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Some aspects of B field and anomalies in string compactifications

*Quelques aspects du champ B et anomalies dans les
compactifications des cordes*

Thèse de doctorat de l'université Paris-Saclay

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sous la direction de **Ruben MINASIAN**, Directeur de Recherche

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Titre : Quelques aspects du champ B et anomalies dans les compactifications des cordes

Mots clés : Champ B, Anomalies, Compactifications de cordes

Résumé : Cette thèse aborde certains aspects des compactifications perturbatives et non-perturbatives des cordes. Pour le côté perturbatif, nous étudions les compactifications de cordes hétérotiques et de type II sur deux classes de variété plates à cinq dimensions (orbifolds) qui préservent la moitié des supersymétries. Nous trouvons que les cordes hétérotiques compactes sur ces deux classes de variété sont équivalentes l'une à l'autre. L'équivalence des théories de cordes de type II compactées sur ces deux classes de variété nécessite des configurations de champs B supplémentaires.

Nous discutons de ces configurations de champs B comme un mécanisme perturbatif de la singularité gelée et nous le comparons avec le mécanisme non-perturbatif de la singularité gelée. Pour le côté non-perturbatif, nous étudions la théorie (F) M compactifiée sur des plis de Calabi-Yau 3-folds (elliptiques) et soulignons les relations entre les contraintes géométriques des plis de Calabi-Yau 3-folds elliptiques et certaines conditions de cohérence récemment proposées pour les théories de gravité quantique.

Title : Some aspects of B field and anomalies in string compactifications

Keywords : B field, Anomalies, String compactifications

Abstract : This dissertation discusses some aspects of perturbative and non-perturbative string compactifications. On the perturbative side, we study Heterotic and type II strings compactifications on flat five dimensional manifolds (orbifolds) preserving half of the supersymmetries. By including non-trivial holonomy for Wilson lines, we demonstrate that Heterotic string compactifications on different classes of internal geometries can be equivalent. Similar equivalence of type II string

theories compactifications requires additional B-field configurations. We further discuss these B-field configurations and show how they lead to a perturbative freezing singularities. On the non-perturbative side, we study (F) M theory compactifications on (elliptic) Calabi-Yau 3-folds and point out the relations between the geometric constraints of elliptic Calabi-Yau 3-folds and recently proposed general consistency conditions for theories of quantum gravity in six dimensions.

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Abstract

This dissertation discusses some aspects of perturbative and non-perturbative string compactifications. On the perturbative side, we study Heterotic and type II strings compactifications on flat five dimensional manifolds (orbifolds) preserving half of the supersymmetries. By including non-trivial holonomy for Wilson lines, we demonstrate that Heterotic string compactifications on different classes of internal geometries can be equivalent. Similar equivalence of type II string theories compactifications requires additional B-field configurations. We further discuss these B-field configurations and show how they lead to a perturbative freezing singularities. On the non-perturbative side, we study (F) M theory compactifications on (elliptic) Calabi-Yau 3-folds and point out the relations between the geometric constraints of elliptic Calabi-Yau 3-folds and recently proposed general consistency conditions for theories of quantum gravity in six dimensions.

Résumé

Cette thèse aborde certains aspects des compactifications perturbatives et non-perturbatives des cordes. Pour le côté perturbatif, nous étudions les compactifications de cordes hétérotiques et de type II sur deux classes de variété plates à cinq dimensions (orbifolds) qui préservent la moitié des supersymétries. Nous trouvons que les cordes hétérotiques compactes sur ces deux classes de variété sont équivalentes l'une à l'autre. L'équivalence des théories de cordes de type II compactées sur ces deux classes de variété nécessite des configurations de champs B supplémentaires. Nous discutons de ces configurations de champs B comme un mécanisme perturbatif de la singularité gelée et nous le comparons avec le mécanisme non-perturbatif de la singularité gelée. Pour le côté non-perturbatif, nous étudions la théorie (F) M compactifiée sur des plis de Calabi-Yau 3-folds (elliptiques) et soulignons les relations entre les contraintes géométriques des plis de Calabi-Yau 3-folds elliptiques et certaines conditions de cohérence récemment proposées pour les théories de gravité quantique.

Introduction

Currently superstring theory is the most promising framework of quantum gravity. The consistent formulation of superstring theory requires a 10 dimensional spacetime so 6 dimensions must be compactified. Furthermore, string compactification provides lots of exotic higher dimensional theories, e.g. 6d (2,0) SCFT, which shed new light on our current understanding of quantum field theory. In this thesis, we investigate both perturbative and non-perturbative aspects of string compactifications.

In the perturbative string compactifications, we apply standard worldsheet CFT analysis. What we find is that stringy T-duality relates two different spacetime backgrounds, one is smooth and the other one is singular: We find that Heterotic string compactified on these two classes of manifolds are equivalent to each other, while equivalence of type II string theories compactified on these two classes of manifolds requires additional flat B-field configurations. We further discuss the connections between these flat B-field configurations and the flat Ramond-Ramond field configurations studied in [24]

In the non-perturbative string setup, we study (F) M theory compactified on (elliptic) Calabi-Yau 3-folds (CY_3), whose low energy supergravity description depends on the internal geometric structure. We first study BPS strings in the supergravity theories and single out the "supergravity strings" from the BPS string spectrum. By applying both the geometric constraints arising from elliptic CY_3 [82] and unitarity constraints proposed in [92] to these "supergravity strings", we find the relations between the geometrical and physical constraints.

Perturbative string compactification

In this part we will be mostly concerned with orbifold conformal theories and the associated string compactifications.

Much of the fascination with orbifold compactification on Y/G comes from the presence of singular points, which correspond to points in Y fixed by the action of G [11, 12]. Despite these singularities, the string theory on Y/G can be perfectly well-defined, and computations of various quantities from the worldsheet perspective have led to important insights into geometry, such as the notion of stringy Hodge numbers and finer cohomological structures on orbifolds and algebraic varieties (e.g. see [13] for an introduction).

The study of orbifolds has also led to insights into the moduli space of string compactifications, for example by showing that certain limits of compactification on smooth manifolds such as K3 or a Calabi-Yau 3-fold where the geometry becomes singular can nevertheless be understood entirely in perturbative string theory. In addition, orbifolds have provided insights into non-trivial dualities such as mirror symmetry. They have also been used extensively in string phenomenology, and there are impressive classification results, such as the description of all symmetric orbifolds of the six-torus T^6 that preserve spacetime supersymmetry [14].

In recent years orbifolds of toroidal compactifications have received renewed interest in relation to the swampland program; see, for example, [15–20]. This recent work involves asymmetric

orbifolds that acts on the toroidal geometry by shifts, while having a non-trivial action on the gauge sector. The results provide important insights into the various disconnected components in the moduli space of toroidal compactification that generalize the CHL construction [21, 22] and are intimately related to the study of heterotic compactifications with non-trivial flat connections [23, 24] and related type II/M-theory/F-theory constructions with background RR fluxes and frozen singularities [25, 26]. It will be interesting to extend these efforts to cases with less space-time supersymmetry, where classification will be even more challenging, but also the physics will be richer. Our work is just a small indication of some of the interesting structures that emerge in this larger setting.

In this part we begin our study with 5-dimensional compactifications on $X_5 = \mathbb{T}^5/G$ that preserve half of spacetime supersymmetry, and where $G = \mathbb{Z}_N$ acts freely on \mathbb{T}^5 .¹ To obtain a free and supersymmetric action, we write $\mathbb{T}^5 = \mathbb{T}^4 \times S^1$, and choose G to have a supersymmetric action on \mathbb{T}^4 , so that \mathbb{T}^4/G has holonomy $\mathbb{Z}_N \subset \text{SU}(2)$. The orbifold X_5 is nevertheless smooth because the action on \mathbb{T}^4 is combined with an order N shift on the 5-th circle.

Since these spaces lack fixed points, we can access their physics from both supergravity and worldsheet perspectives, and that allows us to uncover a number of their properties. For example, we discuss how we can interpret these compactifications from a six-dimensional spacetime point of view: it is possible to think of the theories as S^1 compactifications of a 6-dimensional theory, where a holonomy is turned on for a discrete \mathbb{Z}_N gauge symmetry; the holonomy breaks half of the supersymmetry and modifies the spectrum in other ways. This point of view should be useful in fitting these compactifications in the framework of F-theory/type II — heterotic duality, for example along the lines explored in [28, 29].

Second, we show that in heterotic string theory these compactifications are dual to more familiar compactifications on $\mathbb{T}^4/G \times S^1$ and are therefore connected to conventional compactification on K3. That this should be the case was already suggested in [30], and we are able to establish the claim beyond an analysis of massless spectrum. We also note that the duality exchanges spaces with different topologies, thus providing another example of stringy geometry relating topologically distinct geometries.

The equivalence we establish is based on the observation that there are orbifold actions on the \mathbb{T}^5 worldsheet CFT, $G_1 \simeq G$ and $G_2 \simeq G$ that are related by a conjugation: there is an element $t \in \text{O}(21, 5, \mathbb{Z})$ that gives an isomorphism $t : G_1 \rightarrow G_2$, with $t : g \mapsto t^{-1}gt$. While the quotient by G_1 leads to the smooth geometry X_5 , the quotient by G_2 leads to $\mathbb{T}^4/G \times S^1$. The induced isomorphism on the moduli of the two CFTs acts as a T-duality on the radius of the non-trivial S^1 in combination with a shift of the Wilson line parameters by a lattice vector.

It is then natural to interpret the isomorphism as a T-duality of the X_5 CFT, but there is a subtlety in applying the standard paradigm of Buscher rules for a non-linear sigma model, despite the fact that the X_5 CFT can be perfectly well described by a large radius non-linear sigma model. The trouble is that the natural circle fibration structure

$$\begin{array}{ccc} S^1 & \hookrightarrow & X_5 \\ & & \downarrow \\ & & B \end{array}$$

has $B = \mathbb{T}^4/G$, i.e. the base is singular! Our results indicate that T-duality extends to such singular fibrations, with the benefit of enlarging the possibilities for topology change via T-duality beyond those discussed in the context of principal torus bundles over smooth manifolds, e.g. in [31, 32].

¹As the recent work [27] indicates, if one is willing to abandon spacetime supersymmetry, then compact flat manifolds already become interesting in compactifications to 7 dimensions.

Next, we consider these backgrounds in the context of type II string theory compactification and find that compactification on X_5 is T-dual to compactification on $\mathbb{T}^4/G \times S^1$ equipped with a topologically non-trivial flat B -field gerbe.

At the level of the orbifold CFT this illustrates, along the lines described in [33], that the gerbe data gives a natural geometric meaning to certain shift orbifold phases. One example of such phases is a choice of discrete torsion [34, 35], but these are absent in all of our examples.

Non-perturbative string compactification

Local anomaly cancellation is not the only consistency condition one can impose on six-dimensional minimal supergravities. A more recently developed criterion is based on the completeness conjecture of the spectrum of the charged BPS objects in supergravity theories. In six dimensions, there are string-like BPS objects that cannot be consistently decoupled from gravity (we shall follow [92, 93] and call them “supergravity strings”), provided certain conditions on their charges are satisfied. These will be spelled out in Section 5.4. These BPS strings support two-dimensional $(0, 4)$ superconformal theories on the worldsheet, whose central charges are completely fixed by the bulk anomaly cancellation, i.e. the coefficients of different couplings in GS terms. They couple to the gauge fields in the bulk for gauge group $G = \prod_i G_i$, and hence the unitarity of the worldsheet theory requires that the total central charge associated with the current algebras of G_i is not larger than the left moving central charge:

$$\sum_i c(G_i) \leq c_L \tag{0.0.1}$$

The consequences of this bound for six-dimensional theories have been analysed in [92]. In contrast, the five-dimensional theories, even those that are obtained from a circle reduction, have a different way of packaging the information, and the expression for the $(0, 4)$ central charges is rather different. We find that the five-dimensional view on the supergravity strings is somewhat more convenient and leads to constraints that are slightly more stringent. Interestingly, for the theories that come from a circle reduction the constraints are still associated with the (reduced) six-dimensional supergravity strings rather than strings carrying KK charges.²

Of course, it is natural to compare any ostensibly consistent minimally supersymmetric theory in six dimensions to F-theory constructions. In addition to the requirements imposed by physical considerations, these are subject to additional constraints that are associated with the geometry of elliptic fibrations. These constraints can be formulated either in terms of the data of the effective theory or in geometrical language. In the F-theory picture, the non-Abelian gauge groups G_i arise from D7-branes wrapping singular gauge divisors S_i in the base manifold B . The CY condition, i.e. the triviality of the canonical bundle of the elliptic fibration, relates K , the canonical divisor of B , to the locus of singular fibers. In addition the Kodaira positivity condition (KPC) states that a residual divisor $Y = -12K - \sum_i x_i S_i$ should be effective. The coefficients x_i are given by the vanishing order of the discriminant on S_i . These can be found in Table 1 in Section 6.3. In particular this means the non-negativity of the intersection

$$D \cdot (-12K - \sum_i x_i S_i) = D \cdot Y \geq 0, \tag{0.0.2}$$

²Based on the anomaly inflow and local counterterms in the bulk, we can see a mismatch of BPS string spectrum in 6d and 5d supergravities, as discussed in Section 5.1. As we shall argue this is explained by noticing that the 5d BPS strings, carrying KK charges, are lifted to certain geometric background (Taub-NUT space) that preserves half of the supersymmetry rather than BPS strings in 6d supergravity.

for any nef divisor D (nef divisors, by definition, intersect every effective divisor non-negatively). The supergravity strings in F-theory models originate from D3-branes wrapping D .

The unitarity conditions (UC) are formulated directly in terms of the data of the effective theory. Assuming that there is an underlying elliptic CY_3 the five-dimensional UC that we derive here can be geometrised, and cast as bound on intersection forms with any nef divisor D . It can also be reformulated as an extra constraint on the residual divisor Y :

$$D \cdot Y \geq 3 - \sum_i (x_i - y_i) D \cdot S_i, \quad (0.0.3)$$

where $y_i = \frac{\dim G_i}{1+h_i^\vee}$ and h_i^\vee is the dual Coxeter number of G_i . Comparative values of x_i and y_i (as we shall see for any group G_i , $x_i - y_i \geq 1$) and the details of the analysis of the condition (0.0.6) can be found in Section 6.3.2. A word of caution is due. This is the strongest form of the unitarity constraint, where the value of the coefficients y_i has been computed under the assumption that $D \cdot S_i = 1$ holds. In the vast majority of cases this bound is automatically satisfied if (0.0.5) holds. If it is violated, the validity of $D \cdot S_i = 1$ needs to be checked before concluding that UC indeed imposes additional constraints on the residual divisor Y . In Section 6.3.3.1 we catalogue all the cases where UC imposes extra constraints.

Notice that an example where the implications of six-dimensional UC were stronger than those imposed by KPC was already presented in [92]. We find that in generic situations 5d UC is more constraining than 6d UC and has the advantage of being cast in a form directly comparable to KPC. In general it is less constraining than KPC, and hence can be useful in delineating the boundaries of the region between the six-dimensional F-theory models and the swampland (which is likely to contain a finite number of theories [94]). The fact that it does in special situations impose additional constraints allows for the intriguing possibility of finding more refined structure in elliptic CY_3 with certain singularity structures. Both cases would deserve further study.

Structure

This thesis is divided into three parts and it is organized as follows:

The first part consists of two chapters. In the first chapter we give a brief introduction to perturbative string theory formulation, we discuss the 5 most common superstring theories and address their properties. We point out that these 5 different superstring theories are in fact closely connected to each other by duality web. In the second chapter we introduce two non-perturbative formulations of string theory, M and F theory. We introduce some methods and ideas through examples in M and F theory framework that will be applied and explored in the second and third part, e.g. anomaly inflow, MSW string, frozen singularity, etc..

The second part studies perturbative string compactification. We first introduce two classes of flat five dimensional manifolds (orbifolds) that preserve half supersymmetry. Then we study Heterotic and type II string compactifications on these two classes of manifolds and discuss their connections. We found Heterotic string compactifies on these two classes of manifolds are equivalent to each other through T-duality. This equivalence in type II string compactifications requires additional data, non-trivial flat B-field configuration. We point out that the non-trivial flat B-field configuration may provide one perturbative interpretation of frozen singularity.

The last part focus on non-perturbative string compactifications, i.e. (F) M theory compactified on (elliptic) CY_3 . F theory compactified on elliptic CY_3 gives a sunset of the infinite families of 6d anomaly free supergravity theories with 8 supercharges. Some consistency conditions based on extended objects of these supergravity theories (BPS strings) has been proposed to rule out infinite families of 6d supergravity theories [92, 94]. We examine these conditions in the F theory

compactification framework and point out the relations between this consistency condition and one geometric condition of elliptic CY_3 .

Notes added

This thesis is based on two papers [115, 144]. Part I is a review of relevant background and techniques.

Part II is based on [144]. In this paper we showed the moduli space of Heterotic strings compactified on two different geometries are connected to each other by T-duality, verifies the low energy observations made by [30]. We also studied the relevant T-duality in type IIA framework and demonstrated their T-dual are equivariant gerbes by constructing the precise group actions on the spacetime gerbe background. The applications to freezing singularities and correspondence spaces are discussed.

Part III is based on [115]. In this paper we investigated the perturbative anomalies in odd dimensional theory from circle compactification of an even dimensional anomalous one. Additionally, we also studied the spectrum of BPS strings in 5d/6d supergravity with 8 supercharges and suggested one way to match them in context of circle compactification. We also showed the relationship between the Kodaira condition [82] and the unitarity condition [92] by compactifying 6d supergravity on a circle.

Synthèse en français

De nos jours, la théorie des supercordes est le cadre le plus prometteur de la gravité quantique. La formulation cohérente de la théorie des supercordes nécessite un espace-temps à 10 dimensions, de sorte que 6 dimensions doivent être compactifiées. De plus, la compactification des cordes permet d'obtenir de nombreuses théories exotiques de plus haute dimension, par exemple des théories de champ supraconforme à 6 dimensions avec supersymétrie (2,0), qui jettent de nouvelles lumières sur notre compréhension actuelle de la théorie quantique des champs. Dans cette thèse, nous étudions les aspects perturbatifs et non-perturbatifs des compactifications de cordes.

Dans les compactifications perturbatives des cordes, nous appliquons l'analyse standard de la théorie des champs conformes aux surfaces d'univers. Nous avons découvert que la T-dualité des cordes relie deux arrière-plans spatiaux différents, l'un est lisse et l'autre est singulier : Nous constatons que les compactifications de cordes hétérotiques sur ces deux classes de variété sont équivalentes l'une à l'autre, tandis que l'équivalence des théories de cordes de type II sur ces deux classes de variété nécessite des configurations de champs B plats supplémentaires. Nous discutons en outre des connexions entre ces configurations de champs B plats et les configurations de champs Ramond-Ramond plats étudiées en [24].

Dans le cadre de la configuration non-perturbative des cordes, nous étudions la théorie (F) M compacte sur les (elliptiques) Calabi-Yau de dimension 3 (CY_3), dont la description de supergravité à basse énergie dépend de la structure géométrique interne. Nous étudions d'abord les cordes BPS dans les théories de supergravité, puis nous distinguons les "cordes de supergravité" du spectre des cordes BPS. En appliquant à la fois les contraintes géométriques issues de l'ellipsoïde de CY_3 et de la supergravité [82] et les contraintes d'unitarité proposées dans [92] à ces "cordes de supergravité", nous trouvons les relations entre les contraintes géométriques et physiques.

Compactification perturbative des cordes

Dans cette partie, nous nous intéresserons principalement aux théories conformes aux orbifolds et aux compactifications de cordes associées.

Une grande partie de la fascination exercée par la compactification des orbifolds sur Y/G provient de la présence de points singuliers, qui correspondent aux points de Y fixés par l'action de G [11, 12]. Malgré ces singularités, la théorie des cordes sur Y/G peut être parfaitement bien définie, et les calculs de diverses quantités du point de vue de la surface d'univers ont conduit à d'importantes découvertes en géométrie, comme la notion de nombres de Hodge "stringy" et de structures cohomologiques plus fines sur les orbifolds et les variétés algébriques (voir par exemple [13] pour une introduction).

L'étude des orbifolds a également permis de mieux comprendre l'espace des modules des compactifications de cordes, par exemple en montrant que certaines limites de la compactification sur des variétés lisses tels que K3 ou un Calabi-Yau 3-folds où la géométrie devient singulière peuvent néanmoins être entièrement comprises dans la théorie perturbative des cordes. En outre, les orb-

ifolds ont permis de comprendre des dualités non triviales telles que la symétrie miroir. Ils ont également été largement utilisés en phénoménologie des cordes, et il existe des résultats de classification impressionnants, tels que la description de tous les orbifolds symétriques du six-torus \mathbb{T}^6 qui préservent la supersymétrie de l'espace-temps [14].

Ces dernières années, les orbifolds des compactification toroïdales ont reçu un regain d'intérêt en relation avec le programme de "Swampland" ; voir, par exemple, [15–20]. Ce travail récent implique des orbifolds asymétriques qui agissent sur la géométrie toroïdale par décalages, tout en ayant une action non triviale sur le secteur de jauge. Les résultats fournissent un aperçu important des diverses composantes déconnectées dans l'espace modulaire de la compactification toroïdale qui généralisent la construction CHL [21, 22] et sont intimement liées à l'étude des compactifications hétérotiques avec des connexions plates non triviales [23, 24] et des constructions connexes de type II/M-théorie/F-théorie avec des flux RR d'arrière-plan et des singularités gelées [25, 26]. Il sera intéressant d'étendre ces efforts à des cas avec moins de supersymétrie spatio-temporelle, où la classification sera encore plus difficile, mais aussi la physique sera plus riche. Notre travail n'est qu'une petite indication de certaines des structures intéressantes qui émergent dans ce cadre plus large.

Dans cette partie, nous commençons notre étude avec des compactifications à 5 dimensions sur $X_5 = \mathbb{T}^5/G$ qui préservent la moitié de la supersymétrie de l'espace-temps, et où $G = \mathbb{Z}_N$ agit librement sur \mathbb{T}^5 .³ Pour obtenir une action libre et supersymétrique, nous écrivons $\mathbb{T}^5 = \mathbb{T}^4 \times S^1$, et choisissons G pour avoir une action supersymétrique sur \mathbb{T}^4 , de sorte que \mathbb{T}^4/G a une holonomie $\mathbb{Z}_N \subset \text{SU}(2)$. L'orbifold X_5 est néanmoins lisse car l'action sur \mathbb{T}^4 est combinée avec un décalage d'ordre N sur le 5-ième cercle. Malgré l'absence de points fixes, qui conduisent à une grande partie de la structure intéressante dans la compactification des orbifolds, ces exemples sont des géométries de cordes non triviales, et leur simplicité nous permet d'avoir un aperçu et de tirer des leçons qui devraient s'appliquer à des contextes plus généraux.

Comme ces espaces n'ont pas de points fixes, nous pouvons accéder à leur physique à la fois du point de vue de la supergravité et du point de vue des surface d'univers, ce qui nous permet de découvrir un certain nombre de leurs propriétés. Par exemple, nous discutons de la manière dont nous pouvons interpréter ces compactifications du point de vue d'un espace-temps à six dimensions : il est possible de considérer les théories comme des compactifiées S^1 d'une théorie à 6 dimensions, où une holonomie est activée pour une symétrie de jauge discrète \mathbb{Z}_N ; l'holonomie brise la moitié de la supersymétrie et modifie le spectre d'autres manières. Ce point de vue devrait être utile pour insérer ces compactifications dans le cadre de la F-théorie/type II — dualité hétérotique, par exemple selon les lignes explorées dans [28, 29].

Deuxièmement, nous montrons que dans la théorie hétérotique des cordes, ces compactifications sont duales aux compactifications plus familières sur $\mathbb{T}^4/G \times S^1$ et sont donc connectées à la compactification conventionnelle sur K3. L'idée que ce soit le cas a déjà été suggérée dans [30], et nous sommes capables d'établir cette affirmation au-delà d'une analyse du spectre sans masse. Nous notons également que la dualité échange des espaces avec des topologies différentes, fournissant ainsi un autre exemple de géométrie "stringy" reliant des géométries topologiquement distinctes.

L'équivalence que nous établissons est basée sur l'observation qu'il existe des actions orbitales de \mathbb{T}^5 sur la CFT à surfaces d'univers, $G_1 \simeq G$ et $G_2 \simeq G$ qui sont liées par une conjugaison : il existe un élément $t \in \text{O}(21, 5, \mathbb{Z})$ qui donne un isomorphisme $t : G_1 \rightarrow G_2$, avec $t : g \mapsto t^{-1}gt$. Alors que le quotient par G_1 conduit à la géométrie lisse X_5 , le quotient par G_2 conduit à $\mathbb{T}^4/G \times S^1$. L'isomorphisme induit sur les modules des deux CFTs agit comme une dualité T sur le rayon du S^1 non triviale en combinaison avec un décalage des paramètres de la ligne de Wilson par un vecteur de réseau.

³Comme l'indique le travail récent [27], si l'on est prêt à abandonner la supersymétrie de l'espace-temps, alors les variétés plates compactes deviennent déjà intéressantes dans les compactifications à 7 dimensions.

Il est alors naturel d'interpréter l'isomorphisme comme une T-dualité de la CFT X_5 , mais il y a une subtilité dans l'application du paradigme standard des règles de Buscher pour un modèle sigma non linéaire, malgré le fait que la X_5 peut être parfaitement bien décrite par un modèle sigma non linéaire à grand rayon. Le problème est que la structure de fibration du cercle naturel

$$\begin{array}{ccc} S^1 & \hookrightarrow & X_5 \\ & & \downarrow \\ N & & B \end{array}$$

à $B = \mathbb{T}^4/G$, c'est-à-dire que la base est singulière ! Nos résultats indiquent que la T-dualité s'étend à de telles fibrations singulières, avec l'avantage d'élargir les possibilités de changement de topologie via la T-dualité au-delà de celles discutées dans le contexte des faisceaux de torus principaux sur des variétés lisses, par exemple dans [31, 32].

Ensuite, nous considérons ces fonds dans le contexte de la compactification de la théorie des cordes de type II et trouvons que la compactification sur X_5 est T-dual à la compactification sur $\mathbb{T}^4/G \times S^1$ équipée d'un gerbe de champs B topologiquement non-trivial et plat.

Au niveau de la CFT orbifold, ceci illustre, selon les lignes décrites dans [33], que les données du gerbe donnent une signification géométrique naturelle à certaines phases orbifoldes de décalage. Un exemple de telles phases est un choix de torsion discrète [34, 35], mais celles-ci sont absentes dans tous nos exemples.

Compactification non perturbatrice des cordes

L'annulation de l'anomalie locale n'est pas la seule condition de cohérence que l'on peut imposer aux supergravités minimales à six dimensions. Un critère développé plus récemment est basé sur la conjecture de complétude du spectre des objets BPS chargés dans les théories de supergravité. En six dimensions, il existe des objets BPS de type corde qui ne peuvent pas être découplés de la gravité de manière cohérente. (nous suivrons [92, 93] et les appellerons "cordes de supergravité"), à condition que certaines conditions sur leurs charges soient satisfaites. Celles-ci seront expliquées dans la section 5.4. Ces cordes BPS supportent des théories supraconformes bidimensionnelles $(0, 4)$ sur la surface d'univers. sur le worldsheet, dont les charges centrales sont complètement fixées par l'anomalie intérieure. complètement fixées par l'annulation de l'anomalie de masse, c'est-à-dire les coefficients des différents couplages en termes GS. Elles se couplent aux champs de jauge dans le bulk pour le groupe de jauge $G = \prod_i G_i$, et donc l'unicité de la théorie du worldsheet exige que la charge centrale totale associée aux algèbres de courant de G_i ne soit pas plus grande que la charge centrale "left moving" :

$$\sum_i c(G_i) \leq c_L \quad (0.0.4)$$

Les conséquences de cette limite pour les théories à six dimensions ont été analysées dans [92]. En revanche les théories à cinq dimensions, même celles qui sont obtenues à partir d'une réduction du cercle, ont une manière différente de grouper l'information, et l'expression pour les charges centrales $(0, 4)$ est plutôt différente. Nous constatons que la vision à cinq dimensions des cordes de la supergravité est un peu plus pratique et conduit à des contraintes légèrement plus strictes. Il est intéressant de noter que pour les théories qui proviennent d'une réduction circulaire, les contraintes sont toujours associées aux cordes de supergravité à six dimensions (réduites) plutôt qu'aux cordes portant des charges KK. ⁴

⁴Sur la base du flux entrant d'anomalies et des contre-termes locaux dans la masse, nous pouvons constater une

Bien sûr, il est naturel de comparer toute théorie ostensiblement cohérente minimalement supersymétrique en six dimensions aux constructions de la F-théorie. En plus des exigences imposées par les considérations physiques, celles-ci sont soumises à des contraintes supplémentaires qui sont associées à la géométrie des fibrations elliptiques. Ces contraintes peuvent être formulées soit en termes de données de la théorie effective, soit en langage géométrique. Dans l'image de la théorie F, les groupes de jauge non abéliens G_i proviennent de D7-branes enveloppant des diviseurs de jauge singuliers S_i dans la variété de base B . La condition CY, c'est-à-dire la trivialité du faisceau canonique de la fibration elliptique, relie K , le diviseur canonique de B , au locus des fibres singulières. En outre, la condition de positivité de Kodaira (KPC) stipule qu'un diviseur résiduel $Y = -12K - \sum_i x_i S_i$ doit être effectif. Les coefficients x_i sont donnés par l'ordre d'annulation du discriminant sur S_i . Celles-ci peuvent être trouvées dans le tableau 1 de la section 6.3. En particulier, cela implique la non-négativité de l'intersection

$$D \cdot (-12K - \sum_i x_i S_i) = D \cdot Y \geq 0, \quad (0.0.5)$$

pour tout nef diviseur D (les nef diviseurs, par définition, intersectent chaque diviseur effectif de manière non négative). Les cordes de supergravité dans les modèles de F-théorie proviennent de D3-branes enveloppant D .

Les conditions d'unitarité (UC) sont formulées directement en termes de données de la théorie effective. En supposant qu'il existe un CY_3 elliptique sous-jacent, les UC à cinq dimensions que nous dérivons ici peuvent être géométrisées, et exprimées comme une limite sur les formes d'intersection avec n'importe quel nef diviseur D . Elle peut également être reformulée comme une contrainte supplémentaire sur le diviseur résiduel Y :

$$D \cdot Y \geq 3 - \sum_i (x_i - y_i) D \cdot S_i, \quad (0.0.6)$$

où $y_i = \frac{\dim G_i}{1+h_i^Y}$ et h_i^Y est le nombre de Coxeter dual de G_i . Les valeurs comparées de x_i et y_i (comme on le verra pour tout groupe G_i , $x_i - y_i \geq 1$) et les détails de l'analyse de la condition (0.0.6) peuvent être trouvés dans la section 6.3.2. Un mot d'avertissement s'impose. Il s'agit de la forme la plus forte de la contrainte d'unitarité, où la valeur des coefficients y_i a été calculée en supposant que $D \cdot S_i = 1$ existe. Dans la grande majorité des cas, cette contrainte est automatiquement satisfaite si (0.0.5) tient. Si elle est violée, la validité de $D \cdot S_i = 1$ doit être vérifiée avant de conclure que UC impose effectivement des contraintes supplémentaires sur le diviseur résiduel Y . Dans la section 6.3.3.1, nous répertorions tous les cas où l'UC impose des contraintes supplémentaires.

Remarquez qu'un exemple où les implications de l'UC à six dimensions étaient plus fortes que celles imposées par KPC a déjà été présenté dans [92]. Nous constatons que, dans des situations génériques, les UC 5d sont plus contraignantes que les UC 6d et ont l'avantage d'être formulées dans une forme directement comparable à la KPC. En général, elle est moins contraignante que la KPC, et peuvent donc être utile pour délimiter. Les limites de la région entre les modèles de la théorie F à six dimensions et le "Swampland" (qui est susceptible de contenir un nombre fini de théories). théories [94]). Le fait qu'elle impose, dans des situations particulières, des contraintes additionnelles, permet d'envisager la possibilité de trouver une structure plus raffinée dans les CY_3 elliptiques avec certaines structures de singularités. Les deux cas mériteraient une étude plus approfondie.

discordance du spectre des cordes BPS dans les supergravités 6d et 5d, comme discuté dans la section 5.1. Comme nous le verrons, cela s'explique par remarquant que les cordes BPS 5d, portant des charges KK, sont élevées à un certain fond géométrique (espace Taub-NUT) qui préserve la moitié de la supersymétrie plutôt que la moitié de la supersymétrie. la moitié de la supersymétrie plutôt que les cordes BPS dans la supergravité 6d.

Structure

Cette thèse est divisée en trois parties et s'articule comme suit:

La première partie est composée de deux chapitres. Dans le premier chapitre, nous donnons une brève introduction à la formulation perturbative de la théorie des cordes, nous discutons les 5 théories des supercordes les plus courantes et nous abordons leurs propriétés. Nous soulignons que ces 5 différentes théories des supercordes sont en fait étroitement liées les unes aux autres par le réseau de dualités. Dans le deuxième chapitre, nous introduisons deux formulations non-perturbatives de la théorie des cordes, la théorie M et la théorie F. Nous introduisons quelques méthodes et idées à travers des exemples dans le cadre des théories M et F qui seront appliquées et explorées dans la seconde et troisième partie, par exemple l'influx d'anomalie, la corde MSW, la singularité gelée, etc.

La deuxième partie étudie la compactification perturbative des cordes. Nous introduisons d'abord deux classes de variété plates à cinq dimensions (orbifolds) qui préserveaient la demi-super-symétrie. Nous étudions ensuite les compactifications de cordes hétérotiques et de type II sur ces deux classes de variété et discutons de leurs connexions. Nous avons trouvé que les compacités de cordes hétérotiques sur ces deux classes de variétés sont équivalentes l'une à l'autre grâce à la dualité T. Cette équivalence dans les compacités de cordes de type II nécessite une donnée supplémentaire, la configuration non triviale du champ B plat. Nous soulignons que la configuration plate non triviale du champ B peut fournir une interprétation perturbative de la singularité gelée.

