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Towards quantum gravity as a field theory of discrete geometries

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Par **Matteo Laudonio**

**Towards quantum gravity as a field theory of discrete
geometries**

Sous la direction de : **Adrian TANASA**

Soutenue le 7 juillet 2022

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Résumé: La formulation de Plebanski de la relativité générale montre que la gravité en quatre dimensions peut être étudiée comme un modèle topologique contraint. Cela a conduit une partie de la communauté de la gravité quantique à étudier les théories (quantiques) des champs topologiques. Dans ce contexte, des modèles de somme d'états sont utilisés pour décrire des invariants topologiques. Les états de la théorie quantique sont des décompositions cellulaires d'une variété et la fonction de partition est une somme sur de telles géométries discrètes. Dans cette thèse, je considère différentes approches de la gravité quantique, toutes basées sur la description de géométries discrètes.

Dans la première partie de ma thèse, je résume brièvement les modèles matriciels, les modèles tensoriels et le modèle SYK. L'une des théories les plus réussies qui traite la gravité comme un modèle de géométries discrètes est le modèle matriciel. Il s'agit d'une formulation d'intégrale de chemin dont les champs fondamentaux sont des matrices. Les diagrammes de Feynman sont des cartes planaires interprétées comme des graphes duaux aux géométries discrètes bidimensionnelles. La fonction de partition des modèles matriciels a donc été exprimée comme une somme sur les topologies bidimensionnelles dont l'expansion est dominée par les triangulations de la sphère. Le grand succès des modèles matriciels a suggéré la description de la gravité quantique dans des dimensions supérieures en utilisant des tenseurs. Cependant, contrairement aux modèles matriciels, la fonction de partition des tenseurs aléatoires s'est avérée être dominée par des graphes assez simples qui ne codent pas les géométries appropriées. Néanmoins, les modèles tensoriels ont jeté les bases des théories tensorielles des champs de groupe (GFT) et ont joué un rôle de premier plan dans l'étude du modèle Sachdev-Ye-Kitaev. Il s'agit d'une théorie quantique des champs unidimensionnelle qui décrit l'interaction d'un nombre arbitraire de fermions couplés par un tenseur aléatoire. En ce sens, le modèle SYK est un exemple de système quantique à plusieurs corps, avec une interaction chaotique non locale. Le modèle SYK attire encore plus l'attention en tant que modèle de jouet unidimensionnel pour la dualité AdS/CFT. Je discuterai de la généralisation d'un modèle SYK où les couplages obéissent à une distribution non-gaussienne. Je prouverai l'universalité gaussienne et donnerai l'action effective du modèle, montrant les effets de la non-gaussianité comme une modification de la covariance.

La deuxième partie de ma thèse est consacrée aux modèles de GFT. Le premier exemple de modèle de somme d'états pour les variétés euclidiennes 3d est historiquement le modèle de Ponzano-Regge. Dans ce cas, les représentations du groupe $SU(2)$ sont utilisées comme décorations des cotés dans la variété triangulée. Boulatov a proposé une théorie qui vise à redériver les mousses de spins comme diagrammes de Feynman d'une théorie des champs. Le modèle a été appelé théorie des champs de groupe (GFT) et a ensuite été généralisé en quatre dimensions. Les GFT peuvent être interprétés comme la deuxième description quantifiée d'un système quantique à plusieurs corps où les états sont les éléments constitutifs d'une géométrie discrète; la fonction de partition est donnée en tant que somme sur les amplitudes de tels états (spin foams) qui sont associés aux représentations d'un groupe G donné. Je passerai d'abord brièvement en revue les ingrédients essentiels du modèle, puis je proposerai deux nouvelles généralisations. La première est basée sur les algèbres de Hopf et vise à décrire une 3d GFT où l'espace des configurations et les espaces des moments sont courbes et non commutatifs. La deuxième généralisation est basée sur les 2-groupes. Dans cette partie, je donnerai d'abord quelques arguments selon lesquels 2-groupes sont nécessaires pour décrire des variétés discrètes à quatre dimensions; je discuterai de la construction d'un espace des phases décoré par des éléments de 2-groupe, et puis j'appliquerai ce résultat pour construire la GFT associée, exprimant la fonction de partition sous la forme d'une somme d'états.

Mots-clés: Géométries Discrètes, Gravité Quantique, GFT, Modèle SYK

Abstract: The Plebanski formulation of general relativity suggests the analysis of four dimensional gravity as a constrained topological model. This led the focus of part of the quantum gravity community towards topological (quantum) field theories. In this context, *state sum models* are used to describe manifold invariants. The states of the quantum theory are parameterized by the cellular decompositions of a manifold and the partition function is a sum over such discrete geometries. In this thesis I discuss different approaches based on the field theoretical formulation of discrete geometries.

In the first part of my thesis I briefly summarize matrix models, tensor models and the SYK model.

One of the most successful theories that treats gravity as a model of discrete geometries is the matrix model. This is a path integral formulation whose fundamental fields are matrices. The Feynman diagrams are planar maps interpreted as the graphs dual to $2d$ geometries. The partition function of matrix models can be expressed as a sum over the $2d$ topologies, whose expansion is dominated by triangulations of the sphere. The great success of matrix models suggested the description of quantum gravity in higher dimensions using tensors. However, differently from matrix models, the triangulations obtained using random tensors turned out to be dominated by rather simple graphs which do not encode the proper geometries. Nevertheless, tensor models laid the foundations of *tensorial group field theories* and played a prominent role in the study of the Sachdev-Ye-Kitaev (SYK) model. The latter is a $1d$ quantum field theory that describes the interaction of an arbitrary number of fermions coupled by a random tensor. The SYK model gathered a large attention as a one dimensional toy model for the AdS/CFT duality. I discuss the generalization of a particular case of the Gross-Rosenhaus version of a complex SYK model, where the couplings obey to a non-Gaussian distribution. I prove the Gaussian universality and provide the effective action of the model, showing the effects of the non-Gaussianity as a modification of the covariance.

The second part is dedicated to group field theories. The first example of a state sum model for $3d$ Euclidean manifolds, was the Ponzano-Regge model. In this case, representations of the $SU(2)$ group are used as *decorations* of the edges in the triangulated manifold. In a similar spirit, Penrose constructed what is called now *spin networks*, by associating representations of $SU(2)$ and intertwiners to respectively links and nodes of a graph. Boulatov proposed a theory that aims to recover the Ponzano-Regge state sum as the Feynman diagrams of a field theory. The model was called *group field theory* and was later generalized in four dimensions. Group field theories can be interpreted as the second quantized description of a many body quantum system where the states are the building blocks of a discrete geometry; the partition function is expanded as a sum over the amplitudes of such states (spin foams) which are associated to the representations of the given gauge group. I first provide a short review of the essential ingredients of the model, and then I propose two new generalizations of it. The first is based on Hopf algebras and aims to describe any possible group field theory associated to a phase space where both the configuration and momentum spaces are curved and non-commutative. The second generalization is instead based on 2-groups. In this part, I present first a few arguments according to which 2-groups are the tool needed to describe topological features of four dimensional manifolds; then I discuss the construction of a phase space decorated by 2-group elements and finally I apply this construction to formulate a 2-group field theory, expressing the partition function as a state sum.

Keywords: Discrete Geometries, Quantum Gravity, Group Field Theories, SYK model

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Contents

Introduction	7
1 Random matrix and tensor models	12
1.1 Random matrices	12
1.2 Random tensor models	15
1.3 Remarks	18
2 The Sachdev-Ye-Kitaev model with a non-Gaussian disorder average	19
2.1 Non-Gaussian disorder average	21
2.2 Concluding remarks	27
3 Group field theories	28
3.1 Three dimensional Group Field Theory	29
3.1.1 Spin foams as Feynman diagrams of GFT	29
3.1.2 Dual picture: discrete geometry	32
3.1.3 Peter-Weyl theorem	35
3.2 Higher dimensional GFT and geometric condition	35
3.3 Remarks	36
4 Quantum groups in group field theory	38
4.1 Quantum groups	39
4.2 Hopf algebra field theory	43
4.2.1 Topological invariance	51
4.3 q -Deformed group field theory	52
4.3.1 Plane-wave and discretization of the BF action	56
4.4 Remarks and perspectives	59
5 2-Groups in quantum gravity	61
5.1 2-Groups	64
5.2 Phase space of a three dimensional triangulation	66
5.2.1 Symplectic reduction	69

5.2.2	Polygon phase space	71
5.2.3	Polyhedra phase space	75
5.3	2-Group field theory	89
5.3.1	Basic ingredients of the model	90
5.3.2	Topological invariance	94
5.4	Remarks and perspectives	96
Conclusion and discussions		98
A Lie bi-algebras and Poisson Lie groups		102
A.1	Poincaré and κ -Poincaré Lie bi-algebras	106
B Hopf algebra field theory		109
B.1	Deriving dual bi-algebras	109
C Triangulation phase space		113
C.1	Fundamental fusions	113
D 2-Group field theory		119
D.1	Key Feynman diagrams	119
D.2	Pachner moves	126
E New proposal for a 2-Hopf algebra		131
E.1	2-Bi-algebra	131
E.2	Examples	132
Bibliography		138

Introduction

The problem of quantum gravity

The description of quantum gravitational phenomena as well as the definition of a quantum space-time and ultimately the unification of the standard model with gravity, are some of the main challenges addressed in theoretical physics for many decades. The main issue for the formulation of a proper quantum gravity model is the lack of experimental evidence. Differently from the formulation of past theories, such as thermodynamics, electromagnetism and even quantum mechanics, we can not compare the predictions of any model of quantum gravity with the experimental data. Nowadays, a common approach to formulate models of quantum gravity is to make *assumptions* based on our *intuition*; here I want to focus on one particular intuition: the connection between gravity and topological models.

According to Einstein's formulation, what we call gravity is the dynamics of the space-time's geometry. Space-time is not fixed, it changes its shape, stretches and contracts according to the distribution of energy and matter in it. In turn, its curved shape dictates the trajectories of moving particles and macroscopic bodies. The approach used to accomplish this description, in Einstein's general relativity, is that of a *field theory*. The field is the ingredient that encodes the fundamental degrees of freedom of the model. In the case of gravity, such field is the metric of the space-time, that contains the information about distances and shape of the geometry.

On the other hand, quantum mechanics deals with the description of microscopic phenomena, and one of the initial milestone on which it is funded is the duality between waves and particles. As a consequence, the dynamics of particles ceases to be deterministic, and becomes probabilistic; the evolution of microscopic bodies can not be described as a single *path*, but rather as multiple trajectories that connect the initial to the final state, weighted by different probabilities. This feature is realized by describing particles through some objects known as *wave functions*. Remarkably, also quantum mechanics can be formulated as a field theory, where such wave functions cover the role of fields. The modern formulation of quantum mechanics is due to Feynman, in terms of the so called *path integral*. Here, all the features of the system in consideration are encoded in a single object, called *partition function*, interpreted as the sum over all the possible paths pursued by the evolution of such system. From the partition function is possible to derive all the predictions of the model, which are thus the probabilities that a given system evolves from an initial to a final setting, following one of the possible paths. This reflects the probabilistic nature of quantum mechanics. Feynman also introduced a graphical tool – the *Feynman diagrams* – that allows to compute these predictions in a rather simple way, according to a set of rules.

Although the modern formulation of gravity and quantum mechanics share the same language of fields, their unification in a common theory remains a very complicated task; the true natures of the two models seem to be incompatible. One of the warnings relies in their fundamental formulation: quantum mechanics describes particles moving on a fixed space-time, while gravity describes the dynamics of such background. This is known for instance the *background independence* problem. Even the mere description of the quantum behaviour of gravity turned

out to be highly non-trivial, and determining what are the fundamental ingredients of a quantum theory of gravity is not an obvious job. Since there are no observations of any quantum gravitational phenomena, and since their theoretical formulation seems to be very complex, it is reasonable to wonder whether a theory of quantum gravity is actually necessary. There are some cases, such as the interior region of a black hole or the first moments of life of our universe, where it is expected that quantum gravitational phenomena were dominating. Neither quantum mechanics or general relativity could be able to properly describe such regimes.

So, where do we start from to construct a model of quantum gravity? Over the years different ideas originating from different motivations converged to some common schemes. One possibility is thus to make some assumptions based on these schemes, which will drive us towards the formulation of the model. For example, Penrose proposed to reconstruct space(time) by colliding objects with a rotational symmetry, with a conservation of angular momentum [1]. This can be represented as a graph decorated by representations of some group, such as $SU(2)$, with nodes decorated by interwiners. These graphs were called *spin networks*. At the same time, gauge theory was developed and in particular lattice gauge theories were used to describe topological models. In this context, the invariants of a manifold can be encoded as the global (gauge) degrees of freedom of the lattice, and the symmetries of the manifold can be reconstructed by concatenating some fundamental geometric building blocks. Also the model proposed by Penrose can be expressed in this language, as the spin networks are interpreted to encode the fundamental quantum states of a discrete geometry. Later on, the *spin foams* were constructed as the transition amplitude between such quantum states [2]. From a geometric perspective, spin foams characterize how the building blocks of a given geometry can be glued together to reconstruct the space-time manifold, whereas from a field theoretic point of view, they are the transition amplitudes between quantum states.

The idea of reconstructing the symmetries of a manifold by merging the fundamental pieces of a discrete geometry will be the main direction that I will follow in this thesis. It will be the common thread between different approaches that I am going to discuss. However, gravity is a topological model only in two or three dimensions. The way to reconstruct the gravitational degrees freedom in higher dimensions is to make use of the constrained systems approach, introduced by Dirac [3]. Indeed, the Plebanski formulation of general relativity [4] suggests that gravity in four dimensions can be studied as a constrained topological model (constrained BF model [2]). From this perspective, inspired by topological (quantum) field theories [5, 6], one can construct models of quantum discrete geometries as topological *state sum models*, where the states are amplitudes associated to discrete geometries or cellular decompositions of a manifold (eventually spin foams). Ultimately, the proper dynamics or the proper degrees of freedom can be implemented by imposing the required constraints.

Matrix and tensor models

As discussed above, a fundamental assumption that I will adopt in this thesis, is that we can use (quantum) discrete geometries to characterize the quantum gravity regime. Under this perspective, gravity is thus understood as a lattice (gauge) theory [7]. One of the most successful theory in this direction is the matrix model [8–11]. This is a path integral formulation whose fundamental fields are matrices. The Feynman diagrams are the so called *ribbon graphs*: these are maps drawn on a surface that can be interpreted as the graphs dual to a two dimensional cellular decomposition. The model gathered large notoriety when 't Hooft provided the large N expansion [12] for the related partition function. The idea is to associate to each graph, a weight governed by its genus. This expansion allowed to control the series of ribbon graphs and to express the matrix model partition function as a (state) sum for a model of two dimensional discrete geometries. Here the sum goes over the $2d$ topologies and the expansion is dominated

by triangulations of the sphere. Later on, matrix models have been fully solved with the use of topological recursion [13] and related to two dimensional quantum gravity [14].

The great success of matrix models suggested the description of quantum gravity or discrete geometries in dimension higher than two, using tensors. This new formulation was first introduced in [15] as the natural generalization of matrix models. It still consists in the path integral formulation of a field theory, whose Feynman diagrams are graphs dual to triangulations of a d dimensional manifold. However, differently from matrix models, the large N expansion for the partition function was much harder to derive, it came after more than twenty years for a specific class of these models based on colored tensors [16–18]. Unfortunately, as the large N expansion of matrix models properly describes two dimensional discrete geometries, the expansion of tensors is dominated by rather simple graphs which do not encode the proper degrees of freedom to describe gravity.

Despite the failure of tensor models to describe higher dimensional discrete geometries, they turned out to be precious in the study of renormalization of quantum gravity models [19], they laid the foundations of *tensorial group field theories* (that I will discuss later and consist in the major part of this manuscript) and played a prominent role in the study of the Sachdev-Ye-Kitaev (SYK) model [20,21]. The latter is a one dimensional quantum field theory that describes the interaction of an arbitrary number of fermions. The connection with tensor models relies in the fact that the coupling between such fermions is assumed to be a random tensor. In this sense the SYK model is an example of a many body quantum system, with a non-local chaotic interaction [22]: in terms of Feynman diagrams, the fermions do not interact at a point, but rather in an extended area represented as the graph associated to the tensor coupling. Despite such inspiring interpretation, the SYK model gathered even more attention as a one dimensional toy model for the AdS/CFT duality [23].

Tensorial group field theories

One of the first examples of state sum model for three dimensional Euclidean manifolds, where gravity is naturally a topological theory, was the Ponzano-Regge model [24]. In this case, the states are based on representations of the $SU(2)$ group, which are used as *decorations* of geometric objects in the triangulated manifold, see also [25]. The approach of using group representations to describe discrete geometries, is at the root of spin foam models [26,27]. In this context, graphs dual to the cellular decomposition of a three dimensional manifold (*dual complexes*) are associated to the states of the quantum theory. A similar construction was first introduced by Penrose in [1], where these graphs were called spin networks [28], and later on it was extensively used in Loop Quantum Gravity [29–31]. The dynamical evolution (transition amplitude) from one spin network to another is encoded in the graphs called *spin foams*, which are thus interpreted as the graphs dual to triangulations of four dimensional manifolds [2]. In this sense, spin networks are understood as the boundary (states) of spin foams.

In [32], Boulatov proposed a new theory that aims to recover the graphs generated by the Ponzano-Regge model as the Feynman diagrams of a field theory. This model, called group field theory (GFT), was based on the gauge symmetry group $SU(2)$, and constructed in such a way its Feynman diagrams are complexes dual to three dimensional triangulations. The four dimensional generalization of group field theory was proposed by Ooguri, in [33]. It was then shown that any spin foam amplitude could be recovered as the Feynman diagram of a specific group field theory [34]. Remarkably, group field theories can be interpreted as the second quantized description of a many body quantum system [35], where the states are the building blocks of the cellular decomposition of a manifold (in arbitrary dimension). The partition function of the model can be expanded as sum of the amplitudes associated to such states, which are thus spin foams.

A possible application of group field theories is to recover continuum physics as an effective behaviour of group field theory [36–38]. It should be clear that group field theories are a specific class of tensor models with an enriched set of (algebraic) degrees of freedom associated to the Feynman diagrams. The amplitudes of GFT’s are indeed associated to the representations of a given group G , rather than to the degree of the graph as in the theory of random tensors. Hence, GFT’s turned out to provide a more suitable description of discrete geometries in dimension higher than two, than ordinary tensor models. Lastly, I would like to mention two of the most celebrated examples of models that can be recast as group field theories: the Barrett-Crane model [39] and the EPRL model [40], which is the most advanced version of spin foam models. In both cases a group field theory based on the Lorentz group is considered. Despite group field theory is initially introduced as a topological model, neither the Barrett-Crane or the EPRL models are topological; they can indeed be formulated as constrained version of specific group field theories. The implementation of gravity that breaks the topological invariance is due to the *simplicity constraint* [4]. This condition appears first in the Plebanski formulation of gravity, expressed as a constrained BF model, and relates the metric degrees of freedom to the bi-vector field [30, 41].

Although the gravitational model is clearly the most interesting case, when discussing some tensorial group field theory models in this thesis, I will focus on the topological (unconstrained) formulation of discrete geometries, leaving the implementation of the simplicity constraint for later investigations.

Let me now present the plan of my thesis. In the second chapter I will discuss the basic features of matrix and tensor models and their fundamental ingredients. In this part, I would like to stress their interpretation as field theories generating discrete geometries and their formulation as state sum models. This point is particularly important for my discussion, since it can be seen as the common thread between the different approaches to discrete geometries that I am presenting here. The connection between random matrices and two dimensional quantum gravity will also be emphasized.

Ch. 2 is then focused on the Sachdev-Ye-Kitaev model. I will introduce first its main aspects, pointing out the reasons why it became so popular; I will mention the several versions in which it can be studied and the features in common with tensor models. Then, I will discuss a work done in collaboration with T. Krajewski, R. Pascalie and A. Tanasa [42]. In this project we considered a complex version of a specific Gross-Rosenhaus generalization [43] of the SYK model, where the tensor couplings obey to a general non-Gaussian potential. We prove the Gaussian universality which, for colored tensor models, was shown in [44]; then we provide the effective action of the model and highlight the effects of the the non-Gaussian distribution as a modification of the covariance.

Later, I will move to the discussion of tensorial group field theories. I provide an introductory summary of the three dimensional model in Ch. 3. The purpose of this chapter is twofold: I want to give an overview to the reader who is not familiar with the topic, and mostly, I would like to emphasize the main aspects of a group field theory that I will often recall in the rest of the thesis.

In chapter 4 indeed I want to discuss a generalization of three dimensional group field theory, based on Hopf algebras. Called Hopf algebra field theory, this is a project in collaboration with F. Girelli [45]. First, I will introduce the concepts of Hopf algebras and quantum groups. These can be seen, for now, as a generalization of groups, used for instance as the tool that encodes the proper symmetries of (discrete) three dimensional gravity. I will first introduce the Hopf algebra ingredients extensively used in the formulation of the new model, and then I will provide its main aspects in analogy with ordinary group field theories. In the final part of the chapter I explain

that Hopf algebra field theory can be regarded as the path integral formulation generating the Turaev-Viro amplitude [46].

Ch. 5 is dedicated to another generalization of group field theory, based on 2-groups. The latter is a mathematical tool used to generalize some concepts of differential geometry, such as the parallel transport on a manifold. It was also suggested that 2-groups are the right tool to properly encode the topological features of a four dimensional manifold. This is why, together with F. Girelli and P. Tsimiklis, we provided the construction of the phase space of a three dimensional cellular decomposition, decorated with 2-group elements [47]. This construction, presented in the first part of Ch. 5, aims to provide a basis for the description of the discrete geometries that, upon quantization, will be interpreted as the quantum states of an eventual state sum model. The step further is addressed in a project in collaboration together with F. Girelli, A. Tanasa and P. Tsimiklis in [48]. I will discuss this work in the second part of Ch. 5; here we formulate a group field theory based on 2-groups, where indeed we use the construction provided in [47] to define the kinematical states of the model. Finally, we show that the new model provides a topologically invariant state sum.

Chapter 1

Random matrix and tensor models

In this chapter I briefly review the theory of random matrices and random tensors, and their application to discrete geometries or quantum gravity. This is an overview with a specific focus on the aspects that will be useful for the rest of my thesis. One can find more details on matrix models in [8, 11, 13, 49]; I refer to [14] for a discussion on the relation between random matrices and two dimensional quantum gravity, and [50] for more hints on the combinatorial aspects. Moreover, the books [9, 10] contain a complete overview on the model. General reviews on tensor models instead are [51–55].

1.1 Random matrices

In the past decades, matrix models have been largely studied for their various application in combinatorics, probability and mathematical physics [11]. They have been first introduced by Wigner [56] to study the diffusion of neutrons with heavy atoms. The initial Wigner's intuition was similar to the arguments that led Boltzmann to the formulation of statistical mechanics. The classical deterministic approach was too complicated for systems with large number of particles, or large number of degrees of freedom. Therefore, Boltzmann formulated a probabilistic description of the macroscopic system, in which the probability distribution depends on the symmetries of the system (related to the type of microscopic interactions between the components). Similarly, when the mass of an atom or the energy of a neutron are too high, the diffusion between the two depends on too many degrees of freedom and its description in terms of ordinary quantum mechanics becomes too complicated. Wigner thus proposed to encode the degrees of freedom (the symmetries) of such system in a random matrix. The job then reduces to study the distribution of the eigenvalues of such matrices. Assuming that these matrices follow a Gaussian distribution, the partition function of the system is

$$\mathcal{Z} = \int_{M_{N \times N}} dM e^{-\frac{N}{2} \text{Tr}(M^2)}, \quad (1.1)$$

where M is the spaces of the $N \times N$ Hermitian matrices with probability measure

$$dM = \frac{1}{2^N} (N/\pi)^{\frac{N^2}{2}} \prod_{i=1}^N dM_{ii} \prod_{i < j} d\Re M_{ij} d\Im M_{ij}. \quad (1.2)$$

In this context, the eigenvalues λ_n are distributed according to the function $\rho(x) = \frac{1}{N} \sum_n \langle \delta(x - \lambda_n) \rangle$. Using the resolvent technique (see [9, 10]) one finds the celebrated semi-circle law [56] for the distribution of the resonant energies of the diffused neutron

$$\rho(x) = \frac{1}{\pi} \sqrt{x^2 - 2}. \quad (1.3)$$

Here $\text{Sym}(\Gamma)$ is the order of the symmetry group of the graph Γ and $v_n(\Gamma)$ is the number of n -valent vertices of Γ . Taking the logarithm of eq. (1.10) reduces to the partition function to only connected graphs

$$\mathcal{F}[t_n] = \log \mathcal{Z}[t_n] = \sum_{\Gamma \text{ connected}} \frac{N^{2-2g(\Gamma)}}{|\text{Sym}(\Gamma)|} \prod_n t_n^{v_n(\Gamma)}. \quad (1.11)$$

Since the genus of a graph is always positive or null, it is clear that the above expansion in $1/N$ is bounded by N^2 . This allows the study of matrix models for large values of N . In this case, the dominant ribbon graphs are those with vanishing genus, called *planar graphs*. One observes that the graph expansion of such planar graphs becomes critical when the coupling constant t_n reaches a critical value t_c . Such regime is called *double scaling limit* [54]. It amounts to take the large N limit ($N \rightarrow \infty$) and, at the same time, the limit of the coupling t_n to the singularity t_c , such that the product $x = N(t - t_c)^\alpha$ remains constant. In this limit, the partition function of connected graphs (1.11) assumes the shape of

$$\mathcal{F}[t_n] = \sum_g x^{2g-2} K_g, \quad (1.12)$$

with K_g being some constant that depends on the genus. Since the terms K_g 's are all positive, the sum converges only if one reduces the model to convergent matrix integrals, see [9, 10]. The main reason why the double scaling limit is interesting for our purpose, is that in this regime, the average number of vertices of a graph diverges, which can be interpreted as the recover of a continuum geometry. The double scaling limit can thus be considered as a continuous limit for the two dimensional quantum gravity interpretation of random matrices.

Random matrices and 2d quantum gravity

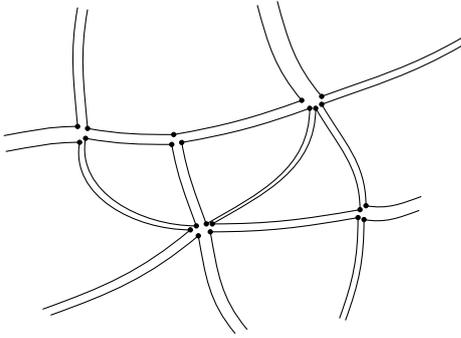
The partition function (1.11) characterizes the field theory based on a single type of Hermitian matrix M randomly distributed. It is possible to further generalize the model, for instance to the multi-matrix case. This lies outside the purpose of this thesis, since we are interested in matrix models for their application to two dimensional quantum gravity [14].

The ribbon graphs introduced above are the Feynman diagrams of a matrix model with partition function (1.4). They can be interpreted as the graphs dual to the cellular decomposition of a two dimensional manifold. The latter is constructed by associating an n -valent polygon to each n -valent vertex of the ribbon graph, with the edges of the polygon being orthogonal to the (double) edges of the ribbon graph. The choice of Hermitian matrices ensures that the cellular decomposition is orientable. An open ribbon graph and the construction of its dual 2d cellular decomposition are represented in Fig. 1.1a and 1.1b. The fact that the Feynman diagrams of a matrix model encode any two dimensional cellular decomposition, should already give an idea of why these models are related to discrete geometries and eventually to quantum gravity. A more precise argument is the following. The Einstein-Hilbert action for $d = 2$ can be computed exactly, and it turns out to be proportional to the Euler characteristic

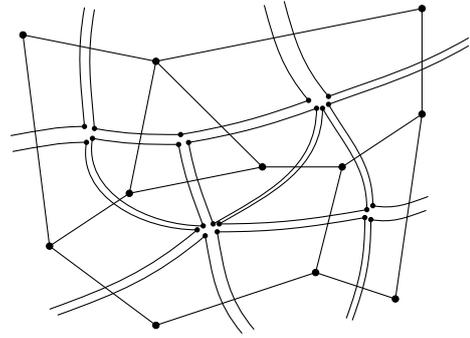
$$\mathcal{S}_{EH}[g] = \frac{c^4}{16\pi G} \int_{\Sigma} dx^2 \sqrt{g} R(g) = \frac{2\pi}{G} \chi(\Sigma). \quad (1.13)$$

Here Σ is a two dimensional manifold and g is the metric on it. One could also insert the contribution of the cosmological constant as

$$\mathcal{S}_{EH}[g, \Lambda] = \frac{c^4}{16\pi G} \int_{\Sigma} dx^2 \sqrt{g} (R(g) - \Lambda) = \frac{2\pi}{G} \chi(\mathcal{M}) - \Lambda A(\Sigma). \quad (1.14)$$



(a) Open ribbon graph made of a single 3-valent vertex, two 4-valent vertices, one 5-valent and one 6-valent vertices.



(b) Open ribbon graph together with its dual $2d$ cellular decomposition: each n -valent vertex is associated to an n -polygon.

The only dynamical variables are the Euler characteristic, coupled to the Newton constant, and eventually the area $A(\Sigma)$ coupled to the cosmological constant. The partition function associated to $2d$ quantum gravity, formulated as a (quantum) field theory, is

$$\mathcal{Z}_{EH} = \int dg e^{-S_{EH}}, \quad (1.15)$$

where for simplicity I considered the Euclidean case. At the discrete level, the integral in the partition function of $2d$ quantum gravity (1.15) can be interpreted as a sum over sufficiently refined $2d$ triangulations, and thus over the ribbon graphs. Expressing the two dimensional Einstein-Hilbert action can be expressed in terms of the Euler characteristic and the area as in (1.14), one obtains the relation between matrix models and two dimensional gravity with the proper map that relates the cosmological constant Λ and the size of a matrix N , resp. to the couplings of the model t_n and to the Newton constant $N = e^{\frac{2\pi}{G}}$ and $t_n = e^\Lambda$ (see also [50]). In this way, the discrete version of the partition function (1.15) is expressed through a sum over ribbon graphs, similar to the one of matrix models (1.11):

$$\mathcal{Z}_{EH} = \int dg e^{-\frac{c^4}{16\pi G} \int_\Sigma dx^2 \sqrt{g}(R(g)-\Lambda)} = \int dM e^{-\frac{N}{2} \text{Tr}(M^2) + N \sum_{n>2} t_n \text{Tr}(M^n)}. \quad (1.16)$$

1.2 Random tensor models

Motivated by the great success of random matrices, random tensors were first proposed in [15] as the natural generalization of matrix models to describe higher dimensional quantum gravity. In this section I will briefly review the main aspects of tensor models, mentioning the various formulations proposed during the years and I will discuss their application to describe discrete geometries in dimension higher than two.

There exist several random tensor models. They usually differ by two aspects: whether the model is real or complex, and whether it is colored or not (see [53] for more details). The Feynman diagrams of the real model are graphs where all the vertices carry the same weight (the coupling constant); whereas in the complex model there are two different couplings associate to black and white vertices of the graph. Normally, such type of graphs are *bipartite*, which means that each black vertex is connected only to white vertices, and vice-versa. The choice of the colored tensors instead leads to the so called *edge colored graphs*, where the edges are labelled by different colors, associated to an extra label of the tensor. There exist several other generalizations of tensor models in the mathematical phsyscs literature, for instance the model based on $O(N)$ [57] invariant tensors, the uncolored version [58] or the multi-orientable model [55,59] attracted much

attention; however, in this manuscript I only provide a brief introduction to the basics features of the QFT model and its combinatorics, to set the ground for discussing the SYK model and group field theories. Therefore, I will restrict this section to the case of complex, colored random tensors, which are naturally associated to orientable discrete cellular decompositions.

Following a similar pattern to matrix models, let us this consider the complex colored tensor model whose fields are given by rank d tensors T_{a_1, \dots, a_d} and $\bar{T}_{a_1, \dots, a_d}$, that transform under the action of a given symmetry group, say (d copies of) the unitary group $U(N)^{\times d}$, in the sense that each index of the tensor transforms under the action of one of the copies of the symmetry group. The model is assumed to be invariant under the action of such group. The *coloring* of the model relies in the fact that the tensor indices are distinguished. The Feynman diagrams of the model are d colored, bipartite graphs, where each tensor T_{a_1, \dots, a_d} is assigned to a black vertex attached to d half edges, and its conjugate tensor $\bar{T}_{a_1, \dots, a_d}$ is assigned to a white vertex attached to d half edges,

$$T_{a_1, \dots, a_d} : \begin{array}{c} \text{---} \frac{1}{\text{---}} \\ \text{---} \frac{2}{\text{---}} \\ \vdots \\ \text{---} \frac{d-1}{\text{---}} \\ \text{---} \frac{d}{\text{---}} \end{array} \quad \bar{T}_{a_1, \dots, a_d} : \begin{array}{c} \text{---} \frac{1}{\text{---}} \\ \text{---} \frac{2}{\text{---}} \\ \vdots \\ \text{---} \frac{d-1}{\text{---}} \\ \text{---} \frac{d}{\text{---}} \end{array} \quad (1.17)$$

The quadratic (or Gaussian) term of the action that governs the model is of the type

$$\mathcal{S}_{\mathcal{K}} = N^{\frac{d}{2}} \text{Tr}(\bar{T}_{a_i} T_{a_i}), \quad (1.18)$$

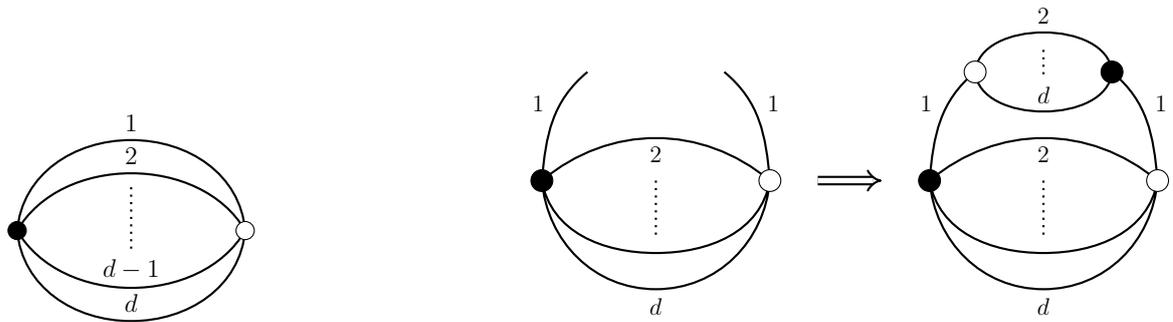
where the trace, that ensures the invariance under $U(N)^{\times d}$, stands for a sum over all the indices, respecting the coloring. The graph associated to the Gaussian term is called *diopole* and it is represented in Fig. 1.2a. Because of the invariance request, all the Feynman diagrams are given as combination of a pair of tensors T and \bar{T} (not necessarily with all the indices contracted); this reflects the *bipartiteness* of the graphs. Given a graph Γ constructed by the combination of a number of tensors T, \bar{T} , the coloring allows to identify the so called *p-bubbles*, with $0 \leq p \leq d-1$. These are sub-graphs of Γ with only p -valent vertices. For instance, a 0-bubble is a vertex, a 1-bubble is a single edge, a 2-bubble is a face made of a combination of two alternated coloring of edges. Without going further in the details, I would like to mention that the concept of bubble turns out to be very useful in the definition of boundary graphs, which is a key aspects in the study of discrete geometries. Moreover, to each d colored graph one can associate $d!/2$ ribbon graphs constructed as the graph whose faces are given by cycles of the colors, modulo their orientation and permutation. These are called *jackets* $\mathcal{J}(\Gamma)$ of the graph Γ [16, 17]. The jackets of a graph are then used to introduce the *degree*:

$$\omega(\Gamma) = \sum_{\mathcal{J}(\Gamma)} g_{\mathcal{J}}, \quad (1.19)$$

where $g_{\mathcal{J}}$ is the genus of the jacket \mathcal{J} . The partition function of random tensor models admits an expansion over the graphs as a power series in $1/N$, governed by the degree ω . Therefore, the general partition function for a complex, colored tensor model turns out to be

$$\mathcal{Z}[t_{\Gamma}] = \int dT d\bar{T} \exp \left(N^{d-1} \sum_{\Gamma} \frac{N^{-\frac{2}{(d-2)!} \omega(\Gamma)}}{|\text{Sym}(\Gamma)|} t_{\Gamma} \langle T, \bar{T} \rangle_{\Gamma} \right). \quad (1.20)$$

Here t_{Γ} is the coupling associated to the graph Γ , and $\langle T, \bar{T} \rangle_{\Gamma}$ is the respective invariant in terms of the tensors T and \bar{T} . Similar to the case of random matrices, taking the logarithm restricts the above partition function to only connected graphs.



(a) Dipole graph: it is constructed as a pair of black and white vertices connected by d edges.

(b) Example of dipole insertion: first cut an edge (color 1), then add a pair of black and white vertices and finally connect them by $d-1$ edges, respecting the coloring.

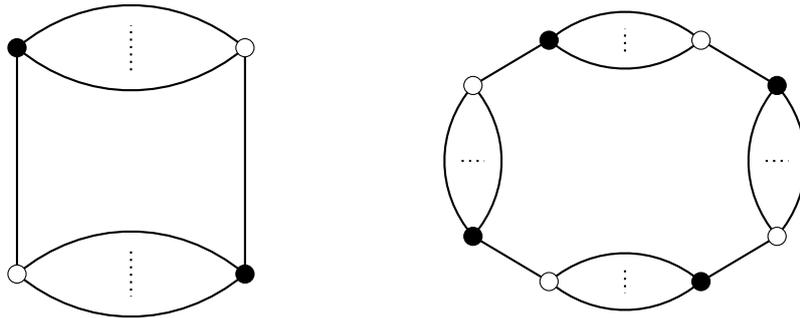


Figure 1.3: Example of bipartite melonic graphs. On the left the quartic (two pairs of black and white vertices) graph, called *pillow*. On the right a graph with four pairs of black and white vertices.

Large N behaviour and double scaling limit

The expansion in $1/N$ of colored tensor models [16–18, 60, 61] (eq. (1.20)) was an important step towards the study of tensor models since it allows a classification of its Feynman graphs. However, while the graph expansion (1.11) of the random matrices, being governed by the genus, is a topological expansion, in the case of random tensors the degree is not a topological invariant. This aspect does not preclude the study of the leading order of tensor models for large values of N . The dominant graphs here are called *melonic graphs*, which are constructed as follows. One can start by the simplest graph, the dipole, made of two vertices connected by $d+1$ edges, see Fig 1.2a. Then, apply a *dipole insertion* to one of its edges. The latter consists in cutting one of the edges of the graph, inserting a pair of black and white vertices attached to the half edges (respecting the bipartiteness) and subsequently connecting the two new vertices through the remaining d edges, respecting the coloring. The dipole insertion is represented in Fig. 1.2b. The family of melonic graphs is isomorphic to the class of trees, indeed all the graphs generated by iterations of such dipole insertion are melonic graphs and have zero degree. Some example of such graphs are given in Fig. 1.3. All the melonic graphs are dual to triangulations of the sphere, even if not all the triangulations of the sphere are dual to melonic graphs (this shows that the expansion (1.20) is not topological).

Similar to matrix models, the expansion (1.20) can be re-written as a power series in $x = N(t - t_c)^\alpha$, for some α that depends on the model, where t_c is the critical point of the coupling t (if there are more coupling constants t_Γ , the limit becomes more complicated). Therefore, one can again perform the double scaling limit [62], taking $N \rightarrow \infty$ and $t \rightarrow t_c$ such that the parameter x remains constant. The case of random tensors is slightly more complicated than random matrix models and one has to use the intermediate field technique to recover the proper

result, see [61] for more details.

1.3 Remarks

Matrix and tensor models played a central role in the past decades in the understanding of quantum gravity as a model of discrete geometries. In particular, these models suggested the possibility of formulating a field theory with Feynman diagrams interpreted as the graphs dual to cellular decompositions of d dimensional manifolds. In this perspective, the $U(N)^{\times d+1}$ invariance of random matrices or random tensors played the role of the discrete version of invariance under the change of coordinate in general relativity. While the single and double scaling limits were interpreted as some sort of continuous limits for the theory. However, if matrix models have a rather sufficiently complicated large N limit, governed by planar graphs, tensor models present a much simplified behaviour dominated by continuous trees (melonic diagrams). This suggested that, despite their suggestive interpretation, random tensor models are not rich enough to describe the quantum version of gravity in dimension higher than two.

One of the most prominent attempts to generalize random tensor models is tensorial group field theory, which I will discuss in more detail in Ch. 3. Before going through GFT's, I would like to discuss another topic that I have been working on during the first period of my Ph.D, still related to tensor models: the Sachdev-Ye-Kitaev model.

Chapter 2

The Sachdev-Ye-Kitaev model with a non-Gaussian disorder average

This chapter is dedicated to the Sachdev-Ye-Kitaev (SYK) model, which reached large audience mostly for its relation to holographic models of gravity such as AdS/CFT. I will also mention the relation between SYK and tensor models, which is a fundamental aspect for the present thesis. After a short summary of the fundamental aspects of the SYK model, I will discuss some results obtained in collaboration with T. Krajewski, A. Tanasa and R. Pascalie in [42].

The Sachdev-Ye-Kitaev model was first introduced by Sachdev and Ye in [20] and later presented by Kitaev during a session of two talks at KITP [21]. The model is a one dimensional quantum field theory, based on N Majorana fermions $\psi_{i_a}(t)$ with $i_a = 1, \dots, N$, coupled by a q -valent random tensor (*quenched disorder*) J_{i_1, \dots, i_q} with entries independently chosen from a Gaussian distribution. The model, governed by the Hamiltonian

$$H = i^{\frac{q}{2}} \sum_{i_1, \dots, i_q} J_{i_1, \dots, i_q} \psi_{i_1} \cdots \psi_{i_q}, \quad (2.1)$$

can thus be interpreted as a many body system, where each particle interacts with each other. The Feynman graphs of the model are made of fermionic legs following the ordinary Feynman rules, that meet at points weighed by the coupling J . Since the coupling is not a constant but rather a random tensor, such points can be represented in turn by sub-graphs exhibiting the disorder. In this sense the interaction between fermions is non-local. Several versions of the SYK model were proposed in the past, such as the higher dimensional generalization [63], its complex version [64], a generalization of the model with several flavours was proposed by Gross and Rosenhaus in [43] and the supersymmetric version was also constructed in [65, 66].

The SYK model gathered a large attention in several areas of physics mostly for its versatility and for the many interesting features that it presents. In particular, one has to mention the main three attractive properties of the model [22, 67, 68].

- *Maximally chaotic*: the study of quantum chaos is a branch of quantum mechanics that quantifies how a given system depends on its initial state. A quantum system is said to be chaotic when it is highly sensitive to the initial conditions. This aspect is measure by the out-of-time order (four point) correlation function, which gives the Lyapunov exponents. If these parameters saturate at a maximum value for large times, the model is said to be maximally chaotic.
- *Integrable at large N* : the SYK model classicalizes in a certain regime (for large values of N). This implies that in this limit the model reduces to a system of classical equations which are exactly solvable (integrable), even for strong couplings.

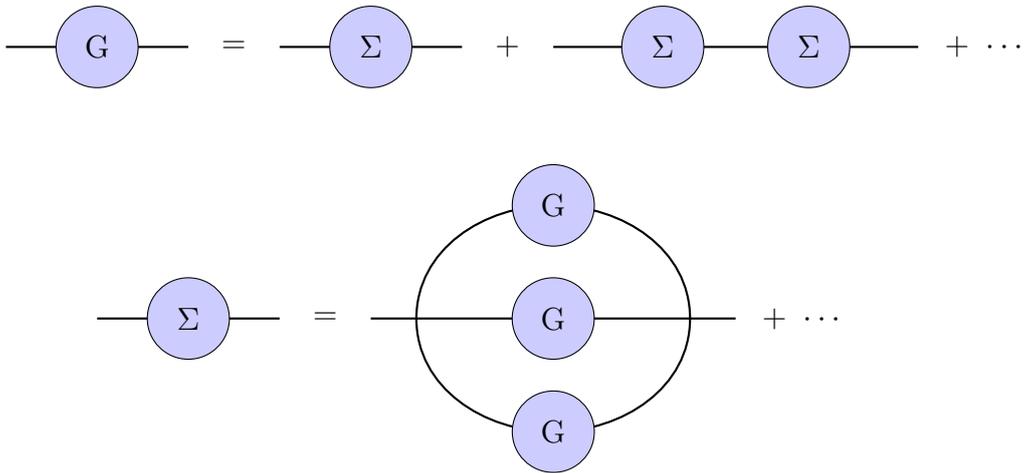


Figure 2.1: Solution of the Schwinger-Dyson equation for the two point function of the SYK model in the large N limit. The two point function G is given as a combination of free energies Σ , which contains all the iterated melonic diagrams.

- *Conformally symmetric in the infrared*: for low energies the two point function of the model presents a one dimensional conformal symmetry, which is spontaneously and explicitly broken by the vacuum.

The first property provides an interesting opportunity to relate many-body systems and black holes, since the out-of-time order correlation function [69], and thus its exponential behaviour governed by the Lyapunov exponents, encodes some properties of the nearly black hole horizon physics [70]. The simultaneous presence of the first two properties is a rather rare aspect and thus gathered some excitement in the community. Usually, quantum chaotic systems are too complicated to be integrable, and vice-versa exactly solvable systems are usually too simple to present a maximally chaotic behaviour.

The second property reflects that the graphs of the couplings J are dominated by the melonic graphs in the large N limit [71]: as Witten showed in [72], the graph expansion of the couplings in the SYK model is exactly the same of tensor models (whereas the other orders are different). The two point function $G(t)$ can be expressed as a re-summation over all the melonic graphs. I give its graphical representation in Fig. 2.1, where the term $\Sigma(t)$ is the self energy of the model, which contains an arbitrary combination of the melonic graphs made by a number of iterations of dipole insertions (shown in Fig. 1.2b). The two point function and the self energy can be seen as the classical solutions of the *effective action*. This is obtained by integrating over the disorder (random tensors J). Upon integration (averaging), the effective action turns out to be a bi-local action for the fermionic fields [22]. Therefore, using a Lagrange multiplier $\tilde{\Sigma}$ one can introduce the bi-local field $\tilde{G}(t_1, t_2)$, so that the effective action writes

$$\frac{\mathcal{S}_{eff}[\tilde{G}, \tilde{\Sigma}]}{N} = -\frac{N}{2} \log \left(\det (\delta(t_1 - t_2) \partial_{t_1} - \tilde{\Sigma}) \right) + \frac{1}{2} \int dt_1 dt_2 \left(\tilde{\Sigma} \tilde{G}(t_1, t_2) - \frac{J^2}{q} \langle \tilde{G}(t_1, t_2)^q \rangle_{\Gamma} \right). \quad (2.2)$$

Where $\langle \tilde{G}(t_1, t_2) \rangle_{\Gamma}$ is the graph Γ expressed as a combination of the bi-local fields \tilde{G} . Finally, the SYK model and its infrared regime are the features that gathered more popularity in the quantum gravity community thanks to their interpretation as a toy model for holography [23, 73]. In particular, in the IR limit, reached for $\omega \ll J$ (where ω is the variable canonically conjugated to the time t), the Schwinger-Dyson equation becomes invariant under

the the reparametrization $t \rightarrow f(T)$. The bi-local fields \tilde{G} and $\tilde{\Sigma}$ transform accordingly

$$\begin{aligned} G(t, t') &\rightarrow \left(f'(t) f'(t') \right)^\Delta G(f(t), f(t')), \\ \Sigma(t, t') &\rightarrow \left(f'(t) f'(t') \right)^{\Delta(q-1)} \Sigma(f(t), f(t')), \end{aligned} \quad (2.3)$$

where $\Delta = 1/q$ is the conformal order. And the effective action (2.2) can be thus expressed as the combination of two contributions

$$\mathcal{S}_{eff} = \mathcal{S}_{CFT} + \mathcal{S}_{Sch}. \quad (2.4)$$

The first, \mathcal{S}_{CFT} , is an explicitly conformally symmetric term (in one dimension), in the sense that it is invariant under the reparametrization $t \rightarrow f(t)$. The second term instead, \mathcal{S}_{Sch} , breaks the conformal symmetry and is governed by the Schwarzian derivative

$$\text{Sch}(f(t), t) = \frac{f'''}{f'} - \frac{3}{2} \left(\frac{f''}{f'} \right)^2. \quad (2.5)$$

The latter is the lowest order of derivatives invariant under the action of the group $\text{SL}(2, \mathbb{C})$. Without going through complicated details, this implies that the IR limit of the SYK model is associated to the description of the nearly horizon behaviour of an anti-de Sitter (AdS_2) black hole in terms of dilaton gravity [22, 73], with almost constant dilaton solutions. The non constant contribution of the dilaton is instead related to Jackiw-Teitelboim (JT) gravity [74, 75]. The latter is a special case of Liouville theory [67, 76], which in turn was related to SYK model in [77]. These features of the SYK model lead the community to consider it a toy model for the AdS/CFT duality in 1 + 1 dimensions.

For a complete review on the SYK model, the interested reader can refer to [22].

2.1 Non-Gaussian disorder average

This section is based on [42], written in collaboration with T. Krajewski, R. Pescalie and A. Tanasa, where we investigate a particular case of the Gross-Rosenhaus model (see also [78]). The generalization of the SYK model proposed by Gross and Rosenhaus was first presented in [43] and is based on f flavours of fermions, with N_a fermions of flavour a appearing q_a times in the interaction. Here we consider the complex version of a specific case of this model, where each fermion of flavor a appears only once. Therefore, we deal with q complex fermions $\psi_{i_a}^a(t)$, where the label $a = 1, \dots, q$ is the flavor and each fermion carries an index $i_a = 1, \dots, N$. The model is governed by the action

$$S_J[\psi, \bar{\psi}] = \int dt \left(\sum_{a, i_a} \bar{\psi}_{i_a}^a \partial_t \psi_{i_a}^a + i^{\frac{q}{2}} \sum_{i_1, \dots, i_q} \bar{J}_{i_1, \dots, i_q} \psi_{i_1}^1 \cdots \psi_{i_q}^q + i^{\frac{q}{2}} \sum_{i_1, \dots, i_q} J_{i_1, \dots, i_q} \bar{\psi}_{i_1}^1 \cdots \bar{\psi}_{i_q}^q \right). \quad (2.6)$$

Here J_{i_1, \dots, i_q} is the rank q tensor that plays the role of a coupling constant. Similarly to the ordinary SYK model, we average the free energy (or connected correlation functions) over the couplings J (this is called average over the *quenched disorder*). In order to perform the average it is convenient to use the replica trick. This consists in adding an extra replica index $r = 1, \dots, n$ to the fermions and taking the average over the replica copies. The averaged logarithm of the partition function is

$$\langle \log Z(J) \rangle_J = \lim_{n \rightarrow 0} \frac{\langle Z^n(J) \rangle_J - 1}{n}, \quad (2.7)$$

with

$$Z^n(J) = \int \prod_{1 \leq r \leq n} [d\psi_r][d\bar{\psi}_r] \exp \sum_r S_J(\psi_r, \bar{\psi}_r). \quad (2.8)$$

In this chapter, I will consider a generalized SYK model in which the averaging over the coupling constant J is performed with a non-Gaussian weight:

$$\langle Z^n(J) \rangle_J = \frac{\int dJ d\bar{J} Z^n(J) \exp \left(- \left(\frac{N^{q-1}}{\sigma^2} J\bar{J} + V_N(J, \bar{J}) \right) \right)}{\int dJ d\bar{J} \exp \left(- \left(\frac{N^{q-1}}{\sigma^2} J\bar{J} + V_N(J, \bar{J}) \right) \right)}. \quad (2.9)$$

Moreover, we demand the potential V_N to be invariant under the unitary transformations

$$J_{i_1, \dots, i_q} \rightarrow \sum_{j_1, \dots, j_q} U_{i_1 j_1}^1 \cdots U_{i_q j_q}^q J_{j_1, \dots, j_q}, \quad \bar{J}_{i_1, \dots, i_q} \rightarrow \sum_{j_1, \dots, j_q} \bar{U}_{i_1 j_1}^1 \cdots \bar{U}_{i_q j_q}^q \bar{J}_{j_1, \dots, j_q}. \quad (2.10)$$

The above invariance condition implies that the potential V_N that governs the distribution of the couplings J and \bar{J} can be expanded as a polynomial in $J\bar{J}$. These polynomials are then associated to q -valent bipartite, edge colored graphs Γ constructed as follows. The tensors J and \bar{J} are associated to black and white vertices. Expanding the potential in a power series of $J\bar{J}$ implies that black vertices are connected to white vertices, and vice-versa, making the graph Γ bipartite. Each vertex is attached to q legs (q -valent), labelled by the fermion flavors associated to an edge coloring of the graph¹. The most general form of the potential expanded over these graphs is

$$V_N(J, \bar{J}) = \sum_{\Gamma} \lambda_{\Gamma} \frac{N^{q-k(\Gamma)}}{|\text{Sym}(\Gamma)|} \langle J, \bar{J} \rangle_{\Gamma}. \quad (2.11)$$

In this expression, the parameter λ_N is a real number and represents the weight of the graph Γ in the expansion, $k(\Gamma)$ is the number of connected components of Γ and $\text{Sym}(\Gamma)$ is its symmetry factor. I have used the shorthand for the contraction of tensors for a given graph Γ

$$\langle J, \bar{J} \rangle_{\Gamma} = \sum_{1 \leq i_{v,a}, \dots, i_{\bar{v},a} \leq N} \prod_{\substack{\text{white} \\ \text{vertices } v}} J_{i_{v,1}, \dots, i_{v,q}} \prod_{\substack{\text{black} \\ \text{vertices } \bar{v}}} \bar{J}_{i_{\bar{v},1}, \dots, i_{\bar{v},q}} \prod_{\substack{\text{edges} \\ e=(v, \bar{v})}} \delta_{i_{v,c(e)}, i_{\bar{v},c(e)}}. \quad (2.12)$$

The contraction of indices through the delta function means that each white vertex v associated to the tensor J is connected to a black vertex \bar{v} associated to a tensor \bar{J} through the edge with coloring $c(e)$. The Gaussian term (the melonic diagram, represented in Fig. 1.2a for a tensor model) corresponds to a dipole graph with a single pair of white and black vertices connected by q edges. This writes

$$\frac{N^{q-1}}{\sigma^2} J\bar{J} = \frac{N^{q-1}}{\sigma^2} \sum_{1 \leq i_1, \dots, i_q \leq N} J_{i_1, \dots, i_q} \bar{J}_{i_1, \dots, i_q}. \quad (2.13)$$

In the following I will compute the effective potential and derive its large N behaviour, showing that the Gaussian term, the dipole, is the dominant graph of the expansion. To do so, we first introduce the pair of complex conjugate tensors K and \bar{K} defined by

$$K_{i_1, \dots, i_q} = i^{\frac{q}{2}} \sum_r \int dt \psi_{i_1, r}^1 \cdots \psi_{i_q, r}^q, \quad \bar{K}_{i_1, \dots, i_q} = i^{\frac{q}{2}} \sum_r \int dt \bar{\psi}_{i_1, r}^1 \cdots \bar{\psi}_{i_q, r}^q. \quad (2.14)$$

¹These graphs are called bubbles in the tensor model literature (see the book [52].)

The averaged partition function (2.8) expressed in the tensors K, \bar{K} is

$$\langle Z^n(J) \rangle_J = \frac{\int [d\psi][d\bar{\psi}] e^{\left(-\int dt \sum_{a,i_a} \bar{\psi}_{i_a}^a \partial_t \psi_{i_a}^a\right)} \int dJ d\bar{J} \exp\left(-\left(\frac{N^{q-1}}{\sigma^2} J\bar{J} + V_N(J, \bar{J}) + J\bar{K} + \bar{J}K\right)\right)}{\int dJ d\bar{J} \exp\left(-\left(\frac{N^{q-1}}{\sigma^2} J\bar{J} + V_N(J, \bar{J})\right)\right)}. \quad (2.15)$$

In order to study the large N limit of the above averaged partition function, I first introduce the background fields $L = -\frac{\sigma^2}{N^{q-1}}K$ and $\bar{L} = -\frac{\sigma^2}{N^{q-1}}\bar{K}$, and then shift the couplings as

$$J \rightarrow J + L, \quad \bar{J} \rightarrow \bar{J} + \bar{L}. \quad (2.16)$$

The effective potential thus becomes

$$V_N(s, L, \bar{L}) = -\log \int dJ d\bar{J} \exp\left(-\frac{N^{q-1}}{s} J\bar{J} - V_N(J + L, \bar{J} + \bar{L})\right) + N^q \log\left(\frac{\pi s}{N^{q-1}}\right). \quad (2.17)$$

Here s is a flowing parameter that interpolates between the values of the integral, from $s = 0$ (where the integration is trivial and $J = \bar{J} = 0$), up to $s = \sigma^2$ (up to a trivial multiplicative constant) which gives the integral that we have to compute. The flowing parameter is fundamental for the study of the large N limit, that I will perform in the next part. The inclusion of the constant term ensures that the effective potential remains zero when we start with a vanishing potential.

Gaussian universality

Here I will derive the large N behavior of the effective potential using a Polchinski-like flow equation, following the approach proposed in [79–82]. Using standard QFT manipulations (see for instance [83]), one can show that the effective potential (2.17) obeys the differential equation

$$\frac{\partial V_N}{\partial s} = \frac{1}{N^{q-1}} \sum_{1 \leq i_1, \dots, i_q \leq N} \left(\frac{\partial^2 V_N}{\partial L_{i_1, \dots, i_q} \partial \bar{L}_{i_1, \dots, i_q}} - \frac{\partial V_N}{\partial L_{i_1, \dots, i_q}} \frac{\partial V_N}{\partial \bar{L}_{i_1, \dots, i_q}} \right). \quad (2.18)$$

This equation is formally a Polchinski-like equation [84], in the sense that there are no short distance degrees of freedom over which one could integrate. I will explain the interpretation of such equation later on, for now it is sufficient to know that I will use it to analyse the large N limit of the effective potential. Since the effective potential is invariant under the unitary transformations defined in eq. (2.10), it may also be expanded over graphs as in (2.11)

$$V_N(s, L, \bar{L}) = \sum_{\Gamma} \lambda_{\Gamma}(s) \frac{N^{q-k(q)}}{|\text{Sym}(\Gamma)|} \langle L, \bar{L} \rangle_{\Gamma}, \quad (2.19)$$

with s dependent weights $\lambda_{\Gamma}(s)$. Inserting this expansion in the differential equation (2.18) one obtains a system of differential equations for the couplings $\lambda_{\Gamma}(s)$:

$$\frac{d\lambda_{\Gamma}}{ds} = \sum_{\Gamma' / (\bar{v}v) = \Gamma} N^{k(\Gamma) - k(\Gamma') + c(v, \bar{v}) - q + 1} \lambda_{\Gamma'} - \sum_{(\Gamma' \cup \Gamma'') / (\bar{v}v) = \Gamma} \lambda_{\Gamma'} \lambda_{\Gamma''}. \quad (2.20)$$

The Polchinski-like equation (2.18) can be represented as in Fig. 2.2. Graphically, a derivation of the potential V with respect to the background fields L and \bar{L} resp. removes a white or a black vertex. Then, the summation over the indices reconnects the edges respecting the coloring. Equation (2.18) gives all the possible ways in which each graph Γ (on the LHS) can be obtained through the operation of removing a pair of black and white vertices (derivation with respect to

$$\frac{\partial}{\partial s} \Gamma = \Gamma' + \Gamma''$$

Figure 2.2: Graphical representation of the Polchinski-like equation (2.18). The derivatives with respect to the background fields L and \bar{L} correspond to the operation of removing a pair of white and black vertices and reconnecting the q edges, from the graph Γ' or from the graphs Γ' and Γ'' on the RHS, such that the resulting graph is equal to Γ on the LHS.

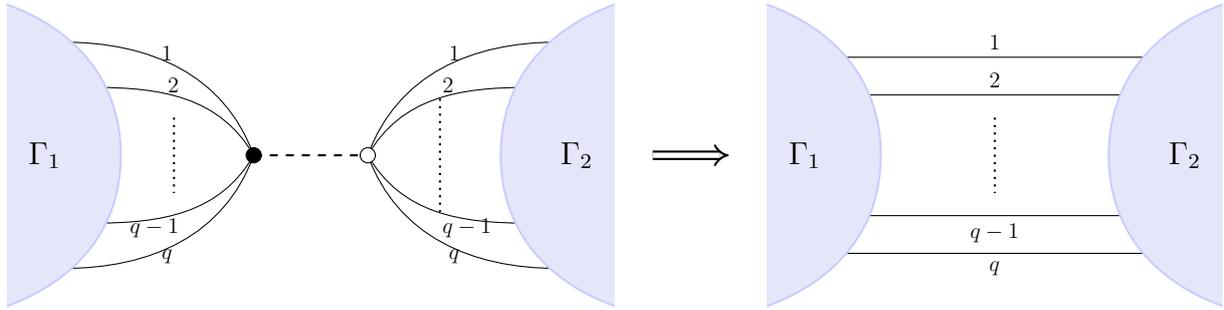


Figure 2.3: Removal of a pair of white and black vertices and reconnection of the edges according to the coloring.

L, \bar{L}) and reconnecting the edges (on the RHS); the two terms on the RHS of the equation take in consideration that the two vertices can be removed either from a single graph Γ' or from two different graphs Γ' and Γ'' . Let me explain in more detail the graphical interpretation of such two terms in the Polchinski-like equation (2.20).

- In the first term on the RHS we remove a pair of white and black vertices v, \bar{v} from a single graph Γ' , then we recombine the edges according to the coloring, such that the resulting graph is equal to Γ . This operation is represented in Fig. 2.3 and Fig. 2.4. The term $e(v, \bar{v})$ in (2.20) is the number of edges connecting v and \bar{v} in Γ' . After summation over the indices, each of these lines yields a power of N in (2.20). Since the operation of removing two vertices and reconnecting the edges can at most increase the number of connected components by $q - 1$, we always have that $k(\Gamma) - k(\Gamma') + e(v, \bar{v}) - q + 1 \leq 0$. One obtains the equality if and only if Γ' is a melonic graph. Therefore, in the large N limit, only melonic graphs survive in the first term on the RHS of (2.20).
- In the second term of (2.20) we remove a pair of white and black vertices from the graphs Γ' and Γ'' , with the condition that the graph obtained after removing the vertices and

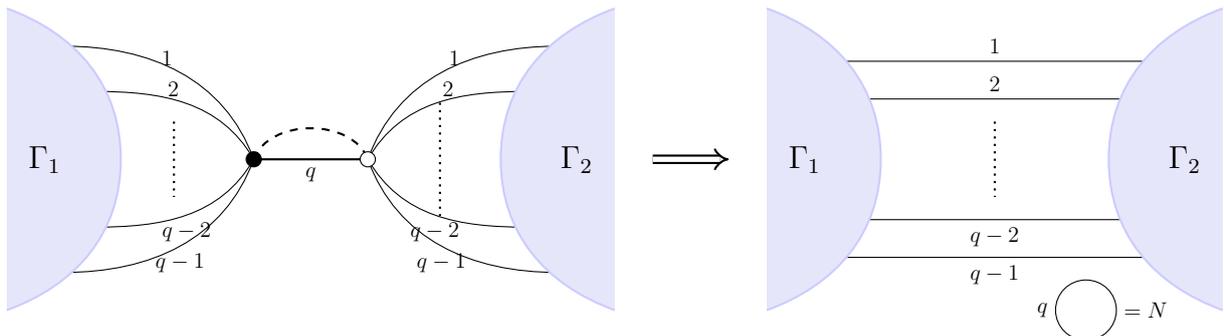


Figure 2.4: Removal of white and black vertices and reconnection of the edges creating a loop.

reconnecting the edges is equal to Γ . In that case, the number of connected components necessarily diminishes by 1, so that all powers of N cancel.

The crucial point in the system (2.20) is that only negative (or null) powers of N appear. This implies that, if $\lambda_\Gamma(s=0)$ is bounded in N , then $\lambda_\Gamma(s)$ is also bounded for all values of s . Once we have checked that the couplings $\lambda_\Gamma(s)$ are bounded for any value of the flowing parameter s , we derive the leading order of the large N expansion of the effective potential (2.19) by setting $s = \sigma^2$ and substituting the values $L = -\frac{\sigma^2}{N^{q-1}}K$ and $\bar{L} = -\frac{\sigma^2}{N^{q-1}}\bar{K}$ of the background fields:

$$V_N(s = \sigma^2, L = -\frac{\sigma^2}{N^{q-1}}K, \bar{L} = -\frac{\sigma^2}{N^{q-1}}\bar{K}) = \sum_{\Gamma} \lambda_\Gamma(\sigma^2) \frac{(-\sigma^2)^{v(\Gamma)} N^{q-k(q)-(q-1)v(\Gamma)}}{|\text{Sym}(\Gamma)|} \langle K, \bar{K} \rangle_{\Gamma}. \quad (2.21)$$

Here $v(\Gamma)$ is the number of vertices of the graph Γ . The exponent of N can be rewritten as $(q-1)(1-v(\Gamma))+1-k(\Gamma)$; it has its maximal value for $v(\Gamma) = 2$ and $k(\Gamma) = 1$, which corresponds to the dipole graph. This proves that the dipole graph (the Gaussian term) is dominant in the large N expansion of the potential V_N of an SYK model based on a non-Gaussian average. This result is a re-expression of the Gaussian universality property of random tensors [44].

Effective action

Let us now derive the effective action of this new model. To this end, we integrate over the random couplings J and \bar{J} using the non-Gaussian probability distribution as in (2.15). The result can be expressed in terms of the graphs Γ , as in the effective potential (2.21). Then, we replace the tensors K and \bar{K} with their expressions (2.14) in terms of the fermionic fields, so that each graph Γ is given by

$$\langle K, \bar{K} \rangle_{\Gamma} = \sum_{1 \leq i_v, a, \dots, i_{\bar{v}}, a \leq N} \prod_{\substack{\text{white} \\ \text{vertices } v}} \sum_{r_v} \int dt_v \psi_{i_v, 1, r_v}^1(t_v) \cdots \psi_{i_v, q, r_v}^q(t_v) \prod_{\substack{\text{black} \\ \text{vertices } \bar{v}}} \sum_{r_{\bar{v}}} \int dt_{\bar{v}} \bar{\psi}_{i_{\bar{v}}, 1, r_{\bar{v}}}^1(t_{\bar{v}}) \cdots \bar{\psi}_{i_{\bar{v}}, q, r_{\bar{v}}}^q(t_{\bar{v}}) \prod_{\substack{\text{edges} \\ e=(v, \bar{v})}} \delta_{i_v, c(e), i_{\bar{v}}, c(e)}. \quad (2.22)$$

We note that, as in the standard SYK model [22], the result is a bi-local effective action that depends on pairs of fermionic fields evaluated in different time t and t' . This allows to use a Lagrangian multiplier $\tilde{\Sigma}$ to introduce the bi-local field below, that carry a single flavour label a and two replica indices r, r' :

$$\tilde{\mathbb{G}}_{r, r'}^a(t, t') = \frac{1}{N} \sum_i \psi_{i, r}^a(t_1) \bar{\psi}_{i, r'}^a(t'). \quad (2.23)$$

Assuming for simplicity a replica symmetric saddle-point we get rid of the replica copies. The resulting effective action in terms of the bi-local fields is thus

$$\frac{\mathcal{S}_{eff}[\tilde{\mathbb{G}}, \tilde{\Sigma}]}{N} = - \sum_{f=1}^q \log \left(\det \left(\delta(t_1 - t_2) \partial_t - \tilde{\Sigma}_f(t_1, t_2) \right) \right) + \int dt \sum_{f=1}^4 \tilde{\Sigma}_f(t) \tilde{\mathbb{G}}_f(t) - \sum_{\Gamma} N^{-(v(\Gamma)-2)(q/2-1)+1-k(\Gamma)} \mu_{\Gamma}(\sigma^2, \{\lambda_{\Gamma^f}\}) \langle \tilde{\mathbb{G}} \rangle_{\Gamma}. \quad (2.24)$$

The term μ_{Γ} is the resulting coupling associated to the term $\langle \tilde{\mathbb{G}} \rangle_{\Gamma}$, which is the graph Γ expressed in terms of the bi-local invariant (2.23). The resulting fermionic Feynman diagrams computed from the effective action are constructed as follows:

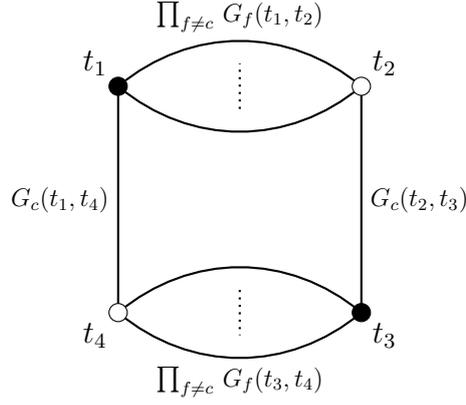


Figure 2.5: Graphical representation of the term $\langle \tilde{G} \rangle_\Gamma$ of a bipartite q -valent quartic (four vertices) melonic graph Γ in terms of the bi-local fields $\tilde{G}(t, t')$.

- associate a real variable t_v to each vertex;
- associate a bi-local field $\tilde{G}_c(t_v, t_{v'})$ to an edge of colour c joining the vertices v and v' ;
- multiply the edge contributions of the graph Γ and integrate over the vertex variables t_v ;
- add up a weight λ_Γ and a power of N equal to the contribution in the effective potential (2.21), plus an extra power of N given by the number of edges $e(\Gamma) = \frac{1}{2}qv(\Gamma)$ of Γ .

In Fig. 2.5 I represent an example of a graph in terms the bi-local invariant constructed as explained above. The resulting N contribution of a given graph Γ is

$$N^{q-k(\Gamma)} \times (N^{-(q-1)v(\Gamma)}) \times N^{e(\Gamma)} = N \times N^{-(v(\Gamma)-2)(q/2-1)+1-k(\Gamma)}. \quad (2.25)$$

At the leading order, only the Gaussian term survives, namely the graphs Γ with $(v(\Gamma) = 2$ and $k(\Gamma) = 1)$). The matrix model case given by $q = 2$ is an exception. In this case, all the terms corresponding to connected graphs survive. Let me emphasize that, similarly to the other couplings μ_Γ , also the covariance associated to the propagator term is modified, as an explicit consequence of the non-Gaussian averaging of our model. Remarkably, for $q > 2$, this is the only modification at leading order in N . It is convenient to use a Schwinger-Dyson equation [60] to compute the actual value of the covariance – denoted σ' – induced by the non-Gaussian disorder. In our case, the Schwinger-Dyson equation arises from the constraint

$$\sum_{i_1 \dots i_q} \int dJ d\bar{J} \frac{\partial}{\partial \bar{J}_{i_1 \dots i_q}} \left(J_{i_1 \dots i_q} \exp \left(- \frac{N^{q-1}}{\sigma^2} J\bar{J} - V_N(J, \bar{J}) \right) \right) = 0, \quad (2.26)$$

which, for large values of N , leads to the algebraic equation

$$1 = \frac{\sigma'^2}{\sigma^2} + \sum_{\Gamma_{\text{melonic}}} \frac{\lambda_\Gamma}{|\text{Sym}(\Gamma)|} (\sigma')^{v(\Gamma)}. \quad (2.27)$$

Finally, it is interesting to note that despite the non-locality of the effective action (2.24), the latter is invariant under re-parametrization below (in the IR) at all orders in $1/N$

$$\tilde{G}(t, t') \rightarrow \left(\frac{d\phi}{dt}(t) \right)^\Delta \left(\frac{d\phi}{dt'}(t') \right)^\Delta \tilde{G}(\phi(t), \phi(t')). \quad (2.28)$$

Indeed, changing the vertex variables as $t_v \rightarrow \phi(t_v)$, the Jacobian cancels with the re-scaling of the bi-locl field \tilde{G} , since Γ is a q -valent graph and $\Delta = 1/q$.

2.2 Concluding remarks

In this chapter I have investigated the effects of non-Gaussian average over the random tensorial couplings J in a particular case of the complex version of the SYK generalization proposed by Gross and Rosenhaus. To our knowledge, this is the first analysis of the effects of the relaxation of the Gaussianity condition in SYK models.

To summarize the results, we first used a Polchinski-like equation to prove the finiteness of the couplings λ_Γ of the non-Gaussian terms of the potential. This allowed to expand the potential as a sum over the graphs (2.11), whose expression was then used to derive the effective action of the model (2.2). At this stage, the main effect of the non-Gaussian potential is reflected in the modification of the couplings λ_Γ into μ_Γ , and in particular in the modification of the covariance, the weight of the Gaussian term.

