

SORBONNE UNIVERSITÉ

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Thesis defended by **Ramon OLIVER-BONAFoux**

Defended on **November 18, 2022**

In order to become Doctor from Sorbonne Université

Academic Field **Mathematics**

Speciality **Partial differential equations**
Calculus of variations

Some existence results for systems of phase transition type

Thesis supervised by Fabrice BETHUEL

Committee members

<i>Referees</i>	Filippo SANTAMBROGIO	Professor at Université Claude Bernard - Lyon 1	Committee President
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COLOPHON

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Unité de recherche Laboratoire Jacques-Louis Lions (UMR 7598)

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Quelques résultats d'existence pour des systèmes du type transition de phase

Thèse dirigée par Fabrice BETHUEL

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SOME EXISTENCE RESULTS FOR SYSTEMS OF PHASE TRANSITION TYPE**Abstract**

In this thesis, we prove several new existence results for gradient systems of phase transition type. Besides mathematics, these systems are also of interest in mechanics, material sciences as well as biology and population dynamics.

In Chapter 1, we give a general introduction on the subject and describe the historical context of this thesis. Chapter 2 contains some results regarding the existence of non-minimizing connecting orbits for multi-well systems.

In Chapter 3, we prove the existence of heteroclinic traveling wave solutions for two-dimensional parabolic Allen-Cahn systems. The main novelty of these solutions is that their speed of propagation is positive and they connect at infinity heteroclinic orbits with different energy.

In Chapter 4, we focus on the problem of existence of heteroclinic traveling waves for parabolic gradient systems in one space dimension. These traveling waves have been shown to exist in several previous works, under non-degeneracy assumptions on the minimizers of the potential. The methods developed in Chapter 3 allow, as a byproduct, to revisit this problem and treat some situations in which the minimizers are degenerate.

All proofs are based on the use of techniques from the calculus of variations.

Keywords: phase transitions, multi-well potentials, allen-cahn systems, connecting orbits, variational methods

QUELQUES RÉSULTATS D'EXISTENCE POUR DES SYSTÈMES DU TYPE TRANSITION DE PHASE**Résumé**

Dans cette thèse, nous démontrons quelques nouveaux résultats d'existence pour des systèmes gradient du type transition de phase. Autre qu'en mathématiques, ces systèmes sont aussi importants en mécanique, sciences des matériaux ainsi qu'en biologie et dynamique des populations.

Dans le Chapitre 1, nous introduisons le sujet et décrivons le contexte historique de cette thèse. Nous donnons aussi une description brève des résultats de cette thèse ainsi que quelques perspectives de recherche. Le Chapitre 2 contient quelques résultats d'existence des orbites de connexion non-minimisantes pour des systèmes de type multi-puits.

Dans le Chapitre 3, nous démontrons l'existence des ondes progressives pour des systèmes Allen-Cahn paraboliques en dimension deux. La nouveauté principale de ces solutions est que leur vitesse de propagation est strictement positive et qu'elles connectent à l'infini des orbites hétéroclines avec des énergies différentes. Dans le Chapitre 4, on considère le problème d'existence des ondes progressives hétéroclines pour des systèmes gradient paraboliques en dimension un. Ces ondes progressives ont été établies dans plusieurs travaux précédents, sous des hypothèses de non-dégénérescence sur les minimiseurs du potentiel. Les méthodes développées au Chapitre 3 permettent de revisiter ce problème et de traiter certaines situations où les minimiseurs sont dégénérés.

Toutes les preuves sont basées sur l'usage des techniques du calcul des variations.

Mots clés : transitions de phase, potentiels multi-puits, systèmes allen-cahn, orbites de connexion, méthodes variationnelles

Preliminary considerations

This thesis has been prepared in Laboratoire Jacques-Louis Lions, Sorbonne Université, located in Paris (France). The thesis supervisor is Fabrice Bethuel. The author has been a PhD student since the 1st of October of 2019.

We should make the reader aware of the following:

- Chapter 1 does not contain any new results. Its purpose is to contextualize the thesis. The main contributions of this thesis can be found in several papers by the author.
- Chapter 2 is an extended version of the article [131], published in *Calculus of Variations and Partial Differential Equations*. The author is indebted to the referee, which provided very useful suggestions and bibliographic references which improved the paper. Chapters 3 and 4 correspond to the preprints [130] and [129], respectively, with modifications on the presentation.
- Each of the Chapters 1, 2 and 3 is self-contained. This is not the case for Chapter 4, in which we refer consistently to material already presented in Chapter 3 in order to reduce repetitions. Each chapter is preceded by its own abstract both in English and French.

Considérations préliminaires

Cette thèse a été préparée au Laboratoire Jacques-Louis Lions, Sorbonne Université, situé à Paris. Le directeur de thèse est Fabrice Béthuel. L'auteur a été doctorant depuis le 1er d'octobre de 2019.

L'auteur souhaite prévenir au lecteur de ce qui suit:

- Le Chapitre 1 ne contient aucun résultat nouveau. Il est consacré à mettre la thèse en contexte. Les contributions principales de cette thèse peuvent être trouvées dans plusieurs papiers écrits par l'auteur.
- Le Chapitre 2 est une version étendue de l'article [131], publié dans *Calculus of Variations and Partial Differential Equations*. L'auteur remercie le rapporteur pour des suggestions, remarques et références qui ont amélioré le papier. Les Chapitres 3 et 4 correspondent à les prépublications [130] et [129], respectivement, avec des modifications dans la présentation.
- Chacun des Chapitres 1, 2 et 3 est essentiellement auto-contenu. Ceci n'est pas le cas pour le Chapitre 4, où nous faisons souvent référence à des contenus déjà présentés au Chapitre 3 pour réduire des répétitions. Chaque chapitre est précédé par son propre résumé en français et anglais.

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Tout d'abord, merci beaucoup à Fabrice pour avoir accepté de diriger ma thèse (après avoir dirigé mon mémoire de M2) et pour m'avoir proposé de travailler sur des problèmes que je trouve très intéressants. Tu m'as laissé une grande liberté pour choisir comment et sur quoi travailler, que j'ai beaucoup appréciée, tout en me donnant l'aide et l'encadrement nécessaires. Tes conseils, les discussions que nous avons eues et ton encouragement n'ont pas seulement été essentielles pendant cette période de doctorat, je suis aussi certain qu'ils vont continuer à m'accompagner et à m'inspirer dans l'avenir.

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Je suis arrivé en France depuis l'Espagne en septembre de 2018, moment auquel j'ai commencé le M2 Mathématiques de la Modélisation à Jussieu. Pour cela, j'ai obtenu la bourse PGSM de la Fondation de Sciences Mathématiques de Paris (FSMP). Cela m'a beaucoup aidé à simplifier mon arrivée et installation en France dans un moment où mon français était encore assez mauvais. Puis, j'ai eu le financement COFUND (aussi donné par la FSMP, de façon conjointe avec la Commission Européenne) pour poursuivre en doctorat à partir d'octobre de 2019. Merci à Ariela Briani, responsable du programme, pour l'aide et l'accompagnement constants pendant ces trois ans. Grâce au programme COFUND j'ai pu financer une visite de deux mois au Technion Institute (Haifa, Israel), que je remercie pour l'hospitalité. J'ai été bien accueilli là-bas par Itai Shafrir, que je remercie pour cela ainsi que pour les discussions que nous avons eues. Pendant ma thèse, j'ai habité dans plusieurs maisons de la Cité Universitaire (Espagne, Maroc et Suisse), dans des conditions que j'ai beaucoup appréciées. Merci au personnel de ces maisons et merci aux voisins pour les conversations et la compagnie.

Je remercie tous les membres du site Jussieu du Laboratoire Jacques-Louis Lions (LJLL), où j'ai fait ma thèse. J'ai trouvé ici une très bonne ambiance dans le sens mathématique et humain. Merci en particulier aux gestionnaires (passées et présentes) qui en plein de situations m'ont aidé: Catherine Drouet, Malika Larcher et Salima Lounici. Je me souviens aussi de presque tous les camarades doctorants (passés et présents, surtout du LJLL mais aussi de quelques uns d'ailleurs) et de certains post-doctorants. Dans beaucoup de cas, il ne s'agit pas simplement de collègues: ils ont devenu aussi des amis. Plus particulièrement, je remercie:

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d'écrire les remerciements génériques, je constate qu'il est difficile d'arriver à mentionner explicitement toutes les personnes concernées. En m'excusant par avance, merci Alexandre, Alexiane, Allen, Anatole, Antoine, Chiara, Eugenio, Fatima, Gontran, Jules, Juliette, Lucas (E. et J.), Ludovic, Katia, Matthieu, Nelly, Nicolas C., Olivier, Pierre, Po-Yi, Rémi, Robin, Thomas, Yipeng, Yvonne...

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Summary

Abstract	ix
Preliminary considerations	xi
Considérations préliminaires	xiii
Remerciements	xv
Summary	xvii
1 General Introduction	1
2 Non-minimizing connecting orbits for multi-well systems	49
3 Heteroclinic traveling waves of 2D parabolic Allen-Cahn systems	81
4 Heteroclinic traveling waves of 1D parabolic systems with degenerate stable states	157
Bibliography	193

General Introduction

Outline of the current chapter

1.1 Historical background	2
1.1.1 Newton's laws	3
1.1.1.1 Conservative and non-conservative forces	3
1.1.2 The simple pendulum	4
1.1.2.1 Setting and immediate properties	4
1.1.2.2 Classifying solutions according to their mechanical energy . .	5
1.1.2.3 Extensions to more general conservative equations and systems	7
1.1.2.4 Extension to non-conservative systems	10
1.1.3 Lagrangian formulation of mechanics: The birth of the calculus of varia-	
tions	11
1.1.3.1 The Lagrangian formalism for the simple pendulum and multi-	
well systems	12
1.1.3.2 The Hamiltonian formalism	13
1.1.3.3 The Maupertuis principle	14
1.1.4 Some contemporary methods of the calculus of variations	16
1.1.4.1 The direct method	16
1.1.4.2 Morse theory	17
1.1.4.3 The mountain pass lemma	18
1.1.4.4 Lack of compactness	20
1.1.4.5 Other problems: Periodic orbits for Hamiltonian systems . . .	21
1.2 State of the art prior to this thesis	22
1.2.1 Multiple well systems	22
1.2.2 Functional setting	22
1.2.3 Existence of connecting orbits: the minimizing case	25
1.2.4 Existence of connecting orbits: the non-minimizing case	26
1.2.5 Behavior of the Palais-Smale sequences of E	27
1.2.6 Two-dimensional heteroclinics	28
1.2.7 One-dimensional heteroclinic traveling waves	29

1.3 Main contributions of this thesis	31
1.3.1 Non-minimizing connecting orbits for multi-well systems	31
1.3.2 Heteroclinic traveling waves of 2D parabolic Allen-Cahn systems . . .	33
1.3.3 Heteroclinic traveling waves of 1D parabolic systems with degenerate stable states	35
1.4 Perspectives and open problems	35
1.4.1 Extensions of the connecting orbits problem	35
1.4.2 The Maupertuis approach to the connecting orbits problem	37
1.4.3 Link between connecting and periodic orbits	40
1.4.4 Long range interaction and critical points at infinity of E	41
1.4.5 Long time behavior in two-dimensional parabolic Allen-Cahn systems	42
1.4.6 Heteroclinic traveling waves in fourth order systems	43
1.4.7 Multi-well potentials and singular limits	44

Abstract. We begin by a general historical introduction and a overview of mathematical areas underlying this thesis. Subsequently, we review the literature most directly related to this thesis and give a brief description of our main results. Finally, we present some open problems and perspectives for future research.

Résumé. Nous commençons par une introduction historique générale et un aperçu sur les domaines des mathématiques sous-jacentes à cette thèse. Ensuite, nous révisons la littérature plus directement liée à cette thèse et nous donnons une brève description de nos résultats principaux. Finalement, nous présentons quelques problèmes ouverts et quelques perspectives de recherche.

1.1 Historical background

The description of the motion of physical objects is certainly one of the oldest problems of the natural sciences. The ancient Greek philosophers were already concerned by this question and so were the astronomers of the Old and Middle ages (Ptolemy, Brahe, Kepler, Galileo, etc). A major breakthrough, which is often considered to be the birth of *classical mechanics*, was the publication of the book *Principia Mathematica* by Isaac Newton in the late 17th century. Newton proposed general principles with the aim of describing the motion of physical objects, which supposed a major unification of many previous scientific observations. These principles were indeed thought to be universal until the emergence of more modern physical theories (relativity theory, quantum physics) and still essentially valid in a wide range of common situations, such as the description of the motion of macroscopic objects which move at not extremely high speed.

The remote motivation of this thesis can be traced back to Newton's mechanics. However, there are other important links between the mathematical problems that we consider in this thesis and the natural sciences (physics, chemistry, biology). Despite the previous fact, we have chosen to begin the presentation of our thesis by a succinct overview on classical and analytical mechanics, starting from Newton. In particular, we will describe the elementary model of the simple pendulum, which is a particular case of the systems of ordinary differential equations which are central in this thesis. Remarkably enough, these classical theories and models already contain most of the essential principles that lie in the (sometimes remote) foundations of our work

as well as most of the basic mathematical objects we are interested in. From this point, we will establish the link with more modern theories (functional analysis, partial differential equations, calculus of variations) and, finally, with the contributions of this thesis.

1.1.1 Newton's laws

The well-known Newton's laws state that the motion of an object (to simplify, a point mass with infinitesimal size) is caused by the action of a force (e. g. gravity) on such object. The previous can be synthesized by means of the so-called *Newton's second law*:

$$\vec{F}(t) = m(t)\vec{a}(t), \quad (1.1.1)$$

where, at some time t , $\vec{a}(t)$ is the *acceleration* of the object, $m(t)$ is its mass and $\vec{F}(t)$ is the force acting on the object.

The language of *differential calculus* allows to conveniently rewrite (1.1.1). Indeed, if $q(t)$ represents the position of the object at time t (we drop the arrow), which is a vector in some *configuration space* \mathcal{M} , we can then write

$$q''(t) = \vec{a}(t) \quad (1.1.2)$$

for all time t . In other words, (1.1.2) is the mathematical description (through the differential calculus due to Newton and Leibniz) of the physical notions which state that the speed is the (infinitesimal) variation of the position and the acceleration is the variation of the speed. Plugging (1.1.2) into (1.1.1) we obtain the system of *ordinary differential equations*

$$q''(t) = \vec{F}(t), \quad (1.1.3)$$

where we have assumed that the mass of the object does not change with time, so that we can renormalize and set $m = 1$. Equation (1.1.3) is in fact the main starting point for the present thesis.

1.1.1.1 Conservative and non-conservative forces

An important subcase of (1.1.3) is that in which \vec{F} is a *conservative* force, i. e., the *work* done for moving the object between two end-points does not depend on the chosen path. Equivalently, there exists a real-valued function U defined in the configuration space such that

$$\vec{F}(t) = -\nabla U(q(t)), \quad (1.1.4)$$

where ∇ stands for the *gradient* and U is the *potential*. That is, conservative forces are those for which there exists a potential function. The simplification (1.1.4) allows us to rewrite (1.1.3) as

$$q''(t) = -\nabla U(q(t)). \quad (1.1.5)$$

System (1.1.5) in particular is central this thesis. We will be also interested in the (more realistic) system (1.1.3) with \vec{F} chosen as

$$\vec{F}(t) = -\nabla U(q(t)) + \tilde{F}(t)$$

where \tilde{F} represents the *friction*, a non-conservative force. With this choice, we obtain the following particular case of (1.1.3)

$$q''(t) = -\nabla U(q(t)) + \tilde{F}(t). \quad (1.1.6)$$

1.1.2 The simple pendulum

One of the most elementary models which can be described with equations (1.1.3) and (1.1.5) is the simple pendulum. The study of the pendulum problem can be traced back to Galileo in the 16th century, we refer to Mawhin [117, 116] for a historical review on the simple pendulum. Its particular relevance for us lies on the fact that it already contains most of the mathematical objects which are fundamental in our work.

1.1.2.1 Setting and immediate properties

We quickly recall the setting of the simple pendulum, inspired by [117, 116]. Consider a point particle of mass 1 attached to a center by a rigid bar of negligible mass and length $L > 0$. We consider that the motion of the particle takes place on a plane, which we identify with \mathbb{R}^2 . Let θ be the angle between the particle and the vertical axis. We wish to use Newton's law (1.1.3) in order to describe the evolution of θ , assuming that the only effect on the particle is that caused by gravity, which is a conservative force. From the physical point of view, the simple pendulum represents a highly restrictive situation, as in most situations plenty of other, non-conservative, forces come into play. However, the simple pendulum is already very interesting for us from the mathematical point of view.

Notice that the system (1.1.3) describes the evolution of the *position* of the particle q . However, the position can be written as a function of the angle, which means that (1.1.3) can be transformed into a single equation for θ , which will be simpler. Indeed, at time t , the position q writes

$$q(t) = (L \sin \theta(t), L \cos \theta(t)) \quad (1.1.7)$$

and the gravity force \vec{F}_g , independent on time, is

$$\vec{F}_g(t) = (0, -g), \quad (1.1.8)$$

where $g > 0$ is the constant of gravitation. See Figure 1.1 for a graphical description of the pendulum. Plugging (1.1.7) and (1.1.8) into (1.1.3) we get a second order differential equation for θ which writes as follows:

$$\theta''(t) = -a^2 \sin \theta(t), \quad (1.1.9)$$

where $a := \sqrt{g/L}$. The simplest solutions to (1.1.9) are those that are constant and coincide identically with a number in $\mathcal{P}_E := \{k\pi\}_{k \in \mathbb{Z}}$. Those correspond to the *equilibrium states* of the physical system, in which the pendulum does not move. Moreover, we have $\mathcal{P}_E := \mathcal{P}_S \sqcup \mathcal{P}_U$, where $\mathcal{P}_S := \{2k\pi\}_{k \in \mathbb{Z}}$ and $\mathcal{P}_U := \{(2k+1)\pi\}_{k \in \mathbb{Z}}$. The elements in \mathcal{P}_S correspond to the *stable* equilibria while the elements in \mathcal{P}_U correspond to the *unstable* equilibria. That is, if we start with an angle close to an element of \mathcal{P}_S and say, zero speed, the angle will remain close for all time. The opposite will happen for initial angle close to an element of \mathcal{P}_U and zero speed. We represent graphically stable and unstable equilibrium states in Figure 1.2.

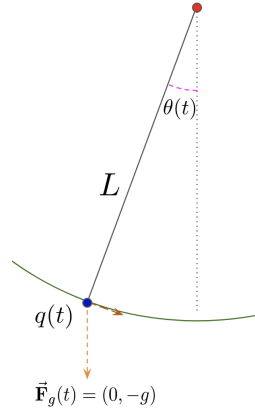


Figure 1.1: Depiction of the simple pendulum at a time t . The gravity force \vec{F}_g causes a motion on the particle which follows the tangent line of the circle centered at the end of the pendulum and radius L , in which the particle moves.

Another important basic fact is that for any θ solving (1.1.9) we have the relation

$$\frac{1}{2}\theta'(t)^2 - a^2(\cos \theta(t) + 1) = E_\theta \quad (1.1.10)$$

for some real constant $E_\theta \in \mathbb{R}$. Notice that if we set

$$V : u \in \mathbb{R} \rightarrow a^2(\cos u + 1) \in \mathbb{R}, \quad (1.1.11)$$

then we have that for all $u \in \mathbb{R}$, $V'(u) = -a^2 \sin u$. According to (1.1.9) this means that $V = -U$, with U a potential function for the conservative force (gravity) acting on the angle θ , see (1.1.4). Therefore, the left-hand side in (1.1.10) is the sum between the kinetic energy and the potential energy, i. e. the mechanical energy of the system, which is independent on time. From the mathematical point of view, notice that (1.1.10) implies that $(\theta')^2$ is bounded by $2(E_\theta + 2a^2)$, which, by elementary ODE theory, means that any solution to (1.1.9) on an interval (t_1, t_2) can be extended to a solution defined on \mathbb{R} . Hence, we will suppose on the latter that all the solutions are defined on \mathbb{R} .

The definition of V in (1.1.11) deserves a comment. Recall that, according to (1.1.4), one can choose the potential up to a constant. We chose U such that its global maxima are attained at zero and then inverted its sign to define V , which is hence non-negative and vanishing exactly on \mathcal{P}_U . The reason for this (a priori arbitrary) choice, will become fully clear later, but at this point it already allows to classify the qualitative behavior of solutions of (1.1.9) according to the sign of the mechanical energy E_θ , as we show in the following paragraph.

1.1.2.2 Classifying solutions according to their mechanical energy

Elementary arguments of ODE theory allow to classify the solutions of (1.1.9) according to the sign of E_θ . Recall that for any pair $(\theta_0, E_0) \in \mathbb{R}^2$ there exists a solution θ of (1.1.9) such that $\theta(0) = \theta_0$ and (1.1.10) holds with $E_\theta = E_0$.

Case 1: $E_\theta > 0$. By (1.1.10) we have that $\inf_{t \in \mathbb{R}} \theta'(t)^2 \geq \sqrt{E_\theta}$, which in particular means that

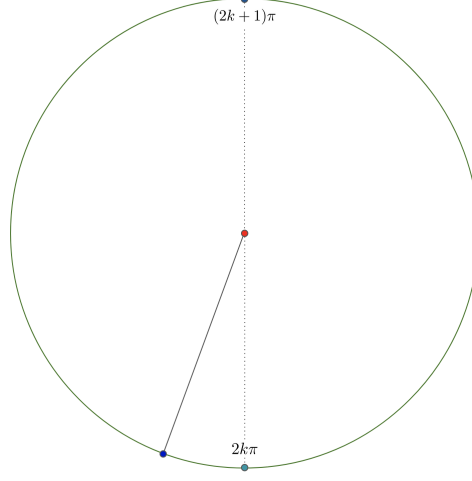


Figure 1.2: Graphical representation of the stable and unstable equilibrium states of the simple pendulum.

$\lim_{t \rightarrow \pm\infty} \theta(t) = \pm\infty$ if $\theta' > 0$ and $\lim_{t \rightarrow \pm\infty} \theta(t) = \mp\infty$ otherwise. Therefore, this case corresponds to the situation in which the particle turns indefinitely around the center of the pendulum. See Figure 1.3.

Case 2: $E_\theta = 0$. There are two possible sub-cases. If we take $\theta_0 = (2k_\theta + 1)\pi \in \mathcal{P}_U$ and impose $\theta(0) = \theta_0$, we then have that $\theta(t) = (2k_\theta + 1)\pi$ for all $t \in \mathbb{R}$. If for $k \in \mathbb{Z}$ we take $\theta_0 \in ((2k + 1)\pi, (2k + 3)\pi)$, then Cauchy-Lipschitz Theorem and other ODE results imply that the associated solution θ verifies

$$\theta(\mathbb{R}) = ((2k + 1)\pi, (2k + 3)\pi), \quad 0 \notin \theta'(\mathbb{R}) \quad (1.1.12)$$

and, in particular,

$$\begin{cases} \lim_{t \rightarrow \pm\infty} \theta(t) = (2k + 2 \pm 1)\pi & \text{if } \theta' > 0, \\ \lim_{t \rightarrow \pm\infty} \theta(t) = (2k + 2 \mp 1)\pi & \text{if } \theta' < 0. \end{cases} \quad (1.1.13)$$

According to (1.1.13) the solution θ is called a *heteroclinic orbit*, terminology which is motivated by the fact that it *connects* (at infinity) the two *different* equilibrium states $(2k + 1)\pi$ and $(2k + 3)\pi$, with orientation determined by the sign of θ' . However, since θ is an angle, one might very well argue that the equilibrium states $(2k + 1)\pi$ and $(2k + 3)\pi$ are in fact the same one. In other words, it makes sense to rather look at $p_\pi \circ \theta$, where $p_\pi : \mathbb{R} \rightarrow \{\pi\} + \mathbb{R}/(2\pi\mathbb{Z})$ is the standard covering map. Projecting a function by p_π might provoke a loss of continuity, as it would happen for the solutions considered in Case 1. But here we have by (1.1.12) that $\theta(\mathbb{R})$ is contained in a single sheet of $\{\pi\} + \mathbb{R}/(2\pi\mathbb{Z})$, so that $p_\pi \circ \theta \cong \theta$ solves (1.1.9) in \mathbb{R} . Furthermore, by (1.1.13) we have

$$\lim_{t \rightarrow \pm\infty} p_\pi \circ \theta(t) = \pi.$$

Hence, the solution $p_\pi \circ \theta$, which is essentially θ seen in a different configuration space, is called a *homoclinic orbit*, meaning that it connects at $\pm\infty$ the *same* equilibrium state. According to the terminology we adopt in this thesis, heteroclinics and homoclinics are the two different types of *connecting orbits*. See Figure 1.4.

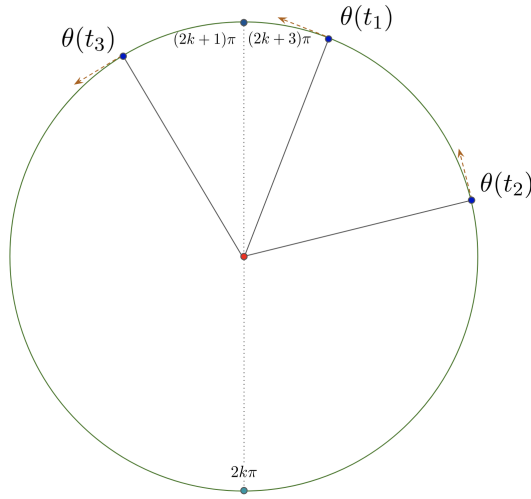


Figure 1.3: Solution corresponding to Case 1 in Section 1.1.2.2. This type of solution turns indefinitely around the center of the pendulum, going through all the equilibrium states.

Case 3: $E_\theta < 0$. In this case, the solutions are periodic and oscillate around a stable equilibrium state. This can be seen by observing that for all $t \in \mathbb{R}$

$$2 \geq \cos \theta(t) + 1 \geq -\frac{E_\theta}{a^2} > 0.$$

In particular, if $E_\theta = -2a^2$, then for some $k \in \mathbb{Z}$ and for all real t it holds $\theta(t) = 2k \in \mathcal{P}_S$, so that the solution is constant. In general, it is an exercise to show that for any $E_\theta \in [-2a^2, 0)$, if θ solves (1.1.9) and has energy E_θ then $\theta(\mathbb{R}) = [2k\pi - \theta_{\max}, 2k\pi + \theta_{\max}]$ for some $k \in \mathbb{Z}$, with $\theta_{\max} \in (0, \pi)$ such that

$$\cos(\pm\theta_{\max}) = -1 - \frac{E_\theta}{a^2}.$$

Moreover, $\theta'(t) = 0$ whenever $\theta(t) \in \{2k\pi - \theta_{\max}, 2k\pi + \theta_{\max}\}$ and for all $\varepsilon > 0$ small enough we have that $\theta'(t + \varepsilon)\theta'(t - \varepsilon) < 0$, i. e., θ' changes sign around t . In other words, θ is a periodic solution of (1.1.9), corresponding to a pendulum oscillating with a maximal angle θ_{\max} around a stable equilibrium at $2k$. See Figure 1.5.

1.1.2.3 Extensions to more general conservative equations and systems

As the previous discussion shows, the pendulum model (1.1.9) possesses periodic solutions and heteroclinic solutions connecting equilibrium states. More precisely, for any $k \in \mathbb{Z}$ there exists a heteroclinic connecting $(2k+1)\pi$ and $(2k+3)\pi$, and conversely by inverting the time. Using the mechanical energy formula (1.1.10), it follows that such a heteroclinic is unique up to translations (notice that (1.1.9) is translation invariant). Moreover, if m_- and m_+ are two odd integers, such that $m_- - m_+ > 2$, then the uniqueness result for ODEs implies that there is no θ solving (1.1.9) and such that

$$\lim_{t \rightarrow \pm\infty} \theta(t) = m_\pm \pi.$$

The reason behind this is that there are other equilibrium states between $m_- \pi$ and $m_+ \pi$ which *obstruct* the existence of a heteroclinic between $m_- \pi$ and $m_+ \pi$.

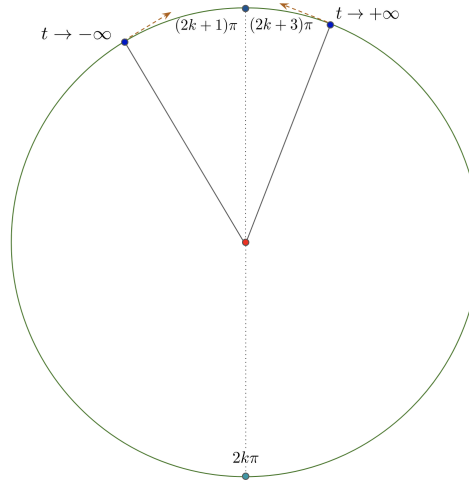


Figure 1.4: Heteroclinic solution which goes from $(2k+1)\pi$ at $-\infty$ to $(2k+3)\pi$ at $+\infty$. Notice that by time reversibility of (1.1.9), we can invert the sign of this solution and obtain an heteroclinic from $(2k+3)\pi$ to $(2k+1)\pi$.

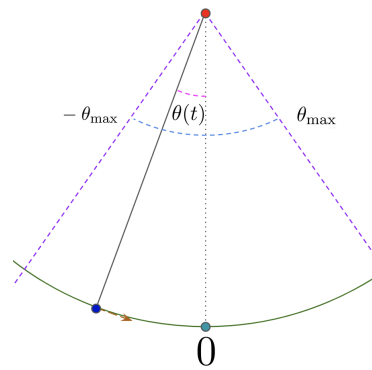


Figure 1.5: Periodic solution that oscillates around the stable equilibrium 0 with maximal angular distance θ_{\max} .

The previous discussion extends to more general ODEs of the type

$$q'' = V'(q) \text{ in } \mathbb{R}, \quad (1.1.14)$$

where V is a (smooth) *scalar multi-well potential*. That is, we have for all $u \in \mathbb{R}$ that $V(u)$ and, moreover, the set

$$\Sigma := \{u \in \mathbb{R} : V(u) = 0\} \quad (1.1.15)$$

contains at least two elements, which we call *wells*. By choosing V as in (1.1.11), we recover the simple pendulum model. However, in this thesis we assume that Σ is finite for simplicity. One standard example in the *Allen-Cahn* (or scalar Ginzburg-Landau) potential:

$$V_{AC} : u \in \mathbb{R} \rightarrow \frac{(1 - u^2)^2}{4},$$

which was introduced in Cahn and Hilliard [62] and also in Allen and Cahn [14] as a model for phase transitions of materials.

We find that for all q solving (1.1.14) it holds

$$\frac{|q'|^2}{2} - V(q) = E_q \text{ in } \mathbb{R}, \quad (1.1.16)$$

for some $E_q \in \mathbb{R}$. Identity (1.1.16) is the equivalent of (1.1.11) for the pendulum. Using (1.1.16) and the same ODE arguments, one also shows the existence and uniqueness of heteroclinics between consecutive wells. More precisely, if we write

$$\Sigma := \{\sigma_i : i \in \{1, \dots, l\}, l := \#\Sigma\},$$

where for all $i \in \{1, \dots, l-1\}$ it holds

$$\sigma_i < \sigma_{i+1}, \quad (1.1.17)$$

then, for any $(i, j) \in \{1, \dots, l\}^2$ there exists a solution q of (1.1.16) connecting σ_i and σ_j if and only if $|i - j| \leq 1$, that is, the wells are either equal or consecutive. Moreover, such solution is unique up to translations. Notice that in the case $i = j$ we are referring to the homoclinic-type solution which is a constant equal to σ_i . If $i \neq j$, we have a heteroclinic solution. Therefore, the picture regarding existence and uniqueness of connecting orbits is quite complete for the equation (1.1.14) and it follows from very classical ODE arguments.

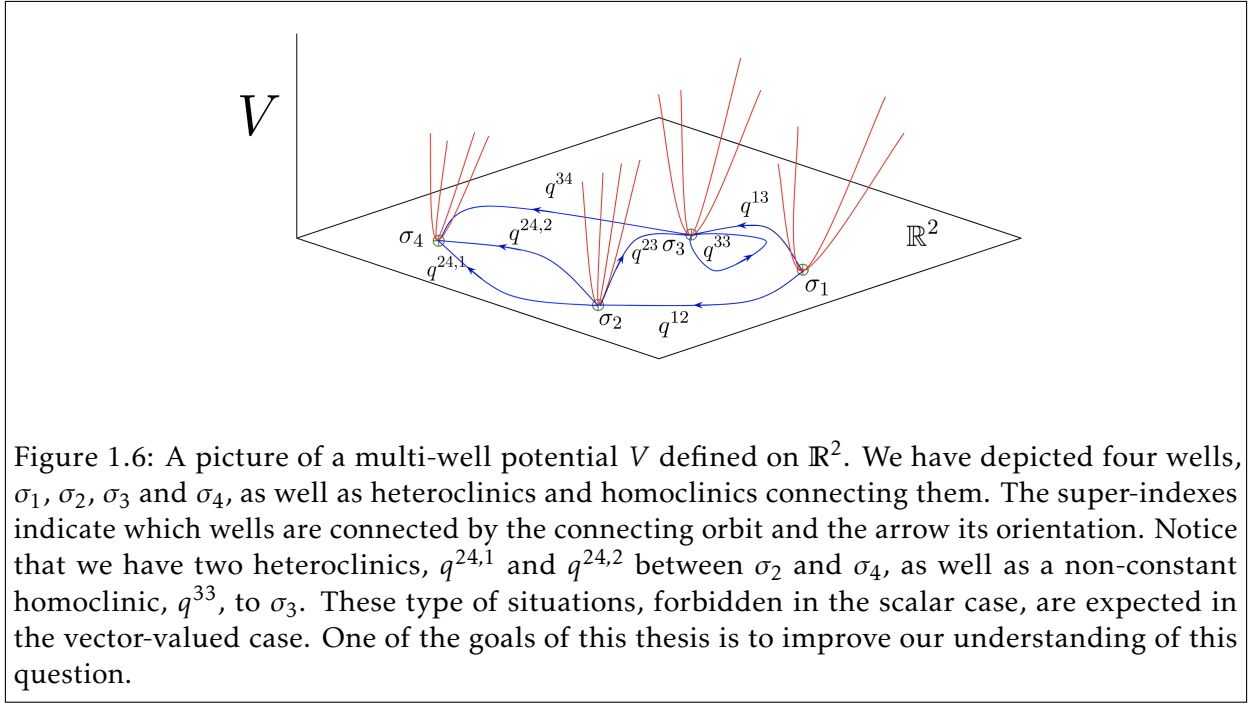
The picture is quite different if one considers the *system*

$$q'' = \nabla V(q) \text{ in } \mathbb{R} \quad (1.1.18)$$

where V is a smooth multi-well potential from \mathbb{R}^k , $k \geq 2$, to \mathbb{R} . That is, we assume that V is smooth and that its set of zeros Σ (see (1.1.15)) is finite. From the physical perspective, (1.1.18) can be seen as a model for a *multiple pendulum*.

It is clear that (1.1.16) still holds for system (1.1.18). However, essentially two new complications arise:

1. There is no notion of *consecutiveness* for elements of \mathbb{R}^k when $k \geq 2$. Therefore, there is no



way of ordering the elements of Σ as in (1.1.17).

2. While there is essentially only one path (up to reparametrizations) that connects two points of \mathbb{R} , this is no longer true in \mathbb{R}^k .

These two new features imply in particular that the uniqueness argument provided in the scalar case does no longer hold for systems. However, as shown for instance by Alikakos, Betelú and Chen [9], one can still have non-existence of heteroclinic solutions for a fixed pair of wells. Hence, even if the notion of consecutive wells does not exist in the case of vector potentials, one still finds obstruction phenomena as in the scalar case. More details will be given later.

In general, one expects that that systems (1.1.18) possess more solutions than the scalar equations (1.1.14). For instance, one might wonder if multiple heteroclinics between two given wells or non-constant homoclinics (which do not exist in the scalar case) exist for some multi-well systems. See Figure 1.6 This is one of the questions that we pose in this thesis, providing new existence results in this direction. We tackle this problem by means of the calculus of variations. We introduce the setting in Section 1.1.3.1.

1.1.2.4 Extension to non-conservative systems

Going back to the simple pendulum problem, one can aim at studying the more realistic situation in which there is a friction force acting on the particle as in (1.1.6). We assume moreover that the friction force is proportional to the speed of the particle through a constant $c > 0$. In this case, the equation for the angle θ writes

$$\theta''(t) + c\theta'(t) = -a^2 \sin \theta(t).$$

The analysis of this ODE is more difficult than in the conservative case, the main reason being the absence of a formula as the mechanical energy formula (1.1.10) for the conservative model. Notice

however that it possesses exactly the same equilibrium states. One can also consider the more general systems

$$q'' + cq' = \nabla W(q) \quad (1.1.19)$$

where $W : \mathbb{R}^k \rightarrow \mathbb{R}$ is a smooth potential which is bounded from below. The equilibrium states of (1.1.19) are exactly the critical points of W and one can study the existence of heteroclinic and homoclinic solutions between such equilibrium states. This question is also posed in this thesis, which is addressed in the more general setting of possibly infinite-dimensional configuration spaces. We again rely on tools from the calculus of variations to treat these problems.

As a first elementary observation for (1.1.19), let q be a solution of (1.1.19) on \mathbb{R} for some $c > 0$ which it is heteroclinic in the following sense

$$\lim_{t \rightarrow \pm\infty} q(t) = a^\pm. \quad (1.1.20)$$

In (1.1.20), $a^\pm \in \mathbb{R}^k$ are two local minimizers of W . We can then easily obtain a formal expression for c . Indeed, if we multiply (1.1.19) by q' and integrate on \mathbb{R} neglecting the limits at infinity we obtain

$$c = \frac{W(a^+) - W(a^-)}{\int_{\mathbb{R}} |q'|^2}. \quad (1.1.21)$$

The formula (1.1.21) suggests that we must have $W(a^+) > W(a^-)$. Therefore, heteroclinics should join stable states with different potential energy. From the physical point of view, the previous means that some energy is lost due to the friction term acting on the particle. From the mathematical point of view, one can give a rigorous justification of the previous fact, see Section 1.2.7.

To conclude this section, we point out that one can consider further generalizations of the pendulum. For instance, the case of the *forced pendulum*, in which there also exists an external force acting on the particle. The corresponding equation or system is then *non-autonomous*, in contrast with the systems we consider in this thesis, which are autonomous. For more details, see Mawhin [117, 116] as well as Mawhin and Willem [118].

1.1.3 Lagrangian formulation of mechanics: The birth of the calculus of variations

For extensive material related to this section, we refer for instance to Dubrovin, Fomenko and Novikov [76], Mawhin and Willem [118]. It is an old idea that some laws of physics can be described in terms of minimization problems. For instance, in the XVII century, Fermat claimed that light reflects following the shortest path. These ideas were generalized and unified by Lagrange in the 18th century, who introduced the *action* or *energy*

$$S_a^b(q) := \int_a^b L(t, q(t), q'(t)) dt, \quad (1.1.22)$$

where, as before, $q : \overline{(a, b)} \rightarrow \mathcal{M}$ (one can have $a = -\infty$ or $b = +\infty$) is the position of a point mass in the configuration space \mathcal{M} and

$$L : (t, q, \xi) \in \overline{(a, b)} \times T\mathcal{M} \rightarrow L(t, q, \xi) \in \mathbb{R},$$

is the (smooth) *Lagrangian*. Here $T\mathcal{M}$ denotes the tangent bundle of \mathcal{M} , so that $T\mathcal{M}$ is diffeomorphic to $\mathbb{R}^k \times \mathbb{R}^k$ when $\mathcal{M} = \mathbb{R}^k$, which, unless otherwise stated, we will assume for the rest of this section for simplicity. Lagrange's theory, known as *analytical* or *Lagrangian* mechanics, states that the position q of the particle must be an *extremal* of the functional S_a^b . That is, the *first variation* of S_a^b (i. e. the differential) at q must vanish. This last condition is equivalent the well known *Euler-Lagrange equations*, which read

$$\frac{d}{dt} \left(\frac{\partial}{\partial \xi} L(t, q(t), q'(t)) \right) = \frac{\partial}{\partial q} L(t, q(t), q'(t)), \quad t \in (a, b). \quad (1.1.23)$$

These ideas gave rise to the calculus of variations, as it was shown that the description of physical principles can be accomplished by studying extremals (called *critical points* in the contemporary terminology) of functionals. Despite the clarity and functionality of this physical interpretation, the basic ideas of the calculus of variations would not be rigorously established from the mathematical point of view up until the beginning of the 20th century, when the fundamentals of functional analysis and differential geometry emerged as we know them today.

In order to conclude this paragraph, we particularize to the framework of conservative mechanical systems. In this case, one has $L = K - P$, where K is the kinetic energy and P is the potential energy. More precisely, L is defined as

$$L(q, \xi) := \frac{|\xi|^2}{2} - U(q) \quad (1.1.24)$$

where $U : \mathbb{R}^k \rightarrow \mathbb{R}_+$ is the potential. With such a choice, one recovers the second Newton's law for conservative forces, written in (1.1.5) above.

1.1.3.1 The Lagrangian formalism for the simple pendulum and multi-well systems

In this thesis, we will look for solutions of multi-well systems (1.1.18) and non-conservative systems as (1.1.19) by studying the associated *Lagrangian action*. According to (1.1.11) as introduced in Section 1.1.2, as well as the identity (1.1.24), we have that the Lagrangian for the simple pendulum model writes

$$L : (q, \xi) \in \mathbb{R}^2 \rightarrow L(q, \xi) = \frac{\xi^2}{2} + a^2(\cos q + 1) \in \mathbb{R}.$$

This gives the Lagrangian action

$$S_a^b(\theta) = \int_a^b \left[\frac{|\theta'(t)|^2}{2} + a^2(\cos \theta(t) + 1) \right] dt, \quad \theta : [a, b] \rightarrow \mathbb{R}.$$

It turns out that the different solutions of the simple pendulum problem, described and classified in Section 1.1.2.2, can be characterized as extremals for S . For instance, periodic solutions of period $T > 0$ are critical points of S_0^T in the class

$$H_{\text{per}, T}^1 := \{u \in H^1([0, T]) : u(0) = u(T)\},$$

where $H^1([0, T])$ stands for the usual Sobolev space. Regarding heteroclinic solutions, they are critical points (in fact, global minimizers) of $E := S_{-\infty}^{+\infty}$ on the classes

$$X_k := \{u \in H_{\text{loc}}^1(\mathbb{R}) : \lim_{t \rightarrow -\infty} u(t) = (2k+1)\pi, \lim_{t \rightarrow +\infty} u(t) = (2k+3)\pi\}$$

for $k \in \mathbb{Z}$. As before, the previous discussion can be extended to multiple pendulum systems as discussed in Section 1.1.2.3. That is, for V a multi-well potential, following (1.1.24) one considers the Lagrangian

$$L : (q, \xi) \in \mathbb{R}^{2k} \rightarrow \frac{|\xi|^2}{2} + V(q) \in \mathbb{R}$$

and the corresponding action

$$S_a^b(q) = \int_a^b \left[\frac{|q'(t)|^2}{2} + V(q(t)) \right] dt, \quad q \in H_{\text{loc}}^1(\mathbb{R}, \mathbb{R}^k).$$

As before, one can find periodic and connecting orbits as critical points of the action, in an analogous functional setting. In particular, heteroclinic orbits can be found by minimizing $E := S_{-\infty}^{+\infty}$ in the spaces

$$X(\sigma, \tilde{\sigma}) := \{q \in H_{\text{loc}}^1(\mathbb{R}, \mathbb{R}^k) : E(q) < +\infty, \lim_{t \rightarrow -\infty} q(t) = \sigma \text{ and } \lim_{t \rightarrow +\infty} q(t) = \tilde{\sigma}\},$$

where σ and $\tilde{\sigma}$ are two different wells in Σ . Recall (see Section 1.1.2.2) that in the scalar case we had existence and uniqueness of heteroclinic orbits between two consecutive wells by purely ODE arguments. However, as discussed in Section 1.1.2.3, such arguments do not extend to multi-well systems. Besides the fact that one might not have uniqueness (up to translations), one still needs to tackle the problem of existence. As it will be described later, the methods of the calculus of variations allow to successfully treat these questions.

Finally, we point for the non-conservative problem described in Section 1.1.2.4, we will look at the family of Lagrangians given by

$$L : (t, q, \xi) \in \mathbb{R}^{2k} \rightarrow e^{ct} \left(\frac{|\xi|^2}{2} + W(q) \right) \in \mathbb{R}.$$

It is immediate to check that the Euler-Lagrange system (1.1.23) for a L as above is exactly a system of the type (1.1.19). Notice however that L is time dependent, which poses several problems and, in particular, does not allow for the simplifications described in the following paragraphs.

1.1.3.2 The Hamiltonian formalism

Given a unit mass particle under a conservative force, its *momentum* p is defined as $p(t) = q'(t)$. According to (1.1.24), one has

$$p(t) = \frac{\partial}{\partial \xi} L(q(t), q'(t)). \quad (1.1.25)$$

In the 19th century, Hamilton reformulated Lagrangian mechanics by replacing the velocity of the particle by its momentum. Moreover, notice that identity (1.1.26) suggests a way of generalizing the notion of momentum for particles under autonomous non-conservative forces. Indeed, one can set

$$p = \frac{\partial}{\partial \xi} L(q, \xi). \quad (1.1.26)$$

If, moreover, it is possible to smoothly invert (1.1.26) and find ξ given p , the *Hamiltonian* is then defined as

$$H(q, p) = \langle p, \xi \rangle - L(q, \xi) \quad (1.1.27)$$

where $\langle \cdot, \cdot \rangle$ denotes the Euclidean inner product in \mathbb{R}^k and q' is obtained from p by inverting (1.1.26). Notice that (1.1.26) and (1.1.27) state that, under the proper conditions, the Hamiltonian and the Lagrangian can be each obtained from the other via the *Legendre-Fenchel transform*. Assuming that the Euler-Lagrange equations (1.1.23) hold for $q : \overline{(a, b)} \rightarrow \mathbb{R}^k$ and choosing p as in (1.1.25), one obtains that

$$\frac{d}{dt}(H(q(t), p(t))) = 0, \quad t \in (a, b),$$

that is, the Hamiltonian is constant on the trajectories of the solutions. Notice that this is the generalization of the conservation of the mechanical energy, which we wrote in (1.1.10) for the simple pendulum and in (1.1.16) for the more general case of multiple well systems. Notice that combining (1.1.26) and (1.1.27) we obtain

$$-\frac{\partial}{\partial q} H = \frac{\partial}{\partial q} L$$

and

$$\frac{\partial}{\partial p} H = \xi.$$

Therefore, for the trajectories, and using again (1.1.23) and (1.1.25), we obtain the first order ODE system

$$\begin{cases} q'(t) = \frac{\partial}{\partial p} H(q(t), p(t)), \\ p'(t) = -\frac{\partial}{\partial q} H(q(t), p(t)). \end{cases} \quad (1.1.28)$$

Therefore, in this paragraph we have recalled that any *Lagrangian* system (as the ones we study in this thesis) can be seen as a *Hamiltonian* system in a canonical way. Hamiltonian formalism is more general than the Lagrangian one, as we can choose a Hamiltonian H and write the system (1.1.28) without referring to a Lagrangian. It is worth mentioning here that the study of existence and multiplicity of periodic orbits for the system (1.1.28), where H is defined on a symplectic manifold, is a question of major importance in calculus of variations and symplectic geometry which originated deep contributions in the late 20th century. Some more details are given later.

1.1.3.3 The Maupertuis principle

Prior to Hamilton, Maupertuis already formulated the action principle in terms of the momentum in the mid 18th century. The ideas of Maupertuis relied (as those of Fermat and others) on the least action principle. From the mathematical perspective, one of the most interesting features of the Maupertuis principle is that it provides, in the conservative case, a one to one correspondence between the extremals of Lagrangian theory and the geodesics of a certain Riemannian manifold (with possibly degenerate metric).

In order to introduce the Maupertuis principle, we do as in [76] and go back to the Hamiltonian formalism. Recall that the Hamiltonian is constant on the trajectories. Therefore, using (1.1.27) and neglecting the constant terms, we find that finding extremals for the Lagrangian action defined

in (1.1.22) is equivalent to finding extremals for the functional

$$SH_a^b(q, p) := \int_a^b \langle p(t), q'(t) \rangle dt, \quad (q, p) \in H_{\text{loc}}^1((a, b), \mathbb{R}^{2k}), \quad (1.1.29)$$

under the constraint

$$p'(t) = -\frac{\partial}{\partial q} H(q(t), p(t)), \quad t \in (a, b).$$

This transformation is essentially the Maupertuis principle.

We now focus on the particular case of conservative systems. Let L be as in (1.1.24) and q be an extremal of the corresponding action S_a^b . According to (1.1.26), the momentum writes $p = q'$ and hence the Hamiltonian action defined in (1.1.29) verifies

$$SH_a^b(q, p) = \int_a^b |p(t)| |q'(t)| dt. \quad (1.1.30)$$

Moreover, recall the choice of Lagrangian (1.1.24) and the identity (1.1.27) for H . These facts imply that

$$|p(t)| = \sqrt{2(E_T - U(q(t)))} \quad (1.1.31)$$

where E_T stands for the total mechanic energy of q , that is, the value of the Hamiltonian on the trajectory of q . Plugging (1.1.31) into (1.1.30) we get

$$SH_a^b(q, p) = \int_a^b \sqrt{2(E_T - U(q(t)))} |q'(t)| dt$$

and this expression is independent on the parametrization of the curve of the particle as well as the momentum p . This justifies the choice $(a, b) = (0, 1)$ and the definition

$$SM(q) := \int_0^1 |q'(t)|_g dt \quad (1.1.32)$$

where $|q(t)|_g = \sqrt{\sum_{i,j=1}^k g_{ij} q'_i(t) q'_j(t)}$, with

$$g_{ij} : q \in \mathbb{R}^k \rightarrow 2(E_T - U(q)) \delta_{ij} \in \mathbb{R}. \quad (1.1.33)$$

The correspondence g defined in (1.1.33) induces a Riemannian structure on the set $U_o^{E_T} := \{q \in \mathbb{R}^k : U(q) < E_T\}$ (here \mathbb{R}^k can be replaced by a manifold \mathcal{M}). The critical points of the functional SM defined in (1.1.32) (which are in bijective correspondence with the extremals of the Lagrangian action) are geodesics for the Riemannian manifold $(U_o^{E_T}, g)$. Hence the link between mechanics and the theory of geodesics.

However, in some cases of interest one might allow $U(q) = E_T$, so that the metric degenerates. For instance, consider the setting of the connecting orbits of multi-well systems. According to the discussion in (1.1.2.2), one has in this case that any connecting orbit q must satisfy $E_T = 0$ and hence the Maupertuis functional writes

$$E_M(q) := \int_0^1 \sqrt{2V(q(t))} |q'(t)| dt$$

so that the metric degenerates if and only if $V(q(t)) = 0$, i. e., when q goes through a well. This implies that the metric does not degenerate on the orbits, but it does on some other cases, e. g. curves formed as concatenation of orbits. In any case, one might aim at showing the existence of connecting orbits by dealing with the Maupertuis functional E_M , instead of the Lagrangian energy described before. More details and references are given in Section 1.2.

1.1.4 Some contemporary methods of the calculus of variations

As mentioned before, while the ideas behind the calculus of variations had been for long present in physics, it was not until the beginning of the 20th century that most of them were rigorously established from the mathematical point of view. More precisely, there was no idea on how to prove the existence of critical points of functionals except on some particular cases. For instance, if critical points of functionals were to solve an ODE (as it is the case for the celebrated brachistocrone problem) then one could aim at using results regarding existence and uniqueness for ODEs (Cauchy-Lipschitz Theorem) in order to prove the existence. However, except for very particular cases, nothing similar was possible for PDEs, meaning that general existence results were lacking even for the simplest linear PDEs such as the Laplace equation.

The major breakthrough, which unblocked the precedent situation, was to incorporate tools from measure theory and functional analysis, which were areas of mathematics that had just emerged, into the calculus of variations. The same was done for the whole field of partial differential equations during the 20th century. Rather than writing a systematic historical survey, in this section we content ourselves with giving a quick overview of some of the techniques (and issues) that appear when one aims at establishing the *existence* of solutions in the calculus of variations (that is, *critical points* of functionals). The primary concern of this thesis is to find solutions for systems of the type (1.1.18) and (1.1.19), with some suitable properties, by looking at the problem in terms of the calculus of variations, using the Lagrangian formalism introduced in Section 1.1.3.1.

Generally speaking, once one proves the existence of a solution for some problem, one wishes naturally to study its properties. In a lot of cases, critical points of functionals are solutions of partial differential equations, meaning that it is an important question to know whether such solutions belong to a function space smaller to that in which the functional is defined, so that they enjoy better properties. The previous question follows into the framework of *regularity* theory for partial differential equations. We point out that while the regularity of the solutions is not an issue in this thesis, the knowledge about other properties (such as their asymptotic behavior) is useful and interesting.

1.1.4.1 The direct method

Due to Hilbert, the so-called direct method of the calculus of variations is a somehow systematic procedure for establishing the existence of global minimizers of functionals. It allowed Hilbert to provide by the first time a rigorous existence result of a minimizer of the Dirichlet functional

$$F_D(u) := \int_{\Omega} |\nabla u|^2,$$

where $\Omega \subset \mathbb{R}^N$ is a smooth domain and u is in the set of smooth real-valued functions on Ω such that $u|_{\partial\Omega} = g$, where $g : \partial\Omega \rightarrow \mathbb{R}$ is smooth¹. Critical points of F_D are solutions of the boundary value problem

$$\begin{cases} \Delta u = 0 & \text{on } \Omega, \\ u = g & \text{on } \partial\Omega, \end{cases} \quad (1.1.34)$$

meaning that the direct method provides a solution for the Laplace equation (1.1.34).

We recall the general scheme of the direct method. It goes as follows: Let (X, \mathcal{T}) be a topological space (rich enough so that it makes sense to define convergence of sequences) and $F : X \rightarrow \mathbb{R}$.

1. Take a *minimizing sequence* for the functional. That is, a sequence $(x_n)_{n \in \mathbb{N}}$ in X such that $F(x_n) \rightarrow m$, where m is the infimum of F in X .
2. Show that, up to a subsequences, one has $x_n \rightarrow x$ in (X, \mathcal{T}) for some $x \in X$. This requires "good" properties on the topology \mathcal{T} .
3. Show that $F(x) = m$, so that x is a global minimizer of F in X . This generally follows from a *lower semicontinuity* property on F , namely that $F(x) \leq \liminf_{n \rightarrow \infty} F(x_n)$.

Loosely speaking, there is a conflict between 2. and 3., in the sense that the weaker is the topology, the more difficult is to have lower semicontinuity for the functionals. As it is well known, in the usual framework of Banach (and, in particular, Sobolev) spaces the good balance is obtained by the use of *weak topologies*, see for instance Brezis [56]. The direct method is already robust enough for treating a wide variety of problems in the calculus of variations. We refer to Dacorogna [74], Mawhin and Willem [118] and Morrey [126]. A question, more general to that of existence of critical points, is that of their *multiplicity*, which is usually a lower bound on the number of critical points. In most cases, the direct method does not allow to fully treat the problem of multiplicity of critical points, as only minimizers can be produced by it. While it is straightforward that *strictly convex* functionals defined on convex spaces (such as the Dirichlet energy F_D defined on H_g^1) possess at most one critical point, which is a global minimum, this is no longer the case for non-convex functionals. In the non-convex framework, one usually expects a much richer structure of critical points that the direct method does not capture properly. This last remark leads to the next paragraph.

1.1.4.2 Morse theory

In the 1930s, Morse [127] found a direct correspondence between the topology (in terms of Betti numbers) of a smooth finite-dimensional compact manifold \mathcal{M} and the (non-degenerate) critical points of any $f : \mathcal{M} \rightarrow \mathbb{R}$ of class C^2 , together with their *Morse indices*. The Morse index of a critical point of f is the number of negative eigenvalues of the Hessian of f at such point. Critical points are non-degenerate if the kernel of the hessian is trivial. The correspondence between these two notions is given by the celebrated Morse inequalities². Morse himself showed that this analysis extends to the study of geodesics between two fixed endpoints in a Riemannian manifold \mathcal{M} . See also the seminal book by Milnor [119]. Almost simultaneously, Lusternik and Schnirelmann [112]

¹As it is well-known from the theory of Sobolev spaces, posterior to the work of Hilbert, one might take $g \in H^{1/2}(\partial\Omega)$ and F_D is usually defined in the class of H_g^1 , consisting of the $H^1(\Omega)$ functions with trace equal to g .

²Assuming that all critical points of f are non-degenerate, which is a generic property.

also made important contribution in this direction, with roots in Birkhoff [49]. This represents a complement to Morse's work.

Morse theory was extended later to the setting of functionals on infinite-dimensional Hilbert spaces (or manifolds). This was done by Rothe [147, 148] in the early 1950s and, in more generality, by Palais and Smale [133, 134, 155] in the 1960s. In order to generalize the ideas of Morse theory to the infinite-dimensional setting, one needs to potentially face the difficulty of critical points possessing infinite Morse index and the lack of compactness for the functional³. While the former is not a problem in this dissertation, the latter is. More details will be given later, see also the book by Chang [67] for additional material.

Roughly speaking, Morse theory reduces the problem of existence of critical points of a functional to the study of the difference of topology of its level sets. That is, if X is a Banach space (or manifold) and $f : X \rightarrow \mathbb{R}$ is C^1 , for $a \in \mathbb{R}$ it is often denoted $f^a := \{x \in X : f(x) \leq a\}$. It turns out that if $a < b$ are such that f^b is not a retract by deformation of f^a , then there is $c \in [a, b]$ for which there exists a *Palais-Smale sequence* at the level c . A Palais-Smale sequence at the level c is a sequence of *almost critical points* around c . That is, a sequence $(x_n)_{n \in \mathbb{N}}$ in X such that $f(x_n) \rightarrow c$ and $Df(x_n) \rightarrow 0$ as $n \rightarrow \infty$, where Df is the differential of f . If one assumes, as in [133, 134, 155], that such sequence possesses a convergent subsequence (the so-called *Palais-Smale condition*, called condition (C) by Palais and Smale), then one finds $\bar{x} \in X$, a critical point of f with $f(\bar{x}) = c$. Notice also, in link with the previous paragraph that if f is convex, then for all $a \in \mathbb{R}$ we have that f^a is contractible, so that there is no change of topology whatsoever.

The previous remarks lead to two questions:

1. How can one find a change of topology in concrete applications?
2. Once this difference of topology is found, what can one say about the associated Palais-Smale sequences?

These two questions are, in fact, the two main questions one needs to ask for proving the existence of critical points. We give some considerations about them in the subsequent paragraph.

1.1.4.3 The mountain pass lemma

The so-called *min-max methods* provide an efficient and practical way to identify changes of topology on the level sets of a functional (question 1. in the previous paragraph), see for instance Rabinowitz [140]. It is also found that the more symmetries enjoys the functional, the more critical points one expects to find.

Here we recall the most celebrated among the min-max result, obtained in 1973 by Ambrosetti and Rabinowitz [16]. Their *mountain pass lemma* soon showed to be applicable to a vast amount of situations. We also make use of it in this thesis. There are many forms and generalizations for this result, which can be essentially thought as the first Morse inequality. We recall one of the possible formulations of the mountain pass lemma. Let X be a Banach space, $f : X \rightarrow \mathbb{R}$ of class C^1 and assume that x_0 and x_1 are two points of X such that

$$c := \inf_{p \in \Lambda} \max f(p) > \max\{f(x_0), f(x_1)\} \quad (1.1.35)$$

³However, lack of compactness can also occur in finite-dimensional problems

where

$$\Lambda := \{p \in C([0, 1], X) : p(i) = x_i \text{ for } i \in \{0, 1\}\}.$$

The property (1.1.35), see Figure 1.7, implies the existence of a Palais-Smale sequence at the level c , as there is a change of the topology of the level sets of f around c . In a wide variety of situations, e. g. when f verifies the Palais-Smale condition, this is already enough for having the existence of a critical point for f at the level c . Let us give an example of one of such situations, which is linked with the problems of this thesis.

Example 1.1.1. Let $V : \mathcal{M} \rightarrow \mathbb{R}$ a non-negative multi-well potential on a k -dimensional Riemannian manifold \mathcal{M} , with non-degenerate set of minimizers Σ . For $T > 0$, consider the problem of finding *periodic orbits* of period T for V . Recall that periodic orbits appeared when studying the simple pendulum in Section 1.1.2.2, so we are just considering the more general situation of a multi-well system.

$$\begin{cases} q'' = \nabla V(q), & \text{in } [-T, T], \\ q(-T) = q(T). \end{cases} \quad (1.1.36)$$

Notice that the constant function $t \in [-T, T] \rightarrow \sigma \in \mathcal{M}$ is a solution to (1.1.36) for any $\sigma \in \Sigma$. The mountain pass lemma allows to easily prove that there exist non-constant solutions of (1.1.36), and that for all $T > 0$. For that purpose, consider the functional

$$E_T(q) := \int_{-T}^T \left[\frac{|q'(t)|^2}{2} + V(q(t)) \right] dt$$

defined on the space of H^1 functions in $[-T, T]$ with values in \mathcal{M} and periodic boundary conditions. We denote such space as $H_{\text{per}}^1([-T, T], \mathcal{M})$. It is standard to show that E_T is a C^1 functional in $H_{\text{per}}^1([-T, T], \mathcal{M})$. Moreover, it is also standard to show that E_T satisfies the Palais-Smale condition. Such facts allow to easily show that there exists a non-constant solution of (1.1.36) by showing that (1.1.35) holds for two well chosen points. Indeed, take $\sigma \in \Sigma$ and consider

$$\mathfrak{c}_T := \inf_{\gamma_T \in \Gamma_T} \max_{s \in [0, 1]} E_T(\gamma_T(s))$$

where

$$\Gamma_T := \{\gamma_T \in C([0, 1], H_{\text{per}}^1([-T, T], \mathcal{M})) : \gamma_T(0) = \sigma, \gamma_T(1) \in \Sigma \setminus \{\sigma\}\}.$$

It is easy to show that $\mathfrak{c}_T > 0$, using that the wells are non-degenerate. Since E_T satisfies the Palais-Smale condition, we obtain the existence of $q_T \in H_{\text{per}}^1([-T, T], \mathcal{M})$ such that $E_T(q_T) = \mathfrak{c}_T$ and $DE_T(q_T) = 0$. The latter implies that q_T solves (1.1.36). Since the only constant solutions of (1.1.36) have zero energy, the fact that $E_T(q_T) > 0$ implies that q_T is not constant.

Notice that this argument is independent of the choice of the manifold \mathcal{M} and in particular holds for manifolds with trivial topology, such as $\mathcal{M} = \mathbb{R}^k$. The reason is that the mountain pass argument only requires the multi-well structure of V , as we see by looking again at the definition of the class of paths Γ_T .

Furthermore, it is known that under suitable non-degeneracy properties and assuming the Palais-Smale condition holds, the mountain pass lemma allows to show the existence of a critical point of Morse index one, Hofer [95] and Solimini [160]. Estimates, or identities, for the Morse

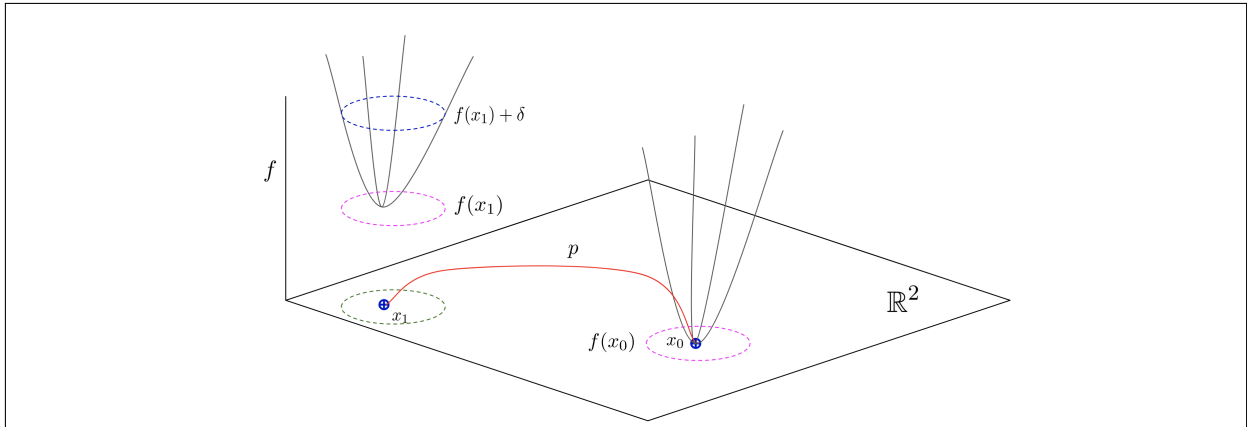


Figure 1.7: Illustration of a typical mountain pass situation, for f defined on a subset of \mathbb{R}^2 . Here, the points x_0 and x_1 are strict local minimizers of f , with $f(x_1) > f(x_0)$ (we also could have chosen $f(x_0) = f(x_1)$). Any path p which goes joins x_0 and x_1 must go through the level $f(x_1) + \delta$, where $\delta > 0$. Therefore, (1.1.35) holds, which means that the mountain pass lemma applies.

indices are also available for critical points obtained by other methods of min-max type. We refer to Lazer and Solimi [107], Ghoussoub [87] and the references therein.

1.1.4.4 Lack of compactness

A significant amount of problems have (some sort of) Palais-Smale condition, which enables for a direct approach to the problem of multiplicity of critical points as in Example 1.1.1. However, soon after the work of Palais and Smale it became clear that an important number of problems, much of them of great interest in mathematics and physics, do not satisfy a compactness condition as the Palais-Smale condition. In other words, there is a *lack of compactness* for these problems. Despite this fact, it is still possible to obtain some information by studying how compactness is lost, that is, how Palais-Smale sequences *diverge*. This is one of the main difficulties of this thesis, as all the problems we face present some sort of default of compactness.

It is worth noticing that when one deals with minimizing sequences, it is sometimes possible to show that compactness is restored by optimality. For instance, by proving that phenomena such as vanishing and mass splitting are not possible or contradict the minimality of the sequence. This type of reasoning was termed by Lions [108] as the *concentration-compactness method*. In contrast, if one deals with an arbitrary Palais-Smale sequence, then one cannot use minimality arguments in order to restore compactness. The analysis becomes then more involved, but the ideas of concentration-compactness extend to some degree. Roughly speaking, one needs to see if it is still possible to establish the existence of new solutions by looking at the asymptotic behavior of the Palais-Smale sequences. Alternatively, one can also perturb the problem in order to restore compactness and then pass to the limit as the perturbation (depending on a parameter) becomes negligible. These ways of arguing work in a lot of situations, as in the Chapter 2 of this thesis. Without being exhaustive, let us cite some of the early papers which rely on this type of reasoning, together with the problem the authors were studying. We have Aubin [21] for the Yamabe problem; Sacks and Uhlenbeck [150] for harmonic maps; Taubes [166, 165] for the Yang-Mills problem; Brezis and Coron [57] for the H-problem (which contains the Plateau problem); Struwe [164], Brezis and Nirenberg [58] for nonlinear boundary value problems with critical Sobolev exponent.

As we can see, these are a wide variety of different problems in mathematics, with roots and important applications in physics.

It is still possible to further refine the previous approach in order to deal with lack of compactness. This idea is due to Bahri [23, 22] and his collaborators. In some sense, it can be rooted back to the work of Conley [70], even though the main aim of Conley was not to deal with compactness issues. Given a Hilbert manifold X and a functional $f : X \rightarrow \mathbb{R}$, one looks at the negative *gradient flow*

$$\begin{cases} x'(t) = -\nabla f(x(t)), & t \in (0, +\infty), \\ x(0) = x_0, \end{cases} \quad (1.1.37)$$

where ∇f stands for the gradient of f . One has that if $a < b$ are such that f^a is not a retract by deformation of f^b , then there exists $x_0 \in f^b \setminus f^a$ such that if $x(\cdot)$ is the associated solution of (1.1.37) it holds that for all $t \in (0, +\infty)$, $f(x(t)) \geq a$. The function $t \in [0, +\infty) \rightarrow f(x(t))$ is decreasing, which means that it has a limit $c \in [a, b]$ as $t \rightarrow +\infty$. Moreover, $|\nabla f(x(\cdot))|^2$ is integrable in $[0, +\infty)$. Therefore, we find a sequence $(t_n)_{n \in \mathbb{N}}$ such that $t_n \rightarrow +\infty$ and $\nabla f(x(t_n)) \rightarrow 0$ as $n \rightarrow \infty$. That is, the sequence $(x(t_n))_{n \in \mathbb{N}}$ is a Palais-Smale sequence for f at the level c . However, it is not just an arbitrary Palais-Smale sequence, as it remains in the same flow line of (1.1.37). As showed by Bahri and collaborators, this additional property can be crucial in order to obtain existence and multiplicity results in situations in which dealing with arbitrary Palais-Smale sequences does not allow to conclude. The asymptotic patterns associated with the Palais-Smale sequences lying in the same flow line were called by Bahri *critical points at infinity*. Clearly, such patterns coincide with a usual critical point in case the sequence is convergent. We also mention that usually, a modified version of the flow, rather than the plain gradient flow, is considered in order to strengthen the argument.

Some examples of (difficult) problems in which this technique is used are Bahri and Coron [25, 26] concerning the nonlinear boundary problem with critical Sobolev exponent and the Kazdan-Warner problem in differential geometry; Bahri and Rabinowitz [27] for the three body problem and Bahri [24] dealing with questions in contact geometry. We do not use these techniques in this thesis, but, as we detail in Section 1.4, we think that ideas in this spirit could be relevant and lead to a possible continuation of this thesis.