La dernière partie se concentre sur la compactification non-perturbative des cordes, c'est-à-dire la compactification de la théorie (F) M sur CY_3 (elliptique). La théorie F compacte sur CY_3 elliptique donne un coucher de soleil des familles infinies de théories de supergravité 6d sans anomalie avec 8 supercharges. Certaines conditions de cohérence basées sur des objets étendus de ces théories de supergravité (cordes BPS) ont été proposées pour exclure les familles infinies de théories de supergravité 6d [92, 94]. Nous examinons ces conditions dans le cadre de la compactification de la théorie F et soulignons les relations entre cette condition de cohérence et une condition géométrique du CY_3 elliptique.

Notes ajoutées

Cette thèse est basée sur deux articles [115, 144]. La partie I est une revue du contexte et des techniques pertinentes.

La partie II est basée sur [144]. Dans cet article, nous avons montré que l'espace mouddi des cordes hétérotiques compactées sur deux géométries différentes sont reliées entre elles par la dualité T, vérifie les observations à basse énergie faites par [30]. Nous avons également étudié les T-dualités pertinentes dans le cadre du type IIA et démontré que leurs T-duels sont des gerbes équivariantes en construisant les actions de groupe précises sur le fond de gerbe de l'espace-temps. Les applications aux singularités gelées ("frozen singularities") et aux espaces de correspondance sont discutées.

La partie III est basée sur [115]. Dans cet article, nous avons étudié les anomalies perturbatives dans la théorie de dimension impaire à partir de la compactification circulaire d'une théorie anormale de dimension paire. De plus, nous avons également étudié le spectre des cordes BPS dans la supergravité 5d/6d avec 8 supercharges et suggéré une façon de les faire correspondre dans le contexte de la compactification circulaire. Nous avons également montré la relation entre la condition de Kodaira [82] et la condition d'unitarité [92] en compactant la supergravité 6d sur un cercle.

Part I

Some perturbative and non-perturbative aspects of string theory

Chapter 1

Perturbative string theory

In this chapter we provide some elementary discussion of the formulation of perturbative string theory, i.e. 2d conformal field theories (CFTs) describe strings propagate in target spacetime. We start with general features of 2d CFT and then briefly introduce the CFTs that give rise to 5 superstring theories. We end this section by pointing out these 5 string theories are connected to each other by duality web.

1.1 2d CFT

1.1.1 Conformal symmetry in 2 dimensions

Generically quantum field theory (QFT) is hard to study roughly due to its infinite degrees of freedom. Here is one easiest example:

Example: ϕ^4 theory on $\mathbb{R}^{3,1}$ Consider a real scalar field theory on Minkowski space time $\mathbb{R}^{3,1}$, its lagrangian is:

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 - \frac{1}{4!} \lambda \phi^4 \quad (1.1.1)$$

canonical quantization tells us for every $x, y \in \mathbb{R}^3$ we have:

$$[\phi(x, t), \dot{\phi}(y, t)] = i \delta^3(x - y) \quad (1.1.2)$$

In other words, every point the space will be associated with a dynamical degree of freedom, hence infinite overall.

Despite general difficulties one may encounter in studying QFT, many times symmetries help us derive non-trivial conclusions. Here we are going to discuss one special kind of symmetry that emerged at the end of many renormalization group flows (RG flow): conformal symmetry.

Conformal symmetry Conformal symmetry acts on the manifold (M, g) (g is the metric on the manifold M) that QFT is formulated on, it acts in the following way:

$$\rho : (M, g) \rightarrow (M, \rho^* g) = (M, e^{2\sigma} g) \quad (1.1.3)$$

where σ is a function on M . Locally this can be expressed as:

$$g'_{\rho\sigma}(x') \frac{\partial x'^\rho}{\partial x^\mu} \frac{\partial x'^\sigma}{\partial x^v} = e^{2\sigma(x)} g_{\mu\nu}(x), x' = \rho(x) \quad (1.1.4)$$

Now put $M = \mathbb{R}^d$ and g is chosen to be the standard Euclidean metric g_E . For an infinitesimal conformal transformation $x' = x + \epsilon(x)$, we have:

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = \frac{2}{d} (\partial \cdot \epsilon) g_{E\mu\nu} \quad (1.1.5)$$

Hit both sides by ∂_ν and sum over index ν , we get:

$$(g_{E\mu\nu} \square + (d-2) \partial_\mu \partial_\nu) (\partial \cdot \epsilon) = 0 \quad (1.1.6)$$

From this, we see conformal symmetry in 2 dimension is special [1]. 1.1.6 tells us that the conformal algebra on space $\mathbb{R}^{p,q}$ with $p+q \geq 2$ is a finite dimensional lie algebra $\mathfrak{so}(p+1, q+1)$. For conformal algebra of \mathbb{R}^2 , it is convenient to treat it as \mathbb{C} by setting $z = x^1 + ix^2$. Then the solution of 1.1.6 is:

$$\epsilon^1(x, y) + i\epsilon^2(x, y) = \epsilon(z) \quad (1.1.7)$$

$$\epsilon(z) = \sum_{n \in \mathbb{Z}} \epsilon_n (-z^{n+1}), \bar{\epsilon}(\bar{z}) = \sum_{n \in \mathbb{Z}} \bar{\epsilon}_n (-\bar{z}^{n+1}) \quad (1.1.8)$$

So the generators of $2d$ conformal algebra are: $l_n = -z^{n+1} \partial_z$ and $\bar{l}_n = -\bar{z}^{n+1} \partial_{\bar{z}}$. This algebra is infinite dimensional and is the double copies (holomorphic and anti-holomorphic) of Witt algebra:

Witt algebra and $2d$ conformal algebra

- Witt algebra:

$$\begin{aligned} [l_m, l_n] &= z^{m+1} \partial_z (z^{n+1} \partial_z) - z^{n+1} \partial_z (z^{m+1} \partial_z) \\ &= (n+1) z^{m+n+1} \partial_z - (m+1) z^{m+n+1} \partial_z \\ &= -(m-n) z^{m+n+1} \partial_z \\ &= (m-n) l_{m+n} \end{aligned} \quad (1.1.9)$$

- 2d conformal algebra:

$$\begin{aligned} [l_m, l_n] &= (m-n) l_{m+n} \\ [\bar{l}_m, \bar{l}_n] &= (m-n) \bar{l}_{m+n} \\ [l_m, \bar{l}_n] &= 0 \end{aligned} \quad (1.1.10)$$

The infinite dimensional nature makes 2d CFT can be studied algebraically, i.e. study the representations of this symmetry algebra (we only consider unitarity representations).

In quantum physics, symmetry group is realized on Hilbert space projectively. Translates into algebra level, the relevant symmetry algebra should be the central extension of Witt algebra.

Central extension of Witt algebra: Virasoro algebra The central extension of an Lie algebra \mathfrak{g} is classified by $H^2(\mathfrak{g}, \mathbb{C})$. For Witt algebra $H^2(\text{Witt}, \mathbb{C}) = \mathbb{C}$, which means there is only one central extension, the Virasoro algebra:

$$[L_m, L_n] = (m-n) L_{m+n} + \delta_{m,-n} \frac{\hat{c}}{12} (m^3 - m) \quad (1.1.11)$$

Here \hat{c} is a generator in the Virasoro algebra, it gives a complex number when it acts on the modules of Virasoro algebra.

We end this subsection by addressing a few points:

- Virasoro algebra is the minimal symmetry algebra for 2d CFTs. Generally 2d CFTs could have a larger symmetry algebra, which the Virasoro algebra is embedded into, e.g. current algebra, superconformal algebra etc...
- The eigenvalues of \hat{c} on physical Hilbert space of 2d CFTs are constrained by certain properties of 2d CFTs. For example, unitarity would require the eigenvalues of \hat{c} is real and non-negative.
- To define a good 2d CFT, Virasoro algebra itself is not enough, we also need to impose modularity.
- For CFTs, there is a nice one to one correspondence between local operators and physical states. This is not true in general QFTs.

1.2 String theory CFTs

String theory requires the central charge of the underlie 2d CFT should be 26 for bosonic case and 15 for fermionic case. In this section we briefly explain the reason [10].

1.2.1 2d CFT for bonsonic strings

Bosonic string theory¹ in Polyakov formulation requires to sum over surfaces:

$$\sum_{g=0}^{\infty} \int Dg_{ab} DX_{\mu} \exp \{-S[X_{\mu}, g_{ab}]\} \quad (1.2.1)$$

From above equation 1.2.1, the path integral in string theory needs to sum over both topological and analytical structures on the Riemann surfaces. Frist, to get rid of the components in Dg_{ab} that come from diffeomorphisms² and Weyl transformation of the metric, i.e.

$$Dg_{ab} = JD\sigma D\delta v Dg_{ab}^{\perp} \quad (1.2.2)$$

Here σ is the Weyl transformation $\delta g_{ab} = \sigma g_{ab}$, δv is the diffeomorphism transformation $\delta g_{ab} = \nabla_a v_b + \nabla_a v_b$ ³. Dg_{ab}^{\perp} is the physical one⁴, it is a $3h - 3$ dimensional complex space \mathcal{M}_h . J is the Jacobian.

To analyze the Jacobian, we choose a complex structure and write the metric in the following form:

$$g_{ab} dx^a dx^b = \rho(z, \bar{z}) dz d\bar{z} \quad (1.2.3)$$

Now we introduce an infinite series of holomorphic bundles equipped with Hermitian metric by tensoring the canonical bundle $K^{\otimes n}$ ($n \in \mathbb{Z}$), as well as the maps between them:

$$\begin{cases} \nabla_{(m)}^z : K^m \rightarrow K^{m-1} & \nabla_{(m)}^z \phi = (g_{z\bar{z}})^{-1} \frac{\partial}{\partial \bar{z}} \phi \otimes (dz)^{-1} \\ \nabla_{(m)}^{\bar{z}} : K^m \rightarrow K^{m+1} & \nabla_{(m)}^{\bar{z}} \phi = (g_{z\bar{z}})^m \frac{\partial}{\partial z} (g_{z\bar{z}})^{-m} \phi \otimes dz \end{cases} \quad (1.2.4)$$

Now we define Laplacian:

$$\Delta_{(m)} = -2\nabla_z^{(m-1)} \nabla_{(m)}^z \quad (1.2.5)$$

and introduce the following objects:

$$Z_{(n)}(g) \equiv \frac{\text{Det}' \Delta_{(n)}}{\det(\phi_j, \phi_k)_g \det(\psi_a, \psi_b)_g} \quad (1.2.6)$$

¹For simplicity, the target spacetime is $\mathbb{R}^{1,d-1}$ with Minkowski metric.

²Here we limit ot Reimann surface with genus $h \geq 2$.

³ ∇ is the LC connection of the metric g_{ab} .

⁴modulo mapping class group, which gives us the moduli of curves \mathcal{M}_h

where $\phi_j \in Ker \nabla_z^{(n-1)}$, $\psi_a \in Ker \nabla_z^{(n)}$, $(,)$ is the standard Hermitian metric ⁵.
With above defined objects, eq 1.2.2 is:

$$Dg_{ab} = Z_{-1}(g) D\sigma D\delta v dy^i d\bar{y}^i, i = 1, \dots, 3h - 3 \quad (1.2.7)$$

where y^i 's are complex coordinates of \mathcal{M}_h .

The matter part of the string CFT in path inetgral gives

$$\int DX_\mu \exp \{-S[X_\mu, g_{ab}]\} = Z_0(g)^{\frac{-d}{2}} det N_1^{\frac{d}{2}} \quad (1.2.8)$$

where:

$$det N_1 = det(\psi_a, \psi_b), \psi_{a,b} \in Ker \nabla_1^z \quad (1.2.9)$$

Using eq 1.2.7,1.2.8, we have

$$\int Dg_{ab} DX_\mu \exp \{-S[X_\mu, g_{ab}]\} = \int_{\mathcal{M}_h} dy^i d\bar{y}^i \left(\int D\sigma D\delta v \right) Z_{-1}(g) Z_0(g)^{\frac{-d}{2}} det N_1^{\frac{d}{2}} \quad (1.2.10)$$

Now $\int D\sigma D\delta v$ cannot be dropped consistently as $Z_{-1,0}$ depend on the Weyl scaling $D\sigma$. So only when the combination $Z_{-1}(g) Z_0(g)^{\frac{-d}{2}}$ is independent of Weyl scaling can $\int D\sigma D\delta v$ be dropped (note $det N_1$ doesn't depend on the Weyl scaling parameter). Heat kernel method gives:

$$\delta_\sigma \log Z_j(g) \propto C(j) \int \delta\sigma \rho^{-1} \partial \bar{\partial} \log \rho d^2 x^i, C(j) = 6j^2 - 6j + 1 \quad (1.2.11)$$

So

$$\delta_\sigma \log Z_{-1}(g) - \frac{d}{2} \delta_\sigma \log Z_0(g) = 0 \rightarrow C(-1) - \frac{d}{2} = 0 \quad (1.2.12)$$

Hence, the consistency of bosonic string requires 26 spacetime dimensions.

For superstrings. The same argument gives consistency condition:

$$C(-1) - C\left(-\frac{1}{2}\right) - \frac{d}{2} \left(C(0) - C\left(\frac{1}{2}\right) \right) = 0 \quad (1.2.13)$$

which gives $d=10$, consistent superstring theories require a 10 diemsnisonal spacetime.

1.2.2 $C(n)$ and Mumford isomorphsim

In above discussions the function $C(n) = 6n^2 - 6n + 1$ plays an important role. In this section ,we discuss another interpretation of this function [2].

To start with, we need one fact that the $Z_{-1} Z_0^{-13} = F(y) \bar{F}(\bar{y})$ admits a holomorphic-antiholomorphic factorization [3]⁶. $F(y)$ is a section of the holomorphic determinant line bundle $det \bar{\partial}_{-1} \otimes det \bar{\partial}_0^{-13}$ over \mathcal{M}_h . Then $det \bar{\partial}_{-1} \otimes det \bar{\partial}_0^{-13}$ should be nonvanishing over the moduli space. Actually this is Mumford theorem:

Mumford Isomorphism Consider the line bundles $\mathcal{L}_n = \wedge^{dim_{\mathbb{C}} H^0(\Sigma_h, K^n)} \omega^n$ where ω^n are the holomorphic sections of K^n over genus h Riemann surface Σ_h ⁷. We have the line bundle isomorphism:

$$\mathcal{L}_n \simeq \mathcal{L}_1^{C(n)} = \mathcal{L}_1^{6n^2 - 6n + 1} \quad (1.2.14)$$

⁵For the sections $\phi_{i,j}$ of K^m , $(\phi_i, \phi_j) = \int \sqrt{g} d^2 x (g^{z\bar{z}})^m \bar{\phi}_i(x)_{\bar{z}, \bar{z}, \dots} \phi_j(x)_{z, z, \dots}$

⁶Only the combination $Z_{-1} Z_0^{-13}$ rather than each single one enjoys such a property

⁷ ω^n 's give $H^0(\Sigma_h, K^n)$ complex dimension vector space for every Σ_h with a complex structure, i.e. every point of moduli \mathcal{M}_h . Therefore they give a holomorphic $H^0(\Sigma_h, K^n)$ complex dimension vector bundle over $H^0(\Sigma_h, K^n)$ from which we can construct $\mathcal{L}_n = \wedge^{dim_{\mathbb{C}} H^0(\Sigma_h, K^n)} \omega^n$.

Using this theorem and identify $\det\bar{\partial}_{-1}$ as \mathcal{L}_2 , $\det\bar{\partial}_0$ as \mathcal{L}_1 , we indeed see that the line bundle $\det\bar{\partial}_{-1} \otimes \det\bar{\partial}_0^{-13} \simeq \mathcal{L}_2 \otimes \mathcal{L}_1^{-13} \simeq \mathcal{L}_1^{13} \otimes \mathcal{L}_1^{-13}$ is trivializable, i.e. it admits a nowhere vanishing holomorphic section. This provides another interpretation of $C(n)$ appeared above.

1.3 Superstring CFTs and their connections

In this section we summarize the 5 superstring theories and their connections, all the details can be found in e.g. [4]

1.3.1 Five different superstring CFTs

Type II strings Type II CFTs are left-right symmetric CFTs. Some of their features are the following:

- There are 8 bosons and 8 Majorana-Weyl fermions on the worldsheet⁸⁹ both for left and right moving part. For the bosonic part, each boson contribute to the vacuum energy $-\frac{1}{24}$. For the fermionic part, we have two sectors, the Ramond (R) sector sets all eight fermions to be periodic ($\psi(\sigma^1 + 2\pi) = \psi(\sigma)$) while Neveu Schwarz (NS) sector sets all eight fermions to be anti-periodic ($\psi(\sigma^1 + 2\pi) = -\psi(\sigma)$). Each R sector fermion contribute $\frac{1}{24}$ while each NS sector contribute $-\frac{1}{48}$. Combine all bosons and fermions, the Hamiltonians of left-moving and right-moving worldsheet CFT are:

$$R - \text{sector} : \mathcal{H}_L = L_0, \mathcal{H}_R = \bar{L}_0 \quad (1.3.1)$$

$$NS - \text{sector} : \mathcal{H}_L = L_0 - \frac{1}{2}, \mathcal{H}_R = \bar{L}_0 - \frac{1}{2} \quad (1.3.2)$$

- R-sector has zero-mode fermions $\psi_0^i (i=1,2,\dots,8)$ due to their periodicity, quantization of them give spacetime fermions:

$$\{\psi_0^i, \psi_0^j\} = \delta^{ij} \quad (1.3.3)$$

This equation 1.3.3 means ψ_0^i s span the Clifford algebra $\text{Cl}(\mathbb{R}^8)$. Hence the quantization gives physical states of a spacetime massless Dirac fermion, i.e. a pair of spacetime Weyl fermions with different chirality. Similarly for the right-moving part.

NS-sector has half-integer mode fermions due to their antiperiodicity, their quantization gives:

$$\{\psi_{n+\frac{1}{2}}^i, \psi_{m-\frac{1}{2}}^j\} = \delta^{n,-m}, n, m \in \mathbb{Z} \quad (1.3.4)$$

From their quantizations we see the NS-sector will give spacetime bosons. As a result, Type II CFT possesses 4 sectors: (Left moving, Right moving)=(NS,NS),(NS,R)(R,NS)(R,R), where (NS,NS),(R,R) sectors give spacetime bosons and (NS,R)(R,NS) sectors give spacetime fermions.

- We further need to impose the GSO projection $\frac{1+\exp(i\pi F)}{2}$ (F is the fermionic number operator) to get rid of the tachyon and achieve spacetime supersymmetry. This is also necessary from the 2d worldsheet CFT point of view, as it is required by mutual locality of the vertex operators. There are two different GSO projections, the type IIB one is left-right symmetric and gives chiral¹⁰ massless spectrum whose low energy limit is type IIB supergravity, while

⁸We choose the light cone gauge

⁹Here we consider the worldsheet to be a cylinder, i.e. spatial dimension $\sigma^1 \sim \sigma^1 + 2\pi$

¹⁰From spacetime point of view

the type IIA one is left-right asymmetric and gives non-chiral massless spectrum whose low energy limit is type IIA supergravity.

- GSO projection projects out the NS-sector vacuum state $|0\rangle_{NS}$ to get rid of the tachyon and requires the physical states in NS-sector has odd number of fermions. As each fermionic mode in NS sector contributes at least $\frac{1}{2}$ to the energy, together with the fact that in NS sector the Hamiltonian is $L_0 - \frac{1}{2}$. GSO projection requires the massless spectrum can only be created by fermionic modes $\psi_{-\frac{1}{2}}^i |0\rangle_{NS}$. Therefore, GSO projection rules out the enhanced non-abelian symmetry at T-dual fixed point of the moduli space as it would be in pure bosonic CFTs.
- Based on the spectrum after GSO projection and the fact that D-branes carry Ramond-Romand charges, we see that Dp branes exist in IIA/B if and only if p is even/odd. In addition to Dp branes, the soliton spectrum of IIA/B also consist of $NS5$ branes and fundamental strings, which are the carrier of magnetic and electric charges of two form B field sits in the NS sector.

Type I strings Type IIB is left-right symmetric even after GSO projection, so it has a parity symmetry Ω on the worldsheet. After gauging this symmetry we get the following features of Type I superstring:

- It is an open string theory, which has 16 spacetime supersymmetry.
- Only D1,D5,D9 could appear in its solitonic spectrum as others are being projected out.
- Tadpole cancellation (vanishing of the amplitudes by summing over Cylinder, Klein bottle and Möbius strip) requires 16 D9 branes on top of O_9^- orientifold plane fill out the spacetime, which gives $SO(32)$ gauge symmetry in the 10d spacetime.

Heterotic strings Heterotic string is a mixture between bosonic string theory and superstring theory. Based on the fact the left-moving and right-moving parts of the worldsheet CFT are decoupled, we can put different theories on left and right moving part of the CFT. Heterotic strings is obtained by putting bosonic string CFT (central charge of the matter part $c_L = 26$) on the left moving part and superstring CFT (central charge of the matter part $c_R = 15$) on the right. The target spacetime is 10d and the left moving part of Heterotic string has an internal CFT with central charge $c_L^{int} = 16$. Here are some of the properties of the Heterotic strings:

- Given modularity and central charge constraints, the internal left moving part can be described by a CFT of 32 Majorana Weyl fermions, or a bosonic CFT attached to rank-16 even self dual Euclidean lattices. These two are equivalent by bosonization and we will use the bosonic description.
- There are only 2 rank-16 even self dual lattices: $Spin(32)/\mathbb{Z}_2$ and $\Gamma_{E_8} \oplus \Gamma_{E_8}$. Hence there are 2 different Heterotic strings.
- Since left moving part is pure bosonic string, the left moving Hamiltonian is:

$$\mathcal{H}_L = L_0 - 1 = N_L + \frac{1}{2} p_L \cdot p_L + \frac{1}{2} p_i \cdot p_i \quad (1.3.5)$$

where p_i is the spacetime momentum and p_L is a point in the internal lattice. Here we see that every root vector (length 2) in the lattice gives a spacetime massless state. Combining with right moving sector, they give spacetime gauge vector bosons and their superpartners (the spacetime supercharges are the same as in Type I) associated with gauge algebra $e_8 \oplus e_8$ or $so(32)$.

- Since there is no R-sector in the left moving, there are no Dp branes sit in the solitonic spectrum of heterotic strings. The solitonic objects in heterotic strings are fundamental strings and NS5 branes.
- One very important property is that due to the chirality of Heterotic worldsheet CFT, there will be 't Hooft anomaly in the 2d worldsheet CFT. As the global symmetry on the worldsheet will be gauge symmetry from the spacetime point of view, such 't Hooft anomaly will destroy the consistency of superstring theory. The way to resolve this problem is called Green Schwarz mechanism, we recall them both from worldsheet and spacetime point of view:

't Hooft Anomaly cancellation on the world-sheet The 't Hooft anomaly polynomial on the worldsheet consists of two part. The right moving fermions has $SO(8)$ flavor symmetry (they carry spacetime index) and the pull back of spacetime tangent bundle connection is the background gauge field for $SO(8)$ on the worldsheet. The related 't Hooft anomaly is:

$$I_4^{Right} = \frac{1}{4}tr(R_{SO(8)}^2) \quad (1.3.6)$$

where $R_{SO(8)}$ is the spacetime curvature.

For the left moving, there is a level-1 $e_8 \oplus e_8$ or $so(8)$ current algebra and the anomaly attached is:

$$I_4^{Left} = -\frac{1}{4h\nu}Tr_{adj}(F^2) = -\frac{1}{120}Tr_{adj}(F^2) \quad (1.3.7)$$

where F the the field strength for spacetime gauge connection. So overall the anomaly is:

$$I_4 = \frac{1}{4}tr(R_{SO(8)}^2) - \frac{1}{120}Tr(F^2) \quad (1.3.8)$$

Under the spacetime gauge and diffeomorphism transformation parametrized by Λ, ϵ , the 't Hooft anomaly gives a term in the Lagrangian:

$$I_2^2 = \frac{1}{4}tr(\Lambda R_{SO(8)}) - \frac{1}{120}Tr_{adj}(\epsilon F) \quad (1.3.9)$$

Note on the worldsheet CFT, there is a topological term $\int \phi^* B$ ($\phi^* B$ is the pull back from spacetime 2-form B field). To cancel this term and make the theory anomaly free, we need to give a non-trivial transformation to the B :

$$\delta B = -I_2^2 = -\frac{1}{4}tr(\Lambda R_{SO(8)}) + \frac{1}{120}Tr_{adj}(\epsilon F) \quad (1.3.10)$$

After this transformation, we see the worldsheet CFT is free of 't Hooft anomaly.

Gauge and gravitational anomaly cancellation in the Spacetime Here we go to the low energy supergravity limit, the related anomaly is:

$$I_{12} = (\frac{1}{4}tr(R_{SO(8)}^2) - \frac{1}{120}Tr_{adj}(F^2)) \wedge X_8 \quad (1.3.11)$$

where X_8 is an eight form constructed from R, F . To cancel this anomaly we need to use one term in the supergravity lagrangian $B \wedge X_8$ and give the following transformation law for B field:

$$\delta B = -\frac{1}{4}tr(\Lambda R_{SO(8)}) + \frac{1}{120}Tr_{adj}(\epsilon F) \quad (1.3.12)$$

Now the anomaly is cancelled and the supergravity is consistent, but the transformation 1.3.12 gives non-trivial Bianchi identity for H -flux:

$$dH = -\frac{1}{4}tr(R_{SO(8)}^2) + \frac{1}{120}Tr_{adj}(F^2) \quad (1.3.13)$$

This Bianchi identity will impose some tadpole cancellation conditions when studying some Heterotic compactification cases.

- Thanks to the fact that the left moving part of Heterotic string is purely bosonic case, the phenomena of enhanced non-abelian symmetry at T-dual fixed point could appear, as in bosonic string theory.

1.3.2 Dualities between the five superstring theories

In this section we discuss one of the most amazing features in superstring theories: all the five superstring theories are related to each other by dualities.

- Type IIA/B are T-dual to each other on a circle.
- Heterotic $Spin(32)/\mathbb{Z}_2$ and $E_8 \times E_8$ are T-dual to each other on a circle.
- Type I string theory is S-dual to Heterotic string theory
- Heterotic string theory on T^4 is dual to Type IIA on $K3$
- Heterotic string on T^2 is dual to type IIB on certain non-perturbative background, i.e. F-theory on elliptic K3 manifolds.
- etc...

To discuss the duality web would require studying string theory at non-perturbative level, which is the topic of next chapter.

Chapter 2

Non-perturbative string theory

String theory coupling g_s is not an input data, instead it is determined by the dynamics of string theory itself as it is characterized by the expectation value of $g_s = \langle \exp(\phi) \rangle$ where ϕ is a dynamical field of string theory. When $g_s \ll 1$, string perturbation theory applied. When $g_s \gg 1$, i.e. strong coupling regime of string theory, non-perturbative effects become important and we need new methods to study them. One way to study them is using BPS objects¹. In string theory, the common BPS objects are Dp branes, $NS5$ branes and fundamental strings $F1$ etc.

The tension of Dp branes have the following nice properties:

$$\tau_p \propto \frac{1}{g_s} \tag{2.0.1}$$

which means that when $g_s \rightarrow \infty$, $\tau_p \rightarrow 0$, the Dp branes become massless hence their dynamics are not frozen out and become important even at IR limit. The physical understanding of these new dynamical degrees of freedom will be one of the topics of this chapter.

2.1 M-theory

11d M-theory originally appeared as the strong coupling limit of 10d type IIA string theory [6].

2.1.1 11d M theory from 10d type IIA string theory

Consider $D0$ particles in type IIA, the mass of $D0$ particle carrying charge n under RR 1-form field is:

$$M_{D0,n} \propto \frac{|n|}{g_s} \tag{2.1.1}$$

When $g_s \rightarrow \infty$, the entire tower of such $D0$ particles go massless.

Now, recall standard Kaluza Klein compactification of a real scalar field on a circle with radius r gives a tower of scalar fields with mass $\frac{2\pi|n|}{r}$. When $r \rightarrow \infty$, a similar situation appears: a tower of scalars go massless. This example suggests the physical interpretation of above case should be a hidden dimension (with circle radius $r \propto g_s$) opening up in strong coupling limit of 10 dimensional IIA string theory: 11 dimensional M theory.

M-theory has the following basic features:

- The low energy description is 11d supergravity, the bosonic field contents are $\phi, g_{\mu\nu}, C_{\mu\nu\rho}$ where $C_{\mu\nu\rho}$ is a three-form field. Compactified on a circle gives type IIA supergravity.

¹In supersymmetric unitary theories, BPS states are the lightest states given fixed quantum number, hence they are stable generically.

- The solitonic spectrum consists of M2 and M5 branes. Put M theory on a circle, M2 brane wraps on the circle gives type IIA elementary string, M2 brane does not have a leg on the circle gives $D2$ brane. Similarly $D4$ and $NS5$ branes can be obtained from $M5$ branes. $D0$ branes are the Kaluza Klein modes from circle compactification. $D6$ branes are lifted to Taub-NUT background in M theory

2.1.2 M5 brane and anomaly inflow

In this section, we give one example of M5 branes and its related anomaly cancellations which will be used later [5, 99, 100].

M5 brane is a BPS soliton in the low energy limit of M theory, i.e. 11d supergravity. The low energy degree of freedom on the M5 brane is a $6d$ $(2, 0)$ tensor multiplet whose field contents are:

- 1 anti self-dual 2-form tensor field $B_{\mu\nu}^-$.
- 5 scalars ϕ . They transform as vector under the $spin(5)$ R-symmetry of the $6d$ $(2, 0)$ super-symmetry.
- 4 Majorana Weyl fermions with negative chirality $4 \psi^-$ transform as $\mathbf{4}$ under the R symmetry $spin(5)$.

The R-symmetry $spin(5)$ comes from the structure group of the normal bundle $SO(5)$. The gravitational and R-symmetry anomaly² for this multiplet is

$$I_8 = -\frac{1}{48}p_2(N) + \frac{1}{96}p_1(N)p_1(TW^6) - \frac{1}{192}(p_1^2(TW^6) + p_1^2(N) - 4p_2(TW^6)) \quad (2.1.2)$$

Here TW^6 is the tangent bundle of the $M5$ brane worldvolume, corresponds to gravitational anomaly for the worldvolume low energy theory. N is the normal bundle of the $M5$ brane worldvolume and signals $spin(5)$ R symmetry anomaly.

In the bulk it is a 11d supergravity with topological coupling:

$$\mathcal{L}_{11d-sugra} = C_3 \wedge X_8 - \frac{1}{6}C \wedge dC \wedge dC \quad (2.1.3)$$

with

$$X_8 = -\frac{1}{48} \left(\frac{p_1(TW^6)^2 + p_1(N)^2 - 2p_1(TW^6)p_1(N)}{4} - p_2(TW^6) - p_2(N) \right) \quad (2.1.4)$$

In the bulk the $M5$ brane is a δ -source for the bianchi identity:

$$dG_4 = \delta_5 (W^6 \hookrightarrow M^{11}) \quad (2.1.5)$$

As a result, C_3 is not well defined with the presence of the $M5$ brane in the bulk and the right way to write the first term in 2.1.3 is:

$$C_3 \wedge X_8 = C_3 \wedge dX_7^{(0)} = G_4 \wedge X_7^{(0)} \quad (2.1.6)$$

But $G_4 \wedge X_7^{(0)}$ is not invariant under the local diffeomorphism transformation in the bulk, instead it picks up a piece:

$$G_4 \wedge \delta X_7^{(0)} = G_4 \wedge dX_6^{(1)} = -dG_4 \wedge X_6^{(1)} \quad (2.1.7)$$

²They are 't Hooft anomaly on the worldvolume.

Due to the $M5$ brane, the Bianchi identity for G_4 is

$$dG_4 = \delta_5 (W^6 \hookrightarrow M^{11}) \quad (2.1.8)$$

From 2.1.7, 2.1.8 we see the anomalous term in the bulk will be localized near the $M5$ brane:

$$\int_{M^{11}} I_{inflow}^{(1)} = - \int_{M^{11}} \delta_5 (W^6 \hookrightarrow M^{11}) \wedge X_6^{(1)} = X_6^{(1)}|_{W^6}, \quad (2.1.9)$$

Hence the anomaly polynomial on the $M5$ brane worldvolume from the bulk will be:

$$I_{inflow} = - \int_{M^{11}} \delta_5 (W^6 \hookrightarrow M^{11}) \wedge X_8 = -X_8|_{W^6} \quad (2.1.10)$$

The entire 11d supergravity should be anomaly free, hence we should have I_8 from eq 2.1.2 completely cancels I_{inflow} from eq 2.1.10. However, we have:

$$I_{inflow} + I_8 = I_8 - X_8|_{W^6} = -\frac{1}{24}p_2(N) \quad (2.1.11)$$

Part of the normal bundle anomaly remains³.

