \|P_M F_N(t)\|_{L_t^\infty L_{x,y}^2} \lesssim (MN)^{-D'} \sum_{\frac{N}{2} \leq k < N} |g_k(\omega)| \|F_{N,k}\|_{B_{k,D''}^{\rho,\gamma}}.$$

Using Lemma 4.4.1 we deduce that

$$C_{N,D'}(M) \|P_M F_N(t)\|_{L_t^\infty L_{x,y}^2} \lesssim (MN)^{-D'} \sum_{\frac{N}{2} \leq k < N} |g_k(\omega)| \|P_{1,k} f_0\|_{L_{x,y}^2}.$$

From the Cauchy-Schwarz inequality and using that D' can be taken large enough to control the extra $N^{\frac{1}{2}}$ -loss coming to the sum over k and the factor M^s , we conclude

$$M^s C_{N,D'}(M) \|P_M F_N(t)\|_{L_{x,y}^2} \lesssim (MN)^{-\frac{D'}{2}} \|\widetilde{P}_N f_0^\omega\|_{L_{x,y}^2}. \quad \square$$

4.4.2 Strichartz estimates for the adapted linear evolution

In this part, we undercut the derivative loss in the Strichartz estimates (4.2.3) for the adapted linear evolution F_N . To do so, we take advantage of the frequency localization of the function $F_{N,k}$ together with probabilistic decoupling.

Lemma 4.4.3 (Strichartz estimate for the individual block). *Let $0 < \rho < \sigma$ as in Lemma 4.4.1, let $\delta = \gamma(\sigma - \rho) > 0$ (arbitrarily small as ρ is close to σ), and let $T_0 > 0$ small enough. For all $D' > 0$, there exists $D'' > D'$ such that for all $\sigma' > \sigma$ and $\frac{N}{2} \leq |k| < N$, $0 < T \leq T_0$, we have*

$$\|\langle D_y \rangle^{\sigma'} F_{N,k}\|_{S_{N,D'}^r[-T,T]} \lesssim_r \|\langle D_y \rangle^{\sigma'} P_{1,k} f_0\|_{L_{x,y}^2} + T^{\frac{1}{2}} N^{\sigma' + \gamma(\frac{1}{2} - \frac{1}{r}) - \gamma\sigma + \delta} \|F_{N,k}\|_{B_{k,D''}^{\rho,\gamma}[-T,T]}.$$

Proof. We need to estimate

$$\sum_M C_{N,D'}(M) M^{\sigma'} \|P_M F_{N,k}\|_{L_t^s([-T,T]; L_x^4 L_y^r(\mathbb{R}^2))}.$$

Given a dyadic integer $M \gg 1$ and $0 \leq t \leq T$, the Duhamel integral formulation of the truncated equation (4.3.7) reads

$$P_M F_{N,k}(t) = e^{itA} P_M P_{1,k} f_0 + i \int_0^t e^{i(t-\tau)A} P_M (\mathcal{N}(F_{N,k}, \phi, \phi))(\tau) d\tau.$$

First, we estimate the linear evolution of the unit-scale block $P_M P_{1,k} f_0$, which is nonzero only when $M \sim N$. To do so, we use the Bernstein-Strichartz estimate (4.2.3) from Lemma 4.2.1 with $(p, q) = (8, 4)$ and $|E| \lesssim 1$. There holds

$$\|P_M e^{itA} P_{1,k} f_0\|_{L_t^s L_x^4 L_y^r} \lesssim \mathbf{1}_{M \sim N} \|P_{1,k} f_0\|_{L_{x,y}^2}.$$

To handle the nonlinear part, we perform a Littlewood-Paley decomposition of ϕ :

$$\phi \sim \sum_{1 \leq L \leq N^\gamma} P_L \phi,$$

and a Littlewood-Paley decomposition of $F_{N,k}$ recentered around k :

$$F_{N,k} \sim \sum_{1 \leq K} P_{K,k} F_{N,k}.$$

In what follows, without loss of generality, we assume that $L_1 \geq L_2$. The frequency support in the variable η of the Duhamel term has length

$$|\text{supp}_\eta \mathcal{F}_{y \rightarrow \eta}(\mathcal{N}(P_{K,k} F_{N,k}, P_{L_1} \phi, P_{L_2} \phi))| \lesssim \max(L_1, K).$$

We use the Bernstein-Strichartz TT^* -estimate (4.2.4) from Lemma 4.2.1 with $(p, q) = (8, 4)$, $(\tilde{p}, \tilde{q}) = (4, \infty)$ and $|E| \lesssim \max(L_1, K)$, and get

$$\begin{aligned} & \sum_{L_2 \leq L_1 \leq N^\gamma} \sum_K \left\| \int_0^t e^{i(t-\tau)A} P_M (\mathcal{N}(P_{K,k} F_{N,k}, P_{L_1} \phi, P_{L_2} \phi))(\tau) d\tau \right\|_{L_t^s L_x^4 L_y^r} \\ & \lesssim \sum_{L_2 \leq L_1 \leq N^\gamma} \sum_K \max(L_1, K)^{\frac{1}{2} - \frac{1}{r}} \|P_M (\mathcal{N}(P_{K,k} F_{N,k}, P_{L_1} \phi, P_{L_2} \phi))(\tau) d\tau\|_{L_t^{\frac{4}{3}} L_x^1 L_y^2}. \end{aligned}$$

Let us first consider the contribution of the terms with $K \ll N$. In such a case, only the terms with $M \sim N$ contribute. Moreover, using the truncation (4.3.8), so that $\|\langle D_y \rangle^\sigma \phi\|_{L_t^8 L_x^4 L_y^\infty} \lesssim 1$,

$$\begin{aligned} M^{\sigma'} & \sum_{L_2 \leq L_1 \leq N^\gamma} \sum_{K \ll N} \max(L_1, K)^{\frac{1}{2}-\delta} \|P_M(\mathcal{N}(P_{K,k} F_{N,k}, P_{L_1} \phi, P_{L_2} \phi))\|_{L_t^{\frac{4}{3}} L_x^1 L_y^2} \\ & \lesssim \mathbf{1}_{M \sim N} N^{\sigma'} \sum_{L_2 \leq L_1 \leq N^\gamma} \sum_{K \ll N} \max(L_1, K)^{\frac{1}{2}-\delta} K^{-\rho} \max\left(1, \frac{K}{N^\gamma}\right)^{-D''} L_1^{-\sigma} L_2^{-\sigma} \\ & \qquad \qquad \qquad c_{k,D''}^{\rho,\gamma}(K) \|P_{K,k} F_{N,k}\|_{L_t^\infty L_{x,y}^2}. \end{aligned}$$

When $K \leq N^\gamma$ we get by comparing K and L_1 that

$$\begin{aligned} \mathbf{1}_{M \sim N} N^{\sigma'} & \sum_{K, L_1, L_2 \leq N^\gamma} \max(L_1, K)^{\frac{1}{2}-\frac{1}{r}} K^{-\rho} L_1^{-\sigma} L_2^{-\sigma} \|F_{N,k}\|_{B_{k,D''}^{\rho,\gamma}} \\ & \lesssim \mathbf{1}_{M \sim N} N^{\sigma'+\gamma(\frac{1}{2}-\frac{1}{r}-\sigma)+\delta} \|F_{N,k}\|_{B_{k,D''}^{\rho,\gamma}}. \end{aligned}$$

On the other hand, when $K \geq N^\gamma$, taking $D'' \gg \frac{1}{2}$ yields

$$\sum_{N^\gamma \leq K \ll N} K^{\frac{1}{2}-\frac{1}{r}-\rho} K^{-D''} \lesssim N^{\gamma(\frac{1}{2}-\frac{1}{r}-\rho-D'')},$$

so that

$$\begin{aligned} \mathbf{1}_{M \sim N} N^{\sigma'} & \left(\sum_{N^\gamma \leq K \ll N} \sum_{L_1, L_2 \leq N^\gamma} K^{\frac{1}{2}-\frac{1}{r}-\rho} \left(\frac{K}{N^\gamma}\right)^{-D''} L_1^{-\sigma} L_2^{-\sigma} \right) \|F_{N,k}\|_{B_{k,D''}^{\rho,\gamma}} \\ & \lesssim \mathbf{1}_{M \sim N} N^{\sigma'+\gamma(\frac{1}{2}-\frac{1}{r}-\sigma)+\delta} \|F_{N,k}\|_{B_{k,D''}^{\rho,\gamma}}. \end{aligned}$$

Summing over $M \sim N$, in which case $C_{N,D'}(M) \sim 1$, and taking the $L^{\frac{4}{3}}$ -norm in time for $T \leq 1$, we obtain the desired bound for the contributions with $K \ll N$:

$$\begin{aligned} \sum_{M \sim N} C_{N,D'}(M) M^{\sigma'} & \sum_{L_1, L_2 \leq N^\gamma} \sum_{K \ll N} \max(L_1, K)^{\frac{1}{2}-\delta} \|P_M(\mathcal{N}(P_{K,k} F_{N,k}, P_{L_1} \phi, P_{L_2} \phi))\|_{L_t^{\frac{4}{3}} L_x^1 L_y^2} \\ & \lesssim T^{\frac{1}{2}} N^{\sigma'+\gamma(\frac{1}{2}-\frac{1}{r}-\sigma)+\delta} \|F_{N,k}\|_{L_t^\infty B_{k,D''}^{\rho,\gamma}}. \end{aligned}$$

Next, we handle the contribution of the terms with $K \gtrsim N$, where only the terms with $K \gtrsim \max(N, M)$ actually contribute. We shall use the localization around frequency $K \sim N^\gamma$ of $F_{N,k}$ imposed by the norm $B_{k,D''}^{\rho,\gamma}$ to prove smallness: for sufficiently large D'' depending on D' ,

there holds

$$\begin{aligned}
M^{\sigma'} & \sum_{L_2 \leq L_1 \leq N^\gamma} \sum_{K \gtrsim N} K^{\frac{1}{2} - \frac{1}{r}} \|P_M(\mathcal{N}(P_{K,k} F_{N,k}, P_{L_1} \phi, P_{L_2} \phi))\|_{L_t^{\frac{4}{3}} L_x^1 L_y^2} \\
& \lesssim T^{\frac{1}{2}} M^{\sigma'} \sum_{K \gtrsim \max(N, M)} K^{\frac{1}{2} - \frac{1}{r}} \|P_{K,k} F_{N,k}\|_{L_t^\infty L_{x,y}^2} \\
& \lesssim T^{\frac{1}{2}} M^{\sigma'} \sum_{K \gtrsim \max(N, M)} K^{\frac{1}{2} - \frac{1}{r}} K^{-\rho} \left(\frac{K}{N}\right)^{-D''} \|F_{N,k}\|_{B_{k,D''}^{\rho,\gamma}} \\
& \lesssim T^{\frac{1}{2}} (NM)^{-10(D'+1)} \|F_{N,k}\|_{B_{k,D''}^{\rho,\gamma}}.
\end{aligned}$$

Hence, the weight $C_{N,D'}(M) = \max(\frac{N}{M}, \frac{M}{N})^{D'}$ and the term $M^{\sigma'}$ are absorbed by the factor $(NM)^{-10(D'+1)}$: we can sum over M to get

$$\begin{aligned}
\sum_M C_{N,D'}(M) M^{\sigma'} & \sum_{L_2 \leq L_1 \leq N^\gamma} \sum_{K \gtrsim N} K^{\frac{1}{2} - \frac{1}{r}} \|P_M(\mathcal{N}(P_{K,k} F_{N,k}, P_{L_1} \phi, P_{L_2} \phi))\|_{L_t^{\frac{4}{3}} L_x^1 L_y^2} \\
& \lesssim T^{\frac{1}{2}} \|F_{N,k}\|_{B_{k,D''}^{\rho,\gamma}}.
\end{aligned}$$

This concludes the proof of the refined Strichartz estimates for the unit blocks $F_{N,k}$. \square

Next, we infer probabilistic Strichartz estimates on the whole adapted linear evolution F_N from the above estimate on the individual blocks $F_{N,k}$, combined with probabilistic decoupling (we recall that the high frequencies of the initial data are independent of the low frequencies). The proof follows the same lines as in [Bri21], Proposition 4.4.

Proposition 4.4.4 (Probabilistic Strichartz estimates). *Let F_N be a solution to the truncated equation (4.3.7), $s > 0$, $D' > 0$ and T_0 be as in Lemmas 4.4.1 and 4.4.2. There holds that for all $0 < T \leq T_0$, $p \geq 1$ and $\sigma' > 0$,*

$$\|\langle D_y \rangle^{\sigma'} F_N\|_{L_\omega^p(\Omega, S_{N,D'}[-T, T])}^+ \lesssim_\delta \sqrt{p} T^{\frac{1}{2}} N^{\sigma' + \frac{\gamma}{2} - s - \gamma\sigma + 2\delta} \|\widetilde{P}_N f_0\|_{L_x^2 H_y^s}, \quad (4.4.4)$$

where $\delta = \gamma(\sigma - \rho) > 0$.

The contributions to the power of N can be explained as follows:

$$N^{\sigma' - s + \frac{\gamma}{2} - \gamma\sigma} = \underbrace{N^{\sigma' - s}}_{\text{difference of derivatives}} \cdot \underbrace{N^{\frac{\gamma}{2}}}_{\text{Strichartz with freq. localization}} \cdot \underbrace{N^{-\gamma\sigma}}_{\text{gain frequency localization}}.$$

Proof. The idea is to use the independence between $\{g_l: \frac{N}{2} \leq |l| < N\}$ and $F_{N,k}$ which is $\mathcal{B}_{\frac{N}{2}} = \sigma(g_l: |l| < \frac{N}{2})$ -measurable thanks to the assumptions on ϕ . In order to use Minkowski's inequality in the large deviation estimates, we first prove the result with the norm $S_{N,D'}^r$ with $r \gg 1$ instead of $S_{N,D'}$ (which denotes $S_{N,D'}^\infty$). We condition on $\mathcal{B}_{\frac{N}{2}}$, take $p \geq r$ and deduce from Minkowski's inequality that

$$\begin{aligned}
\|\langle D_y \rangle^{\sigma'} F_N\|_{L_\omega^p S_{N,D'}^r} & = \mathbb{E} \left[\mathbb{E} \left[\|C_{N,D'}(M) \langle D_y \rangle^{\sigma'} P_M F_N\|_{\ell_M^1 L_t^s L_x^4 L_y^r}^p \middle| \mathcal{B}_{\frac{N}{2}} \right] \right]^{\frac{1}{p}} \\
& \leq \mathbb{E} \left[\left[\mathbb{E} \left[|C_{N,D'}(M) \langle D_y \rangle^{\sigma'} P_M F_N|^p \middle| \mathcal{B}_{\frac{N}{2}} \right] \right]^{\frac{1}{p}} \left\| \ell_M^1 L_t^s L_x^4 L_y^r \right\|^p \right]^{\frac{1}{p}}.
\end{aligned}$$

Since

$$F_N = \sum_{\frac{N}{2} \leq |k| < N} g_k(\omega) F_{N,k},$$

The conditional Khintchine's inequality and the mutual independence between the (g_k) 's and the $(F_{N,k})$'s yield

$$\mathbb{E} \left[\left| \langle D_y \rangle^{\sigma'} P_M F_N \right|^p \middle| \mathcal{B}_{\frac{N}{2}} \right]^{\frac{1}{p}} \leq \sqrt{p} \|\mathbf{1}_{\frac{N}{2} \leq |k| < N} \langle D_y \rangle^{\sigma'} P_M F_{N,k}\|_{\ell_k^2}.$$

Plugging into (4.4.2),

$$\|\langle D_y \rangle^{\sigma'} F_N\|_{L_\omega^p S_{N,D'}^r} \lesssim \|\mathbf{1}_{\frac{N}{2} \leq |k| < N} C_{N,D'}(M) \langle D_y \rangle^{\sigma'} P_M F_{N,k}\|_{L_\omega^p \ell_M^1 L_t^8 L_x^4 L_y^4 \ell_k^2}.$$

Thanks to the Cauchy-Schwarz inequality, we can go from ℓ_M^1 to ℓ_M^2 up to replacing the weight $C_{N,D'}$ by $C_{N,D'+1}$. Using the Minkowski inequality again, we obtain

$$\begin{aligned} \|\langle D_y \rangle^{\sigma'} F_N\|_{L_\omega^p S_{N,D'}^r} &\lesssim \sqrt{p} \|C_{N,D'+1}(M) \langle D_y \rangle^{\sigma'} P_M F_{N,k}\|_{L_\omega^p \ell_M^1 L_t^8 L_x^4 L_y^4 \ell_k^2} \\ &\lesssim \sqrt{p} \|C_{N,D'+1}(M) \langle D_y \rangle^{\sigma'} P_M F_{N,k}\|_{\ell_k^2 L_\omega^p \ell_M^2 L_t^8 L_x^4 L_y^4} \\ &\lesssim \sqrt{p} \|C_{N,D'+1}(M) \langle D_y \rangle^{\sigma'} P_M F_{N,k}\|_{\ell_k^2 L_\omega^p S_{N,D'+1}^r}. \end{aligned}$$

In the last estimate, we used that $\ell_M^1 \subset \ell_M^2$. Finally, we apply the refined Strichartz estimate for the individual blocks $F_{N,k}$ (Lemma 4.4.3). The frequency localization stated in Lemma 4.4.1 yields

$$\begin{aligned} \|\langle D_y \rangle^{\sigma'} F_N\|_{L_\omega^p S_{N,D'}^r} &\lesssim \sqrt{p} \|\langle D_y \rangle^{\sigma'} P_{1,k} f_0\|_{\ell_k^2} + \sqrt{p} T^{\frac{1}{2}} N^{\sigma'+\gamma(\frac{1}{2}-\frac{1}{r})-\gamma\sigma+\delta} \|F_{N,k}\|_{\ell_k^2 B_{k,D''}^{\rho,\gamma}} \\ &\lesssim \sqrt{p} \|\tilde{P}_N f_0\|_{L_x^2 H_y^{\sigma'}} + \sqrt{p} T^{\frac{1}{2}} N^{\sigma'+\gamma(\frac{1}{2}-\frac{1}{r})-\gamma\sigma-s+\delta} \|\tilde{P}_N f_0\|_{L_x^2 H_y^s}. \end{aligned}$$

Meanwhile, the estimate for $p \leq \min(4, r)$ follows from Hölder's inequality. Next, we pass from $S_{N,D'}$ to $S_{N,D'}^r$ with $r \gg 1$ by using the Sobolev embedding in y , and by loosing therefore an arbitrarily small power of $N^{\frac{1}{r}}$ (see Step 3 in the proof of Proposition 4.4 from [Bri21]). Here we can choose N^δ . \square

4.5 Estimates on the nonlinear term w_N

For fixed N , we construct the local solution to the truncated equation on a time interval $[-T_0, T_0]$ that does not depend on N .

Proposition 4.5.1. *Assume that $0 < \sigma < \nu - \frac{1}{2}$, $0 < \sigma < \sigma' < s$ and $0 < \gamma < 1$. Fix $\alpha > 0$ such that $\max(\nu - \sigma', \sigma) < \alpha < \nu$. Let $D = D(s, \nu, \sigma', \sigma, \alpha) > 0$ and $D' \geq D'(s, \nu, \sigma', \sigma, \alpha, D) > 0$ be sufficiently large. There exist $T_0 = T_0(s, \nu, \sigma', \sigma, \alpha, D, D') > 0$ smaller than the T_0 in Lemmas 4.4.1 and 4.4.2, and a unique solution $w_N \in Y_N^\nu([-T_0, T_0])$ to the truncated equation (4.3.10). In addition, for all $0 < T < T_0$,*

$$\|w_N\|_{Y_N^\nu([-T, T])} \lesssim_\delta T^{\frac{1}{2}} N^{\nu-\sigma'-\gamma\nu} \|\langle D_y \rangle^{\sigma'} F_N\|_{S_{N,D'}} + T^{\frac{1}{2}} N^{\nu-s-\gamma\sigma'} \|\langle D_y \rangle^s F_N\|_{X_{N,D'}}. \quad (4.5.1)$$

To prove such a proposition we do a contraction mapping argument, which is a consequence of the a priori estimates stated below.

4.5.1 A priori trilinear estimates

We denote the Duhamel integral by

$$\mathcal{I}f(t) := \int_0^t e^{i(t-\tau)\mathcal{A}} f(\tau) d\tau,$$

and we write

$$\Gamma w(t) := \mathcal{I}\tilde{\mathcal{N}}_\theta(w)(t), \quad (4.5.2)$$

where we recall that $\tilde{\mathcal{N}}_\theta$ was defined in (4.3.11). Applying the trilinear estimates from Section 4.7, together with the truncation of the nonlinearity, we obtain the following a priori estimates on the nonlinear component w_N .

Proposition 4.5.2 (A priori estimate on w_N). *Under the same constraints on the parameters as in the statement of Proposition 4.5.1, for $0 \leq T \leq T_0$, there holds*

$$\|w_N\|_{Y_N^\nu([-T, T])} \lesssim T^{\frac{1}{2}} \|w_N\|_{Y_N^\nu([-T, T])} + T^{\frac{1}{2}} N^{\nu-\sigma'-\gamma\nu} \|\langle D_y \rangle^{\sigma'} F_N\|_{S_{N, D'}} + T^{\frac{1}{2}} N^{\nu-s-\gamma\sigma'} \|\langle D_y \rangle^s F_N\|_{X_{N, D'}}.$$

Let us briefly comment the proof of the above estimate. Notice that the new terms of the approximate solution u_N constructed at step N , which are either $F := F_N$ or $w := w_n$, appear at least once in the nonlinear forcing term (because $\mathcal{N}(u_{\frac{N}{2}})$ is removed from the nonlinearity).

Then, the strategy to obtain optimal a priori estimates on w_N is to use the TT^* estimate from Lemma 4.2.1 and the refined probabilistic estimate from Proposition 4.4.4. Specifically, when we have trilinear mixed terms (i.e some functions of type F_N interact with others of type w_N), we place F_N in L_y^∞ and w_n in L_y^2 , respectively. Surprisingly, we do so even if F_N has the highest frequency, in order to reduce the loss of derivatives from the *deterministic* Strichartz estimates of Lemma 4.2.1. This is implemented precisely in the paracontrolled trilinear estimates collected in Section 4.7.

Proof. Up to permutation, we always assume that a term of type w_N or F_N is at the first position. We collect the different contributions depending on the nature of the terms. We place the terms $u_{N/2}$ (denoted ϕ in Section 4.7) in the last position. Thanks to the truncation, they satisfy $\|\langle D_y \rangle^\sigma \phi\|_{S_{\leq N, D}} \lesssim 1$.