Effect of the non-Gaussian average in other SYK models. An interesting perspective for future works would be the investigation of the effects of such a perturbation from the Gaussian distribution in other versions of SYK model, such as

- real Gross-Rosenhaus generalization of the SYK model;
- super-symmetric SYK model;
- SYK model based on a real and anti-symmetric random tensor J .

Note that in [42] we have already made the first steps towards this direction, where the real Gross-Rosenhaus SYK model was considered with a quartic potential. In this context we proved explicitly the Gaussian universality, we computed the effective action and the first order correction in N of the covariance, due to the quartic contribution in the potential. The main technical complication for generalizing the real version to any arbitrary non-Gaussian potential, comes from the fact that the graphs will not be bipartite. Consequently, the operation of removing the vertices and reconnecting the edges, which played a central role in our proof, would be much more involved.

Chapter 3

Group field theories

The suggestive idea of using group representations as decorations of geometric objects to describe discrete geometries dates back to the Ponzano-Regge formulation [24]. Here the edge vectors of a triangulation of a three dimensional manifold are decorated by $SU(2)$ elements. The partition function of the model is expanded as a state sum, where such $3d$ discrete geometries are encoded in the transition amplitudes between any pair of two dimensional triangulations. Later on, Penrose provided a model of three dimensional gravity in the dual picture in [1], where the graphs dual to three dimensional triangulations (dual complexes) were decorated by holonomies of $SU(2)$ and called *spin networks*. These spin networks appear also in the discretization of three dimensional Euclidean gravity, formulated as a BF model and cover the role of quantum (boundary) states in the spin foam quantization, where the transition amplitudes between any pair of spin networks, called *spin foams*, are interpreted as the complexes dual to the cellular decompositions of a manifold. Boulatov proposed a path integral prescription that generates triangulations of a three dimensional manifold (as in the Ponzano-Regge model) as Feynman diagrams of a field theory [32]. This model, baptized *group field theory*, is a sub-class of tensor models where the (quantum) states are enriched with algebraic data. Its four dimensional generalization [33] is due to Ooguri. For an introductory overview of group field theory, the interested reader can refer to [85, 86].

Group field theory has also been related to the classical action of gravity, when the model proposed by Barrett and Crane in [39], which is linked to the state sum of a specific group field theory based on a Lorentzian manifold, was connected to the discretization of a constrained BF model [41]. In [35], it was suggested the formulation of tensorial group field theories as a quantum gravitational system made of many interacting bodies – the building blocks of a discrete geometry. Under this perspective, the analysis of the hydrodynamical approximation of GFT’s aims to retrieve classical physics, specially cosmology, as an emergent behaviour of group field theory models, [36]. This also suggested a link between GFT’s and quantum cosmology [37]. The renormalization group flow of a GFT model was also analysed in [87–90]. As I will emphasize later on in this chapter, the quantization procedure used in GFT is the geometric quantization, also used in the spin foam approach. The classical phase space underlying such quantization, is the cotangent bundle T^*G , which is given by the group G and its tangent space. At the semi-classical level, the tangent space can be understood as a non-abelian algebra (or non-commutative abelian group). Such formulation sheds light on the relation between group field theories and non-commutative space-times [91].

I recall that models of quantum mechanics can be expressed either in the configuration or in the momentum space pictures. The same principle holds for group field theories. For what concerns us, in the triangulation picture the (quantum) states are simplicial triangulations of a two dimensional manifold, whereas in the dual complex they are two dimensional spin networks. In both cases, the Feynman diagrams are transition amplitudes between any pair of such (quantum)

states, interpreted as three dimensional discrete geometries, and the partition function of the model is expanded as a (state) sum over such geometries. The two formulations are mapped into each other by (the generalization of) a Fourier transform [92–94].

In this chapter I will introduce the main ingredients of a three dimensional group field theory based on a general Lie group G . In the first and second parts of the chapter I will discuss the model in the triangulation and in the dual complex pictures, providing the Fourier transform between them. In the final part of the chapter I will also mention how to generalize the model to the higher dimensional case, how to introduce the simplicity constraint and the application of the Peter-Weyl theorem in group field theory.

Since we are dealing with a simplicial triangulation and its dual complex, it is worth to fix the convention here to avoid any ambiguity along the discussion. I will call *vertices* and *edges* the points and the one dimensional objects of the triangulation, respectively. The zero and one dimensional objects of the dual complex will instead be called resp. *nodes* and *links*.

3.1 Three dimensional Group Field Theory

In this section I briefly review group field theory in 2+1 dimensions [91]. I will introduce the main ingredients, such as the field, the action that governs the model and the Feynman diagram amplitudes, emphasizing the geometric interpretation of each element. I deal with the dual complex picture in the first part, while in the second part I will introduce the Fourier transform and recast the model in the triangulation formulation.

3.1.1 Spin foams as Feynman diagrams of GFT

Invariant field

The fundamental field of 2+1 dimensional group field theory is a function on three copies of a Lie group G , $\phi \in F(G^{\times 3})$. The field is invariant under a gauge transformation of the group G itself. Such symmetry is enforced through the projector $\mathcal{P} : F(G^{\times 3}) \rightarrow F(G^{\times 3})$ defined as

$$(\mathcal{P} \phi)(g_1, g_2, g_3) = \int dh \phi(hg_1, hg_2, hg_3), \quad (3.1)$$

where $g_i, h \in G$ and thus dh is the left Haar measure¹ on G . The above equation is called *gauge averaging* and the invariance of the field under the gauge transformation can be understood as the functional identity

$$(\mathcal{P} \phi) = \phi. \quad (3.3)$$

Action

Similarly to matrix and tensor models, the action of a GFT model is given by a quadratic term plus an interaction term, which will be used to construct transition amplitudes of the model (spin foams [30]). The action is thus

$$\mathcal{S} = \mathcal{S}_{\mathcal{K}} + \mathcal{S}_{\mathcal{V}}. \quad (3.4)$$

¹The left Haar measure dg on G is defined by the invariance property

$$d(\bar{g}g) = dg, \quad \forall \bar{g} \in G. \quad (3.2)$$

Similarly for the right Haar measure.

The quadratic part of the GFT action, called *kinetic term*, is defined as a product of fields

$$\mathcal{S}_{\mathcal{K}} = \int dg^3 (\phi \tau \phi)(g_1, g_2, g_3) = \int dg^6 \mathcal{K}(\{g\}) \phi(g_1, g_2, g_3) \phi(g_4, g_5, g_6), \quad (3.5)$$

where

$$\mathcal{K}(\{g\}) = \delta(g_1 g_6^{-1}) \delta(g_2 g_5^{-1}) \delta(g_3 g_4^{-1}), \quad (3.6)$$

is the kernel of the *propagator amplitude* and can be interpreted as the conservation of the three momenta g_i . The operator $\tau : F(G^3) \rightarrow F(G^3)$ defined as $(\tau \phi)(g_1, g_2, g_3) = \phi(g_3, g_2, g_1)$ is the permutation map and it is used to encode the proper combinatorics of the model². For a 2+1 dimensional GFT, the *interaction term* is as a *non-local*³ product of four projected fields (3.1)

$$\begin{aligned} \mathcal{S}_{\mathcal{V}} &= \int dg^6 (\mathcal{P} \phi)(g_1, g_2, g_3) (\mathcal{P} \phi)(g_3, g_4, g_5) (\mathcal{P} \phi)(g_5, g_6, g_1) (\mathcal{P} \phi)(g_6, g_4, g_2) \\ &= \int dg^{12} dh^4 \mathcal{V}(\{h; g\}) \phi(g_1, g_2, g_3) \phi(g_4, g_5, g_6) \phi(g_7, g_8, g_9) \phi(g_{10}, g_{11}, g_{12}), \end{aligned} \quad (3.7)$$

where

$$\begin{aligned} \mathcal{V}(\{h; g\}) &= \delta(h_1 g_1 g_9^{-1} h_3^{-1}) \delta(h_1 g_2 g_{12}^{-1} h_4^{-1}) \delta(h_1 g_3 g_4^{-1} h_2^{-1}) \\ &\quad \delta(h_2 g_5 g_{11}^{-1} h_4^{-1}) \delta(h_2 g_6 g_7^{-1} h_3^{-1}) \delta(h_3 g_8 g_{10}^{-1} h_4^{-1}), \end{aligned} \quad (3.8)$$

is the kernel of the *tetrahedron amplitude*. Despite the terms \mathcal{K} and \mathcal{V} are the kernels of the amplitudes, I will sometimes call them just amplitudes by abuse of language.

Feynman amplitudes

Similarly to other topological models of discrete geometries, the partition function of a group field theory $\mathcal{Z} = \int d\phi e^{i\mathcal{S}}$ can be expanded as a state sum. Such state sum runs over the three dimensional discrete geometries (with no boundary) represented as dual complexes Γ^* , built by merging an arbitrary number of tetrahedron amplitudes (3.8) through the propagator amplitudes (3.6). Consider a path made of an arbitrary number, say N , of what I will call *bulk links*, and denote it as \mathcal{L}_N . The amplitude of a given complex Γ^* is expressed as a combination of delta functions enforcing the closure of such paths \mathcal{L}_N

$$\mathcal{Z} = \mathcal{N} \sum_{\Gamma^*} \prod_{\mathcal{L}_N \in \Gamma^*} \int \prod dh \delta(h_1 h_2 \dots h_N), \quad (3.9)$$

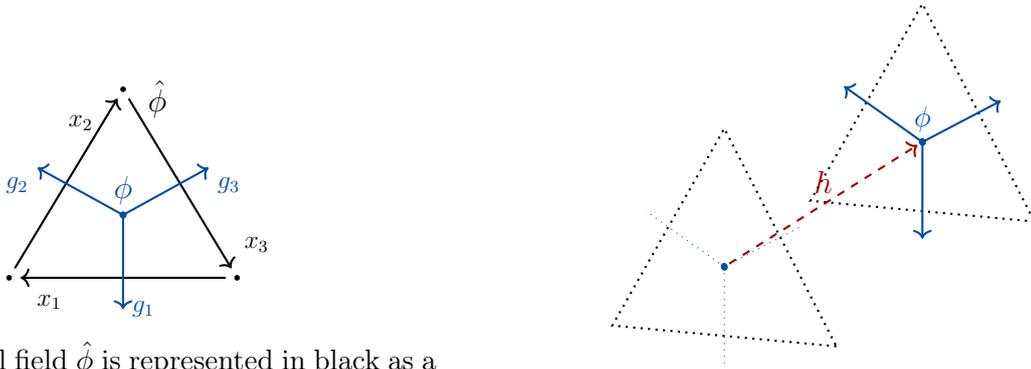
with \mathcal{N} a normalization factor. In the next part I will clarify the notion of bulk and boundary variables, and the geometric interpretation of the partition function above.

Geometric interpretation

The field ϕ encodes the fundamental degrees of freedom of the model; upon quantization indeed, the space of fields (the algebra of functions $F(G^{\times 3})$) would be interpreted as the Hilbert space of the model. As a function on three copies of the Lie group G , the field is represented as the graph dual to a triangle, composed by a central node attached to three links, see Fig. 3.1a. The gauge transformation, is enforced as a group multiplication on each of the three elements g_i

²Normally one should sum over all the permutations of the kinetic term [85, 86], in order to encode all the possible Feynman amplitudes. I will avoid to discuss this subtle issue, which is particularly relevant in dimension higher than three.

³The nomenclature non-local here is used for a product of fields that share only some of their sub-components g , such as $\phi(g, g_i, g_j) \phi(g, g_n, g_m)$.



(a) The dual field $\hat{\phi}$ is represented in black as a triangle, with the three elements $x_i \in G^*$ decorating the three edges. The field ϕ is its dual graph represented in blue, made of a central node and three links decorated by the group elements $g_i \in G$.

(b) The gauge symmetry is interpreted as a translation in the dual complex. The field ϕ , represented as three links (in blue) dual to a triangle, is shifted by an extra link (the dashed red line) decorated by $h \in G$ that represents the (gauge) translation.

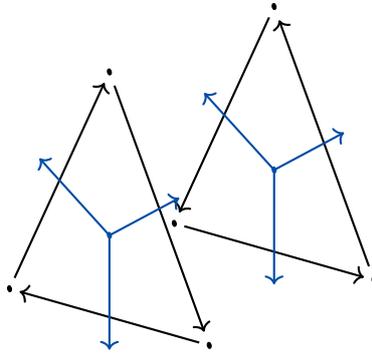
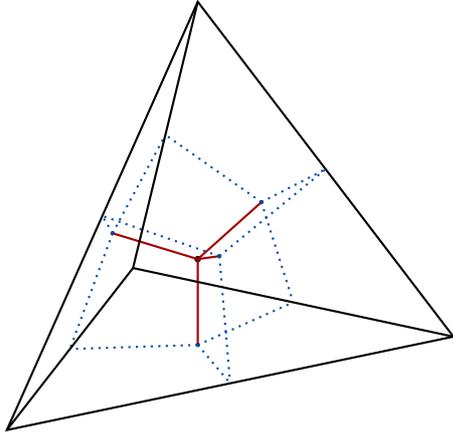


Figure 3.2: The propagator amplitude enforces the identification of a pair of triangles in black (dual fields $\hat{\phi}$) or their dual graphs in blue (fields ϕ). I used arrows to emphasize that edges are identified with opposite orientations while links with the same orientation.

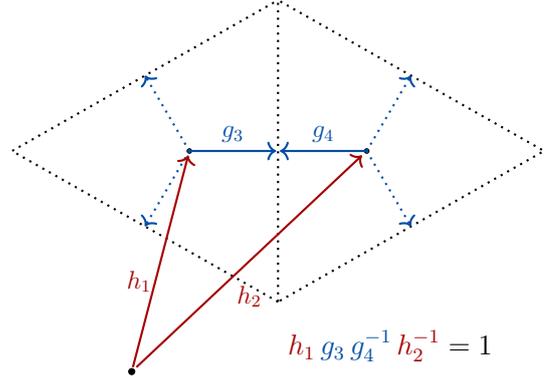
with respect to a single group element h . The transformed field is thus a function of the three elements hg_i , and can be represented as the graph dual to a triangle translated by a single link h , as in Fig. 3.1b. Under this perspective, the gauge transformation thus encodes the notion of parallel transport in the dual complex.

The propagator amplitude (3.6), being given as the product of two fields evaluated at the same point (same triplet of group elements), is represented as an identification of two graphs 3.1a dual to triangles, as in Fig. 3.2. Similarly, the tetrahedron amplitude (3.8) is the combination of four fields, where each of them shares only one variable g_i with each of the other three, respecting the combinatorics of a tetrahedron. The four fields are thus associated to the graph dual to the boundary of a tetrahedron. While the four variables h_i are the decorations of the four links internal to the tetrahedron and dual to its four faces, see Fig. 3.3a. The same construction arises if one represents each of the six delta function in (3.8) as a closed loop of four links (two bulk plus two boundary links), of the type $h_a g_i g_j^{-1} h_b^{-1} = 1$, see Fig. 3.3b. The proper combination of such six delta functions together gives again the graph in Fig. 3.3a.

Note that, despite the elements h and g belong to the same group G , they play a completely different role. The links g , initially used to define a single field, are part of the boundary graph of a tetrahedron and thus *lie* in two dimensional cellular decomposition. The links h instead are an extra labels for the states, introduced through the gauge projector (3.1). They form the graph dual to the bulk of the tetrahedron, hence they *lie* in a three dimensional cellular decomposition. Therefore, one can interpret the links g and h resp. as the boundary and bulk degrees of freedom of the graph dual to the simplicial triangulation of a three dimensional manifold.



(a) The tetrahedron amplitude. In black the four triangles that compose the tetrahedron boundary. The bulk graph dual to the tetrahedron is in red and the graph dual to its boundary are the dotted blue links.



(b) Each of the six delta functions of the tetrahedron amplitude (3.8) enforces a the closure of the combination of two bulk links (in red) and two boundary links (in blue).

Given a complex Γ^* dual to a simplicial triangulation Γ of a three dimensional manifold, its Feynman amplitude is given as a combination of loops made of a number, say N , of bulk links. Each of such loops is the closed path spanning a face dual to an edge (shared by N tetrahedra) of the triangulation. In line with the usual interpretation of discrete geometries, each of these loops encode the local curvature round its dual edge. The dual complexes Γ^* are naturally interpreted as three dimensional spin foams. The partition function (3.9) is thus expanded as the sum over all the amplitudes associated to such dual complexes Γ^* .

3.1.2 Dual picture: discrete geometry

Fourier transform

Given the GFT model introduced in the previous section, characterized by the gauge symmetry group G , one can define a (non-commutative) Fourier transform that maps the functions on such group to the functions on its dual Lie group G^* . The notion of dual Lie group is clarified in App. A, where I discuss the Poisson Lie group construction. To the purpose of this chapter, one can think at the group G^* as the dual Lie algebra \mathfrak{g}^* , which can be regarded as the abelian group \mathbb{R}_*^n , where n is the dimension of G and the functions on G^* obey to a non-commutative product rule denoted by the symbol \star .

The above mentioned Fourier transform is the map $\mathcal{F} : F(G) \rightarrow F(G^*)$ defined as

$$\mathcal{F}[f](x) = \int dg e(g, x) f(g), \quad (3.10)$$

with $x \in G^*$, $g \in G$ and $f \in F(G)$. I will call *plane wave* the element⁴ $e(g, x) \in F(G) \otimes F(G^*)$, in analogy with quantum mechanics, where the kernel of the Fourier transform is indeed an ordinary plane wave. In the Lie group context, the plane wave is defined in terms of its properties

$$\begin{aligned} (e(g_1, \cdot) \star e(g_2, \cdot))(x) &= e(g_1 g_2, x), & (e(\cdot, x_2) e(\cdot, x_1))(g) &= e(g, x_1 + x_2), \\ \int dg e(g, x) &= \hat{\delta}(g), & e(g, 1) &= 1, & \int dx e(g, x) &= \delta(g), & e(1, x) &= 1. \end{aligned} \quad (3.11)$$

⁴The element $e(g, x)$ often carries a subscript \star in literature [93]. This is used to recall that the plane wave is partially defined as a function on $F(G^*)$, and thus it respect non-commutative multiplication.

Here δ and $\hat{\delta}$ are resp. the delta functions on G and G^* . The plane wave has also an inverse element

$$e^{-1}(g, x) := e(g^{-1}, x) = e(g, -x), \quad (3.12)$$

which is used to define the inverse map $\mathcal{F}^{-1} : F(G^*) \rightarrow F(G)$:

$$\mathcal{F}^{-1}[\hat{f}](g) = \int dx (e^{-1}(g, \cdot) \star \hat{f})(x), \quad (3.13)$$

for $\hat{f} \in F(G^*)$. It is not complicated, using the properties (3.11) and (3.12), to show that $\mathcal{F}[\mathcal{F}^{-1}[\hat{f}]] = \hat{f}$ and $\mathcal{F}^{-1}[\mathcal{F}[f]] = f$. The minus sign in the inverse element (3.12) as well as the addition at the place of the product in (3.11) reflect the abelianess of the group G^* .

Dual field and closure constraint

As in the dual complex picture of GFT, where the fundamental degrees of freedom are encoded in the field ϕ , in the transformed formulation these are encoded in its Fourier transform $\hat{\phi} = \mathcal{F}[\phi] \in F(G^{*\times 3})$, which I will call dual field.

The Fourier transform of the gauge averaging (3.1) gives

$$\begin{aligned} \mathcal{F}[(\mathcal{P}\phi)](x_1, x_2, x_3) &= \int dg^3 \prod_{i=1}^3 e(g_i, x_i) (\mathcal{P}\phi)(g_1, g_2, g_3) = \int dh dg^3 \prod_{i=1}^3 e(g_i, x_i) \phi(hg_1, hg_2, hg_3) \\ &= \int dh dg^3 \prod_{i=1}^3 e(h^{-1}g_i, x_i) \phi(g_1, g_2, g_3) \\ &= \int dh \prod_{i=1}^3 e(h^{-1}, x_i) \int dg^3 \prod_{i=1}^3 e(g_i, x_i) \phi(g_1, g_2, g_3) \\ &= \int dh \prod_{i=1}^3 e(h^{-1}, x_1 + x_2 + x_3) \hat{\phi}(x_1, x_2, x_3) = (\hat{\mathcal{C}}\hat{\phi})(x_1, x_2, x_3). \end{aligned} \quad (3.14)$$

Above I used the definition of dual field as the Fourier transform of the field ϕ and the properties (3.11) and (3.12) of the plane wave. The function

$$\hat{\mathcal{C}}(x_1, x_2, x_3) = \hat{\delta}(x_1 + x_2 + x_3), \quad (3.15)$$

is called *closure constraint*, since it is enforcing that the three elements x_i sum up to zero [95]. The invariance of the field under the gauge transformation (3.3) yields to the functional identity

$$\hat{\phi} = (\hat{\mathcal{C}} \star \hat{\phi}). \quad (3.16)$$

Action

Also the action (3.4), and thus the kinetic (3.5) and interaction (3.7) terms, can be expressed in the G^* picture, upon Fourier transform. Let me show in details the steps for the kinetic term, which are rather simple:

$$\begin{aligned} \mathcal{S}_{\mathcal{K}} &= \int dg^3 \phi(g_1, g_2, g_3) \phi(g_3, g_2, g_1) = \int dg^3 \mathcal{F}^{-1}[\hat{\phi}](g_1, g_2, g_3) \mathcal{F}^{-1}[\hat{\phi}](g_3, g_2, g_1) \\ &= \int dg^3 dx^6 \left(\prod_{i=1}^3 e(g_i, \cdot) \star \hat{\phi} \right)(x_1, x_2, x_3) \left(\prod_{i=1}^3 e(g_{4-i}, \cdot) \star \hat{\phi} \right)(x_4, x_5, x_6) \\ &= \int dg^3 dx^6 (e(g_1, x_6 + x_1) e(g_2, x_5 + x_2) e(g_3, x_4 + x_3)) \star (\hat{\phi}(x_1, x_2, x_3) \hat{\phi}(x_4, x_5, x_6)) \\ &= \int dx^6 (\hat{\mathcal{K}} \star (\hat{\phi} \tau \hat{\phi}))(\{x\}) = \int dx^3 (\hat{\phi} \star \tau \hat{\phi}^{-1})(x_1, x_2, x_3), \end{aligned} \quad (3.17)$$

where in the last step the inverse field stands for $\hat{\phi}^{-1}(\{x\}) = \hat{\phi}(\{x^{-1}\})$. To perform the computation, I used the properties (3.11) of the plane wave and the definition of the inverse Fourier transform (3.13). In the third line I wrote the star product between functions $(\hat{\phi} \star \hat{\phi})(x_i)$ has a product of functions already evaluated in the elements, $\hat{\phi}(x_i) \star \hat{\phi}(x_i)$. This is an abuse of notation used to simplify the expression. The term

$$\hat{\mathcal{K}}(\{x\}) = \hat{\delta}(x_1 + x_6) \hat{\delta}(x_2 + x_5) \hat{\delta}(x_3 + x_4), \quad (3.18)$$

is the kernel of the *propagator amplitude* in the triangulation picture. Note that the amplitude (3.18) can be directly derived by the propagator amplitude (3.6) upon Fourier transform. I will not show all the details of the computation for the interaction term (3.7). Its derivation is slightly longer than the propagator amplitude but follows the same steps. Using again an abuse of notation, the interaction term can be re-expressed as

$$\begin{aligned} \mathcal{S}_V &= \int dx^6 (\hat{\mathcal{C}} \hat{\phi})(x_1, x_2, x_3) \star (\hat{\mathcal{C}} \hat{\phi})(-x_3, x_4, x_5) \star (\hat{\mathcal{C}} \hat{\phi})(-x_5, x_6, -x_1) \star (\hat{\mathcal{C}} \hat{\phi})(-x_6, -x_4, -x_2) \\ &= \int dx^{12} \hat{\mathcal{V}}(\{x\}) \star \hat{\phi}(x_1, x_2, x_3) \star \hat{\phi}(x_4, x_5, x_6) \star \hat{\phi}(x_7, x_8, x_9) \star \hat{\phi}(x_{10}, x_{11}, x_{12}), \end{aligned} \quad (3.19)$$

where the kernel of the *tetrahedron amplitude* in the triangulation picture is

$$\begin{aligned} \hat{\mathcal{V}}(\{x\}) &= (\hat{\mathcal{C}}(x_1, x_2, x_3) \hat{\mathcal{C}}(x_4, x_5, x_6) \hat{\mathcal{C}}(x_7, x_8, x_9) \hat{\mathcal{C}}(x_{10}, x_{11}, x_{12})) \\ &\quad \star (\delta(x_1 + x_9) \delta(x_2 + x_{12}) \delta(x_3 + x_4) \delta(x_5 + x_{11}) \delta(x_6 + x_7) \delta(x_8 + x_{10})). \end{aligned} \quad (3.20)$$

Again, the above amplitude (3.19) can be directly derived as the Fourier transform of (3.8).

Feynman amplitudes

Analogous to the GFT model expressed in the dual complex, the building blocks for the Feynman diagrams also in this case are tetrahedron amplitudes (3.20). Once again, the partition function can be expanded as a sum over all the possible Feynman amplitudes associated to the triangulations Γ of three dimensional manifolds. The partition function of a group field theory in the triangulation picture is thus

$$\mathcal{Z} = \hat{\mathcal{N}} \sum_{\Gamma} \int \prod dx \prod_{\tau} \hat{\mathcal{V}}(\{x\}), \quad (3.21)$$

where $\hat{\mathcal{N}}$ is a normalization factor and the product goes over the tetrahedra $\tau \subset \Gamma$.

Geometric interpretation

The fundamental degrees of freedom of the GFT model in the triangulation picture are encoded in the dual field $\hat{\phi}$; again, upon quantization, the space of fields $F(\mathbb{G}^{\ast \times 3})$ would be the Hilbert space of the model. As the field ϕ was interpreted as a graph dual to a triangle, the representation of the dual field is now that of a triangle, as in Fig. 3.1a. This interpretation is enforced by the closure constraint (3.15), which suggests that the three group elements x_i can be interpreted as the (three edge vectors of the) boundary of a triangle.

The propagator amplitude (3.18) is again represented as the identification of two triangles, as in Fig. 3.2. Last, the tetrahedron amplitude (3.20) is still a product of four dual fields, which are now the four triangles that compose the boundary of a tetrahedron, see Fig. 3.3a. This is more clear if one looks at the expression in (3.20): the first four functions are the closure constraints of four triangles, and the remaining six deltas merge the triangles by identifying the six edges pairwise. The amplitudes of the model are constructed as combinations of tetrahedra glued together by identifying their faces (triangles) through the propagator (3.18). The Feynman graphs are thus simplicial triangulations of three dimensional manifolds; as a consequence, the partition function (3.18) is just the sum over all such triangulations (discrete geometries).

3.1.3 Peter-Weyl theorem

It is worth to dedicate a few words on the use of the Peter-Weyl theorem in GFT [91].

Theorem 1 (Peter-Weyl). *Let G be a compact group.*

- *The set of matrix elements of the irreducible representations of G are dense in the space $F(G)$.*
- *Unitary representations of G are completely reducible.*
- *Regular representations of G can be decomposed as a direct sum of irreducible unitary representations, for which the matrix coefficients form an orthonormal basis.*

One can use the Peter-Weyl theorem – in GFT as well as other similar spin foam-like approaches – to expand the amplitudes of the model as a Fourier decomposition in terms of irreducible representations of the group G . Upon quantization, this map, which can be seen as another Fourier transform, turns the Hilbert space of the model into the space of irreducible representations and allows the numerical evaluation of the amplitudes. In the particularly interesting case of $G = \text{SU}(2)$, the partition function (3.9) reduces to the one of Ponzano-Regge model, expressed in terms of the $6j$ symbols:

$$\mathcal{Z} = \sum_{\{j_i\}} \prod_t \{6j\}_t(j_i). \quad (3.22)$$

For non-compact groups the theorem does not hold and the story is slightly more complicated; however, in most of the cases of interest, one can still decompose representations of non-compact groups as direct sum of irreducible ones. I refer to [96–98] for some explicit realizations, when the gauge groups are taken to be $G = \text{SU}(2)$ or the double cover the the Lorentz group, $G = \text{SL}(2, \mathbb{C})$.

3.2 Higher dimensional GFT and geometric condition

In the previous sections I presented a 2+1 dimensional GFT model. The same construction holds for general $d + 1$ dimensions, with a slightly different geometric interpretation:

- * the dual field and the field are functions on $d + 1$ copies of the groups G^* and G , and are resp. represented as a d -simplex or its dual graph;
- * the gauge transformation does not change, and the closure constraint turns into the closure of the boundary of a d -simplex;
- * the propagator remains the identification of two d -simplices or their dual graphs;
- * the building blocks of the Feynman amplitudes (given by the interaction term) are $d + 1$ -simplices or their dual graphs;
- * the partition function is a sum over the $d + 1$ dimensional discrete geometries or over their dual graphs ($d + 1$ dimensional spin foams).

In the previous sections I presented the $d = 2$ case for two reasons:

- the geometric interpretation is easier to visualize;

- in the next chapter I will provide the generalization of a group field theory based on quantum groups. According to the arguments that I will explain in Ch. 5, this generalization is better motivated in 2+1 dimensions, whereas it would require further adjustments in higher dimensions.

One fundamental feature of the general $d + 1$ GFT model is that it is a topological model: as I will show in the following chapters, GFT models with no further constraints, are invariant under the so called Pachner moves. Hence, the Feynman amplitudes (and thus the partition function) are topologically invariant. Since the initial aim of GFT is to describe quantum gravity, not just discrete geometries, the topological invariance makes GFT a model of quantum gravity only in three dimensions.

In four dimensions group field theory does not encode any gravitational degree of freedom. Here, quantum gravity can be formulated as a constrained topological model. GFT is not an exception: one can constrain the model implementing the *simplicity constraint* [99–101]. The latter, in BF theories imposes the B field to be (the two form dual to) a *simple bi-vector*⁵ rather than just a two form. Two celebrated examples of gravitational models re-phrased as constrained four dimensional GFT's are the EPRL [40] and the Barrett-Crane models [39, 102, 103]. A pedagogical discussion of these concepts is presented in [96, 103], where the Barrett-Crane model is recovered from the amplitude of a group field theory, using the double cover of the Lorentz group $G = \text{SL}(2, \mathbb{C})$ as the gauge group. In this case, the restriction from the topological to the gravitational case is performed by the introduction of the projector

$$(\mathcal{P}_S \phi)(g_1, g_2, g_3, g_4) = \int d\gamma^4 \phi(g_1\gamma_1, g_2\gamma_2, g_3\gamma_3, g_4\gamma_4). \quad (3.23)$$

Here $g_i \in \text{SL}(2, \mathbb{C})$ and $\gamma_i \in \text{SU}(2)$. Therefore, the projector reduces the space of fields to the functions invariant under the right translations by any $\text{SU}(2)$ element; namely, the restricted space is the set of functions on the hyperboloid $\mathcal{H} = \text{SL}(2, \mathbb{C})//\text{SU}(2)$. This can be seen even better using the Plancharel theorem⁶: imposing the constraint (3.23) on the field, it is decomposed as a direct sum of time-like irreducible (balanced) representations, that are properly associated to simple (dual) bi-vectors, see [40, 102, 104].

3.3 Remarks

Since I have intensively celebrated some of the most interesting aspects of group field theories, to conclude this chapter I would like to mention two aspects that are particularly important for the current discussion and open the windows for possible improvements.

Quantum groups. Despite the model clearly provides the basis for a quantum theory of gravity, the classical phase space, on which the spin foam quantization would be implemented, is the cotangent bundle T^*G . Since the dual Lie algebra \mathfrak{g}^* is an abelian group, these model can only describe the simplicial triangulations of flat manifolds. The generalization to curved three dimensional manifolds requires the use of Poisson Lie groups as a gauge symmetry group. These allow to have curvature on both the configuration and momentum spaces and are discussed in more detail in App. A.

⁵A simple bi-vector can be written as a wedge (skew symmetric tensor) product of vectors: $b = e \wedge e$.

⁶The Plancharel theorem plays the role of the Peter-Weyl theorem for the Lorentz group. It provides the decomposition of the functions on the group in terms of irreducible (balanced) representations, see [39].

Higher categories. As we will see in Ch. 5, in four dimensions even the use of quantum groups is not enough to describe curved geometries. In this case one needs to generalize even further the concept of group, using what is called a 2-group.

Chapter 4

Quantum groups in group field theory

Group field theories were first introduced with the aim of retrieving the amplitudes of three dimensional Euclidean quantum gravity as the Feynman diagrams of a field theory. Nevertheless, I already emphasized that GFT's can be naturally generalized to the higher dimensional case and even in the Lorentzian setting; another interesting extension is also the inclusion of the cosmological constant [41], which is normally interpreted as the insertion of an homogeneous curvature of the discretized manifold. As I discussed in the previous chapter, the (spin foam) quantization used in group field theories is based on the cotangent bundle of the gauge group, as the classical phase space. This implies that the triangulation associated to the discretized manifold has to be flat (without homogeneous curvature). In [105] the underlying symmetries of three dimensional gravity with non-zero cosmological constant were derived, and it was pointed that these symmetries are realized in terms of *quantum groups* [106], which was well-known since Witten's seminal work [107].

In this line, the proper state sum for three dimensional discrete gravity with non-zero cosmological constant is provided by the Turaev-Viro model [46]. This is a generalization of the Ponzano-Regge amplitude, where the representations of $SU(2)$ are replaced by representations of the quantum group $SU_q(2)$. In the original model, the quantum group is also used as a regulator for the model, as the parameter q is taken as root of unity¹. The Turaev-Viro vertex term is thus a topological invariant associated to $3d$ gravity [6] and it is related to discrete version of Chern-Simons action [108, 109]. The generalization of Turaev-Viro model to the $4d$ case is the Yetter-Crane model [110]. This is a state sum model whose quantum (boundary) states are the Turaev-Viro invariants, and the partition function is interpreted as four dimensional discrete curved geometry.

Quantum groups, or more generally monoidal (spherical) categories, are also the natural tool to describe topological quantum field theories (TQFT) [111], widely known in mathematics and condensed matter. The use of quantum groups to describe the symmetries of quantum gravity are also at the basis of doubly (deformed) special relativity (DSR) [112–114] and non-commutative space-time approaches [115, 116].

I refer to the books [106, 117] as complete introductions to quantum groups. Assuming that not all the readers are familiar with the notion of quantum groups and Hopf algebras, I provide a short overview in the next section. After this, I will introduce the core of this chapter: the Hopf algebra field theory, based on an project in collaboration with F. Girelli [45]. This is a new model constructed in a framework inspired by the group field theory approach in Ch. 3, that,

¹The parameter q is called deformation parameter in quantum group theory and allows to recover the ordinary definition of groups in the limit $q \rightarrow 1$. Taking q to be root of unity consists in assuming that $q^n = 1$ for some $n \in \mathbb{N}$. This assumption is analogous to the choice of a finite group in place of a Lie group in ordinary gauge theories, used to regularize the model.

using co-associative Hopf algebras, aims to provide the description of homogeneously curved discrete geometries. To better understand the goal of Hopf algebra field theory, I recall that the Ponzano-Regge model can be retrieved as the discretization of a three dimensional BF model with vanishing cosmological constant, and its partition function can essentially be seen as a plane-wave or as a Dirac delta function implementing the vanishing of the curvature (with the B field playing the role of a Lagrange multiplier),

$$\begin{aligned} \mathcal{Z}_{BF}^{\Lambda=0} &= \int [dB][dA] e^{i \int_{\mathcal{M}} B^I \wedge F_I} \sim \int [dA] \delta(F) \\ &\rightarrow \mathcal{Z}_d^{\Lambda=0} = \int [dX][dg] \prod_e e^{i \langle X_e, g_e \rangle} = \int [dg] \prod_e \delta(g_e) \approx \mathcal{Z}_{PR}, \end{aligned} \quad (4.1)$$

where $X_e \in \mathbb{R}^3$ is a variable associated to the edge e of a triangulation and g_e is the closed holonomy on the loop of links spanning the surface dual to the edge e . No analogue of such discretization really exists when the cosmological constant is not zero, even if the asymptotic analysis of the TV amplitude shows that it is related to the Regge action with a non vanishing cosmological constant [118]. One can nevertheless wonder whether a similar derivation could exist for the TV model and in particular whether there exists a generalized notion of plane-wave which would be used to encode a discretized BF action.

Using the notion of generalized Drinfeld double [119] that naturally encodes the duality between Hopf algebras, I will provide the formulation of Hopf algebra field theory both in the configuration and in the momentum space. The two formulations are associated to the simplicial triangulation of an homogeneously curved three dimensional manifold and to its dual complex, and are related by a (generalized notion of) Fourier transform. The canonical element of the generalized Drinfeld double plays the role of generalized plane wave; it is used as the kernel for the Fourier transform between dual Hopf algebras, and most importantly it will be the core to construct the amplitudes of the new model. Indeed, I will also show that such plane wave provides the discretization or regularization of a BF amplitude with or without cosmological constant.

4.1 Quantum groups

In this part I will briefly introduce the concept of Hopf algebra and quantum groups. This section is mostly inspired by [106, 117], to which I refer for a more detailed overview. The reader who is familiar with the notion of Hopf algebra can skip this part.

Algebra

Let us start from the definition of an algebra. We call algebra the set $(A, +, \mathbb{K})$ over the field \mathbb{K} , where A is a vector space equipped with a linear² multiplication (or product) $m : A \otimes A \rightarrow A$ and $+$ denotes the vector addition. Without risk of confusion, I will sometimes use the simpler notation $a_1 \cdot a_2$ or even $a_1 a_2$ at the place of $m(a_1 \otimes a_2)$ for the multiplication, with $a_1, a_2 \in A$. I consider algebras whose product is associative and has a unit element, (usually denoted 1 or η). Given any two algebras A, B , one can also define their tensor product $A \otimes B$, which is still an algebra with product $(a_1 \otimes b_1) \cdot (a_2 \otimes b_2) = a_1 a_2 \otimes b_1 b_2$.

Co-algebra

²In the sense that the product is compatible with the sum and the action of \mathbb{K} .

This definition of algebra suggests a dual picture, given by a vector space over \mathbb{K} equipped with a *co-product* $\Delta : A \rightarrow A \otimes A$, for which there exists a linear *co-unit* $\varepsilon : A \rightarrow \mathbb{K}$ defined by

$$(\varepsilon \otimes \text{id}) \circ \Delta = (\text{id} \otimes \varepsilon) \circ \Delta = \text{id}. \quad (4.2)$$

Here id denotes the identity map and \circ the composition of maps. I will consider co-associative co-algebras, for which the identity $(\text{id} \otimes \Delta) \Delta = (\Delta \otimes \text{id}) \Delta$ holds. Later on I will often use the Sweedler notation for the co-product

$$\Delta a = \sum a_{(1)} \otimes a_{(2)}. \quad (4.3)$$

Again, given any two co-algebras A, B , their tensor product $A \otimes B$ is still a co-algebra with co-product $\Delta(a \otimes b) = \sum a_{(1)} b_{(1)} \otimes a_{(2)} b_{(2)}$.

Bi-algebra

A bi-algebra is a vector space equipped with all the maps mentioned above: product, co-product, unit and co-unit. Said otherwise, a bi-algebra A is both an algebra and a co-algebra, where the above mentioned maps are all compatible, in the sense that

$$\Delta(a_1 \cdot a_2) = \Delta(a_1) \cdot \Delta(a_2), \quad \Delta(1) = 1 \otimes 1, \quad \varepsilon(a_1 \cdot a_2) = \varepsilon(a_1) \varepsilon(a_2), \quad \varepsilon(1) = 1. \quad (4.4)$$

The above identities state that the co-product and co-unit are algebra homomorphisms, while the product and the unit are co-algebra homomorphisms.

Hopf algebra

An Hopf algebra over \mathbb{K} is a bi-algebra over \mathbb{K} equipped with an *antipode* map $S : A \rightarrow A$, compatible with product and co-product, as such

$$m \circ (\text{id} \otimes S) \circ \Delta = m \circ (S \otimes \text{id}) \circ \Delta = \eta \varepsilon. \quad (4.5)$$

Even if the role of the antipode can be interpreted as that of an inverse map, the identity $S^2 = \text{id}$ does not always hold; for some Hopf algebra (such as the finite dimensional ones) there also exists an inverse antipode S^{-1} for which $S \circ S^{-1} = S^{-1} \circ S = \text{id}$.

Quantum group

A *quantum group* or *quasitriangular Hopf algebra* is an Hopf algebra A equipped with an element $\mathcal{R} \in A \otimes A$ called quantum \mathcal{R} -matrix. This obeys the axioms

$$\begin{aligned} (\Delta \otimes \text{id}) \mathcal{R} &= \mathcal{R}_{13} \mathcal{R}_{23}, & (\text{id} \otimes \Delta) \mathcal{R} &= \mathcal{R}_{13} \mathcal{R}_{12}, \\ \mathcal{R} \Delta a &= (\tau \circ \Delta a) \mathcal{R}, & \forall a \in A, \end{aligned} \quad (4.6)$$

here I used the tensor product notation $\mathcal{R}_{ij} = \sum 1 \otimes \dots \otimes \mathcal{R}^{(1)} \otimes \dots \otimes \mathcal{R}^{(2)} \otimes \dots \otimes 1$, where $\mathcal{R} = \sum \mathcal{R}^{(1)} \otimes \mathcal{R}^{(2)}$. The quantum \mathcal{R} -matrix has an inverse \mathcal{R}^{-1} and satisfies the further identities

$$(\varepsilon \otimes \text{id}) \mathcal{R} = (\text{id} \otimes \varepsilon) \mathcal{R} = 1, \quad (S \otimes \text{id}) \mathcal{R} = \mathcal{R}^{-1}, \quad (\text{id} \otimes S) \mathcal{R}^{-1} = \mathcal{R}. \quad (4.7)$$

Given the properties (4.6), one can check that the quantum \mathcal{R} -matrix satisfies the Quantum Yang-Baxter equation (QYBE):

$$\mathcal{R}_{12} \mathcal{R}_{13} \mathcal{R}_{23} = \mathcal{R}_{23} \mathcal{R}_{13} \mathcal{R}_{12}. \quad (4.8)$$

Action and co-action

There exists a notion of action and co-action between Hopf algebras. A left action of A on H is a map $\triangleright : A \otimes H \rightarrow H$, denoted $a \triangleright h \in H$ and defined by the axioms

$$(ab) \triangleright h = a \triangleright (b \triangleright h) , \quad 1 \triangleright h = h , \quad (4.9)$$

similar for the right action. If the (left) action is an algebra homomorphism

$$a \triangleright (hg) = \sum (a_{(1)} \triangleright h) (a_{(2)} \triangleright g) , \quad a \triangleright 1 = \varepsilon(a) 1 , \quad (4.10)$$

or a co-algebra homomorphism

$$\Delta(a \triangleright h) = \sum (a_{(1)} \triangleright h_{(1)}) \otimes (a_{(2)} \triangleright h_{(2)}) , \quad \varepsilon(a \triangleright h) = \varepsilon(a) \varepsilon(h) , \quad (4.11)$$

one resp. says that H is a left A module algebra or co-algebra, and similar for the right action. A left co-action of H on A is a map $\beta : A \rightarrow H \otimes A$, denoted $\beta(a) = \sum a^{(\bar{1})} \otimes a^{(\bar{2})} \in H \otimes A$ and defined by the axioms

$$(\text{id} \otimes \beta) \circ \beta = (\Delta \otimes \text{id}) \circ \beta , \quad (\varepsilon \otimes \text{id}) \circ \beta = \text{id} , \quad (4.12)$$

similar for the right co-action. If the (left) co-action is an algebra homomorphism

$$\beta(ab) = \beta(a) \beta(b) , \quad \beta(1) = 1 \otimes 1 , \quad (4.13)$$

or a co-algebra homomorphism

$$(\text{id} \otimes \Delta) \circ \beta(a) = \sum a_{(1)}^{(\bar{1})} a_{(2)}^{(\bar{1})} \otimes a_{(1)}^{(\bar{2})} \otimes a_{(2)}^{(\bar{2})} , \quad (\varepsilon \otimes \text{id}) \circ \beta(a) = 1 \varepsilon(a) , \quad (4.14)$$

one resp. says that A is a left H co-module algebra or co-algebra, and similar for the right co-action. If H and A are both bi-algebras, then one can build a number of bi-algebras on their tensor product $A \otimes H$. Given the left or right actions between A and H , one has the spaces

$$\begin{aligned} \underline{\text{Left cross product}}: & \quad A \rtimes H \\ \underline{\text{Right cross product}}: & \quad A \ltimes H \\ \underline{\text{Double cross product}}: & \quad A \bowtie H \end{aligned}$$

The left or right cross products are bi-algebras with a trivial co-product (that of a tensor product of co-algebras), and a left or right action that induce the respective multiplications

$$\begin{aligned} (a \otimes h) \cdot (b \otimes g) &= a(h \triangleright b) \otimes hg , \\ (a \otimes h) \cdot (b \otimes g) &= ab \otimes (h \triangleleft b)g . \end{aligned} \quad (4.15)$$

The double cross product is a bi-algebra with both a left and right actions that induce the multiplication

$$(a \otimes h) \cdot (b \otimes g) = a(h \triangleright b) \otimes (h \triangleleft b)g , \quad (4.16)$$

while the co-product is again trivial. Given the left or right co-actions between A and H , one has the spaces

$$\begin{aligned} \underline{\text{Left cross co-product}}: & \quad A \rtimes\!\!\blacktriangleleft H \\ \underline{\text{Right cross co-product}}: & \quad A \blacktriangleright\!\!\blacktriangleleft H \\ \underline{\text{Double cross co-product}}: & \quad A \blacktriangleright\!\!\blacktriangleleft H \end{aligned}$$

The left or right cross co-products are bi-algebras with a trivial multiplication (that of a tensor product of algebras), and left or right co-actions (resp. β and α) that induce the respective co-products

$$\begin{aligned}\Delta(a \otimes h) &= (a_{(1)} \otimes \beta(a_{(2)}) \otimes \text{id}) \cdot (\text{id} \otimes h_{(1)} \otimes \text{id} \otimes h_{(2)}), \\ \Delta(a \otimes h) &= (a_{(1)} \otimes \text{id} \otimes a_{(2)} \otimes \text{id}) \cdot (\text{id} \otimes \alpha(h_{(1)}) \otimes h_{(2)}).\end{aligned}\tag{4.17}$$

The double cross co-product is a bi-algebra with both a left and right co-actions that induce the co-product

$$\Delta(a \otimes h) = (a_{(1)} \otimes \beta(a_{(2)}) \otimes \text{id}) \cdot (\text{id} \otimes \alpha(h_{(1)}) \otimes h_{(2)}),\tag{4.18}$$

while the multiplication is again trivial. Given the left or right actions and co-actions between A and H , one has the mixed spaces

$$\begin{aligned}\underline{\text{Left bicrossproduct:}} & \quad A \blacktriangleright H \\ \underline{\text{Right bicrossproduct:}} & \quad A \blacktriangleleft H\end{aligned}$$

The left bicrossproduct is a bi-algebra with multiplication of a left cross product and co-product of a right cross co-product. The right bicrossproduct is a bi-algebra with multiplication of a right cross product and co-product of a left cross co-product.

Duality

Given two Hopf algebras H, A , they are dual each other if there exist a bi-linear map $\langle \cdot, \cdot \rangle : H \otimes A \rightarrow \mathbb{K}$ which implements the notion of duality between the two Hopf algebra structures, as

$$\begin{aligned}\langle h \cdot g, a \rangle &= \langle h \otimes g, \Delta a \rangle, & \langle h, a \cdot b \rangle &= \langle \Delta h, a \otimes b \rangle, \\ \langle h, 1 \rangle &= \varepsilon(h), & \langle 1, a \rangle &= \varepsilon(a), & \langle Sh, a \rangle &= \langle h, Sa \rangle,\end{aligned}\tag{4.19}$$

for all $h, g \in H$ and $a, b \in A$. The same duality can be defined between a left or right actions and a right or left co-actions (resp. α, α' and β, β'):

$$\begin{aligned}\langle h, g \triangleright a \rangle &= \langle \alpha(h), g \otimes a \rangle, & \langle h \triangleleft a, b \rangle &= \langle h \otimes a, \alpha'(b) \rangle, \\ \langle h, a \triangleleft g \rangle &= \langle \beta(h), a \otimes g \rangle, & \langle a \triangleright h, b \rangle &= \langle a \otimes h, \beta'(b) \rangle.\end{aligned}\tag{4.20}$$

Doubles

As the notion of duality (4.19) yields to dual Hopf algebras, the duality between action and co-action (4.20) extends this concept to the Hopf algebras built on the tensor product $A \otimes H$. The Hopf algebra dual to a left or right cross product is resp. a right or left cross co-product, the Hopf algebra dual to a double cross product is a double cross co-product, while the left and right bicrossproduct Hopf algebras are dual other.

This leads to the generalization to the Hopf algebras context, of the double of a group, which is used to construct the space of symmetries (Drinfeld double) or its phase space (Heisenberg double). The quantum double of an Hopf algebra H , denoted $D(H, H^*, \langle \cdot, \cdot \rangle)$, is the double cross product Hopf algebra defined on the tensor product³ $H^{*op} \bowtie H$, where the angle brackets are the bi-linear map that encode the duality between H and H^* .

For the purpose of the Hopf algebra field theory construction, in the next section I will further generalize the concept of quantum double and its dual Hopf algebra, introducing the notion of skew-pairing and co-pairing, see [106, 119].

³The Hopf algebras A^{op} and A^{cop} are the Hopf algebras obtained by taking A with resp. an inverse product or an inverse co-product.

4.2 Hopf algebra field theory

In this part, based on a project in collaboration with F. Girelli [45], I discuss the new model called Hopf algebra field theory (HAFT).

After introducing the main notations and conventions that will be used in the rest of the chapter, I provide some relevant definitions such as the notion of generalized Fourier transform for Hopf algebras and hence the notion of plane-wave that plays a central role in Hopf algebra field theory. Then I introduce the main ingredients of HAFT, as the action that governs the model and the Feynman diagram amplitudes associated to homogeneously curved discrete geometries. I also show the topological invariance of the new model. In the last part, I consider the main example of Hopf algebra field theory, which consists in the specific choice of the quantum group $SU_q(2)$, with q real. In this case, Hopf algebra field theory will be interpreted as the path integral formulation related to TV model. Finally, I discuss how the generalized plane-wave can be related to the discretization of the BF amplitude with a negative cosmological constant.

For the sake of simplicity, I will not go through the details of the the long and tedious computations. I refer to the original work [45] for the interested reader. In [45] we also show how to recover ordinary group field theories based either on finite groups or on the $SU(2)$ Lie group, as sub-cases of Hopf algebra field theory. Since this point is particularly crucial, in App. B.1 I propose a new prescription to construct dual Hopf algebras and the associated quantum double Hopf algebras, which is an highly non-trivial job.

Hopf algebra ingredients

Let me first summarize the most relevant Hopf algebra ingredients that will be needed for the construction of the new model. The expert reader can skip this part, even though I still suggest to check the conventions and the notations, as well as the definition of Fourier transform.

Given the Hopf algebra A , denote $A^n = \bigotimes_i^n A_i$ with $n \geq 2$ its associated tensor product and consider the following maps.

$$\text{Co-product: } \Delta^n : A \rightarrow A^n \quad \text{with} \quad \Delta^n \equiv (\text{id} \otimes \dots \otimes \text{id} \otimes \Delta) \circ \dots \circ \Delta \quad (4.21)$$

$$\text{Product: } m^n : A^n \rightarrow A \quad \text{with} \quad m^n \equiv m \circ \dots \circ (\text{id} \otimes \dots \otimes \text{id} \otimes m) \quad (4.22)$$

$$\text{Permutation: } \tau^n : A^n \rightarrow A^n \quad \text{with} \quad \tau^n(a_1 \otimes \dots \otimes a_n) = a_n \otimes \dots \otimes a_1 \quad (4.23)$$

$$\text{Co-unit: } \varepsilon^n : A^n \rightarrow \mathbb{K} \quad (4.24)$$

$$\text{Unit: } \eta^n : \mathbb{K} \rightarrow A^n \quad (4.25)$$

The tensor product A^n is in turn an Hopf algebra with associated maps

$$\begin{aligned} \Delta^{(n)} : A^n &\rightarrow A^n \otimes A^n \\ \text{with } \Delta^{(n)}(a_1 \otimes \dots \otimes a_n) &= (a_{1(1)} \otimes \dots \otimes a_{n(1)}) \otimes (a_{1(2)} \otimes \dots \otimes a_{n(2)}) \end{aligned} \quad (4.26)$$

$$\begin{aligned} m^{(n)} : A^n \otimes A^n &\rightarrow A^n \\ \text{with } m^{(n)}((a_1 \otimes \dots \otimes a_n) \otimes (b_1 \otimes \dots \otimes b_n)) &= m(a_1 \otimes b_1) \otimes \dots \otimes m(a_n \otimes b_n) \end{aligned} \quad (4.27)$$

$$\begin{aligned} \tau^{(n)} : A^n \otimes A^n &\rightarrow A^n \otimes A^n \\ \text{with } \tau^{(n)}((a_1 \otimes \dots \otimes a_n) \otimes (b_1 \otimes \dots \otimes b_n)) &= (b_1 \otimes \dots \otimes b_n) \otimes (a_1 \otimes \dots \otimes a_n) \end{aligned} \quad (4.28)$$

$$\begin{aligned} S^{(n)} : A^n &\rightarrow A^n \\ \text{with } S^{(n)}(a_1 \otimes \dots \otimes a_n) &= (Sa_1 \otimes \dots \otimes Sa_n) \end{aligned} \quad (4.29)$$

Given a tensor product of Hopf-algebras, I will later use the notation

$$m_{ij}(1 \otimes \dots \otimes a_1 \otimes \dots \otimes a_2 \otimes \dots \otimes 1) = 1 \otimes \dots \otimes m(a_1 \otimes a_2) \otimes \dots \otimes 1, \quad (4.30)$$

$$\Delta_{ij}a = 1 \otimes \dots \otimes a_{(1)} \otimes \dots \otimes a_{(2)} \otimes \dots \otimes 1, \quad (4.31)$$

for $a \in A$. The map m_{ij} stands for a product of two elements resp. living in the i^{th} and j^{th} positions of the tensor product, and the result, $m(a_i \otimes a_j) = a_i \cdot a_j$, is in the i^{th} space, for $i < j$. The map Δ_{ij} stands for usual co-product embedded in a tensor product of Hopf algebras, with the two components resp. placed in the i^{th} and j^{th} positions. I use a minus sign as notation for the antipode, for instance

$$\Delta_{-ij}a = 1 \otimes \dots \otimes Sa_{(1)} \otimes \dots \otimes a_{(2)} \otimes \dots \otimes 1. \quad (4.32)$$

These notations can be generalized for the n -dimensional product (4.22) and co-product (4.21), which I denote $m_{i_1 \dots i_n}^n$ and $\Delta_{i_1 \dots i_n}^n$. Similarly, they can be generalized to the product (4.27) and co-product (4.26) of the tensor space Hopf algebra, denoted $m_{i_1 \dots i_n j_1 \dots j_n}^{(n)}$ and $\Delta_{i_1 \dots i_n j_1 \dots j_n}^{(n)}$.

Definition 1 (Simplex maps). *Consider the maps*

$$\underline{m} : A^{12} \rightarrow A^6, \quad \underline{\Delta} : A^6 \rightarrow A^{12}, \quad \Delta_{\mathcal{V}} : A^6 \rightarrow A^{16}, \quad (4.33)$$

that stand for a combination of products or co-products that have the same pattern of a tetrahedron. The maps are defined as

$$\underline{m}(a_1 \otimes \dots \otimes a_{12}) = (a_1 \cdot a_9) \otimes (a_2 \cdot a_{12}) \otimes (a_3 \cdot a_4) \otimes (a_5 \cdot a_{11}) \otimes (a_6 \cdot a_7) \otimes (a_8 \cdot a_{10}), \quad (4.34)$$

$$\underline{\Delta}(a_1 \otimes \dots \otimes a_6) = \Delta_{19}a_1 \otimes \Delta_{212}a_2 \otimes \Delta_{34}a_3 \otimes \Delta_{511}a_4 \otimes \Delta_{67}a_5 \otimes \Delta_{810}a_6, \quad (4.35)$$

$$\begin{aligned} \Delta_{\mathcal{V}}(a_1 \otimes \dots \otimes a_6) = & \Delta_{-164-16}^4 a_1 \cdot \Delta_{-15429}^4 a_2 \cdot \Delta_{-144-312}^4 a_3 \cdot \Delta_{-113210}^4 a_4 \\ & \cdot \Delta_{-133-15}^4 a_5 \cdot \Delta_{-8-2-17}^4 a_6, \end{aligned} \quad (4.36)$$

where Δ_{ij} , and similarly its generalization $\Delta_{i_1 i_2 i_3 i_4}^4$, are given in (4.31).

Quantum double

Definition 2 (Matched pair bi-algebras). [106, 119] *Two bi-algebras H and A form a matched pair if there exist a pair of actions*

$$\triangleright : A \otimes H \rightarrow H, \quad \triangleleft : A \otimes H \rightarrow A \quad (4.37)$$

that satisfy the compatibility relations

$$\begin{aligned} (ab) \triangleleft h &= (a \triangleleft (b_{(1)} \triangleright h_{(1)})) (b_{(2)} \triangleleft h_{(2)}), & 1 \triangleleft h &= \varepsilon(h), \\ a \triangleright (hg) &= (a_{(1)} \triangleright h_{(1)}) ((a_{(2)} \triangleleft h_{(2)}) \triangleright g), & a \triangleright 1 &= \varepsilon(a), \\ a_{(1)} \triangleleft h_{(1)} \otimes a_{(2)} \triangleright h_{(2)} &= a_{(2)} \triangleleft h_{(2)} \otimes a_{(1)} \triangleright h_{(1)}. \end{aligned} \quad (4.38)$$

Definition 3 (Skew paired bi-algebras). [119] *Two bi-algebras H and A are skew paired if there exists a map called skew pairing $\sigma : A \otimes H \rightarrow \mathbb{K}$, such that*

$$\begin{aligned} \sigma(a \cdot b \otimes h) &= \sigma(a \otimes h_{(1)}) \sigma(b \otimes h_{(2)}), \\ \sigma(a \otimes h \cdot g) &= \sigma(a_{(1)} \otimes g) \sigma(a_{(2)} \otimes h), \end{aligned} \quad (4.39)$$

and

$$\sigma(1 \otimes h) = \varepsilon(h), \quad \sigma(a \otimes 1) = \varepsilon(a), \quad (4.40)$$

for all $a, b \in A$ and $h, g \in H$. I used the symbol 1 as a shorthand to address to the unit in both the bi-algebras H and A . If either A admits an antipode or H admits an inverse-antipode, then there exist a convolution inverse σ^{-1} that obeys

$$\sigma^{-1}(a \otimes h) \equiv \sigma(Sa \otimes h) = \sigma(a \otimes S^{-1}h). \quad (4.41)$$

Definition 4 (Generalized quantum double). [119] Let H and A be skew paired bi-algebras with skew pairing σ which is convolution-invertible. They form a matched pair with mutual actions

$$\begin{aligned} a \triangleleft h &= a_{(2)} \sigma^{-1}(a_{(1)} \otimes h_{(1)}) \sigma(a_{(3)} \otimes h_{(2)}), \\ a \triangleright h &= h_{(2)} \sigma^{-1}(a_{(1)} \otimes h_{(1)}) \sigma(a_{(2)} \otimes h_{(3)}). \end{aligned} \quad (4.42)$$

The generalized quantum double is the double cross product bi-algebra $D(H, A, \sigma) \equiv H \bowtie A$ built on $H \otimes A$, with product

$$\begin{aligned} (h \otimes a) \cdot (g \otimes b) &= h(a_{(1)} \triangleright g_{(1)}) \otimes (a_{(2)} \triangleleft g_{(2)})b = \\ &= hg_{(2)} \otimes a_{(2)}b \sigma^{-1}(a_{(1)} \otimes g_{(1)}) \sigma(a_{(3)} \otimes g_{(3)}), \end{aligned} \quad (4.43)$$

trivial co-product, $\Delta = (\Delta_H \otimes \Delta_A)$, tensor product, unit and co-unit.

Definition 5 (Matched co-pair bi-algebras). [106, 119] Two bi-algebras H and A form a matched co-pair if there exist a pair of co-actions

$$\alpha : H \rightarrow H \otimes A, \quad \beta : A \rightarrow H \otimes A \quad (4.44)$$

that satisfy the compatibility relations

$$\begin{aligned} (\Delta \otimes id) \circ \alpha(h) &= ((id \otimes \beta) \circ \alpha(h_{(1)})) (1 \otimes \alpha(h_{(2)})), \\ (id \otimes \Delta) \circ \beta(a) &= (\beta(a_{(1)}) \otimes 1) ((\alpha \otimes id) \circ \beta(a_{(2)})), \\ \alpha(h) \beta(a) &= \beta(a) \alpha(h). \end{aligned} \quad (4.45)$$

Definition 6 (Skew co-paired bi-algebras). [119] Two bi-algebras H and A are skew co-paired if there exists an element called skew co-pairing $\sigma \in H \otimes A$, such that⁴

$$(id \otimes \Delta_A) \sigma = \sigma_{13} \sigma_{12}, \quad (4.46)$$

$$(\Delta_H \otimes id) \sigma = \sigma_{13} \sigma_{23}. \quad (4.47)$$

If either A admits an antipode or H admits an inverse-antipode, then the skew co-pairing is invertible, with inverse

$$\sigma^{-1} \equiv (id \otimes S) \sigma = (S^{-1} \otimes id) \sigma, \quad (4.48)$$

that satisfies the axioms

$$(id \otimes \Delta_A) \sigma^{-1} = \sigma_{12}^{-1} \sigma_{13}^{-1}, \quad (4.49)$$

$$(\Delta_H \otimes id) \sigma^{-1} = \sigma_{23}^{-1} \sigma_{13}^{-1}. \quad (4.50)$$

Using the co-unit axiom and the co-pairing properties (4.46) and (4.47), one derives the identities

$$(\varepsilon \otimes id) \sigma = (id \otimes \varepsilon) \sigma = 1, \quad (4.51)$$

that I will call co-unit properties of the skew co-pairing element.

Definition 7 (Dual of the generalized quantum double). [119] Let H and A be skew co-paired bi-algebras with invertible skew co-pairing. They form a matched co-pair with mutual co-actions $\alpha : H \rightarrow H \otimes A$ and $\beta : A \rightarrow H \otimes A$ given by

$$\begin{aligned} \alpha(h) &= \sigma^{-1}(h \otimes 1) \sigma, \\ \beta(a) &= \sigma^{-1}(1 \otimes a) \sigma. \end{aligned} \quad (4.52)$$

⁴In eq. (4.46), (4.47) and (4.49), (4.50), for clarity, I used the notation Δ_H and Δ_A to address to the co-products on H and A . In the following, for simplicity, we will drop the indices H and A .

The dual of the generalized quantum double is the double cross co-product bi-algebra $D^*(A, H, \sigma) = A \blacktriangleleft H$ built on $A \otimes H$, with co-product

$$\begin{aligned} \Delta(a \otimes h) &= ((id \otimes \alpha \otimes id) \circ (\Delta_H h \otimes 1)) \cdot ((id \otimes \beta \otimes id) \circ (1 \otimes \Delta_A a)) \\ &= \sigma_{23}^{-1} (\Delta_A a \otimes \Delta_H h) \sigma_{23}, \end{aligned} \quad (4.53)$$

trivial product, $(a \otimes h) \cdot (b \otimes g) = ab \otimes hg$, tensor product, unit and co-unit.

I will later need to extend this construction in the context of tensor product of bi-algebras. Consider two tensor product bi-algebras $H^n = \bigotimes_i^n H_i$ and $A^n = \bigotimes_i^n A_i$ such that each pair of sub bi-algebras H_i and A_i are skew paired (resp. skew co-paired); that is, for each pair H_i, A_i there exists a map $\sigma_i : A_i \otimes H_i \rightarrow \mathbb{K}$ (resp. an element $\sigma_i \in H_i \otimes A_i$). Then the tensor product bi-algebras H^n, A^n are skew paired by the map

$$\Sigma^n : A^n \otimes H^n \rightarrow \mathbb{K} \quad (4.54)$$

or skew co-paired by the element

$$\Sigma^n \in H^n \otimes A^n. \quad (4.55)$$

In particular, since the bi-algebras H^n and A^n are tensor products of independent bi-algebras, the n dimensional skew co-pairing can be written in the tensor notation as

$$\Sigma^n = \sigma_{1n+1} \sigma_{2n+2} \cdots \sigma_{n2n}. \quad (4.56)$$

Integral and Fourier transform

Definition 8 (Integral and co-integral). [106, 117] A left (resp. right) integral **in** A is an element ℓ_L , (resp. ℓ_R) in A such that

$$a \cdot \ell_L = \varepsilon(a) \ell_L, \quad \ell_R \cdot a = \varepsilon(a) \ell_R, \quad \forall a \in A. \quad (4.57)$$

The integral ℓ is normalized if $\varepsilon(\ell) = 1$. A left (resp. right) co-integral **on** A is a map $\int_A^L : A \rightarrow \mathbb{K}$ (resp. $\int_A^R : A \rightarrow \mathbb{K}$) that satisfies the left (resp. right) invariance condition

$$\left(id \otimes \int_A^L \right) \Delta a = 1 \otimes \int_A^L a, \quad \left(\int_A^R \otimes id \right) \Delta a = \int_A^R a \otimes 1, \quad \forall a \in A. \quad (4.58)$$

The co-integral \int_A is normalized if $\int_A 1 = 1$.

In order to keep track of divergencies, in the following I will consider non-normalized co-integrals on H and A obeying the relations

$$\int_H 1 = V_H, \quad \int_A 1 = V_A. \quad (4.59)$$

Definition 9 (Fourier transform). Let H and A be skew co-paired bi-algebras. The Fourier transform from H to A is a map $\mathcal{F} : H \rightarrow A$ defined as⁵

$$\mathcal{F}[h] \equiv \frac{1}{\sqrt{\mu}} \left(\int_H^L \otimes id \right) (\sigma \cdot (h \otimes 1)), \quad (4.60)$$

⁵I used left co-integrals for both the Fourier transform (4.60) and its inverse (4.61). One could alternatively use right co-integrals. In this case the inverse skew co-pairing element σ^{-1} would appear in the definition (4.60) and the skew co-pairing element σ in (4.61).

with inverse map $\mathcal{F}^{-1} : A \rightarrow H$

$$\mathcal{F}^{-1}[a] \equiv \frac{1}{\sqrt{\mu}} \left(id \otimes \int_A^L \right) (\sigma^{-1} \cdot (1 \otimes a)), \quad (4.61)$$

with $h \in H$, $a \in A$, and normalization factor

$$\mu \equiv \left(\int_H^L \otimes \int_A^L \right) \sigma = \left(\int_H^L \otimes \int_A^L \right) \sigma^{-1}. \quad (4.62)$$

Proposition 1. *The Fourier transform (4.60) and the inverse Fourier transform (4.61) are inverse maps, in the sense that*

$$(\mathcal{F} \circ \mathcal{F}^{-1}) = (\mathcal{F}^{-1} \circ \mathcal{F}) = id. \quad (4.63)$$

Definition 10 (Delta function). *Let H and A be skew co-paired bi-algebras. The delta functions of H (denoted δ_H) and that of A (denoted $\hat{\delta}_A$), are respectively defined as the Fourier transform and the inverse Fourier transform of the unit:*

$$\hat{\delta}_A \equiv \mathcal{F}[1] = \frac{1}{\sqrt{\mu}} \left(\int_H^L \otimes id \right) \sigma, \quad \delta_H \equiv \mathcal{F}^{-1}[1] = \frac{1}{\sqrt{\mu}} \left(id \otimes \int_A^L \right) \sigma^{-1}. \quad (4.64)$$

Similarly, I call opposite delta functions the elements

$$\hat{\delta}_A^{-1} = \frac{1}{\sqrt{\mu}} \left(\int_H^L \otimes id \right) \sigma^{-1}, \quad \delta_H^{-1} = \frac{1}{\sqrt{\mu}} \left(id \otimes \int_A^L \right) \sigma. \quad (4.65)$$

Proposition 2 (Properties of the delta function). *Let H and A be skew co-paired Hopf algebras. The delta functions on them satisfy the identities below.*

$$\begin{aligned} (a \otimes 1) \cdot \Delta \hat{\delta}_A &= (1 \otimes Sa) \cdot \Delta \hat{\delta}_A, & \Delta \hat{\delta}_A^{-1} \cdot (a \otimes 1) &= \Delta \hat{\delta}_A^{-1} \cdot (1 \otimes Sa), \\ (h \otimes 1) \cdot \Delta \delta_H &= (1 \otimes S^{-1}h) \cdot \Delta \delta_H, & \Delta \delta_H^{-1} \cdot (h \otimes 1) &= \Delta \delta_H^{-1} \cdot (1 \otimes S^{-1}h). \end{aligned} \quad (4.66)$$

Moreover, the delta functions are normalized in the sense that

$$\frac{1}{\sqrt{\mu}} \int_A^L \hat{\delta}_A = \frac{1}{\sqrt{\mu}} \int_A^L \hat{\delta}_A^{-1} = 1, \quad \frac{1}{\sqrt{\mu}} \int_H^L \delta_H = \frac{1}{\sqrt{\mu}} \int_H^L \delta_H^{-1} = 1. \quad (4.67)$$

Proposition 3 (Delta function as integral in the Hopf algebra). *Consider the skew co-paired Hopf algebras H, A . The delta functions (4.64) of the Hopf algebras H and A are resp. left integrals in A and H :*

$$a \cdot \hat{\delta}_A = \hat{\delta}_A \varepsilon(a), \quad h \cdot \delta_H = \delta_H \varepsilon(h). \quad (4.68)$$

The opposite delta functions (4.65) are resp. right integrals in A and H :

$$\hat{\delta}_A^{-1} \cdot a = \hat{\delta}_A^{-1} \varepsilon(a), \quad \delta_H^{-1} \cdot h = \delta_H^{-1} \varepsilon(h). \quad (4.69)$$

The proofs of the above propositions can be found in [45], where I also provide some further propositions useful to construct the new model. For the sake of clarity, here I have always specified whether the co-integrals are left or right by using the indices L, R . In the following I will omit such index, implying that all the co-integrals are always *left* invariant ones.

As announced at the beginning of this chapter, the canonical element σ of the dual of the generalized Drinfeld double (in Def. 7) is used in (4.60) to construct the general notion of Fourier transform (4.60) between dual Hopf algebras. Therefore, it will often be called *plane wave* and will be the central element to construct the Feynman amplitudes of HAFT, related to the discretization of a BF model with cosmological constant.

We are now ready to provide the main ingredients of Hopf algebra field theory. I will focus on the three dimensional model, where the Feynman diagrams are interpreted as three dimensional (homogeneously curved) geometries. I use the same nomenclature adopted in Ch. 3, where *nodes* and *links* are $0d$ and $1d$ objects living in the dual complex, while *vertices* and *edges* are $0d$ and $1d$ objects in the triangulation.

Invariant field and closure constraint

Let the fundamental degrees of freedom of three dimensional Hopf algebra field theory be the elements of the tensor product bi-algebras, $\Phi \in H^3$ and $\hat{\Phi} \in A^3$. The fields⁶ Φ and $\hat{\Phi}$ are related by the Fourier transform (4.60) and its inverse (4.61), with kernel⁷ Σ^3 .

Geometrically, the dual field $\hat{\Phi} \in A^3$ is associated to a triangle, where each of its sub-components (elements of A , which cover the role of the variables $x_i \in G^*$ in ordinary GFT) decorates one of the edges that compose its boundary. Dually, the field $\Phi \in H^3$ is associated to the graph dual to such triangle and its sub-components (elements of H , which cover the role of the variables $g_i \in G$ in ordinary GFT) are the links outgoing from a single common node, and are orthogonal to the three edges.

The fields Φ and $\hat{\Phi}$ have the same graphical interpretation of those of ordinary GFT, and are represented in Fig. 3.1a. In this context, the skew co-pairing σ encodes the information on both the triangulation and the dual complex: in three dimensions, each element σ can be seen as the decoration of an edge and a dual link.

Let the field Φ be invariant under a gauge symmetry, which is enforced through a projector.