1.1.4.5 Other problems: Periodic orbits for Hamiltonian systems

As a concluding historical remark, tangential to this thesis, we mention that there exists a huge amount of literature devoted to the study of periodic orbits for a wide variety of Hamiltonian systems of the type (1.1.28). See for instance Ekeland [77] and Mawhin and Willem [118]. The existence of periodic orbits is not a question in this thesis, as we restrict ourselves to connecting orbits. However, in order to complete the first part of our introduction, we believe it can be interesting to give a brief and non-exhaustive historic account on the subject of periodic orbits of Hamiltonian systems.

In the case of general Hamiltonian systems defined on symplectic manifolds, the celebrated Arnold's conjecture claims that the number of periodic orbits and their index can be described by means of classical Morse theory. That is, one must have the Morse inequalities for the periodic orbits, as one has for a smooth real function defined on the manifold or for the geodesics of a

Riemannian manifold. For several reasons, such a problem is very degenerate from the point of view of the calculus of variations as, for instance, periodic orbits have infinite Morse index⁴. Hence, Arnold's conjecture was considered as intractable for a number of years and new ideas were needed.

Very deep and influential work by Floer [81] provided a proof of a very general subcase of Arnold's conjecture. His approach relied on completely new tools which, in particular, involve replacing the notion of Morse index by a novel one. A proof of Arnold's conjecture on the torus had been provided a few years before by Conley and Zehnder [72], who also relied on a new definition of index, see also [71] by the same authors. Other important contributions related to Arnold's conjecture are due to Hofer [96] and Sikorav [154]. Floer was also influenced by previous works on the Weinstein conjecture [170], regarding the existence of periodic orbits in the framework of contact geometry. Some of these works are due to Weinstein himself [171] (for the case convex hypersurfaces) and Rabinowitz [141, 139] (for the case of starshaped hypersurfaces). Some other important contributions are due to Viterbo [168] and Hofer [97]. For the details, results, discussions and further references we refer to the book by Hofer and Zehnder [98].

1.2 State of the art prior to this thesis

We review the main literature which precedes this thesis in its most immediate way. We also set up the main notations that will be used consistently on the rest of the thesis.

1.2.1 Multiple well systems

We will consider $V : \mathbb{R}^k \rightarrow \mathbb{R}$ a non-negative multi-well potential with non-degenerate minima. We also assume that V is not zero at infinity and some other technical conditions which will be made explicitly in the each of the chapters. Let $\Sigma := \{V = 0\}$ be the set of wells. Recall that we consider the second order ODE system

$$q'' = \nabla V(q), \text{ in } \mathbb{R}. \quad (1.2.1)$$

As advanced before, we are interested in solutions of (1.2.1) which arise from a variational context. More precisely, as (1.2.1) is formally the *Euler-Langrange* equation of the energy (Lagrangian action)

$$E(q) := \int_{\mathbb{R}} \left[\frac{|q'(t)|^2}{2} + V(q(t)) \right] dt$$

we seek for solutions of (1.2.1) which are *critical points* of E , in a suitable functional setting.

1.2.2 Functional setting

In order to study the critical points of E , we need to know in which functional spaces it is defined as a C^1 functional. Let us recall how this can be worked out. Consider the standard Sobolev space

$$H_{\text{loc}}^1(\mathbb{R}, \mathbb{R}^k) := \{q : \mathbb{R} \rightarrow \mathbb{R}^k : \forall K \subset \mathbb{R} \text{ compact } q \in H^1(K, \mathbb{R}^k)\}.$$

⁴Recall that, as we mentioned earlier, we do not face such a problem in this thesis.

Clearly, the energy E is well-defined as a mapping from $H_{\text{loc}}^1(\mathbb{R}, \mathbb{R}^k)$ to $[0, +\infty]$. Furthermore, if $q \in H_{\text{loc}}^1(\mathbb{R}, \mathbb{R}^k)$ is such that

$$E(q) < +\infty,$$

then it follows easily that $q' \in L^2(\mathbb{R}, \mathbb{R}^k)$. The classical Sobolev embeddings imply that $H_{\text{loc}}^1(\mathbb{R}, \mathbb{R}^k) \subset \mathcal{C}^{0,1/2}(\mathbb{R}, \mathbb{R}^k)$, where the latter is the standard Hölder space with exponent $1/2$. In particular, functions in $H_{\text{loc}}^1(\mathbb{R}, \mathbb{R}^k)$ are continuous, a fact which we will always use implicitly. Under mild assumptions on the potential V (namely, that it does not decay too fast to 0 at ∞), it is easy to show that

$$\lim_{t \rightarrow \pm\infty} V(q(t)) = 0,$$

which, by the continuity of q , is equivalent to $\lim_{t \rightarrow \pm\infty} q(t) = \sigma^\pm(q)$ for some $\sigma^\pm(q) \in \Sigma$. In other words, we have that

$$\mathcal{D} := \{q \in H_{\text{loc}}^1(\mathbb{R}, \mathbb{R}^k) : E(q) < +\infty\} = \bigcup_{(\sigma, \tilde{\sigma}) \in \Sigma^2} X(\sigma, \tilde{\sigma}), \quad (1.2.2)$$

where, for $(\sigma, \tilde{\sigma}) \in \Sigma^2$,

$$X(\sigma, \tilde{\sigma}) := \{q \in H_{\text{loc}}^1(\mathbb{R}, \mathbb{R}^k) : E(q) < +\infty, \lim_{t \rightarrow -\infty} q(t) = \sigma \text{ and } \lim_{t \rightarrow +\infty} q(t) = \tilde{\sigma}\}. \quad (1.2.3)$$

The appearance of the spaces $X(\sigma, \tilde{\sigma})$ can be traced back to Rabinowitz [136]. The importance of identity (1.2.2) is that it provides a decomposition of the energy space in l^2 disjoint connected components (where l is the number of wells) which are exactly the class of $H_{\text{loc}}^1(\mathbb{R}, \mathbb{R}^k)$ curves with finite energy joining two different wells. It also shows that critical points of E (once one is able to define such a notion) are necessarily *connecting orbits*, that already appeared before.

Notice also that each component of \mathcal{D} as well as E are invariant by the action of the translation group. That is, for all $(\sigma, \tilde{\sigma}) \in \Sigma^2$, $\tau \in \mathbb{R}$ and $q \in X(\sigma, \tilde{\sigma})$ we have that $q(\cdot + \tau) : t \in \mathbb{R} \rightarrow q(t + \tau) \in \mathbb{R}^k$ verifies $q(\cdot + \tau) \in X(\sigma, \tilde{\sigma})$ and $E(q(\cdot + \tau)) = E(q)$.

Another important assumption that we make in this thesis is that the elements of Σ are non-degenerate global minimizers. Such an assumption implies that

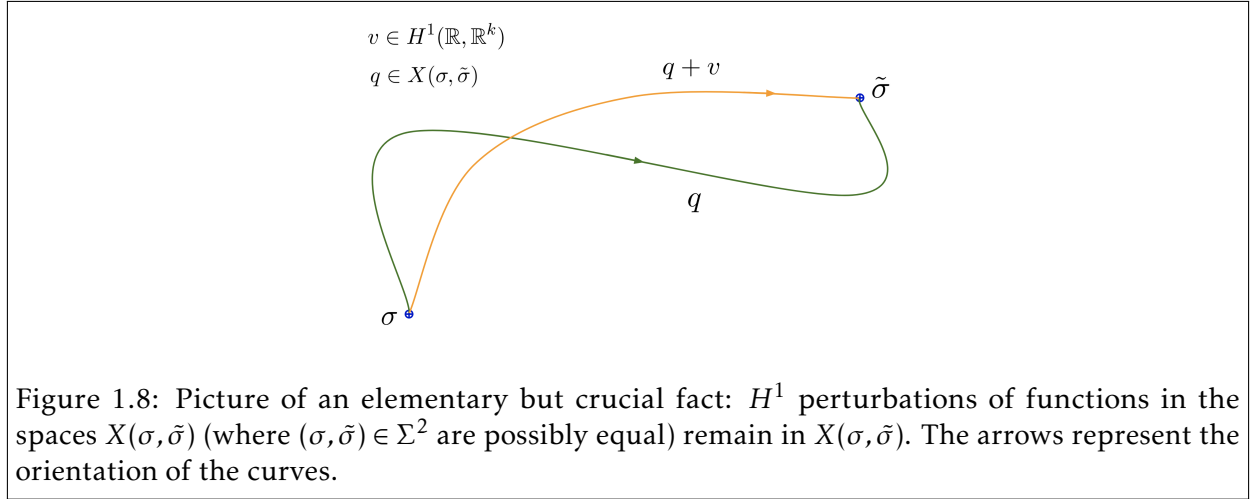
$$V(\mathbf{u}) \sim \beta_\sigma |\mathbf{u} - \sigma|^2 \quad (1.2.4)$$

in a sufficiently small neighborhood of $\sigma \in \Sigma$, with $\beta_\sigma > 0$. A combination of (1.2.3) and (1.2.4) gives that for any $(\sigma, \tilde{\sigma}) \in \Sigma$ and q_1, q_2 in $X(\sigma, \tilde{\sigma})$ it holds $q_1 - q_2 \in H^1(\mathbb{R}, \mathbb{R}^k)$, as well as $q_1 + v \in X(\sigma, \tilde{\sigma})$ whenever $v \in H^1(\mathbb{R}, \mathbb{R}^k)$. It is worth to recall here that if $v \in H^1(\mathbb{R}, \mathbb{R}^k)$ then

$$\lim_{t \rightarrow \pm\infty} v(t) = 0.$$

The previous considerations imply that the spaces $X(\sigma, \tilde{\sigma})$ are *affine spaces* obtained from $H^1(\mathbb{R}, \mathbb{R}^k)$. That is, if one chooses a connected component of the energy space (an element on the right-hand side in (1.2.2)) and moves the center accordingly, then such a component is $H^1(\mathbb{R}, \mathbb{R}^k)$, where E is well-defined and finite. See Figure 1.8.

At this point, we have all the ingredients for completing the description of the functional setting and, in particular, for defining the notion of critical point for E . Indeed, we have that for



any $\chi \in \mathcal{D}$ we can define the *perturbed functional*

$$J_\chi : v \in H^1(\mathbb{R}, \mathbb{R}^k) \rightarrow E(\chi + v) \in [0, +\infty) \quad (1.2.5)$$

and it is routine to check that J is a C^1 (or even C^2 , if V is smooth enough) functional on the linear space $H^1(\mathbb{R}, \mathbb{R}^k)$, see Bisgard [50]; Montecchiari and Rabinowitz [123]. The differential DJ_χ of J_χ at $v \in H^1(\mathbb{R}, \mathbb{R}^k)$ is the bounded linear form given by

$$DJ_\chi(v) : w \in H^1(\mathbb{R}, \mathbb{R}^k) \rightarrow \int_{\mathbb{R}} (\langle v'(t) + \chi'(t), w'(t) \rangle + \langle \nabla V(v(t) + \chi(t)), w(t) \rangle) dt.$$

In particular, the meaning of *critical point of J_χ* is unambiguous and standard. It is also clear that if $v \in H^1(\mathbb{R}, \mathbb{R}^k)$ is a critical point of J_χ for some $\chi \in \mathcal{D}$, then $v + \chi$ solves (1.2.1). This leads to define the notion of critical point of E as follows: for $q \in \mathcal{D}$, the differential of E at q is $DJ_q(0)$, and q is a critical point of E if $DJ_q(0) = 0 \in H^{-1}(\mathbb{R}, \mathbb{R}^k)$ (that is, if $0 \in H^1(\mathbb{R}, \mathbb{R}^k)$ is a critical point of J_q).

As a side note, it is worth recalling here that one does not need a functional to be defined on a linear space in order to have a notion of critical point, nor for using variational methods with the intention of obtaining existence results for such critical points. As for the finite-dimensional realm, there exists a standard notion of *infinite-dimensional manifold* modeled after a base Hilbert space. Naturally, the same goes for the notion of differential and critical points for functionals defined on such manifolds. We refer for instance to Palais [133] and the references therein. In particular, one can see E as a functional defined on the Hilbert manifold \mathcal{D} and consider the derived notions of differential and critical point as given by the theory presented in [133]. The objects that are obtained from such an approach are equivalent to those we defined above via the perturbed functionals. It seems preferable to us to follow the elementary treatment that we presented here instead of invoking the machinery considered in [133], which allows for manifolds and functionals that do not permit the straightforward reduction to a linear space that we are able to perform in our setting.

In order to conclude this paragraph, let us discuss the relevance of the non-degeneracy assumption on the wells in the functional setting for E . It is clear that (1.2.4) is essential in order to show that for all $(\sigma, \tilde{\sigma}) \in \Sigma$ and q_1, q_2 in $X(\sigma, \tilde{\sigma})$ it holds $q_1 - q_2 \in L^2(\mathbb{R}, \mathbb{R}^k)$, which implies that $q_1 - q_2 \in H^1(\mathbb{R}, \mathbb{R}^k)$ because $q' \in L^2(\mathbb{R}, \mathbb{R}^k)$ whenever $q \in \mathcal{D}$. Therefore, it seems natural to wonder

about the natural functional setting for E whenever (1.2.4) is relaxed. In order for (1.2.2) to be valid, a sufficient condition is

$$\inf \left\{ V(\mathbf{u}) : \mathbf{u} \in \mathbb{R}^k \setminus \left(\bigcup_{\sigma \in \Sigma} B(\sigma, \varepsilon) \right) \right\} > 0 \quad (1.2.6)$$

for all $\varepsilon > 0$, where $B(\mathbf{u}, r)$ stands for the open ball of center \mathbf{u} and radius $r > 0$ in \mathbb{R}^k . It is clear that (1.2.6) can hold in case some $\sigma \in \Sigma$ is degenerate. Assume for instance that for all $\sigma \in \Sigma$ there exists $p_\sigma \geq 2$ such that

$$V(\mathbf{u}) \sim \beta_\sigma |\mathbf{u} - \sigma|^{p_\sigma}, \quad (1.2.7)$$

which is more general than (1.2.4). In particular, if (1.2.7) holds for σ , then σ is degenerate if and only if $p_\sigma > 2$. Under the previous assumption, and provided that V has a good behavior at infinity, we have that (1.2.6) holds, which implies that (1.2.2) holds. Hence, if $(\sigma, \tilde{\sigma}) \in \Sigma^2$ and we assume $p_\sigma = p_{\tilde{\sigma}} = p > 2$ for simplicity, we have that for $(q_1, q_2) \in X(\sigma, \tilde{\sigma})^2$ it holds $q_1 - q_2 \in L^p(\mathbb{R}, \mathbb{R}^k)$. As before, we can also show that if we set

$$B_p := \{v \in L^p(\mathbb{R}, \mathbb{R}^k) : v' \in L^2(\mathbb{R}, \mathbb{R}^k)\}$$

we obtain that $X(\sigma, \tilde{\sigma}) = \{q\} + H_p$ for all $q \in X(\sigma, \tilde{\sigma})$. We endow H_p with the norm

$$\|v\|_{B_p} := \|v\|_{L^p(\mathbb{R}, \mathbb{R}^k)} + \|v'\|_{L^2(\mathbb{R}, \mathbb{R}^k)}.$$

It is then clear that if (1.2.7) holds instead of (1.2.4), then one can replace the Hilbert space $H^1(\mathbb{R}, \mathbb{R}^k)$ by the Banach space B_p in the discussion above and obtain an analogous functional setting. In particular, for any $\chi \in \mathcal{D}$ one can consider the perturbed functional J_χ as in (1.2.5), replacing $H^1(\mathbb{R}, \mathbb{R}^k)$ by B_p as space domain. It would be interesting to know to what extent the results concerned by this section (namely, existence of connecting orbits) can be extended to this degenerate framework. For instance, it seems clear for us that most of the results of Chapter 2 (which corresponds to our paper [131]) still hold under (1.2.7), up to possible technical subtleties. The reason is that we do not see why the fact that the functional setting is based on the Hilbert space $H^1(\mathbb{R}, \mathbb{R}^k)$ should play any major role, and hence we conjecture that most arguments still hold if one works in B_p instead of $H^1(\mathbb{R}, \mathbb{R}^k)$.

1.2.3 Existence of connecting orbits: the minimizing case

With identity (1.2.2) in mind, the first natural idea in order to obtain connecting orbits for (1.2.1) is to study the minimization of the energy E in each connected component of the energy space \mathcal{D} . In other words, for each $(\sigma, \tilde{\sigma}) \in \Sigma^2$, one considers the value

$$\mathfrak{m}_{\sigma\tilde{\sigma}} := \inf_{q \in X(\sigma, \tilde{\sigma})} E(q) \quad (1.2.8)$$

and studies existence of the minimization problem. Lack of compactness implies that existence of a minimizer is not obvious, and indeed one might have that (1.2.8) is not attained for some pair $(\sigma, \tilde{\sigma}) \in \Sigma$. Some examples are given in Alikakos, Betelú and Chen [9]. It is clear that $\mathfrak{m}_{\sigma\tilde{\sigma}} \geq 0$ and one shows that $\mathfrak{m}_{\sigma\tilde{\sigma}} = 0$ if and only if $\sigma = \tilde{\sigma}$. In the latter case, the infimum in (1.2.8) is attained by

the constant homoclinic equal to σ . In the case $\sigma \neq \tilde{\sigma}$, a sufficient condition for the infimum in (1.2.8) to be attained is

$$m_{\sigma\tilde{\sigma}} < \min\{m_{\sigma\hat{\sigma}} + m_{\hat{\sigma}\tilde{\sigma}} : \hat{\sigma} \in \Sigma \setminus \{\sigma, \tilde{\sigma}\}\} \quad (1.2.9)$$

with the standard convention $\inf \emptyset = +\infty$. In fact, (1.2.9) is a strict triangle's inequality with respect to the geodesic distance induced by \sqrt{V} , according to the Maupertuis principle. In particular, we have that if V is a double-well potential, then there always exists a globally minimizing heteroclinic orbit.

All the previous facts are well-known since long before this thesis was written. A proof can be given by using concentration-compactness arguments. Roughly speaking, (1.2.9) implies that there is no *dichotomy* (in the sense of Lions [108]) for the minimizing sequences of (1.2.8) and one can rule out *vanishing* even without (1.2.9). It then follows that minimizing sequences are *compact*, which allows to obtain a uniform control on their behavior at infinity and hence conclude.

With respect to the available literature with for this minimization problem, we refer to Bolotin [51], Bolotin and Kozlov [52], Bertotti and Montecchiari [40] and Rabinowitz [138, 142]. It also holds that if (1.2.9) is satisfied then minimizing sequences for (1.2.8) are compact i. e., they possess a strongly convergent subsequence up to translations. This last property has several important consequences in some PDE problems, as we will recall in Section 1.2.6.

More recently, Monteil and Santambrogio [124] (see also [125] by the same authors, in which they extend their results to possibly infinite dimensional settings); Zuñiga and Sternberg [174] showed that rather than solving directly (1.2.8) by a concentration-compactness scheme, one can instead tackle it by using the Maupertuis principle, that we introduced in Section 1.1.3.3. We give more details in Section 1.4.2. It is worth mentioning that, in the multi-well setting, the degenerate geodesic distance induced by \sqrt{V} had been used before by several authors: see Modica and Mortola [121, 120], Baldo [28], Sternberg [162, 163], Fonseca and Tartar [82]. The context of these works is different, as a singular perturbation problem is studied (more details in Section 1.4.7).

Even another approach was taken by Alikakos and Fusco [11], which considered a family of constrained problems depending on a parameter (for which minimizers exists as one has restored compactness) and then showing that for suitable values of the parameter the constraints are not met, meaning that one has a solution for the unconstrained problem (1.2.8). In fact, we have used their idea in Chapters 3 and 4 of this thesis.

1.2.4 Existence of connecting orbits: the non-minimizing case

There exists a vast literature concerning the existence of connecting orbits for general Hamiltonian systems of the type

$$q''(t) = \nabla V_{\star}(t, q(t)), \quad t \in \mathbb{R}, \quad (1.2.10)$$

where the potential V_{\star} might take various forms. A significant part of this literature is based on the use of *variational methods*. Such variational methods are not only used for showing the existence of global minimizers (as we described for instance in the previous section) but have also been extensively applied to the obtention of *non-minimizing* connecting orbits. Generally speaking, such methods are essentially those of Morse theory (in a lot of cases, the mountain pass lemma of Ambrosetti and Rabinowitz [16]), that we briefly introduced before. With these methods, plenty of results concerning the existence of non-minimizing connecting orbits in systems which can be

written as (1.2.10) have been obtained. Some of the earlier references are Ambrosetti and Coti Zelati [15]; Caldirola and Montecchiari [63]; Coti Zelati and Rabinowitz [73] and Rabinowitz [137, 136].

In all the papers we mentioned above, the potential V_\star is not of multi-well type. Therefore, the approach that one needs to take as well as the difficulties present are not the same as the ones we are facing in this thesis. Nevertheless, there exist some works which are closer to the goals of this thesis. Such papers study *non-autonomous* systems as (1.2.10) with potentials V_\star which are 1-periodic in time and of multi-well type, with t -independent zero set. These references are Bisgard [50] and Montecchiari and Rabinowitz [123, 122]. In particular, the second chapter of the PhD thesis [50] is close in spirit to the results we obtained in our thesis. However, as it will be made clear later, in all such previous works is essential that V_\star is non-autonomous, that is, the autonomous case is excluded by the hypothesis that they make.

As a consequence, there are no results prior to this thesis which show the existence of non-minimizing connecting orbits for *autonomous* multi-well systems as (1.2.1). In other words, the results of Chapter 2, in which we show that, under suitable assumptions, there exist non-minimizing connecting orbits for the system (1.2.1), are the first of this kind for (1.2.1). However, the methods of our proof strongly rely on all the vast literature mentioned before and, more particularly, they are closely related to those used in [50]. More details are given in Section 1.3, as well as Chapter 2.

1.2.5 Behavior of the Palais-Smale sequences of E

In [136], *Proposition 3.10*, Rabinowitz performed a concentration-compactness analysis for arbitrary Palais-Smale sequences of the energy. Essentially, he showed that Palais-Smale sequences formally converge to a chain of connecting orbits. Moreover, the sum of the energies of the elements of the chain coincides with the level of the Palais-Smale sequence. In fact, Rabinowitz result concerns non-autonomous potentials V_\star as in Section 1.2.4, but it also applies to our autonomous setting. More precisely, let $\chi \in \mathcal{D}$ and consider the functional J_χ introduced in Section 1.2.2. Let $(v_n)_{n \in \mathbb{N}}$ be a Palais-Smale sequence for J_χ with level $c \geq 0$. Rabinowitz results states that there exist $\ell \geq 1$, $\{q_1, \dots, q_\ell\}$ connecting orbits and $((\tau_{1,n}, \dots, \tau_{\ell,n}))_{n \in \mathbb{N}}$ in \mathbb{R}^ℓ such that, up to an extraction of $(v_n)_{n \in \mathbb{N}}$,

1. For all $i \in \{1, \dots, \ell - 1\}$ we have $\tau_{i+1,n} - \tau_{i,n} \rightarrow +\infty$ as $n \rightarrow +\infty$.
2. There exists a sequence $((t_{0,n}, \dots, t_{\ell,n}))_{n \in \mathbb{N}}$ in $[-\infty, +\infty]^\ell$ such that $t_{0,n} = -\infty$, $t_{\ell,n} = +\infty$, $t_{i+1,n} - t_{i,n} \rightarrow +\infty$ as $n \rightarrow \infty$ and for $i \in \{1, \dots, \ell\}$

$$\lim_{n \rightarrow \infty} \|v_n(\cdot + \tau_{i,n}) + \chi - q_i\|_{H^1([t_{i-1,n}, t_{i,n}], \mathbb{R}^k)} = 0.$$

3. For all $i \in \{1, \dots, \ell\}$ we have

$$\lim_{t \rightarrow -\infty} q_1(t) = \lim_{t \rightarrow -\infty} \chi(t), \quad \lim_{t \rightarrow +\infty} q_i(t) = \lim_{t \rightarrow -\infty} q_{i+1}(t), \quad \lim_{t \rightarrow +\infty} q_\ell(t) = \lim_{t \rightarrow +\infty} \chi(t).$$

4. $c = \sum_{i=1}^{\ell} E(q_i)$.

We call the chain $\{q_1, \dots, q_\ell\}$ a *generalized* critical point for J_χ . This terminology has been used by other authors in different contexts.

To sum up, despite the fact that the Palais-Smale condition is not satisfied by the energy, the behavior of its Palais-Smale sequences is known. Hence, there is some hope for establishing the existence of new connecting orbits by using variational methods. This is the object of Chapter 2 in this thesis.

1.2.6 Two-dimensional heteroclinics

In their celebrated paper, Alama, Bronsard and Gui [3] showed the existence of a solution of the equation

$$\Delta \mathfrak{U} = \nabla V(\mathfrak{U}) \text{ in } \mathbb{R}^2$$

satisfying the conditions at infinity,

$$\begin{cases} \lim_{x_2 \rightarrow \pm\infty} \|\mathfrak{U}(\cdot, x_2) - \sigma^\pm\|_{L^\infty(\mathbb{R}, \mathbb{R}^k)} = 0, \\ \lim_{x_1 \rightarrow \pm\infty} \|\mathfrak{U}(x_1, \cdot) - \mathfrak{q}^\pm\|_{L^\infty(\mathbb{R}, \mathbb{R}^k)} = 0. \end{cases} \quad (1.2.11)$$

In (1.2.11), σ^\pm are two different wells of V and \mathfrak{q}^\pm two globally minimizing heteroclinics connecting σ^- to σ^+ , different up to translations. Hence, it is implicitly assumed that $V : \mathbb{R}^k \rightarrow \mathbb{R}$ with $k \geq 2$, as otherwise heteroclinics between two fixed wells are unique up to translations. Moreover, this is the only point in which the assumption $k \geq 2$ is needed. The terminology *2D heteroclinic* is justified by the fact that, when seen as a curve $x_1 \in \mathbb{R} \rightarrow \mathfrak{U}(x_1, \cdot)$ and $x_2 \in \mathbb{R} \rightarrow \mathfrak{U}(\cdot, x_2)$, \mathfrak{U} connects two different stable states (which are also curves) as $x_1 \rightarrow \pm\infty$ and $x_2 \rightarrow \pm\infty$, respectively.

Among other things, the results in [3] provide, in the framework of Allen-Cahn systems, a counterexample to the so called *De Giorgi conjecture* for the *scalar* Allen-Cahn equation. The De Giorgi conjecture setting considers any entire solution of

$$\Delta u = u^3 - u \text{ in } \mathbb{R}^N$$

with $N \leq 8$ and satisfying some monotonicity conditions. The conjecture claims that the level sets of u are hyperplanes. Partial results have been obtained by several authors, see Ghoussoub and Gui [88] (for $N = 2$) Alberti, Ambrosio and Cabré [5], Ambrosio and Cabré [17] (for $N = 3$), Savin [152] (for any $N \leq 8$, under additional assumptions). Notice that, due to (1.2.11), the level sets of the solution \mathfrak{U} are not hyperplanes.

In [3], the existence of \mathfrak{U} is proven by variational methods. More precisely, \mathfrak{U} minimizes the energy

$$E_2(U) := \int_{\mathbb{R}} \left[\int_{\mathbb{R}} \frac{|\partial_{x_1} U(x_1, x_2)|^2}{2} dx_2 + \int_{\mathbb{R}} \left(\frac{|\partial_{x_2} U(x_1, x_2)|^2}{2} + V(U(x_1, x_2)) - \mathfrak{m} \right) dx_2 \right] dx_1,$$

among the class of functions in $H_{\text{loc}}^1(\mathbb{R}^2, \mathbb{R}^k)$ which verify (1.2.11). The quantity $\mathfrak{m} := \mathfrak{m}_{\sigma^+ - \sigma^-}$ (see (1.2.8)) is subtracted so that E_2 is finite. The invariance by translations of E_2 with respect to the x_2 directions represents a difficulty, which we also face in this thesis. In [3], this is dealt with by imposing a symmetry assumption on V . More precisely, that σ^\pm are exchanged by a reflection with respect to an hyperplane⁵. This symmetry assumption was later replaced by Schatzman [153] by

⁵In fact, in [3] restrict to the case $k = 2$, but the argument readily extends to $k \geq 3$.

the assumption that the minimizers q^\pm are non-degenerate up to translations. That is, that $D^2E(q^+)$ has one-dimensional kernel, which is the minimal dimension due to the invariance by translations. As expected, such an assumption is generic.

A key point, raised by Monteil and Santambrogio [125], is that the previous problem can be formulated as a heteroclinic orbit one for curves which take values in an *infinite-dimensional* configuration space, with $E - m$ as potential. More precisely, \mathfrak{U} is seen as a curve $x_1 \in \mathbb{R} \rightarrow \mathfrak{U}(x_1, \cdot) \in X(\sigma^-, \sigma^+)$ and E_2 rewritten as

$$E_2(U) = \int_{\mathbb{R}} \left[\frac{\|\partial_{x_1} U(x_1)\|_{L^2(\mathbb{R}, \mathbb{R}^k)}^2}{2} + (E(U(x_1)) - m) \right] dx_1 \quad (1.2.12)$$

where we have identified $H_{\text{loc}}^1(\mathbb{R}^2, \mathbb{R}^k)$ with $H_{\text{loc}}^1(\mathbb{R}, L_{\text{loc}}^2(\mathbb{R}, \mathbb{R}^k)) \cap L_{\text{loc}}^2(\mathbb{R}, H_{\text{loc}}^1(\mathbb{R}, \mathbb{R}^k))$. Furthermore, notice that $E - m$ is a nonnegative functional on $X(\sigma^-, \sigma^+)$, vanishing exactly on the corresponding set of globally minimizing heteroclinics (in particular, q^- and q^+). Therefore, \mathfrak{U} can be seen as a heteroclinic in $X(\sigma^-, \sigma^+)$ with $E - m$ as a potential, which is moreover minimizing with respect to the set of all curves in $X(\sigma^-, \sigma^+)$ which connect q^- and q^+ .

As said in Section 1.1.3.3, in [125] the authors follow the Maupertuis principle and formulate an abstract geodesic problem with degenerate Riemannian metric given by $\sqrt{V_X}$ defined on a metric space X (instead of simply \mathbb{R}^k). Here V_X is a non-negative potential with disconnected zero set. Under suitable conditions, including a generalization of the strict triangle's inequality (1.2.9), the authors show the existence of a minimizing geodesic with different endpoints σ_0 and σ_1 in X . The existence of \mathfrak{U} follows then as a particular case, provided than one chooses q^- and q^+ in a way that the strict triangle's inequality is fulfilled. It is however delicate to perform such a passage. The authors recover the results from both [3] and [153]. More recently, Smyrnelis [157] considered the abstract connecting orbits problem for a potential \mathcal{V} defined on an affine Hilbert space. These abstract approaches have a major influence in this thesis, in particular regarding Chapters 3 and 4.

1.2.7 One-dimensional heteroclinic traveling waves

We now focus on the non-conservative system

$$q'' + cq' = \nabla W(q) \text{ in } \mathbb{R}, \quad (1.2.13)$$

which we introduced in Section 1.1.2.4. We seek for heteroclinic solutions, i. e.,

$$\lim_{t \rightarrow \pm\infty} q(t) = a^\pm, \quad (1.2.14)$$

for a^\pm local minimizers of W , which is smooth and bounded below. Recall that, since we impose $c > 0$ and we have the formal expression

$$c = \frac{W(a^+) - W(a^-)}{\int_{\mathbb{R}} |q'|^2}, \quad (1.2.15)$$

so that we must have $W(a^+) > W(a^-)$. Up to translations, we shall assume that $0 = W(a^+)$. That is, q connects two local minimizers of W which are at *different* levels. From (1.2.15), we see that c cannot be fixed freely in (1.2.13) and that it makes also part of the unknown of the equation. That

is, the question we are interested on is that of finding a pair (c, q) such that (1.2.13) and (1.2.14) holds.

Existence of such solutions⁶ has been proven by different authors: Lucia, Muratov and Novaga [110], Risler [145, 144], Alikakos and Katzourakis [13]. In [13] the authors also establish rigorously the formula (1.2.15) for the heteroclinic solution they find. Their proof relies on variational methods and it was later revisited in the book by Alikakos, Fusco and Smyrnelis [12].

The curve q can be also seen as the *profile* of a *traveling wave* to the reaction-diffusion system

$$\partial_t u - \partial_{xx} u = -\nabla W(u) \text{ in } (0, +\infty) \times \mathbb{R}. \quad (1.2.16)$$

Indeed, it suffices to set

$$w(t, x) := q(x - ct)$$

and it is easily verified that w solves (1.2.16). As shown by Risler [144], traveling waves play a central role in the description of the dynamics of (1.2.16), as solutions to (1.2.16) are asymptotic to a pattern composed of stationary and traveling wave solutions connecting local minima of W , each solution going away from the rest. Traveling waves are also important objects in plenty of other reaction-diffusion models. A more detailed review of the literature can be found in Chapters 3 and 4.

Let us now give a few elementary facts regarding the problem considered in [13]. As we already pointed out in Section 1.1.3.1, (1.2.13) is the Euler-Lagrange equation for the action

$$E_c(q) := \int_{\mathbb{R}} \left[\frac{|q'(t)|^2}{2} + W(q(t)) \right] e^{ct} dt. \quad (1.2.17)$$

Notice that, unlike E , E_c is not invariant by translations. Furthermore, for all $\tau \in \mathbb{R}$ we have

$$E_c(q(\cdot + \tau)) = e^{-c\tau} E_c(q), \quad (1.2.18)$$

Formula (1.2.18) implies that

$$\inf E_c(q) \in \{-\infty, 0\}, \quad (1.2.19)$$

where the infimum is taken among all the functions in $H_{\text{loc}}^1(\mathbb{R}, \mathbb{R}^k)$ satisfying (1.2.14). The expression (1.2.18) also implies that it is not possible to solve the minimization problem (1.2.19) by a concentration-compactness approach, even when the infimum is zero (of course, otherwise the problem does not have a solution). In [13], this is solved by applying the technique introduced by Alikakos and Fusco [11] in order to deal with the minimization problem of Section 1.2.3. The idea is to introduce a family of artificial constraints on the minimization problem (1.2.19) so that compactness is restored and a solution exist. Then, for the right choice of c and another parameter, it is shown that the constraints are not met, which means that the solution is in fact unconstrained. The proof of the results we obtain in Chapters 3 and 4 is also based in this technique.

⁶Maybe with conditions at infinity different than (1.2.13).

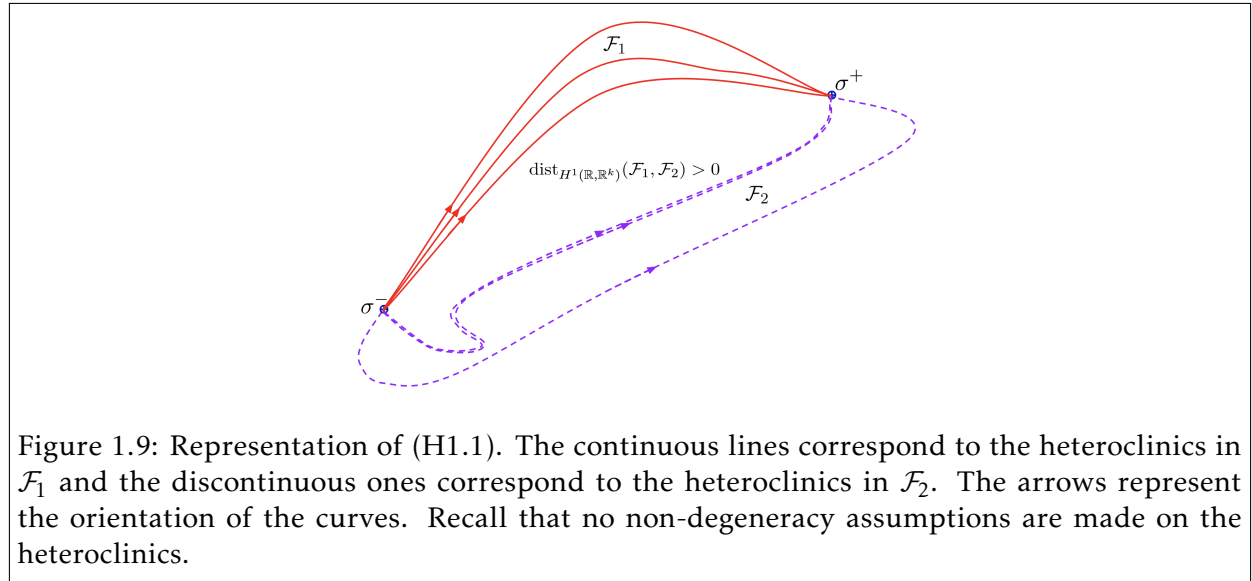


Figure 1.9: Representation of (H1.1). The continuous lines correspond to the heteroclinics in \mathcal{F}_1 and the discontinuous ones correspond to the heteroclinics in \mathcal{F}_2 . The arrows represent the orientation of the curves. Recall that no non-degeneracy assumptions are made on the heteroclinics.

1.3 Main contributions of this thesis

We now describe the main new results presented in this thesis. We follow the notations, terminology and definitions of Section 1.2.

1.3.1 Non-minimizing connecting orbits for multi-well systems

The results of Chapter 2 are mostly those of [131]. We are indebted to the referee of the journal, which pointed us to several important references (for instance, the PhD Thesis of Bisgard [50]) and provided useful remarks which improved the paper.

Essentially, in Chapter 2 we show that, under suitable assumptions, there exist non-minimizing connecting orbits for (1.2.1). Let us give a description of the statement of these results. We start by selecting $(\sigma^-, \sigma^+) \in \Sigma^2$ such that $\sigma^- \neq \sigma^+$ and such that the strict triangle's inequality (1.2.9) holds. The latter implies that the set of globally minimizing heteroclinics

$$\mathcal{F} := \{q \in X(\sigma^-, \sigma^+) : E(q) = m\}, \quad (1.3.1)$$

is non-empty. In (1.3.1) we have set $m := m_{\sigma^-, \sigma^+}$ as in (1.2.8). A crucial assumption for our results is that the set \mathcal{F} has two different connected components such that the H^1 -distance between them is positive

(H1.1). There exist \mathcal{F}_1 and \mathcal{F}_2 subsets of \mathcal{F} such that $\mathcal{F}_1 \cup \mathcal{F}_2 = \mathcal{F}$ and $\text{dist}_{H^1(\mathbb{R}, \mathbb{R}^k)}(\mathcal{F}_1, \mathcal{F}_2) > 0$.

Assumption (H1.1) was introduced by Alessio [6] and is a generalization of the situation in which there exist two different (up to translations) globally minimizing heteroclinics, as was assumed in the pioneering work of Alama, Bronsard and Gui [3]. In particular, if (H1.1) holds then \mathcal{F} is not connected and the minimization problem (1.2.8) for (σ^-, σ^+) has multiple solutions up to translations. See Figure 1.9.

A fundamental idea behind our results is to show that from (H1.1) it follows the existence of a *mountain pass geometry* for E in the set $X(\sigma^-, \sigma^+)$. More precisely, we consider the set of paths that join \mathcal{F}_1 and \mathcal{F}_2 and we take the min-max of E with respect to this set of paths. We denote the

mountain pass value as \mathfrak{c} . Assumption (H1.1) implies then that $\mathfrak{c} > \mathfrak{m}$. In other words, thanks to (H1.1) we are able to identify a change of topology at the level set $\mathfrak{c} > \mathfrak{m}$, which is a first step for showing the existence of new connecting orbits. It is worth mention here that there exists a significant number of previous references in which it is assumed that there exists a *gap* in the set of global minimizers of a variational problem and then the min-max of the associated set of paths is considered. At a latter stage, a non-minimizing critical point is produced. Some of these references are Bolotin and Rabinowitz [53, 54]; de la Llave and Valdinoci [109] and, in more close relation with our results, Bisgard [50]; Montecchiari and Rabinowitz [123, 122].

Once the mountain pass geometry has been established, one needs to study the behavior of the Palais-Smale sequences at the level \mathfrak{c} . As we described in Section 1.2.5, this analysis was performed by Rabinowitz [136]. Unfortunately, such an analysis does not allow to conclude that there exists a non-minimizing connecting orbit, as Palais-Smale sequences could formally converge to a chain composed entirely by elements of \mathcal{F} , up to reversing their orientation. We circumvent this difficulty by imposing the following technical assumption:

(H1.2). Consider the mountain pass family

$$\Gamma := \{\gamma \in C([0, 1], X(\sigma^-, \sigma^+)) : \gamma(i) \in \mathcal{F}_i \text{ for } i \in \{0, 1\}\}$$

and the mountain pass value $\mathfrak{c} := \inf_{\gamma \in \Gamma} \max_{s \in [0, 1]} E(\gamma(s))$. Then, the following holds:

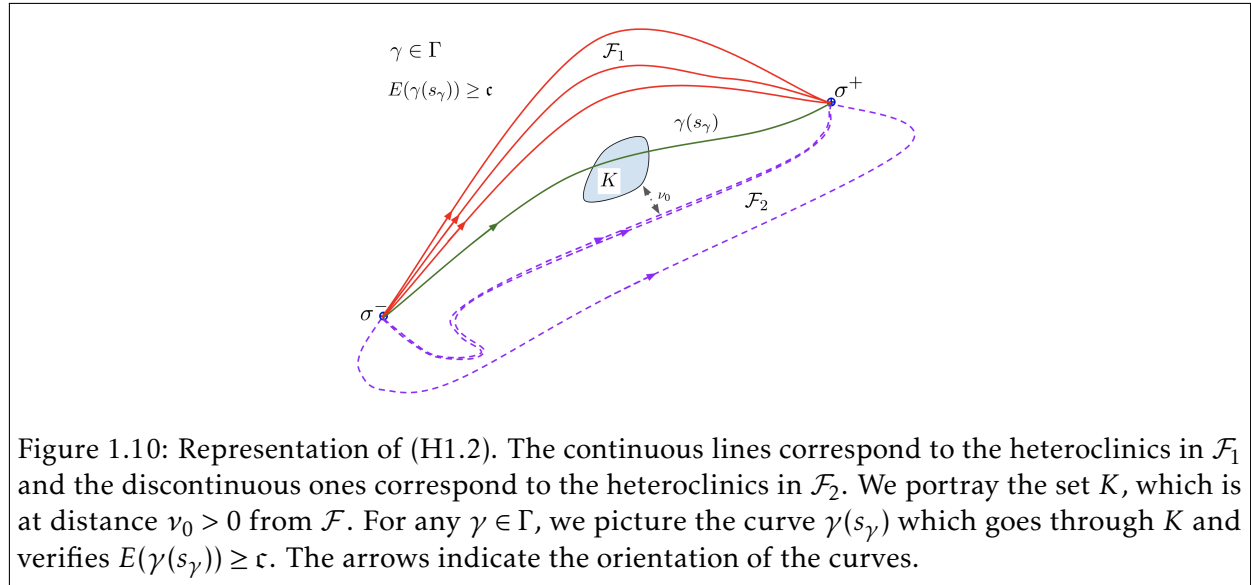
1. There exists $\nu_0 > 0$ and a closed set $K \subset \mathbb{R}^k$ such that for all $q \in \mathcal{F}$ it holds

$$\text{dist}(q(\mathbb{R}), K) \geq \nu_0.$$

2. There exists $M > \mathfrak{c}$ such that for all $\gamma \in \Gamma$ verifying $\max_{s \in [0, 1]} E(\gamma(s)) \leq M$ there exists $s_\gamma \in [0, 1]$ such that $E(\gamma(s_\gamma)) \geq \mathfrak{c}$ and $\gamma(s_\gamma)(\mathbb{R}) \cap K \neq \emptyset$.

See Figure 1.10. In Chapter 2 we give an equivalent version of (H1.2) which addresses J and $H^1(\mathbb{R}, \mathbb{R}^k)$ instead of E and $X(\sigma^-, \sigma^+)$. Essentially, (H1.2) in combination with a deformation result due to Willem [172] allows to show the existence of a Palais-Smale sequence at the level \mathfrak{c} for which every element goes through the set K , which means that at least one of the elements of the limiting chain is not in \mathcal{F} . Hence if (H1.1) and (H1.2) hold then one has a non-minimizing connecting orbit of (1.2.1) which is as follows: either a non-minimizing heteroclinic in $X(\sigma^-, \sigma^+)$ or a non-constant homoclinic in $X(\sigma, \sigma)$ for some $\sigma \in \{\sigma^-, \sigma^+\}$. This is the main result of Chapter 2. We also show that (H1.2) can be replaced by the symmetry assumption of Alama, Bronsard and Gui [3] in order to obtain a stronger result. Finally, we prove that combining the previous symmetry assumption with (H1.2) one obtains the existence of a non-minimizing *heteroclinic*. A better understanding of (H1.2), as well as the possibility of an eventual relaxation, is still an open question.

In order to conclude this paragraph, we mention that the second chapter of the PhD thesis of Bisgard [50] contains results which can be thought as the analogous of ours for the non-autonomous case. In [50], the gap is produced by the explicit periodic time dependence of the potential V_\star (see (1.2.10)), so that the autonomous setting is excluded. Subsequently, he imposes an assumption on the value of the mountain pass level so that the corresponding Palais-Smale sequences do not formally converge to a chain of globally minimizing heteroclinics. A more detailed comparison between our work and his is given in Chapter 2. In particular, we show that our assumption (H1.2)



implies as a particular case that $c \neq (2k+1)m$ for all $k \in \mathbb{N}$, which is essentially the assumption made in [50].

1.3.2 Heteroclinic traveling waves of 2D parabolic Allen-Cahn systems

In Chapter 3 we present the results contained in the preprint [130]. We study the parabolic Allen-Cahn system

$$\partial_t w - \Delta w = -\nabla V(w) \text{ in } (0, +\infty) \times \mathbb{R}^2. \quad (1.3.2)$$

Our main result consists on proving the existence of a new type of traveling wave solution for (1.3.2). This solution is in fact a combination of the solution found by Alama, Bronsard and Gui [3] and Schatzman [153] with that found by Alikakos and Katzourakis [13]. These solutions have been presented in Sections 1.2.6 and 1.2.7 respectively.

Our traveling wave solution w moves along the x_1 -axis with speed $c^* > 0$, so that it has the form

$$w(t, x_1, x_2) = \mathfrak{U}(x_1 - c^*t, x_2),$$

where the profile \mathfrak{U} solves the elliptic system

$$c^* \partial_{x_1} \mathfrak{U} + \Delta \mathfrak{U} = \nabla V(\mathfrak{U}) \text{ in } \mathbb{R}^2.$$

Comparing with the two-dimensional heteroclinic of Section 1.2.6, we see the new “friction” term $c^* \partial_{x_1} \mathfrak{U}$. Recall that in Section 1.2.6 we interpreted the solution of [3, 153] as a heteroclinic orbit for an infinite-dimensional Hamiltonian system (via the works of Monteil and Santambrogio [125] and Smyrnelis [157]), where the potential (the one-dimensional energy E) is defined in $L^2(\mathbb{R}, \mathbb{R}^k)$. We take the same approach here. That is, \mathfrak{U} satisfies the heteroclinic conditions at infinity

$$\lim_{x_1 \rightarrow \pm\infty} \|\mathfrak{U}(x_1) - \mathfrak{q}^\pm\|_{L^\infty(\mathbb{R}, \mathbb{R}^k)} = 0, \quad (1.3.3)$$

where \mathfrak{q}^\pm are two heteroclinic orbits in $X(\sigma^-, \sigma^+)$, for $\sigma^- \neq \sigma^+$. See Figure 1.11. As a consequence, our problem can be seen as an extension to that of Alikakos and Katzourakis [13] in an abstract

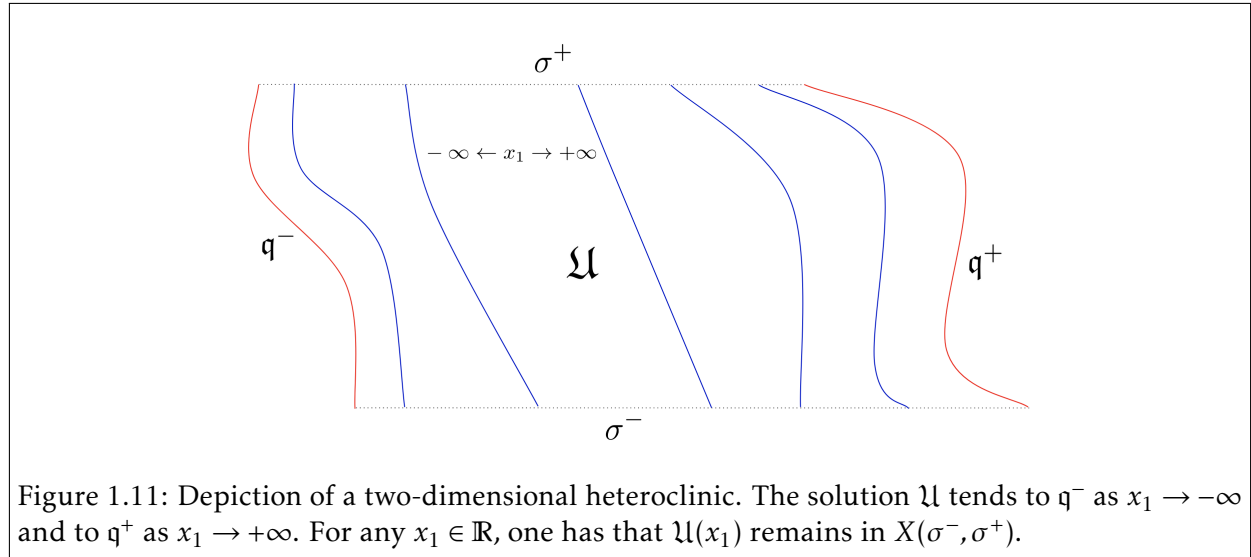


Figure 1.11: Depiction of a two-dimensional heteroclinic. The solution \mathfrak{U} tends to q^- as $x_1 \rightarrow -\infty$ and to q^+ as $x_1 \rightarrow +\infty$. For any $x_1 \in \mathbb{R}$, one has that $\mathfrak{U}(x_1)$ remains in $X(\sigma^-, \sigma^+)$.

framework of infinite-dimensional Hamiltonian systems. As a consequence, we see that the orbits q^\pm must have *different* energy and, due to the sign of the speed, they must be such that $E(q^+) > E(q^-)$. We shall assume that q^\pm are, up to translations, non-degenerate critical points of E in $X(\sigma^-, \sigma^+)$. Hence, in order to prove our main result, it is natural to perform the following steps:

1. Modify the technique of [13] so that it applies to potentials defined in possibly infinite-dimensional Hilbert spaces, with degenerate minimizers (as invariance by translations must be taken into account).
2. Perform the passage from the abstract setting to the setting of (1.3.2).

It turns out that the first step is delicate, as originally the device of [13] requires assumptions of non-degeneracy on the minimizers, convexity of some level sets as well as some radial monotonicity with respect the minima (not only locally). This is due to the fact that the associated Lagrangian action changes sign, recall (1.2.17). This adds a significant difficulty to the *balanced* setting and prevents from applying the same concentration-compactness approach.

The assumptions of [13] turn out to be too restrictive in our setting. Essentially, invariance by translations needs to be taken into account and it is unclear that there exist examples of potentials V so that the energy E has convex level sets for $a > E(q^-)$ large enough. We circumvented this problem by showing that the argument works if one has a certain upper bound on $E(q^+) - E(q^-)$. We also provide examples of potentials for which this bound is met. The drawback of our method is essentially that the upper bound is rather hard to compute and we cannot exclude the possibility that it is “small”.

Once we have established the existence of the traveling wave solution, we show that the convergence (1.3.3) is also exponential with respect to the L^2 -norm. We also show that the analogous of the formula (1.2.15) holds, as well as a variational characterization and a uniqueness property on the speed c^* . We refer to Chapter 3 for more details, where we also provide a detail discussion about the literature regarding reaction-diffusion problems and their traveling waves. The solution \mathfrak{U} is in fact a minimizer of E_{2,c^*} in the class of functions satisfying (1.3.3), where, for

$c > 0$,

$$E_{2,c}(U) := \int_{\mathbb{R}} \left[\frac{\|\partial_{x_1} U(x_1)\|_{L^2(\mathbb{R}, \mathbb{R}^k)}^2}{2} + E(U(x_1)) - E(q^+) \right] e^{cx_1} dx_1.$$

Compare with E_c defined in (1.2.17) and E_2 defined in (1.2.12). In fact, as in the 1D case, we have $E_{2,c}(\mathbb{1}) = 0$.

1.3.3 Heteroclinic traveling waves of 1D parabolic systems with degenerate stable states

The methods developed on Chapter 3 allow us to revisit the one-dimensional parabolic system

$$\partial_t w - \partial_{xx} w = -\nabla W(w) \text{ in } (0, +\infty) \times \mathbb{R} \quad (1.3.4)$$

considered by Alikakos and Katzourakis [13]. This is the object of Chapter 4, based on the preprint [129]. More precisely, our potential W allows for some kind of degenerate minima, which can eventually be a continuum. See Figure 1.12. Naturally, this follows from the fact that in Chapter 3 we had to deal with the invariance by translations.

The abstract result of Chapter 3 readily applies to the framework of (1.3.4). However, in Chapter 4 we replace the key assumption of Chapter 3, which is the upper bound on the difference between the energy of the minima, by a different one. This new assumption is more directly inspired by the work in [13]. Essentially, convexity of suitable level sets is assumed, as well as monotonicity with respect to a chosen direction. Most of the properties on the speed and the profile proven in [13] are recovered.

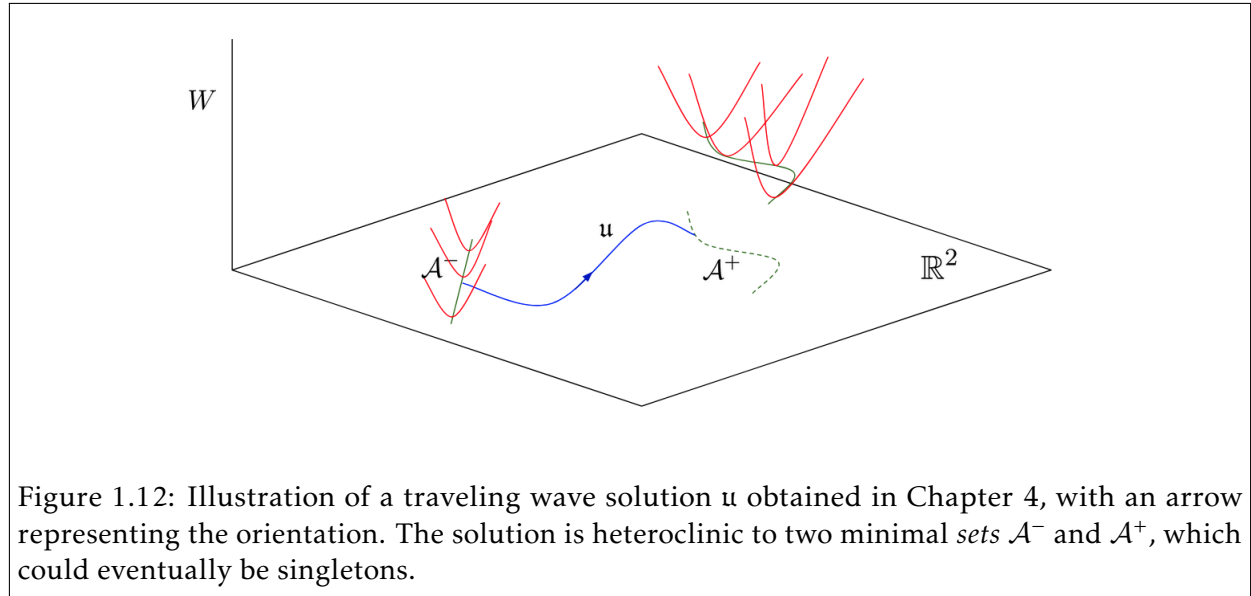
Again, the results in Chapter 4 are proven in an abstract framework. Hence, they could in principle be applied to the 2D setting of Chapter 3. However, the assumptions of Chapter 4 seem to us to be too restrictive for the 2D setting. In any case, we were not able to find any examples, therefore we do not claim applicability of these results to the 2D setting. However, we thought it is worth to prove the results in a more general framework and we hope that this can eventually lead to applications to another problems, if not the 2D one. More details and conjectures are given in Chapter 4.

1.4 Perspectives and open problems

We now describe some lines of prospective research which represent a continuation of the work of this thesis. Some other open problems can be found in the different chapters of the thesis.

1.4.1 Extensions of the connecting orbits problem

There exist several possible extensions of problem we consider in Chapter 2. For instance, we expect the results of Chapter 2 to hold if one replaces \mathbb{R}^k by a k -dimensional Riemannian manifold \mathcal{M} . In Section 1.2.2, we sketched a possible way to treat a class of degenerate potentials. In Section 1.2.4, we discuss related literature regarding the non-autonomous problem. In this direction, we refer also to Sourdis [161]. Another extension concerns the study of solutions of the system (1.2.1) which connect *level sets* of V rather than zeros. Here V is not necessarily of multi-well type, even though the multi-well setting is contained as a particular case. Existence results for these type of



problems have been obtained recently by Antonopoulos and Smyrnelis [19], Fusco, Gronchi and Novaga [84, 85] and Alessio, Montecchiari and Zuñiga [8]. In all these works, the solutions are obtained by minimization arguments. We are not aware of existence results for non-minimizing solutions in this setting, but it is possible that some could be obtained by generalizing and adapting the analysis and the arguments that we present in this thesis for the multi-well case.

In a different direction, Ruan [149] considered generalized multi-well systems of the type

$$(\varphi(|q'|)|q'|^{-1}q')' = \nabla V(q) \text{ in } I, \quad (1.4.1)$$

for some interval $I \subset \mathbb{R}$, where φ is a function which can take certain forms. For instance, it is possible to choose $\varphi = \text{Id}$, so that the classical system (1.2.1) is recovered. But the analysis in [149] also allows for taking more general φ , e. g., of p -Laplace type. In [149], the author extends the minimization results that we recalled in Section 1.2.3 to this more general setting. This is done both by the metric approach and the concentration-compactness approach. The author also shows that the results of Alikakos, Betelu and Chen [9], which specialize to the case $k = 2$ and use complex variables theory, can be extended to the generalized setting of (1.4.1). We expect that, maybe up to restricting to some degree the class of admissible φ , the existence of non-minimizing connecting orbits that we prove in this thesis for (1.2.1) also holds for (1.4.1) under analogous assumptions.

Another possible extension would concern the study of non-minimizing solutions of (1.2.1) obtained by other variational methods. For example, we may think about solutions of *multibump* or *multitransition* type. That is, solutions which go back and forth the neighborhoods of two wells a prescribed number of times. Existence of these solutions has been proven in the non-autonomous setting described in Section 1.2.4 by several authors and also in some related PDE settings. See Alama and Li [4], Byeon, Montecchiari and Rabinowitz [61], Montecchiari and Rabinowitz [122] and the references therein. The problems considered in these references are semilinear and share as a common feature a periodic non-homogeneous dependence on the nonlinear (potential) term which, allows to develop the argument. It would be interesting to know whether these type of solutions exist for the autonomous system (1.2.1). Our question is motivated by the fact that, as we

show in Chapter 2 and already explained in Section 1.3.1, the time dependence on the potential required in Bisgard's work can be replaced by a non-uniqueness assumption (up to translations) on the set of globally minimizing heteroclinics, namely (H1.1). Thus, we wonder if such a replacement (or a similar one) can allow to prove the existence of multibump solutions.

Finally, we mention that here we studied traveling waves and stationary solutions, both of connecting type, for parabolic systems. However, one can also focus on the system

$$Ju_t - u_{xx} = \nabla \tilde{V}(u) \text{ in } \mathbb{R} \times \mathbb{R} \quad (1.4.2)$$

where

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad (1.4.3)$$

is the 2×2 symplectic matrix and $\tilde{V} : \mathbb{R}^2 \rightarrow \mathbb{R}$ is some potential. By identifying \mathbb{R}^2 with \mathbb{C} (after the computation of $\nabla \tilde{V}$), (1.4.2) becomes

$$iu_t + u_{xx} = \nabla \tilde{V}(u) \text{ in } \mathbb{R} \times \mathbb{R}, \quad (1.4.4)$$

which is a nonlinear Schrödinger-type equation. If one chooses $\tilde{V} = V_{GL}$, where $V_{GL}(u) := (1 - |u|^2)^2/4$ is the standard Ginzburg-Landau potential, then (1.4.4) is called the Gross-Pitaevskii equation. The existence of traveling waves for this equation, as well as their dynamical properties, have been widely studied, see for instance Bethuel, Gravejat, Saut and Smets [43, 44, 45, 90]. Similar research has also been conducted in higher space dimensions, see Maris [114] and the references therein. In all these papers, the traveling waves are found by minimizing the energy under the constraint of fixed *momentum*, as both quantities are invariant by the flow (1.4.4).

The potential \tilde{V} can also be chosen equal to a double-well potential V . In this case, heteroclinic traveling wave solutions of (1.4.4) have been established by Alama, Bronsard, Contreras, Dadok and Sternberg [1, 75]. The authors consider the Maupertuis setting of Section 1.1.3.3 and minimize the functional under a constraint which is related to the momentum in [43, 44, 45, 90] for the class of curves which connect the two wells of V . The heteroclinic solution they obtained after solving this minimization problem can also be interpreted in terms of a degenerate isoperimetric problem on \mathbb{R}^2 . The question that could be posed would be that of studying non-minimizing solutions in this setting. We also find interesting to know whether it would make sense to consider an analogous problem in \mathbb{R}^{2k} (replacing J in (1.4.3) by the symplectic matrix in \mathbb{R}^{2k}), at least when it comes to the traveling wave interpretation of it. We also wonder if there can exist a fruitful link with the system of two coupled Gross-Pitaevskii equations studied by Alama, Bronsard, Contreras and Pelinovski [2].

1.4.2 The Maupertuis approach to the connecting orbits problem

As we have pointed out before (see Section 1.1.3.3), the connecting orbits problem is equivalent to the study of the geodesics given by the functional

$$E_M(p) := \int_0^1 |p'(t)| \sqrt{2V(p(t))} dt,$$

for $p \in X(p) := W_{\text{loc}}^{1,\infty}([0,1] \setminus \Sigma(p), \mathbb{R}^k) \cap C^0([0,1], \mathbb{R}^k)$, where $\Sigma(p) := \{s \in [0,1] : p(s) \in \Sigma\}$ is the set in which p intersects Σ . Indeed, recall that in the recent works of Monteil and Santambrogio [124]; Zuñiga and Sternberg [174] they show that by minimizing E_M in a suitable class of curves one obtains the minimization results for E that we presented in Section 1.4.2.

Roughly speaking, there exists a link between E (see Section 1.2.2) and E_M which we interpret as follows (see also Figure 1.13): While critical points of E are connecting orbits, and chains of such orbits emerge as *generalized* critical points due to the lack of compactness of the problem, compactness is restored when we deal with E_M . The critical points are then a gluing of segments with endpoints in Σ and away from it elsewhere. Each of these segments can be conveniently reparametrized into a connecting orbit. That is, the *generalized* critical points for E can be identified to the ordinary critical points of E_M , assuming one successfully deals with the non-smoothness issues related to E_M . The previous remark suggests that any existence result for E has a counterpart for E_M , possibly with differences in the assumptions. In particular, one could aim at obtaining the counterpart of the results of Chapter 2 in this thesis (see Section 1.3.1) for E_M . We conjecture that such a thing is possible, but we fail to see how this would improve our results. In particular, notice that even though there is no loss of compactness for E_M , nothing prevents the critical point generated by the mountain pass approach to be a concatenation of globally minimizing geodesic between the two wells σ^- and σ^+ . That is, the problem we described in Section 1.3.1 is not solved by looking at E_M instead, or at least not in a way that is obvious to us.

In the remainder of this paragraph, we describe the link between E and E_M in a somewhat more precise manner. In particular, we intend to make more clear why we say that critical points of E_M can be identified with generalized for E . Nevertheless, at this point some considerations we make hold only on a formal level.

Essentially, we rely on two observations. Firstly, we see that at least formally we have that for all $\varphi \in C_c^\infty([0,1] \setminus \Sigma(p), \mathbb{R}^k)$ and $\lambda \in \mathbb{R}$ it holds

$$E_M(p + \lambda\varphi) = E_M(p) + \lambda \int_0^1 \left[\frac{\langle p', \varphi' \rangle}{|p'|} \sqrt{2V(p)} + |p'| \frac{\langle \nabla V(p), \varphi \rangle}{\sqrt{2V(p)}} \right] + o(\lambda),$$

that is, again formally, we have that the differential at p with respect to φ is

$$DE_M(p)(\varphi) = \int_0^1 \left[\frac{\langle p', \varphi' \rangle}{|p'|} \sqrt{2V(p)} + |p'| \frac{\langle \nabla V(p), \varphi \rangle}{\sqrt{2V(p)}} \right]. \quad (1.4.5)$$

The second observation is that if q is a classical solution of (1.2.1), then for all $t \in \mathbb{R}$ we have the equipartition formula

$$\frac{d}{dt} \left(\frac{|q'(t)|^2}{2} - V(q(t)) \right) = 0,$$

that is, for all $t \in \mathbb{R}$

$$\frac{|q'(t)|^2}{2} - V(q(t)) = K_q,$$

with $K_q \in \mathbb{R}$. If $q \in \mathcal{D}$, then the left-hand side in the identity above belongs to $L^1(\mathbb{R})$, which in fact means that $K_q = 0$. That is, for all $t \in \mathbb{R}$

$$\frac{|q'(t)|^2}{2} - V(q(t)) = 0, \quad (1.4.6)$$

which is the well-known *equipartition identity* for the finite energy solutions of (1.2.1), that we already introduced in (1.1.16). Furthermore, Young's inequality implies that if $q \in \mathcal{D}$ is such that (1.4.6) holds for q for all $t \in \mathbb{R}$, then

$$E(q) = \tilde{E}(q) := \int_{\mathbb{R}} |q'(t)| \sqrt{2V(q(t))} dt$$

and \tilde{E} does not depend on the parametrization of the trace of q .

Once these observations have been made, assume that $(a, b) \subset \mathbb{R}$ is bounded and $p \in X(p)$ verifies $|p'| > 0$ a. e., is such that $p(a)$ and $p(b)$ can be defined and belong to Σ . Assume also that for all $t \in (a, b)$ we have $p(t) \notin \Sigma$. Then, we can easily show that p can be identified with a curve in $X(p(a), p(b))$ with the same energy. We first recall that can find a surjective $\Phi_p : \mathbb{R} \rightarrow (a, b)$ such that for a. e. $t \in \mathbb{R}$

$$\Phi_p'(t) = \sqrt{2V(p(\Phi_p(t)))}.$$

The previous statement is rigorously justified in [124], *Theorem 3*. Once Φ_p is defined, we can reparametrize p as follows: Up to changing (a, b) by (c, d) , we can assume that $|p'| = 1$ because we assume $|p'| > 0$ a. e.. Then, by setting $q := p \circ \Phi_p$ we check that q satisfies (1.4.6) and

$$E(q) = \tilde{E}(q) = E_M(p)$$

and clearly $q \in X(p(a), p(b))$. In particular, we have that minimizers for E_M can be reparametrized into minimizers of E . More generally, if p is a geodesic for the (Riemannian) metric induced by $\sqrt{2V}$ in the sense that for all $\varphi \in \mathcal{C}_c^1((a, b), \mathbb{R}^k)$ it holds

$$\int_a^b \left[\frac{\langle p', \varphi' \rangle}{|p'|} \sqrt{2V(p)} + |p'| \frac{\langle \nabla V(p), \varphi \rangle}{\sqrt{2V(p)}} \right] = 0,$$

then we have that q is a connecting orbit for E . In order to see this, recall that we assume $|p'| = 1$, so that a change of variables shows that for all $\varphi \in \mathcal{C}_c^1((a, b), \mathbb{R}^k)$

$$0 = \int_{\mathbb{R}} \left[\langle p' \circ \Phi_p, \varphi' \circ \Phi_p \rangle (\Phi_p')^2 + \langle \nabla V(q), \varphi \circ \Phi_p \rangle \right] = \int_{\mathbb{R}} \left[\langle q', (\varphi \circ \Phi_p)' \rangle + \langle \nabla V(q), \varphi \circ \Phi_p \rangle \right]$$

Therefore, since Φ_p is locally Lipschitz with locally Lipschitz inverse, we clearly have by usual arguments that $DE(q) = 0$. That is, q is a connecting orbit. Reciprocally, given a connecting orbit q we can reparametrize it into a geodesic for the metric induced by $\sqrt{2V}$ which intersects Σ only at its endpoints.