The way to solve this problem is to give a more refined interpretation of the Bianchi identity. There is a smoothed-out (UV) solution to the Bianchi identity:

$$dG = \delta_5 (W^6 \hookrightarrow M^{11}) = d\rho(r) \wedge \frac{e}{2} \quad (2.1.12)$$

Here $\rho(r)$ is a function only depends the radial direction and satisfies $\rho(r \leq \frac{\epsilon}{2}) = -1, \rho(r \geq \epsilon) = 0$ (smoothed-out version of the δ source) and e is angular form satisfies $\int_{S_r^4} e = 2$ (S_r^4 is the 4-sphere transverse to the $M5$ brane with radial distance r). Actually e is an equivariant Euler class, so Wess-Zumino decendent procedure can be defined on it. The smooth G solution is:

$$G = dC - d\rho(r) \wedge \frac{e^{(0)}}{2} \quad (2.1.13)$$

Like Green-Schwarz mechanism, G should be invariant under the gauge transformation, hence the gauge transformation for C is:

$$\delta C = -d\rho(r) \wedge \frac{e^{(1)}}{2} \quad (2.1.14)$$

Now we have:

$$G - \rho(r) \frac{e}{2} = d(C - \rho(r) \frac{e^{(0)}}{2}) \quad (2.1.15)$$

As the Chern-Simons term has the form $x \wedge dx \wedge dx$, the modified Chern-Simons term in presence of $M5$ brane is:

$$(C - \rho(r) \frac{e^{(0)}}{2}) \wedge (G - \rho(r) \frac{e}{2}) \wedge (G - \rho(r) \frac{e}{2}) \quad (2.1.16)$$

Now under the gauge transformation $\delta(C - \rho(r) \frac{e^{(0)}}{2}) = -d(\rho(r) \frac{e^{(1)}}{2})$ and the anomalous transformation of the Chern-Simons term on $M^{11} - S_{\epsilon/2}^4(M5)$ ($S_{\epsilon/2}^4(M5)$ is the sphere bundle over $M5$

³Notice one important assumption of using anomaly inflow for $G_4 \wedge X_7^{(0)}$ is the validity of treating $M5$ brane as δ -source in the bulk, i.e. the thickness of $M5$ brane profile should not be able to be probed in the bulk low energy effective theory, i.e. 11d supergravity.

brane at radial distance $\epsilon/2$ gives:

$$-\frac{1}{6} \int_{M^{11}-S_{\epsilon/2}^4(M5)} \delta(C - \rho(r) \frac{e^{(0)}}{2}) \wedge (G - \rho(r) \frac{e}{2}) \wedge (G - \rho(r) \frac{e}{2}) = \frac{1}{48} \int_{S_{\epsilon/2}^4(M5)} e^{(1)} \wedge e \wedge e \quad (2.1.17)$$

Now we can use the Cattaneo-Bott formula

$$\int_{S_{\epsilon/2}(M5)} e_4 \wedge e_4 \wedge e_2^{(1)} = 2 \int_{W^6} p_2^{(1)}(N) \quad (2.1.18)$$

to get the anomaly inflow from the bulk Chern Simons term:

$$I_{inflow,CS} = \frac{1}{24} p_2(N) \quad (2.1.19)$$

Together with 2.1.11, we have:

$$I_{inflow} + I_8 + I_{inflow,CS} = 0 \quad (2.1.20)$$

The entire theory is anomaly free.

The anomaly inflow mechanism can also be used to calculate the 't Hooft anomaly of the theory lives on Q $M5$ branes, under the assumption that the entire system is anomaly-free:

$$I_{QM5} + Q I_{inflow} + Q^3 I_{inflow,CS} = 0 \quad (2.1.21)$$

hence

$$I_{QM5} = -Q I_{inflow} - Q^3 I_{inflow,CS} \quad (2.1.22)$$

2.1.3 M-theory on Calabi-Yau threefold: anomaly inflow and MSW string

In this section we study another example of anomaly inflow: 5d supergravity with 8 supercharges obtained from M theory compactified on Calabi-Yau threefold (CY3) [101].

The low energy spectrum of 5d supergravity from M theory on a generic CY3 has $h^{1,1}(CY3)$ $U(1)$ gauge fields $A^{I=1,\dots,h^{1,1}}$, and the topological coupling descended from 2.1.3 is:

$$\frac{1}{96} A^I \wedge tr(R^2) \left(\frac{1}{4} \int_{CY3} \omega^I \wedge tr(R_{CY3}^2) \right) - \frac{1}{6} A^I \wedge F^J \wedge F^K \left(\int_{CY3} \omega^I \wedge \omega^J \wedge \omega^K \right) \quad (2.1.23)$$

$$= \frac{1}{96} a_I A^I \wedge tr(R^2) - \frac{1}{6} c_{IJK} A^I \wedge F^J \wedge F^K \quad (2.1.24)$$

Now, consider a $M5$ brane wraps an semi-ample divisor $D = \sum Q_I PD(\omega^I)^4$ on CY3, it gives a $(4,0)$ BPS string in 5d supergravity carries the $U(1)$ magnetic charges Q^I . In the low energy, the Bianchi identity will become:

$$dF^I = Q^I \delta_3(W^2 \hookrightarrow M^5) \quad (2.1.25)$$

From 2.1.23, we can calculate anomaly inflow:

$$I_{inflow} = \frac{Q^I a_I}{48} (p_1(W^2) + p_1(N)) + \frac{c_{IJK} Q^I Q^J Q^K}{24} p_1(N) + \frac{1}{6} c_{IJK} Q^I F^J \wedge F^K \quad (2.1.26)$$

then anomaly cancellation gives the 't Hooft anomaly for the 2d $(4,0)$ SCFT lives on the BPS

⁴Here we use the ω^I s whose Poincare dual $PD(\omega^I)$ span the lattice $H^4(CY3, \mathbb{Z})$

string:

$$I_{W^2} = -\frac{Q^I a_I}{48} p_1(W^2) + \left\{ \frac{c_{IJK} Q^I Q^J Q^K}{6} + \frac{Q^I a_I}{12} \right\} c_2(so(3)_R) - \frac{1}{6} c_{IJK} Q^I F^J \wedge F^K \quad (2.1.27)$$

Here $so(3)_R$ is the R-symmetry of the (4, 0) supersymmetric algebra comes from the normal bundle. The (4, 0) supersymmetric algebra also gives

$$c_L = 6k_L \quad (2.1.28)$$

while gravitational anomaly is given by:

$$I_{grav} = \frac{c_L - c_R}{24} p_1(W^2) \quad (2.1.29)$$

2.1.27 gives $k_L = \left(\frac{c_{IJK} Q^I Q^J Q^K}{6} + \frac{Q^I a_I}{12} \right)$ and $c_L - c_R = -\frac{Q^I a_I}{2}$. From 2.1.28 2.1.29 we read out:

$$c_L = c_{IJK} Q^I Q^J Q^K + \frac{Q^I a_I}{2}, c_R = c_{IJK} Q^I Q^J Q^K + Q^I a_I \quad (2.1.30)$$

With the knowledge of the UV origin of the BPS string considered here, we can also derive the central charges in eq 2.1.30 by counting degrees of freedom on the UV side. To do this we need the divisor D to be very ample, i.e. can be represented by smooth four-manifold.

- The moduli \mathcal{M}_D of divisor D insides CY gives both $dim_{\mathbb{R}} \mathcal{M}_D$ left and right moving bosons on the BPS string of 5d sugra. $dim_{\mathbb{R}} \mathcal{M}_D$ is given by Hirzebruch–Riemann–Roch theorem (HRR) together with vanishing theorem associated with the very ampleness of divisor D .

$$dim_{\mathbb{R}} \mathcal{M}_D = 2 \int_{CY3} \left(\frac{P^3}{6} + \frac{1}{12} P c_2(CY3) \right) - 2 \quad (2.1.31)$$

where $P = PD(D) \in H^2(CY3, \mathbb{Z})$.

- There is one anti self-dual tensor on the $M5$ branes, it gives b_{2+} left moving bosons and b_{2-} right moving bosons. $b_{2\pm}$ are calculated by index theorem (This is the requirement of very ampleness for D , that is D should be represented by a smooth four manifold so index theorem can be applied.).

$$\chi = b_{2+} + b_{2-} + 2 = \int_{CY3} (P c_2(CY3) + P^3) \quad (2.1.32)$$

Notice $b_{1,3}(D) = 0$ due to Lefschetz hyperplane theorem.

$$\sigma = b_{2+} - b_{2-} = - \int_{CY3} \left(\frac{1}{3} P^3 + \frac{2}{3} P c_2(CY3) \right) \quad (2.1.33)$$

- Together with bosons correspond to the three transversal modes of the BPS string, we get the number of right and left moving bosons:

$$N_L = \int_{CY3} \frac{2}{3} P^3 + \frac{1}{3} c_2(CY3) P, N_R = \int_{CY3} P^3 + c_2(CY3) P \quad (2.1.34)$$

Together with the (4, 0) supersymmetry, we get

$$c_L = \frac{3}{2} N_L = \int_{CY3} P^3 + \frac{1}{2} c_2(CY3) P, c_R = N_R = \int_{CY3} P^3 + c_2(CY3) P \quad (2.1.35)$$

- Using $P = PD(D) = \sum_I Q^I \omega^I$, we can see eq 2.1.35 and eq 2.1.30 agrees with each other (Notice that the notation we used differs from the notation in the literature up to the notation of the supersymmetry, i.e. $(0, 4)$ or $(4, 0)$, we will switch to the standard notation $(0, 4)$ in part III).

We end this section by pointing out that the anomaly inflow method is more general than microscopic counting (MSW string), as later requires the divisor has some very nice properties: vanishing theorem and can be represented by smooth four manifold.

2.1.4 10d supergravity and swampland

In [92], the authors use anomaly inflow to discuss a swampland condition ⁵. For 10d supergravity, anomaly cancellation gives four possibilities for gauge algebra: $so(32), e_8 \oplus e_8, e_8 \oplus u(1)^{248}, u(1)^{496}$. String theory can only realize the first two. Using anomaly inflow, there is a swampland condition to rule out the later two.

Put a BPS string in the 10d supergravity, which gives a $(0, 8)$ SCFT in the IR on its worldsheet. The bulk supergravity picks up an action due to the BPS string:

$$S_{10d}^{str} = \int_{M^{10}} B \wedge \delta_8 (W^2 \hookrightarrow M^{10}) \quad (2.1.36)$$

This term will cause anomaly inflow under gauge and diffeomorphism transformation according to Green Schwarz mechanism:

$$\delta_{\Lambda, \Theta} S^{str} = - \int_{W^2} \left[\sum_i \frac{1}{4h_i^V} \text{Tr}(\Lambda_i F_{G_i}) - \frac{1}{4} \text{tr}(\Theta R) \right] \quad (2.1.37)$$

't Hooft anomaly of the $(0, 8)$ SCFT can be read out:

$$I_{SCFT} = \sum_i \frac{1}{4h_i^V} \text{Tr}(F_{G_i}^2) - \frac{1}{4} \text{tr}(R^2) = \sum_i \frac{1}{4h_i^V} \text{Tr}(F_{G_i}^2) + \frac{1}{2} p_1(W^2) - c_2(SO(8)_R) \quad (2.1.38)$$

and the central charges c_L, c_R :

$$c_R - c_L = -12, c_R = 12k_R = 12, c_L = 24 \quad (2.1.39)$$

The SCFT on the worldsheet is factorized to two parts, one is a free $(0, 8)$ multiplet from the transversal mode of the BPS string and contribute:

$$c_L^{free} = 8, c_R^{free} = 12 \quad (2.1.40)$$

The other part is an interacting SCFT with central charges

$$c_L^{int} = 16, c_R^{int} = 0 \quad (2.1.41)$$

The gauge anomaly part $\frac{1}{4h_i^V} \text{Tr}(F_{G_i}^2)$ requires there are current algebras with group G_i and level 1 on the left moving side, each will contribute left moving central charge $c_i = \frac{\dim G_i}{1+h^V}$ ($c_i = 1$ for $G_i = U(1)$). Since the overall left moving central charge is 16, unitarity requires:

$$\sum_i \frac{\dim G_i}{1+h^V} \leq 16 \quad (2.1.42)$$

⁵See [[116]-[143]] for part of recent progress of this vast program

This condition is satisfied for $so(32), e_8 \oplus e_8$ but not for $e_8 \oplus u(1)^{248}, u(1)^{496}$. So the later two supergravity theories are not consistent.

2.2 F-theory

Type IIB string theory is believed to have $SL(2, \mathbb{Z})$ duality group under which the NS-NS 2-form B_2 , RR 2-form C_2 and the axiodilaton $\tau = C_0 + ie^{-\phi}$ consists of RR 0-form C_0 and dilation ϕ transform as:

$$\tau \rightarrow \frac{a\tau + b}{c\tau + d}, \quad \begin{pmatrix} C_2 \\ B_2 \end{pmatrix} \rightarrow M \begin{pmatrix} C_2 \\ B_2 \end{pmatrix}, \quad M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}) \quad (2.2.1)$$

F theory [75, 84, 85] is type IIB string theory with 7-branes ($D7$ and its $SL(2, \mathbb{Z})$ duality transformation) source the axiodilaton τ . Hence τ varies in spacetime and can be represented as modulus of an elliptic curve over spacetime and F theory should be compactified on elliptically fibered CY n -fold (we only consider $n = 2, 3$ in this section). Singular fiber signals the location of 7-branes.

M/F theory duality Put M-theory on a elliptically fibered Calabi-Yau n -fold (CY_n), lable the two cycle S^1 on the elliptic curve fiber S^1_A, S^1_B . First compactified on S^1_A gives type IIA, further compactify on S^1_B then T-dualize it gives F-theory compactified on $CY_n \times S^1$, sending $S^1_B \rightarrow 0$ gives F-theory compactified on CY_n . The complex structure of the fiber on M-theory side maps to τ on F-theory side.

Elliptic CY_n We only consider standard conventions for the elliptic fibrations over base B with section (see e.g. [84, 85]). The elliptical fiber on a CY_n is defined by an equation

$$y^2 = x^3 + f(u_i)x + g(u_i) \quad (2.2.2)$$

in an affine patch of the weighted projective space $\mathbb{WP}_{2,3,1}$, with u_i one set of affine coordinates on the base B , fixed. Then $f \in \Gamma(-4K)$ and $g \in \Gamma(-6K)$ (K is the canonical divisor of base B). The degeneration loci of the elliptic fiber is given by zeros of the discriminant:

$$\Delta = 4f^3(u_i) + 27g^2(u_i) \quad (2.2.3)$$

$\Delta \in \Gamma(-12K)$. The enhanced non-Abelian gauge symmetry G_i comes from the singularities of elliptic fibration which gives divisors $n_i D_i$ on the base B with multiplicity n_i determined by the singularity type. As the entire degeneration loci is given by $-12K$, the divisor Y

$$Y = -12K - \sum_i n_i D_i \quad (2.2.4)$$

must be effective. The relation between this geometric condition and the physical condition in sect 2.1.4 will be discussed in part III

2.2.1 F-theory on elliptic CY_2

The only elliptic CY_2 is elliptic $K3$ ⁶, i.e. an elliptic curve fibered over the base \mathbb{P}^1 . The degeneration loci is $-12K_{\mathbb{P}^1} = 24 pt$ (pt is a point on base \mathbb{P}^1), i.e. 24 7-branes.

⁶We ignore the trivial T^4 case

A special CY_2 case: T^4/\mathbb{Z}_2 For T^4/\mathbb{Z}_2 case, the axiodilation τ is a constant on spacetime [7]. So there are two descriptions when string coupling is weak:

- Perturbative analysis: Type IIB compactified on T^2/\mathbb{Z}_2 with 4 O_{7-} orientifold planes and 4 $D7$ branes on top of each orientifold. The gauge algebra is $so(8)^{\oplus 4}$
- Non-perturbative analysis; F-theory on elliptic CY_2 : Since the background has constant τ , the j -invariant $j(\tau(z)) = \frac{4 \cdot (24f)^3}{27g^2 + 4f^3}$ must be a constant. The corresponding CY_2 model is:

$$y^2 = x^3 + f(z)x + g(z) \quad (2.2.5)$$

with

$$g = \phi^3, \quad f = \alpha\phi^2, \quad \phi = \prod_{i=1}^4 (z - z_i) \quad (2.2.6)$$

$z_{i=1,2,3,4}$ are 4 points on \mathbb{P}^1 and $j(\tau) = \frac{4 \cdot (24\alpha)^3}{27 + 4\alpha^3}$ is indeed a constant in this model. This model also gives gauge algebra $so(8)^{\oplus 4}$. But the interpretation now is 6 $D7$ branes on each point z_i .

The way to connect two interpretations is first slightly deforming the system, i.e. pulling the 4 $D7$ branes out of O_{7-} plane, then each O_{7-} plane will be splitted into 2 $D7$ branes by quantum corrections and hence matches the F-theory interpretation. This splitting is very similar to 4d $\mathcal{N} = 2$ Seiberg-Witten theory: Consider the pure $\mathcal{N} = 2$ $SO(3)$ gauge theory. The classical vacua at origin of the moduli space gets splitted to dyon and monopole points by instanton corrections.

Frozen singularity O_{7-} plane with 8 $D7$ branes has the same R-R charge as O_{7+} plane and they cause the same monodromy for τ so they appear in elliptic CY_2 in the same way: I_4^* singularity. But their physical interpretations are different.

- For $O_{7-} + 8D7$, it gives gauge algebra $so(16)$ after compactification. The associated Coulomb branch can be characterized by the positions of 8 $D7$ branes on \mathbb{P}^1 .
- For O_{7+} , it doesn't give rise to any gauge algebra after compactification as there is no Coulomb branch associated to it, i.e. no $D7$ branes.

This phenomena is called frozen of singularity, i.e. moduli space attached with the singularity gets frozen [8]. We will discuss a similar case of frozen singularity in perturbative string regime in part II

2.2.2 F-theory on elliptic CY_3

In the most common cases, the base B of CY_3 is \mathbb{P}^2 or $F^{0 \leq n \leq 12}$ (Hirzebruch surfaces) or blow up thereof [76, 77]. F-theory compactified on elliptic CY_3 gives 6d supergravity with 8 supercharges and it has the following features:

- The low energy spectrum contains $n_T = h^{1,1}(B) - 1$ anti self-dual tensor multiplet (one self-dual tensor in the gravity multiplet). All the tensors come from divisor classes on the base B , which has signature $(1, n_T)$ ⁷
- Non-abelian gauge symmetry comes from the singularity of elliptic fibration, and blow up the singularity still gives a CY_3 but without elliptic fibration structure⁸. So CY_3 after blow up non longer makes sense in F- theory compactification but still makes sense in M-theory compactification. This geometric fact matches with 5d/6d supersymmetry (SUSY) with 8

⁷Note $h^{2,0}(B) = 0$ for base B .

⁸More precisely, we only consider singularity types with this property.

supercharges. 6d vector multiplets with 8 supercharges doesn't have scalars hence doesn't have Coulomb branch attached with it, while 5d vector multiplets do.

We use one example to end this chapter [104]

Phase transition in M/F theory Heterotic $E_8 \times E_8$ compactify on $K3$ gives 6d supergravity with 8 supercharges. Due to tadpole cancellation eq 1.3.13, the gauge bundle on $K3$ is required to have $(12 + n, 12 - n)$ ($0 \leq n \leq 12$) instantons for $E_8 \times E_8$. It is dual to:

- M-theory on $K3 \times I$, together with instanton number $(12 + n, 12 - n)$ at each of the E_8 Hořava-Witten domain wall, i.e. $12 \pm n$ $M5$ branes dissolve into each E_8 domain wall.
- F-theory compactified on elliptic $CY3$ with base Hirzebruch surface F^n .

One fact is F^n and $F^{n\pm 1}$ can be related by blowup-blowdown procedure. In F-theory this geometric transition first introduces one tensor multiplet and then gets rid of one tensor multiplet. In M-theory this is a transition between $(12 + n, 12 - n)$ and $(12 + n \pm 1, 12 - n \mp 1)$:

- One E_8 domain wall emits one $M5$ brane.
- Then the second E_8 domain wall absorbs this $M5$ brane

Notice there is one tensor multiplet lives on $M5$ brane (sect 2.1.2). So the first process adds a tensor multiplet and the second process reduces one. This is the M-theory interpretation of the geometric transition between Hirzebruch surfaces F^n .

Part II

On perturbative side of string compactification

Chapter 3

Heterotic compactification on X_5

In this chapter¹, we study perturbative $E_8 \times E_8$ Heterotic compactification on two classes five dimensional flat manifold X_5 which preserves half supersymmetry:

- Orbifolds $X_5 = \mathbb{T}^4 / \mathbb{Z}_{2,3,4,6} \times S^1$. Using coordinate description the \mathbb{Z} action is:

$$g(z_1, z_2, \theta) = (\zeta_N z_1, \zeta_N^{-1} z_2, \theta) . \quad (3.0.1)$$

- Seifert manifolds $X_5 = (\mathbb{T}^4 \times S^1) / \mathbb{Z}_{2,3,4,6}$. Using coordinate description the \mathbb{Z} action is:

$$g(z_1, z_2, \theta) = \left(\zeta_N z_1, \zeta_N^{-1} z_2, \theta + \frac{2\pi}{N} \right) . \quad (3.0.2)$$

$E_8 \times E_8$ Heterotic string theory compactify on the first class of orbifolds has been very-well studied using standard embedding techniques. We first review relevant results and then discuss connections between $E_8 \times E_8$ Heterotic string compactifications on these two classes of flat manifolds (orbifolds).

3.1 Heterotic compactification to six dimensions: set up

We begin our study of heterotic compactification on X_5 with a review of the well-known construction of heterotic theory compactified on \mathbb{T}^4 . This compactification yields a $d = 6$ (1,1) supergravity theory with massless content consisting of the (1,1) supergravity multiplet and a number of vector multiplets. Each vector multiplet contains 4 scalars that transform in the adjoint of the gauge algebra. At a generic point in the scalar moduli space $\text{Gr}(20, 4) / \text{O}(\Gamma_{20,4})$ the gauge algebra is $\mathfrak{u}(1)^{\oplus 20}$.² At special points the gauge algebra can be enhanced, and the enhanced symmetries and the corresponding loci in the moduli space have been recently studied in [16]. For our purposes it will be sufficient to consider the locus where the gauge algebra is $\mathfrak{e}_8 \oplus \mathfrak{u}(1)^{\oplus 4} \oplus \mathfrak{e}_8$: this locus has a standard RNS worldsheet realization, which will make it easy to describe the constructions we wish to consider. In this section we will summarize some of the relevant details, and in the next section we will apply them to some classic examples.

We work in light-cone gauge on a Euclidean worldsheet with coordinates z, \bar{z} , with worldsheet supersymmetry on the right (anti-holomorphic) side of the string. In addition to the degrees of freedom for the Minkowski directions, the internal CFT consists of the scalar fields $\Phi^i(z, \bar{z})$ and

¹This part is based on [144]

²Here $\text{Gr}(20, 4)$ is the coset $\text{SO}(20, 4, \mathbb{R}) / \text{SO}(20, \mathbb{R}) \times \text{SO}(4, \mathbb{R})$, and $\text{O}(\Gamma_{20,4})$ is the group of lattice isomorphisms of the even self-dual lattice $\Gamma_{20,4}$, often written as $\text{O}(20, 4, \mathbb{Z})$.

their right-moving Majorana-Weyl superpartners, 8 left-moving Weyl fermions, and a level 1 \mathfrak{e}_8 current algebra for the “hidden E_8 ” that will play a spectator role in our analysis.

We find it convenient to break up the left-moving fermions into 2 Weyl fermions $\gamma^{1,2}(z)$ and their conjugates $\bar{\gamma}^{1,2}(z)$, and 6 more Weyl fermions $\xi^I(z)$, $\bar{\xi}^I(z)$; similarly, we organize the right-moving fermions into a Weyl pair $\psi^{1,2}(\bar{z})$, with conjugates $\bar{\psi}^{1,2}(\bar{z})$. This gives a decomposition of the corresponding level 1 current algebras

$$\mathfrak{so}(16)_L \supset \mathfrak{so}(12)_L \oplus \mathfrak{su}(2)_L \oplus \mathfrak{su}(2)'_L, \quad \mathfrak{so}(4)_R \simeq \mathfrak{su}(2)_R \oplus \mathfrak{su}(2)'_R. \quad (3.1.1)$$

The fermions transform in the following representations. On the holomorphic side we have

$$\xi \in (\mathbf{12}, \mathbf{1}, \mathbf{1}), \quad \gamma, \bar{\gamma} \in (\mathbf{1}, \mathbf{2}, \mathbf{2}), \quad (3.1.2)$$

while on the anti-holomorphic side we have $\psi, \bar{\psi} \in (\mathbf{2}, \mathbf{2})$ of $\mathfrak{su}(2)_R \oplus \mathfrak{su}(2)'_R$.

This structure is a special case of a (0,4) SCFT necessary [47] for the preservation of (1,0) supersymmetry in $\mathbb{R}^{1,5}$. As our ultimate aim will be study compactifications with 8 supercharges, we will now describe some features in this more general setting, focusing on the identification of states in the worldsheet theory with massless fermions in spacetime.

The key to making this connection is through the current algebra, which is now of the form $\mathfrak{so}(12)_L \oplus \mathfrak{su}(2)_L$ on the left and $\mathfrak{su}(2)_R$ on the right.³ The latter is the R-symmetry of the N=4 superconformal algebra (SCA), and it contains $\mathfrak{u}(1)_R$ an R-symmetry for an N=2 subalgebra of the N=4 SCA with current \bar{J} and operator charges labeled by \bar{q} . We will also choose $\mathfrak{u}(1)_L \subset \mathfrak{su}(2)_L$ with current J and charges q to label our states, and we similarly define the currents and charges J', q' for $\mathfrak{u}(1)'_L \subset \mathfrak{su}(2)'_L$ and \bar{J}', \bar{q}' for $\mathfrak{u}(1)'_R \subset \mathfrak{su}(2)'_R$. The currents give simple expressions for the left- and right-fermion numbers that are necessary for the GSO projections: we have $(-1)^{F_\gamma} = e^{i\pi J_0}$, while $(-1)^{F_\psi} = e^{i\pi \bar{J}_0}$.

We can now use the familiar rules—see e.g. [48, 49]—to identify worldsheet states with massless multiplets in spacetime.

- The identity operator of the internal CFT gives rise to a (1,0) supergravity multiplet and a (1,0) tensor multiplet.⁴
- The spacetime gauge bosons arise in two ways: every holomorphic current gives rise to a spacetime gauge boson, and every anti-holomorphic operator with weight $\bar{h} = 1/2$ (i.e. a free fermion) gives rise to an abelian gauge boson. The latter, when present, complete the (1,0) gravity multiplet and the (1,0) tensor multiplet to a (1,1) supergravity multiplet.

In all of our examples the holomorphic current algebra will be of the form $\mathfrak{e}_8 \oplus \mathfrak{u}(1)^{\oplus k} \oplus \mathfrak{g}$, and in most of our examples $\mathfrak{g} \supset \mathfrak{e}_7 \oplus \mathfrak{u}(1)'_L$.⁵ Note that the “linearly realized” currents $\mathfrak{so}(12) \oplus \mathfrak{su}(2)_L \oplus \mathfrak{u}(1)'_L$ are completed to \mathfrak{e}_7 by additional currents from the left-moving Ramond sector in the $(\mathbf{32}, \mathbf{1}) \oplus (\mathbf{12}, \mathbf{2})$ representation.

In some of our examples the $\mathfrak{u}(1)'_L$ current will be enhanced to $\mathfrak{su}(2)'_L$, and when there are $h = 1/2$ holomorphic operators transforming in $\mathbf{2}$ of $\mathfrak{su}(2)'_L$, the left-moving Ramond sector will produce currents in

$$(\mathbf{32}, \mathbf{2}, \mathbf{1}) \oplus (\mathbf{12}, \mathbf{2}, \mathbf{1}) \oplus (\mathbf{32}', \mathbf{1}, \mathbf{2}) \oplus (\mathbf{12}, \mathbf{1}, \mathbf{2}), \quad (3.1.3)$$

³In general (0,4) theories the $\mathfrak{su}(2)_L$ symmetry factor will be reduced to $\mathfrak{u}(1)_L$ or be entirely absent. It will be present in all theories we consider and will simplify the analysis.

⁴Details of the multiplet structure in $d = 6$, as well as in the $d = 5$ case of interest to us can be found in the recent pedagogical review [50].

⁵In some of our later discussions we will break the \mathfrak{e}_7 factor further by introducing a Wilson line.

that complete \mathfrak{g} to $\mathfrak{g} = \mathfrak{e}_8$.

Finally, the $\mathfrak{u}(1)^{\oplus k}$ factor arises from additional currents neutral with respect to \mathfrak{g} . For example in \mathbb{T}^4 compactification, $k = 4$ with the currents $i\partial\Phi^i$.

- The remaining spacetime massless fields reside in $(1,0)$ $\frac{1}{2}$ -hypermultiplets. These are in one to one correspondence with the $\overline{\text{NS}}$ chiral primary states with $\bar{q} = 1$ and holomorphic weight $h = 1$. Their gauge transformations are determined by the left-moving sector: each worldsheet state with $q = 0$ leads to an \mathfrak{e}_7 -neutral $\frac{1}{2}$ -hypermultiplet, while each worldsheet state with $q = 1$ leads to a state transforming in **56** of \mathfrak{e}_7 .

Let us apply these rules to the \mathbb{T}^4 example, focusing on the $(1,0)$ vector and hypermultiplets. The R-charge assignments to the right-moving fermions are

$$\begin{array}{cccc} \psi^1 & \psi^2 & \bar{\psi}^1 & \bar{\psi}^2 \\ \bar{q} & +1 & +1 & -1 & -1 \\ \bar{q}' & +1 & -1 & -1 & +1 \end{array} \quad (3.1.4)$$

We can easily see that the spectrum is consistent with $(1,1)$ spacetime supersymmetry: for every holomorphic current J we have the operators $J\psi^1$ and $J\psi^2$ that together give rise to a $(1,0)$ hypermultiplet transforming in the same way as the current J under the gauge symmetry.

3.1.1 $\mathbb{T}^4/\mathbb{Z}_N$ compactifications

We now turn our attention to more interesting $(1,0)$ theories that are built by considering symmetric orbifolds $\mathbb{T}^4/\mathbb{Z}_N$. In each case, we take the \mathbb{Z}_N action to be just the action described in (3.0.2) on the \mathbb{T}^4 part, extending it to act on the worldsheet fermions in the standard left-right-symmetric fashion: the generator g of the \mathbb{Z}_N action is set to be

$$g = \underbrace{\exp\left[\frac{2\pi i}{N} J'_0\right]}_{=g_\gamma} \underbrace{\exp\left[\frac{2\pi i}{N} \bar{J}'_0\right]}_{=g_\psi}. \quad (3.1.5)$$

The \mathbb{Z}_N action is diagonally embedded in $U(1)'_L \times U(1)'_R$. For $N = 2$ the action is contained in the center of each corresponding $SU(2)$ factor, so that the $\mathfrak{su}(2)'_L \oplus \mathfrak{su}(2)'_R$ algebra is left invariant, while for $N > 2$ the invariant subalgebra is $\mathfrak{u}(1)'_L \oplus \mathfrak{u}(1)'_R$. We are of course by no means the first to consider these orbifolds—see, for example [42–44, 51, 52] and references therein for earlier work on these orbifolds and some of their non-symmetric generalizations. However, we will recall some of the details and give a view that will be useful for what follows.

Untwisted massless states

To describe the untwisted massless states, we just need to apply the projection onto \mathbb{Z}_N -invariant states. We first observe that for all $N > 0$ the special anti-holomorphic states with $\bar{h} = 1/2$ are projected out, and therefore the spacetime supersymmetry is reduced. However, since the projection keeps the $\mathfrak{su}(2)_R$ current algebra and leaves the vacuum of the internal theory invariant, we are guaranteed $(1,0)$ supersymmetry. This makes the preceding discussion of the multiplet structure well-adapted to understand the projection.

The invariant content of the $(1,1)$ supergravity multiplet is exactly the $(1,0)$ supergravity and tensor multiplets. To analyze the invariant content of the $(1,1)$ vector multiplets, we use the decomposition

$$(1,1) \text{ vector} = (1,0) \text{ vector} \oplus (1,0) \text{ hyper}. \quad (3.1.6)$$

The \mathbb{Z}_N action on each multiplet is a combination of an $SU(2)'_L$ gauge symmetry, an $SU(2)'_R$ R-symmetry, as well as $G_{\text{ab}} \simeq \mathbb{Z}_N$, which acts on the abelian (1,1) vector multiplets corresponding to the $\mathfrak{u}(1)^{\oplus 4}$ factor according to (3.0.2). To describe that action, we package the 4 vector multiplets into two complex combinations that transform according to

$$\mathbf{1}_{\zeta_N} \oplus \mathbf{1}_{\zeta_N^{-1}} \quad (3.1.7)$$

under the action of G_{ab} .

Consider first the (1,1) vector multiplets corresponding to the hidden \mathfrak{e}_8 symmetry. While the (1,0) vector submultiplets of these are left invariant, every $\frac{1}{2}$ -hyper is projected out, so we are left with just the (1,0) vector multiples of \mathfrak{e}_8 .

Next, we examine the (1,1) abelian vectors: now each (1,0) vector is projected out, but because of the additional action in $SU(2)'_R$, the (1,0) hypermultiplet content is more interesting: under $SU(2)'_R \times G_{\text{ab}}$ the $\frac{1}{2}$ -hypermultiplets transform as

$$\mathbf{2}_{\zeta_N} \oplus \mathbf{2}_{\zeta_N^{-1}}, \quad (3.1.8)$$

or decomposing further with respect to $U(1)'_R$, as

$$\mathbf{1}_{1,\zeta_N} \oplus \mathbf{1}_{-1,\zeta_N} \oplus \mathbf{1}_{1,\zeta_N^{-1}} \oplus \mathbf{1}_{-1,\zeta_N^{-1}}. \quad (3.1.9)$$

Thus, if $N = 2$, then all of these are invariant, while if $N > 2$ then half of these are invariant. So, we find 4 neutral hyps for $N = 2$ and 2 neutral hyps for $N > 2$.