Definition 11 (Gauge projector). *Consider the (left) projector $\mathcal{P}_L : H^3 \rightarrow H^3$ whose action on the field, called gauge averaging, is*

$$(\mathcal{P}_L \Phi) = \frac{1}{V_H} \left(\int_H \otimes id^{\otimes 3} \right) ((m^3 \otimes id^{\otimes 3}) \circ \Delta^{(3)} \Phi). \quad (4.70)$$

I recall that \int_H is the left co-integral on H , while m^3 and $\Delta^{(3)}$ are the maps resp. defined in (4.22) and (4.26).

I demand the field Φ to be invariant under the gauge averaging

$$(\mathcal{P}_L \Phi) = \Phi. \quad (4.71)$$

The element $(m^3 \otimes id^{\otimes 3}) \circ \Delta^{(3)} \Phi$ in (4.70) belongs to the Hopf algebra H^4 : this can be understood as the graph dual to the triangle (represented by the field) plus an extra link. Such extra link is interpreted as a parallel transport in the dual complex, and (4.71) enforces the invariance of the field under any possible translation of this type. Such transformation inherit the name *gauge symmetry* by the direct analogy with ordinary group field theory, and has a graphical representation analogous to the one in Fig. 3.1b. Dually (upon Fourier transform), the gauge symmetry translates into the closure constraint.

Proposition 4 (Closure constraint). *The Fourier transform of the gauged projected field (4.70) gives*

$$\mathcal{F}[(\mathcal{P}_L \Phi)] = \hat{\mathcal{C}} \cdot \hat{\Phi}, \quad (4.72)$$

where the element $\hat{\mathcal{C}} \in A^3$ is called closure constraint and it is given by

$$\hat{\mathcal{C}} = \frac{\sqrt{\mu}}{V_H} \Delta^3 \hat{\delta}_A^{-1}. \quad (4.73)$$

The Fourier transformation of the gauge invariance condition (4.71) gives the identity

$$(\hat{\mathcal{C}} \cdot \hat{\Phi}) = \hat{\Phi}. \quad (4.74)$$

As an element of the tensor product algebra A^3 , the closure constraint is interpreted as the combination of three edges in the triangulation. Being given as the co-product of the delta function, these edges are naturally interpreted as part of a discrete closed path. Therefore, (4.72) implements the closure of the boundary of the (triangle) dual field $\hat{\Phi}$.

⁶For clarity, I call Φ the field and $\hat{\Phi}$ the dual field.

⁷When there is no risk of confusion I will use the symbol Σ at the place of Σ^3

Action

In analogy with ordinary group field theories, the action of Hopf algebra field theory is composed by a kinetic plus an interaction term (with coupling constant set to 1 for simplicity)

$$\mathcal{S} = \mathcal{S}_{\mathcal{K}} + \mathcal{S}_{\mathcal{V}}. \quad (4.75)$$

The interaction term is defined as the proper product of fields respecting the combinatorics of the tetrahedron.

Definition 12 (Interaction term). *The interaction term of three dimensional Hopf algebra field theory is*

$$\mathcal{S}_{\mathcal{V}} = \int_{H^6} (\underline{m}(\mathcal{P}_L \Phi \otimes \mathcal{P}_L \Phi \otimes \mathcal{P}_L \Phi \otimes \mathcal{P}_L \Phi)). \quad (4.76)$$

The map \underline{m} is given in 4.34.

Proposition 5 (Tetrahedron). *The interaction term (4.76) can be expressed as an integral operator in the two forms below*

$$\mathcal{S}_{\mathcal{V}} = \int_{H^{12}} (\mathcal{V} \cdot (\Phi \otimes \Phi \otimes \Phi \otimes \Phi)) = \int_{A^{12}} (\hat{\mathcal{V}} \cdot (\hat{\Phi} \otimes \hat{\Phi} \otimes \hat{\Phi} \otimes \hat{\Phi})), \quad (4.77)$$

where

$$\begin{aligned} \mathcal{V} &= \frac{1}{\mu^6 V_H^4} \left(\int_{H^4} \otimes id^{\otimes 12} \right) \Delta_{\mathcal{V}} \delta_H^{-1}, \\ \hat{\mathcal{V}} &= \frac{1}{\mu^6} \underline{\Delta} \hat{\delta}_A^{-1} \cdot (\hat{\mathcal{C}} \otimes \hat{\mathcal{C}} \otimes \hat{\mathcal{C}} \otimes \hat{\mathcal{C}}), \end{aligned} \quad (4.78)$$

are the kernels of the tetrahedron amplitudes, resp. in the H and A representations. The co-product $\Delta_{\mathcal{V}}$ was introduced in Def. 1, moreover the opposite delta functions δ_H^{-1} and $\hat{\delta}_A^{-1}$ in both the amplitudes (4.78) are six dimensional delta functions.

Since each dual field is associated to a triangle, by construction, the interaction term represents the composition of four triangles, with the combinatorics of a tetrahedron specified by the product (4.34). The associated amplitude is thus interpreted as the boundary of a tetrahedron in the triangulation picture, or as the bulk and boundary of its dual graph in the dual complex picture, as in ordinary GFT. It is illustrated in Fig. 3.3a, and each of the delta functions in (4.36) enforces one of the six loops of links, similar to Fig. 3.3b.

The kinetic term of Hopf algebra field theory is defined as the product of fields Φ .

Definition 13 (Kinetic term). *The kinetic term of three dimensional Hopf algebra field theory is*

$$\mathcal{S}_{\mathcal{K}} = \int_{H^3} \Phi \cdot (\tau \Phi). \quad (4.79)$$

Proposition 6 (Propagator). *The kinetic term (4.79) can be expressed as an integral operator in two different ways,*

$$\mathcal{S}_{\mathcal{K}} = \left(\int_{H^3} \otimes \int_{H^3} \right) (\mathcal{K} \cdot (\Phi \otimes \Phi)) = \left(\int_{A^3} \otimes \int_{A^3} \right) (\hat{\mathcal{K}} \cdot (\hat{\Phi} \otimes \hat{\Phi})), \quad (4.80)$$

where

$$\mathcal{K} = \frac{1}{\sqrt{\mu^3}} (S^{(3)} \otimes \tau^3) \circ \Delta^{(3)} \delta_H, \quad \hat{\mathcal{K}} = \frac{1}{\sqrt{\mu^3}} (id^{\otimes 3} \otimes \tau^3) \circ \Delta^{(3)} \hat{\delta}_A^{-1}, \quad (4.81)$$

are the kernels of the propagator amplitudes, resp. in the H and A representations.

As a product of two fields or dual fields, the propagator represents the identification of two triangles or the identification of their dual graphs, and it has the same graphical representation of the propagator amplitude of ordinary GFT, illustrated in Fig. 3.2.

In analogy with group field theory, the tetrahedron amplitude (4.78) encodes the smallest information (let me call it *building block*) of any Feynman diagram of Hopf algebra field theory and the propagator amplitude (4.81) is used to merge several tetrahedra to construct arbitrary Feynman diagrams.

The proof of propositions 5 and 6 can be found in [45].

Feynman amplitudes and partition function

Similar to standard topological quantum field theories, also Hopf algebra field theory can be formulated as a state sum model, with partition that can be expanded as the *sum* over all the possible discrete geometries weighted by the associated Feynman amplitude. The amplitude of a given discrete geometry can be expressed either as a triangulation Γ or as its dual complex Γ^* .

In particular, the Feynman amplitude of a triangulation Γ in the A representation is simply expressed as a combination of tetrahedron amplitudes (4.78) glued by the proper propagator amplitudes (4.81).

Proposition 7 (Amplitude of a triangulation). *Let Γ be the triangulation of a manifold (with no boundary) built as the combination of M tetrahedra τ . The Feynman amplitude of Hopf algebra field theory associated to Γ is*

$$\mathcal{A}_\Gamma = \frac{1}{\mu^{6M}} \int_{A^{12M}} \prod_{\tau} (\underline{\Delta}^{(3)} \hat{\delta}_A^{-1} \cdot (\hat{\mathcal{C}} \otimes \hat{\mathcal{C}} \otimes \hat{\mathcal{C}} \otimes \hat{\mathcal{C}})). \quad (4.82)$$

On the other hand, the Feynman amplitude associated to a general dual complex Γ^* , can be expressed as a combination of loops made of an arbitrary number of bulk links (part of the bulk graph of a tetrahedron). Each of such loops is the closed path spanning the face dual to an edge (shared by a number, say N , of tetrahedra) in the triangulation. In line with the usual interpretation of models of discrete geometries, each of these loops can be interpreted as probing the *local curvature* around the respective edge, and hence, the Feynman amplitude associated to a general dual complex is expressed as the sum over all such local curvatures.

Proposition 8 (Amplitude of a dual complex). *Consider N tetrahedra of a triangulation Γ sharing a single edge and let \mathcal{L}_N be the closed loop, made of $2N$ half bulk links, that spans the face dual to the edge. Let Γ^* be the complex dual to the three dimensional triangulation Γ , built as the combination of such loops \mathcal{L}_N , with some given N for each loop. The Feynman amplitude of Hopf algebra field theory associated to Γ^* is*

$$\mathcal{A}_{\Gamma^*} = \prod_{\{\mathcal{L}_N\}} \frac{1}{\mu^N V_H^N} \int_{H^{2N}} \Delta_{1-23-4\dots(2N-1)-2N}^{2N} \delta_H^{-1}. \quad (4.83)$$

A similar expansion of the Feynman amplitude can be realized in terms of the plane wave σ , the canonical element of the dual of the generalized quantum double $D^*(A, H, \sigma)$. This explicitly encodes the degrees of freedom of a given edge and the loop of links around it. As I will show in Sec. 4.3.1, this expression is associated to the discretization of a BF theory.

Proposition 9 (Amplitude of a dual complex in terms of the plane-wave). *Given the same setting of the previous proposition, the Feynman amplitude of the dual complex Γ^* can be represented in terms of the plane wave σ , as*

$$\mathcal{A}_{\Gamma^*} = \prod_{\{\mathcal{L}_N\}} \frac{1}{\sqrt{\mu^{2N+1}} V_H^N} \left(\int_{H^{2N}} \otimes \int_A \right) (\Delta_{1-23-4\dots(2N-1)-2N}^{2N} \otimes id) \sigma. \quad (4.84)$$

Therefore, the Hopf algebra partition function can be expressed as

$$\mathcal{Z} = \sum_{\Gamma} \mathcal{A}_{\Gamma} = \sum_{\Gamma^*} \mathcal{A}_{\Gamma^*}. \quad (4.85)$$

The proof of propositions 7, 8, 9 can be found in [45].

4.2.1 Topological invariance

Given two simplicial decompositions of a manifold \mathcal{M} , there exists a finite set of transformations, called Pachner moves, that map one simplicial decomposition into the other [120, 121]. If the Feynman amplitudes of a model of discrete geometries are invariant under the action of these transformations, then the model does not depend on the specific simplicial decomposition chosen and we say that it is *topological invariant*.

Proposition 10. *The Feynman amplitudes of three dimensional Hopf algebra field theory is topological invariant.*

For a three dimensional simplicial triangulation, there exist two Pachner moves, denoted $P_{(1,4)}$ and $P_{(2,3)}$. I provide below the resulting relation between the relevant amplitudes, and I refer to [45] for more details about the proofs.

Pachner move $P_{(1,4)}$. The Pachner move $P_{(1,4)}$ takes the amplitude of one tetrahedron (denoted $\mathcal{A}_{\mathcal{V}}$) into the amplitude of four tetrahedra (denoted $\mathcal{A}_{\mathcal{V}^4}$). The (reduced) amplitudes of one and four tetrahedra are

$$\begin{aligned} \mathcal{A}_{\mathcal{V}} &= \frac{1}{\mu^3} \int_{A^6} (\hat{\mathcal{C}}_{-6-4-2} \cdot \hat{\mathcal{C}}_{-56-1} \cdot \hat{\mathcal{C}}_{-345} \cdot \hat{\mathcal{C}}_{123}), \\ \mathcal{A}_{\mathcal{V}^4} &= \frac{1}{\mu^{12}} \int_{A^6} (\hat{\mathcal{C}}_{-1-3-2} \cdot \hat{\mathcal{C}}_{-51-4} \cdot \hat{\mathcal{C}}_{-642} \cdot \hat{\mathcal{C}}_{356}). \end{aligned} \quad (4.86)$$

The relation between the two amplitudes, encoded by the Pachner move $P_{(1,4)}$, is

$$(P_{(1,4)} \mathcal{A}_{\mathcal{V}}) := \mathcal{A}_{\mathcal{V}^4} = \frac{1}{\mu^9} \mathcal{A}_{\mathcal{V}}. \quad (4.87)$$

Pachner move $P_{(2,3)}$. The Pachner move $P_{(2,3)}$ takes the amplitude of two tetrahedra (denoted $\mathcal{A}_{\mathcal{V}^2}$) into the one of three (denoted $\mathcal{A}_{\mathcal{V}^3}$). The (reduced) amplitudes of two and three tetrahedra are

$$\begin{aligned} \mathcal{A}_{\mathcal{V}^2} &= \frac{1}{\mu^6} \int_{A^9} (\hat{\mathcal{C}}_{-6-4-2} \cdot \hat{\mathcal{C}}_{-56-1} \cdot \hat{\mathcal{C}}_{-345} \cdot \hat{\mathcal{C}}_{-92-7} \cdot \hat{\mathcal{C}}_{-819} \cdot \hat{\mathcal{C}}_{387}), \\ \mathcal{A}_{\mathcal{V}^3} &= \frac{1}{\mu^9} \int_{A^9} (\hat{\mathcal{C}}_{-5-3-2} \cdot \hat{\mathcal{C}}_{-45-1} \cdot \hat{\mathcal{C}}_{-82-6} \cdot \hat{\mathcal{C}}_{-718} \cdot \hat{\mathcal{C}}_{-963} \cdot \hat{\mathcal{C}}_{479}). \end{aligned} \quad (4.88)$$

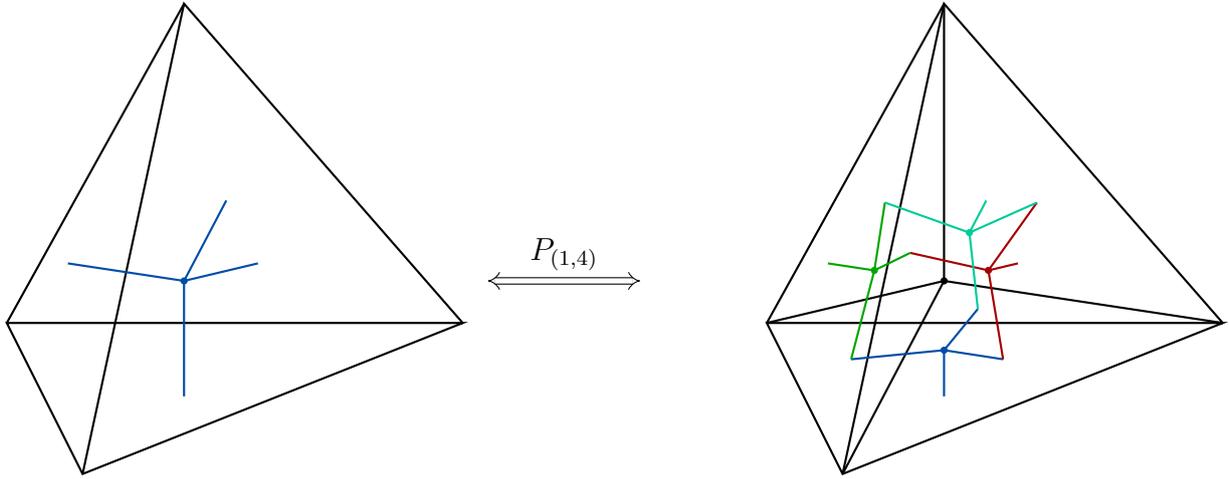


Figure 4.1: The Pachner move $P_{(1,4)}$ takes one tetrahedron in the combination of four. In the triangulation it is realized by connecting the center of the tetrahedron with its four vertices. The surfaces ranging between the center of the tetrahedron and any pair of vertices are internal triangles. The four external faces of the initial tetrahedron thus become the external faces of the four different tetrahedra that share the six internal triangles. In the dual complex, the move is realized by taking four nodes (the colored ones on the right) at the place of the central node (in blue on the left). Each node is connected to one of the four external links and to each of the other nodes.

The relation between the two amplitudes, encoded by the Pachner move $P_{(2,3)}$, is

$$(P_{(2,3)} \mathcal{A}_{\mathcal{V}^2}) := \mathcal{A}_{\mathcal{V}^3} = \frac{1}{\mu^3} \mathcal{A}_{\mathcal{V}^2}. \quad (4.89)$$

In Fig. 4.1 and 4.2 I represent the Pachner moves $P_{(1,4)}$ and $P_{(2,3)}$ in the triangulation picture together with the transformations of the respective bulk graphs. The amplitudes associated to the building blocks of any Feynman diagram of Hopf algebra field theory are invariant (up to some constant factors) under the action of the Pachner moves. Hence, the HAFT proposed in this chapter is a topological invariant model.

Note that, in the standard analysis of the three dimensional Pachner moves, the proportionality constant for the move $P_{(1,4)}$ amounts to the cube of the volume of the group (or Hopf algebra in our case) that decorates the dual complex. Whereas, the proportionality constant for the Pachner move $P_{(2,3)}$ is a single volume term. Geometrically, these terms amount to the difference between the number of (independent) internal edges from the initial and to the final amplitudes of a given move. In (4.87) and (4.89) we do not encounter the (usually) expected proportionality constants. One can retrieve them by setting the normalization factor (4.62) to the identity, $\mu = 1$ (as in the standard case), and by removing the volume term V_H in the definition of gauge projector (4.70). This would prevent the operator \mathcal{P}_L in (4.70) to be a projector, as it would satisfy the identity $\mathcal{P}_L^2 = V_H \mathcal{P}_L$ at the place of $\mathcal{P}_L^2 = \mathcal{P}_L$; but this choice would also lead to the standard proportionality constants in the moves (4.87) and (4.89),

$$(P_{(1,4)} \mathcal{A}_{\mathcal{V}}) := \mathcal{A}_{\mathcal{V}^4} = V_H^3 \mathcal{A}_{\mathcal{V}}, \quad (P_{(2,3)} \mathcal{A}_{\mathcal{V}^2}) := \mathcal{A}_{\mathcal{V}^3} = V_H \mathcal{A}_{\mathcal{V}^2}. \quad (4.90)$$

4.3 q -Deformed group field theory

In this part I consider a specific example of Hopf algebra field theory to construct the standard q -deformation of a group field theory based on the $SU_q(2)$ group, with a real deformation parameter

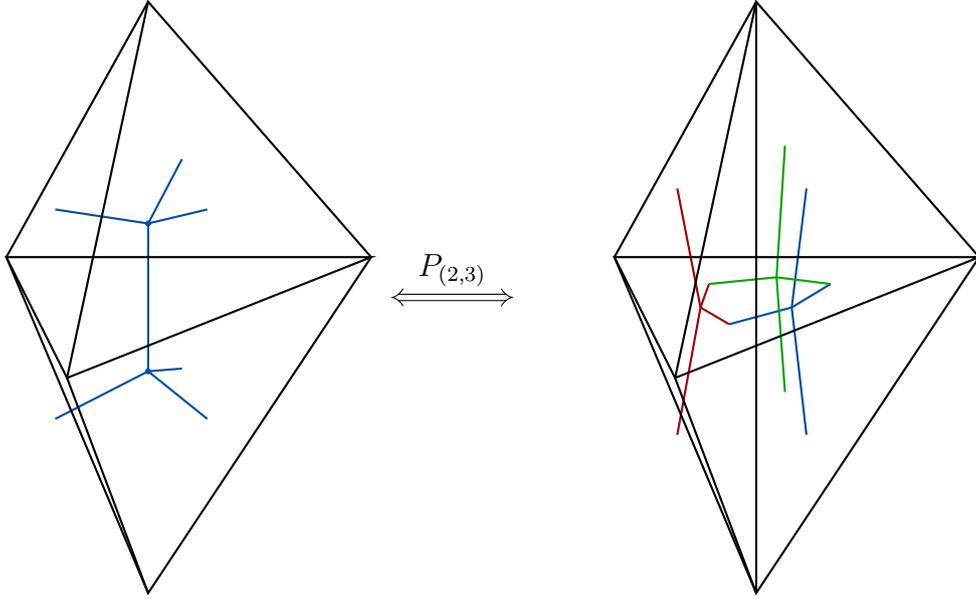


Figure 4.2: The Pachner move $P_{(2,3)}$ takes two tetrahedra in the combination of three. In the triangulation it is realized by connecting the vertices above and below through an internal edge. The surfaces ranging between the internal edge and one of the three vertices shared by the two initial tetrahedra become internal triangles. The three faces above, that initially belonged to a single tetrahedron, now belong to three different tetrahedra. The same for the faces below. In the dual complex, the move is realized by taking three nodes (the colored ones on the right) at the place of the two initial ones (in blue on the left). Each node is thus connected to each other by two of its links, plus to one external link above and one below.

q . After a quick overview of the underlying Hopf algebra ingredients, I show that this example is equivalent to the original Boulatov construction [32], and it thus provides the Turaev-Viro amplitude [46]. Moreover, I show how the plane-wave σ allows to recover the discretization of (Euclidian) BF theory in the presence of a (negative) cosmological constant.

Let the Hopf algebras H and A be the q -deformation of $F(\text{SU}(2))$ and $\mathcal{U}(\mathfrak{su}(2))$ respectively. Such standard deformations [106] are the non commutative and non co-commutative Hopf algebras denoted $\mathcal{U}_q(\mathfrak{an}(2)) \cong F(\text{SU}_q(2))$ and $\mathcal{U}_q(\mathfrak{su}(2)) \cong F(\text{AN}(2))$. I consider the a *real* deformation parameter $q = e^{\ell\lambda}$, where ℓ has the dimension of a length and is the characteristic scale of $\text{SU}(2)$, while λ has the dimension of an inverse length and is the characteristic scale of $\text{AN}(2)$.

Let H, X_{\pm} be the generators of $\mathcal{U}_q(\mathfrak{su}(2))$ with the dimension of a length and obeying Hopf algebra structure

$$\begin{aligned}
\text{Product: } & [H, X_{\pm}] = \pm 2\ell X_{\pm} & \rightarrow & e^{\lambda H} X_{\pm} = q^{\pm 2} X_{\pm} e^{\lambda H}, \\
& [X_+, X_-] = \ell^2 q^{-1} \frac{\sinh(\lambda H)}{\sinh(\ell\lambda)}, \\
\text{Co-product: } & \Delta H = H \otimes 1 + 1 \otimes H \\
& \Delta X_+ = X_+ \otimes 1 + e^{-\lambda H} \otimes X_+ \\
& \Delta X_- = X_- \otimes e^{\lambda H} + 1 \otimes X_-, \\
\text{Co-unit: } & \varepsilon(H) = \varepsilon(X_{\pm}) = 0, \\
\text{Antipode: } & S(H) = -H, \\
& S(X_+) = -e^{\lambda H} X_+, \\
& S(X_-) = -X_- e^{-\lambda H}.
\end{aligned} \tag{4.91}$$

Let ϕ, φ_{\pm} be the generators of $F(SU_q(2))$ with the dimension of an inverse length and with Hopf algebra structure specified by

$$\begin{aligned}
\text{Product: } & [\phi, \varphi_{\pm}] = -i\lambda\varphi_{\pm} \quad \rightarrow \quad e^{i\ell\phi}\varphi_{\pm} = q\varphi_{\pm}e^{i\ell\phi}, \\
& [\varphi_+, \varphi_-] = 0, \\
\text{Co-product: } & \Delta\phi = \frac{i}{\ell} \log \left(\frac{1}{\Delta\varphi_0} (\varphi_0 e^{-i\ell\phi} \otimes \varphi_0 e^{-i\ell\phi} - \ell^2 \varphi_- \otimes \varphi_+) \right), \\
& \Delta\varphi_+ = \varphi_+ \otimes \varphi_0 e^{-i\ell\phi} + e^{i\ell\phi} \varphi_0 \otimes \varphi_+, \\
& \Delta\varphi_- = \varphi_- \otimes e^{i\ell\phi} \varphi_0 + \varphi_0 e^{-i\ell\phi} \otimes \varphi_-, \\
\text{Co-unit: } & \varepsilon(\phi) = \varepsilon(\varphi_{\pm}) = 0, \\
\text{Antipode: } & S(\phi) = -\phi, \\
& S(\varphi_{\pm}) = -q^{\mp} \varphi_{\pm}.
\end{aligned} \tag{4.92}$$

with $\varphi_0 = \sqrt{1 - q^{-1}\ell^2\varphi_-\varphi_+}$. I use the PBW basis [117] for the two Hopf algebras, where the elements are expressed as a linear combination of monomials in the generators:

$$\{X_+^b H^a X_-^c\}_{abc=0}^{\infty} \in \mathcal{U}_q(\mathfrak{su}(2)), \quad \{\varphi_-^b \phi^a \varphi_+^c\}_{abc=0}^{\infty} \in \mathcal{U}_q(\mathfrak{an}(2)). \tag{4.93}$$

Note that the $\mathcal{U}_q(\mathfrak{su}(2))$ basis above is obtained by the standard symmetric deformation (with generators $\{J_{\pm}, J_3\}$) presented in [106], with the re-scaling

$$J_{\pm} \rightarrow e^{\mp\frac{1}{2}\lambda H} X_{\pm}, \quad J_3 \rightarrow H. \tag{4.94}$$

The $F(SU_q(2))$ basis given above is instead obtained by the following change of coordinates on the standard $SU_q(2)$ matrix element [106]

$$g = \begin{pmatrix} a & b \\ -qb^* & a^* \end{pmatrix} \rightarrow g = \begin{pmatrix} \varphi_0 e^{-i\ell\phi} & i\ell\varphi_- \\ i\ell\varphi_+ & e^{i\ell\phi}\varphi_0 \end{pmatrix} \Rightarrow \begin{cases} a = \varphi_0 e^{-i\ell\phi}, \\ b = i\ell\varphi_-. \end{cases} \tag{4.95}$$

The Haar measures on $F(SU_q(2))$ and $F(AN_q(2))$ are resp. given by the standard q -deformation of the Haar measure on $SU(2)$ [106] and by the q -deformation of the Haar measure on $AN(2)$, in the coordinate basis used above.

Let me give here the relevant ingredients to construct the generalized quantum double and its dual, and therefore the plane wave. The standard q -deformation of group field theory is given by the choices $H = F_q(SU(2)) \cong \mathcal{U}_q(\mathfrak{an}(2))$ and $A = \mathcal{U}_q(\mathfrak{su}(2)) \cong F_q(AN(2))$.

Proposition 11 (q -Deformed group field theory: generalized quantum double). *The generalized quantum double (Def. 4) associated to the q -deformation of group field theory is the Drinfeld double of $\mathcal{U}_q(\mathfrak{su}(2))$:*

$$\mathcal{D}(\mathcal{U}_q(\mathfrak{su}(2)), \mathcal{U}_q(\mathfrak{an}(2)), \sigma) = \mathcal{D}(\mathcal{U}_q(\mathfrak{su}(2)) \cong \mathcal{U}_q(\mathfrak{su}(2)) \bowtie_{\sigma} \mathcal{U}_q(\mathfrak{an}(2)), \tag{4.96}$$

with (or canonical) skew pairing

$$\sigma(X_+^j H^i X_-^k, \varphi_-^b \phi^a \varphi_+^c) = i^{a+b+c} \delta_{ai} \delta_{bj} \delta_{ck} a! [b]_{q^2}! [c]_{q^{-2}}!, \tag{4.97}$$

where

$$[n]_f = \sum_{i=0}^{n-1} f^i = \frac{1-f^{n+1}}{1-f}, \quad [n]_f! = \prod_{i=1}^n [i]_f, \tag{4.98}$$

are the q -number and the q -factorial. In [45] also the mutual actions (4.42) are given.

Proposition 12 (*q*-Deformed group field theory: dual of the generalized quantum double). *The dual of generalized quantum double (Def. 7) associated to the q-deformation of group field theory is*

$$\mathcal{D}(\mathcal{U}_q(\mathfrak{an}(2)), \mathcal{U}_q(\mathfrak{su}(2)), \sigma) = \mathcal{U}_q(\mathfrak{an}(2)) \blacktriangleright_{\sigma} \mathcal{U}_q(\mathfrak{su}(2)), \quad (4.99)$$

with skew co-pairing element given by the *q*-star exponential

$$\sigma = e_{\star q^2}^{i\varphi_+ \otimes X_-} e_{\star}^{i\phi \otimes H} e_{\star q^{-2}}^{i\varphi_- \otimes X_+}, \quad (4.100)$$

where the *q*-exponential is defined as

$$e_q^x = \sum_{n=0}^{\infty} \frac{x^n}{[n]_q!} = \exp \left(\sum_{n=1}^{\infty} \frac{x^n (1-q)^n}{n (1-q^n)} \right). \quad (4.101)$$

In [45] also the mutual co-actions (4.52) are given.

Note that the plane wave (4.97) is similar to the exponential map obtained in [122].

Closure constraint and Feynman amplitude

As a concrete example, I derive here the closure constraint (4.73) and the Feynman amplitude associated to a dual complex Γ^* (4.84) of Hopf algebra field theory, using the generalized quantum double of $\mathcal{U}_q(\mathfrak{su}(2))$ above.

The closure constraint (4.73) is given by the delta function on $\mathcal{U}_q(\mathfrak{su}(2))$, which enforces the following co-products to vanish

$$\begin{aligned} H \otimes 1 \otimes 1 + 1 \otimes H \otimes 1 + 1 \otimes 1 \otimes H &= 0, \\ X_+ \otimes 1 \otimes 1 + e^{-\lambda H} \otimes X_+ \otimes 1 + e^{-\lambda H} \otimes e^{-\lambda H} \otimes X_+ &= 0, \\ X_- \otimes e^{\lambda H} \otimes e^{\lambda H} + 1 \otimes X_- \otimes e^{\lambda H} + 1 \otimes 1 \otimes X_- &= 0. \end{aligned} \quad (4.102)$$

Such closure condition can be re-packaged as a product of three $AN_q(2)$ group elements [105]. The Feynman amplitude (4.84) is

$$\mathcal{A}_{\Gamma^*} = \prod_{\{\mathcal{L}_N\}} \frac{1}{V_{\text{SU}_q(2)}^N} \int [d\phi d\varphi_{\pm}]^N [dH dX_{\pm}] e_{\star q^2}^{i\Delta^{2N} \varphi_+ \otimes X_-} e_{\star}^{i\Delta^N \phi \otimes H} e_{\star q^{-2}}^{i\Delta^N \varphi_- \otimes X_+}, \quad (4.103)$$

where I noted $V_{\text{SU}_q(2)}$ the volume of $F(\text{SU}_q(2))$, and for simplicity, I assumed $\mu = 1$.

Relation with Turaev-Viro model (with *q* real)

In this part I will show how one can relate the amplitude (4.103) to the Turaev-Viro invariant [46]. To this scope, it is enough to explain how to recover the original Boulatov model [32]. In his work, Boulatov defines a field theory based on the representations of a given group G , where the fundamental field is expanded in the Fourier decomposition

$$\begin{aligned} \Phi(x, y, z) &= \sum_{i_1, j_2, j_3} \sum_{\{m, n, k\}} \Phi_{j_1 j_2 j_3}^{m_1 m_2 m_3; k_1 k_2 k_3} D_{m_1, n_1}^{j_1}(x) D_{m_2, n_2}^{j_2}(y) D_{m_3, n_3}^{j_3}(z) \\ &\int d\omega D_{n_1, k_1}^{j_1}(\omega) D_{n_2, k_2}^{j_2}(\omega) D_{n_3, k_3}^{j_3}(\omega), \end{aligned} \quad (4.104)$$

where $x, y, z, \omega \in G$ and $D_{m, n}^j(x)$ are matrix elements obeying the orthogonality condition

$$\int dx D_{m_1, n_1}^{j_1}(x) D_{m_2, n_2}^{j_2}(x) = \frac{1}{d_j} \delta_{j_1, j_2} \delta_{m_1, m_2} \delta_{n_1, n_2}, \quad (4.105)$$

with d_j being the dimension of the irreducible representation associated to j . Taking $G = \text{SU}(2)$, the matrix elements $D_{m,n}^j(x)$ are the standard Wigner D -matrices, whereas – as Boulatov explains – for $G = \text{SU}_q(2)$ the matrix elements $D_{m,n}^j(x)$ become the q -deformed Wigner matrices [106]. In Hopf algebra field theory, the fundamental field ϕ is an element of the tensor product Hopf algebra H^3 . I defined the q -deformed group field theory as the Hopf algebra field theory with the specific choice $H = F(\text{SU}_q(2))$, for which we used the parametrization in terms of the coordinates $\{\varphi_+, \phi, \varphi_-\}$ of the $\text{SU}_q(2)$ group element (4.95). Each field Φ in Hopf algebra field theory is thus given by the tensor product of three copies of the linear combination of monomials $\varphi_+^j \phi^i \varphi_-^k$. Therefore, to make contact between Hopf algebra field theory and the original Boulatov model, I define the Fourier expansion for the fields $\Phi \in F(\text{SU}_q(2))^{\times 3}$

$$\begin{aligned} \Phi &= \sum_{i_1, j_2, j_3} \sum_{\{m, n, k\}} \Phi_{j_1 j_2 j_3}^{m_1 m_2 m_3; k_1 k_2 k_3} D_{m_1, n_1}^{j_1}(\{\varphi_{\pm}, \phi\}_1) D_{m_2, n_2}^{j_2}(\{\varphi_{\pm}, \phi\}_2) D_{m_3, n_3}^{j_3}(\{\varphi_{\pm}, \phi\}_3) \\ &\int d\varphi'_+ d\phi' d\varphi'_- D_{n_1, k_1}^{j_1}(\{\varphi'_{\pm}, \phi'\}) D_{n_2, k_2}^{j_2}(\{\varphi'_{\pm}, \phi'\}) D_{n_3, k_3}^{j_3}(\{\varphi'_{\pm}, \phi'\}), \end{aligned} \quad (4.106)$$

where I used the symbol $\{\varphi_{\pm}, \phi\}_i$ for the monomial in the coordinates $\{\varphi_+, \phi, \varphi_-\}$ in the i^{th} tensor space. Once again the $D_{m,n}^j$ are matrix elements obeying the same orthogonality condition. Note that the above Fourier expansion holds for any Hopf algebra of functions on a Lie group G or its deformation $F(G_q)$. However, for what concern us, it is enough to focus on the case of $H = F(\text{SU}_q(2))$. In this case, the partition function of the Boulatov model, and thus the one of Hopf algebra field theory (4.82) as well, is given in terms of the q -deformed 6- j symbols and allows to recover the Turaev-Viro invariant [46] with q real.

4.3.1 Plane-wave and discretization of the BF action

The amplitudes of Hopf algebra field theory, expressed in terms of the plane-wave. Since Hopf algebra field theory is a topological model, one would expect that it should be related to the BF -action. More concretely, I am going to show that the Feynman amplitude (4.103) provides a regularization or discretization of the BF amplitude $e^{i\mathcal{S}_{BF}^{\Lambda}}$ where $\mathcal{S}_{BF}^{\Lambda}$ is the BF action with cosmological constant.

BF action

Let us focus on the Euclidian case with a negative or null cosmological constant $\Lambda \leq 0$, which corresponds to the the deformation with a real q deformation parameter [123]. The fundamental fields are the frame field e and the connection A , which are respectively 1-forms with values in the boosts K and in the Lie algebra $\mathfrak{su}(2)$. Noting \langle, \rangle the Killing form of $\mathfrak{sl}(2, C)$, the BF model is governed by the action

$$\mathcal{S}_{BF}^{\Lambda}[A, e] = \int_{\mathcal{M}} \langle e, \left(F + \frac{\Lambda}{6} (e \times e) \right) \rangle. \quad (4.107)$$

The frame field variables are difficult to quantize since they are valued in the boosts. It was shown in [105] that it is convenient to do a canonical transformation

$$\omega_I = A_I + \epsilon_{IJK} n^J e_K, \quad n^I = (0, 0, \lambda), \quad n^2 = \lambda^2 = -\Lambda, \quad (4.108)$$

which makes the frame field discretizable with values in the Lie algebra $\mathfrak{an}(2)$. With this new connection the action becomes, up to a boundary term,

$$\mathcal{S}_{BF}^{\Lambda}[\omega, e] = \int_{\mathcal{M}} (e^I \wedge F_I + \lambda e_3 \wedge (\omega_+ \wedge e_- + \omega_- \wedge e_+)). \quad (4.109)$$

Let me emphasize that this action depends on two constants: the Newton's constant G (related to the length ℓ) and the cosmological constant (related to λ). As mentioned earlier, the constant ℓ typically encodes curvature of $SU(2)$, while λ encodes the curvature of the group $AN(2)$. Hence, when looking at the action (4.109), one can take two limits, the first is $\lambda \rightarrow 0$ which leads to the ordinary BF theory with zero cosmological constant, and second one, less common, is $\ell \rightarrow 0$, which can be related to a model with zero gravity but a non-commutative space-time [116, 124]. In [105] it was pointed that the underlying symmetry structure of the action (4.109) is the classical Drinfeld double $\mathfrak{sl}(2, C) \cong \mathfrak{su}(2) \bowtie \mathfrak{an}(2)$, which in the two limits above become

$$\mathfrak{su}(2) \bowtie \mathfrak{an}(2) \rightarrow \begin{cases} \mathfrak{su}(2) \times \mathbb{R}^3 & \text{for } \lambda = 0, \\ \mathbb{R}^3 \times \mathfrak{an}(2) & \text{for } \ell = 0. \end{cases} \quad (4.110)$$

In order to highlight the different symmetries, one can re-write the action (4.109) to

$$\mathcal{S}_{BF}^\lambda[\omega, e] = \int_{\mathcal{M}} \left(\langle e, d\omega \rangle + \frac{1}{2} \langle e, [\omega, \omega]_{\mathfrak{su}(2)} \rangle + \frac{1}{2} \langle [e, e]_{\mathfrak{an}(2)}, \omega \rangle \right), \quad (4.111)$$

which thus reduces as

$$\mathcal{S}_{BF}^\lambda[\omega, e] \rightarrow \begin{cases} \mathcal{S}_{BF}^{\mathfrak{su}(2)} = \int_{\mathcal{M}} \langle e, d\omega + \frac{1}{2} [\omega, \omega]_{\mathfrak{su}(2)} \rangle, & \text{for } \lambda = 0, \\ \mathcal{S}_{BF}^{\mathfrak{an}(2)} = \int_{\mathcal{M}} \langle de + \frac{1}{2} [e, e]_{\mathfrak{an}(2)}, \omega \rangle, & \text{for } \ell = 0, \end{cases} \quad (4.112)$$

where a boundary term was omitted in the second expression.

Deiscretization of the BF amplitude

Let me recall the main aspects of the standard discretization procedure of a BF theory with no cosmological constant based on a general symmetry group G , with action $\mathcal{S}_{BF} = \int_{\mathcal{M}} \langle B \wedge F[A] \rangle$. The field B is a \mathfrak{g}^* valued one form and A is a \mathfrak{g} -valued connection. The $3d$ manifold \mathcal{M} is discretized on a cellular decomposition Γ , noting Γ^* its dual complex. According to the Poincaré duality, the fields B and F are discretized on the dual structures. The standard choice is to discretize B on the triangulation Γ , and the curvature $F[A]$ along an holonomy $g_e \in G$ forming a closed loop in Γ^* . In this construction, one can typically attribute to each link l_i an holonomy $g_i \in G$, such that $g_e = \prod_i^n g_i$. Upon quantization, such product is actually characterized by the co-product

$$g_e = \prod_i^n g_i \rightarrow \Delta^n g, \quad (4.113)$$

where the discrete variables of each tensor space of the co-product Δ^n are associated to one of the links of the loop g_e that defines the boundary of a surface dual to an edge. The B field is discretized on such edge as a d dimensional vector $\mathbf{x}_e \in \mathbb{R}_*^d$, where d is the dimension of G . The natural discretization of the three dimensional BF action is thus

$$e^{i\mathcal{S}_{BF}[A,B]} \approx e^{\langle g_e, \mathbf{x} \rangle} \rightarrow e^{\Delta^n g \otimes \mathbf{x}_e} = (\Delta^n \otimes \text{id}) e^{g \otimes \mathbf{x}_e}, \quad (4.114)$$

where $\Delta^n g \otimes \mathbf{x}_e$ is the quantization of $\langle g_e, \mathbf{x} \rangle$. The same discretization procedure can be realized using the coordinates on the groups G and \mathbb{R}_*^d , $p^a(g_e) \in F(G)$ and $x_a(\mathbf{x}_e) \in F(\mathbb{R}_*^d)$:

$$e^{i\mathcal{S}_{BF}[A,B]} \approx e^{\langle g_e, \mathbf{x} \rangle} = e^{ip^a \cdot x_a} \rightarrow e^{i\Delta^n p^a \otimes x_a} = (\Delta^n \otimes \text{id}) e^{ip^a \otimes x_a}. \quad (4.115)$$

This time $\Delta^n p^a \otimes x_a$ is the quantization of $p^a \cdot x_a$. As alluded earlier, I would like to identify the plane wave σ as a generalized notion of the exponential $e^{ip^a \otimes x_a}$, associated to a single link of Γ^* and to a single edge of Γ . Taking the plane wave (4.100) as an example, the main difference is that this is expressed as a product of three (\star - q) exponentials, while $e^{ip^a \otimes x_a}$ is a single exponential. However, one can always use the BCH formula and a change of coordinates to re-package any plane wave, expressed in an arbitrary parametrization, as a single exponential. We do not expect that the discretization of a BF amplitude is affected by such choice. Note that a similar argument for the parametrization of the Lorentz group was already raised in [94]. Therefore, since the Hopf algebras associated to $SU(2)$ and $AN(2)$ can be derived as the limiting cases $\lambda = 0$ or $\ell = 0$ from the q deformed case (as it is shown in [45]), and since the BF actions (4.112) can be obtained from the BF action (4.109) in the same limiting cases, their associated amplitudes are discretized on the q -deformed plane wave (4.100) in such limits:

$$\begin{aligned} e^{i\mathcal{S}_{BF}^{\text{su}(2)}} &\rightarrow e^{i\mathcal{S}_d^{\text{su}(2)}} = (\Delta^n \otimes \text{id}) \left(e_{\star q^2}^{i\varphi_+ \otimes X_-} e_{\star}^{i\phi \otimes H} e_{\star q^{-2}}^{i\varphi_- \otimes X_+} \right) \Big|_{\lambda=0}, \\ e^{i\mathcal{S}_{BF}^{\text{an}(2)}} &\rightarrow e^{i\mathcal{S}_d^{\text{an}(2)}} = (\Delta^n \otimes \text{id}) \left(e_{\star q^2}^{i\varphi_+ \otimes X_-} e_{\star}^{i\phi \otimes H} e_{\star q^{-2}}^{i\varphi_- \otimes X_+} \right) \Big|_{\ell=0}. \end{aligned} \quad (4.116)$$

I recall that for $\ell \neq 0$ and $\lambda \neq 0$, the underlying symmetry structure of the BF model governed by the action (4.109) is the double $\mathfrak{su}(2) \bowtie \mathfrak{an}(2)$. Hence, the plane wave (4.100), which is the canonical element of the generalized quantum double $D^*(\mathcal{U}_q(\mathfrak{su}(2)), \mathcal{U}_q(\mathfrak{an}(2)), \sigma)$, provides a discretization for the amplitude of a BF model with negative cosmological constant governed by the action (4.109):

$$e^{i\mathcal{S}_{BF}^{\lambda}[\omega, e]} \rightarrow e^{i\mathcal{S}_d^{\lambda}} = (\Delta^n \otimes \text{id}) \left(e_{\star q^2}^{i\varphi_+ \otimes X_-} e_{\star}^{i\phi \otimes H} e_{\star q^{-2}}^{i\varphi_- \otimes X_+} \right). \quad (4.117)$$

Discretization of the BF action

The formula above provides the discretization of a BF amplitude. In order to make contact between the classical fields and the discrete variables, one would need the explicit discretization of the BF action as well. To this scope, one would need to use the BCH formula for the q -deformed plane wave (4.100) and then a (simultaneous) change of coordinate that involves the generators of both the Hopf algebras $\mathcal{U}_q(\mathfrak{su}(2))$ and $\mathcal{U}_q(\mathfrak{an}(2))$. As the complexity of the BCH formula prevents us to derive the full expression, we truncate the formula at the first order in ℓ , λ and $\ell\lambda$

$$\begin{aligned} \sigma &= e_{\star q^2}^{i\varphi_+ \otimes X_-} e_{\star}^{i\phi \otimes H} e_{\star q^{-2}}^{i\varphi_- \otimes X_+} \approx e_{\star}^{i\varphi_+ \otimes X_-} e_{\star}^{i\phi \otimes H} e_{\star}^{i\varphi_- \otimes X_+} \\ &\approx e_{\star}^{i\varphi_+ \otimes X_- + i\phi \otimes H + i\varphi_- \otimes X_+ + \frac{1}{2}[\phi \otimes H, \varphi_+ \otimes X_-] - \frac{1}{2}[\phi \otimes H, \varphi_- \otimes X_+] + \frac{1}{2}[\varphi_- \otimes X_+, \varphi_+ \otimes X_-]} \\ &\approx e_{\star}^{i\varphi_+ (1+i\ell\phi) \otimes X_- + i(1+i\ell\phi)\varphi_- \otimes X_+ + i(\phi - \frac{1}{2}\ell\varphi_+ \varphi_-) \otimes H + \frac{i\lambda}{2}(\varphi_- \otimes H X_+ - \varphi_+ \otimes X_- H)}. \end{aligned} \quad (4.118)$$

The discretization of the BF action $\mathcal{S}_{BF}^{\lambda}[\omega, e] \rightarrow \mathcal{S}_d^{\lambda}$ is then realized by taking n^{th} the co-product of such expression. According to the discussion above, for $\ell = 0$ or $\lambda = 0$, one obtains the discretization of the actions (4.112) associated to a BF model based on the Lie groups $AN(2)$ or $SU(2)$ respectively

$$\begin{aligned} \mathcal{S}_{BF}^{\text{su}(2)}[\omega, e] &\rightarrow \mathcal{S}_d^{\lambda=0} = (\Delta^n \otimes \text{id}) \left(\varphi_+ (1 + i\ell\phi) \otimes X_- + (1 + i\ell\phi)\varphi_- \otimes X_+ + (\phi - \frac{i}{2}\ell\varphi_+ \varphi_-) \otimes H \right) + \mathcal{O}(\ell^2). \\ \mathcal{S}_{BF}^{\text{an}(2)}[\omega, e] &\rightarrow \mathcal{S}_d^{\ell=0} = (\Delta^n \otimes \text{id}) \left(\varphi_+ \otimes X_- (1 - \frac{\lambda}{2}H) + \varphi_- \otimes (1 + \frac{\lambda}{2}H)X_+ + \phi \otimes H \right) + \mathcal{O}(\lambda^2). \end{aligned} \quad (4.119)$$

In the general case, for $\ell \neq 0$ and $\lambda \neq 0$, since we discarded all the terms of the order $\ell\lambda$, the deformation parameter q is automatically set to 1 and thus, according to (4.92), the $F(SU_q(2))$

co-product reduces to the $F(\text{SU}(2))$ co-product. Therefore, noting that the first three terms of (4.118) coincide with the discrete action $\mathcal{S}_d^{\lambda=0}$, the BF action (4.109) with negative cosmological constant is discretized, with the precision fixed at the first order in ℓ and λ , as

$$\begin{aligned}\mathcal{S}_{BF}^\lambda[\omega, e] &= \mathcal{S}_{BF}^{\lambda=0}[\omega, e] + \lambda (\omega_- \wedge e_3 \wedge e_+ - \omega_+ \wedge e_- \wedge e_3) \\ &\rightarrow \mathcal{S}_d^\lambda = \mathcal{S}_d^{\lambda=0} + \frac{1}{2}\lambda (\Delta^n \otimes \text{id}) (\varphi_- \otimes HX_+ - \varphi_+ \otimes X_-H).\end{aligned}\quad (4.120)$$

4.4 Remarks and perspectives

In this chapter I discussed a model based on Hopf algebras that provides the description of three dimensional discrete geometries. The formulation of the model is inspired by the notion group field theories in Ch. 3, where the fundamental degrees of freedom are encoded in a gauge invariant field and the partition function is expressed as a sum over the amplitudes associated to cellular decomposition of a three dimensional manifold.

The field theory is discussed both in the configuration and momentum spaces, and a map (generalized Fourier transform) between the two is provided. The object used as the kernel of such transformation is a generalized notion of plane-wave, used as the central object to construct the amplitudes of the model.

The main scope of this work was to provide a field theory generating curved discrete geometries as Feynman diagrams, in such a way that the amplitudes would provide a discretization of the BF amplitude with non-vanishing cosmological constant, which is interpreted as the insertion of an homogeneous curvature of the three dimensional manifold.

Let me now mention several directions that I would find interesting to explore.

Relation with integrable systems. The canonical element associated to the dual Hopf algebras also appeared in the context of integrable systems under the shape of the transfer/-transport matrix, or T-matrix [122, 125, 126]. In this context, one considers a lattice whose sites are associated with a T-matrix σ . It can be written as

$$\sigma = e^{x \cdot k}, \quad (4.121)$$

where $L = x \cdot k$ originates from a Lax pair and the x^i are the dynamical variables, which can be seen as generating a Lie algebra \mathfrak{g}_x , while the k_i are the generators of the Lie algebra \mathfrak{g}_k . One can multiply the T-matrices sitting at different sites, with dynamical variables x and x' to obtain a new one $x'' = x \oplus x'$. This is precisely the structure that one would expect from a plane-wave. The dynamical variables are equipped with a Poisson bracket which induces a Poisson structure for the T-matrix. Such Poisson brackets are expected to be compatible with the product of the T-matrices. The main features of the T-matrix are thus captured by the expected plane-wave properties

$$\begin{aligned}\sigma_{x, k_1} \sigma_{x, k_2} &= e^{ix \cdot k_1} e^{ix \cdot k_2} = e^{ix \cdot (k_1 \oplus k_2)} \equiv \sigma_{x, (k_1 \oplus k_2)}, \\ \sigma_{x_1, k} \sigma_{x_2, k} &= e^{ix_1 \cdot k} e^{ix_2 \cdot k} = e^{i(x_1 \oplus x_2) \cdot k} \equiv \sigma_{(x_1 \oplus x_2) \cdot k}.\end{aligned}\quad (4.122)$$

Note that, at the quantum level, by performing some projection in terms of representations, we can recover from the T-matrix the notion of R -matrix [106] which also play a fundamental role in the study of integrable systems. It would be interesting to explore whether the fact that this common structure appears both for integrable systems and $3d$ gravity could clarify some interesting questions, such as holography for example.

Generalization to quasi Hopf algebras. Hopf algebra field theory relies on associative and co-associative Hopf algebras. The original TV model is defined for quasi Hopf algebras – in which the associativity condition is released – obtained by a deformation of $SU(2)$ with q being the root of unity. These class of Hopf algebras are particularly relevant in physics as they provide a natural regularization in the UV. It would be interesting to explore how the framework proposed in this chapter extends to this case.

Introducing matter. To include matter degrees of freedom (as topological defects) in a $3d$ group field theory, Krasnov pointed that one can use the Drinfeld double of a group G , $D \cong G \times \mathfrak{g}^*$ as the gauge symmetry group of the model [127]. Hopf algebra field theory encodes this case as an example, and sets the ground to describe its q -deformation:

- $H = F(SU(2))$ and $A = \mathcal{U}(\mathfrak{su}(2))$: standard group field theory, with no matter degrees of freedom and $\lambda = 0$;
- $H = F(SU_q(2))$ and $A = \mathcal{U}_q(\mathfrak{su}(2))$: q -deformed group field theory, with no matter degrees of freedom and $\lambda \neq 0$;
- $H = D^*(\mathcal{U}(\mathfrak{su}(2)))$ and $A = D(\mathcal{U}(\mathfrak{su}(2)))$: Krasnov group field theory, with matter degrees of freedom and $\lambda = 0$;
- $H = D^*(\mathcal{U}_q(\mathfrak{su}(2)))$ and $A = D(\mathcal{U}_q(\mathfrak{su}(2)))$: q -deformed Krasnov group field theory, with matter degrees of freedom and $\lambda \neq 0$.

Moreover, as I will discuss in details in the next chapter, in some cases, the Drinfeld double is a semi direct product of groups and thus can be re-phrased in an higher categorical language. I refer to the next chapter for more details, and for now I would like to emphasize that, under this perspective, Krasnov’s model [127] is likely to be an example of a three dimensional group field theory based on 2-groups.

Kitaev model. In [128] it was shown that the Kitaev models are equivalent to the combinatorial quantization of a Chern-Simons theory [108, 109, 129]. The latter is a topological theory of three dimensional lattices whose underlying symmetry is given by quantum groups. The Kitaev model [130] is a topological model that describes two dimensional quadrangulations of a manifold with topological defects [131]. The quadrangulation is normally decorated by elements of a finite group, and was later extended to the case of finite dimensional semi-simple Hopf algebras [132]. The model was then rephrased in a language very similar to Hopf algebra field theory in [128, 133], as a gauge theory based on finite dimensional semi-simple Hopf algebras. In [128] it is also pointed that the Turaev-Viro amplitude [46] is associated to the protected space of the Kitaev model, in the sense that the ground state of the Kitaev model can be seen as the Hamiltonian analogue of the TV model. The fact that also Hopf algebra field theory was shown to be related to the TV model (with q real) suggests a relation between Hopf algebra field theory and the Kitaev model, and I would find interesting to explore this analogy in detail.

Four dimensional model: quantum 2-groups. Despite the HAFT can be easily generalized to the d dimensional case, I discussed the three dimensional version of the model for two reasons. First, in $3d$ it is easier to visualize the underlying geometric objects. Second, in the four dimensional case, instead of using Hopf algebras one should have used a categorified version of Hopf algebras. I will extensively discuss this point in the next chapter, to which I refer for this outlook.

Chapter 5

2-Groups in quantum gravity

Despite the success of (quantum) groups as a tool for three dimensional quantum gravity, it has been argued that they do not encode the correct amount of degrees of freedom to describe the quantum states of four dimensional geometries. I would like to mention two arguments in this regard:

- i. the need of *edge degrees of freedom*
 - * state sum models of four dimensional discrete geometries with no edge decorations contain degenerate discrete geometries [110];
 - * according to the Eckmann-Hilton argument [134], edge degrees of freedom are needed to describe **curved** discrete geometries characterized by non-abelian face decorations;
- ii. the use of *1-holonomies* and *2-holonomies* [135] is necessary to encode all the topological degrees of freedom of a four dimensional geometry.

Let me explain in more details these arguments.

Edges degrees of freedom. In most of the approaches to four dimensional quantum gravity – especially the quantization procedures associated to LQG, spin foam models or group field theories – there is no edge degrees of freedom. From a discrete geometry perspective indeed, the standard approach is to use group elements as the holonomies attached to the *links* ($1d$) of the graph dual to a simplicial decomposition. These are needed as they encode the information about the parallel transport on the manifold. Due to the Poincaré duality, the conjugate momentum of such holonomies is encoded in the dual *face* ($2d$) of the triangulation. The one dimensional information of the triangulation in this picture is clearly lacking.

More explicitly, when formulating gravity as a constrained four dimensional BF model, the fundamental variables are the connection A , a one-form, and the B -field, a two-form. The latter is related to the frame field e through the *simplicity constraint*, which re-writes the B -field as a (dual) bi-vector $B = *e \wedge e$. Nevertheless, from the Hamiltonian analysis, the frame field does not appear as a phase space variable, and the bi-vector remains the fundamental degree of freedom both in the classical and in the quantum theory. Despite some attempts to formulate a quantum gravity model based on the quantization of the frame field [136, 137], the lack of edge degrees of freedom was never considered as a real issue, and most of the times it has been disregarded. Hence, one can reasonably wonder why the lack of edge degrees of freedom should be a major issue and why it is a fundamental argument to believe that (quantum) groups do not encode the necessary information to describe quantum gravity in four dimensions.

According to [110], a state sum model based on just face or bi-vectors decorations contains

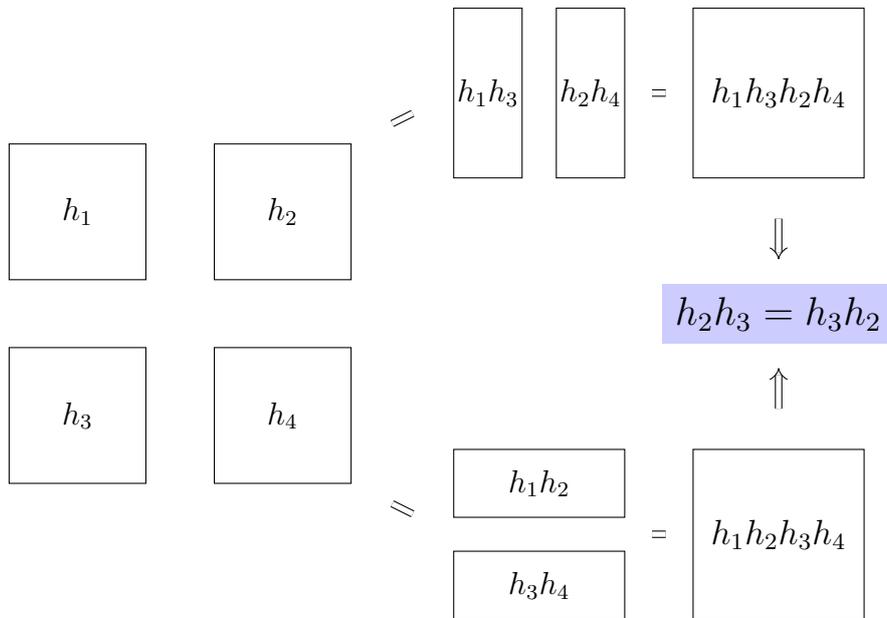


Figure 5.1: The four squares are decorated by the elements $h_i \in H$, for $i = 1, 2, 3, 4$. We merge them in two different ways and demand that the decoration of the resulting surface is the same in the two cases. This implies $h_2h_3 = h_3h_2$, and thus the group H has to be abelian.

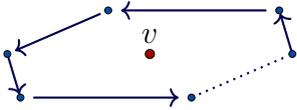
degenerate geometries. One can construct the Hilbert space of a four dimensional geometry using the algebraic data of an abelian group (or a commutative quantum group) to label the surfaces of such discrete geometry. In this way, there is no information about the boundaries of these surfaces. It turns out that the quantum states associated to such discrete geometries are degenerated: different geometries will be associated to the same quantum state. In [110] it is argued that, in order to fix this issue, and thus to properly distinguish different geometries, one needs to be able to differentiate their boundaries. Therefore, having access to edge degrees of freedom would lead to a non-degenerate state sum model.

Moreover, it is now widely accepted that in three dimensions the proper symmetries of a quantum gravity model are governed by quantum groups (see [105] for a recent derivation). As I explained in the previous chapter, the main reason for the appearance of quantum groups is that they allow to describe quantum states of curved configuration spaces and curved momentum spaces at the same time. In four dimensions, even using quantum groups this would not be possible due to the Eckmann-Hilton argument.

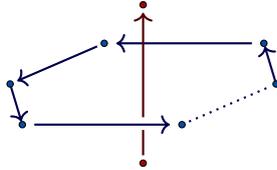
Proposition 13 (Eckmann-Hilton argument). [134] *Consider a cellular decomposition with two dimensional surfaces decorated by the elements of a group H . If there is no decoration on the one-cells, then the group H **must** be abelian.*

A simple representation to explain the Eckmann-Hilton argument is given in Fig. 5.1. By decorating the faces of a cellular decomposition and the links of the dual complex with elements of a (quantum) group, due to the Eckmann-Hilton argument, the face decorations are elements of an abelian group (or commutative quantum group). Therefore the four dimensional geometry under consideration can not be curved.

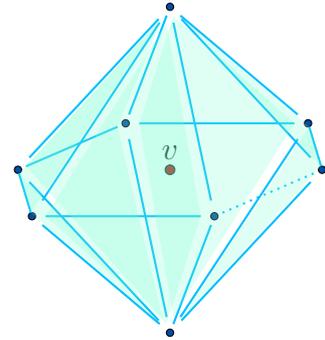
There is a further detail: using (quantum) groups to describe these geometries, we can only decorate either the one dimensional or the two dimensional objects of a given decomposition. Since we have no control (no decoration) on both the one and two dimensional pieces, it is not clear whether the geometry can actually be curved. This issue has been taken in consideration only a few times in quantum gravity, in particular in the models of twisted geometries [138–141].



(a) The one dimensional holonomy encodes the curvature around a point-like defect in 2+1 dimensions.



(b) The one dimensional holonomy encodes the curvature around a string-like defect ($1d$ object) in 3+1 dimensions.



(c) The two dimensional holonomy (closed polyhedron decorated by the 2-connection) encodes the curvature around a point-like defect in 3+1 dimensions.

Topological degrees of freedom. As I emphasized a few times now, one way to construct a model of four dimensional quantum gravity is to describe first a four dimensional geometry, and then implementing the gravitational degrees of freedom by constraining the topological model. With this in mind, one would expect to be able to encode all the topological degrees of freedom, such as the topological defects of a cellular decomposition (I suggest to read [142] for a more detailed explanation). These defects can assume different interpretations in different approaches, see for instance [131, 143]. It is reasonable to assume that their description is necessary to provide a proper model of quantum gravity. In a 2+1 dimensional model, one example is the point-like topological defect, used to introduce matter degrees of freedom. Here the quantum states are parameterized by the two dimensional triangulations or their dual graphs. In order to capture such defect, one has to be able to compute the curvature around it. Usually, one would use the elements of a (quantum) group to define the holonomy associated to the loop of links around the point-like defect (as I did for GFT in Ch. 3, or in in Hopf algebra field theory in Ch. 4). This holonomy encodes the local curvature around this point, as in Fig. 5.2a. In four dimensions the analogous example is slightly more complicated. Here one expects to encode, not only the zero dimensional defects, but also the one dimensional ones, as represented in Fig. 5.2b and 5.2c. In this case, the connection associated to one dimensional links are used to encode the curvature around a one dimensional defect, Fig. 5.2b; to encode the curvature around a single vertex (point-like defect) of the discrete geometry, one needs the information on the surfaces of the dual complex, Fig. 5.2c. By analogy, this suggests the need of one and two dimensional connections, which resp. leads to the notion of *1-* and *2-holonomies*, see [135].

In particular, *2-groups* have been identified as a natural tool to describe topological features of four dimensional manifolds. They can be seen as the natural generalization of groups: they would automatically provide a notion of 1- and 2-holonomies, as well as decoration on both the one and two dimensional objects of a discrete geometry. However, due to their high complexity, a complete theory of (quantum) 2-groups and their representation is still lacking. Indeed so far, the application of 2-categories in quantum gravity has been restricted to a few simplified cases. Nevertheless, the Euclidian and Poincaré 2-groups [144] and their representations [145–147] have successfully been used to construct a state sum in the KBF model [148–151]. A more general state sum model based on 2-groups was proposed by Yetter [152] in three dimensions and generalized by Mackaay in four dimensions [153, 154]; however, due to the lack of a Fourier expansion, the analogue of the Peter-Weyl theorem for 2-categories, it is not possible to express the Yetter amplitude in terms of representations for a general 2-group. As the Yetter model is constructed in the triangulation picture, its dual model has been recently proposed in [140], where the graphs dual to the triangulations are called G-networks and decorated by elements of

the Poincaré 2-group. It was shown that such model is related to the KBF amplitude and to the work [155], where the $BFCG$ theory was shown to provide the proper classical action that gives the Yetter model upon discretization.

In this chapter, I will first provide a brief summary of the main aspects of 2-group theory. Then, I will discuss how to construct the phase space of a triangulation of a three dimensional manifold using (a specific class of) 2-groups. The importance of this preliminary step is twofold:

- to provide a more refined description of the phase space of three dimensional triangulations, that can be used to define the states (Hilbert space) of an eventual 3+1 dimensional model of discrete geometry;
- to derive some insights on the combinatorics and on the proper construction of cellular decompositions with decorations on 1- and 2-holonomies¹.

This part is based on [47], a work in collaboration with F. Girelli and P. Tsimiklis. In the last part of the chapter, I will then move on to the definition of a group field theory based on 2-groups. This part is taken from [48], which is a project in collaboration with F. Girelli, A. Tanasa and P. Tsimiklis. We provide the partition function of the new model as a state sum, proving its topological invariance. These new results agree with the other works, such as the construction of G -networks in [140] and thus the KBF model.

For the sake of clarity, it is worth to fix some conventions here. I will call *2-graph* the graph with decorations on both one and two dimensional objects. Similarly, the nomenclature *2-triangulation* and *dual 2-complex* are used to recall that both 1- and 2-holonomies are decorated. Moreover, I call *vertex*, *edge* and *face* respectively the $0d$, $1d$ and $2d$ dimensional objects of a 2-triangulation. While the $0d$, $1d$ and $2d$ dimensional objects of a dual 2-complex are called *node*, *link* and *wedge*.

5.1 2-Groups

In this section I will give a short overview of the 2-group theory. For the interested reader, I strongly suggest [135] for a deeper review of the main aspects of the theory, specially the proper formulation of 1- and 2-holonomies. See also [156–158] for more details on 2-group theory.

Definition 14 (Strict 2-group). *A strict 2-group, also known as crossed module, is given as the set of*

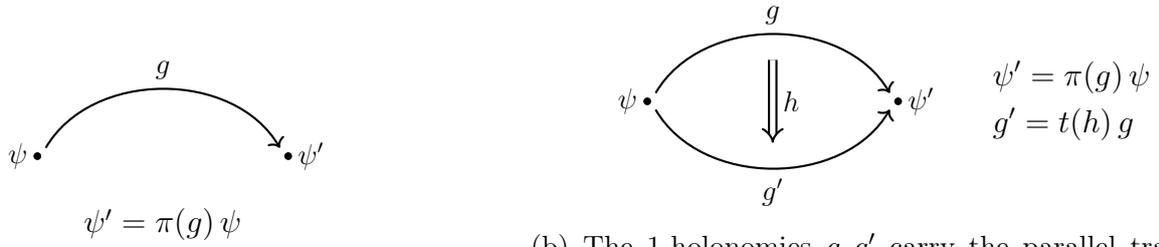
- a pair of groups G and H of 1-morphisms and 2-morphisms;
- a group homomorphism $t : H \rightarrow G$ that sends each surface decoration h ;
- a left action $\triangleright : G \times H \rightarrow H$,

such that

$$\begin{aligned} t \text{ is } G\text{-equivariant: } & t(g \triangleright h) = gt(h)g^{-1}, \\ \text{the Peiffer identity holds: } & t(h) \triangleright h' = hh'h^{-1}. \end{aligned} \tag{5.1}$$

Let the element of the 2-group $\mathcal{G} = (G, H, t, \triangleright)$ be the pair (h, g) , with $h \in H$ and $g \in G$. The 1-morphism g is the source of the 2-morphism h , while its target is $t(h)g$. Group elements can be interpreted as holonomies that carry the parallel transport between two points (states) of a

¹These insights will be fundamental for the construction of the 2-group field theory in Sec. 5.3



(a) The holonomy g carries the parallel transport between points (states) ψ, ψ' of a manifold.

(b) The 1-holonomies g, g' carry the parallel transport between points (states) ψ, ψ' of a manifold. The 2-Holonomy h carries the parallel transport between the 1-holonomies.

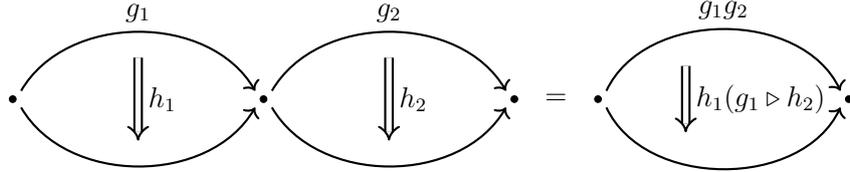


Figure 5.4: Horizontal composition of the 2-group elements (h_1, g_1) and (h_2, g_2) .

manifold, Fig. 5.3a; similarly, 2-group elements, represented as *digons* in Fig. 5.3b, are naturally interpreted as the pair of 1-holonomies encoding the parallel transport between points (states) of a manifold, and the 2-holonomies that carry the parallel transport between 1-holonomies. A pair of 2-group elements (h_1, g_1) and (h_2, g_2) can be composed *horizontally* or *vertically*. They are vertically composable only if the target of the first 2-holonomy coincides with the source of the second 2-holonomy: $t(h_1)g_1 = g_2$. The compositions for the 2-group elements, illustrated in Fig. 5.4 and 5.5, are

$$\text{horizontal composition: } (h_1, g_1) \circ (h_2, g_2) = (h_1(g_1 \triangleright h_2), g_1 g_2), \quad (5.2)$$

$$\text{vertical composition: } (h_1, g_1) \cdot (h_2, g_2) = (h_1 h_2, g_1), \quad t(h_1)g_1 = g_2. \quad (5.3)$$

The horizontal and vertical compositions are compatible, in the sense that they obey to the interchange law

$$(\alpha'_1 \cdot \alpha_1) \circ (\alpha'_2 \cdot \alpha_2) = (\alpha'_1 \circ \alpha'_2) \cdot (\alpha_1 \circ \alpha_2). \quad (5.4)$$

A 2-group is a group under the horizontal composition, namely there are a unit and an inverse elements:

$$\begin{aligned} \text{Horizontal identity: } & (1, 1), \\ \text{Horizontal inverse: } & (h, g)^{-1H} = ((g^{-1} \triangleright h)^{-1}, g^{-1}) = (g^{-1} \triangleright h^{-1}, g^{-1}). \end{aligned} \quad (5.5)$$

The 2-group is instead a groupoid under the vertical composition, which implies that the unit and the inverse are not unique elements. One can indeed identify a pair of left and right vertical

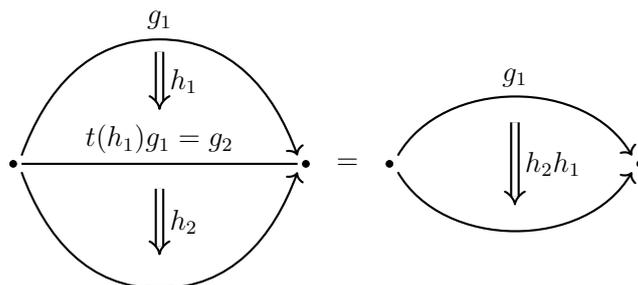


Figure 5.5: Vertical composition of the 2-group elements (h_1, g_1) and (h_2, g_2) , with $g_2 = t(h_1)g_1$.

identities with a single inverse element:

$$\begin{aligned}
\text{Left vertical identity:} & \quad (1, g), \\
\text{Right vertical identity:} & \quad (1, t(h)g), \\
\text{Vertical inverse:} & \quad (h, g)^{-1v} = (h^{-1}, t(h)g).
\end{aligned} \tag{5.6}$$

The left and right unit elements of (h, g) satisfy the identities $(1, g) \cdot (h, g) = (h, g)$ and $(h, g) \cdot (1, t(h)g) = (h, g)$, for all $(h, g) \in \mathcal{G}$. Similarly, the vertical inverse satisfies the left or right identities $(h, g)^{-1v} \cdot (h, g) = (1, t(h)g)$ and $(h, g) \cdot (h, g)^{-1v} = (1, g)$.

Strict 2-groups are thus specified by a pair of groups, an action and the t -map, together with the compatibility relations (5.1). In the definition above, I used a left action of G on H , but one could use equivalently a right action. In this case the G -equivariance and Peiffer identity are

$$\begin{aligned}
\text{G-equivariance:} & \quad t(h \triangleleft g) = g^{-1}t(h)g, \\
\text{Peiffer identity:} & \quad h' \triangleleft t(h) = h^{-1}h'h,
\end{aligned} \tag{5.7}$$

while the horizontal and vertical compositions for a pair of 2-group elements are

$$\text{horizontal composition:} \quad (g_1, h_1) \circ (g_2, h_2) = (g_1g_2, (h_1 \triangleleft g_2)h_2), \tag{5.8}$$

$$\text{vertical composition:} \quad (g_1, h_1) \cdot (g_2, h_2) = (g_2, h_2h_1), \quad g_1 = g_2t(h_2). \tag{5.9}$$

Note that also the expression of the target of the 2-holonomy h is different: in the convention with the left action the target of h is the 1-holonomy $t(h)g$, while in the convention with the right action the target is instead $gt(h)$. This is related with the main difference between the two conventions: the *root* of the 2-holonomy. The root can be seen as the point in the digon where the 2-holonomy is attached. In the first convention the 2-holonomy h is rooted at the source of the 1-holonomy g , while in the second convention it is rooted at its target. I refer to the reference frame part of the next section for a deeper understanding of the rooting and its relation with left or right action.