The previous observations readily apply to critical points of E_M , for which intersections with Σ in the middle points are allowed. Indeed, if one has $p \in W_{\text{loc}}^{1,\infty}((0, 1) \setminus \Sigma(p), \mathbb{R}^k)$ such that $p(a)$ and $p(b)$ belong to Σ , and one has (formally) that $DE_M(p) = 0$ (see (1.4.5)), then one has that

$$\mathbb{R} \setminus \Sigma(p) = \bigcup_{i=1}^{\text{Ob}(p)-1} (a_i, a_{i+1})$$

with $\text{Ob}(p) \geq 1$ and $a_0 = 0$, $a_{\text{Ob}(p)} = 1$. Therefore, one applies the previous reasoning in each interval (a_i, a_{i+1}) . Indeed, we have that $p|_{(a_i, a_{i+1})}$ is a geodesic according to the metric induced by $\sqrt{2V}$ which

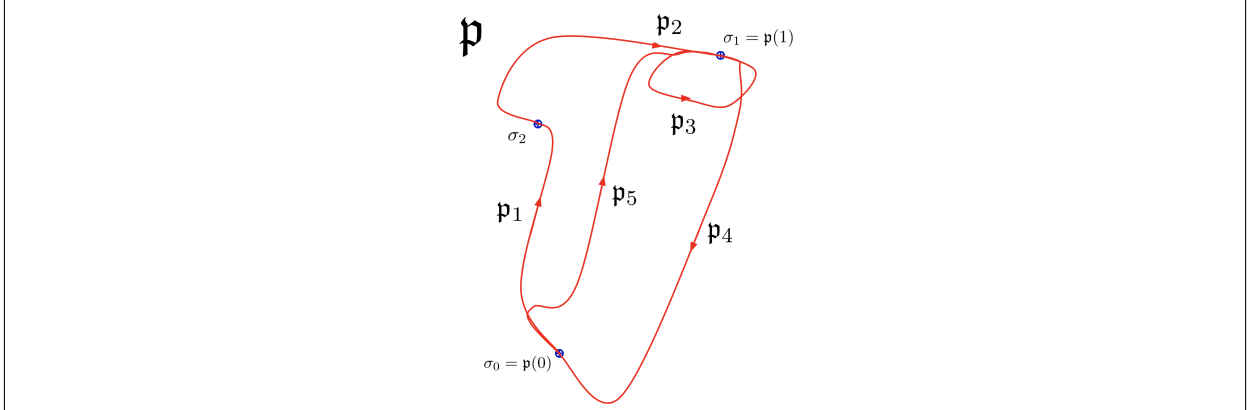


Figure 1.13: Representation of a geodesic p for the Maupertuis functional E_M . The geodesic is such that $\sigma_0 = p(0)$ and $\sigma_1 = p(1)$ are two wells of Σ . There is also another well σ_2 involved. It decomposes into five geodesic segments, p_1 , p_2 , p_3 , p_4 and p_5 . Each of the segments can be reparametrized so that it becomes a connecting orbit solving (1.2.1), joining the two wells which correspond to the endpoints of the segment. The arrows represent the orientation of the geodesic.

connects the two points (in Σ) $\sigma_i := p(a_i)$, $\sigma_{i+1} := p(a_{i+1})$ and can be converted into a connecting orbit $q_i \in X(\sigma_i, \sigma_{i+1})$. Clearly, it is also possible to transform a given chain of connecting orbits (a critical point at infinity for E) into a critical point for E_M .

Notice that we did not rigorously define the notion of differential for E_M , which might pose some difficulties due to the non-smoothness of the integrand in E_M . However, we believe that the difficulties should be only of technical order (as they are for the classical theory of geodesics) and that the considerations of this paragraph can be made completely rigorous. It would be also interesting to replace \mathbb{R}^k by a more general space, maybe infinite-dimensional.

1.4.3 Link between connecting and periodic orbits

In Example 1.1.1, Section 1.1.4.3, we showed that for any multi-well potential V we can find a non-constant periodic orbit for any fixed period $T > 0$. The result follows by an standard application of the mountain pass lemma, as the C^1 functional

$$E_T(q) = \int_{-T}^T \left[\frac{|q'(t)|^2}{2} + V(q(t)) \right] dt,$$

satisfies the Palais-Smale condition on $H_{\text{per}}^1([-T, T], \mathbb{R}^k)$, the set of curves in $H^1([-T, T], \mathbb{R}^k)$ with periodic boundary conditions. The mountain pass geometry follows from the multi-well structure of V , as the curves which coincide identically with a well belong to $H_{\text{per}}^1([-T, T], \mathbb{R}^k)$. However, it is possible to show that

$$c_T \rightarrow +\infty \text{ as } T \rightarrow +\infty,$$

where c_T is the associated mountain pass value. Therefore, if $(q_T)_{T>0}$ is the associated family of mountain pass solutions, it is unclear what to expect at the limit $T \rightarrow +\infty$.

Suppose that, on the contrary, one has a family of T -periodic solutions $(\tilde{q}_T)_{T>0}$ with energies $(\tilde{c}_T)_{T>0}$ satisfying a uniform bound $\sup_{T>0} \tilde{c}_T < +\infty$. Then, it is easy to prove that, up to subsequences, $\tilde{q}_T \rightarrow \tilde{q}$ as $T \rightarrow +\infty$. Moreover, \tilde{q} is a connecting orbit with $E(\tilde{q}) \leq \liminf_{T \rightarrow +\infty} \tilde{c}_T$. However, without additional information, it is not possible to exclude the possibility that \tilde{q} is a

globally minimizing heteroclinic which can be obtained by minimization in the appropriate set of curves. Hence, even if one we to find a family $(\tilde{q}_T)_{T>0}$ of periodic orbits satisfying the assumption above, it is not clear that they would yield the existence of new connecting orbits.

We do not know the study of periodic orbits can shed some light into the problem of connecting orbits, or viceversa. There seems to be a link to keep in mind, but at this stage the question seems too vague to us and we are not able to extract any precise conclusions.

1.4.4 Long range interaction and critical points at infinity of E

Here we focus on the parabolic system

$$\partial_t u - \partial_{xx} u = -\nabla V(u) \text{ in } (0, +\infty) \times \mathbb{R} \quad (1.4.7)$$

which is, at least formally, the L^2 -gradient flow for the energy E . This observation can be made rigorous: If one takes $u_0 \in X(\sigma^-, \sigma^+)$, then the associated solution of u with $u(0) = u_0$ verifies that $u(t) \in X(\sigma^-, \sigma^+)$ for all time $t \in (0, +\infty)$. Moreover, one can prove that the *relaxation* expected from the gradient-flow structure holds in a rigorous level. By relaxation, we mean that as $t \rightarrow +\infty$, u reassembles a chain of *connecting orbits*, which are the stationary solutions of (1.4.7). One can think of a continuous analog to the asymptotic result for the Palais-Smale sequences of the energy, see Section 1.2.5. Relaxation for (1.4.7) has been proven by several authors: Bethuel, Orlandi and Smets [46] proved relaxation, obtained quantitative results and dealt with families of potentials $\varepsilon^{-2}V$ with vanishing parameter ε . It [146], Risler proved relaxation for systems which are more general than (1.4.7), as he allows connecting orbits joining *local* minima of V , which is not required to be multi-well but just bounded from below.

Relaxation being well understood by now⁷, the next step would be to study the so-called *long range interaction* for (1.4.7). That is, how an initial condition close to a chain of connecting orbits behaves. More precisely, how relaxation occurs for such an initial condition. It is expected that some orbits will attract and eventually “collapse” while others will repulse, depending on how they arrive to the wells. Results of this kind have been proven in the multi-well *scalar* case by Bethuel and Smets [47, 48], following previous works by Fusco and Hale [86], Carr and Pego [65, 64]. However, as noticed by Risler [146], this remains an open question for systems.

As it has been explained to us by Emmanuel Risler (private communication), if a description of the long range interaction for the solutions of (1.4.7) was available, it is likely that it could shed some light into the question of multiplicity of connecting orbits for the energy E . A wide variety of different settings could probably be studied. The reason is that, in Chapter 2, we deal with the *generalized* critical points that we describe in Section 1.2.5. However, long range interaction would prove that the class of *critical points at infinity* of E , in the sense of Bahri (see Section 1.1.4.4), is strictly smaller than the class of generalized critical points of E . By critical points at infinity E we mean chains of connecting orbits that can be actually realized as a relaxation of (1.4.7) for some initial condition. By working in a more restrictive class of possible asymptotic behaviors, the analysis would be refined and certainly more results could be obtained. We believe this to be a promising line of research.

⁷At least when the minimizers of V are non-degenerate.

1.4.5 Long time behavior in two-dimensional parabolic Allen-Cahn systems

Consider again a two-dimensional parabolic Allen-Cahn system as in Section 1.3.2,

$$\partial_t w - \Delta w = -\nabla V(w) \text{ in } (0, +\infty) \times \mathbb{R}^2. \quad (1.4.8)$$

Recall the two-dimensional energy

$$E_2(U) := \int_{\mathbb{R}} \left[\frac{\|\partial_{x_1} U(x_1)\|_{L^2(\mathbb{R}, \mathbb{R}^k)}^2}{2} + (E(U(x_1)) - m) \right] dx_1.$$

We refer to Section 1.2.6 for the details. We are interested in initial data w_0 verifying $E(w_0) < +\infty$. This implies the existence of $q^\pm = q^\pm(w_0) \in \mathcal{F}$ (the set of globally minimizing heteroclinics between two fixed wells σ^\pm , which we assume to be non-degenerate up to translations) such that

$$\lim_{x_1 \rightarrow \pm\infty} \inf_{\tau \in \mathbb{R}} \|w_0(x_1) - q^\pm(\cdot + \tau)\|_{L^2(\mathbb{R}, \mathbb{R}^k)} = 0. \quad (1.4.9)$$

An interesting question to us is whether one finds *relaxation*⁸ for the associated solution w to (1.4.8) with $w(0) = w_0$. That is, that w asymptotic as $t \rightarrow +\infty$ to a chain of stationary solutions of (1.4.8) with heteroclinic conditions (1.4.9) as $x_1 \rightarrow \pm\infty$ and going away one from another. This relaxation is well-known in the 1D case, see Section 1.4.4. If such a result holds, in order to prove it we would have to face the difficulty of the invariance by translations. Moreover, there would be problems associated with the fact that the configuration space is infinite-dimensional.

If one is able to prove relaxation, a more general question would be to characterize the behavior for more general conditions as done by Risler [145, 144] in the 1D case. Let us be more precise. Let \mathcal{G} be the set of *locally* minimizing heteroclinics of E for two fixed wells σ^\pm . Assume that all the orbits in \mathcal{G} are non-degenerate, which is a generic property. For $q \in \mathcal{G}$ and $c \geq 0$, one considers the weighted functional

$$E_{2,c}(U) := \int_{\mathbb{R}} \left[\frac{\|\partial_{x_1} U(x_1)\|_{L^2(\mathbb{R}, \mathbb{R}^k)}^2}{2} + E(U(x_1)) - E(q) \right] e^{cx_1} dx_1.$$

which already appeared in Sections 1.2.6 and 1.3.2. It is easy to see that, if $E_{2,c}(U) < +\infty$, U converges to (or *invades* the state) q with respect to the L^2 -convergence as $x_1 \rightarrow +\infty$. The rate of convergence is exponential if $c > 0$. On the contrary, nothing can be said about the behavior as $x_1 \rightarrow -\infty$.

The results by Risler in the 1D case show that for any initial condition with finite weighted energy for some speed $c \geq 0$, the associated solution of the parabolic system converges to a pattern formed by traveling and stationary solutions, going away one from another. If w_0 is such that $E_0(w_0) < +\infty$ then we would expect to find relaxation to a pattern of stationary solutions. However, if $c > 0$, traveling wave solutions should appear. Therefore, the culmination of the present program for the system (1.4.8) would imply the existence of traveling wave solutions very close to those we find in Chapter 3, see Section 1.3.2. That is, we would have obtained a different proof of our result, probably under weaker assumptions but with less information on the behavior of the traveling

⁸Before that one needs to prove that the problem is well-posed and that for all $t \in (0, +\infty)$, $w(t)$ satisfies (1.4.9).

wave as $x_1 \rightarrow -\infty$. This conjecture is natural, as the results of Risler provide an alternative for proving the existence of the traveling wave solution of Alikakos and Katzourakis [13].

We expect that carrying the full program that we describe here is a difficult task. However, we also find this question very interesting. If we were at least able to prove the relaxation statement, it is then likely that the existence of a non-minimizing stationary solution would follow under assumptions analogous to (H1.1) and (H1.2). This would allow to circumvent some issues that one finds if wishes to establish a functional setting for E_2 . Unlike the 1D case, we might have U such that $E_2(U) < +\infty$ and $w \in H^1$ such that $E_2(U + w) = +\infty$. This complicates the structure of the set of finite energy functions and the possible ways of perturbing inside this space. Another interesting question would be the study of the (asymptotic) stability of the minimizing 2D heteroclinics of Alama, Bronsard and Gui [3] and Schatzman [153]. This would follow immediately from relaxation provided that the 2D heteroclinics are isolated.

1.4.6 Heteroclinic traveling waves in fourth order systems

Consider the fourth order system (known as Fisher-Kolmogorov)

$$\Delta^2 \mathfrak{U} - \beta \Delta \mathfrak{U} = -\nabla_u V(\mathfrak{U}) \text{ in } \mathbb{R}^2 \quad (1.4.10)$$

where $\beta > 0$ and Δ^2 is the bi-laplacian. Consider also the associated 1D system

$$\frac{d^4}{dt^4} \mathfrak{q} - \beta \mathfrak{q}'' = -\nabla_u V(\mathfrak{q}) \text{ in } \mathbb{R}. \quad (1.4.11)$$

As for the second order case, equations (1.4.10) and (1.4.11) are variational and the associated energy functionals are (formally)

$$\tilde{E}_2(U) := \int_{\mathbb{R}^2} \left[\frac{|\Delta U(x_1, x_2)|^2}{2} + \frac{\beta |DU(x_1, x_2)|^2}{2} + V(U(x_1, x_2)) \right] dx_1 dx_2, \quad U \in H_{\text{loc}}^2(\mathbb{R}^2, \mathbb{R}^k)$$

and

$$\tilde{E}(q) := \int_{\mathbb{R}} \left[\frac{|q''(t)|^2}{2} + \frac{\beta |q'(t)|^2}{2} + V(q(t)) \right] dt, \quad q \in H_{\text{loc}}^2(\mathbb{R}, \mathbb{R}^k)$$

respectively. Existence results for heteroclinic solutions for (1.4.10) and (1.4.11) have been proven very recently. As in the second order case, one shows that a pair of wells (σ^-, σ^+) can be chosen so that there exists a heteroclinic orbit solving (1.4.11) which is a global minimizers of \tilde{E} in the space of paths joining σ^- and σ^+ at infinity. See Peletier, Troy and Van der Vorst [135], Kalies and Van der Vorst [102] for a proof in the scalar case (where, contrary to the second order case, one might have non-uniqueness) and Smyrnelis [159] for the extension to the vector-valued case. It was recently shown by Smyrnelis [158] that one can find a solution to (1.4.10) joining two different globally minimizing heteroclinics at infinity (assuming they exist), in the same sense than in the second order case with the obvious modifications. The proof is obtained as a byproduct of a more general abstract result, as it was also done by Smyrnelis in the paper [157] that we already discussed in Section 1.2.6. Therefore, a natural question is whether traveling waves for the parabolic gradient flow associated with (1.4.10) exist. More precisely, let

$$\partial_t w + \Delta^2 w - \beta \Delta w = -\nabla_u V(w) \text{ in } [0, +\infty) \times \mathbb{R}^2 \quad (1.4.12)$$

and consider the problem of finding traveling wave solutions to (1.4.12) of the type

$$w : (t, x_1, x_2) \in [0, +\infty) \times \mathbb{R}^2 \rightarrow \mathfrak{U}(x_1 - c^\star t, x_2) \in \mathbb{R}^k$$

with $c^\star > 0$ and \mathfrak{U} converging (in $H^2(\mathbb{R}, \mathbb{R}^k)$) at $-\infty$ to a globally minimizing heteroclinic and at $+\infty$ to a locally minimizing heteroclinic with energy close enough to the global minimum. As in the second order system, potentials verifying the necessary assumptions could be presumably found as a perturbation of those considered by Smyrnelis in [158]. The analogous question can also be posed for the 1D system associated with (1.4.11).

This question can be again formulated in an abstract framework (we refer to [158]) which however does not coincide with the one we consider in Chapters 3 and 4, since the norm of the second derivative appears in the energy functional. Indeed, \mathfrak{U} should be found as a global minimizer (in a suitable space) of

$$U \in H_{\text{loc}}^2(\mathbb{R}, L^2(\mathbb{R}, \mathbb{R}^k)) \rightarrow \int_{\mathbb{R}} \left[\frac{\|U''(t)\|_{L^2(\mathbb{R}, \mathbb{R}^k)}^2}{2} + \frac{\beta \|U'(t)\|_{L^2(\mathbb{R}, \mathbb{R}^k)}^2}{2} + \tilde{\mathcal{W}}(U(t)) \right] e^{ct} dt,$$

with

$$\tilde{\mathcal{W}}(v) = \begin{cases} \tilde{E}(v) & \text{if } v \in H^2(\mathbb{R}, \mathbb{R}^k), \\ +\infty & \text{otherwise.} \end{cases}$$

Therefore, in order to find the traveling waves for (1.4.12) one would need to succeed into adapting our abstract results into a higher order framework, which is essentially what it is done in [158] for the stationary wave. One could also aim, following again Smyrnelis [157], at studying other fourth order systems of the type

$$\partial_t w + \partial_{x_1 x_1} \partial_{x_2 x_2} w - \Delta w = -\nabla_u V(w) \text{ in } [0, +\infty) \times \mathbb{R}^2.$$

In such a case, one might hope to fit this problem into the abstract framework of Chapters 3 and 4, since the energy functionals for the profile are

$$U \in H_{\text{loc}}^1(\mathbb{R}, L_{\text{loc}}^2(\mathbb{R}, \mathbb{R}^k)) \rightarrow \int_{\mathbb{R}} \left[\frac{\|U'(t)\|_{H^1(\mathbb{R}, \mathbb{R}^k)}^2}{2} + \mathcal{W}(U(t)) \right] e^{ct} dt.$$

1.4.7 Multi-well potentials and singular limits

In order to complete the introduction of this thesis, we give a brief overview on an important and interesting question regarding multi-well/Allen-Cahn systems which, unfortunately, finally remained out of the scope of this thesis. Such a problem is that of studying the *singular limit* of a multi-well potential perturbed by a gradient term. It comes motivated by physics, more precisely the study of *phase transitions* in materials, a theory started by Allen and Cahn [14] and Cahn and Hilliard [62]. The definitions and statements we make here are usually not rigorous enough, the reader is referred to the given references for the precise mathematical description.

Consider the *scalar* Allen-Cahn potential

$$V_{AC} : u \in \mathbb{R} \rightarrow \frac{(1 - u^2)^2}{4} \in \mathbb{R},$$

which can be replaced by a scalar double well potential V without substantial alteration of the subsequent discussion. For $\varepsilon > 0$ we define the functional

$$F_{AC,\varepsilon}(u) := \int_{\Omega} \left[\frac{\varepsilon |\nabla u(x)|^2}{2} + \frac{1}{\varepsilon} V_{AC}(u(x)) \right] dx, \quad u : \Omega \rightarrow \mathbb{R}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded, smooth domain. The associated Euler-Lagrange equation is the semilinear PDE

$$\Delta u = \varepsilon^{-2} V'_{AC}(u) \text{ in } \Omega,$$

which is the standard Allen-Cahn equation. Physically speaking, u is an order parameter which represents the distribution of the material between two equally preferred stable states -1 and 1 . These two stable states can for instance represent two immiscible fluids. As detailed in [14, 62], experimentally for $\varepsilon \approx 0$ one observes that the minimizers (under a mass constraint) of $F_{AC,\varepsilon}$ present a phase transition between the two states such that its interfacial area is minimized. Mathematically speaking, the mass constraint on the minimizers forces them to split between the two states -1 and 1 , while the smallness of the parameter ε forces them to stay close to either -1 and 1 . There is an incompatibility between the two previous facts, which implies that when passing to the limit one obtains a singular object in some sense. Roughly speaking, given a family $(u_\varepsilon)_{\varepsilon>0}$ of critical points of $(F_{AC,\varepsilon})_{\varepsilon>0}$ (usually under a mass constraint), one finds (up to subsequences) a step function $u_0 : \Omega \rightarrow \{-1, 1\}$ such that the reduced boundary of $\{u_0 = 1\}$ is a surface of codimension one with constant mean curvature, up to a negligible set with respect to the $(N - 1)$ -dimensional Hausdorff measure. If the mass constraint is removed, one obtains a minimal surface. Therefore, the interfaces of the family $(u_\varepsilon)_{\varepsilon>0}$ *concentrate* in a codimension one (rectifiable) set (more precisely, a stationary varifold).

Rigorous mathematical results of the previous statements were first proven by Modica [120], following a previous work of Modica and Mortola [121], the Γ -convergence theory due to De Giorgi and various tools from geometric measure theory, we refer to [120] for the precise sources. As it is known, Γ -convergence is well-suited for minimizers, but not for general critical points, meaning that Modica's work does not apply to arbitrary families of critical points $(u_\varepsilon)_{\varepsilon>0}$. The extension to arbitrary critical points was performed by Hutchinson and Tonegawa [99], relying on a different approach. Moreover, they showed that for $\varepsilon \approx 0$, u_ε is essentially one-dimensional, close to *the* heteroclinic orbit joining -1 and 1 .

The scalar theory being well understood, the next natural step is to replace V_{AC} by a potential defined in \mathbb{R}^k for $k \geq 2$. At this point, there is a bifurcation. When passing to higher dimensions, one might consider the potentials of the type

$$V_{GL} : u \in \mathbb{R}^k \rightarrow \frac{(1 - |u|^2)^2}{4} \in \mathbb{R},$$

which are those of *Ginzburg-Landau theory*, emanating from the book by Bethuel, Brezis and Hélein [42]. Physically speaking, the Ginzburg-Landau model is used to study phenomena such as

superconductivity. Mathematically speaking, if one considers the asymptotic problem analogous to the Allen-Cahn one, a singular limit is also found. For the case $k = N = 2$ treated in [42], the limiting object of the previous problem is a harmonic map with point singularities. Generalizations to different values of k and N have also been studied. While the singularities of Allen-Cahn theory appear due to the disconnectedness of the zero set, in the classical Ginzburg-Landau theory of [42] they are due to topological obstructions due to degree theoretical reasons (in fact, one readily observes that the zero set of V_{GL} is \mathbb{S}^{k-1} , which is connected).

The other possible path of extending Allen-Cahn theory to systems is to replace V_{AC} by a multi-well potential V as the ones we consider in this thesis. From the physical perspective, this passage is motivated by the study of *multiphase materials*, rather than a situation in which only two phases are present. Let $\varepsilon > 0$, for the system

$$\Delta u_\varepsilon = -\varepsilon^{-2} \nabla V(u_\varepsilon), \text{ in } \Omega, \quad (1.4.13)$$

one aims at studying the behavior of families of solutions $(u_\varepsilon)_{\varepsilon>0}$ as $\varepsilon \rightarrow 0$. For the present problem, the zero set is disconnected unlike in Ginzburg-Landau theory and one also expects to also find minimal surfaces of co-dimension one as a limiting objects, as in the scalar Allen-Cahn theory. Indeed, several authors generalized the work of Modica to different vector-valued settings by proving Γ -convergence results: Baldo [28], Fonseca and Tartar [82], Sternberg [162, 163]. However, it is a very challenging problem to go beyond this Γ -convergence setting for systems in the fully general setting, as the arguments in [99] rely heavily on the scalar nature of the problem. Few partial results are available: see the recent paper by Bethuel [41], where the $N = 2$ case is solved and a more detail account on the problem is given.

An analogous question can be posed for the parabolic gradient flow,

$$\partial_t u_\varepsilon - \Delta u_\varepsilon = -\varepsilon^{-2} \nabla V(u_\varepsilon), \text{ in } (0, +\infty) \times \Omega. \quad (1.4.14)$$

In this framework, one expects that the interface *moves by mean curvature* as $\varepsilon \rightarrow 0$. This has been proven in the scalar case by Ilmanen [100]. For systems, it is a broad open question. Some partial results are due to Bronsard and Reitich [60], Laux and Simon [106]. See also the references therein. As in the elliptic case, the problem in its full generality seems out of the scope for now, but it is possible that further partial questions can be successfully treated.

Some questions reciprocal to the previous ones have been posed and answered. For instance: Given a co-dimension one minimal surface (or, more generally, with constant mean curvature) in \mathbb{R}^N , does there exist a sequence $(u_\varepsilon)_{\varepsilon>0}$ of solutions of (1.4.13) such that their interfaces concentrate on the given surface as $\varepsilon \rightarrow 0$? The answer turns out to be positive in the scalar case under generic non-degeneracy assumptions, see Pacard and Ritoré [132]. See also Kohn and Sternberg [104], where the proof is obtained by using the Γ -convergence structure and only applies to local minimizers. In the vector-valued setting, not much seems to have been written so far, besides a related question treated in the recent paper by Andrade, Conrado, Nardulli and Piccione [18], in which multiplicity results under a volume constraint are obtained for small $\varepsilon > 0$. The Γ -convergence results in [28, 82] allow to treat the minimizing case, but not the general one. In [101], Jerrard and Sternberg give general abstract conditions for which, given a critical point of

the Γ -limit⁹, there exist a sequence of critical points of the Γ -converging sequence such that the corresponding critical *values* converge to the *value* of the initial critical point. They show that this result applies to the 2D Modica-Mortola functional and the 3D Ginzburg-Landau functional. In general, the Γ -convergence framework is not sufficient for ensuring convergence of the critical points themselves; examples are provided in [101]. However, in the recent paper [69], the two previous authors together with Colinet showed that one has actual convergence of critical points in the 3D (Riemmanian) Ginzburg-Landau setting, under suitable non-degeneracy assumptions. Since the limit object in the Ginzburg-Landau setting is a co-dimension *two* surface (therefore, a geodesic in dimension 3), one sees that the 3D Ginzburg-Landau problem is somewhat linked to the 2D vector Allen-Cahn problem. Hence, it is natural to wonder if some sort of analogous of the results in [69] holds for 2D Allen-Cahn systems, at least for a subclass of potentials V , providing with a converse for the recent result by Bethuel [41].

Moreover, we think that a question analogous to the previous one could also be posed in the parabolic setting. That is, given a co-dimension one surface *moving by mean curvature* (e. g. in the Brakke sense), does there exist a family $(u_\varepsilon)_{\varepsilon>0}$ of solutions of (1.4.14) such that their interfaces concentrate on the initial surface as $\varepsilon \rightarrow 0$? We are not aware of works dealing with this question, even in the scalar case. Pursuing the analogy with [69, 101], it is possible that the adaptation of the Γ -convergence scheme for gradient flows due to Sandier and Serfaty [151] (applied by the authors to the Ginzburg-Landau setting) could be of some use here.

As we see, the questions and problems we mention in this paragraph differ with respect to the ones we concerned ourselves with in this thesis. These differences are due to both the nature of the problems and the mathematical techniques involved. Indeed, while geometric measure theory is almost not present in this thesis, it is central in the problems we bring up in this paragraph. We think it is possible that the results we obtained in this thesis can shed some light into the problems we described in this paragraph, but at this point we fail to see any concrete link.

⁹In a sense to precise, as usually the Γ -limit is not smooth

Non-minimizing connecting orbits for multi-well systems

Outline of the current chapter

2.1 Introduction	50
2.1.1 Goal of the chapter and statement of the main results	52
2.1.1.1 The general case	53
2.1.1.2 The symmetric case	54
2.2 Proofs of the results	56
2.2.1 Scheme of the proofs and comparison with the previous literature . .	58
2.2.1.1 The mountain pass geometry	58
2.2.1.2 The analysis of the Palais-Smale sequences	59
2.2.2 Preliminary results	60
2.2.3 Existence of a mountain pass geometry	65
2.2.4 An abstract deformation lemma	67
2.2.5 The proof of Theorem 2.1	68
2.2.6 The proofs of Theorems 2.2 and 2.3	69
2.2.6.1 The proof of Theorem 2.2	71
2.2.6.2 The proof of Theorem 2.3	72
2.3 On the assumptions (H2.7) and (H2.10)	73
2.4 Link with a continuous selection problem	75

This chapter is an extension of the article [131]. More precisely, [131] corresponds essentially to Sections 2.1, 2.2 and 2.3 in this chapter, with some minor style modifications. Section 2.4 contains some material which was finally removed from the published version [131] due to the partial and incomplete state of the results contained therein. However, we consider that Section 2.4 contains interesting points and this is the reason why we decided to include it in this thesis. We are indebted to the referee of the journal for all the remarks and suggestions which lead to an important improvement on the paper, as well as for pointing to us important sources such as some unpublished results contained in Bisgard's PhD Thesis [50].

Abstract. Given a nonnegative, smooth potential $V : \mathbb{R}^k \rightarrow \mathbb{R}$ ($k \geq 2$) with multiple zeros, we say that a curve $q : \mathbb{R} \rightarrow \mathbb{R}^k$ is a *connecting orbit* if it solves the autonomous system of ordinary differential equations

$$q'' = \nabla_u V(q), \quad \text{in } \mathbb{R}$$

and tends to a zero of V at $\pm\infty$. Broadly, our goal is to study the existence of connecting orbits for the problem above using variational methods. Despite the rich previous literature concerning the existence of connecting orbits for other types of second order systems, to our knowledge only connecting orbits which minimize the associated energy functional in a suitable function space were proven to exist for autonomous multi-well potentials. The contribution of this chapter is to provide, for a class of such potentials, some existence results regarding *non-minimizing* connecting orbits. Our results are closely related to the ones in the same spirit obtained by J. Bisgard in his PhD thesis (University of Wisconsin-Madison, 2005), where non-autonomous periodic multi-well potentials (ultimately excluding autonomous potentials) are considered. Our approach is based on several refined versions of the classical Mountain Pass Lemma and *concentration-compactness* arguments.

Résumé. Soit $V : \mathbb{R}^k \rightarrow \mathbb{R}$ ($k \geq 2$) une fonction (que l'on appelle *potentiel*) positive, régulière qui possède plusieurs zéros (que l'on appelle *puits*). On dit qu'une courbe $q : \mathbb{R} \rightarrow \mathbb{R}^k$ est une *orbite de connexion* si elle résout le système autonome d'équations différentielles ordinaires

$$q'' = \nabla_u V(q), \quad \text{in } \mathbb{R}$$

et converge vers un zéro de V lorsque $\pm \rightarrow +\infty$. Globalement, nous étudions l'existence d'orbites de connexion pour le problème ci-dessus en faisant appel à des méthodes variationnelles. Il existe une littérature très riche concernant l'existence d'orbites de connexion pour des autres systèmes de second ordre. Cependant, à notre connaissance seulement des orbites de connexion qui minimisent la fonctionnelle d'énergie associée dans un espace fonctionnel bien choisi avaient été trouvées pour des potentiels multi-puits autonomes. La contribution de cet article consiste à démontrer, pour cette classe de potentiels, quelques résultats d'existence concernant les orbites de connexion *non-minimisantes*. Nos résultats sont proches de ceux dans le même esprit obtenus par J. Bisgard dans sa thèse de doctorat (University of Wisconsin-Madison, 2005), où des potentiels non-autonomes et périodiques sont considérés. Ces résultats ne s'appliquent pas au cadre autonome de cet article. L'approche de nos preuves est basée sur plusieurs raffinements du Lemme du Col classique et des arguments du type *concentration-compactité*.

2.1 Introduction

The focus of this chapter is to find solutions $q : \mathbb{R} \rightarrow \mathbb{R}^k$ to the second order ordinary differential equation

$$q'' = \nabla_u V(q), \quad \text{in } \mathbb{R} \tag{2.1.1}$$

verifying the conditions at infinity

$$\lim_{t \rightarrow \pm\infty} q(t) = \sigma_{\pm}.$$

If $\sigma_- = \sigma_+$, we say that the solution q is a *homoclinic orbit*. If $\sigma_- \neq \sigma_+$, we say that q is a *heteroclinic orbit*. The function V is a standard multi-well potential. That is, a non negative function vanishing in a finite set Σ , with non degenerate global minima. The elements σ_- and σ_+ belong to the set Σ . If $\sigma \in \Sigma$, we say that σ is a *well* of V . More precisely, V is as follows:

(H2.1). $V \in \mathcal{C}_{\text{loc}}^2(\mathbb{R}^k)$ and $V \geq 0$ in \mathbb{R}^k . Moreover, $V(\mathbf{u}) = 0$ if and only if $\mathbf{u} \in \Sigma$, where, for some $l \geq 2$

$$\Sigma := \{\sigma_1, \dots, \sigma_l\}.$$

(H2.2). There exist $\alpha_0, \beta_0, R_0 > 0$ such that for all $\mathbf{u} \in \mathbb{R}^k$ with $|\mathbf{u}| \geq R_0$ it holds $\langle \nabla_{\mathbf{u}} V(\mathbf{u}), \mathbf{u} \rangle \geq \alpha_0 |\mathbf{u}|^2$ and $V(\mathbf{u}) \geq \beta_0$.

(H2.3). For all $\sigma \in \Sigma$, the matrix $D^2 V(\sigma)$ is positive definite.

One formally checks that critical points of the functional

$$E(q) := \int_{\mathbb{R}} e(q)(t) dt := \int_{\mathbb{R}} \left[\frac{1}{2} |q'(t)|^2 + V(q(t)) \right] dt, \quad q \in H_{\text{loc}}^1(\mathbb{R}, \mathbb{R}^k),$$

solve equation (2.1.1). For any $(\sigma_i, \sigma_j) \in \Sigma^2$ we consider as in Rabinowitz [136] the function space

$$X(\sigma_i, \sigma_j) := \left\{ q \in H_{\text{loc}}^1(\mathbb{R}, \mathbb{R}^k) : E(q) < +\infty \text{ and } \lim_{t \rightarrow -\infty} q(t) = \sigma_i, \lim_{t \rightarrow +\infty} q(t) = \sigma_j \right\},$$

and seek for critical points inside these spaces, as one easily shows that any finite energy curve in $H_{\text{loc}}^1(\mathbb{R}, \mathbb{R}^k)$ must belong to $X(\sigma_i, \sigma_j)$ for some $(\sigma_i, \sigma_j) \in \Sigma^2$. We first define the infimum value

$$\mathfrak{m}_{\sigma_i \sigma_j} := \inf \{ E(q) : q \in X(\sigma_i, \sigma_j) \}. \quad (2.1.2)$$

The minimization problem in (2.1.2) is well understood. Indeed, if $\sigma_i = \sigma_j$, then (2.1.2) is attained by the constant curve σ_i . Otherwise, the problem is more involved but still well known (see Bolotin [51], Bolotin and Kozlov [52], Bertotti and Montecchiari [40] and Rabinowitz [138, 142]). Its lack of compactness implies that (2.1.2) does not always have a solution if Σ possesses at least three elements. Let us fix once and for all $(\sigma^-, \sigma^+) \in \Sigma^2$, $\sigma^- \neq \sigma^+$ and set

$$\mathfrak{m} := \mathfrak{m}_{\sigma^- \sigma^+}. \quad (2.1.3)$$

We will assume that the following strict triangle's inequality holds:

(H2.4). We have that

$$\forall \sigma \in \Sigma \setminus \{\sigma^-, \sigma^+\}, \quad \mathfrak{m} < \mathfrak{m}_{\sigma^- \sigma} + \mathfrak{m}_{\sigma \sigma^+}.$$

Under assumption (H2.4), it is well known that by concentration-compactness arguments (Lions [108]) there exists a globally minimizing heteroclinic in $X(\sigma^-, \sigma^+)$. See Theorem 0.1 later for a precise statement.

We finally recall that the Sobolev embeddings imply that curves in $H_{\text{loc}}^1(\mathbb{R}, \mathbb{R}^k)$ are continuous. This classical fact is used implicitly along the chapter.

2.1.1 Goal of the chapter and statement of the main results

The goal of this chapter is to show that for a class of multi-well potentials V , there exist connecting orbits (either heteroclinic or homoclinic) which are not global minimizers in their natural spaces. We obtain several such results using variational methods. In particular our proof is based on a mountain pass argument (see Ambrosetti and Rabinowitz [16]).

There exists a vast literature concerning the existence of non-minimizing heteroclinics or homoclinic orbits for second order ordinary differential systems using variational methods. Some early references are Ambrosetti and Coti Zelati [15], Coti Zelati and Rabinowitz [73], Rabinowitz [137, 136]. Despite this fact, this question had not been addressed for the case of the autonomous multi-well potentials that we consider in this chapter. However, the case of time-periodic multi-well potentials has been studied by Montecchiari and Rabinowitz in [123, 122] as well as by Bisgard in the second chapter of his PhD Thesis [50]. The present chapter deals with a problem which is analogous to that in [50]. It is worth mentioning that while most of Bisgard's technical results also apply to the autonomous problem, his main results ultimately exclude such a possibility. The reason is that his key assumption is never satisfied by autonomous potentials due to the translation invariance of the associated problem. Roughly speaking, our Theorem 2.1 shows that the ideas and arguments of Bisgard, as well as his key assumption, can be adapted to the autonomous setting. Nevertheless, our strategy and assumptions present some difference with respect to his. A detailed account regarding the main differences and similarities between the proofs is given in Section 2.2.1. We also provide the proof of other results, which are Theorems 2.2 and 2.3, using for them a symmetry assumption on V . These results do not have a counterpart in Bisgard's work.

Our mountain pass argument is carried out under a multiplicity assumption (up to translations) on the set of globally minimizing heteroclinics joining the two fixed wells σ^- and σ^+ . More precisely, the natural idea is to suppose that there exists a gap in the set of global minimizers and consider the family of paths that join two disconnected components. Subsequently, one shows that the associated min-max value is strictly larger than the minimum value, so that the existence of a mountain pass geometry has been established. Examples of earlier papers in which this approach is used are Bolotin and Rabinowitz [53, 54], de la Llave and Valdinoci [109] as well as the above mentioned [50, 123, 122]. In our precise context, we work under assumption (H2.5). This assumption was introduced by Alessio [6] and it has been used under different forms for proving existence of solutions for Allen-Cahn systems, see the recent paper by Alessio and Montecchiari [7] for a survey. It is the natural generalization of the assumption introduced by Alama, Bronsard and Gui [3] in their celebrated paper concerning entire solutions for two-dimensional Allen-Cahn systems.

We write $\mathcal{H} := H^1(\mathbb{R}, \mathbb{R}^k)$ and $\mathcal{L} := L^2(\mathbb{R}, \mathbb{R}^k)$. We define

$$\mathcal{F} := \{q : q \in X(\sigma^-, \sigma^+) \text{ and } E(q) = m\},$$

the set of globally minimizing heteroclinics. The quantity m is as in (2.1.3). The invariance by translations of the problem implies that if $q \in \mathcal{F}$, then for all $\tau \in \mathbb{R}$ we have $q(\cdot + \tau) \in \mathcal{F}$. It is well-known (see Lemma 2.2.1) that $X(\sigma^-, \sigma^+)$ has the structure of an affine space in $H_{\text{loc}}^1(\mathbb{R}, \mathbb{R}^k)$

and it is a metric space when endowed with the natural distance

$$d : (q, \bar{q})^2 \in X(\sigma^-, \sigma^+) \rightarrow \|q - \bar{q}\|_{\mathcal{H}}. \quad (2.1.4)$$

We can now state the following assumption:

(H2.5). It holds $\mathcal{F} := \mathcal{F}_0 \cup \mathcal{F}_1$ where \mathcal{F}_0 and \mathcal{F}_1 are not empty and such that

$$d(\mathcal{F}_0, \mathcal{F}_1) > 0,$$

where d is the distance defined in (2.1.4).

As stated before Assumption (H2.5) is the gap condition which permits the mountain pass approach. Implicitly, it implies that $k \geq 2$, as it is well-known that heteroclinics are unique in the scalar case $k = 1$. As it was pointed out before, (H2.5) was already considered in [6] and it generalizes the one made in the previous work [3]. Let us now define

$$\psi(t) := \begin{cases} \sigma^- & \text{if } t \leq -1, \\ \frac{t+1}{2}\sigma^+ + \frac{1-t}{2}\sigma^- & \text{if } -1 \leq t \leq 1, \\ \sigma^+ & \text{if } t \geq 1. \end{cases} \quad (2.1.5)$$

We have that for all $v \in \mathcal{H}$ it holds that $v + \psi \in X(\sigma^-, \sigma^+)$ (see Lemma 2.2.1 for a proof). As in the earlier works [50, 123] we define the functional

$$J : v \in \mathcal{H} \rightarrow E(v + \psi) \in \mathbb{R}, \quad (2.1.6)$$

which presents the advantage of being defined in a linear space. We also point out that the choice of the function ψ is arbitrary.

2.1.1.1 The general case

We set $\mathcal{V} := \mathcal{F} - \{\psi\}$, and for $i \in \{0, 1\}$, $\mathcal{V}_i := \mathcal{F}_i - \{\psi\}$. Those are nonempty subsets of \mathcal{H} . We can now define the mountain pass family:

$$\Gamma := \{\gamma \in C([0, 1], \mathcal{H}) : \forall i \in \{0, 1\}, \gamma(i) \in \mathcal{V}_i\} \quad (2.1.7)$$

and the corresponding mountain pass value

$$\mathfrak{c} := \inf_{\gamma \in \Gamma} \max_{s \in [0, 1]} J(\gamma(s)) < +\infty. \quad (2.1.8)$$

In this chapter we show that $\mathfrak{c} > \mathfrak{m}$ (see Proposition 2.2.2 later). Therefore, \mathfrak{c} is a *mountain pass value* for J . As it is well known, this is generally not sufficient to ensure the existence of new solutions. In order to prove our first result, we will need two more assumptions:

(H2.6). It holds that $\mathfrak{c} < \mathfrak{m}^*$, where

$$\mathfrak{m}^* := \min\{\mathfrak{m}_{\sigma^- \sigma} + \mathfrak{m}_{\sigma \sigma^+} : \sigma \in \Sigma \setminus \{\sigma^-, \sigma^+\}\}. \quad (2.1.9)$$

It is clear that (H2.6) is stronger than (H2.4) and weaker than $\Sigma = \{\sigma^-, \sigma^+\}$. It is used in order to prevent that curves with energy close to \mathfrak{c} go trough a well in $\Sigma \setminus \{\sigma^-, \sigma^+\}$, in case there are any.

(H2.7). There exists a closed set $K \subset \mathbb{R}^k$ such that:

1. There exists $\nu_0 > 0$ such that

$$\forall q \in \mathcal{F}, \quad \text{dist}(q(\mathbb{R}), K) \geq \nu_0$$

where dist stands for the usual Euclidean distance between two sets in \mathbb{R}^k .

2. There exists $M > \mathfrak{c}$ such that for any $\gamma \in \Gamma$ (Γ is defined in (2.1.7)) such that $\max_{s \in [0,1]} J(\gamma(s)) \leq M$, there exists $s_\gamma \in [0,1]$ such that $J(\gamma(s_\gamma)) \geq \mathfrak{c}$ and $(\gamma(s_\gamma) + \psi)(\mathbb{R}) \cap K \neq \emptyset$, where \mathfrak{c} is the mountain pass value defined in (2.1.8).

Assumption (H2.7) is more technical and as we show in Lemma 2.3.1 it is satisfied if $\mathfrak{c} \notin \{(2j+1)\mathfrak{m} : j \in \mathbb{N}^*\}$, or more particularly if $\mathfrak{c} < 3\mathfrak{m}$. An analogous assumption was made by Bisgard in [50] with the same purpose. The comparison is made in Section 2.2.1. Our first result then is as follows:

Theorem 2.1. *Assume that (H2.1), (H2.2), (H2.3), (H2.5), (H2.6) and (H2.7) hold. Then, there exists $u \in H_{\text{loc}}^1(\mathbb{R}, \mathbb{R}^k) \cap C^2(\mathbb{R}, \mathbb{R}^k)$ a solution of (2.1.1) that satisfies one of the two following conditions:*

1. *u is not constant, $E(u) \leq \mathfrak{c}$ and u is homoclinic to σ^- or σ^+ , that is, there exists $\sigma \in \{\sigma^-, \sigma^+\}$ such that*

$$\lim_{t \rightarrow \pm\infty} u(t) = \sigma.$$

2. *$u \in X(\sigma^-, \sigma^+)$ and $\mathfrak{c} \geq E(u) > \mathfrak{m}$.*

Moreover, $u(0) \in K$.

That is, Theorem 2.1 shows that, under the previous assumptions, there exists a non-minimizing solution which might be either heteroclinic or homoclinic. As it will be made clear later, Theorem 2.1 is strongly related to *Theorem 2.3* by Bisgard [50].

Remark 2.1.1. Following the arguments by Bisgard [50] which give rise to his *Theorem 2.2*, we also have that there exists a (possibly small) constant $\eta_{\min} > 0$ such that if $\mathfrak{c} < \mathfrak{m} + \eta_{\min}$ ($\eta_{\min} < \mathfrak{m}$), then u is heteroclinic and $J(u) = \mathfrak{c}$. See Corollary 2.2.1.

2.1.1.2 The symmetric case

In [3], Alama, Bronsard and Gui considered potentials which are symmetric with respect to a reflection:

(H2.8). We have that $\sigma^- = (-1, 0, \dots, 0)$ and $\sigma^+ = (+1, 0, \dots, 0)$. Moreover, we have for all $u \in \mathbb{R}^k$, $V(\mathfrak{s}(u)) = V(u)$, where

$$\mathfrak{s} : u = (u_1, u_2, \dots, u_k) \in \mathbb{R}^k \rightarrow (-u_1, u_2, \dots, u_k) \in \mathbb{R}^k.$$

Such condition eliminates the degeneracy due to invariance by translations (when restricted to the set of symmetric curves) and, hence, allows to restore some compactness. The first remark is that condition (H2.8) allows to look for solutions which belong to the equivariant space:

$$X_{\text{sym},+} := \{q \in X(\sigma^-, \sigma^+) : \forall t \geq 0, q_1(t) \geq 0 \text{ and } \mathfrak{s}(q(t)) = q(-t)\}.$$

The purpose of the symmetry assumption (H2.8) is to replace (H2.7) in order to obtain a slightly better result. Moreover, we show that the combination of both hypothesis permits to ensure the existence of a non-minimizing heteroclinic in $X(\sigma^-, \sigma^+)$, while the general setting of Theorem 2.1 does not allow us to claim such a thing (see however Remark 2.1.1). Firstly, we recall that assumption (H2.8) shows that energy decreases by symmetrization, see Lemma 2.2.9 later. Therefore, we have that the sets

$$\forall i \in \{0, 1\}, \mathcal{F}_{\text{sym},i} := \mathcal{F}_i \cap X_{\text{sym},+}$$

are non-empty by (H2.5). Moreover, $d(\mathcal{F}_{\text{sym},0}, \mathcal{F}_{\text{sym},1}) \geq d(\mathcal{F}_0, \mathcal{F}_1) > 0$, again by assumption (H2.5). We write $\mathcal{F}_{\text{sym}} := \mathcal{F} \cap X_{\text{sym},+}$, notice that $\mathcal{F}_{\text{sym}} = \mathcal{F}_{\text{sym},0} \cup \mathcal{F}_{\text{sym},1}$. We see that the function ψ defined in (2.1.5) belongs to $X_{\text{sym},+}$. Hence, we can do as before and define:

$$\mathcal{H}_{\text{sym}} := \{v \in \mathcal{H} : \forall t \geq 0, \mathfrak{s}(v(t)) = v(-t)\},$$

which is a closed subspace of \mathcal{H} , thus we will regard it as a Hilbert space itself. Notice that by Lemma 2.2.1 and the linearity of the symmetry, we have as before that $\{\psi\} + \mathcal{H}_{\text{sym}} = X_{\text{sym}}$. We set $\mathcal{V}_{\text{sym}} := \mathcal{F}_{\text{sym}} - \{\psi\}$ and for $i \in \{0, 1\}$

$$\mathcal{V}_{\text{sym},i} := \mathcal{F}_{\text{sym},i} - \{\psi\}, \quad (2.1.10)$$

which are subsets of \mathcal{H} . We now have all the ingredients to define the *symmetric mountain pass family*

$$\Gamma_{\text{sym}} := \{\gamma \in C([0, 1], \mathcal{H}_{\text{sym}}) : \forall i \in \{0, 1\}, \gamma(i) \in \mathcal{V}_{\text{sym},i}\}. \quad (2.1.11)$$

As we will see later, the possibility of considering only the paths contained in \mathcal{H}_{sym} will be the key of our argument. Now, define the corresponding mountain pass value:

$$\mathfrak{c}_{\text{sym}} := \inf_{\gamma \in \Gamma_{\text{sym}}} \max_{s \in [0, 1]} J(\gamma(s)) < +\infty. \quad (2.1.12)$$

As before, we show that $\mathfrak{c}_{\text{sym}} > \mathfrak{m}$ (Proposition 2.2.3). Subsequently, we write the analogous of (H2.6) for $\mathfrak{c}_{\text{sym}}$:

(H2.9). It holds that $\mathfrak{c}_{\text{sym}} < \mathfrak{m}^*$, where \mathfrak{m}^* is introduced in (2.1.9).

We can finally state the first result in the symmetric setting:

Theorem 2.2. Assume that (H2.1), (H2.2), (H2.3), (H2.5), (H2.8) and (H2.9) hold. Then, we have one of the two following scenarios:

1. There exist \mathfrak{u}_+ and \mathfrak{u}_- in $H_{\text{loc}}^1(\mathbb{R}, \mathbb{R}^k) \cap C^2(\mathbb{R}, \mathbb{R}^k)$ two non constant functions such that $E(\mathfrak{u}_+) \leq \mathfrak{c}_{\text{sym}}$, $\lim_{t \rightarrow \pm\infty} \mathfrak{u}_+(t) = \sigma^+$ and \mathfrak{u}_- is obtained by reflecting \mathfrak{u}_+ , that is

$$\forall t \in \mathbb{R}, \mathfrak{u}_-(t) = \mathfrak{s}(\mathfrak{u}_+(t)).$$

In particular, $\lim_{t \rightarrow \pm\infty} u_-(t) = \sigma^-$.

2. There exists $u \in X_{\text{sym},+}$ such that $E(u) = c_{\text{sym}}$. In particular, $u \notin \mathcal{F}_{\text{sym}}$.

Remark 2.1.2. Notice that in the first case in Theorem 2.2, the solution u_- is obtained for free from u_+ . Indeed, it suffices to check that, due to (H2.8), any q solution of (2.1.1) gives rise to a reflected solution \hat{q} defined as $\hat{q}: t \in \mathbb{R} \rightarrow s(q(t))$.

Finally, we show that under an assumption which combines (H2.7) and (H2.8) we can be sure to obtain a non-minimizing heteroclinic joining σ^- and σ^+ . Such assumption writes as follows:

(H2.10). Assumption (H2.8) holds. Moreover, there exists a closed set $K_{\text{sym}} \subset \mathbb{R}^k$ such that:

1. There exists $\nu_0 > 0$ such that

$$\forall q \in \mathcal{F}_{\text{sym}}, \quad \text{dist}(q(0), K_{\text{sym}}) \geq \nu_0$$

where dist stands for the usual Euclidean distance between two sets in \mathbb{R}^k .

2. Let Γ_{sym} be as in (2.1.11) and c_{sym} be as in (2.1.12). There exists $M > c_{\text{sym}}$ such that for any $\gamma_+ \in \Gamma_{\text{sym}}$ with $(\psi + \gamma_+)([0, 1]) \subset X_{\text{sym},+}$ and $\max_{s \in [0, 1]} J(\gamma_+(s)) \leq M$, there exists $s_\gamma \in [0, 1]$ such that $J(\gamma_+(s_\gamma)) \geq c_{\text{sym}}$ and $\gamma_+(s_\gamma)(0) \in K_{\text{sym}}$.

Assumption (H2.10) is nothing but the symmetric version of (H2.7). Notice that we also need to ask that $\gamma(s_\gamma)(0) \in K_{\text{sym}}$, which is stronger than the condition $(\gamma(s_\gamma) + \psi)(\mathbb{R}) \cap K \neq \emptyset$ required in (H2.7). We can then state the following result:

Theorem 2.3. Assume that (H2.1), (H2.2), (H2.3), (H2.5), (H2.9) and (H2.10) hold. Then, there exists a solution $u \in X_{\text{sym},+}$ such that $c_{\text{sym}} \geq E(u) > m$.

Remark 2.1.3. Notice that the hypothesis of Theorem 2.2 are contained in those of Theorem 2.3. Therefore, if u is the solution given by Theorem 2.3, then by Theorem 2.2 either $E(u) = c_{\text{sym}}$ or there exist u_+, u_- a pair of non constant homoclinics.

The chapter is organized as follows: Section 2.2 is devoted to the proofs of Theorems 2.1, 2.2 and 2.3. Section 2.3 is devoted to some comments and results regarding the assumptions (H2.7) and (H2.10).

2.2 Proofs of the results

This section is devoted to the proofs of Theorems 2.1, 2.2 and 2.3. The organisation goes as follows: In Section 2.2.1, we give the overall scheme of the proofs and compare it with the previous literature. In Section 2.2.2, we state the preliminary results which are needed, most of which are well-known. In Section 2.2.3, we prove the existence of the mountain pass geometry. In Section 2.2.4, we state an abstract deformation result from Willem [172] which is used after. In Section 2.2.5, we provide the proof of Theorem 2.1. Finally, Section 2.2.6 is devoted to the proofs of Theorems 2.2 and 2.3.

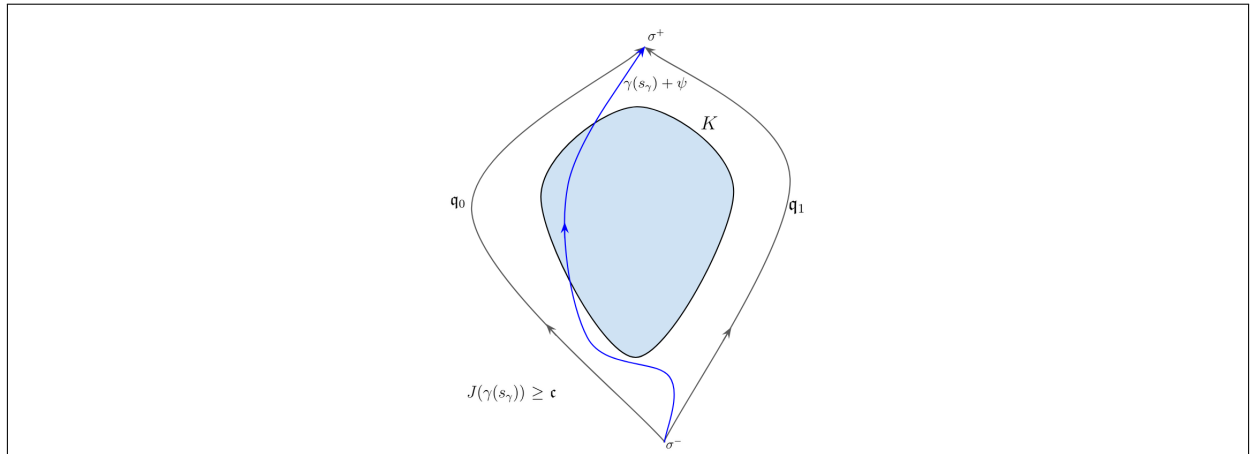


Figure 2.1: Illustration of (H2.7) for the particular case of a potential with exactly two distinct (up to translations) globally minimizing heteroclinics (q_0 and q_1) between σ^- and σ^+ .

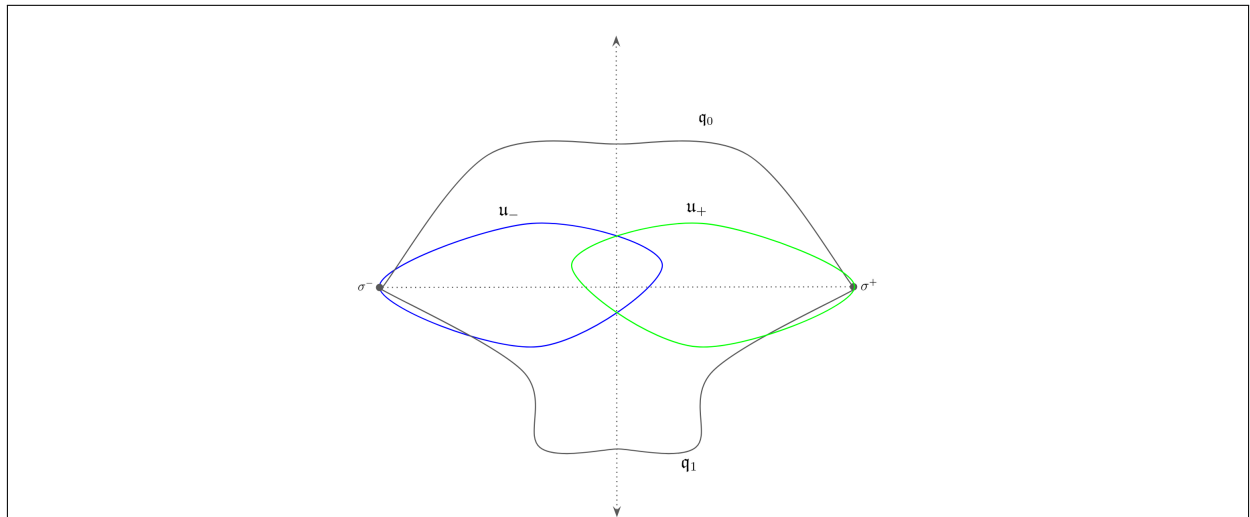


Figure 2.2: Illustration of one of the two possible scenarios described in Theorem 2.2. The heteroclinic orbits q_0 and q_1 represent two distinct symmetric minimizing heteroclinics. The two homoclinic orbits u_- and u_+ are related by the reflection s .

2.2.1 Scheme of the proofs and comparison with the previous literature

As stated in the introduction, it is worth recalling that the problem of the existence of homoclinic and heteroclinic solutions for the second-order system of ODEs

$$q''(t) = \nabla_{\mathbf{u}} V_{\star}(t, q(t)), \quad \forall t \in \mathbb{R}, \quad (2.2.1)$$

using variational methods has been extensively studied during the past decades. In (2.2.1), $V_{\star} : \mathbb{R} \times \mathbb{R}^k \rightarrow \mathbb{R}$ is the potential, usually T -periodic in time. Some examples of early papers which use a mountain pass approach to find such solutions are Caldirola and Montecchiari [63], Coti Zelati and Rabinowitz [73] and Rabinowitz [137] (where the autonomous case is also treated). In those papers, the potential considered is quite far from being of multi-well type, meaning that the geometry of the associated functional is substantially different to the one considered in the present chapter. On the contrary, in the papers Montecchiari and Rabinowitz [123, 122] as well as Bisgard [50], T -periodic multi-well potentials V_{\star} (with explicit time dependence) are considered. In this chapter, we prove results which are very close (but not included) to those in [50] following an equivalent scheme of proof. More precisely, we rely on the following natural approach (as for instance in the seminal paper by Brézis and Nirenberg [58]):

1. We prove the existence of a min-max value. In our case, we show in Proposition 2.2.2 that there exists a mountain pass value using the gap condition (H2.5). The same is shown in Proposition 2.2.3 for the symmetric setting
2. We analyze the behavior of the associated *Palais-Smale sequences* in order to establish the existence of non-minimizing solutions from this analysis. This is the purpose of assumptions (H2.7), (H2.8) and (H2.10), which give rise to Theorems 2.1, 2.2 and 2.3 respectively.

We now detail the previous steps of the proof and compare with [50].

2.2.1.1 The mountain pass geometry

In order to obtain a mountain pass geometry, Bisgard and the other authors consider q a globally minimizing heteroclinic joining two wells σ^- and σ^+ . If V_{\star} is say 1-periodic in time and the set $\Sigma := \{V_{\star} = 0\}$ is t -independent, this implies that for any $n \in \mathbb{Z}$, $q(\cdot + n)$ is also a globally minimizing heteroclinic. In order to establish the mountain pass geometry, Bisgard and the other authors define the family of paths

$$\Gamma_{\star} := \{\gamma \in C([0, 1], \mathcal{H}) : \gamma(0) = q - \psi \text{ and } \gamma(1) = q(\cdot + 1) - \psi\},$$

where ψ is an interpolating function between σ^- and σ^+ as in (2.1.5). If one considers the min-max value

$$c_{\star} := \inf_{\gamma \in \Gamma_{\star}} \max_{s \in [0, 1]} J_{\star}(\gamma(s)),$$

where

$$J_{\star} : v \in \mathcal{H} \rightarrow \int_{\mathbb{R}} \left[\frac{|v'(t) + \psi'(t)|^2}{2} + V_{\star}(t, v(t) + \psi(t)) \right] dt,$$

then $c_\star > J_\star(q - \psi) = J_\star(q(\cdot + 1) - \psi)$ implies

$$\{q(\cdot + \tau) : \tau \in [0, 1]\} \text{ is not a continuum of globally minimizing heteroclinics,} \quad (2.2.2)$$

see *Proposition 2.1* in [50]. Since (2.2.2) is never fulfilled if V_\star is autonomous due to translation invariance, autonomous potentials are excluded from Bisgard's approach. Hence, in order to find a mountain pass value of this type for the case of autonomous potentials (that is, for the functional J defined in (2.1.6)), we need then to add an additional assumption which produces a mountain pass geometry by playing a role analogous to (2.2.2). As explained before, we do so by considering the natural candidate (H2.5) introduced in [6]. Indeed, in *Proposition 2.2.2* we show that such an assumption implies the existence of a mountain pass geometry for the autonomous case. Notice that (2.2.2) only requires an explicit time dependence on the potential and, therefore, it does not exclude the scalar case. On the contrary, assumption (H2.5) for the autonomous problem is more restrictive and completely rules out scalar potentials.

2.2.1.2 The analysis of the Palais-Smale sequences

Once the mountain pass geometry has been established, the next natural step is to analyze the behavior of the Palais-Smale sequences at the mountain pass level, as the classical Palais-Smale condition is not satisfied by J nor J_\star . For J_\star , this analysis is known and it can be found in *Proposition 3.10* in Rabinowitz [136], as well as the results in Bisgard [50], especially *Theorem 1.21*. Condition (2.2.2) is not necessary for proving those results, meaning that, in particular, they apply to our J , see *Proposition 2.2.1*. From this analysis it follows that Palais-Smale sequences (both for J_\star and J) split into a chain of connecting orbits solving (2.1.1) and that the sum of the energies of the elements of the chain is equal to the level of the Palais-Smale sequence. Using (H2.6), we find that if one of the elements of the chain is not a globally minimizing heteroclinic between σ^- and σ^+ , then *Theorem 2.3* in [50] or *Theorem 2.1* here is established. Nevertheless, there is still the possibility that each element of the limiting chain is a globally minimizing heteroclinic joining σ^- and σ^+ . In such a case, no new solution is produced by the mountain pass argument. Therefore, one needs to rule out this possibility by examining more closely the behavior of the Palais-Smale sequences at the mountain pass level. In [50], the possibility of a chain of minimizing heteroclinics is excluded by imposing an assumption on the mountain pass level c_\star . More precisely, by setting

$$m_\star^\pm := \inf_{v \in \mathcal{H}} J_\star(v + \psi(\pm \cdot)) \quad (2.2.3)$$

if we have that

$$c_\star \notin \{k_1 m_\star^- + k_2 m_\star^+ : k_1 + k_2 = 2j + 1, j \in \mathbb{N}^*\}, \quad (2.2.4)$$

then one of the elements of the limiting chain satisfies the requirements. This is essentially the assumption imposed by Bisgard in *Theorem 2.3* [50]. In our case, assumption (H2.7) serves the same purpose. The difference is that our argument is slightly more involved, as (H2.7) does not allow to claim the desired conclusion in such a direct fashion. Instead, we show by a deformation procedure¹ based on a result by Willem [172] that (H2.7) implies that there exists a Palais-Smale

¹We owe this idea to the referee. In previous versions of this work, we relied instead on a lengthier and less direct argument based on a localized version of the mountain pass lemma due to Ghoussoub and Preis [89, 87].

sequence at the mountain pass level for which each element of the sequence goes through the set K , so it cannot be asymptotic to a formal chain of globally minimizing heteroclinics. The purpose of this approach is the following: as we show in Lemma 2.3.1, if \mathfrak{c} satisfies

$$\mathfrak{c} \notin \{(2j+1)\mathfrak{m} : j \in \mathbb{N}^+\} \quad (2.2.5)$$

then (H2.7) holds. Relation (2.2.5) is nothing but the reformulation of (2.2.4) for the autonomous case. Indeed, in the autonomous setting, the values \mathfrak{m}_\star^\pm defined in (2.2.3) coincide (while they do not necessarily do in the non-autonomous case) meaning that (2.2.4) and (2.2.5) are the same. Therefore, one could assume (2.2.5) instead of (H2.7) and obtain Theorem 2.1 by the same way that in [50]. Nevertheless, as shown in Lemma 2.3.1 we have that (H2.7) can be more general, so we worked under it instead of (2.2.5). In particular, the possibility $\mathfrak{c} \in \{(2j+1)\mathfrak{m} : j \in \mathbb{N}^+\}$ is not excluded by (H2.7). We think that this feature is relevant as some addition phenomenon among the energies of several non-minimizing solutions in the chain could happen so that the total sum of the energies would be in $(2j+1)\mathbb{N}$. In this case, (2.2.4) would not allow to conclude while (H2.7) would.

Another assumption is made by Bisgard in [50], which leads to the stronger result *Theorem 2.2*, where existence of an heteroclinic at the mountain pass level is shown. It consists on supposing that the mountain pass value is close enough to the minimum. The proof follows from the fact that for a range of values close enough to the minimum, no splitting on the Palais-Smale sequences can occur, meaning that they converge strongly. As we pointed out in Remark 2.1.1, the same result holds for our problem. The precise statement is given in Corollary 2.2.1.

In any case, all the assumptions discussed before can be difficult to verify in applications. For this reason, we consider the more explicit symmetry assumption (H2.8) in order to remove the degeneracy due to invariance by translations and recover some compactness. Under this assumption and (H2.9), we show that if we have dichotomy of the Palais-Smale sequence (which can be chosen such that it belongs to the appropriate symmetrized space $X_{\text{sym},+}$) then there exists a pair of non-constant homoclinic solutions. Theorem 2.2 is then deduced. The idea of using the symmetries in order to recover compactness and subsequently establishing existence and multiplicity results has been extensively used in the previous research, we refer for instance to the seminal paper by Berestycki and Lions [36] as well as Van Schaftingen [167] which contains some of the key ideas that we use in our approach and other material. Assumption (H2.8) has the advantage of being more explicit than (H2.7), (2.2.4) and (2.2.5), but it rules out a wide class of interesting non-symmetric potentials. We can also combine (H2.8) with (H2.10), which is the symmetrized version of (H2.10), in order to show the existence of a non-minimizing heteroclinic, which is Theorem 2.3. This is done by relying again on the deformation argument.

2.2.2 Preliminary results

In this subsection, we state the technical preliminary results which will be used for establishing the main Theorems. They are for the most part essentially known and a few others are proven by classical arguments. Some relevant references which contain them (or close versions of them) are Rabinowitz [136], Bisgard [50], Montecchiari and Rabinowitz [123], Bertotti and Montecchiari [40], Alama, Bronsard and Gui [3], Bronsard, Gui and Schatzman [59]. In several cases, we take

results from those references and we rephrase them in order to be coherent with our setting.

We begin by recalling some basic properties on the potential V . These properties are easy to prove and well known, so the proofs are skipped. We refer, for instance, to [50] and see also [2] for a particularization to the autonomous case. We first recall the following:

Lemma 2.2.1. *Assume that (H2.1) and (H2.3) hold. Let $(\sigma_i, \sigma_j) \in \Sigma^2$. Let q and \tilde{q} be two elements in $X(\sigma_i, \sigma_j)$. Then $q - \tilde{q} \in \mathcal{H}$. Similarly, if $q \in X(\sigma_i, \sigma_j)$ and $v \in \mathcal{H}$ then $v + q \in X(\sigma_i, \sigma_j)$.*

We refer for instance to Lemma 1.4 in [50] for a proof of this fact.

Lemma 2.2.2. *Assume that (H2.1) and (H2.3) hold. Then, there exist two positive constants δ and β such that for all $\sigma \in \Sigma$*

$$\forall \mathbf{u} \in B(\sigma, \delta), \quad \beta^{-1} V(\mathbf{u}) \leq |\mathbf{u} - \sigma|^2 \leq \beta V(\mathbf{u}).$$

and

$$\forall \mathbf{u} \in B(\sigma, \delta), \quad \beta^{-1} \langle \nabla V(\mathbf{u}), \mathbf{u} - \sigma \rangle \leq |\mathbf{u} - \sigma|^2 \leq \beta \langle \nabla V(\mathbf{u}), \mathbf{u} - \sigma \rangle.$$

The constants δ and β will be fixed for the latter.

Lemma 2.2.3. *Assume that (H2.1) and (H2.3) hold. Let $q \in H_{\text{loc}}^1(\mathbb{R}, \mathbb{R}^k)$ satisfy $E(q) < +\infty$. Then*

$$\lim_{t \rightarrow \pm\infty} V(q(t)) = 0.$$

In order to apply the mountain pass lemma, we need to show that J is a C^1 functional. This is done in [50] and [123]. Let $(\sigma_i, \sigma_j) \in \Sigma^2$, following [50], take $\chi \in X(\sigma_i, \sigma_j)$ and define

$$J_\chi : v \in \mathcal{H} \rightarrow E(\chi + v)$$

which is well-defined by Lemma 2.2.1. Under these notations, we have that the functional J defined in (2.1.6) is $J = J_\psi$, with ψ as in (2.1.5).