Contribution of $F_N F_N F_N$. We have from (4.7.3), (4.7.8), (4.7.12) and (4.7.16) that

$$\|\mathcal{I}(\theta_{F; N}^2 \mathcal{N}(F_N, F_N, F_N))\|_{Y_N^\nu} \lesssim T^{\frac{1}{2}} N^{\nu-s-\sigma'} \|\langle D_y \rangle^s F_N\|_{X_{N, D'}}.$$

Contribution of $w_N F_N F_N$. We have from (4.7.4) and (4.7.6), (4.7.10), (4.7.14) that

$$\|\mathcal{I}(\theta_{F; N}^2 \mathcal{N}(w_N, F_N, F_N))\|_{Y_N^\nu} \lesssim T^{\frac{1}{2}} \|w_N\|_{Y_N^\nu}.$$

Contribution of $w_N w_N F_N$. We have from (4.7.5), (4.7.7), (4.7.11) and (4.7.15) that

$$\|\mathcal{I}(\theta_{w; N} \theta_{F; N} \mathcal{N}(w_N, w_N, F_N))\|_{Y_N^\nu} \lesssim T^{\frac{1}{2}} \|w_N\|_{Y_N^\nu}.$$

Contribution of $w_N w_N w_N$. We have from (4.7.5), (4.7.7), (4.7.11) and (4.7.15) that

$$\|\mathcal{I}(\theta_{w;N}^2 \mathcal{N}(w_N, w_N, w_N))\|_{Y_N^\nu} \lesssim T^{\frac{1}{2}} \|w_N\|_{Y_N^\nu}.$$

Contribution of $w_N u_{\frac{N}{2}} u_{\frac{N}{2}}$. We have from (4.7.4), (4.7.6), (4.7.10), (4.7.14) or from (4.7.5), (4.7.7), (4.7.11), (4.7.15) depending on the nature of the first factor $u_{\frac{N}{2}}$ that

$$\|\mathcal{I}(\theta_{F,w;\leq \frac{N}{2}} \mathcal{N}(w_N, u_{\frac{N}{2}}, u_{\frac{N}{2}}))\|_{Y_N^\nu} \lesssim T^{\frac{1}{2}} \|w_N\|_{Y_N^\nu}.$$

Contribution of $w_N w_N u_{\frac{N}{2}}$. We have from (4.7.5), (4.7.6), (4.7.10) and (4.7.14) that

$$\|\mathcal{I}(\theta_{w;N} \theta_{F,w;\leq \frac{N}{2}} \mathcal{N}(w_N, w_N, u_{\frac{N}{2}}))\|_{Y_N^\nu} \lesssim T^{\frac{1}{2}} \|w_N\|_{Y_N^\nu}.$$

Contribution of $w_N F_N^\omega u_{\frac{N}{2}}$. We have from (4.7.4), (4.7.6), (4.7.10) and (4.7.14) that

$$\|\mathcal{I}(\theta_{F;N} \theta_{F,w;\leq \frac{N}{2}} \mathcal{N}(w_N, F_N, u_{\frac{N}{2}}))\|_{Y_N^\nu} \lesssim T^{\frac{1}{2}} \|w_N\|_{Y_N^\nu}.$$

Contribution of $F_N F_N^\omega u_{\frac{N}{2}}$. We have from (4.7.3), (4.7.8), (4.7.12) and (4.7.16) that

$$\|\mathcal{I}(\theta_{F;N} \theta_{F,w;\leq \frac{N}{2}} \mathcal{N}(F_N, F_N, u_{\frac{N}{2}}))\|_{Y_N^\nu} \lesssim T^{\frac{1}{2}} N^{\nu-s-\sigma'} \|\langle D_y \rangle^s F_N\|_{X_{N,D'}}.$$

Contribution of $F_N u_{n-1} u_{\frac{N}{2}}$. It follows from the definition of the adapted linear evolution F_N that these terms only contribute to the equation satisfied by w_N when one of the two terms $u_{N/2}$ is truncated at frequencies $> N^\gamma$. We split this term into

$$P_{>N^\gamma} u_{\frac{N}{2}} = P_{>N^\gamma} w_{\leq \frac{N}{2}} + P_{>N^\gamma} F_{\leq \frac{N}{2}}.$$

There holds

$$\|\mathcal{I}(\theta_{F,w;\leq \frac{N}{2}}^2 \mathcal{N}(F_N, P_{>N^\gamma} w_{\leq \frac{N}{2}}, u_{\frac{N}{2}}))\|_{Y_N^\nu} \lesssim T^{\frac{1}{2}} N^{\nu-\sigma'-\gamma\nu} \|\langle D_y \rangle^{\sigma'} F_N\|_{S_{N,D'}}.$$

Indeed, we use (4.7.1) for the *high-low-low*-type interactions. For the other interactions, we use (4.7.8), (4.7.12), (4.7.16) when $u_{\frac{N}{2}}$ is of type G , and (4.7.9), (4.7.13), (4.7.17) when $u_{\frac{N}{2}}$ is of type v , respectively. Similarly, we have

$$\|\mathcal{I}(\theta_{F,w;\leq \frac{N}{2}}^2 \mathcal{N}(F_N, P_{>N^\gamma} F_{\leq \frac{N}{2}}, u_{\frac{N}{2}}))\|_{Y_N^\nu} \lesssim T^{\frac{1}{2}} N^{\nu-s-\gamma\sigma'} \|\langle D_y \rangle^s F_N\|_{X_{N,D'}},$$

which follows from (4.7.2) for the *high-low-low* type interactions, and (4.7.8), (4.7.12), (4.7.16), or (4.7.9), (4.7.13), (4.7.17) depending on the nature of $u_{\frac{N}{2}}$. \square

4.5.2 Contraction mapping argument

As a consequence of the a priori estimates (4.5.1), we see that Γ (defined in (4.5.2)) maps Y_N^ν into Y_N^ν . We now prove that it is a contraction mapping in the Banach space Y_N^ν i.e. that for all $z, w \in Y_N^\nu[-T, T]$ when $T \leq T_0$,

$$\|\Gamma(z) - \Gamma(w)\|_{Y_N^\nu} \leq \frac{1}{2} \|z - w\|_{Y_N^\nu}.$$

To achieve this goal, we adapt the proof in [Bri21], inspired by [BD99], to the trilinear case. Since the linear terms are handled as above, it only remains to consider the quadratic and cubic terms. Let

$$t_z := \sup\{0 \leq t \leq T : \|z\|_{Y_N^\nu[-t, t]} \leq 2\}.$$

The time t_w is defined similarly. We observe from a continuity argument (both on the norm and on z, w) that for all $t > t_z$ (resp. $t > t_w$), we have $\theta_{z;N}(t) = 0$ (resp. $\theta_{w;N}(t) = 0$). Without loss of generality, we assume that $t_z \leq t_w$ and we encapsulate the nonlinear terms under consideration in Λ :

$$\begin{aligned} \Lambda(z, w)(t) &= \Lambda_1(z, w)(t) + \Lambda_2(z, w)(t) + \Lambda_3(z, w)(t), \\ \Lambda_1(z, w)(t) &= \theta_{z;N}(t)^2 |z|^2 z(t) - \theta_{w;N}(t)^2 |w|^2 w(t), \\ \Lambda_2(z, w)(t) &= \theta_{F;N}(t) \left(\theta_{z;N}(t) \mathcal{N}(z, z, F_N)(t) - \theta_{w;N}(t) \mathcal{N}(w, w, F_N)(t) \right), \\ \Lambda_3(z, w)(t) &= \theta_{F, w; \leq \frac{N}{2}}(t) \left(\theta_{z;N}(t) \mathcal{N}(z, z, u_{\frac{N}{2}})(t) - \theta_{w;N}(t) \mathcal{N}(w, w, u_{\frac{N}{2}})(t) \right). \end{aligned}$$

Let us consider the cubic term $\Lambda_1(z, w)$. We decompose the nonlinear interactions as in [BD99]:

$$\begin{aligned} \left\| \int_0^t e^{i(t-\tau)\mathcal{A}} \Lambda_1(\tau) d\tau \right\|_{Y_N^\nu} &\leq \left\| \int_0^t e^{i(t-\tau)\mathcal{A}} \mathbf{1}_{[0, t_z]}(\tau) (\theta_{z;N}(\tau) - \theta_{w;N}(\tau)) |z|^2 z(\tau) d\tau \right\|_{Y_N^\nu} \\ &\quad + \left\| \int_0^t e^{i(t-\tau)\mathcal{A}} \mathbf{1}_{[0, t_z]}(\tau) \theta_{w;N}(\tau) (|z|^2 z(\tau) - |w|^2 w(\tau)) d\tau \right\|_{Y_N^\nu} \\ &\quad + \left\| \int_0^t e^{i(t-\tau)\mathcal{A}} \mathbf{1}_{(t_z, t_w]}(\tau) (\theta_{z;N}(\tau) - \theta_{w;N}(\tau)) |w|^2 w(\tau) d\tau \right\|_{Y_N^\nu}. \end{aligned}$$

Here, we used that $\theta_{z;N}(\tau) = 0$ when $\tau \in (t_z, t_w]$. Then, we deduce from the multilinear estimates (4.7.5), (4.7.7), (4.7.11) and (4.7.15) that

$$\begin{aligned} \left\| \int_0^t e^{i(t-\tau)\mathcal{A}} \Lambda_1(\tau) d\tau \right\|_{Y_N^\nu} &\lesssim T^{\frac{1}{2}} \|\theta_{z;N} - \theta_{w;N}\|_{L_t^\infty} \|\mathbf{1}_{[0, t_z]} z\|_{Y_N^\nu}^3 \\ &\quad + T^{\frac{1}{2}} (\|\mathbf{1}_{[0, t_z]} z\|_{Y_N^\nu}^2 + \|\mathbf{1}_{[0, t_z]} w\|_{Y_N^\nu}^2) \|z - w\|_{Y_N^\nu} \\ &\quad + T^{\frac{1}{2}} \|\theta_{z;N} - \theta_{w;N}\|_{L_t^\infty} \|\mathbf{1}_{[t_z, t_w]} w\|_{Y_N^\nu}^3 \\ &\lesssim T^{\frac{1}{2}} \|z - w\|_{Y_N^\nu}. \end{aligned}$$

The quadratic interactions are controlled analogously. Hence, Γ is a contraction mapping on Y_N^ν for sufficiently small T_0 , and this concludes the proof of Proposition 4.5.1.

4.6 Conclusion

We now show that the sequence $(u_N)_N$ tends to a limit u on some time interval $[-T_0, T_0]$, that u is solution to equation (NLS-HW) on a random time interval $[-T, T]$, and therefore finish the proof of Theorem 4.1.1. We refer to [Bri21], Appendix A regarding the measurability properties of the functions F_N and w_N .

4.6.1 Optimizing the constraints

We collect the main constraints resulting from the nonlinear analysis.

- Constraint coming from the adapted linear evolution F_N in Strichartz type spaces $S_{N,D'}$:

$$\sigma' + \frac{\gamma}{2} - s - \gamma\sigma < 0.$$

- Contribution of term $F_N(P_{>N\gamma}F_{\leq \frac{N}{2}})u_{\frac{N}{2}}$ in Y_N^ν :

$$\nu - s - \gamma\sigma' < 0.$$

- Contribution of term $F_N(P_{>N\gamma}w_{\leq \frac{N}{2}})u_{\frac{N}{2}}$ in Y_N^ν :

$$\nu - \sigma' - \gamma\sigma < 0.$$

This reduces to the minimization problem for s under the constraints

$$\begin{cases} \sigma' - s - \gamma\sigma + \frac{\gamma}{2} < 0 \\ \nu - \gamma\sigma' - s < 0 \\ \nu - \sigma' - \gamma\nu < 0 \\ \sigma - \nu < -\frac{1}{2} \\ \sigma - \sigma' < 0. \end{cases}$$

Discretizing $\gamma \in (0, 1)$ and using a linear programming solver leads to an approximate optimal solution

$$(s_0, \sigma'_0, \sigma_0, \nu_0) = (0.464131, 0.154581, 0.077290, 0.577291), \quad \gamma = 0.732232. \quad (4.6.1)$$

We chose $s > s_0$, $\sigma = \sigma_0 -$, $\sigma' = \sigma'_0 -$ and $\nu = \nu_0 -$, and we now establish Theorem 4.1.1.

4.6.2 Convergence of $(u_N)_{N \in 2^{\mathbb{N}}}$

First, we prove the convergence (4.1.3) in expectation along the dyadic subsequence $(u_N)_{N \in 2^{\mathbb{N}}}$ up to the time $T_0 > 0$ which is chosen so that Proposition 4.5.1 holds true, and yields the a priori estimate on the truncated solution. In what follows T is fixed, with $0 < T \leq T_0$, and the norms are

taken on the slab $\Omega \times [-T, T] \times \mathbb{R}^2$. We recall from (4.3.2) that for some N_0 sufficiently large, the approximate solution reads

$$u_N = u_{N_0} + \sum_{M=N_0}^N (w_M + F_M) , ,$$

We first prove the convergence of the series $\sum F_M$ and $\sum w_M$ in the space $L_\omega^2(\Omega; X_T^{s,\sigma})$, where the Sobolev-Strichartz space $X_T^{s,\sigma}$ is defined in (4.1.2). Thanks to a completeness argument, it is enough to prove that the partial sums form a Cauchy sequence. Let $0 \leq N_- \leq N_+ < \infty$. We first address the adapted linear evolution, controlled in $L_\omega^2(\Omega; \mathcal{C}_t([-T_0, T_0]; L_x^2 H_y^s))$:

$$\left\| \sum_{M=N_-}^{N_+} \langle D_y \rangle^s F_M \right\|_{L_\omega^2 L_t^\infty L_{x,y}^2} \lesssim \left\| \sum_{M=N_-}^{N_+} \langle D_y \rangle^s P_N F_M \right\|_{L_\omega^2 L_t^\infty \ell_N^2 L_{x,y}^2} \lesssim \left\| \sum_{M=N_-}^{N_+} \langle D_y \rangle^s P_N F_M \right\|_{L_\omega^2 \ell_N^2 L_t^\infty L_{x,y}^2} .$$

From the definition (4.3.3) of the spaces $X_{M,D'}$, we have

$$\left\| \sum_{m=N_-}^{N_+} \langle D_y \rangle^s F_M \right\|_{L_\omega^2 L_t^\infty L_{x,y}^2} \lesssim \left\| \sum_{m=N_-}^{N_+} \max\left(\frac{N}{M}, \frac{M}{N}\right)^{-D'} \|\langle D_y \rangle^s F_M\|_{X_{M,D'}} \right\|_{L_\omega^2 \ell_N^2} .$$

Applying Cauchy-Schwarz's inequality, and then Lemma 4.4.2, the right-hand-side of the above inequality is bounded by

$$\text{r.h.s} \lesssim \left\| \sum_{M=N_-}^{N_+} \max\left(\frac{N}{M}, \frac{M}{N}\right)^{-D'} \|\widetilde{P}_M f_0^\omega\|_{L_x^2 H_y^s} \right\|_{L_\omega^2 \ell_N^2} \lesssim \left(\sum_{M=N_-}^{N_+} \|\widetilde{P}_M f_0\|_{L_x^2 H_y^s}^2 \right)^{\frac{1}{2}} .$$

Similarly, convergence in $L_\omega^2(\Omega; L_t^8([-T_0, T_0]; L_x^4 W_y^{\sigma,\infty}))$ follows from the probabilistic Strichartz estimate in Proposition 4.4.4

$$\|\langle D_y \rangle^\sigma F_M\|_{L_\omega^2 L_t^8 L_x^4 L_y^\infty} \lesssim T^{\frac{1}{2}} M^{\sigma + \frac{\gamma}{2} - s - \gamma\sigma + 2\delta} \|P_M f_0\|_{\mathcal{H}^s} ,$$

where $\sigma + \frac{\gamma}{2} - s - \gamma\sigma + 2\delta < 0$ when ρ is chosen close enough to σ since $\sigma < \sigma'$. To establish the convergence of the nonlinear remainders $(w_N)_N$, we use the a priori estimate from Proposition 4.5.1. Namely,

$$\|w_M\|_{Y_M^\nu} \lesssim T^{\frac{1}{2}} M^{\nu - \sigma' - \gamma\nu} \|\langle D_y \rangle^{\sigma'} F_M\|_{S_{M,D'}} + T^{\frac{1}{2}} M^{\nu - s - \gamma\sigma'} \|\langle D_y \rangle^s F_M\|_{X_{M,D'}} ,$$

where $\nu - \sigma' - \gamma\nu < 0$ and $\nu - s - \gamma\sigma' < 0$. Using again Lemma 4.4.2 and Proposition 4.4.4, we get that for a very small parameter $\varepsilon > 0$,

$$\|\langle D_y \rangle^\sigma w_M\|_{L_\omega^2 L_t^8 L_x^4 L_y^\infty} + \|\langle D_y \rangle^\nu w_M\|_{L_\omega^2 L_t^\infty L_{x,y}^2} \lesssim \|w_N\|_{Y_M^\nu} \lesssim T^{\frac{1}{2}} M^{-\varepsilon} \|\widetilde{P}_M f_0\|_{L_x^2 H_y^s} ,$$

so that the sum over M converges.

4.6.3 The limit is a strong solution

We now show that almost-surely in ω there exists $0 < T^\omega \leq T_0$ such that the limit u satisfies the Duhamel formula for equation (NLS-HW) on $[-T^\omega, T^\omega]$. To achieve such a goal, we prove that the truncation functions $\theta_{F,w;\leq \frac{N}{2}}$, $\theta_{F;N}$ and $\theta_{w;N}$ are actually equal to 1 at least until a time $T^\omega > 0$ almost-surely. First, we observe that the map

$$t \in [-T_0, T_0] \mapsto \sum_{M \geq N_0} \|\langle D_y \rangle^{\sigma'} F_M\|_{S_{M,D'}([-t,t])}$$

is almost-surely finite thanks to Proposition 4.4.4, in which case it is continuous. Moreover, it is equal to zero at $t = 0$. Therefore, the random time $T_1(\omega)$, defined by

$$T_1(\omega) = \sup \left\{ t \in [-T_0, T_0] : \sum_{M \geq 0} \|\langle D_y \rangle^{\sigma'} F_M\|_{S_{M,D'}([-t,t])} \leq \frac{1}{2} \right\},$$

is almost-surely positive. Then we deduce from Proposition 4.5.1 that for a very small parameter $\varepsilon > 0$, we have

$$\|w_M\|_{Y_M^\nu([-t,t])} \lesssim T^{\frac{1}{2}} M^{-\varepsilon} (\|\langle D_y \rangle^{\sigma'} F_M\|_{S_{M,D'}([-t,t])} + \|\langle D_y \rangle^s F_M\|_{X_{M,D'}([-t,t])}),$$

the series being almost surely finite at least until the time T_0 thanks to Lemma 4.4.2 and Proposition 4.4.4, and it is equal to zero at $t = 0$. Therefore, the random time T^ω defined by

$$T^\omega = \sup \left\{ t \in [-T_1(\omega), T_1(\omega)] : \sum_{M \geq N_0} \|w_M\|_{Y_M^\nu([-t,t])} \leq \frac{1}{2} \right\}$$

is almost-surely nonnegative. Therefore, on the time interval $[-T^\omega, T^\omega]$ one has $\theta_{F,w;\leq \frac{N}{2}}(t) = \theta_{F;N}(t) = \theta_{w;N}(t) = 1$, so that for every N , u_N is an actual solution to (NLS-HW) with initial data $P_{\leq N} f_0^\omega$. We conclude by passing to the limit $N \rightarrow \infty$ in the Duhamel formula.

4.6.4 The $H_x^{2s} L_y^2$ part

We prove that for each $N \in 2^{\mathbb{N}}$ the sequence (u_N) , a priori constructed in $\mathcal{C}([-T_0, T_0]; L_x^2 H_y^s)$, actually converges to u in $\mathcal{C}([-T_0, T_0]; \mathcal{H}^s)$. We simply use the TT^* -estimate to have a priori estimates of the $H_x^{2s} L_y^2$ -norm of u_N :

$$\begin{aligned} \|u_N\|_{L_t^\infty H_x^{2s} L_y^2} &\lesssim \|P_N f_0^\omega\|_{H_x^{2s} L_y^2} + \|\langle D_x \rangle^{2s} |u_N|^2 u_N\|_{L_t^{\frac{4}{3}}([-T_0, T_0]; L_x^1 L_y^2)} \\ &\lesssim \|P_N f_0^\omega\|_{H_x^{2s} L_y^2} + T_0^{\frac{1}{2}} \|u_N\|_{L_t^\infty H_x^{2s} L_y^2} \|u_N\|_{L_t^8 L_x^4 L_y^\infty}^2 \\ &\leq C \|P_N f_0^\omega\|_{H_x^{2s} L_y^2} + \frac{1}{2} \|u_N\|_{L_t^\infty H_x^{2s} L_y^2}, \end{aligned}$$

for T_0 sufficiently small, and where we used the truncation argument of De Bouard-Debussche to control the $L_t^8 L_x^4 L_y^\infty$ -norm of $u_N = u_{\frac{N}{2}} + F_N + w_N$. The rest of the argument follows as above.

4.6.5 Limit of the whole sequence

Once the sequence $(u_N)_{N \in 2^{\mathbb{N}}}$ is constructed and its convergence established, we define the general approximating sequence $(u_n)_{n \in \mathbb{N}}$ by iteration: given $n \in \mathbb{N}$ and N such that $\frac{N}{2} < n \leq N$, we set

$$u_n = u_{\frac{N}{2}} + F_n + w_n,$$

where the adapted linear evolution F_n solves

$$i\partial_t F_n + \mathcal{A}F_n = \mathcal{N}(F_n, P_{\leq N^\gamma} u_{\frac{N}{2}}, P_{\leq N^\gamma} u_{\frac{N}{2}}),$$

with initial data

$$F_n(0) = P_{\leq n} f_0^\omega - P_{\leq \frac{N}{2}} f_0^\omega = \sum_{\frac{N}{2} < |k| \leq N} g_k(\omega) P_{1;k} f_0.$$

Note that $(u_N)_N$ is a subsequence of $(u_n)_n$. The truncated versions $F_{n,\theta}$ and $w_{n,\theta}$ are defined as in (4.3.7) and (4.3.10), respectively. For fixed $n \in \mathbb{N}$ and $N \in 2^{\mathbb{N}}$ such that $\frac{N}{2} < n \leq N$, we can prove the same estimates for $F_{n,\theta}$ and $w_{n,\theta}$ that we get for $F_{N,\theta}$ (in Section 4.4) and $w_{N,\theta}$ (in Section 4.5). We deduce from these estimates and from the convergence of $(u_{N,\theta})_{N \in 2^{\mathbb{N}}}$ that $(u_{n,\theta})_{n \in \mathbb{N}}$ is a Cauchy sequence. Finally we conclude that the whole sequence $(u_n)_n$ is convergent to the limit u of $(u_N)_N$ in $X_{T_0}^{s,\sigma}$, moreover it is a solution to equation (NLS-HW) with initial data $P_{\leq n} f_0^\omega$ on $[-T^\omega, T^\omega]$ by construction.