5.2 Phase space of a three dimensional triangulation

This section is based on [47], a project in collaboration with F. Girelli and P. Tsimiklis. I will introduce the concept of phase space for a simplicial three dimensional 2-triangulation, even if the construction that I will introduce holds for a general cellular decomposition. The symplectic reduction is the mathematical tool used for this construction, and it will be discussed in the first part of the section. As I already mentioned, the use 2-groups in place of ordinary groups allows to describe curved geometries. However, we did not consider the most general problem of a fully curved geometry. Instead, we focused on a specific class of strict 2-groups, called *skeletal 2-group* [135]. As I will show in a moment, skeletal 2-groups are naturally interpreted as decorations of cellular decompositions with (possibly) non-abelian (curved) 1-holonomies and abelian (**flat**) 2-holonomies. Despite this choice precludes the description of a completely curved geometry, which is the main argument to use 2-groups at the place of groups, skeletal 2-groups encompass the Poincaré 2-group [144] as a key example, and its natural generalization, known as κ -Poincaré 2-group, see [159].

Moreover, this construction will still bring several interesting intuitions on how to deal with the combinatorics of a 2-triangulation and a phase space with decorations on links, edges, wedges and faces. These insights will be used in the next section to build a 3+1 dimensional field theory based on more general 2-groups. The states (three dimensional geometries) of such field theory will be the 2-triangulations or the dual 2-complexes.

Skeletal 2-groups and semi-dualized Lie (bi-)algebras

As I already mentioned, this work provides the construction of the phase space of a 2-triangulation decorated by a *skeletal 2-group*.

Definition 15 (Skeletal 2-group). [135] *A skeletal 2-group $\mathcal{G} = (G, H, t = id, \triangleright)$ is a special class of crossed modules with a trivial t map, namely $t(h) = 1$, for all $h \in H$.*

The triviality of the t map for skeletal 2-groups has three main implications:

- * due to the Peiffer identity (5.1), the group H is abelian;
- * the holonomy around a single 2-group element (boundary of a digon) is flat, $g'g^{-1} = 1$, where g and g' are resp. the source and target of a given 2-holonomy h ;
- * the vertical composition reduces to the (opposite) multiplication of the group H .

According to the above points, the notion of skeletal 2-group does not stray too far from the concept of group. Indeed, besides the second point, according to which the parallel transport along the closed holonomy $g'g^{-1}$ around any surface is flat, the algebraic structure of a skeletal 2-group $\mathcal{G} = (G, H, t = id, \triangleright)$ is similar to that of a semi-direct product of groups $\mathcal{G} = H \rtimes G$, with H abelian. In the following, I will extensively make use of this similarity to deal with skeletal 2-groups.

According to this perspective, we will discuss the construction of phase spaces of skeletal 2-groups bearing in mind they behave as semi-direct product of groups. There exists a natural notion of phase space when dealing with groups, which is the associated cotangent bundle. As already mentioned in Ch. 3 and 4, the cotangent bundle of a group does not allow to include curvature on both the configuration and momentum spaces. To have curvature everywhere, one has to use quantum groups (as we did in the previous chapter) or eventually Poisson Lie groups, which I review in App. A. The natural generalization of cotangent bundle in the case of Poisson Lie groups is known as Heisenberg double [160, 161]. It is closely related to the notion of symmetries on the phase space, provided by the Drinfeld double. I discuss their definitions in App. A as well. For the reader who is not familiar with these concepts, I suggest to go through this appendix to have a complete understanding of this section.

The core idea behind the notion of Heisenberg and Drinfeld double is that, given a Lie group G and its Lie algebra \mathfrak{g} , there exists a natural notion of dual Lie algebra \mathfrak{g}^* ; this notion is associated to the concept of Lie bi-algebra. The integration (exponentiation) of the dual Lie algebra leads to the *dual Lie group* G^* . The external (Cartesian) product of a Lie group G and its dual group G^* in general is equipped with a pair of mutual actions; moreover, it can be further endowed with a bi-linear map, known as Poisson bracket. There are two standard choices for such Poisson structures [162]:

- *symplectic Poisson structure*: the group $\mathcal{H} = G \bowtie G^* \cong G^* \bowtie G$ is called Heisenberg double and has a natural interpretation as the (curved) phase space of G . It is equipped with a *ribbon equation*

$$d = \ell h = \tilde{h} \tilde{\ell} \quad \text{with} \quad d \in \mathcal{H}, h, \tilde{h} \in G, \ell, \tilde{\ell} \in G^*. \quad (5.10)$$

The ribbon relation, allows to define the mutual left and right actions between G and G^*

$$\ell \triangleright h = \tilde{h}, \quad \ell \triangleleft h = \tilde{\ell}, \quad \tilde{h} \triangleright \tilde{\ell} = \ell, \quad \tilde{h} \triangleleft \tilde{\ell} = h. \quad (5.11)$$

➤ *Poisson structure compatible with the group product:* the group $\mathcal{D} = \mathbb{G} \bowtie \mathbb{G}^* \cong \mathbb{G}^* \bowtie \mathbb{G}$ turns into a Poisson Lie group, it is called Drinfeld double and has a natural interpretation as the group of symmetries on the phase space.

One can read [161, 163] for more details. In the case of 2-groups, these concepts should be extended to include the compatibility relations with both the horizontal and vertical products. For what concern us, by viewing skeletal 2-groups as semi-direct product of groups, the notions of Heisenberg and Drinfeld doubles are trivially generalized. In order to fix the notation, let me summarize how to construct the tangent space associated to a semi-direct product of groups and its dual Lie algebra. The full details of this construction are given in [47], and the main steps are summarized in App. A.

Consider a pair of Lie bi-algebras \mathfrak{g}_1 and \mathfrak{g}_2 with generators $e^i \in \mathfrak{g}_1$ and $f^i \in \mathfrak{g}_2$. Let both these Lie bi-algebras have non-trivial Lie brackets with resp. structure constant c^{ij}_k and d^{ij}_k and trivial Lie co-algebra structures. As a consequence, their dual Lie bi-algebras \mathfrak{g}_1^* and \mathfrak{g}_2^* have trivial algebra sector (trivial Lie brackets) and non-trivial Lie co-cycles. One can further define a pair of mutual actions between the Lie bi-algebras \mathfrak{g}_1 and \mathfrak{g}_2 defined in terms of the structure constants α^{ij}_k and β^{ij}_k , such that they form a matched pair of Lie bi-algebras. Dually, the Lie bi-algebras \mathfrak{g}_1^* and \mathfrak{g}_2^* are equipped with a pair of mutual co-actions that make them a matched co-pair of Lie bi-algebras. By construction, the double cross sum $\mathfrak{g}_1 \bowtie \mathfrak{g}_2$ and its dual double cross co-sum $\mathfrak{g}_1^* \blacktriangleright \mathfrak{g}_2^*$ are Lie bi-algebras. Putting the two Lie bi-algebras together one gets the classical double

$$\mathfrak{d} = (\mathfrak{g}_1 \bowtie \mathfrak{g}_2)^{cop} \times (\mathfrak{g}_1^* \blacktriangleright \mathfrak{g}_2^*), \quad (5.12)$$

where *cop* stands for an opposite Lie co-cycle. However, this is not yet the structure of interest. We consider the *semi-dualization* that takes the double cross sum $\mathfrak{g}_1 \bowtie \mathfrak{g}_2$ into the bicross sum $\mathfrak{g}_2 \blacktriangleright \mathfrak{g}_1^*$. The semi-dualization amounts to exchange some of the configuration variables with the associated momentum variables. Denote \mathfrak{b} the resulting classical double

$$\mathfrak{b} = (\mathfrak{g}_2 \blacktriangleright \mathfrak{g}_1^*)^{cop} \bowtie (\mathfrak{g}_2^* \blacktriangleright \mathfrak{g}_1). \quad (5.13)$$

The Lie brackets and co-cycles of the Lie bi-algebras $(\mathfrak{g}_2 \blacktriangleright \mathfrak{g}_1^*)^{cop}$ and $(\mathfrak{g}_2^* \blacktriangleright \mathfrak{g}_1)$ are

$$\begin{aligned} [f^i, f^j] &= d^{ij}_k f^k, & [e_i^*, e_j^*] &= 0, & [e^i, e^j] &= c^{ij}_k e^k, & [f_i^*, f_j^*] &= 0, \\ [e_i^*, f^j] &= \beta^{jk}_i e_k^*, & [e^i, f_j^*] &= \alpha^{ki}_j f_k^*, \end{aligned} \quad (5.14)$$

$$\begin{aligned} \delta_{\mathfrak{g}_2 \blacktriangleright \mathfrak{g}_1^*}(f^i) &= -\alpha^{ij}_k e_j^* \wedge f^k, & \delta_{\mathfrak{g}_2^* \blacktriangleright \mathfrak{g}_1}(e^i) &= \beta^{ki}_j e_j \wedge f_k^*, \\ \delta_{\mathfrak{g}_2 \blacktriangleright \mathfrak{g}_1^*}(e_i^*) &= -\frac{1}{2} c^{jk}_i e_j^* \wedge e_k^*, & \delta_{\mathfrak{g}_2^* \blacktriangleright \mathfrak{g}_1}(f_i^*) &= \frac{1}{2} d^{jk}_i f_j^* \wedge f_k^*. \end{aligned} \quad (5.15)$$

Since the Lie bi-algebras $\mathfrak{g}_2 \blacktriangleright \mathfrak{g}_1^*$ and $\mathfrak{g}_2^* \blacktriangleright \mathfrak{g}_1$ are dual each other, their natural mutual actions are encoded in the respective co-adjoint actions, that induce the cross Lie brackets

$$\begin{aligned} [f_i^*, f^j] &= d^{jk}_i f_k^* + \alpha^{jk}_i e_k^*, & [e^i, e_j^*] &= c^{ki}_j e_k^* + \beta^{ki}_j f_k^*, \\ [e^i, f^j] &= -\alpha^{ji}_k f_k^* - \beta^{ji}_k e_k^*, & [e_i^*, f_j^*] &= 0. \end{aligned} \quad (5.16)$$

An explicit realization of this construction are the Poincaré and κ -Poincaré bi-algebras, presented in App. A. Note that there exist a notion of Lie 2-bi-algebra [164, 165] (and skeletal Lie 2-bi-algebra), and in [47] we show that it coincides with the above construction. The classical double \mathfrak{b} is thus equivalent to the one defined in [166] for skeletal Lie 2-bi-algebras. Upon exponentiation of such Lie bi-algebras, one recovers a pair of skeletal crossed modules

$$\mathfrak{g}_2^* \times \mathfrak{g}_1 \rightarrow \mathcal{G} = \mathbb{G}_2^* \times \mathbb{G}_1, \quad \mathfrak{g}_2 \times \mathfrak{g}_1^* \rightarrow \mathcal{G}^* = \mathbb{G}_2 \times \mathbb{G}_1^*. \quad (5.17)$$

Accordingly, the classical double \mathfrak{b} becomes the double cross product $\mathcal{B} = (\mathbb{G}_1 \times \mathbb{G}_2^*) \bowtie (\mathbb{G}_2 \times \mathbb{G}_1^*)$, which can be equipped with a symplectic Poisson bracket (that turns it into an Heisenberg

double) or with a Poisson bracket compatible with the group multiplication (that turns \mathcal{B} into a Drinfeld double). Since both G_2^* and G_1^* are abelian groups, both $(G_2^* \times G_1)$ and $(G_2 \times G_1^*)$ can be viewed as skeletal 2-groups (more precisely, as skeletal Poisson crossed modules [165]). Therefore, the double cross product \mathcal{B} , equipped with a symplectic Poisson structure, becomes the natural notion of phase space for the skeletal 2-group \mathcal{G} . This will be the fundamental object (the building block) to construct the phase space for our three dimensional 2-triangulation.

5.2.1 Symplectic reduction

In this section I shortly review the main concepts of momentum maps and symplectic reduction, see [167] for the general set up. The symplectic reduction is the essential tool to build phase spaces by *fusing* together smaller ones. In this section, the Heisenberg double will thus be considered the smallest phase space possible, the brick, and the symplectic reduction can be interpreted as the cement that glues the bricks.

Momentum maps

Momentum maps can be seen as functions on phase space that generate the infinitesimal symmetry transformations. The Heisenberg double \mathcal{H} is taken to be the phase space, and the Drinfeld double is the group that acts on it², generating such infinitesimal symmetry transformations. The simplest example is the phase space $T^*\mathbb{R} = \mathbb{R} \times \mathbb{R}$ with coordinates (x, p) ; the momentum p generates the infinitesimal translations in x , while x generates the infinitesimal translations in p . Hence these coordinates themselves can be seen as momentum maps. More explicitly, let the infinitesimal transformations be of the type

$$\delta_\epsilon x = \epsilon, \quad \delta_\epsilon p = 0, \quad \delta_\eta x = 0, \quad \delta_\eta p = \eta, \quad (5.18)$$

where $\delta_{\epsilon, \eta}$ is the vector field generating the transformation, with $\epsilon \in \mathbb{R}$ and $\eta \in \mathbb{R}^*$. The symplectic form associated to $T^*\mathbb{R}$ is $\Omega = \delta x \wedge \delta p$. Plugging the infinitesimal symmetries in it, one gets

$$\delta_\epsilon \lrcorner \Omega = \epsilon \delta p = \langle \epsilon, \delta p \rangle, \quad \delta_\eta \lrcorner \Omega = -\eta \delta x = -\langle \eta, \delta x \rangle, \quad (5.19)$$

where the angle brackets stands for the canonical pairing between \mathbb{R} and $\mathbb{R}^* \cong \mathbb{R}$. Here the coordinates p and x are those momentum maps just given by the projection of the phase space coordinates either on configuration or momentum spaces. Let us expend this concept to the more general Heisenberg double. Let σ be the left action of the Poisson Lie group G on a phase space \mathcal{P} , X^L be the left invariant 1-form on G^* which takes the value α at the identity, and δ_X be the associated vector field generating the infinitesimal version of σ .

Definition 16 (Momentum map). [167] *The momentum map of the action σ is the C^∞ map $\mathcal{J} : \mathcal{P} \rightarrow G^*$ with*

$$\delta_X \lrcorner \Omega = \mathcal{J}^*(X^L). \quad (5.20)$$

The momentum map is said equivariant if it commutes with the symmetry action.

Proposition 14. *Consider the Heisenberg double \mathcal{H} of the group G , with associated ribbon equation $d = \tilde{h} \ell = \tilde{\ell} h \in \mathcal{H}$. The G^* (resp. G) group element ℓ (resp. \tilde{h}) generates the infinitesimal left G transformation (A.24) (resp. the infinitesimal left G^* transformation (A.26))*

$$\delta_\alpha^L \lrcorner \Omega = \langle \alpha, \Delta \ell \rangle, \quad \delta_{\tilde{\phi}}^L \lrcorner \Omega = -\langle \tilde{\phi}, \Delta \tilde{h} \rangle, \quad (5.21)$$

²I will restrict the action to the left or right multiplication by the groups G or G^* , and of their sub-groups.

with $\alpha \in \mathfrak{g}$ and $\phi \in \mathfrak{g}^*$. The associated momentum maps are therefore $\ell := \mathcal{J}_{G^*}^L(d)$ and $\tilde{h} := \mathcal{J}_G^L(d)$. Similarly, h and $\tilde{\ell}$ are resp. the momentum maps for the right transformations of G^* and G .

The proof of this proposition is given in [105]. The momentum maps associated to the right of left multiplication of the Drinfeld double on the Heisenberg double are thus given by the projections

$$\begin{aligned} \mathcal{J}_G^R : \mathcal{H} = G^* \bowtie G &\rightarrow G, & \mathcal{J}_{G^*}^R : \mathcal{H} = G \bowtie G^* &\rightarrow G^*, \\ \mathcal{J}_G^L : \mathcal{H} = G \bowtie G^* &\rightarrow G, & \mathcal{J}_{G^*}^L : \mathcal{H} = G^* \bowtie G &\rightarrow G^*. \end{aligned} \quad (5.22)$$

One can check that these momentum maps are equivariant. For example, consider the map $\mathcal{J}_{G^*}^L$ and use its definition given in Prop. 14:

$$\mathcal{J}_{G^*}^L(h'd) = \mathcal{J}_{G^*}^L(h'\ell h) = \mathcal{J}_{G^*}^L((h' \triangleright \ell) ((h' \triangleleft \ell) h)) = h' \triangleright \ell = h' \triangleright \mathcal{J}_{G^*}^L(d), \quad (5.23)$$

for $d \in \mathcal{H}$, $h', h \in G$ and $\ell \in G^*$. Let me summarize here some properties of momentum maps that will be relevant for the construction (all these properties are discussed in [167]).

- If the (Poisson Lie) group G (resp. its dual G^*) can be decomposed into subgroups $G = G_1 \bowtie G_2 \cong G_2 \bowtie G_1$ (resp. $G^* = G_1^* \bowtie G_2^* \cong G_2^* \bowtie G_1^*$), then the projection of \mathcal{H} into the subgroups G_i also defines a momentum map and generates the infinitesimal right G_i^* transformations.
- If the momentum map given by the coordinate $\ell \in G^*$ generates the infinitesimal right G translations through the symplectic form Ω (as above), then ℓ^{-1} is a momentum map for the infinitesimal left G translations through the symplectic form $-\Omega$. Similarly for the other momentum maps $\tilde{\ell}, h, \tilde{h}$.
- Consider the phase space $\mathcal{H}^{(n)}$ made of n (independent) copies of \mathcal{H} , $\mathcal{H}^{(n)} = \mathcal{H} \times \cdots \times \mathcal{H}$ with symplectic form $\Omega^{(n)} = \sum_i \Omega_i$. Then the global right infinitesimal G translation is generated by the momentum map $\ell_1 \dots \ell_n$. Similarly for the other coordinates $\tilde{\ell}, h, \tilde{h}$.

Symplectic reduction

Consider a symplectic space with a set of symmetries; the *symplectic reduction* is the process of reducing the space by implementing the symmetries, such that the resulting space is still symplectic. For reducing the symmetries, I mean to restrict the set of functions of the initial symplectic space, to those that are invariant under the symmetries. The invariant functions are those that commute with the momentum map associated to the symmetry in consideration³.

Theorem 2 (Symplectic reduction). [163, 167] *Let $\sigma : G \times \mathcal{H} \rightarrow \mathcal{H}$ be a Poisson action of a Poisson Lie group G on its phase space \mathcal{H} . The equivariant momentum map associated to σ is \mathcal{J}_{G^*} . Consider an element $\ell_0 \in G^*$ and G_{ℓ_0} its isotropy subgroup under the action⁴ of G on G^* . If G acts freely and properly on $\mathcal{J}^{-1}(\ell_0)$, then the symplectic form Ω of \mathcal{H} is pulled back to the quotient space $\mathcal{J}^{-1}(\ell_0)/G_{\ell_0}$ for $d \in \mathcal{H}$, which is denoted⁵ \mathcal{H}/G_{ℓ_0} .*

The general proof of this theorem can be found in [163]. In our case, the use of symplectic reduction to fuse phase spaces is justified by the fact that the actions in consideration – left and right translations – are free and proper and thus the associated momentum maps are equivariant.

³This is just a different language to discuss Dirac's approach to constrained systems [3].

⁴ G_{ℓ_0} is the set of elements of G whose action leaves $\ell_0 \in G^*$ invariant.

⁵If ℓ_0 is the unit, then I will simply write \mathcal{H}/G .

5.2.2 Polygon phase space

Before going through the construction of a three dimensional simplicial 2-triangulation, let me summarize the simplest case of a two dimensional flat or curved simplicial triangulation with groups as decorations. In both cases the construction extends naturally to a general cellular decomposition. Let me start with the seminal work by Kapovich and Millson [168], dealing with the Euclidian case. Discussion of a Lorentzian picture can be found in [169]. Note that the discussion below is probably well known by most of the readers, but it is still very instructive to fully understand the more complicated case.

Consider three vector (edges) all lying on the same plane decorated by \mathbb{R}^3 variables in the $\mathfrak{su}(2)$ basis: $\ell \equiv \vec{\ell} \cdot \vec{\sigma}$, where σ_I , $I = 1, 2, 3$ are the Pauli matrices. The three vectors are the edges of a triangle if they obey to the closure relation

$$\mathcal{C} = \ell_1 + \ell_2 + \ell_3 = 0. \quad (5.24)$$

The idea is to describe the phase space of such triangle using a Poisson structure invariant under the constraint above, so that \mathcal{C} would be considered as the momentum map that generates the symmetry of the triangle. At a first level, Kapovich and Millson proposed the Poisson structure

$$\{\ell_i^I, \ell_j^J\} = \delta_{ij} \epsilon^{IJ} \ell_i^K. \quad (5.25)$$

This refers to a triangle whose edges all have fixed lengths, so that each vector ℓ_i is an element of the sphere \mathcal{S}^2 . The above choice of Poisson bracket is symplectic on such sphere, hence, the total phase space \mathcal{P}^{KM} for a triangle with fixed edge lengths would be given by the symplectic reduction

$$\mathcal{P}^{KM} = (S^2 \times S^2 \times S^2) // \text{SO}(3). \quad (5.26)$$

A possible generalization of the Kapovich and Millson construction is accomplished by adding some extra variables. The interest of doing so is twofold:

- an further geometric information would allow to extend the phase space to a full $2d$ cellular decomposition;
- the constraint \mathcal{C} only encodes the rotational symmetry of a triangle, and an extra variable would allow the realization of the *translational* symmetry as well.

In order to generalize the phase space, one can thus add some holonomy information, and extend \mathbb{R}^3 for each edge to the cotangent bundle $T^*(\text{SO}(3)) \cong \text{SO}(3) \times \mathfrak{so}^*(3) \cong \text{SO}(3) \times \mathbb{R}^3$. The holonomy is thus an element of $\text{SO}(3)$ and (its action on an edge) provides a way to transport vectors along a path dual to the edge itself. In this sense, the holonomy contains some non-local information, which allows to discuss about adjacent triangles sharing the edge dual to it⁶

Flat triangle

As before, we associate a vector $\ell_i \in \mathbb{R}^3$ to the three edges of the triangle, with center denoted by c . Then, consider holonomies $h_{ci} \in \text{SO}(3)$ joining the center to a point on the edge i . Such path is called *half link*. For each pair of half link and edge, we associate the phase space given by the cotangent bundle $T^*(\text{SO}(3))$ given by

$$\{\ell_i^I, \ell_j^J\} = \delta_{ij} \epsilon^{IJ} \ell_i^K, \quad \{h_{ci}, h_{cj}\} = 0, \quad \{\ell_i^I, h_{cj}\} = \delta_{ij} h_i \sigma^I. \quad (5.27)$$

⁶Alternatively to the introduction of the holonomy, one could have extended the phase space by taking \mathbb{R}^3 to $\mathbb{R}^4 \sim \mathbb{C}^2$, with elements (z^-, z^+) called spinor variables. This is a more local picture at the root of the *spinor approach* [170, 171] and it is very convenient to describe the algebra of observables for a given polygon [123, 172, 173].

The rotational symmetry of the triangle is still given by the closure constraint (5.24). The total angular momentum is thus given by three copies of the phase space constrained by the momentum map⁷ (5.24):

$$\mathcal{P}_t = (T^*(\text{SO}(3)))^{\times 3} // \text{SO}^t(3). \quad (5.28)$$

This approach can naturally be extended to the case of curved triangles.

Curved triangle

Let me focus here on the triangle with negative curvature, as discussed in [141, 162]. I refer to [174] for the case of positive curvature. The edges of a triangle are given in terms of geodesics in the hyperboloid, described as the quotient space $\text{SL}(2, \mathbb{C}) // \text{SU}(2) \cong \text{AN}(2)$. Such geodesics can thus be characterized by elements (holonomies) ℓ of the (Poisson) Lie group $\text{AN}(2)$. By deforming \mathbb{R}^3 into $\text{AN}(2)$, the cotangent bundle $T^*(\text{SO}(3))$ is also deformed into the Heisenberg double that includes curvature both in momentum and configuration spaces:

$$T^*(\text{SO}(3)) \cong \text{SO}(3) \times \mathbb{R}^3 \quad \rightarrow \quad \mathcal{H} = \text{SL}(2, \mathbb{C}) \cong \text{SU}(2) \bowtie \text{AN}(2). \quad (5.29)$$

The momentum map (5.24) generating the global (left) rotations ($\text{SU}^t(2)$ transformations) that describes the closure of the three edges of the triangle, is now modified in terms of non-abelian group elements

$$\ell_1 \ell_2 \ell_3 = 1. \quad (5.30)$$

The total phase space for the curved triangle is thus

$$\mathcal{P} = \text{SL}(2, \mathbb{C})^{\times 3} // \text{SU}^t(2) \cong \mathcal{H}^{(3)} // \text{SU}^t(2). \quad (5.31)$$

One could further extend the construction to the case where the holonomies are elements of a general Poisson Lie group G and the edges are decorated by elements of its dual G^* ; the phase space for every pair of edge/link is thus the general Heisenberg double $\mathcal{H} = G \bowtie G^* \cong G^* \bowtie G$, endowed with the ribbon factorization (5.10) that reflects the symplectic structure. Extending the construction to a polygon with n edges, the phase space would be given by the symplectic reduction

$$\mathcal{P} = \mathcal{H}^{(n)} // G, \quad (5.32)$$

where the constraint $G^* \ni \ell_1 \dots \ell_n = 1$ is the momentum map generating the global left G transformations.

Reference frame for 2d geometries

A fundamental point of the symplectic reduction that has been veiled on purpose so far, is *how to choose the correct momentum maps*. Indeed, to properly glue triangles or polygons together, one has to make sure that the quantities associated to different triangles are identified in a consistent way. To this scope, it is useful to introduce the concept of *reference frame*.

As mentioned earlier, elements of the (Poisson) Lie groups G and G^* are resp. used to decorate links and edges, so that the fundamental building block of the phase space, the Heisenberg double $\mathcal{H} = G \bowtie G^* \cong G^* \bowtie G$, is represented by the pair edge/link. Both of them are one dimensional objects, depicted as arrows, in the sense that they both carry an orientation, with an origin called *source*, and a final point called *target*. Consider here an edge e with source and target resp. at the vertices v_1 and v_2 , and a link l with source and target resp. at the nodes c_1 and c_2 . One can use the Heisenberg double \mathcal{H} and the associated ribbon equation (5.10) to induce representations of the groups G and G^* .

⁷Denote $\text{SO}^t(3)$ the group which encodes the momentum map that generates rigid rotations of the triangle t .

- On the source c_1 of the link, consider a vector space V_{c_1} which carries a representation ρ^{c_1} of the group G^* , coming from the right decomposition $\mathcal{H} = G^* \bowtie G$.

$$e \rightarrow \rho^{c_1}(\ell) \equiv \ell^{c_1}. \quad (5.33)$$

- On the target c_2 of the link, consider a vector space V_{c_2} which carries a representation ρ^{c_2} of the group G^* , coming from the left decomposition $\mathcal{H} = G \bowtie G^*$.

$$e \rightarrow \rho^{c_2}(\tilde{\ell}) \equiv \tilde{\ell}^{c_2}. \quad (5.34)$$

- On the source v_1 of the edge, consider a vector space W_{v_1} which carries a representation φ^{v_1} of the group G , coming from the right decomposition $\mathcal{H} = G \bowtie G^*$.

$$l \rightarrow \varphi^{v_1}(\tilde{h}) \equiv \tilde{h}^{v_1}. \quad (5.35)$$

- On the target v_2 of the edge, consider a vector space W_{v_2} which carries a representation φ^{v_2} of the group G , coming from the left decomposition $\mathcal{H} = G^* \bowtie G$.

$$l \rightarrow \varphi^{v_2}(h) \equiv h^{v_2}. \quad (5.36)$$

Hence, the idea is to associate the elements $\ell, \tilde{\ell}, h, \tilde{h}$ to different representations of the edge e and link l , placed at different vertices or nodes. According to the ribbon decomposition (5.10), such group elements are related each other by the left and right mutual actions (5.11) between G and G^* . These actions, that relate different representations at different points, can thus be interpreted as parallel transport between such points. For example, the relation

$$\ell = \tilde{h} \triangleright \tilde{\ell} = \tilde{h} \tilde{\ell} h^{-1} \quad \Leftrightarrow \quad \rho^{c_1}(\ell) = \varphi^{v_1}(\tilde{h}) \triangleright \rho^{c_2}(\tilde{\ell}) \quad (5.37)$$

encodes how the edge variable is transported from the representation at c_2 to the representation at c_1 , through the representation h of the link placed at v_1 . According to this interpretation, we deduces the following rules:

- * the left action of G on G^* transports the representation of the edge from the target c_2 to the source c_1 of the link;
- * the left action of G^* on G transports the representation of the link from the target v_2 to the source v_1 of the edge;
- * the right action of G on G^* transports the representation of the edge from the source c_1 to the target c_2 of the link;
- * the right action of G^* on G transports the representation of the link from the source v_1 to the target v_2 of the edge.

Since this point is particularly important, let me summarize the role of left and right actions also in Table 5.1. Moreover, note that the left action of \tilde{h} (resp. ℓ) is the inverse of the right action of h (resp. $\tilde{\ell}$):

$$\begin{aligned} \ell = \tilde{h} \triangleright \tilde{\ell} = \tilde{h} \triangleright (\ell \triangleleft h), & \quad h = \tilde{h} \triangleleft \tilde{\ell} = (\ell \triangleright h) \triangleleft \tilde{\ell}, \\ \tilde{\ell} = \ell \triangleleft h = (\tilde{h} \triangleright \tilde{\ell}) \triangleleft h, & \quad \tilde{h} = \ell \triangleright h = \ell \triangleright (\tilde{h} \triangleleft \tilde{\ell}). \end{aligned} \quad (5.38)$$

This interpretation of reference frame as representations and parallel transport is important to understand how to implement the correct geometric identification to properly fuse the phase spaces. Indeed one needs to identify objects living in the same representation.

	G on G*	G* on G
Left action \triangleright	$c_2 \rightarrow c_1$	$v_2 \rightarrow v_1$
Right action \triangleleft	$c_1 \rightarrow c_2$	$v_1 \rightarrow v_2$

Table 5.1: Geometric interpretation of the left and right mutual actions between G and G*. Geometrically, the left action of a link (resp. edge) on the edge (resp. link) transports the representation of the edge (resp. link) from the node c_2 to c_1 (resp. from the vertex v_2 to v_1); the right action of a link (resp. edge) on the edge (resp. link) transports the representation from c_1 to c_2 (resp. from v_1 to v_2).

Phase space of a 2d triangulation

As a final part for the two dimensional case, I will explain how to recover the phase space for a general triangulation \mathcal{T} , by fusing together what I will call *atomic phase spaces*, namely the Heisenberg doubles of a general (Poisson) Lie group G. The setting is the following: consider two triangles t_i with centers c_i , for $i = 1, 2$. In particular, focus on one edges e_i of each triangle with resp. sources and targets at the vertices $v_{i,1}$ and $v_{i,2}$. The idea is to glue the two triangles by identifying these two edges. The result is the fusion of the associated phase spaces \mathcal{H}_i , represented as the pairs of edge/half link (e_i, l_i) , where for simplicity, I denoted l_i the half link that has c_i as source and a point on the edge e_i as target. The ribbon equations of the two phase space are

$$\mathcal{H}_i \ni l_i h_i = \tilde{h}_i \tilde{l}_i. \quad (5.39)$$

Now it is fundamental to have in mind where each of the eight variables that appear in the two ribbon equations are represented. Since the two triangles have the same orientation convention, the two edges e_i are identified with opposite orientation, $e_1 = -e_2$. In particular, this implies the vertices are identified as $v_{1,1} = v_{2,2}$ and $v_{1,2} = v_{2,1}$. Most importantly, one has to demand that the targets of the two half links l_i coincide. This requirement is essential since the identification of the edge decorations has to be enforced at this common point. According to the reference frame interpretation, the proper identification is

$$\tilde{l}_1 = \tilde{l}_2^{-1}, \quad (5.40)$$

where the inverse was taken for the opposite orientation⁸. This constraint is the momentum map that generates left global rotations with respect to G. Putting the two ribbon equations together with such constraint, one obtains

$$\left. \begin{aligned} \tilde{l}_1 &= \tilde{h}_1^{-1} l_1 h_1 \\ \tilde{l}_2^{-1} &= h_2^{-1} l_2^{-1} \tilde{h}_2 \end{aligned} \right\} \Rightarrow \begin{aligned} \tilde{h}_1^{-1} l_1 h_1 &= h_2^{-1} l_2^{-1} \tilde{h}_2 \\ l_1 h_{12} &= \tilde{h}_{12} l_2^{-1}. \end{aligned} \quad (5.41)$$

The fusion of phase spaces is represented in Fig. 5.6. In the last step of eq. (5.41), I introduced the variables $h_{12} = h_1 \tilde{h}_2^{-1}$ and $\tilde{h}_{12} = \tilde{h}_1 h_2^{-1}$. As one can see from Fig. 5.6, these are decorations of the link with source at the center of triangle t_1 , node c_1 , and target at the center of triangle t_2 , node c_2 . Call this path a *full link*. The phase space obtained after the fusion is isomorphic to an Heisenberg double with G decorations on the (full) link and G* decorations on the common

⁸One can use a more self-explaining notation for the variables \tilde{l}_i , such as $\tilde{l}_{v_{i,1}v_{i,2}}^{c'_i}$; this emphasizes that each variable decorates the edge with source and target resp. at the vertices $v_{i,1}$ and $v_{i,2}$, represented at the node c'_i , which is the common node on the two edges and the target of the half links l_i . Using this notation, the inverse stands for the variable $\tilde{l}_i^{-1} \equiv (\tilde{l}_{v_{i,1}v_{i,2}}^{c'_i})^{-1} = \tilde{l}_{v_{i,2}v_{i,1}}^{c'_i}$.

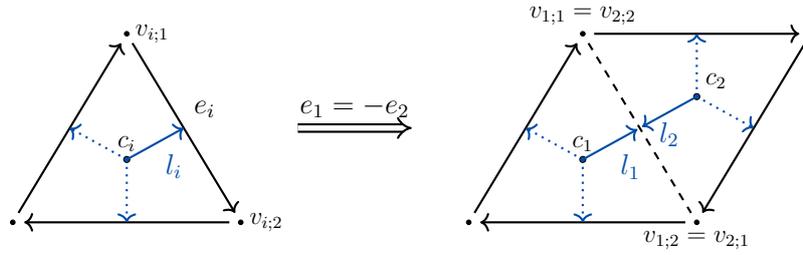


Figure 5.6: Fusion of two triangle phase spaces by identifying the edges with opposite orientation $e_1 = -e_2$. As a consequence, the dual half links l_i are composed in a full link connecting the centers c_i of the two triangles.

edge⁹. By repeating such triangle gluing we can recover the full phase space $\mathcal{P}_{\mathcal{T}}$ for any given two dimensional simplicial triangulation \mathcal{T} ; the general symplectic reduction is

$$\mathcal{P}_{\mathcal{T}} = \left(\prod_{(e,l)} \mathcal{H} \right) // \left(\prod_t G^t \right). \quad (5.42)$$

Where the external product of Heisenberg doubles (atomic phase spaces) runs over all the pairs of edge/link (e, l) , while the product of groups G^t runs over the triangles and it stands for the the global G transformation (momentum map) induced by the closure constraint of each triangle t of the triangulation \mathcal{T} .

5.2.3 Polyhedra phase space

The natural way to generalize the phase space construction explained in the previous section to the three dimensional case, would be to use elements $h \in G$ as decorations of (one dimensional) holonomies (links) and elements $\ell \in G^*$ as (two dimensional) face decorations of a cellular decomposition. The closure of an n polyhedron would be given by the constraint

$$\sum_i^n \ell_i = 0. \quad (5.43)$$

Note that such constraint is completely analogous to the closure of a polygon in $2d$. Indeed, at this level, the difference between the two cases is a mere geometric interpretation. This ambiguity still holds for intertwiners and thus for spin networks, where the geometric interpretation is necessary to precise whether the network is dual to a two or three dimensional geometry. On the top of such ambiguity, I recall the arguments presented at the beginning of this chapter, according to which, it is not possible to describe curved three dimensional geometries using ordinary (Poisson Lie) groups. This is why in the three dimensional case we deal instead with 2-groups. In this section, I will generalize the phase space construction explained previously for a two dimensional triangulation. The atomic phase space is now the Heisenberg double constructed at the beginning of the section,

$$\mathcal{B} = \mathcal{G} \bowtie \mathcal{G}^* \cong \mathcal{G}^* \bowtie \mathcal{G}, \quad (5.44)$$

where \mathcal{G}^* and \mathcal{G} are dual skeletal 2-groups resp. decorating the 2-triangulation and its dual 2-complex. As both \mathcal{G} and \mathcal{G}^* are seen as skeletal crossed modules, I will treat them as semi direct product of groups with left or right decompositions

$$\mathcal{G} = G_1 \times G_2^* \cong G_2^* \times G_1, \quad \mathcal{G}^* = G_1^* \times G_2 \cong G_2 \times G_1^*. \quad (5.45)$$

⁹Note that a similar construction was done at the *quantum* level using fusion products in [175]. Moreover, this construction agrees with the one obtained after discretization of $3d$ gravity with a non-zero cosmological constant [105].

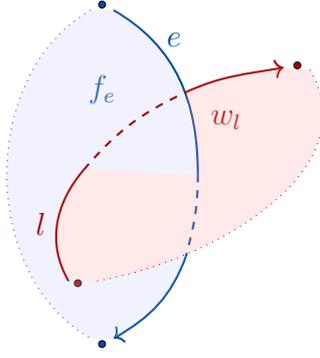


Figure 5.7: Graphical representation of the atomic phase space \mathcal{B} used to reconstruct a full $3d$ simplicial 2-triangulation. The link and wedge in red are decorated by \mathcal{G} elements (resp. G_1 and G_2^*), while the edge and face in blue are decorated by elements in \mathcal{G}^* (resp. G_2 and G_1^*).

Reference frame for $3d$ geometries

The symplectic reduction will be used again as a tool to properly fuse the Heisenberg doubles \mathcal{B} ; however, as I stressed in the previous section, a non-trivial step is to understand which is the correct momentum map to implement. To this scope, the notion of reference frame was introduced in the two dimensional case. Let me show how to generalize the same idea for the phase space of a three dimensional cellular decomposition using skeletal 2-groups.

Reference frame: 2-representation and rooting

Consider an edge e with source and target resp. at the vertices v_1 and v_2 , and a face f which contains the vertices v_1 and v_2 in its boundary. Dually, the link l is orthogonal to the face and has the nodes c_1 and c_2 resp. as source and target. Let w be the wedge orthogonal to the edge e that contains the nodes c_1, c_2 in its boundary. Let the link l and the wedge w be resp. decorated by the elements of the groups G_1 and G_2^* , so that the pair wedge/link is an element of the 2-group \mathcal{G} ; on the other hand, the edge e and the face f are resp. decorated by elements of the groups G_2 and G_1^* , so that the pair edge/face is an element of the 2-group \mathcal{G}^* . The set up is shown in Fig. 5.7. I recall that both the 2-groups are skeletal crossed modules, which implies that the groups G_2^* and G_1^* are abelian. In the $2d$ case, sources and targets of holonomies carry a representation of the dual group. Let us see how this idea can be generalized for skeletal 2-groups¹⁰ \mathcal{G} and \mathcal{G}^* .

- The source c_1 of the link carries a 2-representation $\rho^{c_1}(\ell) = \ell^{c_1}$ of the skeletal crossed module \mathcal{G}^* , coming from the right decomposition $\mathcal{B} = \mathcal{G}^* \bowtie \mathcal{G}$.
- The target c_2 of the link carries a 2-representation $\rho^{c_2}(\tilde{\ell}) = \tilde{\ell}^{c_2}$ of the skeletal crossed module \mathcal{G}^* , coming from the left decomposition $\mathcal{B} = \mathcal{G} \bowtie \mathcal{G}^*$.
- The source v_1 of the edge carries a 2-representation $\varphi^{v_1}(\tilde{h}) = \tilde{h}^{v_1}$ of the skeletal crossed module \mathcal{G} , coming from the left decomposition $\mathcal{B} = \mathcal{G} \bowtie \mathcal{G}^*$.
- The target v_2 of the edge carries a 2-representation $\varphi^{v_2}(h) = h^{v_2}$ of the skeletal crossed module \mathcal{G} , coming from the right decomposition $\mathcal{B} = \mathcal{G}^* \bowtie \mathcal{G}$.

¹⁰As in the group case we used vector spaces and representations, in the 2-group case one should use 2-vector spaces and 2-representations [145, 146, 156, 157]. For the purpose of this work, it is not necessary to go through the details of a 2-representation. We mostly use it as a device to keep track of where each object is. It is thus enough to say that nodes and vertices carry 2-representations of the dual skeletal crossed module.

	<i>Source of l: c_1</i>	<i>Target of l: c_t</i>
<i>Source of e: v_1</i>	$\ell^{c_1} = \beta_{v_1}^{c_1} \lambda^{c_1}$, $\tilde{h}^{v_1} = \tilde{y}_{c_1}^{v_1} \tilde{u}^{v_1}$	$\tilde{\ell}^{c_t} = \tilde{\beta}_{v_1}^{c_t} \tilde{\lambda}^{c_t}$, $\tilde{h}^{v_1} = \tilde{u}^{v_1} \tilde{y}_{c_t}^{v_1}$
<i>Target of e: v_2</i>	$\ell^{c_1} = \lambda^{c_1} \bar{\beta}_{v_2}^{c_1}$, $h^{v_2} = \bar{y}_{c_1}^{v_2} u^{v_2}$	$\tilde{\ell}^{c_t} = \tilde{\lambda}^{c_t} \tilde{\bar{\beta}}_{v_2}^{c_t}$, $h^{v_2} = u^{v_2} \bar{y}_{c_t}^{v_2}$

Table 5.2: List of the decorations of edges, links, faces and wedges. Decorations of links and edges are labelled by a single index, related to vertex or node where they are represented. Decorations of wedges and faces are labelled by an index that specifies where they are represented (resp. vertex or node) and a subscript that specifies the root (resp. node or vertex).

The skeletal crossed modules \mathcal{G} and \mathcal{G}^* , in turn, can be split in the left or right decompositions. According to the 2-group theory interpretation, different decompositions of a 2-group stand for alternative conventions of a 2-group in which the 2-holonomy is rooted either at the source or at the target of the 1-holonomy. Given the left and right decompositions of the full phase space \mathcal{B} , one has the following sub-decompositions:

$$\mathcal{B} = \mathcal{G}^* \bowtie \mathcal{G} = \mathcal{G} \bowtie \mathcal{G}^* \quad \rightarrow \quad \ell h = \tilde{h} \tilde{\ell}$$

$$\mathcal{G} \ni h = \begin{cases} uy \in G_1 \times G_2^* \\ \bar{y}u \in G_2^* \times G_1 \end{cases} \quad \mathcal{G} \ni \tilde{h} = \begin{cases} \tilde{u}\tilde{y} \in G_1 \times G_2^* \\ \tilde{y}\tilde{u} \in G_2^* \times G_1 \end{cases}$$

$$\mathcal{G}^* \ni \ell = \begin{cases} \beta\lambda \in G_1^* \times G_2 \\ \lambda\bar{\beta} \in G_2 \times G_1^* \end{cases} \quad \mathcal{G}^* \ni \tilde{\ell} = \begin{cases} \tilde{\beta}\tilde{\lambda} \in G_1^* \times G_2 \\ \tilde{\lambda}\tilde{\bar{\beta}} \in G_2 \times G_1^* \end{cases}$$

I recall below the geometric interpretation of the sub-decompositions.

- The skeletal crossed module \mathcal{G} decorates a pair link/wedge:
 - The right decomposition $\mathcal{G} = G_2^* \times G_1$ implies that the wedge (G_2^* element) is rooted at the source of the link, node c_1 .
 - The left decomposition $\mathcal{G} = G_1 \times G_2^*$ implies that the wedge (G_2^* element) is rooted at the target of the link, node c_2 .
- The skeletal crossed module \mathcal{G}^* decorates a pair edge/face.
 - The right decomposition $\mathcal{G} = G_1^* \times G_2$ implies that the face (G_1^* element) is rooted at the source of the edge, vertex v_1 .
 - The left decomposition $\mathcal{G} = G_2 \times G_1^*$ implies that the face (G_1^* element) is rooted at the target of the edge, vertex v_2 .

Therefore, as the choice of decorations of edges or links are resp. specified by a single node or vertex (as in the $2d$ case), decorations of faces and wedges are instead specified by a pair of node and vertex. In Table 5.2 I list the decorations of the pairs edge/face and wedge/link, according to the choice of representation and root.

Action and parallel transport

In analogy with the geometric interpretation given in the two dimensional case, here the mutual actions between \mathcal{G} and \mathcal{G}^* transport the 2-representations and the roots. Since both 2-representations and roots are specified by vertices or nodes (sources and targets of edges and links), the action responsible for the parallel transport between nodes is that of the group $G_1 \subset \mathcal{G}$ and the action responsible for the parallel transport between vertices is that of the group $G_2 \subset \mathcal{G}^*$. I list below the geometric interpretations of the actions

	G_1 <i>on</i> $\left\{ \begin{array}{l} G_2^* \\ \mathcal{G}^* \end{array} \right\}$	G_2 <i>on</i> $\left\{ \begin{array}{l} G_1^* \\ \mathcal{G} \end{array} \right\}$
Left action \triangleright	$c_2 \rightarrow c_1$	$v_2 \rightarrow v_1$
Right action \triangleleft	$c_1 \rightarrow c_2$	$v_1 \rightarrow v_2$

Table 5.3: Geometrically, the left action of G_1 (resp. G_2) on the other groups, transports the decorations from the node c_2 to c_1 (resp. from the vertex v_2 to v_1); the right action of G_1 (resp. G_2) on the other groups, transports the decorations from c_1 to c_2 (resp. from v_1 to v_2).

- * the left action of G_1 on \mathcal{G}^* transports the 2-representation of the pair edge/face from the target c_2 to the source c_1 ;
- * the right action of G_1 on \mathcal{G}^* transports the 2-representation of the pair edge/face from the source c_1 to the target c_2 ;
- * the left action of G_2 on \mathcal{G} transports the 2-representation of the pair link/wedge from the target v_2 to the source v_1 ;
- * the right action of G_2 on \mathcal{G} transports the 2-representation of the pair link/wedge from the source v_1 to the target v_2 .

Moreover,

- the left action of G_1 on G_2^* transports the root of the wedge from c_2 to c_1 ;
- the right action of G_1 on G_2^* transports the root of the wedge from c_1 to c_2 ;
- the left action of G_2 on G_1^* transports the root of the face from v_2 to v_1 ;
- the right action of G_2 on G_1^* transports the root of the face from v_1 to v_2 .

For clarity, I summarize the geometric role of left and right actions in Table 5.3. Note that the geometric interpretation given in Table 5.3 agrees with the convention in Table 5.2.

Symplectic reduction rules

According to the discussion on symplectic reduction and reference frames, the momentum maps used to properly fuse phase spaces are imposed on geometric objects that lie in the same reference frame. Namely, momentum maps are imposed on decorations represented and rooted at the same point, whose indices therefore have to match. This is the fundamental rule to derive the appropriate momentum map for fusing atomic phase spaces \mathcal{B} . The fusion is then performed by merging the related ribbon equations through the momentum map chosen. Note that from the ribbon equation for the Heisenberg double \mathcal{B} and from its sub-decompositions, one can derive a number (sixteen) of different ribbon equations for the sub-components. Each time, one has to choose the proper ribbon equation to use, depending on the momentum map to impose.

Phase space construction

Having now all the necessary ingredients, let me explain the details of the construction of a general simplicial 2-triangulation. As I said a few times now, the idea is to take several atomic phase spaces and perform the proper symplectic reductions, so that the result is a new phase

space with resulting symplectic structure being expressed through the fused ribbon equation. It is important to emphasize that the ribbon equation is mostly useful to express the variables in the correct reference frame and, after the fusion, to derive the glued variables. However, by keeping on repeating several fusions, the expression of the ribbon equation will become very complicated. Fortunately, the symplectic reduction is *associative, commutative* [167] and most importantly, *local*. In the sense that it is possible to apply several reductions to build up arbitrary complicated spaces, and the order in which we implement the momentum maps does not matter. For local, I mean that the fusion of two arbitrary complicated phase spaces, does not depend on all the variables: one can forget about those that are not involved in the identification (momentum map). This implies that it is enough to learn how to fuse atomic phase spaces together, and the same gluings will hold also for more complicated cases. In App. C I listed the four fundamental phase space fusions that one can perform by merging two atomic phase spaces. Now I will use them to list the essential fusions that are needed to build up any arbitrary three dimensional simplicial 2-triangulation. In particular, one needs three symplectic reductions:

➤ the *triangle* construction: symplectic reduction with respect to the group

$$H^t = (G_1^* \times G_1^*) \times G_2^*; \quad (5.46)$$

➤ the *tetrahedron* construction, symplectic reduction with respect to the group

$$H^\tau = G_2^* \times G_1; \quad (5.47)$$

➤ the *gluing of two tetrahedra*, symplectic reduction with respect to the group

$$H^\partial = G_1 \times G_2^*. \quad (5.48)$$

Definition 17. *The phase space \mathcal{P} of a simplicial 2-triangulation \mathcal{T} is given by the symplectic reduction*

$$\mathcal{P} = \left(\times_{\{w_i, l_i, e_i, f_i\}} \mathcal{B} \right) // \left(\times_{t \in \mathcal{T}} H^t \times_{\tau \in \mathcal{T}} H^\tau \times_{t \in \mathcal{T}} H^\partial \right). \quad (5.49)$$

Let me give below the details of these essential three fusions.

Triangle

To determine the phase space of a triangle, take three Heisenberg doubles \mathcal{B}_i , with $i = 1, 2, 3$: three atomic phase spaces associated to the (half) links l_i (with sources at the nodes c_i and targets at $c_{i,f}$), wedges w_i , edges e_i (with sources and targets resp. at the vertices $v_{i,1}$ and $v_{i,2}$) and faces f_i . The idea is to fuse first the three atomic phase spaces into a single surface and then require the closure of its boundary. See Fig. 5.8.

Fuse the first two phase spaces by identifying the links l_1 and l_2 , represented at the resp. vertices $v_{1,2}$ and $v_{2,1}$, by demanding the vertices to match. I refer to the *face gluing* in App. C for more details. The geometric identification induces the momentum constraint

$$u_1 = \tilde{u}_2. \quad (5.50)$$

Add the third phase space, by identifying the links l_2 and l_3 , resp. represented at the vertices $v_{2,2}$ and $v_{3,1}$, and again, let the vertices match. This is another *face gluing* with geometric identification that induces the momentum constraint

$$u_2 = \tilde{u}_3. \quad (5.51)$$

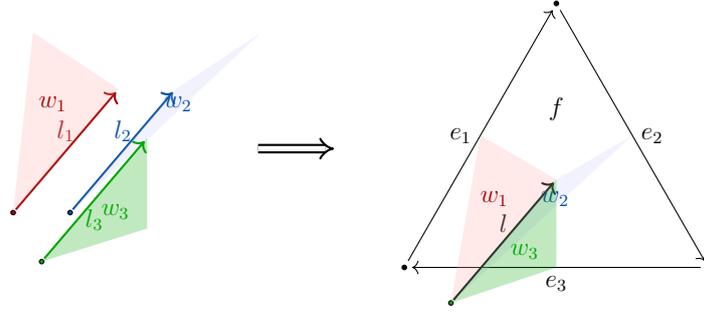


Figure 5.8: The phase space of a triangle is constructed by fusing three atomic phase spaces, by identifying the three links and by closing the triangle boundary. The link identification dually leads to a fused face variable.

Putting everything together, one gets the full ribbon equation

$$\begin{aligned} \tilde{u}_1(\tilde{y}_1\tilde{y}'\tilde{y}'')(\tilde{\beta}_1(\lambda_1 \triangleright \tilde{\beta}_2)\tilde{\beta}'((\lambda_1\lambda_2) \triangleright \tilde{\beta}_3)\tilde{\beta}'')(\tilde{\lambda}_1\tilde{\lambda}_2\tilde{\lambda}_3) \\ = (\lambda_1\lambda_2\lambda_3)((\bar{\beta}_1 \triangleleft (\lambda_2\lambda_3))((\bar{\beta}_2\beta') \triangleleft \lambda_3)\bar{\beta}_3\beta'')(y''\bar{y}_3)u_3, \end{aligned} \quad (5.52)$$

with $\tilde{\lambda}_1\tilde{y}_2\tilde{\lambda}_1^{-1} = \tilde{y}'\tilde{\beta}'$, $\lambda_2^{-1}\bar{y}_1\lambda_2 = y'\beta'$, $(\tilde{\lambda}_1\tilde{\lambda}_2)\tilde{y}_3(\tilde{\lambda}_1\tilde{\lambda}_2)^{-1} = \tilde{y}''\tilde{\beta}''$ and $\lambda_3^{-1}(y'\bar{y}_2)\lambda_3 = y''\beta''$. Last, in order to close the triangle, identify the vertices $v_{3;2}$ and $v_{1;1}$. This identification is implemented through the momentum map (closure constraint)

$$G_2 \ni \lambda_1\lambda_2\lambda_3 = 1. \quad (5.53)$$

The symplectic reduction for a triangle is thus

$$\mathcal{P}_t = (\mathcal{B}_1 \times \mathcal{B}_2 \times \mathcal{B}_3) // ((G_1^* \times G_1^*) \times G_2^*), \quad (5.54)$$

with momentum maps $\{u_1\tilde{u}_2^{-1}, u_2\tilde{u}_3^{-1}, \lambda_1\lambda_2\lambda_3\}$ and fused face variable

$$\bar{\beta}_t = \bar{\beta}_1 \triangleleft (\lambda_2\lambda_3) + \bar{\beta}_2 \triangleleft \lambda_3 + \bar{\beta}_3 + (\lambda_3^{-1}\bar{y}_2\lambda_3)|_{G_1^*} + ((\lambda_2\lambda_3)^{-1}\bar{y}_1(\lambda_2\lambda_3))|_{G_1^*}, \quad (5.55)$$

deduced from the fused ribbon equation.

Tetrahedron

In order to derive the phase space of a tetrahedron, one needs twelve copies of the Heisenberg double, $\mathcal{B}_{a;i}$ for $a = 1, 2, 3, 4$ labelling the four triangles and $i = 1, 2, 3$ labelling the three edges. Each of these atomic phase spaces is associated to the (half) link $l_{a;i}$ (with source at the node $c_{a;i}$ and target at $c_{a;i;f}$), the wedge $w_{a;i}$, the edge $e_{a;i}$ (with source and target at the nodes $v_{a;i;1}$ and $v_{a;i;2}$) and the face $f_{a;i}$.

The first step is to re-construct the four triangles. According to the triangle phase space construction explained in the previous paragraph, one needs to identify the links $l_{a;1} = l_{a;2} = l_{a;3}$, and identify the vertices $v_{a;1;2} = v_{a;2;1}$, $v_{a;2;2} = v_{a;3;1}$ and $v_{a;3;2} = v_{a;1;1}$, for each of the four triangles $a = 1, 2, 3, 4$. These geometric identifications are given by the twelve momentum maps

$$u_{a;1} = \tilde{u}_{a;2}, \quad u_{a;2} = \tilde{u}_{a;3}, \quad \lambda_{a;1}\lambda_{a;2}\lambda_{a;3} = 1, \quad \text{for } a = 1, 2, 3, 4. \quad (5.56)$$

These four phase space fusions lead to four (triangle) ribbon equations of the type (5.52), from which one can deduce the respective four fused face decorations

$$\begin{aligned} \bar{\beta}_{t_1} &= \beta_{1;1} + \bar{\beta}_{1;2} \triangleleft \lambda_{1;3} + \bar{\beta}_{1;3} + (\lambda_{1;3}^{-1}\bar{y}_{1;2}\lambda_{1;3})|_{G_1^*} + ((\lambda_{1;2}\lambda_{1;3})^{-1}\bar{y}_{1;1}(\lambda_{1;2}\lambda_{1;3}))|_{G_1^*}, \\ \bar{\beta}_{t_2} &= \beta_{2;1} + \bar{\beta}_{2;2} \triangleleft \lambda_{2;3} + \bar{\beta}_{2;3} + (\lambda_{2;3}^{-1}\bar{y}_{2;2}\lambda_{2;3})|_{G_1^*} + ((\lambda_{2;2}\lambda_{2;3})^{-1}\bar{y}_{2;1}(\lambda_{2;2}\lambda_{2;3}))|_{G_1^*}, \\ \bar{\beta}_{t_3} &= \beta_{3;1} + \bar{\beta}_{3;2} \triangleleft \lambda_{3;3} + \bar{\beta}_{3;3} + (\lambda_{3;3}^{-1}\bar{y}_{3;2}\lambda_{3;3})|_{G_1^*} + ((\lambda_{3;2}\lambda_{3;3})^{-1}\bar{y}_{3;1}(\lambda_{3;2}\lambda_{3;3}))|_{G_1^*}, \\ \bar{\beta}_{t_4} &= \beta_{4;1} + \bar{\beta}_{4;2} \triangleleft \lambda_{4;3} + \bar{\beta}_{4;3} + (\lambda_{4;3}^{-1}\bar{y}_{4;2}\lambda_{4;3})|_{G_1^*} + ((\lambda_{4;2}\lambda_{4;3})^{-1}\bar{y}_{4;1}(\lambda_{4;2}\lambda_{4;3}))|_{G_1^*}. \end{aligned} \quad (5.57)$$

Each of them is rooted at the vertex $v_{a;1} = v_{a;3;2}$ and represented at the source of the resp. link, node $c_{a;1}$. To construct the tetrahedron phase space, one has to merge the triangles, by identifying each edge of each triangle with one of the edges of each of the other three triangles. Identify the sources of the four links, and denote this common node $c \equiv c_{1;1} = c_{2;1} = c_{3;1} = c_{4;1}$, that will be the center of the tetrahedron. See Fig. 5.9 for the proper combinatorics. I refer to the *wedge gluing* in App. C for more details about this phase space fusions. According to Fig. 5.9, the proper six momentum maps are

$$\lambda_{1;1} = \lambda_{3;3}^{-1}, \quad \lambda_{1;2} = \lambda_{4;3}^{-1}, \quad \lambda_{1;3} = \lambda_{2;1}^{-1}, \quad \lambda_{2;2} = \lambda_{4;2}^{-1}, \quad \lambda_{2;3} = \lambda_{3;1}^{-1}, \quad \lambda_{3;2} = \lambda_{4;1}^{-1}. \quad (5.58)$$

The edge identifications dually leads to the six fused wedge variables

$$\begin{aligned} \bar{y}_{12} &= -\bar{y}_{2;1} + (\lambda_{1;3} \bar{y}_{1;3} \lambda_{1;3}^{-1})|_{G_2^*}, & \bar{y}_{23} &= -\bar{y}_{3;1} + (\lambda_{2;3} \bar{y}_{2;3} \lambda_{2;3}^{-1})|_{G_2^*}, \\ \bar{y}_{13} &= \bar{y}_{1;1} - (\lambda_{3;3} \bar{y}_{3;3} \lambda_{3;3}^{-1})|_{G_2^*}, & \bar{y}_{24} &= \bar{y}_{2;2} - (\lambda_{4;2} \bar{y}_{4;2} \lambda_{4;2}^{-1})|_{G_2^*}, \\ \bar{y}_{14} &= \bar{y}_{1;2} - (\lambda_{4;3} \bar{y}_{4;3} \lambda_{4;3}^{-1})|_{G_2^*}, & \bar{y}_{34} &= \bar{y}_{3;2} - (\lambda_{4;1} \bar{y}_{4;1} \lambda_{4;1}^{-1})|_{G_2^*}. \end{aligned} \quad (5.59)$$

where I denoted \bar{y}_{ab} the fused wedge decoration shared by triangles a and b .

The construction of the tetrahedron phase space is completed with the last momentum map, known as 1-Gauss constraint, that enforces the closure of the four faces. According to Fig. 5.9, only three of the four triangles can share the same vertex (call it v , in red in the figure). To correctly impose the 1-Gauss constraint, the associated momentum map has to be enforced on variables that all lie in the same reference frame. Let the common reference frame be placed at the center of the tetrahedron c and at the vertex v . The variables that decorate the fourth triangle thus have to be transported from the vertex $v_{4;1}$ (in blue) to v , through the edge decorated by $\lambda_{1;1}$. According to the reference frame and parallel transport interpretation, we need to move the reference frame of the fourth triangle from the target to the source of the edge $e_{1;1}$. Such parallel transport is thus implemented as the left translation with respect to the group G_2 (as a sub-group of the Drinfeld double). Once we have the expressions of the four face decorations all in the same reference frame, one can correctly impose the closure of the tetrahedron boundary

$$\bar{\beta}_{t_1} + \bar{\beta}_{t_2} + \bar{\beta}_{t_3} + \lambda_{1;1} \triangleright \bar{\beta}_{t_4} + (\lambda_{1;1} \bar{y}_{t_4} \lambda_{1;1}^{-1})|_{G_1^*} = 0, \quad (5.60)$$

where the face decorations $\bar{\beta}_{t_i}$ are given in (5.57) and the wedge variable \bar{y}_{t_4} is

$$\bar{y}_{t_4} = \bar{y}_{4;3} + (\lambda_{4;3}^{-1} \bar{y}_{4;2} \lambda_{4;3})|_{G_2^*} + ((\lambda_{4;2} \lambda_{4;3})^{-1} \bar{y}_{4;1} (\lambda_{4;2} \lambda_{4;3}))|_{G_2^*}. \quad (5.61)$$

The symplectic reduction for a tetrahedron phase space is thus

$$\begin{aligned} \mathcal{P}_\tau &= (\mathcal{P}_{t_1} \times \mathcal{P}_{t_2} \times \mathcal{P}_{t_3} \times \mathcal{P}_{t_4}) // ((G_2^*)^{\times 6} \times G_1) \\ &= \mathcal{B}_{a;i}^{(12)} // (((G_1^* \times G_1^*) \times G_2^*)^{\times 4} \times ((G_2^*)^{\times 6} \times G_1)), \end{aligned} \quad (5.62)$$

with momentum maps (5.56), (5.58) and (5.60). For the next part, it will be relevant to express the 1-Gauss constraint (5.60) in terms of the fused wedge variables (5.59). To this scope, denote λ_{ab} the edge shared by triangles a and b , as we did for the fused wedges

$$\lambda_{12} \equiv \lambda_{2;1}, \quad \lambda_{13} \equiv \lambda_{1;1}, \quad \lambda_{14} \equiv \lambda_{1;2}, \quad \lambda_{23} \equiv \lambda_{3;1}, \quad \lambda_{24} \equiv \lambda_{2;2}, \quad \lambda_{34} \equiv \lambda_{3;2}, \quad (5.63)$$

so that the constraint (5.60) can be written as

$$\begin{aligned} b_1 + b_2 + b_3 + \lambda_{13} \triangleright b_4 + (\lambda_{12} \bar{y}_{12} \lambda_{12}^{-1})|_{G_1^*} + (\lambda_{13} \bar{y}_{13} \lambda_{13}^{-1})|_{G_1^*} + \lambda_{13} \triangleright (\lambda_{14} \bar{y}_{14} \lambda_{14}^{-1})|_{G_1^*} \\ + (\lambda_{23} \bar{y}_{23} \lambda_{23}^{-1})|_{G_1^*} + \lambda_{12} \triangleright (\lambda_{24} \bar{y}_{24} \lambda_{24}^{-1})|_{G_1^*} + \lambda_{23} \triangleright (\lambda_{34} \bar{y}_{34} \lambda_{34}^{-1})|_{G_1^*} = 0, \end{aligned} \quad (5.64)$$

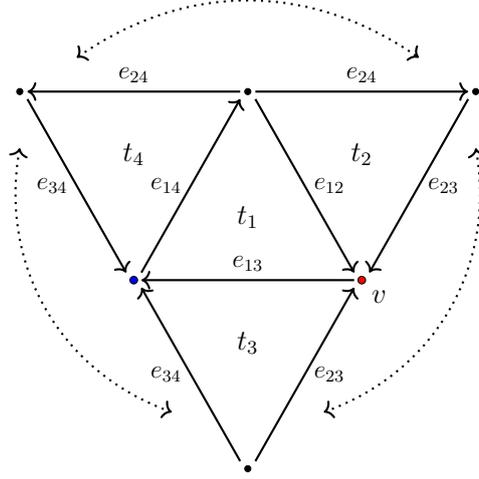


Figure 5.9: Combinatorics of a tetrahedron. The phase space of a tetrahedron is constructed by fusing four triangle phase spaces, by identifying the edges pairwise and by closing the tetrahedron boundary. To properly fuse the edges, they have to lie in the same reference frame: use the tetrahedron center c as common node, and the vertex common to the first three triangles v , in red. Let the fourth triangle be rooted at the vertex in blue; its reference frame thus needs to be transported along the edge $e_{1;1}$.

with the face decorations

$$\begin{aligned}
b_1 &= \beta_{1;1} + \lambda_{13} \triangleright \beta_{1;2} + \bar{\beta}_{1;3} + \lambda_{12} \triangleright (\lambda_{21}^{-1} \bar{y}_{1;3} \lambda_{21})|_{G_1^*} + (\lambda_{13} (\lambda_{14} \bar{y}_{1;2} \lambda_{14}^{-1})|_{G_2^*} \lambda_{13}^{-1})|_{G_1^*}, \\
b_2 &= -\beta_{2;1} + \lambda_{12} \triangleright \beta_{2;2} + \bar{\beta}_{2;3} + \lambda_{34}^{-1} \triangleright (\lambda_{32}^{-1} \bar{y}_{2;3} \lambda_{32})|_{G_1^*} + (\lambda_{12} (\lambda_{24} \bar{y}_{2;2} \lambda_{24}^{-1})|_{G_2^*} \lambda_{12}^{-1})|_{G_1^*}, \\
b_3 &= -\beta_{3;1} + \lambda_{32} \triangleright \beta_{3;2} - \bar{\beta}_{3;3} - \lambda_{13} \triangleright (\lambda_{13}^{-1} \bar{y}_{3;3} \lambda_{13})|_{G_1^*} + (\lambda_{23} (\lambda_{34} \bar{y}_{3;2} \lambda_{34}^{-1})|_{G_2^*} \lambda_{23}^{-1})|_{G_1^*}, \\
b_4 &= -\beta_{4;1} - \lambda_{34}^{-1} \triangleright \beta_{4;2} - \bar{\beta}_{4;3} - \lambda_{13}^{-1} \triangleright ((\lambda_{13} \bar{y}_{4;3} \lambda_{13}^{-1})|_{G_2^*} + \lambda_{12}^{-1} \triangleright (\lambda_{14}^{-1} \bar{y}_{4;3} \lambda_{14})|_{G_1^*} \\
&\quad + (\lambda_{12} \bar{y}_{4;2} \lambda_{12}^{-1})|_{G_2^*} + \lambda_{23} \triangleright (\lambda_{24}^{-1} \bar{y}_{4;2} \lambda_{24})|_{G_1^*} + (\lambda_{23} \bar{y}_{4;1} \lambda_{23}^{-1})|_{G_2^*} + \lambda_{13} \triangleright (\lambda_{34}^{-1} \bar{y}_{4;1} \lambda_{34})|_{G_1^*}).
\end{aligned} \tag{5.65}$$

Note that, each fused wedge decoration \bar{y}_{ab} in eq. (5.64) is subject to a conjugation of the respective dual edge decoration λ_{ab} .

Tetrahedron gluing

In the previous parts I explained how to construct first the phase space of a triangle and then that of a tetrahedron, which is the building block for a three dimensional simplicial 2-triangulation. The last symplectic reduction left needed to fully construct any arbitrary 2-triangulation is the one that allows to merge these building blocks. In order to glue two tetrahedra, one has to identify two of their faces together and the respective boundaries. Consider then two triangle phase spaces \mathcal{P}_a (or equivalently six atomic phase spaces with two sets of momentum maps (5.56)) for $a = 1, 2$. Each phase space is associated to the (half) link l_a (with source at the node c_a and target at $c_{a;f}$), a triplet of wedges w_a , a triplet of boundary edges $e_{a;1}, e_{a;2}, e_{a;3}$ (with resp. sources and targets at the vertices $v_{a;i;1}$ and $v_{a;i;2}$) and a single total face f_a . In this phase space fusion, both the two faces and the pairs of three boundary edges are identified with opposite orientations.

Identify the two faces $f_1 = -f_2$ and the boundary edges as $e_{1;1} = -e_{2;3}$, $e_{1;2} = -e_{2;2}$ and $e_{1;3} = -e_{2;1}$. As reference frame, I will use the targets of the two links (points on the resp. faces) $c_{a;f}$ and the vertices $v_{1;1;1} = v_{1;3;2}$ and $v_{2;1;1} = v_{2;3;2}$. Then, let the two nodes and the two vertices match. I refer to the *link gluing* and *wedge gluing* in App. C for more details. The geometric

identification induces the momentum constraints

$$\tilde{\beta}_1 = \tilde{\beta}_2^{-1}, \quad \lambda_{1;1} = \lambda_{2;3}^{-1}, \quad \lambda_{1;2} = \lambda_{2;2}^{-1}, \quad \lambda_{1;3} = \lambda_{2;1}^{-1}. \quad (5.66)$$

Here the face decorations $\tilde{\beta}_a$ are the fused face variables (5.55) expressed in the reference frame at the target of the link rather than at its source. The total symplectic reduction is thus

$$\begin{aligned} \mathcal{P}_\partial &= (\mathcal{P}_{t_1} \times \mathcal{P}_{t_2}) // (G_1 \times (G_2^*)^{\times 3}) \\ &= \mathcal{B}_{a;i}^{(6)} // (((G_1^* \times G_1^*) \times G_2^*)^{\times 2} \times (G_1 \times (G_2^*)^{\times 3})), \end{aligned} \quad (5.67)$$

with momentum maps (5.66) and fused wedge variables

$$\bar{y}_i = -\bar{y}_{1;i} + (\lambda_{1;i} \bar{y}_{2;i} \lambda_{1;i}^{-1})|_{G_2^*}, \quad (5.68)$$

for $i = 1, 2, 3$. Note that both the in symplectic reduction just presented and in the one related to the tetrahedron phase space construction, I used the *Wedge Gluing* of App. C. Even though the above wedge decorations and those in eq. (5.64) have a similar expression, their geometric interpretation is different. The latter are decorations of the wedges fused *internally* to the tetrahedron, dual to each of its six edges. While the variables derived in this part are decorations of wedges fused *externally* between two different tetrahedra. Note also that, given a pair of atomic phase spaces \mathcal{B} , applying the symplectic reduction with respect to the groups G_1 and G_2^* , leads to a fused phase space isomorphic to the Heisenberg double \mathcal{B} . This is indeed somehow equivalent to take the symplectic reduction with respect to the whole 2-group \mathcal{G} . The geometric interpretation is that each of the initial atomic phase spaces were associated to an half link, a wedge, an edge and a face, while the resulting phase space is associated to a full link, a fused wedge a single (identified) edge and a single (identified) face. In a similar way, the symplectic reduction just explained that provides the gluing of two tetrahedra, takes two triangle phase spaces into a fused face space which is isomorphic to a single triangle phase space. Again, the final phase space will be associated to a full link, a fused (external) wedge, a single triplet of boundary edges and a single face (triangle) decoration.

Applications and examples

1-flatness and edge simplicity

Suppose to have constructed the phase space of n tetrahedra and to glue them in such a way they all share a single edge denoted e . All the fused wedges internal to each of the n tetrahedra are thus fused together into a single closed wedge, dual to the edge e , with fused decoration

$$\bar{y} = \bar{y}_1 + u_1 \triangleright \bar{y}_2 + \cdots + (u_1 \cdots u_{n-1}) \triangleright \bar{y}_n + (u_1 \bar{\beta}_2 u_1^{-1})|_{G_2^*} + ((u_1 \cdots u_{n_1}) \bar{\beta}_n (u_1 \cdots u_{n_1})^{-1})|_{G_2^*}. \quad (5.69)$$

One can compute the ribbon equation associated to this symplectic reduction. I will not do it because its expression would be long and complicated, but given the fundamental gluings presented above, one has all the necessary tools to derive the decoration of the composition of links around such a fused wedge. Dealing with skeletal 2-groups, the total holonomy turns out to be flat:

$$u_1 \cdots u_n = 1. \quad (5.70)$$

Such condition is called 1-flatness constraint [140]. The edge simplicity constraint discussed in [140, 176, 177] is instead a weaker condition: it enforces the triviality of the parallel transport of the edge data around a closed loop of links. If the 1-flatness condition already holds as in our case, then the edge simplicity is automatically implemented.