Lemma 2.2.4. *Assume that (H2.1) and (H2.3) hold. Then, we have:*

i) *For any $(\sigma_i, \sigma_j) \in \Sigma^2$ and $\chi \in X(\sigma_i, \sigma_j)$, J_χ is a C^1 functional on \mathcal{H} with derivative:*

$$\forall v \in \mathcal{H}, \quad DJ_\chi(v) : w \in \mathcal{H} \rightarrow \int_{\mathbb{R}} (\langle \chi' + v', w' \rangle + \langle \nabla V(\chi + v), w \rangle) \in \mathbb{R}. \quad (2.2.6)$$

In particular, if (H2.1) and (H2.3) hold and $DJ_\chi(v) = 0$ for $v \in \mathcal{H}$, then $v + \chi$ solves (2.1.1).

ii) *J is C^1 as a functional restricted to \mathcal{H}_{sym} and its differential is as in (2.2.6) with the proper modifications. If, moreover, we add the symmetry assumption (H2.8) and $v \in \mathcal{H}_{\text{sym}}$ is such that $DJ(v) = 0$ in \mathcal{H}_{sym} , then $v + \psi$ solves (2.1.1).*

Item i) in Lemma 2.2.4 is essentially Proposition 1.6 in [50], for the particular case of autonomous potentials. The proof of item ii) follows from classical arguments using assumption (H2.8), so we skip it. Next, we recall the following general property for sequences with uniformly bounded energy:

Lemma 2.2.5. *Assume that (H2.1), (H2.2) and (H2.3) hold. Let $(\sigma_i, \sigma_j) \in \Sigma^2$. Let $(q_n)_{n \in \mathbb{N}}$ be a sequence in $X(\sigma_i, \sigma_j)$ such that $\sup_{n \in \mathbb{N}} E(q_n) < +\infty$. Then, up to an extraction, there exists $q \in H_{\text{loc}}^1(\mathbb{R}, \mathbb{R}^k)$ such that $q_n \rightarrow q$ locally uniformly and $q'_n \rightharpoonup q'$ weakly in \mathcal{L} . Moreover, $E(q) \leq \liminf_{n \rightarrow \infty} E(q_n)$.*

The property given by Lemma 2.2.5 is certainly well-known and the proof is classical, so we omit it. As we see, a uniform bound on the energy is not sufficient to obtain control on the behavior of the sequence of infinity. This is due to the fact that V possesses more than one zero and it is the cause of non-existence phenomena already when dealing with the minimization problem. Using Lemma 2.2.5, we obtain by classical arguments the following property for arbitrary Palais-Smale sequences:

Lemma 2.2.6. *Assume that (H2.1), (H2.2) and (H2.3) hold. Let $(v_n)_{n \in \mathbb{N}}$ be Palais Smale sequence at $c \geq m$, i.e.,*

$$\lim_{n \rightarrow \infty} J(v_n) = c \text{ and } \lim_{n \rightarrow \infty} DJ(v_n) = 0 \text{ in } \mathcal{H}. \quad (2.2.7)$$

Then, the following holds:

1. *There exists a subsequence of $(v_n)_{n \in \mathbb{N}}$ (not relabeled) and $q \in H_{\text{loc}}^1(\mathbb{R}, \mathbb{R}^k)$ such that*

$$\forall S_K \subset \mathbb{R} \text{ compact, } \psi + v_n \rightarrow_{n \rightarrow \infty} q \text{ strongly in } H^1(S_K, \mathbb{R}^k).$$

Moreover, $E(q) \leq c$ and $q \in C^2(\mathbb{R}, \mathbb{R}^k)$ solves (2.1.1).

2. *For any $(\tau_n)_{n \in \mathbb{N}}$ a sequence of real numbers, the sequence $(v_n^{\tau_n})_{n \in \mathbb{N}}$ defined as*

$$\forall n \in \mathbb{N}, \quad v_n^{\tau_n} := \psi(\cdot + \tau_n) + v_n(\cdot + \tau_n) - \psi$$

is a Palais-Smale sequence at the level c as in (2.2.7).

Proof. We show the first part. Define $(q_n)_{n \in \mathbb{N}} := (\psi + v_n)_{n \in \mathbb{N}}$, which is a sequence contained in $X(\sigma^-, \sigma^+)$. Using Lemma 2.2.5 and the first part of the Palais-Smale condition (2.2.7), we find $q \in H_{\text{loc}}^1(\mathbb{R}, \mathbb{R}^k)$ and a subsequence (not relabeled) such that $E(q) \leq c$, $q_n \rightarrow q$ locally uniformly and $q'_n \rightarrow q'$ in \mathcal{L} . We show the local convergence with respect to the H^1 norm. Let $S_K \subset \mathbb{R}$ be compact and $v_{S_K} \in \mathcal{H}$ with $\text{supp}(v_K) \subset S_K$. Using Cauchy-Schwartz inequality, we have

$$\begin{aligned} \left| \int_{\mathbb{R}} \langle \nabla V(q_n) - \nabla V(q), v_K \rangle \right| &= \left| \int_K \int_0^1 \langle D^2 V(\lambda q + (1 - \lambda)q_n)(q_n - q), v_K \rangle d\lambda \right| \\ &\leq C_K \|q_n - q\|_{L^2(K, \mathbb{R}^k)} \|v_K\|_{L^2(K, \mathbb{R}^k)}, \end{aligned} \quad (2.2.8)$$

where $C_K := \max_K (D^2 V(\lambda q + (1 - \lambda)q_n))$. We have that $q_n \rightarrow q$ uniformly in S_K , so due to the continuity of $D^2 V$ we have that $C_K < \infty$ and C_K independent on the sequence $(q_n)_{n \in \mathbb{N}}$. Using (2.2.6) and (2.2.8), we write

$$\left| \int_K \langle q'_n - q', v'_K \rangle \right| \leq C_K \|q_n - q\|_{L^2(K, \mathbb{R}^k)} \|v_K\|_{L^2(K, \mathbb{R}^k)} + DJ(v_n)(v_K). \quad (2.2.9)$$

Taking the supremum in (2.2.9) for $v_K \in \mathcal{H}$ with $\text{supp}(v_K) \subset S_K$ and $\|v_K\|_{\mathcal{H}} \leq 1$, by the dual characterization of the norm of a Hilbert space we get

$$\|q'_n - q'\|_{L^2(K, \mathbb{R}^k)} \leq C_K \|q_n - q\|_{L^2(K, \mathbb{R}^k)} + \|DJ(v_n)\|_{\mathcal{H}}.$$

Since $q_n \rightarrow q$ uniformly in K , we have $q_n \rightarrow q$ in $L^2(S_K, \mathbb{R}^k)$. In addition, the Palais-Smale condition

(2.2.7) implies $\|DJ(v_n)\|_{\mathcal{H}} \rightarrow 0$. Therefore, we have

$$\|q'_n - q'\|_{L^2(S_K, \mathbb{R}^k)} \rightarrow 0,$$

meaning that $q_n \rightarrow q$ in $H^1(S_K, \mathbb{R}^k)$, as we wanted to show. It only remains to show that q solves (2.1.1). Take $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}, \mathbb{R}^k)$. The convergence of the sequence inside $H^1(\text{supp}(\varphi), \mathbb{R}^k)$ is strong, meaning that we can show

$$DJ(v)(\varphi) = \lim_{n \rightarrow \infty} DJ(v_n)(\varphi) = 0.$$

In conclusion

$$\forall \varphi \in \mathcal{C}_c(\mathbb{R}, \mathbb{R}^k), \quad \int_{\mathbb{R}} [\langle q', \varphi' \rangle + \langle \nabla V(q), \varphi \rangle] = 0,$$

which by classical regularity arguments means that q is a solution of (2.1.1) which belongs to $\mathcal{C}^2(\mathbb{R}, \mathbb{R}^k)$.

For proving part 2, it suffices to write for any $n \in \mathbb{N}$ and $\varphi \in \mathcal{H}$

$$DJ(v_n^{\tau_n})(\varphi) = DJ(v_n)(\varphi(\cdot - \tau_n)),$$

which by taking the supremum in the unit ball of \mathcal{H} gives

$$\|DJ(v_n^{\tau_n})\|_{\mathcal{H}} = \|DJ(v_n)\|_{\mathcal{H}}.$$

□

As in Lemma 2.2.5, Lemma 2.2.6 gives no control on the convergence of the elements of the sequence at infinity. In particular, in general the functional J *does not* satisfy the so-called Palais-Smale condition², at least for arbitrary $c \geq m$. The problem is not fixed even if we use the translation invariance property from the second part of Lemma 2.2.6. As explained already, assumptions (H2.7), (H2.8) and (H2.4) are introduced in order to circumvent this issue. Assumptions (H2.6) and (H2.9) are made in order to exclude the possibility that the Palais-Smale sequences at the mountain pass levels originate a globally minimizing connecting orbit joining a well in $\{\sigma^-, \sigma^+\}$ and a well in $\Sigma \setminus \{\sigma^-, \sigma^+\}$. This is shown by the following:

Lemma 2.2.7. *Assume that (H2.1), (H2.2) and (H2.3) hold. Let $q \in X(\sigma^-, \sigma^+)$ be such that*

$$E(q) \leq C,$$

where $C < m^$, where m^* is as in (2.1.9). There exists $\rho_2(C) > 0$, depending only on V and C , such that*

$$\forall \sigma \in \Sigma \setminus \{\sigma^-, \sigma^+\}, \forall t \in \mathbb{R}, \quad |q(t) - \sigma| \geq \rho_2(C). \quad (2.2.10)$$

Lemma 2.2.7 is a straightforward generalization of results which were known previously, see [3] and [59]. The proof is skipped.

We conclude this paragraph by recalling that the complete asymptotic analysis of the Palais-Smale sequences and some of the consequences that follow are available in [136] and [50]. Such

²We say that J satisfies the Palais-Smale condition at the level $c \geq m$ if every sequence satisfying (2.2.7) possesses a convergent subsequence in \mathcal{H} .

properties do not play a major role in our argument³ the reason being that we find Lemma 2.2.6 is better adapted to our purposes. The main result can be stated as follows for our setting:

Proposition 2.2.1. *Assume that (H2.1), (H2.2) and (H2.3) hold. Let $(\sigma_i, \sigma_j) \in \Sigma^2$, $c \in \mathbb{R}$, $\chi \in X(\sigma_i, \sigma_j)$ and $(v_n)_{n \in \mathbb{N}}$ be a Palais-Smale sequence for J_χ at the level c . Then, up to an extraction there exists $j \in \mathbb{N}^*$, such that there is $(A_n^i)_{n \in \mathbb{N}, i \in \{1, \dots, j\}}$ a sequence of adjacent sub-intervals of \mathbb{R} , $(\tau_n^i)_{n \in \mathbb{N}, i \in \{1, \dots, j\}}$ a sequence of translates in \mathbb{R} and q^1, \dots, q^j solutions of (2.1.1) such that:*

1. *For all $n \in \mathbb{N}$, $\cup_{i=1}^j A_n^i = \mathbb{R}$.*
2. *For all $i \in \{1, \dots, j-1\}$, we have*

$$\lim_{t \rightarrow -\infty} q^{i+1}(t) = \lim_{t \rightarrow +\infty} q^i(t).$$

Moreover,

$$\lim_{t \rightarrow -\infty} q^1(t) = \sigma_i \text{ and } \lim_{t \rightarrow +\infty} q^j(t) = \sigma_j.$$

3. *For all $i \in \{1, \dots, j\}$ we have that*

$$\lim_{n \rightarrow \infty} \|v_n + \chi - q^i(\cdot - \tau_n^i)\|_{H^1(A_n^i, \mathbb{R}^k)} = 0$$

4. *For all $i \in \{1, \dots, j-1\}$, it holds that $\tau_n^{i+1} - \tau_n^i \rightarrow +\infty$ as $n \rightarrow \infty$.*
5. *$c = \sum_{i=1}^j E(q^i)$.*

Proposition 2.2.1 is essentially *Proposition 3.10* by Rabinowitz [136], with the main difference that we do not restrict to double-well potentials and we particularize to the autonomous case. The modifications needed in order to adapt the proof in [136] are minor, so we do not include them. Proposition 2.2.1 can also be deduced from the results in [50]. As already explained, in [50] this analysis is used to obtain existence results for non-minimizing connecting orbits under an assumption on the mountain pass value. We briefly recall the procedure. We first recall the following property, which is equivalent to *Corollary 1.18* in [50] and *Lemma 3.6* in [136]. It states that there exists an inferior bound depending only on V for the energy of non-constant connecting orbits:

Lemma 2.2.8. *Assume that (H2.1), (H2.2) and (H2.3) hold. There exists $\eta_{\min} > 0$ such that for any $(\sigma_i, \sigma_j) \in \Sigma^2$, if $q \in X(\sigma_i, \sigma_j)$ solves (2.1.1) then either $E(q) \geq \eta_{\min}$ or q is constant.*

The proof of Lemma 2.2.8 follows from the fact that V is strictly convex in a neighbourhood of the wells. We refer to the references mentioned before for a proof. Inspecting the proof of those results, we see that η_{\min} is of the order of δ from Lemma 2.2.2, which can be very small. Proposition 2.2.1 and Lemma 2.2.8 can be combined in order to easily obtain the following existence principle, which is essentially the result by Bisgard:

Corollary 2.2.1. *Assume that (H2.1), (H2.2) and (H2.3) hold. Let $(\sigma_i, \sigma_j) \in \Sigma^2$, $c \in \mathbb{R}$, $\chi \in X(\sigma_i, \sigma_j)$ and $(v_n)_{n \in \mathbb{N}}$ a Palais-Smale sequence for J_χ at the level c . Then, we have:*

³Proposition 2.2.1 is only invoked once, in the proof of Lemma 2.3.1 and Corollary 2.2.1 is brought into account in Remark 2.1.1

- i) If $c < \mathfrak{m}_{ij} + \eta_{\min}$, where \mathfrak{m}_{ij} is defined in (2.1.2) and η_{\min} is the constant from Lemma 2.2.8, then there exists $\mathfrak{q}_c \in X(\sigma_i, \sigma_j)$ and a sequence of real numbers $(\tau_n)_{n \in \mathbb{N}}$ such that $v_n + \chi - \mathfrak{q}_c(\cdot - \tau_n) \rightarrow 0$ strongly in \mathcal{H} up to subsequences. In particular, \mathfrak{q}_c solves (2.1.1) and $E(v_c + \chi) = c$.
- ii) If $c \notin \{(2l+1)\mathfrak{m}_{ij} : l \in \mathbb{N}^*\}$ there exists $\tilde{\mathfrak{u}}_c$ a solution to (2.1.1) which is not a globally minimizing connecting orbit joining σ_i and σ_j .

Up to the obvious minor modifications, i) in Corollary 2.2.1 corresponds to Theorem 2.2 in [50] and ii) is Theorem 2.3 in the same reference. While in [50] those results are particularized to $\sigma_i = \sigma^-$, $\sigma_j = \sigma^+$ and $c = \mathfrak{c}$ as in (2.1.8), an examination of the arguments shows that it also applies to the case $\sigma_i = \sigma_j$ and for any level c possessing a Palais-Smale sequence, so there is no obstacle for this more general statement. Nevertheless, it is important to notice as we already did in Remark 2.1.1 that by i) we have that if $\mathfrak{c} < \mathfrak{m} + \eta_{\min}$, then there exists a mountain pass heteroclinic in $X(\sigma^-, \sigma^+)$ with energy \mathfrak{c} . The counterpart of this statement is that the value η_{\min} can be very small, as we point out after the statement of Lemma 2.2.8. Notice also that by combining Lemma 2.2.8 and i) in Corollary 2.2.1 we have that for any $c \in (0, \eta_{\min})$ there is not any Palais-Smale sequence for J_χ at the level c , where $\chi \in X(\sigma, \sigma)$ and $\sigma \in \Sigma$.

2.2.3 Existence of a mountain pass geometry

The existence of a mountain pass geometry is proven by combining (H2.5) with the last part of the following well-known result:

Theorem 0.1. *Assume that (H2.1), (H2.2), (H2.3) and (H2.4) hold. Then, there exists $\mathfrak{q} \in X(\sigma^-, \sigma^+)$ such that $E(\mathfrak{q}) = \mathfrak{m}$, where \mathfrak{m} is as in (2.1.3). Moreover, if $(q_n)_{n \in \mathbb{N}}$ is a minimizing sequence in $X(\sigma^-, \sigma^+)$, there exists a subsequence (not relabeled) and a sequence $(\tau_n)_{n \in \mathbb{N}}$ of real numbers such that $q_n(\cdot + \tau_n) - \tilde{\mathfrak{q}} \rightarrow 0$ strongly in \mathcal{H} , for some $\tilde{\mathfrak{q}} \in X(\sigma^-, \sigma^+)$ such that $E(\tilde{\mathfrak{q}}) = \mathfrak{m}$.*

The existence part in Theorem 0.1, under different forms but using analogous arguments, can be found in several references. See for instance Bolotin [51], Bolotin and Kozlov [52], Bertotti and Montecchiari [40] and Rabinowitz [138, 142]. Proofs which use other type of arguments can be also found in Alikakos and Fusco [11], Monteil and Santambrogio [124], Zuñiga and Sternberg [174]. Regarding the compactness of the minimizing sequences and the applications of this property to some PDE problems, see Alama, Bronsard and Gui [3], Alama et. al. [2] and Schatzman [153]. As it is well known, (H2.4) might not be necessary but it cannot be removed, see Alikakos, Betelú and Chen [9] for some counterexamples. We can now establish the existence of a mountain pass geometry:

Proposition 2.2.2. *Assume that (H2.1), (H2.2), (H2.3) and (H2.5) hold. Let \mathfrak{c} be as in (2.1.8). Then, we have $\mathfrak{c} > \mathfrak{m}$.*

Proof. Let $\gamma \in \Gamma$. By (H2.5) and using the definition of \mathcal{V}_0 and \mathcal{V}_1 , we have that

$$\rho := \text{dist}_{\mathcal{H}}(\mathcal{V}_0, \mathcal{V}_1) = d(\mathcal{F}_0, \mathcal{F}_1) > 0,$$

where $\text{dist}_{\mathcal{H}}$ denotes the distance between two sets in \mathcal{H} . Since $\mathcal{V}_0 \cup \mathcal{V}_1 = \mathcal{V}$ and γ is a continuous path which joins \mathcal{V}_0 and \mathcal{V}_1 , we have that there exists $s^* \in [0, 1]$ such that

$$\text{dist}_{\mathcal{H}}(\gamma(s^*), \mathcal{V}) \geq \frac{\rho}{4}. \quad (2.2.11)$$

We claim that there exists $c(\rho) > 0$ such that for all $v \in \mathcal{H}$ verifying

$$\text{dist}(v, \mathcal{V}) \geq \frac{\rho}{4}$$

we have $J(v) \geq m + c(\rho)$. This is actually a well know result (see [2, 153]), which is a straightforward consequence of the compactness property for minimizing sequences given by Theorem 0.1. Thus, by (2.2.11) we obtain $\mathfrak{c} \geq m + c(\rho)$, which concludes the proof. \square

Subsequently, we establish the existence of a mountain pass geometry under the symmetry assumption. We begin by the following preliminary result:

Lemma 2.2.9. *Assume that (H2.1), (H2.2), (H2.3) and (H2.8) hold. Let $q \in X(\sigma^-, \sigma^+)$. Then, there exist $q_{\text{sym}} \in X_{\text{sym}}$ and $q_{\text{sym},+} \in X_{\text{sym},+}$ such that we have*

$$E(q_{\text{sym},+}) \leq E(q_{\text{sym}}) \leq E(q).$$

Proof. Let $q \in X(\sigma^-, \sigma^+)$. By the intermediate value Theorem, there exists $\tau \in \mathbb{R}$ such that

$$q_1(\tau) = 0.$$

Due to the translation invariance of the energy, we can assume that $\tau = 0$ (otherwise, replace q by $q(\cdot + \tau)$). Without loss of generality, assume that

$$\int_0^{+\infty} e(q) \leq \int_{-\infty}^0 e(q). \quad (2.2.12)$$

We define q_{sym} as

$$q_{\text{sym}}(t) := \begin{cases} q(t) & \text{if } t \geq 0, \\ \mathfrak{s}(q(-t)) & \text{if } t \leq 0, \end{cases}$$

which is well defined and belongs to X_{sym} . Notice that, due to this last fact, assumption (H2.8) and (2.2.12)

$$E(q_{\text{sym}}) = 2 \int_0^{+\infty} e(q) \leq E(q).$$

Subsequently, we set

$$q_{\text{sym},+}(t) := \begin{cases} (|(q_{\text{sym}})_1(t)|, (q_{\text{sym}})_2(t), \dots, (q_{\text{sym}})_k(t)) & \text{if } t \geq 0, \\ (-|(q_{\text{sym}})_1(t)|, (q_{\text{sym}})_2(t), \dots, (q_{\text{sym}})_k(t)) & \text{if } t \leq 0. \end{cases}$$

The function $q_{\text{sym},+}$ is also well defined and belongs to $X_{\text{sym},+}$. By assumption (H2.8), we have for all $t \in \mathbb{R}$ that $V(q_{\text{sym},+}(t)) = V(q_{\text{sym}}(t))$ and, by definition, we also have $|q'_{\text{sym},+}| \leq |q'_{\text{sym}}|$, a.e. in \mathbb{R} . Therefore,

$$E(q_{\text{sym},+}) \leq E(q_{\text{sym}}),$$

which establishes the proof. \square

Proposition 2.2.3. *Assume that (H2.1), (H2.2), (H2.3), (H2.5) and (H2.8) hold. Let $\mathfrak{c}_{\text{sym}}$ be as in (2.1.12). Then, we have $\mathfrak{c}_{\text{sym}} > m$.*

Proof. We have the following result which shows that coercivity also holds in the equivariant setting (see [3] for a proof):

Lemma 2.2.10 (Alama-Bronsard-Gui [3], Lemma 2.4). *For any $\varepsilon > 0$, there exists $c(\varepsilon) > 0$ such that for any $q \in X_{\text{sym}}$ such that $E(q) < m + c(\varepsilon)$ we have $\|q - q\|_{\mathcal{H}} < \varepsilon$ for some $q \in \mathcal{F}_{\text{sym}}$.*

Using Lemma 2.2.10 as well as Lemma 2.2.9, it suffices to apply the argument given in the proof of Proposition 2.2.2 to conclude. \square

Combining Lemma 2.2.4 and Proposition 2.2.2, the classical mountain pass lemma states that there exists a Palais-Smale sequence at a level \mathfrak{c} , i. e., a sequence $(v_n)_{n \in \mathbb{N}}$ in \mathcal{H} such that

$$\lim_{n \rightarrow \infty} J(v_n) = \mathfrak{c} \text{ and } \lim_{n \rightarrow \infty} DJ(v_n) = 0 \text{ in } \mathcal{H}.$$

Similarly, by Proposition 2.2.3 we find a sequence $(v'_n)_{n \in \mathbb{N}}$ in \mathcal{H}_{sym} such that

$$\lim_{n \rightarrow \infty} J(v'_n) = \mathfrak{c}_{\text{sym}} \text{ and } \lim_{n \rightarrow \infty} DJ(v'_n) = 0 \text{ in } \mathcal{H}_{\text{sym}}.$$

2.2.4 An abstract deformation lemma

As explained before, assumptions (H2.7) and (H2.10) are used in order to produce Palais Smale sequences at the mountain pass levels such that each element of the sequences goes through a suitable subset of \mathbb{R}^k . In order to show the existence of these sequences, we will use a deformation lemma due to Willem. Let us recall some standard terminology. Given a Banach space X we denote by X' its topological dual and given $I \in C^1(X)$, DI is its derivative and for $c \in \mathbb{R}$, $I^c := \{x \in X : I(x) \leq c\}$. Given $S \subset X$ and $\rho > 0$, we write $S_\rho := \{x \in X : \text{dist}_X(x, S) \leq \rho\}$. The result we will invoke is as follows:

Lemma 2.2.11 (Willem, Lemma 2.3 [172]). *Let X be a Banach space, $I \in C^1(X)$, $S \subset X$, $c \in \mathbb{R}$, $\varepsilon, \rho > 0$ such that*

$$\forall x \in I^{-1}([c - 2\varepsilon, c + 2\varepsilon]) \cap S_{2\rho}, \quad \|DI(x)\|_{X'} \geq 8\varepsilon/\rho \quad (2.2.13)$$

Then, there exists $\eta \in C([0, 1] \times X, X)$ such that

$$(i) \quad \eta(t, u) = u \text{ if } t = 0 \text{ or if } u \notin I^{-1}([c - 2\varepsilon, c + 2\varepsilon]) \cap S_{2\rho}.$$

$$(ii) \quad \eta(1, I^{c+\varepsilon} \cap S) \subset I^{c-\varepsilon}.$$

$$(iii) \quad \text{For all } t \in [0, 1], \eta(t, \cdot) \text{ is an homeomorphism of } X.$$

$$(iv) \quad \text{For all } x \in X \text{ and } t \in [0, 1], \|\eta(t, x) - x\|_X \leq \delta.$$

$$(v) \quad \text{For all } x \in X, I(\eta(\cdot, x)) \text{ is non increasing.}$$

$$(vi) \quad \text{For all } x \in \varphi^c \cap S_\rho \text{ and } t \in (0, 1], I(\eta(t, u)) < c.$$

Roughly speaking, the key point of Lemma 2.2.11 is that if (2.2.13) holds then there exists a homotopy equivalence between $I^{c+\varepsilon} \cap S$ and a subset of $I^{c-\varepsilon}$. Equivalently, if we can find S such that there is not any homotopy equivalence between $I^{c+\varepsilon} \cap S$ and any $S' \subset I^{c-\varepsilon}$, then (2.2.13) does not hold. The purpose of properties such as (H2.7) or (H2.10) is to provide such a set S .

2.2.5 The proof of Theorem 2.1

The idea of the proof of Theorem 2.1 is to show the existence of a Palais-Smale sequence at the level \mathfrak{c} (\mathfrak{c} as in (2.1.8)) which produces a solution u such that $u(0) \in K$, which is hence not in \mathcal{F} . It is here when (H2.7) enters. We define the set

$$F := \{v \in \mathcal{H} : (v + \psi)(\mathbb{R}) \cap K \neq \emptyset\} \quad (2.2.14)$$

with K as in (H2.7). We show the following:

Proposition 2.2.4. *There exists sequences, $(u_n)_{n \in \mathbb{N}}$ in \mathcal{H} and $(\tau_n)_{n \in \mathbb{N}}$ in \mathbb{R} , such that*

1. $J(u_n) \rightarrow \mathfrak{c}$ as $n \rightarrow \infty$.
2. $DJ(u_n) \rightarrow 0$ in \mathcal{H} as $n \rightarrow \infty$.
3. For all $n \in \mathbb{N}$, there exists $\tau_n \in \mathbb{R}$

$$\lim_{n \rightarrow \infty} \text{dist}_{\mathcal{H}}(u_n(\tau_n) + \psi(\tau_n), F) = 0.$$

Proof. We prove the result by contradiction. If a sequence as in the statement does not exist, then we can find $h \in (0, \frac{1}{2} \min\{M - \mathfrak{c}, \mathfrak{c} - \mathfrak{m}\})$ (M as in (H2.7)). Recall also that $\mathfrak{c} > \mathfrak{m}$ due to Proposition 2.2.2), $\mu > 0$ and $\nu > 0$ such that

$$\forall v \in J^{-1}([c - h, c + h]) \cap F_\nu, \quad \|DJ(v)\|_{\mathcal{H}} \geq \mu \quad (2.2.15)$$

with F as in (2.2.14) and $F_\nu := \{v \in \mathcal{H}, \text{dist}(v, F) \leq \nu\}$. We have that (2.2.15) is (2.2.13) in Lemma 2.2.11 with $X = \mathcal{H}$, $I = J$, $c = \mathfrak{c}$, $\varepsilon = h/2$, $\rho = \nu/2$ (we decrease the value of h if necessary so that $\mu \geq 8h/\nu$). Therefore, there exists $\eta \in C([0, 1] \times \mathcal{H}, \mathcal{H})$ satisfying the properties of Lemma 2.2.11. Let $\bar{\gamma} \in \Gamma$ be such that

$$\max_{s \in [0, 1]} J(\bar{\gamma}(s)) \leq \mathfrak{c} + \frac{1}{4}h. \quad (2.2.16)$$

Let us set $\hat{\gamma} : s \in [0, 1] \rightarrow \eta(1, \bar{\gamma}(s)) \in \mathcal{H}$. Since $\eta(1, \cdot)$ is a homeomorphism by (iii) in Lemma 2.2.11, we have that $\hat{\gamma} \in C([0, 1], \mathcal{H})$. Moreover, by the definition of h we have that $\mathfrak{c} - h > \mathfrak{m}$. Therefore, (i) in Lemma 2.2.11 implies that for $i \in \{0, 1\}$ we have $\hat{\gamma}(i) = \bar{\gamma}(i) \in \mathcal{V}_i$. As a consequence, $\hat{\gamma} \in \Gamma$. Moreover, by (v) in Lemma 2.2.11 and (2.2.16) we have that

$$\max_{s \in [0, 1]} J(\hat{\gamma}(s)) \leq \mathfrak{c} + \frac{1}{4}h \quad (2.2.17)$$

which means by (ii) in Lemma 2.2.11 that if $\hat{s} \in [0, 1]$ is such that $J(\hat{\gamma}(\hat{s})) \geq \mathfrak{c}$, then $\hat{\gamma}(\hat{s}) \notin F$, meaning that $(\psi + \hat{\gamma}(s))(\mathbb{R}) \cap K = \emptyset$. But since $\max_{s \in [0, 1]} J(\hat{\gamma}(s)) < M$ by (2.2.17) and the definition of h , we get a contradiction with 2. in (H2.7), which we assume to hold true. Therefore, the proof is completed. \square

Proposition 2.2.4 along with Lemmas 2.2.6 and 2.2.7 allows to finish the proof of Theorem 2.1 as follows:

Proof of Theorem 2.1 completed. Assume that the hypothesis made for Theorem 2.1 hold. Let $(u_n)_{n \in \mathbb{N}}$ and $(\tau_n)_{n \in \mathbb{N}}$ be the sequences given by Proposition 2.2.4. By part 2 in Lemma 2.2.6, the sequence $(\tilde{u}_n) := (t_{\tau_n}(u_n))$ is a Palais-Smale sequence and it also satisfies

$$\lim_{n \rightarrow \infty} \text{dist}_{\mathcal{H}}(\tilde{u}_n(0) + \psi(0), K) = 0. \quad (2.2.18)$$

Up to an extraction, we have by (H2.6) that for all $n \in \mathbb{N}$ we have $J(\tilde{u}_n) \leq \tilde{C} := (m^* - \mathfrak{c})/2 + \mathfrak{c}$. Therefore, by applying Lemma 2.2.7, we obtain $\rho_2 := \rho_2(\tilde{C})$ such that

$$\forall n \in \mathbb{N}, \forall \sigma \in \Sigma \setminus \{\sigma^-, \sigma^+\}, \forall t \in \mathbb{R}, |\tilde{u}_n(t) + \psi(t) - \sigma| \geq \rho_2. \quad (2.2.19)$$

Using now part 1 of Lemma 2.2.6, we find $u \in H_{\text{loc}}^1(\mathbb{R}, \mathbb{R}^k) \cap \mathcal{C}^2(\mathbb{R}, \mathbb{R}^k)$ such that u solves (2.1.1), $E(u) \leq \mathfrak{c}$ and for all $S_K \subset \mathbb{R}$ compact, $\tilde{u}_n + \psi \rightarrow u$ in $H^1(S_K, \mathbb{R}^k)$ (in particular, $\tilde{u}_n \rightarrow u$ pointwise in \mathbb{R}). Using (2.2.18), the fact that K is closed and pointwise convergence, we find $u(0) \in K$. By assumption (H2.7), we have that u does not coincide with any minimizing heteroclinic in \mathcal{F} . By (2.2.19) and pointwise convergence, we have that

$$\forall \sigma \in \Sigma \setminus \{\sigma^-, \sigma^+\}, \forall t \in \mathbb{R}, |u(t) - \sigma| \geq \rho_2, \quad (2.2.20)$$

meaning in particular that u cannot be a minimizing connecting orbit between $\sigma \in \Sigma \setminus \{\sigma^-, \sigma^+\}$ and $\sigma' \in \{\sigma^-, \sigma^+\}$. Assume now that $u \in X(\sigma^-, \sigma^+)$. Due to the previous discussion, we must have $E(u) > m$. If u does not belong to $X(\sigma^-, \sigma^+)$, by Lemma 2.2.3 we have

$$\exists \sigma \in \{\sigma^-, \sigma^+\} : \lim_{t \rightarrow \pm\infty} u(t) = \sigma$$

and $u(0) \notin \{\sigma^-, \sigma^+\}$ because $u(0) \in K$ and $K \cap \{\sigma^-, \sigma^+\} = \emptyset$ due to the first part of (H2.7). We also have that $u(0) \notin \Sigma \setminus \{\sigma^-, \sigma^+\}$ due to (2.2.20). Therefore, $u(0) \in \Sigma$. Hence, u is not constant. \square

2.2.6 The proofs of Theorems 2.2 and 2.3

The first step of the proof of both Theorems consists on showing that there exists a Palais-Smale sequence $(u_n)_{n \in \mathbb{N}}$ at the level $\mathfrak{c}_{\text{sym}}$ such that $(\psi + u_n)_{n \in \mathbb{N}}$ approaches $X_{\text{sym},+}$. The existence of such sequence follows from the fact that we can map X_{sym} into $X_{\text{sym},+}$ continuously and leaving $X_{\text{sym},+}$ invariant and that such mapping does not increase the energy due to the symmetry assumption (H2.8). The idea then is to show that a nontrivial solution is produced even if we have dichotomy of the Palais-Smale sequence. This proves Theorem 2.2. More precisely, if a Palais-Smale sequence in $\mathcal{H}_{\text{sym},+}$ is not compact, then we are in the situation 1. of Theorem 2.2 and we find a pair of nontrivial homoclinic solutions. Of course, if such a Palais-Smale sequence is compact, we recover a solution in $X_{\text{sym},+}$ with energy $\mathfrak{c}_{\text{sym}}$, thus also nontrivial. Subsequently, for proving Theorem 2.3 under the additional assumption (H2.10), the argument is supplemented with a deformation argument analogous to that in the proof of Theorem 2.1.

We begin by showing the following:

Lemma 2.2.12. *Assume that (H2.1), (H2.2), (H2.3) and (H2.8) hold. Let d be as in (2.1.4) and*

$F_+ : (X_{\text{sym}}, d) \rightarrow (X_{\text{sym},+}, d)$ be such that

$$\forall q \in X_{\text{sym}}, \quad F_+(q)(t) := \begin{cases} (|q_1(t)|, q_2(t), \dots, q_k(t)) & \text{if } t \geq 0, \\ (-|q_1(t)|, q_2(t), \dots, q_k(t)) & \text{if } t \leq 0. \end{cases}$$

Then for all $q \in X_{\text{sym}}$ we have $E(F_+(q)) \leq E(q)$, $F_+|_{X_{\text{sym},+}} = \text{Id}|_{X_{\text{sym},+}}$ and F_+ is continuous.

Proof. Let $q \in X_{\text{sym}}$, notice that repeating the argument in the proof of Lemma 2.2.9 shows that $E(F_+(q)) \leq E(q)$. Notice also that in case $q \in X_{\text{sym},+}$ then $F_+(q) = q$. Therefore, it only remains to show that F_+ is continuous. Let $(q_n)_{n \in \mathbb{N}}$ be a sequence in X_{sym} and $q \in X_{\text{sym}}$ such that

$$\lim_{n \rightarrow \infty} \|q_n - q\|_{\mathcal{H}} = 0. \quad (2.2.21)$$

For each $n \in \mathbb{N}$ set $q_n^+ := F_+(q_n) \in X_{\text{sym},+}$ and $q^+ := F_+(q) \in X_{\text{sym},+}$. We need to show that

$$\lim_{n \rightarrow \infty} \|q_n^+ - q^+\|_{\mathcal{H}} = 0. \quad (2.2.22)$$

Let $\kappa \leq \frac{1}{4}$ be arbitrary and take $t_q^+ \in \mathbb{R}$ such that

$$\forall t \geq t_q^+, \quad |q(t) - \sigma^+| \leq \kappa \quad (2.2.23)$$

and $t_q^- < t_q^+$ such that

$$\forall t \leq t_q^-, \quad |q(t) - \sigma^-| \leq \kappa. \quad (2.2.24)$$

We set $I := [t_q^-, t_q^+]$. By (2.2.21), we have that $q_n \rightarrow q$ uniformly, so in particular there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ it holds $\|q_n - q\|_{L^\infty(\mathbb{R}, \mathbb{R}^k)} \leq \kappa$. This fact along with (2.2.23), the definition of κ and (2.2.24) allow us to say that

$$\forall n \geq n_0, \forall t \in \mathbb{R} \setminus I, \quad q_n(t) = q_n^+(t) \text{ and } q(t) = q^+(t),$$

which means that $(q_n^+ - q^+)_{n \in \mathbb{N}}$ converges to 0 in $H^1(\mathbb{R} \setminus I, \mathbb{R}^k)$ by (2.2.21). Hence, in order to establish (2.2.22) we only need to show that $(q_n^+ - q^+)_{n \in \mathbb{N}}$ converges to 0 in $H^1(I, \mathbb{R}^k)$. Notice that in fact all functions belong now to $H^1(I, \mathbb{R}^k)$ because I is bounded. Let $f_+ : H^1(I) \rightarrow H^1(I)$ the application such that

$$\forall v \in H^1(I), \forall t \in I, \quad f_+(v)(t) := |v(t)|.$$

We have that the absolute value function is Lipschitz as a function from \mathbb{R} to \mathbb{R} and, moreover, the interval I is bounded. Therefore, f_+ is continuous due to *Theorem 1* in Marcus and Mizel [113]. As a consequence, we have

$$\lim_{n \rightarrow \infty} \|q_{n,1}^+ - q_1^+\|_{H^1(I \cap [0, +\infty))} = \lim_{n \rightarrow \infty} \|f_+(q_{n,1}) - f_+(q_1)\|_{H^1(I \cap [0, +\infty))} = 0$$

and

$$\lim_{n \rightarrow \infty} \|q_{n,1}^+ - q_1^+\|_{H^1(I \cap (-\infty, 0])} = \lim_{n \rightarrow \infty} \|-f_+(q_{n,1}) + f_+(q_1)\|_{H^1(I \cap (-\infty, 0])} = 0,$$

that is

$$\lim_{n \rightarrow \infty} \|q_{n,1}^+ - q_1^+\|_{H^1(I)} = 0.$$

Since all the other components were not modified, (2.2.22) has been proven and the proof is concluded. \square

Lemma 2.2.12 implies the following:

Lemma 2.2.13. *Assume that (H2.1), (H2.2), (H2.3), (H2.5) and (H2.8) hold. Let $h_{\text{sym}} : \mathcal{H}_{\text{sym}} \rightarrow \mathcal{H}_{\text{sym}}$ be defined as*

$$h_{\text{sym}} : v \in \mathcal{H}_{\text{sym}} \rightarrow F_+(v + \psi) - \psi \in \mathcal{H}_{\text{sym}}.$$

Then for all $v \in \mathcal{H}_{\text{sym}}$ we have $h_{\text{sym}}(v) + \psi \in X_{\text{sym},+}$, $J(h_{\text{sym}}(v)) \leq J(v)$ and for all $\gamma \in \Gamma_{\text{sym}}$ it holds that the composed path $h_{\text{sym}} \circ \gamma$ belongs to Γ_{sym} .

Proof. Let $v \in \mathcal{H}$. By Lemma 2.2.12 we have that $h_{\text{sym}}(v) + \psi = F_+(v + \psi) \in X_{\text{sym},+}$ and $J(h_{\text{sym}}(v)) = E(F_+(v + \psi)) \leq E(v + \psi) = J(v)$. It is straightforward to show that h_{sym} is continuous. Notice that if $v \in \mathcal{V}_{\text{sym}}$ then $\mathcal{V}_{\text{sym}} + \{\psi\} = \mathcal{F}_{\text{sym}} \subset X_{\text{sym},+}$ by definition. Therefore, using again Lemma 2.2.12 we have $h_{\text{sym}}(v) = F_+(v + \psi) - \psi = v + \psi - \psi = v = \text{Id}_{\mathcal{H}_{\text{sym}}}(v)$. \square

2.2.6.1 The proof of Theorem 2.2

We have the following result:

Proposition 2.2.5. *Assume that (H2.1), (H2.2), (H2.3), (H2.5) and (H2.8) hold. Then, there exists a sequence $(u_n)_{n \in \mathbb{N}}$ in \mathcal{H}_{sym} such that*

$$\lim_{n \rightarrow +\infty} J(u_n) \rightarrow c_{\text{sym}} \text{ and } DJ(u_n) \rightarrow 0 \text{ in } \mathcal{H}_{\text{sym}}.$$

and, moreover,

$$\lim_{n \rightarrow +\infty} d(u_n + \psi, X_{\text{sym},+}) = 0, \quad (2.2.25)$$

where d is as in (2.1.4).

The proof of Proposition 2.2.5 is a direct consequence of Proposition 2.2.3 and Lemma 2.2.13 along with a usual variant of the mountain pass lemma (see for instance *Corollary 4.3* in Mawhin and Willem [118]) which allows to find a Palais-Smale sequence associated with any given minimizing sequence of paths. We can now tackle the final part of the proof of Theorem 2.2.

Proof of Theorem 2.2 completed. Assume that the hypothesis of Theorem 2.2 hold. Let $(u_n)_{n \in \mathbb{N}}$ be the Palais-Smale sequence provided by Proposition 2.2.5. By assumption (H2.9), up to an extraction we have

$$\sup_{n \in \mathbb{N}} J(u_n) \leq C < m^*$$

for an arbitrary $C \in (c_{\text{sym}}, m^*)$. We can then use Lemma 2.2.7 to find $\rho_2 > 0$ such that

$$\forall n \in \mathbb{N}, \forall \sigma \in \Sigma \setminus \{\sigma^-, \sigma^+\}, \forall t \in \mathbb{R}, |u_n(t) + \psi(t) - \sigma| \geq \rho_2. \quad (2.2.26)$$

We divide the proof according to the two possible scenarios (dichotomy or compactness):

Case 1. Dichotomy. Assume that there exist $c_0 > 0$, $c_1 > 0$ and a sequence $t_n \rightarrow \infty$ such that, up to an extraction

$$\forall n \in \mathbb{N}, \int_{t_n - c_1}^{t_n + c_1} e(u_n + \psi) \geq c_0. \quad (2.2.27)$$

Since $(u_n + \psi)_{n \in \mathbb{N}}$ approaches $X_{\text{sym},+}$ due to (2.2.25), up to an extraction we can suppose

$$\forall n \in \mathbb{N}, \forall t \geq 0, |u_n(t) + \psi(t) - \sigma^-| \geq \rho_2. \quad (2.2.28)$$

For each $n \in \mathbb{N}$, we can define $\tilde{q}_n := u_n(\cdot + t_n) + \psi(\cdot + t_n) \in X(\sigma^-, \sigma^+)$ and $\tilde{u}_n := \tilde{q}_n - \psi$. We can regard $(u_n)_{n \in \mathbb{N}}$ as a Palais-Smale sequence in \mathcal{H} because \mathcal{H}_{sym} is a closed subspace of \mathcal{H} . Part 2 in Lemma 2.2.6 implies then that $(\tilde{u}_n)_{n \in \mathbb{N}}$ is a Palais-Smale sequence in \mathcal{H} . By using now part 1 of Lemma 2.2.6, we find $u_+ \in H_{\text{loc}}^1(\mathbb{R}, \mathbb{R}^k)$ such that for all $S_K \subset \mathbb{R}$ compact, $\tilde{q}_n \rightarrow u_+$ in $H^1(S_K, \mathbb{R}^k)$. Moreover, $u_+ \in \mathcal{C}^2(\mathbb{R}, \mathbb{R}^k)$ solves (2.1.1) and $E(u_+) \leq c_{\text{sym}}$. By (2.2.27) and the convergence, we have

$$\int_{-c_1}^{c_1} e(u_+) = \lim_{n \rightarrow +\infty} \int_{-c_1}^{c_1} e(\tilde{q}_n) \geq c_0,$$

meaning that $E(u_+) \geq c_0$, so in particular u_+ is not constant. We now show that u_+ converges to σ^+ at infinity. Rewriting (2.2.28) for $(\tilde{q}_n)_{n \in \mathbb{N}}$, we have

$$\forall n \in \mathbb{N}, \forall t \geq -t_n, |\tilde{q}_n(t) - \sigma^-| \geq \rho_2,$$

which combined with (2.2.26), Lemma 2.2.3 and pointwise convergence $\tilde{q}_n \rightarrow u_+$ gives that $\lim_{t \rightarrow \pm\infty} u_+(t) = \sigma^+$ as we wanted. Finally, notice that by symmetry we have that the function

$$u_- : t \in \mathbb{R} \rightarrow \mathfrak{s}(u_+(t)),$$

is a non constant solution of (2.1.1) such that $\lim_{t \rightarrow \pm\infty} u_-(t) = \sigma^-$.

Case 2. Compactness. The hypothesis made for Case 1 is not satisfied. Then, for all $c_2 > 0$ there exists $t(c_2) > 0$ such that

$$\forall n \in \mathbb{N}, \int_{t(c_2)}^{+\infty} e(u_n + \psi) \leq c_2$$

and, by symmetry

$$\forall n \in \mathbb{N}, \int_{-\infty}^{-t(c_2)} e(u_n + \psi) \leq c_2.$$

Equivalently, up to taking a diagonal extraction, for each $m \geq 1$ we can find $t(m) \geq 0$ such that

$$\forall n \in \mathbb{N}, \int_{-t(m)}^{t(m)} e(u_n + \psi) \geq c_{\text{sym}} - \frac{1}{m}. \quad (2.2.29)$$

Using again Lemma 2.2.6, we find $u \in H_{\text{loc}}^1(\mathbb{R}, \mathbb{R}^k)$ a solution to (2.1.1) such that $u_n + \psi \rightarrow u$ strongly in $H^1(S_K, \mathbb{R}^k)$ for each compact interval I . Moreover, by (2.2.26) and (2.2.25) we have $u \in X_{\text{sym},+}$. Finally, using (2.2.29) we get $E(u) = c_{\text{sym}}$, which concludes the proof. \square

2.2.6.2 The proof of Theorem 2.3

We will use (H2.10) and Lemma 2.2.11. Define

$$A_{\text{sym}} := \{q \in X_{\text{sym}} : q(0) \in K_{\text{sym}} \text{ and } E(q) \geq c_{\text{sym}}\}$$

and

$$F_{\text{sym}} := A_{\text{sym}} - \{\psi\} \in \mathcal{H}_{\text{sym}}.$$

We have the following, which is the analogous of Proposition 2.2.4:

Proposition 2.2.6. *Assume that (H2.1), (H2.2), (H2.3), (H2.5), (H2.8) and (H2.10) hold. Then, there exists a sequence $(u_n)_{n \in \mathbb{N}}$ in \mathcal{H}_{sym} such that*

$$\lim_{n \rightarrow +\infty} J(u_n) \rightarrow c_{\text{sym}} \text{ and } DJ(u_n) \rightarrow 0 \text{ in } \mathcal{H}_{\text{sym}}$$

and, moreover,

$$\lim_{n \rightarrow +\infty} \text{dist}_{\mathcal{H}}(u_n + \psi, X_{\text{sym},+} \cap A_{\text{sym}}) = 0. \quad (2.2.30)$$

The proof of Proposition 2.2.6 is analogous to the proof of Proposition 2.2.4. The only significant difference is that the path which is obtained from the deformation provided by Lemma 2.2.11 must be contained in $X_{\text{sym},+}$ in order to get the contradiction with (H2.10). However, this can be assumed by Lemma 2.2.13. Hence, we do not include the proof of Proposition 2.2.6 here.

Proof of Theorem 2.3 completed. We now suppose that the assumptions of Theorem 2.3 are satisfied. Let $(u_n)_{n \in \mathbb{N}}$ be the Palais-Smale sequence given by Proposition 2.2.6. As done before, up to an extraction we can use (H2.9) and Lemma 2.2.7 to find $\rho_2 > 0$ such that

$$\forall n \in \mathbb{N}, \forall \sigma \in \Sigma \setminus \{\sigma^-, \sigma^+\}, \forall t \in \mathbb{R}, |u_n(t) + \psi(t) - \sigma| \geq \rho_2. \quad (2.2.31)$$

Regarding $(u_n)_{n \in \mathbb{N}}$ as a Palais-Smale sequence in \mathcal{H}_{sym} and using Lemma 2.2.6, we find $u \in H_{\text{loc}}^1(\mathbb{R}, \mathbb{R}^k)$ such that $E(u) \leq c_{\text{sym}}$, $u_n + \psi \rightarrow u$ strongly in $H^1(S_K, \mathbb{R}^k)$ (S_K compact). By (2.2.31), we have that

$$\forall \sigma \in \Sigma \setminus \{\sigma^-, \sigma^+\}, \forall t \in \mathbb{R}, |u(t) - \sigma| \geq \rho_2. \quad (2.2.32)$$

By pointwise convergence, we have for all $t \in \mathbb{R}$, that $u(-t) = s(u(t))$. Since $u_n + \psi$ approaches $X_{\text{sym},+}$ due to (2.2.30), we have for all $t \geq 0$, $u_1(t) \geq 0$ and analogously for $t \leq 0$. These facts along with (2.2.32) give $\lim_{t \rightarrow \pm\infty} u(t) = \sigma^\pm$, which all together implies $u \in X_{\text{sym},+}$. Finally, using again (2.2.25) we have $u(0) \in K_{\text{sym}}$, which by (H2.10) means that $u \notin \mathcal{F}_{\text{sym}}$, i.e., $E(u) > m$. \square

2.3 On the assumptions (H2.7) and (H2.10)

As commented in Section 2.2.1, assumptions (H2.7) and (H2.10) might appear as rather artificial and, moreover, difficult to verify in hypothetical applications. Despite the fact that in Theorem 2.2 we show that (H2.7) can be removed if we restrict to potentials which are symmetric as in (H2.8), we believe that a better understanding of (H2.7) is still an interesting open question. Indeed, even though adding symmetry is a natural procedure in order to simplify a problem, it can be found to be too restrictive in many applications. In this direction, we show in Lemma 2.3.1 that (H2.7) holds if the mountain pass value c lies outside some known countable subset of $(m, +\infty)$, and in particular if it is smaller than $3m$. As explained in Section 2.2.1, this requirement is equivalent to the assumption made by Bisgard in [50]. In any case, a better understanding of hypothesis (H2.7) and (H2.10) remains an open problem. Geometric intuition suggests that such hypothesis should always (or close) hold, but we do not have a proof of such a fact. The same type of comment is

made by Bisgard in [50], where he states (see the Remark after his *Theorem 2.3*) that he expects his assumption on \mathfrak{c} to be generic (that is, valid for a dense class of potentials). We also think that this is the natural conjecture as the set of *bad* values for \mathfrak{c} is discrete. We believe that a starting point to aim at understanding this question better would be to try to understand the relation between the mountain pass value and the geometry of V in a deeper fashion.

We now state the result which links (H2.7) and Bisgard's assumption:

Lemma 2.3.1. *Let V be a potential satisfying (H2.1), (H2.2), (H2.3) and (H2.5). Let \mathfrak{c} be the mountain pass value defined in (2.1.8). Then, if we have*

$$\mathfrak{c} \in (\mathfrak{m}, +\infty) \setminus \{(2j+1)\mathfrak{m} : j \in \mathbb{N}^*\}, \quad (2.3.1)$$

there exists $K \subset \mathbb{R}^k$ such that assumption (H2.7) is satisfied for some constants $\nu_0 > 0$ and $M > \mathfrak{c}$.

Proof. For each $\varepsilon > 0$, define

$$\tilde{K}_\varepsilon := \bigcup_{\mathbf{q} \in \mathcal{F}} \{\mathbf{u} \in \mathbb{R}^k : \text{dist}(\mathbf{u}, \mathbf{q}(\mathbb{R})) < \varepsilon\}$$

and $K_\varepsilon := \mathbb{R}^k \setminus \tilde{K}_\varepsilon$. The proof will be concluded if we show the existence of $\nu_0 > 0$ and $M > 0$ such that for any $\gamma \in \Gamma$, with $\max_{s \in [0,1]} J(\gamma(s)) \leq M$ there exists $s_\gamma \in [0, 1]$ such that $(\gamma(s_\gamma) + \psi)(\mathbb{R}) \cap K_{\nu_0} \neq \emptyset$ and $J(\gamma(s_\gamma)) \geq \mathfrak{c}$. By contradiction, assume that for any $\varepsilon > 0$ and $M > \mathfrak{c}$, there exists $\gamma_\varepsilon \in \Gamma$ with $\max_{s \in [0,1]} J(\gamma_\varepsilon(s)) \leq M$ such that for all $s \in [0, 1]$ satisfying $(\gamma_\varepsilon(s) + \psi)(\mathbb{R}) \cap K_\varepsilon \neq \emptyset$ we have $J(\gamma_\varepsilon(s)) < \mathfrak{c}$. Otherwise stated, if $s \in [0, 1]$ is such that $J(\gamma_\varepsilon(s)) \geq \mathfrak{c}$, then $(\gamma_\varepsilon(s) + \psi)(\mathbb{R}) \subset \tilde{K}_\varepsilon$. Taking subsequences $(\varepsilon_n)_{n \in \mathbb{N}}$ and $(M_n)_{n \in \mathbb{N}}$ such that $\varepsilon_n \rightarrow 0^+$ and $M_n \rightarrow \mathfrak{c}^+$ as $n \rightarrow \infty$, we have found a sequence of paths $(\gamma_{\varepsilon_n})_{n \in \mathbb{N}}$ such that $\max_{s \in [0,1]} J(\gamma_{\varepsilon_n}(s)) \rightarrow \mathfrak{c}$. By usual arguments (for instance *Corollary 4.3* in Mawhin and Willem [118]), we find a Palais-Smale sequence $(v_{\varepsilon_n})_{n \in \mathbb{N}}$ at the level \mathfrak{c} such that

$$\lim_{n \rightarrow \infty} \text{dist}_{\mathcal{H}}(v_{\varepsilon_n}, \{v \in \mathcal{H} : J(v) \geq \mathfrak{c}\} \cap \gamma_{\varepsilon_n}([0, 1])) \rightarrow 0. \quad (2.3.2)$$

Due to the contradiction assumption stated above, we have that if

$$v \in \{v \in \mathcal{H} : J(v) \geq \mathfrak{c}\} \cap \gamma_{\varepsilon_n}([0, 1]),$$

then $(v + \psi)(\mathbb{R}) \subset \tilde{K}_{\varepsilon_n}$. The goal now is to obtain that $\mathfrak{c} = (2j+1)\mathfrak{m}$ for some $j \in \mathbb{N}^*$, which will give the desired contradiction since we assume (2.3.1). Let $s_\tau := (\tau_n)_{n \in \mathbb{N}}$ be an arbitrary sequence in \mathbb{R} . Using Lemma 2.2.6, we have that $(v_n^{\tau_n})_{n \in \mathbb{N}}$ (with the notations as in the second part of Lemma 2.2.6) is a Palais-Smale sequence at the level \mathfrak{c} converging (up to subsequences) locally in H^1 to \mathfrak{q}^{s_τ} a solution of (2.1.1) with $E(\mathfrak{q}^{s_\tau}) \leq \mathfrak{c}$. Using (2.3.2), we have that in fact \mathfrak{q}^{s_τ} is either a constant equal to σ^- or σ^+ , $\mathfrak{q}^{s_\tau} \in \mathcal{F}$ or $\mathfrak{q}^{s_\tau}(\cdot) \in \mathcal{F}$. Therefore, by Proposition 2.2.1 it follows that there exists $j \in \mathbb{N}^*$ and sequences $((t_n^0, \dots, t_n^{2j+1}))_{n \in \mathbb{N}}$, $(\tau_n^1, \dots, \tau_n^{2j+2})_{n \in \mathbb{N}}$ in \mathbb{R}^{2j+2} and \mathbb{R}^{2j+1} respectively such that (up to an extraction)

$$\forall j' \in \{1, \dots, 2j+2\}, \quad t_n^{j'-1} - t_n^{j'} \rightarrow +\infty \text{ as } n \rightarrow \infty,$$

$$\begin{aligned} \forall j' \in \{0, \dots, j\}, \quad \lim_{n \rightarrow \infty} v_n(t_n^{2j'}) + \psi(t_n^{2j'}) &= \sigma^-, \\ \lim_{n \rightarrow \infty} v_n(t_n^{2j'+1}) + \psi(t_n^{2j'+1}) &= \sigma^+, \end{aligned}$$

$$\begin{aligned} \forall j' \in \{1, \dots, 2j+2\}, \quad (v_n + \psi) - q^{j'}(\cdot - \tau_n^{j'}) &\rightarrow 0 \\ \text{strongly in } H^1([t_n^{j'}, t_n^{j'+1}], \mathbb{R}^k) &\text{ as } n \rightarrow \infty \end{aligned}$$

with $q^{j'} \in \mathcal{F}$ if j' is even and $q^{j'}(-\cdot) \in \mathcal{F}$ if j' is odd. Moreover

$$\lim_{n \rightarrow \infty} E(v_n + \psi) = \lim_{n \rightarrow +\infty} \sum_{j'=1}^{2j+2} \int_{t_n^{j'-1}}^{t_n^{j'}} e(v_n + \psi) = (2j+1)m$$

which gives the desired contradiction. \square

Notice that if $\mathfrak{c} < 3m$, then (2.3.1) holds.

Remark 2.3.1. An interpretation of Lemma 2.3.1 can be given as follows: Take a function q which has energy strictly greater than m , DJ applied to $q - \psi$ has *small* norm and the trace of q is *close* enough to the traces of the elements of \mathcal{F} . Then, q must look close to one element of \mathcal{F} which is glued to $j \geq 1$ cycles in \mathcal{F} . Such cycles are as follows: take an element of \mathcal{F} and glue it to an element of \mathcal{F} with reversed sign to obtain a connecting orbit joining σ^- and σ^+ . The energy of q must be then close to $(2j+1)m$. This argument is the key of the proof of Lemma 2.3.1. An illustration is shown in Figure 2.3. In different words words, Palais-Smale sequences which have the type of behavior described above yield only trivial solutions. The point of assumptions (H2.7) and (H2.10) is to exclude such type of behaviors for Palais-Smale sequences.

We obtain the analogous result for symmetric potentials, with an identical proof:

Lemma 2.3.2. *Let V be a potential satisfying (H2.1), (H2.2), (H2.3), (H2.5) and (H2.8). Let $\mathfrak{c}_{\text{sym}}$ be the mountain pass value defined in (2.1.12). Then, if we have*

$$\mathfrak{c}_{\text{sym}} \in (m, +\infty) \setminus \{(2j+1)m : j \in \mathbb{N}^*\},$$

there exists $K_{\text{sym}} \subset \mathbb{R}^k$ such that assumption (H2.10) is satisfied for some constants $\nu_0 > 0$ and $M > \mathfrak{c}$.

2.4 Link with a continuous selection problem

In an ideal situation with no loss of compactness, once the mountain pass levels \mathfrak{c} and $\mathfrak{c}_{\text{sym}}$ (under symmetry) are established, one would find heteroclinic solutions from σ^- and σ^+ at the level \mathfrak{c} or $\mathfrak{c}_{\text{sym}}$. As it has been made clear, this is not what we are able to prove in our main Theorems, except for when the mountain pass value is small enough (see Remark 2.1.1). The purpose of this section is to show how results of this type could be established by proving some connections of a constrained minimization problem with a problem of *continuous selections*, see the book by Repovs and Semenov [143] and the references therein. Indeed, we show that under the symmetry assumption (H2.8) one can find a convergent Palais-Smale sequence at the level $\mathfrak{c}_{\text{sym}}$ provided that one is capable to select continuously on a set of minimizing 1D solutions in the half line, with a

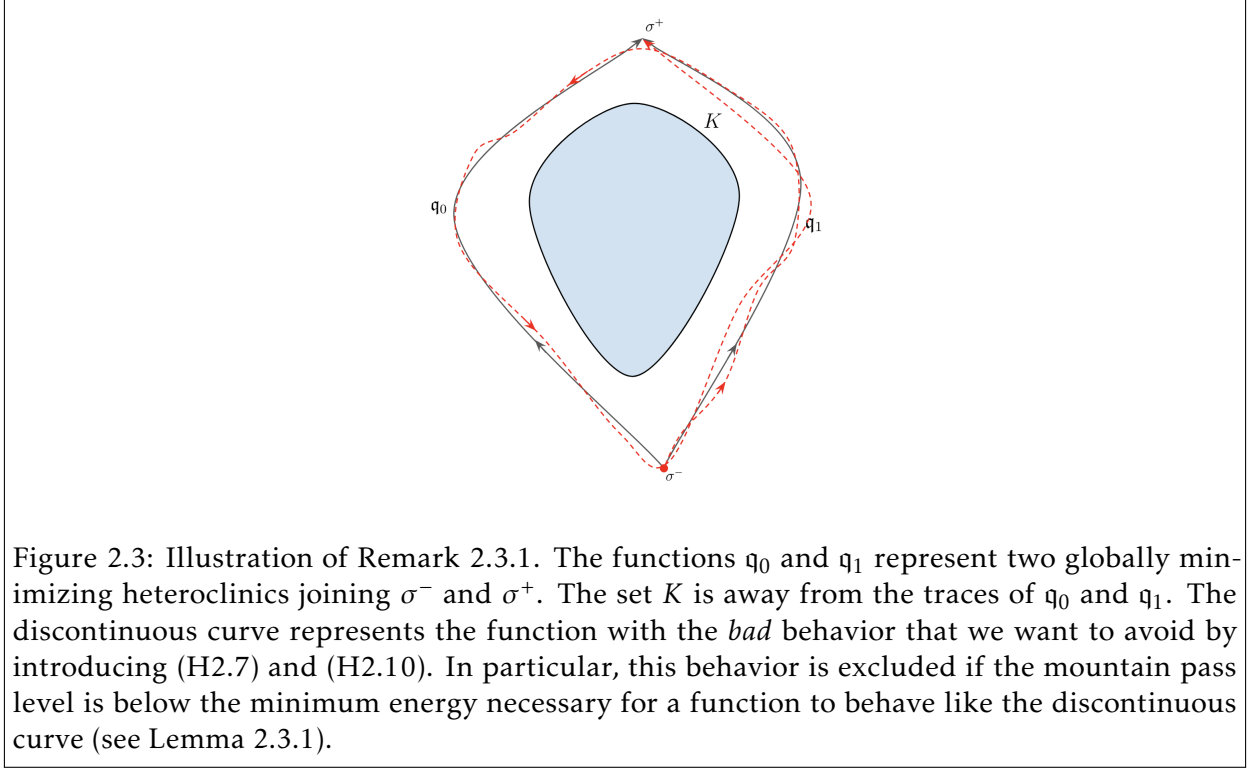


Figure 2.3: Illustration of Remark 2.3.1. The functions q_0 and q_1 represent two globally minimizing heteroclinics joining σ^- and σ^+ . The set K is away from the traces of q_0 and q_1 . The discontinuous curve represents the function with the *bad* behavior that we want to avoid by introducing (H2.7) and (H2.10). In particular, this behavior is excluded if the mountain pass level is below the minimum energy necessary for a function to behave like the discontinuous curve (see Lemma 2.3.1).

constraint on the initial position. Minimization imposes that this solutions have a convergence at $+\infty$ which depends only on the energy. This allows to find a suitable sequence of min-max paths inside this set of constrained solutions, in a fashion that the associated Palais-Smale sequence converges. The main issue with this approach is that we are not able to show that we can perform such a selection, even in particular simple cases. Nevertheless, we think that this question might have a positive answer and, hence, could be used to establish existence of heteroclinics at the mountain pass level. For this reason, we think that the content of this section is of interest.

We begin by stating results regarding a weak Palais-Smale property for J . Let $f \in L^1(\mathbb{R})$ and t_0^+, t_0^- two real numbers such that $t_0^+ - t_0^- > 0$. Define the set

$$F_{t_0^-, t_0^+, f} := \left\{ v \in \mathcal{H} : \forall t \in \mathbb{R} \setminus (t_0^-, t_0^+), \left| v(t) + \psi(t) - \sigma^{\text{sign}(t-t_0^+)} \right|^2 \leq |f(t)| \right\}. \quad (2.4.1)$$

Proposition 2.4.1. *Let t_0^+, t_0^- two real numbers such that $t_0^+ - t_0^- > 0$ and $f \in L^1(\mathbb{R})$. Then, the associated set $F_{t_0^-, t_0^+, f}$ defined in (2.4.1) is closed and convex in \mathcal{H} . Assume that there exist $c \geq 0$ and $(v_n)_{n \in \mathbb{N}}$ a sequence in \mathcal{H} satisfying*

$$\begin{aligned} \lim_{n \rightarrow \infty} J(v_n) &= c, \\ \lim_{n \rightarrow \infty} DJ(v_n) &= 0 \text{ in } \mathcal{H} \end{aligned}$$

and

$$\lim_{n \rightarrow \infty} \text{dist}_{\mathcal{H}}(v_n, F_{t_0^-, t_0^+, f}) = 0.$$

Then there exists $v \in \mathcal{H}$ such that $v_n \rightarrow v$ in \mathcal{H} up to an extraction.

Proposition 2.4.1 follows from Corollary 1.7 in [50], so we skip the proof. Notice that Proposition 2.4.1 can be adapted almost for free to the symmetric setting. Let now \mathfrak{M} be an arbitrary number

such that

$$\mathfrak{M} \in (\mathfrak{c}^\star, \mathfrak{m}^\star).$$

As we already emphasized before, Lemma 2.2.7 implies that all min-maxing sequences of paths in Γ_{sym} (2.1.11) will eventually enter the energy level \mathfrak{M} where “splitting” with respect to any third well is not allowed, which will remove the first possible source of degeneracy. We put $P := \{0\} \times \mathbb{R}^{k-1}$. Given $\mathbf{u} \in P$, we consider the associated constrained minimization problem:

$$\mathfrak{m}(\mathbf{u}) := \inf\{E(q) : q \in \mathcal{A}(\mathbf{u})\}, \quad (2.4.2)$$

where

$$\mathcal{A}(\mathbf{u}) := \{q \in X_{\text{sym},+} : q(0) = \mathbf{u}\}. \quad (2.4.3)$$

We also define

$$P_{\mathfrak{M}} := \{\mathbf{u} \in P : \mathfrak{m}(\mathbf{u}) < \mathfrak{M}\}. \quad (2.4.4)$$

The most immediate properties of such problem are the following:

Lemma 2.4.1. *Assume that (H2.1), (H2.2), (H2.3), (H2.5), (H2.8) and (H2.9) hold. Let $\mathbf{u} \in P_{\mathfrak{M}}$, with $P_{\mathfrak{M}}$ as in (2.4.4). Then, there exists $\bar{t} > 0$ and $\mathfrak{f} \in L^1(\mathbb{R})$, depending only on \mathfrak{M} (so, in particular, independent of \mathbf{u}) and $\mathfrak{q}(\mathbf{u}) - \psi \in F_{-\bar{t}, \bar{t}, \mathfrak{f}}$ such that $E(\mathfrak{q}(\mathbf{u})) = \mathfrak{m}(\mathbf{u})$, see (2.4.1). Moreover, any solution $\tilde{\mathfrak{q}}(\mathbf{u})$ of (2.4.2) belongs to $F_{-\bar{t}, \bar{t}, \mathfrak{f}}$.*

In the proof of Lemma 2.4.1, we make use of the following result:

Lemma 2.4.2. *Assume that (H2.1), (H2.2) and (H2.3) hold. Let $q \in H_{\text{loc}}^1(\mathbb{R}, \mathbb{R}^k)$ be such that $E(q) \leq C$ for some $C > 0$. For each $\nu > 0$, there exists $K_{C,\nu} > 0$ independent of q such that the set*

$$D_{q,\nu} := \{t \in \mathbb{R} : \min_{\sigma \in \Sigma} |q(t) - \sigma| \geq \nu\}$$

satisfies $\mathcal{L}^1(D_{q,\nu}) \leq K_{C,\nu}$.

The proof of Lemma 2.4.2 is skipped, because it is immediate. We will also use the following result due Monteil and Santambrogio [125] (rephrased according to our setting):

Lemma 2.4.3. *There exist constants $\eta_0 > 0$, $\nu > 0$ such that for any $q \in X(\sigma^-, \sigma^+)$, and t^+, t^- such that $t^+ - t^- > 0$ and $|q(t_0^\pm) - \sigma^\pm| \leq \eta_0$, then we have $E(P^\pm[q]) \leq E(q)$ where*

$$P^\pm[q](t) := \begin{cases} \sigma^\pm + \mathfrak{f}(t)|q(t) - \sigma^\pm| & \text{if } t \pm t^\pm < 0 \text{ and } |q(t) - \sigma^\pm| > \mathfrak{f}(t), \\ q(t) & \text{otherwise,} \end{cases}$$

where $\mathfrak{f} \in L^1(\mathbb{R})$.

We refer to [125] for a proof.

Proof of Lemma 2.4.1. Let $(q_n)_{n \in \mathbb{N}}$ be a minimizing sequence for the problem (2.4.2). By Lemma 2.2.5, we get $\mathfrak{q}(\mathbf{u}) \in H_{\text{loc}}^1(\mathbb{R}, \mathbb{R}^k)$ such that $q_n \rightarrow \mathfrak{q}(\mathbf{u})$ locally uniformly, $q_n \rightarrow \mathfrak{q}(\mathbf{u})$ pointwise in \mathbb{R} and $q'_n \rightarrow \mathfrak{q}(\mathbf{u})'$ weakly in \mathcal{L} , up to an extraction.

Step 1. $\mathfrak{q}(\mathbf{u}) \in X(\sigma^-, \sigma^+)$.

Since $m(\mathbf{u}) < \mathfrak{M}$, we can assume up to an extraction that for all $n \in \mathbb{N}$ we have $E(q_n) \leq \mathfrak{M}$. Applying the local uniform convergence and Lemma 2.2.7, we get that $q(\mathbf{u})$ satisfies (2.2.10). Moreover, we have for all $n \in \mathbb{N}$, $q_n \in X_{\text{sym},+}$, which implies that

$$\forall n \in \mathbb{N}, \quad \inf\{|q_n(t) - \sigma^-| : t \geq 0\} \geq \inf\{|q_{n,1}(t) + 1| : t \geq 0\} = 1 > 0,$$

where $q_{n,1}$ is the first component of q_n . Using the local uniform convergence we obtain that

$$\inf\{|q(\mathbf{u}) - \sigma^-| : t \geq 0\} > 0,$$

which along with (2.2.10) implies that $\lim_{t \rightarrow +\infty} q(\mathbf{u})(x) = \sigma^+$. Similarly, we obtain $\lim_{t \rightarrow -\infty} q(\mathbf{u})(t) = \sigma^-$.

Step 2. $q(\mathbf{u}) \in X_{\text{sym},+}$, $q(\mathbf{u})(0) = u$ and $E(q(\mathbf{u})) = m(\mathbf{u})$.