4.7 Paracontrolled trilinear estimates

In order to establish the trilinear estimates, we will perform a Littlewood-Paley decomposition. After we cut in frequencies, we use in various situations the following decomposition.

Lemma 4.7.1. *For every $\phi^{(1)}, \phi^{(2)}, \phi^{(3)}$ such that the upper bound is finite, we have*

$$\begin{aligned} \|\mathcal{I}(\mathcal{N}(\phi^{(1)}, \phi^{(2)}, \phi^{(3)}))\|_{Y_N^\nu} &\lesssim \sum_M (C_{\leq N, D}(M) + C_{N, \alpha}(M)) M^\nu \|P_M \mathcal{N}(\phi^{(1)}, \phi^{(2)}, \phi^{(3)})\|_{L_t^{\frac{4}{3}} L_x^1 L_y^2}. \end{aligned}$$

Proof of Lemma 4.7.1. This follows from the definition of the norms in Section 4.3. To control the Strichartz-norms we use the TT^* -Bernstein-Strichartz estimate from Lemma 4.2.1 with $(\tilde{p}, \tilde{q}) = (4, \infty)$, and either $(p, q) = (\infty, 2)$ and $r = 2$ or $(p, q) = (8, 4)$ and $r = \infty$ respectively. We have the y -frequency localization on an interval $E = \{\frac{M}{2} \leq |\eta| \leq 2M\}$ of length $|E| \lesssim M$. Moreover, we use that $\nu = \sigma + \frac{1}{2} + 0$ in the second case. \square

We later use this Lemma for some given functions $\phi^{(1)}, \phi^{(2)}, \phi^{(3)}$ of type F, G, w, v , where

$$F = F_N, \quad G = F_N \text{ or } F_{\leq \frac{N}{2}}, \quad w = w_N, \quad v = w_N \text{ or } w_{\leq \frac{N}{2}}.$$

Next, we perform a Littlewood-Paley decomposition of each function, and we conduct a case-by-case analysis of each paracontrolled term defined below, depending on the nature of the functions

$(\phi^{(i)})_{1 \leq i \leq 3}$ occurring in the equation (4.3.10) satisfied by w_n .

$$\begin{aligned}\Pi_{\text{hi,lo,lo}}(\phi^{(1)}, \phi^{(2)}, \phi^{(3)}) &= \sum_{N_2, N_3 \ll N_1} \mathcal{N}(P_{N_1} \phi^{(1)}, P_{N_2} \phi^{(2)}, P_{N_3} \phi^{(3)}), \\ \Pi_{\text{lo,hi,lo}}(\phi^{(1)}, \phi^{(2)}, \phi^{(3)}) &= \sum_{N_1, N_3 \ll N_2} \mathcal{N}(P_{N_1} \phi^{(1)}, P_{N_2} \phi^{(2)}, P_{N_3} \phi^{(3)}), \\ \Pi_{\text{lo,hi,hi}}(\phi^{(1)}, \phi^{(2)}, \phi^{(3)}) &= \sum_{N_1 \ll N_2 \sim N_3} \mathcal{N}(P_{N_1} \phi^{(1)}, P_{N_2} \phi^{(2)}, P_{N_3} \phi^{(3)}), \\ \Pi_{\text{hi,hi}}(\phi^{(1)}, \phi^{(2)}, \phi^{(3)}) &= \sum_{N_3 \lesssim N_1 \sim N_2} \mathcal{N}(P_{N_1} \phi^{(1)}, P_{N_2} \phi^{(2)}, P_{N_3} \phi^{(3)}).\end{aligned}$$

To estimate the norm $L_t^{\frac{4}{3}} L_x^1 L_y^2$ of the Duhamel term, we use the Hölder inequality, putting one term in $L_t^\infty L_{x,y}^2$, the other two in $L_t^8 L_x^4 L_y^\infty$, and gaining a factor $T^{\frac{1}{2}}$. Recall that any small loss of derivative in y is harmless, since our goal is to prove a result for some $s > s_0$. We multiply by N^{0^+} to account for such a 0^+ -loss. At the end the loss are collected in the factor N^δ , where $\delta > \sigma - \rho > 0$ is arbitrarily small.

Proposition 4.7.2 (Trilinear estimates with *high-low-low* type interactions). *Let ϕ be of type v or G , and assume that $\max(\nu - \sigma', \sigma) < \alpha < \nu$, $D > \alpha$ and $2D + s + \sigma' < D'$. Assuming that $\|\langle D_y \rangle^\sigma \phi\|_{S_{\leq N, D}} \lesssim 1$, we have*

$$\|\mathcal{I}(\Pi_{\text{hi,lo,lo}}(F, P_{>N^\gamma} v, \phi))\|_{Y_N^\nu} \lesssim T^{\frac{1}{2}} N^{\nu - \sigma' - \gamma\nu + \delta} \|\langle D_y \rangle^{\sigma'} F\|_{S_{N, D'}} \|\langle D_y \rangle^\nu v\|_{X_{\leq N, D}}, \quad (4.7.1)$$

$$\|\mathcal{I}(\Pi_{\text{hi,lo,lo}}(F, P_{>N^\gamma} G, \phi))\|_{Y_N^\nu} \lesssim T^{\frac{1}{2}} N^{\nu - s - \gamma\sigma'} \|\langle D_y \rangle^s F\|_{X_{N, D'}} \|\langle D_y \rangle^{\sigma'} G\|_{S_{\leq N, D'}}, \quad (4.7.2)$$

$$\|\mathcal{I}(\Pi_{\text{hi,lo,lo}}(F, F, \phi))\|_{Y_N^\nu} \lesssim T^{\frac{1}{2}} N^{\nu - s - \sigma'} \|\langle D_y \rangle^s F\|_{X_{N, D'}} \|\langle D_y \rangle^{\sigma'} F\|_{S_{N, D'}}, \quad (4.7.3)$$

$$\|\mathcal{I}(\Pi_{\text{hi,lo,lo}}(w, G, \phi))\|_{Y_N^\nu} \lesssim T^{\frac{1}{2}} \|\langle D_y \rangle^\nu w\|_{X_{N, \alpha} \cap X_{\leq N, D}} \|\langle D_y \rangle^{\sigma'} G\|_{S_{\leq N, D'}}, \quad (4.7.4)$$

$$\|\mathcal{I}(\Pi_{\text{hi,lo,lo}}(w, v, \phi))\|_{Y_N^\nu} \lesssim T^{\frac{1}{2}} \|\langle D_y \rangle^\nu w\|_{X_{N, \alpha} \cap X_{\leq N, D}} \|\langle D_y \rangle^\sigma v\|_{S_{\leq N, D}}. \quad (4.7.5)$$

Note that by homogeneity, the above inequalities are still true without the assumption $\|\langle D_y \rangle^\sigma \phi\|_{S_{\leq N, D}} \lesssim 1$, up to multiplying the upper bound by $\|\langle D_y \rangle^\sigma \phi\|_{S_{\leq N, D}}$. We only make this assumption for shortness of notation.

Proposition 4.7.3 (Trilinear estimates with *low-high-low* type interactions). *Let ϕ be of type v or G , and assume that $\max(\nu - \sigma', \sigma) < \alpha < \nu$, $D > \alpha$ and $2D + s + \sigma' < D'$. Assuming that $\|\langle D_y \rangle^\sigma \phi\|_{S_{\leq N, D}} \lesssim 1$, we have*

$$\|\mathcal{I}(\Pi_{\text{lo,hi,lo}}(w, G, \phi))\|_{Y_N^\nu} \lesssim T^{\frac{1}{2}} \|\langle D_y \rangle^\nu w\|_{X_{N, \alpha}} \|\langle D_y \rangle^{\sigma'} G\|_{S_{\leq N, D'}}, \quad (4.7.6)$$

$$\|\mathcal{I}(\Pi_{\text{lo,hi,lo}}(w, v, \phi))\|_{Y_N^\nu} \lesssim T^{\frac{1}{2}} \|\langle D_y \rangle^\sigma w\|_{S_{N, \alpha}} \|\langle D_y \rangle^\nu v\|_{X_{\leq N, D}}, \quad (4.7.7)$$

$$\|\mathcal{I}(\Pi_{\text{lo,hi,lo}}(F, G, \phi))\|_{Y_N^\nu} \lesssim T^{\frac{1}{2}} N^{\nu - \sigma' - s} \|\langle D_y \rangle^s F\|_{X_{N, D'}} \|\langle D_y \rangle^{\sigma'} G\|_{S_{\leq N, D'}}, \quad (4.7.8)$$

$$\|\mathcal{I}(\Pi_{\text{lo,hi,lo}}(F, v, \phi))\|_{Y_N^\nu} \lesssim T^{\frac{1}{2}} N^{-\sigma' + 0} \|\langle D_y \rangle^{\sigma'} F\|_{S_{N, D'}} \|\langle D_y \rangle^\nu v\|_{X_{\leq N, D}}. \quad (4.7.9)$$

Proposition 4.7.4 (Trilinear estimates with *low-high-high* type interactions). *Let ϕ be of type v or G , and assume that $\max(\nu - \sigma', \sigma) < \alpha < \nu$ and $2D + s + \sigma' < D'$. Assuming that $\|\langle D_y \rangle^\sigma \phi\|_{S_{\leq N, D}} \lesssim 1$, we have*

$$\|\mathcal{I}(\Pi_{\text{lo,hi,hi}}(w, G, \phi))\|_{Y_N^\nu} \lesssim T^{\frac{1}{2}} \|\langle D_y \rangle^\nu w\|_{X_{N, \alpha}} \|\langle D_y \rangle^{\sigma'} G\|_{S_{\leq N, D'}}, \quad (4.7.10)$$

$$\|\mathcal{I}(\Pi_{\text{lo,hi,hi}}(w, v, \phi))\|_{Y_N^\nu} \lesssim T^{\frac{1}{2}} \|\langle D_y \rangle^\sigma w\|_{S_{N, \alpha}} \|\langle D_y \rangle^\nu v\|_{X_{\leq N, D}}, \quad (4.7.11)$$

$$\|\mathcal{I}(\Pi_{\text{lo,hi,hi}}(F, G, \phi))\|_{Y_N^\nu} \lesssim T^{\frac{1}{2}} N^{\nu - \sigma' - s + 0} \|\langle D_y \rangle^s F\|_{X_{N, D'}} \|\langle D_y \rangle^{\sigma'} G\|_{S_{\leq N, D'}}, \quad (4.7.12)$$

$$\|\mathcal{I}(\Pi_{\text{lo,hi,hi}}(F, v, \phi))\|_{Y_N^\nu} \lesssim T^{\frac{1}{2}} N^{-\sigma' + 0} \|\langle D_y \rangle^{\sigma'} F\|_{S_{N, D'}} \|\langle D_y \rangle^\nu v\|_{X_{\leq N, D}}. \quad (4.7.13)$$

Proposition 4.7.5 (Trilinear estimates with *high-high* interactions type). *Let ϕ be of type v or G , and assume that $\max(\nu - \sigma', \sigma) < \alpha < \nu$ and $2D + \nu + s + \sigma' < D'$. Assuming that $\|\langle D_y \rangle^\sigma \phi\|_{S_{\leq N, D}} \lesssim 1$, we have*

$$\|\mathcal{I}(\Pi_{\text{hi,hi}}(w, G, \phi))\|_{Y_N^\nu} \lesssim T^{\frac{1}{2}} \|\langle D_y \rangle^\nu w\|_{X_{N, \alpha}} \|\langle D_y \rangle^{\sigma'} G\|_{S_{\leq N, D'}}, \quad (4.7.14)$$

$$\|\mathcal{I}(\Pi_{\text{hi,hi}}(w, v, \phi))\|_{Y_N^\nu} \lesssim T^{\frac{1}{2}} \|\langle D_y \rangle^\nu w\|_{X_{N, \alpha} \cap X_{\leq N, D}} \|\langle D_y \rangle^\sigma v\|_{S_{\leq N, D}}, \quad (4.7.15)$$

$$\|\mathcal{I}(\Pi_{\text{hi,hi}}(F, G, \phi))\|_{Y_N^\nu} \lesssim T^{\frac{1}{2}} N^{\nu - \sigma' - s + 0} \|\langle D_y \rangle^s F\|_{X_{N, D'}} \|\langle D_y \rangle^{\sigma'} G\|_{S_{\leq N, D'}}, \quad (4.7.16)$$

$$\|\mathcal{I}(\Pi_{\text{hi,hi}}(F, v, \phi))\|_{Y_N^\nu} \lesssim T^{\frac{1}{2}} N^{-\sigma' + 0} \|\langle D_y \rangle^{\sigma'} F\|_{S_{N, D'}} \|\langle D_y \rangle^\nu v\|_{X_{\leq N, D}}. \quad (4.7.17)$$

The general strategy is as follows. The first term $\phi^{(1)}$ is of type w or F . Hence, it is always localized at frequencies $\sim N$, in the sense that they belong to the spaces $X_{N, \alpha}$ and $S_{N, \alpha}$. When it is of type F , and the other terms are of type w , we place F in L^∞ to exploit the improved stochastic bounds from Proposition 4.4.4.

Proof of Proposition 4.7.2. It is sufficient to put the third term in $S_{\leq N, D}$ with σ derivatives, and to perform the same analysis as in [Bri21] Section 7 to handle the first two terms. We only detail the first case.

• *Proof of (4.7.1).* For fixed $M \geq 1$, we have

$$\begin{aligned} & M^\nu \|P_M(\Pi_{\text{hi,lo,lo}}(F, P_{>N^\gamma} v, \phi))\|_{L_t^{\frac{4}{3}} L_x^1 L_y^2} \\ & \lesssim M^\nu \sum_{K \sim M} \sum_{N^\gamma < L_1 \ll K} \sum_{L_2 \ll K} \|P_M(P_K F_N P_{L_1} w_{\leq \frac{N}{2}} P_{L_2} u_{\frac{N}{2}})\|_{L_t^{\frac{4}{3}} L_x^1 L_y^2} \\ & \lesssim M^\nu T^{\frac{1}{2}} \sum_{K \sim M} \sum_{N^\gamma < L_1 \ll M} \sum_{L_2 \ll M} L_1^{-\nu} L_2^{-\sigma} K^{-\sigma'} \min\left(\frac{N}{K}, \frac{K}{N}\right)^{-D'} \\ & \quad C_{N, D'}(K) \|\langle D_y \rangle^{\sigma'} P_K F\|_{L_t^8 L_x^4 L_y^\infty} \|\langle D_y \rangle^\nu P_{L_1} v\|_{L_t^\infty L_{x, y}^2} \|\langle D_y \rangle^\sigma P_{L_2} \phi\|_{L_t^8 L_x^4 L_y^\infty}. \end{aligned}$$

Therefore, we obtain from the definitions of the norms and the assumption $D' \gg \max(\alpha + \sigma' + \gamma\nu, D + \nu)$, that

$$\begin{aligned} & M^\nu \|P_M \Pi_{\text{hi,lo,lo}}(F, P_{>N^\gamma} v, \phi)\|_{L_t^{\frac{4}{3}} L_x^1 L_y^2} \\ & \lesssim T^{\frac{1}{2}} M^{\nu - \sigma' - \gamma\nu + 0} \min\left(\frac{N}{M}, \frac{M}{N}\right)^{-D'} \|\langle D_y \rangle^{\sigma'} F\|_{S_{N, D'}} \|\langle D_y \rangle^\nu v\|_{X_{\leq N, D}} \|\langle D_y \rangle^\sigma \phi\|_{S_{\leq N, D}}. \end{aligned}$$

It remains to multiply by $C_{N,\alpha}(M) + C_{\leq N,D}(M)$, and to sum over M . Since $D > \alpha$, we get

$$\begin{aligned} & \sum_M M^{\nu-\sigma'-\gamma\nu+0} (C_{N,\alpha}(M) + C_{\leq N,D}(M)) \left(1_{M \lesssim N} \left(\frac{N}{M} \right)^{-D'} + 1_{M \gg N} \left(\frac{M}{N} \right)^{-D'} \right) \\ &= \sum_M M^{\nu-\sigma'-\gamma\nu+0} \left(1_{M \lesssim N} \left(\frac{N}{M} \right)^{-D'+\alpha} + 1_{M \gg N} \left(\frac{M}{N} \right)^{-D'+D} \right) \\ &= \sum_M N^{\nu-\sigma'-\gamma\nu+0} \left(1_{M \lesssim N} \left(\frac{N}{M} \right)^{-D'+\alpha-\nu+\sigma'+\gamma\nu} + 1_{M \gg N} \left(\frac{M}{N} \right)^{-D'+D+\nu-\sigma'-\gamma\nu} \right) \\ &\lesssim N^{\nu-\sigma'-\gamma\nu+0}, \end{aligned}$$

Similarly, in all the other *high-low-low*-type frequency interactions estimated in Proposition 4.7.2, we put the third term in $L_t^8 L_x^4 L_y^\infty$, and we gain nothing from it. As for the quadratic interaction of the first two terms, the analysis is exactly as in [Bri21], and we do not write it here. \square

Proof of Proposition 4.7.3. For fixed $M \in 2^N$, note that we have by definition

$$\begin{aligned} M^\nu \|P_M \Pi_{\text{lo,hi,lo}}(\phi^{(1)}, \phi^{(2)}, \phi^{(3)})\|_{L_t^{\frac{4}{3}} L_x^1 L_y^2} & \\ & \lesssim M^\nu \sum_{N_2 \sim M} \sum_{N_1, N_3 \ll N_2} \|P_{N_1} \phi^{(1)} P_{N_2} \phi^{(2)} P_{N_3} \phi^{(3)}\|_{L_t^{\frac{4}{3}} L_x^1 L_y^2}. \end{aligned} \quad (4.7.18)$$

•*Proof of (4.7.6).* In this case, $\phi^{(2)} = G$ and $\phi^{(1)} = w$. On the other hand, the second term $\phi^{(3)} := \phi$, localized at frequencies $\lesssim N$, has no importance, and we always place it in L^∞ with σ derivatives in y . We do not gain anything from it. There holds

$$\begin{aligned} & \sum_{N_2 \sim M} \sum_{N_1, N_3 \ll N_2} \|P_{N_1} w P_{N_2} G P_{N_3} \phi^{(3)}\|_{L_t^{\frac{4}{3}} L_x^1 L_y^2} \\ & \lesssim T^{\frac{1}{2}} \|\langle D_y \rangle^\nu w\|_{X_{N,\alpha}} \left(\sum_{N_1 \ll M} \max\left(\frac{N}{N_1}, \frac{N_1}{N}\right)^{-\alpha} N_1^{-\nu} \right) \\ & \quad \sum_{N_2 \sim M} N_2^{-\sigma'} \|\langle D_y \rangle^{\sigma'} P_{N_2} G\|_{L_t^8 L_x^4 L_y^\infty} \|\langle D_y \rangle^\sigma \phi^{(3)}\|_{S_{\leq N,D}} \\ & \lesssim T^{\frac{1}{2}} M^{-\sigma'} N^{-\alpha} \max\left(1, \frac{M}{N}\right)^{-D'} \|\langle D_y \rangle^\nu w\|_{X_{N,\alpha}} \|\langle D_y \rangle^{\sigma'} G\|_{S_{\leq N,D'}} \|\langle D_y \rangle^\sigma \phi^{(3)}\|_{S_{\leq N,D}}. \end{aligned}$$

Note that we used the assumption $\alpha < \nu$ to obtain

$$\sum_{N_1} \max\left(\frac{N_1}{N}, \frac{N}{N_1}\right)^{-\alpha} N_1^{-\nu} \lesssim N^{-\alpha}. \quad (4.7.19)$$

Then, we multiply by $(C_{\leq N,D}(M) + C_{N,\alpha}(M))M^\nu$, and using that $\alpha > \nu - \sigma'$, $D > \alpha$ and $D' > D + \nu$,

we sum over M to conclude

$$\begin{aligned} \sum_M (C_{N,\alpha}(M) + C_{\leq N,D}(M)) M^{\nu-\sigma'+0} N^{-\alpha} \max\left(1, \frac{M}{N}\right)^{-D'} \\ \lesssim \sum_{M \lesssim N} M^{\nu-\sigma'-\alpha+0} + N^{-\alpha} \sum_{M \gg N} M^{\nu-\sigma'+0} \left(\frac{M}{N}\right)^{D-D'} \lesssim 1. \end{aligned}$$

•*Proof of (4.7.8).* We proceed similarly in this case, where $\phi^{(1)} = F$ and $\phi^{(2)} = G$. Once again, we use the localization of F at frequencies $\sim N$, which is stronger than the localization of w . There holds

$$\begin{aligned} \sum_{N_2 \sim M} \sum_{N_1, N_3 \ll N_2} \|P_{N_1} F P_{N_2} G P_{N_3} \phi^{(3)}\|_{L_t^{\frac{4}{3}} L_x^1 L_y^2} \\ \lesssim T^{\frac{1}{2}} \left(\sum_{N_1 \ll M} \max\left(\frac{N}{N_1}, \frac{N_1}{N}\right)^{-D'} N_1^{-s} \right) \|\langle D_y \rangle^s F\|_{X_{N,D'}} \\ \sum_{N_2 \sim M} N_2^{-\sigma'+0} \|\langle D_y \rangle^{\sigma'} P_{N_2} G\|_{L_t^8 L_x^4 L_y^\infty} \|\langle D_y \rangle^\sigma \phi^{(3)}\|_{S_{\leq N,D}}. \end{aligned}$$

Then, we observe that

$$\begin{aligned} M^\nu \left(\sum_{N_1 \ll M} \max\left(\frac{N}{N_1}, \frac{N_1}{N}\right)^{-D'} N_1^{-s} \right) \sum_{N_2 \sim M} N_2^{-\sigma'+0} \max\left(1, \frac{N_2}{N}\right)^{-D'} \\ \lesssim M^{\nu-\sigma'+0} \left(\mathbf{1}_{M \lesssim N} \left(\frac{N}{M}\right)^{-D'} M^{-s} + \mathbf{1}_{M \gg N} N^{-s} \left(\frac{M}{N}\right)^{-D'} \right) \\ \lesssim N^{\nu-\sigma'-s} \left(\mathbf{1}_{M \lesssim N} \left(\frac{N}{M}\right)^{s-D'-\nu+\sigma'-0} + \mathbf{1}_{M \gg N} \left(\frac{M}{N}\right)^{-D'+\nu-\sigma'+0} \right) \\ \lesssim N^{\nu-\sigma'-s} \max\left(\frac{M}{N}, \frac{N}{M}\right)^{-2D}, \end{aligned}$$

provided $D' > 2D + \nu + \sigma' + s$. Hence, we can multiply by $(C_{\leq N,D}(M) + C_{N,\alpha}(M))M^\nu$, sum over M and obtain (4.7.8).