4-simplex boundary

Having defined the phase space for an arbitrary three dimensional phase space, it is worth to give the explicit construction of one of the most interesting examples (also considered in [140]): the triangulation of a sphere, or equivalently the boundary of a 4-simplex, given as the combination of five tetrahedra.

The starting point are then sixty copies of the Heisenberg double $\mathcal{B}_{A;a;i}$, with $A = 1, 2, 3, 4, 5$ labelling the five tetrahedra, $a = 1, 2, 3, 4$ labelling the four triangles of each tetrahedron and $i = 1, 2, 3$ labelling the three edges of each triangle. The idea is to use the symplectic reduction with respect H^t to construct the twenty triangles, then the symplectic reduction with respect to H^r to construct the five tetrahedra; and finally, ten times the symplectic reduction H^∂ to identify the faces of each tetrahedron and their boundary edges, with a face and its corresponding boundary edges of each of the other four tetrahedra. I refer to Fig. 5.10 for the combinatorics of the 4-simplex boundary. Denote c_A the center of the tetrahedron τ_A . Let $t_{A;a}$ be the triangle a of tetrahedron A and $w_{A;ab}$ the fused wedge shared by triangles a and b of tetrahedron A . By convenience, I will use either the variable $\bar{\beta}_{t_{A;a}}$ of eq. (5.57) or $b_{A;a}$ of eq. (5.65) as decorations of the triangle $t_{A;a}$; while as decoration of the wedge $w_{A;ab}$ I will use the variable $\bar{y}_{A;ab}$.

Let me now illustrate the main steps of the symplectic reduction in consideration. The starting point are thus five tetrahedron phase spaces on which we now implement the tetrahedron gluings. Therefore, first identify the faces according to the Fig. 5.10, and denote t_{AB} the face shared by tetrahedra A and B :

$$\begin{aligned} t_{12} \equiv t_{1;1} &= -t_{2;4} \ , \quad t_{13} \equiv t_{1;2} = -t_{3;3} \ , \quad t_{14} \equiv t_{1;3} = -t_{4;2} \ , \quad t_{15} \equiv t_{1;4} = -t_{5;1} \ , \\ t_{23} \equiv t_{2;1} &= -t_{3;4} \ , \quad t_{24} \equiv t_{2;2} = -t_{4;3} \ , \quad t_{25} \equiv t_{2;3} = -t_{5;2} \ , \quad t_{34} \equiv t_{3;1} = -t_{4;4} \ , \\ t_{35} \equiv t_{3;2} &= -t_{5;3} \ , \quad t_{45} \equiv t_{4;1} = -t_{5;4} \ . \end{aligned} \quad (5.71)$$

These geometric conditions are enforced on the variables $\bar{\beta}_{t_{A;a}}$ through the ten momentum maps

$$\begin{aligned} \bar{\beta}_{12} \equiv \bar{\beta}_{t_{1;1}} &= -(u_{12} \bar{\beta}_{t_{2;4}} u_{12}^{-1})|_{G_1^*} \ , \quad \bar{\beta}_{31} \equiv \bar{\beta}_{t_{3;3}} = -(u_{13}^{-1} \bar{\beta}_{t_{1;2}} u_{13})|_{G_1^*} \ , \\ \bar{\beta}_{14} \equiv \bar{\beta}_{t_{1;3}} &= -(u_{14} \bar{\beta}_{t_{4;2}} u_{14}^{-1})|_{G_1^*} \ , \quad \bar{\beta}_{51} \equiv \bar{\beta}_{t_{5;1}} = -(u_{15}^{-1} \bar{\beta}_{t_{1;4}} u_{15})|_{G_1^*} \ , \\ \bar{\beta}_{23} \equiv \bar{\beta}_{t_{2;1}} &= -(u_{23} \bar{\beta}_{t_{3;4}} u_{23}^{-1})|_{G_1^*} \ , \quad \bar{\beta}_{42} \equiv \bar{\beta}_{t_{4;3}} = -(u_{24}^{-1} \bar{\beta}_{t_{2;2}} u_{24})|_{G_1^*} \ , \\ \bar{\beta}_{25} \equiv \bar{\beta}_{t_{2;3}} &= -(u_{25} \bar{\beta}_{t_{5;2}} u_{25}^{-1})|_{G_1^*} \ , \quad \bar{\beta}_{34} \equiv \bar{\beta}_{t_{3;1}} = -(u_{34} \bar{\beta}_{t_{4;4}} u_{34}^{-1})|_{G_1^*} \ , \\ \bar{\beta}_{53} \equiv \bar{\beta}_{t_{5;3}} &= -(u_{35}^{-1} \bar{\beta}_{t_{3;2}} u_{35})|_{G_1^*} \ , \quad \bar{\beta}_{45} \equiv \bar{\beta}_{t_{4;1}} = -(u_{45} \bar{\beta}_{t_{5;4}} u_{45}^{-1})|_{G_1^*} \ , \end{aligned} \quad (5.72)$$

where I re-named $\bar{\beta}_{AB}$ the decoration of the fused face shared by tetrahedra A and B , rooted at the center of tetrahedron A . Using a similar notation, let me introduce the variable b_{AB} :

$$\begin{aligned} b_{12} \equiv b_{1;1} &= -(u_{12} b_{2;4} u_{12}^{-1})|_{G_1^*} \ , \quad b_{31} \equiv b_{3;3} = -(u_{13}^{-1} b_{1;2} u_{13})|_{G_1^*} \ , \\ b_{14} \equiv b_{1;3} &= -(u_{14} b_{4;2} u_{14}^{-1})|_{G_1^*} \ , \quad b_{51} \equiv b_{5;1} = -(u_{15}^{-1} b_{1;4} u_{15})|_{G_1^*} \ , \\ b_{23} \equiv b_{2;1} &= -(u_{23} b_{3;4} u_{23}^{-1})|_{G_1^*} \ , \quad b_{42} \equiv b_{4;3} = -(u_{24}^{-1} b_{2;2} u_{24})|_{G_1^*} \ , \\ b_{25} \equiv b_{2;3} &= -(u_{25} b_{5;2} u_{25}^{-1})|_{G_1^*} \ , \quad b_{34} \equiv b_{3;1} = -(u_{34} b_{4;4} u_{34}^{-1})|_{G_1^*} \ , \\ b_{53} \equiv b_{5;3} &= -(u_{35}^{-1} b_{3;2} u_{35})|_{G_1^*} \ , \quad b_{45} \equiv b_{4;1} = -(u_{45} b_{5;4} u_{45}^{-1})|_{G_1^*} \ . \end{aligned} \quad (5.73)$$

Dually, each of these G_1^* momentum maps gives rise to a fused link decoration

$$\begin{aligned} u_{12} &= u_{1;1} u_{2;4}^{-1} \ , \quad u_{13} = u_{1;2} u_{3;3}^{-1} \ , \quad u_{14} = u_{1;3} u_{4;2}^{-1} \ , \quad u_{15} = u_{1;4} u_{5;1}^{-1} \ , \quad u_{23} = u_{2;1} u_{3;4}^{-1} \ , \\ u_{24} &= u_{2;2} u_{4;3}^{-1} \ , \quad u_{25} = u_{2;3} u_{5;2}^{-1} \ , \quad u_{34} = u_{3;1} u_{4;4}^{-1} \ , \quad u_{35} = u_{3;2} u_{5;3}^{-1} \ , \quad u_{45} = u_{4;1} u_{5;4}^{-1} \ , \end{aligned} \quad (5.74)$$

where u_{AB} decorates the fused (full) link with source at the center of tetrahedron A and target at the center of tetrahedron B , with inverse $u_{AB}^{-1} = u_{BA}$. According to the tetrahedron gluing, also

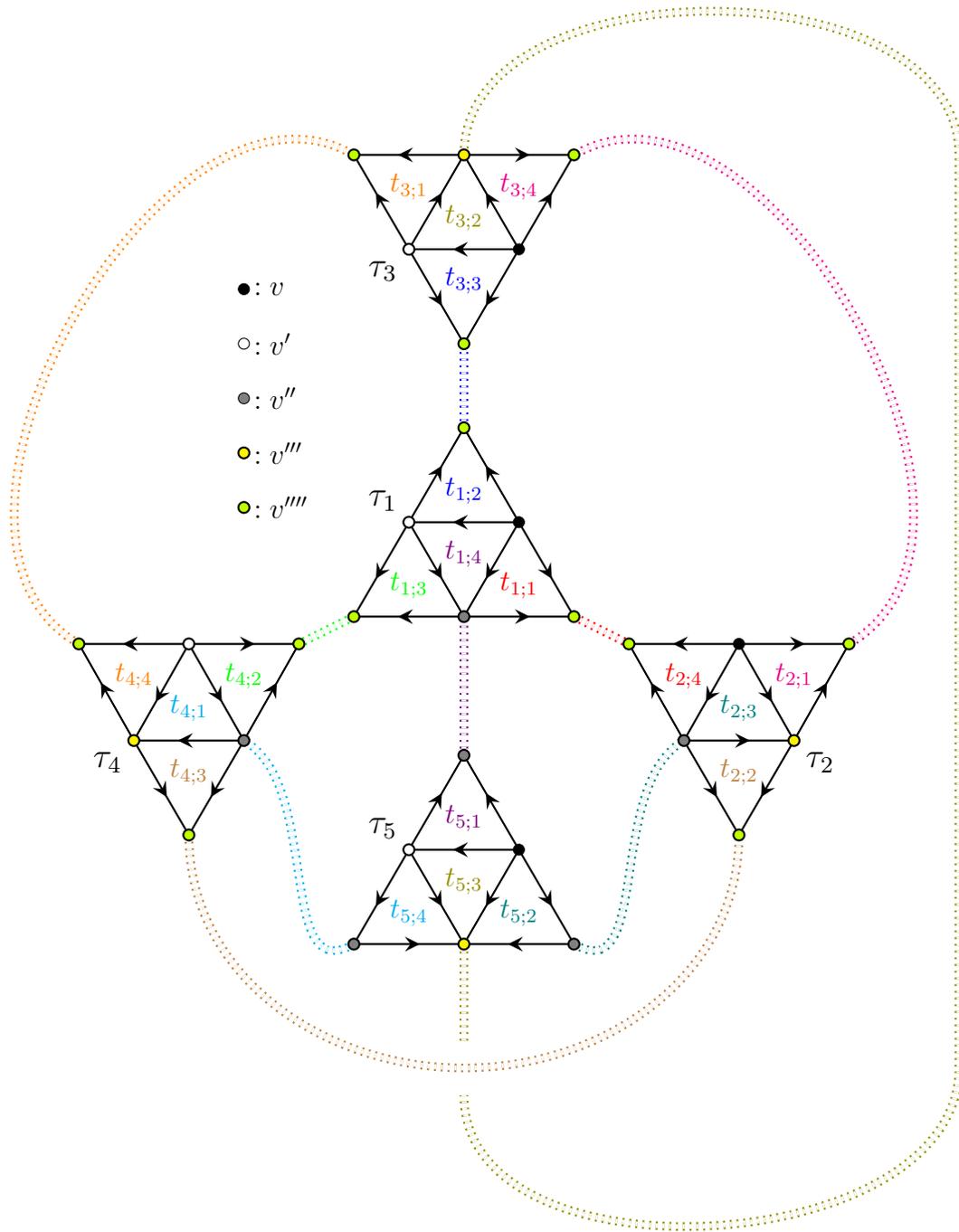


Figure 5.10: 4-simplex boundary construction. Five tetrahedra share five vertices: v (in black) is shared by tetrahedra 1,2,3,5, v' (in white) is shared by tetrahedra 1,3,4,5, v'' (in gray) is shared by tetrahedra 1,2,4,5, v''' (in yellow) is shared by tetrahedra 2,3,4,5 and v'''' (in green) is shared by tetrahedra 1,2,3,4. Each of the four faces of each tetrahedron is identified with one of the faces of the other four tetrahedra. I use the same color and a double dotted line for the identified faces.

the triangle boundaries have to be identified. Use again Fig. 5.10 for the proper combinatorics of the identifications. Denote $\lambda_{A,ab}$ the edge shared by faces a and b of tetrahedron A , represented at the center of it. Let e_{ABC} be the edge shared by tetrahedra A, B, C and λ_{ABC}^D be the respective edge decoration represented at the node c_D defined through the following momentum maps

$$\begin{aligned}
\lambda_{123}^1 &\equiv \lambda_{1;12} = u_{12} \triangleright \lambda_{2;14} = u_{13} \triangleright \lambda_{3;34} , & \lambda_{124}^1 &\equiv \lambda_{1;13} = u_{12} \triangleright \lambda_{2;24} = u_{14} \triangleright \lambda_{4;23} , \\
\lambda_{125}^1 &\equiv \lambda_{1;14} = u_{12} \triangleright \lambda_{2;34} = u_{15} \triangleright \lambda_{5;12} , & \lambda_{134}^1 &\equiv \lambda_{1;23} = u_{13} \triangleright \lambda_{3;13} = u_{14} \triangleright \lambda_{4;24} , \\
\lambda_{135}^1 &\equiv \lambda_{1;24} = u_{13} \triangleright \lambda_{3;23} = u_{15} \triangleright \lambda_{5;13} , & \lambda_{145}^1 &\equiv \lambda_{1;34} = u_{14} \triangleright \lambda_{4;12} = u_{15} \triangleright \lambda_{5;14} , \\
\lambda_{234}^2 &\equiv \lambda_{2;12} = u_{23} \triangleright \lambda_{3;14} = u_{24} \triangleright \lambda_{4;34} , & \lambda_{235}^2 &\equiv \lambda_{2;13} = u_{23} \triangleright \lambda_{3;24} = u_{25} \triangleright \lambda_{5;23} , \\
\lambda_{245}^2 &\equiv \lambda_{2;23} = u_{24} \triangleright \lambda_{4;13} = u_{25} \triangleright \lambda_{5;24} , & \lambda_{345}^3 &\equiv \lambda_{3;12} = u_{34} \triangleright \lambda_{4;14} = u_{35} \triangleright \lambda_{5;34} .
\end{aligned} \tag{5.75}$$

Dually, one gets a fused wedge decoration for each of the ten identified edges

$$\begin{aligned}
\bar{y}_{123}^1 &= \bar{y}_{1;12} + u_{12} \triangleright \bar{y}_{2;14} + u_{13} \triangleright \bar{y}_{3;34} + (u_{12} \bar{\beta}_{23} u_{12}^{-1})|_{G_2^*} + (u_{13} \bar{\beta}_{31} u_{13}^{-1})|_{G_2^*} , \\
\bar{y}_{124}^1 &= \bar{y}_{1;13} + u_{12} \triangleright \bar{y}_{2;24} + u_{14} \triangleright \bar{y}_{4;23} + (u_{14} \bar{\beta}_{42} u_{14}^{-1})|_{G_2^*} , \\
\bar{y}_{125}^1 &= \bar{y}_{1;14} + u_{12} \triangleright \bar{y}_{2;34} + u_{15} \triangleright \bar{y}_{5;12} + (u_{12} \bar{\beta}_{25} u_{12}^{-1})|_{G_2^*} + (u_{15} \bar{\beta}_{51} u_{15}^{-1})|_{G_2^*} , \\
\bar{y}_{134}^1 &= \bar{y}_{1;23} + u_{13} \triangleright \bar{y}_{3;14} + u_{14} \triangleright \bar{y}_{4;24} + (u_{13} \bar{\beta}_{34} u_{13}^{-1})|_{G_2^*} + (u_{14} \bar{\beta}_{41} u_{14}^{-1})|_{G_2^*} , \\
\bar{y}_{135}^1 &= \bar{y}_{1;24} + u_{13} \triangleright \bar{y}_{3;23} + u_{15} \triangleright \bar{y}_{5;13} + (u_{35} \bar{\beta}_{51} u_{35}^{-1})|_{G_2^*} + (u_{35} \bar{\beta}_{53} u_{35}^{-1})|_{G_2^*} , \\
\bar{y}_{145}^1 &= \bar{y}_{1;34} + u_{14} \triangleright \bar{y}_{4;12} + u_{15} \triangleright \bar{y}_{5;14} + (u_{14} \bar{\beta}_{45} u_{14}^{-1})|_{G_2^*} + (u_{15} \bar{\beta}_{51} u_{15}^{-1})|_{G_2^*} , \\
\bar{y}_{235}^2 &= \bar{y}_{2;13} + u_{23} \triangleright \bar{y}_{3;24} + u_{25} \triangleright \bar{y}_{5;23} + (u_{25} \bar{\beta}_{53} u_{25}^{-1})|_{G_2^*} , \\
\bar{y}_{245}^2 &= \bar{y}_{2;23} + u_{24} \triangleright \bar{y}_{4;13} + u_{25} \triangleright \bar{y}_{5;24} + (u_{24} \bar{\beta}_{45} u_{24}^{-1})|_{G_2^*} + (u_{24} \bar{\beta}_{42} u_{24}^{-1})|_{G_2^*} , \\
\bar{y}_{234}^2 &= \bar{y}_{2;12} + u_{23} \triangleright \bar{y}_{3;14} + u_{24} \triangleright \bar{y}_{4;34} + (u_{23} \bar{\beta}_{34} u_{23}^{-1})|_{G_2^*} + (u_{24} \bar{\beta}_{42} u_{24}^{-1})|_{G_2^*} , \\
\bar{y}_{345}^3 &= \bar{y}_{3;12} + u_{34} \triangleright \bar{y}_{4;14} + u_{35} \triangleright \bar{y}_{5;34} + (u_{34} \bar{\beta}_{45} u_{34}^{-1})|_{G_2^*} + (u_{35} \bar{\beta}_{53} u_{35}^{-1})|_{G_2^*} ,
\end{aligned} \tag{5.76}$$

where the variable \bar{y}_{ABC}^D decorates the fused wedge dual to the edge e_{ABC} , rooted at the center of tetrahedron D . The symplectic reduction for the phase space of a 4-simplex boundary is thus

$$\begin{aligned}
\mathcal{P}_{\tau^4} &= \mathcal{B}^{(60)} // \left((((G_1^* \times G_1^*) \times G_2^*)^{\times 4} \times ((G_2^*)^{\times 6} \times G_1))^{ \times 5} \times (G_1^{\times 10} \times (G_2^*)^{\times 10}) \right) \\
&= \mathcal{P}_{\tau}^{(5)} // (G_1^{\times 10} \times (G_2^*)^{\times 10}) ,
\end{aligned} \tag{5.77}$$

with momentum maps (5.72) and (5.75). Last, let us express the 1-Gauss constraints of the five

tetrahedra, in terms of the variables (5.73) as in (5.64)

$$\begin{aligned} \tau_1 : \quad & b_{12} - (u_{13} b_{31} u_{13}^{-1})|_{G_1^*} + \lambda_{135}^1 \triangleright b_{14} - (u_{15} b_{51} u_{15}^{-1})|_{G_1^*} \\ & + (\lambda_{123}^1 \bar{y}_{1;12} (\lambda_{123}^1)^{-1})|_{G_1^*} + \lambda_{125}^1 \triangleright (\lambda_{124}^1 \bar{y}_{1;13} (\lambda_{124}^1)^{-1})|_{G_1^*} + (\lambda_{125}^1 \bar{y}_{1;14} (\lambda_{125}^1)^{-1})|_{G_1^*} \\ & + \lambda_{135}^1 \triangleright (\lambda_{134}^1 \bar{y}_{1;23} (\lambda_{234}^1)^{-1})|_{G_1^*} + (\lambda_{135}^1 \bar{y}_{1;24} (\lambda_{135}^1)^{-1})|_{G_1^*} + \lambda_{135}^1 \triangleright (\lambda_{145}^1 \bar{y}_{1;34} (\lambda_{145}^1)^{-1})|_{G_1^*} = 0, \end{aligned} \quad (5.78)$$

$$\begin{aligned} \tau_2 : \quad & b_{23} - \lambda_{125}^2 \triangleright (u_{24} b_{42} u_{24}^{-1})|_{G_1^*} + b_{25} - (u_{12}^{-1} b_{12} u_{12})|_{G_1^*} \\ & + \lambda_{235}^2 \triangleright (\lambda_{234}^2 \bar{y}_{2;12} (\lambda_{234}^2)^{-1})|_{G_1^*} + (\lambda_{235}^2 \bar{y}_{2;13} (\lambda_{235}^2)^{-1})|_{G_1^*} + (\lambda_{123}^2 \bar{y}_{2;14} (\lambda_{123}^2)^{-1})|_{G_1^*} \\ & + \lambda_{125}^2 \triangleright (\lambda_{245}^2 \bar{y}_{2;23} (\lambda_{245}^2)^{-1})|_{G_1^*} + \lambda_{125}^2 \triangleright (\lambda_{124}^2 \bar{y}_{2;24} (\lambda_{124}^2)^{-1})|_{G_1^*} + (\lambda_{125}^2 \bar{y}_{2;34} (\lambda_{125}^2)^{-1})|_{G_1^*} = 0, \end{aligned} \quad (5.79)$$

$$\begin{aligned} \tau_3 : \quad & \lambda_{135}^3 \triangleright b_{34} - (u_{35} b_{53} u_{35}^{-1})|_{G_1^*} + b_{31} - (u_{23}^{-1} b_{23} u_{23})|_{G_1^*} \\ & + \lambda_{135}^3 \triangleright (\lambda_{345}^3 \bar{y}_{3;12} (\lambda_{345}^3)^{-1})|_{G_1^*} + \lambda_{135}^3 \triangleright (\lambda_{134}^3 \bar{y}_{3;13} (\lambda_{134}^3)^{-1})|_{G_1^*} + \lambda_{235}^3 \triangleright (\lambda_{234}^3 \bar{y}_{3;14} (\lambda_{234}^3)^{-1})|_{G_1^*} \\ & + (\lambda_{135}^3 \bar{y}_{3;23} (\lambda_{135}^3)^{-1})|_{G_1^*} + (\lambda_{235}^3 \bar{y}_{3;24} (\lambda_{235}^3)^{-1})|_{G_1^*} + (\lambda_{123}^3 \bar{y}_{3;34} (\lambda_{123}^3)^{-1})|_{G_1^*} = 0, \end{aligned} \quad (5.80)$$

$$\begin{aligned} \tau_4 : \quad & \lambda_{135}^4 \triangleright (b_{45} - (u_{14}^{-1} b_{14} u_{14})|_{G_1^*} + \lambda_{145}^4 \triangleright b_{42} - (u_{34}^{-1} b_{34} u_{34})|_{G_1^*}) \\ & + \lambda_{135}^4 \triangleright (\lambda_{145}^4 \bar{y}_{4;12} (\lambda_{145}^4)^{-1})|_{G_1^*} + \lambda_{125}^4 \triangleright (\lambda_{245}^4 \bar{y}_{4;13} (\lambda_{245}^4)^{-1})|_{G_1^*} + \lambda_{135}^4 \triangleright (\lambda_{345}^4 \bar{y}_{4;14} (\lambda_{345}^4)^{-1})|_{G_1^*} \\ & + \lambda_{125}^4 \triangleright (\lambda_{124}^4 \bar{y}_{4;23} (\lambda_{124}^4)^{-1})|_{G_1^*} + \lambda_{135}^4 \triangleright (\lambda_{134}^4 \bar{y}_{4;24} (\lambda_{134}^4)^{-1})|_{G_1^*} + \lambda_{235}^4 \triangleright (\lambda_{234}^4 \bar{y}_{4;34} (\lambda_{234}^4)^{-1})|_{G_1^*} = 0, \end{aligned} \quad (5.81)$$

$$\begin{aligned} \tau_5 : \quad & b_{51} - (u_{25}^{-1} b_{25} u_{25})|_{G_1^*} + b_{53} - \lambda_{135}^5 \triangleright (u_{45}^{-1} b_{45} u_{45})|_{G_1^*} \\ & + (\lambda_{125}^5 \bar{y}_{5;12} (\lambda_{125}^5)^{-1})|_{G_1^*} + (\lambda_{135}^5 \bar{y}_{5;13} (\lambda_{135}^5)^{-1})|_{G_1^*} + \lambda_{135}^5 \triangleright (\lambda_{145}^5 \bar{y}_{5;14} (\lambda_{145}^5)^{-1})|_{G_1^*} \\ & + (\lambda_{235}^5 \bar{y}_{5;23} (\lambda_{235}^5)^{-1})|_{G_1^*} + \lambda_{125}^5 \triangleright (\lambda_{245}^5 \bar{y}_{5;24} (\lambda_{245}^5)^{-1})|_{G_1^*} + \lambda_{135}^5 \triangleright (\lambda_{345}^5 \bar{y}_{5;34} (\lambda_{345}^5)^{-1})|_{G_1^*} = 0. \end{aligned} \quad (5.82)$$

Examples

As announced, this construction encodes the particularly interesting case of the Euclidean Poincaré 2-group (with a trivial Poisson structure). I am going to show that, with such choice, one recovers the phase space that was introduced in [140] and again in [178].

Let $\mathcal{G} = \text{ISO}(4) \cong \text{SO}(4) \times \mathbb{R}^4 \cong \mathbb{R}^4 \rtimes \text{SO}(4)$ and $\mathcal{G}^* = \text{ISO}(4)^* \cong \text{SO}^*(4) \times \mathbb{R}^4 \cong \mathbb{R}^4 \times \text{SO}^*(4)$, where $\text{SO}^*(4) \cong \mathbb{R}^6$ as Lie groups. The simplicial 2-triangulation has edges and faces decorated by elements of \mathbb{R}^4 and $\text{SO}^*(4)$ respectively; its dual 2-complex has links and wedges decorated by holonomies of $\text{SO}(4)$ and $\mathbb{R}^{*4} \cong \mathbb{R}^4$ respectively:

$$G_1 = \text{SO}(4) \ni u, \quad G_2^* = \mathbb{R}^4 \ni y, \bar{y}, \quad G_2 = \mathbb{R}^4 \ni \lambda, \tilde{\lambda}, \quad G_1^* = \text{SO}^*(4) \ni \beta, \tilde{\beta}. \quad (5.83)$$

Note that there is no action of \mathbb{R}^4 on $\text{SO}(4)$ nor on $\text{SO}^*(4)$ and thus the conjugations of $\text{SO}(4)$ on $\text{SO}^*(4)$ and of \mathbb{R}^4 on \mathbb{R}^4 , take the simpler shape

$$\begin{aligned} u\beta u^{-1} = \beta' y' & \Leftrightarrow \begin{cases} \beta' = (u\beta u^{-1})|_{\text{SO}^*(4)}, \\ y' = (u\beta u^{-1})|_{\mathbb{R}^4} = 1, \end{cases} \\ \lambda y \lambda^{-1} = y'' \beta'' & \Leftrightarrow \begin{cases} \beta' = (\lambda y \lambda^{-1})|_{\text{SO}^*(4)} \equiv [\lambda, y], \\ y' = (\lambda y \lambda^{-1})|_{\mathbb{R}^4} = y. \end{cases} \end{aligned} \quad (5.84)$$

The triangle phase space is recovered by imposing the momentum maps $u_1 = \tilde{u}_2$ and $u_2 = \tilde{u}_3$, plus the closure constraint $\lambda_1 + \lambda_2 + \lambda_3 = 0$, and leads to the fused face decoration (5.55)

$$\beta_t = \beta_1 + \beta_2 + \beta_3 + \beta_4 - [\lambda_2, \bar{y}_1] - [\lambda_3, \bar{y}_1] - [\lambda_3, \bar{y}_2]. \quad (5.85)$$

Similarly, the tetrahedron phase space is given by four triangle phase spaces fused together by the six momentum maps (5.58), from which one derives the six fused wedge decorations (5.59)

$$\begin{aligned}\bar{y}_{12} &= -\bar{y}_{2;1} + \bar{y}_{1;3}, & \bar{y}_{13} &= \bar{y}_{1;1} - \bar{y}_{3;3}, & \bar{y}_{14} &= \bar{y}_{1;2} - \bar{y}_{4;3}, \\ \bar{y}_{23} &= -\bar{y}_{3;1} + \bar{y}_{2;3}, & \bar{y}_{24} &= \bar{y}_{2;2} - \bar{y}_{4;2}, & \bar{y}_{34} &= \bar{y}_{3;2} - \bar{y}_{4;1}.\end{aligned}\quad (5.86)$$

The last momentum map needed to construct the tetrahedron phase space is the 1-Gauss constraint (5.64), that now reduces to

$$b_1 + b_2 + b_3 + b_4 + [\lambda_{12}, \bar{y}_{12}] + [\lambda_{13}, \bar{y}_{13}] + [\lambda_{14}, \bar{y}_{14}] + [\lambda_{23}, \bar{y}_{23}] + [\lambda_{24}, \bar{y}_{24}] + [\lambda_{34}, \bar{y}_{34}] = 0, \quad (5.87)$$

where

$$\begin{aligned}b_1 &= \beta_{1;1} + \beta_{1;2} + \beta_{1;3} - [\lambda_{21}, \bar{y}_{1;3}] + [\lambda_{13}, \bar{y}_{1;2}], \\ b_2 &= -\beta_{2;1} + \beta_{2;2} + \beta_{2;3} - [\lambda_{32}, \bar{y}_{2;3}] + [\lambda_{12}, \bar{y}_{2;2}], \\ b_3 &= -\beta_{3;1} + \beta_{3;2} - \beta_{3;3} - [\lambda_{13}, \bar{y}_{3;3}] + [\lambda_{23}, \bar{y}_{3;2}], \\ b_4 &= -\beta_{4;1} - \beta_{4;2} - \beta_{4;3} - [(\lambda_{13} - \lambda_{14}), \bar{y}_{4;3}] - [(\lambda_{12} - \lambda_{24}), \bar{y}_{4;2}] - [(\lambda_{23} - \lambda_{34}), \bar{y}_{4;1}].\end{aligned}\quad (5.88)$$

It is now interesting to derive again the phase space of the 4-simplex boundary (the five 1-Gauss constraints (5.82)) to compare the result with the G-network construction of [140]. Follow the steps explained in the previous part, and then consider the change of variables for the face decorations below

$$\begin{aligned}b'_{12} &= b_{12} - u_{12} [\lambda_{123}^2, \bar{y}_{2;14}] u_{12}^{-1} - u_{12} [\lambda_{124}^2, \bar{y}_{2;24}] u_{12}^{-1} - u_{12} [\lambda_{125}^2, \bar{y}_{2;34}] u_{12}^{-1}, \\ b'_{31} &= b_{31} + [\lambda_{134}^3, \bar{y}_{3;13}] + [\lambda_{123}^3, \bar{y}_{3;34}] - u_{13}^{-1} [\lambda_{135}^1, \bar{y}_{1;24}] u_{13}, \\ b'_{14} &= b_{14} - u_{14} [\lambda_{145}^4, \bar{y}_{4;12}] u_{14}^{-1} - u_{14} [\lambda_{124}^4, \bar{y}_{4;23}] u_{14}^{-1} - u_{14} [\lambda_{134}^4, \bar{y}_{4;24}] u_{14}^{-1}, \\ b'_{51} &= b_{51} + [\lambda_{125}^5, \bar{y}_{5;12}] + [\lambda_{145}^5, \bar{y}_{5;14}], \\ b'_{23} &= b_{23} - u_{23} [\lambda_{234}^3, \bar{y}_{3;14}] u_{23}^{-1} - u_{23} [\lambda_{235}^3, \bar{y}_{3;24}] u_{23}^{-1}, \\ b'_{42} &= b_{42} + [\lambda_{245}^4, \bar{y}_{4;13}] + [\lambda_{234}^4, \bar{y}_{4;34}], \\ b'_{25} &= b_{25} - u_{25} [\lambda_{235}^5, \bar{y}_{5;23}] u_{25}^{-1} - u_{25} [\lambda_{245}^5, \bar{y}_{5;24}] u_{25}^{-1}, \\ b'_{34} &= b_{34} - u_{34} [\lambda_{345}^4, \bar{y}_{4;14}] u_{34}^{-1}, \\ b'_{53} &= b_{53} + [\lambda_{135}^5, \bar{y}_{5;13}] + [\lambda_{345}^5, \bar{y}_{5;34}].\end{aligned}\quad (5.89)$$

The five 1-Gauss constraints (5.82) expressed through new variables and in terms of the fused wedge variables (5.76) write

$$\tau_1 : \quad b'_{12} - (u_{13} b'_{31} u_{13}^{-1}) + b'_{14} - (u_{15} b'_{51} u_{15}^{-1}) + [\lambda_{123}^1, \bar{y}_{123}^1] + [\lambda_{124}^1, \bar{y}_{124}^1] + [\lambda_{125}^1, \bar{y}_{125}^1] \\ + [\lambda_{134}^1, \bar{y}_{134}^1] + [\lambda_{145}^1, \bar{y}_{145}^1] = 0, \quad (5.90)$$

$$\tau_2 : \quad b'_{23} - (u_{24} b'_{42} u_{24}^{-1}) + b'_{25} - (u_{12}^{-1} b'_{12} u_{12}) + [\lambda_{235}^2, \bar{y}_{235}^2] + [\lambda_{245}^2, \bar{y}_{245}^2] + [\lambda_{234}^2, \bar{y}_{234}^2] = 0, \quad (5.91)$$

$$\tau_3 : \quad b'_{34} - (u_{35} b'_{53} u_{35}^{-1}) + b'_{31} - (u_{23}^{-1} b'_{23} u_{23}) + [\lambda_{135}^3, \bar{y}_{135}^3] + [\lambda_{345}^3, \bar{y}_{345}^3] = 0, \quad (5.92)$$

$$\tau_4 : \quad b_{45} - (u_{14}^{-1} b'_{14} u_{14}) + b'_{42} - (u_{34}^{-1} b'_{34} u_{34}) = 0, \quad (5.93)$$

$$\tau_5 : \quad b'_{51} - (u_{25}^{-1} b'_{25} u_{25}) + b'_{53} - (u_{45}^{-1} b_{45} u_{45}) = 0. \quad (5.94)$$

Comparing this result with that of [178], one derives the dictionary between the variables here and those in [178], summarized in Table 5.4. This shows that the construction just presented allows to recover exactly the phase space of [140, 178] in the case of the Euclidean Poisson Lie 2-group with a trivial Poisson structure.

The above choice of Euclidean Poincaré 2-group with trivial Poisson structure describes a flat 2-triangulation, with curved the 1-holonomies of its dual 2-complex. Going further, one can input curvature on the edges of the 2-triangulation. Let me mention two interesting realizations of 2-triangulations with curved edges:

	Our notation	Notation in [178]
Face (triangle)	b'_{ij}	$b_{(ij)^*}$
Link	u_{ij}	h_{ij}
Edge e	$\lambda_{e^*}^i$	ℓ_i^e
Wedge/dual face	$Y_{e^*}^i$	$V_i^{e^*}$

Table 5.4: Dictionary between the variables used in this construction and the ones in [178].

- *κ -Poincaré deformation*: consider the choice of Poisson Lie groups

$$G_1 = \text{SO} , \quad G_2 = \text{AN} , \quad G_1^* = \text{SO}^* , \quad G_2^* = \text{AN}^* . \quad (5.95)$$

The underlying Lie bi-algebra is the one described in App. A. This type of deformation would work for any sign of the cosmological constant in the Lorentzian case, and for a negative cosmological constant in the Euclidian case.

- *Self dual case*: consider the choice of Poisson Lie groups

$$G_1 = \text{SU} , \quad G_2 = \text{SU} , \quad G_1^* = \mathbb{R}^3 , \quad G_2^* = \mathbb{R}^{*3} . \quad (5.96)$$

This deformation can be seen as the double of the Drinfeld double of $\text{SU}(2)$ (as a Poisson Lie group) and can be recovered by semi-dualizing the co-tangent bundle $T^*\text{Spin}(4)$.

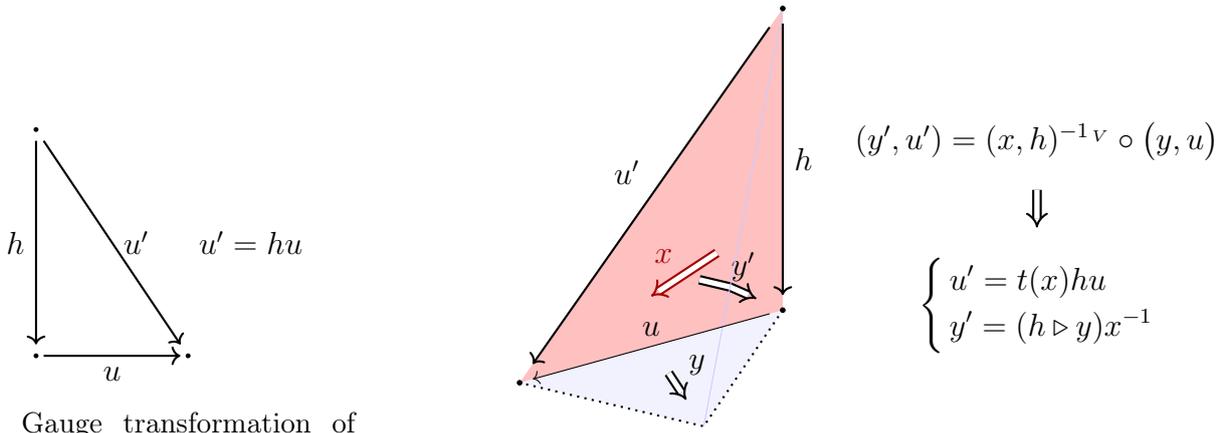
5.3 2-Group field theory

In this section I will provide a model of four dimensional group field theory based on 2-groups, based on [48]. I present the main ingredients of the model in the 2-complex picture following the pattern used in Ch. 3 for ordinary GFT and in Ch. 4 for HAFT. In particular, I will show that this new model, which can be called 2-group field theory, is a topological invariant; moreover, the 2-complexes generated by the 2-group field theory are the same 2-graphs introduced in [47] and presented in the previous section. This allows to relate the 2-group field theory to the G-networks of [140] and thus to the KBF model.

Notation. Consider the 2-complex dual to a three dimensional simplicial 2-triangulation and decorate it with the crossed module $\mathcal{G} = (G_1, G_2, t, \triangleright)$, where the groups G_i can be finite groups or Lie groups. In the latter case, consider only the class of crossed modules where the groups G_2 is unimodular¹¹. Moreover, the Haar measure on the 2-group \mathcal{G} is taken to be the one of the semi-direct product $G_2 \rtimes G_1$, where the measure on G_2 is invariant under the left action of G_1 . Assume further that both the groups G_i have a trivial Poisson structure, so that the dual 2-group $\mathcal{G}^* = (G_1^*, G_2^*, t, \triangleleft)$ is characterized by a pair of abelian groups. As a consequence, the model will only describe flat simplicial 2-triangulations, with non-trivial 1- and 2-holonomies.

It is worth to fix notations here once for all, this also provides a better geometric understanding. In the following, 4-simplices will be labelled by greek letters α , tetrahedra τ_A by capital latin letters $A = 1, \dots, 5$, triangles t_i by $i = 1, \dots, 4$ and edges e_a by $a = 1, 2, 3$. The wedges of a tetrahedron – decorated by $y \in G_2$ – will thus be labelled by quadruplets $\{\alpha; A; i; a\}$; while the

¹¹While the construction can be probably extended to a more general Lie groups, it would complicate the presentation and it would not bring any worth additional value, since the main goal here is just the introduction of a GFT based on 2-groups, with no over complicated requirements.



(a) Gauge transformation of $g \in \mathcal{G}$ with respect to $h \in \mathcal{G}$, represented as a closed loop in the gauge fiber.

(b) 2-Gauge transformation of $(y, u) \in \mathcal{G}$ with respect $(x, h) \in \mathcal{G}$, again represented as a loop in the gauge fiber, which in this case is a closed surface.

links – resp. decorated by $u \in G_1$ – will be labelled by triplets $\{\alpha; A; i\}$. A similar notation will be used for the gauge variables that will be introduced later on, and for the nodes: center of 4-simplices are labelled by c_α , centers of tetrahedra are $c_{\alpha;A}$ and centers of triangles are $c_{\alpha;A;i}$. In all these cases, when it will not generate ambiguities, the 4-simplex index α will be omitted.

Remarks on lattice 2-gauge theories. similarly to ordinary GFT's, 2-group field theory is a 2-gauge theory, hence before going into its details, let me give an overview of (lattice) 2-gauge theories. Suppose to have a cellular decomposition (such as the complex dual to a simplicial triangulation), and decorate the links with holonomies of a group G , say elements g . Gauge theories are thus based on the internal symmetry of the model under the action of a gauge group. Taking the gauge group to be G itself, such action is simply given by the left and/or right multiplication; take for instance the left one. In this case, The gauge transformation

$$g \rightarrow g' = hg, \quad (5.97)$$

can be represented as a closed loop generated by the gauge fiber, see Fig. 5.11a. The picture for 2-groups is analogous. In this case, one would talk about 2-gauge theories. These can be seen as the straight generalization of gauge theories. Consider a 2-group \mathcal{G} with element (y, u) . The 2-gauge transformation parameterized by a 2-group with element (x, h) . If the gauge 2-group is \mathcal{G} itself, its action is given by the composition of 2-group elements in the gauge fiber, which is now a closed surface. can be realized as the horizontal composition

$$(y, u) \rightarrow (y', u') = (x, h)^{-1v} \circ (y, u), \quad (5.98)$$

and is represented in Fig. 5.11b.

5.3.1 Basic ingredients of the model

Field and 2-gauge transformation

In analogy with others GFT models, the fundamental degrees of freedom are encoded in the field. The latter, for a 3+1 dimensional 2-group field theory, is the 2-graph dual to a tetrahedron. While for ordinary GFT's the graph dual to a d -simplex would be simply given by $d+1$ independent links sharing a common node, in this case, the 2-graph dual to a d -simplex is given by the $d+1$ links, plus the $\frac{1}{2}d(d+1)$ wedges shared by each pair of links. In the $d = 3$ case, the 2-graph

dual to a tetrahedron is indeed given by four links (dual to the faces) and six wedges (dual to the edges). This is the fundamental information that has to be encoded in the field of 2-GFT. We immediately face the first difference between GFT and 2-GFT. Note that 2-group elements are given as pairs $(y, u) \in \mathcal{G}$ decorating a single wedge and a single link. While in the ordinary group case there are no ambiguities in defining the field as a function on four copies of a group (four links), in this case the number of links (four) and that of the wedges (six) do not match, and this prevents to define the field as a function on a number of copies of \mathcal{G} .

To solve this issue, the phase space construction presented in the previous section comes to help. In 2-group field theory, instead of dealing with phase spaces, one has to take the polarization of that construction in the dual 2-complex, but the idea does not change. According to the phase space construction, the graph dual to a tetrahedron is given by twelve fundamental (atomic) pairs of wedge/link, say elements $(y_{i;a}, u_{i;a}) \in \mathcal{G}$, fused together to construct first the four triangles and then by merging them to properly build the tetrahedron boundary. I will thus introduce the field of 2-group field theory in two steps. Let the *fundamental field* be a function $\phi \in F(\mathcal{G}^{\times 12})$ on twelve copies of the 2-group \mathcal{G} . Then, the *geometric field* Φ is constructed as the fundamental field plus the proper geometric constraints that turn it into the correct decoration of the 2-graph dual to a tetrahedron.

Definition 18 (Geometric field). *The geometric field is a function of four copies of the group G_1 and six copies of the group G_2 , defined as*

$$\Phi(\{Y_{i,j}; u_i\}) = \int du^{12} dy^{12} \prod_{i=1}^4 (\delta_{G_1}(u_i^{-1} u_{i;1}) \delta_{G_1}(u_i^{-1} u_{i;2}) \delta_{G_1}(u_i^{-1} u_{i;3})) \delta_{\{Y\}}(Y_{i,j}; y_{i;a}) \phi(\{(y_{i;a}, u_{i;a})\}), \quad (5.99)$$

where ϕ is the fundamental field and the symbol $\delta_{\{Y\}}$ stands for the combination of six delta functions on the group G_2 that enforce the definition of six fused internal wedges $Y_{i,j}$ (see also (5.59) in the construction of the tetrahedron phase space) in terms of the twelve wedges $y_{i;a}$,

$$\begin{aligned} Y_{1,2} &= y_{1;3}^{-1} y_{2;1}, & Y_{1,3} &= y_{1;1}^{-1} y_{3;3}, & Y_{1,4} &= y_{1;2}^{-1} y_{4;3}, \\ Y_{2,3} &= y_{2;3}^{-1} y_{3;1}, & Y_{2,4} &= y_{2;2}^{-1} y_{4;2}, & Y_{3,4} &= y_{3;2}^{-1} y_{4;1}. \end{aligned} \quad (5.100)$$

As emphasized before, the geometric field is not a function on a 2-group, but rather a function on the groups G_1 and G_2 separately. Nevertheless, thanks to its definition in terms of the constraints on links and wedges, it is possible to keep track of the initial 2-group elements.

The geometric field is invariant under the 2-gauge transformation (5.98). Let the projected geometric field be

$$\begin{aligned} (\mathcal{P}\Phi)(\{Y_{i,j}; u_i\}) &= \int dx^4 dh dy^{12} du^{12} \prod_{i=1}^4 \left(\delta_{G_1}(u_i^{-1} t(x_i) h u_{i;1}) \delta_{G_1}(u_i^{-1} t(x_i) h u_{i;2}) \right. \\ &\quad \left. \delta_{G_1}(u_i^{-1} t(x_i) h u_{i;3}) \right) \delta_{\{Y\}}(Y_{i,j}; x_i^{-1} y_{i;a}) \phi(\{(x_i^{-1} h \triangleright y_{i;a}, t(x_i) h u_{i;a})\}). \end{aligned} \quad (5.101)$$

The 2-gauge transformation for a single 2-group element $(y_{i;a}, u_{i;a})$ is illustrated in Fig. 5.11b. The above equation is called *gauge averaging*, and the invariance of the field under the 2-gauge transformation is

$$\Phi = (\mathcal{P}\Phi). \quad (5.102)$$

The geometric interpretation is the same given in Ch. 3 for ordinary 3d GFT and Ch. 4 for HAFT; hence, without repeating the same concepts too many times, here I would just like to

emphasize that the elements Y and u will be part of the 2-graph dual to the boundary of the four dimensional Feynman diagrams of 2-GFT, while the wedges x and the link h will decorate the bulk 2-graph. I will thus call them respectively boundary and bulk (or eventually gauge) variables. According to the combinatorics of a tetrahedron, here there is a single bulk wedge decoration x_i for all the three wedges $y_{i;a}$ dual to the three edges of a triangle, and a single bulk link decoration h for all the elements.

Note that, in order to construct the 2-graphs with the proper combinatorics, one needs to demand the field to be invariant under permutations $\sigma \in S_4$, up to the parity of the permutation. In the following, I will typically assume this symmetry without making it explicit in order to not burden the notations.

Action and partition function

The action of 2-GFT is given by the contribution of a kinematic and an interaction term

$$\mathcal{S} = \mathcal{S}_{\mathcal{K}} + \mathcal{S}_{\mathcal{V}}. \quad (5.103)$$

The kinetic term is given by the product of a pair of fields

$$\mathcal{S}_{\mathcal{K}} = \int dy^{12} du^{12} \phi(\{(y_{i;a}, u_{i;a})\}) \phi(\{(y'_{i;a}, u'_{i;a})\}). \quad (5.104)$$

It can be written as an integral operator

$$\mathcal{S}_{\mathcal{K}} = \int dy^{24} du^{24} \mathcal{K} \phi(\{(y_{i;a}, u_{i;a})\}) \phi(\{(y'_{i;a}, u'_{i;a})\}), \quad (5.105)$$

where

$$\mathcal{K} = \prod_{i=1}^4 \prod_{a=1}^3 \delta_{G_2}(y_{i;a}^{-1} y'_{i;a}) \delta_{G_1}(u_{i;a}^{-1} u'_{i;a}) \quad (5.106)$$

is the kernel of the *propagator amplitude*. The interaction term is given by the proper non-local product of five projected geometric fields $\mathcal{P}\Phi$, such that they respect the combinatorics of a 4-simplex. Once again, such product can be written as an integral operator

$$\mathcal{S}_{\mathcal{V}} = \int dY^{30} du^{20} dX^{10} dh^5 \mathcal{V} \phi^{\times 5}, \quad (5.107)$$

where the term \mathcal{V} is the kernel of the *4-simplex amplitude* in the dual 2-complex. In analogy with ordinary GFT's, one expects the interaction term to encode two types of holonomies:

- * a generalization of the flat 1-holonomies given as combination of bulk variables h 's and boundary data u 's, plus eventual contributions from the t -map of the bulk wedges x ;
- * a flat 2-holonomy generated by the bulk and boundary data x and y .

These terms can be viewed as the generalization of the closure of Fig. 3.3b, and are represented in Fig. 5.12. The kernel \mathcal{V} amounts to

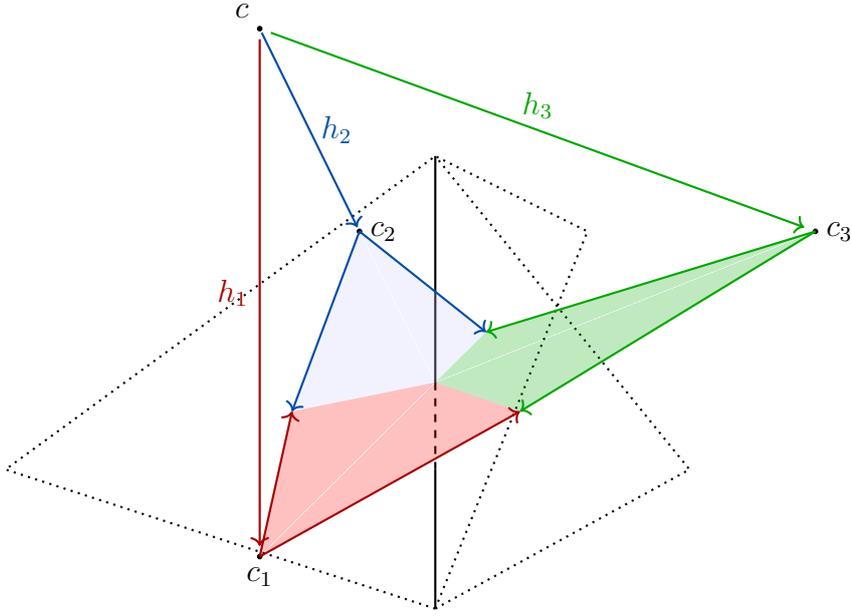


Figure 5.12: A portion of the interaction term. The dotted lines are the combination of three faces shared by three tetrahedra and sharing a common edge (the black solid line). The three solid lines h_i are three of the five bulk links of the 4-simplex, with source at the node c (center of the 4-simplex), and targets resp. at c_A (centers of the three tetrahedra). In different colors the three fused internal boundary wedges Y and their respective boundary links. The three fused boundary wedges (six half wedges $u_{a;i}$) combine into the total wedge dual the the central edge.

$$\begin{aligned}
\mathcal{V} = & \delta_{G_1}(t(X_{2,1})h_1u_{1;1}u_{2;4}^{-1}h_2^{-1}) \delta_{G_1}(t(X_{3,1})h_1u_{1;2}u_{3;3}^{-1}h_3^{-1}) \delta_{G_1}(t(X_{4,1})h_1u_{1;3}u_{4;2}^{-1}h_4^{-1}) \\
& \delta_{G_1}(t(X_{5,1})h_1u_{1;4}u_{5;1}^{-1}h_5^{-1}) \delta_{G_1}(t(X_{3,2})h_2u_{2;1}u_{3;4}^{-1}h_3^{-1}) \delta_{G_1}(t(X_{4,2})h_2u_{2;2}u_{4;3}^{-1}h_4^{-1}) \\
& \delta_{G_1}(t(X_{5,2})h_2u_{2;3}u_{5;2}^{-1}h_5^{-1}) \delta_{G_1}(t(X_{4,3})h_3u_{3;1}u_{4;4}^{-1}h_4^{-1}) \delta_{G_1}(t(X_{5,3})h_3u_{3;2}u_{5;3}^{-1}h_5^{-1}) \\
& \delta_{G_1}(t(X_{5,4})h_4u_{4;1}u_{5;4}^{-1}h_5^{-1}) \\
& \delta_{G_2}((h_1 \triangleright Y_{1;2,1}) X_{1,2} (h_2 \triangleright Y_{2;4,1}) X_{2,3} (h_3 \triangleright Y_{3;4,3}) X_{3,1}) \\
& \delta_{G_2}((h_1 \triangleright Y_{1;3,1}) X_{1,2} (h_2 \triangleright Y_{2;4,2}) X_{2,4} (h_4 \triangleright Y_{4;3,2}) X_{4,1}) \\
& \delta_{G_2}((h_1 \triangleright Y_{1;4,1}) X_{1,2} (h_2 \triangleright Y_{2;4,3}) X_{2,5} (h_5 \triangleright Y_{5;2,1}) X_{5,1}) \\
& \delta_{G_2}((h_1 \triangleright Y_{1;3,2}) X_{1,3} (h_3 \triangleright Y_{3;3,1}) X_{3,4} (h_4 \triangleright Y_{4;4,2}) X_{4,1}) \\
& \delta_{G_2}((h_1 \triangleright Y_{1;4,2}) X_{1,3} (h_3 \triangleright Y_{3;3,2}) X_{3,5} (h_5 \triangleright Y_{5;3,1}) X_{5,1}) \\
& \delta_{G_2}((h_1 \triangleright Y_{1;3,4}) X_{1,4} (h_4 \triangleright Y_{4;2,1}) X_{4,5} (h_5 \triangleright Y_{5;4,1}) X_{5,1}) \\
& \delta_{G_2}((h_2 \triangleright Y_{2;2,1}) X_{2,3} (h_3 \triangleright Y_{3;4,1}) X_{3,4} (h_4 \triangleright Y_{4;4,3}) X_{4,2}) \\
& \delta_{G_2}((h_2 \triangleright Y_{2;3,1}) X_{2,3} (h_3 \triangleright Y_{3;4,2}) X_{3,5} (h_5 \triangleright Y_{5;3,2}) X_{5,2}) \\
& \delta_{G_2}((h_2 \triangleright Y_{2;3,2}) X_{2,4} (h_4 \triangleright Y_{4;3,1}) X_{4,5} (h_5 \triangleright Y_{5;4,2}) X_{5,2}) \\
& \delta_{G_2}((h_3 \triangleright Y_{3;2,1}) X_{3,4} (h_4 \triangleright Y_{4;4,1}) X_{4,5} (h_5 \triangleright Y_{5;4,3}) X_{5,3}). \tag{5.108}
\end{aligned}$$

The first ten deltas (on the group G_1) enforce the closure of the links dual to the ten identified faces of a 4-simplex while the other ten delta functions (on the group G_2) enforce the closure of the combination of bulk and boundary wedges around the ten edges of the 4-simplex. In order to express such closures in a compact way, I introduced the decorations of the fused gauge wedges $X_{A,B}$ between tetrahedra A, B :

$$\begin{aligned}
X_{1,2} &= x_{1;1}^{-1} x_{2;4}, & X_{1,3} &= x_{1;2}^{-1} x_{3;3}, & X_{1,4} &= x_{1;3}^{-1} x_{4;2}, & X_{1,5} &= x_{1;4}^{-1} x_{5;1}, \\
X_{2,3} &= x_{2;1}^{-1} x_{3;4}, & X_{2,4} &= x_{2;2}^{-1} x_{4;3}, & X_{2,5} &= x_{2;3}^{-1} x_{5;2}, & X_{3,4} &= x_{3;1}^{-1} x_{4;4}, \\
X_{3,5} &= x_{3;2}^{-1} x_{5;3}, & X_{4,5} &= x_{4;1}^{-1} x_{5;4}, & & & &
\end{aligned} \tag{5.109}$$

with inverses $X_{A,B}^{-1} = X_{B,A}$. I refer again to Fig. 5.10 for the 4-simplex combinatorics. 2-Group field theory can be formulated as a state sum model, with partition function given by the sum over all the 2-complexes Γ^* dual to simplicial four dimensional 2-triangulations Γ :

$$\mathcal{Z} = \sum_{\Gamma^*} \int dX dh \prod_t \delta_{G_1}(h_{tt}(X_t)) \prod_e \delta_{G_2}(X_e), \quad (5.110)$$

where $X_t \in G_2$ is the bulk wedge dual to the triangle $t \subset \Gamma$ and $h_t \in G_1$ decorates the boundary of this face; while X_e stands for the combination of closed bulk wedges dual to a single edge $e \subset \Gamma$. Note that this is a natural generalization of the amplitude that would be obtained in an ordinary four dimensional GFT, with the extra information coming from the 2-holonomies $X_e \in G_2$ that encode the curvature around a single edge. As stated at the beginning of this chapter, a (topological) model based on 2-groups encodes both the curvature around an edge and that around a vertex of a cellular decomposition.

5.3.2 Topological invariance

2-Group field theory is a topological model. As stated for for HAFT of Ch. 4, the topological invariance emerges from the invariance of the partition function (5.110) under the Pachner moves. The latter are transformations that map (the amplitude associated to) a given cellular decomposition to (the amplitude of) another topologically equivalent cellular decomposition. In four dimensions there exist three Pachner moves.

- * $P_{1,5}$: relates the amplitude of one 4-simplex \mathcal{V} to the amplitude of the combination of five 4-simplices \mathcal{V}_5 ;
- * $P_{2,4}$: relates the amplitude of the combination of two 4-simplices \mathcal{V}_2 to the amplitude of the combination of four 4-simplices \mathcal{V}_4 ;
- * $P_{3,3}$: relates the amplitude of the combination of three 4-simplices \mathcal{V}_3 to the amplitude of the combination of three 4-simplices \mathcal{V}_3 .

The key Feynman diagrams for the four dimensional Pachner moves are thus the 2-graphs dual to a single 4-simplex, and the ones dual to a combination of two, three, four or five 4-simplices. I listed the explicit expression of the associated five amplitudes in App. D.1, denoted $\mathcal{A}_{\mathcal{V}_N}$ for $N = 1, 2, 3, 4, 5$. Given these five amplitudes, one can check the invariance of the model by computing the Pachner moves. I give a few details of the computation in App. D.2. Let me summarize here the results. Let V_{G_i} be the volume of the group G_i .

- * **Pachner move** $P_{1,5}$ is illustrated in Fig. 5.13. The amplitude of five 4-simplices $\mathcal{A}_{\mathcal{V}_5}$, upon integration of the bulk variables, is proportional to the amplitude of a single 4-simplex $\mathcal{A}_{\mathcal{V}}$:

$$\mathcal{A}_{\mathcal{V}_5} = (V_{G_1}^4 V_{G_2}^{37}) \mathcal{A}_{\mathcal{V}}. \quad (5.111)$$

- * **Pachner move** $P_{2,4}$ is illustrated in Fig. 5.14. The amplitude of four 4-simplices $\mathcal{A}_{\mathcal{V}_4}$, upon integration of the bulk variables, is proportional to the amplitude of two 4-simplices $\mathcal{A}_{\mathcal{V}_2}$:

$$\mathcal{A}_{\mathcal{V}_4} = (V_{G_1}^2 V_{G_2}^{19}) \mathcal{A}_{\mathcal{V}_2}. \quad (5.112)$$

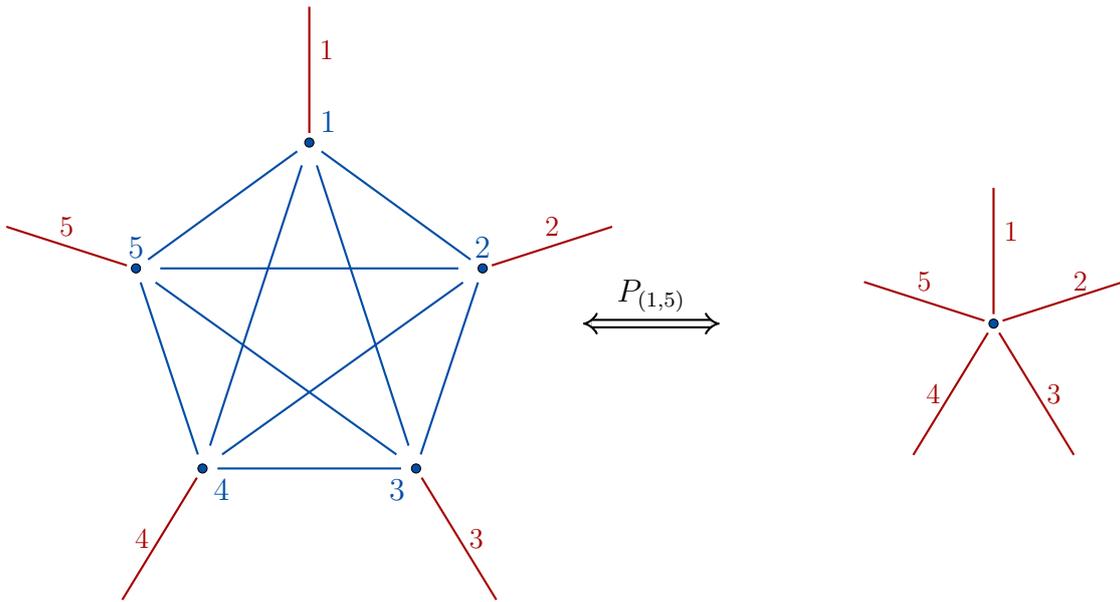


Figure 5.13: Pachner move $P_{(1,5)}$. On the left the graph dual to five 4-simplices: in blue the bulk links shared by pairs of tetrahedra and in red the bulk links dual to the five tetrahedra of the boundary. Upon integration over the internal (blue) links, only the external (red) links remain and the graph reduces to the one dual to a single 4-simplex, on the right.

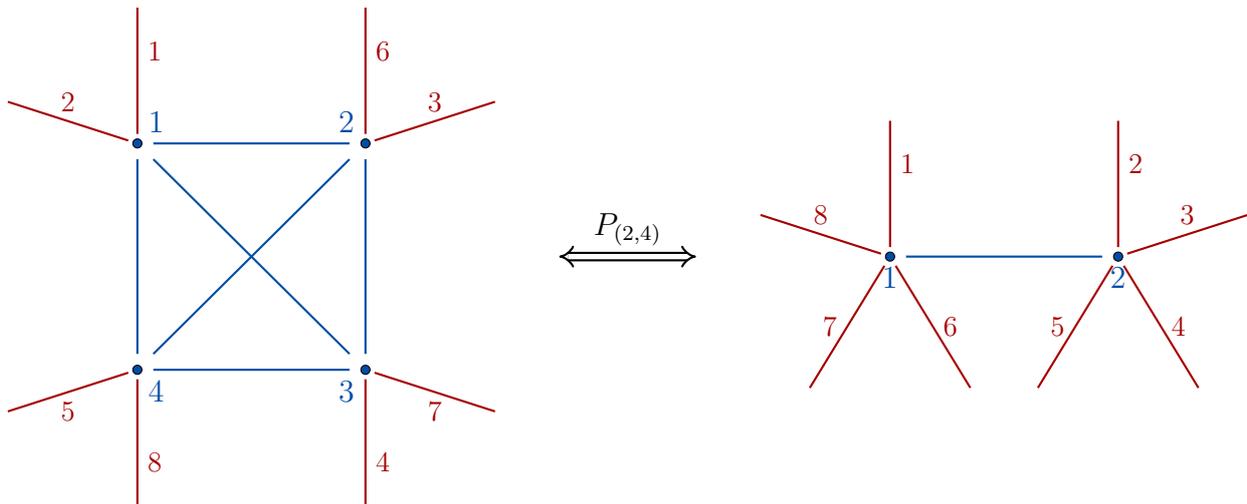


Figure 5.14: Pachner move $P_{(2,4)}$. On the left the graph dual to four 4-simplices: in blue the bulk links shared by pairs of tetrahedra and in red the bulk links dual to the eight tetrahedra of the boundary. Upon integration over the internal (blue) links, only the external (red) links remain and the graph reduces to the one dual to a pair of 4-simplices with a single bulk link (in blue) connecting their centers, on the right.

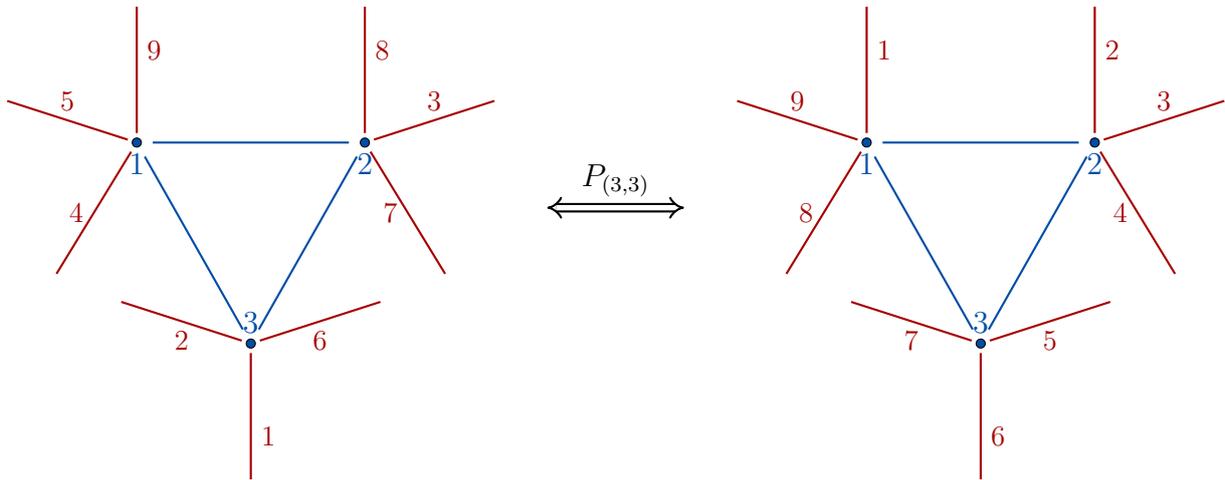


Figure 5.15: Pachner move $P_{(3,3)}$. On the left the graph dual to three 4-simplices: in blue the bulk links dual to the nine boundary tetrahedra. Upon a change of variables, the three boundary tetrahedra of each of the three 4-simplices become the boundary tetrahedra of three different (new) 4-simplices, on the right.

* **Pachner move** $P_{3,3}$ is illustrated in Fig. 5.14. The amplitude of three 4-simplices $\mathcal{A}_{\mathcal{V}_3}$ is equivalent to the amplitude of three 4-simplices $\mathcal{A}_{\mathcal{V}_3}$ with a different combinatorics:

$$\mathcal{A}_{\mathcal{V}_3} = \mathcal{A}_{\mathcal{V}_3}. \quad (5.113)$$

5.4 Remarks and perspectives

In this final chapter I discussed 2-group theory and an eventual applications of its concepts in a theory of four dimensional discrete geometries. The first step was the construction of the phase space of a cellular decomposition of a three dimensional manifold with decorations on 1- and 2-holonomies; these cellular decompositions are interpreted as the states that would be used to construct the eventual Hilbert space at the quantum level. I discussed how to construct such phase space where the decorations of 1- and 2-holonomies are associated to the elements of a skeletal 2-group, regarded as the semi direct product of groups; the symplectic reduction was used as basic tool to merge several phase spaces, where three particular reductions are identified as the essential ones to build up any arbitrary phase space. In the second part of the chapter I defined a field theory, based on 2-groups, where the states of the new model are defined accordingly to the previous construction. The Feynman diagrams of 2-group field theory are the 2-complexes dual to the cellular decompositions of a four dimensional manifold. The model was proven to be topologically invariant.

I want to mention a few possible related future works.

Phase space for a general 2-group. The initial phase space construction was restricted to the case of skeletal 2-groups. Here it would be interesting to repeat the same construction for a general pair of dual 2-groups; this would require a more appropriate definition of 2-Heisenberg double and 2-Drinfeld double [166].

Quantum 2-groups. The 2-group field theory instead is focused on a 2-group with trivial Poisson structure. This is in direct analogy with ordinary GFT's and implies that we can only

describe flat discrete geometries. As in Ch. 4 I provided a generalization of group field theory based on quantum groups, one should also generalize the 2-group field theory to the quantum 2-group case. However, only a few proposals for quantum 2-groups have been made so far [179, 180], and it is not clear whether and how they are compatible each other. In App. E I also provide a new proposal for a 2-Hopf algebra of functions on a 2-group, that might be compatible with 2-group field theory. Unfortunately, the lack of a complete and widely accepted theory of 2-Hopf algebras prevents the rigorous formulation of a 2-group field theory that describes curved geometries.

Fourier transform for 2-groups. The 2-group field theory is only presented in the dual 2-complex picture. Similarly to ordinary GFT's or to HAFT, one would like to define a Fourier transform between (quantum) 2-groups, which allows to map the 2-group field theory just presented, to a 2-group field theory expressed in the 2-triangulation picture. The latter is then expected to coincide with the Mackaay-Yetter model. [154]. The phase space construction presented in the first part of this chapter should also provide an intuition of the properties that such Fourier transform is expected to satisfy. Indeed, it has to reflect the proper combinatorics of a 2-triangulation and its dual 2-complex, and it has to provide a map between the momentum maps (identification of geometric objects) and the dual gluing of variables. This type of duality is exactly the one that I emphasized in the phase space construction.

Introducing matter. One of the motivations to use higher categories to describe discrete geometries is that they allow to encode the proper topological degrees of freedom in higher dimensions. As hinted in the introduction of this chapter, such topological defects can sometimes be interpreted as matter degrees of freedom. As 2-groups are used to encode the proper topological features in four dimensions, in $3d$ they can be used to implement matter degrees of freedom; similarly, in four dimensions one should use 3-groups [181] for the same scope. It would be interesting to construct field theories based on 2- or 3-groups in three and four dimensions, with the same spirit of the 2-group field theory proposed in this chapter, to introduce matter degrees of freedom in a theory of discrete geometry. As I emphasized in the concluding part of Ch. 4, there already exists a group field theory where matter is introduced in a way that can be attributed to 2-groups. Krasnov indeed proposed a model of group field theory based on the Drinfeld double of $SU(2)$, [127]. According to the assumption made in Sec. 5.2, where skeletal 2-groups were regarded as semi direct product of groups, also the Drinfeld double of a Lie group can be analysed in a similar way. Under this perspective, Krasnov model can be seen as a three dimensional 2-group field theory based on the Euclidean 2-group $\mathcal{G} = SU(2) \times \mathbb{R}_*^3$, where matter degrees of freedom are introduced as zero dimensional topological defects, encoded by the 2-curvature.

Gravitational degrees of freedom: simplicity constraint. Finally, I recall that the model presented does not include any gravitational degrees of freedom. To this scope one should implement the simplicity constraint, which in a 2-group theory seems to be a subtle issue [178]. The simplicity constraint, in a model of discrete geometries, enforces all the flux decorations of the faces of a triangulation, to be given as a wedge product of the elements of the translation group and its dual, $\mathfrak{so}^*(p, q) \cong \mathbb{R}^{p+q} \wedge \mathbb{R}^{(p+q)*}$. This suggests that, in a model of four dimensional geometries, it might be necessary to use group decorations on both the edges and the wedges to properly impose the simplicity constraint. This would require the use of a phase space based on a 2-group structure. I leave this interesting and important question for later investigations.

Conclusion and discussions

Driven by the aim of analysing quantum gravity as a constrained topological model of quantum geometries, in this thesis I discuss some field theoretic approaches to discrete geometries.

In the first part I focused on some models closely related to the combinatorial aspects of a quantum geometry, such as matrix or tensor models, with an emphasis on the SYK model. In particular, I analysed the consequences of a non-Gaussian disorder average in a complex version of the SYK model [42].

The second part of the thesis was instead devoted to the group field theory approach to quantum gravity, which allows a closer contact between discrete or quantum geometries and a discrete version of gravity. In this part I focused on some possible extensions of these models that probe the properties of homogeneously curved discrete geometries. Following the usual pattern of the group field theory approach, I discussed first its generalization to the case of Hopf algebras [45], which allows to include the cosmological constant and thus to discuss three dimensional curved geometries (seen as triangulations of homogeneously curved manifolds). Similarly, the construction of curved geometries in four dimension requires the use of a different mathematical tool, identified as a 2-group. Therefore, after a short introduction on the main aspects of 2-group theory, I discussed the construction of a phase space based on 2-group decorations, which is the preparatory step to construct a quantum geometry [47]. Using this construction, I discussed, in the last part, the formulation of a four dimensional group field theory based on 2-groups [48].

SYK model with non-Gaussian disorder average. The first original work that to which contributed during my research, aims to explore the consequences of a non-Gaussian distribution in the SYK model.

For the analysis of an SYK model with a non-Gaussian potential, I used some standard techniques, well known in the study of these type of field theories, such as the replica trick or the use of intermediate fields. Similarly to the standard SYK model, the aim is to average the partition function over the disorder to obtain the one dimensional effective theory in terms of the fermionic fields. The complication due to the non-Gaussianity was that the graph expansion of the potential may not have been finite (the couplings could have been unbounded) in the large N limit. Moreover, it is important to check whether the usual properties of the SYK model, such as the melonic dominance for the bubbles and in particular the Gaussian universality still hold in presence of the non-Gaussian disorder. In order to address these questions I used a Polchinski-like equation for the potential, obtaining the following results:

- finiteness of the graph expansion;
- melonic dominance for the bubbles;
- Gaussian universality (dominance of the dipole term).