For the first two identities, it suffices to use that for all $n \in \mathbb{N}$, $q_n \in X_{\text{sym},+}$ and the local uniform convergence. The third one is due to the fact that $(q_n)_{n \in \mathbb{N}}$ is a minimizing sequence and the usual lower semicontinuity arguments.

Step 3. There exists $\bar{t} > 0$ and $\mathfrak{f} \in L^1(\mathbb{R})$, depending only on \mathfrak{M} , such that if $\tilde{q}(\mathbf{u})$ is a solution to (2.4.2), then $\tilde{q}(\mathbf{u}) \in F_{-\bar{t}, \bar{t}, \mathfrak{f}}$.

We apply Lemma 2.4.2 to $\tilde{q}(\mathbf{u})$. Since $E(\tilde{q}(\mathbf{u})) = m(\mathbf{u}) < \mathfrak{M}$, there exists $\bar{t} > 0$ depending only on \mathfrak{M} and the fixed constant η_0 (see Lemma 2.4.3) such that

$$|\tilde{q}(\mathbf{u})(\bar{t}) - \sigma^+| \leq \eta_0. \quad (2.4.5)$$

Notice also that, by minimality $\tilde{q}(\mathbf{u})$ solves the ODE

$$\tilde{q}(\mathbf{u})'' = \nabla V(\tilde{q}(\mathbf{u})) \text{ in } (0, +\infty). \quad (2.4.6)$$

Lemma 2.4.3 and (2.4.5) allow us to project $\tilde{q}(\mathbf{u})$ into the funnels in the interval $(\bar{t}, +\infty)$ (and $(-\infty, -\bar{t})$ by symmetry). This gives a function $\tilde{q} \in F_{-\bar{t}, \bar{t}, \mathfrak{f}}$ as in (2.4.1) such that

$$\forall t \in [0, \bar{t}], \quad \tilde{q}(t) = \tilde{q}(\mathbf{u})(t) \quad (2.4.7)$$

and

$$E(\tilde{q}) \leq E(\tilde{q}(\mathbf{u})). \quad (2.4.8)$$

Equality (2.4.7) implies that $\tilde{q}(0) = \tilde{q}(\mathbf{u})(0) = u$. Therefore, by (2.4.8) we have that \tilde{q} is a solution for the minimization problem (2.4.2). Again by minimality, we have that \tilde{q} solves the same second order ODE (2.4.6) in $(0, +\infty)$. Since both \tilde{q} and $\tilde{q}(\mathbf{u})$ belong to $\mathcal{C}^2((0, +\infty), \mathbb{R}^k)$, equality (2.4.7) implies

$$\forall t \in (0, \bar{t}), \quad \tilde{q}'(t) = \tilde{q}(\mathbf{u})'(t). \quad (2.4.9)$$

Hence, we deduce by (2.4.7) and (2.4.9) that for all $t > 0$, we have $\tilde{q}(\mathbf{u})(t) = \tilde{q}(t)$, due to Cauchy-Lipschitz Theorem. By symmetry, we have in fact $\tilde{q}(\mathbf{u}) = \tilde{q} \in F_{-\bar{t}, \bar{t}, \mathfrak{f}}$ in \mathbb{R} , which concludes the proof. \square

The function \mathfrak{f} is in fact exponential and it has an explicit form, see Lemma 19 in [125].

Remark 2.4.1. As one may see by checking the proof of Lemma 2.4.1, the symmetry assumption (H2.8) is necessary for working out such proof. In particular, one needs to exclude that the limit of a minimizing sequence is a function $q \in H_{\text{loc}}^1(\mathbb{R}, \mathbb{R}^k)$ of the type

$$\lim_{x \rightarrow \pm\infty} q(t) = \sigma \in \{\sigma^-, \sigma^+\},$$

which would not belong to the admissible set. Symmetry is a possible way to exclude such a possibility. For general potentials, we cannot find a way to avoid the previous behavior for minimizing sequences corresponding to any $m(\mathbf{u}) < \mathfrak{M}$. We only know that the q obtained above would verify $E(q) > m$, which is, in fact, what allows to establish (by lower semicontinuity) the existence of globally minimizing solutions in the general case. Nevertheless, the value c_{sym} being rather unknown, we are not able to establish an inequality of the type “ $E(q) > m(\mathbf{u})$ ” for $m(\mathbf{u})$ “close” to c_{sym} , which leaves us one argument short.

We fix the constant \bar{t} for the latter. For $\mathbf{u} \in P$, we define the set of solutions

$$\mathfrak{S}(\mathbf{u}) := \{q \in X_{\text{sym},+} : q \in \mathcal{A}(\mathbf{u}), E(q) = m(\mathbf{u})\},$$

where $\mathcal{A}(\mathbf{u})$ was defined in (2.4.3). Notice that Lemma 2.4.1 can be shortly rewritten as

$$\forall \mathbf{u} \in P_{\mathfrak{M}}, \quad \mathfrak{S}(\mathbf{u}) \neq \emptyset \text{ and } \mathfrak{S}(\mathbf{u}) - \psi \subset F_{-\bar{t}, \bar{t}, \bar{f}}.$$

Remark 2.4.2. If $\mathbf{u} \in P$ and $q(\mathbf{u}) \in X_{\bar{x}, \bar{f}}$ is such that $E(q(\mathbf{u})) = m(\mathbf{u})$, then $q(\mathbf{u})$ solves

$$q(\mathbf{u})'' = \nabla V(q(\mathbf{u})) \text{ in } (-\infty, 0) \cup (0, +\infty).$$

In generic situations $\lim_{t \rightarrow 0^-} q(\mathbf{u})'(0)$ will not coincide $\lim_{t \rightarrow 0^+} q(\mathbf{u})'(0)$, meaning that $q(\mathbf{u})$ does not necessarily extend to a solution of the system in the whole \mathbb{R} .

To sum up, we found that the set-valued map

$$\mathfrak{S}_{\mathcal{H}} : \mathbf{u} \in P_{\mathfrak{M}} \rightarrow \mathfrak{S}(\mathbf{u}) - \psi \tag{2.4.10}$$

is such that for any $\mathbf{u} \in P_{\mathfrak{M}}$, $\mathfrak{S}_{\mathcal{H}}(\mathbf{u})$ is a non-empty subset of $F_{-\bar{t}, \bar{t}, \bar{f}} \cap (X_{\text{sym},+} - \psi)$. It can now be easily seen that Proposition 2.4.1 and Lemma 2.4.1 imply that the existence of a mountain pass heteroclinic solution in $X_{\text{sym},+}$ as long as we are able to solve a continuous selection problem. For that purpose, let us define the class of paths

$$\mathcal{P}_{\mathfrak{M}} := \{p \in \mathcal{C}([0, 1], P_{\mathfrak{M}}) : \forall i \in \{0, 1\}, \quad p(i) \in \{q(0) : q \in \mathcal{V}_{\text{sym}, i}\}\} \tag{2.4.11}$$

where the sets $\mathcal{V}_{\text{sym}, i}$ are as in (2.1.10). The paths in $\mathcal{P}_{\mathfrak{M}}$ can be seen as the trace in P of the paths in the mountain pass family Γ_{sym} , which leads to the following formulation of the result:

Corollary 2.4.1. *Assume (H2.1), (H2.2), (H2.3), (H2.5), (H2.8) and (H2.9) hold. Assume moreover that for any $p \in \mathcal{P}_{\mathfrak{M}}$, $\mathcal{P}_{\mathfrak{M}}$ as in (2.4.11), there exists $\gamma_p \in \Gamma_{\text{sym}}$ such that*

$$\forall s \in [0, 1], \quad \gamma_p(s) \in \mathfrak{S}_{\mathcal{H}}(p(s)), \tag{2.4.12}$$

where the map $\mathfrak{S}_{\mathcal{H}}$ is as in (2.4.10). Then, there exists $u \in X_{\text{sym},+}$ a solution to (2.1.1) such that $E(u) = c_{\text{sym}}$.

Proof. Due to the definition of c_{sym} in (2.1.12), there exists $(\gamma_n)_{n \in \mathbb{N}}$ a sequence in Γ_{sym} such that

$$\lim_{n \rightarrow +\infty} \max_{s \in [0,1]} J(\gamma_n(s)) = c_{\text{sym}}. \quad (2.4.13)$$

Since $c_{\text{sym}} < \mathfrak{M}$, we can assume without loss of generality that

$$\sup_{n \in \mathbb{N}} \max_{s \in [0,1]} J(\gamma_n(s)) < \mathfrak{M}. \quad (2.4.14)$$

Consider the associated sequence

$$p_n : s \in [0,1] \rightarrow \gamma_n(s)(0) \in P.$$

Notice that the fact that $(\gamma_n)_{n \in \mathbb{N}}$ is a sequence in Γ_{sym} and (2.4.14) imply that for all $n \in \mathbb{N}$, $p_n \in \mathcal{P}_{\mathfrak{M}}$ as defined in (2.4.11). Therefore, for each $n \in \mathbb{N}$ there exists $\hat{\gamma}_n := \gamma_{p_n} \in \Gamma_{\text{sym}}$ such that $\hat{\gamma}_n$ fulfills (2.4.12) for p_n . As a consequence, Lemma 2.4.1 implies

$$\forall n \in \mathbb{N}, \forall s \in [0,1], J(\hat{\gamma}_n(s)) \leq J(\gamma_n(s)) \quad (2.4.15)$$

and

$$\forall n \in \mathbb{N}, \forall s \in [0,1], \hat{\gamma}_n(s) \in F_{-\bar{t}, \bar{t}, \bar{f}} \cap (X_{\text{sym},+} - \psi). \quad (2.4.16)$$

Using (2.4.13) and (2.4.15), we have that

$$\lim_{n \rightarrow +\infty} \max_{s \in [0,1]} J(\hat{\gamma}_n(s)) = c_{\text{sym}}$$

where $\bar{t} > 0$ and $\bar{f} \in L^1(\mathbb{R})$ are given by Lemma 2.4.1. Therefore, by classical results already used in this chapter (e. g. *Corollary 4.3* in Mawhin and Willem [118]) we obtain a Palais-Smale sequence $(u_n)_{n \in \mathbb{N}}$ the associated with min-maxing sequence of paths $(\hat{\gamma}_n)_{n \in \mathbb{N}}$. Due to (2.4.16), $(u_n)_{n \in \mathbb{N}}$ fulfills the additional assumptions of Proposition 2.4.1. Moreover, the sequence $(u_n)_{n \in \mathbb{N}}$ is such that

$$\lim_{n \rightarrow +\infty} \text{dist}_{\mathcal{H}}(u_n + \psi, X_{\text{sym},+}) = 0. \quad (2.4.17)$$

Therefore, Proposition 2.4.1 yields the existence of $u - \psi \in \mathcal{H}$ such that $u_n \rightarrow u - \psi$ strongly in \mathcal{H} up to an extraction. The strong convergence of the Palais-Smale sequence implies that $E(u) = c_{\text{sym}}$ and u solves (2.1.1). By (2.4.17), we have that $u \in X_{\text{sym},+}$, which concludes the proof. \square

We do not see how the existence of the selection maps required by Corollary 2.4.1 could be proven, as the results we can find on continuous selections (see [143] and the references therein) do not seem to apply to the set-valued map \mathfrak{S} , the main issue being its potentially complicated structure. Therefore, so far we are only able to prove an “if” result such as Corollary 2.4.1.

Heteroclinic traveling waves of 2D parabolic Allen-Cahn systems

Outline of the current chapter

3.1 Introduction	82
3.2 The main results: Statements and discussions	84
3.2.1 Basic assumptions and definitions	85
3.2.2 Existence	86
3.2.3 Conditions at infinity	89
3.2.4 Min-max characterization of the speed	90
3.2.5 Definition of the upper bounds	91
3.2.6 Methods and ideas of the proofs	92
3.2.7 Examples of potentials verifying the assumptions	95
3.3 Discussion on the previous literature and open problems	99
3.3.1 Related reaction-diffusion models and the question of stability	99
3.3.2 The heteroclinic stationary wave for 2D Allen-Cahn systems	101
3.3.3 Traveling waves for 1D parabolic systems of gradient type	103
3.3.4 Link with the singular limit problem	104
3.4 The abstract setting	106
3.4.1 Main definitions and notations	106
3.4.2 Statement of the abstract results	111
3.5 Proof of the abstract results	112
3.5.1 Scheme of the proofs	112
3.5.2 Preliminaries	115
3.5.3 The infima of E_c in X_T are well defined	117
3.5.4 General continuity and semi-continuity results	118
3.5.5 Existence of an infimum for E_c in X_T	121
3.5.6 The comparison result	124
3.5.7 Existence of the unconstrained solutions	133

3.5.8	Uniqueness of the speed	137
3.5.9	Proof of Theorem 3.5 completed	139
3.5.10	Asymptotic behavior of the constrained solutions at $-\infty$	139
3.5.11	Proof of Theorem 3.6 completed	142
3.5.12	Proof of Theorem 3.7 completed	143
3.6	Proofs of the main results completed	144
3.6.1	Proving the link between the main setting and the abstract setting . .	144
3.6.2	Proof of Theorem 3.1 completed	149
3.6.3	Proof of Theorem 3.2 completed	150
3.6.4	Proof of Theorem 3.3	150
3.6.5	Proof of Theorem 3.4 completed	155

This chapter corresponds to the preprint [130], with some minor presentation modifications.

Abstract. In this chapter we show the existence of traveling waves $w : [0, +\infty) \times \mathbb{R}^2 \rightarrow \mathbb{R}^k$ ($k \geq 2$) for the parabolic Allen-Cahn system

$$\partial_t w - \Delta w = -\nabla_u V(w) \text{ in } [0, +\infty) \times \mathbb{R}^2,$$

satisfying some *heteroclinic* conditions at infinity. The potential V is a non-negative and smooth multi-well potential, which means that its null set is finite and contains at least two elements. The traveling wave w propagates along the horizontal axis according to a speed $c^\star > 0$ and a profile \mathfrak{U} . The profile \mathfrak{U} joins as $x_1 \rightarrow \pm\infty$ (in a suitable sense) two locally minimizing 1D heteroclinics which have *different* energies and the speed c^\star satisfies certain uniqueness properties. The proof is variational and, in particular, it requires the assumption of an upper bound, depending on V , on the difference between the energies of the 1D heteroclinics.

Résumé. Dans ce chapitre, nous établissons l'existence d'ondes progressives $w : [0, +\infty) \times \mathbb{R}^2 \rightarrow \mathbb{R}^k$ ($k \geq 2$) pour le système du type Allen-Cahn parabolique

$$\partial_t w - \Delta w = -\nabla_u V(w) \text{ in } [0, +\infty) \times \mathbb{R}^2,$$

qui satisfont certaines conditions *hétéroclines* à l'infini. L'onde progressive w se propage au long de l'axe horizontal suivant une vitesse c^\star et un profil \mathfrak{U} . Lorsque $x_1 \rightarrow \pm\infty$, le profil \mathfrak{U} connecte (dans un certain sens) deux orbites hétéroclines unidimensionnelles qui ont des énergies *différentes*. La vitesse c^\star satisfait certaines propriétés d'unicité. La preuve est variationnelle et, en particulier, nous faisons l'hypothèse que la différence entre les énergies des hétéroclines est bornée supérieurement par une constante dépendante de V .

3.1 Introduction

Consider the parabolic system of equations

$$\partial_t w - \Delta w = -\nabla_u V(w) \text{ in } [0, +\infty) \times \mathbb{R}^2, \quad (3.1.1)$$

where $V : \mathbb{R}^k \rightarrow \mathbb{R}$ is a smooth, non-negative, multi-well potential (see assumptions (H3.1), (H3.2), (H3.3) later) and $w : [0, +\infty) \times \mathbb{R}^2 \rightarrow \mathbb{R}^k$, with $k \geq 2$. We seek for *traveling wave* solutions to (3.1.1). That is, we impose on w

$$\forall (t, x_1, x_2) \in [0, +\infty) \times \mathbb{R}^2, \quad w(t, x_1, x_2) = \mathfrak{U}(x_1 - c^*t, x_2),$$

where $\mathfrak{U} : \mathbb{R}^2 \rightarrow \mathbb{R}^k$ is the *profile* of the wave and $c^* > 0$ is the *speed* of propagation of the wave, which occurs in the x_1 -direction. Both the profile and the speed are the unknowns of the problem. Replacing in (3.1.1), we find that the profile \mathfrak{U} and c^* must satisfy the elliptic system

$$-c^* \partial_{x_1} \mathfrak{U} - \Delta \mathfrak{U} = -\nabla_u V(\mathfrak{U}) \text{ in } \mathbb{R}^2. \quad (3.1.2)$$

The system (3.1.1) can be seen as a reaction-diffusion system. Since the early works, motivated by questions from population dynamics, of Fisher [80] and Kolmogorov, Petrovsky and Piskunov [105], devoted to a scalar reaction-diffusion equation in one space dimension known today as the Fisher-KPP equation, traveling and stationary waves are known to play a major role in the dynamics of reaction-diffusion problems. For instance, in [79, 78], Fife and McLeod proved stability results for the equations considered in [80, 105]. Regarding higher dimensional problems (but always in the scalar case), existence results for traveling waves were obtained by Aronson and Weinberger [20] for equations with \mathbb{R}^N as space domain and by Berestycki, Larrouturou and Lions [35], Berestycki and Nirenberg [39] for unbounded cylinders of the type $\mathbb{R} \times \omega$, with $\omega \subset \mathbb{R}^{N-1}$ a bounded domain. We also mention that asymptotic stability results (for a suitable class of perturbations) for traveling waves in the scalar Allen-Cahn equation in \mathbb{R}^N were obtained by Matano, Nara and Taniguchi [115].

All the papers mentioned above are devoted to scalar equations and they rely on the application of the maximum principle and its related tools. As it is well-known, the maximum principle does not apply in general to systems of equations, meaning that other techniques are needed in order to study the existence of traveling waves (and their properties in case they exist) for systems. Different, more general, approaches had been taken in order to circumvent the lack of the maximum principle when dealing with parabolic systems. We refer to the books by Smoller [156] and Volpert, Volpert and Volpert [169]. One of these approaches consists on the use of variational methods. In the context of reaction-diffusion equations, this approach seems to appear for the first time in Heinze's PhD thesis [93] (even though the existence of a variational framework for reaction diffusion problems was known since [79, 78]) and subsequently carried on also by Muratov [128], Lucia, Muratov and Novaga [110], Alikakos and Katzourakis [13] (see also Alikakos, Fusco and Smyrnelis [12]), Risler [145, 146, 144] and, more recently, by Chen, Chien and Huang [68]. In the latter, the authors consider a parabolic Allen-Cahn system in a two dimensional strip $\mathbb{R} \times (-l, l)$ and find traveling waves which join a well and an approximation of an heteroclinic orbit in $(-l, l)$, for a class of symmetric triple-well potentials. Lastly, we mention that variational methods have also been applied to scalar reaction-diffusion equations, see for instance Bouhours and Nadin [55] for the case of heterogeneous equations as well as Lucia, Muratov and Novaga [111]. In this chapter we shall also take a variational approach for dealing with the following question:

Question: Assuming that there exist two *heteroclinic orbits*, joining two fixed wells, with *different* energy (defined in (3.2.1)) levels, does there exist a solution (c, \mathfrak{U}) to (3.1.2) such that \mathfrak{U}

joins the two heteroclinic orbits at infinity, uniformly in x_1 ?

Heteroclinic orbits are curves $q : \mathbb{R} \rightarrow \mathbb{R}^k$ which solve the equation

$$q'' = \nabla_u V(q) \text{ in } \mathbb{R}$$

and join two *different* wells of Σ at $\pm\infty$. Moreover, one asks that the *1D energy* (i. e., the functional associated with the previous equation, see (3.2.1)) is finite. We show that, under the proper assumptions, the question we posed has an affirmative answer. Our motivation comes from two different sides:

1. Stationary heteroclinic-type solutions of (3.1.1) have been known to exist in several situations for a long time. Indeed, for a class of symmetric potentials, Alama, Bronsard and Gui in [3] showed the existence of a stationary wave (that is, a solution to (3.1.2) with $c = 0$) in the situation such that two heteroclinics with *equal* energy levels exist and are global minimizers of the 1D energy. Their analysis was later extended to potentials without symmetry in several papers, which in some cases obtained similar results by means of different techniques. See Fusco [83], Monteil and Santambrogio [125] (an extension of the previous work by the same authors [124] for the finite dimensional problem), Schatzman [153], Smyrnelis [157]. A key observation is that this problem can be seen as a heteroclinic orbit problem for a potential (the 1D energy, see (3.2.1)) defined in the infinite-dimensional space $L^2(\mathbb{R}, \mathbb{R}^k)$. Therefore, it is natural to aim at solving a connecting orbit problem for potentials defined in, say, Hilbert spaces and then deduce the original problem as a particular case. This is the approach taken in [125] (in the metric space setting) and in [157] (in the Hilbert space setting).
2. Alikakos and Katzourakis [13], showed the existence of traveling waves for a class of 1D parabolic systems of gradient type. Essentially, they assume that the potential possesses two local minima (one of them global) at *different* levels. Hence, their potential is not of multi-well type in general. The profile of the traveling waves connects the two local minima at infinity and the determination of the speed becomes also part of the problem.

The results of this chapter follow by suitably merging the ideas of the previous items. More precisely, we formulate and provide solutions for a *heteroclinic traveling problem* as that in [13] for potentials defined in an abstract Hilbert space. Then, we recover as a particular case the existence of a traveling wave solution for (3.1.1) with heteroclinic behavior at infinity.

3.2 The main results: Statements and discussions

We now state the results of this chapter. In Theorem 3.1, which is the main result, existence of a traveling wave solution with speed c^* and profile \mathfrak{u} is established as well as the uniqueness (in some sense) of c^* and the L^2 exponential convergence of \mathfrak{u} at the limit $x_1 \rightarrow +\infty$. For proving such a result, we use the bound assumption (H3.6). In Theorem 3.2, we show that under the additional assumption (H3.7)) the condition at infinity as $x_1 \rightarrow -\infty$ can be strengthened with respect to that given in Theorem 3.1 and in particular we show that the solution converges at $-\infty$ at an exponential rate. In Theorem 3.3, we show that under the previous assumptions we have uniform convergence of the solution in the x_1 and the x_2 direction. Assumption (H3.7) is also used

for proving Theorem 3.4, which gives further properties on the speed c^\star . We conclude this section by describing the outline and main ideas of our proofs (Section 3.2.6) as well as giving examples of potentials that verify the assumptions of this chapter (Section 3.2.7).

3.2.1 Basic assumptions and definitions

Before stating the results, we recall some standard assumptions, definitions and results and we introduce some notation. The multi-well potentials V considered in this chapter satisfy the following:

(H3.1). $V \in C_{\text{loc}}^2(\mathbb{R}^k)$ and $V \geq 0$ in \mathbb{R}^k . Moreover, $V(u) = 0$ if and only if $u \in \Sigma$, where, for some $l \geq 2$

$$\Sigma := \{\sigma_1, \dots, \sigma_l\}.$$

(H3.2). There exist $\alpha_0, \beta_0, R_0 > 0$ such that for all $u \in \mathbb{R}^k$ with $|u| \geq R_0$ it holds $\langle \nabla_u V(u), u \rangle \geq \alpha_0 |u|^2$ and $V(u) \geq \beta_0$.

(H3.3). For all $\sigma \in \Sigma$, the matrix $D^2V(\sigma)$ is positive definite.

As we advanced before, one considers the 1D energy functional

$$E(q) := \int_{\mathbb{R}} e(q)(t) dt := \int_{\mathbb{R}} \left[\frac{1}{2} |q'(t)|^2 + V(q(t)) \right] dt, \quad q \in H_{\text{loc}}^1(\mathbb{R}, \mathbb{R}^k). \quad (3.2.1)$$

Given a pair of wells $(\sigma_i, \sigma_j) \in \Sigma^2$, as done for instance in Rabinowitz [136] we define

$$X(\sigma_i, \sigma_j) := \left\{ q \in H_{\text{loc}}^1(\mathbb{R}, \mathbb{R}^k) : E(q) < +\infty \text{ and } \lim_{t \rightarrow -\infty} q(t) = \sigma_i, \lim_{t \rightarrow +\infty} q(t) = \sigma_j \right\},$$

the set of curves in \mathbb{R}^k connecting σ_i and σ_j . The space $X(\sigma_i, \sigma_j)$ is a metric space when it is endowed with the L^2 and the H^1 distances, since $q - \tilde{q} \in H^1(\mathbb{R}, \mathbb{R}^k)$ whenever q and \tilde{q} belong to $X(\sigma_i, \sigma_j)$. If q is a critical point of the energy E in $X(\sigma_i, \sigma_j)$, we say that q is an *homoclinic orbit* when $\sigma_i = \sigma_j$ and that q is an *heteroclinic orbit* when $\sigma_i \neq \sigma_j$. Define as well the corresponding infimum value

$$m_{\sigma_i \sigma_j} := \inf \{ E(q) : q \in X(\sigma_i, \sigma_j) \}. \quad (3.2.2)$$

If σ^- and σ^+ are two distinct wells in Σ , it turns out that $m_{\sigma^- \sigma^+}$ is not attained in general. We need to add the following assumption:

(H3.4). We have that

$$\forall \sigma \in \Sigma \setminus \{\sigma^-, \sigma^+\}, \quad m_{\sigma^- \sigma^+} < m_{\sigma^- \sigma} + m_{\sigma \sigma^+}.$$

Notice that one can always find a pair $(\sigma^-, \sigma^+) \in \Sigma^2$ such that (H3.4) holds. Assuming that (H3.1), (H3.2), (H3.3) and (H3.4) hold, it is well known that there exists a minimizer of E in $X(\sigma^-, \sigma^+)$. Moreover, we have the compactness of minimizing sequences as follows: For any $(q_n)_{n \in \mathbb{N}}$ in $X(\sigma^-, \sigma^+)$ such that $E(q_n) \rightarrow m$, there exists $(\tau_n)_{n \in \mathbb{N}}$ in \mathbb{R} and $q \in X(\sigma^-, \sigma^+)$ such that $E(q) = m$ and, up to subsequences

$$\|q_n(\cdot + \tau_n) - q\|_{H^1(\mathbb{R}, \mathbb{R}^k)} \rightarrow 0 \text{ as } n \rightarrow +\infty. \quad (3.2.3)$$

As we said before, this result is well-known. The earlier references are Bolotin [51], Bolotin and Kozlov [52], Bertotti and Montecchiari [40] and Rabinowitz [138, 142], sometimes in a slightly different setting. Proofs and applications of the compactness property (3.2.3) are also given in Alama, Bronsard and Gui [3], Alama et al [2] and Schatzman [153].

We fix the two wells σ^- and σ^+ for the rest of the chapter as well as $m := m_{\sigma^-\sigma^+}$. According to the previous discussion, we have that the set

$$\mathcal{F} := \{q : q \in X(\sigma^-, \sigma^+) \text{ and } E(q) = m\}, \quad (3.2.4)$$

is not empty. We term the elements of \mathcal{F} as *globally minimizing heteroclinics* between σ^- and σ^+ . The term *heteroclinics* comes from the fact that σ^- and σ^+ are different. An important fact is that, due to the translation invariance of E and $X(\sigma^-, \sigma^+)$, we have that if $q \in \mathcal{F}$, then for all $\tau \in \mathbb{R}$ it holds $q(\cdot + \tau) \in \mathcal{F}$.

3.2.2 Existence

The assumptions (H3.1), (H3.2), (H3.3) and (H3.4) stated before are classical. In order to obtain our results, we shall supplement them with the following one, which is more specific to the setting of this chapter:

(H3.5). Assume that (H3.1), (H3.2), (H3.3) and (H3.4) hold for the potential V . We keep the previous notations. We assume the following:

1. It holds that $\mathcal{F}^- := \mathcal{F} = \{q^-(\cdot + \tau) : \tau \in \mathbb{R}\}$ for some $q^- \in X(\sigma^-, \sigma^+)$, where \mathcal{F} was defined in (3.2.4). We also set $m^- := m$.
2. There exists $m^+ > m^-$ and $q^+ \in X(\sigma^-, \sigma^+)$ such that $E(q^+) = m^+$ and q^+ is a local minimizer of E with respect to the H^1 norm. We denote $\mathcal{F}^+ := \{q^+(\cdot + \tau) : \tau \in \mathbb{R}\}$.
3. We have the spectral nondegeneracy assumption due to Schatzman ([153]): For all $q \in X(\sigma^-, \sigma^+)$, let $A(q)$ be the unbounded linear operator in $L^2(\mathbb{R}, \mathbb{R}^k)$ with domain $H^2(\mathbb{R}, \mathbb{R}^k)$ defined as

$$A(q) : v \rightarrow -v'' + D^2V(q)v,$$

then, it holds that for any $q \in \mathcal{F}^- \cup \mathcal{F}^+$ we have $\text{Ker}(A(q)) = \{q'\}$. The fact that $q' \in H^2(\mathbb{R}, \mathbb{R}^k)$ follows from the identity $q''' = D^2V(q)q'$.

Notice that if we had $m^+ = m^-$ we would be in the framework of Alama, Bronsard and Gui [3], for which the 2D solution connecting q^- and q^+ is stationary. Essentially, conditions 1. and 2. in (H3.5) imply that q^- is a globally minimizing heteroclinic and q^+ is a locally (but not globally) minimizing heteroclinic. Regarding assumption 3., introduced in [153], it must be seen as a generalized non-degeneracy assumption for the minima. They are still degenerate critical points because every critical point of E is degenerate due to the invariance by translations. Nevertheless, the assumption 3. implies that they are *non-degenerate up to invariance by translations*. As shown in [153], such a condition is generic in the sense that given a potential satisfying 1. and 2. one can always find a potential which verifies 1. 2. (with the same minimizers) and 3. and it is arbitrarily

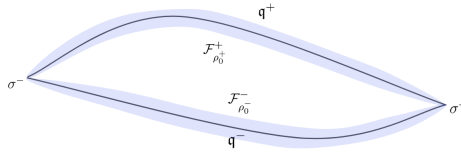


Figure 3.1: Situation described by (H3.5). The curves correspond to the traces of q^- and q^+ as indicated. The shadowed regions correspond to the traces of the functions in $\mathcal{F}_{\rho_0^-}^-$ and $\mathcal{F}_{\rho_0^+}^+$, which are a neighborhood of \mathcal{F}^- and \mathcal{F}^+ respectively.

close to the given potential¹. The most important consequence of this assumption, as proven in [153], is the existence of two constants $\rho_0^+ > 0$ and ρ_0^- such that

$$\begin{aligned} \forall q \in L_{\text{loc}}^2(\mathbb{R}, \mathbb{R}^k), \quad \text{dist}_{L^2(\mathbb{R}, \mathbb{R}^k)}(q, \mathcal{F}^\pm) \leq \rho_0^\pm \\ \Rightarrow \exists! \tau^\pm(q) \in \mathbb{R} : \|q - q^\pm(\cdot + \tau^\pm(q))\|_{L^2(\mathbb{R}, \mathbb{R}^k)} = \text{dist}_{L^2(\mathbb{R}, \mathbb{R}^k)}(q, \mathcal{F}^\pm) \end{aligned} \quad (3.2.5)$$

and for some constant β^\pm we have

$$\forall q \in X(\sigma^-, \sigma^+), \quad \text{dist}_{H^1(\mathbb{R}, \mathbb{R}^k)}(q, \mathcal{F}^\pm) \leq \rho_0^\pm \Rightarrow \text{dist}_{H^1(\mathbb{R}, \mathbb{R}^k)}(q, \mathcal{F}^\pm)^2 \leq \beta^\pm(E(q) - m^\pm). \quad (3.2.6)$$

In fact, in [153] this is only proven for global minimizers but the proof readily extends to local ones as well. Notice that (3.2.5) and (3.2.6) state that the energy is *quadratic* around \mathcal{F}^- and \mathcal{F}^+ , which is the infinite-dimensional analogue of (H3.3), taking into account the degeneracy generated by the group of translations. We will define for $r > 0$ the sets

$$\mathcal{F}_r^\pm := \{q \in L_{\text{loc}}^2(\mathbb{R}, \mathbb{R}^k) : \text{dist}_{L^2(\mathbb{R}, \mathbb{R}^k)}(q, \mathcal{F}^\pm) \leq r\}, \quad (3.2.7)$$

so that we can assume without loss of generality that $\mathcal{F}_{\rho_0^+}^+ \cap \mathcal{F}_{\rho_0^-}^- = \emptyset$. See Figure 3.1 for an explanatory design of (H3.5). Let us now assume that $m^+ - m^-$ is bounded above as follows:

(H3.6). Assume that (H3.5) holds and, moreover,

$$0 < m^+ - m^- < E_{\max},$$

where E_{\max} will be defined later in (3.2.23). Moreover, assume that

$$\{q \in X(\sigma^-, \sigma^+) : E(q) < m^+\} \subset \mathcal{F}_{\rho_0^-/2}^-,$$

with $\mathcal{F}_{\rho_0^-/2}^-$ as in (3.2.7).

See also Figure 3.2. Essentially, (H3.6) requires that $m^- - m^+$ is not too large and the bound is given by a constant E_{\max} that can be computed through the constants produced in (3.2.5) and (3.2.6) as a consequence of (H3.5). If (H3.6) holds, then we are able to answer the question that we

¹One can think about the analogy between this property and the classical results for Morse functions (i. e., functions without degenerate critical points), which state that such type of functions are generic.

posed at the beginning of the chapter in a positive way. More precisely, recall the equation of the profile:

$$-c\partial_{x_1}\mathfrak{U} - \Delta\mathfrak{U} = -\nabla_u V(\mathfrak{U}) \text{ in } \mathbb{R}^2 \quad (3.2.8)$$

and consider the conditions at infinity

$$\exists L^- \in \mathbb{R}, \forall x_1 \leq L^-, \mathfrak{U}(x_1, \cdot) \in \mathcal{F}_{\rho_0^-/2}^- \quad (3.2.9)$$

$$\exists L^+ \in \mathbb{R}, \forall x_1 \geq L^+, \mathfrak{U}(x_1, \cdot) \in \mathcal{F}_{\rho_0^+/2}^+ \quad (3.2.10)$$

As stated before, our proof is variational, which implies that the profile \mathfrak{U} can be characterized as a critical point of a functional. The variational framework is as follows: assume that (H3.6) holds and set

$$S := \{U \in H_{\text{loc}}^1(\mathbb{R}, L^2(\mathbb{R}, \mathbb{R}^k)) : \exists L \geq 1, \forall x_1 \geq L, U(x_1, \cdot) \in \mathcal{F}_{\rho_0^+/2}^+ \\ \forall x_1 \leq -L, U(x_1, \cdot) \in \mathcal{F}_{\rho_0^-/2}^-\}.$$

For $U \in S$ and $c > 0$ we define the energy

$$E_{2,c}(U) := \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \frac{|\partial_{x_1} U(x_1, x_2)|^2}{2} dx_2 + (E(U(x_1, \cdot)) - m^+) \right) e^{cx_1} dx_1.$$

Formally, critical points of $E_{2,c}$ give rise to solutions of (3.2.8). If $U \in S$, we can define the translated function $U^\tau := U(\cdot + \tau, \cdot)$ for $\tau \in \mathbb{R}$. Then, for all $c > 0$ we have

$$E_{2,c}(U^\tau) = e^{-c\tau} E_{2,c}(U)$$

which implies that

$$\forall c > 0, \inf_{U \in S} E_{2,c}(U) \in \{-\infty, 0\}.$$

We have by now introduced the notations which allow us to state the main result of this chapter:

Theorem 3.1 (Main Theorem). *Assume that (H3.6) holds. Then, we have:*

1. **Existence.** *There exist $c^\star > 0$ and $\mathfrak{U} \in C^{2,\alpha}(\mathbb{R}^2, \mathbb{R}^k) \cap S$, $\alpha \in (0, 1)$, which fulfill (3.2.8). The profile \mathfrak{U} satisfies the conditions at infinity (3.2.9) and (3.2.10) as well as the variational characterization*

$$E_{2,c^\star}(\mathfrak{U}) = 0 = \inf_{U \in S} E_{2,c^\star}(U). \quad (3.2.11)$$

2. **Uniqueness of the speed.** *The speed c^\star is unique in the following sense: Assume that $\overline{c^\star} > 0$ is such that*

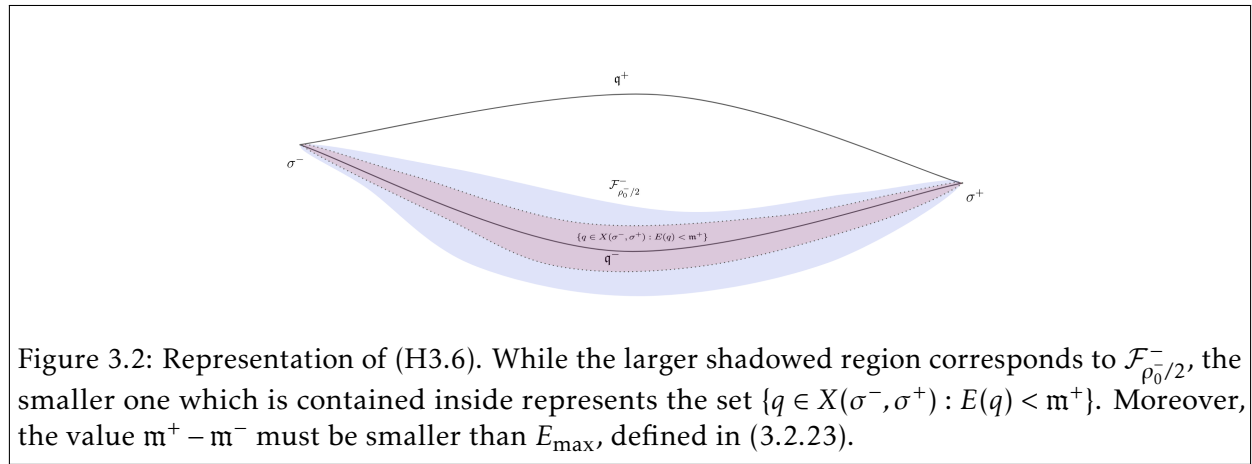
$$\inf_{U \in S} E_{2,\overline{c^\star}}(U) = 0$$

and that $\overline{\mathfrak{U}} \in S$ is such that $(\overline{c^\star}, \overline{\mathfrak{U}})$ solves (3.2.8) and $E_{2,\overline{c^\star}}(\overline{\mathfrak{U}}) < +\infty$. Then, $\overline{c^\star} = c^\star$.

3. **Exponential convergence.** *The convergence of \mathfrak{U} at $+\infty$ is exponential with respect to the L^2 -norm. More precisely, there exists $\mathfrak{M}^+ > 0$ and $\tau^+ \in \mathbb{R}$ such that for all $x_1 \in \mathbb{R}$*

$$\|\mathfrak{U}(x_1, \cdot) - \mathfrak{q}^+(\cdot + \tau^+)\|_{L^2(\mathbb{R}, \mathbb{R}^k)} \leq \mathfrak{M}^+ e^{-c^\star t}. \quad (3.2.12)$$

Remark 3.2.1. The existence part of Theorem 3.1 states that there exists a solution (c^*, \mathfrak{U}) such that \mathfrak{U} is a global minimizer of E_{c^*} in S . We also have that the speed c^* is unique for some class of solutions, namely for finite energy solutions and speeds for which the corresponding energy is bounded below in S . In particular, c^* is unique among the class of globally minimizing profiles. In other words, if $c > 0$ is such that the infimum of E_c in S is attained, then $c = c^*$. This is analogous to what it was shown in Alikakos and Katzourakis [13]. As explained in the introduction, the main drawback of our approach is the existence assumption (H3.6). In particular, the definition of the upper bound E_{\max} is technical and it is possible that in several situations it could be small. Nevertheless, in Section 3.2.7 we show that there exists examples of potentials for which (H3.6) holds.



3.2.3 Conditions at infinity

The solutions given by Theorem 3.1 satisfy the conditions at infinity (3.2.9) and (3.2.10). As we can see, condition (3.2.9) is more imprecise than expected, as it only states that $\mathfrak{U}(x_1, \cdot)$ is not far from \mathcal{F}^- with respect to the L^2 distance when x_1 is close enough to $-\infty$. In particular, we cannot ensure that \mathfrak{U} is really heteroclinic, in the sense of connecting two stable states as $x_1 \rightarrow \pm\infty$. Therefore, it is reasonable to wonder if we can establish a behavior of the type

$$\exists \tau^- \in \mathbb{R}, \quad \|\mathfrak{U}(x_1, \cdot) - q^-(\cdot + \tau^-)\|_{L^2(\mathbb{R}, \mathbb{R}^k)} \rightarrow 0 \text{ as } x_1 \rightarrow -\infty, \quad (3.2.13)$$

so that \mathfrak{U} is an actual heteroclinic. While (3.2.13) was established in [13], their argument does not seem to apply to the infinite-dimensional setting, which means that new ideas are needed. We have been able to show that (3.2.13) holds under the following additional assumption:

(H3.7). We have that assumption (H3.6) holds and, additionally:

$$m^+ - m^- < \frac{(\mu^- \mathfrak{d}_0)^2}{2},$$

where the constants \mathfrak{d}_0 and μ^- are defined later in (3.2.21) and (3.2.22) respectively.

Assumption (H3.7) is not too restrictive (at least with respect to the assumptions we already have), since an upper bound on $m^+ - m^-$ is already imposed in (H3.6), meaning that at worst one

only needs to lower it. The definition of the constant \mathfrak{d}_0 is essentially technical and depends only on the distance between the sets \mathcal{F}^- and \mathcal{F}^+ , while μ^- depends only on local information around \mathcal{F}^- . Anyway, the result given by (H3.7) writes as follows:

Theorem 3.2. *Assume that (H3.6) and (H3.7) hold. Let (c^\star, \mathfrak{U}) be the solution given by Theorem 3.1. Then, \mathfrak{U} satisfies the stronger condition (3.2.13). Moreover, it holds that $c^\star < \mu^-$, μ^- to be defined later in (3.2.22), and there exists $\mathfrak{N}^- > 0$ such that for all $x_1 \in \mathbb{R}$*

$$\|\mathfrak{U}(x_1, \cdot) - \mathfrak{q}^-(\cdot + \tau^-)\|_{L^2(\mathbb{R}, \mathbb{R}^k)} \leq \mathfrak{N}^- e^{(\mu^- - c^\star)x_1} \quad (3.2.14)$$

The natural question is whether (3.2.12) in Theorem 3.1 and (3.2.14) in Theorem 3.2 can be improved. In particular, whether the L^2 -norm can be replaced by the H^1 -norm. We conjecture that the answer to this question is positive, but we do not have a proof of this fact. However, as one can check in Smyrnelis [157] and Fusco [83], such a fact holds for the balanced 2D heteroclinic solution. They obtain these properties by combining standard elliptic estimates with some properties which are intrinsic to minimal solutions of the elliptic system (3.1.2) with $c^\star = 0$. See the results of Section 4 in Alikakos, Fusco and Smyrnelis [12], mainly based on Alikakos and Fusco [10]. The main obstacle is that even if one was able to extend their analysis to the case $c^\star > 0$, a crucial hypothesis of in their results is that solutions are minimal with respect to compactly supported perturbations, but the solution of Theorem 3.1 is only *locally* minimizing (due to the fact that \mathfrak{q}^+ is a local minimizer of the 1D energy). Therefore, we leave this question open. Nevertheless, besides the L^2 -convergence rates (3.2.12) and (3.2.14) one can prove uniform convergence both in the x_1 and the x_2 direction:

Theorem 3.3. *Assume that (H3.6) holds. Let (c^\star, \mathfrak{U}) be the solution given by Theorem 3.1. Then, we have that*

$$\lim_{x_1 \rightarrow +\infty} \|\mathfrak{U}(x_1, \cdot) - \mathfrak{q}^+(\cdot + \tau^+)\|_{L^\infty(\mathbb{R}, \mathbb{R}^k)} = 0 \quad (3.2.15)$$

and for all $L \in \mathbb{R}$ we have

$$\lim_{x_2 \rightarrow \pm\infty} \|\mathfrak{U}(\cdot, x_2) - \sigma^\pm\|_{L^\infty([L, +\infty), \mathbb{R}^k)} = 0. \quad (3.2.16)$$

If, moreover, (H3.7) holds, then we have

$$\lim_{x_1 \rightarrow -\infty} \|\mathfrak{U}(x_1, \cdot) - \mathfrak{q}^-(\cdot + \tau^-)\|_{L^\infty(\mathbb{R}, \mathbb{R}^k)} = 0 \quad (3.2.17)$$

and (3.2.16) can be improved into

$$\lim_{x_2 \rightarrow \pm\infty} \|\mathfrak{U}(\cdot, x_2) - \sigma^\pm\|_{L^\infty(\mathbb{R}, \mathbb{R}^k)} = 0. \quad (3.2.18)$$

3.2.4 Min-max characterization of the speed

We provide here a min-max characterization of the speed c^\star and other related properties which are summarized in Theorem 3.4. The idea of providing a variational characterization for the speed of traveling waves in reaction-diffusion systems can be traced back to Heinze [93], Heinze, Papanicolau and Stevens [94] and it was used later in several other papers [13, 55, 110, 111, 128].

Theorem 3.4. *Assume that (H3.6) and (H3.7) hold. Let (c^\star, \mathfrak{l}) be the solution given by Theorem 3.1. Then for any $\tilde{\mathfrak{l}} \in S_i$ such that*

$$E_{2,c^\star}(\tilde{\mathfrak{l}}) = 0$$

we have that $(c^\star, \tilde{\mathfrak{l}})$ solves (3.2.8) and

$$c^\star = \frac{\mathfrak{m}^+ - \mathfrak{m}^-}{\int_{\mathbb{R}^2} |\partial_{x_1} \tilde{\mathfrak{l}}(x_1, x_2)|^2 dx_2 dx_1}. \quad (3.2.19)$$

In particular, the quantity $\int_{\mathbb{R}^2} |\partial_{x_1} \tilde{\mathfrak{l}}(x_1, x_2)|^2 dx_2 dx_1$ is well-defined and constant among the set of minimizers of $E_{2,c}$ in S . Moreover, it holds

$$c^\star = \sup\{c > 0 : \inf_{U \in S} E_{2,c}(U) = -\infty\} = \inf\{c > 0 : \inf_{U \in S} E_{2,c}(U) = 0\} \quad (3.2.20)$$

and we have the bound

$$c^\star \leq \frac{\sqrt{2(\mathfrak{m}^+ - \mathfrak{m}^-)}}{\mathfrak{d}_0} < \min \left\{ \frac{\sqrt{2E_{\max}}}{\mathfrak{d}_0}, \mu^- \right\}$$

where \mathfrak{d}_0 , μ^- and E_{\max} will be defined later in (3.2.21), (3.2.22) and (3.2.23) respectively and the second inequality follows from the bound on $\mathfrak{m}^+ - \mathfrak{m}^-$ given by (H3.6) and (H3.7).

Remark 3.2.2. Notice that the conditions at infinity imply that any $U \in S$ is such that

$$\int_{\mathbb{R}^2} \frac{|\partial_{x_1} U(x_1, x_2)|^2}{2} dx_2 dx_1 > 0$$

As it can be seen, Theorem 3.4 shows that the speed c^\star is characterized by the explicit formula (3.2.19), which nevertheless requires knowledge on a profile $\tilde{\mathfrak{l}}$. However, one also has the variational characterization (3.2.20), which does not involve any information on the profiles. Indeed, one only needs to be able to compute the infimum of the energies with $c > 0$ as a parameter. Moreover, notice that combining (3.2.20) with the uniqueness part of Theorem 3.1, we obtain that if $\bar{c} > c^\star$ and $(\bar{c}, \bar{\mathfrak{l}})$, with $\bar{\mathfrak{l}} \in S$, solves (3.2.8), then $E_{2,\bar{c}}(\bar{\mathfrak{l}}) = +\infty$, which is actually a contradiction. On the contrary, if we take $\bar{c} < c^\star$, then (3.2.20) implies that $\inf_{U \in S} E_{2,\bar{c}}(U) = -\infty$, meaning that Theorem 3.1 does not apply and nothing else can be said.

3.2.5 Definition of the upper bounds

We will now define some important numerical constants which are necessary in order to formulate assumptions (H3.6) and (H3.7). Assume first that (H3.5) holds. Let ρ_0^\pm as in (3.2.5) and (3.2.6). Recall that we chose ρ_0^+ and ρ_0^- such that

$$\mathcal{F}_{\rho_0^+}^+ \cap \mathcal{F}_{\rho_0^-}^- = \emptyset$$

and, since those two sets (see the definition in (3.2.7)) are L^2 -closed due to the local compactness of the sets \mathcal{F}^- and \mathcal{F}^+ , we have that

$$\mathfrak{d}_0 := \text{dist}_{L^2(\mathbb{R}, \mathbb{R}^k)}(\mathcal{F}_{\rho_0^+/2}^+, \mathcal{F}_{\rho_0^-/2}^-) \quad (3.2.21)$$

is positive. Therefore, as we advanced before, one can see that the constant \mathfrak{d}_0 depends only on the distance between the two families of minimizing heteroclinics. Next, under (H3.5), recall the constants β^\pm from (3.2.6). Set

$$\overline{\beta^\pm} := \frac{1}{2}(\beta^\pm)^2((\beta^\pm)^2 + (\beta^\pm + 1)^2) > 0$$

and, subsequently

$$\mu^- := \frac{1}{\beta^- + \overline{\beta^-}} > 0 \quad (3.2.22)$$

which is the constant appearing in (H3.7). Of course, the nature of the definition given in (3.2.22) obeys to technical considerations. But μ^- should be thought as a constant depending only on the local behavior of the energy around \mathcal{F}^- and, in particular, independent on the behavior of the energy near \mathcal{F}^+ . Now let for $r \in (0, \rho_0^\pm]$

$$\mathfrak{e}_r^\pm := \inf\{E(q) : q \in X(\sigma^-, \sigma^+), \text{dist}_{L^2(\mathbb{R}, \mathbb{R}^k)}(q, \mathcal{F}^\pm) \in [r, \rho_0^\pm]\}.$$

Known results which follow from compactness of minimizing sequences (see for instance Schatzman [153]) imply that $\mathfrak{e}_r^\pm > 0$. Moreover, we also have that for $r \in (0, \rho_0^\pm]$ there exists $\nu^\pm(r) > 0$ such that

$$\forall q \in \mathcal{F}_{\rho_0^\pm/2}^\pm, \quad E(q) - \mathfrak{m}^\pm \leq \nu^\pm(r) \Rightarrow \text{dist}_{H^1(\mathbb{R}, \mathbb{R}^k)}(q, \mathcal{F}^\pm) \leq r.$$

This leads to define the constants

$$\delta_0^- := \min \left\{ \sqrt{e^{-1} \frac{\rho_0^-}{4} \sqrt{2(\mathfrak{e}_{\rho_0^-/4}^- - \mathfrak{m}^-)}}, \frac{\rho_0^-}{4} \right\} > 0,$$

$$\mathfrak{r}^- := \frac{\rho_0^-}{\beta^- + 1} > 0$$

and

$$E_{\max} := \frac{1}{(\beta^-)^2(\beta^- + 1)} \min \left\{ \frac{(\delta_0^-)^2}{4}, \mathfrak{e}_{\delta_0^-}^- - \mathfrak{m}^-, \nu^-(\mathfrak{r}^-), \nu^-(\delta_0^-) \right\} > 0 \quad (3.2.23)$$

which is the constant appearing in (H3.6). Again, the definition on E_{\max} is essentially due to technical reasons, but it must be thought as a constant which only depends on local information around \mathcal{F}^- .

3.2.6 Methods and ideas of the proofs

The main result of this chapter is Theorem 3.1, which establishes the existence of a solution (c^*, \mathfrak{U}) , with the profile \mathfrak{U} satisfying the heteroclinic asymptotic conditions (3.2.9), (3.2.10). We also prove an exponential rate of convergence for the profile at $+\infty$ (with respect to the L^2 -norm). We finally show that the speed c^* has some uniqueness properties. Important properties on the profile and the speed, as well as improvements on the results under additional assumptions, are also established in Theorems 3.2, 3.3 and 3.4.

As already stated, the proof of our results follows by bringing together two different lines of research, see items 1. and 2. in the introduction. More precisely, in the spirit of [125, 153], we adapt the result of Alikakos and Katzourakis [13] (actually, we rather follow more closely the

simplified version given in Alikakos, Fusco and Smyrnelis [12]) to potentials defined in an abstract, possibly infinite-dimensional, Hilbert space and possessing two local minima at different levels. This abstract setting is established in Section 3.4 and the main abstract results are Theorems 3.5, 3.6 and 3.7. The proof of these results is found in Section 3.5. Assumption (H3.5) guarantees that our main results (Theorems 3.1-3.4) are a particular case of the abstract results. Naturally, the advantage of proving the results in an abstract framework is that one can apply them to several problems different than the original one. In our case, the results in this chapter apply to the 1D system

$$\partial_t w - \partial_x^2 w = -\nabla_u W(w) \text{ in } [0, +\infty) \times \mathbb{R},$$

where W is a smooth potential bounded below possessing two local and non-degenerate minima at different levels. As said before, this is formally the system considered in [13], but the results of this chapter allow to somewhat relax the non-degeneracy assumption used in [13]. More details, as well as other extensions, are given in Chapter 4.

Generalizing the result from [13] for curves taking values in a more general, possibly infinite-dimensional, Hilbert space raises several additional difficulties. A detailed outline of our proof is given in Section 3.5.1, but let us here try to motivate the main difficulties of the problem we are facing.

As pointed out before, the approach in [13] is variational. A family of weighted energy functionals (essentially those introduced in Fife and McLeod [79, 78]) depending on a speed parameter $c > 0$ is considered. In order to make the functionals well defined in the space of curves that connect the minima, the global minimum of the potential must be negative and the local one must be zero, which is always true up to an additive constant. As a consequence, one deals with an energy density which changes sign, which is in contrast with the equal depth (balanced) case, for which the energy is always non-negative. Recall that finite energy 1D connecting heteroclinics between two wells at the same level must be stationary. Another difficulty of the heteroclinic traveling wave existence problem comes from the fact that not only the profile but also the speed of the wave is an unknown as well.

The method used in [13] is an adaptation of that introduced in Alikakos and Fusco [11] for the equal depth case. This method consists on considering families of solutions with prescribed behavior outside an interval of length $2T$ (namely, they are forced to stay close to the respective minimum) and minimizing the weighted energy functionals seeing the speed as parameter. Since compactness is restored due to the constraints, the problem has a solution for each $c > 0$ and $T \geq 1$. These ideas can be adapted to our setting without major difficulty. The next step consists on determining the solution speed c^* and then showing that for c^* and a suitable T the corresponding constrained minimizer does not meet the constraints, meaning that it is an actual solution. Since the energy functionals change sign, one needs to show that the constrained minimizers do not oscillate between positive and negative regions of the energy (which would produce compensations) inside arbitrarily long intervals as T goes to infinity. In order to show that, the authors in [13] assume that the local minima are isolated and that in the negative region of the functional one has strict radial monotonicity with respect to the global minimum. Subsequently, they use this property in combination with the ODE system and minimality arguments in order to exclude oscillations.

For several reasons, the previous idea does not seem available in our setting without substantial

modifications. Despite the fact that, as we show in Chapter 3, one can adapt the assumption of [13] for potentials in infinite-dimensional spaces with possibly degenerate minima, in this case we have trouble showing that our original problem can be put as a particular case of the abstract one. In other words, it does not seem reasonable to expect that such adaptation of the radial monotonicity assumption of [13] would be met in our original problem. Indeed, one would need to prove some kind of radial monotonicity for the energy E (see (3.2.1)), in some suitable subset. We think that this might be too restrictive and we cannot prove it even for simple explicit examples. The difficulty comes from the fact that, while in the finite-dimensional case one can directly modify the potential, the level sets of E depend on a rather indirect way on the potential V and they are infinite-dimensional manifolds. Therefore, the most important difficulty of our problem is to replace the radial monotonicity assumption of [13] by another one which can be met in our situation and which allows to obtain a similar type of conclusion (namely, exclude oscillatory behavior for the constrained minimizers in arbitrarily large intervals). We have been able to provide one assumption, (H3.6), which plays this role. It consists on imposing an upper bound on the difference between the energy levels. This upper bound is (the abstract version of) the constant \mathcal{E}_{\max} , defined in (3.2.23), Section 3.2.5. It enables us to exclude oscillations on the minimizers because the (renormalized) energy is positive outside the region in which the solution is constrained. Once oscillations are excluded, we conclude the proof as in [13].

The main drawback of our proof is that the computation of the upper bound in (H3.6) is not straightforward and it obeys technical considerations, as the definition of \mathcal{E}_{\max} (3.2.23) in Section 3.2.5 shows. In particular, we cannot exclude the possibility that \mathcal{E}_{\max} is *small*. However, in Section 3.2.7 we give a general method in order to obtain potentials for which the corresponding energy functional satisfies the bound assumption (H3.6). Essentially, one considers a potential for which two different globally minimizing heteroclinics exist (which implies $k \geq 2$) and then modifies it in a suitable manner. It would be also interesting to know whether our assumption (H3.6) is only technical or rather there is some kind of obstruction for existence when the difference between the energy of the heteroclinics is too large. We think that the answer to this question is possibly related to the loss of compactness for the 1D energy functional (see Chapter 3) and hence we conjecture that (H3.6) is not only technical (although it is likely non-optimal) and that in its absence some counterexamples might be found. It is also reasonable to conjecture that the removal of (H3.6) would imply the existence of traveling waves with more complicated behavior at infinity, for example approaching chains of connecting orbits (heteroclinic or homoclinic) stable in some suitable sense.

At the final stage of the proof, one needs to ensure that the solution obtained presents the suitable heteroclinic behavior at infinity. Moreover, we want to obtain more refined convergence results, see Theorems 3.1 and 3.2. This asymptotic analysis is delicate, as finite energy functions do not necessarily converge at all $-\infty$ and they converge exponentially at $+\infty$ but only with respect to the L^2 -norm. Moreover, one needs to distinguish between L^2 convergence (which is weaker and does not imply convergence of the energy) and H^1 convergence. We deal with all these difficulties by using minimality of the solution (inspired for instance by [157]), which allows us to obtain the convergence at $-\infty$, exponentially with respect to the L^2 -norm. Assumption (H3.7) is needed in order to obtain the convergence at $-\infty$. Moreover, working in the main setting, we obtain Theorem 3.3, which shows that convergence is not only L^2 but also L^∞ , and not only according to x_1 but

also x_2 , the limit in this case being the wells σ^\pm .

We will use extensively the fact that the energy has *good* properties at least on a neighborhood of the 1D minimizing heteroclinics. Essentially, those are the results and assumptions made by Schatzman [153], which in our case are given in (H3.5) and the discussion that follows. To our knowledge, these properties have not been shown to hold for the 2D heteroclinic solutions of [3, 153] and one could expect that some of them do not hold. This represents, in our opinion, a (momentary) obstruction to establishing the existence of 3D heteroclinic traveling waves connecting two 2D heteroclinics. Moreover, notice that the fact that V is multi-well implies that 1D heteroclinic traveling waves do not exist in general, unless one imposes the existence of a local minimum at a level higher than 0 and some other properties are verified.

3.2.7 Examples of potentials verifying the assumptions

In order to conclude this section, we exhibit a rather general and elementary method in order to produce examples of potentials for which the assumptions we make in this chapter are satisfied. As we advanced before, the idea is to modify a given multi-well potential $V_0 : \mathbb{R}^k \rightarrow \mathbb{R}$ satisfying (H3.1), (H3.2) and (H3.3) such that the associated energy possesses two minimizing heteroclinics (up to translations) for two given wells σ^- , σ^+ in a finite set Σ . We also assume that the strict triangle's inequality (H3.4) is met for V_0 with respect to (σ^-, σ^+) . Furthermore, we assume that the generic Schatzman's spectral assumption ([153]) is satisfied for those heteroclinics, meaning that the constants defined in Section 3.2.5 (with the obvious modifications) also make sense here. That is, one can think of any potential V_0 satisfying the assumptions of Schatzman's paper [153]. For the reader's convenience, we shall give here some explicit examples of such potentials which we found on the literature.

The first of the examples we give was found by Antonopoulos and Smyrnelis, see *Remark 3.6* in [19]. Consider the case $k = 2$. Let V_{GL} be the Ginzburg-Landau potential

$$V_{GL} : u = (u_1, u_2) \in \mathbb{R}^2 \rightarrow \frac{(1 - |u|^2)^2}{4} \in \mathbb{R}$$

and consider the corresponding energy

$$E_{GL}(q) := \int_{\mathbb{R}} \left[\frac{|q'(t)|^2}{2} + \frac{(1 - |q(t)|^2)^2}{4} \right] dt, \quad q \in H_{\text{loc}}^1(\mathbb{R}, \mathbb{R}^2).$$

The idea is to perturb V_{GL} in order to obtain a double-well potential with zero set $\{(-1, 0), (1, 0)\}$ and symmetric with respect to the axis $\{u_2 = 0\}$. Such a potential will possess two heteroclinics provided that any curve with trace in $\{u_2 = 0\}$ can be beaten by a competitor with a trace that is not contained in this set. Notice that for all $(u_1, 0) \in \mathbb{R} \times \{0\}$ we have that $V_{GL}(u_1, 0) = (1 - u_1^2)^2/4$, which is the standard scalar double-well potential. As it is well known, the (unique) heteroclinic for such potential is given by the odd function $q_{AC} : t \in \mathbb{R} \rightarrow \tanh(t/\sqrt{2}) \in \mathbb{R}$. Therefore, each curve $q = (q_1, q_2)$ in $H_{\text{loc}}^1(\mathbb{R}, \mathbb{R}^2)$ with $q_2 = 0$ and $\lim_{t \rightarrow \pm\infty} q_1(t) = \pm 1$ verifies

$$E_{GL}(q) \geq E_{GL}(q_{AC}) = \frac{2\sqrt{2}}{3}. \quad (3.2.24)$$

For $T > 0$, define

$$q_T(t) := \begin{cases} (-1, 0) & \text{if } t \leq -T-1, \\ ((t+T) + (t+T+1)q_{AC}(-T), 0) & \text{if } -T-1 \leq t \leq -T, \\ -q_{AC}(T)(\cos(\pi(t+T)/(2T)), \sin(\pi(t+T)/(2T))) & \text{if } -T \leq t \leq T, \\ ((t-T) - (t-T-1)q_{AC}(T), 0) & \text{if } T \leq t \leq T+1, \\ (1, 0) & \text{if } T+1 \leq t. \end{cases}$$

A modification of the computations made in [19] shows that

$$\lim_{T \rightarrow +\infty} E_{GL}(q_T) = 0$$

meaning that by (3.2.24) there exists $\bar{T} > 0$ such that $E_{GL}(q_{\bar{T}}) < E_{GL}(q_{AC})$. Then, given $\varepsilon := (1 - |q_{AC}(\bar{T})|^2)/4$, consider $\phi \in \mathcal{C}^\infty(\mathbb{R}, [0, +\infty))$ such that

$$\phi(t) = \begin{cases} 0 & \text{if } t \leq |q_{AC}(\bar{T})|^2 + \varepsilon \\ 1 & \text{if } t \geq 1 - \varepsilon \end{cases}$$

and define $\tilde{V}_0 : u = (u_1, u_2) \in \mathbb{R}^2 \rightarrow V_{GL}(u) + u_2^2 \phi(|u|^2)$. Let \tilde{E}_0 be the corresponding energy. Notice that \tilde{V}_0 is a double-well potential verifying (H3.1), (H3.2) and (H3.3). By definition, we have that if $q = (q_1, q_2) \in H^1([-R, R], \mathbb{R}^2)$ is such that $q_2 = 0$, then $\tilde{E}_0(q) = E_{GL}(q)$. Moreover, we also have $\tilde{E}_0(q_{\bar{T}}) = E_{GL}(q_{\bar{T}}) < E_{GL}(q_{AC})$. As a consequence, the minimizer $\mathbf{q} = (q_1, q_2)$ of \tilde{E}_0 in the class of curves in $H_{loc}^1(\mathbb{R}, \mathbb{R}^k)$ which tend to $(\pm 1, 0)$ at $\pm\infty$ satisfies $q_2 \neq 0$, which means that $\hat{\mathbf{q}} := (q_1, -q_2)$ is also a minimizer due to the symmetry of \tilde{V}_0 and $\hat{\mathbf{q}}$ is not a translation of \mathbf{q} . Therefore, \tilde{V}_0 possesses two geometrically distinct globally minimizing heteroclinics. In order to find our example of potential, we need that such heteroclinics are non-degenerate in the sense asked by Schatzman in [153], see our 3. in (H3.5). However, as shown in her *Theorem 4.3* such assumption is generic, i. e., we can find V_0 arbitrarily close to \tilde{V}_0 which is still a double-well potential with wells $(-1, 0)$, $(1, 0)$ and with \mathbf{q} and $\hat{\mathbf{q}}$ non-degenerate globally minimizing heteroclinics which satisfy the spectral assumptions.

Another example, this time in dimension $k = 3$, is provided by Zuñiga and Sternberg [174]. They consider the potential

$$\tilde{V}_0 : u = (u_1, u_2, u_3) \rightarrow u_1^2(1 - u_1^2)^2 + \left(u_2^2 - \frac{1}{2}(1 - u_1^2)^2\right)^2 + \left(u_3^2 - \frac{1}{2}(1 - u_1^2)^2\right)^2 \in \mathbb{R},$$

which vanishes exactly on the points

$$(-1, 0, 0), (1, 0, 0), \left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right), \left(0, -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right).$$

By explicit computations, they show that the potential \tilde{V}_0 satisfies (H3.1), (H3.2), (H3.3) and (H3.4) with $\sigma^\pm := (\pm 1, 0, 0)$ and, moreover, that the infimum of the corresponding energy \tilde{E}_0 in $X(\sigma^-, \sigma^+)$ is not attained by a curve with trace contained in $\{u_2 = u_3 = 0\}$. Using the reflections

$(0, u_2, 0) \rightarrow (0, -u_2, 0)$ and $(0, 0, u_3) \rightarrow (0, 0, -u_3)$, one deduces the multiplicity up to translations of the globally minimizing heteroclinics for \tilde{E}_0 in $X(\sigma^-, \sigma^+)$. As above, one can obtain V_0 arbitrarily close to \tilde{V}_0 such that the globally minimizing heteroclinics satisfy the spectral assumption.