To discuss the \mathfrak{e}_8 vectors, we first decompose

$$\begin{aligned} \mathfrak{e}_8 &\supset \mathfrak{e}_7 \oplus \mathfrak{u}(1)'_L \\ \mathbf{248} &= \mathbf{133}_0 \oplus \mathbf{1}_0 \oplus \mathbf{56}_{+1} \oplus \mathbf{56}_{-1} \oplus \mathbf{1}_{+2} \oplus \mathbf{1}_{-2}. \end{aligned} \quad (3.1.10)$$

From this we see that for $N > 2$ the invariant (1,0) vectors transform in the adjoint of $\mathfrak{e}_7 \oplus \mathfrak{u}(1)'$, while for $N = 2$ we find the adjoint of $\mathfrak{e}_7 \oplus \mathfrak{su}(2)'_L$. To obtain the invariant $\frac{1}{2}$ -hypers, we tensor this with $\mathbf{2}$ of $\mathfrak{su}(2)'_R$, or equivalently with $\mathbf{1}_{+1} \oplus \mathbf{1}_{-1}$ of $\mathfrak{u}(1)'_R$:

$$\begin{aligned} \mathbf{248} &= \mathbf{133}_{0,+1} \oplus \mathbf{133}_{0,-1} \oplus \mathbf{1}_{0,+1} \oplus \mathbf{1}_{0,-1} \\ &\oplus \mathbf{56}_{+1,+1} \oplus \mathbf{56}_{+1,-1} \oplus \mathbf{56}_{-1,+1} \oplus \mathbf{56}_{-1,-1} \\ &\oplus \mathbf{1}_{+2,+1} \oplus \mathbf{1}_{+2,-1} \oplus \mathbf{1}_{-2,+1} \oplus \mathbf{1}_{-2,-1}. \end{aligned} \quad (3.1.11)$$

The invariant $\frac{1}{2}$ -hypers are those with $q' + \bar{q}' = 0 \pmod N$. The result is the following invariant spectrum from the untwisted sector (as promised, we ignore the hidden \mathfrak{e}_8):

\mathbb{Z}_N	gauge symmetry	hypers
\mathbb{Z}_2	$\mathfrak{e}_7 \oplus \mathfrak{su}(2)'_L$	$\mathbf{1}_0^{\oplus 4} \oplus (\mathbf{56}, \mathbf{2})$
\mathbb{Z}_3	$\mathfrak{e}_7 \oplus \mathfrak{u}(1)'_L$	$\mathbf{1}_0^{\oplus 2} \oplus \mathbf{1}_2 \oplus \mathbf{56}_1$
$\mathbb{Z}_{4,6}$	$\mathfrak{e}_7 \oplus \mathfrak{u}(1)'_L$	$\mathbf{1}_0^{\oplus 2} \oplus \mathbf{56}_1$

(3.1.12)

Twisted sector contributions

A simple approach to work out the twisted sector contributions is to model each fixed point of the orbifold by $\mathbb{C}^2/\mathbb{Z}_N$ and use free field techniques to calculate the quantum numbers of states in the twisted sector. It is then not too difficult to read off the twisted sector states that yield spacetime massless states. We will not have need for details of the construction, so we will simply

quote the results from the literature [42, 44]. The twisted sectors make no contribution to the gauge symmetry, but they do produce additional hypermultiplets arranged as follows:

$$\begin{aligned}
\mathbb{Z}_N & \text{ twisted hypers} \\
\mathbb{Z}_2 & (\mathbf{2}, \mathbf{1})^{\oplus 32} \oplus (\mathbf{1}, \mathbf{56})^{\oplus 8} \\
\mathbb{Z}_3 & \mathbf{56}_{-1/3}^{\oplus 9} \oplus \mathbf{1}_{2/3}^{\oplus 45} \oplus \mathbf{1}_{-4/3}^{\oplus 18} \\
\mathbb{Z}_4 & \mathbf{56}_{-1/2}^{\oplus 4} \oplus \mathbf{1}_{3/2}^{\oplus 8} \oplus \mathbf{1}_{1/2}^{\oplus 24} \oplus \mathbf{56}_0^{\oplus 5} \oplus \mathbf{1}_1^{\oplus 32} \\
\mathbb{Z}_6 & \mathbf{1}_{1/3}^{\oplus 8} \oplus \mathbf{1}_{-5/3}^{\oplus 2} \oplus \mathbf{56}_{-2/3} \oplus \mathbf{1}_{2/3}^{\oplus 22} \oplus \mathbf{1}_{-4/3}^{\oplus 10} \oplus \mathbf{56}_{-1/3}^{\oplus 5} \oplus \mathbf{1}_1^{\oplus 22} \oplus \mathbf{56}_0^{\oplus 3}
\end{aligned}$$

The reader can check that in all cases the six-dimensional anomaly cancelation condition $N_{\text{hyper}} - N_{\text{vector}} = 244$ is satisfied.

Each of these theories has a Higgs branch: we can Higgs the $\mathfrak{su}(2)'_L$ for $N = 2$ and the $\mathfrak{u}(1)'_L$ for $N > 2$ to obtain a spectrum with gauge algebra \mathfrak{e}_7 and hypermultiplets in $(\mathbf{56})^{\oplus 10}$ —which is exactly the massless spectrum of heterotic compactification on K3 with standard embedding. We expect this result since each of these singularities is a degeneration limit of the K3 surface.⁶

3.1.2 Compactification on $\mathbb{T}^4/\mathbb{Z}_N \times S^1$

Having obtained the $d = 6$ theory, it is a simple matter to compactify further on a circle. If we treat the circle as completely decoupled, then the massless spectrum is obtained by standard Kaluza-Klein reduction [50]:

- the reduction of the (1,0) supergravity and tensor multiplets leads to a (minimal) $d = 5$ supergravity multiplet and 2 abelian vector multiplets⁷;
- each $d = 6$ vector multiplet in representation \mathbf{r} of the gauge algebra reduces to a $d = 5$ vector multiplet, which now has a real scalar transforming in the same representation;
- similarly, each $d = 6$ hypermultiplet reduces to a $d = 5$ hypermultiplet in the same representation of the gauge algebra.

All in all, the gauge algebra is now $\mathfrak{u}(1)^{\oplus 2} \oplus \mathfrak{e}_8 \oplus \mathfrak{g}$, where the first factor is due to the vector multiplets obtained from the reduction of supergravity and tensor (1,0) multiplets, and the hypermultiplet spectrum is unmodified.

The new feature in $d = 5$ is the existence of a Coulomb branch. Giving a generic set of expectation values to the scalars in the vector multiplets breaks the gauge group to $\mathfrak{u}(1)^{\oplus 18}$ and leaves 4 neutral hypermultiplets in the \mathbb{Z}_2 orbifold and just 2 neutral hypermultiplets for $\mathbb{Z}_{3,4,6}$. Observing that under $\mathfrak{e}_7 \supset \mathfrak{e}_6 \oplus \mathfrak{u}(1)$

$$\mathbf{56} = \mathbf{27}_{+1} \oplus \overline{\mathbf{27}}_{-1} \oplus \mathbf{1}_{+3} \oplus \mathbf{1}_{-3} , \quad (3.1.13)$$

we see that for the $\mathbb{Z}_{3,4,6}$ compactifications there is a Coulomb branch with unbroken $\mathfrak{g} = \mathfrak{u}(1)^2 \oplus \mathfrak{e}_8 \oplus \mathfrak{e}_6 \oplus \mathfrak{u}(1)'$ where all of the twisted sector matter is lifted, and there are 2 neutral massless hypermultiplets. A special feature of the \mathbb{Z}_3 example is that it is possible to lift all of the twisted sector states by just going on the $\mathfrak{u}(1)'$ Coulomb branch, where the full gauge algebra \mathfrak{g} is preserved.

⁶The limit in the non-linear sigma model's moduli space is subtle due to a choice of B -field on the collapsing two-cycles of the degenerating K3 surface: see [45, 46] for further discussion.

⁷One of these is obtained by dualizing a $d = 5$ abelian tensor multiplet.

3.1.3 Compactification on X_5

Having reviewed the well-known compactifications on $\mathbb{T}^4/\mathbb{Z}_N \times S^1$, we now consider compactification on X_5 . From the worldsheet point of view, the new ingredient relative to the preceding discussion is a \mathbb{Z}_N shift orbifold of the circle. By itself this theory is easy to understand: starting with a CFT for a circle of radius r , the shift orbifold is a CFT of radius r/N [53]. In the full quotient CFT for $X_5 = (\mathbb{T}^4 \times S^1)/\mathbb{Z}_N$, we just need to combine the twisted sectors of the $\mathbb{T}^4/\mathbb{Z}_N$ and S^1/\mathbb{Z}_N CFTs and impose the orbifold projection. With a view to later developments, we will first describe the construction of the S^1 shift orbifold in some detail.

The \mathbb{Z}_N shift orbifold of the circle

Consider a compact $c = 1$ boson at a generic radius r . Splitting the worldsheet field $\Phi(z, \bar{z})$ into left- and right-moving components, we have the defining OPEs

$$\Phi_L(z)\Phi_L(w) \sim -\frac{1}{r^2} \log(z-w), \quad \Phi_R(\bar{z})\Phi_R(\bar{w}) \sim -\frac{1}{r^2} \log(\bar{z}-\bar{w}). \quad (3.1.14)$$

The theory enjoys a Kac-Moody $U(1)_L^{\text{shift}} \times U(1)_R^{\text{shift}}$ symmetry with currents

$$J = ir\partial\Phi_L, \quad \bar{J} = ir\bar{\partial}\Phi_R, \quad (3.1.15)$$

and the (Kac-Moody) primary states $|p\rangle$ are labeled by the momentum and winding modes $n, w \in \mathbb{Z}$. More precisely,

$$p = we + ne^* \in \Gamma_{1,1}, \quad (3.1.16)$$

with $\Gamma_{1,1} \subset \mathbb{R}^{1,1}$ the even self-dual lattice spanned by two lattice vectors e and e^* satisfying $e \cdot e = e^* \cdot e^* = 0$ and $e \cdot e^* = 1$. Here $a \cdot b$ denotes the inner product on $\mathbb{R}^{1,1}$ induced by the bilinear pairing on the lattice. Note that for our discussions of lattices here and in what follows, we will take \mathbb{R}^{n_L, n_R} to have Lorentzian metric

$$\eta = \begin{pmatrix} -\mathbb{1}_{n_L} & 0 \\ 0 & +\mathbb{1}_{n_R} \end{pmatrix}. \quad (3.1.17)$$

Our theory has a one-dimensional moduli space, which we can think of as a specification of a spacelike 1-plane $\Pi \subset \mathbb{R}^{1,1}$. We can always choose a basis vector for Π

$$\pi = e + r^2 e^*, \quad (3.1.18)$$

so that $\tilde{\pi} = e - r^2 e^*$ spans the orthogonal complement Π_\perp .

With that preparation, we write the operator corresponding to the state $|p\rangle$ as

$$\mathcal{V}_p = : \mathbf{C}(p) \exp \left[\frac{i}{\sqrt{2}} (\pi \cdot p) \Phi_R(\bar{z}) + \frac{i}{\sqrt{2}} (\tilde{\pi} \cdot p) \Phi_L(z) \right] : . \quad (3.1.19)$$

Here $\mathbf{C}(p)$ is a ‘‘cocycle operator’’ in the literature, is necessary to ensure proper commutation relations of the vertex operators. Using the OPEs, it is now straightforward to see that \mathcal{V}_p carries $U(1)_L^{\text{shift}} \times U(1)_R^{\text{shift}}$ charges

$$q_L^{\text{sh}} = \frac{\tilde{\pi} \cdot p}{r\sqrt{2}}, \quad q_R^{\text{sh}} = \frac{\pi \cdot p}{r\sqrt{2}} \quad (3.1.20)$$

and weights

$$h_L = \frac{1}{2}(q_L^{\text{sh}})^2 = \frac{1}{4r^2} (n - wr^2)^2, \quad h_R = \frac{1}{2}(q_R^{\text{sh}})^2 = \frac{1}{4r^2} (n + wr^2)^2. \quad (3.1.21)$$

The spin of the operator is

$$s(p) = h_L - h_R = -\frac{1}{2}p \cdot p = -nw. \quad (3.1.22)$$

Note that in these conventions the self-dual radius is $r = 1$, with the T-duality map being $r \mapsto 1/r$ and $(n, w) \mapsto (w, n)$.

With this preparation, we define the action of the shift symmetry $\mathbb{Z}_N^{\text{shift}}$, taking its generator g_{sh} to act as

$$g_{\text{sh}}|p\rangle = e^{2\pi i n/N}|p\rangle. \quad (3.1.23)$$

Since the momentum quantum number n is conserved, this is clearly a symmetry of the spectrum, of the OPE, and of the correlation functions. Moreover, we can represent this action in terms of the conserved shift currents (or, rather, the corresponding conserved charges):

$$g_{\text{sh}} = e^{\frac{2\pi i}{N}Q}, \quad (3.1.24)$$

with the charge Q given by

$$Q = \frac{r}{\sqrt{2}} (J_{L,0}^{\text{sh}} + J_{R,0}^{\text{sh}}). \quad (3.1.25)$$

The orbifold projection is then onto states with $Q \in N\mathbb{Z}$.

This is a key simplification in the orbifold analysis, since it allows us to directly construct the twisted Hilbert space—see [49] for a recent pedagogical discussion.⁸ To carry this out, we find a twist field of the form

$$\Sigma_k(z, \bar{z}) = \exp \left[-i \frac{r^2}{\sqrt{2}} \tau_k \Phi_R - i \frac{r^2}{\sqrt{2}} \tilde{\tau}_k \Phi_L \right], \quad (3.1.26)$$

where the parameters τ_k and $\tilde{\tau}_k$ are chosen so that the OPE $\mathcal{V}_p(z, \bar{z})$ and $\Sigma_k(z, \bar{z})$ has the correct monodromy, i.e. so that under a continuous rotation $z \rightarrow e^{i\theta}z$, we obtain

$$\mathcal{V}_p(e^{2\pi i}z, e^{-2\pi i}\bar{z})\Sigma_k(0) = e^{\frac{2\pi i k n}{N}} \mathcal{V}_p(z, \bar{z})\Sigma_k(0) \quad (3.1.27)$$

The field Σ then gives the k -th twisted ground state $|0; k\rangle = \lim_{z, \bar{z} \rightarrow 0} \Sigma(z, \bar{z})|0\rangle$, and the full Hilbert space in the k -th twisted sector is constructed from $|p; k\rangle$ —obtained by acting further with the \mathcal{V}_p on the $|0; k\rangle$ —by acting with all possible oscillators.

In our example a quick computation shows that choosing

$$\tau_k = -\frac{k}{N}, \quad \tilde{\tau}_k = \frac{k}{N} \quad (3.1.28)$$

produces the correct monodromy. Moreover, it is straightforward to read off the shifts in the weights and charges due to the twist, i.e. the weights and charges of the state $|p; k\rangle$:

$$q_L^{\text{sh}}(p; k) = \frac{\tilde{\pi} \cdot p - \tilde{\tau}_k r^2}{r\sqrt{2}}, \quad q_R^{\text{sh}}(p; k) = \frac{\pi \cdot p - \tau_k r^2}{r\sqrt{2}}, \quad (3.1.29)$$

⁸A recent discussion of the more general situation is given in [54].

and the weights are again

$$h_L(p; k) = \frac{1}{2}(q_L^{\text{sh}}(p; k))^2 \quad h_R(p; k) = \frac{1}{2}(q_R^{\text{sh}}(p; k))^2 . \quad (3.1.30)$$

Not only do we know the twisted Hilbert spaces, but we also know how to implement the projection onto invariant states: we just need to project onto states with

$$Q(p; k) = \frac{r}{\sqrt{2}} (q_L^{\text{sh}}(p; k) + q_R^{\text{sh}}(p; k)) \in N\mathbb{Z} . \quad (3.1.31)$$

For our circle shift orbifold it is easy to check that $Q(p; k) = Q(p)$, and also that the spin satisfies

$$s(p; k) = h_L(p; k) - h_R(p; k) = s(p) - \frac{k}{N}Q(p; k) , \quad (3.1.32)$$

so that \mathbb{Z}_N -invariant states are guaranteed to have integer spin.

Carrying out the construction for the $N - 1$ twisted sectors, we find the expected structure: the projection sets the momentum modes to be valued in $N\mathbb{Z}$, as is consistent with a circle of radius r/N , while the k -th twisted sector adds in the winding modes with $w \in \frac{k}{N} + \mathbb{Z}$, which are the “extra” winding modes for a circle of radius r/N relative to that of radius r .

The X_5 orbifold

Having understood the shift orbifold in detail, we could easily construct the full partition function for the X_5 CFT. The only modification to our previous discussion of the $\mathbb{T}^4/\mathbb{Z}_N$ compactification is to treat the right-moving superpartner of $\Phi(z, \bar{z})$ as part of the internal CFT rather than belonging to the CFT describing the $\mathbb{R}^{1,5}$ degrees of freedom.⁹

If our interest is in the massless spectrum, then the effect of the extra shift at generic radius is simple to understand: the untwisted massless states are exactly those of the $\mathbb{T}^4/\mathbb{Z}_N \times S^1$ CFT, while all of the twisted sector states are massive. This observation was the starting point for our study, since it suggests a close relationship between the two theories.

A spacetime picture

There is another way to think about the X_5 compactification:

$$\begin{array}{ccc} \mathbb{T}^4 & \hookrightarrow & X_5 \\ & & \downarrow \mu_{\text{cir}} \\ & & S^1 \end{array} .$$

We start in the $d = 6$ (1,1) obtained by heterotic compactification on \mathbb{T}^4 . The worldsheet symmetries we identified in our discussion of the $\mathbb{T}^4/\mathbb{Z}_N$ orbifold are interpreted as spacetime gauge symmetries, and in particular there is a discrete gauge symmetry $G = \mathbb{Z}_N$ which is a subgroup

$$G \subset G_{\text{ab}} \times \text{SU}(2)'_L \times \text{SU}(2)'_R , \quad (3.1.33)$$

the latter action being part of the spacetime R-symmetry group of the $d = 6$ (1,1) theory. Given the presence of this discrete spacetime gauge symmetry, when we compactify the theory on a circle with coordinate $\theta \sim \theta + 2\pi$, we can choose to turn on a holonomy for G . That is, we modify the

⁹There are some subtleties in applying the usual RNS rules in odd-dimensional compactification, but they can be easily avoided by introducing an additional spectator circle with its superpartner; see e.g. [55].

periodicity conditions on the fields as

$$\phi(x, \theta + 2\pi) = g \cdot \phi(x, \theta) . \quad (3.1.34)$$

The holonomy will have the effect that only G -invariant fields will have zero modes in the θ expansion, so that the low energy theory obtained in this way will have the same massless spectrum as compactification on X_5 . Because G involves an $SU(2)'_R$ action, half of the gravitinos will be lifted in the process, reducing supersymmetry from 16 to 8 supercharges.

Our point of view is that the orbifold construction of the heterotic string on X_5 is a UV completion of circle compactification of the spacetime theory with a G -holonomy. Note we are not suggesting a spacetime interpretation relating compactification based on a CFT to that based on the orbifold CFT. Instead, the relationship is between a compactification based on a freely acting orbifold and a circle compactification of a theory from one dimension higher.

It is clear that the existence of a UV completion is a non-trivial condition because not every global discrete symmetry of the heterotic CFT leads to a modular-invariant orbifold, yet we expect such symmetries to still correspond to gauge symmetries in the spacetime theory. Thus, there should in general be obstructions to turning on non-trivial holonomy for a discrete spacetime gauge symmetry G . We hope to return to a study of these in the future.

3.2 The heterotic worldsheet and duality

In the previous section we presented two classes of $d = 5$ heterotic compactifications that preserve 8 supercharges. While we saw that their spectra are closely related, and in particular are identical when restricted to the massless untwisted states, in general the theories appear to be distinct. As a stark difference we noted that $\mathbb{T}^4/\mathbb{Z}_N \times S^1$ compactifications can be deformed to $K3 \times S^1$ by going on the Higgs branch, while no such Higgs branch appears to be present in the X_5 compactifications.

Nevertheless, we will now show that by enlarging the moduli space to include Wilson lines we can establish an isomorphism between these classes of theories. Thus, not only are these compactifications connected in moduli space, they are in fact identical.

3.2.1 Bosonic construction

Turning on Wilson line parameters is best described in the bosonized construction of the heterotic worldsheet CFT on \mathbb{T}^d [56, 57]. Grouping the left-moving fermions into 16 Weyl fermions $\xi^a, \bar{\xi}^a$, so that the currents $J^a = \xi^a \bar{\xi}^a$ generate a Cartan algebra of $\mathfrak{e}_8 \oplus \mathfrak{e}_8$, we bosonize these as $J^a = i\partial\mathcal{X}_L^a$. The bosonic OPEs now depend on the metric g_{ij} on \mathbb{T}^d :

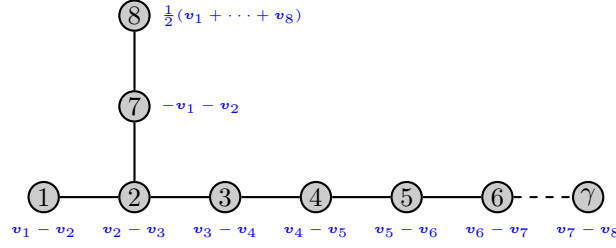
$$\begin{aligned} \Phi_L^i(z)\Phi_L^j(w) &\sim -g^{ij}\log(z-w) , & \Phi_R^i(\bar{z})\Phi_R^j(\bar{w}) &\sim -g^{ij}\log(\bar{z}-\bar{w}) , \\ \mathcal{X}_L^a(z)\mathcal{X}_L^b(w) &\sim -\delta^{ab}\log(z-w) . \end{aligned} \quad (3.2.1)$$

The vertex operators are now labeled by $p \in \Gamma_{d+16,d} \subset \mathbb{R}^{d+16,d}$. The lattice isomorphism $\Gamma_{d+16,d} \simeq (\Gamma_{1,1})^5 + \Gamma_8 + \Gamma_8$ —see e.g. [16, 58] for details of the isomorphism relevant to the relation between the $E_8 \times E_8$ and $\text{Spin}(32)/\mathbb{Z}_2$ heterotic strings—induces an isomorphism $\mathbb{R}^{d+16,d} \simeq R^{d,d} \oplus \mathbb{R}^8 \oplus \mathbb{R}^8$, and we can use this to pick a basis that isometrically respects this splitting. For the $\mathbb{R}^{d,d}$ factor we choose lattice vectors e_i, e^{*i} satisfying

$$e_i \cdot e^{*j} = \delta_i^j , \quad e_i \cdot e_j = 0 , \quad e^{*i} \cdot e^{*j} = 0 , \quad (3.2.2)$$

while for each of the Γ_8 factors we choose the set of simple roots $\alpha_I, I = 1, \dots, 8$, of \mathfrak{e}_8 , with $\alpha_I \cdot \alpha_J$ the Killing metric, taken to be negative in our conventions. The α_I are encoded in the

Dynkin diagram written in terms of the standard orthonormal basis for \mathbb{R}^8 , denoted by \mathbf{v}_a , with $a = 1, \dots, 8$ and $\mathbf{v}_a \cdot \mathbf{v}_b = -\delta_{ab}$:



To accommodate the second (our hidden) \mathfrak{e}_8 factor, we will let the indices I, a run through $9, \dots, 16$ as well, but this will play no role in our analysis.

With the lattice set up complete, a point in the Narain moduli space corresponds to a choice of spacelike d -plane Π spanned by the vectors π_i , which can be taken to be of the form¹⁰

$$\pi_i = \mathbf{e}_i + (g_{ij} + b_{ij} - \frac{1}{2}A_i \cdot A_j)\mathbf{e}^{*j} + A_i, \quad A_i = A_i^I \alpha_I = A_i^a \mathbf{v}_a. \quad (3.2.3)$$

In addition to the torus metric g_{ij} , this also encodes the choice of constant B -field b_{ij} and Wilson lines A_i^a valued the Cartan subalgebra of \mathfrak{e}_8 .

The spacelike vectors satisfy $\pi_i \cdot \pi_j = 2g_{ij}$, and it is easy to find a basis for the orthogonal complement Π_\perp , a timelike $(d + 16)$ -plane. We take it to be spanned by

$$\tilde{\pi}_i = \pi_i - 2g_{ij}\mathbf{e}^{*j}, \quad \tilde{\pi}_a = \mathbf{v}_a + A_i^a \mathbf{e}^{*i}. \quad (3.2.4)$$

These two sets of vectors are orthogonal to the π_i and each other, and satisfy $\tilde{\pi}_i \cdot \tilde{\pi}_j = -2g_{ij}$ and $\tilde{\pi}_a \cdot \tilde{\pi}_b = -\delta_{ab}$. Extending our discussion of the compact boson, we arrive at the vertex operators

$$\mathcal{V}_p = : \mathcal{C}(p) \exp \left[\frac{i}{\sqrt{2}}(\pi_i \cdot p) \Phi_R^i(\bar{z}) + \frac{i}{\sqrt{2}}(\tilde{\pi}_i \cdot p) \Phi_L^i(z) + i(\tilde{\pi}_a \cdot p) \mathcal{X}_L^a(z) \right] : \quad (3.2.5)$$

for all

$$p = w^i \mathbf{e}_i + n_i \mathbf{e}^{*i} + \lambda \in \Gamma_{16+d,d}, \quad \lambda \in \Gamma_8 + \Gamma_8. \quad (3.2.6)$$

The weights of these operators¹¹ are given by a generalization of (3.1.21):

$$h_R(p) = \frac{1}{4}(\pi_i \cdot p)g^{ij}(\pi_j \cdot p), \quad h_L(p) = \frac{1}{4}(\tilde{\pi}_i \cdot p)g^{ij}(\tilde{\pi}_j \cdot p) + \sum_a \frac{1}{2}(\tilde{\pi}_a \cdot p)^2, \quad (3.2.7)$$

and the spin satisfies

$$s(p) = h_L(p) - h_R(p) = -\frac{1}{2}p \cdot p = -n^i w_i - \frac{1}{2}\lambda \cdot \lambda. \quad (3.2.8)$$

Since Γ_8 is an even lattice the last term is an integer for all λ , so that $s(p) \in \mathbb{Z}$.

Finally, to complete the heterotic construction we need to add the right-moving superpartners of the Φ^i , the fermions $\psi^i(\bar{z})$. For all of our applications we will be able to group these into the two Weyl fermions $\psi^{1,2}, \bar{\psi}^{1,2}$ for the \mathbb{T}^4 directions, and the extra Majorana-Weyl ψ^5 for the additional circle direction.

¹⁰We follow, with some small adjustments, the presentation given in [46]

¹¹The factor $\mathcal{C}(p)$ will be addressed later.

3.2.2 Two orbifolds

Starting with this presentation of the heterotic string on \mathbb{T}^5 , we now consider the special locus where the \mathbb{T}^4 factor admits a $G = \mathbb{Z}_N$ symmetry with generator g . The \mathbb{Z}_N symmetry leads to significant simplification in the CFT moduli: both g_{i5} and b_{i5} must be zero for $i \neq 5$, and the $A_{i \neq 5}$ may be set to zero as well.¹²

To understand how G acts on the states, we reconsider the actions described in our RNS discussion. These were the geometric orbifold action on \mathbb{T}^4 , accompanied by its supersymmetric extension to the right-moving fermions, the translation on S^1 , and the action on the left-moving fermions $\gamma^{1,2}$. We will now translate each of these into the bosonic description.

First, the geometric action on the \mathbb{T}^4 bosons induces an action on the associated vectors $\vec{p} \in \Gamma_{d,d}$. If the Φ^i coordinates transform as

$$g_T \cdot \Phi^i = R_j^i \Phi^j, \quad (3.2.9)$$

then we have

$$g_T \cdot w^i = R_j^i w^j, \quad g_T \cdot n_i = (R^{-1})_i^j n_j. \quad (3.2.10)$$

This clearly preserves the spin of every \mathcal{V}_p , and, because it is a torus isometry, it preserves the weights as well. The right-moving fermions also transform accordingly, and as we discussed around equation (3.1.5), it is convenient to represent the action by $g_\psi = \exp[\frac{2\pi i}{N} \bar{J}'_0]$.

Second, the translation in the circle direction is precisely the shift symmetry g_{sh} described in section 3.1.3. It leaves the lattice vectors invariant and acts on states by a \mathbb{Z}_N -valued phase.

Finally, we have the action on the γ fermions, represented by $g_\gamma = \exp[\frac{2\pi i}{N} J'_0]$. This is the only one that is not quite obvious in the bosonized description, but because we can write it in terms of the current J' , finding the corresponding action on the \mathcal{X}^a is not too difficult. Taking a look at our Dynkin diagram, we see that

$$J' = i\partial\mathcal{X}^7 - i\partial\mathcal{X}^8, \quad (3.2.11)$$

i.e. it is the current that corresponds to the extended root of the diagram. Thus, g_γ turns out to be another shift action in the bosonized description:

$$g_\gamma |p\rangle = e^{\frac{2\pi i}{N} (\mathbf{v}_7 - \mathbf{v}_8) \cdot p} |p\rangle. \quad (3.2.12)$$

We can see this gives the expected structure for the unbroken symmetry with $A = 0$. In this case the \mathfrak{e}_8 currents that survive the projection are the Cartan currents $i\partial\mathcal{X}^a$, as well as all the roots α_I orthogonal to $\mathbf{v}_7 - \mathbf{v}_8$. For $N > 2$ this gives rise to the $\mathfrak{e}_7 \oplus \mathfrak{u}(1)'$ current algebra, and in the special case of $N = 2$ the state corresponding to the $\mathbf{v}_7 - \mathbf{v}_8$ root is left invariant as well, leading to the $\mathfrak{u}(1)' \rightarrow \mathfrak{su}(2)'$ enhancement.

As above, we have two \mathbb{Z}_N symmetries by which we can orbifold: G , generated by $g = g_T g_\psi g_\gamma g_{\text{sh}}$, leads to the X_5 orbifold, while G' , generated by $g' = g_T g_\psi g_\gamma$, produces the $\mathbb{T}^4/\mathbb{Z}_N \times S^1$ orbifold.

Both of the symmetries are consistent with turning on a Wilson line A_5 along the shift circle, and by including this degree of freedom, we will demonstrate that, despite appearances, the two orbifolds are equivalent. We will do this by finding an element t of the T-duality group $O(\Gamma_{17,1}) \subset O(\Gamma_{21,5})$

¹²Note the $A_{i \neq 5}$ Wilson lines are not required to be set to zero by our \mathbb{Z}_N symmetry because there is an accompanying action on the left-moving fermions, or equivalently on the \mathcal{X}_a . In the orbifold theory these symmetry-preserving Wilson line parameters describe hypermultiplet expectation values.

such that

$$g' = t^{-1}gt . \quad (3.2.13)$$

To give this construction, we will have to delve a little bit into the structure of the heterotic T-duality group.

3.2.3 Elements of $O(\Gamma_{17,1})$

The T-duality group $O(\Gamma_{16+d,d})$ arises from lattice isomorphisms, and each such isomorphism induces a (possibly trivial) action on the moduli. A discussion of its generators is given in [16, 59]. In our discussion two elements will play a role: the T-duality transformation on the 5-th circle, as well as a shift of the Wilson line associated to the circle by a lattice vector in Γ_8 .

Typically, the induced action on the moduli is complicated. For example, a T-duality on a single circle usually involves a non-trivial action all of the g_{ij} , b_{ij} and A_i components [16, 59]. However, since we assume the \mathbb{Z}_N symmetry, the moduli are restricted, so that we may as well think of our transformations as living in $O(\Gamma_{17,1})$.

A vector in $\Gamma_{17,1}$ has the form

$$p = we + ne^* + \lambda , \quad (3.2.14)$$

and using (3.2.7) with $g_{55} = r^2$, and $A_5 = A$, we find the weight and spin

$$h_R(p) = \frac{1}{4r^2} (\pi \cdot p)^2 , \quad s(p) = h_L(p) - h_R(p) = -\frac{1}{2}p \cdot p = -nw - \frac{1}{2}\lambda \cdot \lambda , \quad (3.2.15)$$

with

$$\pi \cdot p = n + \left(r^2 - \frac{1}{2}A \cdot A\right)w + A \cdot \lambda . \quad (3.2.16)$$

The T-duality action on the lattice is simply the exchange $(n, w) \rightarrow (w, n)$. Let us call this action $g_1(p)$. Clearly $g_1(p) \cdot g_1(p) = p \cdot p$ for all p , which is necessary and sufficient for g_1 to be a lattice isomorphism. Moreover, we can find an induced action the moduli (r, A) , so that

$$(g_1^* h_R)(p) = h_R(g_1(p)) : \quad (3.2.17)$$

$$(r', A') = g_1^*(r, A) , \quad r' = \frac{r}{r^2 - \frac{1}{2}A \cdot A} , \quad A' = \frac{A}{r^2 - \frac{1}{2}A \cdot A} . \quad (3.2.18)$$

The lattice vector shift of the Wilson line arises through a more elaborate lattice isomorphism, depending on a choice of lattice vector $\rho \in \Gamma_8 + \Gamma_8$:

$$g_2(p) = p + \left(\rho \cdot \lambda - \frac{1}{2}\rho \cdot \rho w\right)e^* - \rho w , \quad (3.2.19)$$

or more explicitly,

$$g_2(n, w, \lambda) = \left(n + \rho \cdot \lambda - \frac{1}{2}\rho \cdot \rho w, w, \lambda - \rho w\right) . \quad (3.2.20)$$

This action also preserves the lattice since $g_2(p) \cdot g_2(p) = p \cdot p$, and the induced map on the moduli is simply

$$(r', A') = g_2^*(r, A) = (r, A + \rho) . \quad (3.2.21)$$

Having reviewed the form of these basic maps, we will now use them to construct an isomorphism between the two orbifold theories, i.e. the compactification on X_5 and the $\mathbb{T}^4/\mathbb{Z}_N \times S^1$ orbifold.

3.2.4 Heterotic isomorphism

Let $t = g_1 g_2$. We claim that this combination of $O(\Gamma_{17,1})$ elements leads to the desired equivalence (3.2.13) for an appropriate choice of lattice vector ρ . Since t does not act on the \mathbb{T}^4 bosons or right-moving fermions, and our two actions g and g' act in the same way on those degrees of freedom, it is sufficient to check the claim for $p \in \Gamma_{17,1}$.

We set the ρ to be proportional to the root $\alpha_6 = \mathbf{v}_6 - \mathbf{v}_7$:

$$\rho = (1 - N)\alpha_6, \quad (3.2.22)$$

and we define for convenience $\gamma = \mathbf{v}_7 - \mathbf{v}_8$. Note that $\gamma \cdot \rho = 1 - N$. We now compute

$$t|n, w, \lambda\rangle = |w, n + \rho \cdot \lambda - \frac{1}{2}\rho \cdot \rho w, \lambda - \rho w\rangle, \quad (3.2.23)$$

and therefore

$$t^{-1}gt|n, w, \lambda\rangle = \exp\left[\frac{2\pi i}{N}(w + \gamma \cdot (\lambda - \rho w))\right]|n, w, \lambda\rangle = e^{\frac{2\pi i}{N}\gamma \cdot \lambda}|n, w, \lambda\rangle = g'|n, w, \lambda\rangle. \quad (3.2.24)$$

In the second equality the phase dependence on the winding mode w drops out precisely because $\gamma \cdot \rho = 1 - N$. Thus, despite appearances, the two orbifold CFTs are isomorphic!

Special features of $N = 3$

As we remarked above, the correspondence between the theories is particularly nice when $N = 3$, since there it is possible to match the massless spectra by turning on a single Wilson line for the $\mathbf{u}(1)'$ vector multiplet. We can see these special features from the point of view of our isomorphism as well. Precisely when $N = 3$ we can set the Willson line shift to be $\rho = \gamma$, and the winding mode will again drop out from the phase factor.