Given these results, it was then possible to compute the effective theory, using the same techniques of the standard SYK model. From the analysis of the effective action one can observe two results:

- modification of the covariance;
- conformal invariance of the effective action in the infrared.

The first in particular is the main consequence of the non-Gaussianity condition and thus it is the main result of this work.

Inspired to this work, it would be interesting to explore the effects of a non-Gaussian distribution in other SYK models:

- * the real Gross-Rosenhaus SYK model;
- * based on a real anti-symmetric random tensor J ;
- * super-symmetric SYK model.

Hopf algebra field theory. The first extension of a group field theory that aims to describe the properties of a three dimensional homogeneously curved discrete geometry is a topological field theory based on Hopf algebras.

All the mathematical ingredients necessary to construct such Hopf algebra field theory are encoded in the dual of the generalized quantum double $D^*(A, H, \sigma)$. Here, elements of the (dual) Hopf algebras A and H are used to decorate geometric objects of a triangulation and its dual complex. The notion of duality is encoded in the canonical element σ , which was called plane wave as it has been used to construct the Fourier transform that relates the two dual Hopf algebras. From a mathematical point of view, the main technical difficulties rely in the non-commutativity and non-cocommutativity of the two Hopf algebras. However, the main aspects of the model can be derived using the standard Hopf algebra structures and the properties of the generalized notion of plane wave. Among these mathematical results, we have

- definition of a Fourier transform (4.60) between dual Hopf algebras, in terms of the plane wave σ ;
- proposition 7 and 8 that provide the amplitudes of a triangulation and a dual complex respectively;
- proof of the topological invariance of the model, manifested as the invariance under the action of the Pachner moves (4.87) and (4.89).

In the second point, it is particularly important to emphasize that the plane wave (or a combination of plane waves) is sufficient to provide the amplitude of any Feynman diagram of the model. This makes such canonical element the core object of the Hopf algebra field theory. From a physical point of view, the main relevance of the model is its relation with the discretization of three dimensional gravity:

- when the standard q -deformation of $SU(2)$ is considered, the Hopf algebra Hopf algebra field theory reduces to the original Boulatov model, and thus it can be seen a path integral formulation for the Turaev-Viro model;
- the plane wave provides the discretization (or quantization) for the amplitude of a topological model. In the case of $SU_q(2)$ it thus provides the discretization of a $3d$ Euclidean BF theory with negative cosmological constant.

The Hopf algebra language used in the model and its relation with the Turaev-Viro amplitude suggest some interesting perspectives:

- * the relation with integrable systems, as the plane wave can be recast as the transfer matrix generated from a Lax pair;
- * a generalization to quasi Hopf algebras would allow to include Hopf algebras based on the q root of unity deformation. These Hopf algebras are particularly important in physical models as they provide a natural regularization in the UV;
- * matter degrees of freedom can be included as topological defects in a $3d$ spin foam or group field theory model by using the Drinfeld double of a Lie group at the place of the group itself. Hopf algebra field theory allows to discuss the inclusion of such topological defects in homogeneously curved geometries (matter degrees of freedom in presence of a cosmological constant).

Phase space based on 2-groups and 2-group field theory. The analysis of four dimensional homogeneously curved geometries requires the use of 2-groups. In this line of research I worked on the construction of the phase space of a general triangulation based on 2-group decorations and to the definition of a field theory based on 2-groups that properly encodes the topological features of four dimensional geometries.

In order to build a general phase space of a three dimensional triangulation with decorations on one and two dimensional geometric objects, I used 2-group elements as decorations. The main obstacle was the lack of several concepts of 2-group theory, such as the notion of an Heisenberg double or the lack of a general representation theory. To avoid these issues, I focused on a restricted class of 2-groups, called skeletal, and I used the similarity between such skeletal 2-groups and semi-direct product of groups. This allowed to treat them as ordinary groups and thus to use the Heisenberg double as a natural notion of phase space. An fundamental step in this part was the derivation of Poisson Lie groups and their doubles as the exponentiation of the classical double of the semi-direct sum of Lie bi-algebras. Then the symplectic reduction technique was used to merge such phase spaces with decorations on 1- and 2-holonomies. A crucial point here was how to use the symplectic reduction in a geometrically meaningful way. To address this question I introduced a notion of reference frame for a cellular decomposition. Hence, in this first part I obtained the following results:

- definition 17 provides the fundamental ingredients for the construction of the phase space of a three dimensional triangulation with elements of a skeletal 2-group as decorations;
- the recover of the G-networks and of the KBF model when the Poincaré 2-group is considered and their extension to the κ deformed case.

Some of the results obtained in this part were subsequently used to construct a group field theory based on (general) 2-groups. Once again, the lack of a general representation theory and thus of a Fourier expansion enforced several restrictions on the model. Since there is not a widely accepted notion of quantum 2-groups, not even the notion of dual 2-groups is not known. These issues prevented the definition of the 2-group field theory in the triangulation or in the representation pictures. Nevertheless

- the 2-group field theory model was consistently constructed, where (5.110) provides the amplitude of a general dual complex. This encodes the correct topological features of a four dimensional geometry, as it is given by the combination of 1- and 2-curvatures;
- the model was proven to be topologically invariant, manifested as the invariance under the three Pachner moves.

It would be particularly useful to have a clear formulation of quantum 2-groups. This would suggest the notion of dual 2-groups and thus the construction of a Fourier transform between them. Indeed, following the discussion of the Hopf algebra field theory, one can use the canonical element of the quantum double between 2-groups as the generalized notion of plane wave to construct the Fourier transform. According to the results obtained, such plane wave would also provide a natural discretization of the amplitude of a *BFCG* theory. Having such a Fourier transform, one could naturally extend the 2-group field theory to the triangulation picture, and check whether the model is equivalent to the Yetter and Mackaay model. However, even without the notion of quantum 2-groups there are some interesting possible works related to the model of 2-group field theory:

- * the formulation of a 2-group field theory in three dimensions should amount to a model of three dimensional geometries with $(0d)$ topological defects interpreted as matter degrees of freedom;
- * generalization to 3-groups that allows to discuss zero dimensional topological defects in four dimensions;
- * introduction of the simplicity constraint, inspired to the Crane-Yetter generalization of relativistic spin networks.

Appendix A

Lie bi-algebras and Poisson Lie groups

One of the mathematical tools that I largely used in this thesis are Lie (bi-)algebras and their exponentiation: the (Poisson) Lie groups. I give here a short summary of the basic definitions related to the notion of a Lie bi-algebra, Poisson Lie groups, what is called classical double of a Lie group and I will give some inputs on the generalization of these structures to Hopf algebras. What follows is just a short review of the main concepts that I will need, I refer to [106] for more details on Lie bi-algebras and their relation with Hopf algebras.

Lie bi-algebra. Let \mathfrak{g} be a vector space with generators $e^i \in \mathfrak{g}$. The left and right adjoint actions $ad^{L/R} : \mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathfrak{g}$ of \mathfrak{g} on itself are

$$ad_{e^i}^L(e^j) = \frac{1}{2}c^{ij}{}_k e^k, \quad ad_{e^i}^R(e^j) = \frac{1}{2}c^{ji}{}_k e^k, \quad \text{with} \quad c^{ij}{}_k = -c^{ji}{}_k. \quad (\text{A.1})$$

Similarly, the left and right adjoint co-action $\underline{ad}^{L/R} : \mathfrak{g} \rightarrow \mathfrak{g} \otimes \mathfrak{g}$ of \mathfrak{g} on itself are

$$\begin{aligned} \underline{ad}^L(e^i) &= e^j \otimes e^k \langle e^i, ad_{e_j^*}^L(e_k^*) \rangle = \frac{1}{2}d_{jk}{}^i e^j \otimes e^k, \\ \underline{ad}^R(e^i) &= e^j \otimes e^k \langle e^i, ad_{e_k^*}^R(e_j^*) \rangle = \frac{1}{2}d_{jk}{}^i e^j \otimes e^k. \end{aligned} \quad (\text{A.2})$$

The vector space \mathfrak{g} equipped with the Lie bracket $[\cdot, \cdot] : \mathfrak{g} \wedge \mathfrak{g} \rightarrow \mathfrak{g}$, defined by the left and right adjoint actions as $[e^i, e^j] = ad_{e^i}^L(e^j) + ad_{e^j}^R(e^i) = c^{ij}{}_k e^k$, is a Lie algebra.

The vector space \mathfrak{g} equipped with the Lie co-cycle $\delta_{\mathfrak{g}} : \mathfrak{g} \rightarrow \mathfrak{g} \wedge \mathfrak{g}$, defined by the left and right adjoint co-actions as $\delta_{\mathfrak{g}}(e^i) = \underline{ad}^L(e^i) + \underline{ad}^R(e^i) = \frac{1}{2}d_{jk}{}^i e^j \wedge e^k$, is a Lie co-algebra.

The vector space \mathfrak{g} equipped with both the Lie bracket and the Lie co-cycle in a compatible way

$$\delta_{\mathfrak{g}}([e^i, e^j]) = [\delta_{\mathfrak{g}}(e^i), \delta_{\mathfrak{g}}(e^j)], \quad (\text{A.3})$$

is a Lie bi-algebra.

Dual Lie bi-algebra. Let \mathfrak{g} and \mathfrak{g}^* be a pair of Lie bi-algebras with generators $e^i \in \mathfrak{g}$ and $e_i^* \in \mathfrak{g}^*$. They are said to be dual each other if there exists a bi-linear map $\langle \cdot, \cdot \rangle : \mathfrak{g} \otimes \mathfrak{g}^* \rightarrow \mathbb{K}$ such that

$$\langle e^i, e_j^* \rangle = \delta_j^i. \quad (\text{A.4})$$

The above bi-linear map further enforces the duality between the left and right adjoint actions of \mathfrak{g} resp. with the left and right adjoint co-action of \mathfrak{g}^* , and vice-versa:

$$\begin{aligned} \langle ad_{e^i}^L(e^j), e_k^* \rangle &= \langle e^i \otimes e^j, \underline{ad}^L(e_k^*) \rangle, & \langle ad_{e^i}^R(e^j), e_k^* \rangle &= \langle e^j \otimes e^i, \underline{ad}^R(e_k^*) \rangle, \\ \langle e^i, ad_{e_j^*}^L(e_k^*) \rangle &= \langle \underline{ad}^L(e^i), e_j^* \otimes e_k^* \rangle, & \langle e^i, ad_{e_j^*}^R(e_k^*) \rangle &= \langle \underline{ad}^R(e^i), e_k^* \otimes e_j^* \rangle. \end{aligned} \quad (\text{A.5})$$

This implies that the Lie brackets of \mathfrak{g} are dual to the co-cycle of \mathfrak{g}^* and vice-versa:

$$\begin{aligned} \langle [e^i, e^j], e_k^* \rangle &= \langle e^i \otimes e^j, \delta_{\mathfrak{g}^*}(e_k^*) \rangle = c^{ij}_k, \\ \langle e^i, [e_j^*, e_k^*] \rangle &= \langle \delta_{\mathfrak{g}}(e^i), e_j^* \otimes e_k^* \rangle = d_{jk}^i. \end{aligned} \quad (\text{A.6})$$

A pair of dual Lie bi-algebras \mathfrak{g} and \mathfrak{g}^* are thus defined by the respective structures

$$\mathfrak{g} : \begin{cases} [e^i, e^j] = c^{ij}_k e^k, \\ \delta_{\mathfrak{g}}(e^i) = \frac{1}{2} d_{jk}^i e^j \wedge e^k, \end{cases} \quad \mathfrak{g}^* : \begin{cases} [e_i^*, e_j^*] = d_{ij}^k e_k^*, \\ \delta_{\mathfrak{g}^*}(e_i^*) = \frac{1}{2} c^{jk}_i e_j^* \wedge e_k^*. \end{cases} \quad (\text{A.7})$$

Classical double. Let \mathfrak{g} and \mathfrak{g}^* be a pair of dual Lie bi-algebras as above. The left and right co-adjoint actions $ad^{*L} : \mathfrak{g} \otimes \mathfrak{g}^* \rightarrow \mathfrak{g}^*$ and $ad^{*R} : \mathfrak{g}^* \otimes \mathfrak{g} \rightarrow \mathfrak{g}^*$ of \mathfrak{g} on \mathfrak{g}^* are

$$\begin{aligned} ad_{e_i^*}^{*L}(e_j^*) &= \langle ad_{e_i^*}^L(e^k), e_j^* \rangle e_k^* = c^{ik}_j e_k^*, \\ ad_{e_i^*}^{*R}(e_j^*) &= \langle ad_{e_i^*}^R(e^k), e_j^* \rangle e_k^* = c^{ki}_j e_k^*. \end{aligned} \quad (\text{A.8})$$

Similarly, the left and right co-adjoint co-actions $\underline{ad}^{*L} : \mathfrak{g} \rightarrow \mathfrak{g}^* \otimes \mathfrak{g}$ and $\underline{ad}^{*R} : \mathfrak{g} \rightarrow \mathfrak{g} \otimes \mathfrak{g}^*$ of \mathfrak{g} on \mathfrak{g} are

$$\begin{aligned} \underline{ad}^{*L}(e^i) &= e_j^* \otimes e^k \langle e^i, ad_{e_j^*}^{*L}(e_k^*) \rangle = c^{ij}_k e_j^* \otimes e^k, \\ \underline{ad}^{*R}(e^i) &= e^j \otimes e_k^* \langle e^i, ad_{e_k^*}^{*R}(e_j^*) \rangle = c^{ki}_j e^j \otimes e_k^*. \end{aligned} \quad (\text{A.9})$$

Let the Lie bi-algebra \mathfrak{g}^{cop} be given by the Lie algebra of \mathfrak{g} (with Lie brackets $[\cdot, \cdot]_{\mathfrak{g}^{cop}} = [\cdot, \cdot]_{\mathfrak{g}}$) and opposite \mathfrak{g} co-algebra, with $\delta_{\mathfrak{g}^{cop}} = -\delta_{\mathfrak{g}}$; similarly, let the Lie bi-algebra \mathfrak{g}^{op} be given by the Lie co-algebra of \mathfrak{g} (with Lie co-cycles $\delta_{\mathfrak{g}^{op}} = \delta_{\mathfrak{g}}$) and opposite \mathfrak{g} algebra, with $[\cdot, \cdot]_{\mathfrak{g}^{op}} = -[\cdot, \cdot]_{\mathfrak{g}}$. The left classical double $\mathfrak{d}_L = \mathfrak{g}^{cop} \bowtie \mathfrak{g}^*$ is the Lie bi-algebra given by the Lie bi-algebras \mathfrak{g}^{cop} and \mathfrak{g}^* , equipped with the cross Lie brackets

$$[e_i^*, e^j] = ad_{e_i^*}^{*L}(e^j) + ad_{e_j^*}^{*R}(e_i^*) = d_{ki}^j e^k + c^{jk}_i e_k^*. \quad (\text{A.10})$$

Similarly, the right classical double $\mathfrak{d}_R = \mathfrak{g}^* \bowtie \mathfrak{g}^{cop}$ is the Lie bi-algebra given by the Lie bi-algebras \mathfrak{g}^{cop} and \mathfrak{g}^* , equipped with the cross Lie brackets

$$[e^i, e_j^*] = ad_{e_i^*}^{*L}(e_j^*) + ad_{e_j^*}^{*R}(e^i) = c^{ki}_j e_k^* + d_{jk}^i e^k. \quad (\text{A.11})$$

The left and right classical doubles are isomorphic, let me denote them as $\mathfrak{d} \equiv \mathfrak{d}_L \cong \mathfrak{d}_R$. The classical double is also equipped with the element $r = e_i^* \otimes e^i$ called *classical r-matrix*. The latter can be split into its symmetric and antisymmetric parts

$$r_- = \frac{1}{2}(r - r^t) = \frac{1}{2} \sum_i e_i^* \wedge e^i, \quad r_+ = \frac{1}{2}(r + r^t). \quad (\text{A.12})$$

The co-cycles of the Lie bi-algebras \mathfrak{g} and \mathfrak{g}^* can equivalently be expressed through the co-boundary condition,

$$\delta_{\mathfrak{d}}(x) = [x \otimes 1 + 1 \otimes x, r_-], \quad \forall x = e^i, e_i^* \in \mathfrak{d}, \quad (\text{A.13})$$

making \mathfrak{d} a *co-boundary Lie bi-algebra*. The classical r -matrix also satisfies the so called classical Yang-Baxter equation (CYBE),

$$[[r, r]] = [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0, \quad (\text{A.14})$$

where I used the tensor notation $r_{12} = e_i^* \otimes e^i \otimes 1$, $r_{13} = e_i^* \otimes 1 \otimes e^i$, $r_{23} = 1 \otimes e_i^* \otimes e^i$.

There exists a notion of generalized classical double. Consider a pair of any two Lie bi-algebras

\mathfrak{g}_1 and \mathfrak{g}_2 with resp. generators $e^i \in \mathfrak{g}_1$ and $f^i \in \mathfrak{g}_2$ and mutual actions $\triangleright : \mathfrak{g}_2 \otimes \mathfrak{g}_1 \rightarrow \mathfrak{g}_2$ and $\triangleleft : \mathfrak{g}_2 \otimes \mathfrak{g}_1 \rightarrow \mathfrak{g}_1$. If the actions satisfy the identities

$$\begin{aligned} [f^i, f^j] \triangleright e^k &= f^i \triangleright (f^j \triangleright e^k) - f^j \triangleright (f^i \triangleright e^k), \\ f^i \triangleleft [e^j, e^k] &= (f^i \triangleleft e^j) \triangleleft e^k - (f^i \triangleleft e^k) \triangleleft e^j, \\ f^i \triangleright [e^j, e^k] &= [f^i \triangleright e^j, e^k] + [e^j, f^i \triangleright e^k] + (f^i \triangleleft e^j) \triangleright e^k - (f^i \triangleleft e^k) \triangleright e^j, \\ [f^i, f^j] \triangleleft e^k &= [f^i \triangleleft e^k, f^j] + [f^i, f^j \triangleleft e^k] + f^i \triangleleft (f^j \triangleright e^k) - f^j \triangleleft (f^i \triangleright e^k), \end{aligned} \quad (\text{A.15})$$

the Lie bi-algebras \mathfrak{g}_1 and \mathfrak{g}_2 form a *matched pair*. Dually, the Lie bi-algebras \mathfrak{g}_1^* and \mathfrak{g}_2^* with generators $e_i^* \in \mathfrak{g}_1^*$ and $f_i^* \in \mathfrak{g}_2^*$ are equipped with a pair of mutual co-actions¹ $\alpha : \mathfrak{g}_2^* \rightarrow \mathfrak{g}_2^* \otimes \mathfrak{g}_1^*$ and $\beta : \mathfrak{g}_1^* \rightarrow \mathfrak{g}_2^* \otimes \mathfrak{g}_1^*$ derived, by dualization of the actions as

$$\langle f^i \triangleleft e^j, f_k^* \rangle = \langle f^i \otimes e^j, \alpha(f_k^*) \rangle, \quad \langle f^i \triangleright e^j, e_k^* \rangle = \langle f^i \otimes e^j, \beta(e_k^*) \rangle. \quad (\text{A.16})$$

If the Lie bi-algebras \mathfrak{g}_1 and \mathfrak{g}_2 form a matched pair, then their dual Lie bi-algebras \mathfrak{g}_1^* and \mathfrak{g}_2^* form a *matched co-pair*. The (double cross sum) Lie bi-algebra $(\mathfrak{g}_1^{\text{cop}} \bowtie \mathfrak{g}_2)$ built on the matched pair and the (double cross co-sum) Lie bi-algebra $(\mathfrak{g}_1^* \blacktriangleright \mathfrak{g}_2^*)$ built on the matched co-pair are resp. the generalized notions of classical double and its dual Lie bi-algebra. The choice $\mathfrak{g}_1 = \mathfrak{g}$ and $\mathfrak{g}_2 = \mathfrak{g}^*$ is a special case of matched pair, with actions given by mutual co-adjoint actions.

Poisson Lie groups. A Lie group G is obtained through the exponential map (integration) of its Lie algebra (its tangent space) \mathfrak{g} . One can equip a Lie group with a further bi-linear map $\{, \} : G \wedge G \rightarrow G$, called Poisson bracket. A Poisson Lie group is a Lie group equipped with such map, where the group multiplication is a Poisson map, and one says that the Poisson brackets are compatible with the group multiplication. Such compatibility reads

$$\{f_1, f_2\}(g_1 \cdot g_2) = \{f_1 \circ R_{g_2}, f_2 \circ R_{g_2}\}(g_1) + \{f_1 \circ L_{g_1}, f_2 \circ L_{g_1}\}(g_2), \quad (\text{A.17})$$

where R_g and L_g resp. denote the right and left translation by g , with $g_1, g_2 \in G$ and $f_1, f_2 \in F(G)$ are functions on G . As the tangent space associated to a Lie group is a Lie algebra, the tangent space associated to a Poisson Lie group is a Lie bi-algebra. The infinitesimal limit of the group multiplication gives the Lie brackets, and the infinitesimal limit of the Poisson brackets gives the Lie co-cycles. Just as Lie brackets and Lie co-cycles of a Lie bi-algebra are compatible, so the group multiplication and the Poisson brackets of a Poisson Lie group are compatible as well. Note that the notion of dual Lie bi-algebras \mathfrak{g} and \mathfrak{g}^* naturally leads – upon exponentiation – to the notion of dual Poisson Lie groups G and G^* .

Classical doubles of a Lie group. The Poisson structure of a given Lie group (that is not necessarily a Poisson Lie group) can be given in terms of a Poisson bi-vector π , as $\{f_1, f_2\}(g) = \pi(df_1 \wedge df_2)$. Given a Lie bi-algebra \mathfrak{g} and the associated classical double, one can define two relevant Poisson bi-vectors in terms of the classical r -matrix.

Heisenberg double. The exponentiation of the classical double \mathfrak{d} equipped with the Poisson brackets given by the bi-vector

$$\pi_+(d) = -[d \otimes d, r_-]_+, \quad (\text{A.18})$$

is called *Heisenberg double* of G and denoted \mathcal{H} , with $d \in \mathcal{H}$, [161, 162]. Here the brackets $[,]_+$ are the anticommutators. As a Lie group, the Heisenberg double is built on the Cartesian

¹As the symbols for the left and right actions are usually triangles $\triangleright, \triangleleft$, the symbols used for the left or right co-actions are black triangles $\blacktriangleleft, \blacktriangleright$. I will also alternatively use α and β .

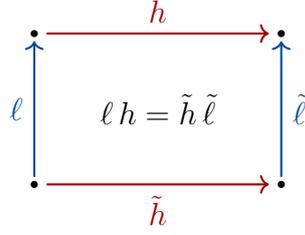


Figure A.1: Ribbon equation associated to the decomposition of the Heisenberg double.

product of Lie groups G and its dual G^* . The splitting $\mathcal{H} \equiv G \bowtie G^* \cong G^* \bowtie G$, in terms of group elements, leads to the so called *ribbon equation*

$$d = \ell h = \tilde{h} \tilde{\ell}. \quad (\text{A.19})$$

The ribbon relation, illustrated in Fig. A.1, allows to define the mutual left and right actions between G and G^*

$$\ell \triangleright h = \tilde{h}, \quad \ell \triangleleft h = \tilde{\ell}, \quad \tilde{h} \triangleright \tilde{\ell} = \ell, \quad \tilde{h} \triangleleft \tilde{\ell} = h. \quad (\text{A.20})$$

The Poisson brackets given in terms of the Poisson bi-vector (A.18) are *symplectic* and thus are not compatible with the group multiplication. Therefore the Heisenberg double is not a Poisson Lie group, but it has a natural interpretation as the phase space associated to the group G , with symplectic form

$$\Omega = \frac{1}{2} (\langle \Delta \ell \wedge \Delta \tilde{h} \rangle + \langle \underline{\Delta} \tilde{\ell} \wedge \underline{\Delta} h \rangle), \quad (\text{A.21})$$

induced by its symplectic Poisson structure, where $\Delta g = \delta g g^{-1}$ and $\underline{\Delta} g = g^{-1} \delta g$ are resp. the right and left Maurer Cartan forms.

Drinfeld double. The exponentiation of the classical double \mathfrak{d} equipped with the Poisson brackets given by the bi-vector

$$\pi_{-}(d) = [d \otimes d, r_{-}]_{-}, \quad (\text{A.22})$$

is called *Drinfeld double* of G and denoted \mathcal{D} , with $d \in \mathcal{D}$, [162, 182]. Here the brackets $[\cdot, \cdot]_{-}$ stand for the usual commutators. The Poisson brackets given in terms of the Poisson bi-vector (A.22) are compatible with the group product, hence the Drinfeld double is a Poisson Lie group. As a Lie group, the Drinfeld double is built on the Cartesian product of the Lie groups G with its dual G^* , similarly to the Heisenberg double; therefore, there is a natural action of the Drinfeld double on the Heisenberg double, given by the left or right group multiplication. Such action is a Poisson map, which means that the symplectic Poisson brackets of the Heisenberg double are invariant under the action of the Drinfeld double [161]. Let me give the explicit example of left translation on the Heisenberg double with respect to the sub-groups (of the Drinfeld double) G and G^* . This type of action is the one used in Sec. 5.2 for the symplectic reduction.

- *Left G transformations.* Let $\alpha \in \mathfrak{g}$. The infinitesimal (left) action δ_{α}^L of $G \ni h' \sim 1 + \alpha$ on $\mathcal{H} \ni d$, induced by the left group multiplication is

$$h' d \sim (1 + \alpha) d \quad \text{with} \quad d = \ell h = \tilde{h} \tilde{\ell}. \quad (\text{A.23})$$

From which one deduces the transformations for the sub-components [105]

$$\delta_{\alpha}^L g = \alpha g \quad \rightarrow \quad \left\{ \begin{array}{l} \delta_{\alpha}^L \tilde{h} = \alpha \tilde{h}, \quad \delta_{\alpha}^L \tilde{\ell} = 0, \\ \delta_{\alpha}^L \ell = \alpha \ell - \ell(\alpha \triangleleft \ell), \quad \delta_{\alpha}^L h = (\alpha \triangleleft h) h. \end{array} \right. \quad (\text{A.24})$$

- *Left G^* transformations.* A similar calculation can be performed for the infinitesimal (left) transformations δ_ϕ^L of $G^* \ni \ell' \sim 1 + \phi$, with $\phi \in \mathfrak{g}^*$,

$$\ell'd \sim (1 + \phi)d \quad \text{with} \quad d = \ell h = \tilde{h}\tilde{\ell}, \quad (\text{A.25})$$

from which, the transformations for the sub-components are [105]

$$\delta_\phi^L d = \phi d \quad \rightarrow \quad \left\{ \begin{array}{l} \delta_\phi^L \ell = \phi \ell, \quad \delta_\phi^L h = 0, \\ \delta_\phi^L \tilde{h} = \phi \tilde{h} - \tilde{h}(\tilde{h}^{-1} \triangleright \phi) = \phi \tilde{h} - \tilde{h}(\phi \triangleleft \tilde{h}), \quad \delta_\phi^L \tilde{\ell} = (\phi \triangleleft \tilde{h})\tilde{\ell}. \end{array} \right. \quad (\text{A.26})$$

From (Poisson) Lie groups to Hopf algebras. Quantum groups or Hopf algebras can be recovered as the *quantization* of Poisson Lie groups. Without going into details, I would like to give here an intuition of this concept. As I mentioned in the first part of Ch. 4, I recall that an Hopf algebra is equipped with two maps: the product and the co-product. Given a Poisson Lie group G and its tangent space, the Lie bi-algebra \mathfrak{g} , one can define two associative Hopf algebras: the *Hopf algebra of functions on the group* and the *universal enveloping Hopf algebra* of \mathfrak{g} .

Universal enveloping Hopf algebra. The co-product of the universal enveloping Hopf algebra $\mathcal{U}(\mathfrak{g})$ can be regarded as the quantization or deformation of the co-cycle of \mathfrak{g} . While the product of $\mathcal{U}(\mathfrak{g})$ is the deformation of the product of the universal enveloping algebra of \mathfrak{g} , such that it is compatible with the co-product. If \mathfrak{g} is an ordinary Lie algebra (with trivial co-cycle), then the co-product called *primitive* (it is always of the shape $\Delta x = x \otimes 1 + 1 \otimes x$, for $x \in \mathcal{U}(\mathfrak{g})$) and the Hopf algebra $\mathcal{U}(\mathfrak{g})$ automatically reduces to the usual enveloping algebra equipped with such trivial co-product. In this case it is said to be co-commutative. I refer to [183] for the quantization of the universal enveloping algebra $\mathcal{U}(\mathfrak{su}(2))$ into its standard q deformation $\mathcal{U}_q(\mathfrak{su}(2))$.

Hopf algebra of functions on a Lie group. Similarly, the product of the algebra of functions on G is the deformation of its Poisson brackets, in the sense that the product (or its quadratic anti-symmetric part, called commutator) is regarded as an expansion in terms of a deformation parameter, the brackets are the zero term in such expansion. The co-product of the Hopf algebra of functions $F(G)$ is instead derived from the group product. The simplest way to derive the co-product of the Hopf algebra of functions on a Lie group, is to choose a coordinate parametrization for its group element g , which always satisfies the group-like co-product $\Delta g = g \otimes g$, from which one can deduce the co-product for the coordinates. If G is an ordinary Lie group (with trivial Poisson structure), then the product of the functions on it is commutative. I refer to [106] for a general discussion, and to [159] for an explicit realization of the standard κ deformation of the Hopf algebra of functions on the Poincaré group.

In these cases a deformation parameter is often used to deform or quantize the ordinary structures into Hopf algebra ones. The most celebrated examples of Hopf algebra are the deformations $\mathcal{U}_q(\mathfrak{sl}_2)$ and $F(\text{SL}_q)$, see [106, 184] for more details. I discuss and use their Hopf algebra structures in Ch. 4.

A.1 Poincaré and κ -Poincaré Lie bi-algebras

In this part I provide an explicit example of Lie bi-algebra, whose general setting was introduced in the previous section. I will discuss the Lie bi-algebra underlying the n dimensional Poioncaré (Poisson) Lie group and its κ deformation [159]. This example will be one of the main applications of the phase space construction provided in Sec. 5.2.

Let $\eta^{\mu\nu}$ be the Minkowski metric; consider the generalization of the classical double introduced in the previous section, based on the matched pair of Lie bi-algebras \mathfrak{g}_1 and \mathfrak{g}_2 . Take the Lie bi-algebra \mathfrak{g}_2 to be the n dimensional Lie bi-algebra \mathfrak{an} with generators P^μ having the same dimension as the scale κ^{-1} , satisfying²

$$[P^\mu, P^\nu] = \kappa^{-1}(\eta^{\mu 0} P^\nu - \eta^{\nu 0} P^\mu), \quad \delta_{\mathfrak{an}}(P^\mu) = 0. \quad (\text{A.27})$$

Let \mathfrak{g}_1 be the Lorentz Lie bi-algebra \mathfrak{so} of dimension $\frac{n(n-1)}{2}$, generated by $J^{\mu\nu}$, obeying

$$[J^{\mu\nu}, J^{\rho\sigma}] = \eta^{\mu\rho} J^{\nu\sigma} + \eta^{\nu\sigma} J^{\mu\rho} - \eta^{\mu\sigma} J^{\nu\rho} - \eta^{\nu\rho} J^{\mu\sigma}, \quad \delta_{\mathfrak{so}}(J^{\mu\nu}) = 0. \quad (\text{A.28})$$

The Lie bi-algebras \mathfrak{an} and \mathfrak{so} , when equipped with the mutual actions

$$\begin{aligned} \triangleleft: \mathfrak{so} \otimes \mathfrak{an} &\rightarrow \mathfrak{so} & \triangleleft: \mathfrak{so} \otimes \mathfrak{an} &\rightarrow \mathfrak{an} \\ J^{\mu\nu} \triangleleft P^\rho &= \kappa^{-1}(\eta^{\nu 0} J^{\mu\rho} - \eta^{\mu 0} J^{\nu\rho}), & J^{\mu\nu} \triangleright P^\rho &= \eta^{\mu\rho} P^\nu - \eta^{\nu\rho} P^\mu, \end{aligned} \quad (\text{A.29})$$

form a matched pair. Consider the dual Lie bi-algebras $\mathfrak{g}_1^* \equiv \mathfrak{an}^*$ and $\mathfrak{g}_2^* \equiv \mathfrak{so}^*$ generated by P_μ^* and $J_{\mu\nu}^*$ respectively, with the bi-linear pairing defined as

$$\langle P^\mu, P_\nu^* \rangle = \eta^\mu{}_\nu, \quad \langle J^{\mu\nu}, J_{\rho\sigma}^* \rangle = \eta^\mu{}_\rho \eta^\nu{}_\sigma - \eta^\nu{}_\rho \eta^\mu{}_\sigma. \quad (\text{A.30})$$

By dualization, the Lie bi-algebra structures of \mathfrak{an}^* and \mathfrak{so}^* are

$$[P_\mu^*, P_\nu^*] = 0, \quad \delta_{\mathfrak{an}^*}(P_\mu^*) = \kappa^{-1} P_\mu^* \wedge P_0^*, \quad (\text{A.31})$$

$$[J_{\mu\nu}^*, J_{\rho\sigma}^*] = 0, \quad \delta_{\mathfrak{so}^*}(J_{\mu\nu}^*) = J_{\mu\rho}^* \wedge J_\nu^{\rho\sigma}, \quad (\text{A.32})$$

with mutual co-actions

$$\begin{aligned} \alpha: \mathfrak{so}^* &\rightarrow \mathfrak{so}^* \otimes \mathfrak{an}^* & \beta: \mathfrak{an}^* &\rightarrow \mathfrak{so}^* \otimes \mathfrak{an}^* \\ \alpha(J_{\mu\nu}^*) &= \kappa^{-1}(J_{0\mu}^* \otimes P_\nu^* - J_{0\nu}^* \otimes P_\mu^*), & \beta(P_\mu^*) &= -J_\mu^{*\rho} \otimes P_\rho^*. \end{aligned} \quad (\text{A.33})$$

Classical double $\mathfrak{an} \bowtie \mathfrak{so}$ and semi-dualization

From the matched pair $(\mathfrak{an}, \mathfrak{so})$ one can construct the cotangent bundle of the double cross sum $\mathfrak{an} \bowtie \mathfrak{so}$ as a classical double

$$\mathfrak{d} = (\mathfrak{an} \bowtie \mathfrak{so})^{\text{cop}} \ltimes (\mathfrak{an}^* \blacktriangleright \mathfrak{so}^*), \quad (\text{A.34})$$

equipped with the classical r -matrix

$$r = P_\mu^* \otimes P^\mu + \frac{1}{2} J_{\mu\nu}^* \otimes J^{\mu\nu}. \quad (\text{A.35})$$

Another classical double, which is used in Sec. 5.2, is obtained from the one constructed above \mathfrak{d} through a process known as *semi-dualization*, realized by exchanging the coordinates $P^\mu \in \mathfrak{an}$ with their dual ones $P_\mu^* \in \mathfrak{an}^*$. The resulting classical double is

$$\mathfrak{b} = (\mathfrak{an}^* \blacktriangleright \mathfrak{so})^{\text{cop}} \bowtie (\mathfrak{an} \blacktriangleleft \mathfrak{so}^*), \quad (\text{A.36})$$

with associated classical r -matrix

$$r = P^\mu \otimes P_\mu^* + \frac{1}{2} J_{\mu\nu}^* \otimes J^{\mu\nu}. \quad (\text{A.37})$$

²I am using the time-like Lorentzian deformation here; but other deformations, such as space or light-like are also possible [185, 186]. The Euclidean signature can also be considered in a similar manner.

Their Lie bi-algebra structure is given by

($\mathfrak{an}^* \bowtie \mathfrak{so}$) Lie algebra:

$$\begin{aligned} [P_\mu^*, P_\nu^*] &= 0, \\ [J^{\mu\nu}, J^{\rho\sigma}] &= \eta^{\mu\rho} J^{\nu\sigma} + \eta^{\nu\sigma} J^{\mu\rho} - \eta^{\mu\sigma} J^{\nu\rho} - \eta^{\nu\rho} J^{\mu\sigma}, \\ [P_\rho^*, J^{\mu\nu}] &= \eta^\nu_\rho P^{*\mu} - \eta^\mu_\rho P^{*\nu}, \end{aligned} \tag{A.38}$$

($\mathfrak{an} \bowtie \mathfrak{so}^*$) Lie algebra:

$$\begin{aligned} [P^\mu, P^\nu] &= \kappa^{-1}(\eta^{\mu 0} P^\nu - \eta^{\nu 0} P^\mu), \\ [J_{\mu\nu}^*, J_{\rho\sigma}^*] &= 0, \\ [P^\rho, J_{\mu\nu}^*] &= \kappa^{-1}(\eta^\rho_\nu J_{0\mu}^* - \eta^\rho_\mu J_{0\nu}^*), \end{aligned} \tag{A.39}$$

($\mathfrak{an}^* \bowtie \mathfrak{so}$) Lie co-algebra:

$$\begin{aligned} \delta_{\mathfrak{an}^* \bowtie \mathfrak{so}}(P_\mu^*) &= \kappa^{-1} P_\mu^* \wedge P_0^*, \\ \delta_{\mathfrak{an}^* \bowtie \mathfrak{so}}(J^{\mu\nu}) &= \kappa^{-1} P_\rho^* \wedge (\eta^{\nu 0} J^{\mu\rho} - \eta^{\mu 0} J^{\nu\rho}), \end{aligned}$$

($\mathfrak{an} \bowtie \mathfrak{so}^*$) Lie co-algebra:

$$\begin{aligned} \delta_{\mathfrak{an} \bowtie \mathfrak{so}^*}(P^\mu) &= P^\rho \wedge J^{*\mu}_\rho, \\ \delta_{\mathfrak{an} \bowtie \mathfrak{so}^*}(J_{\mu\nu}^*) &= J_{\mu\rho}^* \wedge J_{\nu}^{*\rho}. \end{aligned} \tag{A.40}$$

with cross Lie brackets, induced by the actions of the matched pair and by the co-adjoint actions

Cross Lie brackets:

$$\begin{aligned} [P^\rho, J^{\mu\nu}] &= \kappa^{-1}(\eta^{\mu 0} J^{\nu\rho} - \eta^{\nu 0} J^{\mu\rho}) + (\eta^{\nu\rho} P^\mu - \eta^{\mu\rho} P^\nu), \\ [J_{\mu\nu}^*, P_\rho^*] &= 0, \\ [J_{\rho\sigma}^*, J^{\mu\nu}] &= (\eta^\nu_\rho J^{*\mu}_\sigma + \eta^\mu_\sigma J^{*\nu}_\rho - \eta^\nu_\sigma J^{*\mu}_\rho - \eta^\mu_\rho J^{*\nu}_\sigma) \\ &\quad + \kappa^{-1}(\eta^{\nu 0}(\eta^\mu_\rho P_\sigma^* - \eta^\mu_\sigma P_\rho^*) - \eta^{\mu 0}(\eta^\nu_\rho P_\sigma^* - \eta^\nu_\sigma P_\rho^*)), \\ [P^\mu, P_\nu^*] &= -\kappa^{-1}(\eta^{\mu 0} P_\nu^* + \eta^\mu_\nu P_0^*) + J^{*\mu}_\nu. \end{aligned}$$

As a Lie algebra, $\mathfrak{so} \bowtie \mathfrak{an}^* \cong \mathfrak{so} \ltimes \mathbb{R}^n$ is the Poincaré (or Euclidean) Lie algebra. So upon exponentiation it would give the Poincaré (or Euclidean) Lie group. Since its co-cycle is non trivial for $\kappa^{-1} \neq 0$, we would get the Poincaré group equipped with a non-trivial Poisson bracket. This is the classical version of the so-called κ -Poincaré group [106, 159]. Similarly, the Lie bi-algebra $\mathfrak{so}^* \bowtie \mathfrak{an}$ upon exponentiation gives rise to the group $\text{AN} \ltimes \text{SO}^* \cong \text{AN} \ltimes \mathbb{R}^{(n^2-n)/2}$ equipped with a non-trivial Poisson structure. This is the group dual to the κ -Poincaré group, also called, by abuse of nomenclature, κ -Poincaré algebra.

According to the analogy between Lie bi-algebras or Poisson Lie groups with skeletal Lie 2-bi-algebras or skeletal Poisson 2-groups (see Ch. 5), the above construction, upon exponentiation, gives rise to the κ -Poincaré (skeletal) 2-group and its dual (skeletal) 2-group that I will call κ -Poincaré 2-algebra. Upon quantization, these would eventually lead to a pair of quantum strict 2-groups [179].

Appendix B

Hopf algebra field theory

B.1 Deriving dual bi-algebras

Here I propose a procedure to derive skew symmetric bi-algebras, as in Def. 3.

Identification of the problem. Given any pair of bi-algebras H and A , we would like to derive the skew pairing map σ that satisfies the properties (4.39) to construct the generalized quantum double, or dually the skew co-pairing element σ that satisfies the properties (4.46), (4.47) to construct the dual of the quantum double. However, this derivation is highly non trivial, since the explicit expressions of both the map and the element σ strongly depend on the choice of coordinates and on the normal order of the basis of each bi-algebra.

Solution: reverse problem. Here I will propose a possible solution to this problem; the idea is to consider first a pair of co-algebras, choose the two sets of coordinates and the normal order for both of them. Then we define the skew pairing map and the skew co-pairing element, and derive the product of the two co-algebras in such way the map and element σ satisfy the required properties.

More specifically, consider a pair of co-algebras C_A and C_H , and denote $a \in C_A$ and $h \in C_H$ their respective basis. Consider further the map $\sigma : C_H \otimes C_A \rightarrow \mathbb{C}$, such that

$$\sigma(a_i, h_j) = \delta_{ij}. \quad (\text{B.1})$$

Let us use such map σ to define the firther maps

$$\begin{aligned} \mathcal{L}_h : C_H &\rightarrow \mathbb{C}, & \mathcal{L}_h &\equiv \sigma(\cdot, h), \\ \mathcal{L}_a : C_A &\rightarrow \mathbb{C}, & \mathcal{L}_a &\equiv \sigma(a, \cdot). \end{aligned} \quad (\text{B.2})$$

Consider further an element $\sigma \in C_H \otimes C_A$ ¹ such that

$$\sigma(\tau \circ \sigma^{-1}) = 1. \quad (\text{B.3})$$

Last, let us consider two more maps, called *star products*

$$\star : C_A \otimes C_A \rightarrow C_A, \quad \star : C_H \otimes C_H \rightarrow C_H, \quad (\text{B.4})$$

defined as

$$\begin{aligned} a \star b &= (\mathcal{L}_a \otimes \mathcal{L}_b \otimes \text{id})(\tau \circ \Delta_H \otimes \text{id})\sigma, \\ h \star g &= (\text{id} \otimes \mathcal{L}_h \otimes \mathcal{L}_g)(\text{id} \otimes \Delta_A)\sigma, \end{aligned} \quad (\text{B.5})$$

¹By abuse of notation, I use the same symbol σ for the map.

for all $h, g \in C_H$ and $a, b \in C_A$. The two star products are associative by co-associativity of the co-products.

Proposition 15 (Bi-algebra compatibility). *Consider two co-algebras C_H and C_A equipped with the star products (B.5) as above. Denote the two resulting structures resp. H and A , they are bi-algebras.*

Proof. Let me give a short proof of this proposition. In order for H and A to be bi-algebras, the two star products have to be compatible with the respective co-product

$$\Delta_A(a \star b) = \Delta_A a \star \Delta_A b, \quad \Delta_H(h \star g) = \Delta_H h \star \Delta_H g. \quad (\text{B.6})$$

Let me prove the first identity, the second is completely analogous. Let us suppose that the canonical element $\sigma \in C_H \otimes C_A$ can be decomposed as $\sigma = \sum \sigma_H \otimes \sigma_A$. Then, using the Sweedler notation, the property (4.46) can be expressed as

$$(\text{id} \otimes \Delta_A) \sigma := \sum \sigma_H \otimes \sigma_{A(1)} \otimes \sigma_{A(2)} = \sum \sigma_{H_1} \sigma_{H_2} \otimes \sigma_{A_2} \otimes \sigma_{A_1} := \sigma_{13} \sigma_{12}. \quad (\text{B.7})$$

Using this identity and the star product of H , the compatibility condition for the Hopf algebra A follows as a direct computation.

$$\begin{aligned} \Delta_A a \star \Delta_A b &:= (\mathcal{L}_{\Delta_A a} \otimes \mathcal{L}_{\Delta_A b} \otimes \text{id}^{\otimes 2}) (\tau \circ \Delta_{H13} \otimes \tau \circ \Delta_{H24} \otimes \text{id}^{\otimes 2}) (\sigma \otimes \sigma) \\ &= (\mathcal{L}_{a(1)} \otimes \mathcal{L}_{a(2)} \otimes \mathcal{L}_{b(1)} \otimes \mathcal{L}_{b(2)} \otimes \text{id}^{\otimes 2}) ((\Delta_{H31} \otimes \text{id}) \sigma) (\Delta_{H42} \otimes \text{id}) \sigma) \\ &= \sum \sigma(a(1), \sigma_{H_1(2)}) \sigma(a(2), \sigma_{H_2(2)}) \sigma(b(1), \sigma_{H_1(1)}) \sigma(b(2), \sigma_{H_2(1)}) \sigma_{A_1} \otimes \sigma_{A_2} \\ &= \sum \sigma(a, \sigma_{H_2(2)} \star \sigma_{H_1(2)}) \sigma(b, \sigma_{H_2(1)} \star \sigma_{H_1(1)}) \sigma_{A(1)} \otimes \sigma_{A(2)} \\ &= (\mathcal{L}_a \otimes \mathcal{L}_b \otimes \Delta_A) (\tau \circ \Delta_H \otimes \text{id}) \sigma := \Delta_A(a \star b). \end{aligned} \quad (\text{B.8})$$

□

The two bi-algebras H and A are automatically skew paired by the map σ , which by construction of the star products, satisfies the axioms (4.39), and they are also skew co-paired by the element σ , which by construction of the star products, satisfies the axioms (4.46), (4.47).

Note that, taking C_A to be a trivial co-algebra (with a primitive co-product) the star product \star becomes a commutative pointwise product. As a particular example, the bi-algebras H and A can be taken respectively the bi-algebras $F(G)$ and $\mathcal{U}(\mathfrak{g})$, with G being a Lie group and \mathfrak{g} its Lie algebra. In this case the map \mathcal{L}_a can be seen as the ordinary Lie derivative on G .

Proposition 16 (Generalized quantum double of $\mathcal{U}_q(\mathfrak{su}(2))$ and its dual). *Consider the $\mathcal{U}_q(\mathfrak{su}(2)) \cong F(\text{AN}_q(2))$ and $\mathcal{U}_q(\mathfrak{an}(2)) \cong F(\text{SU}_q(2))$ co-algebras given in (4.91) and (4.92). The full bi-algebra structures as well as the generalized quantum double of $\mathcal{U}_q(\mathfrak{su}(2))$ in Prop. 11 and its dual in Prop. 11 can be deduced by the skew pairing map (4.97)*

$$\sigma(X_-^j H^i X_+^k, \varphi_+^b \phi^a \varphi_-^c) = i^{a+b+c} \delta_{ai} \delta_{bj} \delta_{ck} a! [b]_{q^2}! [c]_{q^{-2}}!, \quad (\text{B.9})$$

and the skew co-pairing element (4.100)

$$\sigma = e_{\star q^2}^{i\varphi_+ \otimes X_-} e_{\star}^{i\phi \otimes H} e_{\star q^{-2}}^{i\varphi_- \otimes X_+}. \quad (\text{B.10})$$

Proof. In the propositions 11 and 12 I have already proved that the multiplication on $F(\text{SU}_q(2))$ (in (4.92)) can be derived as a \star -product, from the co-algebra sector of $\mathcal{U}_q(\mathfrak{su}(2))$ in (4.91). It

remains to show that the multiplication of $\mathcal{U}_q(\mathfrak{su}(2))$ (in (4.91)) can be derived as a \star -product from the co-algebra sector of $F(\mathrm{SU}_q(2))$ in (4.92). We show how to derive the most complicated commutator

$$[X_+, X_-] = \ell^2 q^{-1} \frac{\sinh(\lambda H)}{\sinh(\ell \lambda)}. \quad (\text{B.11})$$

The computation of the commutators $[H, X_\pm]$ follows a similar and much simpler pattern. According to the definition (B.5), we consider

$$\begin{aligned} [X_+ \star, X_-] &:= (\mathcal{L}_{X_+} \wedge \mathcal{L}_{X_-} \otimes \mathrm{id}) (\Delta^{op} \otimes \mathrm{id}) \sigma \\ &= (\sigma(X_+ \wedge X_-, \) \otimes \mathrm{id}) (\Delta^{op} \otimes \mathrm{id}) \sigma. \end{aligned} \quad (\text{B.12})$$

Note that, as we evaluate the co-product of the skew co-pairing element in $\sigma(X_+, \)$ and $\sigma(X_-, \)$, only the terms with a single power of $\varphi_- \otimes \varphi_+$ or $\varphi_+ \otimes \varphi_-$ survive. Hence, in the following computation I will use the symbol \approx to discard all such non-relevant contributions. The co-product of the skew co-pairing element is

$$\begin{aligned} (\Delta^{op} \otimes \mathrm{id}) \sigma &= e_{\star q^2}^{i\Delta^{op}\varphi_+ \otimes X_-} e_{\star}^{i\Delta^{op}\phi \otimes H} e_{\star q^{-2}}^{i\Delta^{op}\varphi_- \otimes X_+} \\ &= \sum_{u,v,w=0}^{\infty} \frac{i^{u+v+w}}{u![v]_{q^2}![w]_{q^{-2}!}} ((\Delta^{op}\varphi_+)^v (\Delta^{op}\phi)^u (\Delta^{op}\varphi_-)^w) \otimes X_-^v H^u X_+^w. \end{aligned} \quad (\text{B.13})$$

The co-products of φ_\pm can be easily computed using the generalized binomial theorem [106]:

$$(\Delta^{op}\varphi_+)^v = \sum_{i=0}^v \begin{bmatrix} v \\ i \end{bmatrix}_{q^{-2}} (\varphi_0 e^{-i\ell\phi} \otimes \varphi_+)^i (\varphi_+ \otimes e^{i\ell\phi} \varphi_0)^{v-i} \approx \delta_{v,0} 1 \otimes 1 + \delta_{v,1} (\varphi_+ \otimes 1 + 1 \otimes \varphi_+), \quad (\text{B.14})$$

$$(\Delta^{op}\varphi_+)^w = \sum_{j=0}^w \begin{bmatrix} w \\ j \end{bmatrix}_{q^2} (e^{i\ell\phi} \varphi_0 \otimes \varphi_-)^j (\varphi_- \otimes \varphi_0 e^{-i\ell\phi})^{w-j} \approx \delta_{w,0} 1 \otimes 1 + \delta_{w,1} (\varphi_- \otimes 1 + 1 \otimes \varphi_-). \quad (\text{B.15})$$

The computation for the co-product of ϕ is more involved. Let us first simplify it by discarding all the unnecessary contributions.

$$\begin{aligned} \Delta^{op}\phi &= \frac{i}{\ell} \log \left(\frac{1}{\Delta^{op}\sqrt{1 - q^{-1}\ell^2\varphi_- \varphi_+}} (\varphi_0 e^{-i\ell\phi} \otimes \varphi_0 e^{-i\ell\phi} - \ell^2 \varphi_+ \otimes \varphi_-) \right) \\ &\approx \frac{i}{\ell} \log \left(\frac{1}{\sqrt{1 \otimes 1 - q^{-1}\ell^2(\varphi_- \otimes e^{-i\ell\phi} + e^{i\ell\phi} \otimes \varphi_-)} (\varphi_+ \otimes e^{i\ell\phi} + e^{-i\ell\phi} \otimes \varphi_+)} \right. \\ &\quad \left. (e^{-i\ell\phi} \otimes e^{-i\ell\phi} - \ell^2 \varphi_+ \otimes \varphi_-) \right) \\ &\approx \frac{i}{\ell} \log \left((1 \otimes 1 + \frac{1}{2} q^{-1} \ell^2 (e^{-i\ell\phi} \varphi_- \otimes \varphi_+ e^{-i\ell\phi} + \varphi_+ e^{-i\ell\phi} \otimes e^{-i\ell\phi} \varphi_-)) (e^{-i\ell\phi} \otimes e^{-i\ell\phi} - \ell^2 \varphi_+ \otimes \varphi_-) \right) \\ &\approx \frac{i}{\ell} \log \left(e^{-i\ell\phi} \otimes e^{-i\ell\phi} + \frac{1}{2} \ell^2 (e^{-2i\ell\phi} \varphi_- \otimes \varphi_+ e^{-2i\ell\phi} - \varphi_+ \otimes \varphi_-) \right) \\ &\approx \frac{i}{\ell} \log \left(e^{-i\ell\phi} \otimes e^{-i\ell\phi} + \frac{1}{2} \ell^2 \varphi_- \wedge \varphi_+ \right). \end{aligned} \quad (\text{B.16})$$

For simplicity, call $\Phi = \phi \otimes 1 + 1 \otimes \phi$ and $\psi = \frac{1}{2} \ell^2 (\varphi_- \wedge \varphi_+)$, and notice that

$$\Phi^n \psi^m = \psi^m (\Phi - 2i\lambda m)^n, \quad e^{i\ell\Phi} \psi = q^2 \psi e^{i\ell\Phi}. \quad (\text{B.17})$$

The co-product of ϕ then reduces to

$$\begin{aligned}
\Delta^{op}\phi &= \frac{i}{\ell} \log(1 \otimes 1 + (e^{-i\ell\Phi} - 1 \otimes 1 + \psi)) = \frac{i}{\ell} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} (e^{-i\ell\Phi} - 1 \otimes 1 + \psi)^n \\
&= \frac{i}{\ell} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sum_{a=0}^n \binom{n}{a} (-1)^{n-a} (e^{-i\ell\Phi} + \psi)^a \\
&= \frac{i}{\ell} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sum_{a=0}^n \binom{n}{a} (-1)^{n-a} \sum_{k=0}^a \begin{bmatrix} a \\ k \end{bmatrix}_{q^{-2}} e^{-i(a-k)\ell\Phi} \psi^k \\
&\approx \frac{i}{\ell} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sum_{a=0}^n \binom{n}{a} (-1)^{n-a} (e^{-ia\ell\Phi} + [a]_{q^{-2}} e^{-i(a-1)\ell\Phi} \psi) \\
&= \frac{i}{\ell} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sum_{a=0}^n \binom{n}{a} (-1)^{n-a} e^{-ia\ell\Phi} \left(1 + \frac{1 - q^{-2a-2}}{1 - q^{-2}} e^{i\ell\Phi} \psi\right) \\
&= \Phi \left(1 + e^{i\ell\Phi} \psi\right) + \frac{2i\lambda}{q^2 - 1} e^{i\ell\Phi} \psi \approx \Phi + \frac{2i\lambda}{q^2 - 1} e^{i\ell\Phi} \psi. \tag{B.18}
\end{aligned}$$

According to the product rule between Φ and ψ , one finally obtains

$$\begin{aligned}
(\Delta^{op}\phi)^u &= \left(\Phi + \frac{2i\lambda}{q^2 - 1} e^{i\ell\Phi} \psi\right)^u \\
&= \Phi^u + \left(\Phi^{u-1} \frac{2i\lambda\psi}{q^2 - 1} + \Phi^{u-2} \frac{2i\lambda\psi}{q^2 - 1} \Phi + \dots\right) + \left(\Phi^{u-2} \left(\frac{2i\lambda\psi}{q^2 - 1}\right)^2 + \Phi^{u-3} \left(\frac{2i\lambda\psi}{q^2 - 1}\right)^2 \Phi + \dots\right) + \dots \\
&\approx \Phi^u + \frac{2i\lambda}{q^2 - 1} \sum_{a=0}^{u-1} \Phi^{u-1-a} \psi \Phi^a \\
&= \Phi^u + \frac{2i\lambda}{q^2 - 1} \sum_{a=0}^{u-1} \sum_{\alpha=0}^{u-1-a} \sum_{\beta=0}^a \binom{u-1-a}{\alpha} \binom{a}{\beta} (-i\lambda)^{u-1-a-\alpha} (i\lambda)^{a-\beta} : \Phi^{\alpha+\beta} \psi : \\
&\approx \Phi^u + \frac{2i\lambda}{q^2 - 1} \sum_{a=0}^{u-1} (-i\lambda)^{u-1} (-1)^a \psi = \Phi^u - 2 \frac{(-i\lambda)^u}{q^2 - 1} \frac{1 - (-1)^u}{2} \psi. \tag{B.19}
\end{aligned}$$

Above I used the symbol $::$ to address at the normal order of the basis. Plugging the results in the $[X_+ * X_-]$ commutator, we obtain the result

$$\begin{aligned}
[X_+ * X_-] &= \delta_{v,1} \delta_{w,1} \delta_{u,0} \sigma(X_+ \wedge X_-, (\varphi_+ \otimes \varphi_- + \varphi_- \otimes \varphi_+)) X_-^v H^u X_+^w \\
&\quad - \delta_{v,0} \delta_{w,0} \sum_{u=0}^{\infty} \frac{i^u}{u!} \frac{(-i\lambda)^u \ell^2}{q^2 - 1} \frac{1 - (-1)^u}{2} \sigma(X_+ \wedge X_-, \varphi_- \wedge \varphi_+) X_-^v H^u X_+^w \\
&= \ell^2 q^{-1} \frac{\sinh(\lambda H)}{\sinh(\ell\lambda)}. \tag{B.20}
\end{aligned}$$

□

Appendix C

Triangulation phase space

C.1 Fundamental fusions

In this part I will present the possible ways that one has to fuse together a pair atomic phase spaces \mathcal{B} constructed in Sec. 5.2. I recall that the fusion of phase spaces comes from the identification (momentum map) of some variables, and as a consequence, the dual variables will be glued. Having decorations on four objects – links, edges, wedges and faces – it is natural to fuse the phase spaces by identifying one of these four elements.

- *Link gluing.* Links are decorated by G_1 elements, so this gluing is performed by identifying elements in G_1^* : a symplectic reduction with respect to the G_1 symmetry.
- *Face gluing.* Faces are decorated by G_1^* elements, so this gluing is performed by identifying elements in G_1 : a symplectic reduction with respect to the G_1^* symmetry.
- *Wedge gluing.* Wedges are decorated by G_2^* elements, so this gluing is performed by identifying elements in G_2 : a symplectic reduction with respect to the G_2^* symmetry.
- *Edge gluing.* Edges are decorated by G_2 elements, so this gluing is performed by identifying elements in G_2^* : a symplectic reduction with respect to the G_2 symmetry.

Let me now illustrate explicitly these *fundamental gluings*. Consider the atomic phase space \mathcal{B}_i associated to the edge e_i (with $v_{i;1}$ and $v_{i;2}$ as source and target), face f_i , (half) link l_i (with c_i and $c_{i,f}$ as source and target, where $c_{i,f}$ is a point on the face f_i) and wedge w_i . Note that, unlike represented in Fig. 5.7, here I considered an half link. I will comment more on this aspect at the end of this part.

For the sake of clarity, it is useful to introduce the following notation. According to the Lie algebra structures associated to the Lie groups equivalent to the skeletal 2-groups in consideration, the Lie bracket between elements of \mathfrak{g}_1 , \mathfrak{g}_1^* and between \mathfrak{g}_2 , \mathfrak{g}_2^* are not stable in \mathfrak{g}_1^* or \mathfrak{g}_2^* . At the group level, the exponentiation of such Lie brackets leads to the group conjugations

$$\begin{aligned} u\beta u^{-1} = \beta' y' &\Leftrightarrow \begin{cases} \beta' = (u\beta u^{-1})|_{G_1^*}, \\ y' = (u\beta u^{-1})|_{G_2^*}, \end{cases} \\ \lambda y \lambda^{-1} = y'' \beta'' &\Leftrightarrow \begin{cases} \beta'' = (\lambda y \lambda^{-1})|_{G_1^*}, \\ y'' = (\lambda y \lambda^{-1})|_{G_2^*}. \end{cases} \end{aligned} \tag{C.1}$$

Therefore, in order to express the conjugations $u\beta u^{-1}$ and $\lambda y \lambda^{-1}$, I will use their projections into the groups G_1^* and G_2^* .

Link gluing. The gluing of two links, as illustrated in Fig. C.1, is done by identifying the face data dual to each link.

The geometric identification consists in setting¹

$$f_1 = -f_2. \quad (\text{C.2})$$

Enforce this geometric constraint as a condition on G_1^* elements: represent the decorations of the faces f_i at the nodes $c_{i,f}$. Let them be resp. rooted at the vertices $v_{1;2}$ and $v_{2;2}$.

$$\mathcal{B}_1 : \begin{cases} \mathcal{G}^* = G_2 \times G_1^* \ni \tilde{\ell}^{c_{1;f}} \equiv \tilde{\ell}_1 = \tilde{\lambda}_1 \tilde{\beta}_1 \equiv \tilde{\lambda}^{c_{1;f}} \tilde{\beta}_{v_{1;2}}^{c_{1;f}}, \\ \mathcal{G} = G_1 \times G_2^* \ni h^{v_{1;2}} \equiv h_1 = u_1 y_1 \equiv u^{v_{1;2}} y_{c_{1;f}}^{v_{1;2}}, \end{cases} \quad (\text{C.3})$$

$$\mathcal{B}_2 : \begin{cases} \mathcal{G}^* = G_2 \times G_1^* \ni \tilde{\ell}^{c_{2;f}} \equiv \tilde{\ell}_2 = \tilde{\lambda}_2 \tilde{\beta}_2 \equiv \tilde{\lambda}^{c_{2;f}} \tilde{\beta}_{v_{2;2}}^{c_{2;f}}, \\ \mathcal{G} = G_1 \times G_2^* \ni h^{v_{2;2}} \equiv h_2 = u_2 y_2 \equiv u^{v_{2;2}} y_{c_{2;f}}^{v_{2;2}}. \end{cases}$$

In order to express the face decorations in the same reference frame, their indices have to match, hence further demand that

$$c_{1;f} = c_{2;f}, \quad v_{1;2} = v_{2;2}. \quad (\text{C.4})$$

The condition $c_{1;f} = c_{2;f}$ provides the gluing the half links. The proper geometric condition for the face decorations is encoded in the identification

$$\tilde{\beta}_{v_{2;2}}^{c_{1;f}} = (\tilde{\beta}_{v_{2;2}}^{c_{2;f}})^{-1} \Leftrightarrow \tilde{\beta}_1 = \tilde{\beta}_2^{-1}. \quad (\text{C.5})$$

Using the above constraint and the Heisenberg ribbon equations, one obtains the extended ribbon equation:

$$\begin{aligned} \ell_i h_i = \tilde{h}_i \tilde{\ell}_i = \tilde{h}_i (\tilde{\lambda}_i \tilde{\beta}_i) &\Leftrightarrow \tilde{\beta}_i = \tilde{h}_i^{-1} \tilde{\lambda}_i^{-1} \ell_i h_i \xrightarrow{\tilde{\beta}_1 = \tilde{\beta}_2^{-1}} \tilde{h}_1^{-1} \tilde{\lambda}_1^{-1} \ell_1 h_1 = h_2^{-1} \ell_2^{-1} \tilde{\lambda}_2 \tilde{h}_2 \\ \Rightarrow &\tilde{\lambda}_1^{-1} (\tilde{u}_1 \tilde{y}_1)^{-1} (\lambda_1 \bar{\beta}_1) (\bar{y}_1 u_1) = (\bar{y}_2 u_2)^{-1} (\lambda_2 \bar{\beta}_2)^{-1} (\tilde{u}_2 \tilde{y}_2) \tilde{\lambda}_2 \\ &(\lambda_1 \bar{\beta}_1) (\bar{y}_1 u_1) \tilde{\lambda}_2^{-1} (\tilde{u}_2 \tilde{y}_2)^{-1} = (\tilde{u}_1 \tilde{y}_1) \tilde{\lambda}_1 (\bar{y}_2 u_2)^{-1} (\lambda_2 \bar{\beta}_2)^{-1} \\ &\lambda_1 \bar{\beta}_1 \tilde{y}_1 (u_1 \triangleright \tilde{\lambda}_2^{-1}) (u_1 \triangleleft \tilde{\lambda}_2^{-1}) \tilde{y}_2^{-1} \tilde{u}_2^{-1} = \tilde{u}_1 \bar{y}_1 (\tilde{\lambda}_1 \triangleright u_2^{-1}) (\tilde{\lambda}_1 \triangleleft u_2^{-1}) \bar{y}_2^{-1} \bar{\beta}_2^{-1} \lambda_2^{-1} \\ \lambda_1 \bar{\beta}_1 (u_1 \triangleright \tilde{\lambda}_2^{-1}) y' \beta' ((u_1 \triangleleft \tilde{\lambda}_2^{-1}) \triangleright \tilde{y}_2^{-1}) (u_1 \triangleleft \tilde{\lambda}_2^{-1}) \tilde{u}_2^{-1} &= \tilde{u}_1 (\tilde{\lambda}_1 \triangleright u_2^{-1}) (\tilde{y}_1 \triangleleft (\tilde{\lambda}_1 \triangleright u_2^{-1})) y'' \beta'' (\tilde{\lambda}_1 \triangleleft u_2^{-1}) \bar{\beta}_2^{-1} \lambda_2^{-1} \\ (\lambda_1 (u_1 \triangleright \tilde{\lambda}_2^{-1})) (\bar{\beta}_1 \triangleleft (u_1 \triangleright \tilde{\lambda}_2^{-1}) \beta') (y' ((u_1 \triangleleft \tilde{\lambda}_2^{-1}) \triangleright \tilde{y}_2^{-1})) &((u_1 \triangleleft \tilde{\lambda}_2^{-1}) \tilde{u}_2^{-1}) \\ = (\tilde{u}_1 (\tilde{\lambda}_1 \triangleright u_2^{-1})) ((\tilde{y}_1 \triangleleft (\tilde{\lambda}_1 \triangleright u_2^{-1})) y'') (\beta'' (\tilde{\lambda}_1 \triangleleft u_2^{-1}) \triangleright \bar{\beta}_2^{-1}) &((\tilde{\lambda}_1 \triangleleft u_2^{-1}) \lambda_2^{-1}), \end{aligned} \quad (\text{C.6})$$

with $(u_1 \triangleright \tilde{\lambda}_2^{-1})^{-1} \bar{y}_1 (u_1 \triangleright \tilde{\lambda}_2^{-1}) = y' \beta'$ and $(\tilde{\lambda}_1 \triangleleft u_2^{-1}) \bar{y}_2 (\tilde{\lambda}_1 \triangleleft u_2^{-1})^{-1} = y'' \beta''$. The link gluing is therefore obtained through the symplectic reduction

$$\mathcal{P}_l = (\mathcal{B}_1 \times \mathcal{B}_2) // G_1, \quad (\text{C.7})$$

with momentum map $\tilde{\beta}_1 \tilde{\beta}_2 \sim \tilde{\beta}_1 + \tilde{\beta}_2 \in G_1^*$ and fused link variable $\tilde{u} = \tilde{u}_1 (\tilde{\lambda}_1 \triangleright u_2^{-1}) \in G_1$.

Wedge gluing. The gluing of two wedges, as illustrated in Fig. C.2, is done by identifying the edge data dual to each wedge.

The geometric identification consists in setting²

$$e_1 = -e_2 \quad \Leftrightarrow \quad v_{1;1} = v_{2;2}, \quad v_{1;2} = v_{2;1}. \quad (\text{C.8})$$

¹Faces can be identified with same or opposite direction. In this example I consider the gluing with opposite orientation, since it is the one relevant for the phase space construction.

²Edges can be identified with same or opposite orientation. In this example I consider the identification with opposite orientation, since it is more relevant for the phase space construction.

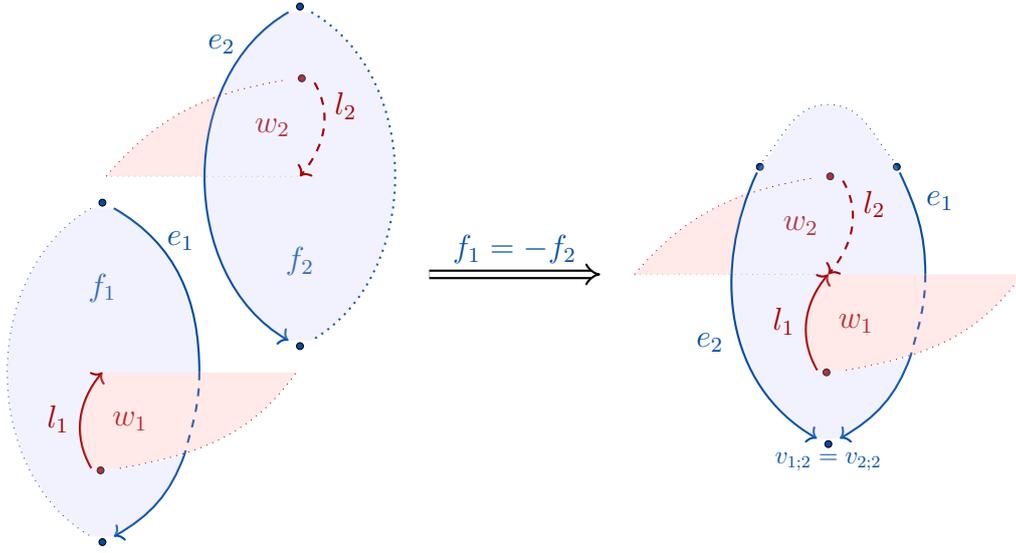


Figure C.1: Gluing of two links by identifying the dual faces f_1 and f_2 . Impose the points $c_{1,f}$ and $c_{2,f}$ in the two faces to match. Root the faces resp. at the vertices $v_{1,2}$ and $v_{2,2}$ and let them match $v_{1,2} = v_{2,2}$, so that the faces share the same root.