•*Proof of (4.7.7).* This case is easier since the derivatives fall onto the term v which is in \mathcal{H}^ν . We have $\phi^{(2)} = v$ and $\phi^{(1)} = w$. We place the high frequency term in L^2 in order to absorb M^σ , and we end up with

$$\begin{aligned} \sum_{N_2 \sim M} \sum_{N_1, N_3 \ll N_2} \|P_{N_1} w P_{N_2} v P_{N_3} \phi^{(3)}\|_{L_t^{\frac{4}{3}} L_x^1 L_y^2} \\ \lesssim T^{\frac{1}{2}} \sum_{N_1, N_3 \ll M} \sum_{N_2 \sim M} \|P_{N_1} w\|_{L_t^8 L_x^4 L_y^\infty} \|P_{N_3} \phi^{(3)}\|_{L_t^8 L_x^4 L_y^\infty} \|P_{N_2} v\|_{L_t^\infty L_{x,y}^2} \\ \lesssim T^{\frac{1}{2}} \left(\sum_{N_1 \ll M} \max\left(\frac{N}{N_1}, \frac{N_1}{N}\right)^{-\alpha} N_1^{-\sigma} \right) \sum_{N_2 \sim M} \max\left(1, \frac{N_2}{N}\right)^{-D} N_2^{-\nu} \\ \|\langle D_y \rangle^\sigma w\|_{S_{N,\alpha}} \|\langle D_y \rangle^\sigma \phi^{(3)}\|_{S_{\leq N,D}} \|\langle D_y \rangle^\nu P_{N_2} v\|_{X_{\leq N,D}}. \end{aligned}$$

Since $\alpha > \sigma$, we observe that the sum over N_1 is bounded by $N^{-\alpha}M^{\alpha-\sigma}$ when $M \lesssim N$ and $N^{-\sigma}$ when $M \gg N$, so that

$$\begin{aligned} M^\nu \sum_{N_1 \ll M} \max\left(\frac{N}{N_1}, \frac{N_1}{N}\right)^{-\alpha} N_1^{-\sigma} \sum_{N_2 \sim M} \max\left(1, \frac{N_2}{N}\right)^{-D} N_2^{-\nu} \\ \lesssim \mathbf{1}_{M \lesssim N} N^{-\alpha} M^{\alpha-\sigma} + \mathbf{1}_{M \gg N} \left(\frac{M}{N}\right)^{-D} N^{-\sigma} \\ \lesssim \mathbf{1}_{M \lesssim N} \left(\frac{N}{M}\right)^{-\alpha} M^{-\sigma} + \mathbf{1}_{M \gg N} \left(\frac{M}{N}\right)^{-D}. \end{aligned}$$

We deduce (4.7.7) after multiplying by $(C_{\leq N, D}(M) + C_{N, \alpha}(M))$ and summing over M , making the norm $\|\langle D_y \rangle^\nu v\|_{X_{\leq N, D}}$ appear.

•*Proof of (4.7.9).* This proof is identical than the one above. Observe that we only used the localization of w at frequency N in the Strichartz space $S_{N, \alpha}$, and the localization of F is even better. Indeed, we have

$$\begin{aligned} \sum_{N_2 \sim M} \sum_{N_1, N_3 \ll N_2} \|P_{N_1} F P_{N_2} v P_{N_3} \phi^{(3)}\|_{L_t^{\frac{4}{3}} L_x^1 L_y^2} \\ \lesssim T^{\frac{1}{2}} \left(\sum_{N_1 \ll M} \max\left(\frac{N}{N_1}, \frac{N_1}{N}\right)^{-D'} N_1^{-\sigma'} \right) \sum_{N_2 \sim M} \max\left(1, \frac{N_2}{N}\right)^{-D} N_2^{-\nu} \\ \|\langle D_y \rangle^{\sigma'} F\|_{S_{N, D'}} \|\langle D_y \rangle^\sigma \phi^{(3)}\|_{S_{\leq N, D}} \|\langle D_y \rangle^\nu P_{N_2} v\|_{X_{\leq N, D}}. \end{aligned}$$

Since $D' > \sigma'$, we have

$$\begin{aligned} \mathbf{1}_{M \gg N} \sum_{N_1 \ll N} \left(\frac{N}{N_1}\right)^{-D'} N_1^{-\sigma'} &\lesssim N^{-\sigma'}, \\ \mathbf{1}_{M \gg N} \sum_{N \lesssim N_1 \ll M} \left(\frac{N_1}{N}\right)^{-D'} N_1^{-\sigma'} &\lesssim N^{-\sigma'}, \\ \mathbf{1}_{M \lesssim N} \sum_{N_1 \ll M} \left(\frac{N}{N_1}\right)^{-D'} N_1^{-\sigma'} &\lesssim N^{-D'} M^{D'-\sigma'}, \end{aligned}$$

so that

$$\begin{aligned} M^\nu \left(\sum_{N_1 \ll M} \max\left(\frac{N}{N_1}, \frac{N_1}{N}\right)^{-D'} N_1^{-\sigma'} \right) \sum_{N_2 \sim M} \|\langle D_y \rangle^\nu P_{N_2} v\|_{X_{\leq N, D}} N_2^{-\nu} \\ \lesssim \mathbf{1}_{M \lesssim N} \left(\frac{N}{M}\right)^{-D'+\sigma'-0} N^{-\sigma'} + \mathbf{1}_{M \gg N} \left(\frac{M}{N}\right)^{-D} N^{-\sigma'}. \end{aligned}$$

Since $D' > \sigma' + \alpha$, we deduce (4.7.9) after multiplying by $(C_{\leq N, D}(M) + C_{N, \alpha}(M))$ and summing over M , making the norm $\|\langle D_y \rangle^\nu v\|_{X_{\leq N, D}}$ appear. \square

Proof of Proposition 4.7.4. In this situation, inequality (4.7.18) becomes restricted to the indices $N_2 \sim N_3 \gtrsim M$ such that $N_1 \ll N_2$:

$$M^\nu \|P_M \Pi_{\text{lo,hi,hi}}(\phi^{(1)}, \phi^{(2)}, \phi^{(3)})\|_{L_t^{\frac{4}{3}} L_x^1 L_y^2} \lesssim M^\nu \sum_{N_2 \sim N_3 \gtrsim M} \sum_{N_1 \ll N_2} \|P_{N_1} \phi^{(1)} P_{N_2} \phi^{(2)} P_{N_3} \phi^{(3)}\|_{L_t^{\frac{4}{3}} L_x^1 L_y^2}.$$

•*Proof of (4.7.10).* Now, we observe that

$$\begin{aligned} & \sum_{N_2 \sim N_3 \gtrsim M} \sum_{N_1 \ll N_2} \|P_{N_1} w P_{N_2} G P_{N_3} \phi^{(3)}\|_{L_t^{\frac{4}{3}} L_x^1 L_y^2} \\ & \lesssim T^{\frac{1}{2}} \sum_{N_2 \gtrsim M} \sum_{N_1 \ll N_2} \max\left(\frac{N}{N_1}, \frac{N_1}{N}\right)^{-\alpha} \max\left(1, \frac{N_2}{N}\right)^{-D'} N_1^{-\nu} N_2^{-\sigma'} \\ & \qquad \|\langle D_y \rangle^\nu w\|_{X_{N,\alpha}} \|\langle D_y \rangle^\sigma \phi^{(3)}\|_{S_{\leq N,D}} \|\langle D_y \rangle^{\sigma'} G\|_{S_{\leq N,D'}}. \end{aligned}$$

Then, according to (4.7.19), the sum over N_1 is bounded by $N^{-\alpha}$. Moreover, one can check that

$$\sum_{M \lesssim N_2} \max\left(1, \frac{N_2}{N}\right)^{-D'} N_2^{-\sigma'} \lesssim \begin{cases} M^{-\sigma'} & \text{if } M \lesssim N, \\ \left(\frac{M}{N}\right)^{-D'} M^{-\sigma'} & \text{if } M \gg N, \end{cases} \quad (4.7.20)$$

therefore we have

$$\begin{aligned} & M^\nu \sum_{N_2 \gtrsim M} \sum_{N_1 \ll N_2} \max\left(\frac{N}{N_1}, \frac{N_1}{N}\right)^{-\alpha} \max\left(1, \frac{N_2}{N}\right)^{-D'} N_1^{-\nu} N_2^{-\sigma'} \\ & \lesssim \mathbf{1}_{M \lesssim N} N^{-\alpha} M^{\nu-\sigma'} + \mathbf{1}_{M \gg N} \left(\frac{M}{N}\right)^{-D'} N^{-\alpha} M^{\nu-\sigma'} \\ & \lesssim \mathbf{1}_{M \lesssim N} \left(\frac{N}{M}\right)^{-\alpha} M^{\nu-\sigma'-\alpha} + \mathbf{1}_{M \gg N} \left(\frac{M}{N}\right)^{-D'+\alpha} M^{\nu-\sigma'-\alpha}. \end{aligned}$$

Estimate (4.7.14) comes from multiplying by $(C_{\leq N,D}(M) + C_{N,\alpha}(M))$ and summing over M , using that $\alpha > \nu - \sigma'$.

•*Proof of (4.7.11).* Similarly, we write

$$\begin{aligned} & \sum_{N_2 \sim N_3 \gtrsim M} \sum_{N_1 \ll N_2} \|P_{N_1} w P_{N_2} v P_{N_3} \phi^{(3)}\|_{L_t^{\frac{4}{3}} L_x^1 L_y^2} \\ & \lesssim T^{\frac{1}{2}} \sum_{N_2 \gtrsim M} \sum_{N_1 \ll N_2} \max\left(\frac{N}{N_1}, \frac{N_1}{N}\right)^{-\alpha} \max\left(1, \frac{N_2}{N}\right)^{-D} N_1^{-\sigma} N_2^{-\nu} \\ & \qquad \|\langle D_y \rangle^\sigma w\|_{S_{N,\alpha}} \|\langle D_y \rangle^\sigma \phi^{(3)}\|_{S_{\leq N,D}} \|\langle D_y \rangle^\nu P_{N_2} v\|_{X_{\leq N,D}}. \end{aligned}$$

Since $\alpha > \sigma$, we observe that the sum over N_1 is bounded by $\left(\frac{N}{M}\right)^{-\alpha} M^{-\sigma}$ when $M \lesssim N$ and $N^{-\sigma}$ when $M \gg N$. Moreover, the sum over N_2 is bounded as in (4.7.20) by replacing D' by D and

$\sigma' - 0$ by ν , which leads to

$$\begin{aligned} M^\nu \sum_{N_2 \gtrsim M} \sum_{N_1 \lesssim N_2} \max\left(\frac{N}{N_1}, \frac{N_1}{N}\right)^{-\alpha} \max\left(1, \frac{N_2}{N}\right)^{-D} N_1^{-\sigma} N_2^{-\nu} \\ \lesssim \mathbf{1}_{M \lesssim N} \left(\frac{N}{M}\right)^{-\alpha} M^{-\sigma} + \mathbf{1}_{M \gg N} \left(\frac{M}{N}\right)^{-D} N^{-\sigma}. \end{aligned}$$

It remains to multiply by $(C_{\leq N, D}(M) + C_{N, \alpha}(M))$ and sum over M to get the norm $\|\langle D_y \rangle^\nu v\|_{X_{\leq N, D}}$.

•*Proof of (4.7.12).* We adopt the same strategy but we put the first term in the adapted space $X_{N, D'}$ after taking s derivatives:

$$\begin{aligned} \sum_{N_2 \sim N_3 \gtrsim M} \sum_{N_1 \lesssim N_2} \|P_{N_1} F P_{N_2} G P_{N_3} \phi^{(3)}\|_{L_t^{\frac{4}{3}} L_x^1 L_y^2} \\ \lesssim T^{\frac{1}{2}} \sum_{N_2 \gtrsim M} \sum_{N_1 \lesssim N_2} \max\left(\frac{N}{N_1}, \frac{N_1}{N}\right)^{-D'} \max\left(1, \frac{N_2}{N}\right)^{-D'} N_1^{-s} N_2^{-\sigma'} \\ \|\langle D_y \rangle^s F\|_{X_{N, D'}} \|\langle D_y \rangle^\sigma \phi^{(3)}\|_{S_{\leq N, D}} \|\langle D_y \rangle^{\sigma'} G\|_{S_{\leq N, D'}}. \end{aligned}$$

Since $D' > s$, the sum over N_1 is bounded by $\left(\frac{N}{M}\right)^{-D'} M^{-s}$ when $M \lesssim N$ and N^{-s} when $M \gg N$. Moreover, the sum over N_2 is bounded as in (4.7.20) and we get

$$\begin{aligned} M^\nu \sum_{N_2 \gtrsim M} \sum_{N_1 \lesssim N_2} \max\left(\frac{N}{N_1}, \frac{N_1}{N}\right)^{-D'} \max\left(1, \frac{N_2}{N}\right)^{-D'} N_1^{-s} N_2^{-\sigma'} \\ \lesssim \mathbf{1}_{M \lesssim N} \left(\frac{N}{M}\right)^{-D'} M^{\nu-s-\sigma'} + \mathbf{1}_{M \gg N} \left(\frac{M}{N}\right)^{-D'} N^{-s} M^{\nu-\sigma'} \\ \lesssim \mathbf{1}_{M \lesssim N} \left(\frac{N}{M}\right)^{-D'-\nu+s+\sigma'-0} N^{\nu-s-\sigma'} + \mathbf{1}_{M \gg N} \left(\frac{M}{N}\right)^{-D'-\nu+\sigma'} N^{\nu-s-\sigma'}. \end{aligned}$$

Since $D' > \alpha + s + \sigma'$ and $D' > D + \sigma'$, multiplying by $(C_{\leq N, D}(M) + C_{N, \alpha}(M))$ the sum over M is bounded by $N^{\nu-s-\sigma'}$.

•*Proof of (4.7.13).* Finally, we put the term F in $S_{N, D'}$:

$$\begin{aligned} \sum_{N_2 \sim N_3 \gtrsim M} \sum_{N_1 \lesssim N_2} \|P_{N_1} F P_{N_2} v P_{N_3} \phi^{(3)}\|_{L_t^{\frac{4}{3}} L_x^1 L_y^2} \\ \lesssim T^{\frac{1}{2}} \sum_{N_2 \gtrsim M} \sum_{N_1 \lesssim N_2} \max\left(\frac{N}{N_1}, \frac{N_1}{N}\right)^{-D'} \max\left(1, \frac{N_2}{N}\right)^{-D} N_1^{-\sigma'} N_2^{-\nu} \\ \|\langle D_y \rangle^{\sigma'} F\|_{S_{N, D'}} \|\langle D_y \rangle^\sigma \phi^{(3)}\|_{S_{\leq N, D}} \|\langle D_y \rangle^\nu v\|_{X_{\leq N, D}}. \end{aligned}$$

The sum over N_1 is bounded as above by $\left(\frac{N}{M}\right)^{-D'} M^{-\sigma'}$ when $M \lesssim N$ and $N^{-\sigma'}$ when $M \gg N$, and the sum over N_2 is bounded as in (4.7.20) by $M^{-\nu}$ if $M \lesssim N$ and $\left(\frac{M}{N}\right)^{-D} M^{-\nu}$ if $M \gg N$. We get

$$\begin{aligned} M^\nu \sum_{N_2 \gtrsim M} \sum_{N_1 \lesssim N_2} \max\left(\frac{N}{N_1}, \frac{N_1}{N}\right)^{-D'} \max\left(1, \frac{N_2}{N}\right)^{-D} N_1^{-\sigma'} N_2^{-\nu} \\ \lesssim \mathbf{1}_{M \lesssim N} \left(\frac{N}{M}\right)^{-D'} M^{-\sigma'} + \mathbf{1}_{M \gg N} \left(\frac{M}{N}\right)^{-D} N^{-\sigma'}. \end{aligned}$$

Since $D' > \alpha + \sigma'$, after multiplying by $(C_{\leq N, D}(M) + C_{N, \alpha}(M))$ the sum over M is bounded by $N^{-\sigma'}$. \square

Proof of Proposition 4.7.5. In this situation, inequality (4.7.18) becomes restricted to the indices $N_1 \sim N_2 \gtrsim M$ such that $N_3 \lesssim N_1$:

$$M^\nu \|P_M \Pi_{\text{hi, hi}}(\phi^{(1)}, \phi^{(2)}, \phi^{(3)})\|_{L_t^{\frac{4}{3}} L_x^1 L_y^2} \lesssim M^\nu \sum_{N_1 \sim N_2 \gtrsim M} \sum_{N_3 \lesssim N_1} \|P_{N_1} \phi^{(1)} P_{N_2} \phi^{(2)} P_{N_3} \phi^{(3)}\|_{L_t^{\frac{4}{3}} L_x^1 L_y^2}.$$

•*Proof of (4.7.14).* Now, we observe that

$$\begin{aligned} \sum_{N_1 \sim N_2 \gtrsim M} \sum_{N_3 \lesssim N_1} \|P_{N_1} w P_{N_2} G P_{N_3} \phi^{(3)}\|_{L_t^{\frac{4}{3}} L_x^1 L_y^2} \\ \lesssim T^{\frac{1}{2}} \sum_{N_1 \sim N_2 \gtrsim M} \max\left(\frac{N}{N_1}, \frac{N_1}{N}\right)^{-\alpha} \max\left(1, \frac{N_2}{N}\right)^{-D'} N_1^{-\nu} N_2^{-\sigma'} \\ \|\langle D_y \rangle^\nu w\|_{X_{N, \alpha}} \|\langle D_y \rangle^\sigma \phi^{(3)}\|_{S_{\leq N, D}} \|\langle D_y \rangle^{\sigma'} G\|_{S_{\leq N, D'}}. \end{aligned}$$

Then, since $N_2 \sim N_1$ and $\alpha < \nu$, we observe that

$$\begin{aligned} \sum_{M \lesssim N_1 \ll N} \left(\frac{N}{N_1}\right)^{-\alpha} N_1^{-\nu-\sigma'} \lesssim N^{-\alpha} M^{\alpha-\nu-\sigma'} = \left(\frac{N}{M}\right)^{-\alpha} M^{-\nu-\sigma'}, \\ \sum_{M \lesssim N \lesssim N_1} \left(\frac{N_1}{N}\right)^{-\alpha-D'} N_1^{-\nu-\sigma'} \lesssim N^{-\nu-\sigma'} = \left(\frac{N}{M}\right)^{-\nu} M^{-\nu} N^{-\sigma'}, \\ \sum_{N_1 \gtrsim M \gg N} \left(\frac{N_1}{N}\right)^{-\alpha-D'} N_1^{-\nu-\sigma'} \lesssim \left(\frac{M}{N}\right)^{-\alpha-D'} M^{-\nu-\sigma'}, \end{aligned}$$

therefore since $D' > D$, we have

$$\begin{aligned} M^\nu \sum_{N_1 \gtrsim M} \max\left(\frac{N}{N_1}, \frac{N_1}{N}\right)^{-\alpha} \max\left(1, \frac{N_1}{N}\right)^{-D'} N_1^{-\nu-\sigma'} \\ \lesssim \left(\mathbf{1}_{M \lesssim N} \left(\frac{N}{M}\right)^{-\alpha} + \mathbf{1}_{M \gg N} \left(\frac{M}{N}\right)^{-D'}\right) M^{-\sigma'}. \end{aligned}$$

Estimate (4.7.14) comes from multiplying by $(C_{\leq N,D}(M) + C_{N,\alpha}(M))$ and summing over M .

•*Proof of (4.7.15)*. The proof is identical as the proof of (4.7.14), indeed we only need to replace G by v , D' by D and σ' by σ in the argument.

•*Proof of (4.7.16)*. We adopt the same strategy but we put the first term in the adapted space $X_{N,D'}$ after taking s derivatives:

$$\begin{aligned} & \sum_{N_1 \sim N_2 \gtrsim M} \sum_{N_3 \lesssim N_1} \|P_{N_1} F P_{N_2} G P_{N_3} \phi^{(3)}\|_{L_t^{\frac{4}{3}} L_x^1 L_y^2} \\ & \lesssim T^{\frac{1}{2}} \sum_{N_1 \sim N_2 \gtrsim M} \max\left(\frac{N}{N_1}, \frac{N_1}{N}\right)^{-D'} \max\left(1, \frac{N_2}{N}\right)^{-D'} N_1^{-s} N_2^{-\sigma'} \\ & \quad \|\langle D_y \rangle^s F\|_{X_{N,D'}} \|\langle D_y \rangle^\sigma \phi^{(3)}\|_{S_{\leq N,D}} \|\langle D_y \rangle^{\sigma'} G\|_{S_{\leq N,D'}}. \end{aligned}$$

Then we have

$$\begin{aligned} \mathbf{1}_{M \lesssim N} & \sum_{M \lesssim N_1 \sim N_2 \lesssim N} \left(\frac{N}{N_1}\right)^{-D'} N_1^{-s-\sigma'} \lesssim N^{-s-\sigma'}, \\ \mathbf{1}_{M \lesssim N} & \sum_{N \lesssim N_1 \sim N_2} \left(\frac{N_1}{N}\right)^{-2D'} N_1^{-s-\sigma'} \lesssim N^{-s-\sigma'}, \\ \mathbf{1}_{M \gg N} & \sum_{M \lesssim N_1 \sim N_2} \left(\frac{N_1}{N}\right)^{-2D'} N_1^{-s-\sigma'} \lesssim \left(\frac{M}{N}\right)^{-2D'} M^{-s-\sigma'}, \end{aligned}$$

so that

$$\begin{aligned} M^\nu & \sum_{N_1 \sim N_2 \gtrsim M} \max\left(\frac{N}{N_1}, \frac{N_1}{N}\right)^{-D'} \max\left(1, \frac{N_2}{N}\right)^{-D'} N_1^{-s} N_2^{-\sigma'} \\ & \lesssim \mathbf{1}_{M \lesssim N} \left(\frac{N}{M}\right)^{-\alpha} M^{\nu-\alpha} N^{\alpha-s-\sigma'} + \mathbf{1}_{M \gg N} \left(\frac{M}{N}\right)^{-2D'+\nu-s-\sigma'} N^{\nu-s-\sigma'}. \end{aligned}$$

After multiplying by $(C_{\leq N,D}(M) + C_{N,\alpha}(M))$ the sum over M is bounded by $N^{\nu-s-\sigma'}$ since $2D' > D + \nu - s - \sigma'$ by assumption.