Let us now return to the initial problem, and let V_0 be any potential satisfying the previous assumptions. In order to obtain a potential which satisfies the requirements of our setting, the idea is to make arbitrarily small smooth perturbations of V_0 around the trace of one of the heteroclinics, in such a way that its energy increases but a locally minimizing heteroclinic still exists (at least for small perturbations), which must necessarily have larger energy. One then chooses a perturbation which is not too large so that the upper bound on the difference of the energies is met. The idea is pictured in Figure 3.3. We now show how to rigorously implement this idea. Let q^- and q^+ in $X(\sigma^-, \sigma^+)$ be different up to translations and such that

$$E_0(q^-) = E_0(q^+) = m_0 := \inf_{q \in X(\sigma^-, \sigma^+)} E_0(q),$$

where, for $q \in H_{\text{loc}}^1(\mathbb{R}, \mathbb{R}^k)$

$$E_0(q) := \int_{\mathbb{R}} \left[\frac{|q'(t)|^2}{2} + V_0(q(t)) \right] dt.$$

Recall that there exist ρ_0^\pm such that

$$\forall q \in X(\sigma^-, \sigma^+), \quad \text{dist}_{H^1(\mathbb{R}, \mathbb{R}^k)}(q, \mathcal{F}^\pm) \leq \rho_0^\pm \Rightarrow \text{dist}_{H^1(\mathbb{R}, \mathbb{R}^k)}(q, \mathcal{F}^\pm)^2 \leq \beta^\pm (E_0(q) - m_0)$$

where

$$\mathcal{F}^\pm := \{q^\pm(\cdot + \tau) : \tau \in \mathbb{R}\}.$$

Let $t_0 \in \mathbb{R}$ be such that $\text{dist}(q^+(t_0), \Sigma) = \max_{t \in \mathbb{R}} \text{dist}(q^+(t), \Sigma)$ for some $q^+ \in \mathcal{F}^+$ and set $u_0 := q^+(t_0)$. Let

$$r := \min\{\rho_0^+/2, \text{dist}(q^+(t_0), \Sigma)/2\} > 0.$$

Define $\chi \in C_c^\infty(\mathbb{R}^k)$ be such that $0 \leq \chi \leq 1$, $\chi = 1$ on $B(u_0, r)$ and $\text{supp}(\chi) \subset B(u_0, 2r)$. For each $\delta > 0$, consider the potential $V_\delta := V + \delta\chi \geq 0$. Define

$$E_\delta(q) := \int_{\mathbb{R}} \left[\frac{|q'(t)|^2}{2} + V_\delta(q(t)) \right] dt$$

Notice that, by the choice of χ , V_δ vanishes exactly in Σ . Let now be $q \in X(\sigma^-, \sigma^+)$ such that $\text{dist}_{H^1(\mathbb{R}, \mathbb{R}^k)}(q, \mathcal{F}^+) \leq \rho_0^+/2$. We have that

$$m_0 + \frac{1}{\beta^+} \text{dist}_{H^1(\mathbb{R}, \mathbb{R}^k)}(q, \mathcal{F}^+)^2 \leq E_0(q) < E_0(q) + \delta \int_{\mathbb{R}} \chi(q) = E_\delta(q) \quad (3.2.25)$$

and notice that for $q \in \mathcal{F}^+$ we have that $E_\delta(q) = m_0 + \delta i$ with

$$i := \int_{\mathbb{R}} \chi(q^+(t)) dt > 0.$$

A contradiction argument shows that

$$m_\delta^+ := \inf\{E_\delta(q) : \text{dist}_{H^1(\mathbb{R}, \mathbb{R}^k)}(q, \mathcal{F}^+) \leq \rho_0^+/2\} > m_0$$

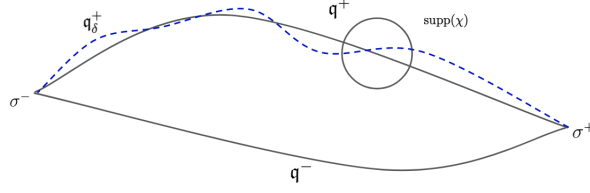


Figure 3.3: Representation of the cut-off function χ used in order to produce the family of perturbed functionals V_δ . We also draw the corresponding local minimizer q_δ^+ (discontinuous curve).

and we have $m_\delta^+ \leq E_\delta(q^+) = m_0 + \delta i$. Since the cut-off function is supported away from Σ , we can show by the usual concentration-compactness arguments that there exists $q_\delta^+ \in X(\sigma^-, \sigma^+)$ such that $\text{dist}_{H^1(\mathbb{R}, \mathbb{R}^k)}(q_\delta^+, \mathcal{F}^+) \leq \rho_0^+/2$ and $E_\delta(q_\delta^+) = m_\delta^+$. If we show that $\text{dist}_{H^1(\mathbb{R}, \mathbb{R}^k)}(q_\delta^+, \mathcal{F}^+) < \rho_0^+/2$, then the constraints of the minimization problem are not saturated and q_δ^+ is an actual critical point. Notice that if $q \in X(\sigma^-, \sigma^+)$ is such that $\text{dist}_{H^1(\mathbb{R}, \mathbb{R}^k)}(q, \mathcal{F}^+) = \rho_0^+/2$, then by (3.2.25) we obtain $E_0(q) \geq m_0 + (\rho_0^+)^2/(4\beta^+) > m_0$. Then, if we take $\delta < \delta_1$ with

$$\delta_1 := \frac{(\rho_0^+)^2}{4\beta^+ i} > 0,$$

it holds $E_\delta(q) > E_0(q) \geq m_0 + \delta i \geq m_\delta^+$, so that q cannot be a minimum. Therefore, for such δ items 1. and 2. in (H3.5) are satisfied for E_δ with minimizing heteroclinics q^- and q_δ^+ , with the obvious modifications on the notations. Regarding item 3., which is the spectral assumption of Schatzman [153], it is a generic assumption, meaning that, arguing as it is done in her *Theorem 4.3*, we find that V_δ can be modified with an arbitrary small perturbation away from the traces of q_δ^+ and q^- so that 3. holds. As a consequence, we can assume that (H3.5) holds for all $\delta \in (0, \delta_1)$. Regarding (H3.6), compute the constant E_{\max} as in (3.2.23), which by the choice of r and χ does not depend on δ , and set

$$\delta_2 := \frac{E_{\max}}{i} > 0,$$

so that for all $\delta \in (0, \delta_2)$ we have $m_\delta^+ - m_0 < E_{\max}$. Define now $\mathcal{F}_{\rho_0^+/2}^-$ as in (3.2.7). The choice of r and χ implies that $\mathcal{F}_{\rho_0^+/2}^-$ does not depend on δ , meaning that we can find δ_3 such that for all $\delta \in (0, \delta_3)$ it holds

$$\{q \in X(\sigma^-, \sigma^+) : E_\delta(q) < m_\delta^+\} \subset \mathcal{F}_{\rho_0^+/2}^-,$$

meaning that (H3.6) holds for E_δ provided that $\delta \in (0, \delta_{\max})$ with $\delta_{\max} := \min\{\delta_1, \delta_2, \delta_3\} > 0$. As a consequence, we have found a family $\{V_\delta\}_{\delta \in (0, \delta_{\max})}$ of potentials which are in the framework of Theorem 3.1 and we obtain a heteroclinic traveling wave with limits q^- and q^+ . Moreover, recall that if we put when $\delta = 0$ we recover the *classical* potentials considered in [3, 83, 125, 153, 157], meaning that in this setting one can prove convergence results of the traveling waves toward stationary waves as $\delta \rightarrow 0^+$. Moreover, we see that is possible to decrease the value of δ even more so that the convergence assumption (H3.7) holds and Theorems 3.2 and 3.4 also apply.

3.3 Discussion on the previous literature and open problems

3.3.1 Related reaction-diffusion models and the question of stability

As we said in the introduction, the problem of existence of traveling waves for reaction-diffusion systems as well as their qualitative properties has been widely studied since the early works of Fisher [80], Kolmogorov, Petrovsky and Piskunov [105] regarding the equation today known as the *Fisher-KPP equation*. From the modeling perspective, while the aim of these authors was to describe of the dynamics of a given population, reaction-diffusion systems have also been proposed as models in other domains of the natural and social sciences. For example, applications in chemistry were given by Zeldovich [173] and Kanel [103] (see also Berestycki, Nicolaenko and Scheurer [38]) and the same Allen-Cahn model that we consider in this thesis was proposed by Allen and Cahn [14], following Cahn and Hilliard [62], for describing phase transition problems in material physics. It is also worth mentioning that for the most classical studies for traveling waves in reaction-diffusion problems the profile tends at infinity to two (possibly equal) constant stable states. However, other type of stable states (in particular, non constant) can be considered as conditions at infinity (as we do here). Moreover, the notion of traveling wave can be generalized in order to contain and describe similar structures. We refer to the papers by Berestycki and Hamel [32, 31] and the references therein.

Going back to the model Fisher-KPP equation, it can be written as follows:

$$\partial_t w - \partial_x^2 w = f(w), \text{ in } [0, +\infty) \times \mathbb{R}$$

where $f : \mathbb{R} \rightarrow \mathbb{R}$ is such that $f(0) = 0$, $f(1) = 0$, $f > 0$ in $(0, 1)$, $f < 0$ in $(-\infty, 0)$ and $f(u) < f'(0)u$ for $u > 0$. Traveling waves for this equation are solutions of the type

$$w(t, x) = U(x - ct)$$

with $c > 0$ and $U : \mathbb{R} \rightarrow \mathbb{R}$ satisfies

$$\lim_{x \rightarrow -\infty} U(x) = 0 \text{ and } \lim_{x \rightarrow +\infty} U(x) = 1.$$

From the point of view of modelling, traveling waves intend, for instance, to describe the invasion from a stable state to another one. An important feature of the Fisher-KPP equation is the existence of an important speed parameter, $c_{KPP} > 0$, usually called the *invasion speed* which can be explicitly computed as follows:

$$c_{KPP} := 2\sqrt{f'(0)}.$$

The previous problem, and related ones, is studied by means of the maximum principle and comparison results. Using these methods, one proves existence and uniqueness of a traveling wave with fixed speed $c > 0$ if and only if $c \geq c_{KPP}$. This seems to be an important contrast with respect to the model that we consider here. Indeed, recall that our Theorem 3.1 states that the threshold speed c^* is, in particular, unique among the class of profiles which are minimizers in our variational setting. In fact, the same phenomenon is observed in earlier papers which also establish existence of traveling waves for reaction-diffusion systems by a variational procedure:

Muratov [128], Lucia, Muratov and Novaga [110], Alikakos and Katzourakis [13], Chen, Chien and Huang [68]. Nevertheless, we point out that our results (the same as the ones we cite) do not exclude the possibility of other type of traveling wave solutions with speed different than c^* . In particular, there could exist traveling waves with heteroclinic profiles which are obtained from a different variational setting than ours, or even from non-variational methods.

The analysis in [80, 105] was substantially extended in subsequent works. Fife and McLeod [79, 78] established stability properties for traveling waves in the Fisher-KPP equation. Generalizations bringing into consideration higher-dimensional equations (in space) were also made. For instance, Aronson and Weinberger [20] (see also Hamel and Nadirashvili [91] and the references therein) considered the case of \mathbb{R}^N as space domain. We also mention the work of Berestycki, Larrouturou and Lions [35] Berestycki and Nirenberg [39] for the case of a cylinder $\mathbb{R} \times \omega$, with $\omega \subset \mathbb{R}^{N-1}$ a bounded domain. For the case of periodic domains, see Berestycki and Hamel [30], Berestycki, Hamel and Nadirashvili [33]. The case of more general domains is addressed in Berestycki, Hamel and Nadirashvili [34]. For the non-local problem see Berestycki et. al. [37].

The family of non-linear functions f which are admissible for the Fisher-KPP model does not contain non-linearities of Allen-Cahn type. Indeed, such non-linearities are written as $f = -V'_{AC}$ where V_{AC} is a non-negative double-well potential, the prototypical case being

$$V_{AC} : u \in \mathbb{R} \rightarrow \frac{(1 - u^2)^2}{4} \in \mathbb{R},$$

which does not satisfy the assumptions for required the equations of Fisher-KPP type, written above. For the scalar Allen-Cahn equation, we mention the result due to Matano, Nara and Taniguchi regarding the stability for traveling waves with \mathbb{R}^N as space domain. In this case, the waves propagate according to one direction and connect the stable states ± 1 at infinity. The case of traveling waves that connect one stable state with one unstable, non-constant periodic 1D solution was studied by Hamel and Roquejoffre [92]. The non-local case was addressed in Bates et. al. [29].

Many results are available for Allen-Cahn *systems*, but mostly in one space dimension. In this case, the (negative) gradient flow structure implies that for initial data of finite energy which connects two different wells, the corresponding solution at long time the solution should look as a chain of glued 1D connecting orbits. More generally, if the initial condition connects at infinity two local minima at *possibly different* levels (and a suitable weighted energy is finite), then traveling waves should also appear in the asymptotic pattern. Proofs of these facts, even in a more general framework, can be found in Risler [145, 146, 144]. Moreover, one can aim at obtaining quantitative results which describe more precisely the previous qualitative behavior and also introduce the problem of considering the system as a singular perturbation. That is, one considers a coefficient ε^{-2} multiplying the non-linear term and passes to the limit $\varepsilon \rightarrow 0^+$. In this direction, it has been shown that the *fronts* (that is, the regions in which the solution is far from the set of wells) of the solution of the gradient flow problem move at slow motion. The first rigorous proofs of this fact was given by Carr and Pego [65, 64], Fusco and Hale [86], for the Allen-Cahn equation. This analysis was later extended to multi-well systems by Bethuel, Orlandi and Smets [46] for multi-well systems. Bethuel and Smets [47, 48] obtained results regarding the motion law and the long-time interaction between stationary solutions in the multi-well *scalar* case, allowing also for degenerate wells, but for the moment their work has not been extended to systems.

Regarding Allen-Cahn systems in higher dimensions, besides the classical articles regarding the stationary wave and this chapter, we are only aware of the recent work of Chen, Chien and Huang [68]. In the latter, the authors consider the strip $\mathbb{R} \times (-l, l)$ as space domain and, for a class of symmetric triple-well potentials on the plane (similar to that from Bronsard, Gui and Schatzman [59]), show the existence of traveling wave solutions connecting at infinity a well and an approximation in $(-l, l)$ of a globally minimizing heteroclinic, which they assume to be unique. Their proof follows by a suitable application of the variational device of Muratov [128], which differs from that by Alikakos and Katzourakis [13] mainly on the fact that a different constrained minimization problem is considered.

A discussion concerning the mathematical methods used for addressing these problems is in order. As it is well known, while the maximum principle and the comparison theorems play a key role in the study of most *scalar* reaction-diffusion equations (such as the Fisher-KPP equation), those tools are no longer available for systems except in some particular classes, for instance when dealing with the so-called *monotone systems*, see Volpert, Volpert and Volpert [169]. As a consequence, for more general classes of systems one needs other (more general) tools. Several approaches were developed, for example the use of Leray-Schauder degree [169] or Conley theory as discussed in Smoller [156]. We refer to the reader to the sources given in [156, 169]. While the gradient structure of some reaction-diffusion equations enables the application of variational methods (see the already cited references [11, 55, 68, 93, 111, 110, 128, 145, 146, 144]), these methods have not been extensively used in this context. This is some kind of contrast with respect to the case of dispersive equations where, since the seminal work of Cazenave and Lions [66], a large amount of results regarding the existence and orbital stability of traveling waves and solitons has been produced. For instance, this has been done for Gross-Pitaevskii equations and systems, which are in some sense the dispersive counterpart of the parabolic problems of Allen-Cahn type. See Bethuel, Gravejat and Saut [43], Bethuel et. al. [44] for the orbital stability of traveling waves for the 1D Gross-Pitaevskii equation, Maris [114] for the Gross-Pitaevskii equation in \mathbb{R}^N , $N \geq 3$. The orbital stability of stationary waves (of heteroclinic type) for two coupled Gross-Pitaevskii equations was proven by Alama et. al. [2].

In order to conclude this section, we mention that the question of the local stability for the parabolic system of this chapter is wide open. Even for the minimizing stationary wave obtained by Alama, Bronsard and Gui [3], stability properties have not been studied to our knowledge. Of course, the question is also open for the traveling wave solutions that we obtain here.

3.3.2 The heteroclinic stationary wave for 2D Allen-Cahn systems

As pointed out before, the profile of the traveling wave solution that we obtain of this chapter behaves at infinity as the stationary waves obtained in [3]. We briefly recall here how the existence of these solutions is shown, which we hope will make the links with our problem clearer. Consider the elliptic system

$$-\Delta \mathfrak{U} = \nabla_u V(\mathfrak{U}) \text{ in } \mathbb{R}^2, \quad (3.3.1)$$

which corresponds to stationary solutions of (3.1.1). The main result obtained in [3] states that if one assumes the existence of two distinct globally minimizing heteroclinics up to translations, q_- and q_+ , and adds a symmetry assumption on the potential, there exists a solution \mathfrak{U} to (3.3.1)

satisfying the conditions at infinity

$$\begin{cases} \mathfrak{U}(x_1, x_2) \rightarrow \sigma^\pm & \text{as } x_2 \rightarrow \pm\infty, \text{ uniformly in } x_1, \\ \mathfrak{U}(x_1, x_2) \rightarrow \mathfrak{q}_\pm(\cdot + \tau^\pm) & \text{as } x_1 \rightarrow \pm\infty, \text{ uniformly in } x_2, \end{cases} \quad (3.3.2)$$

for some translation parameters $(\tau^-, \tau^+) \in \mathbb{R}^2$ (in the symmetric case $\tau^- = \tau^+ = 0$). The symmetry assumption was later removed by Schatzman in [153], so that the parameters (τ^-, τ^+) are part of the solution as well. While the existence proofs in [3] and [153] are (roughly speaking) addressed by addressing directly a functional associated with (3.3.1), a more general approach was carried out successfully in a more recent paper by Monteil and Santambrogio [125], later by Smyrnelis in [157]. Their approach follows from the key observation (for more details see Alessio and Montecchiari [7] and the references therein) that \mathfrak{U} can be seen as a curve in the space

$$Y(\mathfrak{q}_-, \mathfrak{q}_+) := \left\{ U : H_{\text{loc}}^1(\mathbb{R}, (X(\sigma^-, \sigma^+), \|\cdot\|_{L^2(\mathbb{R}, \mathbb{R}^k)})) : \lim_{x_1 \rightarrow \pm\infty} \inf_{\tau \in \mathbb{R}} \|U(x_1) - \mathfrak{q}_\pm(\cdot + \tau)\|_{L^2(\mathbb{R}, \mathbb{R}^k)} = 0 \right\}.$$

The previous leads to consider the functional

$$E_2 : U \in Y(\mathfrak{q}_-, \mathfrak{q}_+) \rightarrow E_2(U) := \int_{\mathbb{R}} \left[\frac{\|U'(x_1)\|_{L^2(\mathbb{R}, \mathbb{R}^k)}^2}{2} + \mathcal{V}(U(x_1)) \right] dx_1 \in \mathbb{R}, \quad (3.3.3)$$

where $\mathcal{V} : X(\sigma^-, \sigma^+) \rightarrow \mathbb{R}$ is the normalized energy

$$\mathcal{V} : q \in X(\sigma^-, \sigma^+) \rightarrow E(q) - \mathfrak{m} \in \mathbb{R},$$

so that \mathcal{V} is a non-negative functional in $X(\sigma^-, \sigma^+)$ with zero set equal to \mathcal{F} . Therefore, \mathcal{V} can be thought as a multi-well potential (modulo translations) in an infinite dimensional space. Identifying U with a function in $H_{\text{loc}}^1(\mathbb{R}^2, \mathbb{R}^k)$ in the obvious way, we rewrite from (3.3.3)

$$\begin{aligned} E_2(U) = & \int_{\mathbb{R}} \left[\int_{\mathbb{R}} \frac{|\partial_{x_1} U(x_1, x_2)|^2}{2} dx_2 \right. \\ & \left. + \left[\int_{\mathbb{R}} \left(\frac{|\partial_{x_2} U(x_1, x_2)|^2}{2} + V(U(x_1, x_2)) \right) dx_2 - \mathfrak{m} \right] \right] dx_1, \end{aligned}$$

so, formally, critical points \mathfrak{U} of E_2 in $Y(\mathfrak{q}_-, \mathfrak{q}_+)$ are solutions to (3.3.1) satisfying the conditions at infinity

$$\lim_{x_1 \rightarrow \pm\infty} \inf_{\tau \in \mathbb{R}} \|\mathfrak{U}(x_1) - \mathfrak{q}(\cdot + \tau)\|_{L^2(\mathbb{R}, \mathbb{R}^k)} = 0$$

In particular, the solution found in [3] is a global minimizer of E_2 in $Y(\mathfrak{q}_-, \mathfrak{q}_+)$. For the case of global minimizers, one shows that the stronger condition (3.3.2) holds, which we suspect might not be true for other critical points. From this starting point, the authors in [125] generalize the one-dimensional problem (3.2.2) seeing it as a problem of finding geodesics in a metric space (more general than \mathbb{R}^k), which can be chosen to be equal to $Y(\mathfrak{q}_-, \mathfrak{q}_+)$. Subsequently, they are able to deduce the results of [3] as particular cases. These type of ideas inspired us for proving the results of this chapter, which we do in the framework of abstract Hilbert spaces similar to that in [157].

3.3.3 Traveling waves for 1D parabolic systems of gradient type

For the reader's convenience, we provide some more details on the result which was proven by Alikakos and Katzourakis [13] and how it links to our problem. Consider the 1D parabolic system

$$\partial_t w - \partial_x^2 w = -\nabla_u W(w) \text{ in } [0, +\infty) \times \mathbb{R}. \quad (3.3.4)$$

Here $W : \mathbb{R}^k \rightarrow \mathbb{R}$ is an *unbalanced* double-well potential. The existence of traveling wave solutions for (3.3.4) has been addressed by several authors, see for instance Risler [145], Lucia, Muratov, Novaga [110] as well as Alikakos and Katzourakis [13]. As we mentioned earlier, in this chapter we look closely to the proof given in [13] (see also the book by Alikakos, Fusco and Smyrnelis [12]). In order to be more precise, one looks for a pair (c^*, u) such that the function

$$w : (t, x) \in [0, +\infty) \times \mathbb{R} \rightarrow u(x - ct) \in \mathbb{R}^k$$

solves (3.3.4) and the profile u joins at infinity two local minimizers of W at *different* levels. The solution w is then a traveling wave solution. More precisely, the profile u solves the system

$$-c^* u' - u'' = -\nabla_u W(u) \text{ in } \mathbb{R}, \quad (3.3.5)$$

and it satisfies at infinity

$$\lim_{t \rightarrow \pm\infty} u(t) = a^\pm$$

where $a^- \in \mathbb{R}^k$ is a global minimum of W with $W(a^-) < 0$ and a^+ is a local minimum of W and $W(a^+) = 0$. Moreover, in [13] it is also shown that the speed c^* is unique (a property to which our Proposition 3.5.3 is analogous), while the profile does not need to be.

The approach in [13, 12] is variational and uses some previous ideas from Muratov [128]. More precisely, they study the family of weighted functionals introduced by Fife and McLeod [79, 78]

$$E_c(q) := \int_{\mathbb{R}} \left(\frac{|q'(t)|^2}{2} + W(q(t)) \right) e^{ct} dt,$$

where q belongs to a suitable subspace in $H_{\text{loc}}^1(\mathbb{R}, \mathbb{R}^k)$ such that $\lim_{t \rightarrow \pm\infty} q(t) = a^\pm$. We formally check that critical points of E_{c^*} solve (3.3.5). The strategy of the proof in [13] was introduced before in Alikakos and Fusco [11] and it can be summarized as follows: First, one solves a family of constrained minimization problems for E_c , where $c > 0$ is at this point thought just as parameter. Once these problems have been solved one needs to find proper speed c^* . Finally, one needs to “remove” the constraints, that is, to show that for c^* one can find a constrained minimizer which does not saturate the constraints, meaning that it is an actual solution to (3.3.5). These last two steps are accomplished by showing that constrained minimizers exhibit a suitable asymptotic behavior (more precisely, that they do not present an oscillatory behavior in arbitrarily large regions).

Therefore, the idea is to follow Monteil and Santambrogio [125], Smyrnelis [157] and adapt the result of Alikakos and Katzourakis for infinite-dimensional ODE systems, in which curves take

values on an abstract Hilbert space. More precisely, we consider for $c > 0$ the functional

$$E_{2,c}(U) := \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \frac{|\partial_{x_1} U(x_1, x_2)|^2}{2} dx_2 + (E(U(x_1, \cdot)) - \mathfrak{m}^+) \right) e^{cx_1} dx_1$$

which we already introduced before. We then see U (with the proper identifications) as a mapping $U : x_1 \in \mathbb{R} \rightarrow U(x_1, \cdot) \in L^2(\mathbb{R}, \mathbb{R}^k)$. For $v \in L^2(\mathbb{R}, \mathbb{R}^k)$, we set

$$\mathcal{W}(v) := \begin{cases} E(U(x_1, \cdot)) - \mathfrak{m}^+ & \text{if } v \in H^1(\mathbb{R}, \mathbb{R}^k), \\ +\infty & \text{otherwise.} \end{cases}$$

Then, under assumption (H3.5) we have that \mathcal{W} is an unbalanced double well potential in $L^2(\mathbb{R}, \mathbb{R}^k)$ and $E_{2,c}$ can be rewritten as

$$E_{2,c}(U) = \int_{\mathbb{R}} \left[\frac{\|U'(x_1)\|_{L^2(\mathbb{R}, \mathbb{R}^k)}}{2} + \mathcal{W}(U(x_1)) \right] e^{cx_1} dx_1,$$

which is as E_c but for curves taking values in $L^2(\mathbb{R}, \mathbb{R}^k)$ instead of \mathbb{R}^k . Therefore, the main issue here is to adapt the result of [13] for curves which take values in a possibly infinite-dimensional Hilbert space \mathcal{L} (to be thought as $L^2(\mathbb{R}, \mathbb{R}^k)$) and possessing a proper subspace \mathcal{H} (to be thought as $H^1(\mathbb{R}, \mathbb{R}^k)$) satisfying suitable properties with respect to \mathcal{W} . A difficulty arises, since the minima of \mathcal{W} are non-isolated due to the invariance by translations of E . Nevertheless, this difficulty can be circumvented, and otherwise we could always restrict to potentials which are symmetric as Alama, Bronsard and Gui [3] and working in the resulting space of equivariant curves, in which invariance by translations disappears. The major difficulty comes from the fact that in [13] the authors impose some non-degeneracy and radial monotonicity assumptions which prevent the constrained minimizers for exhibiting a degenerate, oscillatory behavior. This assumption can be, in some sense, weakened in order to allow degenerate minima (we prove this in Chapter 4), but we cannot prove that \mathcal{W} fulfills them even for simple examples and we think it can be too restrictive. The reason is that the geometry of the level sets \mathcal{W} is difficult to understand, as it depends indirectly on V . For this reason, a new type of assumption, which in our case is (H3.6), is needed to replace the one from [13].

Finally, we point out that since our results are proven in an abstract setting, they also apply to 1D systems as (3.3.4) by seeing this time \mathcal{L} and \mathcal{H} as \mathbb{R}^k . Moreover, in Chapter 4 we show that the abstract approach allows to modify the non-degeneracy assumptions on the minimizers used in [13] and consider classes of potentials with some kind of degenerate minima.

3.3.4 Link with the singular limit problem

The asymptotic behavior as $\varepsilon \rightarrow 0$ for families of solutions $(u_\varepsilon)_{\varepsilon>0}$ of

$$\partial_t u_\varepsilon - \Delta u_\varepsilon = -\varepsilon^{-2} \nabla_u V(u_\varepsilon), \quad u_\varepsilon : [0, T] \times \Omega \rightarrow \mathbb{R}^k \quad (3.3.6)$$

has been extensively studied for bounded domains $\Omega \subset \mathbb{R}^N$ and $T > 0$. Concerning the scalar case $k = 1$, Ilmanen showed in [100] that the equation above converges to *Brakke's motion by mean curvature* as $\varepsilon \rightarrow 0$. Regarding the vectorial case, while analogous results are established under

several additional assumptions, little is proven regarding the general picture. We refer to Bronsard and Reitich [60] as well as the more recent Laux and Simon [106] and the references therein. A state of the art regarding the elliptic problem can be found in Bethuel [41]. We will now briefly comment on how the results obtained in this chapter can be linked to (and hopefully shed some light into) to asymptotic problem introduced above. For $\varepsilon > 0$, consider (c^\star, \mathfrak{U}) the solution given by Theorem 3.1 and let

$$w_\varepsilon(t, x_1, x_2) := \mathfrak{U}_\varepsilon(x_1 - c_\varepsilon t, x_2), \text{ for } (t, (x_1, x_2)) \in [0, +\infty) \times \mathbb{R}^2 \quad (3.3.7)$$

where for $(\tilde{x}_1, \tilde{x}_2) \in \mathbb{R}^2$

$$\mathfrak{U}_\varepsilon(\tilde{x}_1, \tilde{x}_2) := \mathfrak{U}(\varepsilon^{-1}\tilde{x}_1, \varepsilon^{-1}\tilde{x}_2) \quad (3.3.8)$$

and

$$c_\varepsilon := \varepsilon^{-1}c. \quad (3.3.9)$$

Combining (3.3.7), (3.3.8) and (3.3.9) we have that for $(t, x_1, x_2) \in [0, +\infty) \times \mathbb{R}^2$

$$w_\varepsilon(t, x_1, x_2) = \mathfrak{U}(\varepsilon^{-1}x_1 - \varepsilon^{-2}ct, \varepsilon^{-1}x_2) \quad (3.3.10)$$

and recall that by Theorem 3.1 we have

$$-c\partial_{x_1}\mathfrak{U} - \Delta\mathfrak{U} = -\nabla_u V(\mathfrak{U}) \text{ in } \mathbb{R}^2 \quad (3.3.11)$$

which implies that for all $\varepsilon > 0$

$$\partial_t w_\varepsilon - \Delta w_\varepsilon = \varepsilon^{-2}(-c\partial_{x_1}\mathfrak{U} - \Delta\mathfrak{U}) = -\varepsilon^{-2}\nabla_u V(\mathfrak{U}) = -\varepsilon^{-2}\nabla_u V(w_\varepsilon) \text{ in } [0, +\infty) \times \mathbb{R}^2.$$

Therefore, for any bounded domain $\Omega \subset \mathbb{R}^2$, $T > 0$ and $\varepsilon > 0$, w_ε solves (3.3.6). That is, to sum up, w_ε is a traveling wave for the re-scaled potential $V_\varepsilon := \varepsilon^{-2}V$, with profile \mathfrak{U}_ε as in (3.3.8) and with speed c_ε as in (3.3.9). Notice that $c_\varepsilon \rightarrow +\infty$ as $\varepsilon \rightarrow 0$. Regarding the asymptotics of $(w_\varepsilon)_{\varepsilon>0}$, let $\Omega = (-1, 1)^2 \subset \mathbb{R}^2$ for simplification and consider a time interval $[0, T]$, $T > 0$. Assume also that (H3.7) holds, so that by Theorem 3.2 we have

$$\lim_{x_1 \rightarrow \pm\infty} \|\mathfrak{U}(x_1, \cdot) - \mathfrak{q}^\pm(\cdot + \tau^\pm)\|_{H^1(\mathbb{R}, \mathbb{R}^k)} = 0. \quad (3.3.12)$$

In combination with (3.3.10), (3.3.12) implies that for all $x^+ \in (0, 1)$

$$\lim_{\varepsilon \rightarrow 0} \sup_{x_1 \in [x^+, 1)} \|w_\varepsilon(0, x_1, \cdot) - \mathfrak{q}^+(\varepsilon^{-1}(\cdot + \tau^+))\|_{L^\infty((-1, 1), \mathbb{R}^k)} = 0$$

and for all $x^- \in (-1, 0)$

$$\lim_{\varepsilon \rightarrow 0} \sup_{x_1 \in (-1, x^-]} \|w_\varepsilon(0, x_1, \cdot) - \mathfrak{q}^-(\varepsilon^{-1}(\cdot + \tau^-))\|_{L^\infty((-1, 1), \mathbb{R}^k)} = 0.$$

Therefore, we find a phase transition on the line $\{(x_1, x_2) \in \mathbb{R}^2 : x_1 = 0\}$, as it happens in the elliptic case with the rescaling of the stationary wave. On the contrary, for any $t > 0$ we have

$$\lim_{\varepsilon \rightarrow 0} \sup_{x_1 \in (-1, 1)} \|w_\varepsilon(t, x_1, \cdot) - q^-(\varepsilon^{-1}(\cdot + \tau^-))\|_{L^\infty((-1, 1), \mathbb{R}^k)} = 0.$$

That is, for positive time and small ε , the rescaled solutions tend to look like the globally minimizing heteroclinic at the limit $x_1 \rightarrow \pm\infty$, in contrast with what is observed for $t = 0$. In terms of the *interfacial density*, the previous means that an initial condition with non-constant density gives a solution with constant density for $t > 0$. This phenomenon is probably explained by some kind of parabolic regularization effects. To conclude this paragraph, notice that the considerations presented here are obtained by direct scaling computations. That is, they do not depend on the way \mathfrak{U} is obtained. It is only required that \mathfrak{U} solves (3.3.11) with conditions (3.3.12). In particular, the assumption (H3.6) is not relevant here and the same would apply to profiles obtained by different means under some other type of assumptions.

3.4 The abstract setting

3.4.1 Main definitions and notations

As we advanced in the introduction, instead of proving directly Theorems 3.1, 3.2 and 3.4, we will prove a set of more general results which will allow us to deduce the original ones as particular cases. In particular, we introduce an abstract setting similar to the ones considered in [125] and specially [157]. The proof of the main abstract results, Theorems 3.5, 3.6 and 3.7 below, are thus the core of the chapter. The passage between the abstract and the original setting is established in Section 3.6, which in turn proves Theorems 3.1, 3.2 and 3.4.

As we said before, the abstract results should be thought as an extension of the work by Alikakos and Katzourakis [13] to curves taking values in a more general Hilbert space and with minimum *sets* instead of isolated minimum *points*. In fact, we essentially perform an adaptation of their strategy of proof, which turns out to carry on to our setting. That is, our approach will consist on establishing existence of a pair (c, \mathbf{U}) in $(0, +\infty) \times X$ which fulfills

$$\mathbf{U}'' - D_{\mathcal{L}}\mathcal{E}(\mathbf{U}) = -c\mathbf{U}' \text{ in } \mathbb{R} \quad (3.4.1)$$

and satisfies the conditions at infinity

$$\exists T^- \in \mathbb{R} : \forall t \leq T^-, \mathbf{U}(t) \in \mathcal{F}_{r_0^-/2}^-, \quad (3.4.2)$$

$$\exists T^+ \in \mathbb{R} : \forall t \geq T^+, \mathbf{U}(t) \in \mathcal{F}_{r_0^+/2}^+. \quad (3.4.3)$$

Notice that this problem can also be thought as a heteroclinic connection problem on Hilbert spaces for a second order potential system with friction term. Such a problem could have its own interest besides the main application to the existence of traveling waves that we give here. Of course, analogous considerations can be also applied to the results in [13] as well as Chapter 4.

The nature of the objects introduced above will be made precise along this paragraph. Let \mathcal{L} be a Hilbert space with inner product $\langle \cdot, \cdot \rangle_{\mathcal{L}}$ and induced norm $\|\cdot\|_{\mathcal{L}}$. Let $\mathcal{H} \subset \mathcal{L}$ a Hilbert space with

inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}}$. In the original setting, \mathcal{L} is $L^2(\mathbb{R}, \mathbb{R}^k)$ and \mathcal{H} is $H^1(\mathbb{R}, \mathbb{R}^k)$, both endowed with their natural inner products. We will take $\mathcal{E} : \mathcal{L} \rightarrow (-\infty, +\infty]$ an *unbalanced* potential. In the setting of Theorem 3.1, \mathcal{E} will *essentially* coincide with $E - m^+$ in $H^1(\mathbb{R}, \mathbb{R}^k)$ and with $+\infty$ elsewhere². Here we just impose a set of abstract assumptions on \mathcal{E} . Most of those assumptions follow by combining ideas in [13] with ideas in Schatzman [153] and Smyrnelis [157]. We will begin by fixing two sets \mathcal{F}^- and \mathcal{F}^+ in \mathcal{L} . For $r > 0$, we define

$$\mathcal{F}_r^\pm := \left\{ v \in \mathcal{L} : \inf_{\mathbf{v} \in \mathcal{F}^\pm} \|v - \mathbf{v}\|_{\mathcal{L}} \leq r \right\}, \quad (3.4.4)$$

and

$$\mathcal{F}_{\mathcal{H},r}^\pm := \left\{ v \in \mathcal{H} : \inf_{\mathbf{v} \in \mathcal{F}^\pm} \|v - \mathbf{v}\|_{\mathcal{H}} \leq r \right\}, \quad (3.4.5)$$

that is, the closed balls in \mathcal{L} and \mathcal{H} respectively, with radius $r > 0$ and center \mathcal{F}^\pm . The main assumption reads as follows:

(H3.1'). The potential \mathcal{E} is weakly lower semicontinuous in \mathcal{L} . The sets \mathcal{F}^- and \mathcal{F}^+ are closed in \mathcal{L} . There exists a constant $a < 0$ such that

$$\forall v \in \mathcal{L}, \forall \mathbf{v}^- \in \mathcal{F}^-, \quad \mathcal{E}(v) \geq \mathcal{E}(\mathbf{v}^-) = a$$

and each $\mathbf{v}^+ \in \mathcal{F}^+$ is a local minimizer satisfying $\mathcal{E}(\mathbf{v}^+) = 0$. Moreover, there exist two positive constants r_0^-, r_0^+ such that $\mathcal{F}_{r_0^-}^+ \cap \mathcal{F}_{r_0^+}^- = \emptyset$ (see (3.4.4)). There also exist $C^\pm > 1$ such that

$$\forall v \in \mathcal{F}_{r_0^\pm}^\pm, \quad (C^\pm)^{-1} \text{dist}_{\mathcal{L}}(v, \mathcal{F}^\pm)^2 \leq \mathcal{E}(v) - \min\{\pm(-a), 0\}. \quad (3.4.6)$$

Moreover, for any $v \in \mathcal{F}_{r_0^\pm}^\pm$, there exists a unique $\mathbf{v}^\pm(v) \in \mathcal{F}^\pm$ such that

$$\|v - \mathbf{v}^\pm(v)\|_{\mathcal{L}} = \inf_{\mathbf{v}^\pm \in \mathcal{F}^\pm} \|v - \mathbf{v}^\pm\|_{\mathcal{L}}.$$

Moreover, the projection maps

$$P^\pm : v \in \mathcal{F}_{r_0^\pm}^\pm \rightarrow \mathbf{v}^\pm(v) \in \mathcal{F}^\pm$$

are C^2 with respect to the \mathcal{L} -norm.

Hypothesis (H3.1') defines \mathcal{E} as an unbalanced double well potential with respect to \mathcal{F}^- and \mathcal{F}^+ and gives local information of the minimizing sets. Compare with (H3.5) and the remarks that follow. We have the following immediate consequence, which will be useful in the sequel:

Lemma 3.4.1. Assume that (H3.1') holds. If we define for $r \in (0, r_0^\pm]$ we define

$$\kappa_r^\pm := \inf\{\mathcal{E}(v) : \text{dist}_{\mathcal{L}}(v, \mathcal{F}^\pm) \in [r, r_0^\pm]\} \quad (3.4.7)$$

then we have $\kappa_r^\pm > \min\{\pm(-a), 0\}$. Moreover,

$$\forall v \in \mathcal{F}_{r_0^\pm/2}^\pm, \quad \mathcal{E}(v) \geq 0. \quad (3.4.8)$$

²this statement is not exact as the energy E is not defined in $H^1(\mathbb{R}, \mathbb{R}^k)$, but on an affine space based on $H^1(\mathbb{R}, \mathbb{R}^k)$. However, we can trivially obtain a functional defined on $H^1(\mathbb{R}, \mathbb{R}^k)$ from E . See Section 3.6.1.

Proof. It follows directly from (3.4.6) in (H3.1'). \square

We now impose the following regarding the relationship between \mathcal{L} and \mathcal{H} :

(H3.2'). We have that $\mathcal{H} = \{v \in \mathcal{L} : \mathcal{E}(v) < +\infty\}$ and $\|\cdot\|_{\mathcal{L}} \leq \|\cdot\|_{\mathcal{H}}$. In particular, $\mathcal{F}^{\pm} \subset \mathcal{H}$. Moreover, \mathcal{E} is a C^1 functional on $(\mathcal{H}, \|\cdot\|_{\mathcal{H}})$ with differential $D\mathcal{E} : v \in \mathcal{H} \rightarrow D\mathcal{E}(v) \in \mathcal{H}'$, where \mathcal{H}' is the (topological) dual of \mathcal{H} . Furthermore, there exists an even smaller space $\tilde{\mathcal{H}}$ with an inner product $\langle \cdot, \cdot \rangle_{\tilde{\mathcal{H}}}$ and associated norm $\|\cdot\|_{\tilde{\mathcal{H}}} \geq \|\cdot\|_{\mathcal{H}}$ such that we can find a continuous correspondence

$$D_{\mathcal{L}}\mathcal{E} : v \in (\tilde{\mathcal{H}}, \|\cdot\|_{\tilde{\mathcal{H}}}) \rightarrow D_{\mathcal{L}}\mathcal{E}(v) \in (\mathcal{L}, \|\cdot\|_{\mathcal{L}}) \quad (3.4.9)$$

such that

$$\forall v \in \tilde{\mathcal{H}}, \forall w \in \mathcal{H}, \quad D_{\mathcal{L}}\mathcal{E}(v)(w) = D\mathcal{E}(v)(w). \quad (3.4.10)$$

Notice that in the context of Theorem 3.1 assumption (H3.2') is easily verified. The space $\tilde{\mathcal{H}}$ will be chosen $H^2(\mathbb{R}, \mathbb{R}^k)$ and (3.4.10) is no other than integration by parts. The notation $D_{\mathcal{L}}\mathcal{E}$ is chosen to emphasize the formal \mathcal{L} -gradient flow structure of the corresponding abstract evolution equation. We now continue by imposing a compactness assumption on \mathcal{F}^{\pm} :

(H3.3'). \mathcal{L} -bounded subsets of \mathcal{F}^{\pm} are compact with respect to \mathcal{H} -convergence.³

Assumption (H3.3') readily implies the following:

Lemma 3.4.2. *Assume that (H3.1') and (H3.3') hold. Then, the sets $\mathcal{F}_{r_0/2}^{\pm}$ defined in (3.4.4) are closed in \mathcal{L} .*

Assumption (H3.3') is necessary in order to establish the conditions at infinity. In the main context, it is a straightforward consequence of the compactness of the minimizing sequences. Subsequently, we impose the following:

(H3.4'). Assume that (H3.1') holds. For \mathcal{F}^{\pm} , one of the two following alternatives holds:

1. \mathcal{F}^{\pm} is \mathcal{L} -bounded.
2. For all $(v, \mathbf{v}^{\pm}) \in \mathcal{F}_{r_0/2}^{\pm} \times \mathcal{F}^{\pm}$, there exists an associated map $\hat{P}_{(v, \mathbf{v}^{\pm})}^{\pm} : \mathcal{L} \rightarrow \mathcal{L}$ such that

$$P^{\pm}(\hat{P}_{(v, \mathbf{v}^{\pm})}^{\pm}(v)) = \mathbf{v}^{\pm} \quad (3.4.11)$$

and

$$\text{dist}_{\mathcal{L}}(\hat{P}_{(v, \mathbf{v}^{\pm})}^{\pm}(v), \mathcal{F}^{\pm}) = \text{dist}_{\mathcal{L}}(v, \mathcal{F}^{\pm}). \quad (3.4.12)$$

Moreover, $\hat{P}_{(v, \mathbf{v}^{\pm})}^{\pm} : \mathcal{L} \rightarrow \mathcal{L}$ is differentiable and

$$\forall (w_1, w_2) \in \mathcal{L}^2, \quad \|D(\hat{P}_{(v, \mathbf{v}^{\pm})}^{\pm})(w_1, w_2)\|_{\mathcal{L}} = \|w_2\|_{\mathcal{L}} \quad (3.4.13)$$

$$\mathcal{E}(\hat{P}_{(v, \mathbf{v}^{\pm})}^{\pm}(v)) = \mathcal{E}(v). \quad (3.4.14)$$

Essentially, in 2. we impose that the projections P^{\pm} from (H3.1') are, in some sense, invertible. Again, this is straightforward in the concrete setting, as the projections P^{\pm} consist on performing a translation. We now impose an assumption for the sets $\mathcal{F}_{\mathcal{H}, r_0}^{\pm}$:

³hence, they are in particular compact with respect to \mathcal{L} -convergence

(H3.5'). For any $v \in \mathcal{F}_{\mathcal{H}, r_0^\pm}^\pm$, as defined in (3.4.5), there exists a unique $\mathbf{v}_{\mathcal{H}}^\pm(v) \in \mathcal{F}^\pm$ such that

$$\|v - \mathbf{v}_{\mathcal{H}}^\pm(v)\|_{\mathcal{L}} = \inf_{\mathbf{v}^\pm \in \mathcal{F}^\pm} \|v - \mathbf{v}^\pm\|_{\mathcal{L}}.$$

Moreover, the projection maps

$$P_{\mathcal{H}}^\pm : v \in \mathcal{F}_{\mathcal{H}, r_0^\pm}^\pm \rightarrow \mathbf{v}_{\mathcal{H}}^\pm(v) \in \mathcal{F}^\pm$$

are C^1 with respect to the \mathcal{H} -norm. Moreover, if $C^\pm > 1$ is the constant from (H3.1'), we have

$$\forall v \in \mathcal{F}_{\mathcal{H}, r_0^\pm}^\pm, \quad \|P^\pm(v) - P_{\mathcal{H}}^\pm(v)\|_{\mathcal{H}} \leq C^\pm \|v - P_{\mathcal{H}}^\pm(v)\|_{\mathcal{H}}. \quad (3.4.15)$$

Furthermore, for each $r^\pm \in (0, r_0^\pm]$ there exist constants $\beta^\pm(r^\pm) > 0$ such that in case that $v \in \mathcal{F}_{r_0^\pm}^\pm$ satisfies

$$\mathcal{E}(v) \leq \min\{\pm(-a), 0\} + \beta^\pm(r^\pm), \quad (3.4.16)$$

then $v \in \mathcal{F}_{\mathcal{H}, r}^\pm$. Finally, we have the following

$$\forall v \in \mathcal{F}_{\mathcal{H}, r_0^\pm}^\pm, \quad (C^\pm)^{-2} \|v - P_{\mathcal{H}}^\pm(v)\|_{\mathcal{H}}^2 \leq \mathcal{E}(v) - \min\{\pm(-a), 0\} \leq (C^\pm)^2 \|v - P_{\mathcal{H}}^\pm(v)\|_{\mathcal{H}}^2. \quad (3.4.17)$$

Assumption (H3.5') is made in order to ensure the suitable local properties around \mathcal{F}^\pm in \mathcal{H} . In the main setting, those are known results which follow essentially from the spectral assumption by Schatzman [153]. Before introducing the last assumptions, we need some additional notation. For $U \in H_{\text{loc}}^1(\mathbb{R}, \mathcal{L})$ and $c > 0$, we (formally) define

$$\mathbf{E}_c(U) := \int_{\mathbb{R}} \mathbf{e}_c(U)(t) dt := \int_{\mathbb{R}} \left[\frac{\|U'(t)\|_{\mathcal{L}}^2}{2} + \mathcal{E}(U(t)) \right] e^{ct} dt. \quad (3.4.18)$$

More generally, for $I \subset \mathbb{R}$ a non-empty interval and $U \in H_{\text{loc}}^1(I, \mathcal{L})$, put

$$\mathbf{E}_c(U; I) := \int_I \mathbf{e}_c(U)(t) dt. \quad (3.4.19)$$

Notice that the integrals defined in (3.4.18) and (3.4.19) might not even make sense in general due to the fact that \mathcal{E} has a sign. Nevertheless, we can define the notion of *local minimizer* of $\mathbf{E}_c(\cdot; I)$ as follows:

Definition 3.4.1. Assume that (H3.1') and (H3.2') hold. Let $I \subset \mathbb{R}$ be a bounded, non-empty interval. Assume that $U \in H_{\text{loc}}^1(I, \mathcal{L})$ is such that $\mathbf{E}_c(U; I)$ is well-defined and finite. Assume also that there exists $C > 0$ such that for any $\phi \in C_c^1(\text{int}(I), (\mathcal{H}, \|\cdot\|_{\mathcal{H}}))$ such that

$$\max_{t \in I} \|\phi(t)\|_{\mathcal{H}} < C,$$

the quantity $\mathbf{E}_c(U + \phi; I)$ is well-defined and larger than $\mathbf{E}_c(U; I)$. Then, we say that U is a local minimizer of $\mathbf{E}_c(\cdot; I)$.

We assume the following property for local minimizers:

(H3.6'). Assume that (H3.1') and (H3.2') hold. There exists a map $\mathfrak{P} : \mathcal{L} \rightarrow \mathcal{L}$ such that

$$\forall v \in \mathcal{L}, \quad \mathcal{E}(\mathfrak{P}(v)) \leq \mathcal{E}(v) \text{ and } \mathcal{E}(\mathfrak{P}(v)) = \mathcal{E}(v) \Leftrightarrow \mathfrak{P}(v) = v, \quad (3.4.20)$$

$$\forall (v_1, v_2) \in \mathcal{L}^2, \quad \|\mathfrak{P}(v_1) - \mathfrak{P}(v_2)\|_{\mathcal{L}} \leq \|v_1 - v_2\|_{\mathcal{L}}, \quad (3.4.21)$$

and

$$\mathfrak{P}|_{\mathcal{G}^\pm} = \text{Id}|_{\mathcal{G}^\pm}. \quad (3.4.22)$$

Let $I \subset \mathbb{R}$, possibly unbounded and non-empty. Let $c > 0$. If $\mathbf{W} \in H_{\text{loc}}^1(I, \mathcal{L})$ is a local minimizer of $\mathbf{E}_c(\cdot; I)$ in the sense of Definition 3.4.1, which, additionally, is such that for all $t \in I$, $\mathbf{W}(t) = \mathfrak{P}(\mathbf{W}(t))$, then $\mathbf{W} \in \mathcal{A}(I)$ where for any open set $O \subset \mathbb{R}$, $\mathcal{A}(O)$ is defined as

$$\mathcal{A}(O) := \mathcal{C}_{\text{loc}}^2(O, \mathcal{L}) \cap \mathcal{C}_{\text{loc}}^1(O, (\mathcal{H}, \|\cdot\|_{\mathcal{H}})) \cap \mathcal{C}_{\text{loc}}^0(O, (\tilde{\mathcal{H}}, \|\cdot\|_{\tilde{\mathcal{H}}})) \quad (3.4.23)$$

and \mathbf{W} solves

$$\mathbf{W}'' - D_{\mathcal{L}}\mathcal{E}(\mathbf{W}) = -c\mathbf{W}' \text{ in } I,$$

where $D_{\mathcal{L}}\mathcal{E}$ was introduced in (3.4.9).

In the context of Theorem 3.1, (H3.6') is a consequence of classical elliptic regularity results as well as properties on the energy functional. The purpose of the projection \mathfrak{P} is technical, and in the main setting it will mean that constrained minimizers are bounded with respect to the L^∞ norm. Before stating the abstract result, we introduce the following constants (assuming that all the previous assumptions hold) which are obviously analogous with those introduced in Section 3.2.5:

$$\eta_0^- := \min \left\{ \sqrt{e^{-1} \frac{r_0^-}{4} \sqrt{2(\kappa_{r_0^-/4}^- - a)}}, \frac{r_0^-}{4} \right\} > 0, \quad (3.4.24)$$

$$\hat{r}^- := \frac{r_0^-}{C^- + 1} > 0 \quad (3.4.25)$$

$$\mathcal{E}_{\max}^- := \frac{1}{(C^-)^2(C^- + 1)} \min \left\{ \frac{(\eta_0^-)^2}{4}, \kappa_{\eta_0^-}^- - a, \beta^-(\hat{r}^-), \beta^-(\eta_0^-) \right\} > 0, \quad (3.4.26)$$

$$C^\pm := \frac{1}{2}(C^\pm)^2((C^\pm)^2 + (C^\pm + 1)^2) > 0, \quad (3.4.27)$$

$$\gamma^- := \frac{1}{C^- + C^-} > 0 \quad (3.4.28)$$

and

$$d_0 := \text{dist}_{\mathcal{L}}(\mathcal{F}_{r_0^+/2}^+, \mathcal{F}_{r_0^-/2}^-) > 0, \quad (3.4.29)$$

where the constants C^- , $\beta^-(\hat{r}^-)$, $\beta^-(\eta_0^-)$ are those from (H3.5') and κ_r^\pm for $r > 0$ are defined in (3.4.7). The fact that $d_0 > 0$ follows from Lemma 3.4.2 and (H3.1'). We can finally state the following assumption:

(H3.7'). Assume that (H3.1') and (H3.2') hold. Moreover, assume that

$$-a < \mathcal{E}_{\max}^-$$

and

$$\{v \in \mathcal{H} : \mathcal{E}(v) < 0\} \subset \mathcal{F}_{r_0/2}^-. \quad (3.4.30)$$

Assumption (H3.7') is essentially the abstract version of (H3.6).

3.4.2 Statement of the abstract results

Let us define the space

$$X := \left\{ U \in H_{\text{loc}}^1(\mathbb{R}, \mathcal{L}) : \exists T \geq 1, \forall t \geq T, \text{dist}_{\mathcal{L}}(U(t), \mathcal{F}^+) \leq \frac{r_0^+}{2}, \right. \\ \left. \forall t \leq -T, \text{dist}_{\mathcal{L}}(U(t), \mathcal{F}^-) \leq \frac{r_0^-}{2} \right\}. \quad (3.4.31)$$

The statement of the main abstract result is as follows:

Theorem 3.5 (Main abstract result). *Assume that (H3.3'), (H3.4'), (H3.5'), (H3.6') and (H3.7') hold. Then, the following holds:*

1. **Existence.** *There exists $c^* > 0$ and $\mathbf{U} \in \mathcal{A}(\mathbb{R}) \cap X$, $\mathcal{A}(\mathbb{R})$ as in (3.4.23) and X as in (3.4.31), such that (c^*, \mathbf{U}) solves (3.4.1) with conditions at infinity (3.4.2), (3.4.3) and \mathbf{U} is a global minimizer of \mathbf{E}_c in X , that is, $\mathbf{E}_c(\mathbf{U}) = 0$. Moreover, for all $t \in \mathbb{R}$, $\mathbf{U}(t) = \mathfrak{P}(\mathbf{U}(t))$, where \mathfrak{P} is as in (H3.6').*
2. **Uniqueness of the speed.** *The speed c^* is unique in the following sense: if $\overline{c^*} > 0$ is such that*

$$\inf_{U \in X} \mathbf{E}_{\overline{c^*}}(U) = 0$$

and there exists $\overline{\mathbf{U}} \in \mathcal{A}(\mathbb{R}) \cap X$ such that $(\overline{c^}, \overline{\mathbf{U}})$ solves (3.4.1) and $\mathbf{E}_{\overline{c^*}}(\overline{\mathbf{U}}) < +\infty$, then $\overline{c^*} = c^*$.*

3. **Exponential convergence.** *There exists a constant $M^+ > 0$ such that for all $t \in \mathbb{R}$ we have*

$$\|\mathbf{U}(t) - \mathbf{v}^+(\mathbf{U})\|_{\mathcal{L}} \leq M^+ e^{-ct}, \quad (3.4.32)$$

for some $\mathbf{v}^+(\mathbf{U}) \in \mathcal{F}^+$.

Remark 3.4.1. Given the definition of X in (3.4.31), we have that for any $U \in X$ and $\tau \in \mathbb{R}$ it holds $U(\cdot + \tau) \in X$ and for any $c > 0$ it holds $\mathbf{E}_c(U(\cdot + \tau)) = e^{-c\tau} \mathbf{E}_c(U)$. Such a thing implies

$$\forall c > 0, \quad \inf_{U \in X} \mathbf{E}_c(U) \in \{-\infty, 0\}.$$

Moreover, we see that in case $c > 0$ is such that $\inf_{U \in X} \mathbf{E}_c(U) = 0$ one can find plenty of examples of minimizing sequences in X which cannot ever reasonably produce a global minimizer. Indeed, consider any function $\tilde{U} \in X$ such that $\mathbf{E}_c(\tilde{U}) > 0$ and then take the minimizing sequence $(\tilde{U}(\cdot + n))_{n \in \mathbb{N}}$.

Remark 3.4.2. A more general statement can be given about the uniqueness of the speed, which in particular works for eventual non-minimizing solutions. See Proposition 3.5.3.

Theorem 3.5 will be shown to contain Theorem 3.1 in Section 3.6. Notice that, as before, the conditions at infinity (3.4.2) are rather weak (and not really of heteroclinic type), since we do not

have convergence to an element of \mathcal{F}^- as $t \rightarrow -\infty$. It is however clear that the conditions at infinity (3.4.2), (3.4.3) are enough to ensure that the solution given by Theorem 3.5 is not constant. In any case, we can impose an additional assumption in order to obtain stronger conditions at $-\infty$ on the solution:

(H3.8'). Hypothesis (H3.7') is fulfilled and, additionally:

$$-a < \frac{(d_0 \gamma^-)^2}{2}, \quad (3.4.33)$$

where d_0 and γ^- were defined in (3.4.29) and (3.4.28) respectively.

Then we can show the following exponential convergence result

Theorem 3.6. *Assume that (H3.2'), (H3.3'), (H3.4'), (H3.5'), (H3.6'), (H3.7') and (H3.8') hold. Then, if (c^*, \mathbf{U}) is the solution given by Theorem 3.5, it holds that $\gamma^- > c^*$ (γ^- as in (3.4.28)) and there exists $\mathfrak{M}^- > 0$ such that for all $t \in \mathbb{R}$*

$$\|\mathbf{U}(t) - \mathbf{v}^-(\mathbf{U})\|_{\mathcal{L}} \leq M^- e^{(\gamma^- - c^*)t} \quad (3.4.34)$$

for some $\mathbf{v}^-(U) \in \mathcal{F}^-$.

Theorem 3.6 corresponds to Theorem 3.2. Finally, we will prove the following result:

Theorem 3.7. *Assume that (H3.2'), (H3.3'), (H3.4'), (H3.5'), (H3.6'), (H3.7') and (H3.8') hold. Let (c^*, \mathbf{U}) be the solution given by Theorem 3.5. Then, if $\tilde{\mathbf{U}} \in \mathcal{A}(\mathbb{R}) \cap X$ is such that*

$$\mathbf{E}_{c^*}(\tilde{\mathbf{U}}) = 0$$

then we have that $(c^*, \tilde{\mathbf{U}})$ solves (3.4.1) and

$$c^* = \frac{-a}{\int_{\mathbb{R}} \|\tilde{\mathbf{U}}'(t)\|_{\mathcal{L}}^2 dt}. \quad (3.4.35)$$

In particular, the quantity $\int_{\mathbb{R}} \|\tilde{\mathbf{U}}'(t)\|_{\mathcal{L}}^2 dt$ is finite. Moreover, we have

$$c^* = \sup\{c > 0 : \inf_{U \in X} \mathbf{E}_c(U) = -\infty\} = \inf\{c > 0 : \inf_{U \in X} \mathbf{E}_c(U) = 0\} \quad (3.4.36)$$

as well as the bound

$$c^* \leq \frac{\sqrt{-2a}}{d_0} < \min \left\{ \frac{\sqrt{2\mathcal{E}_{\max}^-}}{d_0}, \gamma^- \right\} \quad (3.4.37)$$

with \mathcal{E}_{\max}^- as in (3.4.26), d_0 as in (3.4.29) and γ^- as in (3.4.28). The second inequality follows from the bounds on $-a$ given by (H3.7') and (H3.8').

Theorem 3.7 corresponds to Theorem 3.4.

3.5 Proof of the abstract results

3.5.1 Scheme of the proofs

As pointed out several times, the structure of the proofs of our abstract results, Theorems 3.5, 3.6 and 3.7, is analogous to that in Alikakos and Katzourakis [13], which has its roots in Alikakos

and Fusco [11]. In fact, most of their results also carry into the abstract setting with the suitable modifications. In fact, the structure of our proofs should be rather compared with *subsection 2.6* in the book by Alikakos, Fusco and Smyrnelis [12], which slightly modifies and simplifies the argument in [13]. We will also rely on some arguments provided in Smyrnelis [157], when an analogous abstract approach is taken for the stationary problem. As usual, most of the intermediate results we prove hold under smaller subsets of assumptions (with respect to the set of all assumptions that we dropped in the previous section). Therefore, for the sake of clarity and generality, the necessary assumptions (and only these) that we use to prove a result are specified in its statement.

Despite the previous facts, and as pointed before, several important difficulties not present in [13] arise when one tries to tackle the same problem in the abstract setting we introduced in the previous section. One of those extra difficulties is due to the fact that, in our setting, we need deal with two different norms in the configuration space of the curves, \mathcal{L} and \mathcal{H} (to be thought as L^2 and H^1 respectively, for simplification) and that the potential \mathcal{E} is only lower semicontinuous with respect to \mathcal{L} -convergence. An additional difficulty comes from the fact that, due to the requirements of our original problem, we are not looking at curves that join two *isolated minimum points*, but rather two *isolated minimum sets*. This turns out to be an obstacle when one tries to adapt argument in [13], even if one were to restrict to finite-dimensional configuration spaces. However, this difficulty is successfully dealt with using the precise knowledge about the *projection mappings* (namely assumptions (H3.1'), (H3.4') and (H3.5')) is available. That is, one uses that, for a suitable neighborhood of the minimum sets, the projection onto the sets (both with the \mathcal{L} and \mathcal{H} norms) is well defined and enjoys some type of continuity and differentiability properties. This idea, in the Allen-Cahn systems setting, has to be traced back to Schatzman [153].

We will now briefly sketch the scheme of the proof of Theorem 3.5. Recall that, according to Remark 3.4.1, direct minimization of E_c in X cannot yield solutions to the problem, the reason being the action of the group of translations. The spaces X_T , which were introduced in [13] (also in [11] for the equal-depth case) and will be precisely presented in (3.5.1), are defined in order to overcome this source of degeneracy, as they are no longer invariant by the action of the group of translations. See the design in Figure 3.4. As a consequence, compactness is restored and the corresponding minimization problem has a solution for all $c > 0$ and $T \geq 1$. See Lemma 3.5.7 later on. In general, minimizers in X_T solve the profile equation on a (possibly proper) subset of \mathbb{R} (see Lemma 3.5.8), meaning that they are in general not solutions of (3.4.1). However, such constrained minimizers are in fact solutions of (3.4.1) in the case they do not saturate the constraints. Therefore, the goal will be to show the existence of the speed c^* such that, for some $T \geq 1$, there exists a constrained minimizer in X_T which does not saturate the constraints. For that purpose, a careful analysis of the behavior of the constrained minimizers is needed. Indeed, one needs a uniform bound (independent on T and continuous on c) on the distance between the *entry times*, i. e. the times in which the constrained minimizers enter $\mathcal{F}_{r_0^\pm/2}^\pm$. In the balanced case this follows from the fact that the energy density is bounded below by a positive constant outside $\mathcal{F}_{r_0/2}^- \cup \mathcal{F}_{r_0/2}^+$ (see for instance Smyrnelis [157]). However this is no longer true for our unbalanced problem, which makes it more involved: If one does not have the positivity of the energy density, the constraint solutions can oscillate between the regions $\mathcal{F}_{r_0^\pm/2}^\pm$ (producing energy compensations) in larger and larger intervals as $T \rightarrow \infty$, so that no T -independent bound can be found. This is the main new

difficulty with respect to the balanced setting, as one needs new ideas in order to obtain a uniform bound on the distance between the entry times. Our assumption (H3.7') provides this control because the energy density of the constrained minimizers is bounded below by a positive constant in the interval given by the two entry times mentioned before, meaning that we can argue as in the balanced case. The precise result is Corollary 3.5.1. This is the main step in which our proof differs with that in [13].

The natural question is what happens if we remove (H3.7'). A natural approach is to replace (H3.7') by an assumption more closely related to the one used in [13] and [12]. This would lead to introduce a convexity assumption on the level sets of \mathcal{E} , as well as some sort of strict monotonicity on well-chosen segments. While this assumption can be worked out in the abstract setting and it is applicable for the finite-dimensional situation considered in [13] (as we show in Chapter 4), we believe it to be too restrictive to be applied to our original problem.

In any case, after the uniform bound on the entry times of constrained minimizers is obtained, one needs to find the speed c^* as, until this point, the speed $c > 0$ has been only considered as a parameter of the problem without any special role. Our arguments adapt without major difficulty from [12] and it goes as follows: One introduces a set which classifies the speeds according to the value of the infimum of the corresponding energy on X (which, due to the weight and the invariance by translations, is either $-\infty$ or 0). Such a set is \mathcal{C} , defined in (3.5.86). Subsequently, one shows (Lemma 3.5.11) that \mathcal{C} is open, bounded, non-empty and that its positive limit points give rise to entire minimizing solutions of the equations (since for those points one can find corresponding constrained minimizers which do not saturate the constraints). The speed c^* is then defined as the supremum of \mathcal{C} , which is in fact the unique positive limit point of the set, as shown in Corollary 3.5.3. At this point, the process of the proof of Theorem 3.5 is completed. Later on, we show that the asymptotic behavior of the constrained solutions can be improved under an additional assumption, namely an upper bound on the speed. This is Proposition 3.5.4. Theorems 3.6 and 3.7 can be then proven.

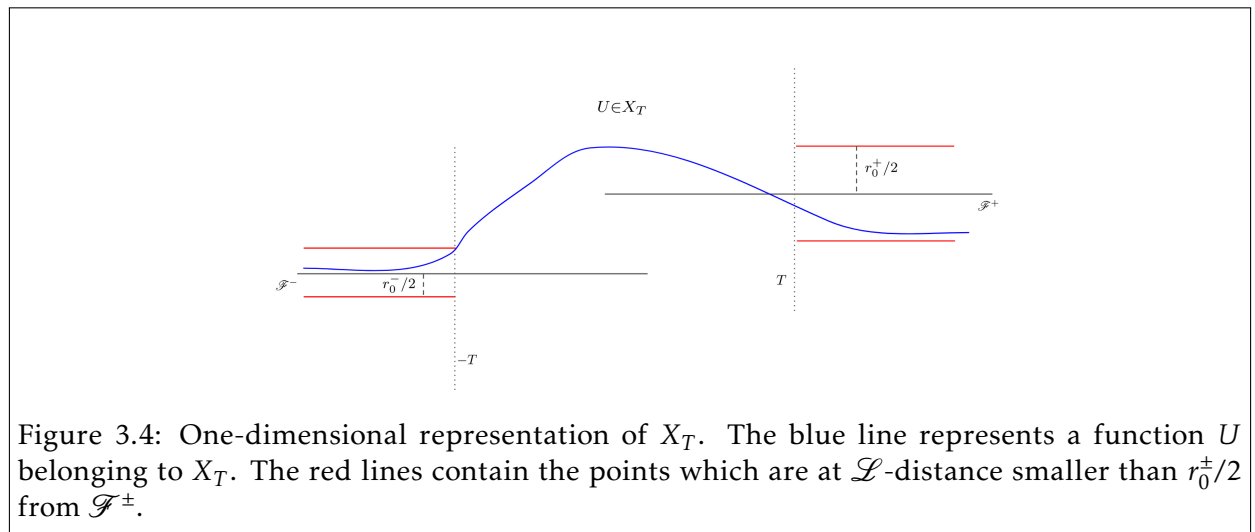


Figure 3.4: One-dimensional representation of X_T . The blue line represents a function U belonging to X_T . The red lines contain the points which are at \mathcal{L} -distance smaller than $r_0^\pm/2$ from \mathcal{S}^\pm .

3.5.2 Preliminaries

Let r_0^- and r_0^+ be the constants introduced in Section 3.4 and $\mathcal{F}_{r_0^\pm/2}^\pm$ be the corresponding closed balls as in (3.4.4). Assume that (H3.1') holds. For $T \geq 1$, we define the sets

$$X_T^- := \left\{ U \in H_{\text{loc}}^1(\mathbb{R}, \mathcal{L}) : \forall t \leq -T, U(t) \in \mathcal{F}_{r_0^-/2}^- \right\},$$

$$X_T^+ := \left\{ U \in H_{\text{loc}}^1(\mathbb{R}, \mathcal{L}) : \forall t \geq T, U(t) \in \mathcal{F}_{r_0^+/2}^+ \right\}.$$

Subsequently, we set

$$X_T := X_T^- \cap X_T^+. \quad (3.5.1)$$

Recall the space X introduced in (3.4.31). Notice that

$$X = \bigcup_{T \geq 1} X_T.$$

We have the following preliminary properties on the spaces X_T :

Lemma 3.5.1. *Assume that (H3.1') holds. Let $c > 0$ and $T \geq 1$. For any $U \in X_T$, we have that*

$$\forall t \geq T, \quad \mathcal{E}(U(t)) \geq 0. \quad (3.5.2)$$

Moreover, the quantity $\mathbf{E}_c(U)$ as introduced in (3.4.18) is well defined in $(-\infty, +\infty]$.

Proof. Let $U \in X_T$. Notice that for $t \geq T$, we have that $U(t) \in \mathcal{F}_{r_0^+/2}^+$. Therefore, (3.5.2) follows directly from (3.4.8) in Lemma 3.4.1.

Let now $\mathcal{E}^+(U) \geq 0$ and $\mathcal{E}^-(U) \geq 0$ be, respectively, the non-negative and the non-positive part of $\mathcal{E}(U)$, so that $\mathcal{E}(U) = \mathcal{E}^+(U) - \mathcal{E}^-(U)$. We have that $\mathcal{E}^-(U)$ is null on $[T, +\infty)$. That is

$$\int_{-\infty}^{+\infty} \mathcal{E}^-(U(t)) e^{ct} dt = \int_{-\infty}^T \mathcal{E}^-(U(t)) e^{ct} dt \leq -\frac{a}{c} e^{cT} < +\infty,$$

where a is the minimum value from (H3.1'). Therefore, the negative part of the energy density $\mathbf{e}_c(U)$ (see (3.4.18)) belongs to $L^1(\mathbb{R})$, which establishes the result. \square

Lemma 3.5.1 shows that for any $T \geq 1$ and $c > 0$, \mathbf{E}_c is well defined as an extended functional on X_T , at least if sufficient hypothesis are made. Moreover, it gives the following useful inequalities:

Lemma 3.5.2. *Assume that (H3.1') holds. Let $c > 0$ and $T \geq 1$. For any $U \in X_T$, we have that*

$$\int_{\mathbb{R}} \frac{\|U'(t)\|_{\mathcal{L}}^2}{2} e^{ct} dt \leq \mathbf{E}_c(U) - \frac{a}{c} e^{cT} \quad (3.5.3)$$

and

$$\int_{\mathbb{R}} |\mathcal{E}(U(t))| e^{ct} dt \leq \mathbf{E}_c(U) - \frac{a}{c} e^{cT}. \quad (3.5.4)$$

Finally, we have that for all $t \in \mathbb{R}$,

$$\int_t^{+\infty} \|U'(s)\|_{\mathcal{L}} ds \leq \left(\left(\mathbf{E}_c(U) - \frac{a}{c} e^{cT} \right) \frac{e^{-ct}}{c} \right)^{\frac{1}{2}} \quad (3.5.5)$$

Proof. Using (3.5.2) in Lemma 3.5.1, we get that

$$\int_{\mathbb{R}} \frac{\|U'(t)\|_{\mathcal{L}}^2}{2} e^{ct} dt \leq \mathbf{E}_c(U) - \int_{-\infty}^T \mathcal{E}(U(t)) e^{ct},$$

which, by (H3.1'), implies that (3.5.3) holds. Inequality (3.5.4) is obtained in the same fashion. Finally, we have that (3.5.5) follows by combining (3.5.3) with Cauchy-Schwartz inequality. \square

The previous results allow to prove the following convergence properties at $+\infty$ for finite energy functions in X_T :

Lemma 3.5.3. *Assume that (H3.1') and (H3.5') hold. Let $c > 0$ and $T \geq 1$. Take $U \in X_T$ such that $\mathbf{E}_c(U) < +\infty$. Then, we have that there exists a subsequence $(t_n)_{n \in \mathbb{N}}$ in \mathbb{R} such that $t_n \rightarrow +\infty$ as $n \rightarrow \infty$ and*

$$\lim_{n \rightarrow \infty} \mathcal{E}(U(t_n)) e^{ct_n} = 0. \quad (3.5.6)$$

Moreover, there exists $\mathbf{v}^+(U) \in \mathcal{F}^+$ such that for all $t \in \mathbb{R}$ it holds

$$\|U(t) - \mathbf{v}^+(U)\|_{\mathcal{L}}^2 \leq \left(\frac{\mathbf{E}_c(U) - \frac{a}{c} e^{cT}}{c} \right) e^{-ct}. \quad (3.5.7)$$

That is, U tends to $\mathbf{v}^+(U)$ at $+\infty$ with an exponential rate of convergence and with respect to the \mathcal{L} -norm.