In this case, then, we can identify the X_5 theory with radius r and circle Wilson line $A_5 = a\gamma$ with the $\mathbb{T}^4/\mathbb{Z}_3 \times S^1$ theory with radius r' and circle Wilson line $A_5 = a'\gamma$ via

$$r = \frac{r'}{r'^2 + (a'-1)^2}, \quad a = \frac{a'-1}{r'^2 + (a'-1)^2}. \quad (3.2.25)$$

As expected, the duality is a stringy one. For example, setting $a' = 1$, which corresponds to the $a = 0$ X_5 theory analyzed above, we find that a large radius circle on the $\mathbb{T}^4/\mathbb{Z}_3 \times S^1$ side corresponds to X_5 with a small shift circle.

Note that $a' = 1$ is equivalent to $a' = 0$ in the \mathbb{T}^5 theory, but that is not the case in the orbifold, where the Wilson line compactness is instead through the equivalence $a' \sim a' + 3$. This is closely related to the appearance of the fractional $U(1)'_L$ charges in the twisted sector states, and it is thus not surprising that such massless states are lifted for $a' = 1$.

3.3 Cocycle subtleties and their resolution

We now explain the need for the factor $C(p)$ in eq 3.2.5 and point out that in general their presence can lead to subtleties in the analysis of CFT symmetries. We will also see, however, that in this case we are lucky: all of the potential subtle factors drop out, and the duality conclusions reached in the previous section remain unmodified. Nevertheless, since the general observations here may lead to subtleties in closely related discussions, we will include them. In addition to the standard

textbook references [60, 61], which supply some of the background, our thinking about these issues was guided by the work [62].

A class of CFT symmetries

We are interested in discussing a class of CFT symmetries of Narain compactification that are realized as the following G action on the vertex operators. For every $g \in G$ there is a map $\varphi_g : \Gamma \rightarrow \Gamma$ and a factor $U(g, p) \in \mathbb{C}$ such that

$$g \circ \mathcal{V}_p = U(g, p) \mathcal{V}_{\varphi_g(p)} . \quad (3.3.1)$$

Not all CFT symmetries can be realized in this fashion. For example, the $SU(2) \times SU(2)$ symmetries of the compact boson at self-dual radius take a more general form and also mix the \mathcal{V}_p with non-Kac-Moody primary operators such as $\partial\Phi$ and $\bar{\partial}\Phi$.

We wish the g -action to be invertible and to be consistent with the OPE, and it must preserve the weights of the operators. Therefore it must be that for every g the factor $U(g, p) \in \mathbb{C}^*$, and φ_g is a lattice isomorphism. We will insist that the action is unitary, which means $U(g, p)$ is a pure phase. Since we also want composition to be consistent with the group product structure, i.e.

$$(g_2 g_1) \circ \mathcal{V}_p = g_2 \circ (g_1 \circ \mathcal{V}_p) , \quad (3.3.2)$$

We also learn that the phases must obey

$$U(g_2 g_1, p) = U(g_2, g_1(p)) U(g_1, p) , \quad (3.3.3)$$

and similarly the φ_g should satisfy $\varphi_{g_1 g_2}(p) = \varphi_{g_1}(\varphi_{g_2}(p))$. We now see that φ must be a map to the group of automorphisms of the lattice, i.e. $\varphi : G \rightarrow O(\Gamma)$. The resulting subgroup of $O(\Gamma)$

$$G_\Gamma \simeq G / \ker(\varphi) \quad (3.3.4)$$

is in general smaller than G : for example, the circle shift symmetry has $G = \mathbb{Z}_N$ and $G_\Gamma = 1$.

These are sensible constraints determined by the group structure, but there are further constraints on the factors $U(g, p)$, and this is where the cocycles make an appearance.

A look at the cocycles

The factors $\mathbf{C}(p)$ are introduced to resolve an issue with commutation properties of the naive vertex operators $\mathcal{V}_p^{\text{naive}}$ which lack these factors: the OPE of two such operators has a non-trivial monodromy as we transport one operator around the other, which leads to

$$: \mathcal{V}_{p_1}^{\text{naive}}(-z/2) : : \mathcal{V}_{p_2}^{\text{naive}}(z/2) : = e^{i\pi p_1 \cdot p_2} : \mathcal{V}_{p_2}^{\text{naive}}(z/2) : : \mathcal{V}_{p_1}^{\text{naive}}(-z/2) : . \quad (3.3.5)$$

Thus, the operators appear to anticommute whenever $p_1 \cdot p_2$ is odd. To resolve this, the operators are modified to include the $\mathbf{C}(p)$, which are chosen so that

$$\mathcal{V}_q \mathbf{C}_p = \varepsilon(q, p) \mathbf{C}_p \mathcal{V}_q , \quad \mathbf{C}_p \mathbf{C}_q = \mathbf{C}_{p+q} \quad (3.3.6)$$

for some phase $\varepsilon(q, p)$. Including these factors will remove the unwanted factor $e^{i\pi p_1 \cdot p_2}$ if the phases obey

$$\varepsilon(p_2, p_1) = (-1)^{p_1 \cdot p_2} \varepsilon(p_1, p_2) . \quad (3.3.7)$$

Associativity of the OPE places further non-trivial conditions on the phases:

$$\varepsilon(p_1, p_3)\varepsilon(p_2, p_3) = \varepsilon(p_1 + p_2, p_3) , \quad (3.3.8)$$

and

$$\varepsilon(p_1, p_2)\varepsilon(p_1 + p_2, p_3) = \varepsilon(p_1, p_2 + p_3)\varepsilon(p_2, p_3) . \quad (3.3.9)$$

The second of these justifies the name of cocycle, i.e. $\varepsilon \in H^2(\Gamma, U(1))$. Consider the maps $\Gamma \times \Gamma \rightarrow U(1)$, denoted by $c(p_1, p_2)$. The coboundary map to the group of homomorphisms $\Gamma \times \Gamma \times \Gamma \rightarrow U(1)$ is taken to be

$$\delta_{32}c(p_1, p_2, p_3) = \frac{c(p_2, p_3)}{c(p_1 + p_2, p_3)} \frac{c(p_1, p_2 + p_3)}{c(p_1, p_2)} , \quad (3.3.10)$$

Similarly, starting with maps $\Gamma \rightarrow U(1)$ denoted by $f(p)$, we have

$$\delta_{21}f(p_1, p_2) = \frac{f(p_1)f(p_2)}{f(p_1 + p_2)} , \quad (3.3.11)$$

and the reader can check $\delta_{32}(\delta_{21}f) = 1$. The condition (3.3.9) is the statement that $\delta_{32}\varepsilon = 1$, i.e. that ε is a cocycle that defines a class in $H^2(\Gamma, U(1))$. On the other hand, the condition (3.3.7) shows that $\varepsilon(p_1, p_2)$ cannot be a coboundary.

Cocycles and symmetry phases

Finally we come to our key point: the cocycle factors and the phases $U(g, p)$ are intertwined, and the consistency of the OPE with the symmetry action, i.e.

$$: g \circ \mathcal{V}_p(z_1) : : g \circ \mathcal{V}_q(z_2) : = g \circ (: \mathcal{V}_p(z_1) : : \mathcal{V}_q(z_2) :) \quad (3.3.12)$$

require the phase factors to obey [62]

$$\frac{U(g, p+q)}{U(g, p)U(g, q)} = \frac{\varepsilon(\varphi_g(p), \varphi_g(q))}{\varepsilon(p, q)} . \quad (3.3.13)$$

There are two ways to read this equation. On one hand, it shows that the ratio of cocycles $\frac{\varepsilon(\varphi_g(p), \varphi_g(q))}{\varepsilon(p, q)}$ is a coboundary, and $U(g, p)$ is the trivializing cycle. On the other hand, whenever the ratio is not 1, it gives an obstruction to choosing $U(g, p)$ to be a group homomorphism from the lattice to $U(1)$.

There are two obvious questions about this result: how does it depend on the choice of cocycle? to what extent does it determine the phases $U(g, p)$? We leave it to the reader to check that a modification of ε by a coboundary $f(p)$ modifies $U(g, p) \rightarrow U(g, p)f(g(p))/f(p)$, and the new $U(g, p)$ so obtained is consistent with the group structure. As far as the second question goes, it is clear that if $U_1(g, p)$ and $U_2(g, p)$ both solve (3.3.13) with the same ε , then their ratio U_1/U_2 is a homomorphism from Γ to $U(1)$ for all g .

Duality unmodified

We just discussed at length that symmetry actions of the sort we used to prove the duality between X_5 and $\mathbb{T}^4/\mathbb{Z}_N \times S^1$ heterotic compactifications in general come with extra phase factors caused by T-duality action t . Since the whole point of our duality claim was a match of phase factors for two group actions, this is a non-trivial concern.

To allay this worry, we write the correct form for the symmetry action on the states, taking into account the possible extra phases. In keeping with the spirit of this section, we distinguish in our notation between the action on the states and the action on the lattice vectors, so that we have, coming back to (3.2.13),

$$t^{-1}gt|p\rangle = U(t,p)U(g,\varphi_t(p))U(t^{-1},\varphi_{gt}(p))|\varphi_{t^{-1}gt}(p)\rangle . \quad (3.3.14)$$

The group multiplication properties imply the relation

$$1 = U(t^{-1}t,\varphi_{t^{-1}}(p)) = U(t^{-1},p)U(t,\varphi_t^{-1}(p)) , \quad (3.3.15)$$

so that

$$U(t^{-1},p) = \frac{1}{U(t,\varphi_t^{-1}(p))} . \quad (3.3.16)$$

But, since we have $\varphi_{t^{-1}gt}(p) = p$, the phase factor becomes

$$t^{-1}gt|p\rangle = U(t,p)U(g,\varphi_t(p))\frac{1}{U(t,p)}|\varphi_{t^{-1}gt}(p)\rangle = U(g,\varphi_t(p))|p\rangle , \quad (3.3.17)$$

but that is exactly the factor we showed to be equal to $U(g',p)$ in eq 3.2.24.

Chapter 4

Type II string compactification on X_5

We have seen Heterotic string compactified on orbifolds $\mathbb{T}^4/\mathbb{Z}_{2,3,4,6} \times S^1$ are equivalent to Heterotic string compactified on smooth Seifert manifolds $(\mathbb{T}^4 \times S^1)/\mathbb{Z}_{2,3,4,6}$. In this chapter we study perturbative Type II string compactification on these two classes of flat manifolds (orbifolds), whose low energy limit is 5d supergravity with 16 supercharges.

4.1 T-dual of Type II string compactifies on Seifert manifolds: a space-time point of view

Without loss of generality, we study type IIA on Seifert manifold T^5/\mathbb{Z}_2 . This Seifert manifold can be considered as S^1 fibered over orbifold T^4/\mathbb{Z}_2 . One fact about this Seifert manifold is that when one moves around a non-trivial loop around the 16 fixed points on the base caused by the orbifold \mathbb{Z}_2 action, a half circle shift on the S^1 fiber accompanied. So if we compactify the type IIA on the S^1 fiber first, we get a 9d theory on the orbifold T^4/\mathbb{Z}_2 together with a Wilson line of 9d KK vector $\int_{\pi_1} g_{\mu 9} = \frac{1}{2}$ ¹ around each fixed point (π_1 corresponds to the nontrivial loop due to the orbifold action)

T dual the S^1 fiber exchanges $B_{\mu 9}$ and $g_{\mu 9}$. Hence T dual gives a 9d theory (IIB on the dual S^1) on orbifold T^4/\mathbb{Z}_2 with Wilson line configuration $\int_{\pi_1} B_{\mu 9} = \frac{1}{2}$. Now if we lift to 10d, the internal configuration of compactification would be a S^1 bundle over T^4/\mathbb{Z}_2 together with non-trivial B-flux $\int_{\pi_1 \times S^1} B = \frac{1}{2}$ centered around the fixed points on the base. The S^1 fiber is trivial and the geometric picture on the IIB side is $S^1 \times T^4/\mathbb{Z}_2$ ²

This spacetime viewpoint also suggests one possible mechanism of frozen singularity that is discussed in 2.2.2. Notice a generic case of type IIA on Seifert manifold $(\mathbb{T}^4 \times S^1)/\mathbb{Z}_2$ gives a 5d supergravity with 4 $U(1)$ vector multiplets, while type IIB on $\mathbb{T}^4/\mathbb{Z}_2 \times S^1$ gives 20 $U(1)$ vector multiplets with 16 come from 16 singularities on T^4/\mathbb{Z}_2 . Remember these 5d supergravities have 16 supercharges so the moduli space of two supergravity theories cannot be connected to each other, as 5d supergravities have 16 supercharges only consist of Coulomb branch. So the non-trivial B field configuration around singularities obtained from T-dualizing the Seifert circle fibration, freezes these singularities out. It is also interesting to notice that this frozen singularity mechanism is purely perturbative.³

¹The quantity $\int_{\pi_1} g_{\mu 9}$ should be understood as holonomy

²Recall the standard T-duality argument: Starts with a non trivial S^1 fibration and trivial B field configuration, T-dual gives a trivial S^1 fibration with non trivial B field configuration.

³This mechanism is related to the known non-perturbative mechanism of frozen singularity [24] through $SL(2, \mathbb{Z})$ duality, as we will discuss later.

4.2 Equivariant gerbes and orbifold CFT

In this section we connect the spacetime perspective with an explicit worldsheet computation. Equivariant gerbes is used to treat the non-trivial B-field configuration. Equivariant gerbes mean that the orbifold group acts on the entire space of spacetime manifold together with B field structure, rather than just spacetime manifold (see appendix B for an introduction).

The notion that equivariant gerbes enter the construction of the orbifold partition function was developed in [33, 35], and the second reference showed that pure holonomy equivariant gerbes produce the kinds of shift orbifold factors that are necessary to relate the X_5 and $\mathbb{T}^4/\mathbb{Z}_N \times S^1$ orbifolds. We will now review those ideas and illustrate in an example that they do lead to the expected phases and partition functions.

Let $Z_{0,0}$ denote the genus 1 partition function of a CFT which admits the action of a discrete abelian group G with 0 the additive identity element. The orbifold partition function then takes the familiar form

$$\mathcal{Z} = \frac{1}{|G|} \sum_{a,b \in G} Z_{a,b}, \quad Z_{a,b} = \text{Tr}_{\mathcal{H}_b} \rho_b(a) \mathbf{q}^{L_0 - c/24} \bar{\mathbf{q}}^{\bar{L}_0 - \bar{c}/24}, \quad (4.2.1)$$

where $\rho_b(a)$ is the representation of the G -action on the Hilbert space \mathcal{H}_b , i.e. the space of states twisted by the element b . In general the choice of representation ρ_b is not unique. For example, if $\rho_b(a)$ leads to a consistent choice of modular-invariant and unitary partition function, then so does

$$\tilde{\rho}_b(a) = \rho_b(a) \frac{\omega^{a,b}}{\omega^{b,a}} \quad (4.2.2)$$

for any representative of a class $[\omega] \in H^2(G, \text{U}(1))$ —this is the discrete torsion of [34].

The observation from [35] is that the equivariant gerbe data for a $G = \mathbb{Z}_N$ quotient enters the choice of $\rho_b(a)$ through a factor⁴

$$P_{a,b} = \exp \left[i \int \Phi^*(B) \right] \exp \left[i \int_0^1 \Phi^*(\mathcal{A}^b) - i \int_0^\tau \Phi^*(\mathcal{A}^a) \right] \quad (4.2.3)$$

in the Lagrangian computation of $Z_{a,b}$. This is also counterpart of B-field holonomy around singularities in the covering space $X = \mathbb{T}^5$. Here $\Phi : T^2 \rightarrow X$ is the map from the worldsheet torus with complex structure $\tau = \tau_1 + i\tau_2$ to the targetspace with

$$\Phi(0) = x, \quad \Phi(1) = \varphi_a(x), \quad \Phi(\tau) = \varphi_b(x), \quad \Phi(1 + \tau) = \varphi_{ab}(x). \quad (4.2.4)$$

Here $\varphi_a(x)$ is the the image of x under action of group element a in the covering space \mathbb{T}^5 . The integration contours in the z plane of the worldsheet torus are along the left and bottom boundaries of the parallelogram:



$$(4.2.5)$$

The expression $P_{a,b}$ is not easy to interpret on a general targetspace. However, our interest is in $X = \mathbb{T}^d/\mathbb{Z}_N \times S^1$, where the interpretation is straightforward in the situation when B-field configuration is flat, i.e. fully characterized by its holonomies. Focusing on the bosonic part of the

⁴When $H^2(G, \text{U}(1)) \neq 1$, there is in general an extra term in the expression that accounts for the choice of discrete torsion: $P_{a,b}$ should be further multiplied by $\omega^{a,b}/\omega^{b,a}$.

CFT, the original theory's configuration space decomposes into sectors labeled by their periodicities under shifts of $z \rightarrow z + 1$ and $z \rightarrow z + \tau$. The x_5 field corresponding to S^1 , which is neutral under the orbifold action, has configurations

$$x_5(z, \bar{z}) = \frac{i}{2\tau_2} (w - \tilde{n}\bar{\tau}) z + \frac{i}{2\tau_2} (-w + \tilde{n}\tau) \bar{z} . \quad (4.2.6)$$

While w has the interpretation as the winding quantum number, \tilde{n} is not the momentum mode n of the Hamiltonian formulation—the two are related through a Poisson resummation. In constructing the path integral of $Z_{a,b}$ we must also consider fluctuations around this classical solution, but those do not enter the phase factor $P_{a,b}$.

In our examples above, the gerbes with $B = 0$ have $\mathcal{A}^a = \frac{2\pi a}{N} dx^{55}$, and these lead to phases

$$P_{a,b} = \exp \left[\frac{2\pi i}{N} (aw - b\tilde{n}) \right] . \quad (4.2.7)$$

The contribution to the circle partition function can be evaluated by including the integration over the fluctuations around the semi-classical solution [53]. Including the phase $P_{a,b}$ the result is (our conventions are given in appendix A)

$$Z_{a,b}^{\text{cir}} = \frac{1}{\eta\bar{\eta}} \sum_w e^{2\pi i \frac{aw}{N}} \frac{\sqrt{\tau_2}}{r} \sum_{\tilde{n}} \exp \left[-S_0(\tilde{n}, w, \tau, r) - 2\pi i \frac{b}{N} \tilde{n} \right] , \quad (4.2.8)$$

where S_0 is the classical action evaluated on the solution (4.2.6). Making a Poisson resummation on \tilde{n} , we therefore obtain

$$Z_{a,b}^{\text{cir}} = \frac{1}{\eta\bar{\eta}} \sum_w e^{2\pi i \frac{aw}{N}} \frac{\sqrt{\tau_2}}{r} \sum_n \int_{-\infty}^{\infty} dx \exp \left[-S_0(x, w, \tau, r) - 2\pi i \frac{b}{N} x - 2\pi i x n \right] . \quad (4.2.9)$$

When $b = 0$, this leads to the usual expression for the circle partition function with an extra phase:

$$Z_{a,0}^{\text{cir}} = \frac{1}{\eta\bar{\eta}} \sum_{n,w} e^{2\pi i \frac{aw}{N}} \mathbf{q}^{h_{n,w}} \bar{\mathbf{q}}^{\bar{h}_{n,w}} , \quad (4.2.10)$$

but now the form of the b -dependence under the x integral shows that more generally including b just leads to the shift $n \rightarrow n + b/N$ in the weights $h_{n,w}$ and $\bar{h}_{n,w}$. But this is precisely the partition function of the T-dual of the shift orbifold of the circle! Since the remaining degrees of freedom, i.e. the fermions and the bosons of the \mathbb{T}^d theory, are treated in the same way in both the X_5 and $\mathbb{T}^d/\mathbb{Z}_N \times S^1$ orbifolds, we therefore see that the gerbe data and the associated phase correctly reproduce the T-dual of the \mathbb{Z}_N quotient that leads to the smooth X_5 geometries.

4.3 Frozen singularities and a dual perspective

The geometry $\mathbb{T}^4/G \times S^1$ is familiar to every string theorist as a limit of compactification on $\text{K3} \times S^1$. The resulting spacetime physics depends on the way in which the limit is taken. For example, if we focus on type II compactification, then the choice of B -field determines whether the limiting theory will lead to additional massless gauge degrees of freedom (in IIA) or tensionless strings (in IIB) [45, 64, 65]. These arise at points in the moduli space where the CFT breaks down, and worldsheet non-perturbative effects are crucial. On the other hand, there is a choice of B -field such that the limit leads precisely to the orbifold CFT for \mathbb{T}^4/G . In each of these cases it is possible to perturb the moduli to a more generic point and arrive at a smooth K3 geometry with some

⁵This is the data in the covering space \mathbb{T}^5 to produce the holonomy around singularities on $\mathbb{T}^4/\mathbb{Z}_2 \times S^1$

small cycles. From the point of view of the orbifold CFT these deformations correspond to massless spacetime fields that arise from the twisted sectors.

As we have seen, in 5-dimensional compactification there is a new possibility: by turning on a pure holonomy gerbe with support on the full $K3 \times S^1$ geometry, we can obtain a theory without moduli associated to any of the blow-up modes. The disappearance of the moduli is easy to understand in the T-dual picture, since the dual geometry is smooth, and every twisted sector is massive. On the other hand, in the original formulation the singularity is frozen by the holonomy of the gerbe.

In terms of the local structure of the geometry, we can think of this as follows.⁶ The equivariant gerbe implies that the orbifold geometry $\mathbb{T}^4/G \times S^1$ carries non-trivial gerbe holonomy: a curve $\tilde{\gamma}$ in \mathbb{T}^4 with $\tilde{\gamma}(0) = x$ and $\tilde{\gamma}(1) = \varphi_a(x)$ projects to a closed curve γ , and the cycle $\gamma \times S^1$ then carries a gerbe holonomy $e^{2\pi ia/N}$, and this topological feature is responsible for freezing the moduli.

This local discussion has a string-dual description: starting with IIB compactified on $\mathbb{T}^4/G \times S^1$ with a pure holonomy gerbe, we use $SL(2, \mathbb{Z})$ duality to obtain IIB compactified on the same space, but now with a holonomy for the Ramond-Ramond C_2 field on the same cycle $\gamma \times S^1$. Taking another T-dual, we obtain IIA on $\mathbb{T}^4/G \times S^1$, but now with holonomy for C_1 on the loop γ . These different descriptions can be understood by compactifying M-theory on $X_5 \times S^1$ and then reducing either on the trivial circle or on the circle of the X_5 fibration. This is exactly the picture described in [24] for the freezing of the singularities, and we see that our construction gives the same mechanism at the level of spacetime physics, but one accessible in standard worldsheet analysis.

If we reconsider the $\mathbb{T}^4/\mathbb{Z}_2 \times S^1$ case, we see that all the B-field holonomy are localized around the singularities. This leads to a natural question: is it possible to turn on different gerbe holonomies at different singularities to freeze an arbitrary number of singularities? As pointed out in [24], there are global constraints that prevent us from turning on such holonomies independently: some combinations of the cycles are homologous, and unless the holonomies are chosen in a consistent manner, they would necessarily lead to non-zero curvature for dC_1 . The number of singularity points that are allowed to turn on non-trivial gerbe holonomy can only be 0, 8, 16.

As we argued the two frozen singularity mechanisms are equivalent by $SL(2, \mathbb{Z})$. These global constraints are also reflected in our worldsheet analysis. This can be seen explicitly by focusing on the bosonic sector of the theory. For example, the partition function for $\mathbb{T}^4/\mathbb{Z}_2 \times S^1$ —see appendix A for the relevant CFT details—has twisted sector contributions that come with an overall factor of $16 = 2^4$, one for each of the orbifold fixed points:

$$\mathcal{Z}_{\text{twist}} = 2^4 Z^{\text{cir}}(r) \left\{ \left(\frac{\eta\bar{\eta}}{\vartheta_4\bar{\vartheta}_4} \right)^2 + \left(\frac{\eta\bar{\eta}}{\vartheta_3\bar{\vartheta}_3} \right)^2 \right\}. \quad (4.3.1)$$

A CFT model of turning on a local gerbe on k of the singularities is to split the twisted sector contribution into k terms for which we introduce the phases associated to the gerbe, and the remaining $16 - k$ terms which we leave untouched. Thus, we have

$$\begin{aligned} \mathcal{Z}_{\text{twist}}^{(k)} &= (16 - k) Z^{\text{cir}}(r) \left\{ \left(\frac{\eta\bar{\eta}}{\vartheta_4\bar{\vartheta}_4} \right)^2 + \left(\frac{\eta\bar{\eta}}{\vartheta_3\bar{\vartheta}_3} \right)^2 \right\} \\ &\quad + k \left\{ Z_{\text{sh}^-}^+ \left(\frac{\eta\bar{\eta}}{\vartheta_4\bar{\vartheta}_4} \right)^2 + Z_{\text{sh}^-}^- \left(\frac{\eta\bar{\eta}}{\vartheta_3\bar{\vartheta}_3} \right)^2 \right\}. \end{aligned} \quad (4.3.2)$$

Can this be the twisted sector of a worldsheet \mathbb{Z}_2 orbifold CFT? If so, we should be able to obtain the untwisted sector partition function by applying worldsheet $SL(2, \mathbb{Z})$ transformations. However,

⁶In this discussion we stick to the pure holonomy equivariant gerbe with $B = 0$.

when we do this, we find that the resulting partition function fails to have an integral \mathbf{q} expansion unless $k = 0 \pmod{8}$.

A similar analysis can be carried out for the \mathbb{Z}_3 example: there the number of frozen singularities can be 0, 6, 9 [24], and those are precisely the values for which the procedure just outline yields a well-behaved worldsheet partition function.

Part III

On non-perturbative side of string compactification

Chapter 5

BPS strings in six and five dimensions

We now turn to supergravity theories in 5 and 6 dimensions following chapter 2¹.

Six and five-dimensional theories - even those related by a simple circle reduction - have a rather different way of packaging geometric information. For example, for reductions of F and M theory on elliptic Calabi-Yau manifolds, the trilinear couplings of the former correspond to only a part of the intersection form of the CY manifold (where one of the two-forms is necessarily related to the pull-back from the base of the elliptic fibration), while the latter sees the entire intersection form. In a similar way, we shall argue that five-dimensional theories offer a better (and more geometric) view on the consistency of six dimensional theories (after compactifying on a circle).

In this chapter, we study the spectrums of BPS strings in 6d and 5d minimal supergravity (eight supercharges) and point out some of the differences between them. Then we offer one way to relate the BPS strings in 6d and their counterpart in $d = 5$ after compactification on a circle.

5.1 BPS strings in six and five-dimensional theories with 8 supercharges

We consider six-dimensional theories with minimal $\mathcal{N} = 1$ supersymmetry with n_T tensor multiplets, Yang-Mills multiplets with a group $G = \prod_i G_i$ and hypermultiplets in different representations of the gauge group. A necessary condition for the Green-Schwarz anomaly cancellation mechanism is the sum-factorisation of 6d $\mathcal{N} = 1$ anomaly polynomial:

$$I_8 = \frac{1}{2} \Omega_{\alpha\beta} X_4^\alpha X_4^\beta \quad (5.1.1)$$

where $\alpha, \beta = 0, 1, \dots, n_T$ and $\Omega_{\alpha\beta}$ is the symmetric inner product on the space of tensors with signature $(1, n_T)$, and²

$$X_4^\alpha = \frac{1}{8} a^\alpha \text{tr} R^2 + \sum_i b_i^\alpha \frac{1}{4h_i^\vee} \text{Tr}_{\text{Adj}} F_i^2 \quad (5.1.2)$$

The vectors $a, b_i \in \mathbb{R}^{1, n_T}$ are determined by the field content of the theory. The anomaly cancellation condition ensures that all mutual inner products are integers. A GS term is added to the six-dimensional action to cancel the anomaly encoded in I_8 via the descent formalism.

¹This part is based on [115]

²Our normalisations of the curvatures R and F are such that they contain a factor $1/2\pi$.

In the presence of solitonic strings, which are the dyonic sources for self dual tensor fields, both the Green-Schwarz couplings and the Bianchi identities for the tensor fields are modified:

$$dH^\alpha = X_4^\alpha + Q^\alpha \prod_{a=1}^4 \delta(x^a) dx^a, \quad (5.1.3)$$

where H^α satisfy a self-duality condition. The 4-form distribution is the Poincaré dual to the string source and Q^α are string charges.

In addition to the standard lack of invariance under gauge transformations and diffeomorphisms, the GS term will lead to anomalous terms restricted to the string worldsheet W_2 in the presence of such a BPS solitonic string. They must cancel the anomaly of the worldsheet theory. One should bear in mind that in (5.1.3) the string source term is given in a particular representation of the Thom class Φ for $i : W_2 \hookrightarrow M_6$, and in general it follows from the Thom isomorphism that the pull-back $i^*\Phi = \chi(N)$, where for the $SO(4) \simeq SU(2)_1 \times SU(2)_2$ structure group of the normal bundle $\chi(N) = c_2(SU(2)_1) - c_2(SU(2)_2)$ is the Euler class of the normal bundle N of the string. Using $\text{tr}R^2|_{TW_2} = -2p_1(TW_2) - 2p_1(N)$ and $p_1(N) = -2(c_2(SU(2)_1) + c_2(SU(2)_2))$, one infers that the anomaly two-form on W_2 is obtained via descent from

$$\begin{aligned} I_4 &= -\Omega_{\alpha\beta} Q^\alpha \left(X_4^\beta(M_6)|_{W_2} + \frac{1}{2} Q^\beta \chi(N) \right) \\ &= -\frac{1}{4} \Omega_{\alpha\beta} Q^\alpha (a^\beta p_1(TW_2) - 2(Q^\beta + a^\beta) c_2(SU(2)_1) + 2(Q^\beta - a^\beta) c_2(SU(2)_2) + \dots) \end{aligned} \quad (5.1.4)$$

The ellipsis stands for the pullback of the YM part in (5.1.2) which is not needed for the following analysis.

The theory on the worldsheet flows in the IR to a $(0, 4)$ SCFT and the information about the left and right central charges as well as the level of the $SU(2)$ \mathcal{R} -symmetry current algebra is contained in I_4 . As discussed in sect 2.1.4, the SCFT splits into a free center of mass SCFT and an interacting SCFT. The former consists of a hypermultiplet with left and right central charges 4 and 6, respectively. Its \mathcal{R} -symmetry group is not contained in the $SO(4)$ from the normal bundle as the four scalars, which are neutral under the \mathcal{R} -symmetry, transform as a vector of $SO(4)$. From the point of view of the worldsheet theory it is an accidental symmetry. The contribution of the c.o.m. part to I_4 is $-\frac{1}{12}p_1(TW_2) - c_2(SU(2)_1)$. In particular it does not interfere with the \mathcal{R} -symmetry of the interaction part of the SCFT, which is $SU(2)_2$. Using the $(0, 4)$ relation $c_R = 6k_{\mathcal{R}}$ between the central charge and the level of the \mathcal{R} -current algebra, we can read off c_R^{int} from the $c_2(SU(2)_2)$ part of I_4 and $c_R^{int} - c_L^{int}$ from the coefficient $p_1(TW_2)$ of the gravitational anomaly. Adding the contribution of the c.o.m. part one finds

$$\begin{aligned} c_L - c_R &= -6\Omega_{\alpha\beta} a^\alpha Q^\beta \equiv -6Q \cdot a \\ c_R &= 3\Omega_{\alpha\beta} Q^\alpha Q^\beta - 6\Omega_{\alpha\beta} a^\alpha Q^\beta + 6 \equiv 3Q \cdot Q - 6Q \cdot a + 6 \end{aligned} \quad (5.1.5)$$

We have defined here an inner product denoted by \cdot using the metric on the space of tensors $\Omega_{\alpha\beta}$.

Following [92, 93] we shall be interested in supergravity strings,³ whose worldsheet \mathcal{R} -symmetry descends from the structure (sub)group of the normal bundle. This condition restrains the values of the admissible Q charges. Once such restrictions are imposed, the worldsheet SCFT should be unitary, i.e. the central charge c_L should serve as a bound for the contribution of the left moving

³i.e. BPS strings that cannot be consistently decoupled from gravity.

current algebra for G at level k :

$$\sum_i \frac{k_i \cdot \dim G_i}{k_i + h_i^\vee} \leq c_L - 4 \quad (5.1.6)$$

where for Abelian gauge factors $h^\vee = 0$.

So far we have discussed solitonic strings in $d = 6$. Most of the subsequent analysis will be five-dimensional, and we shall in particular be interested in the five-dimensional solitonic strings obtained via circle reduction, when the S^1 is transverse to the six-dimensional string. To get the anomaly formula of the resulting $(0, 4)$ SCFT on the string, we simply let the normal bundle be $\mathbb{R}^3 \times S^1$. To go to five dimensions, we take $c_2(SU(2)_1) = c_2(SU(2)_2) = c_2(N)$, where N is the S^2 normal bundle fiber inside \mathbb{R}^3 . Imposing this in (5.1.4) leads to

$$c_L = 2c_R = -12\Omega_{\alpha\beta} a^\alpha Q^\beta \equiv -12Q \cdot a \quad (5.1.7)$$

preserving the difference $c_L - c_R$ (5.1.5). Such five-dimensional solitonic strings with central charges linear in Q are magnetic sources for the $U(1)$ gauge fields obtained from the reduction of the six-dimensional tensor fields.

We now turn to the generic string sources in five-dimensional $\mathcal{N} = 1$ supergravity. Such a BPS string also hosts a $(0, 4)$ 2d SCFT on its worldsheet, hence we can obtain c_L, c_R for this 2d SCFT via anomaly inflow caused by 5d bulk Chern-Simons terms:

$$-\frac{1}{96}a_I A^I \text{tr}(R \wedge R) + \frac{1}{6}C_{IJK} A^I \wedge F^J \wedge F^K \quad (5.1.8)$$

From these Chern-Simons terms we obtain [93, 99, 100]

$$\begin{aligned} c_R &= C_{IJK} Q^I Q^J Q^K + \frac{1}{2}a_I Q^I \\ c_L &= C_{IJK} Q^I Q^J Q^K + a_I Q^I \end{aligned} \quad (5.1.9)$$

The index I runs over all $d = 5$ vectors. In 6d language, $I = 1, \dots, n_T + n_V + 1$.