•*Proof of (4.7.17)*. Finally, we put the term F in $S_{N,D'}$:

$$\begin{aligned} & \sum_{N_1 \sim N_2 \gtrsim M} \sum_{N_3 \lesssim N_1} \|P_{N_1} F P_{N_2} v P_{N_3} \phi^{(3)}\|_{L_t^{\frac{4}{3}} L_x^1 L_y^2} \\ & \lesssim T^{\frac{1}{2}} \sum_{N_1 \sim N_2 \gtrsim M} \max\left(\frac{N}{N_1}, \frac{N_1}{N}\right)^{-D'} \max\left(1, \frac{N_2}{N}\right)^{-D'} N_1^{-\sigma'} N_2^{-\nu} \\ & \quad \|\langle D_y \rangle^{\sigma'} F\|_{S_{N,D'}} \|\langle D_y \rangle^\sigma \phi^{(3)}\|_{S_{\leq N,D}} \|\langle D_y \rangle^\nu v\|_{X_{\leq N,D}}. \end{aligned}$$

Then we have

$$\mathbf{1}_{M \lesssim N} \sum_{M \lesssim N_1 \sim N_2 \lesssim N} \left(\frac{N}{N_1}\right)^{-D'} N_1^{-\sigma'-\nu} \lesssim N^{-\sigma'-\nu},$$

$$\begin{aligned} \mathbf{1}_{M \lesssim N} \sum_{N \lesssim N_1 \sim N_2} \left(\frac{N_1}{N} \right)^{-D'-D} N_1^{-\sigma'-\nu} &\lesssim N^{-\sigma'-\nu}, \\ \mathbf{1}_{M \gg N} \sum_{M \lesssim N_1 \sim N_2} \left(\frac{N_1}{N} \right)^{-D'-D} N_1^{-\sigma'-\nu} &\lesssim \left(\frac{M}{N} \right)^{-D'-D} M^{-\sigma'-\nu}, \end{aligned}$$

so that

$$\begin{aligned} M^\nu \sum_{N_1 \sim N_2 \gtrsim M} \max\left(\frac{N}{N_1}, \frac{N_1}{N}\right)^{-D'} \max\left(1, \frac{N_2}{N}\right)^{-D} N_1^{-\sigma'} N_2^{-\nu} \\ \lesssim \mathbf{1}_{M \lesssim N} \left(\frac{N}{M}\right)^{-\alpha} M^{\nu-\alpha} N^{\alpha-\nu-\sigma'} + \mathbf{1}_{M \gg N} \left(\frac{M}{N}\right)^{-D'-D-\sigma'} N^{\sigma'}. \end{aligned}$$

After multiplying by $(C_{\leq N, D}(M) + C_{N, \alpha}(M))$ the sum over M is bounded by $N^{-\sigma'}$. □

4.8 Remarks on ill-posedness

4.8.1 Semilinear ill-posedness

From the traveling wave profiles for the Szegő equation, we cook up a one-parameter family of profiles from which we deduce that the bilinear estimate only holds when $s \geq \frac{1}{2}$. Then, we proceed as in Remark 2.12 from [BGT05a] to deduce that the flow map cannot be of class \mathcal{C}^3 at the origin when $\frac{1}{4} < s < \frac{1}{2}$. We recall that $\mathcal{H}^s = H_x^{2s} L_y^2 \cap L_x^2 H_y^s$.

Theorem 4.8.1 (Semilinear ill-posedness). *If there exists a local in time flow map on \mathcal{H}^s with regularity \mathcal{C}^3 at the origin, then $s \geq \frac{1}{2}$.*

Proof. As a corollary of [BGT05a], Remark 2.12, if there exists a \mathcal{C}^3 local in time flow map at the vicinity of the origin in the space \mathcal{H}^s , then the following bilinear estimate holds:

$$\|e^{itA} f\|_{L^4([0,1] \times \mathbb{R}^2)} \lesssim \|f\|_{\mathcal{H}^{\frac{s}{2}}}. \quad (4.8.1)$$

As usual, we use a one-parameter family of stationary solutions to evidence the instabilities of the flow-map. Namely, we consider a Gaussian distribution G , a family of traveling waves profiles for the Szegő equation $K_\rho(y)$, for $\rho \in (0, +\infty)$, and set

$$f(x, y) = G(x)K_\rho(y), \quad K_\rho(y) = \frac{1}{y + i\rho}.$$

First, let us look at the scaling of $L_{t,x}^4$ -norm of f_ρ as ρ goes to 0. Since K_ρ is a traveling wave for the Szegő equation on the line [Poc11], and in particular a traveling wave for equation (NLS-HW), one can see that for every t , there holds

$$\|e^{it|D_y|} K_\rho\|_{L^4(\mathbb{R}_y)} = \|K_\rho\|_{L^4(\mathbb{R}_y)}$$

with

$$\|K_\rho\|_{L^4(\mathbb{R}_y)}^4 = \int_{\mathbb{R}} \frac{1}{(y^2 + \rho^2)^2} dy = \rho^{-3} \int_{\mathbb{R}} \frac{1}{(v^2 + 1)^2} dv.$$

This implies that as $\rho \rightarrow 0$,

$$\|e^{it\mathcal{A}}f\|_{L^4([0,1] \times \mathbb{R}_{x,y}^2)} = \|e^{it\partial_{xx}}G\|_{L^4([0,1] \times \mathbb{R}_x)} \|K_\rho\|_{L^4(\mathbb{R}_y)} \sim C\rho^{-\frac{3}{4}}.$$

Next, let us look at the $L_x^2 \dot{H}_y^{\frac{s}{2}}$ component of the \mathcal{H}^s -norm of f_ρ . It reads

$$\|f\|_{L_x^2 \dot{H}_y^{\frac{s}{2}}} = \|G\|_{L_x^2} \|K_\rho\|_{\dot{H}_y^{\frac{s}{2}}},$$

and we compute

$$\|K_\rho\|_{\dot{H}_y^{\frac{s}{2}}}^2 = \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|K_\rho(y+h) - K_\rho(y)|^2}{|h|^{1+s}} dh dy.$$

Since

$$K_\rho(y+h) - K_\rho(y) = \frac{-h}{(y+h+i\rho)(y+i\rho)},$$

we deduce that

$$\|K_\rho\|_{\dot{H}_y^{\frac{s}{2}}}^2 = \int_{\mathbb{R}} \int_{\mathbb{R}} |h|^{1-\frac{s}{2}} \frac{1}{((y+h)^2 + \rho^2)(y^2 + \rho^2)} dh dy.$$

We make the change of variable $y = \rho u$ and $h = \rho v$ to get

$$\|K_\rho\|_{\dot{H}_y^{\frac{s}{2}}}^2 = \rho^{-1-s} \int_{\mathbb{R}} \int_{\mathbb{R}} |v|^{1-s} \frac{1}{((u+v)^2 + 1)(u^2 + 1)} du dv \sim C'\rho^{-1-s},$$

Moreover,

$$\|f\|_{\dot{H}_x^s L_y^2} = \|e^{it\partial_{xx}}G\|_{\dot{H}_x^s} \|K_\rho\|_{L_y^2} = \rho^{-\frac{1}{2}} \left(\int \frac{1}{1+y^2} dy \right)^{\frac{1}{2}} \sim C''\rho^{-\frac{1}{2}},$$

so that

$$\|f\|_{\mathcal{H}^s} \underset{\rho \rightarrow 0}{\sim} C'\rho^{-\frac{1}{2}-\frac{s}{2}}.$$

Since inequality (4.8.1) has to hold as $\rho \rightarrow 0$, we see that necessarily, $\rho^{-\frac{3}{4}} \lesssim \rho^{-\frac{1}{2}-s}$, therefore

$$\frac{3}{4} \leq \frac{1}{2} + \frac{s}{2},$$

leading to the result $\frac{1}{2} \leq s$. □

4.8.2 Pathological set

An adaptation of the arguments from [BGT05a] implemented in [Kat21] implies the following ill-posedness result.

Theorem 4.8.2 (Norm inflation [Kat21]). *Let $s < \frac{1}{4}$. For every bounded set B of \mathcal{H}^s and for every $T > 0$, the flow map cannot be extended as a continuous map from B to $\mathcal{C}([-T, T], \mathcal{H}^s)$. More precisely, there exists a sequence $(t_n)_{n \in \mathbb{N}}$ of positive numbers tending to zero and a sequence $(u_N(t))_{n \gg 1}$ of $C^\infty(\mathbb{R}^2)$ solutions of (NLS-HW) defined for $t \in [0, t_n]$, and as $n \rightarrow \infty$*

$$\|u_n(0)\|_{\mathcal{H}^s} \rightarrow 0,$$

$$\|u_n(t_n)\|_{\mathcal{H}^s} \rightarrow \infty.$$

Non-uniform continuity of the flow map for $s = \frac{1}{4}$. has also been investigated in [Kat21]. Using the method of Sun and Tzvetkov [ST20a] adapted to Schrödinger-type equations in [CG22], we make precise this norm inflation result in the context of probabilistic well-posedness. We fix $\rho \in C_c^\infty(\mathbb{R}^2)$, valued in $[0, 1]$, such that $\int_{\mathbb{R}^2} \rho(x) dx = 1$ and ρ vanishes for $x^2 + y^2 \geq \frac{1}{10^4}$. Due to the anisotropy, we define an approximate identity $(\rho_\varepsilon)_{\varepsilon > 0}$ of the form

$$\rho_\varepsilon(x, y) := \frac{1}{\varepsilon^3} \rho\left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon^2}\right).$$

Theorem 4.8.3 (Generic ill-posedness for (NLS-HW)). *Let $s < 1/2$. There exists a dense set $S \subset X^s$ such that for every $f \in S$, the family of local solutions $(u^\varepsilon)_{\varepsilon > 0}$ of (NLS-HW) with initial data $\rho_\varepsilon * f$ does not converge. More precisely, there exist $\varepsilon_n \rightarrow 0$ and $t_n \rightarrow 0$ such that $u^{\varepsilon_n}(t_n)$ is well-defined and*

$$\lim_{n \rightarrow \infty} \|u^{\varepsilon_n}(t_n)\|_{\mathcal{H}^s} = \infty.$$

Since the argument is the same as in [CG22], we only make a sketch of the proof adapted to the anisotropy. We construct unstable profiles with growing \mathcal{H}^s norm based on the bounded time-periodic solution $V(t) = e^{it}$ to the ODE

$$\begin{cases} iV' + |V|^2V = 0, \\ V(0) = 1. \end{cases}$$

Let $\varphi \in C_c^\infty(\{x^2 + y^2 \leq 1\})$, radial such that $0 \leq \varphi \leq 1$. Let γ to be chosen later, and

$$\kappa_n := (\log(n))^{-\gamma}, \quad \lambda_n := \kappa_n n^{\frac{3}{2}-2s}.$$

We define

$$v_n(0, x, y) := \lambda_n \varphi(nx, n^2y).$$

We recall that

$$\rho_\varepsilon(x, y) := \frac{1}{\varepsilon^3} \rho\left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon^2}\right),$$

and define

$$v_n^\varepsilon(0) := \rho_\varepsilon * v_n(0).$$

The profile

$$v_n^\varepsilon(t, x, y) := v_n^\varepsilon(0, x, y) V(t|v_n^\varepsilon(0, x, y)|^2)$$

is solution with initial data $v_n^\varepsilon(0)$ to

$$i\partial_t v_n^\varepsilon + |v_n^\varepsilon|^2 v_n^\varepsilon = 0.$$

We now fix the parameters

$$\varepsilon_n := \frac{1}{100n}, \quad t_n := \log(n)^{2\beta} n^{2(2s-3/2)} = \lambda_n^{-2} \log(n)^{2(\beta-\gamma)}, \quad 0 < \gamma < \beta < 1.$$

Lemma 2.1 in [CG22] becomes as follows.

Lemma 4.8.4 (Growth of the ODE profile). *Let $0 \leq s < \frac{1}{2}$.*

1. *For every $\varepsilon \leq \varepsilon_n$, there holds*

$$\|v_n^\varepsilon(t_n)\|_{H_x^{2s} L_y^2} \gtrsim \kappa_n (\lambda_n^2 t_n)^{2s}, \quad \|v_n^\varepsilon(t_n)\|_{L_x^2 H_y^s} \gtrsim \kappa_n (\lambda_n^2 t_n)^s.$$

2. *For every $k, l \in \mathbb{N}$, there exists $C > 0$ such that for every $n \in \mathbb{N}$, $t \in \mathbb{R}$ and $\varepsilon > 0$,*

$$\|v_n^\varepsilon(t)\|_{H_x^k H_y^l} \leq C \kappa_n n^{k+2l-2s} (1 + (t\lambda_n^2)^{k+l}) \min\left(1, \frac{1}{\varepsilon n}\right)^{k+2l+3}.$$

3. *Moreover, with the same notation, we have*

$$\|\partial_x^k \partial_y^l v_n^\varepsilon(t)\|_{L^\infty} \leq C \lambda_n n^{k+2l} (1 + (t\lambda_n^2)^{k+l}) \min\left(1, \frac{1}{\varepsilon n}\right)^{k+2l+3}.$$

Let us denote by $t \mapsto \Phi(t)(f)$ the local maximal solution to (NLS-HW) with initial data f in \mathcal{H}^m when $m > \frac{1}{2}$. In order to define a dense subset S of the pathological set

$$\mathcal{P} = \left\{ f \in \mathcal{H}^s(\mathbb{R}^2) \mid \limsup_{\varepsilon \rightarrow 0, t \rightarrow 0} \|\Phi(t)(\rho_\varepsilon * f)\|_{\mathcal{H}^s} \rightarrow \infty \right\},$$

we apply the “*tanghuru*” construction. The k -th bubble $v_{0,k}$ is centered at any point $y_k \in \mathbb{R}^2$, with scaling parameter $n_k = e^{a^k}$ for some $1 \ll a$:

$$v_{0,k}(x, y) := v_{n_k}(x, y - y_k) = \log(n_k)^{-\gamma} n_k^{3/2-2s} \varphi(n_k x, n_k^2(y - y_k)).$$

Then for $k, l \geq k_0$, we have that $v_{0,k}$ and $v_{0,l}$ have disjoint supports with radius of respective orders n_k^{-1} and n_l^{-1} , moreover,

$$d(\text{supp}(v_{0,k}), \text{supp}(v_{0,l})) \sim \frac{1}{k} - \frac{1}{l}.$$

Applying the convolution by the function $\rho_{\varepsilon_{n_k}}$ which has a support of size $\varepsilon_{n_k}^{-1} = 100/n_k$, we know that

$$\text{supp}(\rho_{\varepsilon_{n_k}} * v_{0,k}) \subset \left\{ (x, y) \in \mathbb{R}^2 \mid |y - y_k| \leq C e^{-a^k} \right\}.$$

Definition 4.8.5 (Dense subset of the pathological set). We denote by S the set of initial data f which can be decomposed under the form

$$f = u_0 + \sum_{k=k_1}^{\infty} v_{0,k}, \quad \text{for } k_1 \geq 1 \text{ and } u_0 \in \mathcal{C}_c^\infty(\mathbb{R}^2).$$

One can see that S defines a dense subset of \mathcal{H}^s . We make use of the following precise upper bounds in H^m , $m \geq 0$, of the initial data regularized by convolution, see Lemma 2.4 in [CG22].

Lemma 4.8.6 (Norm inflation of the lollipop by convolution). *Let $m_1, m_2 \in \mathbb{N}$, $k \geq k_0 \geq 1$. If $m_1 = m_2 = 0$, there holds*

$$\sum_{l=k_0}^{k-1} \|\rho_{\varepsilon_{n_k}} * v_{0,l}\|_{H_x^{m_1} H_y^{m_2}} \lesssim 1, \quad \sum_{l=k}^{\infty} \|\rho_{\varepsilon_{n_k}} * v_{0,l}\|_{H_x^{m_1} H_y^{m_2}} \lesssim n_{k+1}^{m_1+2m_2-s} \left(\frac{n_k}{n_{k+1}} \right)^{\frac{3}{2}}.$$

If $m_1 + 2m_2 \geq 1$, we have

$$\sum_{l=k_0}^{k-1} \|\rho_{\varepsilon_{n_k}} * v_{0,l}\|_{H_x^{m_1} H_y^{m_2}} \lesssim n_{k-1}^{m_1+2m_2-s}, \quad \sum_{l=k}^{\infty} \|\rho_{\varepsilon_{n_k}} * v_{0,l}\|_{H_x^{m_1} H_y^{m_2}} \lesssim n_k^{m_1+2m_2-s} n_{k+1}^{-s} \left(\frac{n_k}{n_{k+1}} \right)^{\frac{3}{2}}.$$

Note that this estimate is not valid for every $\varepsilon < \varepsilon_{n_k}$, but only when $\varepsilon \geq \varepsilon_{n_k}$. As a consequence, we do not have a uniform control on the time of existence of the smooth solution with initial data $\rho_\varepsilon * f$ in $L_x^2 H_y^m$ for $m > 1/2$.

It remains to study the time evolution of $\rho_{\varepsilon_{n_k}} * f$, by comparing it to the ODE profile $\rho_{\varepsilon_{n_k}} * v_{0,k}$ only. An adaptation of the proof of Proposition 2.6 in [CG22] leads to the following result. Let $u^{\varepsilon_{n_k}}$ be the maximal solution of (NLS-HW) with initial data $\rho_{\varepsilon_{n_k}} * f$ in $L_x^2 H_y^1$, and $T_*(\varepsilon_{n_k})$ the maximal time of existence. Note that according to [BIK20], Theorem 1.6, the Cauchy problem for (NLS-HW) is locally well-posed in this space. Moreover, $T_*(\varepsilon_{n_k})$ only depends on the norm of the initial data in $L_x^2 H_y^1$. Therefore, the norm of the solution must blow up when $t \rightarrow T_*(\varepsilon_{n_k})$. We establish that this does not happen before the norm inflation time t_{n_k} by running a perturbative analysis.

Proposition 4.8.7 (Growth of perturbed initial data). *Fix $0 < \gamma < \beta < 1$. Let $0 \leq s < \frac{1}{4}$. Let $n = n_k$. Then the solution u^{ε_n} is well-defined in $\mathcal{C}([0, t_n], L_x^2 H_y^1)$. Moreover, there holds*

$$\|u^{\varepsilon_n}(t_n) - \rho_{\varepsilon_n} * e^{it\mathcal{A}} u_0 - v_n^{\varepsilon_n}(t_n)\|_{L_x^2 H_y^s} \leq C,$$

as a consequence,

$$\|u^{\varepsilon_n}(t_n)\|_{L_x^2 H_y^s} \gtrsim \log(n)^{2s(\beta-\gamma)-\gamma}.$$

As a consequence, the set S is a subset of the pathological set \mathcal{P} .

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Chapter 5

Lack of Regularisation in the second Picard iteration for NLS on \mathbb{S}^2 with random initial data

joint work with Nicolas Burq, Mickaël Latocca, Chenmin Sun and Nikolay Tzvetkov [Bur+22].

Résumé. Nous considérons l'équation de Schrödinger cubique avec une renormalisation de Wick (NLS) sur la sphère bidimensionnelle, avec des données initiales aléatoires qui peuvent être distribuées selon la mesure de Gibbs. Nous montrons que la seconde itération de Picard n'améliore pas la régularité des données initiales, à l'échelle des espaces de Sobolev classiques. Ceci contraste fortement avec la même équation de Schrödinger renormalisée sur des tores bidimensionnel, pour laquelle nous savons grâce aux travaux de Bourgain que la seconde itération de Picard gagne une demi-dérivée. Notre preuve repose sur l'identification d'une partie singulière de la non-linéarité. Nous montrons que cette partie singulière est responsable d'un phénomène de concentration sur un grand cercle (à savoir une géodésique fermée stable), qui tout effet régularisant non linéaire dans la seconde itération de Picard.

Abstract. We consider the Wick ordered NLS on the two-dimensional sphere, with random initial data, possibly distributed according to the Gibbs measure. We show that the first nontrivial Picard iteration does not improve the regularity of the initial data in the scale of the classical Sobolev spaces. This is in sharp contrast with the Wick ordered NLS on the two-dimensional tori, a model for which we know from the work of Bourgain that the second Picard iteration gains one half derivative. Our proof relies on identifying a singular part of the nonlinearity. We show that this singular part is responsible for a concentration phenomenon on a large circle (i.e. a stable closed geodesic), which prevents any regularization in the second Picard iteration.