Proof. We have by (3.5.4) in Lemma 3.5.2 that $t \in \mathbb{R} \rightarrow \mathcal{E}(U(t)) e^{ct} \in \mathbb{R}$ belongs to $L^1(\mathbb{R})$ because $\mathbf{E}_c(U) < +\infty$. Therefore, combining with (3.5.2) in Lemma 3.4.1, we obtain (3.5.6).

Subsequently, notice that (3.5.5) in Lemma 3.5.2 and the fact that $\mathbf{E}_c(U) < +\infty$ gives the existence of $\mathbf{v}^+(U) \in \mathcal{F}^+$ such that $\lim_{t \rightarrow +\infty} \|U(t) - \mathbf{v}^+(U)\|_{\mathcal{L}} = 0$. Therefore, fix $t \in \mathbb{R}$ and notice that for any $\tilde{t} > t$ we have

$$\|U(\tilde{t}) - U(t)\|_{\mathcal{L}} \leq \int_t^{\tilde{t}} \|U'(s)\|_{\mathcal{L}} ds \leq \int_t^{+\infty} \|U'(s)\|_{\mathcal{L}} ds,$$

which by (3.5.5) in Lemma 3.5.2 means that

$$\|U(\tilde{t}) - U(t)\|_{\mathcal{L}}^2 \leq \left(\frac{\mathbf{E}_c(U) - \frac{a}{c} e^{cT}}{c} \right) e^{-ct}.$$

Therefore, passing to the limit $\tilde{t} \rightarrow +\infty$ we obtain (3.5.7), also due to the fact that U is continuous with respect to the \mathcal{L} -norm. \square

Remark 3.5.1. Notice that (3.5.7) in Lemma 3.5.3 does not imply convergence of $\mathcal{E}(U)$ towards 0 at $+\infty$, due to the fact that \mathcal{E} is not continuous with respect to the \mathcal{L} norm.

Remark 3.5.2. Regarding the behavior at $-\infty$, notice that we can only say that if $U \in X_T$ is such that $\mathbf{E}_c(U) < +\infty$, then $\mathcal{E}(U)$ does not go to $+\infty$ faster than e^{ct} at the limit $t \rightarrow -\infty$. That is, almost nothing can be said for generic finite energy solutions regarding their behavior at $-\infty$.

3.5.3 The infima of E_c in X_T are well defined

Once we have defined the spaces X_T , we show that the corresponding infimum of E_c is well defined as a real number for all $c > 0$. Set

$$\mathbf{m}_{c,T} := \inf_{U \in X_T} E_c(U) \in [-\infty, +\infty). \quad (3.5.8)$$

We have the following:

Lemma 3.5.4. *Assume that (H3.1') and (H3.2') hold. Fix $\hat{\mathbf{v}}^\pm \in \mathcal{F}^\pm$. Let $c > 0$ and $T \geq 1$. For all $T \geq 1$ the function*

$$\Psi(t) := \begin{cases} \hat{\mathbf{v}}^- & \text{if } t \leq -1, \\ \frac{1-t}{2}\hat{\mathbf{v}}^- + \frac{t+1}{2}\hat{\mathbf{v}}^+ & \text{if } -1 \leq t \leq 1, \\ \hat{\mathbf{v}}^+ & \text{if } t \geq 1, \end{cases} \quad (3.5.9)$$

belongs to X_T . Moreover, for all $c > 0$

$$E_c(\Psi) < +\infty. \quad (3.5.10)$$

Furthermore, we have

$$-\infty < \mathbf{m}_{c,T} < +\infty. \quad (3.5.11)$$

Proof. It is clear that $\Psi \in X_T$. We now show that (3.5.10) holds. Notice first that

$$\int_{-\infty}^1 \mathbf{e}_c(\Psi) = \int_{-\infty}^1 a e^{ct} dt = \frac{a}{c} e^c,$$

where a is the minimum value from (H3.1'). Subsequently, we have

$$\int_1^{+\infty} \mathbf{e}_c(\Psi) = 0$$

and

$$\begin{aligned} \int_{-1}^1 \mathbf{e}_c(\Psi) &= \int_{-1}^1 \left[\frac{\|\hat{\mathbf{v}}^+ - \hat{\mathbf{v}}^-\|_{\mathcal{L}}^2}{8} + \mathcal{E}\left(\frac{1-t}{2}\hat{\mathbf{v}}^- + \frac{t+1}{2}\hat{\mathbf{v}}^+\right) \right] e^{ct} dt \\ &\leq \left[\frac{\|\hat{\mathbf{v}}^+ - \hat{\mathbf{v}}^-\|_{\mathcal{L}}^2}{4} + 2 \max_{t \in [-1,1]} \mathcal{E}\left(\frac{1-t}{2}\hat{\mathbf{v}}^- + \frac{t+1}{2}\hat{\mathbf{v}}^+\right) \right] \frac{e^c - e^{-c}}{c} \end{aligned}$$

and we have

$$\max_{t \in [-1,1]} \mathcal{E}\left(\frac{1-t}{2}\hat{\mathbf{v}}^- + \frac{t+1}{2}\hat{\mathbf{v}}^+\right) < +\infty,$$

by (H3.2'). Therefore, we have obtained $E_c(\Psi) < +\infty$, which readily implies that $\mathbf{m}_{c,T} < +\infty$. In order to establish (3.5.11), we still need to show that $\mathbf{m}_{c,T} > -\infty$. For that purpose, let $U \in X_T$. By (3.5.2) in Lemma 3.5.1, we have

$$\int_T^{+\infty} \mathbf{e}_c(U) \geq 0.$$

We also have

$$\int_{-\infty}^T \mathbf{e}_c(U) \geq \int_{-\infty}^T a e^{ct} dt = \frac{a}{c} e^{cT}.$$

That is

$$\forall U \in X_T, \mathbf{E}_c(U) \geq \frac{a}{c} e^{cT} > -\infty,$$

which means that $\mathbf{m}_{c,T} > -\infty$. □

The next goal will be to show that, under the proper assumptions, we have that for any $c > 0$ and $T \geq 1$, the infimum values defined in (3.5.8) are attained. Such a fact is not hard to prove since the constraints that define the spaces X_T allow to restore compactness. It relies on some properties that will be proven in the next section.

3.5.4 General continuity and semi-continuity results

We now provide some results which address continuity and semicontinuity properties of the energies \mathbf{E}_c in the spaces X_T . Such properties will allow us to show that the infimum values defined in (3.5.8) are attained under the proper assumptions. They will be also be useful in a more advanced stage of the proof, when the constraints will be removed. For now, we essentially adapt some results from [13] to our setting.

Our first result is essentially *Lemma 26* in [13]:

Lemma 3.5.5. *Assume that (H3.1') holds. Fix $T \geq 1$ and $U \in X_T$. Consider the set*

$$A_{T,U} := \{c > 0 : \mathbf{E}_c(U) < +\infty\}.$$

Then, if $c \in A_{T,U}$, then $(0, c] \subset A_{T,U}$. Moreover, the correspondence

$$c \in A_{T,U} \rightarrow \mathbf{E}_c(U) \in \mathbb{R}$$

is continuous.

Proof. Let $c \in A_{T,U}$. On the one hand, inequality (3.5.2) in Lemma 3.5.1 gives

$$0 \leq \int_T^{+\infty} \mathbf{e}_c(U(t)) dt \leq \mathbf{E}_c(U) - \int_{-\infty}^T \mathcal{E}^-(U(t)) e^{ct} dt \leq \mathbf{E}_c(U) - a e^{cT} < +\infty,$$

which implies that a. e. in $(T, +\infty)$

$$0 \leq \mathbf{e}_c(U(\cdot)) \in L^1((T, +\infty)).$$

Therefore, if $c' \leq c$ we have that a. e. in $(T, +\infty)$

$$0 \leq \mathbf{e}_{c'}(U(\cdot)) \leq \mathbf{e}_c(U(\cdot)) \in L^1(T, +\infty). \quad (3.5.12)$$

On the other hand,

$$\int_{-\infty}^T |\mathbf{e}_c(U(t))| dt \leq \mathbf{E}_c(U) + 2a e^{cT} < +\infty$$

because $\mathbf{e}_c(U(\cdot))$ is non-negative a. e. in $[T, +\infty)$. The previous inequality shows

$$|\mathbf{e}_c(U(\cdot))| \in L^1((-\infty, T)),$$

and we have that a. e. in $(-\infty, T)$

$$|\mathbf{e}_{c'}(U(\cdot))| \leq |\mathbf{e}_c(U(\cdot))| \in L^1((-\infty, T)). \quad (3.5.13)$$

Combining (3.5.12) and (3.5.13), we obtain that $|\mathbf{e}_c(U(\cdot))| \in L^1(\mathbb{R})$, meaning that $c' \in A_{T,U}$. Hence, we have $(0, c] \subset A_{T,U}$ as we wanted to show.

Consider now a sequence $(c_n)_{n \in \mathbb{N}}$ in $A_{T,U}$ such that $c_n \rightarrow c_\infty \in A_{T,U}$. The sequence $(c_n)_{n \in \mathbb{N}}$ is convergent (so in particular it is bounded), meaning that in case it does not attain its sup we must have $c_\infty = \sup_{n \in \mathbb{N}} c_n$. Therefore, we can set

$$\hat{c} := \begin{cases} c_\infty & \text{if } c_\infty = \sup_{n \in \mathbb{N}} c_n, \\ \max_{n \in \mathbb{N}} c_n & \text{otherwise,} \end{cases}$$

and we obviously have $\hat{c} \in A_{T,U}$. As a consequence, (3.5.12) and (3.5.13) imply that a. e. in \mathbb{R}

$$\forall n \in \mathbb{N}, \quad |\mathbf{e}_{c_n}(U(\cdot))| \leq |\mathbf{e}_{\hat{c}}(U(\cdot))| \in L^1(\mathbb{R}).$$

Since we also have $\mathbf{e}_{c_n}(U(\cdot)) \rightarrow \mathbf{e}_{c_\infty}(U(\cdot))$ pointwise a. e. in \mathbb{R} , the Dominated Convergence Theorem gives the result. \square

We now show a lower semicontinuity result, which in particular will imply the existence of the constrained solutions:

Lemma 3.5.6. *Assume that (H3.1'), (H3.3') and (H3.4') hold. Let $T \geq 1$ be fixed. Let $(U_n^i)_{n \in \mathbb{N}}$ be a sequence in X_T and $(c_n)_{n \in \mathbb{N}}$ a convergent sequence of positive real numbers such that*

$$\sup_{n \in \mathbb{N}} \mathbf{E}_{c_n}(U_n^i) < +\infty. \quad (3.5.14)$$

Then, there exists a sequence $(U_n)_{n \in \mathbb{N}}$ in X_T and $U_\infty \in X_T$ such that up to extracting a subsequence in $(U_n, c_n)_{n \in \mathbb{N}}$ it holds

$$\forall n \in \mathbb{N}, \quad \mathbf{E}_{c_n}(U_n) = \mathbf{E}_{c_n}(U_n^i), \quad (3.5.15)$$

$$\forall t \in \mathbb{R}, \quad U_n(t) \rightharpoonup U_\infty(t) \text{ weakly in } \mathcal{L} \quad (3.5.16)$$

and

$$U_n' h_{c_n} \rightharpoonup U_\infty' h_{c_\infty} \text{ weakly in } L^2(\mathbb{R}, \mathcal{L}) \quad (3.5.17)$$

where, for $k \in \mathbb{R}$, $h_k : t \in \mathbb{R} \rightarrow e^{kt/2} \in \mathbb{R}$ and $c_\infty := \lim_{n \rightarrow \infty} c_n$. Moreover,

$$\mathbf{E}_{c_\infty}(U_\infty) \leq \liminf_{n \rightarrow \infty} \mathbf{E}_{c_n}(U_n). \quad (3.5.18)$$

Proof. Denote $M := \sup_{n \in \mathbb{N}} \mathbf{E}_{c_n}(U_n^i)$, which is finite by (3.5.14). We will now use (H3.4'). We assume that 2. holds, the argument when 1. holds being similar and easier. Fix any $\mathbf{v}^+ \in \mathcal{F}^+$ and for all $n \in \mathbb{N}$, set $v_n := U_n^i(T) \in \mathcal{F}^+$. Define

$$U_n : t \in \mathbb{R} \rightarrow \hat{P}_{(v_n, \mathbf{v}^+)}(U_n^i(t)),$$

where $\hat{P}_{\mathbf{v}^+}$ is the differentiable operator introduced in (H3.4'). We apply the properties summarized

in 2. of (H3.4'). Notice that for all $n \in \mathbb{N}$ we have $U_n \in X_T$ due to (3.4.12). The energy equality (3.5.15) follows from (3.4.13) and (3.4.14). Moreover, (3.4.11) implies that for all $n \in \mathbb{N}$

$$P^+(U_n(T)) = P^+(\hat{P}_{(v_n, \mathbf{v}^+)}(U_n^i(T))) = P^+(\hat{P}_{(v_n, \mathbf{v}^+)}(v_n)) = \mathbf{v}^+,$$

which in particular means

$$\|U_n(T) - \mathbf{v}^+\|_{\mathcal{L}} \leq \frac{r_0^+}{2}. \quad (3.5.19)$$

Notice now that $\mathcal{E}(U(\cdot))$ is non-negative in $[T, +\infty)$ as $U \in X_T$ by (3.5.2) in Lemma 3.5.1, therefore

$$\begin{aligned} \forall n \in \mathbb{N}, \quad \frac{1}{2} \int_{\mathbb{R}} \|U_n'(t)\|_{\mathcal{L}}^2 e^{c_n t} dt &\leq M - \int_{\mathbb{R}} \mathcal{E}(U_n(t)) e^{c_n t} dt \\ &\leq M - \int_{-\infty}^T \mathcal{E}(U_n(t)) e^{c_n t} dt \leq \sup_{n \in \mathbb{N}} \left\{ M - \frac{a}{c_n} e^{c_n T} \right\} < +\infty. \end{aligned} \quad (3.5.20)$$

That is, we have that $(U_n' h_{c_n})_{n \in \mathbb{N}}$ is uniformly bounded in $L^2(\mathbb{R}, \mathcal{L})$. Therefore, there exists $\tilde{U} \in L^2(\mathbb{R}, \mathcal{L})$ such that

$$U_n' h_{c_n} \rightharpoonup \tilde{U} \text{ weakly in } L^2(\mathbb{R}, \mathcal{L}) \quad (3.5.21)$$

up to subsequences. Such a thing implies

$$\int_{\mathbb{R}} \|\tilde{U}(t)\|_{\mathcal{L}}^2 dt \leq \liminf_{n \rightarrow \infty} \int_{\mathbb{R}} \|U_n'(t)\|_{\mathcal{L}}^2 e^{c_n t} dt. \quad (3.5.22)$$

Now, notice that by (3.5.19) we have that $(U_n(T))_{n \in \mathbb{N}}$ is bounded in \mathcal{L} . Therefore, up to an extraction there exists $v_\infty \in \mathcal{L}$ such that

$$U_n(T) \rightharpoonup v_\infty \text{ in } \mathcal{L}. \quad (3.5.23)$$

As in [157], we point out that

$$\forall t \in \mathbb{R}, \forall n \in \mathbb{N}, \quad U_n(t) = U_n(T) + \int_T^t U_n'(s) ds.$$

Now, notice that for all $t \in \mathbb{R}$ we have $\mathbf{1}_{(0,t)} h_{-c_n} \rightarrow \mathbf{1}_{(0,t)} h_{-c_\infty}$ in $L^\infty(\mathbb{R})$, where $\mathbf{1}$ states for the indicator function of a set. Therefore, we obtain by (3.5.21) and (3.5.23)

$$\forall t \in \mathbb{R}, \quad U_n(t) \rightharpoonup U_\infty(t) := v_\infty + \int_T^t \tilde{U}(s) e^{-c_\infty s/2} ds,$$

which gives (3.5.16). Moreover, we have that $U_\infty \in H_{\text{loc}}^1(\mathbb{R}, \mathcal{L})$ and $U_\infty' = \tilde{U} h_{-c_\infty}$, meaning by (3.5.21) that (3.5.17) also holds.

Recall now that \mathcal{E} is lower semicontinuous on \mathcal{L} by (H3.1'), so that (3.5.16) gives

$$\forall t \in \mathbb{R}, \quad \mathcal{E}(U_\infty(t)) \leq \liminf_{n \rightarrow \infty} \mathcal{E}(U_n(t)). \quad (3.5.24)$$

We need to show that $U_\infty \in X_T$ and to establish the inequality (3.5.18).

- We begin by showing that $U_\infty \in X_T$. We need to show that for all $t \in [T, +\infty)$, it holds $U_\infty(t) \in \mathcal{F}_{r_0^+/2}^+$ and similarly for $(-\infty, -T]$. Fix $t \in [T, +\infty)$. We have that $U_n(t) \in \mathcal{F}_{r_0^+/2}^+$, so we

can define the sequence $(\mathbf{v}_n^+(t))_{n \in \mathbb{N}}$ in \mathcal{F}^+ as $\mathbf{v}_n^+(t) := P^+(U_n(t))$. We show that such a sequence is bounded. Indeed, we have

$$\forall n \in \mathbb{N}, \quad \|\mathbf{v}_n^+(t)\|_{\mathcal{L}} \leq \frac{r_0^+}{2} + \|U_n(t)\|_{\mathcal{L}}$$

and $(U_n(t))_{\mathcal{L}}$ converges weakly in \mathcal{L} , so in particular it is bounded. Therefore, up to an extraction we can assume that $\mathbf{v}_n^+(t) \rightharpoonup \mathbf{v}_\infty^+(t) \in \mathcal{L}$ and by (H3.3') we have $\mathbf{v}_\infty^+(t) \in \mathcal{F}^+$. Using now the convergence properties we get the inequality

$$\|U_\infty(t) - \mathbf{v}_\infty^+(t)\|_{\mathcal{L}} \leq \liminf_{n \rightarrow \infty} \|U_n(t) - \mathbf{v}_n^+(t)\|_{\mathcal{L}} \leq \frac{r_0^+}{2},$$

so that $U_n(t) \in \mathcal{F}_{r_0^+/2}^+$. An identical argument shows that for all $t \in (-\infty, -T]$ we have $U_n(t) \in \mathcal{F}_{r_0^+/2}^-$. Therefore, we have shown that $U_\infty \in X_T$.

- Next, we prove (3.5.18). We have

$$\sup_{n \in \mathbb{N}} \int_{\mathbb{R}} \mathcal{E}(U_n(t)) e^{c_n t} dt \leq M - \sup_{n \in \mathbb{N}} \int_{\mathbb{R}} \frac{\|U_n'(t)\|_{\mathcal{L}}^2}{2} e^{c_n t} dt < +\infty,$$

by (3.5.20). Hence, we can apply Fatou's Lemma to $(t \in \mathbb{R} \rightarrow \mathcal{E}(U_n(t)) e^{c_n t})_{n \in \mathbb{N}}$ (a sequence of functions uniformly bounded below by a) to show

$$\int_{\mathbb{R}} \liminf_{n \rightarrow +\infty} \mathcal{E}(U_n(t)) e^{c_n t} dt \leq \liminf_{n \rightarrow \infty} \int_{\mathbb{R}} \mathcal{E}(U_n(t)) e^{c_n t} dt,$$

which, combined with (3.5.24) implies

$$\int_{\mathbb{R}} \mathcal{E}(U_\infty(t)) e^{c t} dt \leq \liminf_{n \rightarrow \infty} \int_{\mathbb{R}} \mathcal{E}(U_n(t)) e^{c_n t} dt. \quad (3.5.25)$$

Combining (3.5.22) and (3.5.25) we get

$$\mathbf{E}_c(U_\infty) \leq \liminf_{n \rightarrow \infty} \int_{\mathbb{R}} \frac{\|U_n'(t)\|_{\mathcal{L}}^2}{2} e^{c_n t} dt + \liminf_{n \rightarrow \infty} \int_{\mathbb{R}} \mathcal{E}(U_n(t)) e^{c_n t} dt,$$

which, by superadditivity of the limit inferior gives (3.5.18).

□

3.5.5 Existence of an infimum for \mathbf{E}_c in X_T

The goal now is to show that, for $T \geq 1$ and $c > 0$ fixed, the infimum $\mathbf{m}_{c,T}$ as defined in (3.5.8) is attained by a function in X_T . This will actually follow easily from Lemma 3.5.6.

Lemma 3.5.7. *Assume that (H3.1'), (H3.2'), (H3.3') and (H3.4') hold. Let $c > 0$, $T \geq 1$ and $\mathbf{m}_{c,T}$ be as in (3.5.8). Then, $\mathbf{m}_{c,T}$ is attained for some $\mathbf{U}_{c,T} \in X_T$.*

Proof. By (3.5.11) in Lemma 3.5.4, we have that there exists a minimizing sequence $(U_n)_{n \in \mathbb{N}}$ for \mathbf{E}_c in X_T . We apply Lemma 3.5.6 to $(U_n)_{n \in \mathbb{N}}$ and the sequence of speeds constantly equal to c . We

obtain a function $\mathbf{U}_{c,T} \in X_T$ such that

$$\mathbf{E}_c(\mathbf{U}_{c,T}) \leq \liminf_{n \rightarrow \infty} \mathbf{E}_c(U_n) = \mathbf{m}_{c,T},$$

due to (3.5.18). Therefore, $\mathbf{m}_{c,T}$ is attained by $\mathbf{U}_{c,T}$ in X_T . \square

Subsequently, we show that assumption (H3.6') implies that the constrained minimizers are solutions of the equation in a certain set containing $(-T, T)$, with the proper regularity.

Lemma 3.5.8. *Assume that (H3.6') holds. Let $c > 0$, $T \geq 1$ and $\mathbf{m}_{c,T}$ be as in (3.5.8). Let $\mathbf{U}_{c,T} \in X_T$ be such that $\mathbf{E}_c(\mathbf{U}_{c,T}) = \mathbf{m}_{c,T}$. Then, $\mathbf{U}_{c,T} \in \mathcal{A}((-T, T))$, $\mathcal{A}((-T, T))$ as in (3.4.23) and*

$$\mathbf{U}_{c,T}'' - D_{\mathcal{L}}\mathcal{E}(\mathbf{U}_{c,T}) = -c\mathbf{U}_{c,T}' \text{ in } (-T, T). \quad (3.5.26)$$

Moreover, if $t \geq T$ is such that

$$\text{dist}_{\mathcal{L}}(\mathbf{U}_{c,T}(t), \mathcal{F}^+) < \frac{r_0^+}{2}, \quad (3.5.27)$$

then, there exists $\delta^+(t) > 0$ such that $\mathbf{U}_{c,T} \in \mathcal{A}((t - \delta^+(t), t + \delta^+(t)))$ and

$$\mathbf{U}_{c,T}'' - D_{\mathcal{L}}\mathcal{E}(\mathbf{U}_{c,T}) = -c\mathbf{U}_{c,T}' \text{ in } (t - \delta^+(t), t + \delta^+(t)). \quad (3.5.28)$$

Similarly, if $t \leq -T$ is such that

$$\text{dist}_{\mathcal{L}}(\mathbf{U}_{c,T}(t), \mathcal{F}^-) < \frac{r_0^-}{2}, \quad (3.5.29)$$

then, there exists $\delta^-(t) > 0$ such that $\mathbf{U}_{c,T} \in \mathcal{A}((t - \delta^-(t), t + \delta^-(t)))$ and

$$\mathbf{U}_{c,T}'' - D_{\mathcal{L}}\mathcal{E}(\mathbf{U}_{c,T}) = -c\mathbf{U}_{c,T}' \text{ in } (t - \delta^-(t), t + \delta^-(t)).$$

Proof. We first show that

$$\forall t \in \mathbb{R}, \quad \mathfrak{P}(\mathbf{U}_{c,T}(t)) = \mathbf{U}_{c,T}(t), \quad (3.5.30)$$

where \mathfrak{P} is the map from (H3.6'). We claim that the function

$$\mathbf{U}_{c,T}^{\mathfrak{P}} : t \in \mathbb{R} \rightarrow \mathfrak{P}(\mathbf{U}_{c,T}(t))$$

belongs to X_T . Indeed, this follows from (3.4.21) and (3.4.22). Property (3.4.20) implies that

$$\forall t \in \mathbb{R}, \quad \mathcal{E}(\mathbf{U}_{c,T}^{\mathfrak{P}}(t)) \leq \mathcal{E}(\mathbf{U}_{c,T}(t)). \quad (3.5.31)$$

Take now $t \in \mathbb{R}$ and $s \in \mathbb{R} \setminus \{t\}$. Property (3.4.21) implies that

$$\left\| \frac{\mathbf{U}_{c,T}^{\mathfrak{P}}(t) - \mathbf{U}_{c,T}^{\mathfrak{P}}(s)}{t - s} \right\|_{\mathcal{L}} \leq \left\| \frac{\mathbf{U}_{c,T}(t) - \mathbf{U}_{c,T}(s)}{t - s} \right\|_{\mathcal{L}},$$

which, by Lebesgue's differentiation Theorem implies that

$$\text{for a. e. } t \in \mathbb{R}, \quad \|(\mathbf{U}_{c,T}^{\mathfrak{P}})'(t)\|_{\mathcal{L}} \leq \|\mathbf{U}_{c,T}'(t)\|_{\mathcal{L}}. \quad (3.5.32)$$

By contradiction, assume now that there exists $t \in \mathbb{R}$ such that $\mathbf{U}_{c,T}(t) \neq \mathfrak{P}(\mathbf{U}_{c,T}(t)) = \mathbf{U}_{c,T}^{\mathfrak{P}}(t)$. Prop-

erty (3.4.21) implies that \mathfrak{P} is a \mathcal{L} -continuous map. Therefore, since $\mathbf{U}_{c,T}$ is also \mathcal{L} -continuous, we must have that for some non-empty interval $I_t \ni t$, it holds

$$\forall s \in I_t, \quad \mathbf{U}_{c,T}(s) \neq \mathfrak{P}(\mathbf{U}_{c,T}(s)) = \mathbf{U}_{c,T}^{\mathfrak{P}}(s),$$

so that, using (3.4.20) we get

$$\forall s \in I_t, \quad \mathcal{E}(\mathbf{U}_{c,T}^{\mathfrak{P}}(s)) < \mathcal{E}(\mathbf{U}_{c,T}(s))$$

so that, combining with (3.5.31) and (3.5.32) we obtain

$$\mathbf{E}_c(\mathbf{U}_{c,T}^{\mathfrak{P}}) < \mathbf{E}_c(\mathbf{U}_{c,T}) = \mathbf{m}_{c,T},$$

which contradicts the definition of $\mathbf{m}_{c,T}$ (3.5.8) since $\mathbf{U}_{c,T}^{\mathfrak{P}} \in X_T$. Therefore, we have shown that (3.5.30) holds. Next, notice that

$$\mathbf{E}_c(\mathbf{U}_{c,T}; [-T, T]) \leq \mathbf{m}_{c,T} - \frac{a}{c} e^{-cT} < +\infty$$

and for any $\phi \in \mathcal{C}_c^1((-T, T), (\mathcal{H}, \|\cdot\|_{\mathcal{H}}))$ we have $\mathbf{U}_{c,T} + \phi \in X_T$, so that $\mathbf{E}_c(\mathbf{U}_{c,T}) \leq \mathbf{E}_c(\mathbf{U}_{c,T} + \phi)$. Therefore, the restriction of $\mathbf{U}_{c,T}$ in $(-T, T)$ is a local minimizer of $\mathbf{E}_c(\cdot, [-T, T])$ in the sense of Definition 3.4.1. Since $\mathbf{U}_{c,T}$ also verifies (3.5.30), we can apply the regularity assumption (H3.6'). Therefore, $\mathbf{U}_{c,T} \in \mathcal{A}((-T, T))$ and (3.5.26) holds. Assume now that there exists $t \geq T$ such that (3.5.27) holds. Then, there exists $\mathbf{v}^+(t) \in \mathcal{F}^+$ such that

$$\|\mathbf{U}_{c,T}(t) - \mathbf{v}^+(t)\|_{\mathcal{L}} < \frac{r_0^+}{2}$$

which, since $\mathbf{U}_{c,T}$ is \mathcal{L} -continuous, implies that there exists $\delta^+(t) > 0$ such that

$$\forall s \in (t - \delta^+(t), t + \delta^+(t)), \quad \|\mathbf{U}_{c,T}(s) - \mathbf{v}^+(t)\|_{\mathcal{L}} < \frac{r_0^+}{2} - d^+(t)$$

where

$$d^+(t) := \frac{1}{2} \left(\frac{r_0^+}{2} - \|\mathbf{U}_{c,T}(t) - \mathbf{v}^+(t)\|_{\mathcal{L}} \right) > 0.$$

Therefore, if $\phi \in \mathcal{C}_c^1((t - \delta^+(t), t + \delta^+(t)), (\mathcal{H}, \|\cdot\|_{\mathcal{H}}))$ is such that

$$\max_{t \in [t - \delta^+(t), t + \delta^+(t)]} \|\phi(t)\|_{\mathcal{H}} \leq \frac{d^+(t)}{2}$$

we have that

$$\forall s \in (t - \delta^+(t), t + \delta^+(t)), \quad \|\mathbf{U}_{c,T}(s) + \phi(s) - \mathbf{v}^+(t)\|_{\mathcal{L}} < \frac{r_0^+}{2} - \frac{d^+(t)}{2}$$

so that $\mathbf{U}_{c,T} + \phi \in X_T$. Meaning that $\mathbf{E}_{c,T}(\mathbf{U}_{c,T}) \leq \mathbf{E}_{c,T}(\mathbf{U}_{c,T} + \phi)$. Since ϕ is supported on $[t - \delta^+(t), t + \delta^+(t)]$, the previous implies that

$$\mathbf{E}_{c,T}(\mathbf{U}_{c,T}; [t - \delta^+(t), t + \delta^+(t)]) \leq \mathbf{E}_{c,T}(\mathbf{U}_{c,T} + \phi; [t - \delta^+(t), t + \delta^+(t)]),$$

so that $\mathbf{U}_{c,T}$ is a local minimizer of $\mathbf{E}_c(\cdot; [t - \delta^+(t), t + \delta^+(t)])$ in the sense of Definition 3.4.1. Since (3.5.30) also holds, we can apply (H3.6') and obtain that $\mathbf{U}_{c,T} \in \mathcal{A}((t - \delta^+(t), t + \delta^+(t)))$ and equation

(3.5.28) holds. If $t \leq -T$ is such that (3.5.29) holds, the same reasoning shows that for some $\delta^-(t) > 0$, $\mathbf{U}_{c,T} \in \mathcal{A}((t - \delta^-(t), t + \delta^-(t)))$ and (3.5.27) holds, which concludes the proof of the result. \square

3.5.6 The comparison result

The goal of this section is to obtain relevant information on the behavior of the constrained minimizers. Such information is contained in Corollary 3.5.1 and it will allow us to remove the constraints later on. In order to carry on these arguments, assumption (H3.7') will become necessary since it will show that our problem can be somehow dealt with as in the balanced one, which will allow us to argue in a fashion similar to Smyrnelis [157]. We begin by introducing some constants. For $0 < r \leq r_0^\pm$, recall the definition of κ_r^\pm introduced in (3.4.7), Lemma 3.4.1. We define

$$\eta_0^+ := \min \left\{ \sqrt{e^{-1} \frac{r_0^+}{4} \sqrt{2\kappa_{r_0^+/4}^+}}, \frac{r_0^+}{4} \right\} > 0, \quad (3.5.33)$$

$$\hat{r}^+ := \frac{r_0^+}{C^+ + 1} > 0, \quad (3.5.34)$$

$$\mathcal{E}_{\max}^+ := \frac{1}{(C^+)^2 (C^+ + 1)} \min \left\{ \frac{(\eta_0^+)^2}{4}, \kappa_{\eta_0^+}^+, \beta^+(\hat{r}^+), \beta^+(\eta_0^+) \right\} > 0, \quad (3.5.35)$$

where the constants C^\pm , $\beta^\pm(\hat{r}^\pm)$, $\beta(\eta_0^\pm)$ were introduced in (H3.5'). Recall that in (3.4.24), (3.4.25), (3.4.26) we introduced the analogous constants

$$\eta_0^- := \min \left\{ \sqrt{e^{-1} \frac{r_0^-}{4} \sqrt{2(\kappa_{r_0^-/4}^- - a)}}, \frac{r_0^-}{4} \right\} > 0, \quad (3.5.36)$$

$$\hat{r}^- := \frac{r_0^-}{C^- + 1} > 0 \quad (3.5.37)$$

and

$$\mathcal{E}_{\max}^- := \frac{1}{(C^-)^2 (C^- + 1)} \min \left\{ \frac{(\eta_0^-)^2}{4}, \kappa_{\eta_0^-}^- - a, \beta^-(\hat{r}^-), \beta^-(\eta_0^-) \right\} > 0. \quad (3.5.38)$$

For any $U \in X_T$, define

$$t^-(U, \mathcal{E}_{\max}^-) := \sup \left\{ t \in \mathbb{R} : \mathcal{E}(U(t)) \leq a + \mathcal{E}_{\max}^- \text{ and } \text{dist}_{\mathcal{L}}(U(t), \mathcal{F}^-) \leq \frac{r_0^-}{2} \right\} \quad (3.5.39)$$

and

$$t^+(U, \mathcal{E}_{\max}^+) := \inf \left\{ t \in \mathbb{R} : \mathcal{E}(U(t)) \leq \mathcal{E}_{\max}^+ \text{ and } \text{dist}_{\mathcal{L}}(U(t), \mathcal{F}^+) \leq \frac{r_0^+}{2} \right\}. \quad (3.5.40)$$

We have the following technical property:

Lemma 3.5.9. *Assume that (H3.1') and (H3.5') hold. Let $\hat{r}^\pm > 0$ be as in (3.5.34), (3.5.37) and \mathcal{E}_{\max}^\pm be as in (3.5.35), (3.5.38). Then, if $v \in \mathcal{F}_{\hat{r}^\pm}^\pm$ is such that*

$$\mathcal{E}(v) \leq \min\{\pm(-a), 0\} + \beta^\pm(\hat{r}^\pm), \quad (3.5.41)$$

then

$$\forall \lambda \in [0, 1], \quad \mathcal{E}(\lambda v + (1 - \lambda)P^\pm(v)) \leq \min\{\pm(-a), 0\} + C^\pm(\mathcal{E}(v) - \min\{\pm(-a), 0\}). \quad (3.5.42)$$

In particular, if

$$\mathcal{E}(u) \leq \min\{\pm(-a), 0\} + \mathcal{E}_{\max}^\pm, \quad (3.5.43)$$

then

$$\forall \lambda \in [0, 1], \quad \mathcal{E}(\lambda v + (1 - \lambda)P^\pm(v)) \leq \min\{\pm(-a), 0\} + C^\pm \mathcal{E}_{\max}^\pm. \quad (3.5.44)$$

The constants \mathcal{E}_{\max}^\pm , C^\pm where defined in (3.5.38), (3.5.35) and (3.4.27) respectively. $P^\pm(u)$ is the projection introduced in (H3.1').

Proof. Assume that (3.5.41) holds for $v \in \mathcal{F}_{r_0^\pm}^\pm$. Then, invoking (H3.5') we have that $v \in \mathcal{F}_{\hat{r}^\pm}$, so in particular the projection $P_{\mathcal{H}}^\pm(u)$ is well defined. Fix $\lambda \in [0, 1]$. Since $v \in \mathcal{F}_{r_0^\pm}^\pm$, the projection $P^\pm(v)$ is well defined by (H3.1'). Using (3.4.15) we obtain

$$\begin{aligned} \|\lambda v + (1 - \lambda)P^\pm(v) - P^\pm(v)\|_{\mathcal{H}} &= \lambda \|v - P^\pm(v)\|_{\mathcal{H}} \\ &\leq (C^\pm + 1) \|v - P_{\mathcal{H}}^\pm(v)\|_{\mathcal{H}} \leq (C^\pm + 1) \hat{r}^\pm \end{aligned} \quad (3.5.45)$$

so that $\lambda v + (1 - \lambda)P^\pm(v) \in \mathcal{F}_{\mathcal{H}, r_0^\pm}^\pm$ by the definition of \hat{r}^\pm in (3.5.34), (3.5.37). Using now again (3.4.15) along with the estimate (3.4.17) in (H3.5'), we get

$$\|P^\pm(v) - P_{\mathcal{H}}^\pm(v)\|_{\mathcal{H}}^2 \leq (C^\pm)^2 (\mathcal{E}(v) - \min\{\pm(-a), 0\})$$

which, plugging into (3.5.45), gives

$$\|\lambda v + (1 - \lambda)P^\pm(v) - P_{\mathcal{H}}^\pm(v)\|_{\mathcal{H}}^2 \leq \frac{1}{2} ((C^\pm)^2 + (C^\pm + 1)^2) (\mathcal{E}(v) - \min\{\pm(-a), 0\}),$$

that, using again (3.4.17), implies exactly (3.5.42). Assuming now that (3.5.43) holds, we have by (3.5.38), (3.5.35) that in particular (3.5.41) holds. Therefore, (3.5.44) follows from (3.5.42). \square

Next, we have the following property:

Lemma 3.5.10. *Assume that (H3.5') and (H3.7') hold. Let $c > 0$ and $T \geq 1$. Assume that $U \in X_T$ is such that $\mathbf{E}_c(U) \leq 0$. Then the quantities $t^-(U, \mathcal{E}_{\max}^-)$ and $t^+(U, \mathcal{E}_{\max}^+)$ defined in (3.5.39) and (3.5.40), respectively, are well defined as real numbers. Moreover, it holds that*

$$\mathcal{E}(U(t^-(U, \mathcal{E}_{\max}^-))) \leq a + \mathcal{E}_{\max}^-, \quad \text{dist}_{\mathcal{L}}(U(t^-(U, \mathcal{E}_{\max}^-)), \mathcal{F}^-) \leq \frac{r_0^-}{2} \quad (3.5.46)$$

and

$$\mathcal{E}(U(t^+(U, \mathcal{E}_{\max}^+))) \leq \mathcal{E}_{\max}^+, \quad \text{dist}_{\mathcal{L}}(U(t^+(U, \mathcal{E}_{\max}^+)), \mathcal{F}^+) \leq \frac{r_0^+}{2}. \quad (3.5.47)$$

Proof. Using that $\mathbf{E}_c(U) \leq 0$ and the fact that $\{t \in \mathbb{R} : \mathcal{E}(U(t)) > 0\}$ is nonempty since $U \in X_T$, we must have that

$$\{t \in \mathbb{R} : \mathcal{E}(U(t)) < 0\} \neq \emptyset$$

and if $v \in \mathcal{L}$ is such that $\mathcal{E}(v) < 0$, then we have $\text{dist}_{\mathcal{L}}(v, \mathcal{F}^-) \leq r_0^-/2$ by (3.4.30) in (H3.7') and $\mathcal{E}(v) < a + \mathcal{E}_{\max}^-$ since we assume $a + \mathcal{E}_{\max}^- > 0$. Therefore, $t^-(U, \mathcal{E}_{\max}^-)$ is well defined, as we have

shown that

$$\left\{ t \in \mathbb{R} : \mathcal{E}(U(t)) \leq a + \mathcal{E}_{\max}^- \text{ and } \text{dist}_{\mathcal{L}}(U(t), \mathcal{F}^-) \leq \frac{r_0^-}{2} \right\} \neq \emptyset$$

and such set is bounded above by T , because $\mathcal{F}_{r_0^-/2}^- \cap \mathcal{F}_{r_0^+/2}^+ = \emptyset$. Using Lemma 3.5.3, we have that $t^+(U, \mathcal{E}_{\max}^+)$ is well defined. Finally, inequalities (3.5.46) and (3.5.47) follow because $t \in \mathbb{R} \rightarrow \mathcal{E}(U(t)) \in \mathbb{R}$ is lower semicontinuous by (H3.1') and $t \rightarrow \text{dist}_{\mathcal{L}}(U(t), \mathcal{F}^\pm)$ is continuous whenever $U(t) \in \mathcal{F}_{r_0^\pm/2}^\pm$ by (H3.1') (recall that $t \in \mathbb{R} \rightarrow U(t) \in \mathcal{L}$ is continuous because $U \in H_{\text{loc}}^1(\mathbb{R}, \mathcal{L})$). \square

The main work is done by the following result:

Proposition 3.5.1. *Assume that (H3.5') and (H3.7') hold. Let $c > 0$ and $T \geq 1$. Consider $U \in X_T$ with $\mathbf{E}_c(U) \leq 0$. Let $t^\pm := t^\pm(U, \mathcal{E}_{\max}^\pm, \eta_0^\pm)$ be as in (3.5.39) and (3.5.40). Then, t^\pm are well defined by Lemma 3.5.10. Moreover, if there exists $\tilde{t}^- < t^-$ such that*

$$r_0^- \geq \text{dist}_{\mathcal{L}}(U(\tilde{t}^-), \mathcal{F}^-) \geq \frac{r_0^-}{2} \quad (3.5.48)$$

then, we can find $\tilde{U}^- \in X_T$ such that

$$\forall t \leq t^-, \quad \text{dist}_{\mathcal{L}}(\tilde{U}^-(t), \mathcal{F}^-) < \frac{r_0^-}{2}, \quad (3.5.49)$$

and

$$\mathbf{E}_c(\tilde{U}^-) < \mathbf{E}_c(U). \quad (3.5.50)$$

Analogously, if there exists $\tilde{t}^+ > t^+$ such that

$$r_0^+ \geq \text{dist}_{\mathcal{L}}(U(\tilde{t}^+), \mathcal{F}^+) \geq \frac{r_0^+}{2}, \quad (3.5.51)$$

then we can find $\tilde{U}^+ \in X_T$ such that

$$\forall t \geq t^+, \quad \text{dist}_{\mathcal{L}}(\tilde{U}^+(t), \mathcal{F}^+) < \frac{r_0^+}{2} \quad (3.5.52)$$

and

$$\mathbf{E}_c(\tilde{U}^+) < \mathbf{E}_c(U). \quad (3.5.53)$$

Furthermore, we have that

$$0 < t^+ - t^- \leq T_\star(c), \quad (3.5.54)$$

where

$$T_\star(c) := \frac{1}{c} \ln \left(\frac{-a}{\alpha_\star} + 1 \right), \quad (3.5.55)$$

with $\alpha_\star > 0$ a constant which is independent from c , T and U .

The idea of the proof of Proposition 3.5.1 is pictured in Figure 3.5.

Proof of Proposition 3.5.1. We begin by proving the first part of the result for \mathcal{F}^- . Recall that Lemma 3.5.10 gives

$$\mathcal{E}(U(t^-)) \leq a + \mathcal{E}_{\max}^-. \quad (3.5.56)$$

and $U(t^-) \in \mathcal{F}_{r_0^-/2}^-$. Since $a + \mathcal{E}_{\max}^- \leq \beta(\mathcal{E}_{\max}^-)$ by the definition of \mathcal{E}_{\max}^- , (3.5.38), we have by (H3.5') that

$$\text{dist}_{\mathcal{L}}(U(t^-), \mathcal{F}^-) \leq \eta_0^-. \quad (3.5.57)$$

Assume that there exists $\tilde{t}^- < t^-$ such that (3.5.48) is satisfied. Moreover, we assume, as we can, that

$$\tilde{t}^- := \max \left\{ t \leq t^- : \text{dist}_{\mathcal{L}}(U(t), \mathcal{F}^-) \geq \frac{r_0^-}{2} \right\} \quad (3.5.58)$$

(the sup can be replaced by a max by continuity). Define

$$t_0^- := \inf \{ t \in [\tilde{t}^-, t^-] : \mathcal{E}(U(t)) \leq a + \mathcal{E}_{\max}^- \text{ and } \text{dist}_{\mathcal{L}}(U(t)) \leq \eta_0^- \}. \quad (3.5.59)$$

Let $\mathbf{v}^- := P^-(U(t_0^-)) \in \mathcal{F}^-$, with P^- as in (H3.1'). Notice that due to (3.5.58), we have

$$\forall t \in [t_0^-, t^-], \quad \text{dist}(U(t), \mathcal{F}^-) < \frac{r_0^-}{2}. \quad (3.5.60)$$

The proof now bifurcates according to two possible cases:

- $t_0^- \leq \tilde{t}^- + 1$. In that case, set

$$\tilde{U}^-(t) := \begin{cases} \mathbf{v}^- & \text{if } t \leq t_0^- - 1, \\ (t_0^- - t)\mathbf{v}^- + (t - t_0^- + 1)U(t_0^-) & \text{if } t_0^- - 1 \leq t \leq t_0^-, \\ U(t) & \text{if } t \geq t_0^-, \end{cases}$$

which belongs to X_T . Due to the definition of \tilde{U}^- and (3.5.60), we have that \tilde{U}^- satisfies (3.5.49). It remains to check (3.5.50). We have

$$\int_{t_0^- - 1}^{t_0^-} \mathbf{e}_c(\tilde{U}^-(t)) dt \leq \int_{t_0^- - 1}^{t_0^-} \left[\frac{\|U(t_0^-) - \mathbf{v}^-\|_{\mathcal{L}}^2}{2} + \mathcal{E}(\tilde{U}^-(t)) \right] e^{ct} dt.$$

Fix $t \in [t_0^- - 1, t_0^-]$. Choosing $\lambda = t - t_0^- + 1 \in [0, 1]$ and applying (3.5.44) in Lemma 3.5.9 and (3.5.56), we have that

$$\mathcal{E}(\tilde{U}^-(t)) \leq a + C^- \mathcal{E}_{\max}^-.$$

The previous fact combined with (3.5.57), (3.5.56) and 3.5.6, gives

$$\int_{t_0^- - 1}^{t_0^-} \mathbf{e}_c(\tilde{U}^-(t)) dt \leq \left(\frac{(\eta_0^-)^2}{2} + C^- \mathcal{E}_{\max}^- \right) e^{ct_0^-} + \frac{a(e^{ct_0^-} - e^{c(t_0^- - 1)})}{c}. \quad (3.5.61)$$

The continuity of U and (3.5.48) implies that there exists $\tilde{t}_2^- \in (\tilde{t}^-, t_0^-)$ such that

$$\text{dist}(U(\tilde{t}_2^-), \mathcal{F}^-) = \frac{r_0^-}{4} \text{ and } \forall t \in [\tilde{t}^-, \tilde{t}_2^-], \quad \text{dist}_{\mathcal{L}}(U(t), \mathcal{F}^-) \geq \frac{r_0^-}{4}. \quad (3.5.62)$$

Using (3.5.62), we get

$$\int_{\tilde{t}^-}^{\tilde{t}_2^-} \|U'(t)\|_{\mathcal{L}} e^{ct} dt \geq \frac{r_0^- e^{c\tilde{t}^-}}{4} \quad (3.5.63)$$

and (3.5.62) also implies

$$\forall t \in [\tilde{t}^-, \tilde{t}_2^-], \quad \mathcal{E}(U(t)) \geq \kappa_{r_0^-/4}^-. \quad (3.5.64)$$

Inequalities (3.5.63) and (3.5.64) along with the definition of η_0^- in (3.5.36) and Young's inequality give

$$\begin{aligned} \int_{\tilde{t}^-}^{\tilde{t}_2^-} \mathbf{e}_c(U(t))dt &\geq \frac{r_0^- e^{c\tilde{t}_0^-}}{4} \sqrt{2(\kappa_{r_0^-/4}^- - a)} + a \frac{e^{c\tilde{t}_2^-} - e^{c\tilde{t}^-}}{c} \\ &= e(\eta_0^-)^2 e^{c\tilde{t}^-} + a \frac{e^{c\tilde{t}_2^-} - e^{c\tilde{t}^-}}{c}, \end{aligned}$$

which gets to, using also that $\tilde{t}^- \geq t_0^- - 1$,

$$\begin{aligned} \int_{-\infty}^{t_0^-} \mathbf{e}_c(U(t))dt &= \int_{(-\infty, t_0^-] \setminus [\tilde{t}^-, \tilde{t}_2^-]} \mathbf{e}_c(U(t))dt + \int_{\tilde{t}^-}^{\tilde{t}_2^-} \mathbf{e}_c(U(t))dt \\ &\geq e(\eta_0^-)^2 e^{c\tilde{t}^-} + a \frac{e^{c\tilde{t}_0^-}}{c} \geq (\eta_0^-)^2 e^{ct_0^-} + a \frac{e^{ct_0^-}}{c}. \end{aligned} \quad (3.5.65)$$

Using now (3.5.61) we get

$$\begin{aligned} \int_{-\infty}^{t_0^-} \mathbf{e}_c(\tilde{U}^-(t))dt &= \int_{-\infty}^{t_0^- - 1} \mathbf{e}_c(\tilde{U}^-(t))dt + \int_{t_0^- - 1}^{t_0^-} \mathbf{e}_c(\tilde{U}^-(t))dt \\ &\leq \left(\frac{(\eta_0^-)^2}{2} + C^- \mathcal{E}_{\max}^- \right) e^{ct_0^-} + \frac{ae^{ct_0^-}}{c}. \end{aligned} \quad (3.5.66)$$

Therefore, subtracting (3.5.66) from (3.5.65), we get

$$\begin{aligned} \int_{-\infty}^{t_0^-} \mathbf{e}_c(U(t))dt - \int_{-\infty}^{t_0^-} \mathbf{e}_c(\tilde{U}^-(t))dt \\ \geq \left(\frac{(\eta_0^-)^2}{2} + C^- \mathcal{E}_{\max}^- \right) e^{ct_0^-}, \end{aligned}$$

which is positive because (3.5.38) implies

$$C^- \mathcal{E}_{\max}^- \leq \frac{(\eta_0^-)^2}{4}.$$

Since \tilde{U}^- and U coincide in $[t_0^-, +\infty)$, the proof of the first case is concluded.

- $t_0^- > \tilde{t} + 1$. In such a case, set

$$\tilde{U}^-(t) := \begin{cases} \mathbf{v}^- & \text{if } t \leq \tilde{t}^-, \\ (t - \tilde{t}^-)U(t_0^-) + (\tilde{t}^- + 1 - t)\mathbf{v}^- & \text{if } \tilde{t}^- \leq t \leq \tilde{t}^- + 1, \\ U(t_0^-) & \text{if } \tilde{t}^- + 1 \leq t \leq t_0^-, \\ U(t) & \text{if } t_0^- \leq t, \end{cases}$$

which clearly belongs to X_T and for all $t \leq t^-$, $U(t) \in \mathcal{F}_{r_0^-/2}^-$ by (3.5.60). We have that \tilde{U}^- is

constant in $[\tilde{t}^- + 1, t_0^-]$, therefore

$$\int_{\tilde{t}^-+1}^{t_0^-} \mathbf{e}_c(\tilde{U}^-(t))dt \leq (a + \mathcal{E}_{\max}^-) \frac{e^{ct_0^-} - e^{c\tilde{t}^-+1}}{c}$$

and, due to the definitions of \mathcal{E}_{\max}^- in (3.5.38) and t_0^- in (3.5.59)

$$\int_{\tilde{t}^-+1}^{t_0^-} \mathbf{e}_c(U(t))dt \geq \min\{a + \mathcal{E}_{\max}^-, \kappa_{\eta_0^-}\} \frac{e^{ct_0^-} - e^{c\tilde{t}^-+1}}{c} \geq \int_{\tilde{t}^-+1}^{t_0^-} \mathbf{e}_c(\tilde{U}^-(t))dt,$$

because $\mathcal{E}_{\max}^- + a > 0$ by (H3.7') and $t_0 \geq \tilde{t}^- + 1$ by assumption. Hence

$$\int_{\tilde{t}^-+1}^{+\infty} \mathbf{e}_c(\tilde{U}(t))dt \leq \int_{\tilde{t}^-+1}^{+\infty} \mathbf{e}_c(U(t))dt.$$

Arguing as in the first case scenario, we can prove that

$$\int_{-\infty}^{\tilde{t}^-+1} \mathbf{e}_c(\tilde{U}(t))dt < \int_{-\infty}^{\tilde{t}^-+1} \mathbf{e}_c(U(t))dt,$$

which concludes the proof of the second case.

To sum up, we have shown that if (3.5.48) is satisfied, then there exists \tilde{U}^- such that (3.5.49) and (3.5.50) hold, as we wanted.

Assume now that there exists $\tilde{t}^+ > t^+$ such that (3.5.51) holds. As before, Lemma 3.5.10 and the definition of \mathcal{E}_{\max}^+ , (3.5.35), imply that $t^+ := t^+(U, \mathcal{E}_{\max}^+)$ is such that

$$\text{dist}_{\mathcal{F}}(U(t^+), \mathcal{F}^+) \leq \eta_0^+ \quad (3.5.67)$$

and

$$\mathcal{E}(U(t^+)) \leq \mathcal{E}_{\max}^+. \quad (3.5.68)$$

We claim that we can assume without loss of generality that

$$\forall t \in [t^+, +\infty), \quad \mathcal{E}(U(t)) \geq 0. \quad (3.5.69)$$

Indeed, if we can find $t_0 \in (t^+, +\infty)$ such that $\mathcal{E}(U(t)) < 0$, then by (H3.7') we have that $\mathcal{E}(U(t_0)) \leq a + \mathcal{E}_{\max}^-$ and by (3.4.30) in (H3.7') we also have that $\text{dist}_{\mathcal{F}}(U(t_0), \mathcal{F}^-) \leq r_0^-/2$. Therefore, we have by the definitions (3.5.39) and (3.5.40) that $t^- \geq t_0$ and $t^+ > t^-$, a contradiction since we assume $t_0 > t^+$.

For the positive case, the proof is simpler as it suffices to define $\mathbf{v}^+ := P^+(U(t^+))$ and

$$\tilde{U}^+(t) := \begin{cases} \mathbf{v}^+ & \text{if } t \geq t^+ + 1 \\ (t - t^+)\mathbf{v}^+ + (t^+ + 1 - t)U(t^+) & \text{if } t^+ + 1 \geq t \geq t^+, \\ U(t) & \text{if } t^+ \geq t, \end{cases}$$

which is such that $U \in X_T$. Moreover, it holds that for all $t \geq t^+$, we have $\tilde{U}^+(t) \in \mathcal{F}_{r_0^+/2}^+$. Therefore, the requirements (3.5.52) and (3.5.53) hold for \tilde{U}^+ . It remains to check that (3.5.53) is also fulfilled.

We have that

$$\int_{t^+}^{t^++1} \mathbf{e}_c(\tilde{U}^+(t)) dt = \int_{t^+}^{t^++1} \left[\frac{\|U(t^+) - \mathbf{v}^+\|_{\mathcal{L}}^2}{2} + \mathcal{E}(\tilde{U}^+(t)) \right] e^{ct} dt. \quad (3.5.70)$$

Using (3.5.44) in Lemma 3.5.9 and (3.5.68), we get

$$\int_{t^+}^{t^++1} \mathcal{E}(\tilde{U}^+(t)) e^{ct} dt \leq C^+ \mathcal{E}_{\max}^+ e^{c(t^++1)}. \quad (3.5.71)$$

Using now (3.5.67), we get

$$\int_{t^+}^{t^++1} \frac{\|U(t^+) - \mathbf{v}^+\|_{\mathcal{L}}^2}{2} e^{ct} dt \leq \frac{(\eta_0^+)^2}{2} e^{c(t^++1)}. \quad (3.5.72)$$

Plugging (3.5.71) and (3.5.72) into (3.5.70), we get

$$\int_{t^+}^{t^++1} \mathbf{e}_c(\tilde{U}^+(t)) dt \leq \left(\frac{(\eta_0^+)^2}{2} + C^+ \mathcal{E}_{\max}^+ \right) e^{ct^++1}. \quad (3.5.73)$$

Since for all $t \geq t^+ + 1$ we have $\tilde{U}^+(t) = \mathbf{v}^+$, we obtain from (3.5.73)

$$\int_{t^+}^{+\infty} \mathbf{e}_c(\tilde{U}^+(t)) dt \leq \left(\frac{(\eta_0^+)^2}{2} + C^+ \mathcal{E}_{\max}^+ \right) e^{ct^++1}. \quad (3.5.74)$$

Next, notice that by continuity we can find $\tilde{t}_2^+ \in (t^+, \tilde{t}^+)$ such that

$$\text{dist}(U(\tilde{t}_2^+), \mathcal{F}^+) = \frac{r_0^+}{4} \text{ and } \forall t \in [\tilde{t}^+, \tilde{t}_2^+], \quad \frac{r_0^+}{2} \geq \text{dist}_{\mathcal{L}}(U(t), \mathcal{F}^+) \geq \frac{r_0^+}{4}. \quad (3.5.75)$$

Therefore, using (3.5.51) and (3.5.75), we obtain

$$\int_{\tilde{t}_2^+}^{\tilde{t}^+} \|U'(t)\|_{\mathcal{L}} e^{ct} dt \geq \frac{r_0^+}{4} e^{ct^++1} e^{-1} \quad (3.5.76)$$

and (3.5.75), (3.4.6) in (H3.1') imply

$$\forall t \in [\tilde{t}^+, \tilde{t}_2^+], \quad \mathcal{E}(U(t)) \geq \kappa_{r_0^+/4}^+. \quad (3.5.77)$$

Inequalities (3.5.76), (3.5.77) yield by Young's inequality

$$\int_{\tilde{t}_2^+}^{\tilde{t}^+} \mathbf{e}_c(U(t)) dt \geq \frac{r_0^+}{4} e^{ct^++1} e^{-1} \sqrt{2\kappa_{r_0^+/4}^+} = (\eta_0^+)^2 e^{t^++1},$$

where the last equality is due to the definition of η_0^+ in (3.5.33). Combining with (3.5.69), we get

$$\int_{t^+}^{+\infty} \mathbf{e}_c(U(t)) dt \geq (\eta_0^+)^2 e^{t^++1}.$$

The definition of \mathcal{E}_{\max}^+ in (3.5.35) and (3.5.74) imply then that

$$\int_{t^+}^{+\infty} \mathbf{e}_c(U(t)) dt > \int_{t^+}^{+\infty} \mathbf{e}_c(\tilde{U}^-(t)) dt,$$

which establishes (3.5.53).

We now show the last part of the proof: we show that (3.5.54) holds with the constant $T_\star(c)$ defined in (3.5.55). The argument is the same as in [12], *Lemma 2.10*. Assume by contradiction that there exists $t \in (t^-, +\infty)$ such that $\mathcal{E}(U(t)) < 0$. Then, arguing as above we must have $t < t^-$ by the definition of t^- in (3.5.39), a contradiction. Therefore, we can write

$$\begin{aligned} \mathbf{E}_c(U) &= \frac{1}{2} \int_{\mathbb{R}} \|U'(t)\|_{\mathcal{L}}^2 e^{ct} dt - \int_{-\infty}^{t^-} \mathcal{E}^-(U(t)) e^{ct} dt \\ &\quad + \int_{\mathbb{R}} \mathcal{E}^+(U(t)) e^{ct} dt, \end{aligned} \quad (3.5.78)$$

where \mathcal{E}^- and \mathcal{E}^+ stand for the positive and the negative part of \mathcal{E} , respectively. We have that

$$\int_{-\infty}^{t^-} \mathcal{E}^-(U(t)) e^{ct} dt \leq \frac{-a}{c} e^{ct^-}. \quad (3.5.79)$$

Set $\alpha_\star := \min\{\mathcal{E}_{\max}^+, \mathcal{E}_{\max}^- + a\} > 0$, which is independent on U , c and T . Notice that for all $t \in (t^-, t^+)$ we have that $\mathcal{E}(U(t)) \geq \alpha_\star$. Indeed, if $\mathcal{E}(U(t)) < \alpha_\star$, then by the definition of t^- and t^+ in (3.5.39) and (3.5.40) we get

$$\text{dist}_{\mathcal{L}}(U(t), \mathcal{F}^\pm) \geq \frac{r_0^\pm}{2},$$

which implies that

$$\mathcal{E}(U(t)) \geq \min\{\kappa_{r_0/2}^+, \kappa_{r_0/2}^- + a\} \geq \alpha_\star,$$

by (3.5.38) and (3.5.35), a contradiction. Therefore,

$$\int_{\mathbb{R}} \mathcal{E}^+(U(t)) e^{ct} dt \geq \int_{t^-}^{t^+} \mathcal{E}^+(U(t)) e^{ct} dt \geq \frac{\alpha_\star}{c} (e^{ct^+} - e^{ct^-}). \quad (3.5.80)$$

Plugging (3.5.79) and (3.5.80) into (3.5.78) and using that $\mathbf{E}_c(U) \leq 0$, we obtain

$$0 \geq \frac{a}{c} e^{ct^-} + \frac{\alpha_\star}{c} (e^{ct^+} - e^{ct^-}) \geq \left(\frac{a}{c} + \frac{\alpha_\star}{c} (e^{c(t^+ - t^-)} - 1) \right) e^{ct^-},$$

that is,

$$0 \geq -\left(\frac{-a}{\alpha_\star} + 1 \right) + e^{c(t^+ - t^-)},$$

which implies

$$0 < t^+ - t^- \leq \frac{1}{c} \ln \left(\frac{-a}{\alpha_\star} + 1 \right) = T_\star(c),$$

which is exactly (3.5.54) according to the definition (3.5.55). \square

The importance of Proposition 3.5.1 is summarized by the following result, which gives important information on the behavior of the constrained minimizers:

Corollary 3.5.1. *Assume that (H3.3'), (H3.4'), (H3.5') and (H3.7') hold. Let $c > 0$ and $T \geq 1$. Let $\mathbf{U}_{c,T}$ be an associated minimizer of \mathbf{E}_c in X_T given by Lemma 3.5.7. Then, if $t^\pm := t^\pm(\mathbf{U}_{c,T}, \mathcal{E}_{\max}^\pm)$ are as in (3.5.39), (3.5.40) it holds that*

$$\forall t \leq t^-, \quad \text{dist}_{\mathcal{L}}(\mathbf{U}_{c,T}(t), \mathcal{F}^-) < \frac{r_0^-}{2} \quad (3.5.81)$$

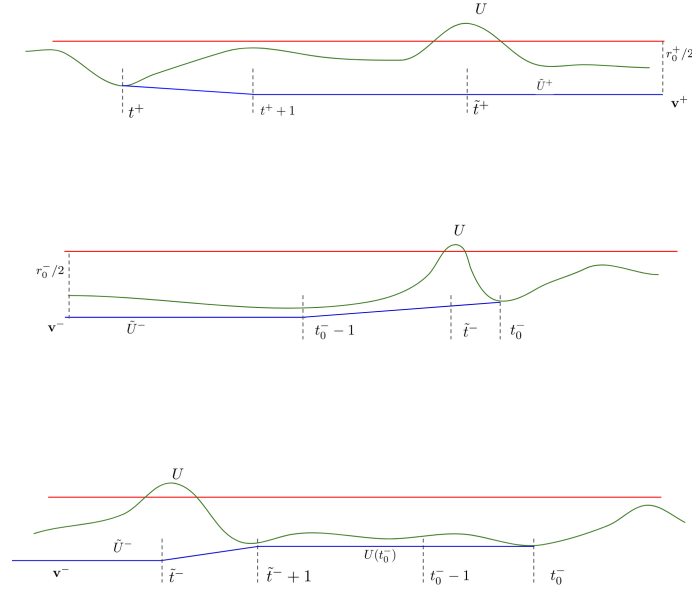


Figure 3.5: As it has been shown, the proof of Proposition 3.5.1 consists on showing that if the function U gets too far from \mathcal{F}^\pm after getting too close, then we can find a suitable competitor with strictly less energy. Above, we see a design for the positive case (the competitor \tilde{U}^+ is represented in blue). The second and third picture correspond to the two possible scenarios for the negative case (the competitor \tilde{U}^- is represented in blue).

and

$$\forall t \geq t^+, \quad \text{dist}_{\mathcal{L}}(\mathbf{U}_{c,T}(t), \mathcal{F}^+) < \frac{r_0^+}{2}, \quad (3.5.82)$$

Moreover, we have

$$\forall t \geq t^-, \quad \mathcal{E}(\mathbf{U}_{c,T}(t)) \geq 0. \quad (3.5.83)$$

Finally, we have that if $\mathbf{E}_c(\mathbf{U}_{c,T}) \leq 0$, then

$$0 < t^+ - t^- \leq T_\star(c), \quad (3.5.84)$$

where $T_\star(c)$ is as in (3.5.55). In particular, the function

$$c \in (0, +\infty) \rightarrow T_\star(c)$$

is continuous.

Proof. If we assume by contradiction that (3.5.82) does not hold, then we necessarily have that there exists $\tilde{t}^- < t^-$ such that (3.5.48) holds. Proposition 3.5.1 implies then the existence of $\tilde{U} \in X_T$ such that $\mathbf{E}_c(\tilde{U}) < \mathbf{E}_c(\mathbf{U}_{c,T}) = \mathbf{m}_{c,T}$, a contradiction. Therefore, (3.5.82) holds. Similarly, we can show that (3.5.81) also holds. Finally, in order to establish (3.5.83), we argue as in the proof Proposition 3.5.1. Indeed, due to the definition of t^- , we have that for $t \geq t^-$ it holds that either

$$\mathcal{E}(\mathbf{U}_{c,T}(t)) \geq a + \mathcal{E}_{\max}^- > 0$$

(which is (H3.7')) or

$$\text{dist}_{\mathcal{L}}(\mathbf{U}_{c,T}(t), \mathcal{F}^-) \geq \frac{r_0^-}{2}$$

which by (3.4.30) in (H3.7') implies that $\mathcal{E}(\mathbf{U}_{c,T}(t)) \geq 0$. Therefore, (3.5.83) holds and the proof is concluded. \square

Moreover, Lemma 3.5.8 applies to $\mathbf{V}_{c,T}$ as follows:

Corollary 3.5.2. *Assume that (H3.3'), (H3.4'), (H3.5') and (H3.7') hold. Let $c > 0$ and $T \geq 1$. Let $\mathbf{U}_{c,T}$ be an associated minimizer of \mathbf{E}_c in X_T given by Lemma 3.5.7. Then, if $t^\pm := t^\pm(\mathbf{U}_{c,T}, \mathcal{E}_{\max}^\pm)$ are as in (3.5.39), (3.5.40) it holds that there exists $\delta_{c,T} > 0$ such that the set*

$$S_{c,T} := (-\infty, t^- + \delta_{c,T}) \cup (-T, T) \cup (t^+ - \delta_{c,T}, +\infty)$$

is such that $\mathbf{U}_{c,T} \in \mathcal{A}(S_{c,T})$ (see (3.4.23)) and

$$\mathbf{U}_{c,T}'' - D_{\mathcal{L}}\mathcal{E}(\mathbf{U}_{c,T}) = -c\mathbf{U}_{c,T}' \text{ in } S_{c,T}. \quad (3.5.85)$$

The proof of 3.5.2 is obtained in a straightforward manner by combining Lemma 3.5.8 with the informations given by Corollary 3.5.1. Notice that 3.5.2 implies that constrained solutions are piecewise solutions and, in particular, they solve the equation for times with large absolute value.

3.5.7 Existence of the unconstrained solutions

We now establish the existence of the unconstrained solutions making use of the previous comparison results. As in [12] and [13], we define the set

$$\mathcal{C} := \{c > 0 : \exists T \geq 1 \text{ and } U \in X_T \text{ such that } \mathbf{E}_c(U) < 0\}. \quad (3.5.86)$$

We first prove some important properties for \mathcal{C} which are the same to Lemma 2.12 in [12] and Lemma 27 in [13]:

Lemma 3.5.11. *Assume that (H3.3'), (H3.4') and (H3.5') hold. Let \mathcal{C} be the set defined in (3.5.86). Then, \mathcal{C} is open and non-empty. Moreover, if we assume that (H3.7') holds, then \mathcal{C} is also bounded with*

$$\sup \mathcal{C} \leq \frac{\sqrt{-2a}}{d_0}, \quad (3.5.87)$$

where d_0 was defined in (3.4.29).

Proof. Firstly, we show that $\mathcal{C} \neq \emptyset$. For that purpose, consider the function Ψ introduced in (3.5.9). Consider the function

$$f : c \in (0, +\infty) \rightarrow e^{-c} \left(\frac{a}{c} + e^{2c} \int_{-1}^1 \left(\frac{\|\Psi'(t)\|_{\mathcal{L}}^2}{2} + \mathcal{E}(\Psi(t)) \right) dt \right) \in \mathbb{R},$$

which is well defined by Lemma 3.5.4. We have that for all $c > 0$

$$\mathbf{E}_c(\Psi) = \frac{-a}{c} e^{-c} + \int_{-1}^1 \left(\frac{\|\Psi'(t)\|_{\mathcal{L}}^2}{2} + \mathcal{E}(\Psi(t)) \right) e^{ct} dt \leq f(c) \quad (3.5.88)$$

and f is a continuous function such that $\lim_{c \rightarrow 0} f(c) = -\infty$ because $a < 0$. Moreover, we have that for all $c > 0$,

$$f'(c) = -e^{-c}a + ce^{2c} \int_{-1}^1 \left(\frac{\|\Psi'(t)\|_{\mathcal{L}}^2}{2} + \mathcal{E}(\Psi(t)) \right) dt > 0$$

and $\lim_{c \rightarrow +\infty} f(c) = +\infty$. Therefore, there exists a unique $c_\Psi > 0$ such that $f(c_\Psi) = 0$ and for all $c < c_\Psi$ we have $\mathbf{E}_c(\Psi) < 0$ by (3.5.88). Therefore, $(0, c_\Psi) \subset \mathcal{C}$, meaning that $\mathcal{C} \neq \emptyset$ as we wanted to show.