The structure of central charges of $(0, 4)$ SCFTs hosted on 5d BPS strings is very different from 6d ones. While in general for 6d strings the leading behaviour for both c_L and c_R is quadratic in Q , due the quadratic terms in the anomaly polynomial (5.1.1), in five dimensions it is generally cubic. Moreover, in five dimensions the anomaly inflow cannot produce central charges with quadratic scaling in Q .

For the vector fields originating from six-dimensional tensors, the coefficient of the gravitational coupling does not renormalise upon reduction and the triple self-intersection does not get generated. One recovers the central charges as in (5.1.7) linear in Q and with $c_L = 2c_R$. So the conclusion would be that for the 5d BPS strings from 6d BPS string compactified on a transverse circle, the central charge c_L, c_R on the $(0, 4)$ SCFT it hosted will have vanishing cubic term (i.e. $C_{IJK} Q^I Q^J Q^K = 0$ in (5.1.9)).

For the remaining $U(1)$ vectors in 5d $\mathcal{N} = 1$ supergravity, including the graviphoton A^0 , integrating out of the massive KK tower in general cases generates the gravitational couplings with coefficients a_I and the trilinear self-intersections with (non-zero) coefficients C_{IJK} . The central charge of these strings in general have a cubic dependence on Q . We shall refer to these types of BPS strings as linear(central charge with vanishing cubic term $C_{IJK} Q^I Q^J Q^K = 0$) and cubic for the cases in the subsequent discussion.⁴

⁴A little clarification is due. ‘‘Linear strings’’ can have trilinear dependence in the central charges which can however be set to zero by appropriate choices of the charge vector. This is the case with the self-dual string in $\mathcal{N} = 1$

In next chapter we shall concentrate on the linear strings, and re-examine the unitarity constraints in sect 2.1.4 of the six-dimensional theories from five-dimensional view-point. Given the change in the nature of c_L in passage from six to five dimensions, the unitarity constraints, as we shall see, are different both in substance (they are in general a bit stronger) and in form (they appear to be more geometric). We did not find the cubic strings to be amenable to such analysis and to produce useful constraints. However in the remainder of this chapter we shall elucidate their six-dimensional origin.

5.2 5d strings from 6d geometry

We will now argue that the five-dimensional cubic strings originate from the six-dimensional geometry $\mathbb{R}^{1,1} \times M_{\text{TN}}$, i.e. when the circle on which the theory is reduced is non-trivially fibered. Moreover, every cubic string should carry some KK (magnetic) charge. As we shall see this argument is consistent with F-theoretic considerations.

We have already seen that the reduction of six-dimensional strings, which are charged under the tensor fields, yields only linear strings. Hence the cubic strings can only be charged under the vectors that come from the reduction of six-dimensional vector multiplets or under the KK vector $g_{\mu 5}$. One could wonder if there is a solitonic object (a membrane) in six dimensions that is charged under the $U(1)$ fields and whose reduction yields the cubic strings, but such a BPS object is not allowed by supersymmetric algebra. Also, if this is the case, the $(0, 4)$ SCFT on the 5d string should arise from 3d $\mathcal{N} = 2$ QFT on the membrane compactified on a S^1 . This generally cannot produce a chiral theory in two dimensions (notice that our 2d theory is obtained from a compactification of a 3d theory on a circle, not via restriction to the boundary of a 3d theory). Also, obviously the magnetic sources for the KK vector after circle compactification do not arise from any wrapped object in 6d either as the 6d theory itself does not have the KK vector.

To find the 6d origin for the cubic BPS strings after circle compactification, let us recall that for five-dimensional supergravities obtained from the compactifications of M-theory on an elliptically fibered CY_3 , the $(0, 4)$ cubic strings arise from M5 branes wrapping a smooth ample divisor⁵. So let us have a closer look at ample divisors in a smooth elliptically fibered CY_3 :

$$E_\tau \rightarrow \text{CY}_3 \rightarrow B. \quad (5.2.1)$$

These can comprise the base B and $\pi^{-1}(\Sigma_i)$, which are pullbacks of curves in the base, and an exceptional divisor X . Hence the generic ample divisor D can be written as

$$D = aB + b\pi^{-1}(\Sigma) + X. \quad (5.2.2)$$

It follows from the Nakai-Moishezon ampleness condition for D , which implies

$$D \cdot D \cdot D > 0 \quad \text{and} \quad D \cdot C > 0 \quad (5.2.3)$$

for any effective curve C , that $a \neq 0$, i.e. any ample divisor in a smooth CY_3 necessarily contains some copies of the base. Indeed, this follows immediately if we take C to be the intersection of two $\pi^{-1}(\Sigma_i)$, and use that

$$\pi^{-1}(\Sigma_i) \cdot \pi^{-1}(\Sigma_j) \cdot \pi^{-1}(\Sigma_k) = 0 \quad \text{and} \quad \pi^{-1}(\Sigma_i) \cdot \pi^{-1}(\Sigma_j) \cdot X = 0. \quad (5.2.4)$$

theory after circle reduction. As we shall see, their central charges can acquire contributions $\sim Q_{\text{KK}}^2 Q$. However Q_{KK} can be consistently taken to zero. The cubic strings, on the contrary, are charged with respect to vector fields that have a cubic *self*-coupling.

⁵In order to see the microscopic origin on the central charge formula in terms of the zero modes of the fields on M5 one should assume that the divisor is very ample [101].

Moreover, in the M theory picture, an M5 brane wrapping the base is a magnetic source for the KK vector. So from the M/F theory points of view, the cubic string should carry some magnetic charge of the KK vectors. In general this implies that the six-dimensional counterpart of these strings should contain the KK monopole configuration, which is naturally given by Euclidean Taub-NUT geometry (see e.g. [102]):

$$ds_6^2 = -dt^2 + dy^2 + ds_{\text{TN}}^2 \quad (5.2.5)$$

with

$$ds_{\text{TN}}^2 = \left(1 + \frac{QR_0}{r}\right) (dr^2 + r^2 d\Omega_2) + R_0^2 \left(1 + \frac{QR_0}{r}\right)^{-1} (2d\psi + QA)^2 \quad (5.2.6)$$

Here $Q \equiv Q_{\text{KK}}$, the KK monopole charge, is a integer; we will restrict to the positive integer case for simplicity and without loss of generality. $dA = d\Omega_2$ is the volume element on the unit 2-sphere and $\psi \simeq \psi + 2\pi$. The TN space is a S^1 fibration over \mathbb{R}^3 (except the locus where the S^1 fiber shrinks to zero size). Far away from the origin the space is $S^1 \times \mathbb{R}^3$ where the radius of the circle is $2R_0$. This is the circle we want to compactify on. It shrinks to zero size at the origin at $r = 0$ where the space has a A_{Q-1} singularity. This is the limit of an Q -centered TN space where all centers coincide (here at the origin $r = 0$).

For a fixed small distance $r = \epsilon$, we can neglect the constant in the harmonic function and the metric becomes that of an S^1 fibration over S^2 (a cyclic Lens-space)

$$S^1 \rightarrow S^3 \rightarrow S^2 \quad (5.2.7)$$

It is characterised by the first Chern number of the KK vector

$$\lim_{\epsilon \rightarrow 0} \int_{S^2} \frac{F^{\text{KK}}}{2\pi} = Q_{\text{KK}} \quad (5.2.8)$$

The argument that 5d cubic strings should come from the 6d theory on a Taub-NUT background after compactification on the circle fiber, can be generalised to include six-dimensional $U(1)$ vector fields following the generalised Taub-NUT solution in [103]. These will give solitonic string-like objects which carry both KK as well as the related $U(1)$ magnetic charges after compactification on the circle. As the Taub-NUT metric is a gravitational instanton, half of the supersymmetry is preserved by this background, just as it is expected for the string solitons with $(0, 4)$ worldsheet supersymmetry.

Finally this picture also accounts qualitatively for the chirality of the theory on the string worldsheet. Given that the six-dimensional theory has a self-dual tensor field in the gravity multiplets and n_T anti-self-dual tensors in tensor multiplets, their decomposition along the basis of self-dual and anti-self-dual two forms on M_{TN} yield two-dimensional modes $b(t, y)$, where (t, y) denote the coordinates along $R^{1,1}$, i.e. the string worldsheet, such that

$$(\partial_t \mp \partial_y) b(t, y) = 0. \quad (5.2.9)$$

Note this is only part of the spectrum and this analysis is on the 6d UV side. So we cannot use this argument to determine c_L and c_R of the resulting $(0, 4)$ SCFT individually. However the chirality of spectrum implies 't Hooft anomalies, which match between the UV and the IR. Hence the resulting solitonic string from Taub-NUT reduction should support a chiral spectrum in the IR. The more direct argument is using anomaly inflow of the compactified five-dimensional theory, as we did before.

We can consider more general configurations. Six-dimensional $\mathcal{N} = 1$ supergravity theory in a (generalised) Taub-NUT background and a BPS string at the locus where the S^1 fiber shrinks

to zero size (the two objects preserve the same set of supercharges), after compactification yields five-dimensional supergravity with solitonic BPS strings. Moreover, these 5d BPS strings carry magnetic charges for the $U(1)$ gauge fields as well as the KK charge. Since upon such reduction cubic self-couplings of the $U(1)$ fields are generated these string configuration will, in general, have cubic central charges.

In summary, we have argued that cubic BPS strings in 5d $\mathcal{N} = 1$ supergravity obtained from minimal 6d supergravity originate from a (generalised) Taub-NUT background.

5.3 On graviphoton couplings in five dimensions

The claim that in five-dimensional theories, obtained via circle reduction of six-dimensional $\mathcal{N} = 1$ supergravity, the cubic solitonic strings arise from non-trivial geometric backgrounds, immediately leads to the following requirement:

Since we can always turn on a purely geometric Taub-NUT background with arbitrary KK monopole charge, there should always be a solitonic string which only carries KK magnetic charge and supports a $(0, 4)$ or $(4, 0)$ SCFT.⁶ The superconformal algebra and unitarity then require c_R (or c_L) = $6k_{SU(2)\mathcal{R}} \in \mathbb{Z}_+$.

To this end, it suffices to consider the Chern-Simons-like couplings to the KK vector in five dimensions. For the S^1 reduction of the Taub-NUT background, the magnetic string charged under the KK vector is at the position where the S^1 shrinks to zero size. Far away from this string, the five-dimensional physics can be derived by just putting a $(0, 1)$ theory on a circle. So the corresponding Chern-Simons level can be obtained by integrating out the massive charged modes in a one-loop Feynman diagram calculation [87]. The relevant couplings are given by

$$\mathcal{L}_{\text{CS}} = -\frac{k_0}{6} A^{\text{KK}} \wedge F^{\text{KK}} \wedge F^{\text{KK}} + \frac{k_R}{96} A^{\text{KK}} \wedge \text{tr} R^2 \quad (5.3.1)$$

and the ensuing central charges c_L and c_R are obtained from the inflow arguments (for a $(0, 4)$ SCFT on the string) as

$$c_R = k_0 Q_{\text{KK}}^3 + \frac{k_R}{2} Q_{\text{KK}} \quad c_L = k_0 Q_{\text{KK}}^3 + k_R Q_{\text{KK}} \quad (5.3.2)$$

where Q_{KK} is the KK string charge.

The coefficients k_0, k_R depend on the six-dimensional field content. They have been essentially calculated in [87]:

$$k_0 = 2(9 - n_T), \quad k_R = 8(12 - n_T) \quad (5.3.3)$$

Obviously c_R (or c_L) = $6k_{SU(2)\mathcal{R}} \in 6\mathbb{Z}_+$ is satisfied.

5.4 Five-dimensional view on the unitarity condition

Six-dimensional gravitational and gauge anomalies in 6d minimal supersymmetric theories allow not only to read off the central charges of stringy objects with $(0, 4)$ worldsheet supersymmetry, but also the level k_L of the current algebra that couples to the left-movers. The condition that the left-moving central charge is large enough to allow for a unitary representation of the current

⁶To determine $(0, 4)$ vs $(4, 0)$ is by looking at the $SU(2)$ \mathcal{R} -symmetry part of the anomaly polynomial which is $\pm k_{\mathcal{R}} c_2(SU(2)\mathcal{R})$. Here $k_{\mathcal{R}}$ is the level of the $SU(2)\mathcal{R}$ current algebra, which unitarity requires to be positive. For the minus sign we have a $(0, 4)$ and for the plus sign a $(4, 0)$ SCFT on the string worldsheet.

algebra at level k_L was used in [92] as a consistency condition of quantum gravity in order to rule out some anomaly-free 6d minimal supergravity theories (as reviewed in sect 2.1.4). We shall re-examine these constraints, for which we shall use the shorthand “unitarity conditions”, from a five-dimensional perspective.

As mentioned previously, the chiral two-dimensional theories that live on the string worldsheet are $(0, 4)$ (or $(4, 0)$) SCFT, i.e the $SU(2)_{\mathcal{R}}$ symmetry inherited from the normal bundle of this string belongs to the right-moving (left-moving) sector. Unitarity of the worldsheet theory requires that the central charges are positive. This should in particular be true for the string charged under the KK vector for which $c_R = 6 k_{\mathcal{R}}$ (or $c_L = 6 k_{\mathcal{R}}$) is given in (5.3.2). However a closer look at this expression seems to lead to a puzzle:

- for $n_T \leq 9$, the string SCFT has $(0, 4)$ supersymmetry and $c_R = 6 k_{\mathcal{R}} = 2(9 - n_T)Q_{\text{KK}}^3 + 4(12 - n_T)Q_{\text{KK}} > 0$ for $Q_{\text{KK}} > 0$;
- for $n_T \geq 12$, the string has a $(4, 0)$ worldsheet SCFT and $c_L = 6 k_{\mathcal{R}} = 2(n_T - 9)Q_{\text{KK}}^3 + 4(n_T - 12)Q_{\text{KK}} > 0$ for $Q_{\text{KK}} > 0$;
- for $n_T = 10, 11$, something unpleasant happens. Take $n_T = 10$ for example, then $2(9 - n_T)Q_{\text{KK}}^3 + 4(12 - n_T)Q_{\text{KK}} = -2Q_{\text{KK}}^3 + 8Q_{\text{KK}}$, which gives $6, 0, -30$ for $Q_{\text{KK}} = 1, 2, 3$, respectively. This would seem to indicate that the string has $(0, 4)$ supersymmetry for $Q_{\text{KK}} = 1$ and $(4, 0)$ supersymmetry for $Q_{\text{KK}} = 3$. But if this KK monopole string indeed originates in the Taub-NUT background in six dimensions, all positive values of Q_{KK} should be allowed, and it is hard to imagine such changes from a change in the value of Q_{KK} .

The puzzle is resolved by realising that our considerations of the BPS strings have implicitly assumed that Q_{KK} is sufficiently large. Indeed, the Taub-NUT metric (5.2.6) has an intrinsic scale, the radius of the compactification circle $2R_0$. Therefore, the five-dimensional supergravity description can only be trusted below the energy scale $\Lambda_{5\text{d-SUGRA}} \simeq \frac{1}{2R_0}$. On the other hand, the anomaly inflow calculation leading to (5.1.8) and (5.1.9) required a smeared-out version of the Bianchi identity $dF = \frac{Q}{2}d\rho(r) \wedge e_2$, see [99, 100] for relevant details, which involves a function $\rho(r)$ of the distance away from the string. As this bump function, which interpolates between -1 and 0 , hides UV physics which is not visible in the 5d supergravity description, its radial compact support should be of the order $2R_0$. On the other hand, in the 5d supergravity description which we used above, the string source should be treated as a δ -function in the directions transverse to its worldsheet. In other words its thickness δr should go to zero. Using the explicit form of the TN metric (5.2.6), this translates into the condition

$$\int_0^{\delta r} \sqrt{1 + \frac{R_0 Q_{\text{KK}}}{x}} dx = 2R_0 \quad \text{with} \quad \delta r \rightarrow 0 \quad (5.4.1)$$

This leads to $\delta r \sim \frac{R_0}{Q_{\text{KK}}} \rightarrow 0$ for fixed R_0 , i.e. Q_{KK} should be large. It is under this condition that the values of the central charge derived from bulk anomaly inflow can be trusted.

In the M-theory picture, where the string arises from an M5-brane wrapping a divisor, this translates into the very ampleness condition on the divisor [101].

This requirement is obviously too strong. Heterotic string on K3 with 9 or 10 five-branes respectively and an $SU(2)$ instanton (with instanton number 15 or 14) easily provides counterexamples to this. Given the cubic dependance of c_L on Q_{KK} this requirement is easily satisfied for larger charges.

As a result, we have shown that we only have to distinguish two situations depending on the value of n_T

- $n_T \leq 9$, KK monopole string supports $(0, 4)$ supersymmetry

- $n_T > 9$, KK monopole supports (4,0) supersymmetry

Note that the value of $n_T = 9$ is somewhat special. For the F-theory models on elliptically-fibered CY_3 X , $n_T = 9 + \chi(X)/60$ where $\chi(X)$ is the CY Euler number. When χ vanishes, i.e. $n_T = 9$, the effective theory has another set of hidden supersymmetries (and can be thought of as a gauged supergravity theory with 16 supercharges) [105]. Correspondingly one would expect that the solitonic supergravity string may also display extra worldsheet supersymmetry and be enhanced to (0, 8). If so the superconformal algebra will require $c_R \in 12\mathbb{Z}$. The KK monopole strings satisfies this requirement, as one easily sees from (5.3.2) and (5.3.3) with $n_T = 9$.

There is an immediate consequence of the large Q_{KK} requirement for the unitarity analysis. Due to the presence of Q_{KK}^3 , the left-moving central charge c_L grows very fast, and hence does not give strong constraints. As we have argued, every cubic string in five-dimensional theories (obtained from a circle compactification of 6d supergravity) carries KK charge. It being large renders a generic cubic string rather useless as far as the unitarity constraints go. Of course this is not the case for the linear strings that come from the six-dimensional supergravity strings. Hence our five-dimensional unitarity analysis will be applied to the very same objects that have been analysed in [92].

One can argue quite generally that the dimensional reduction should not be imposing any new consistency conditions (even if, as it is the case here, it can repackage these in a new and useful fashion). Although we know that sometimes IR properties can be used to constrain the possible UV completion (e.g. c -theorem in 2d, or a -theorem in 4d, or the obstructions of liftability discussed previously), this is not the case in the current $5d/6d$ context. Here we know both the 6d UV side and 5d IR side, as well as the correspondence of the extended objects on both sides. Since the Taub-NUT background does not cause any inconsistencies on the six-dimensional UV side, no inconsistencies should be generated along the RG flow.

Chapter 6

Unitarity condition for linear BPS strings

In this chapter we study consistency conditions (see sect 2.1.4) in 5d and 6d supergravity .

6.1 Unitarity condition for 6d supergravity

Before considering five-dimensional theories in detail, we recall the unitarity condition for 6d $\mathcal{N} = 1$ supergravity theories proposed in [92]. The anomaly polynomial for the worldsheet theory, which can be computed from anomaly inflow from the bulk, was already given in eq. (5.1.4). From this we need to subtract the contribution of a free (0,4) hypermultiplet, whose bosonic components describe the position of the string in the four transverse directions:

$$I_4^{free} = -\frac{1}{12}p_1(TW_2) - c_2(1) \quad (6.1.1)$$

Recall that $c_2(1)$ and $c_2(2)$ correspond to the subbundles of the normal bundle $SO(4) \cong SU(2)_1 \times SU(2)_2$. We identify $SU(2)_1$ with the $SU(2)_{\mathcal{R}}$ -symmetry of the interacting (0,4) SCFT in the IR. The anomaly polynomial of the interacting theory is then (cf. also Section 5.1 for further details on the notation)

$$\begin{aligned} I_4^{int} &= -\frac{1}{12}(3Q \cdot a - 1)p_1(TW_2) + \sum_i Q \cdot b_i \frac{1}{4h_i^\vee} \text{Tr}_{\text{Adj}}(F_{G_i}^2) \\ &\quad - \frac{1}{2}(Q \cdot Q - Q \cdot a)c_2(1) + \frac{1}{2}(Q \cdot Q + Q \cdot a + 2)c_2(2) \\ &\supset -\frac{1}{24}(c_R^{int} - c_L^{int})p_1(TW_2) + \sum_i k_i \frac{1}{4h_i^\vee} \text{Tr}_{\text{Adj}}(F_{G_i}^2) - k_{\mathcal{R}}c_2(SU(2)_{\mathcal{R}}) \end{aligned} \quad (6.1.2)$$

Note that the positivity of the central charge of the $SU(2)_2$ current algebra requires $Q \cdot Q + Q \cdot a + 2 \geq 0$.

This leads to the expression for the central charges of the interacting SCFT

$$\begin{aligned} c_L^{int} - c_R^{int} &= -6Q \cdot a + 2 \\ c_R^{int} &= 6k_{\mathcal{R}} = 3(Q \cdot Q - Q \cdot a) \end{aligned}$$

and to the unitarity requirement

$$\sum_i \frac{(Q \cdot b_i) \cdot \dim G_i}{Q \cdot b_i + h_i^\vee} \leq c_L^{int} = 3Q \cdot Q - 9Q \cdot a + 2 \quad (6.1.3)$$

where we have used the relation between the levels $k_i = Q \cdot b_i$ of the left-moving current algebras and their contribution to the central charge. In general, Eq. (6.1.3) gives strong constraints when the charge Q is small.

As discussed in Section 5.1, when putting this 6d supergravity on a circle transverse to the string, we identify $c_2(1) = c_2(2) = c_2(SU(2)_{\mathcal{R}})$. The resulting central charges are then:

$$c_R = 6k_R = -6Q \cdot a \quad \text{and} \quad c_L = -12Q \cdot a$$

Again, subtracting the free part of the central charge, we have

$$c_R^{int} = -6Q \cdot a - 6, \quad c_L^{int} = -12Q \cdot a - 3 \quad (6.1.4)$$

On the other hand, the gauge anomaly should not be changed by compactifying our theory on a circle¹. Since if a 6d theory is good, it should also be good after S^1 compactification, so we derive the following unitarity condition:

$$\sum_i \frac{(Q \cdot b_i) \cdot \dim G_i}{Q \cdot b_i + h_i^\vee} \leq c_L^{int} = -12Q \cdot a - 3 \quad (6.1.5)$$

A remark is in order here. Notice that the above central charge calculation in 5d differs from the 6d case. First, in 6d there is a second $SU(2)$ on the right moving side while in 5d generically we only have $SU(2)_{\mathcal{R}}$ symmetry. Second, the -3 contribution which appears in 5d central charge c_L^{int} is due to the fact that for the 5d strings we only have three transverse bosons on the left moving side. The left moving compact boson from compact transverse circle may belong to the interacting part of the CFT. However, due to the (0,4) supersymmetry, the right moving compact boson should sit in the free hypermultiplet together with the other three right moving transverse bosons.

6.2 Charges of supergravity strings

In order to use the unitarity condition (6.1.5), we must find a way to single out supergravity strings [93] (i.e. strings that cannot be consistently decoupled from gravity). In order to read off the central charge of the (0,4) SCFT on the BPS string, it is essential to identify the $SU(2)_{\mathcal{R}}$ symmetry of the (0,4) 2d SCFT with the structure (sub)group from the normal bundle. However, as [92, 93] already pointed out that for the BPS strings that can be consistently decoupled from gravity (i.e. BPS strings in 6d/5d SCFT), the $SU(2)_{\mathcal{R}}$ symmetry of the (0,4) 2d SCFT may no longer come from the structure (sub)group of the normal bundle (for example, it may be mixed with the $SU(2)$ R-symmetry from the bulk in the SCFT limit).

The conditions for the six-dimensional $\mathcal{N} = 1$ theory to have a well defined moduli space were analysed in [92], and can be summarised using a $(1, n_T)$ vector j (related to the Kähler form on the base of the elliptic fibration B) on the tensor branch of the theory

$$j \cdot j > 0, \quad j \cdot b_i > 0, \quad j \cdot a < 0. \quad (6.2.1)$$

¹No Wilson line is turned on on the circle

In order for the string to have a non-negative tension, $j \cdot Q \geq 0$ also needs to be imposed. Finally, unitarity of the (0,4) 2d SCFT hosted on the BPS string imposed

$$Q \cdot Q + Q \cdot a \geq -2, \quad Q \cdot a < 0 \quad \text{and} \quad Q \cdot b_i \geq 0 \quad (6.2.2)$$

as $k_i = Q \cdot b_i$ is the level of affine current algebra on the left moving side of the (0,4) SCFT and $c_R = -6Q \cdot a$ for the (0,4) 5d BPS strings that comes from (0,4) BPS strings in 6d after circle compactification. Any five-dimensional theory obtained from a circle reduction should also be subject to these constraints. As our main interest is in BPS strings in 6d $\mathcal{N} = 1$ supergravity and their counterpart in 5d $\mathcal{N} = 1$ supergravity after circle compactification, we shall impose the above conditions (6.2.1) and (6.2.2).

Now we shall argue that non-negative $Q \cdot Q$ is a sufficient condition for a BPS string to be identified as a supergravity string. This argument is carried out in two steps:

- First, notice that the strings which can be decoupled from gravity (i.e. *not* supergravity strings) must go tensionless at some point of the Kähler moduli space.
- Then we argue the strings with $Q \cdot Q \geq 0$ can never go tensionless on the Kähler moduli space at any finite distance point.

As a byproduct of this discussion, we can show that all the 5d $U(1)$ gauge fields from 6d tensors associated with supergravity strings, can never be enhanced to non-Abelian gauge fields in 5d supergravity. When gravity is decoupled, the $U(1)$'s related to the supergravity strings will also decouple. The $U(1)$'s sourced by the other strings may be enhanced to non-Abelian gauge fields in the field theory limit.

Which strings can be consistently decoupled from gravity? The energy scale associated to a magnetic string is given by its tension T , while gravity sets the energy scale M_{Pl} . In order for a string to decouple from gravity, it should be possible to take the limit $\frac{T}{M_{\text{Pl}}} \rightarrow 0$, where the backreaction of the string can be neglected. Working in the supergravity regime, we may chose to keep M_{Pl} fixed and, as a result, need to have $T \rightarrow 0$ in the decouplings limit. One may equivalently state:

Any string that can be decoupled from gravity, must go tensionless at some point of the Kähler moduli space.

For six-dimensional theories obtained from F-theory on a Kähler base B of a elliptically fibered Calabi-Yau manifold, this can be also understood geometrically. The string source is given by a D3-brane wrapping a curve $D \subseteq B$, and the two energy scales

$$T \sim \text{vol}(D), \quad M_{\text{Pl}} \sim \text{vol}(B) \quad (6.2.3)$$

Usually in order to go to the field theory (decoupling) limit, one takes $\text{vol}(B) \rightarrow \infty$, i.e. the internal manifold is taken to be non-compact. Here instead we take $\text{vol}(B) = 1$ (which is $j \cdot j = 1$ for j a $(1, n_T)$ vector which parameterizes the Kähler moduli space in 6d $\mathcal{N} = 1$ supergravity language). Then the decoupling is achieved by

$$\text{vol}(D) \rightarrow 0, \quad (6.2.4)$$

which indicates that the submanifold D on which D3 wraps should be be shrinkable. This is equivalent to $D \cdot D < 0$ and translates into the condition $Q \cdot Q < 0$ for the BPS string charge Q . Such strings should be excluded from our analysis.

On the contrary, when a D3 brane wraps a semi-ample divisor, we expect to have a supergravity string [93] that cannot be decoupled from gravity consistently and is subject to the unitarity constraints. An semi-ample divisor is not shrinkable while keeping the base being an algebraic surface, and has the property $D \cdot D \geq 0 \iff Q \cdot Q \geq 0$. We also assume the divisor which the D3 brane wraps is irreducible.

Strings with $Q \cdot Q \geq 0$ are supergravity strings. In order to see that strings with $Q \cdot Q \geq 0$ will not become tensionless on the Kähler moduli space, first recall that the string tension is given by $j \cdot Q$. We have fixed $j \cdot j = 1$, and can now choose the inner product and Kähler parameter to be respectively:

$$\Omega = \text{diag}(1, -1, \dots, -1), \quad \text{and} \quad j = (\sqrt{|\vec{j}|^2 + 1}, \vec{j}) \quad (6.2.5)$$

If $Q \cdot Q \geq 0$, we may take:

$$Q = (\sqrt{|\vec{Q}|^2 + m}, \vec{Q}) \quad (6.2.6)$$

where m is a non-negative integer.

Now the tension can be evaluated directly:

$$j \cdot Q = \sqrt{|\vec{j}|^2 + 1} \cdot \sqrt{|\vec{Q}|^2 + m} - \vec{Q} \cdot \vec{j} \geq \sqrt{|\vec{j}|^2 + 1} \cdot \sqrt{|\vec{Q}|^2 + m} - |\vec{j}| \cdot |\vec{Q}| > 0 \quad (6.2.7)$$

and is strictly positive on the Kähler moduli space. One may, of course, have $|\vec{j}| \rightarrow \infty$ at infinite distance at the boundary of the moduli space. However, there the entire effective supergravity description may break down and the full stringy picture needs to be considered, very much in analogy with the distance conjecture. As a result, in the supergravity theory, that we are considering, these BPS strings cannot go to tensionless limit and cannot be consistently decoupled from gravity.

As we shall see, due to the absence of the quadratic piece $Q \cdot Q$ in the unitarity condition, the five-dimensional unitarity condition is in general stronger than the six-dimensional condition of [92]. The only exception to this is when $Q \cdot Q + Q \cdot a = -2$. When this holds, the 6d unitarity condition imposes slightly stronger constraints than the 5d one.

6.3 Unitarity condition as a weak Kodaira positivity condition

In the F-theory framework, the upper bound on the rank and the type of non-Abelian gauge groups in six-dimensional $\mathcal{N} = 1$ theories arises naturally, and is due to the purely geometric condition, the Kodaira positivity (KPC), on the elliptically fibered threefold. The purpose of this section is to compare the implications of the KPC with the unitarity condition (UC) discussed in Section 5.4. This comparison will be complete for the theories without Abelian gauge groups. This does not lead to a significant loss of generality due to the fact that generally $U(1)$ factors in F-theory models appear due to Higgsing of a non-Abelian gauge group as argued in [107].

In order to carry out this comparison we should rewrite the UC (6.1.5)

$$\sum_i \frac{(Q \cdot b_i) \cdot \dim G_i}{Q \cdot b_i + h_i^\vee} \leq -12Q \cdot a - 3 \quad (6.3.1)$$

in a more convenient form. This is possible due to the fact that it involves $-12Q \cdot a$. In the F-theoretic models, when mapping the anomaly data to the geometric data of elliptically-fibered CY_3 , a is mapped to the canonical divisor K . The fact, that the elliptic fibration requires that all the gauge divisors should be contained in the effective divisor $-12K$, hints at a possible interpretation of the UC as a physical counterpart of the purely geometrical KPC. If the six-dimensional minimally-supersymmetric theory is obtained from F-theory on an elliptically-fibered CY_3 , the comparison is

direct (as we shall see in Section 6.3.2). However, the UC should apply without any assumption on the model having a F-theory realisation.

A remark on notation: In F-theoretic models, we are interested in the BPS strings that originate from D3-branes wrapping effective divisors in the base manifold B . As we shall see these BPS objects correspond to supergravity strings when the divisor in question is semi-ample, i.e. the linear system associated to a positive power of this divisor is base-point free. On the other hand, all effective nef divisors, i.e. the divisors that have a nonnegative intersection with every curve in B , are semi-ample. Since we are discussing only the effective divisors wrapped by D3-branes we just use the label nef divisors, hopefully without causing any confusion.

6.3.1 Rewriting the unitarity condition

The Kodaira positivity condition states that all singular divisors should be contained in the divisor of the discriminant of the Weierstrass model.² This requires the residual divisor Y given by

$$Y = -12K - \sum_i x_i S_i \quad (6.3.2)$$

to be effective.

Here S_i are the divisors with singular elliptic curve that host non-Abelian gauge groups [108] and x_i is the vanishing order of the discriminant on S_i (i.e. $\text{ord}(\Delta)$ in Table 1). Every effective divisor satisfies

$$j_B \cdot (-12K - \sum_i x_i S_i) = j_B \cdot Y \geq 0, \quad (6.3.3)$$

where j_B is the Kähler form on the base B . In fact, the following

$$D \cdot (-12K - \sum_i x_i S_i) = D \cdot Y \geq 0, \quad (6.3.4)$$

holds for any nef (or semi-ample) divisor D , as nef divisor should intersect every effective divisor non-negatively.

One can recast this condition in a form that just uses the data of six-dimensional supergravity, notably the four-form X_4^α entering the anomaly polynomial, and does not invoke the elliptically fibered CY_3 explicitly [82]

$$j \cdot (-12a - \sum_i x_i S_i) \geq 0 \quad (6.3.5)$$

where j is a $(1, n_T)$ vector on the tensor branch of our six-dimensional theory which satisfies $j \cdot j > 0$, $j \cdot b_i \geq 0$, $j \cdot a < 0$. For any 6d minimal supergravity theory not obtained from an elliptically-fibered CY_3 , condition (6.3.5) would appear to be not physically motivated and does not have to be satisfied.

On the other hand, the condition (6.3.1) follows from the worldsheet unitarity of supergravity strings and is expected to hold for all 6d minimal supergravity theories that are consistent at

²We are using the standard conventions for the elliptic fibrations with section (see e.g. [84, 85]). The elliptical fiber on a CY_3 is defined by an equation

$$y^2 = x^3 + f(u, v)x + g(u, v)$$

in an affine patch of the weighted projective space $\mathbb{WP}_{2,3,1}$, with u and v , one set of affine coordinates on the base B , fixed. Note, $f \in \Gamma(-4K)$ and $g \in \Gamma(-6K)$. The degeneration loci of the elliptic fiber are given by zeros of the discriminant:

$$\Delta = 4f^3(u, v) + 27g^2(u, v),$$

and, $\Delta \in \Gamma(-12K)$.

quantum level. When applied to an F-theoretic model, it can be rewritten as

$$D \cdot (-12K - \sum_i S_i \frac{\dim G_i}{D \cdot S_i + h_i^\vee}) \geq 3. \quad (6.3.6)$$

One has to bear in mind that the divisor D is wrapped by a D3-brane, and is required to be nef as we are talking about supergravity strings. To see why this is so, recall that the direct analogue of $Q \cdot Q \geq 0$ for the supergravity charges is given by $D \cdot D \geq 0$ for an irreducible effective divisor D . Irreducibility of D will be assumed throughout this paper.