Contents

5.1	Introduction	246
5.2	Notation and preliminaries	249
5.2.1	Spherical harmonics	249
5.2.2	Randomization : Gaussian measure on spherical harmonics of degree n	250
5.3	Concentration of spherical Harmonics	251
5.4	Proof of the main theorem in the case of \mathbb{S}^2	255
5.5	First iteration for the torus	261

5.1 Introduction

We consider the cubic Schrödinger equation on a compact surface without boundary (\mathcal{M}, g) , which reads

$$\begin{cases} i\partial_t u + \Delta_g u = |u|^2 u, & (t, x) \in \mathbb{R} \times \mathcal{M}. \\ u|_{t=0} = u_0, \end{cases} \quad (\text{NLS})$$

The dynamics associated with (NLS) formally conserves the energy and the mass of the solution

$$\mathcal{H}(v) = \frac{1}{2} \|\nabla v\|_{L^2(\mathcal{M})}^2 + \frac{1}{4} \|v\|_{L^4(\mathcal{M})}^4 \quad \text{and} \quad M(u) = \|v\|_{L^2(\mathcal{M})}^2.$$

The Gibbs measure, which is formally associated with the Hamiltonian structure of (NLS), reads

$$d\mu(u) = \exp(-\mathcal{H}(u)) du.$$

From the Liouville theorem and the conservation of the Hamiltonian, we expect the Gibbs measure to be invariant under the dynamics of (NLS). In particular, by using the so-called Bourgain globalization argument from [Bou94], we can deduce global existence of solutions and Poincaré recurrence property from a local Cauchy theory on the support of the Gibbs measure. In order to make sense of this measure, and to realize a typical function on the support of this measure, we see it as a density measure with respect to the Gaussian free field:

$$\begin{aligned} d\mu(u) &= \exp\left(-\frac{1}{4}\|u\|_{L^4}^4 + \frac{1}{2}\|u\|_{L^2}^2\right) \exp\left(-\frac{1}{2}(\|\nabla u\|_{L^2}^2 + \|u\|_{L^2}^2)\right) du \\ &=: \exp\left(-\frac{1}{4}\|u\|_{L^4}^4 + \frac{1}{2}\|u\|_{L^2}^2\right) d\rho(u). \end{aligned}$$

The Gaussian free field $d\rho$ is induced by random series

$$\omega \in \Omega \mapsto \phi^\omega(x) = \sum_{n \in \mathbb{N}} \frac{g_n(\omega)}{\langle \lambda_n \rangle^{\frac{1}{2}}} \varphi_n(x), \quad (5.1.1)$$

where $\{\varphi_n\}_{n \geq 0}$ form an orthonormal basis of $L^2(M)$ made eigenfunctions of $-\Delta_g$, and $\{\lambda_n^2\}_{n \geq 0}$ are the associated eigenvalues, possibly with multiplicity

$$-\Delta_g \varphi_n = \lambda_n^2 \varphi_n, \quad 0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots$$

The $\{g_n\}_{n \geq 0}$ are identically independent normalized Gaussian variables with complex values. According to the Weyl's asymptotics, we know that $\lambda_n = O(n)$ and, as a consequence, ϕ^ω defined by (5.1.1) converges in $L^2(\Omega; H^{0-}(\mathcal{M}))$.

On the other-hand, local well-posedness a priori holds in $H^s(M)$ for $s > \frac{1}{2}$, and we do not know how to construct a dynamics on the support of the Gibbs measure. In the case of the torus \mathbb{T}^2 , Bourgain prove local well-posedness up to scaling-critical space, namely for all $s > 0$, by using bilinear estimates and Fourier restriction spaces. Then, Bourgain proved in [Bou96] well-posedness for almost-every initial data on the support of the Gibbs measure, by using a linear-nonlinear decomposition, and by observing a probabilistic nonlinear smoothing effect. Specifically, Bourgain showed that the second Picard's iteration has regularity $H^{\frac{1}{2}-}(\mathbb{T}^2)$.

The cubic Schrödinger equation on the sphere \mathbb{S}^2

In a series of works, Burq, Gérard and Tzvetkov [BGT02; BGT04; BGT05a; BGT05b; BGT10] investigated the Cauchy problem for the cubic Schrödinger equation on \mathbb{S}^2 . They proved the following.

Theorem 5.1.1 ([BGT05a]). *The Cauchy problem (NLS) is uniformly locally well-posed in $H^s(\mathbb{S}^2)$ when $s > \frac{1}{4}$. When $s \leq \frac{1}{4}$, however, instabilities occurs and the flow-map fails to be of class \mathcal{C}^3 .*

The positive result follows from a bilinear estimate that requires $\frac{1}{4}$ -derivatives, and which is optimal for the certain spherical harmonics. In the other hand, the instabilities are due to the presence of spherical harmonics which concentrated on a great circle. The quasi-linearity of (NLS) was further studied in [Ban04; BGT10], proving that oscillating H^s -normalized sequences of initial data are not linearly propagated.

Regularity of the second Picard's iteration

In light of the above discussion, the deterministic threshold for the local Cauchy theory is far from the regularity of the Gibbs measure. Moreover, it is not known how to construct strong solutions in $H^s(\mathbb{S}^2)$ when $0 < s < \frac{1}{4}$. In this work, we consider randomized initial data, and we prove that the second Picard's iteration does not gain any regularity. This is in sharp contrast with the case of the torus, and precludes the construction of strong solutions by using the Bourgain linear-nonlinear decomposition.

On the other hand, we propose a self-contained proof of the gain of one-half derivative of the second Picard's iteration in the case of irrational tori.

In order to make our statement precise, we introduce some notation. Denote by

$$\Pi_N u = \sum_{n: \lambda_n \leq N} (u, \varphi_n)_{L^2(M)} \varphi_n$$

the orthogonal projection on eigenfunctions with eigenvalues $\leq N$. We shall write

$$: |u|^2 u := \left(|u|^2 u - 2 \|u\|_{L^2(M)}^2 u \right)$$

the Wick-ordering nonlinearity in a renormalized version of (NLS). Note that we pass from the Wick-ordering equation to (NLS) from a gauge transformation $e^{it\|u\|_{L^2}^2}$, by using the conservation of

the mass. The randomized initial data

$$\phi^\omega = \sum_n \frac{g_n(\omega)}{\langle \lambda_n \rangle^\alpha} \varphi_n$$

lies in $\mathcal{H}^{\alpha-\frac{1}{2}-}(\mathcal{M})$ almost-surely. The case of the Gaussian free field, and hence of the Gibbs measure, corresponds to $\alpha = \frac{1}{2}$. Then, the *first Picard iterate* of ϕ^ω reads

$$\mathcal{I}_{\mathcal{M}}(t, \phi^\omega) = \int_0^t e^{i(t-t')\Delta_g} \left(: |e^{it'\Delta_g} \phi^\omega|^2 e^{it'\Delta_g} \phi^\omega : \right) dt',$$

where ϕ^ω is given by (5.1.1). The subscript \mathcal{M} marks the dependence of this object on the underlying manifold considered. Let us denote by \mathbb{T}_β^2 the torus \mathbb{T}^2 endowed with the metric $dx^2 + \beta^{-2}dy^2$, and by \mathbb{S}^2 the Euclidean sphere in \mathbb{R}^3 . With these notations at hand, we prove the following theorem.

Theorem 5.1.2. *Fix $t \in \mathbb{R}$. There exist $N_0 \in \mathbb{N}$ and $\eta > 0$ such that for all $N \geq N_0$ and $\alpha \geq \frac{1}{2}$,*

$$\eta|t| \ln(N) \leq \mathbb{E} \left[\left\| \mathcal{I}_{\mathbb{S}^2}(t, \Pi_N \phi^\omega) \right\|_{H^{\alpha-\frac{1}{2}}(\mathbb{S}^2)}^2 \right]. \quad (5.1.2)$$

On the other hand, for all $\varepsilon > 0$ there exists $C > 0$ such that

$$\sup_{N \in \mathbb{N}} \mathbb{E} \left[\left\| \mathcal{I}_{\mathbb{T}_\beta^2}(t, \Pi_N \phi^\omega) \right\|_{H^{\alpha-\varepsilon}(\mathbb{T}_\beta^2)}^2 \right] \leq C. \quad (5.1.3)$$

We give some comments.

- The bound (5.1.3) should be viewed as a gain of 1/2-derivative by the Picard iteration, compared to the H^{0-} regularity of the initial data ϕ^ω . Even if this result is contained in the work of [Fan+21], it allows us to stress out the existence of radically different behaviors for (NLS) with random initial data below L^2 on \mathbb{S}^2 and on \mathbb{T}^2 . We propose a self-contained proof of (5.1.3) in Section 5.5.
- In contrast to the case of the tori, the second Picard iterate for (NLS) on the sphere does not improve the regularity of initial data in $H^{\alpha-\frac{1}{2}-0}(\mathbb{S}^2)$. Moreover, the divergence for the quadratic moment of $\left\{ \left\| \Pi_N \mathcal{I}_{\mathbb{S}^2}^2(t, \phi^\omega) \right\|_{L^2(\mathbb{S}^2)} \right\}_{N \geq 1}$ is likely to yield the almost sure divergence of this random process, but we do not have a self-contained elementary proof of this result.

Our result indicates that proving the existence of a dynamics below $H^{\frac{1}{4}}(\mathbb{S}^2)$ does not follow from a linear-nonlinear decomposition. In addition, the structure of the solution associated to ϕ^ω would not be

$$u(t) = e^{it\Delta} \phi^\omega + \{\text{more regular terms}\}, \quad (5.1.4)$$

which is in sharp contrast with the work of [DNY19]. Nevertheless, we have hope to use the random averaging operators to paracontrol the worst frequency interactions, and to prove almost-sure local well-posedness for some $\alpha < \frac{3}{4}$, breaking therefore the deterministic threshold.

Organization of the article

The proof of Theorem 5.1.2 in the case of the sphere is given in Section 5.4 and mostly relies on a concentration property of Proposition 5.3.1, which is proven in Section 5.3. We also recall preliminary estimates and notations in Section 5.2. Finally, we prove (5.1.3) in Section 5.5.

5.2 Notation and preliminaries

5.2.1 Spherical harmonics

Spherical coordinates (ρ, θ, φ) are defined by

$$(x_1, x_2, x_3) = (\rho \sin(\theta) \cos(\varphi), \rho \sin(\theta) \sin(\varphi), \rho \cos(\theta)) .$$

The polar coordinate $\theta \in (0, \pi)$ is the *colatitude*, $\varphi \in (0, 2\pi)$ is the *azimut* and $\rho \in \mathbb{R}^+$ is the radial distance, so that $\mathbb{S}^2 = \{x \in \mathbb{R}^3 \mid |x| = 1\}$. With this coordinates system, we have

$$(f \mid g)_{L^2(\mathbb{S}^2)} = \int_0^\pi \int_0^{2\pi} f(\theta, \varphi) \overline{g(\theta, \varphi)} \sin(\theta) d\theta d\varphi . \quad (5.2.1)$$

Note that this system of coordinate is singular on the axis (Ox_3) . The Laplace operator on the sphere writes

$$-\Delta_{\mathbb{S}^2} = -\frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \varphi^2} - \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial}{\partial \theta} \right) . \quad (5.2.2)$$

It is a selfadjoint positive operator that admits the following spectral resolution. For $n \in \mathbb{N}$ and $k \in \{-n, \dots, n\}$, we denote by $Y_{n,k}$ the spherical harmonics of degree n and order k , namely

$$Y_{n,k}(\theta, \varphi) = e^{ik\varphi} v_{n,k}(\theta), \quad v_{n,k}(\theta) = c_{n,k} L_{n,k}(\cos(\theta)),$$

where $L_{n,k}(\cos(\theta))$ is the associated Legendre function of degree n and order k , and $c_{n,k}$ is a normalization constant.

Proposition 5.2.1 (Spherical harmonics, see [SW71]). *Spherical harmonics $\{Y_{n,k}\}_{1 \leq n, |k| \leq n}$ form an orthonormal basis of $L^2(\mathbb{S}^2)$ made of eigenfunctions of the Laplace operator on \mathbb{S}^2 . They satisfy*

$$-\Delta_{\mathbb{S}^2} Y_{n,k} = \lambda_n^2 Y_{n,k}, \quad n \in \mathbb{N}^*, \quad k \in \{-n, \dots, n\},$$

with $\lambda_n = \sqrt{n(n+1)}$. The multiplicity of the eigenvalue λ_n^2 is $[n] = 2n + 1$.

Let us now gather some results that we will use in the proof of Theorem 5.1.2.

Lemma 5.2.2. *For any $x \in \mathbb{S}^2$ there holds*

$$\sum_{|k| \leq n} |Y_{n,k}(x)|^2 = [n]. \quad (5.2.3)$$

See Lemma 2.1 in [BL14] for a proof. This crucially exploits the invariance by rotation of the sphere \mathbb{S}^2 . This indeed implies that the kernel

$$K_n(x, y) := \sum_{|k| \leq n} Y_{n,k}(x) \overline{Y_{n,k}(y)}$$

of the orthogonal projector on E_n is invariant under the action of any rotation. Since rotations act transitively on \mathbb{S}^2 we deduce that $K_n(x, x)$ does not depend on x , and the result follows from integrating in x over \mathbb{S}^2 .

5.2.2 Randomization : Gaussian measure on spherical harmonics of degree n

Fix $n \in \mathbb{N}$ and let E_n be the space of spherical harmonics of degree n :

$$E_n = \text{Span}_{\mathbb{C}} \{ Y_{n,k} \mid k \in \{-n, \dots, n\} \}.$$

Recall that we denoted by $\lambda_n = n(n+1)$ the n^{th} eigenvalue of $\Delta_{\mathbb{S}^2}$ with multiplicity $[n] = 2n+1$. Given $\{g_{n,k}(\omega)\}_{1 \leq n, |k| \leq n}$ a sequence of identically distributed standard complex Gaussian random variables, the randomization

$$e_n^\omega = \frac{1}{\sqrt{2n+1}} \sum_{k=-n}^n g_{n,k}(\omega) Y_{n,k} \quad (5.2.4)$$

induces a Gaussian measure on E_n . This randomization also yields a Gaussian measure on $H^{\alpha-\frac{1}{2}}(\mathbb{S}^2)$ induced by the random variable

$$\omega \mapsto \phi^\omega = \sum_{n=1}^{\infty} \frac{1}{\lambda_n^\alpha} e_n^\omega. \quad (5.2.5)$$

In what follows, and without any loss of generality, we do as if $\lambda_n = [n] = n$. By doing so, the random initial data reads

$$\phi^\omega = \sum_{n=1}^{\infty} \frac{1}{n^\alpha} \sum_{k=-n}^n \frac{g_{n,k}(\omega)}{n^{\frac{1}{2}}} Y_{n,k}.$$

Next we will prove that random spherical harmonics $\{e_n^\omega\}_{1 \leq n}$ defined above satisfy some uniform bounds in Lebesgue spaces.

Lemma 5.2.3. *Let $q \geq 2$, then there exists a constant C_q , not depending on n , such that for all $n \geq 1$ there holds*

$$\mathbb{E} \left[\|e_n^\omega\|_{L_x^q}^2 \right] \leq C_q. \quad (5.2.6)$$

Proof. Let $n \geq 0$. One observes that for fixed $x \in \mathbb{S}^2$,

$$\sum_{|k| \leq n} g_{n,k}(\omega) Y_{n,k}(x)$$

is a Gaussian variable of mean 0 and variance $\sum_{k \leq n} |Y_{n,k}(x)|^2$. It follows from (5.2.3) that $e_n^\omega(x)$ is a normalized Gaussian variable. Therefore, there exists $C > 0$ independent of q, n, x such that

$$\|e_n^\omega(x)\|_{L_\Omega^q} \leq C\sqrt{q}.$$

Then thanks to the Hölder inequality, Fubini theorem and that \mathbb{S}^2 has finite volume, we obtain

$$\mathbb{E} \left[\|e_n^\omega\|_{L_x^q}^2 \right] \leq \|e_n^\omega(x)\|_{L^q(\Omega; L^q(\mathbb{S}^2))}^2 \leq C\sqrt{q}.$$

This concludes the proof of (5.2.6) with $C_q = C\sqrt{q}$. \square

5.3 Concentration of spherical Harmonics

It follows from the expression of the Laplace operator on the sphere (5.2.2) that $v_{n,k}$ is solution on $(0, \pi)$ to

$$-\left(\sin(\theta)\frac{d}{d\theta}\right)^2 v_{n,k}(\theta) + (k^2 - n(n+1)\sin^2(\theta))v_{n,k}(\theta) = 0. \quad (5.3.1)$$

This equation turns out to display some ellipticity away from the equator, in the regime where $|k|$ is close to n . We shall exploit this property in a quantitative way by the use of semiclassical functional calculus, and get a large family of spherical harmonics that are concentrated near the equator.

Proposition 5.3.1. *Let $0 < \delta \ll 1$. There exists $C_\delta > 0$ such that for all $k \in \{-n, \dots, n\}$ with $n\sqrt{1-\delta^2} \leq |k|$ we have*

$$\|Y_{n,k}\|_{L^2(|x_3|>2\delta)} \leq \frac{C_\delta}{n}.$$

Remark 5.3.2. In fact, one has $\|Y_{n,k}\|_{L^2(|x_3|>2\delta)} \leq C_{\delta,K}n^{-K}$ for all $K \in \mathbb{N}$, but we choose to present a self-contained proof for the crude bound $O(n^{-1})$, which already fits our needs.

Proof. In order to use semiclassical functional calculus and prove this proposition, we shall extend $v_{n,k}$ to a function defined on the whole real line \mathbb{R} . One way of doing so would be to use the symmetries of $Y_{n,k}$ to periodize $v_{n,k}$. Still, we prefer to avoid boundary conditions and we rather make the change of variable¹

$$y(\theta) = \operatorname{arctanh}(\cos(\theta)).$$

Note that $y : (0, \pi) \rightarrow \mathbb{R}$ is a C^∞ -diffeomorphism and that

$$\sin^2(\theta) = 1 - \cos^2(\theta) = 1 - \tanh^2(y) = \cosh(y)^{-2}.$$

This yields

$$dy = -\sin(\theta)^{-1}d\theta = -\cosh^2(y)\sin(\theta)d\theta. \quad (5.3.2)$$

In light of the definition of the scalar product (5.2.1) on \mathbb{S}^2 , we define

$$\tilde{v}_{n,k}(y) = \cosh(y)^{-1}v_{n,k}(\theta(y)),$$

and (5.3.2) yields that $\tilde{v}_{n,k}$ belongs to $L^2(\mathbb{R})$ with

$$\int_{-\infty}^{\infty} |\tilde{v}_{n,k}(y)|^2 dy = \int_0^\pi |v_{n,k}(\theta)|^2 \sin(\theta) d\theta = 1. \quad (5.3.3)$$

¹This change of variable can be deduced from the requirement of $\sin \theta \partial_\theta = -\partial_y$.

We deduce from (5.3.1), the chain rule and (5.3.2) that $\tilde{v}_{n,k}$ is solution in $C^\infty(\mathbb{R})$ to the equation

$$-\frac{d^2}{dy^2}\tilde{v}_{n,k} - (1 + 2 \tanh(y) \frac{d}{dy})\tilde{v}_{n,k} + (k^2 - n(n+1) \cosh(y)^{-2})\tilde{v}_{n,k} = 0. \quad (5.3.4)$$

Indeed, $\sin(\theta)\partial_\theta = -\partial_y$ and it follows from (5.3.1) and from the relation $v_{n,k} = \cosh(y)\tilde{v}_{n,k}$ that

$$-\partial_y^2(\cosh(y)\tilde{v}_{n,k}) + (k^2 - n(n+1) \cosh(y)^{-2})(\cosh(y)\tilde{v}_{n,k}) = 0.$$

Then, we obtain (5.3.4) by dividing the above equation by $\cosh(y)$ and by noting that

$$[\partial_y^2, \cosh(y)] = \cosh(y)'' + 2 \cosh(y)' \partial_y = \cosh(y) + 2 \sinh(y) \partial_y.$$

We introduce the *semiclassical parameter* $h \sim n^{-1}$ and we multiply the above equation by h^2 to get

$$-h^2 \frac{d^2}{dy^2} \tilde{v}_{n,k} + ((kh)^2 - \cosh(y)^{-2}) \tilde{v}_{n,k} + h 2i \tanh(y) i h \frac{d}{dy} \tilde{v}_{n,k} - h \cosh(y)^{-2} \tilde{v}_{n,k} + h^2 \tilde{v}_{n,k} = 0. \quad (5.3.5)$$

Thus, we introduce $\alpha = \frac{|k|}{n} = |k|h$ and we write $P_{h,\alpha} = p(y, hD_y, h, \alpha)$ the differential operator of order 2 with symbol

$$p(y, \xi, \alpha, h) = p_0(y, \xi, \alpha) + h p_1(y, \xi) + h^2,$$

where

$$p_0(y, \xi, \alpha) = \xi^2 + \alpha^2 - \cosh(y)^{-2}, \quad p_1(y, \xi) = 2i\xi \tanh(y) - \cosh(y)^{-2}.$$

Note that in the regime where $\sqrt{1 - \delta^2} \leq \alpha$, we have

$$\text{Char}(P_{h,\alpha}) = \{(y, \xi) \in \mathbb{R}^2 \mid p_0(y, \xi, \alpha) = 0\} \subset \{(y, \xi) \in \mathbb{R}^2 \mid |\tanh(y)| \leq \delta\}.$$

Indeed, if $(y, \xi) \in \text{Char}(P_{h,\alpha})$ then

$$p_0(y, \xi, \alpha) = \xi^2 + \alpha^2 - \cosh(y)^{-2} = 0.$$

By the assumption that $\sqrt{1 - \delta^2} \leq \alpha$ we get

$$1 - \delta^2 \leq \alpha^2 \leq \cosh(y)^{-2},$$

and this leads to the condition

$$|\tanh(y)| \leq \delta.$$

Besides, observe that

$$|\tanh(y)| \leq \delta \iff |\cos \theta| \leq \delta \iff |x_3| \leq \delta.$$

Hence, $\text{Char}(P_{h,\alpha})$ concentrates near the equator as δ goes to 0. By the use of semiclassical functional calculus, we will deduce from this that $\tilde{v}_{n,k}$, which is solution to $P_{h,\alpha} f = 0$ in $L^2(\mathbb{R})$, also concentrates near the equator as δ goes to 0. Denote by $P_{0,h,\alpha} = p_0(y, hD_y, \alpha)$.