Enforce this geometric constraint as a condition on G_2 elements: represent the decorations of the edges e_i resp. at the nodes c_i .

$$\begin{aligned} \mathcal{B}_1 : & \begin{cases} \mathcal{G}^* = G_2 \times G_1^* \ni \ell^{c_1} \equiv \ell_1 = \lambda_1 \bar{\beta}_1 \equiv \lambda^{c_1} \bar{\beta}_{v_{1,2}}^{c_1}, \\ \mathcal{G} = G_2^* \times G_1 \ni h^{v_{1,2}} \equiv h_1 = \bar{y}_1 u_1 \equiv \bar{y}_{c_1}^{v_{1,2}} u^{v_{1,2}}, \end{cases} \\ \mathcal{B}_2 : & \begin{cases} \mathcal{G}^* = G_1^* \times G_2 \ni \ell^{c_2} \equiv \ell_2 = \beta_2 \lambda_2 \equiv \beta_{v_{2,1}}^{c_2} \lambda^{c_2}, \\ \mathcal{G} = G_2^* \times G_1 \ni \tilde{h}^{v_{2,1}} \equiv \tilde{h}_2 = \tilde{y}_2 \tilde{u}_2 \equiv \tilde{y}_{c_2}^{v_{2,1}} \tilde{u}^{v_{2,1}}. \end{cases} \end{aligned} \quad (\text{C.9})$$

In order to express the edge decorations in the same reference frame, their indices have to match, hence further demand that

$$c_1 = c_2. \quad (\text{C.10})$$

The proper geometric condition for the edge decorations is encoded in the identification

$$\lambda^{c_1} = (\lambda^{c_2})^{-1} \Leftrightarrow \lambda_1 = \lambda_2^{-1}. \quad (\text{C.11})$$

Using the above constraint and the Heisenberg ribbon equations, one obtains the extended ribbon equation:

$$\begin{cases} \tilde{h}_1 \tilde{\ell}_1 = \ell_1 h_1 = (\lambda_1 \bar{\beta}_1) h_1 & \Leftrightarrow \lambda_1 = \tilde{h}_1 \tilde{\ell}_1 h_1^{-1} \bar{\beta}_1^{-1} \\ \tilde{h}_2 \tilde{\ell}_2 = \ell_2 h_2 = (\beta_2 \lambda_2) h_2 & \Leftrightarrow \lambda_2 = \beta_2^{-1} \tilde{h}_2 \tilde{\ell}_2 h_2^{-1} \end{cases} \xrightarrow{\lambda_1 = \lambda_2^{-1}} \tilde{h}_1 \tilde{\ell}_1 h_1^{-1} \bar{\beta}_1^{-1} = h_2^{-1} \tilde{\ell}_2^{-1} \tilde{h}_2^{-1} \beta_2$$

$$\begin{aligned} \Rightarrow & (\tilde{u}_1 \tilde{y}_1) (\tilde{\beta}_1 \tilde{\lambda}_1) (\bar{y}_1 u_1)^{-1} \bar{\beta}_1^{-1} = \bar{\beta}_2 (\bar{y}_2 u_2) (\tilde{\beta}_2 \tilde{\lambda}_2)^{-1} (\tilde{u}_2 \tilde{y}_2)^{-1} \\ & (\bar{y}_1 u_1)^{-1} \bar{\beta}_1^{-1} (\tilde{u}_2 \tilde{y}_2) (\tilde{\beta}_2 \tilde{\lambda}_2) = (\tilde{\beta}_1 \tilde{\lambda}_1)^{-1} (\tilde{u}_1 \tilde{y}_1)^{-1} \bar{\beta}_2 (\bar{y}_2 u_2) \\ & u_1^{-1} \bar{y}_1^{-1} \tilde{u}_2 y' \beta' \tilde{y}_2 \tilde{\beta}_2 \tilde{\lambda}_2 = \tilde{\lambda}_1^{-1} \bar{\beta}_1^{-1} \tilde{y}_1^{-1} y'' \beta'' \tilde{u}_1^{-1} \bar{y}_2 u_2 \\ & (u_1^{-1} \tilde{u}_2) ((\bar{y}_1^{-1} \triangleleft \tilde{u}_2) y' \tilde{y}_2) (\beta' \tilde{\beta}_2) \tilde{\lambda}_2 = \tilde{\lambda}_1^{-1} (\bar{\beta}_1^{-1} \beta'') (\tilde{y}_1^{-1} y'' (\tilde{u}_1^{-1} \triangleright \bar{y}_2)) (\tilde{u}_1^{-1} u_2), \end{aligned} \quad (\text{C.12})$$

with $\tilde{u}_2^{-1} \bar{\beta}_1 \tilde{u}_2 = (y' \beta')^{-1}$ and $\tilde{u}_1^{-1} \bar{\beta}_2 \tilde{u}_1 = y'' \beta''$. The wedge gluing is therefore obtained through the symplectic reduction

$$\mathcal{P}_w = (\mathcal{B}_1 \times \mathcal{B}_2) // G_2^*, \quad (\text{C.13})$$

with momentum map $\lambda_1 \lambda_2 \in G_2$ and fused wedge variable $\tilde{y} = -\bar{y}_1 \triangleleft \tilde{u}_2 + \tilde{y}_2 - (\tilde{u}_1^{-1} \bar{\beta}_1 \tilde{u}_2) |_{G_2^*} \in G_2^*$.

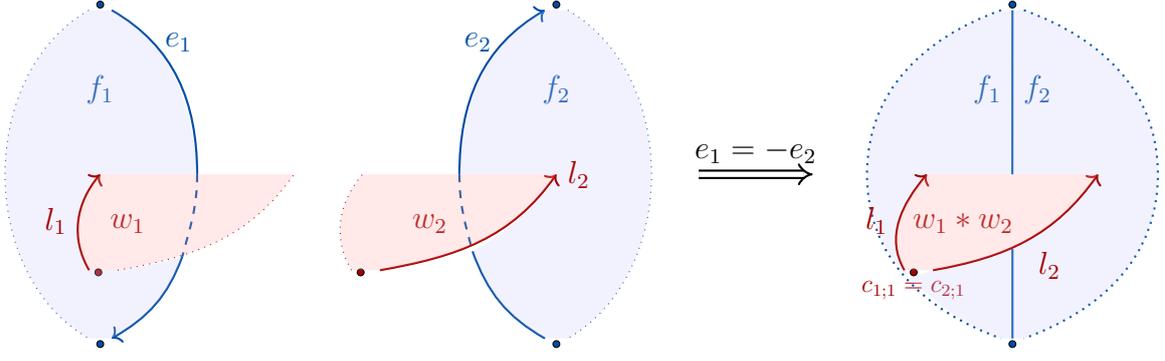


Figure C.2: Gluing of two wedges by identifying the dual edges e_1 and e_2 with opposite orientation. Impose the identification of the nodes $c_{1;1}$ and $c_{2;1}$, so that the wedges share the same root.

Face gluing. The gluing of two faces, as illustrated in Fig. C.3, is done by identifying the link data dual to each face.

The geometric identification consists in setting³

$$l_1 = l_2 \quad \leftrightarrow \quad c_1 = c_2, \quad c_{1;f} = c_{2;f}. \quad (\text{C.14})$$

Enforce this geometric constraint as a condition on G_1 elements: represent the decorations of the links l_i resp. at the vertices $v_{1;2}$ and $v_{2;1}$.

$$\mathcal{B}_1 : \begin{cases} \mathcal{G}^* = G_2 \times G_1^* \ni \tilde{\ell}^{c_{1;f}} \equiv \tilde{\ell}_1 = \tilde{\lambda}_1 \tilde{\beta}_1 \equiv \tilde{\lambda}^{c_{1;f}} \tilde{\beta}_{v_{1;2}}^{c_{1;f}}, \\ \mathcal{G} = G_1 \times G_2^* \ni h^{v_{1;2}} \equiv h_1 = u_1 y_1 \equiv u^{v_{1;2}} y_{c_{1;f}}^{v_{1;2}}, \end{cases} \quad (\text{C.15})$$

$$\mathcal{B}_2 : \begin{cases} \mathcal{G}^* = G_1^* \times G_2 \ni \tilde{\ell}^{c_{2;f}} \equiv \tilde{\ell}_2 = \tilde{\beta}_2 \tilde{\lambda}_2 \equiv \tilde{\beta}_{v_{2;1}}^{c_{2;f}} \tilde{\lambda}^{c_{2;f}}, \\ \mathcal{G} = G_1 \times G_2^* \ni \tilde{h}^{v_{2;1}} \equiv \tilde{h}_2 = \tilde{u}_2 \tilde{y}_2 \equiv \tilde{u}^{v_{2;1}} \tilde{y}_{c_{2;f}}^{v_{2;1}}. \end{cases}$$

In order to express the link decorations in the same reference frame, their indices have to match, hence further demand that

$$v_{1;2} = v_{2;1}. \quad (\text{C.16})$$

The proper geometric condition is encoded in the identification

$$u^{v_{1;2}} = \tilde{u}^{v_{2;1}} \quad \leftrightarrow \quad u_1 = \tilde{u}_2. \quad (\text{C.17})$$

Using the above constraint and the Heisenberg ribbon equations, we obtain the extended ribbon equation:

$$\begin{cases} \tilde{h}_1 \tilde{\ell}_1 = \ell_1 h_1 = \ell_1 (u_1 y_1) & \Leftrightarrow & u_1 = \ell_1^{-1} \tilde{h}_1 \tilde{\ell}_1 y_1^{-1} \\ \ell_2 h_2 = \tilde{h}_2 \tilde{\ell}_2 = (\tilde{u}_2 \tilde{y}_2) \tilde{\ell}_2 & \Leftrightarrow & \tilde{u}_2 = \ell_2 h_2 \tilde{\ell}_2^{-1} \tilde{y}_2^{-1} \end{cases} \quad \xrightarrow{u_1 = \tilde{u}_2} \quad \ell_1^{-1} \tilde{h}_1 \tilde{\ell}_1 y_1^{-1} = \ell_2 h_2 \tilde{\ell}_2^{-1} \tilde{y}_2^{-1}$$

$$\begin{aligned} \Rightarrow \quad & \bar{y}_1^{-1} (\lambda_1 \bar{\beta}_1)^{-1} (\tilde{u}_1 \tilde{y}_1) (\tilde{\beta}_1 \tilde{\lambda}_1) = (\lambda_2 \bar{\beta}_2) (\tilde{y}_2 u_2) (\tilde{\beta}_2 \tilde{\lambda}_2)^{-1} \tilde{y}_2^{-1} \\ & (\tilde{u}_1 \tilde{y}_1) (\tilde{\beta}_1 \tilde{\lambda}_1) \tilde{y}_2 (\tilde{\beta}_2 \tilde{\lambda}_2) = (\lambda_1 \bar{\beta}_1) \bar{y}_1 (\lambda_2 \bar{\beta}_2) (\tilde{y}_2 u_2) \\ & \tilde{u}_1 \tilde{y}_1 \tilde{\beta}_1 \tilde{y}' \tilde{\beta}' \tilde{\lambda}_1 \tilde{\beta}_2 \tilde{\lambda}_2 = \lambda_1 \bar{\beta}_1 \lambda_2 \beta' y' \bar{\beta}_2 \bar{y}_2 u_2 \\ & \tilde{u}_1 (\tilde{y}_1 \tilde{y}') (\tilde{\beta}_1 \tilde{\beta}' (\tilde{\lambda}_1 \triangleright \tilde{\beta}_2)) (\tilde{\lambda}_1 \tilde{\lambda}_2) = (\lambda_1 \lambda_2) ((\bar{\beta}_1 \triangleleft \lambda_2) \beta' \bar{\beta}_2) (y' \bar{y}_2) u_2, \end{aligned} \quad (\text{C.18})$$

³Links can be identified with same or opposite orientation. In this example I consider the identification with same orientation, since it is the one relevant for the phase space construction.

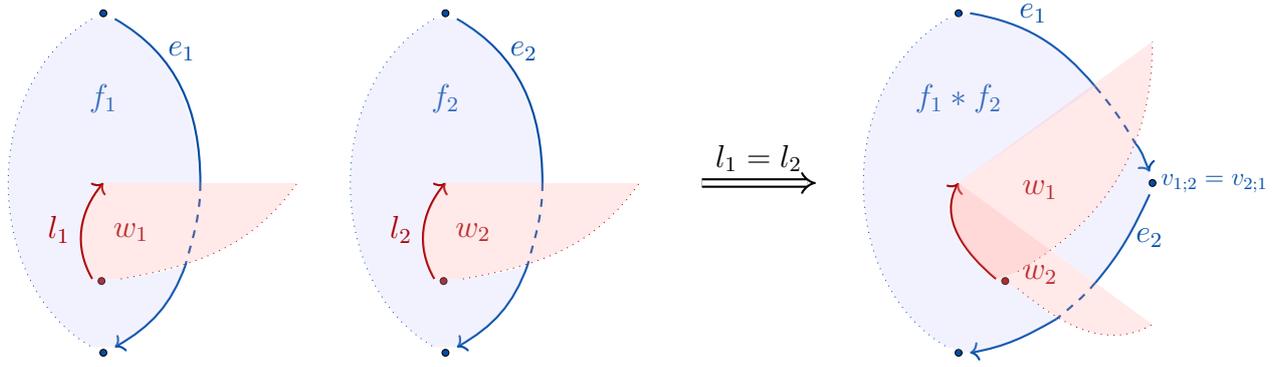


Figure C.3: Gluing of two faces by identifying the dual links l_1 and l_2 with same orientation. Impose the identification of the vertices $v_{1;2} = v_{2;1}$, so that the two faces share the same root.

with $\tilde{\lambda}_1 \tilde{y}_2 \tilde{\lambda}_1^{-1} = \tilde{y}' \tilde{\beta}'$ and $\lambda_2^{-1} \bar{y}_1 \lambda_2 = \beta' y'$. The face gluing is therefore obtained through the symplectic reduction

$$\mathcal{P}_f = (\mathcal{B}_1 \times \mathcal{B}_2) // G_1^*, \quad (\text{C.19})$$

with momentum map $u_1 \tilde{u}_2^{-1} \in G_1$ and fused face variable $\tilde{\beta} = \tilde{\beta}_1 + \lambda_1 \triangleright \tilde{\beta}_2 + (\tilde{\lambda}_1 \tilde{y}_2 \tilde{\lambda}_1^{-1})|_{G_1^*} \in G_1^*$.

Edge gluing. As the link gluing fused a pair of phase spaces associated to half links into a since phase space associated to a full link, the edge gluing is expected to do the same with edges. In this case, I will thus start by a pair of phase spaces associated to half links l_i , wedges w_i , faces f_i and half edges e_i , with sources and targets resp. at the vertices $v_{i;1}$ and $v_{i;w}$, where the second is a point on the wedge w .

The gluing of two edges, as illustrated in Fig. C.4, is done by identifying the wedge data, dual to each edge.

The geometric identification consists in setting⁴

$$w_1 = -w_2. \quad (\text{C.20})$$

Enforce this geometric constraint as a condition on G_2^* elements: represent the decorations of the wedges w_i at the vertices $v_{i;w}$. Let them be resp. rooted at the nodes c_i .

$$\mathcal{B}_1 : \begin{cases} \mathcal{G}^* = G_2 \times G_1^* \ni \ell^{c_1} \equiv l_1 = \lambda_1 \bar{\beta}_1 \equiv \lambda^{c_1} \bar{\beta}_{v_{1;w}}^{c_1}, \\ \mathcal{G} = G_2^* \times G_1 \ni h^{v_{1;w}} \equiv h_1 = \bar{y}_1 u_1 \equiv \bar{y}_{c_1}^{v_{1;w}} u^{v_{1;w}}, \end{cases} \quad (\text{C.21})$$

$$\mathcal{B}_{l',e'} : \begin{cases} \mathcal{G}^* = G_2 \times G_1^* \ni \ell^{c_2} \equiv l_2 = \lambda_2 \bar{\beta}_2 \equiv \lambda^{c_2} \bar{\beta}_{v_{2;w}}^{c_2}, \\ \mathcal{G} = G_2^* \times G_1 \ni h^{v_{2;w}} \equiv h_2 = \bar{y}_2 u_2 \equiv \bar{y}_{c_2}^{v_{2;w}} u^{v_{2;w}}. \end{cases}$$

In order to express the wedge decorations in the same reference frame, their indices have to match, hence further demand that

$$c_1 = c_2, \quad v_{1;w} = v_{2;w}. \quad (\text{C.22})$$

The condition $v_{1;w} = v_{2;w}$ provides the gluing of the edges. The proper geometric condition is encoded in the identification

$$\bar{y}_{c_1}^{v_{1;w}} = (\bar{y}_{c_2}^{v_{2;w}})^{-1} \Leftrightarrow \bar{y}_1 = \bar{y}_2^{-1}. \quad (\text{C.23})$$

⁴Wedges can be identified with same or opposite direction. In this example I consider the gluing with opposite orientation, simply because it is easier to represent and understand graphically.

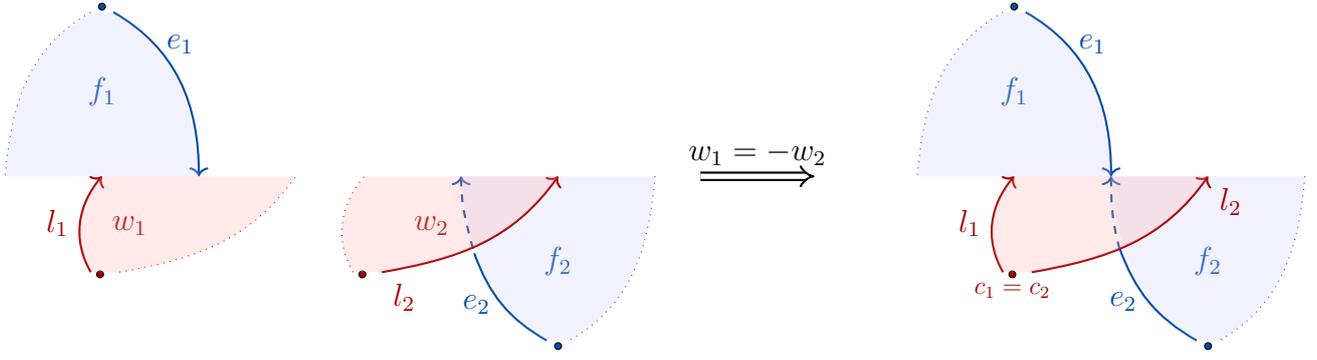


Figure C.4: Gluing of two edges by identifying the dual wedges w_1 and w_2 . Impose the vertices $v_{1;w}$ and $v_{2;w}$ in the two wedges to match. Root the wedges resp. at the nodes c_1 and c_2 and let them match $c_1 = c_2$, so that the wedges share the same root.

Using the above constraint and the Heisenberg ribbon equations, one obtains the extended ribbon equation:

$$\begin{aligned}
\tilde{h}_i \tilde{\ell}_i = \ell_i h_i = \ell_i (\bar{y}_i u_i) &\Leftrightarrow \bar{y}_i = \ell_i^{-1} \tilde{h}_i \tilde{\ell}_i u_i^{-1} \xrightarrow{\bar{y}_1 = \bar{y}_2^{-1}} \ell_1^{-1} \tilde{h}_1 \tilde{\ell}_1 u_1^{-1} = u_2 \tilde{\ell}_2^{-1} \tilde{h}_2^{-1} \ell_2 \\
\Rightarrow (\beta_1 \lambda_1)^{-1} (\tilde{y}_1 \tilde{u}_1) (\tilde{\lambda}_1 \tilde{\beta}_1) u_1^{-1} &= u_2 (\tilde{\lambda}_2 \tilde{\beta}_2)^{-1} (\tilde{y}_2 \tilde{u}_2)^{-1} (\beta_2 \lambda_2) \\
(\tilde{\lambda}_1 \tilde{\beta}_1) u_1^{-1} (\beta_2 \lambda_2)^{-1} (\tilde{y}_2 \tilde{u}_2) &= (\tilde{y}_1 \tilde{u}_1)^{-1} (\beta_1 \lambda_1) u_2 (\tilde{\lambda}_2 \tilde{\beta}_2)^{-1} \\
\tilde{\lambda}_1 \tilde{\beta}_1 (u_1^{-1} \triangleright \lambda_2^{-1}) (u_1^{-1} \triangleleft \lambda_2^{-1}) \beta_2^{-1} \tilde{y}_2 \tilde{u}_2 &= \tilde{u}_1^{-1} \tilde{y}_1^{-1} \beta_1 (\lambda_1 \triangleright u_2) (\lambda_1 \triangleleft u_2) \tilde{\beta}_2^{-1} \tilde{\lambda}_2^{-1} \\
(\tilde{\lambda}_1 (u_1^{-1} \triangleright \lambda_2^{-1})) ((\tilde{\beta}_1 \triangleleft (u_1^{-1} \triangleright \lambda_2^{-1})) \beta') &(y' ((u_1^{-1} \triangleleft \lambda_2^{-1}) \triangleright \tilde{y}_2)) ((u_1^{-1} \triangleleft \lambda_2^{-1}) \tilde{u}_2) \\
= (\tilde{u}_1^{-1} (\lambda_1 \triangleright u_2)) ((\tilde{y}_1 \triangleleft (\lambda_1 \triangleright u_2)) y'') &(\beta'' ((\lambda_1 \triangleleft u_2) \triangleright \tilde{\beta}_2^{-1})) ((\lambda_1 \triangleleft u_2) \tilde{\lambda}_2^{-1}), \tag{C.24}
\end{aligned}$$

with $(u_1^{-1} \triangleleft \lambda_2^{-1}) \beta_2^{-1} (u_1^{-1} \triangleleft \lambda_2^{-1})^{-1} = \beta' y'$ and $(\lambda_1 \triangleright u_2)^{-1} \beta_1 (\lambda_1 \triangleright u_2) = \beta'' y''$. The edge gluing is therefore obtained through the symplectic reduction

$$\mathcal{P}_e = (\mathcal{B}_1 \times \mathcal{B}_2) // G_2, \tag{C.25}$$

with momentum map $\bar{y}_1 \bar{y}_2 \sim \bar{y}_1 + \bar{y}_2 \in G_2^*$ and fused edge variable $\lambda = (\lambda_1 \triangleleft u_2) \tilde{\lambda}_2^{-1} \in G_2$.

Note that one can start by Heisenberg doubles associated to half links and half edges as in the last example. Then, by using the edge and face gluings one recovers the phase space associated to a full edge and an half link, or using the link and wedge gluings one recovers the phase space associated to a full link and an half edge. Therefore, Heisenberg doubles used for the first three gluings can be thought to be constructed according to the first option. In the full construction, we will need to use all the four fundamental gluings to retrieve the Heisenberg double associated to a full link and to a full edge.

Appendix D

2-Group field theory

D.1 Key Feynman diagrams

Here I will list the generating functions associated to the 2-graphs dual to a combination of N 4-simplices, with $N = 1, 2, 3, 4, 5$. These are the key Feynman diagrams involved in the four dimensional Pachner moves. Denote $\delta_{G_1}^{\{\mathcal{V}_N\}}$ $\delta_{G_2}^{\{\mathcal{V}_N\}}$ the sets of constraints on links (group G_1) and wedges (group G_2) in the amplitude of N independent 4-simplices.

i. **Feynman diagram associated to the graph dual to a 4-simplex**, illustrated in Fig. D.1a, is given by the amplitude (see also eq. (5.108))

$$\mathcal{A}_{\mathcal{V}} = \int dX^{10} dh^5 dY^{30} du^{20} \mathcal{V} = \int dX^{10} dh^5 dY^{30} du^{20} \delta_{G_1}^{\{\mathcal{V}\}}, \delta_{G_2}^{\{\mathcal{V}\}}. \quad (D.1)$$

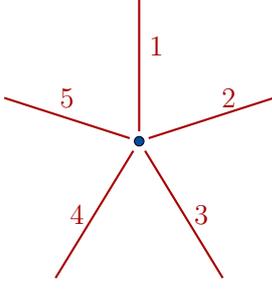
I denoted

$$\begin{aligned} \delta_{G_1}^{\{\mathcal{V}\}} = & \delta_{G_1}(t(X_{2,1})h_1u_{1;1}u_{2;4}^{-1}h_2^{-1}) \delta_{G_1}(t(X_{3,1})h_1u_{1;2}u_{3;3}^{-1}h_3^{-1}) \delta_{G_1}(t(X_{4,1})h_1u_{1;3}u_{4;2}^{-1}h_4^{-1}) \\ & \delta_{G_1}(t(X_{5,1})h_1u_{1;4}u_{5;1}^{-1}h_5^{-1}) \delta_{G_1}(t(X_{3,2})h_2u_{2;1}u_{3;4}^{-1}h_3^{-1}) \delta_{G_1}(t(X_{4,2})h_2u_{2;2}u_{4;3}^{-1}h_4^{-1}) \\ & \delta_{G_1}(t(X_{5,2})h_2u_{2;3}u_{5;2}^{-1}h_5^{-1}) \delta_{G_1}(t(X_{4,3})h_3u_{3;1}u_{4;4}^{-1}h_4^{-1}) \delta_{G_1}(t(X_{5,3})h_3u_{3;2}u_{5;3}^{-1}h_5^{-1}) \\ & \delta_{G_1}(t(X_{5,4})h_4u_{4;1}u_{5;4}^{-1}h_5^{-1}), \end{aligned} \quad (D.2)$$

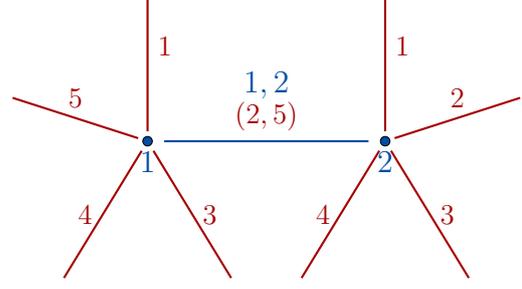
the set of delta functions on the group G_1 (on the links) and

$$\begin{aligned} \delta_{G_2}^{\{\mathcal{V}\}} = & \delta_H((h_1 \triangleright Y_{1;2,1}) X_{1,2} (h_2 \triangleright Y_{2;4,1}) X_{2,3} (h_3 \triangleright Y_{3;4,3}) X_{3,1}) \\ & \delta_{G_2}((h_1 \triangleright Y_{1;3,1}) X_{1,2} (h_2 \triangleright Y_{2;4,2}) X_{2,4} (h_4 \triangleright Y_{4;3,2}) X_{4,1}) \\ & \delta_{G_2}((h_1 \triangleright Y_{1;4,1}) X_{1,2} (h_2 \triangleright Y_{2;4,3}) X_{2,5} (h_5 \triangleright Y_{5;2,1}) X_{5,1}) \\ & \delta_{G_2}((h_1 \triangleright Y_{1;3,2}) X_{1,3} (h_3 \triangleright Y_{3;3,1}) X_{3,4} (h_4 \triangleright Y_{4;4,2}) X_{4,1}) \\ & \delta_{G_2}((h_1 \triangleright Y_{1;4,2}) X_{1,3} (h_3 \triangleright Y_{3;3,2}) X_{3,5} (h_5 \triangleright Y_{5;3,1}) X_{5,1}) \\ & \delta_{G_2}((h_1 \triangleright Y_{1;3,4}) X_{1,4} (h_4 \triangleright Y_{4;2,1}) X_{4,5} (h_5 \triangleright Y_{5;4,1}) X_{5,1}) \\ & \delta_{G_2}((h_2 \triangleright Y_{2;2,1}) X_{2,3} (h_3 \triangleright Y_{3;4,1}) X_{3,4} (h_4 \triangleright Y_{4;4,3}) X_{4,2}) \\ & \delta_{G_2}((h_2 \triangleright Y_{2;3,1}) X_{2,3} (h_3 \triangleright Y_{3;4,2}) X_{3,5} (h_5 \triangleright Y_{5;3,2}) X_{5,2}) \\ & \delta_{G_2}((h_2 \triangleright Y_{2;3,2}) X_{2,4} (h_4 \triangleright Y_{4;3,1}) X_{4,5} (h_5 \triangleright Y_{5;4,2}) X_{5,2}) \\ & \delta_{G_2}((h_3 \triangleright Y_{3;2,1}) X_{3,4} (h_4 \triangleright Y_{4;4,1}) X_{4,5} (h_5 \triangleright Y_{5;4,3}) X_{5,3}). \end{aligned} \quad (D.3)$$

the set of delta functions on the group G_2 (on the edges).



(a) Simplest graph dual to a 4-simplex.



(b) Graph dual to two 4-simplices sharing the tetrahedron labelled 2 in the 4-simplex 1 and labelled 5 in the 4-simplex 2.

ii. Feynman diagram associated to the graph dual to two 4-simplices, illustrated in Fig. D.1b, is given by the amplitude

$$\mathcal{A}_{\mathcal{V}_2} = \int dX^{20} dh^{10} dY^{60} du^{40} \mathcal{K}_{1,2} (\mathcal{V}_1 \mathcal{V}_2) = \int dX^{20} dh^9 dY^{48} du^{32} \delta_{G_1}^{\{\mathcal{V}_2\}} \delta_{G_2}^{\{\mathcal{V}_2\}}. \quad (\text{D.4})$$

I denoted

$$\begin{aligned} \delta_{G_1}^{\{\mathcal{V}_2\}} = & \left(\delta_{G_1} (t(X_{1;3,2}) h_{1,2} t(X_{2;5,2}) h_{2,2} u_{2;2,3} u_{1;3,4}^{-1} h_{1;3}^{-1}) \delta_{G_1} (t(X_{1;4,2}) h_{1,2} t(X_{2;5,3}) h_{2,3} u_{2;3,2} u_{1;4,3}^{-1} h_{1;4}^{-1}) \right. \\ & \left. \delta_{G_1} (t(X_{1;5,2}) h_{1,2} t(X_{2;5,4}) h_{2,4} u_{2;4,1} u_{1;5,2}^{-1} h_{1;5}^{-1}) \delta_{G_1} (t(X_{1;2,1}) h_{1,1} u_{1;1,1} u_{2;1,4}^{-1} h_{2;1}^{-1} t(X_{2;1,5}) h_{2,1}) \right) \\ & \left(\delta_{G_1} (t(X_{1;3,1}) h_{1,1} u_{1;1,2} u_{1;3,3}^{-1} h_{1;3}^{-1}) \delta_{G_1} (t(X_{1;4,1}) h_{1,1} u_{1;1,3} u_{1;4,2}^{-1} h_{1;4}^{-1}) \right. \\ & \delta_{G_1} (t(X_{1;5,1}) h_{1,1} u_{1;1,4} u_{1;5,1}^{-1} h_{1;5}^{-1}) \delta_{G_1} (t(X_{1;4,3}) h_{1,3} u_{1;3,1} u_{1;4,4}^{-1} h_{1;4}^{-1}) \\ & \delta_{G_1} (t(X_{1;5,3}) h_{1,3} u_{1;3,2} u_{1;5,3}^{-1} h_{1;5}^{-1}) \delta_{G_1} (t(X_{1;5,4}) h_{1,4} u_{1;4,1} u_{1;5,4}^{-1} h_{1;5}^{-1}) \\ & \delta_{G_1} (t(X_{2;2,1}) h_{2,1} u_{2;1,1} u_{2;2,4}^{-1} h_{2;2}^{-1}) \delta_{G_1} (t(X_{2;3,1}) h_{2,1} u_{2;1,2} u_{2;3,3}^{-1} h_{2;3}^{-1}) \\ & \delta_{G_1} (t(X_{2;4,1}) h_{2,1} u_{2;1,3} u_{2;4,2}^{-1} h_{2;4}^{-1}) \delta_{G_1} (t(X_{2;3,2}) h_{2,2} u_{2;2,1} u_{2;3,4}^{-1} h_{2;3}^{-1}) \\ & \left. \delta_{G_1} (t(X_{2;4,2}) h_{2,2} u_{2;2,2} u_{2;4,3}^{-1} h_{2;4}^{-1}) \delta_{G_1} (t(X_{2;4,3}) h_{2,3} u_{2;3,1} u_{2;4,4}^{-1} h_{2;4}^{-1}) \right), \end{aligned} \quad (\text{D.5})$$

the set of delta functions on the group G_1 (on the links) and

$$\begin{aligned} \delta_{G_2}^{\{\mathcal{V}_2\}} = & \delta_{G_2} (h_{1,2} \triangleright (X_{2;5,3} (h_{2,3} \triangleright Y_{2;3,2,4}) X_{2;3,2} (h_{2,2} \triangleright Y_{2;2,1,3}) X_{2;2,5}) X_{1;2,3} (h_{1,3} \triangleright Y_{1;3,4,1}) X_{1;3,4} (h_{1,4} \triangleright Y_{1;4,4,3}) X_{1;4,2}) \\ & \delta_{G_2} (h_{1,2} \triangleright (X_{2;5,4} (h_{2,4} \triangleright Y_{2;4,1,3}) X_{2;4,2} (h_{2,2} \triangleright Y_{2;2,2,3}) X_{2;2,5}) X_{1;2,3} (h_{1,3} \triangleright Y_{1;3,4,2}) X_{1;3,5} (h_{1,5} \triangleright Y_{1;5,3,2}) X_{1;5,2}) \\ & \delta_{G_2} (h_{1,2} \triangleright (X_{2;5,1} (h_{2,1} \triangleright Y_{2;1,4,1}) X_{2;1,2} (h_{2,2} \triangleright Y_{2;2,4,3}) X_{2;2,5}) X_{1;2,3} (h_{1,3} \triangleright Y_{1;3,4,3}) X_{1;3,1} (h_{1,1} \triangleright Y_{1;1,2,1}) X_{1;1,2}) \\ & \delta_{G_2} (h_{1,2} \triangleright (X_{2;5,4} (h_{2,4} \triangleright Y_{2;4,1,4}) X_{2;4,3} (h_{2,3} \triangleright Y_{2;3,1,2}) X_{2;3,5}) X_{1;2,4} (h_{1,4} \triangleright Y_{1;4,3,1}) X_{1;4,5} (h_{1,5} \triangleright Y_{1;5,4,2}) X_{1;5,2}) \\ & \delta_{G_2} (h_{1,2} \triangleright (X_{2;5,1} (h_{2,1} \triangleright Y_{2;1,4,2}) X_{2;1,3} (h_{2,3} \triangleright Y_{2;3,3,2}) X_{2;3,5}) X_{1;2,4} (h_{1,4} \triangleright Y_{1;4,3,2}) X_{1;4,1} (h_{1,1} \triangleright Y_{1;1,3,1}) X_{1;1,2}) \\ & \delta_{G_2} (h_{1,2} \triangleright (X_{2;5,1} (h_{2,1} \triangleright Y_{2;1,3,4}) X_{2;1,4} (h_{2,4} \triangleright Y_{2;4,2,1}) X_{2;4,5}) X_{1;2,5} (h_{1,5} \triangleright Y_{1;5,2,1}) X_{1;5,1} (h_{1,1} \triangleright Y_{1;1,4,1}) X_{1;1,2}) \\ & \delta_{G_2} ((h_{1;1} \triangleright Y_{1;1,3,2}) X_{1;1,3} (h_{1;3} \triangleright Y_{1;3,3,1}) X_{1;3,4} (h_{1;4} \triangleright Y_{1;4,4,2}) X_{1;4,1}) \\ & \delta_{G_2} ((h_{1;1} \triangleright Y_{1;1,4,2}) X_{1;1,3} (h_{1;3} \triangleright Y_{1;3,3,2}) X_{1;3,5} (h_{1;5} \triangleright Y_{1;5,3,1}) X_{1;5,1}) \\ & \delta_{G_2} ((h_{1;1} \triangleright Y_{1;1,3,4}) X_{1;1,4} (h_{1;4} \triangleright Y_{1;4,2,1}) X_{1;4,5} (h_{1;5} \triangleright Y_{1;5,4,1}) X_{1;5,1}) \\ & \delta_{G_2} ((h_{1;3} \triangleright Y_{1;3,2,1}) X_{1;3,4} (h_{1;4} \triangleright Y_{1;4,4,1}) X_{1;4,5} (h_{1;5} \triangleright Y_{1;5,4,3}) X_{1;5,3}) \\ & \delta_{G_2} ((h_{2;1} \triangleright Y_{2;1,2,1}) X_{2;1,2} (h_{2;2} \triangleright Y_{2;2,4,1}) X_{2;2,3} (h_{2;3} \triangleright Y_{2;3,4,3}) X_{2;3,1}) \\ & \delta_{G_2} ((h_{2;1} \triangleright Y_{2;1,3,1}) X_{2;1,2} (h_{2;2} \triangleright Y_{2;2,4,2}) X_{2;2,4} (h_{2;4} \triangleright Y_{2;4,3,2}) X_{2;4,1}) \\ & \delta_{G_2} ((h_{2;1} \triangleright Y_{2;1,3,2}) X_{2;1,3} (h_{2;3} \triangleright Y_{2;3,3,1}) X_{2;3,4} (h_{2;4} \triangleright Y_{2;4,4,2}) X_{2;4,1}) \\ & \delta_{G_2} ((h_{2;2} \triangleright Y_{2;2,2,1}) X_{2;2,3} (h_{2;3} \triangleright Y_{2;3,4,1}) X_{2;3,4} (h_{2;4} \triangleright Y_{2;4,4,3}) X_{2;4,2}), \end{aligned} \quad (\text{D.6})$$

the set of delta functions on the group G_2 (on the wedges). I defined the composed link $h_{1,2} \equiv h_{1;2}h_{2;5}^{-1}$ with inverse $h_{1,2}^{-1} = h_{2,1}$.

The first four delta functions in (D.5) involve a combination of bulk and boundary links of both the two 4-simplices, while the remaining twelve deltas involve bulk and boundary links of the first or second 4-simplices separately. Similarly, the first six deltas in (D.6) involve a combination of bulk and boundary wedges of both the 4-simplices, while the remaining eight deltas involve bulk and boundary wedges of the first or the second 4-simplices separately.

iii. **Feynman diagram associated to the graph dual to three 4-simplices**, illustrated in Fig. D.2a, is given by the amplitude

$$\mathcal{A}_{\mathcal{V}_3} = \int dX^{30} dh^{15} dY^{90} du^{60} (\mathcal{K}_{1,2} \mathcal{K}_{1,3} \mathcal{K}_{2,3}) (\mathcal{V}_1 \mathcal{V}_2 \mathcal{V}_3) = \int dX^{30} dh^{12} dY^{54} du^{36} \delta_{G_1}^{\{\mathcal{V}_3\}} \delta_{G_2}^{\{\mathcal{V}_3\}}. \quad (\text{D.7})$$

I denoted

$$\begin{aligned} \delta_{G_1}^{\{\mathcal{V}_3\}} = & \delta_{G_1} (t(X_{1;3,2})h_{1,2}t(X_{2;5,2})h_{2,3}t(X_{3;5,4})h_{3,1}) \\ & \left(\delta_{G_1} (t(X_{1;4,2})h_{1,2}t(X_{2;5,3})h_{2,3}u_{2;3;2}u_{1;4;3}^{-1}h_{1;4}^{-1}) \delta_{G_1} (t(X_{1;5,2})h_{1,2}t(X_{2;5,4})h_{2,4}u_{2;4;1}u_{1;5;2}^{-1}h_{1;5}^{-1}) \right. \\ & \delta_{G_1} (t(X_{1;2,1})h_{1,1}u_{1;1;1}u_{2;1;4}^{-1}h_{2;1}^{-1}t(X_{2;1,5})h_{2,1}) \delta_{G_1} (t(X_{1;3,1})h_{1,1}u_{1;1;2}u_{3;1;3}^{-1}h_{3;1}^{-1}t(X_{3;1,4})h_{3,1}) \\ & \delta_{G_1} (t(X_{1;4,3})h_{1,3}t(X_{3;4,2})h_{3,2}u_{3;2;2}u_{1;4;4}^{-1}h_{1;4}^{-1}) \delta_{G_1} (t(X_{1;5,3})h_{1,3}t(X_{3;4,3})h_{3,3}u_{3;3;1}u_{1;5;3}^{-1}h_{1;5}^{-1}) \\ & \delta_{G_1} (t(X_{2;2,1})h_{2,1}u_{2;1;1}u_{3;1;4}^{-1}h_{3;1}^{-1}t(X_{3;1,5})h_{3,2}) \delta_{G_1} (t(X_{2;3,2})h_{2,3}t(X_{3;5,2})h_{3,2}u_{3;2;3}u_{2;3;4}^{-1}h_{2;3}^{-1}) \\ & \left. \delta_{G_1} (t(X_{2;4,2})h_{2,3}t(X_{3;5,3})h_{3,3}u_{3;3;2}u_{2;4;3}^{-1}h_{2;4}^{-1}) \right) \\ & \left(\delta_{G_1} (t(X_{1;5,4})h_{1,1}u_{1;1;3}u_{1;4;2}^{-1}h_{1;4}^{-1}) \delta_{G_1} (t(X_{1;5,1})h_{1,1}u_{1;1;4}u_{1;5;1}^{-1}h_{1;5}^{-1}) \delta_{G_1} (t(X_{1;4,1})h_{1,4}u_{1;4;1}u_{1;5;4}^{-1}h_{1;5}^{-1}) \right. \\ & \delta_{G_1} (t(X_{2;3,1})h_{2,1}u_{2;1;2}u_{2;3;3}^{-1}h_{2;3}^{-1}) \delta_{G_1} (t(X_{2;4,1})h_{2,1}u_{2;1;3}u_{2;4;2}^{-1}h_{2;4}^{-1}) \delta_{G_1} (t(X_{2;4,3})h_{2,3}u_{2;3;1}u_{2;4;4}^{-1}h_{2;4}^{-1}) \\ & \left. \delta_{G_1} (t(X_{3;2,1})h_{3,1}u_{3;1;1}u_{3;2;4}^{-1}h_{3;2}^{-1}) \delta_{G_1} (t(X_{3;3,1})h_{3,1}u_{3;1;2}u_{3;3;3}^{-1}h_{3;3}^{-1}) \delta_{G_1} (t(X_{3;3,2})h_{3,2}u_{3;2;1}u_{3;3;4}^{-1}h_{3;3}^{-1}) \right), \quad (\text{D.8}) \end{aligned}$$

the set of delta functions on the group G_1 (on the links) and

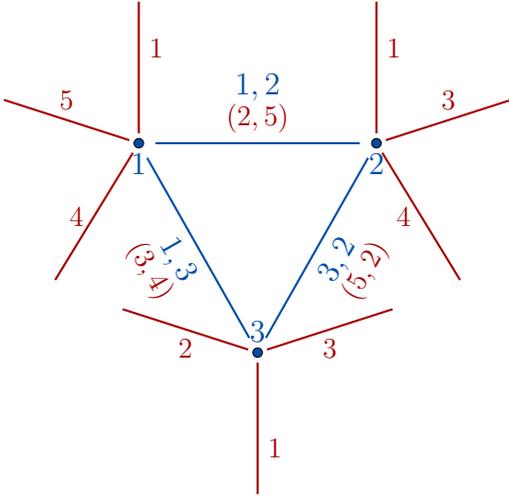
$$\begin{aligned}
\delta_{G_2}^{\{\mathcal{V}_3\}} = & \delta_{G_2} \left(h_{1,2} \triangleright (X_{2;5,3} (h_{2,3} \triangleright Y_{2;3,2,4}) X_{2;3,2} X_{2;2,5}) X_{1;2,3} h_{1,3} \triangleright (X_{3;4,5} X_{3;5,2} (h_{3,2} \triangleright Y_{3;2,3,2}) X_{3;2,4}) X_{1;3,4} (h_{1,4} \triangleright Y_{1;4,4,3}) X_{1;4,2} \right) \\
& \delta_{G_2} \left(h_{1,2} \triangleright (X_{2;5,4} (h_{2,4} \triangleright Y_{2;4,1,3}) X_{2;4,2} X_{2;2,5}) X_{1;2,3} h_{1,3} \triangleright (X_{3;4,5} X_{3;5,3} (h_{3,3} \triangleright Y_{3;3,2,1}) X_{3;3,4}) X_{1;3,5} (h_{1,5} \triangleright Y_{1;5,3,2}) X_{1;5,2} \right) \\
& \delta_{G_2} \left(h_{1,2} \triangleright (X_{2;5,1} (h_{2,1} \triangleright Y_{2;1,4,1}) X_{2;1,2} X_{2;2,5}) X_{1;2,3} h_{1,3} \triangleright (X_{3;4,5} X_{3;5,1} (h_{3,1} \triangleright Y_{3;1,3,4}) X_{3;1,4}) X_{1;3,1} (h_{1,1} \triangleright Y_{1;1,2,1}) X_{1;1,2} \right) \\
& \delta_{G_2} \left(h_{1,2} \triangleright (X_{2;5,4} (h_{2,4} \triangleright Y_{2;4,1,4}) X_{2;4,3} (h_{2,3} \triangleright Y_{2;3,1,2}) X_{2;3,5}) X_{1;2,4} (h_{1,4} \triangleright Y_{1;4,3,1}) X_{1;4,5} (h_{1,5} \triangleright Y_{1;5,4,2}) X_{1;5,2} \right) \\
& \delta_{G_2} \left((h_{1,1} \triangleright Y_{1;1,3,1}) X_{1;1,2} h_{1,2} \triangleright (X_{2;5,1} (h_{2,1} \triangleright Y_{2;1,4,2}) X_{2;1,3} (h_{2,3} \triangleright Y_{2;3,3,2}) X_{2;3,5}) X_{1;2,4} (h_{1,4} \triangleright Y_{1;4,3,2}) X_{1;4,1} \right) \\
& \delta_{G_2} \left((h_{1,1} \triangleright Y_{1;1,4,1}) X_{1;1,2} h_{1,2} \triangleright (X_{2;5,1} (h_{2,1} \triangleright Y_{2;1,3,4}) X_{2;1,4} (h_{2,4} \triangleright Y_{2;4,2,1}) X_{2;4,5}) X_{1;2,5} (h_{1,5} \triangleright Y_{1;5,2,1}) X_{1;5,1} \right) \\
& \delta_{G_2} \left(h_{1,3} \triangleright (X_{3;4,3} (h_{3,3} \triangleright Y_{3;3,1,4}) X_{3;3,2} (h_{3,2} \triangleright Y_{3;2,1,2}) X_{3;2,4}) X_{1;3,4} (h_{1,4} \triangleright Y_{1;4,4,1}) X_{1;4,5} (h_{1,5} \triangleright Y_{1;5,4,3}) X_{1;5,3} \right) \\
& \delta_{G_2} \left((h_{1,1} \triangleright Y_{1;1,3,2}) X_{1;1,3} h_{1,3} \triangleright (X_{3;4,1} (h_{3,1} \triangleright Y_{3;1,3,1}) X_{3;1,2} (h_{3,2} \triangleright Y_{3;2,4,2}) X_{3;2,4}) X_{1;3,4} (h_{1,4} \triangleright Y_{1;4,4,2}) X_{1;4,1} \right) \\
& \delta_{G_2} \left((h_{1,1} \triangleright Y_{1;1,4,2}) X_{1;1,3} h_{1,3} \triangleright (X_{3;4,1} (h_{3,1} \triangleright Y_{3;1,3,2}) X_{3;1,3} (h_{3,3} \triangleright Y_{3;3,3,1}) X_{3;3,4}) X_{1;3,5} (h_{1,5} \triangleright Y_{1;5,3,1}) X_{1;5,1} \right) \\
& \delta_{G_2} \left((h_{2,1} \triangleright Y_{2;1,2,1}) X_{2;1,2} h_{2,3} \triangleright (X_{3;5,1} (h_{3,1} \triangleright Y_{3;1,4,1}) X_{3;1,2} (h_{3,2} \triangleright Y_{3;2,4,3}) X_{3;2,5}) X_{2;2,3} (h_{2,3} \triangleright Y_{2;3,4,3}) X_{2;3,1} \right) \\
& \delta_{G_2} \left((h_{2,1} \triangleright Y_{2;1,3,1}) X_{2;1,2} h_{2,3} \triangleright (X_{3;5,1} (h_{3,1} \triangleright Y_{3;1,4,2}) X_{3;1,3} (h_{3,3} \triangleright Y_{3;3,3,2}) X_{3;3,5}) X_{2;2,4} (h_{2,4} \triangleright Y_{2;4,3,2}) X_{2;4,1} \right) \\
& \delta_{G_2} \left(h_{2,3} \triangleright (X_{3;5,3} (h_{3,3} \triangleright Y_{3;3,2,4}) X_{3;3,2} (h_{3,2} \triangleright Y_{3;2,1,3}) X_{3;2,5}) X_{2;2,3} (h_{2,3} \triangleright Y_{2;3,4,1}) X_{2;3,4} (h_{2,4} \triangleright Y_{2;4,4,3}) X_{2;4,2} \right) \\
& \delta_{G_2} \left((h_{1,1} \triangleright Y_{1;1,3,4}) X_{1;1,4} (h_{1,4} \triangleright Y_{1;4,2,1}) X_{1;4,5} (h_{1,5} \triangleright Y_{1;5,4,1}) X_{1;5,1} \right) \\
& \delta_{G_2} \left((h_{2,1} \triangleright Y_{2;1,3,2}) X_{2;1,3} (h_{2,3} \triangleright Y_{2;3,3,1}) X_{2;3,4} (h_{2,4} \triangleright Y_{2;4,4,2}) X_{2;4,1} \right) \\
& \delta_{G_2} \left((h_{3,1} \triangleright Y_{3;1,2,1}) X_{3;1,2} (h_{3,2} \triangleright Y_{3;2,4,1}) X_{3;2,3} (h_{3,3} \triangleright Y_{3;3,4,3}) X_{3;3,1} \right), \tag{D.9}
\end{aligned}$$

the set of delta functions on the group G_2 (on the wedges). I defined the composed links $h_{1,3} \equiv h_{1,3} h_{3,4}^{-1}$ and $h_{2,3} \equiv h_{2,2} h_{3,5}^{-1}$ with inverses $h_{1,3}^{-1} = h_{3,1}$ and $h_{2,3}^{-1} = h_{3,2}$.

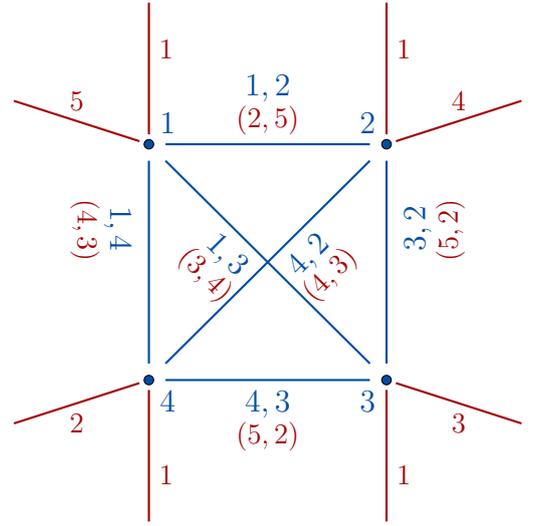
The first delta in (D.8) enforces a closed path of only bulk links of the three 4-simplices; this is the loop of links dual to the single face shared by the three 4-simplices. The following nine delta functions involve a combination of bulk and boundary links of the 4-simplices $\{1, 2\}$, $\{1, 3\}$ and $\{2, 3\}$, while the remaining nine deltas involve bulk and boundary links of the first, the second or the third 4-simplices separately. Similarly, the first three deltas in (D.9) involve a combination of bulk and boundary wedges of the three 4-simplices. The following nine deltas involve a combination of bulk and boundary wedges of the three 4-simplices pairwise, $\{1, 2\}$, $\{1, 3\}$ and $\{2, 3\}$. The remaining three deltas involve bulk and boundary wedges of the first, second or third 4-simplices separately.

iv. **Feynman diagram associated to the graph dual to four 4-simplices**, illustrated in Fig. D.2b, is given by the amplitude

$$\begin{aligned}
\mathcal{A}_{\mathcal{V}_4} = & \int dX^{40} dh^{20} dY^{120} du^{80} (\mathcal{K}_{1,2} \mathcal{K}_{1,3} \mathcal{K}_{1,4} \mathcal{K}_{2,3} \mathcal{K}_{2,4} \mathcal{K}_{3,4}) (\mathcal{V}_1 \mathcal{V}_2 \mathcal{V}_3 \mathcal{V}_4) \\
= & \int dX^{40} dh^{14} dY^{48} du^{32} \delta_{G_1}^{\{\mathcal{V}_4\}} \delta_{G_2}^{\{\mathcal{V}_4\}}. \tag{D.10}
\end{aligned}$$



(a) Graph dual to three 4-simplices.



(b) Graph dual to four 4-simplices.

I denoted

$$\begin{aligned}
\delta_{G_1}^{\{\mathcal{V}_4\}} = & \delta_{G_1} (t(X_{1;3,2})h_{1,2}t(X_{2;5,2})h_{2,3}t(X_{3;5,4})h_{3,1}) \delta_{G_1} (t(X_{1;4,2})h_{1,2}t(X_{2;5,3})h_{2,4}t(X_{4;4,3})h_{4,1}) \\
& \delta_{G_1} (t(X_{1;4,3})h_{1,3}t(X_{3;4,2})h_{3,4}t(X_{4;5,3})h_{4,1}) \delta_{G_1} (t(X_{2;3,2})h_{2,3}t(X_{3;5,2})h_{3,4}t(X_{4;5,4})h_{4,2}) \Big) \\
& \left(\delta_{G_1} (t(X_{1;5,2})h_{1,2}t(X_{2;5,4})h_{2,4}u_{2;4;1}u_{1;5;2}^{-1}h_{1;5}^{-1}) \delta_{G_1} (t(X_{1;2,1})h_{1,1}u_{1;1;1}u_{2;1;4}^{-1}h_{2;1}^{-1}t(X_{2;1,5})h_{2,1}) \right. \\
& \delta_{G_1} (t(X_{1;3,1})h_{1,1}u_{1;1;2}u_{3;1;3}^{-1}h_{3;1}^{-1}t(X_{3;1,4})h_{3,1}) \delta_{G_1} (t(X_{1;5,3})h_{1,3}t(X_{3;4,3})h_{3,3}u_{3;3;1}u_{1;5;3}^{-1}h_{1;5}^{-1}) \\
& \delta_{G_1} (t(X_{2;2,1})h_{2,1}u_{2;1;1}u_{3;1;4}^{-1}h_{3;1}^{-1}t(X_{3;1,5})h_{3,2}) \delta_{G_1} (t(X_{2;4,2})h_{2,3}t(X_{3;5,3})h_{3,3}u_{3;3;2}u_{2;4;3}^{-1}h_{2;4}^{-1}) \\
& \delta_{G_1} (t(X_{1;5,4})h_{1,1}u_{1;1;3}u_{4;1;2}^{-1}h_{4;1}^{-1}t(X_{4;1,3})h_{4,1}) \delta_{G_1} (t(X_{1;4;1})h_{1,4}t(X_{4;3,2})h_{4,2}u_{4;2;1}u_{1;5;4}^{-1}h_{1;5}^{-1}) \\
& \delta_{G_1} (t(X_{2;3,1})h_{2,1}u_{2;1;2}u_{4;1;3}^{-1}h_{4;1}^{-1}t(X_{4;1,4})h_{4,2}) \delta_{G_1} (t(X_{2;4,3})h_{2,4}t(X_{4;4,2})h_{4,2}u_{4;2;2}u_{2;4;4}^{-1}h_{2;4}^{-1}) \\
& \delta_{G_1} (t(X_{3;2,1})h_{3,1}u_{3;1;1}u_{4;1;4}^{-1}h_{4;1}^{-1}t(X_{4;1,5})h_{4,3}) \delta_{G_1} (t(X_{3;3,2})h_{3,4}t(X_{4;5,2})h_{4,2}u_{4;2;3}u_{3;3;4}^{-1}h_{3;3}^{-1}) \Big) \\
& \left(\delta_{G_1} (t(X_{1;5,1})h_{1,1}u_{1;1;4}u_{1;5;1}^{-1}h_{1;5}^{-1}) \delta_{G_1} (t(X_{2;4,1})h_{2,1}u_{2;1;3}u_{2;4;2}^{-1}h_{2;4}^{-1}) \delta_{G_1} (t(X_{3;3,1})h_{3,1}u_{3;1;2}u_{3;3;3}^{-1}h_{3;3}^{-1}) \right. \\
& \left. \delta_{G_1} (t(X_{4;2,1})h_{4,1}u_{4;1;1}u_{4;2;4}^{-1}h_{4;2}^{-1}) \right), \tag{D.11}
\end{aligned}$$

the set of delta functions on the group G_1 (on the links) and

$$\begin{aligned}
\delta_{G_2}^{\{\mathcal{V}_4\}} = & \delta_{G_2} \left(X_{1;4,2} h_{1,2} \triangleright (X_{2;5,3} X_{2;3,2} X_{2;2,5}) X_{1;2,3} h_{1,3} \triangleright (X_{3;4,5} X_{3;5,2} X_{3;2,4}) X_{1;3,4} h_{1,4} \triangleright (X_{4;3,5} X_{4;5,4} X_{4;4,3}) \right) \\
& \delta_{G_2} \left(h_{1,2} \triangleright (X_{2;5,4} (h_{2;4} \triangleright Y_{2;4,1,3}) X_{2;4,2} X_{2;2,5}) X_{1;2,3} h_{1,3} \triangleright (X_{3;4,5} X_{3;5,3} (h_{3;3} \triangleright Y_{3;3,2,1}) X_{3;3,4}) X_{1;3,5} (h_{1;5} \triangleright Y_{1;5,3,2}) X_{1;5,2} \right) \\
& \delta_{G_2} \left(h_{1,2} \triangleright (X_{2;5,1} (h_{2;1} \triangleright Y_{2;1,4,1}) X_{2;1,2} X_{2;2,5}) X_{1;2,3} h_{1,3} \triangleright (X_{3;4,5} X_{3;5,1} (h_{3;1} \triangleright Y_{3;1,3,4}) X_{3;1,4}) X_{1;3,1} (h_{1;1} \triangleright Y_{1;1,2,1}) X_{1;1,2} \right) \\
& \delta_{G_2} \left(h_{1,2} \triangleright (X_{2;5,4} (h_{2;4} \triangleright Y_{2;4,1,4}) X_{2;4,3} X_{2;3,5}) X_{1;2,4} h_{1,4} \triangleright (X_{4;3,4} X_{4;4,2} (h_{4;2} \triangleright Y_{4;2,2,1}) X_{4;2,3}) X_{1;4,5} (h_{1;5} \triangleright Y_{1;5,4,2}) X_{1;5,2} \right) \\
& \delta_{G_2} \left(h_{1,2} \triangleright (X_{2;5,1} (h_{2;1} \triangleright Y_{2;1,4,2}) X_{2;1,3} X_{2;3,5}) X_{1;2,4} h_{1,4} \triangleright (X_{3;3,4} X_{4;4,1} (h_{4;1} \triangleright Y_{4;1,3,2}) X_{4;1,3}) X_{1;4,1} (h_{1;1} \triangleright Y_{1;1,3,1}) X_{1;1,2} \right) \\
& \delta_{G_2} \left(h_{1,3} \triangleright (X_{3;4,3} (h_{3;3} \triangleright Y_{3;3,1,4}) X_{3;3,2} X_{3;2,4}) X_{1;3,4} h_{1,4} \triangleright (X_{4;3,5} X_{4;5,2} (h_{4;2} \triangleright Y_{4;2,3,1}) X_{4;2,3}) X_{1;4,5} (h_{1;5} \triangleright Y_{1;5,4,3}) X_{1;5,3} \right) \\
& \delta_{G_2} \left(h_{1,3} \triangleright (X_{3;4,1} (h_{3;1} \triangleright Y_{3;1,3,1}) X_{3;1,2} X_{3;2,4}) X_{1;3,4} h_{1,4} \triangleright (X_{4;3,5} X_{4;5,1} (h_{4;1} \triangleright Y_{4;1,4,2}) X_{4;1,3}) X_{1;4,1} (h_{1;1} \triangleright Y_{1;1,3,2}) X_{1;1,3} \right) \\
& \delta_{G_2} \left(h_{2,3} \triangleright (X_{3;5,1} (h_{3;1} \triangleright Y_{3;1,4,1}) X_{3;1,2} X_{3;2,5}) X_{2;2,3} h_{2,4} \triangleright (X_{4;4,5} X_{4;5,1} (h_{4;1} \triangleright Y_{4;1,3,4}) X_{4;1,4}) X_{2;3,1} (h_{2;1} \triangleright Y_{2;1,2,1}) X_{2;1,2} \right) \\
& \delta_{G_2} \left(h_{2,3} \triangleright (X_{3;5,3} (h_{3;3} \triangleright Y_{3;3,2,4}) X_{3;3,2} X_{3;2,5}) X_{2;2,3} h_{2,4} \triangleright (X_{4;4,5} X_{4;5,2} (h_{4;2} \triangleright Y_{4;2,3,2}) X_{4;2,4}) X_{2;3,4} (h_{2;4} \triangleright Y_{2;4,4,3}) X_{2;4,2} \right) \\
& \delta_{G_2} \left(h_{1,2} \triangleright (X_{2;5,1} (h_{2;1} \triangleright Y_{2;1,3,4}) X_{2;1,4} (h_{2;4} \triangleright Y_{2;4,2,1}) X_{2;4,5}) X_{1;2,5} (h_{1;5} \triangleright Y_{1;5,2,1}) X_{1;5,1} (h_{1;1} \triangleright Y_{1;1,4,1}) X_{1;1,2} \right) \\
& \delta_{G_2} \left(h_{1,3} \triangleright (X_{3;4,1} (h_{3;1} \triangleright Y_{3;1,3,2}) X_{3;1,3} (h_{3;3} \triangleright Y_{3;3,3,1}) X_{3;3,4}) X_{1;3,5} (h_{1;5} \triangleright Y_{1;5,3,1}) X_{1;5,1} (h_{1;1} \triangleright Y_{1;1,4,2}) X_{1;1,3} \right) \\
& \delta_{G_2} \left(h_{2,3} \triangleright (X_{3;5,1} (h_{3;1} \triangleright Y_{3;1,4,2}) X_{3;1,3} (h_{3;3} \triangleright Y_{3;3,3,2}) X_{3;3,5}) X_{2;2,4} (h_{2;4} \triangleright Y_{2;4,3,2}) X_{2;4,1} (h_{2;1} \triangleright Y_{2;1,3,1}) X_{2;1,2} \right) \\
& \delta_{G_2} \left(h_{1,4} \triangleright (X_{4;3,1} (h_{4;1} \triangleright Y_{4;1,2,1}) X_{4;1,2} (h_{4;2} \triangleright Y_{4;2,4,1}) X_{4;2,3}) X_{1;4,5} (h_{1;5} \triangleright Y_{1;5,4,1}) X_{1;5,1} (h_{1;1} \triangleright Y_{1;1,3,4}) X_{1;1,4} \right) \\
& \delta_{G_2} \left(h_{2,4} \triangleright (X_{4;4,1} (h_{4;1} \triangleright Y_{4;1,3,1}) X_{4;1,2} (h_{4;2} \triangleright Y_{4;2,4,2}) X_{4;2,4}) X_{2;3,4} (h_{2;4} \triangleright Y_{2;4,4,2}) X_{2;4,1} (h_{2;1} \triangleright Y_{2;1,3,2}) X_{2;1,3} \right) \\
& \delta_{G_2} \left(h_{3,4} \triangleright (X_{4;5,1} (h_{4;1} \triangleright Y_{4;1,4,1}) X_{4;1,2} (h_{4;2} \triangleright Y_{4;2,4,3}) X_{4;2,5}) X_{3;2,3} (h_{3;3} \triangleright Y_{3;3,4,3}) X_{3;3,1} (h_{3;1} \triangleright Y_{3;1,2,1}) X_{3;1,2} \right),
\end{aligned} \tag{D.12}$$

the set of delta functions on the group G_2 (on the wedges). I defined the composed links $h_{1,4} \equiv h_{1;4} h_{4;3}^{-1}$, $h_{2,4} \equiv h_{2;3} h_{4;4}^{-1}$ and $h_{3,4} \equiv h_{3;2} h_{4;5}^{-1}$, with inverses $h_{1,4}^{-1} = h_{4,1}$, $h_{2,4}^{-1} = h_{4,2}$ and $h_{3,4}^{-1} = h_{4,3}$.

The first four delta functions in (D.11) enforces a closed path of only bulk links of the three 4-simplices pairwise. These are the loops of links dual to the four faces shared by triplets of the four 4-simplices, $\{1, 2, 3\}$, $\{1, 2, 4\}$, $\{1, 3, 4\}$ and $\{2, 3, 4\}$. The following twelve delta functions involve a combination of bulk and boundary links of the 4-simplices pairwise, $\{A, B\}$, with $A, B = 1, 2, 3, 4$ and $B > A$; while the remaining four deltas involve bulk and boundary links of the four 4-simplices separately. Similarly, the first delta functions in (D.12) is a closed 2-path of only bulk wedges shared by the four 4-simplices; this is the closed surface in the dual complex dual to the single edge shared by the four 4-simplices. The following eight delta functions involve a combination of bulk and boundary wedges of triplets of 4-simplices, $\{A, B, C\}$ for $A, B, C = 1, 2, 3, 4$ and $C > B > A$; the last six delta functions involve a combination of bulk and boundary wedges of the 4-simplices pairwise.

v. **Feynman diagram associated to the graph dual to five 4-simplices**, illustrated in Fig. D.3, is given by the amplitude

$$\begin{aligned}
\mathcal{Z}_{\mathcal{V}_5} = & \int dX^{50} dh^{25} dY^{150} du^{100} (\mathcal{K}_{1,2} \mathcal{K}_{1,3} \mathcal{K}_{1,4} \mathcal{K}_{1,5} \mathcal{K}_{2,3} \mathcal{K}_{2,4} \mathcal{K}_{2,5} \mathcal{K}_{3,4} \mathcal{K}_{3,5} \mathcal{K}_{4,5}) (\mathcal{V}_1 \mathcal{V}_2 \mathcal{V}_3 \mathcal{V}_4 \mathcal{V}_5) \\
= & \int dX^{50} dh^{15} dY^{30} du^{20} \delta_{G_1}^{\{\mathcal{V}_5\}} \delta_{G_2}^{\{\mathcal{V}_5\}}.
\end{aligned} \tag{D.13}$$

I denoted

$$\begin{aligned}
\delta_{G_1}^{\{\mathcal{V}_5\}} = & \left(\delta_{G_1} (t(X_{1;3,2})h_{1,2}t(X_{2;5,2})h_{2,3}t(X_{3;5,4})h_{3,1}) \delta_{G_1} (t(X_{1;4,2})h_{1,2}t(X_{2;5,3})h_{2,4}t(X_{4;4,3})h_{4,1}) \right. \\
& \delta_{G_1} (t(X_{1;4,3})h_{1,3}t(X_{3;4,2})h_{3,4}t(X_{4;5,3})h_{4,1}) \delta_{G_1} (t(X_{2;3,2})h_{2,3}t(X_{3;5,2})h_{3,4}t(X_{4;5,4})h_{4,2}) \\
& \delta_{G_1} (t(X_{1;5,2})h_{1,2}t(X_{2;5,4})h_{2,5}t(X_{5;3,2})h_{5,1}) \delta_{G_1} (t(X_{1;5,3})h_{1,3}t(X_{3;4,3})h_{3,5}t(X_{5;4,2})h_{5,1}) \\
& \delta_{G_1} (t(X_{2;4,2})h_{2,3}t(X_{3;5,3})h_{3,5}t(X_{5;4,3})h_{5,2}) \delta_{G_1} (t(X_{1;5,4})h_{1,4}t(X_{4;3,2})h_{4,5}t(X_{5;5,2})h_{5,1}) \\
& \delta_{G_1} (t(X_{2;4,3})h_{2,4}t(X_{4;4,2})h_{4,5}t(X_{5;5,3})h_{5,2}) \delta_{G_1} (t(X_{3;3,2})h_{3,4}t(X_{4;5,2})h_{4,5}t(X_{5;5,4})h_{5,3}) \left. \right) \\
& \left(\delta_{G_1} (t(X_{1;2,1})h_{1,1}u_{1;1;1}u_{2;1;4}^{-1}h_{2;1}^{-1}t(X_{2;1,5})h_{2,1}) \delta_{G_1} (t(X_{1;3,1})h_{1,1}u_{1;1;2}u_{3;1;3}^{-1}h_{3;1}^{-1}t(X_{3;1,4})h_{3,1}) \right. \\
& \delta_{G_1} (t(X_{1;4,1})h_{1,1}u_{1;1;3}u_{4;1;2}^{-1}h_{4;1}^{-1}t(X_{4;1,3})h_{4,1}) \delta_{G_1} (t(X_{2;2,1})h_{2,1}u_{2;1;1}u_{3;1;4}^{-1}h_{3;1}^{-1}t(X_{3;1,5})h_{3,2}) \\
& \delta_{G_1} (t(X_{2;3,1})h_{2,1}u_{2;1;2}u_{4;1;3}^{-1}h_{4;1}^{-1}t(X_{4;1,4})h_{4,2}) \delta_{G_1} (t(X_{3;2,1})h_{3,1}u_{3;1;1}u_{4;1;4}^{-1}h_{4;1}^{-1}t(X_{4;1,5})h_{4,3}) \\
& \delta_{G_1} (t(X_{1;5,1})h_{1,1}u_{1;1;4}u_{5;1;1}^{-1}h_{5;1}^{-1}t(X_{5;1,2})h_{5,1}) \delta_{G_1} (t(X_{2;4,1})h_{2,1}u_{2;1;3}u_{5;1;2}^{-1}h_{5;1}^{-1}t(X_{5;1,3})h_{5,2}) \\
& \delta_{G_1} (t(X_{3;3,1})h_{3,1}u_{3;1;2}u_{5;1;3}^{-1}h_{5;1}^{-1}t(X_{5;1,4})h_{5,3}) \delta_{G_1} (t(X_{4;2,1})h_{4,1}u_{4;1;1}u_{5;1;4}^{-1}h_{5;1}^{-1}t(X_{5;1,5})h_{5,4}) \left. \right), \tag{D.14}
\end{aligned}$$

the set of delta functions on the group G_1 (on the links) and

$$\begin{aligned}
\delta_{G_2}^{\{\mathcal{V}_5\}} = & \delta_{G_2} \left(X_{1;4,2} h_{1,2} \triangleright (X_{2;5,3} X_{2;3,2} X_{2;2,5}) X_{1;2,3} h_{1,3} \triangleright (X_{3;4,5} X_{3;5,2} X_{3;2,4}) X_{1;3,4} h_{1,4} \triangleright (X_{4;3,5} X_{4;5,4} X_{4;4,3}) \right) \\
& \delta_{G_2} \left(X_{1;5,2} h_{1,2} \triangleright (X_{2;5,4} X_{2;4,2} X_{2;2,5}) X_{1;2,3} h_{1,3} \triangleright (X_{3;4,5} X_{3;5,3} X_{3;3,4}) X_{1;3,5} h_{1,5} \triangleright (X_{5;2,4} X_{5;4,3} X_{5;3,2}) \right) \\
& \delta_{G_2} \left(X_{1;5,2} h_{1,2} \triangleright (X_{2;5,4} X_{2;4,3} X_{2;3,5}) X_{1;2,4} h_{1,4} \triangleright (X_{4;4,3} X_{4;3,2} X_{4;2,4}) X_{1;4,5} h_{1,5} \triangleright (X_{5;2,5} X_{5;5,3} X_{5;3,2}) \right) \\
& \delta_{G_2} \left(X_{1;5,3} h_{1,3} \triangleright (X_{3;4,3} X_{3;3,2} X_{3;2,4}) X_{1;3,4} h_{1,4} \triangleright (X_{4;3,5} X_{4;5,2} X_{4;2,3}) X_{1;4,5} h_{1,5} \triangleright (X_{5;2,5} X_{5;5,4} X_{5;4,2}) \right) \\
& \delta_{G_2} \left(X_{2;4,2} h_{2,3} \triangleright (X_{3;5,3} X_{3;3,2} X_{3;2,5}) X_{2;2,3} h_{2,4} \triangleright (X_{4;4,5} X_{4;5,2} X_{4;2,4}) X_{2;3,4} h_{2,5} \triangleright (X_{5;3,5} X_{5;5,4} X_{5;4,3}) \right) \\
& \delta_{G_2} \left(h_{1,2} \triangleright (X_{2;5,1} (h_{2;1} \triangleright Y_{2;1;4,1}) X_{2;1,2} X_{2;2,5}) X_{1;2,3} h_{1,3} \triangleright (X_{3;4,5} X_{3;5,1} (h_{3;1} \triangleright Y_{3;1;3,4}) X_{3;1,4}) X_{1;3,1} (h_{1;1} \triangleright Y_{1;1;2,1}) X_{1;1,2}) \right) \\
& \delta_{G_2} \left(h_{1,2} \triangleright (X_{2;5,1} (h_{2;1} \triangleright Y_{2;1;4,2}) X_{2;1,3} X_{2;3,5}) X_{1;2,4} h_{1,4} \triangleright (X_{3;3,4} X_{4;4,1} (h_{4;1} \triangleright Y_{4;1;3,2}) X_{4;1,3}) X_{1;4,1} (h_{1;1} \triangleright Y_{1;1;3,1}) X_{1;1,2}) \right) \\
& \delta_{G_2} \left(h_{1,3} \triangleright (X_{3;4,1} (h_{3;1} \triangleright Y_{3;1;3,1}) X_{3;1,2} X_{3;2,4}) X_{1;3,4} h_{1,4} \triangleright (X_{4;3,5} X_{4;5,1} (h_{4;1} \triangleright Y_{4;1;4,2}) X_{4;1,3}) X_{1;4,1} (h_{1;1} \triangleright Y_{1;1;3,2}) X_{1;1,3}) \right) \\
& \delta_{G_2} \left(h_{2,3} \triangleright (X_{3;5,1} (h_{3;1} \triangleright Y_{3;1;4,1}) X_{3;1,2} X_{3;2,5}) X_{2;2,3} h_{2,4} \triangleright (X_{4;4,5} X_{4;5,1} (h_{4;1} \triangleright Y_{4;1;3,4}) X_{4;1,4}) X_{2;3,1} (h_{2;1} \triangleright Y_{2;1;2,1}) X_{2;1,2}) \right) \\
& \delta_{G_2} \left(h_{1,2} \triangleright (X_{2;5,1} (h_{2;1} \triangleright Y_{2;1;3,4}) X_{2;1,4} X_{2;4,5}) X_{1;2,5} h_{1,5} \triangleright (X_{5;2,3} X_{5;3,1} (h_{5;1} \triangleright Y_{5;1;2,1}) X_{5;1,2}) X_{1;5,1} (h_{1;1} \triangleright Y_{1;1;4,1}) X_{1;1,2}) \right) \\
& \delta_{G_2} \left(h_{1,3} \triangleright (X_{3;4,1} (h_{3;1} \triangleright Y_{3;1;3,2}) X_{3;1,3} X_{3;3,4}) X_{1;3,5} h_{1,5} \triangleright (X_{5;4,3} X_{5;3,1} (h_{5;1} \triangleright Y_{5;1;2,3}) X_{5;1,4}) X_{1;5,1} (h_{1;1} \triangleright Y_{1;1;4,2}) X_{1;1,3}) \right) \\
& \delta_{G_2} \left(h_{2,3} \triangleright (X_{3;5,1} (h_{3;1} \triangleright Y_{3;1;4,2}) X_{3;1,3} X_{3;3,5}) X_{2;2,4} h_{2,5} \triangleright (X_{5;5,3} X_{5;3,1} (h_{5;1} \triangleright Y_{5;1;2,4}) X_{5;1,5}) X_{2;4,1} (h_{2;1} \triangleright Y_{2;1;3,1}) X_{2;1,2}) \right) \\
& \delta_{G_2} \left(h_{1,4} \triangleright (X_{4;3,1} (h_{4;1} \triangleright Y_{4;1;2,1}) X_{4;1,2} X_{4;2,3}) X_{1;4,5} h_{1,5} \triangleright (X_{5;2,5} X_{5;5,1} (h_{5;1} \triangleright Y_{5;1;4,1}) X_{5;1,2}) X_{1;5,1} (h_{1;1} \triangleright Y_{1;1;3,4}) X_{1;1,4}) \right) \\
& \delta_{G_2} \left(h_{2,4} \triangleright (X_{4;4,1} (h_{4;1} \triangleright Y_{4;1;3,1}) X_{4;1,2} X_{4;2,4}) X_{2;3,4} h_{2,5} \triangleright (X_{5;3,5} X_{5;5,1} (h_{5;1} \triangleright Y_{5;1;4,2}) X_{5;1,3}) X_{2;4,1} (h_{2;1} \triangleright Y_{2;1;3,2}) X_{2;1,3}) \right) \\
& \delta_{G_2} \left(h_{3,4} \triangleright (X_{4;5,1} (h_{4;1} \triangleright Y_{4;1;4,1}) X_{4;1,2} X_{4;2,5}) X_{3;2,3} h_{3,5} \triangleright (X_{5;4,5} X_{5;5,1} (h_{5;1} \triangleright Y_{5;1;3,4}) X_{5;1,4}) X_{3;3,1} (h_{3;1} \triangleright Y_{3;1;2,1}) X_{3;1,2}) \right), \tag{D.15}
\end{aligned}$$

the set of delta functions on the group G_2 (on the wedges). We defined the composed links $h_{1,5} \equiv h_{1;5}h_{5;2}^{-1}$, $h_{2,5} \equiv h_{2;4}h_{5;3}^{-1}$, $h_{3,5} \equiv h_{2;3}h_{5;4}^{-1}$ and $h_{4,5} \equiv h_{3;2}h_{5;5}^{-1}$, with inverses $h_{1,5}^{-1} = h_{5,1}$, $h_{2,5}^{-1} = h_{5,2}$, $h_{3,5}^{-1} = h_{5,3}$ and $h_{4,5}^{-1} = h_{5,4}$.