Without assuming that the 6d minimal supergravity theory has an F-theoretic origin, one still needs to augment (6.3.1) by the following:

$$Q \cdot Q + Q \cdot a + 2 \geq 0, \quad k_i = Q \cdot b_i \geq 0 \quad \text{and} \quad -Q \cdot a > 0 \quad (6.3.7)$$

These can be interpreted as constraints on admissible values of the charge Q , in addition to $Q \cdot Q \geq 0$. The first two conditions are the requirements that the levels of current algebras are larger than 0, while the last one is the positivity of the right-moving central charge of the $(0, 4)$ worldsheet theory (recalling $c_R = -6Q \cdot a$ after the circle compactification).

It is not hard to see that the strongest constraints following from (6.3.1) are when $Q \cdot b_i = 1$ in the denominator (although $Q \cdot b_i = 1$ may not be achieved as intersection of divisors of a base B when we consider F theory model). In the following we shall compare the KPC with the following

$$Q \cdot (-12a - \sum_i b_i (\frac{\dim G_i}{1 + h_i^\vee})) \geq 3. \quad (6.3.8)$$

Indeed when this condition is satisfied, (6.3.1) will hold also for $Q \cdot b_i > 1$.

The failure of the (6.3.8) to hold does not immediately signify any inconsistency. Indeed, one has to first verify that $Q \cdot b_i = 1$ is possible.³ We shall see that in general (6.3.8) is weaker than KPC, and hence it may serve as a useful measure for the 6d minimal supergravity theories that have no F-theoretic realisation. On the contrary, for the 6d theories originating from F theory, (6.3.8) may provide finer information about the effective divisor $Y = -12K - \sum_i x_i S_i$ in some special cases, where the constraints imposed by the UC turn out to be stronger than those following from KPC.

6.3.2 Comparing KPC and UC

It is useful to recall the types of singularities present in the elliptically fibered CY_3 and the ensuing local gauge groups. These are conveniently summarised by the Kodaira data and can be found in Table 1, which we have augmented by some data entering the UC.

We can directly compare the quantity $y_i = \frac{\dim G_i}{1 + h_i^\vee}$ and the Kodaira multiplicity $x_i = \text{ord}(\Delta)$ one by one:

- For $E_{6,7,8}$, we have $y_{E_{6,7,8}} = x_{E_{6,7,8}} - 2 = 6, 7, 8$
- For $SU(n \geq 2)$, we have $y_{SU(n)} = n - 1 = x_{SU(n)} - 1$ for type I_n and $y_{SU(2)} = 1 = x_{SU(2)} - 2$ for type III, IV
- For F_4, G_2 , we have $y_{F_4, G_2} = \frac{14}{5}, \frac{52}{10}$ while $x_{F_4, G_2} = 6, 8$
- For $SO(2n + 1), n \geq 3$, we have $y_{SO(2n+1)} = n + \frac{1}{2}$, while $x_{SO(2n+1)} = n + 3$

³We will do the full comparison between UC (rather than its strongest version as here) and KPC in 6.3.3. So the UC in the strong form (6.3.8) serves as a red flag: if the strong condition fails, UC as given in (6.3.1) should be checked. In fact we have found situations where it fails, but $Q \cdot b_i = 1$ fails as well.

type	ord(f)	ord(g)	ord(Δ)	sing.	\mathfrak{g}	split	$y = \frac{\dim G}{1+h^\vee}$	K -type
I_0	≥ 0	≥ 0	0	—	—		—	—
I_1	0	0	1	—	—		—	—
II	≥ 1	1	2	—	—		—	—
III	1	≥ 2	3	A_1	$su(2)$		1	K_2
IV	≥ 2	2	4	A_2	$sp(1)$ $su(3)$	IV^{ns} IV^s	1 2	K_2
I_m	0	0	m	A_m	$sp(\lfloor \frac{m}{2} \rfloor)$ $su(m)$	I_m^{ns} I_m^s	$2\lfloor \frac{m}{2} \rfloor - \frac{3\lfloor \frac{m}{2} \rfloor}{\lfloor \frac{m}{2} \rfloor + 2}$ $m - 1$	K_1/K_2 K_1
I_0^*	≥ 2	≥ 3	6	D_4	g_2 $so(7)$ $so(8)$	I_0^{*ns} I_0^{*ss} I_0^{*s}	14/5 7/2 4	K_2
I_{2n-5}^* , $n \geq 3$	2	3	$2n + 1$	D_{2n-1}	$so(4n - 3)$ $so(4n - 2)$	I_{2n-5}^{*ns} I_{2n-5}^{*s}	$2n - 3/2$ $2n - 1$	K_2
I_{2n-4}^* , $n \geq 3$	2	3	$2n + 2$	D_{2n}	$so(4n - 1)$ $so(4n)$	I_{2n-4}^{*ns} I_{2n-4}^{*s}	$2n - 1/2$ $2n$	K_2
IV^*	≥ 3	4	8	E_6	f_4 e_6	IV^{*ns} IV^{*s}	52/10 6	K_2
III^*	3	≥ 5	9	E_7	e_7		7	K_2
II^*	≥ 4	5	10	E_8	e_8		8	K_2
non-min.	≥ 4	≥ 6	≥ 12	non-can.	—		—	—

Table 6.1. The left side of this Table summarises the Kodaira-Tate data for singular fibers of the Weierstrass model. The Weierstrass data f , g and Δ define the type of singularity. Some of the singularities can lead to different gauge algebras. This is governed by the refined Tate fiber type (see e.g. [85] for details). In the last column of the left side of the Table, ns , s and ss stand for non-split, split and semi-split respectively. In our context the most important column is $\text{ord}(\Delta)$ which defines the x_i multiplicities of the divisors with singular fibers S_i . The right side of the Table summarises the values of y_i multiplicities that appear in the UC. The last column, K -type, is determined by the difference $x_i - y_i$ (see also Table 2).

- For $SO(2n)$, $n \geq 4$, we have $y_{SO(2n)} = n$, while $x_{SO(2n)} = n + 2$
- For $Sp(k)$, we have $y_{Sp(k)} = 2k - \frac{3k}{k+2}$, $x_{Sp(k)} = 2k, 2k + 1$

From above, we see $x_i > y_i = \frac{\dim G^i}{1+h_i^\vee}$, hence we naturally have on any elliptic CY_3 :

$$D \cdot (-12K - \sum_i y_i S_i) > D \cdot (-12K - \sum_i x_i S_i) \quad (6.3.9)$$

for any nef divisor D .

Given the respective forms of our unitarity condition

$$D \cdot (-12K - \sum_i y_i S_i) \geq 3 \quad (6.3.10)$$

and the Kodaira positivity condition :

$$D \cdot (-12K - \sum_i x_i S_i) = D \cdot Y \geq 0 \quad (6.3.11)$$

few more steps are needed to see which one leads to stronger constraints. The Kodaira positivity is a necessary condition that is satisfied in all elliptically fibered CY_3 . In cases where our unitarity constraints turns out to be weaker, we are not learning much new in the context of elliptically

Type of gauge algebra	$x_i - y_i$	Gauge algebra
K_1	< 2	$su(m), sp(1), sp(2), sp(3)$ in Kodaira type I
K_2	≥ 2	All other groups in Table 1

Table 6.2. Classification of gauge groups in Table 1 based on the minimal value of $x_i - y_i$. Notice that $sp(1), sp(2), sp(3)$ in the first row come from I_2, I_4, I_6 respectively.

fibered CY_3 .⁴ When they are stronger, it should follow that the CY_3 in question should satisfy extra hidden conditions.

To proceed, notice that when the gauge algebra is $su(m)$, as well $sp(1), sp(2), sp(3)$ when these are in Kodaira type I (as opposed to $sp(1)$ in Kodaira type IV , $su(2)$ in type III and $su(3)$ in type IV), we have $y_i + 1 \leq x_i < y_i + 2$. We label gauge groups of this type as K_1 . For all other gauge groups we have $x_i \geq y_i + 2$, and we label these as type K_2 . In the subsequent analysis we shall label the gauge group as $G = \{K_1, K_2\}$ when it can be of any type, either K_1 or K_2 (see Table 2).

The UC applies to any supergravity theory, but the comparison to KPC requires to adapt it to the elliptically fibered CY_3 , where it can be formulated as a condition on divisor D in the base B , wrapped by a D3-brane. B is a smooth algebraic surface. In addition to an irreducible effective divisor D it has the gauge divisors S_i . The gauge divisor S_i should also be an effective divisor so that it can be wrapped by $D7$ branes.

We may recall that the charges for the supergravity strings Q should satisfy $j \cdot Q > 0$, $Q \cdot b_i \geq 0$ and $Q \cdot a < 0$. We shall also impose $Q \cdot Q \geq 0$ (and comment on $Q \cdot Q = -1$ case momentarily). These conditions can be translated into geometric statements for the D

$$D \cdot D \geq 0, \quad D \cdot S_i \geq 0, \quad D \cdot K < 0. \quad (6.3.12)$$

These conditions already contain a great deal of information: D is a nef divisor, and hence it intersects any effective divisor E on the base non-negatively $D \cdot E \geq 0$.⁵

$Q \cdot Q = -1$ case: Notice that we have restricted Q so that $Q \cdot Q \geq 0$. Before turning to the analysis of the conditions on D , we comment on $Q \cdot Q = -1$ case. For this case, we have $D \cdot D + D \cdot K = -2$ as $D \cdot D = -1$ and $D \cdot K < 0$, and hence D is a rational curve with self intersection -1 . So it corresponds to blowing up a point on a smooth base $\sigma : B \rightarrow B'$, which means that this exceptional divisor can be smoothly shrunk to zero size. As a result, the corresponding string could be tensionless and be consistently decoupled from gravity. (Actually the metric on B which gives zero size for this exceptional divisor can be interpreted as a metric on B' . See [109, 110] and also [111] for a related physical discussion)

6.3.3 General nef divisor D

We can now turn to the general case, where we require only that D is a nef divisor and analyse KPC and UC more carefully with a purpose of singling out the cases which KPC is satisfied coarsely (if we just ignore the effective divisor $Y = -12K - \sum_i x_i S_i$) while UC is violated. These are the cases where UC should be revealing a hidden finer structure in the elliptic CY_3 involved.

Let us start by collecting a slightly rewriting KPC and UC (in its general form, and not the

⁴These cases are important however for understanding the part of the not-swamped landscape of theories not covered by F-theory constructions.

⁵Note that the condition $Q \cdot Q + Q \cdot a + 2 \geq 0$ is automatically satisfied in F-theory models due to the adjunction formula.

strong form (6.3.8)):

$$\begin{aligned} -12D \cdot K &= D \cdot Y + \sum_i x_i D \cdot S_i \\ -12D \cdot K &\geq 3 + \sum_i \frac{\dim G_i}{D \cdot S_i + h_i^\vee} D \cdot S_i = 3 + \sum_i \mu_i D \cdot S_i \end{aligned} \quad (6.3.13)$$

where as before G_i is the (non-Abelian) gauge group hosted on (singular) gauge divisor S_i , and we have defined $\mu_i = \frac{\dim G_i}{D \cdot S_i + h_i^\vee}$. When $D \cdot S_i = 1$, $\mu_i = y_i$, otherwise $\mu_i < y_i$. Replacing μ_i by y_i result in the strongest version of UC (as already notice, in some cases this strong version may fail, while the UC (6.3.8) actually holds).

Note that $-12D \cdot K \in 12\mathbb{Z}_+$, and hence:

$$\begin{aligned} \sum_i x_i D \cdot S_i &\leq D \cdot Y + \sum_i x_i D \cdot S_i = -12D \cdot K = 12n \\ 3 + \sum_i \mu_i D \cdot S_i &\leq -12D \cdot K = 12n \end{aligned} \quad (6.3.14)$$

for a positive integer n . It is not hard to see that there are three possibilities

- If for all gauge divisors, $D \cdot S_i = 0$, the two conditions are equivalent trivially. This is very unlikely to happen in a base B .
- If at least three gauge divisors $S_{1,2,3}$ have the property that $D \cdot S_{1,2,3} > 0$ (this holds for a generic S_i), then even the strongest version of UC is weaker than KPC. The same conclusion holds for the case of at least two gauge divisors where at least one yields a K_2 type gauge group (due to $x_i - \mu_i \geq x_i - y_i \geq 2$ for K_2 type).
- For the remaining cases, let us notice that positive integer solution for $D \cdot S_i$ exist only when

$$12n - 3 < \sum_i \mu_i D \cdot S_i \leq 12n - \sum_i (x_i - \mu_i) D \cdot S_i \quad (6.3.15)$$

is satisfied. This condition implies that while KPC is respected, UC is violated. For these cases, UC can lead to stronger constraints than KPC and $Y = -12K - \sum_i x_i S_i$ cannot be numerically 0 in order for F-theory models not to violate UC.

Obviously we are interested only in the last situation, where we can divide the nontrivial solutions of condition (6.3.15) into two cases:

- There are two gauge divisors S_1 and S_2 in gauge groups of type K_1 , and $D \cdot S_{1,2} > 0$. The cases are where UC is more constraining than KPC are:
 - $SU(n) \times SU(m)$ with $m + n \in 12\mathbb{Z}$ and $D \cdot S_{1,2} = 1$.
 - $SU(12n - 2) \times Sp(1)$, $SU(12n - 4) \times Sp(2)$ and $SU(12n - 6) \times Sp(3)$ with $D \cdot S_{1,2} = 1$. Here $Sp(1)$, $Sp(2)$ and $Sp(3)$ should come from I_2 , I_4 and I_6 type singularities (see Table 1).
For these gauge groups we need to impose a further condition on the effective divisor $D \cdot Y \geq 1$ in order for UC not be violated by F-theory models.
- There is only a single gauge divisor S with the property $D \cdot S > 0$. UC can be more constraining than KPC only for the gauge groups
 - $SU(12n)$, $SU(12n - 1)$ with $D \cdot S = 1$.

– $SO(24n-5)$, $SO(24n-4)$ and $Sp(6n)$ with $D \cdot S = 1$. Here $Sp(6n)$ should come from I_{12n} type singularity (see Table 1).

In all these cases $\mu_i = y_i$. Only when the divisor $Y = -12K - \sum_i x_i S_i$ satisfies $D \cdot Y \geq 2$ for $SU(12n-1)$ and $D \cdot Y \geq 1$ for the rest, UC is not violated in F-theory models.

For completeness, we can present an example of a group where no extra constraints emerge. For E_6 , UC would be stronger than KPC only if we have a solution for

$$12n - 3 < \mu_{E_6} D \cdot S_{E_6} \leq 12n - (x_{E_6} - \mu_{E_6}) D \cdot S_{E_6}. \quad (6.3.16)$$

This would require

$$3 > (x_{E_6} - \mu_{E_6}) D \cdot S_{E_6} \geq (x_{E_6} - y_{E_6}) D \cdot S_{E_6} = 2D \cdot S_{E_6} \rightarrow D \cdot S_{E_6} = 1 \quad (6.3.17)$$

leading to $\mu_{E_6} = y_{E_6} = 6$. However, then $12n - 3 < \mu_{E_6} D \cdot S_{E_6} \leq 12n - (x_{E_6} - \mu_{E_6}) D \cdot S_{E_6}$ becomes $12n - 3 < 6 \leq 12n - 2$ which doesn't have a solution! Similar arguments can be applied to other cases.

6.3.3.1 Special cases where UC is stronger than KPC

Following the discussion in the previous section, we can give a precise statement about what UC may teach us about elliptic Calabi-Yau threefolds through F-theory models:

For F-theory on an elliptic CY_3 over base B , only when there exist gauge (singular) divisors $\{S_i\}$ and a nef divisor D on the base B , which satisfy some (very special) numerical conditions, UC hints at a finer information than contained in KPC, on the effective divisor $Y = -12K - x_i S_i$.

There are three types of models where this can happen:

- A:** There exists one gauge divisor $S_1 \in \{S_i\}$ hosting a gauge group $SU(12n)$ or $SU(12n-1)$, and a nef divisor D satisfying $D \cdot S_1 = 1$ and $D \cdot S_i = 0$ for all other $i \neq 1$, as well as $-D \cdot K \in \mathbb{Z}_+$. Then such a nef divisor D should satisfy $D \cdot Y \geq 1$ for $SU(12n)$ and $D \cdot Y \geq 2$ for $SU(12n-1)$ in order for UC to be satisfied by F-theory models
- B:** There exists one gauge divisor $S_1 \in \{S_i\}$ hosting a gauge group $SO(24n-5)$, $SO(24n-4)$ or $Sp(6n)$ (which comes from I_{12n} type singularity), and a nef divisor D satisfying $D \cdot S_1 = 1$ and $D \cdot S_i = 0$ for all $i \neq 1$, as well as $-D \cdot K \in \mathbb{Z}_+$. Then such a nef divisor D should satisfy $D \cdot Y \geq 1$ in order for UC to be satisfied by F-theory models
- C:** There exist two gauge divisors $S_1, S_2 \in \{S_i\}$ hosting gauge group $SU(a) \times SU(12n-a)$, $Sp(1) \times SU(12n-2)$, $Sp(2) \times SU(12n-4)$ or $SU(12n-6) \times Sp(3)$ (where $Sp(1)$, $Sp(2)$ and $Sp(3)$ come from I_2 , I_4 and I_6 type singularities) and a nef divisor D satisfying $D \cdot S_{1,2} = 1$ and $D \cdot S_i = 0$ for all $i \neq 1, 2$, as well as $-D \cdot K \in \mathbb{Z}_+$. Then such a nef divisor D should satisfy $D \cdot Y \geq 1$ in order for UC to be satisfied by F-theory models

Note that an example in class C has been discussed in Section 6.3.3.2, and has appeared previously in [92], where it was pointed out that (six-dimensional) unitarity condition can lead to stronger constraints than KPC.

To conclude, in a generic F-theory model UC leads to weaker constraints than KPC. Under some special conditions UC hints at finer information about the possible elliptic CY_3 than KPC on the remaining effective divisor $Y = -12K - \sum_i x_i S_i$ on the base.

In the next section we shall briefly discuss some examples, but we finish this section with some remarks.

- In all special cases, where UC is stronger than KPC, the numerical constraints on the gauge divisors and on the residual divisor Y are rather strong. We have not studied if and how many non-trivial realisations of these conditions exist in elliptic CY threefolds.
- A general lesson provided by UC in all above special cases for elliptic CY_3 is that the residual divisor Y on the base of the elliptic CY_3 should have nontrivial numerical properties and not be numerically equivalent to 0, and in general the gauge divisors do not sweep out the entire $-12K$.
- All cases where UC is stronger than KPC involve at most two gauge divisors (of fixed type) intersecting the nef divisor D . The corresponding supergravity models can however contain more than two gauge factors. The extra gauge groups should come from singular divisors that do not intersect D .
- Only in one special case, 6d UC in [92] is stronger than the 5d UC discussed in this paper. This happens when an additional condition $D \cdot D + D \cdot K = -2$ is satisfied, and the nef divisor D is a genus 0 curve. For this very special situation, we need to decrease the upper bound of UC by 1.⁶
- Further compactification on a circle to four dimensions does not lead to further unitarity constraints.

6.3.3.2 Examples

In order to illustrate the previous discussion, we may consider three examples of elliptic CY_3 which are fibrations over Hirzebruch surfaces \mathbb{F}_n (the details of the geometry of these examples can be found in e.g. [84]). For all these examples, we shall see that the residual divisor Y is indeed numerically nontrivial (its intersections with all nef divisors are strictly positive). Our forth example has already appeared in the text and in [92], and, to the best of our knowledge, has no known F-theoretic realisation. We shall see that if such realisation exists, it would require Y to be numerically nontrivial.

First we collect some data on \mathbb{F}_n , which will be useful in the first three examples. The effective divisor is spanned by D_v and D_s . Their intersection data are

$$D_v \cdot D_v = -n, \quad D_v \cdot D_s = 1, \quad D_s \cdot D_s = 0 \quad (6.3.18)$$

and the canonical divisor K satisfies

$$-12K = 24D_v + 12(m+2)D_s. \quad (6.3.19)$$

Example 1. 6d supergravity with a single $SU(N)$ can be modelled on base \mathbb{F}_2 . These types of F-theory models have some overlap with special case A in Section 6.3.3.1. In these cases, gravity anomaly cancellation requires $N \leq 15$. The gauge divisor on \mathbb{F}_2 is $S = D_v$, and the residual effective divisor Y is given by

$$Y = -12K - ND_v = (24 - N)D_v + 48D_s \quad (6.3.20)$$

⁶As in the case when the nef divisor is a rational curve, it is not hard to see from (6.1.3) that the upper bound set by 6d UC is $-12D \cdot K - 4$ rather than $-12D \cdot K - 3$ set by 5d UC. In this case, UC is stronger than KPC only when there exist positive integer solutions for $D \cdot S_i$ satisfying $12n - 4 < \sum_i \mu_i D \cdot S_i \leq 12n - \sum_i (x_i - \mu_i) D \cdot S_i$. As a result, since the lower bound is relaxed by 1 when compared with (6.3.15), some new special cases will appear. The conclusion will still be the same: for these cases $Y = -12K - \sum_i x_i S_i$ should satisfy some numerical properties in order for F-theory models not to violate UC.

Hence any nef divisor D with $D \cdot D_v > 0$ has to have the form $\alpha D_v + \beta D_s$ for some integers α and β with $\beta > 2\alpha$. Requiring in addition that $D \cdot K < 0$ and $D \cdot D \geq 0$ yields $\alpha \geq 0$, $\beta > 0$. So any such nef divisor will have $D \cdot Y = (24 - N)\beta + 2N\alpha \geq 9$.

Example 2. 6d supergravity with gauge group $SO(16) \times SU(4) \times SU(4)$ is modelled over \mathbb{F}_4 . The gauge divisors are $S_1 = D_v$, $S_2 = D_v + 4D_s$ and $S_3 = D_v + 8D_s$. The residual effective divisor is given by:

$$Y = -12K - 10D_v - 4(D_v + 4D_s) - 4(D_v + 8D_s) = 6D_v + 24D_s \quad (6.3.21)$$

It is not hard to see that the conditions $\alpha \geq 0$, $\beta > 0$, $\beta \geq 2\alpha$ are required in order for any divisor $D = \alpha D_v + \beta D_s$ to satisfy $D \cdot D \geq 0$, $D \cdot K < 0$ and $D \cdot S_i \geq 0$. As a result, $D \cdot Y = 6\beta \geq 6$.

Example 3. 6d supergravity with gauge group $U(1) \times SU(8)$ is modelled on \mathbb{F}_0 . For our purposes, we can ignore the Abelian factor. The relevant gauge divisor is $S = 2D_v + 2D_s$, and the residual effective divisor Y is given by

$$Y = -12K - 8(2D_v + 2D_s) = 8D_v + 8D_s \quad (6.3.22)$$

Any nef divisor $D = \alpha D_v + \beta D_s$ satisfies $D \cdot K < 0$, $D \cdot (2D_v + 2D_s) \geq 0$, $D \cdot D \geq 0$ only provided that $\alpha + \beta > 0$ and $\alpha\beta \geq 0$. As a result, $D \cdot Y = 8\alpha + 8\beta \geq 8$. Notice that this theory passes the unitarity test even with the additional $U(1)$ included (since any $U(1)$ factor can only contribute 1 to the central charge).

UC and KPC criteria can also be translated to conditions on some physical data of general 6d anomaly free minimal supergravity without knowing whether it can be realised in F-theory or not. The following is one example of applying UC and KPC to a 6d anomaly free minimal supergravity model.

Example 4. 6d supergravity with $SU(N) \times SU(N)$ with two bifundamentals and 9 tensor multiplets is an anomaly free theory [92]. The relevant data (the constant vectors in the GS couplings) are given by:⁷

$$\begin{aligned} \Omega &= \text{diag} (+1, (-1)^9), & a &= (-3, (+1)^9) \\ b_1 &= (1, -1, -1, -1, 0^6), & b_2 &= (2, 0, 0, 0, (-1)^6) \end{aligned} \quad (6.3.23)$$

Choosing the string charge as $Q = (1, 0, 0, 0, -1, 0, \dots, 0)$, we obtain $Q \cdot Q = 0$, $Q \cdot a = -2$ and $Q \cdot b_1 = Q \cdot b_2 = 1$. The unitarity condition (6.3.8) gives us:

$$2(N - 1) \leq 24 - 3 \quad (6.3.24)$$

Note that in this case $Q \cdot Q + Q \cdot a = -2$, and thus the 6d unitarity condition of [92] is slightly stronger than 5d UC: a shift by 1 on right hand side is needed and $2(N - 1) \leq 24 - 4$. Either way, the bound is $N \leq 11$, while the Kodaira positivity condition yields:

$$2N \leq 24 \rightarrow N \leq 12 \quad (6.3.25)$$

We see that in this case UC is slightly stronger than KPC. Also notice that $SU(12) \times SU(12)$ case, which satisfies KPC but violates UC, belongs to case C, enumerated in Section 6.3.3.1.

⁷Note that the string charge here is different from the one used in [92]. The choice of Q in [92] leads to $Q \cdot K = Q \cdot Q = -1$ and the putative corresponding divisor D on the base B of elliptic Calabi-Yau side would no longer be nef.

Assuming that this theory has an F-theoretic realisation and that there is an underlying elliptic CY_3 , the residual effective divisor would be

$$-12K = NS_1 + NS_2 + Y \tag{6.3.26}$$

Restricting for simplicity to the case $N \geq 4$, the singular divisors are of type I_N (see Table 1).⁸ Then we have $S_1 \cdot K = S_2 \cdot K = 0$. Since there are two hypers in the bifundamental and no hypers in the adjoint, we should take $S_1 \cdot S_1 = -2 = S_2 \cdot S_2$ and $S_1 \cdot S_2 = 2$, while $n_T = 9$ translates into $K \cdot K = 0$. Based on the above discussion, we can see that the new lesson UC offered in this case is that the relation

$$-12K = 12S_1 + 12S_2 \tag{6.3.27}$$

cannot be realised on the base B of an elliptic Calabi-Yau threefold with the required singularity structure, and the residual effective divisor $Y = -12K - 12S_1 - 12S_2$ has to be numerically non-trivial.

⁸We exclude $SU(2)$ and $SU(3)$ gauge groups since they may also be realised by III, IV -type singularities.

Conclusion and future directions

In this thesis, we discussed some aspects of string compactifications, both in perturbative and non-perturbative framework. We see the prominent role played by string dualities in these areas.

We point out several possible directions to conclude this thesis:

- As mentioned in part I, there is a duality between Heterotic string on \mathbb{T}^4 and Type IIA on $K3$. Applying adiabatic duality argument for Heterotic string on Seifert manifold X_5 would require studying "quantum $K3$ " geometry on type IIA side [114]. A question is can we have a geometric picture if we lift the IIA theory to M-theory and what is the M-theory background that gives the dual of Heterotic string on X_5 ? This direction currently is under investigation.
- In part II, we discussed the role of B -field in the worldsheet CFT: it gives a phase to each (g, h) sector $Z_{g,h}$ of the partition function in a way that satisfies modularity. However, its relation to second group cohomology $H^2(G, U(1))$ is not very clear. Could these phases given by B -field be realized by $H^2(G, U(1))$ if we go to a specific covering space? Or some of these phases could not be realized purely by $H^2(G, U(1))$ in any way? These questions are under investigation.
- In part III, we mentioned in some very special cases, unitarity condition hints finer information about elliptic fibration structure of Calabi-Yau threefold. It would be interesting to study these finer structures directly from geometry of elliptic Calabi-Yau threefold and derive these conditions in a geometric way.
- From the discussion of equivariant gerbes, it is clear that spacetime singularities may or may not affect the low energy effective theory, i.e. it depends on additional stringy background. In the duality between IIA/ $CY3$ and Het/ $K3 \times \mathbb{T}^2$, a similar situation appears as the flat C_3 holonomy could be turned on over the three cycles of the $CY3$ and obstructs the related extremal transitions ([145]). Moreover, in [145] the authors argued the local picture of $CY3$ dual of Heterotic standard embedding involves non-trivial C_3 holonomy. It would be interesting to generalize the local argument to global ones and find the exact type IIA background dual to Heterotic on $K3 \times \mathbb{T}^2$ with standard embedding.

We end this thesis with some general (and rough) remarks. Notice we mainly discussed string theory compactified on compact manifolds. We could also consider the non compact manifolds hence decouple gravity in the low energy effective theory. A natural question is how the stringy background (non-trivial B -field and RR -field holonomies) affects the resulting quantum field theories and the implications on their dynamics. We would like to explore these directions in the future.

Appendix A

Bosonic toy model

In this appendix we discuss the bosonic partition function for compactification on X_5 in the case of the \mathbb{Z}_2 quotient. This will illustrate some features of the decompactification limit to six dimensions discussed in section 3.1.3, and it will also set up conventions for some later developments. While the \mathbb{Z}_2 quotient is particularly simple, there is a straightforward to the remaining \mathbb{Z}_N actions.

A.1 Shift orbifold of S^1 : partition function

We described the construction of the shift orbifold of the circle in section 3.1.3. Starting with the circle partition function

$$Z(r) = \frac{1}{\eta\bar{\eta}} \sum_{n,w \in \mathbb{Z}} \mathbf{q}^{h_{n,w}} \bar{\mathbf{q}}^{\bar{h}_{n,w}} , \quad \eta = \mathbf{q}^{1/24} \prod_{n=1}^{\infty} (1 - \mathbf{q}^n) , \quad (\text{A.1.1})$$

where

$$\bar{h}_{n,w} = h_R(p) = \left(\frac{n}{2r} + \frac{rw}{2}\right)^2 , \quad h_{n,w} = h_L(p) = \left(\frac{n}{2r} - \frac{rw}{2}\right)^2 , \quad (\text{A.1.2})$$

the partition function of the shift orbifold for the \mathbb{Z}_2 quotient is

$$Z_{\text{sh}}(r) = \frac{1}{2} (Z_{\text{sh}+}^+ + Z_{\text{sh}+}^- + Z_{\text{sh}-}^+ + Z_{\text{sh}-}^-) , \quad (\text{A.1.3})$$

with

$$\begin{aligned} Z_{\text{sh}+}^+ &= Z(r) , & Z_{\text{sh}+}^- &= \frac{1}{\eta\bar{\eta}} \sum_{n,w \in \mathbb{Z}} e^{i\pi n} \mathbf{q}^{h_{n,w}} \bar{\mathbf{q}}^{\bar{h}_{n,w}} , \\ Z_{\text{sh}-}^+ &= \frac{1}{\eta\bar{\eta}} \sum_{n,w \in \mathbb{Z}} \mathbf{q}^{h_{n,w}^-} \bar{\mathbf{q}}^{\bar{h}_{n,w}^-} , & Z_{\text{sh}-}^- &= \frac{1}{\eta\bar{\eta}} \sum_{n,w \in \mathbb{Z}} e^{i\pi n} \mathbf{q}^{h_{n,w}^-} \bar{\mathbf{q}}^{\bar{h}_{n,w}^-} , \end{aligned} \quad (\text{A.1.4})$$

where the weights in the twisted sector are

$$\bar{h}_{n,w}^- = \left(\frac{n}{2r} + \frac{r}{2}(w + \frac{1}{2})\right)^2 , \quad h_{n,w}^- = \left(\frac{n}{2r} - \frac{r}{2}(w + \frac{1}{2})\right)^2 . \quad (\text{A.1.5})$$

This is just a fancy rewriting of $Z(r/2)$, but the split into the sectors will be useful for building the X_5 orbifold CFT.

A.2 A reflection orbifold of S^1

Another familiar quotient of the compact boson is the $x \rightarrow -x$ orbifold. In this case the action does affect the oscillators, and it acts on the states $|p\rangle$ by $g'|p\rangle = |-p\rangle$. As discussed in [53], the orbifold partition function can be obtained by explicitly constructing the untwisted sector with the g' projection, and the the twisted sector contributions are obtained by taking orbits of $SL(2, \mathbb{Z})$. The result is

$$Z_{\text{re}}(r) = \frac{1}{2} (Z_{\text{re}+}^+ + Z_{\text{re}+}^- + Z_{\text{re}-}^+ + Z_{\text{re}-}^-) , \quad (\text{A.2.1})$$

with

$$Z_{\text{re}+}^+ = Z(r) , \quad Z_{\text{re}+}^- = 2\sqrt{\frac{\eta\bar{\eta}}{\vartheta_2\bar{\vartheta}_2}} , \quad Z_{\text{re}-}^+ = 2\sqrt{\frac{\eta\bar{\eta}}{\vartheta_4\bar{\vartheta}_4}} , \quad Z_{\text{re}-}^- = 2\sqrt{\frac{\eta\bar{\eta}}{\vartheta_3\bar{\vartheta}_3}} . \quad (\text{A.2.2})$$

Our conventions for the Jacobi theta functions ϑ_i are as in [53]. The crucial factors of 2 in the twisted sector contributions reflect the presence to two fixed points for the orbifold action, at $x = 0$ and $x = 1/2$.

A.3 Bosonic CFT for X_5

To make contact with our compactification of X_5 , we first generalize the S^1 reflection orbifold to a reflection orbifold of \mathbb{T}^d . Since the reflection symmetry is present for all parameters of the \mathbb{T}^d CFT, we have

$$Z_{\text{re}+}^+ = Z_{\mathbb{T}^d} , \quad Z_{\text{re}+}^- = 2^d \left(\frac{\eta\bar{\eta}}{\vartheta_2\bar{\vartheta}_2} \right)^{d/2} , \quad Z_{\text{re}-}^+ = 2^d \left(\frac{\eta\bar{\eta}}{\vartheta_4\bar{\vartheta}_4} \right)^{d/2} , \quad Z_{\text{re}-}^- = 2^d \left(\frac{\eta\bar{\eta}}{\vartheta_3\bar{\vartheta}_3} \right)^{d/2} . \quad (\text{A.3.1})$$

Finally, to obtain the bosonic CFT for X_5 , we combine the orbifold actions on $\mathbb{T}^d \times S^1$ and find the orbifold partition function

$$\mathcal{Z} = \frac{1}{2} (\mathcal{Z}_+^+ + \mathcal{Z}_+^- + \mathcal{Z}_-^+ + \mathcal{Z}_-^-) , \quad (\text{A.3.2})$$

with

$$\mathcal{Z}_+^+ = Z(r)Z_{\mathbb{T}^d} , \quad \mathcal{Z}_+^- = Z_{\text{sh}+}^- Z_{\text{re}+}^- , \quad \mathcal{Z}_-^+ = Z_{\text{sh}-}^+ Z_{\text{re}-}^+ , \quad \mathcal{Z}_-^- = Z_{\text{sh}-}^- Z_{\text{re}-}^- . \quad (\text{A.3.3})$$

Appendix B

Equivariant flat gerbes

In this appendix we review aspects of abelian gerbes and equivariant structures on gerbes, following [33, 35, 72].