Lemma 5.3.3. *Let $\chi \in C^\infty(\mathbb{R})$ such that $\chi(y) = 1$ when $|\tanh(y)| \geq 2\delta$, and $\text{supp } \chi \subset \{y \in \mathbb{R} \mid |\tanh(y)| \geq \sqrt{\frac{3}{2}}\delta\}$. There exists a uniformly in h bounded operator Q_h such that*

$$\|Q_h \circ P_{0,h,\alpha} - \chi\|_{L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})} \leq C_\delta h. \quad (5.3.6)$$

Proof. The key point is that $P_{0,h,\alpha}$ is elliptic in the region $|\tanh(y)| > 2\delta$ and the proof follows from the standard parametrix construction of elliptic operators. Since we only need to invert $P_{h,\alpha}$ in the elliptic region up to order 1, the proof is more straightforward and self-contained. Recall that for all $y \in \text{supp}(\chi)$, $\xi \in \mathbb{R}$ and $\sqrt{1-\delta^2} < \alpha$ we have

$$p_0(y, \xi, \alpha) = \xi^2 + \alpha^2 - \cosh(y)^{-2} \geq \frac{\delta^2}{2}. \quad (5.3.7)$$

Let us write

$$\phi_\alpha(y) = \alpha^2 - \cosh^{-2}(y), \quad \varphi_\alpha(y, z) = \int_0^1 \phi'_\alpha(tz + (1-t)y) dt$$

so that

$$p_0(y, \xi, \alpha) = \xi^2 + \phi_\alpha(y), \quad \phi_\alpha(y) - \phi_\alpha(z) = (y-z)\varphi_\alpha(y, z).$$

We introduce the symbol

$$q(y, \xi, \alpha) := \frac{\chi(y)}{p_0(y, \xi, \alpha)}.$$

It follows from (5.3.7) that for all $k \in \mathbb{N}$ there exists $0 < C_k$ such that for all $\sqrt{1-\delta^2} < \alpha \leq 1$ and $\xi \in \mathbb{R}$,

$$\sup_{y \in \text{supp}(\chi)} |\partial_\xi^k q(y, \xi, \alpha)| \leq C_k \delta^{-(2+k)} \langle \xi \rangle^{-(2+k)}. \quad (5.3.8)$$

The semiclassical pseudodifferential operator $Q_{h,\alpha}$ associated with q is defined via its Schwartz kernel

$$Q_h f(y) = \int_{\mathbb{R}} K_q(y, z) f(z) dz, \quad K_q(y, z) = \frac{1}{2\pi h} \int_{\mathbb{R}} e^{\frac{i(y-z)\xi}{h}} q(y, \xi) d\xi.$$

Observe that

$$(z-y)e^{\frac{i(y-z)\xi}{h}} = ih \partial_\xi (e^{\frac{i(y-z)\xi}{h}}). \quad (5.3.9)$$

Hence, integrating by parts twice yields that

$$(y-z)^2 K(y, z) = -\frac{h}{2\pi} \int_{\mathbb{R}} \partial_\xi^2 q(y, \xi, \alpha) e^{\frac{i(y-z)\xi}{h}} d\xi,$$

and we deduce from (5.3.8) that

$$|K(y, z)| \leq C\delta^{-4} \frac{h}{(y-z)^2} \left(\int_{\mathbb{R}} \langle \xi \rangle^{-4} d\xi \right) \leq C\delta^{-4} \frac{h}{(y-z)^2}.$$

Moreover, we have the trivial bound

$$|K(y, z)| \leq h^{-1} \delta^{-2} \left(\int_{\mathbb{R}} \langle \xi \rangle^{-2} d\xi \right) \leq C\delta^{-4} h^{-1}.$$

Therefore, there exists $0 < C$ such that for all $z \in \mathbb{R}$

$$\begin{aligned} \int_{\mathbb{R}} |K(y, z)| dy &\leq C\delta^{-4} \int_{\mathbb{R}} \min\left\{ \frac{h}{(y-z)^2}, h^{-1} \right\} dy \\ &\leq C\delta^{-4} \left(h \int_{|s| \geq h} \frac{ds}{s^2} + h^{-1} \int_{|s| \leq h} ds \right) \\ &\leq C\delta^{-4}. \end{aligned}$$

Since the computations are the same when changing the role of y and z , we prove that there exists $C > 0$ such that for all $0 < h \leq 1$ and $0 < \delta < 1$,

$$\sup_{y \in \mathbb{R}} \int_{\mathbb{R}} |K(y, z)| dz + \sup_{z \in \mathbb{R}} \int_{\mathbb{R}} |K(y, z)| dy \leq C\delta^{-4}.$$

It follows from Schur's test that Q_h is uniformly bounded in $L^2(\mathbb{R})$ for $0 < h \leq 1$, with

$$\|Q_h\|_{L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})} \leq C\delta^{-4}. \quad (5.3.10)$$

Then we check directly from integration by part that the Schwartz kernel of the operator

$$R_h := Q_h \circ P_{0,h,\alpha} - \chi$$

is given by

$$K(y, z) = \frac{1}{2\pi h} \int_{\mathbb{R}} \frac{\chi(y)(\phi_\alpha(z) - \phi_\alpha(y))}{\xi^2 + \phi_\alpha(y)} e^{\frac{i(y-z)\xi}{h}} d\xi.$$

Integrating by parts according to (5.3.9) provides

$$K(y, z) = -\frac{i\varphi_\alpha(y, z)\chi(y)}{2\pi} \int_{\mathbb{R}} \partial_\xi \left(\frac{1}{\xi^2 + \phi_\alpha(y)} \right) e^{\frac{i(y-z)\xi}{h}}.$$

Besides, iterating the integration by parts leads to

$$|K(y, z)| \leq C\delta^{-4} \min\left\{1, \frac{h^2}{|y-z|^2}\right\}.$$

Hence,

$$\sup_{z \in \mathbb{R}} \int_{\mathbb{R}} |K(y, z)| dy + \sup_{y \in \mathbb{R}} \int_{\mathbb{R}} |K(y, z)| dz \leq C\delta^{-4}h$$

and it follows from Schur's test that

$$\|R_h\|_{L^2 \rightarrow L^2} \leq C_\delta h.$$

This concludes the proof of Lemma 5.3.3 with $C_\delta \leq C\delta^{-4}$. \square

Now we finish the proof of Proposition 5.3.1. Note that $\tilde{v}_{n,k}$ solves the equation

$$P_{0,h}\tilde{v}_{n,k} = -(hp_1(y, hD_y) + h^2)\tilde{v}_{n,k}.$$

Recall that $p_1(y, hD_y) = \tanh(y)h\partial_y - \cosh(y)^{-2}$. From the a priori estimate on $\tilde{v}_{n,k}$, obtained by multiplying (5.3.5) by the complex conjugation of $\tilde{v}_{n,k}$ and integrating over \mathbb{R} , we have

$$\|p_1(y, hD_y)\tilde{v}_{n,k}\|_{L^2(\mathbb{R})} \leq Ch.$$

for all $0 < h \leq 1$. Therefore,

$$\|P_{0,h}\tilde{v}_{n,k}\|_{L^2(\mathbb{R})} \leq Ch.$$

From Lemma 5.3.3, there exists C_δ such that

$$\|\chi\tilde{v}_{n,k}\|_{L^2(\mathbb{R})} = \|Q_h P_{0,h}\tilde{v}_{n,k} - R_h\tilde{v}_{n,k}\|_{L^2(\mathbb{R})} \leq (C_\delta + C)h.$$

It follows that

$$\|Y_{n,k}\|_{L^2(|x_3| > 2\delta)} = \|\tilde{v}_{n,k}\|_{L^2(|\tanh(y)| > 2\delta)} \leq \|\chi\tilde{v}_{n,k}\|_{L^2(\mathbb{R})} \leq C_\delta h,$$

which completes the proof of Proposition 5.3.1. \square

5.4 Proof of the main theorem in the case of \mathbb{S}^2

Recall that $\lambda_n^2 = n(n+1)$ is the n^{th} eigenvalue of $-\Delta_{\mathbb{S}^2}$ with multiplicity $[n] = 2n+1$, and that we assumed $[n] = \lambda_n = n$. Following the notations introduced in part 5.2.2, we write

$$u^\omega(t) = \sum_{n \geq 1} e^{itn^2} \frac{1}{n^\alpha} e_n^\omega =: \sum_{n \geq 1} u_n^\omega(t). \quad (5.4.1)$$

To give sense to the above series in $L^\infty(\mathbb{R}; L^2(\mathbb{S}^2))$ we consider

$$\Pi_N u^\omega(t) = \sum_{1 \leq n \leq N} e^{itn^2} \frac{1}{n^\alpha} e_n^\omega.$$

Let us expand the nonlinearity as follows :

$$\begin{aligned} \mathcal{N}(\Pi_N u^\omega)(t) &= |\Pi_N u^\omega|^2 \Pi_N u^\omega - 2 \|\Pi_N u^\omega\|_{L^2(\mathbb{S}^2)}^2 \Pi_N u^\omega \\ &= \sum_{1 \leq n_1, n_2, n_3 \leq N} e^{it(n_1^2 - n_2^2 + n_3^2)} \frac{1}{(n_1 n_2 n_3)^{1/2}} e_{n_1}^\omega \overline{e_{n_2}^\omega} e_{n_3}^\omega - 2 \sum_{1 \leq n_1, n_2 \leq N} \frac{\|e_{n_2}^\omega\|_{L^2(\mathbb{S}^2)}^2}{n_2^{2\alpha}} \frac{e_{n_1}^\omega}{n_1^\alpha}. \end{aligned}$$

We decompose $\mathcal{N}(\Pi_N u^\omega)$ into three different regimes :

$$\begin{aligned} \mathcal{N}(\Pi_N u^\omega) &= \text{I} + \text{II} + \text{III} \\ &= \sum_{\substack{1 \leq n_1, n_2, n_3 \leq N \\ n_2 \neq n_1, n_3}} e^{it(n_1^2 - n_2^2 + n_3^2)} \frac{1}{(n_1 n_2 n_3)^\alpha} e_{n_1}^\omega \overline{e_{n_2}^\omega} e_{n_3}^\omega \\ &\quad + 2 \sum_{1 \leq n_1, n_2 \leq N} e^{itn_1^2} \frac{(|e_{n_2}^\omega|^2 - \|e_{n_2}^\omega\|_{L^2(\mathbb{S}^2)}^2)}{n_2^{2\alpha}} \frac{e_{n_1}^\omega}{n_1^\alpha} \\ &\quad - \sum_{1 \leq n \leq N} \frac{e^{itn^2} |e_n^\omega|^2 e_n^\omega}{n^{3\alpha}}. \end{aligned}$$

Remark 5.4.1. We will see that the term II is responsible for the divergence of the first Picard iteration claimed in (5.1.2). We stress out that this term vanishes in the case of the tori since the eigenfunctions of $\Delta_{\mathbb{T}^2}$ are the plain waves $e_n(x) = e^{in \cdot x}$, and satisfy $|e_n(x)| = 1$ for all $x \in \mathbb{T}^2$.

To prove Theorem 5.1.2 we proceed as follows. In Lemma 5.4.2 we use independence to reduce the proof of Theorem 5.1.2 to estimates for the terms II and III. Then, we prove in Lemma 5.4.3 that III is in $L^2(\mathbb{S}^2)$ a.s. and uniformly in N . On the other hand, we prove in Proposition 5.4.4 that the quadratic moment of the $L^2(\mathbb{S}^2)$ -norm of the Duhamel integral for II is bounded from below by $\sim \ln(N)$ for large N .

Lemma 5.4.2. *Given $v \in L^\infty(\mathbb{R}; L^2(\mathbb{S}^2))$ we denote the truncated Duhamel integral representation for v at time t by $\mathcal{I}_N(t, v) = \Pi_N \int_0^t e^{i(t-t')\Delta_{\mathbb{S}^2}} v(t') dt'$. We have*

$$\left| \|\mathcal{I}_N(t, \text{II})\|_{L^2(\Omega; H^{\alpha-\frac{1}{2}}(\mathbb{S}^2))}^2 - \|\mathcal{I}_N(t, \text{III})\|_{L^2(\Omega; H^{\alpha-\frac{1}{2}}(\mathbb{S}^2))}^2 \right| \leq \|\Pi_N \mathcal{I}_{\mathbb{S}^2}^2(t, \phi^\omega)\|_{L^2(\Omega; H^{\alpha-\frac{1}{2}}(\mathbb{S}^2))}. \quad (5.4.2)$$

Proof. Recall that if g is a standard complex Gaussian variable then

$$\mathbb{E}[g^2] = \mathbb{E}[\bar{g}^2] = \mathbb{E}[|g|^2 g] = \mathbb{E}[|g|^2 \bar{g}] = 0.$$

Since the u_n^ω 's are linear combinations of independent standard complex Gaussian variables, we also have

$$\mathbb{E}[(u_n^\omega)^2] = \mathbb{E}[(\overline{u_n^\omega})^2] = \mathbb{E}[|u_n^\omega|^2 u_n^\omega] = \mathbb{E}[|u_n^\omega|^2 \overline{u_n^\omega}] = 0. \quad (5.4.3)$$

Moreover, we see from the definition of the random spherical harmonics $\{e_n^\omega\}_{n \geq 1}$ that

$$\mathbb{E}[|u_n^\omega|^2 - \|u_n^\omega\|_{L^2(\mathbb{S}^2)}^2] = 0. \quad (5.4.4)$$

Therefore, we claim that for all $t \in \mathbb{R}$,

$$\mathbb{E}[(\text{I} | \text{II})_{L^2(\mathbb{S}^2)}] = \mathbb{E}[(\text{I} | \text{III})_{L^2(\mathbb{S}^2)}] = 0. \quad (5.4.5)$$

To prove this, we expand

$$\mathbb{E}[(\text{I} | \text{II})_{L^2(\mathbb{S}^2)}] = \sum_{\substack{1 \leq n_1, n_2, n_3, q, m \leq N \\ n_2 \neq n_1, n_3}} \int_{\mathbb{S}} \mathbb{E} \left[u_{n_1}^\omega \overline{u_{n_2}^\omega} u_{n_3}^\omega \overline{u_m^\omega} (|u_q^\omega|^2 - \|u_q^\omega\|_{L^2(\mathbb{S}^2)}^2) \right] dx.$$

In light of (5.4.3), (5.4.4) and from the independence of the $\{u_n^\omega\}_{n \geq 1}$, we see from a case by case analysis that expectation of each term appearing in the above sum vanishes provided that $n_2 \neq n_1$ and $n_2 \neq n_3$. Similarly,

$$\mathbb{E}[(\text{I} | \text{III})_{L^2(\mathbb{S}^2)}] = \sum_{\substack{1 \leq n_1, n_2, n_3, m \leq N \\ n_2 \neq n_1, n_3}} \int_{\mathbb{S}} \mathbb{E} [u_{n_1}^\omega \overline{u_{n_2}^\omega} u_{n_3}^\omega \overline{u_m^\omega} |u_m^\omega|^2] dx = 0.$$

This proves (5.4.5). We deduce from this that

$$\mathbb{E}[(\mathcal{I}_N(t, \text{I}) | \mathcal{I}_N(t, \text{II}))_{L^2(\mathbb{S}^2)}] = \mathbb{E}[(\mathcal{I}_N(t, \text{I}) | \mathcal{I}_N(t, \text{III}))_{L^2(\mathbb{S}^2)}] = 0.$$

Thus

$$\|\Pi_N \mathcal{I}_{\mathbb{S}^2}(t, \phi^\omega)\|_{L^2(\Omega; H^{\alpha-\frac{1}{2}}(\mathbb{S}^2))}^2 = \|\mathcal{I}_N(t, \text{I}) + \mathcal{I}_N(t, \text{II})\|_{L^2(\Omega; H^{\alpha-\frac{1}{2}}(\mathbb{S}^2))}^2$$

and (5.4.2) follows from the triangle inequality. \square

Lemma 5.4.3. *There exists $C > 0$ such that for all $N \geq 1$ and $t \in \mathbb{R}$,*

$$\mathbb{E} \left[\|\Pi_N \text{III}(t)\|_{H^{\alpha-\frac{1}{2}}(\mathbb{S}^2)}^2 \right] \leq C.$$

Proof. Recall that

$$\text{III} = - \sum_{1 \leq n_1 \leq N} e^{itn_1^2} \frac{|e_{n_1}^\omega|^2 e_{n_1}^\omega}{n_1^{3\alpha}}.$$

We use the orthogonality of the spherical harmonics $\{Y_{n,k}\}$ to expand

$$\|\Pi_N(\text{III})\|_{L^2(\Omega; H^s(\mathbb{S}^2))}^2 = \sum_{\substack{1 \leq n_1 \leq N \\ |k| \leq n}} n^{2s} \mathbb{E} \left| \sum_{n_1=1}^N \int_{\mathbb{S}^2} \frac{|e_{n_1}^\omega(x)|^2 e_{n_1}^\omega(x) \overline{Y_{n,k}(x)}}{n_1^{3\alpha}} dx \right|^2,$$

and further expand the square, which after an application of Fubini's theorems leads to

$$\|\Pi_N(\text{III})\|_{L^2(\Omega; H^s(\mathbb{S}^2))}^2 = \sum_{\substack{1 \leq n \leq N \\ |k| \leq n}} \sum_{1 \leq n_1, n'_1} \frac{n^{2s}}{(n_1 n'_1)^{3\alpha}} \int_{\mathbb{S}^2} \mathbb{E} \left[|e_{n_1}^\omega(x)|^2 e_{n_1}^\omega(x) |e_{n'_1}^\omega(x) \overline{e_{n'_1}^\omega(y)}| \right] \overline{Y_{n,k}(x)} Y_{n,k}(y) dx dy.$$

By rotational invariance the law of $e_{n_1}^\omega(x)$ and $e_{n'_1}^\omega(y)$ are the same. Hence, we obtain from the independence of $e_{n_1}^\omega(x)$ and $e_{n'_1}^\omega(y)$ when $n_1 \neq n'_1$ that

$$\begin{aligned} \|\Pi_N(\text{III})\|_{L^2(\Omega; H^s(\mathbb{S}^2))}^2 &= \sum_{\substack{1 \leq n \leq N \\ |k| \leq n}} \sum_{1 \leq n_1} \frac{n^{2s}}{n_1^{6\alpha}} \mathbb{E} \left[\left| \int_{\mathbb{S}^2} |e_{n_1}^\omega(x)|^2 e_{n_1}^\omega(x) \overline{Y_{n,k}(x)} dx \right|^2 \right] \\ &= \sum_{1 \leq n_1} \sum_{\substack{1 \leq n \leq n_1 \\ |k| \leq n}} \frac{n^{2s}}{n_1^{6\alpha}} \mathbb{E} [|(e_{n_1}^\omega)^2 | Y_{n,k} |_{L^2(\mathbb{S}^2)}|^2]. \end{aligned}$$

We used that the contribution of the terms when $n \gg n_1$, in the semiclassical regime, is negligible. Then, it follows from the Plancherel's formula that

$$\begin{aligned} \|\Pi_N(\text{III})\|_{L^2(\Omega; H^s(\mathbb{S}^2))}^2 &= \sum_{1 \leq n_1} n_1^{2s-6\alpha} \mathbb{E} \left[\| |e_{n_1}^\omega|^2 e_{n_1}^\omega \|_{H^{\alpha-\frac{1}{2}}(\mathbb{S}^2)}^2 \right] \\ &\leq \sum_{1 \leq n_1} n_1^{2s-6\alpha} \mathbb{E} \left[\| e_{n_1}^\omega \|_{L^6(\mathbb{S}^2)}^6 \right]. \end{aligned}$$

As a consequence of Lemma 5.2.3, there exists C not depending on p such that

$$\mathbb{E} \left[\| e_{n_1}^\omega \|_{L^6(\mathbb{S}^2)}^6 \right] \leq C.$$

Therefore the series converges provided that $s < \alpha - \frac{1}{2} + 2\alpha$, and the lemma is proved. Note that this term gains one derivative when the initial data is supported on the Gibbs measure. \square

At this stage, we proved in Lemma 5.4.2 and in Lemma 5.4.3 that there exists $C > 0$ such that for all $N \geq 1$, $t \in \mathbb{R}$ and $s \leq \alpha - \frac{1}{2}$,

$$\|\mathcal{I}_N(t, \text{II})\|_{L^2(\Omega; H^s(\mathbb{S}^2))}^2 - C \leq \|\Pi_N \mathcal{I}_S^2(t, \phi^\omega)\|_{L^2(\Omega; H^s(\mathbb{S}^2))}^2.$$

To get the divergence claimed in Theorem 5.1.2 it remains to estimate from below the quadratic moment of the $H^{\alpha-\frac{1}{2}}(\mathbb{S}^2)$ -norm of the Duhamel integral $\mathcal{I}_N(t, \text{II})$. To do so we shall use the concentration property for high order spherical harmonics of Proposition 5.3.1.