The first six delta functions in (D.14) enforces a closed path of only bulk links of the 4-simplices pairwise; these are the loops of links dual to the six faces shared by triplets of four 4-simplices. The remaining ten delta functions involve a combination of bulk and boundary links of the 4-simplices pairwise. Similarly, the first five delta functions in (D.15) enforce the closure of five

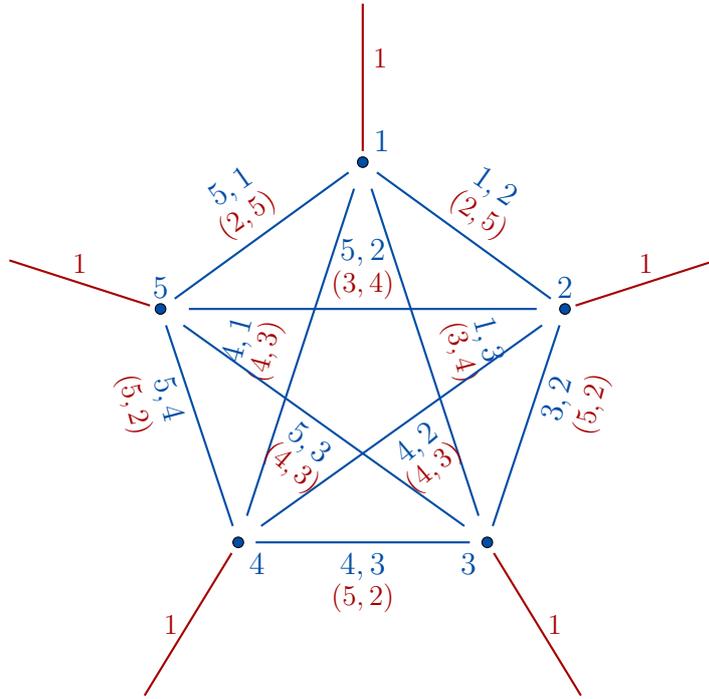


Figure D.3: Graphs dual to five 4-simplices.

2-path of only bulk wedges shared by the 4-simplices four by four; these are closed surfaces in the dual complex dual to the edges shared by the four 4-simplices. The remaining ten delta functions involve a combination of bulk and boundary wedges of triplets of 4-simplices.

D.2 Pachner moves

Let me give here a few details of the computation of the four dimensional Pachner moves for a 2-group field theory.

Pachner move $P_{1,5}$. Consider the amplitude of five 4-simplices (D.13) and integrate over six bulk links ($h_{1,2}, h_{1,3}, h_{1,4}, h_{2,3}, h_{2,4}, h_{3,4}$) and over three bulk wedges ($X_{4;3,4}, X_{4;3,5}, X_{4;4,5}$). Note that, upon integration, four delta functions on the group G_1 and one on the group G_2 are automatically satisfied. This lead to the type of divergence that is encountered very often in group field theories. Such divergencies have to be regularized by hand, by dividing for the proper volume term. The remaining integrals give instead several volume terms of the groups G_1 and

G_2 . Upon integration, consider the map below for the remaining terms of the amplitude.

	Bulk wedges
Boundary links	
$u_{A;1;i} \rightarrow u_{A;i}$ for $A = 1, 2, 3, 4, 5$ $i = 1, 2, 3, 4$	$h_{5,2} \triangleright (X_{2;1,5} X_{2;5,4}) X_{5;3,2} h_{5,1} \triangleright (X_{1;5,2} X_{1;2,1}) \rightarrow X_{2,1}$ $h_{5,3} \triangleright (X_{3;1,4} X_{3;4,3}) X_{5;4,2} h_{5,1} \triangleright (X_{1;5,3} X_{1;3,1}) \rightarrow X_{3,1}$ $h_{5,4} \triangleright (X_{4;1,3} X_{4;3,2}) X_{5;5,2} h_{5,1} \triangleright (X_{1;5,4} X_{1;4,1}) \rightarrow X_{4,1}$
Boundary wedges	
$Y_{A;1;i,j} \rightarrow Y_{A;i,j}$ for $A = 1, 2, 3, 4, 5$ $i, j = 1, 2, 3, 4$	$h_{5,3} \triangleright (X_{3;1,5} X_{3;5,3}) X_{5;4,3} h_{5,2} \triangleright (X_{2;4,2} X_{2;2,1}) \rightarrow X_{3,2}$ $h_{5,4} \triangleright (X_{4;1,4} X_{4;4,2}) X_{5;5,3} h_{5,2} \triangleright (X_{2;4,3} X_{2;3,1}) \rightarrow X_{4,2}$ $h_{5,4} \triangleright (X_{4;1,5} X_{4;5,2}) X_{5;5,4} h_{5,3} \triangleright (X_{3;3,2} X_{3;2,1}) \rightarrow X_{4,3}$
Bulk links	
$h_{5,A} h_{A;1} \rightarrow h_A$ for $A = 1, 2, 3, 4$ $h_{5;1} \rightarrow h_5$	$X_{5;1,2} h_{5,1} \triangleright (X_{1;5,1}) \rightarrow X_{5,1}$ $X_{5;1,3} h_{5,2} \triangleright (X_{2;4,1}) \rightarrow X_{5,2}$ $X_{5;1,4} h_{5,3} \triangleright (X_{3;3,1}) \rightarrow X_{5,3}$ $X_{5;1,5} h_{5,4} \triangleright (X_{4;2,1}) \rightarrow X_{5,4}$

Under such change of variables, the amplitude of five 4-simplices (D.13) turns out to be proportional to that of a single one (D.1):

$$\mathcal{A}_{\mathcal{V}_5} = (V_{G_1}^4 V_{G_2}^{37}) \mathcal{A}_{\mathcal{V}_1}. \quad (\text{D.16})$$

This transformation is represented in Fig. 5.13.

Pachner move $P_{2,4}$. Consider the amplitude of four 4-simplices (D.10) and integrate over three bulk links ($h_{1,2}, h_{1,3}, h_{2,3}$). Again, upon integration, there will be one divergence (a constraint in the group G_1) that has to be regularized. The other integration lead again to some volume terms, and for the remaining terms consider the map below.

Boundary links	Bulk links
$u_{1;1;1} \rightarrow u_{1;1;2}$ $u_{2;1;1} \rightarrow u_{1;3;1}$ $u_{3;1;1} \rightarrow u_{1;4;1}$ $u_{4;1;1} \rightarrow u_{1;5;2}$	$h_{4,1} h_{1;1} \rightarrow h_{1,1}$
$u_{1;1;2} \rightarrow u_{1;1;3}$ $u_{2;1;2} \rightarrow u_{1;3;2}$ $u_{3;1;2} \rightarrow u_{1;4;3}$ $u_{4;1;2} \rightarrow u_{1;5;1}$	$h_{4,2} h_{2;1} \rightarrow h_{1,3}$
$u_{1;1;3} \rightarrow u_{1;1;4}$ $u_{2;1;3} \rightarrow u_{1;3;4}$ $u_{3;1;3} \rightarrow u_{1;4;2}$ $u_{4;1;3} \rightarrow u_{1;5;3}$	$h_{4,3} h_{3;1} \rightarrow h_{1,4}$
$u_{1;1;4} \rightarrow u_{1;1;1}$ $u_{2;1;4} \rightarrow u_{1;3;3}$ $u_{3;1;4} \rightarrow u_{1;4;4}$ $u_{4;1;4} \rightarrow u_{1;5;4}$	$h_{4,1} \rightarrow h_{1,5}$
$u_{1;5;1} \rightarrow u_{2;1;4}$ $u_{2;4;1} \rightarrow u_{2;2;4}$ $u_{3;3;1} \rightarrow u_{2;3;3}$ $u_{4;2;1} \rightarrow u_{2;4;2}$	$h_{4,1} h_{1;5} \rightarrow h_{1,2} h_{2,1}$
$u_{1;5;2} \rightarrow u_{2;1;1}$ $u_{2;4;2} \rightarrow u_{2;2;3}$ $u_{3;3;2} \rightarrow u_{2;3;4}$ $u_{4;2;2} \rightarrow u_{2;4;3}$	$h_{4,2} h_{2;4} \rightarrow h_{1,2} h_{2,2}$
$u_{1;5;3} \rightarrow u_{2;1;2}$ $u_{2;4;3} \rightarrow u_{2;2;1}$ $u_{3;3;3} \rightarrow u_{2;3;2}$ $u_{4;2;3} \rightarrow u_{2;4;4}$	$h_{4,3} h_{3;3} \rightarrow h_{1,2} h_{2,3}$
$u_{1;5;4} \rightarrow u_{2;1;3}$ $u_{2;4;4} \rightarrow u_{2;2;2}$ $u_{3;3;4} \rightarrow u_{2;3;1}$ $u_{4;2;4} \rightarrow u_{2;4;1}$	$h_{4,2} \rightarrow h_{1,2} h_{2,4}$

Bulk wedges

$h_{4,1} \triangleright X_{1;1,5} \rightarrow X_{1;1,2} h_{1,2} \triangleright X_{2;5,1}$	$h_{4,2} \triangleright X_{2;1,4} \rightarrow X_{1;3,2} h_{1,2} \triangleright X_{2;5,2}$
$h_{4,3} \triangleright X_{3;1,3} \rightarrow X_{1;4,2} h_{1,2} \triangleright X_{2;5,3}$	$X_{4;1,2} \rightarrow X_{1;5,2} h_{1,2} \triangleright X_{2;5,4}$
$h_{4,1} \triangleright (X_{1;5,2} X_{1;2,4}) X_{4;3,4} h_{4,2} \triangleright (X_{2;3,5} X_{2;5,4}) \rightarrow h_{1,2} \triangleright X_{2;1,2}$	$X_{4;1,3} h_{4,1} \triangleright X_{1;4,1} \rightarrow X_{1;5,1}$
$h_{4,1} \triangleright (X_{1;1,2} X_{1;2,4}) X_{4;3,4} h_{4,2} \triangleright (X_{2;3,5} X_{2;5,1}) \rightarrow X_{1;1,3}$	$X_{4;2,3} h_{4,1} \triangleright X_{1;4,5} \rightarrow h_{1,2} \triangleright X_{2;4,1}$
$h_{4,1} \triangleright (X_{1;1,3} X_{1;3,4}) X_{4;3,5} h_{4,3} \triangleright (X_{3;2,4} X_{3;4,1}) \rightarrow X_{1;1,4}$	$X_{4;1,4} h_{4,2} \triangleright X_{2;3,1} \rightarrow X_{1;5,3}$
$h_{4,1} \triangleright (X_{1;5,3} X_{1;3,4}) X_{4;3,5} h_{4,3} \triangleright (X_{3;2,4} X_{3;4,3}) \rightarrow h_{1,2} \triangleright X_{2;1,3}$	$X_{4;2,4} h_{4,2} \triangleright X_{2;3,4} \rightarrow h_{1,2} \triangleright X_{2;4,2}$
$h_{4,2} \triangleright (X_{2;4,2} X_{2;2,3}) X_{4;4,5} h_{4,3} \triangleright (X_{3;2,5} X_{3;5,3}) \rightarrow h_{1,2} \triangleright X_{2;2,3}$	$X_{4;1,5} h_{4,3} \triangleright X_{3;2,1} \rightarrow X_{1;5,4}$
$h_{4,2} \triangleright (X_{2;1,2} X_{2;2,3}) X_{4;4,5} h_{4,3} \triangleright (X_{3;2,5} X_{3;5,1}) \rightarrow X_{1;3,4}$	$X_{4;2,5} h_{4,3} \triangleright X_{3;2,3} \rightarrow h_{1,2} \triangleright X_{2;4,3}$

Boundary wedges

$$\begin{aligned}
h_{4,1} \triangleright (h_{1,1} \triangleright Y_{1,1;2,1}) &\rightarrow h_{1,1} \triangleright Y_{1,1;3,2} & h_{4,2} \triangleright (h_{2,1} \triangleright Y_{2,1;4,1}) &\rightarrow h_{1,3} \triangleright Y_{1,3;3,1} \\
h_{4,1} \triangleright (h_{1,1} \triangleright Y_{1,1;3,1}) &\rightarrow h_{1,1} \triangleright Y_{1,1;4,2} & h_{4,2} \triangleright (h_{2,1} \triangleright Y_{2,1;4,2}) &\rightarrow h_{1,3} \triangleright Y_{1,3;3,2} \\
h_{4,1} \triangleright (h_{1,1} \triangleright Y_{1,1;3,2}) &\rightarrow h_{1,1} \triangleright Y_{1,1;3,4} & h_{4,2} \triangleright (h_{2,1} \triangleright Y_{2,1;2,1}) &\rightarrow h_{1,3} \triangleright Y_{1,3;2,1} \\
h_{4,1} \triangleright (h_{1,5} \triangleright Y_{1,5;3,2}) &\rightarrow (h_{1,2}h_{2,1}) \triangleright Y_{2,1;2,1} & h_{4,2} \triangleright (h_{2,4} \triangleright Y_{2,4;1,3}) &\rightarrow (h_{1,2}h_{2,2}) \triangleright Y_{2,2;4,1} \\
h_{4,1} \triangleright (h_{5,1} \triangleright Y_{5,1;4,2}) &\rightarrow (h_{1,2}h_{2,1}) \triangleright Y_{2,1;3,1} & h_{4,2} \triangleright (h_{2,4} \triangleright Y_{2,4;1,4}) &\rightarrow (h_{1,2}h_{2,2}) \triangleright Y_{2,2;4,2} \\
h_{4,1} \triangleright (h_{1,5} \triangleright Y_{1,5;4,3}) &\rightarrow (h_{1,2}h_{2,1}) \triangleright Y_{2,1;3,2} & h_{4,2} \triangleright (h_{2,4} \triangleright Y_{2,4;4,3}) &\rightarrow (h_{1,2}h_{2,2}) \triangleright Y_{2,2;2,1} \\
h_{4,1} \triangleright (h_{1,1} \triangleright Y_{1,1;4,1}) &\rightarrow h_{1,1} \triangleright Y_{1,1;1,2} & h_{4,2} \triangleright (h_{2,1} \triangleright Y_{2,1;3,4}) &\rightarrow h_{1,3} \triangleright Y_{1,3;3,4} \\
h_{4,1} \triangleright (h_{1,5} \triangleright Y_{1,5;2,1}) &\rightarrow (h_{1,2}h_{2,1}) \triangleright Y_{2,1;1,4} & h_{4,2} \triangleright (h_{2,4} \triangleright Y_{2,4;4,2}) &\rightarrow (h_{1,2}h_{2,2}) \triangleright Y_{2,2;3,4} \\
h_{4,1} \triangleright (h_{1,5} \triangleright Y_{1,5;3,1}) &\rightarrow (h_{1,2}h_{2,1}) \triangleright Y_{2,1;2,4} & h_{4,2} \triangleright (h_{2,1} \triangleright Y_{2,1;3,1}) &\rightarrow h_{1,3} \triangleright Y_{1,3;4,1} \\
h_{4,1} \triangleright (h_{1,1} \triangleright Y_{1,1;4,2}) &\rightarrow h_{1,1} \triangleright Y_{1,1;1,3} & h_{4,2} \triangleright (h_{2,4} \triangleright Y_{2,4;3,2}) &\rightarrow (h_{1,2}h_{2,2}) \triangleright Y_{2,2;1,3} \\
h_{4,1} \triangleright (h_{1,1} \triangleright Y_{1,1;3,4}) &\rightarrow h_{1,1} \triangleright Y_{1,1;1,4} & h_{4,2} \triangleright Y_{4,2;4,1} &\rightarrow (h_{1,2}h_{2,4}) \triangleright Y_{2,4;1,2} \\
h_{4,1} \triangleright (h_{1,5} \triangleright Y_{1,5;4,1}) &\rightarrow (h_{1,2}h_{2,1}) \triangleright Y_{2,1;4,3} & h_{4,2} \triangleright (h_{2,1} \triangleright Y_{2,1;3,2}) &\rightarrow h_{1,3} \triangleright Y_{1,3;4,2} \\
h_{4,3} \triangleright (h_{3,1} \triangleright Y_{3,1;3,4}) &\rightarrow h_{1,4} \triangleright Y_{1,4;4,2} & h_{4,2} \triangleright (h_{2,4} \triangleright Y_{2,4;4,2}) &\rightarrow (h_{1,2}h_{2,2}) \triangleright Y_{2,2;2,3} \\
h_{4,3} \triangleright (h_{3,1} \triangleright Y_{3,1;3,1}) &\rightarrow h_{1,4} \triangleright Y_{1,4;2,1} & h_{4,1} \triangleright Y_{4,1;4,2} &\rightarrow h_{1,5} \triangleright Y_{1,5;4,1} \\
h_{4,3} \triangleright (h_{3,1} \triangleright Y_{3,1;4,1}) &\rightarrow h_{1,4} \triangleright Y_{1,4;4,1} & h_{4,1} \triangleright Y_{4,1;3,4} &\rightarrow h_{1,5} \triangleright Y_{1,5;4,3} \\
h_{4,3} \triangleright (h_{3,3} \triangleright Y_{3,3;2,1}) &\rightarrow (h_{1,2}h_{2,3}) \triangleright Y_{2,3;4,3} & h_{4,1} \triangleright Y_{4,1;3,2} &\rightarrow h_{1,5} \triangleright Y_{1,5;3,1} \\
h_{4,3} \triangleright (h_{3,3} \triangleright Y_{3,3;1,4}) &\rightarrow (h_{1,2}h_{2,3}) \triangleright Y_{2,3;3,1} & h_{4,2} \triangleright Y_{4,2;2,1} &\rightarrow (h_{1,2}h_{2,4}) \triangleright Y_{2,4;3,2} \\
h_{4,3} \triangleright (h_{3,3} \triangleright Y_{3,3;2,4}) &\rightarrow (h_{1,2}h_{2,3}) \triangleright Y_{2,3;4,1} & h_{4,2} \triangleright Y_{4,2;3,1} &\rightarrow (h_{1,2}h_{2,4}) \triangleright Y_{2,4;4,2} \\
h_{4,3} \triangleright (h_{3,1} \triangleright Y_{3,1;3,2}) &\rightarrow h_{1,4} \triangleright Y_{1,4;2,3} & h_{4,2} \triangleright Y_{4,2;3,2} &\rightarrow (h_{1,2}h_{2,4}) \triangleright Y_{2,4;4,3} \\
h_{4,3} \triangleright (h_{3,3} \triangleright Y_{3,3;3,1}) &\rightarrow (h_{1,2}h_{2,3}) \triangleright Y_{2,3;2,3} & h_{4,1} \triangleright Y_{4,1;2,1} &\rightarrow h_{1,5} \triangleright Y_{1,5;1,2} \\
h_{4,3} \triangleright (h_{3,1} \triangleright Y_{3,1;4,2}) &\rightarrow h_{1,4} \triangleright Y_{1,4;4,3} & h_{4,1} \triangleright Y_{4,1;3,1} &\rightarrow h_{1,5} \triangleright Y_{1,5;3,2} \\
h_{4,3} \triangleright (h_{3,3} \triangleright Y_{3,3;3,2}) &\rightarrow (h_{1,2}h_{2,3}) \triangleright Y_{2,3;2,4} & h_{4,2} \triangleright Y_{4,2;4,2} &\rightarrow (h_{1,2}h_{2,4}) \triangleright Y_{2,4;1,3} \\
h_{4,3} \triangleright (h_{3,1} \triangleright Y_{3,1;2,1}) &\rightarrow h_{1,4} \triangleright Y_{1,4;3,1} & h_{4,1} \triangleright Y_{4,1;4,1} &\rightarrow h_{1,5} \triangleright Y_{1,5;4,2} \\
h_{4,3} \triangleright (h_{3,3} \triangleright Y_{3,3;4,3}) &\rightarrow (h_{1,2}h_{2,3}) \triangleright Y_{2,3;1,2} & h_{4,2} \triangleright Y_{4,2;4,3} &\rightarrow (h_{1,2}h_{2,4}) \triangleright Y_{2,4;1,4}
\end{aligned}$$

Under such change of variables, the amplitude of four 4-simplices (D.13) turns out to be proportional to that of two (D.4):

$$\mathcal{A}_{\mathcal{V}_4} = (V_{G_1}^2 V_{G_2}^{19}) \mathcal{A}_{\mathcal{V}_2}. \quad (\text{D.17})$$

This Pachner move is illustrated in Fig. 5.14.

Pachner move $P_{3,3}$. Consider the amplitude of three 4-simplices (D.7). The last transformation can be simply checked by applying the following change of variables.

Boundary links

$$\begin{array}{lll}
u_{1;1;1} \rightarrow u_{1;5;4} & u_{2;1;1} \rightarrow u_{1;4;2} & u_{3;1;1} \rightarrow u_{1;1;1} \\
u_{1;1;2} \rightarrow u_{1;5;1} & u_{2;1;2} \rightarrow u_{1;4;3} & u_{3;1;2} \rightarrow u_{1;1;2} \\
u_{1;1;3} \rightarrow u_{1;5;2} & u_{2;1;3} \rightarrow u_{1;4;4} & u_{3;1;3} \rightarrow u_{1;1;4} \\
u_{1;1;4} \rightarrow u_{1;5;3} & u_{2;1;4} \rightarrow u_{1;4;1} & u_{3;1;4} \rightarrow u_{1;1;3} \\
u_{1;4;1} \rightarrow u_{2;4;3} & u_{2;3;1} \rightarrow u_{2;3;4} & u_{3;2;1} \rightarrow u_{2;1;1} \\
u_{1;4;2} \rightarrow u_{2;4;1} & u_{2;3;2} \rightarrow u_{2;3;1} & u_{3;2;2} \rightarrow u_{2;1;3} \\
u_{1;4;3} \rightarrow u_{2;4;4} & u_{2;3;3} \rightarrow u_{2;3;2} & u_{3;2;3} \rightarrow u_{2;1;2} \\
u_{1;4;4} \rightarrow u_{2;4;2} & u_{2;3;4} \rightarrow u_{2;3;3} & u_{3;2;4} \rightarrow u_{2;1;4} \\
u_{1;5;1} \rightarrow u_{3;3;1} & u_{2;4;1} \rightarrow u_{3;2;1} & u_{3;3;1} \rightarrow u_{3;1;2} \\
u_{1;5;2} \rightarrow u_{3;3;4} & u_{2;4;2} \rightarrow u_{3;2;2} & u_{3;3;2} \rightarrow u_{3;1;1} \\
u_{1;5;3} \rightarrow u_{3;3;3} & u_{2;4;3} \rightarrow u_{3;2;4} & u_{3;3;3} \rightarrow u_{3;1;3} \\
u_{1;5;4} \rightarrow u_{3;3;2} & u_{2;4;4} \rightarrow u_{3;2;3} & u_{3;3;4} \rightarrow u_{3;1;4}
\end{array}$$

Bulk links

$$\begin{array}{l}
h_{1;1} \rightarrow h_{1;5} \\
h_{1;4} \rightarrow h_{1,2}h_{2;4} \\
h_{1;5} \rightarrow h_{1,3}h_{3;3} \\
h_{1,2}h_{2;1} \rightarrow h_{1;4} \\
h_{1,2}h_{2;3} \rightarrow h_{1,2}h_{2;3} \\
h_{1,2}h_{2;4} \rightarrow h_{1,3}h_{3;2} \\
h_{1,3}h_{3;1} \rightarrow h_{1;1} \\
h_{1,3}h_{3;2} \rightarrow h_{1,2}h_{2;1} \\
h_{1,3}h_{3;3} \rightarrow h_{1,3}h_{3;1}
\end{array}$$

Bulk wedges

$$\begin{array}{l}
X_{1;1,2}h_{1,2} \triangleright X_{2;5,1} \rightarrow X_{1;5,4} \\
X_{1;1,3}h_{1,3} \triangleright X_{3;4,1} \rightarrow X_{1;5,1} \\
X_{1;1,4} \rightarrow X_{1;5,2}h_{1,2} \triangleright X_{2;5,4} \\
X_{1;1,5} \rightarrow X_{1;5,3}h_{1,3} \triangleright X_{3;4,3} \\
X_{1;4,2}h_{1,2} \triangleright X_{2;5,3} \rightarrow h_{1,2} \triangleright X_{2;4,3} \\
X_{1;4,3}h_{1,3} \triangleright X_{3;4,2} \rightarrow h_{1,2} \triangleright X_{2;4,1} \\
X_{1;4,5} \rightarrow h_{1,2} \triangleright (X_{2;4,2}X_{2;2,5})X_{1;2,3}h_{1,3} \triangleright (X_{3;4,5}X_{3;5,3}) \\
X_{1;5,2}h_{1,2} \triangleright X_{2;5,4} \rightarrow h_{1,3} \triangleright X_{3;3,2} \\
X_{1;5,3}h_{1,3} \triangleright X_{3;4,3} \rightarrow h_{1,2} \triangleright X_{3;3,1} \\
h_{1,2} \triangleright X_{2;1,3} \rightarrow X_{1;4,2}h_{1,2} \triangleright X_{2;5,3} \\
h_{1,2} \triangleright X_{2;1,4} \rightarrow X_{1;4,3}h_{1,3} \triangleright X_{3;4,2} \\
h_{1,2} \triangleright X_{2;3,4} \rightarrow h_{1,2} \triangleright (X_{2;3,2}X_{2;2,5})X_{1;2,3}h_{1,3} \triangleright (X_{3;4,5}X_{3;5,2}) \\
h_{1,3} \triangleright X_{3;1,2} \rightarrow X_{1;1,2}h_{1,2} \triangleright X_{2;5,1} \\
h_{1,3} \triangleright X_{3;1,3} \rightarrow X_{1;1,3}h_{1,3} \triangleright X_{3;4,1} \\
h_{1,3} \triangleright X_{3;2,3} \rightarrow h_{1,2} \triangleright (X_{2;1,2}X_{2;2,5})X_{1;2,3}h_{1,3} \triangleright (X_{3;4,5}X_{3;5,1}) \\
h_{1,2} \triangleright (X_{2;1,2}X_{2;2,5})X_{1;2,3}h_{1,3} \triangleright (X_{3;4,5}X_{3;5,1}) \rightarrow X_{1;4,1} \\
h_{1,2} \triangleright (X_{2;3,2}X_{2;2,5})X_{1;2,3}h_{1,3} \triangleright (X_{3;4,5}X_{3;5,2}) \rightarrow h_{1,2} \triangleright X_{2;3,1} \\
h_{1,2} \triangleright (X_{2;4,2}X_{2;2,5})X_{1;2,3}h_{1,3} \triangleright (X_{3;4,5}X_{3;5,3}) \rightarrow h_{1,3} \triangleright X_{3;2,1}
\end{array}$$

Boundary wedges

$$\begin{aligned}
(h_{1,1} \triangleright Y_{1;1;1,2}) &\rightarrow (h_{1,5} \triangleright Y_{1;5;4,1}) & h_{1,2} \triangleright (h_{2,1} \triangleright Y_{2;1;1,2}) &\rightarrow (h_{1,4} \triangleright Y_{1;4;2,3}) \\
(h_{1,1} \triangleright Y_{1;1;1,3}) &\rightarrow (h_{1,5} \triangleright Y_{1;5;4,2}) & h_{1,2} \triangleright (h_{2,1} \triangleright Y_{2;1;1,3}) &\rightarrow (h_{1,4} \triangleright Y_{1;4;2,4}) \\
(h_{1,1} \triangleright Y_{1;1;1,4}) &\rightarrow (h_{1,5} \triangleright Y_{1;5;4,3}) & h_{1,2} \triangleright (h_{2,1} \triangleright Y_{2;1;1,4}) &\rightarrow (h_{1,4} \triangleright Y_{1;4;2,1}) \\
(h_{1,1} \triangleright Y_{1;1;2,3}) &\rightarrow (h_{1,5} \triangleright Y_{1;5;1,2}) & h_{1,2} \triangleright (h_{2,1} \triangleright Y_{2;1;2,4}) &\rightarrow (h_{1,4} \triangleright Y_{1;4;3,1}) \\
(h_{1,1} \triangleright Y_{1;1;2,4}) &\rightarrow (h_{1,5} \triangleright Y_{1;5;1,3}) & h_{1,2} \triangleright (h_{2,1} \triangleright Y_{2;1;2,3}) &\rightarrow (h_{1,4} \triangleright Y_{1;4;3,4}) \\
(h_{1,1} \triangleright Y_{1;1;4,3}) &\rightarrow (h_{1,5} \triangleright Y_{1;5;2,3}) & h_{1,2} \triangleright (h_{2,1} \triangleright Y_{2;1;4,3}) &\rightarrow (h_{1,4} \triangleright Y_{1;4;4,1}) \\
(h_{1,4} \triangleright Y_{1;4;1,2}) &\rightarrow h_{1,2} \triangleright (h_{2,4} \triangleright Y_{2;4;3,1}) & h_{1,2} \triangleright (h_{2,3} \triangleright Y_{2;3;2,1}) &\rightarrow h_{1,2} \triangleright (h_{2,3} \triangleright Y_{2;3;1,4}) \\
(h_{1,4} \triangleright Y_{1;4;1,3}) &\rightarrow h_{1,2} \triangleright (h_{2,4} \triangleright Y_{2;4;3,4}) & h_{1,2} \triangleright (h_{2,3} \triangleright Y_{2;3;1,3}) &\rightarrow h_{1,2} \triangleright (h_{2,3} \triangleright Y_{2;3;4,2}) \\
(h_{1,4} \triangleright Y_{1;4;1,4}) &\rightarrow h_{1,2} \triangleright (h_{2,4} \triangleright Y_{2;4;3,2}) & h_{1,2} \triangleright (h_{2,3} \triangleright Y_{2;3;1,4}) &\rightarrow h_{1,2} \triangleright (h_{2,3} \triangleright Y_{2;3;4,3}) \\
(h_{1,4} \triangleright Y_{1;4;2,3}) &\rightarrow h_{1,2} \triangleright (h_{2,4} \triangleright Y_{2;4;1,4}) & h_{1,2} \triangleright (h_{2,3} \triangleright Y_{2;3;2,3}) &\rightarrow h_{1,2} \triangleright (h_{2,3} \triangleright Y_{2;3;1,2}) \\
(h_{1,4} \triangleright Y_{1;4;2,4}) &\rightarrow h_{1,2} \triangleright (h_{2,4} \triangleright Y_{2;4;1,2}) & h_{1,2} \triangleright (h_{2,3} \triangleright Y_{2;3;3,4}) &\rightarrow h_{1,2} \triangleright (h_{2,3} \triangleright Y_{2;3;2,3}) \\
(h_{1,4} \triangleright Y_{1;4;3,4}) &\rightarrow h_{1,2} \triangleright (h_{2,4} \triangleright Y_{2;4;4,2}) & h_{1,2} \triangleright (h_{2,3} \triangleright Y_{2;3;4,2}) &\rightarrow h_{1,2} \triangleright (h_{2,3} \triangleright Y_{2;3;3,1}) \\
(h_{1,5} \triangleright Y_{1;5;1,2}) &\rightarrow h_{1,3} \triangleright (h_{3,3} \triangleright Y_{3;3;1,4}) & h_{1,2} \triangleright (h_{2,4} \triangleright Y_{2;4;1,2}) &\rightarrow h_{1,3} \triangleright (h_{3,2} \triangleright Y_{3;2;1,2}) \\
(h_{1,5} \triangleright Y_{1;5;1,3}) &\rightarrow h_{1,3} \triangleright (h_{3,3} \triangleright Y_{3;3;1,3}) & h_{1,2} \triangleright (h_{2,4} \triangleright Y_{2;4;3,1}) &\rightarrow h_{1,3} \triangleright (h_{3,2} \triangleright Y_{3;2;4,1}) \\
(h_{1,5} \triangleright Y_{1;5;1,4}) &\rightarrow h_{1,3} \triangleright (h_{3,3} \triangleright Y_{3;3;1,2}) & h_{1,2} \triangleright (h_{2,4} \triangleright Y_{2;4;2,3}) &\rightarrow h_{1,3} \triangleright (h_{3,2} \triangleright Y_{3;2;2,4}) \\
(h_{1,5} \triangleright Y_{1;5;2,3}) &\rightarrow h_{1,3} \triangleright (h_{3,3} \triangleright Y_{3;3;4,3}) & h_{1,2} \triangleright (h_{2,4} \triangleright Y_{2;4;2,4}) &\rightarrow h_{1,3} \triangleright (h_{3,2} \triangleright Y_{3;2;2,3}) \\
(h_{1,5} \triangleright Y_{1;5;2,4}) &\rightarrow h_{1,3} \triangleright (h_{3,3} \triangleright Y_{3;3;4,2}) & h_{1,2} \triangleright (h_{2,4} \triangleright Y_{2;4;4,1}) &\rightarrow h_{1,3} \triangleright (h_{3,2} \triangleright Y_{3;2;3,1}) \\
(h_{1,5} \triangleright Y_{1;5;3,4}) &\rightarrow h_{1,3} \triangleright (h_{3,3} \triangleright Y_{3;3;3,2}) & h_{1,2} \triangleright (h_{2,4} \triangleright Y_{2;4;3,4}) &\rightarrow h_{1,3} \triangleright (h_{3,2} \triangleright Y_{3;2;4,3}) \\
\\
h_{1,3} \triangleright (h_{3;1} \triangleright Y_{3;1;1,2}) &\rightarrow (h_{1,1} \triangleright Y_{1;1;1,2}) & h_{1,3} \triangleright (h_{3;2} \triangleright Y_{3;2;2,4}) &\rightarrow h_{1,2} \triangleright (h_{2,1} \triangleright Y_{2;1;4,3}) \\
h_{1,3} \triangleright (h_{3;1} \triangleright Y_{3;1;1,3}) &\rightarrow (h_{1,1} \triangleright Y_{1;1;1,4}) & h_{1,3} \triangleright (h_{3;2} \triangleright Y_{3;2;3,1}) &\rightarrow h_{1,2} \triangleright (h_{2,1} \triangleright Y_{2;1;2,1}) \\
h_{1,3} \triangleright (h_{3;1} \triangleright Y_{3;1;1,4}) &\rightarrow (h_{1,1} \triangleright Y_{1;1;1,3}) & h_{1,3} \triangleright (h_{3;2} \triangleright Y_{3;2;3,4}) &\rightarrow h_{1,2} \triangleright (h_{2,1} \triangleright Y_{2;1;2,4}) \\
h_{1,3} \triangleright (h_{3;1} \triangleright Y_{3;1;2,3}) &\rightarrow (h_{1,1} \triangleright Y_{1;1;2,4}) & h_{1,3} \triangleright (h_{3;3} \triangleright Y_{3;3;1,2}) &\rightarrow h_{1,3} \triangleright (h_{3;1} \triangleright Y_{3;1;2,1}) \\
h_{1,3} \triangleright (h_{3;1} \triangleright Y_{3;1;2,4}) &\rightarrow (h_{1,1} \triangleright Y_{1;1;2,3}) & h_{1,3} \triangleright (h_{3;3} \triangleright Y_{3;3;1,3}) &\rightarrow h_{1,3} \triangleright (h_{3;1} \triangleright Y_{3;1;2,3}) \\
h_{1,3} \triangleright (h_{3;1} \triangleright Y_{3;1;4,3}) &\rightarrow (h_{1,1} \triangleright Y_{1;1;3,4}) & h_{1,3} \triangleright (h_{3;3} \triangleright Y_{3;3;4,1}) &\rightarrow h_{1,3} \triangleright (h_{3;1} \triangleright Y_{3;1;4,2}) \\
h_{1,3} \triangleright (h_{3;2} \triangleright Y_{3;2;1,2}) &\rightarrow h_{1,2} \triangleright (h_{2,1} \triangleright Y_{2;1;1,3}) & h_{1,3} \triangleright (h_{3;3} \triangleright Y_{3;3;2,3}) &\rightarrow h_{1,3} \triangleright (h_{3;1} \triangleright Y_{3;1;1,3}) \\
h_{1,3} \triangleright (h_{3;2} \triangleright Y_{3;2;1,4}) &\rightarrow h_{1,2} \triangleright (h_{2,1} \triangleright Y_{2;1;1,4}) & h_{1,3} \triangleright (h_{3;3} \triangleright Y_{3;3;4,2}) &\rightarrow h_{1,3} \triangleright (h_{3;1} \triangleright Y_{3;1;4,1}) \\
h_{1,3} \triangleright (h_{3;2} \triangleright Y_{3;2;2,3}) &\rightarrow h_{1,2} \triangleright (h_{2,1} \triangleright Y_{2;1;3,2}) & h_{1,3} \triangleright (h_{3;3} \triangleright Y_{3;3;3,4}) &\rightarrow h_{1,3} \triangleright (h_{3;1} \triangleright Y_{3;1;4,3})
\end{aligned}$$

Under such change of variables the amplitude of the three 4-simplices (D.7) goes in itself. Both the amplitude describe three 4-simplices glued each other in such a way that each of them shares one tetrahedron with each of the other. Therefore, each 4-simplex has 2 tetrahedra shared with the other two 4-simplices and three boundary tetrahedra. The change of variable, and thus the Pachner move, takes this combination, into an equivalent one, where each of the three boundary tetrahedra of each of the three initial 4-simplices, becomes the boundary tetrahedron of each of the new three 4-simplices. Even though the combinatorics is different, the two amplitudes are clearly equivalent

$$\mathcal{A}_{\mathcal{V}_3} = \mathcal{A}'_{\mathcal{V}_3}. \quad (\text{D.18})$$

This move is represented in Fig. 5.15.

Appendix E

New proposal for a 2-Hopf algebra

Here I present a new proposal for a 2-Hopf algebra. I will first introduce the fundamental ingredients and definitions. Then I provide the concrete examples of a 2-Hopf algebra of functions on a 2-group.

E.1 2-Bi-algebra

2-Hopf algebra

Definition 19 (Constrained tensor product). *Consider the bi-algebra A . Let $A \otimes A$ be the usual tensor product on it. Consider the constraint $\mathcal{C} \in A \otimes A$, and use it to define the restricted tensor product $\square_{\mathcal{C}}$ given as the usual tensor product \otimes defined on the subspace $A \otimes A$ that satisfy the constraint \mathcal{C} :*

$$A \square_{\mathcal{C}} A = \{a_1 \otimes a_2 \in A \otimes A \mid \mathcal{C} = 0\}. \quad (\text{E.1})$$

Definition 20 (2-Bi-algebra). *A 2-bi-algebra is given by the set of*

- *a bi-algebra G ;*
- *the semi-direct co-product of bi-algebras $H \rtimes G$. The bi-algebra maps (product \circ , co-product Δ_H and co-unit ε_H) on $H \rtimes G$ are called horizontal;*
- *a pair of constraints $\mathcal{C}_1, \mathcal{C}_2 \in (H \rtimes G) \otimes (H \rtimes G)$, used to define the pair of restricted tensor products $\square_{\mathcal{C}_1}$ and $\square_{\mathcal{C}_2}$;*
- *the set of bi-algebra maps on $H \rtimes G$, called vertical:*

$$\begin{aligned} \text{Product: } & \cdot : (H \rtimes G) \square_{\mathcal{C}_1} (H \rtimes G) \rightarrow (H \rtimes G), \\ \text{Co-product: } & \Delta_V : (H \rtimes G) \rightarrow (H \rtimes G) \square_{\mathcal{C}_2} (H \rtimes G), \\ \text{Co-unit: } & \varepsilon_V : (H \rtimes G) \rightarrow \mathbb{C}. \end{aligned} \quad (\text{E.2})$$

The horizontal and vertical maps have to satisfy the compatibility relations

$$\begin{aligned} \circ(\cdot \otimes \cdot) &= \cdot(\circ \otimes \circ)(id \otimes \tau \otimes id), \\ (\Delta_V \otimes \Delta_V) \Delta_H &= (id \otimes \tau \otimes id)(\Delta_H \otimes \Delta_H) \Delta_V, \end{aligned} \quad (\text{E.3})$$

where τ denotes the permutation map on $H \rtimes G$.

Definition 21 (2-Hopf algebra). Consider a 2-bi-algebra and suppose that there exist a pair of antipodes $S_G : G \rightarrow G$ and $S : (H \bowtie G) \rightarrow (H \bowtie G)$ so that G and $H \bowtie G$ are Hopf algebras. The 2-bi-algebra turns into a 2-Hopf algebra if the map $S : (H \bowtie G) \rightarrow (H \bowtie G)$ is an antipode also for the vertical maps.

Pairing 2-bi-algebras

Definition 22 (Skew paired 2-bi-algebras). Given two 2-bi-algebras $\mathcal{G} \ni \phi$ and $\mathcal{G}^* \ni \hat{\phi}$, they are skew paired if there exists a map $\sigma : \mathcal{G}^* \otimes \mathcal{G} \rightarrow \mathbb{C}$ such that

$$\begin{aligned} \sigma(\hat{\phi}, (\phi_1 \cdot \phi_2)) &= \sigma(\Delta_V^{op} \hat{\phi}, (\phi_1 \otimes \phi_2)), & \sigma((\hat{\phi}_1 \cdot \hat{\phi}_2), \phi) &= \sigma((\hat{\phi}_1 \otimes \hat{\phi}_2), \Delta_V \phi), \\ \sigma(\hat{\phi}, (\phi_1 \circ \phi_2)) &= \sigma(\Delta_H^{op} \hat{\phi}, (\phi_1 \otimes \phi_2)), & \sigma((\hat{\phi}_1 \circ \hat{\phi}_2), \phi) &= \sigma((\hat{\phi}_1 \otimes \hat{\phi}_2), \Delta_H \phi). \end{aligned} \quad (\text{E.4})$$

The skew pairing has to satisfy the unit relations

$$\begin{aligned} \sigma(\hat{\eta}_H, \phi) &= \varepsilon_H \phi, & \sigma(\hat{\eta}_V, \phi) &= \varepsilon_V \phi, \\ \sigma(\hat{\phi}, \eta_H) &= \varepsilon_H \hat{\phi}, & \sigma(\hat{\phi}, \eta_V) &= \varepsilon_V \hat{\phi}. \end{aligned} \quad (\text{E.5})$$

Definition 23 (2-Plane wave). Given two 2-bi-algebras \mathcal{G} and \mathcal{G}^* , the 2-plane-wave is an element of their tensor product $E \in \mathcal{G} \otimes \mathcal{G}^*$, which satisfies the following set of properties.

Product/co-product relations:

$$\begin{aligned} (\Delta_H \otimes id)E &= (id^{\otimes 2} \otimes \circ)(E \otimes E), & (id \otimes \Delta_H)E &= (\circ^{op} \otimes id^{\otimes 2})(E \otimes E), \\ (\Delta_V \otimes id)E &= (id^{\otimes 2} \otimes \cdot)(E \otimes E), & (id \otimes \Delta_V)E &= (\cdot^{op} \otimes id^{\otimes 2})(E \otimes E). \end{aligned} \quad (\text{E.6})$$

Unit/co-unit relations:

$$\begin{aligned} (\varepsilon_H \otimes id)E &= \hat{\eta}_H, & (id \otimes \varepsilon_H)E &= \eta_H, \\ (\varepsilon_V \otimes id)E &= \hat{\eta}_V, & (id \otimes \varepsilon_V)E &= \eta_V. \end{aligned} \quad (\text{E.7})$$

There also exist a pair of inverse plane waves, called horizontal and vertical inverses and resp. denoted E^{-1_H} and E^{-1_V} .

E.2 Examples

Finite 2-groups

Example 1 (2-Bi-algebra of functions on a finite 2-group). Consider the finite 2-group \mathcal{G} given by the finite group of morphisms G and the finite group of 2-morphisms $H \times G$. Consider the bi-algebra of functions on them: $F(G)$ with elements $f : G \rightarrow \mathbb{C}$ and $F(H \times G)$ with elements $\phi : H \times G \rightarrow \mathbb{C}$.

$F(G)$ is the usual bi-algebra of functions on a finite group (with pointwise product), while the bi-algebra $F(H \times G)$ is given below.

First, I introduce the source and target maps

$$\begin{aligned} \text{s} : F(H \times G) &\rightarrow F(G) & \text{t} : F(H \times G) &\rightarrow F(G) \\ (\text{s} \phi)(h, g) &= \phi_0(g) & (\text{t} \phi)(h, g) &= \phi_0(\text{t}(h)g), \end{aligned} \quad (\text{E.8})$$

where $\phi_0 \in F(G)$ is the projection of the function $\phi \in F(H \rtimes G)$ into $F(G)$. Consider then the constraint

$$F(H_1 \rtimes G_1) \otimes F(H_2 \rtimes G_2) \ni \mathcal{C} : \quad (t\phi)(h_1, g_1) = (s\phi)(h_2, g_2). \quad (\text{E.9})$$

Use the constraint \mathcal{C} to define the restricted tensor product $\square_{\mathcal{C}}$. Moreover, I use the subscript V to denote

$$((H_1 \rtimes G_1) \times (H_2 \rtimes G_2))_V \equiv \cdot ((H_1 \rtimes G_1) \square_{\mathcal{C}} (H_2 \rtimes G_2)), \quad (\text{E.10})$$

the 2-group with elements given by the vertical composition of elements in the constrained tensor product $\square_{\mathcal{C}}$. This construction ensures that the elements in $H_1 \rtimes G_1$ and $H_2 \rtimes G_2$ are vertically composable.

The 2-bi-algebra of functions $F(H \rtimes G)$, is specified by the horizontal structure

$$\begin{aligned} \text{Horizontal product: } (\phi \circ \varphi)(h, g) &= \frac{1}{\sqrt{|H|}} \sum_{h', g'} \phi(h', g') \varphi((h', g')^{-1_V} \cdot (h, g)) \\ &= \frac{1}{\sqrt{|H|}} \sum_{h', g'} \phi(h', g') \varphi(hh'^{-1}, g') \delta_G(t(h')g'g^{-1}), \\ \text{Horizontal co-product: } \Delta_H \phi((h_1, g_1), (h_2, g_2)) &= \frac{1}{\sqrt{|H|}} \phi((h_1, g_1) \circ (h_2, g_2)) \\ &= \frac{1}{\sqrt{|H|}} \phi(h_1(g_1 \triangleright h_2), g_1 g_2), \end{aligned}$$

$$\text{Horizontal unit: } \eta_H = \sqrt{|H|} \alpha,$$

$$\text{Horizontal co-unit: } \varepsilon_H \phi = \sqrt{|H|} \phi(1, 1),$$

Here $|G| = \sum_g$ and $|H| = \sum_h$ denote resp. the number of elements in the finite groups G and H , and the unit is defined as the function

$$F(\mathcal{G}) \ni \alpha(h, g) = \begin{cases} 1 & \text{if } h = 1 \\ 0 & \text{otherwise.} \end{cases} \quad (\text{E.11})$$

The pair of bi-algebras $F(G)$ and $F(H \rtimes G)$ form a 2-bi-algebra when it is equipped with the vertical structure

$$\begin{aligned} \text{Vertical product: } (\phi \cdot \varphi)(h, g) &= \frac{1}{\sqrt{|H|}} \sum_{h', g'} \phi(h', g') \varphi((h', g')^{-1_H} \circ (h, g)) \\ &= \frac{1}{\sqrt{|H|}} \sum_{h', g'} \phi(h', g') \varphi(g'^{-1} \triangleright (h'^{-1}h), g'^{-1}g), \\ \text{Vertical co-product: } \Delta_V \phi((h_1, g_1), (h_2, g_2)) &= \frac{1}{\sqrt{|H|}} \phi((h_1, g_1) \cdot (h_2, g_2)), \\ &= \frac{1}{\sqrt{|H|}} \phi(h_2 h_1, g_1) \delta_G(t(h_1)g_1 g_2^{-1}), \\ \text{Vertical unit: } \eta_V &= \sqrt{|H|} \delta, \\ \text{Vertical co-unit: } \varepsilon_V \phi &= \sqrt{|H|} \sum_{h, g} \phi(h, g) \alpha(h, g) \end{aligned}$$

Here I denoted resp. δ_G and δ the deltas in the bi-algebras $F(G)$ and $F(H \rtimes G)$, and I used the delta in $F(G)$ to define a function on the vertical composition of 2-group elements. Moreover,

the function α in the vertical unit was defined in (E.11). Note that the horizontal and vertical structures, as maps, are

$$\begin{aligned} \circ &: F(H_1 \rtimes G_1) \otimes F((H_1 \rtimes G_1) \times (H_2 \rtimes G_2))_V \rightarrow F(H \rtimes G), \\ \Delta_H &: F(H \rtimes G) \rightarrow F((H \rtimes G) \otimes (H \rtimes G)), \\ \varepsilon_H &: (H \rtimes G) \rightarrow \mathbb{C}, \end{aligned} \tag{E.12}$$

$$\begin{aligned} \cdot &: F(H \rtimes G) \otimes F(H \rtimes G) \rightarrow F(H \rtimes G), \\ \Delta_V &: F((H_1 \rtimes G_1) \times (H_2 \rtimes G_2))_V \rightarrow F((H \rtimes G) \otimes (H \rtimes G)), \\ \varepsilon_V &: F(H \rtimes G) \rightarrow \mathbb{C}. \end{aligned} \tag{E.13}$$

One can check that the vertical structure form a bi-algebra and that is is compatible with the horizontal one, according to (E.3).

Continuous group

Definition 24 (Haar measure). *Given a 2-group, we denote*

- $dg^{L/R}$ resp. the left or right Haar measure on G . They resp. satisfy the left or right invariance properties

$$dg^L = d[g'g]^L, \quad dg^R = d[gg']^R, \tag{E.14}$$

for all $g' \in G$;

- $d[(h, g)]^{L/R}$ resp. the left or right Haar measure on the semi-direct product of groups $H \rtimes G$. They resp. satisfy the left or right invariance properties

$$d[(h, g)]^L = d[(h', g') \circ (h, g)]^L, \quad d[(h, g)]^R = d[(h, g) \circ (h', g')]^R, \tag{E.15}$$

for all $(h', g') \in H \rtimes G$.

Note that, since the convention $H \rtimes G$ for the semi-direct product of groups implies a left action, the left and right Haar measures on it can be written as¹

$$d[(h, g)]^L \equiv dh dg, \quad d[(h, g)]^R \equiv dh dg \frac{1}{D(a_g)}. \tag{E.17}$$

Proposition 17 (Vertical measure). *Consider the 2-group \mathcal{G} and the left Haar measure on it $d[(h, g)]$ (drop the index L). Let me refer to it as the left horizontal measure. Call*

$$d[(h, g)] \delta_G((t(h')g')^{-1}g), \tag{E.18}$$

the left vertical measure, since it satisfies the left invariance property

$$d[(h, g)] \delta_G((t(h')g')^{-1}g) = d[(h', g') \cdot (h, g)]. \tag{E.19}$$

¹Using instead the convention $G \rtimes H$ for the semi-direct product of groups, since here there is a right action, the left and right Haar measure on it can be written as

$$d[(h, g)]^L \equiv dh dg \frac{1}{D'(a_g)}, \quad d[(h, g)]^R \equiv dh dg. \tag{E.16}$$

Proof. To prove the proposition I simply use the expression of the function on the vertical composition and the left horizontal invariance of the measure $d[(h, g)]$.

$$\begin{aligned} d[(h', g') \cdot (h, g)] &= d[(hh', g')] \delta_G((t(h')g')^{-1}g) = d[(hh', t(h')^{-1}g)] \delta_G((t(h')g')^{-1}g) \\ &= d[(h', t(h')^{-1}) \circ (h, g)] \delta_G((t(h')g')^{-1}g) \\ &= d[(h, g)] \delta_G((t(h')g')^{-1}g). \end{aligned} \quad (\text{E.20})$$

□

Example 2 (2-Bi-algebra of functions on a continuous 2-group). Consider the 2-group \mathcal{G} given by the group of morphisms G and the group of 2-morphisms $H \rtimes G$. Consider the bi-algebra of functions on them: $F(G)$ with elements $f : G \rightarrow \mathbb{C}$ and $F(H \rtimes G)$ with elements $\phi : H \rtimes G \rightarrow \mathbb{C}$. $F(G)$ is the usual bi-algebra of functions on a continuous group (with pointwise product), while the bi-algebra $F(H \rtimes G)$ is given below.

First, I introduce the source and target maps

$$\begin{aligned} \text{s} : F(H \rtimes G) &\rightarrow F(G) & \text{t} : F(H \rtimes G) &\rightarrow F(G) \\ (\text{s}\phi)(h, g) &= \phi_0(g) & (\text{t}\phi)(h, g) &= \phi_0(t(h)g), \end{aligned} \quad (\text{E.21})$$

where $\phi_0 \in F(G)$ is the projection of the function $\phi \in F(H \rtimes G)$ into $F(G)$. Consider then the constraint

$$F(H_1 \rtimes G_1) \otimes F(H_2 \rtimes G_2) \ni \mathcal{C} : \quad (\text{t}\phi)(h_1, g_1) = (\text{s}\phi)(h_2, g_2). \quad (\text{E.22})$$

Use the constraint \mathcal{C} to define the restricted tensor product $\square_{\mathcal{C}}$. Moreover, we denote

$$((H_1 \rtimes G_1) \times (H_2 \rtimes G_2))_{\mathcal{V}} \equiv \cdot ((H_1 \rtimes G_1) \square_{\mathcal{C}} (H_2 \rtimes G_2)), \quad (\text{E.23})$$

the 2-group with elements given by the vertical composition of elements in the constrained tensor product $\square_{\mathcal{C}}$. This construction ensures that the elements in $H_1 \rtimes G_1$ and $H_2 \rtimes G_2$ are vertically composable.

The 2-bi-algebra of functions $F(H \rtimes G)$, is specified by the horizontal structure

$$\begin{aligned} \text{Horizontal product:} \quad (\phi \circ \varphi)(h, g) &= \frac{1}{\sqrt{V_H}} \int d[(h', g')] \phi(h', g') \varphi((h', g')^{-1}v \cdot (h, g)) \\ &= \frac{1}{\sqrt{V_H}} \int d[(h', g')] \phi(h', g') \varphi(hh'^{-1}, g') \delta_G(t(h')g'g^{-1}), \end{aligned}$$

$$\begin{aligned} \text{Horizontal co-product:} \quad \Delta_H \phi((h_1, g_1), (h_2, g_2)) &= \frac{1}{\sqrt{V_H}} \phi((h_1, g_1) \circ (h_2, g_2)) \\ &= \frac{1}{\sqrt{V_H}} \phi(h_1(g_1 \triangleright h_2), g_1 g_2), \end{aligned}$$

$$\text{Horizontal unit:} \quad \eta_H = \sqrt{V_H} \alpha,$$

$$\text{Horizontal co-unit:} \quad \varepsilon_H \phi = \sqrt{V_H} \phi(1, 1),$$

Here $V_G = \int dg$ and $V_H = \int dh$ denote resp. the volumes of the groups G and H , and the unit is defined similarly to the function (E.11) for the finite group case, where the Kronecker delta is replaced by a Dirac delta. The pair of bi-algebras $F(G)$ and $F(H \rtimes G)$ form a 2-bi-algebra when

it is equipped with the vertical structure

$$\begin{aligned}
\text{Vertical product: } (\phi \cdot \varphi)(h, g) &= \frac{1}{\sqrt{V_H}} \int d[(h', g')] \phi(h', g') \varphi((h', g')^{-1_H} \circ (h, g)) \\
&= \frac{1}{\sqrt{V_H}} \int d[(h', g')] \phi(h', g') \varphi(g'^{-1} \triangleright (h'^{-1}h), g'^{-1}g), \\
\text{Vertical co-product: } \Delta_V \phi((h_1, g_1), (h_2, g_2)) &= \frac{1}{\sqrt{V_H}} \phi((h_1, g_1) \cdot (h_2, g_2)), \\
&= \frac{1}{\sqrt{V_H}} \phi(h_2 h_1, g_1) \delta_G(t(h_1)g_1 g_2^{-1}), \\
\text{Vertical unit: } \eta_V &= \sqrt{V_H} \delta, \\
\text{Vertical co-unit: } \varepsilon_V \phi &= \sqrt{V_H} \int d[(h, g)] \phi(h, g) \alpha(h, g)
\end{aligned}$$

Here I denoted resp. δ_G and δ the Dirac deltas in the bi-algebras $F(G)$ and $F(H \rtimes G)$, and I used the delta in $F(G)$ to define a function on the vertical composition of 2-group elements. One can check that the vertical structure form a bi-algebra and that is is compatible with the horizontal one, according to (E.3).

Dual 2-bi-algebra and 2-plane wave

Proposition 18. Consider the 2-bi-algebra $F(\mathcal{G})$ with elements $\phi(y, u)$ with structure given in example 2 and the 2-bi-algebra $F(\mathcal{G}^*)$ specified by the horizontal and vertical structures below.

$$\begin{aligned}
& \text{Horizontal product: } (\hat{\phi}_1 \circ \hat{\phi}_2)(\lambda, \beta) = \frac{1}{\sqrt{V_{G^*}}} \int d[(\lambda', \beta')] \hat{\phi}_1(\lambda', \beta') \hat{\phi}_2((\lambda, \beta) \cdot (\lambda', \beta')^{-1_V}), \\
& \text{Vertical product: } (\hat{\phi}_1 \cdot \hat{\phi}_2)(\lambda, \beta) = \frac{1}{\sqrt{V_{G^*}}} \int d[(\lambda', \beta')] \hat{\phi}_1(\lambda', \beta') \hat{\phi}_2((\lambda, \beta) \circ (\lambda', \beta')^{-1_H}), \\
& \text{Horizontal co-product: } \Delta_H \hat{\phi}((\lambda_1, \beta_1), (\lambda_2, \beta_2)) = \frac{1}{\sqrt{V_{G^*}}} \hat{\phi}((\lambda_1, \beta_1) \circ (\lambda_2, \beta_2)), \\
F(\mathcal{G}^*) : & \text{Vertical co-product: } \frac{1}{\sqrt{V_{G^*}}} \hat{\phi}((\lambda_1, \beta_1), (\lambda_2, \beta_2)) = \frac{1}{V_{G^*}} \hat{\phi}((\lambda_1, \beta_1) \cdot (\lambda_2, \beta_2)), \\
& \text{Horizontal unit: } \hat{\eta}_H = \sqrt{V_{G^*}} \hat{\alpha}, \\
& \text{Vertical unit: } \hat{\eta}_V = \sqrt{V_{G^*}} \hat{\delta}, \\
& \text{Horizontal co-unit: } \varepsilon_H \hat{\phi} = \sqrt{V_{G^*}} \hat{\phi}(1, 1), \\
& \text{Vertical co-units: } \varepsilon_V \hat{\phi} = \sqrt{V_{G^*}} \int d[(\lambda, \beta)] \hat{\phi}(\lambda, \beta) \hat{\alpha}(\lambda, \beta).
\end{aligned} \tag{E.24}$$

Where the function $\hat{\alpha}$ is similar to the function α used in Ex. 2 and the term $V_{G^*} = \int d\beta$ is the volume of the group G^* .

The plane wave $E \in F(\mathcal{G}) \otimes F(\mathcal{G}^*)$ satisfies the properties below $\forall(y, u) \in G_2 \rtimes G_1$ and $\forall(\lambda, \beta) \in G_2^* \times G_1^*$.

Product/co-product relations:

$$\begin{aligned}
E((y_1, u_1) \circ (y_2, u_2), (\lambda, \beta)) &= \int d[(\lambda', \beta')] E((y_1, u_1), (\lambda', \beta')) E((y_2, u_2), (\lambda, \beta) \cdot (\lambda', \beta')^{-1\nu}), \\
E((y_1, u_1) \cdot (y_2, u_2), (\lambda, \beta)) &= \int d[(\lambda', \beta')] E((y_1, u_1), (\lambda', \beta')) E((y_2, u_2), (\lambda, \beta) \circ (\lambda', \beta')^{-1H}), \\
E((y, u), (\lambda_1, \beta_1) \circ (\lambda_2, \beta_2)) &= \int d[(y', u')] E((u', y'), (\lambda_2, \beta_2)) E((y', u')^{-1\nu} \cdot (y, u), (\lambda_1, \beta_1)), \\
E((y, u), (\lambda_1, \beta_1) \cdot (\lambda_2, \beta_2)) &= \int d[(y', u')] E((u', y'), (\lambda_2, \beta_2)) E((y', u')^{-1H} \circ (y, u), (\lambda_1, \beta_1)).
\end{aligned} \tag{E.25}$$

Unit/co-unit relations:

$$\begin{aligned}
E((1, 1), (\lambda, \beta)) &= \hat{\alpha}(\lambda, \beta), & E((y, u), (1, 1)) &= \alpha(y, u), \\
\int d[(y, u)] E((y, u), (\lambda, \beta)) \alpha(y, u) &= \hat{\delta}(\lambda, \beta), & \int d[(\lambda, \beta)] E((y, u), (\lambda, \beta)) \hat{\alpha}(\lambda, \beta) &= \delta(y, u).
\end{aligned} \tag{E.26}$$

Horizontal and vertical inverses:

$$\begin{aligned}
E^{-1H}((y, u), (\lambda, \beta)) &= E((y, u)^{-1H}, (\lambda, \beta)) = E((y, u), (\lambda, \beta)^{-1H}), \\
E^{-1\nu}((y, u), (\lambda, \beta)) &= E((y, u)^{-1\nu}, (\lambda, \beta)) = E((y, u), (\lambda, \beta)^{-1\nu}).
\end{aligned} \tag{E.27}$$

Such 2-bi-algebras $F(\mathcal{G})$ and $F(\mathcal{G}^*)$ are dual to each other, with the map σ given in terms of the plane wave:

$$\sigma(\hat{\phi}, \phi) = \int d[(y, u)] d[(\lambda, \beta)] E((y, u), (\lambda, \beta)) \hat{\phi}(\lambda, \beta) \phi(y, u). \tag{E.28}$$

$$(y_1^{-1}, t(y_1)u) \cdot (y_2, u_2) = (y_2 y_1^{-1}, t(y_1)u_1) \quad u_1 = u_2 \tag{E.29}$$

Lemma 1. *Given the dual 2-bi-algebras in Prop. 18, the following identities hold.*

$$\begin{aligned}
\int d[(y, u)] E((y, u), (\lambda_1, \beta_1)) E^{-1}((y, u), (\lambda_2, \beta_2)) &= \hat{\delta}((\lambda_1, \beta_1) \circ (\lambda_2, \beta_2)^{-1H}), \\
\int d[(\lambda, \beta)] E((y_1, u_1), (\lambda, \beta)) E^{-1}((y_2, u_2), (\lambda, \beta)) &= \delta((y_1, u_1) \circ (y_2, u_2)^{-1H}).
\end{aligned} \tag{E.30}$$

Fourier transform

Definition 25 (Fourier transform). *Given the dual 2-bi-algebras $F(\mathcal{G})$ and $F(\mathcal{G}^*)$, the Fourier transform $\mathcal{F} : F(\mathcal{G}) \rightarrow F(\mathcal{G}^*)$ is defined as*

$$\mathcal{F}[\phi](\lambda, \beta) = \int d[(y, u)] E^{-1}((y, u), (\lambda, \beta)) \phi(y, u). \tag{E.31}$$

Proposition 19 (Inverse Fourier transform). *The inverse Fourier transform is the map $\mathcal{F}^{-1} : F(\mathcal{G}^*) \rightarrow F(\mathcal{G})$ defined by*

$$\mathcal{F}^{-1}[\hat{\phi}](y, u) = \int d[(\lambda, \beta)] E((y, u), (\lambda, \beta)) \hat{\phi}(\lambda, \beta). \tag{E.32}$$

and satisfies

$$\mathcal{F}[\mathcal{F}^{-1}] = id, \quad \mathcal{F}^{-1}[\mathcal{F}] = id. \tag{E.33}$$

Theorem 3 (Generalized convolution theorem). *Given the dual 2-bi-algebras $F(\mathcal{G})$ and $F(\mathcal{G}^*)$, the following identities hold*

$$\begin{aligned}\mathcal{F}[(\phi_1 \circ \phi_2)] &= \mathcal{F}[\phi_1] \cdot \mathcal{F}[\phi_2], & \mathcal{F}^{-1}[(\hat{\phi}_1 \circ \hat{\phi}_2)] &= \mathcal{F}[\hat{\phi}_1] \cdot \mathcal{F}[\hat{\phi}_2], \\ \mathcal{F}[(\phi_1 \cdot \phi_2)] &= \mathcal{F}[\phi_1] \circ \mathcal{F}[\phi_2], & \mathcal{F}^{-1}[(\hat{\phi}_1 \cdot \hat{\phi}_2)] &= \mathcal{F}^{-1}[\hat{\phi}_1] \circ \mathcal{F}^{-1}[\hat{\phi}_2].\end{aligned}\quad (\text{E.34})$$

Theorem 4 (Co-convolution theorem). *Given the dual 2-bi-algebras $F(\mathcal{G})$ and $F(\mathcal{G}^*)$, the following identities hold*

$$\begin{aligned}\mathcal{F}[\Delta_H \phi] &= \Delta_V^{op} \mathcal{F}[\phi], & \mathcal{F}^{-1}[\Delta_H^{op} \hat{\phi}] &= \Delta_V^{op} \mathcal{F}[\hat{\phi}], \\ \mathcal{F}[\Delta_V \phi] &= \Delta_H^{op} \mathcal{F}[\phi], & \mathcal{F}^{-1}[\Delta_V^{op} \hat{\phi}] &= \Delta_H^{op} \mathcal{F}^{-1}[\hat{\phi}].\end{aligned}\quad (\text{E.35})$$

Pills of 2-group field theory

In this part I show how to apply the abstract non-sense concepts of dual 2-bi-algebras to formalize the notion of field and dual field for a 2-group field theory given in Sec. 5.3.

Definition 26 (Field). *Consider the external product of four copies of the 2-group \mathcal{G} and let $\phi \in F(\mathcal{G}^{\times 4})$ be a function on them. The field in 2-GFT is the function*

$$\Phi((y_{1,2}, u_{1,2}), \dots, (y_{3,4}, u_{3,4})) = (\cdot_{19} \cdot_{212} \cdot_{34} \cdot_{511} \cdot_{67} \cdot_{810}) (\Delta_V^3 \otimes \Delta_V^3 \otimes \Delta_V^3 \otimes \Delta_V^3) \phi, \quad (\text{E.36})$$

where \cdot_{ij} is a vertical product between the tensor spaces i and j . Explicitly it writes

$$\begin{aligned}\Phi &= \int d[(y, u)]^{12} \phi((y_1, u_1) \cdot (y_2, u_2) \cdot (y_3, u_3), (y_4, u_4) \cdot (y_5, u_5) \cdot (y_6, u_6), \\ &\quad (y_7, u_7) \cdot (y_8, u_8) \cdot (y_9, u_9), (y_{10}, u_{10}) \cdot (y_{11}, u_{11}) \cdot (y_{12}, u_{12})) \\ &\quad \delta((y_1, u_1) \circ (y_9, u_9) \circ (y_{1,9}, u_{1,9})^{-1H}) \delta((y_2, u_2) \circ (y_{12}, u_{12}) \circ (y_{2,12}, u_{2,12})^{-1H}) \\ &\quad \delta((y_3, u_3) \circ (y_4, u_4) \circ (y_{3,4}, u_{3,4})^{-1H}) \delta((y_5, u_5) \circ (y_{11}, u_{11}) \circ (y_{5,11}, u_{5,11})^{-1H}) \\ &\quad \delta((y_6, u_6) \circ (y_7, u_7) \circ (y_{6,7}, u_{6,7})^{-1H}) \delta((y_8, u_8) \circ (y_{10}, u_{10}) \circ (y_{8,10}, u_{8,10})^{-1H}).\end{aligned}\quad (\text{E.37})$$

Definition 27 (Dual field). *Consider the external product of four copies of the 2-group \mathcal{G}^* and let $\hat{\phi} \in F(\mathcal{G}^{*\times 4})$ be a function on them. The dual field in 2-GFT is the function*

$$\hat{\Phi}((\lambda_{1,2}, \beta_{1,2}), \dots, (\lambda_{3,4}, \beta_{3,4})) = (\circ_{19} \circ_{212} \circ_{34} \circ_{511} \circ_{67} \circ_{810}) (\Delta_H^3 \otimes \Delta_H^3 \otimes \Delta_H^3 \otimes \Delta_H^3) \phi, \quad (\text{E.38})$$

where \circ_{ij} is an horizontal product between the tensor spaces i and j . Explicitly it writes

$$\begin{aligned}\hat{\Phi} &= \int d[(\lambda, \beta)]^{12} \hat{\phi}((\lambda_1, \beta_1) \circ (\lambda_2, \beta_2) \circ (\lambda_3, \beta_3), (\lambda_4, \beta_4) \circ (\lambda_5, \beta_5) \circ (\lambda_6, \beta_6), \\ &\quad (\lambda_7, \beta_7) \circ (\lambda_8, \beta_8) \circ (\lambda_9, \beta_9), (\lambda_{10}, \beta_{10}) \circ (\lambda_{11}, \beta_{11}) \circ (\lambda_{12}, \beta_{12})) \\ &\quad \delta(((\lambda_1, \beta_1) \cdot (\lambda_9, \beta_9)) \circ (\lambda_{1,9}, \beta_{1,9})^{-1V}) \delta(((\lambda_2, \beta_2) \cdot (\lambda_{12}, \beta_{12})) \circ (\lambda_{2,12}, \beta_{2,12})^{-1V}) \\ &\quad \delta(((\lambda_3, \beta_3) \cdot (\lambda_4, \beta_4)) \circ (\lambda_{3,4}, \beta_{3,4})^{-1V}) \delta(((\lambda_5, \beta_5) \cdot (\lambda_{11}, \beta_{11})) \circ (\lambda_{5,11}, \beta_{5,11})^{-1V}) \\ &\quad \delta(((\lambda_6, \beta_6) \cdot (\lambda_7, \beta_7)) \circ (\lambda_{6,7}, \beta_{6,7})^{-1V}) \delta(((\lambda_8, \beta_8) \cdot (\lambda_{10}, \beta_{10})) \circ (\lambda_{8,10}, \beta_{8,10})^{-1V}).\end{aligned}\quad (\text{E.39})$$

Note that, the definition of dual field $\hat{\Phi}$ can be derived by that of the field Φ using the Generalized Convolution and Co-convolution Theorems 3 and 4.

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