Consider a compact smooth Riemannian manifold X that admits an action of a finite group G : for every $a \in G$ there is a diffeomorphism $\varphi_a : X \rightarrow X$, and the composition respects the group structure: $\varphi_a(\varphi_b(x)) = \varphi_{ab}(x)$ for all $x \in X$. It is possible to choose a good cover $\mathfrak{U} = \{U_\alpha\}_{\alpha \in I}$ such that G has an action on the indexing set I , with $a : \alpha \rightarrow a(\alpha)$, and $\varphi_a(U_\alpha) = U_{a(\alpha)}$ and similarly for all non-empty intersections $U_{\alpha\beta}$, $U_{\alpha\beta\gamma}$, etc. [73]. We will assume that such a cover has been chosen, so that the various Čech cochain manipulations that are to follow have a simple interpretation.¹

B.1 Čech cochains and Hermitian line bundles

For what follows, it will be convenient to work in the language of Čech cochains and the coboundary operator δ , so we will take a moment to review that language and fix our conventions.² We consider a sheaf valued in an abelian group \mathcal{F} defined over our cover \mathfrak{U} . We will denote the cochains by $C^k(X, \mathcal{F})$, with $k = 0$ denoting the space of sections defined on each U_α , $k = 1$ on the $U_{\alpha\beta}$, and so on. The coboundary operator δ then takes $\sigma \in C^k(X, \mathcal{F})$ to $(\delta\sigma) \in C^{k+1}(X, \mathcal{F})$ in the familiar way: for example, if $\sigma^0 \in C^0(X, \mathcal{F})$, and $\sigma^1 \in C^1(X, \mathcal{F})$, then we have

$$(\delta\sigma^0)_{\alpha\beta} = \sigma_\alpha^0 - \sigma_\beta^0, \quad (\delta\sigma^1)_{\alpha\beta\gamma} = \sigma_{\alpha\beta}^1 + \sigma_{\beta\gamma}^1 + \sigma_{\gamma\alpha}^1. \quad (\text{B.1.1})$$

The signs and ordering are chosen so that $\delta^2 = 0$. We say a section σ is a cocycle if $\delta\sigma = 0$, and it is a coboundary if it can be written as $\sigma = \delta\lambda$. Note that we will have occasion to use both additive and multiplicative abelian groups; in the latter case the cocycle condition is written as $\delta\sigma = 1$.

The cochains that will show up in our discussion are:

- $C^k(X, \Omega^p)$, where Ω^p denotes smooth p -forms;
- $C^k(X, S^1)$, where $S^1 = \text{U}(1)$ denotes circle-valued constants;
- $C^k(X, \underline{S}^1)$, where \underline{S}^1 denotes smooth circle-valued functions.

A key result in Čech cohomology is that for sheaves that admit partitions of unity, such as $C^k(X, \Omega^p)$, the Čech cohomology groups $\check{H}^k(X, \Omega^p)$ are trivial for $k > 0$: every cocycle is a coboundary.

¹This is not strictly speaking necessary, but it makes for simpler arguments. A complete treatment would involve also proving that the results are independent of the choice of cover; we will leave that to the references [72].

²A readable introduction is given in [74].

Hermitian line bundles: conventions

Consider now a Hermitian line bundle $\pi : L \rightarrow X$ with transition functions $g_{\alpha\beta} : U_{\alpha\beta} \rightarrow S^1$ obeying

$$g_{\alpha\beta}g_{\beta\gamma}g_{\gamma\alpha} = 1 \quad (\text{B.1.2})$$

on all non-empty triple overlaps $U_{\alpha\beta\gamma}$, as well as connection 1-forms A_α defined on each U_α and satisfying

$$A_\alpha = A_\beta - id \log g_{\alpha\beta} \quad (\text{B.1.3})$$

on each $U_{\alpha\beta}$. Gauge transformations are encoded by functions $h_\alpha : U_\alpha \rightarrow S^1$, which act by

$$g_{\alpha\beta} \rightarrow g_{\alpha\beta}h_\alpha(h_\beta)^{-1}, \quad A_\alpha \rightarrow A_\alpha - id \log h_\alpha. \quad (\text{B.1.4})$$

A gauge transformation is global if on every overlap $h_\alpha = h_\beta$, so that h_α is a restriction of a circle-valued function defined on X to the set U_α .

This data is elegantly presented in the Čech language: the data for a line bundle with connection is a pair (g, A) with $g \in C^1(X, \underline{S}^1)$, $A \in C^0(X, \Omega^1)$ subject to

$$\delta g = 1, \quad \delta A = -id \log g, \quad (\text{B.1.5})$$

and gauge transformations are encoded by $h \in C^0(X, \underline{S}^1)$, and they act by

$$g \rightarrow g\delta h, \quad A \rightarrow A - id \log h. \quad (\text{B.1.6})$$

A global gauge transformation satisfies $\delta h = 1$.

Two bundles $L \rightarrow X$, $L' \rightarrow X$ with data (g, A) and (g', A') respectively, are isomorphic if and only if it is possible (after a suitable refinement of the covers) to find a gauge transformation h such that $(g', A') = (g\delta h, A - id \log h)$.

In what follows, we will take all of our line bundles to be Hermitian, so that the transition functions are circle-valued.

Equivariant line bundles

Given a line bundle $L \rightarrow X$ with transition functions g and connection A , we would like to lift the action of G on X to an action on L . For every $a \in G$ we can use φ_a to construct the pull-back bundle $\varphi_a^*(L)$. We say $L \rightarrow X$ is G -equivariant if and only if $\varphi_a^*(L) \simeq L$, and the isomorphisms, which we denote by $R^a \in C^0(X, \underline{S}^1)$ are compatible with the group multiplication law. That is, we have the diagram

$$\begin{array}{ccc} \varphi_a^*(L) & \simeq & L \\ \downarrow & & \downarrow \\ X & \xrightarrow{\varphi_a} & X \end{array} \quad (\text{B.1.7})$$

and for any section of L , $s \in C^0(X, \Omega^0)$, we have

$$\varphi_a^*(s) = s(\varphi_a(x)) = R^a(x)s(x). \quad (\text{B.1.8})$$

Since $\varphi_{ab}^*(s) = \varphi_b^*\varphi_a^*(s)$,

$$R^{ab}s = R^b\varphi_b^*(R^a)s, \quad (\text{B.1.9})$$

for any section s , and we conclude that for all $a, b \in G$

$$R^{ab} = R^b \varphi_b^*(R^a) . \quad (\text{B.1.10})$$

Since s also satisfies $\delta s = g$ —this is the Čech language for the perhaps more familiar relation $s_\alpha = g_{\alpha\beta} s_\beta$ for all $U_{\alpha\beta}$ —consistency with the pull-back requires

$$\varphi_a^*(g) = g \delta R^a . \quad (\text{B.1.11})$$

This implies that the G -action preserves the first Chern class of the bundle, i.e. $\varphi_a^*(c_1(L)) = c_1(L)$, which is a necessary condition for the lift to exist. We will be interested in lifts that also preserve the connection, meaning that the covariant derivative $Ds = ds - iAs$ satisfies $\varphi_a^*(Ds) = R^a(Ds)$. This requires

$$\varphi_a^*(A) = A - id \log R^a . \quad (\text{B.1.12})$$

Notice that the G -action preserves the curvature $F = dA$: $\varphi_a^*(F) = F$.

It is not hard to check that the definition is consistent with gauge transformations: if R^a provide the lift for the bundle data (g, A) , then

$$R_{\text{new}}^a = R^a \frac{\varphi_a^*(f)}{f} \quad (\text{B.1.13})$$

provide a lift for the gauge equivalent data $(g_{\text{new}}, A_{\text{new}}) = (g\delta f, A - id \log f)$.

Geometrically, all of this amounts to finding a G -action on the total space of the line bundle that is consistent with the projection and choice of connection.

B.2 Gerbes: basic structure

Having reviewed the case of line bundles, we now extend the discussion to G -equivariant gerbes. We begin with the defining data of a gerbe over X with connection: (ϑ, β, B) ,

$$\vartheta \in C^2(X, \underline{S}^1) , \quad \beta \in C^1(X, \Omega^1) , \quad B \in C^0(X, \Omega^2) \quad (\text{B.2.1})$$

satisfying

$$\delta\vartheta = 1 , \quad \delta\beta = id \log \vartheta , \quad \delta B = d\beta . \quad (\text{B.2.2})$$

Two gerbes with connections are equivalent if and only if (after a suitable refinement of cover) they are related by a 0- and 1-gauge transformations, with parameters $f \in C^1(X, \underline{S}^1)$ and $\eta \in C^0(X, \Omega^1)$, which act by

$$\vartheta \rightarrow \vartheta \delta f , \quad \beta \rightarrow \beta + id \log f + \delta\eta , \quad B \rightarrow B + d\eta . \quad (\text{B.2.3})$$

We will call the full transformation (f, η) a gerbe gauge transformation. Note that (f, η) leave the gerbe data invariant if and only if f defines transition functions for a flat line bundle over X with a compatible flat connection η .

The gerbe curvature $H = \frac{1}{2\pi} dB$ is a closed 3-form on X . Like the curvature of a line bundle, the curvature H is gerbe gauge-invariant, and its cohomology class characterizes gerbes at the level of topology.

B.3 G -action on the gerbe data

In the situation of a line bundle, we had a clear geometric perspective on finding a lift of the G -action to L : we were essentially finding a set of diffeomorphisms on the total space of the line bundle compatible with the projection to X . Such a perspective is not immediately available for a gerbe, but we can, as in [35], study the action on (ϑ, β, B) , and we will show that the solution to the consistency requirements of that action allows for more general results than those obtained in [35].

Starting with the defining relations of the gerbe we demand that for every $a \in G$ we have (f^a, η^a) such that

$$\varphi_a^*(\vartheta) = \vartheta \delta f^a, \quad \varphi_a^*(\beta) = \beta + id \log f^a + \delta \eta^a, \quad \varphi_a^*(B) = B + d\eta^a. \quad (\text{B.3.1})$$

Applying φ_b^* , we find that consistency with the group law require

$$\begin{aligned} \vartheta \delta f^{ab} &= \vartheta \delta f^b \varphi_b^*(\delta f^a), \\ -id \log \frac{f^{ab}}{f^b \varphi_b^*(f^a)} &= \delta(\eta^{ab} - \eta^b - \varphi_b^*(\eta^a)), \\ 0 &= d(\eta^{ab} - \eta^b - \varphi_b^*(\eta^a)). \end{aligned} \quad (\text{B.3.2})$$

These conditions are solved by

$$f^{ab} = f^b \varphi_b^*(f^a) k^{a,b}, \quad \eta^{ab} = \eta^b + \varphi_b^*(\eta^a) + \tau^{a,b}, \quad (\text{B.3.3})$$

where $(k^{a,b}, \tau^{a,b})$ is the data for a flat line bundle $T^{a,b} \rightarrow X$.

Associativity

There are non-trivial conditions from associativity of the group product. Imposing $f^{a(bc)} = f^{(ab)c}$ and $\eta^{a(bc)} = \eta^{(ab)c}$ in (B.3.3), requires

$$T^{a,bc} \otimes T^{b,c} \simeq T^{ab,c} \otimes \varphi_c^*(T^{a,b}), \quad (\text{B.3.4})$$

with

$$k^{a,bc} k^{b,c} = k^{ab,c} \varphi_c^*(k^{a,b}), \quad \tau^{a,bc} + \tau^{b,c} = \tau^{ab,c} + \varphi_c^*(\tau^{a,b}). \quad (\text{B.3.5})$$

Any flat connection $\tau^{a,b}$ can be written as

$$\tau^{a,b} = id \log h^{a,b} \quad (\text{B.3.6})$$

for some $h^{a,b} \in C^0(X, \underline{S}^1)$ satisfying

$$\delta h^{a,b} = t^{a,b} (k^{a,b})^{-1} \quad (\text{B.3.7})$$

for some Čech cocycle $t^{a,b} \in C^1(X, S^1)$. Recall that any flat line bundle admits a trivialization with constant transition functions, and the choice of such constant transition functions is a flat structure on the bundle [63]; $t^{a,b}$ is such a flat structure on the line bundle $T^{a,b} \rightarrow X$.

The relations (B.3.5) are equivalent to

$$\delta \left(\frac{h^{ab,c} \varphi_c^*(h^{a,b})}{h^{a,bc} h^{b,c}} \right) = \frac{t^{ab,c} \varphi_c^*(t^{a,b})}{t^{a,bc} t^{b,c}}, \quad d \left(\frac{h^{ab,c} \varphi_c^*(h^{a,b})}{h^{a,bc} h^{b,c}} \right) = 0. \quad (\text{B.3.8})$$

Therefore for every $a, b, c \in G$ there exists a locally constant phase $\psi^{a,b;c} \in C^0(X, S^1)$ such that

$$\varphi_c^*(h^{a,b}) = \psi^{a,b;c} \frac{h^{a,bc} h^{b,c}}{h^{ab,c}}, \quad t^{a,bc} t^{b,c} = \delta(\psi^{a,b;c}) t^{ab,c} \varphi_c^*(t^{a,b}). \quad (\text{B.3.9})$$

So, we see that $\psi^{a,b;c}$ is a constant gauge transformation relating the flat structures on the flat bundles in (B.3.4). Letting φ_d^* act on both sides of the first equation in (B.3.9), we find that ψ is a group cocycle, i.e. for each patch U_α , the constants $\psi_\alpha^{a,b;c}$ define a class in the group cohomology $H^3(G, \text{U}(1))$.

The appearance of the line bundles $T^{a,b}$ is a new feature compared to the analysis of [35], and it in principle allows for more general G -actions. As with many matters involving non-trivial gerbe structures, it would be useful to have concrete classes of examples that realize these seemingly more exotic possibilities.

Discrete torsion

From the first of the expressions in (B.3.9) we can already see the possibility of discrete torsion of [34]. Let $\lambda^{a,b}$ be a G 2-cocycle, i.e. $\lambda : G \times G \rightarrow \text{U}(1)$, such that

$$\frac{\lambda^{a,bc} \lambda^{b,c}}{\lambda^{ab,c} \lambda^{a,b}} = 1. \quad (\text{B.3.10})$$

In that case, if we have found $h^{a,b}$ that satisfy (B.3.9) with some $\psi^{a,b;c}$, then we obtain a new solution to (B.3.9) with the same $\psi^{a,b;c}$ by setting

$$h_{\text{new}}^{a,b} = h^{a,b} \lambda^{a,b}. \quad (\text{B.3.11})$$

As we will see below, the factors $h^{a,b}$ only enter the orbifold CFT partition function through combinations $h^{a,b}/h^{b,a}$ for commuting elements $a, b \in G$. Therefore, shifting $\lambda^{a,b} \rightarrow \lambda^{a,b} \xi^a \xi^b / \xi^{ab}$ for any G 1-cochain ξ leave the partition function invariant; said another way, the partition function only depends on the cohomology class $[\lambda] \in H^2(G, \text{U}(1))$.

Action on a topologically trivial gerbe

As for line bundles, the familiar exponential short exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{R} \xrightarrow{e^{2\pi i \cdot}} \text{U}(1) \longrightarrow 0 \quad (\text{B.3.12})$$

and the associated long exact sequence in cohomology can be used to relate differential data encoded in the connections to topological information. For example, the cohomology class of the curvature $[H] \in H^3(X, \mathbb{R})$ of a gerbe is the image of a class in $H^3(X, \mathbb{Z})$, and this latter class characterizes the gerbe at the level of topology.

A gerbe is flat if and only if its curvature vanishes: $H = 0$. Flat gerbes are classified by the cohomology group $H^2(X, \text{U}(1))$ [113], which encodes the holonomy of the B -field on 2-cycles. Using the long exact sequence associated to (B.3.12), we find

$$0 \longrightarrow H^2(X, \mathbb{R})/H^2(X, \mathbb{Z}) \longrightarrow H^2(X, \text{U}(1)) \longrightarrow \{H^3(X, \mathbb{Z})\}^{\text{tors}} \longrightarrow 0, \quad (\text{B.3.13})$$

where the last term is the torsion subgroup of $H^3(X, \mathbb{Z})$. The right-hand-side encodes a possible non-trivial topology of the flat gerbe, while the left-hand-side encodes the choice of B up to gauge transformations. This is analogous to the characterization of connections on flat line bundles, where the same description holds with cohomology degrees reduced by 1.

A gerbe with data (ϑ, β, B) is topologically trivial if and only if ϑ is a Čech coboundary. For any topologically trivial gerbe it is possible to make a gerbe gauge transformation that sets $\vartheta = 1$ and $\beta = 0$. This a partial gauge fixing, and the gerbe gauge transformations that preserve the choice are $(f, \eta) = (g, \mathcal{A})$, where (g, \mathcal{A}) is data for a line bundle $\mathcal{L} \rightarrow X$; these act by

$$B \rightarrow B + d\mathcal{A} . \quad (\text{B.3.14})$$

We now consider the G -action on a topologically trivial gerbe. Following our previous line of reasoning, we specify line bundles $\mathcal{L}^a \rightarrow X$ with data (g^a, \mathcal{A}^a) for every $a \in G$ such that

$$\varphi_a^*(B) = B + d\mathcal{A}^a . \quad (\text{B.3.15})$$

The group structure imposes further requirements on this data. Specializing (B.3.3) to the topologically trivial gerbe, we find that the data for the flat $T^{a,b}$ bundles is determined by the data for the \mathcal{L}^a bundles that specify the G -action. Denoting the dual bundle to L by L^\vee ,

$$T^{a,b} \simeq \mathcal{L}^{ab} \otimes (\mathcal{L}^b \otimes \varphi_b^*(\mathcal{L}^a))^\vee , \quad t^{a,b} = \frac{g^{ab}}{g^b \varphi_b^*(g^a)} \delta h^{a,b} , \quad id \log h^{a,b} = \mathcal{A}^{ab} - \mathcal{A}^b - \varphi_b^*(\mathcal{A}^a) . \quad (\text{B.3.16})$$

Bundle and gerbe gauge transformations

Are these structures are consistent with gauge transformations of the gerbe and of the individual line bundles \mathcal{L}^a ? If the latter does not hold, then we could not speak of the G -action as encoded in a choice of line bundles; if the former does not hold, then our G -action would depend on a particular representative B . We dispel both of these concerns, starting with bundle gauge transformations.

Given (g^a, \mathcal{A}^a) and $h^{a,b}$ satisfying (B.3.9), (B.3.15), and (B.3.16), we can pick gauge-equivalent data for the \mathcal{L}^a

$$g_{\text{new}}^a = g^a f^a , \quad \mathcal{A}_{\text{new}}^a = \mathcal{A}^a - id \log f^a , \quad (\text{B.3.17})$$

and set

$$h_{\text{new}}^{a,b} = h^{a,b} f^b \varphi_b^*(f^a) f^{ab-1} . \quad (\text{B.3.18})$$

It is straightforward to check that the factor $\xi^{a,b} = f^b \varphi_b^*(f^a) (f^{ab})^{-1}$ —a gauge transformation on $T^{a,b}$ —satisfies

$$\frac{\xi^{ab,c} \varphi_c^*(\xi^{a,b})}{\xi^{a,bc} \xi^{b,c}} = 1 , \quad (\text{B.3.19})$$

So, we see that $(g_{\text{new}}^a, \mathcal{A}_{\text{new}}^a)$ give a consistent G -action.

Next, consider a gerbe gauge transformation specified by a line bundle $L \rightarrow X$ with data (g_L, Λ) and

$$B_{\text{new}} = B + d\Lambda . \quad (\text{B.3.20})$$

Suppose that we have a G -action on the gerbe with connection B specified by line bundles \mathcal{L}^a such that

$$\varphi_a^*(B) = B + d\mathcal{A}^a . \quad (\text{B.3.21})$$

We can then obtain a G -action on the gerbe with connection B_{new} by taking new bundles

$$\mathcal{L}_{\text{new}}^a = \mathcal{L}^a \otimes \varphi_a^*(L) \otimes L^\vee, \quad (\text{B.3.22})$$

with data accordingly satisfying

$$g_{\text{new}}^a = g^a \varphi_a^*(g_L)(g_L)^{-1}, \quad \mathcal{A}_{\text{new}}^a = \mathcal{A}^a + \varphi_a^*(\Lambda) - \Lambda. \quad (\text{B.3.23})$$

Trivial $T^{a,b}$

In the special situation that $t^{a,b} = 1$, (B.3.16) gives isomorphisms

$$\mathcal{L}^b \otimes \varphi_b^*(\mathcal{L}^a) \simeq \mathcal{L}^{ab}, \quad g^b \varphi_b^*(g^a) = g^{ab} \delta h^{a,b}, \quad \mathcal{A}^b + \varphi_b^*(\mathcal{A}^a) = \mathcal{A}^{ab} - id \log h^{a,b}. \quad (\text{B.3.24})$$

As we discussed in section B.1, such isomorphisms have a geometric interpretation at the level of sections, or, equivalently, as diffeomorphisms on the total space of the bundle that commute with projection to X : for a point (x, ξ) in the total space of $\mathcal{L}^b \otimes \varphi_b^*(\mathcal{L}^a)$ we have

$$\mathcal{L}^b \otimes \varphi_b^*(\mathcal{L}^a) \rightarrow \mathcal{L}^{ab}, \quad (x, \xi) \mapsto (x, h^{a,b}(x)\xi). \quad (\text{B.3.25})$$

The associativity of the group multiplication law then requires the diffeomorphisms to satisfy

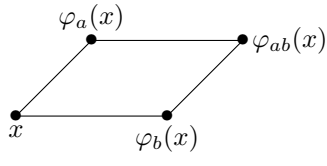
$$\varphi_c^*(h^{a,b})h^{ab,c} = h^{a,bc}h^{b,c}. \quad (\text{B.3.26})$$

Using these relations in (B.3.9), we see that $\psi^{a,b;c} = 1$. A bit more generally, if $t^{a,b} = \delta s^{a,b}$ for some $s^{a,b} \in C^0(X, S^1)$, then we can absorb the $s^{a,b}$ into a redefinition of $h^{a,b}$ while preserving (B.3.24). The associativity of the bundle isomorphisms then again implies (B.3.26), so that the $\psi^{a,b;c} = 1$. We also see that (B.3.26) is consistent with multiplying $h^{a,b}$ by a representative of $H^2(G, \text{U}(1))$, and modifying it by a group coboundary can be absorbed into the gauge transformations of the individual bundles \mathcal{L}^a —again, this is the discrete torsion of [34]. In this way, we recover the results of [35] when $T^{a,b}$ are trivial bundles. The nice properties³ given by trivial $T^{a,b}$ also suggests trivial $T^{a,b}$ should be encoded in the possible definition of equivariant gerbes, i.e. the G -action on gerbes should give isomorphism $\mathcal{L}^b \otimes \varphi_b^*(\mathcal{L}^a) \simeq \mathcal{L}^{ab}$.

B.4 A trivial flat gerbe on \mathbb{R}^d and orbifold CFT

Following [35], we now discuss how the equivariant structure just defined allows us to define the contribution of the B -field to the orbifold partition function for the theory on X/G . This is not easy for a general X equipped with a gerbe, but it is manageable and instructive in the special case that $X = \mathbb{R}^d$, G is abelian, and the gerbe is trivial and flat.⁴

Taking the worldsheet to be a torus, we fix a map from $T^2 \rightarrow X$ with image $S_{a,b}(x)$:



$$\begin{array}{ccc} & \varphi_a(x) & \\ & \bullet & \text{---} \bullet \varphi_{ab}(x) \\ & \nearrow & \searrow \\ x & \bullet & \bullet \varphi_b(x) \end{array} \quad (\text{B.4.1})$$

³see also next section

⁴The results are also relevant for non-abelian G , since the orbifold construction restricts the a, b to mutually commuting elements in G .

We then define the phase factor

$$P_{a,b} = \exp i \left[\int_{S_{a,b}(x)} B \right] \exp i \left[\int_x^{\varphi_a(x)} \mathcal{A}^b - \int_x^{\varphi_b(x)} \mathcal{A}^a \right] \frac{h^{a,b}(x)}{h^{b,a}(x)}. \quad (\text{B.4.2})$$

This phase factor is precisely the holonomy of the gerbe on X/G associated to the cycle that lifts to $S_{a,b}(x)$, and it enjoys three important properties.

- It is independent of the basepoint x .

First observe that $\int_{S_{a,b}(x)} B$ is base-point independent because $S_{a,b}(x)$ and $S_{a,b}(x+v)$ are homologous for any $v \in \mathbb{R}^d$ and $H = 0$. So, under a variation of $x \rightarrow x+v$, we have

$$\begin{aligned} -i\delta_v \log P_{a,b} &= \text{Lie}_v \left[\int_x^{\varphi_a(x)} \mathcal{A}^b - \int_x^{\varphi_b(x)} \mathcal{A}^a \right] - i \text{Lie}_v \log \frac{h^{a,b}(x)}{h^{b,a}(x)} \\ &= v \lrcorner (\varphi_a^*(\mathcal{A}^b) - \mathcal{A}^b - \varphi_b^*(\mathcal{A}^a) + \mathcal{A}^a) - iv \lrcorner d \log \frac{h^{a,b}}{h^{b,a}} = 0. \end{aligned} \quad (\text{B.4.3})$$

- It is invariant under gerbe gauge transformations.

This follows because

$$\begin{aligned} \int_{S_{a,b}} B_{\text{new}} &= \int_{S_{a,b}} B + \left(\int_x^{\varphi_a(x)} - \int_x^{\varphi_b(x)} + \int_{\varphi_a(x)}^{\varphi_{ab}(x)} - \int_{\varphi_b(x)}^{\varphi_{ab}(x)} \right) \Lambda \\ &= \int_{S_{a,b}} B + \int_x^{\varphi_a(x)} (\Lambda - \varphi_b^*(\Lambda)) - \int_x^{\varphi_b(x)} (\Lambda - \varphi_a^*(\Lambda)), \end{aligned} \quad (\text{B.4.4})$$

and the last two terms are canceled by the Λ transformations of the \mathcal{A}^a from (B.3.23).

- It is invariant under bundle gauge transformations.

To see this, we note that B does not transform under (B.3.17), while

$$\left[\int_x^{\varphi_a(x)} \mathcal{A}^b - \int_x^{\varphi_b(x)} \mathcal{A}^a \right] \rightarrow \left[\int_x^{\varphi_a(x)} \mathcal{A}^b - \int_x^{\varphi_b(x)} \mathcal{A}^a \right] \times \frac{f^a \varphi_a^*(f^b)}{f^b \varphi_b^*(f^a)}. \quad (\text{B.4.5})$$

The last factor is then canceled by the transformation of the ratio $h^{a,b}/h^{b,a}$.

While these properties make the factor well-defined, it is not obvious that including such a factor leads to a well-behaved orbifold CFT. Fortunately, if we assume that the X/G orbifold CFT is well-behaved without this phase factor, then the necessary and sufficient conditions for a well-defined partition function at any genus are well-known from the classic work [34]: the phases should satisfy

$$P_{a,b} P_{b,a} = 1, \quad P_{a,a} = 1, \quad P_{ab,c} = P_{a,c} P_{b,c}. \quad (\text{B.4.6})$$

Using (B.4.2) we see that

$$P_{a,b} P_{b,a} = \exp i \left[\int_{S_{a,b}(x)} B + \int_{S_{b,a}(x)} B \right]. \quad (\text{B.4.7})$$

The two integrals cancel because $S_{a,b}$ and $-S_{b,a}$ are homologous, and $H = 0$. Moreover, since the integrals cancel exactly, we also see that $P_{a,a} = 1$.

To study the last condition, we write

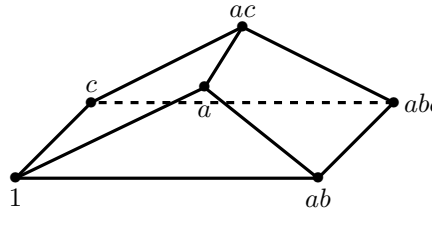
$$P_{a,b} = P_{a,b}^B P_{a,b}^A P_{a,b}^h \quad (\text{B.4.8})$$

and study the ratios for the different terms in turn.

The ratio of the B - factors,

$$\frac{P_{ab,c}^B}{P_{a,c}^B P_{b,c}^B}, \quad (\text{B.4.9})$$

involves the integral $\int_{S_{ab,c}} B$. We can relate this to the other integrals by constructing a closed surface as follows (we drop the x dependence and just label the points by elements of G):



$$(\text{B.4.10})$$

Because $H = 0$,

$$\int_{S_{ab,c}(x)} B = \int_{S_{a,c}(x)} B + \int_{S_{b,c}(\varphi_a(x))} B - \int_{\Sigma_{a,b}(x)} B + \int_{\Sigma_{a,b}(\varphi_c(x))} B, \quad (\text{B.4.11})$$

where $\Sigma_{a,b}$ is the oriented surface with ordered vertices $1, a, ab$. We then obtain

$$\begin{aligned} \int_{S_{ab,c}(x)} B - \int_{S_{a,c}(x)} B - \int_{S_{b,c}(x)} B &= \int_{S_{b,c}(x)} (\varphi_a^*(B) - B) + \int_{\Sigma_{a,b}} (\varphi_c^*(B) - B) \\ &= \int_{S_{b,c}(x)} d\mathcal{A}^a + \int_{\Sigma_{a,b}} d\mathcal{A}^c. \end{aligned} \quad (\text{B.4.12})$$

So, we conclude

$$\frac{P_{ab,c}^B}{P_{a,c}^B P_{b,c}^B} = \exp i \left[\int_{S_{b,c}(x)} d\mathcal{A}^a + \int_{\Sigma_{a,b}} d\mathcal{A}^c \right]. \quad (\text{B.4.13})$$

Next we tackle the $P_{ab,c}^A$ factor. Using (B.3.16) to eliminate the \mathcal{A}^{ab} term in $P_{ab,c}^A$ we arrive at

$$\frac{P_{ab,c}^A}{P_{a,c}^A P_{b,c}^A} = \exp i \left[\left(\int_x^{\varphi_{ab}(x)} - \int_x^{\varphi_a(x)} - \int_x^{\varphi_b(x)} \right) \mathcal{A}^c + \int_x^{\varphi_c(x)} (\mathcal{A}^a - \varphi_b^*(\mathcal{A}^a)) \right] \frac{\varphi_c^*(h^{a,b})}{h^{a,b}}. \quad (\text{B.4.14})$$

We rewrite the first term as

$$\left(\int_x^{\varphi_{ab}(x)} - \int_x^{\varphi_a(x)} - \int_x^{\varphi_b(x)} \right) \mathcal{A}^c = - \int_{\Sigma_{a,b}} d\mathcal{A}^c - \int_x^{\varphi_b(x)} (\mathcal{A}^c - \varphi_a^*(\mathcal{A}^c)), \quad (\text{B.4.15})$$

Because G is abelian, we have the relation

$$\mathcal{A}^a - \varphi_c^*(\mathcal{A}^a) + id \log h^{c,a} = \mathcal{A}^c - \varphi_a^*(\mathcal{A}^c) + id \log h^{a,c}, \quad (\text{B.4.16})$$

so that

$$\left(\int_x^{\varphi_{ab}(x)} - \int_x^{\varphi_a(x)} - \int_x^{\varphi_b(x)} \right) \mathcal{A}^c = - \int_{\Sigma_{a,b}} d\mathcal{A}^c - \int_x^{\varphi_b(x)} (\mathcal{A}^a - \varphi_c^*(\mathcal{A}^a) + id \log \frac{h^{c,a}}{h^{a,c}}), \quad (\text{B.4.17})$$

and combining factors we learn that

$$\begin{aligned} \frac{P_{ab,c}^B}{P_{a,c}^B P_{b,c}^B} \frac{P_{ab,c}^A}{P_{a,c}^A P_{b,c}^A} &= \exp i \left[\int_{S_{b,c}(x)} d\mathcal{A}^a + \int_x^{\varphi_c(x)} (\mathcal{A}^a - \varphi_b^*(\mathcal{A}^a) - \int_x^{\varphi_b(x)} (\mathcal{A}^a - \varphi_c^*(\mathcal{A}^a)) \right] \\ &\times \frac{\varphi_c^*(h^{a,b})}{h^{a,b}} \varphi_b^* \left(\frac{h^{c,a}}{h^{a,c}} \right) \frac{h^{a,c}}{h^{c,a}} \end{aligned} \quad (\text{B.4.18})$$

The square bracket is zero, and we have

$$\frac{P_{ab,c}^B}{P_{a,c}^B P_{b,c}^B} \frac{P_{ab,c}^A}{P_{a,c}^A P_{b,c}^A} = \frac{\varphi_c^*(h^{a,b})}{h^{a,b}} \varphi_b^* \left(\frac{h^{c,a}}{h^{a,c}} \right) \frac{h^{a,c}}{h^{c,a}}. \quad (\text{B.4.19})$$

Combining this with the ratio $P_{ab,c}^h/P_{a,c}^h P_{b,c}^h$, we find

$$\frac{P_{ab,c}}{P_{a,c} P_{b,c}} = \frac{h^{ab,c}}{h^{c,ab}} \frac{h^{c,b}}{h^{b,c}} \frac{\varphi_c^*(h^{a,b})}{h^{a,b}} \varphi_b^* \left(\frac{h^{c,a}}{h^{a,c}} \right). \quad (\text{B.4.20})$$

Using the constraint from (B.3.9) we can eliminate the pullbacks from this expression, and we find the necessary and sufficient condition for a well-defined CFT partition function at any genus:

$$\frac{P_{ab,c}}{P_{a,c} P_{b,c}} = \frac{\psi^{a,b;c} \psi^{c,a;b}}{\psi^{a,c;b}} = 1. \quad (\text{B.4.21})$$

We saw above that in the case $t^{a,b} = 1$ we have $\psi^{a,b;c} = 1$, so the orbifold partition function is consistent for every equivariant gerbe with trivial $T^{a,b}$.

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