Proposition 5.4.4. *There exist $\eta > 0$ and $N_0 \geq 0$ such that for all $N \geq N_0$ and $t \in \mathbb{R}$ we have*

$$\eta \sum_{N_0 \leq n \leq N} \frac{1}{n} \leq \|\mathcal{I}_N(t, \Pi)\|_{L^2(\Omega; H^{\alpha-\frac{1}{2}}(\mathbb{S}^2))}. \quad (5.4.6)$$

Proof. Recall that we defined

$$\|\mathcal{I}_N(t, \Pi)\|_{L^2(\Omega; H^s(\mathbb{S}^2))} = \mathbb{E} \left[\left\| \Pi_N \int_0^t e^{i(t-t')\Delta_{\mathbb{S}^2}}(\Pi)(t') dt' \right\|_{H^s(\mathbb{S}^2)}^2 \right]^{1/2}.$$

We expand $e^{-it'\Delta_{\mathbb{S}^2}}(\Pi)$ on the orthonormal basis of $L^2(\mathbb{S}^2)$ made of spherical harmonics :

$$\begin{aligned} \Pi_N(e^{-it'\Delta_{\mathbb{S}^2}}(\Pi)(t)) &= \sum_{\substack{1 \leq n \leq N \\ |k| \leq n}} \sum_{1 \leq n_1, n_2 \leq N} e^{it'(n^2 - n_1^2)} \\ &\quad \left(\int_{\mathbb{S}^2} \frac{|e_{n_2}^\omega|^2 - \|e_{n_2}^\omega(x)\|_{L^2(\mathbb{S}^2)}^2}{n_2^{2\alpha}} \frac{e_{n_1}^\omega(x)}{n_1^\alpha} \overline{Y_{n,k}(x)} dx \right) Y_{n,k}. \end{aligned} \quad (5.4.7)$$

By applying the Plancherel formula and using the fact that $e^{it\Delta_{\mathbb{S}^2}}$ is a unitary operator on $L^2(\mathbb{S}^2)$, we have

$$\begin{aligned} \|\Pi_N \int_0^t e^{i(t-t')\Delta}(\Pi)(t) dt\|_{L^2(\mathbb{S}^2)}^2 &= \|\Pi_N \int_0^t e^{-it\Delta}(\Pi)(t) dt\|_{H^s(\mathbb{S}^2)}^2 \\ &= \sum_{\substack{n \leq N \\ |k| \leq n}} n^{2s} \left| \sum_{1 \leq n_1, n_2 \leq N} \frac{e^{it(n^2 - n_1^2)} - 1}{i(n^2 - n_1^2)} \int_{\mathbb{S}^2} \frac{|e_{n_2}^\omega|^2 - \|e_{n_2}^\omega(x)\|_{L^2(\mathbb{S}^2)}^2}{n_2^{2\alpha}} \frac{e_{n_1}^\omega(x)}{n_1^\alpha} \overline{Y_{n,k}(x)} dx \right|^2. \end{aligned} \quad (5.4.8)$$

Then, we expand $e_{n_1}^\omega = \frac{1}{n_1^{\frac{1}{2}}} \sum_{|r| \leq n_1} g_{n_1,r}(\omega) Y_{n_1,r}$ and we develop the absolute value in (5.4.8) :

$$\begin{aligned} & \left| \sum_{1 \leq n_1, n_2 \leq N} \frac{e^{it(n^2 - n_1^2)} - 1}{i(n^2 - n_1^2)} \int_{\mathbb{S}^2} \frac{|e_{n_2}^\omega|^2 - \|e_{n_2}^\omega(x)\|_{L^2(\mathbb{S}^2)}^2}{n_2^{2\alpha}} \frac{e_{n_1}^\omega(x)}{n_1^\alpha} \overline{Y_{n,k}(x)} dx \right|^2 \\ & \sim \sum_{\substack{1 \leq n_1, n_2, n'_1, n'_2 \leq N \\ |r| \leq n_1, |r'| \leq n'_1}} \frac{1}{\langle n - n_1 \rangle^2} \left(\int_{\mathbb{S}^2} \frac{|e_{n_2}^\omega|^2 - \|e_{n_2}^\omega(x)\|_{L^2(\mathbb{S}^2)}^2}{n_2^{2\alpha}} \frac{g_{n_2,r}(\omega)}{n_2^{\frac{1}{2}}} \frac{Y_{n_1,r}(x)}{n_1^{\frac{1}{2}}} \overline{Y_{n,k}(x)} dx \right) \\ & \times \frac{1}{\langle n^2 - n_1'^2 \rangle} \left(\int_{\mathbb{S}^2} \frac{|e_{n_2}'^\omega|^2 - \|e_{n_2}'^\omega(y)\|_{L^2(\mathbb{S}^2)}^2}{n_2'^{2\alpha}} \frac{g_{n_1',r'}(\omega)}{n_1'^{\frac{1}{2}}} \frac{e_{n_1',r'}(y)}{n_1'^\alpha} \overline{Y_{n,k}(y)} dy \right). \end{aligned} \quad (5.4.9)$$

Observe that

$$\mathbb{E}(g_{m,r}) = 0, \quad \mathbb{E}(g_{m,r}^2) = 0, \quad \mathbb{E}(|e_m^\omega|^2 - \|e_m^\omega\|_{L^2(\mathbb{S}^2)}^2) = 0.$$

and that the laws of $e_n^\omega(x)$ and $e_n^\omega(y)$ are the same as a consequence of rotational invariance. Hence, it follows from the independence of the $\{g_{m,r}\}$ that the only terms that remain in the series (5.4.9)

after taking the expectation are the ones with $n_1 = n'_1$, $n_2 = n'_2$, $r = r'$. To estimate from below the right-hand side from (5.4.8), we are thus reduced to handle the series

$$\begin{aligned} & \sum_{\substack{1 \leq n \leq N \\ |k| \leq n}} \sum_{\substack{1 \leq n_1, n_2 \leq N \\ |r| \leq n_1}} \frac{n^{2s}}{n_1^{2(\alpha + \frac{1}{2})}} \frac{1}{\langle n^2 - n_1^2 \rangle^2} \mathbb{E} \left[\left| \int_{\mathbb{S}^2} \frac{|e_{n_2}^\omega|^2 - \|e_{n_2}^\omega\|_{L^2(\mathbb{S}^2)}^2}{n_2^{2\alpha}} g_{n_1, r}(\omega) Y_{n_1, r}(x) \overline{Y_{n, k}(x)} dx \right|^2 \right] \\ & \gtrsim \sum_{\substack{2 \leq n \leq N \\ |k| \leq n}} n^{2(s - \alpha - \frac{1}{2})} \mathbb{E} \left[|g_{n, k}(\omega)|^2 \left| \int_{\mathbb{S}^2} (|e_1^\omega|^2 - \|e_1^\omega\|_{L^2(\mathbb{S}^2)}^2) |Y_{n, k}(x)|^2 dx \right|^2 \right], \end{aligned} \quad (5.4.10)$$

where we made a crude estimate by keeping in the series the terms where $n_1 = n$, $r = k$ and $n_2 = 1$. Moreover, we only keep the terms where $2 \leq n$ to exploit the independence between e_1^ω and $g_{n, k}(\omega)$. Since the $\{g_{n, k}\}$ are standard Gaussian variables, we have $\mathbb{E}[|g_{n, k}(\omega)|^2] = 1$ and taking $s = \alpha - \frac{1}{2}$ yields

$$\|\mathcal{I}_N(t, \Pi)\|_{L^2(\Omega; H^s(\mathbb{S}^2))}^2 \gtrsim \sum_{\substack{2 \leq n \leq N \\ |k| \leq n}} \frac{1}{n^2} \mathbb{E} \left[\left| \int_{\mathbb{S}^2} (|e_1^\omega|^2 - \|e_1^\omega\|_{L^2(\mathbb{S}^2)}^2) |Y_{n, k}(x)|^2 dx \right|^2 \right]. \quad (5.4.11)$$

Let us now estimate from below the right hand side of (5.4.11). We use the explicit expression of e_1^ω , and we get rid of a well-chosen set in order to exploit the concentration of $Y_{1, 1}$ near the equator. Besides, recall that

$$e_1^\omega = \frac{1}{\sqrt{3}} (g_{1, 1}(\omega) \beta(x_1 + ix_2) + g_{1, -1}(\omega) Y_{1, -1} + g_{1, 0}(\omega) Y_{1, 0}),$$

where $\beta \geq 1$ is the renormalization constant such that $\beta^2 \int_{\mathbb{S}^2} |x_1 + ix_2|^2 dx = 1$. Then, given $0 \leq \varepsilon \ll 1$ we define

$$S_\varepsilon = \{\omega \in \Omega \mid 18 \leq |g_{1, 1}(\omega)|^2, \quad |g_{1, -1}(\omega)|^2 + |g_{1, 0}(\omega)|^2 \leq \varepsilon\}.$$

Note that $\mathbb{P}(S_{A, \varepsilon}) > 0$. Moreover, for $\omega \in S_\varepsilon$ we have

$$\begin{aligned} |e_1^\omega|^2 - \|e_1^\omega\|_{L^2(\mathbb{S}^2)}^2 &= \frac{1}{3} |g_{1, 1}^\omega|^2 \beta^2 |x_1 + ix_2|^2 + O(\varepsilon) - \frac{1}{3} (|g_{1, 1}^\omega|^2 + |g_{1, -1}^\omega|^2 + |g_{1, 0}^\omega|^2) \\ &= \frac{1}{3} |g_{1, 1}^\omega|^2 (\beta^2 |x_1 + ix_2|^2 - 1) + O(\varepsilon). \end{aligned}$$

Write

$$a = \int_{\mathbb{S}} (|e_1^\omega(x)|^2 - \|e_1^\omega\|_{L^2(\mathbb{S}^2)}^2) |Y_{n, k}(x)|^2 dx, \quad b = O(\varepsilon) \int_{\mathbb{S}} |Y_{n, k}(x)|^2 dx = O(\varepsilon).$$

Then, for $\omega \in S_\varepsilon$ the inequality $|a + b|^2 \geq \frac{|a|^2}{2} - |b|^2$ yields

$$\begin{aligned} & \left| \int_{\mathbb{S}} (|e_1^\omega(x)|^2 - \|e_1^\omega\|_{L^2(\mathbb{S}^2)}^2) |Y_{n, k}(x)|^2 dx \right|^2 \\ & \geq \frac{|g_{1, 1}(\omega)|^2}{18} \left| \int_{\mathbb{S}} (\beta^2 |x_1 + ix_2|^2 - 1) |Y_{n, k}(x)|^2 dx \right|^2 \\ & \quad - O(\varepsilon^2) \left| \int_{\mathbb{S}} |Y_{n, k}(x)|^2 dx \right|^2. \end{aligned} \quad (5.4.12)$$

To handle the first term on the right hand side of (5.4.12), we observe that

$$\beta^2|x_1 + ix_2|^2 - 1 = \beta^2(1 - x_3^2) - 1 \geq \frac{\beta^2 - 1}{2} > 0$$

provided that

$$|x_3| \leq \frac{1}{\beta} \sqrt{\frac{\beta^2 - 1}{2}} =: 2\delta. \quad (5.4.13)$$

In section 5.3, we prove that in the regime where n is large and $\sqrt{1 - \delta^2}n \leq |k| \leq n$, the spherical harmonics $\{Y_{n,k}\}$ are concentrated near the equator, precisely in the region where $\{x \in \mathbb{S}^2 \mid |x_3| \leq 2\delta\}$ (see Proposition 5.3.1). We write

$$\begin{aligned} \left| \int_{\mathbb{S}} (\beta^2|x_1 + ix_2|^2 - 1) |Y_{n,k}(x)|^2 dx \right| &\geq \frac{\beta^2 - 1}{2} \int_{\mathbb{S}} \mathbf{1}_{|x_3| \leq 2\delta} |Y_{n,k}(x)|^2 dx \\ &\quad - C \int_{\mathbb{S}} \mathbf{1}_{|x_3| > 2\delta} |Y_{n,k}(x)|^2 dx, \end{aligned} \quad (5.4.14)$$

where $C = \sup\{|\beta^2(1 - x_3^2) - 1| \mid 2\delta < |x_3| \leq 1\}$. Then, Proposition 5.3.1 yields that there exist C_δ , and $N_0(\delta) \in \mathbb{N}$ such that for all (n, k) in the set

$$\Lambda_{N_0} = \{(n, k) \in \mathbb{N}^2 \mid N_0 \leq n, \sqrt{1 - \delta^2} \leq \frac{|k|}{n} \leq 1\},$$

we have

$$\int_{\mathbb{S}} \mathbf{1}_{|x_3| > 2\delta} |Y_{n,k}(x)|^2 dx \leq C_\delta n^{-1}.$$

Hence, it follows from (5.4.14) that for all (n, k) in Λ_{N_0} ,

$$\begin{aligned} \left| \int_{\mathbb{S}} (\beta^2|x_1 + ix_2|^2 - 1) |Y_{n,k}(x)|^2 dx \right| &\geq \frac{\beta^2 - 1}{2} - \left(C + \frac{\beta^2 - 1}{2}\right) \int_{\mathbb{S}} \mathbf{1}_{|x_3| > 2\delta} |Y_{n,k}(x)|^2 dx \\ &\geq \frac{\beta^2 - 1}{2} - C_\delta \left(C + \frac{\beta^2 - 1}{2}\right) n^{-1} \\ &\geq \frac{\beta^2 - 1}{4}, \end{aligned}$$

provided that n is large enough. Taking ε small enough such that $O(\varepsilon^2) \leq \frac{1}{2} \left(\frac{\beta^2 - 1}{4}\right)^2$, we deduce from (5.4.12) and from (5.4.14) that

$$\begin{aligned} \mathbb{E} \left[\mathbf{1}_{S_\varepsilon(\omega)} \left| \int_{\mathbb{S}} (|e_1^\omega(x)|^2 - \|e_1^\omega\|_{L^2(\mathbb{S}^2)}^2) |Y_{n,k}(x)|^2 dx \right|^2 \right] &\geq \frac{1}{2} \left(\frac{\beta^2 - 1}{4}\right)^2 \mathbb{P}(S_\varepsilon) \\ &=: \alpha \mathbb{P}(S_\varepsilon) > 0. \end{aligned} \quad (5.4.15)$$

By collecting estimates (5.4.11) and (5.4.15), we obtain that

$$\begin{aligned} \|\mathcal{I}_N(t, \Pi)\|_{L^2(\Omega; H^s(\mathbb{S}^2))}^2 &\gtrsim \mathbb{E} \left[\mathbf{1}_{S_\varepsilon(\omega)} \|\Pi_N \int_0^t e^{i(t-t')\Delta_{\mathbb{S}^2}}(\Pi)(t') dt'\|_{L^2(\mathbb{S}^2)}^2 \right] \\ &\gtrsim \sum_{\substack{2 \leq n \leq N \\ |k| \leq n}} \frac{1}{n^2} \mathbb{E} \left[\mathbf{1}_{S_\varepsilon(\omega)} \left| \int_{\mathbb{S}^2} (|e_1^\omega|^2 - \|e_1^\omega\|_{L^2(\mathbb{S}^2)}^2) |Y_{n,k}(x)|^2 dx \right|^2 \right] \\ &\gtrsim \sum_{\substack{(n,k) \in \Lambda_{N_0} \\ n \leq N}} \frac{1}{n^2} \mathbb{E} \left[\mathbf{1}_{S_\varepsilon(\omega)} \left| \int_{\mathbb{S}^2} (|e_1^\omega|^2 - \|e_1^\omega\|_{L^2(\mathbb{S}^2)}^2) |Y_{n,k}(x)|^2 dx \right|^2 \right]. \end{aligned}$$

It follows from (5.4.15) that

$$\begin{aligned} \|\mathcal{I}_N(t, \Pi)\|_{L^2(\Omega; H^s(\mathbb{S}^2))}^2 &\gtrsim \alpha \mathbb{P}(S_\varepsilon) \sum_{\substack{(n,k) \in \Lambda_{N_0} \\ n \leq N}} \frac{1}{n^2} \\ &\gtrsim \alpha \mathbb{P}(S_\varepsilon) \delta^2 \sum_{N_0 \leq n \leq N} \frac{1}{n}. \end{aligned}$$

Indeed, we see from the definition of Λ_{N_0} that for each $n \geq N_0$,

$$\#\{k \mid (n, k) \in \Lambda_{N_0}\} \geq n(1 - \sqrt{1 - \delta^2}) \gtrsim n \delta^2.$$

This proves that the right hand side in the above estimate displays a logarithmic divergence, and concludes the proof of Proposition 5.4.4 by taking $\eta \sim \alpha \mathbb{P}(S_\varepsilon) \delta^2 > 0$. \square

5.5 First iteration for the torus

In this appendix, we consider the cubic NLS on the general torus

$$\mathbb{T}_\beta^2 = \mathbb{R}^2 / (2\pi\mathbb{Z})^2,$$

endowed with the metric $g = dx_1^2 + \beta^{-2} dx_2^2$ for some $\beta > 0$. The cubic NLS on \mathbb{T}_β^2 is written as

$$i\partial_t v + \Delta_\beta v = |v|^2 v, \tag{5.5.1}$$

where $\Delta_\beta = \partial_{x_1}^2 + \beta^2 \partial_{x_2}^2$. Hence from now on we will only concentrate on (5.5.1). For $n = (k, m)$, $n' = (k', m') \in \mathbb{Z}^2$, we denote by $Q(n, n') = kk' + \beta^2 mm'$ the associate quadratic form and $Q(n) := Q(n, n)$. Then

$$e^{it\Delta_\beta} = \mathcal{F}_x^{-1} e^{-itQ(\cdot)} \mathcal{F}_x.$$

In light of (5.1.1), we consider initial data of the form

$$\phi^\omega(x) = \sum_{n \in \mathbb{Z}^2} \frac{1}{\langle n \rangle} g_n^\omega e^{in \cdot x},$$

where $\langle n \rangle = (1 + Q(n))^{1/2}$, and

$$u^\omega(t, x) = \sum_{n \in \mathbb{Z}^2} \frac{e^{itQ(n)}}{\langle n \rangle} g_n^\omega e^{in \cdot x}$$

We write the Wick-ordered nonlinearity

$$\begin{aligned} \mathcal{N}(u^\omega)(t) &= \sum_{\substack{(n_1, n_2, n_3) \in \mathbb{Z}^2 \\ n_2 \neq n_1, n_3}} \frac{e^{it(Q(n_1) - Q(n_2) + Q(n_3))}}{\langle n_1 \rangle \langle n_2 \rangle \langle n_3 \rangle} \overline{g_{n_1}(\omega)} g_{n_2}(\omega) g_{n_3}(\omega) e^{i(n_1 - n_2 + n_3) \cdot x} \\ &= \sum_{(n_1, n_2, n_3, n) \in \Gamma} \frac{e^{it\Phi(n)}}{\langle n_1 \rangle \langle n_2 \rangle \langle n_3 \rangle} \overline{g_{n_1}(\omega)} g_{n_2}(\omega) g_{n_3}(\omega) e^{in \cdot x}, \end{aligned}$$

where the constraint set is

$$\Gamma = \{(n_1, n_2, n_3, n) \in (\mathbb{Z}^2)^4 : n = n_1 - n_2 + n_3, n_2 \neq n_1, n_2 \neq n_3\}.$$

and the resonant function Φ defined on Γ is

$$\Phi(n) := Q(n_1) - Q(n_2) + Q(n_3) - Q(n_1 - n_2 + n_3) = 2Q(n_1 - n_2, n_2 - n_3).$$

The first Picard iteration is written as

$$\mathcal{I}_{\mathbb{T}_\beta^2}(t, \phi^\omega) = \int_0^t e^{i(t-t')\Delta_\beta} \mathcal{N}(u^\omega(t')) dt',$$

We shall now prove the gain of regularity (5.1.3) claimed in Theorem 5.1.2.

Proof. For fixed t , direct computation, independence assumption as well as the observation that $\mathbb{E}[g^2] = 0$ for standard complex gaussian functions yields that

$$\mathbb{E} \left[\|\mathcal{I}_{\mathbb{T}_\beta^2}(t, \phi^\omega)\|_{H_x^s}^2 \right] \sim \sum_{n=(n_1, n_2, n_3) \in \Gamma} \frac{\langle n_1 - n_2 + n_3 \rangle^{2s}}{\langle n_1 \rangle^2 \langle n_2 \rangle^2 \langle n_3 \rangle^3 \langle \Phi(n) \rangle^2}, \quad (5.5.2)$$

To compute the right side of (5.5.2), we decompose dyadically $|n_j| \in N_j$, and assume that $N^{(1)} \geq N^{(2)} \geq N^{(3)}$ is the non-increasing order of N_1, N_2, N_3 . It would be sufficient to obtain an inequality of the form

$$(N_1 N_2 N_3)^{-2} \sum_{\substack{|n_j| \sim N_j, j=1,2,3 \\ n_2 \neq n_1, n_3}} \frac{\langle n_1 - n_2 + n_3 \rangle^{2s}}{\langle Q(n_1 - n_2, n_2 - n_3) \rangle^2} \lesssim (N^{(1)})^{-\delta}. \quad (5.5.3)$$

The left side of (5.5.3) can be majorized by

$$(N_1 N_2 N_3)^{-2} \sum_{l \in \mathbb{Z}} \frac{(N^{(1)})^{2s}}{\langle l \rangle^2} \sum_{\substack{|n_j| \in N_j \\ n_2 \neq n_1, n_3}} \mathbf{1}_{|Q(n_1 - n_2, n_2 - n_3) - l| \leq \delta} \quad (5.5.4)$$

$$\leq (N_1 N_2 N_3)^{-2} (N^{(1)})^{2s} \sup_{l \in \mathbb{Z}} \sum_{\substack{|n_j| \in N_j \\ n_2 \neq n_1, n_3}} \mathbf{1}_{|Q(n_1 - n_2, n_2 - n_3) - l| \leq \delta}, \quad (5.5.5)$$

for some number $0 < \delta < 1$ to be fixed later. Now let us estimate the quantity (for fixed $l \in \mathbb{Z}$ and $\delta < 1/4$)

$$M_{N_1, N_2, N_3} := \sum_{\substack{|n_j| \sim N_j \\ n_2 \neq n_1, n_3}} \mathbf{1}_{|Q(n_1 - n_2, n_2 - n_3) - l| \leq \delta} \quad (5.5.6)$$

- **Case 1** $N_1 \gg N_2, N_3$: in this case, (5.5.6) can be majorized by

$$(N_2 N_3)^2 \sup_{n_2, n_3: n_2 \neq n_3} \sum_{n_1: |n_1| \sim N_1, n_1 \neq n_2} \mathbf{1}_{|Q(n_1 - n_2, n_2 - n_3) - l| \leq \delta}.$$

For fixed n_2, n_3 , we denote by $m_1 = n_1 - n_2, m_0 = n_2 - n_3, |m_1| \sim N_1 \gg |m_0|$. It is sufficient to estimate the number of $m_1 \in \mathbb{Z}^2, |m_1| \sim N_1$ such that $|Q(m_1, m_0) - l| \leq \delta$. Denote by $m_0 = (\xi_0, \eta_0)$, if $\eta_0 = 0$ or $\xi_0 = 0$, $Q(m_1, m_0) \in \beta^2 \mathbb{Z}$ or $Q(m_1, m_0) \in \mathbb{Z}$, with respectively. Then $Q(m_1, m_0)$ can only take a discrete number of values. In these situations, we have (the same dimension treatment as in [Bou96])

$$\#\{m_1 \in \mathbb{Z}^2 : |m_1| \sim N_1, |Q(m_1, m_0) - l| \leq \delta\} \lesssim N_1.$$

Therefore, without loss of generality, we may assume that both ξ_0 and η_0 are non zero. Denote by $m_\beta = (\xi_0, \beta^2 \eta_0) = \text{diag}(1, \beta^2) m_0$. Then it is reduced to estimate the cardinality of the set

$$S_\delta := \left\{ z \in \mathbb{Z}^2 : \left| z \cdot \frac{m_\beta}{|m_\beta|} - \frac{l}{|m_\beta|} \right| \leq \frac{\delta}{|m_\beta|} \right\}.$$

We observe that S_δ is a rectangle with side length $\sim N_1$ and width $\leq 2\delta$. Since $2\delta < 1/2$, we see that

$$\#S_\delta \lesssim N_1.$$

Thus the contribution of (5.5.4) in the sum is less than N_1^{2s-1} , and the associated dyadic summation over $N_1 \gg N_2, N_3$ converges provided that $s < \frac{1}{2}$.

- **Case 2:** $N_2 \gg N_1, N_3$:

In this case, $|Q(n_1 - n_2, n_2 - n_3)| \sim N_2^2$. Coming back to (5.5.4), the range of the sum of l is $|l| \geq N_2^2$. Hence by the crude estimate $\mathcal{M}_{N_1, N_2, N_3} \lesssim (N_1 N_2 N_3)^3$, the contribution of (5.5.4) is bounded by

$$\sum_{|l| \geq N_2^2} \frac{N_2^{2s}}{l^2} \lesssim N_2^{2s-2}.$$

Then associated dyadic summation over $N_2 \gg N_1, N_3$ converges, provided that $s < 1$.

For the rest situations, the argument is the same as for **Case 1**. For example, if $N_1 \sim N_2 \gg N_3$ ($N_2 \sim N_3 \gg N_1$), we can fix n_2, n_3 (n_1, n_2), and do the same manipulation as in **Case 1**.

This completes the proof of (5.1.3) in Theorem 5.1.2. \square

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