We next prove that \mathcal{C} is open. Let $c \in \mathcal{C}$, we have that $\mathbf{E}_c(\mathbf{U}_{c,T}) < 0$, where $\mathbf{U}_{c,T}$ is a minimizer of \mathbf{E}_c in X_T given by Lemma 3.5.7. By (3.5.6) in Lemma 3.5.3, there exists a sequence $(t_n)_{n \in \mathbb{N}}$ in $[T, +\infty)$ such that $t_n \rightarrow +\infty$ and

$$\lim_{n \rightarrow \infty} \mathcal{E}(\mathbf{U}_{c,T}(t_n)) = 0. \quad (3.5.89)$$

Up to subsequences, we have that for all $n \in \mathbb{N}$, $\mathbf{U}_{c,T}(t_n) \in \mathcal{F}_{r_0^+}^+$. Hence, we can define

$$\mathbf{U}_{c,T}^n(s) := \begin{cases} \mathbf{U}_{c,T}(s) & \text{if } s \leq t_n, \\ (1 + t_n - s)\mathbf{U}_{c,T}(t_n) + (s - t_n)P^+(\mathbf{U}_{c,T}(t_n)) & \text{if } t_n \leq s \leq t_n + 1, \\ P^+(\mathbf{U}_{c,T}(t_n)) & \text{if } t_n + 1 \leq s. \end{cases}$$

We have that for all $n \in \mathbb{N}$,

$$\begin{aligned} \mathbf{E}_c(\mathbf{U}_{c,T}^n(s)) &= \int_{-\infty}^{t_n} \mathbf{e}_c(\mathbf{U}_{c,T}(s)) ds + \frac{\|\mathbf{U}_{c,T}(t_n) - P^+(\mathbf{U}_{c,T}(t_n))\|_{\mathcal{L}}^2}{2} \\ &\quad + \int_{t_n}^{t_n+1} \mathcal{E}(\mathbf{U}_{c,T}^n(s)) ds \\ &\leq \mathbf{E}_c(\mathbf{U}_{c,T}) + \frac{\|\mathbf{U}_{c,T}(t_n) - P^+(\mathbf{U}_{c,T}(t_n))\|_{\mathcal{L}}^2}{2} \\ &\quad + \int_{t_n}^{t_n+1} \mathcal{E}((1 + t_n - s)\mathbf{U}_{c,T}(t_n) + (s - t_n)P^+(\mathbf{U}_{c,T}(t_n))) ds, \end{aligned} \quad (3.5.90)$$

where we have used that $t_n \geq T$ in order to obtain the inequality. Let $\beta^+(\hat{r}^+)$ be as in Lemma 3.5.9. Up to a subsequence, we have that for all $n \in \mathbb{N}$ it holds $\mathcal{E}(\mathbf{U}_{c,T}(t_n)) \leq \beta^+(\hat{r}^+)$. Therefore, by Lemma 3.5.9 we have that for all $\lambda \in [0, 1]$ and $n \in \mathbb{N}$ it holds

$$\mathcal{E}(\lambda \mathbf{U}_{c,T}(t_n) + (1 - \lambda)P^+(\mathbf{U}_{c,T}(t_n))) \leq C^+ \mathcal{E}(\mathbf{U}_{c,T}(t_n)),$$

where $C^+ > 0$ is independent on n (see (3.4.27)). Plugging into (3.5.90), we obtain that for all $n \in \mathbb{N}$ it holds

$$\mathbf{E}_c(\mathbf{U}_{c,T}^n(s)) \leq \mathbf{E}_c(\mathbf{U}_{c,T}) + \frac{\|\mathbf{U}_{c,T}(t_n) - P^+(\mathbf{U}_{c,T}(t_n))\|_{\mathcal{L}}^2}{2} + C^+ \mathcal{E}(\mathbf{U}_{c,T}(t_n)). \quad (3.5.91)$$

Recall that we assume Notice also that (3.5.7) implies in particular that

$$\lim_{n \rightarrow +\infty} \|\mathbf{U}_{c,T}(t_n) - P^+(\mathbf{U}_{c,T}(t_n))\|_{\mathcal{L}}^2 = 0$$

which, in combination with inequalities (3.5.89) and (3.5.91) together with the fact that $\mathbf{E}_c(\mathbf{U}_{c,T}) < 0$, gives that there exists $N \in \mathbb{N}$ such that $\mathbf{E}_c(\mathbf{U}_{c,T}^N) < 0$. Since $\mathbf{U}_{c,T}^N$ is constant in $[t_N + 1, +\infty)$, we have

that for all $\tilde{c} > 0$, $\mathbf{E}_{\tilde{c}}(\mathbf{U}_{c,T}^N) < +\infty$. Therefore, by Lemma 3.5.5 we have that

$$\tilde{c} \in (0, +\infty) \rightarrow \mathbf{E}_{\tilde{c}}(\mathbf{U}_{c,T}^N) \in \mathbb{R}$$

is well defined and continuous. Therefore, we can find some $\delta > 0$ such that for all $\tilde{c} \in (c - \delta, c + \delta)$, $\mathbf{E}_{\tilde{c}}(\mathbf{U}_{c,T}^N) < 0$. As a consequence, we have that $(c - \delta, c + \delta) \subset \mathcal{C}$, which shows that \mathcal{C} is open.

We now assume that (H3.7') holds and we use it to establish the bound (3.5.87). In particular, we can apply Proposition 3.5.1. Let $c > 0$ and $T \geq 1$ be such that $\mathbf{E}_c(\mathbf{U}_{c,T}) < 0$ with $\mathbf{U}_{c,T} \in X_T$ a minimizing solution given by Lemma 3.5.7. Let $t^\pm := t^\pm(\mathbf{U}_{c,T}, \mathcal{E}_{\max}^\pm)$ be as in (3.5.39), (3.5.40). Inequality (3.5.84) in Proposition 3.5.1 implies that $t^- < t^+$. Recall the definition of d_0 in (3.4.29) and the fact that $\mathbf{U}_{c,T}(t^\pm) \in \mathcal{F}_{r_0^\pm/2}^\pm$. Those facts imply

$$d_0 \leq \|\mathbf{U}_{c,T}(t^+) - \mathbf{U}_{c,T}(t^-)\|_{\mathcal{L}}. \quad (3.5.92)$$

Since (H3.7') holds, we can use (3.5.83) in Corollary 3.5.1 to obtain

$$\begin{aligned} \|\mathbf{U}_{c,T}(t^+) - \mathbf{U}_{c,T}(t^-)\|_{\mathcal{L}}^2 &\leq 2 \int_{\mathbb{R}} \frac{\|\mathbf{U}'_{c,T}(t)\|^2}{2} e^{ct} dt \left(\frac{e^{-ct^-} - e^{-ct^+}}{c} \right) \\ &\leq 2 \left(\mathbf{E}_c(\mathbf{U}_{c,T}) - \frac{a}{c} e^{ct^-} \right) \left(\frac{e^{-ct^-} - e^{-ct^+}}{c} \right). \end{aligned}$$

Using now that $\mathbf{E}_c(\mathbf{U}_{c,T}) \leq 0$, the fact that $t^- < t^+$ and (3.5.92), the inequality above becomes

$$d_0^2 \leq -2a \frac{1 - e^{c(t^- - t^+)}}{c^2} \leq \frac{-2a}{c^2},$$

so that (3.5.87) follows. \square

We now have all the ingredients for establishing the existence of the unconstrained solutions:

Proposition 3.5.2. *Assume that (H3.3'), (H3.4'), (H3.5'), (H3.6') and (H3.7') hold. Let $\bar{c} \in \partial(\mathcal{C}) \cap (0, +\infty)$, where $\partial(\mathcal{C})$ stands for the boundary of the set \mathcal{C} defined in (3.5.86). Then, there exists $\bar{T} \geq 1$ such that $\mathbf{m}_{\bar{c}, \bar{T}} = 0$ ($\mathbf{m}_{\bar{c}, \bar{T}}$ as in (3.5.8)) and $\bar{\mathbf{U}} \in X_{\bar{T}}$ an associated minimizer of $\mathbf{E}_{\bar{c}}$ in $X_{\bar{T}}$ which does not saturate the constraints, i. e.*

$$\forall t \geq \bar{T}, \quad \text{dist}_{\mathcal{L}}(\bar{\mathbf{U}}(t), \mathcal{F}^+) < \frac{r_0^+}{2} \quad (3.5.93)$$

and

$$\forall t \leq -\bar{T}, \quad \text{dist}_{\mathcal{L}}(\bar{\mathbf{U}}(t), \mathcal{F}^-) < \frac{r_0^-}{2}. \quad (3.5.94)$$

Moreover, $\bar{\mathbf{U}} \in \mathcal{A}(\mathbb{R})$ and the pair $(\bar{c}, \bar{\mathbf{U}})$ solves (3.4.1).

Remark 3.5.3. Notice that Lemma 3.5.11 implies that (under the necessary assumptions) the set \mathcal{C} is bounded, meaning that $\partial(\mathcal{C}) \cap (0, +\infty) \neq \emptyset$. Such a fact, in combination with Proposition 3.5.2 shows the existence of the unconstrained solutions.

Proof of Proposition 3.5.2. By Lemma 3.5.11, we have that $\mathcal{C} \neq \emptyset$ is open, which implies that $\partial(\mathcal{C}) \subset \mathbb{R} \setminus \mathcal{C}$. Therefore, we have $\bar{c} \notin \mathcal{C}$. Recall that due to the definition of \mathcal{C} in (3.5.86), we have that

$$\forall T \geq 1, \quad \mathbf{m}_{\bar{c}, T} \geq 0. \quad (3.5.95)$$

The definition of the boundary allows to consider a sequence $(c_n)_{n \in \mathbb{N}}$ contained in \mathcal{C} such that $c_n \rightarrow \bar{c}$. Then, for each $n \in \mathbb{N}$ there exists $T_n \geq 1$ such that $\mathbf{E}_{c_n}(\mathbf{U}_{c_n, T_n}) < 0$, where, for each $n \in \mathbb{N}$, \mathbf{U}_{c_n, T_n} is a minimizer of \mathbf{E}_{c_n} in X_{T_n} . For each $n \in \mathbb{N}$, set $t_n^\pm := t^\pm(\mathbf{U}_{c_n, T_n}, \mathcal{E}_{\max}^\pm)$ as in (3.5.39), (3.5.40). Using (3.5.84) in Corollary 3.5.1 we have that that

$$\forall n \in \mathbb{N}, \quad 0 < t_n^+ - t_n^- \leq T_\star(c_n),$$

and the function

$$c \in (0, +\infty) \rightarrow T_\star(c) \in (0, +\infty)$$

is continuous. Since the sequence $(c_n)_{n \in \mathbb{N}}$ is bounded, we have that

$$T_\star := \max \left\{ 1, \sup_{n \in \mathbb{N}} T_\star(c_n) \right\} < +\infty$$

and

$$\forall n \in \mathbb{N}, \quad 0 < t_n^+ - t_n^- \leq T_\star, \quad (3.5.96)$$

so that we have a bound on $(t_n^+ - t_n^-)_{n \in \mathbb{N}}$. Moreover, (3.5.81) and (3.5.82) in Corollary 3.5.1 imply

$$\forall n \in \mathbb{N}, \forall t \geq t_n^+, \quad \text{dist}_{\mathcal{L}}(\mathbf{U}_{c_n, T_n}(t), \mathcal{F}^+) < \frac{r_0^+}{2} \quad (3.5.97)$$

and

$$\forall n \in \mathbb{N}, \forall t \leq t_n^-, \quad \text{dist}_{\mathcal{L}}(\mathbf{U}_{c_n, T_n}(t), \mathcal{F}^-) < \frac{r_0^-}{2}. \quad (3.5.98)$$

For each $n \in \mathbb{N}$, define the function $\mathbf{U}_{c_n, T_n}^{t_n^+} := \mathbf{U}_{c_n, T_n}(\cdot + t_n^+)$. Then, (3.5.96) implies that (3.5.97) and (3.5.98) write as

$$\forall n \in \mathbb{N}, \forall t \geq 0, \quad \text{dist}_{\mathcal{L}}(\mathbf{U}_{c_n, T_n}^{t_n^+}(t), \mathcal{F}^-) < \frac{r_0^-}{2}.$$

and

$$\forall n \in \mathbb{N}, \forall t \leq -T_\star, \quad \text{dist}_{\mathcal{L}}(\mathbf{U}_{c_n, T_n}(t), \mathcal{F}^-) < \frac{r_0^-}{2}.$$

so that for all $n \in \mathbb{N}$ we have $\mathbf{U}_{c_n, T_n}^{t_n^+} \in X_{T_\star}$. Moreover, a computation shows

$$\forall n \in \mathbb{N}, \quad \mathbf{E}_{c_n}(\mathbf{U}_{c_n, T_n}^{t_n^+}) = e^{-c_n t_n^+} \mathbf{E}_{c_n}(\mathbf{U}_{c_n, T_n}) < 0.$$

Therefore, if we apply Lemma 3.5.6 with sequence of speeds $(c_n)_{n \in \mathbb{N}}$ and the sequence $(\mathbf{U}_{c_n, T_n}^{t_n^+})_{n \in \mathbb{N}}$ in X_{T_\star} , we obtain $\bar{\mathbf{U}} \in X_{T_\star}$ such that

$$\mathbf{E}_{\bar{c}}(\bar{\mathbf{U}}) \leq \liminf_{n \rightarrow \infty} \mathbf{E}_{c_n}(\mathbf{U}_{c_n, T_n}^{t_n^+}) \leq 0,$$

which in combination with (3.5.95) implies that $\mathbf{m}_{\bar{c}, T_\star} = 0$. Therefore, we have $\mathbf{E}_{\bar{c}}(\bar{\mathbf{U}}) = 0$, so that $\bar{\mathbf{U}}$ is a minimizer of $\mathbf{E}_{\bar{c}}$ in X_{T_\star} . Set $t_\star^\pm := t^\pm(\bar{\mathbf{U}}, \mathcal{E}_{\max}^\pm)$ as in (3.5.39), (3.5.40). Invoking (3.5.95) and Corollary 3.5.2, we obtain that for all $T \geq 1$ such that $\mathbf{m}_{\bar{c}, T} = 0$ and $\bar{\mathbf{U}} \in X_T$, we have $\bar{\mathbf{U}} \in \mathcal{A}(S_{\bar{c}, T})$ with

$$S_{\bar{c}, T} := (-\infty, t_\star^- + \delta_\star(T)) \cup (-T, T) \cup (t_\star^+ - \delta_\star(T), +\infty) \quad (3.5.99)$$

for some $\delta_\star(T) > 0$ and

$$\bar{\mathbf{U}}'' - D_{\mathcal{L}}\mathcal{E}(\bar{\mathbf{U}}) = -\bar{c}\bar{\mathbf{U}}' \text{ in } S_{\bar{c},T}.$$

Moreover, using (3.5.81) and (3.5.82) in Corollary 3.5.1, we obtain as before that

$$\forall t \geq t_\star^+, \quad \text{dist}_{\mathcal{L}}(\bar{\mathbf{U}}(t), \mathcal{F}^+) < \frac{r_0^+}{2} \quad (3.5.100)$$

and

$$\forall t \leq t_\star^-, \quad \text{dist}_{\mathcal{L}}(\bar{\mathbf{U}}(t), \mathcal{F}^-) < \frac{r_0^-}{2}. \quad (3.5.101)$$

Therefore, if we set $\bar{T} = \max\{1, t_\star^+, -t_\star^-\}$, then (3.5.100) and (3.5.101) imply that $\bar{\mathbf{U}} \in X_{\bar{T}}$ and that (3.5.93), (3.5.94) hold. Moreover, we have that $\mathbf{E}_{\bar{c}}(\bar{\mathbf{U}}) = 0$, so that $\bar{\mathbf{U}}$ is a minimizer of $\mathbf{E}_{\bar{c}}$ in $X_{\bar{T}}$ by (3.5.95). Therefore, we obtain that $\bar{\mathbf{U}} \in \mathcal{A}(S_{\bar{c},\bar{T}})$ and

$$\bar{\mathbf{U}}'' - D_{\mathcal{L}}\mathcal{E}(\bar{\mathbf{U}}) = -\bar{c}\bar{\mathbf{U}}' \text{ in } S_{\bar{c},\bar{T}},$$

with $S_{\bar{c},\bar{T}}$ as in (3.5.99). The choice of \bar{T} implies that $S_{\bar{c},\bar{T}} = \mathbb{R}$. Therefore, $\bar{\mathbf{U}} \in \mathcal{A}(\mathbb{R})$ and $(\bar{c}, \bar{\mathbf{U}})$ solves (3.4.1), which finishes the proof. \square

Notice that our Proposition 3.5.2 follows very similar lines than the analogous results in [12] and [13].

3.5.8 Uniqueness of the speed

The precise statement of the uniqueness result is as follows:

Proposition 3.5.3. *Assume that (H3.6') holds. Let X be the set defined in (3.4.31). Let $(c_1, c_2) \in (0, +\infty)^2$ be such that there exist \mathbf{U}_1 and \mathbf{U}_2 in $X \cap \mathcal{A}(\mathbb{R})$ such that (c_1, \mathbf{U}_1) and (c_2, \mathbf{U}_2) solve (3.4.1) and for each $i \in \{1, 2\}$, $\mathbf{E}_{c_i}(\mathbf{U}_i) < +\infty$. Assume moreover that*

$$\forall i \in \{1, 2\}, \forall j \in \{1, 2\} \setminus \{i\}, \quad \mathbf{E}_{c_i}(\mathbf{U}_j) \geq 0. \quad (3.5.102)$$

Then, we have $c_1 = c_2$.

Proof. We prove the result by contradiction. Hence, we can assume without loss of generality that $c_1 < c_2$. A direct computation shows that for every $(c, U) \in (0, +\infty) \times (X \cap \mathcal{A}(\mathbb{R}))$ a solution to (3.4.1), we have

$$\forall t \in \mathbb{R}, \quad \frac{\|U'(t)\|_{\mathcal{L}}^2}{2} + \mathcal{E}(U(t)) = e^{-ct} \left(\frac{e^{ct}}{c} \left(\mathcal{E}(U(t)) - \frac{\|U'(t)\|_{\mathcal{L}}^2}{2} \right) \right)'. \quad (3.5.103)$$

Replacing (c_2, U_2) in (3.5.103) and multiplying for each $t \in \mathbb{R}$ by $e^{c_1 t}$, computations show that

$$\begin{aligned} \forall t_1 < t_2, \quad c_1 \mathbf{E}_{c_1}(\mathbf{U}_2; (t_1, t_2)) &= (c_1 - c_2) \int_{t_1}^{t_2} \|\mathbf{U}_2'(t)\|_{\mathcal{L}}^2 e^{c_1 t} dt \\ &\quad + \left[e^{c_1 t} \left(\mathcal{E}(\mathbf{U}_2(t)) - \frac{\|\mathbf{U}_2'(t)\|_{\mathcal{L}}^2}{2} \right) \right]_{t_1}^{t_2}. \end{aligned} \quad (3.5.104)$$

Notice now that the definition of X in (3.4.31) implies that

$$X = \bigcup_{T \geq 1} X_T,$$

which means that there exists $T \geq 1$ such that $\mathbf{U}_2 \in X_T$. Combining then Lemma 3.5.1 and the fact that $\mathbf{E}_{c_2}(\mathbf{U}_2) < +\infty$, we get that $\mathbf{e}_{c_2}(\mathbf{U}_2(\cdot)) \in L^1(\mathbb{R})$. Therefore, we can find two sequences $(t_n^+)_{n \in \mathbb{N}}$ and $(t_n^-)_{n \in \mathbb{N}}$ such that $t_n^\pm \rightarrow \pm\infty$ and

$$\lim_{n \rightarrow \infty} \mathbf{e}_{c_2}(\mathbf{U}_2(t_n^\pm)) = 0. \quad (3.5.105)$$

Since we have $c_1 < c_2$, it holds

$$\forall t \in \mathbb{R}, \quad e^{c_1 t} \left| \mathcal{E}(\mathbf{U}_2(t)) - \frac{\|\mathbf{U}_2'(t)\|_{\mathcal{L}}^2}{2} \right| \leq |\mathbf{e}_{c_2}(\mathbf{U}_2(t))|$$

which in combination with (3.5.105) implies

$$\lim_{n \rightarrow \infty} e^{c_1 t_n^\pm} \left(\mathcal{E}(\mathbf{U}_2(t_n^\pm)) - \frac{\|\mathbf{U}_2'(t_n^\pm)\|_{\mathcal{L}}^2}{2} \right) = 0. \quad (3.5.106)$$

Therefore, if we replace for each $n \in \mathbb{N}$ in (3.5.104) with $a = t_n^-$ and $b = t_n^+$, we can then pass to the limit (3.5.106) and obtain

$$c_1 \mathbf{E}_{c_1}(\mathbf{U}_2) = (c_1 - c_2) \int_{\mathbb{R}} \|\mathbf{U}_2'(t)\|_{\mathcal{L}}^2 e^{c_1 t} dt < 0$$

because we assume $c_1 < c_2$. However, by (3.5.102) we have $\mathbf{E}_{c_1}(\mathbf{U}_2) \geq 0$, which is a contradiction. \square

Remark 3.5.4. Again, the proof of Proposition 3.5.3 is essentially a direct adaptation of that given in [12] and [13]. Our hypothesis are slightly weaker, since we only assume that the solutions have finite energies and (3.5.102), while in [12] and [13] it is assumed that the solutions are global minimizers of the corresponding energy functionals. Notice also that (H3.7') is not needed for proving Proposition 3.5.3, which holds in a more general setting.

Proposition 3.5.3 along with Proposition 3.5.2 allows to show that the set \mathcal{C} defined in (3.5.86) is in fact an open interval:

Corollary 3.5.3. *Assume that (H3.3'), (H3.4'), (H3.5'), (H3.6') and (H3.7') hold. Let*

$$c(\mathcal{C}) := \sup \mathcal{C}.$$

Then, we have $\mathcal{C} = (0, c(\mathcal{C}))$.

Proof. The statement of the result is equivalent to showing that

$$\partial(\mathcal{C}) \cap (0, +\infty) = \{c(\mathcal{C})\}$$

The quantity $c(\mathcal{C})$ is well defined in $(0, +\infty)$ because \mathcal{C} is non-empty and bounded by Lemma 3.5.11. Therefore, we have $c(\mathcal{C}) \in \partial(\mathcal{C}) \cap (0, +\infty)$ because \mathcal{C} is open, so it does not contain its limit points. By Proposition 3.5.2, we find $\mathbf{U}^{\mathcal{C}} \in X$ such that $(c(\mathcal{C}), \mathbf{U}^{\mathcal{C}})$ solves (3.4.1). Let now $\bar{c} \in \partial(\mathcal{C}) \cap (0, +\infty)$. If

we show that $\bar{c} = c(\mathcal{C})$, the proof will be finished. Applying Proposition 3.5.2 with \bar{c} , we find $\bar{U} \in X$ such that (\bar{c}, \bar{U}) solves (3.4.1). Proposition 3.5.2, along with the fact that \bar{c} and $c(\mathcal{C})$ do not belong to \mathcal{C} , also implies that

$$\inf_{U \in X} \mathbf{E}_{\bar{c}}(U) = \mathbf{E}_{\bar{c}}(\bar{U}) = 0 = \mathbf{E}_{c(\mathcal{C})}(\mathbf{U}^{\mathcal{C}}) = \inf_{U \in X} \mathbf{E}_{c(\mathcal{C})}(U)$$

so that

$$\mathbf{E}_{c(\mathcal{C})}(\bar{U}) \geq 0 \text{ and } \mathbf{E}_{\bar{c}}(\mathbf{U}^{\mathcal{C}}) \geq 0$$

meaning that we can apply Proposition 3.5.3 to $(c(\mathcal{C}), \mathbf{U}^{\mathcal{C}})$, (\bar{c}, \bar{U}) . As a consequence, we have $\bar{c} = c(\mathcal{C})$, which concludes the proof. \square

3.5.9 Proof of Theorem 3.5 completed

All the elements of the proof of Theorem 3.5 are already present in previous result. Indeed, Proposition 3.5.2 along with Corollary 3.5.3 implies the existence of $(c^*, \mathbf{U}) \in (0, +\infty) \times X_{T^*}$ a solution to (3.4.1) with $c^* = c(\mathcal{C})$. Conditions (3.4.2) and (3.4.3) are satisfied due to the fact that $\mathbf{U} \in X_{T^*}$. The statement regarding the uniqueness of the speed c^* follows from Proposition 3.5.3. Finally, we have that (3.5.7) is exactly the exponential rate of convergence (3.4.32), which completes the proof. \square

3.5.10 Asymptotic behavior of the constrained solutions at $-\infty$

As it has been pointed out before, almost nothing can be said about the behavior of arbitrary function in X_T at $-\infty$. However, it turns out that constrained minimizers converge exponentially at $-\infty$ with respect to the \mathcal{L} -norm provided that the speed fulfills an explicit upper bound, see Proposition 3.5.4. Such an upper bound also allows to establish some other properties. Once Proposition 3.5.4 will have been established, we will be able to complete the proofs of Theorems 3.6 and 3.7. The results of this section are obtained by combining ideas from Smyrnelis [157], Alikakos and Katzourakis [13], Alikakos, Fusco and Smyrnelis [12]. It is worth to point out that the arguments we present here strongly rely on the fact that the solutions considered are minimizers and that we do not expect them to hold for more general critical points.

We begin by showing the following preliminary result, which follows by a direct computation:

Lemma 3.5.12. *Assume that (H3.6') holds. Let $c > 0$, $t_1 < t_2$ and $U \in \mathcal{A}((t_1, t_2))$ such that*

$$U'' - D_{\mathcal{L}}\mathcal{E}(U) = -cU' \text{ in } (t_1, t_2).$$

Then, we have the formula

$$\forall t \in (t_1, t_2), \quad \frac{d}{dt} \left(\mathcal{E}(U(t)) - \frac{\|U'(t)\|_{\mathcal{L}}^2}{2} \right) = c\|U'(t)\|_{\mathcal{L}}^2. \quad (3.5.107)$$

Lemma 3.5.12 gives the following pointwise bounds for constrained solutions:

Lemma 3.5.13. *Assume that (H3.3'), (H3.4'), (H3.5'), (H3.6') and (H3.7') hold. Let $\mathbf{U}_{c,T}$ be a constrained solution given by Lemma 3.5.7 and $t^- := t^-(\mathbf{U}_{c,T}, \mathcal{E}_{\max}^-)$ be as in (3.5.39). Then for all $t < t^-$*

we have the inequality

$$\frac{\|\mathbf{U}'_{c,T}(t)\|_{\mathcal{L}}^2}{2} \leq \mathcal{E}(\mathbf{U}_{c,T}(t)) - a. \quad (3.5.108)$$

Similarly, it holds that for all $t > t^+$

$$\mathcal{E}(\mathbf{U}_{c,T}(t)) \leq \frac{\|\mathbf{U}'_{c,T}(t)\|_{\mathcal{L}}^2}{2}, \quad (3.5.109)$$

where $t^+ := t^+(\mathbf{U}_{c,T}, \mathcal{E}_{\max}^+)$ is as in (3.5.40).

Proof. Notice that (3.5.85) in Corollary 3.5.2 implies that $\mathbf{U}_{c,T}$ solves

$$\mathbf{U}_{c,T}'' - D_{\mathcal{L}} \mathcal{E}(\mathbf{U}_{c,T}) = -c\mathbf{U}'_{c,T} \text{ in } (-\infty, t^-).$$

Therefore, the function

$$f_{c,T} : t \in (-\infty, t^-] \rightarrow e^{ct} \left(\mathcal{E}(\mathbf{U}_{c,T}(t)) - a - \frac{\|\mathbf{U}'_{c,T}(t)\|_{\mathcal{L}}^2}{2} \right),$$

is C^1 and we clearly have that $f_{c,T} \in L^1((-\infty, t^-])$. By (3.5.107) in Lemma 3.5.12, we have

$$\forall t \in (-\infty, t^-), \quad f'_{c,T}(t) = c f_{c,T}(t) + c e^{ct} \|\mathbf{U}'_{c,T}(t)\|_{\mathcal{L}}^2 \geq 0, \quad (3.5.110)$$

and we also have $f'_{c,T} \in L^1((-\infty, t^-))$. Therefore, it holds that

$$\lim_{t \rightarrow -\infty} f_{c,T}(t) = 0. \quad (3.5.111)$$

Fix $t_1 < t_2 \leq t^-$. Integrating (3.5.110) in $[t_1, t_2]$ we get

$$f_{c,T}(t_2) \geq f_{c,T}(t_1),$$

which in combination with (3.5.111) gives

$$\forall t < t^-, \quad f_{c,T}(t) \geq 0,$$

which is (3.5.108). Inequality (3.5.109) is obtained in an identical fashion. \square

We conclude this section by proving the exponential convergence result, which is inspired by the ideas in *Proof of (28)* in Smyrnelis [157].

Proposition 3.5.4. *Assume that (H3.3'), (H3.4'), (H3.5'), (H3.6') and (H3.7') hold. Let $c > 0$ and $T \geq 1$. Assume moreover that $c < \gamma^-$, where γ^- is defined in (3.4.28). Let $\mathbf{U}_{c,T}$ be a constrained solution given by Lemma 3.5.7. Then, there exists $\overline{M} > 0$ such that for all $\varepsilon \in (0, \gamma^- - c)$ and $t \in \mathbb{R}$ it holds*

$$\int_{-\infty}^t (\mathcal{E}(\mathbf{U}_{c,T}(s)) - a) e^{-\varepsilon s} ds \leq \overline{M} e^{(\gamma^- - c - \varepsilon)t}. \quad (3.5.112)$$

Furthermore, there exist $M^- > 0$ and $\mathbf{v}_{c,T}^- \in \mathcal{F}^-$ such that for all $t \in \mathbb{R}$

$$\|\mathbf{U}_{c,T}(t) - \mathbf{v}_{c,T}^-\|_{\mathcal{L}}^2 \leq M^- e^{(\gamma^- - c)t}. \quad (3.5.113)$$

Proof. Let $t^- := t^-(\mathbf{U}_{c,T}, \mathcal{E}_{\max}^-)$ be as in (3.5.39). By applying (3.5.81) in Corollary 3.5.1, we obtain that for all $t \leq t^-$, $\mathbf{U}_{c,T}(t) \in \mathcal{F}_{r_0/2}^-$. For all $t \leq t^-$, define $\mathbf{v}^-(t) := P^-(\mathbf{U}_{c,T}(t))$. Consider the function

$$\tilde{U}_t^-(s) := \begin{cases} \mathbf{v}^-(t) & \text{if } s \leq t-1, \\ (t-s)\mathbf{v}^-(t) + (s-t+1)\mathbf{U}_{c,T}(t) & \text{if } t-1 \leq s \leq t, \\ \mathbf{U}_{c,T}(s) & \text{if } t \leq s, \end{cases}$$

which belongs to X_T . Therefore,

$$\mathbf{E}_c(\mathbf{U}_{c,T}) \leq \mathbf{E}_c(\tilde{U}_t^-)$$

and, equivalently

$$\int_{-\infty}^t \mathbf{e}_c(\mathbf{U}_{c,T}(s)) ds \leq \frac{a}{c} e^{ct} + \int_{t-1}^t \left(\frac{\|\mathbf{U}_{c,T}(t) - \mathbf{v}^-(t)\|_{\mathcal{L}}^2}{2} + (\mathcal{E}(\tilde{U}_t^-(s)) - a) \right) e^{cs} ds. \quad (3.5.114)$$

Using Lemma 3.5.9 and (3.4.6) in (H3.1'), (3.5.114) becomes

$$\int_{-\infty}^t \mathbf{e}_c(\mathbf{U}_{c,T}(s)) ds \leq \frac{a}{c} e^{ct} + (C^- + C^-)(\mathcal{E}(\mathbf{U}_{c,T}(t)) - a) e^{ct},$$

which gives

$$\int_{-\infty}^t (\mathcal{E}(\mathbf{U}_{c,T}(s)) - a) e^{cs} ds \leq \frac{1}{\gamma^-} (\mathcal{E}(\mathbf{U}_{c,T}(t)) - a) e^{ct}, \quad (3.5.115)$$

where γ^- was defined in (3.4.28). Define the function

$$\theta_{c,T}^- : t \in (-\infty, t^-] \rightarrow \int_{-\infty}^t (\mathcal{E}(\mathbf{U}_{c,T}(s)) - a) e^{cs} ds \in \mathbb{R}. \quad (3.5.116)$$

By (H3.6'), the function $\theta_{c,T}^-$ defined in (3.5.116) verifies that for all $t \in (-\infty, t^-)$

$$(\theta_{c,T}^-)'(t) = (\mathcal{E}(\mathbf{U}_{c,T}(t)) - a) e^{ct}$$

which, by (3.5.115) implies

$$\forall t \leq t^-, \quad \gamma^- \theta_{c,T}^-(t) \leq (\theta_{c,T}^-)'(t).$$

Fix now $t \in (-\infty, t^-)$ and assume that $\theta_{c,T}^-(t) > 0$. The previous inequality is equivalent to

$$\gamma^- \leq (\ln(\theta_{c,T}^-(t)))'.$$

which, by integrating in $[t, t^-]$ becomes

$$\gamma^-(t^- - t) \leq \ln(\theta_{c,T}^-(t^-)) - \ln(\theta_{c,T}^-(t)),$$

hence

$$e^{\gamma^-(t^- - t)} \leq \frac{\theta_{c,T}^-(t^-)}{\theta_{c,T}^-(t)},$$

that is

$$\theta_{c,T}^-(t) e^{\gamma^-(t^- - t)} \leq \theta_{c,T}^-(t^-),$$

which clearly also holds if we drop the assumption $\theta_{c,T}^-(t) > 0$, as $\theta_{c,T}^-$ is a non-negative function. Thus, we have shown that

$$\forall t \leq t^-, \quad \theta_{c,T}^-(t) \leq \theta_{c,T}^-(t^-) e^{-\gamma^-(t^- - t)}. \quad (3.5.117)$$

Now, we have that using (3.5.117) we get for any fixed $t \leq t^- - 1$, $\varepsilon > 0$ and $i \in \mathbb{N}$

$$\begin{aligned} \int_{t-i-1}^{t-i} (\mathcal{E}(\mathbf{U}_{c,T}(s)) - a) e^{-\varepsilon s} ds &\leq e^{-(c+\varepsilon)(t-i-1)} \int_{t-i-1}^{t-i} (\mathcal{E}(\mathbf{U}_{c,T}(s)) - a) e^{cs} ds \\ &\leq e^{-(c+\varepsilon)(t-i-1)} \theta_{c,T}^-(t^-) e^{-\gamma^-(t^- - t + i)} \\ &= e^{(c+\varepsilon)(1-t^-)} \theta_{c,T}^-(t^-) e^{(\gamma^- - c - \varepsilon)(t-t^-)} e^{(c+\varepsilon-\gamma^-)i}. \end{aligned} \quad (3.5.118)$$

Since we assume that $c < \gamma^-$, we have that by choosing any $\varepsilon \in (0, \gamma^- - c)$ it holds

$$\sum_{i \in \mathbb{N}} e^{(c+\varepsilon-\gamma^-)i} = \frac{1}{1 - e^{(c+\varepsilon-\gamma^-)}},$$

which, in combination with (3.5.118) gives (3.5.112) (notice that the case $t > t^- - 1$ presents no problem, as $e^{(\gamma^- - c - \varepsilon)t}$ is then large). Therefore, by (3.5.108) in Lemma 3.5.13 we have that for all $\varepsilon \in (0, \gamma^- - c)$ and $t \in \mathbb{R}$

$$\int_{-\infty}^t \frac{\|U'(s)\|_{\mathcal{L}}^2}{2} e^{-\varepsilon s} ds \leq \overline{M}^- e^{(\gamma^- - c - \varepsilon)t},$$

which, by Cauchy-Schwarz inequality, means that

$$\int_{-\infty}^t \|\mathbf{U}'_{c,T}(s)\|_{\mathcal{L}} ds \leq \left(\frac{e^{\varepsilon t}}{\varepsilon} \int_{-\infty}^t \|\mathbf{U}'_{c,T}(s)\|_{\mathcal{L}}^2 e^{-\varepsilon s} ds \right)^{\frac{1}{2}} \leq \frac{2\overline{M}^-}{\varepsilon} e^{(\gamma^- - c)t}, \quad (3.5.119)$$

where we have used that $\lim_{s \rightarrow -\infty} e^{\varepsilon s} = 0$, because $\varepsilon > 0$. Since $c < \gamma^-$, in particular inequality (3.5.119) implies the existence of some $\tilde{v}^- \in \mathcal{L}$ such that

$$\lim_{t \rightarrow -\infty} \|\mathbf{U}_{c,T}(t) - \tilde{v}^-\|_{\mathcal{L}} = 0. \quad (3.5.120)$$

Inequality (3.5.119) also implies that for all $\tilde{t} < t \in \mathbb{R}$ we have

$$\|\mathbf{U}_{c,T}(t) - \mathbf{U}_{c,T}(\tilde{t})\|_{\mathcal{L}} \leq \frac{2\overline{M}^-}{\varepsilon} e^{(\gamma^- - c)t},$$

which by taking the limit $\tilde{t} \rightarrow -\infty$ and using (3.5.120) gives (3.5.113), by choosing for instance $\varepsilon = (\gamma^- - c)/2 \in (0, \gamma^- - c)$ and $M^- = \frac{2\overline{M}^-}{\varepsilon} > 0$. □

Remark 3.5.5. Notice that combining (3.5.112) in Proposition 3.5.4 with (3.5.108) in Lemma 3.5.13, we obtain in particular that $\mathbf{U}'_{c,T} \in L^2(\mathbb{R}, \mathcal{L})$ provided that $c < \gamma^-$ (see the statements of the results for the notations).

3.5.11 Proof of Theorem 3.6 completed

Assume first that (H3.7') holds. Let (c^*, \mathbf{U}) be the solution to (3.4.1) with conditions at infinity (3.4.2) and (3.4.32) given by Proposition 3.5.2 and Lemma 3.5.3. Since we took $c^* = \sup \mathcal{C}$ with \mathcal{C}

as in (3.5.86), inequality (3.5.87) in Lemma 3.5.11 implies that

$$c^\star \leq \frac{\sqrt{-2a}}{d_0}$$

which by (3.4.33) in (H3.8') implies that

$$c^\star < \gamma^-$$

so that we can apply (3.5.113) in Proposition 3.5.4 to \mathbf{U} , as it is a minimizer of \mathbf{E}_{c^\star} in X_{T^\star} for some $T^\star \geq 1$. Therefore, (3.4.34) holds for \mathbf{U} , which completes the proof. \square

3.5.12 Proof of Theorem 3.7 completed

Since we assume that (H3.8') holds and $\tilde{\mathbf{U}}$ is such that $\tilde{\mathbf{U}} \in X_T$ for some $T \geq 1$ and $\mathbf{E}_{c^\star}(\tilde{\mathbf{U}}) = 0$, then by Proposition 3.5.2, we can apply Proposition 3.5.4 to \mathbf{U} . We recall that by Remark 3.5.5 we have that $\mathbf{U}' \in L^2(\mathbb{R}, \mathcal{L})$ and by (3.5.112) in Proposition 3.5.4 we have that $\mathcal{E} \circ \mathbf{U} \in L^1((-\infty, t])$ for all $t \in \mathbb{R}$. Therefore, we can find a sequence $(t_n^-)_{n \in \mathbb{N}}$ in \mathbb{R} such that

$$\lim_{n \rightarrow \infty} t_n^- = -\infty \quad (3.5.121)$$

and

$$\lim_{n \rightarrow \infty} \left(\mathcal{E}(\mathbf{U}(t_n^-)) - a - \frac{\|\mathbf{U}'(t_n^-)\|_{\mathcal{L}}^2}{2} \right) = 0. \quad (3.5.122)$$

Similarly, since $\mathbf{E}_c(\mathbf{U}) = 0 < +\infty$, we have that $\mathcal{E} \circ \mathbf{U} \in L^1([t, +\infty))$ for all $t \in \mathbb{R}$, which means that we can find $(t_n^+)_{n \in \mathbb{N}}$ a sequence of real numbers such that

$$\lim_{n \rightarrow \infty} t_n^+ = +\infty \quad (3.5.123)$$

and

$$\lim_{n \rightarrow \infty} \left(\frac{\|\mathbf{U}'(t_n^+)\|_{\mathcal{L}}^2}{2} - \mathcal{E}(\mathbf{U}(t_n^+)) \right) = 0. \quad (3.5.124)$$

Taking the scalar product in \mathcal{L} between equation (3.4.1) and \mathbf{U}' , we obtain

$$\forall t \in \mathbb{R}, \quad \langle \mathbf{U}''(t), \mathbf{U}'(t) \rangle_{\mathcal{L}} - \langle D_{\mathcal{L}} \mathcal{E}(\mathbf{U}(t)), \mathbf{U}'(t) \rangle_{\mathcal{L}} = -c \|\mathbf{U}'(t)\|_{\mathcal{L}}^2$$

so that

$$\forall t \in \mathbb{R}, \quad \langle \mathbf{U}''(t), \mathbf{U}'(t) \rangle_{\mathcal{L}} - (\mathcal{E}(\mathbf{U}(t)))' = -c^\star \|\mathbf{U}'(t)\|_{\mathcal{L}}^2.$$

Fix $n \in \mathbb{N}$. Integrating above in $[t_n^-, t_n^+]$ (which is non-empty up to an extraction) we obtain

$$\int_{t_n^-}^{t_n^+} \langle \mathbf{U}''(t), \mathbf{U}'(t) \rangle_{\mathcal{L}} dt - \mathcal{E}(\mathbf{U}(t_n^+)) + \mathcal{E}(\mathbf{U}(t_n^-)) = -c^\star \int_{t_n^-}^{t_n^+} \|\mathbf{U}'(t)\|_{\mathcal{L}}^2 dt. \quad (3.5.125)$$

Integrating by parts we obtain

$$\int_{t_n^-}^{t_n^+} \langle \mathbf{U}''(t), \mathbf{U}'(t) \rangle_{\mathcal{L}} dt = \|\mathbf{U}'(t_n^+)\|_{\mathcal{L}}^2 - \|\mathbf{U}'(t_n^-)\|_{\mathcal{L}}^2 - \int_{t_n^-}^{t_n^+} \langle \mathbf{U}'(t), \mathbf{U}''(t) \rangle_{\mathcal{L}} dt,$$

which means

$$\int_{t_n^-}^{t_n^+} \langle \mathbf{U}''(t), \mathbf{U}'(t) \rangle_{\mathcal{L}} dt = \frac{1}{2} (\|\mathbf{U}'(t_n^+)\|_{\mathcal{L}}^2 - \|\mathbf{U}'(t_n^-)\|_{\mathcal{L}}^2).$$

Plugging into (3.5.125) we obtain

$$a + \left(\mathcal{E}(\mathbf{U}(t_n^-)) - a - \frac{\|\mathbf{U}'(t_n^-)\|_{\mathcal{L}}^2}{2} \right) + \left(\frac{\|\mathbf{U}'(t_n^+)\|_{\mathcal{L}}^2}{2} - \mathcal{E}(\mathbf{U}(t_n^+)) \right) = -c^* \int_{t_n^-}^{t_n^+} \|\mathbf{U}'(t)\|_{\mathcal{L}}^2 dt.$$

Using (3.5.121), (3.5.122), (3.5.123) and (3.5.124), along with the fact that $\mathbf{U}' \in L^2(\mathbb{R}, \mathcal{L})$, we can pass to the limit $n \rightarrow \infty$ and we get that

$$a = -c^* \int_{\mathbb{R}} \|\mathbf{U}'(t)\|_{\mathcal{L}}^2 dt,$$

which shows (3.4.35). We now show that (3.4.36) holds. Inspecting again the proof of Theorem 3.5, we have that c^* is equal to $c(\mathcal{C})$ as in Corollary 3.5.3. Take $c < c^*$, then by Corollary 3.5.3 we have that $c \in \mathcal{C}$. The definitions of \mathcal{C} in (3.5.86) implies then that

$$\exists T \geq 1, \inf_{U \in X_T} \mathbf{E}_c(U) < 0$$

which, by considering $\tilde{U} \in X_T$ such that $\mathbf{E}_c(\tilde{U}) < 0$ and then the sequence $(\tilde{U}(\cdot + n))_{n \in \mathbb{N}}$ which is contained in X , implies that $\inf_{U \in X} \mathbf{E}_c(U) = -\infty$. If we now take $c > c^*$, we have again by Corollary 3.5.3 that

$$\forall T \geq 1, \inf_{U \in X_T} \mathbf{E}_c(U) \geq 0$$

which means

$$\inf_{U \in X} \mathbf{E}_c(U) = 0.$$

Therefore, (3.4.36) follows. Finally, we have that (3.4.37) is exactly (3.5.87) in Lemma 3.5.11. \square

3.6 Proofs of the main results completed

Once we have proved the abstract results, we are ready to prove the main ones. In order to do such a thing, we need to show that the main problem can be put into the abstract framework. This is shown in Lemma 3.6.1 which is in Section 3.6.1. The next sections are then devoted to the conclusion of the proofs of the main results, which are Theorems 3.1, 3.2 and 3.4. However, as pointed out before, we do not have a counterpart of Theorem 3.3 in the abstract setting, which means that we prove it using arguments relative to the main setting.

3.6.1 Proving the link between the main setting and the abstract setting

The following result establishes the link between the main assumptions and the abstract ones. As a consequence, the main results can be deduced from the abstract framework, which we have already established.

Lemma 3.6.1. *Assume that (H3.5) holds. Set*

$$\mathcal{L} := L^2(\mathbb{R}, \mathbb{R}^k), \quad \mathcal{H} := H^1(\mathbb{R}, \mathbb{R}^k), \quad \tilde{\mathcal{H}} := H^2(\mathbb{R}, \mathbb{R}^k), \quad (3.6.1)$$

$$r_0^\pm := \rho_0^\pm \quad (3.6.2)$$

and

$$\forall v \in \mathcal{L}, \quad \mathcal{E}(v) := \begin{cases} E(\psi + v) - \mathfrak{m}^+ & \text{if } v \in \mathcal{H}, \\ +\infty & \text{otherwise,} \end{cases}$$

where \mathfrak{m}^+ was introduced in (H3.5), the constants ρ_0^\pm are those from (3.2.5) and the function ψ is any smooth function in $X(\sigma^-, \sigma^+)$ converging to σ^\pm at $\pm\infty$ at an exponential rate and such that $\psi' \in H^2(\mathbb{R}, \mathbb{R}^k)$. Under this choice, assumptions (H3.1'), (H3.2'), (H3.3'), (H3.4'), (H3.5') and (H3.6') hold. Moreover, we have:

- If (H3.6) holds, then (H3.7') holds.
- If (H3.7) holds, then (H3.8') holds.

Proof. The fact that the functional

$$v \in \mathcal{H} \rightarrow E(\psi + v)$$

is well defined and, moreover, is a C^1 functional on $(\mathcal{H}, \|\cdot\|_{\mathcal{H}})$ is proven by classical arguments. See for instance Bisgard [50], Montecchiari and Rabinowitz [123]. See also Chapter 2 for the precise statement in this setting. We obviously have $\mathcal{F}^\pm = \mathcal{F}^\pm - \psi$. We now pass to prove that the assumptions are satisfied.

Assumption (H3.1') is satisfied:

The fact that \mathcal{E} is weakly lower semicontinuous in \mathcal{L} is standard, see Lemma 3.1 in [157]. We already invoked Lemma 2.1 in [153] in Schatzmann so that (3.2.5) and (3.2.6) hold. That is, due to (3.6.2) we have that if

$$\inf_{\tau \in \mathbb{R}} \|v + \psi - \mathfrak{q}^\pm(\cdot + \tau)\|_{\mathcal{L}} \leq r_0^\pm \quad (3.6.3)$$

there is a unique $\tau(v) \in \mathbb{R}$ which attains the infimum in (3.6.3). Moreover, the correspondence $v \rightarrow \tau(v)$ defined on the subset of \mathcal{L} composed of functions that verify (3.6.3) is of class C^2 . Therefore, the applications

$$P^\pm : v \in \mathcal{F}_{r_0^\pm/2}^\pm \rightarrow \mathfrak{q}^\pm(\cdot + \tau(v)) - \psi \in \mathcal{F}^\pm, \quad (3.6.4)$$

satisfy the properties required. Finally, we have that estimate (3.4.6) follows by Lemma 3.2 in [125], up to modifying the choice of the constants ρ_0^\pm, β_0^\pm .

Assumption (H3.2') is satisfied:

By (3.6.1), we have that $\tilde{\mathcal{H}} \subset \mathcal{H} \subset \mathcal{L}$ and the associated norms verify

$$\|\cdot\|_{\mathcal{L}} \leq \|\cdot\|_{\mathcal{H}} \leq \|\cdot\|_{\tilde{\mathcal{H}}}.$$

As we pointed out before, \mathcal{E} restricted to $(\mathcal{H}, \|\cdot\|_{\mathcal{H}})$ is a C^1 functional. Moreover, as shown in [50, 123], we have that the differential is given by

$$\forall v \in \mathcal{H}, \quad D\mathcal{E}(v) : w \in \mathcal{H} \rightarrow \int_{\mathbb{R}} (\langle \psi' + v', w' \rangle + \langle \nabla V(\psi + v), w \rangle) \in \mathbb{R}. \quad (3.6.5)$$

Let now $v \in \tilde{\mathcal{H}}$, since ψ is smooth with good behavior at infinity we can integrate by parts to get

$$\begin{aligned} \forall w \in \tilde{\mathcal{H}}, \quad D\mathcal{E}(v)(w) &= \int_{\mathbb{R}} \langle -(\psi'' + v'') + \nabla V(\psi + v), w \rangle \\ &= \langle D_{\mathcal{L}}\mathcal{E}(v), w \rangle_{\mathcal{L}}, \end{aligned} \quad (3.6.6)$$

where we have set

$$D_{\mathcal{L}}\mathcal{E} : v \in (\tilde{\mathcal{H}}, \|\cdot\|_{\tilde{\mathcal{H}}}) \rightarrow -(\psi'' + v'') + \nabla V(\psi + v) \in (\mathcal{L}, \|\cdot\|_{\mathcal{L}}),$$

which, by standard arguments, can be shown to be continuous. Notice that (3.4.10) in (H3.2') is exactly (3.6.6) above, which concludes this part of the proof.

Assumption (H3.3') is satisfied:

Let $(\mathbf{v}_n^-)_{n \in \mathbb{N}}$ be a \mathcal{L} -bounded sequence in \mathcal{F}^- . We want to show the existence of a subsequence of $(\mathbf{v}_n^-)_{n \in \mathbb{N}}$ strongly convergent in \mathcal{H} . Since

$$\mathcal{F}^- = \mathcal{F}^- - \psi = \{\mathbf{q}^-(\cdot + \tau) - \psi : \tau \in \mathbb{R}\},$$

we have that $(\mathbf{v}_n^-)_{n \in \mathbb{N}} = (\mathbf{q}^-(\cdot + \tau_n) - \psi)_{n \in \mathbb{N}}$ with $(\tau_n)_{n \in \mathbb{N}}$ a bounded sequence of real numbers. Since such a sequence is bounded in \mathcal{L} , we know that, up to an extraction, there exists $\tilde{v} \in \mathcal{L}$ such that $\mathbf{q}^-(\cdot + \tau_n) - \psi \rightharpoonup \tilde{v}$ weakly in \mathcal{L} . Due to the weak lower semicontinuity of \mathcal{E} , we have that

$$\mathcal{E}(\tilde{v}) \leq \liminf_{n \rightarrow +\infty} \mathcal{E}(\mathbf{q}^-(\cdot + \tau_n) - \psi) = a,$$

which, by minimality, implies that $\mathcal{E}(\tilde{v}) = 0$, that is, $\tilde{v} \in \mathcal{F}^-$. We can then write $\tilde{v} = \mathbf{q}^-(\cdot + \tau) - \psi$ for some $\tau \in \mathbb{R}$. Now, notice that, by the compactness of minimizing sequences (3.2.3), there exists a sequence $(\tau'_n)_{n \in \mathbb{N}}$ of real numbers such that, up to an extraction

$$\mathbf{q}^-(\cdot + \tau_n + \tau'_n) - \mathbf{q}^- \rightarrow 0 \text{ strongly in } \mathcal{H} \quad (3.6.7)$$

which necessarily implies that

$$\tau_n + \tau'_n \rightarrow 0$$

and, since $(\tau_n)_{n \in \mathbb{N}}$ is bounded, we have that $(\tau'_n)_{n \in \mathbb{N}}$ is a bounded sequence as well. Therefore, we can assume, up to an extraction, that $\tau'_n \rightarrow \tau$. Combining this information with (3.6.7), we obtain

$$\mathbf{q}^-(\cdot + \tau_n) - \mathbf{q}^-(\cdot - \tau^-) \rightarrow 0 \text{ strongly in } \mathcal{H},$$

which establishes the claim.

We need to show the same for \mathcal{F}^+ . The argument is identical to the one above, except for the fact that the compactness of minimizing sequences is replaced by 3. in assumption (H3.5), which is in fact stronger, and we use that the elements in \mathcal{F}^+ are local minimizers (instead of global ones), which does not require any modification of the reasoning.

Assumption (H3.4') is satisfied:

More precisely, we show that 2. in (H3.4') holds. Notice that since the results are local in nature and (H3.1') implies that locally the situation does not change between \mathcal{F}^- and \mathcal{F}^+ , we can treat

both cases together. Let $(v, \mathbf{v}^\pm) \in \mathcal{F}_{r_0^\pm}^\pm$. Let $\tau(v)$ be given by the projection map defined in (3.6.4). We have that $\mathbf{v}^\pm = \mathbf{q}^\pm(\cdot + \tau) - \psi$ for some $\tau \in \mathbb{R}$. Define

$$\hat{P}_{(v, \mathbf{v}^\pm)}^\pm : w \in \mathcal{L} \rightarrow w(\cdot - \tau(v) + \tau) - \psi + \psi(\cdot - \tau(v) + \tau) \in \mathcal{L}.$$

Clearly, using the definition of the projection in (3.6.4) and $\tau(v)$

$$\begin{aligned} \|\hat{P}_{(v, \mathbf{v}^\pm)}^\pm(v) - \mathbf{v}^\pm\|_{\mathcal{L}} &= \|v(\cdot - \tau(v) + \tau) - (\mathbf{q}^\pm(\cdot + \tau) - \psi(\cdot - \tau(v) + \tau))\|_{\mathcal{L}} \\ &= \|v - (\mathbf{q}^\pm(\cdot + \tau(v)) - \psi)\|_{\mathcal{L}} = \inf_{\tilde{\tau} \in \mathbb{R}} \|v - (\mathbf{q}^\pm(\cdot + \tilde{\tau}) - \psi)\|_{\mathcal{L}}, \end{aligned}$$

meaning that

$$P^\pm(\hat{P}_{(v, \mathbf{v}^\pm)}^\pm(v)) = \mathbf{v}^\pm$$

and

$$\text{dist}_{\mathcal{L}}(\hat{P}_{(v, \mathbf{v}^\pm)}^\pm(v), \mathcal{F}^\pm) = \text{dist}_{\mathcal{L}}(v, \mathcal{F}^\pm)$$

which are (3.4.11) and (3.4.12) respectively. Next, notice that for $(w_1, w_2) \in \mathcal{L}^2$ and $h \in \mathbb{R}$ we have

$$\hat{P}_{(v, \mathbf{v}^\pm)}^\pm(w_1 + hw_2) = \hat{P}_{(v, \mathbf{v}^\pm)}^\pm(w_1) + hw_2(\cdot - \tau(v) + \tau)$$

so that $\hat{P}_{(v, \mathbf{v}^\pm)}^\pm$ is differentiable and

$$\forall (w_1, w_2) \in \mathcal{L}^2, \quad D(\hat{P}_{(v, \mathbf{v}^\pm)}^\pm)(w_1, w_2) = w_2(\cdot - \tau(v) + \tau)$$

so that

$$\forall (w_1, w_2) \in \mathcal{L}^2, \quad \|D(\hat{P}_{(v, \mathbf{v}^\pm)}^\pm)(w_1, w_2)\|_{\mathcal{L}} = \|w_2\|_{\mathcal{L}},$$

which is (3.4.13). Finally, notice that $v \in \mathcal{H}$ if and only if $\hat{P}_{(v, \mathbf{v}^\pm)}^\pm \in \mathcal{H}$. Assuming that $v \in \mathcal{H}$ we have

$$\begin{aligned} \mathcal{E}(\hat{P}_{(v, \mathbf{v}^\pm)}^\pm(v)) &= \mathcal{E}(v(\cdot - \tau(v) + \tau) - \psi + \psi(\cdot - \tau(v) + \tau)) \\ &= E(v(\cdot - \tau(v) + \tau) + \psi(\cdot - \tau(v) + \tau)) = E(\psi + v) = \mathcal{E}(v) \end{aligned}$$

and if $v \in \mathcal{L} \setminus \mathcal{H}$, we have $\mathcal{E}(\hat{P}_{(v, \mathbf{v}^\pm)}^\pm(v)) = +\infty = \mathcal{E}(v)$. Therefore, (3.4.14) holds. We have then showed that (H3.4') holds.

Assumption (H3.5') is satisfied:

Lemma 2.1 in Schatzmann states that for $v \in \mathcal{F}_{\mathcal{H}, r_0^\pm}^\pm$ the problem

$$\inf_{\tau \in \mathbb{R}} \|v + \psi - \mathbf{q}^\pm(\cdot + \tau)\|_{\mathcal{H}}$$

has a unique solution $\tau_{\mathcal{H}}(v) \in \mathbb{R}$ and the projection map

$$P_{\mathcal{H}}^\pm : v \in \mathcal{F}_{\mathcal{H}, r_0^\pm}^\pm \rightarrow \mathbf{q}^\pm(\cdot + \tau_{\mathcal{H}}(v)) \in \mathcal{F}^\pm$$

is C^1 with respect to the \mathcal{H} -norm. Next, we have that (3.4.15) is *Corollary 2.3* in [153]. Finally, the fact that (3.4.16) implies (3.4.17) for the constants C^\pm (up to possibly increasing) is a consequence of the compactness of the minimizing sequences. See for example *Corollary 3.2* in [153].

Assumption (H3.6') is satisfied:

We show the existence of the map \mathfrak{P} . We follow *Lemma 3.3* in [153]. Let $R_0 > 0$ be the constant from (H3.2). For $R \geq R_0$ define in \mathbb{R}^k

$$f_R(u) := \begin{cases} u & \text{if } |u| \leq R, \\ R \frac{u}{|u|} & \text{otherwise,} \end{cases}$$

where R_0 is the constant from (H3.2). For $u \in \mathbb{R}^k$ such that $|u| \leq R$, we have $f_R(u) = u$. Assume that $u \in \mathbb{R}^k$ is such that $|u| > R$. In that case, there exists $\xi \in \left(\frac{R}{|u|}, 1\right)$ such that

$$V(u) = V(f_R(u)) + \langle \nabla_u V(\xi u), u - f_R(u) \rangle = V(f_R(u)) + \frac{1}{\xi} \left(1 - \frac{R}{|u|}\right) \langle \nabla_u V(\xi u), \xi u \rangle$$

which, by (H3.2) implies

$$\forall R \geq R_0, \forall u \in \mathbb{R}^k : |u| > R, \quad V(u) \geq V(f_R(u)) + \frac{1}{\xi} \left(1 - \frac{R}{|u|}\right) \nu_0 |\xi u|^2 > V(f_R(u)). \quad (3.6.8)$$

In particular, we have shown

$$\forall R \geq R_0, \forall u \in \mathbb{R}^k, \quad V(u) \geq V(f_R(u)). \quad (3.6.9)$$

Next, let $J \subset \mathbb{R}$ be a compact interval and $v \in H^1(J, \mathbb{R}^k)$. For $R \geq R_0$, consider the function $v_R := f_R \circ v$. Since we clearly have that for all $u \in \mathbb{R}^k$, $|f_R(u)| \leq |u|$, it holds that $v_R \in L^2(J, \mathbb{R}^k)$. Next, we have that f_R is the projection onto the closed ball of center 0 and radius R , so that it is non-expansive. As a consequence, we have

$$\forall R \geq R_0, \forall u \in \mathbb{R}^k, \quad |Df_R(u)| \leq 1. \quad (3.6.10)$$

Therefore, applying the chain rule we obtain

$$\text{for a. e. } t \in J, \quad |v'_R(t)| \leq |v'(t)|,$$

which means that $v_R \in H^1(J, \mathbb{R}^k)$ and, combining with (3.6.9) we obtain

$$E(v_R; J) \leq E(v; J) \quad (3.6.11)$$

and, by (3.6.8) the equality above holds if and only if $v_R = v$. Let now

$$R_{\max} := 2 \max\{R_0, \|\mathfrak{q}^-\|_{L^\infty(\mathbb{R}, \mathbb{R}^k)}, \|\mathfrak{q}^+\|_{L^\infty(\mathbb{R}, \mathbb{R}^k)}\}.$$

Consider now the application

$$\mathfrak{P} : v \in \mathcal{L} \rightarrow f_{R_{\max}} \circ (v + \psi) - \psi \in \mathcal{L} \quad (3.6.12)$$

which is well-defined due to the previous considerations. Moreover, the choice of R_{\max} implies that \mathfrak{P} equals the identity on $\{\mathfrak{q}^-(\cdot + \tau) - \psi : \tau \in \mathbb{R}\}$ and $\{\mathfrak{q}^+(\cdot + \tau) - \psi : \tau \in \mathbb{R}\}$, which is exactly (3.4.22).

Inequality (3.6.11) gives (3.4.20). Finally, using (3.6.10) we have that

$$\begin{aligned} \forall (v_1, v_2) \in \mathcal{L}^2, \quad \|\mathfrak{P}(v_1) - \mathfrak{P}(v_2)\|_{\mathcal{L}}^2 &= \int_{\mathbb{R}} |f_{R_{\max}} \circ (v_1 + \psi) - f_{R_{\max}} \circ (v_2 + \psi)|^2 \\ &\leq \int_{\mathbb{R}} \sup_{u \in \mathbb{R}^k} |Df_{R_{\max}}(u)|^2 |v_1 - v_2|^2 \\ &\leq \int_{\mathbb{R}} |v_1 - v_2|^2 = \|v_1 - v_2\|_{\mathcal{L}}^2, \end{aligned}$$

which is (3.4.21). Therefore, our map \mathfrak{P} satisfies the required properties. Next, let \mathbf{W} be a local minimizer of \mathbf{E}_c . We show that \mathbf{W} satisfies the desired regularity properties, that is, $\mathbf{W} \in \mathcal{A}(I)$ with $\mathcal{A}(I)$ as in (3.4.23). Write $\overline{\mathbf{W}} := \mathbf{W} + \psi$. We assume that for all $t \in I$, $\mathbf{W}(t) = \mathfrak{P}(\mathbf{W})$. The definition of \mathfrak{P} in (3.6.12) implies that

$$\forall (x_1, x_2) \in I \times \mathbb{R}, \quad \overline{\mathbf{W}}(x_1, x_2) = f_{R_{\max}}(\overline{\mathbf{W}}(x_1, x_2))$$

so that

$$\|\overline{\mathbf{W}}\|_{L^\infty(I \times \mathbb{R}, \mathbb{R}^k)} \leq R_{\max}.$$

Therefore, by classical elliptic regularity arguments, we have that, with the obvious identifications, $\overline{\mathbf{W}}$ solves

$$-c \partial_{x_1} \overline{\mathbf{W}} - \Delta \overline{\mathbf{W}} = -\nabla_u V(\overline{\mathbf{W}}) \text{ in } I \times \mathbb{R}$$

and for all $\alpha \in (0, 1)$ we have that $\overline{\mathbf{W}} \in \mathcal{C}^{3, \alpha}(I_C \times \mathbb{R}, \mathbb{R}^k)$ for any compact $I_C \subset I$. It is then clear that

$$\mathbf{W} \in \mathcal{C}^2(I_C, L^2(\mathbb{R}, \mathbb{R}^k)) \cap \mathcal{C}^1(I_C, H^1(\mathbb{R}, \mathbb{R}^k)) \cap \mathcal{C}^0(I_C, H^2(\mathbb{R}, \mathbb{R}^k))$$

for any $I_C \subset I$ compact, which means that $\mathbf{W} \in \mathcal{A}(I)$.

Assumption (H3.6) implies (H3.7'): Immediate.

Assumption (H3.7) implies (H3.8'): Immediate. □

Once Lemma 3.6.1 has been established, the main results are easily obtained by rephrasing the abstract ones.

3.6.2 Proof of Theorem 3.1 completed

Assume that (H3.6) holds. Notice that (H3.6) implies that (H3.1), (H3.2), (H3.3), (H3.4) and (H3.5) hold. Therefore, applying Lemma 3.6.1 we have that, choosing the objects as in its statement, we get that (H3.3'), (H3.4'), (H3.5'), (H3.6') and (H3.7') hold. Those are exactly the assumptions which are needed for Theorem 3.5 to hold, meaning that we obtain (c^*, \mathbf{U}) with $c^* > 0$ and $\mathbf{U} \in \mathcal{A}(\mathbb{R}) \cap X$, with $\mathcal{A}(\mathbb{R})$ as in (3.4.23) and X as in (3.4.31), which solves

$$\mathbf{U}'' - D_{\mathcal{L}} \mathcal{E}(\mathbf{U}) = -c\mathbf{U}' \text{ in } \mathbb{R} \tag{3.6.13}$$

and satisfies the conditions at infinity

$$\exists T^- \leq 0 : \forall t \leq T^-, \quad \mathbf{U}(t) \in \mathcal{F}_{r_0/2}^- \text{ and } \exists \mathbf{v}^+(\mathbf{U}) \in \mathcal{F}^+ : \lim_{t \rightarrow +\infty} \|\mathbf{U}(t) - \mathbf{v}^+(\mathbf{U})\|_{\mathcal{H}} = 0. \tag{3.6.14}$$

We now pass to prove each of the three statements of Theorem 3.1 separately:

1. **Existence.** Recall that for all $t \in \mathbb{R}$ we have $\mathbf{U}(t) \in \mathcal{L} = L^2(\mathbb{R}, \mathbb{R}^k)$. Let us then define

$$\mathfrak{U} : (x_1, x_2) \in \mathbb{R}^2 \rightarrow \mathbf{U}(x_1)(x_2) \in \mathbb{R}^k. \quad (3.6.15)$$

It is clear then that since $\mathbf{U} \in \mathcal{A}(\mathbb{R})$ we have that $\mathfrak{U} \in \mathcal{C}_{\text{loc}}^2(\mathbb{R}, \mathbb{R}^k)$ and, moreover for all (x_1, x_2) and any pair of index $(i, j) \in \{0, 1, 2\}^2$ such that $i + j \leq 2$ we have

$$\partial_{x_1}^i \partial_{x_2}^j \mathfrak{U}(x_1, x_2) = (\mathbf{U}^{(i)}(x_1))^{(j)}(x_2) \quad (3.6.16)$$

where for a curve f taking values in a Hilbert space we denote by $f^{(i)}$ its i -th derivative, $i \in \mathbb{N}$. As a consequence of (3.6.13), (3.6.16) and the formula for $D\mathcal{E}$ when we make $\mathcal{E} = E - \mathfrak{m}^+$ (see (3.6.5)) we obtain that

$$-c \partial_{x_1} \mathfrak{U} - \Delta \mathfrak{U} = -\nabla_u V(\mathfrak{U}) \text{ in } \mathbb{R}^2$$

and by (3.6.14) we obtain that for some $L \in \mathbb{R}$ we have for some $x_1 \leq L$ that $\mathfrak{U}(x_1, \cdot) \in \mathcal{F}_{\rho^-/2}^-$, since we choose $r_0^\pm = \rho^\pm$, so that $\mathcal{F}_{\rho^\pm/2}^\pm = \mathcal{F}_{r_0^\pm/2}^\pm$. The variational characterization (3.2.11) follows directly from Theorem 3.1, using the fact that we have $X = S$ and $\mathbf{E}_c = E_{2,c}$ for all $c > 0$ (again we implicitly identify \mathfrak{U} with \mathbf{U} via (3.6.15)). Finally, we have that for all $t \in \mathbb{R}$, $\mathbf{U}(t) = \mathfrak{P}(\mathbf{U}(t))$. According to the choice of \mathfrak{P} made in Lemma 3.6.1, this implies that $\|\mathfrak{U}\|_{L^\infty(\mathbb{R}, \mathbb{R}^k)} < +\infty$, which by classical Schauder theory and the smoothness properties of V implies that for all $\alpha \in (0, 1)$, $\mathfrak{U} \in \mathcal{C}^{2,\alpha}(\mathbb{R}^2, \mathbb{R}^k)$. The proof of the existence part of Theorem 3.1 is hence completed.

2. **Uniqueness of the speed.** Again, we have that $X = S$ and $\mathbf{E}_c = E_{2,c}$ for all $c > 0$, meaning that the proof of this statement follows from the analogous one in Theorem 3.5.
3. **Exponential convergence.** Using the exponential rate of convergence of \mathbf{U} given by Theorem 3.5, which is (3.4.32), we obtain that for some $b > c^*/2$ it holds

$$\lim_{x_1 \rightarrow +\infty} \|\mathfrak{U}(x_1, \cdot) - \mathfrak{q}^+(\cdot + \tau^+)\|_{H^1(\mathbb{R}, \mathbb{R}^k)} e^{bx_1} = 0$$

for some $\tau^+ \in \mathbb{R}$. This concludes the proof of the statement.

The proof of Theorem 3.1 is concluded. □

3.6.3 Proof of Theorem 3.2 completed

Assume that (H3.6) and (H3.7) hold. Arguing as in the proof of Theorem 3.1, we have that the assumptions of Theorem 3.6 are fulfilled if we choose as in Lemma 3.6.1. Therefore, Theorem 3.2 is readily obtained by a straightforward rewriting of Theorem 3.6. □

3.6.4 Proof of Theorem 3.3

We now provide the proof of Theorem 3.3, which is a consequence of the following results, which are more general as required by Theorem 3.3 and might be of independent interest:

Lemma 3.6.2. Assume that (H3.1), (H3.2) and (H3.3) hold. Let $(\hat{\sigma}_-, \hat{\sigma}_+) \in \Sigma^2$ (possibly equal) and $q \in X(\hat{\sigma}_-, \hat{\sigma}_+)$. Assume moreover that there exists $L^+ \in \mathbb{R}$ and $U \in H_{\text{loc}}^1([L^+, +\infty) \times \mathbb{R}, \mathbb{R}^k)$ uniformly continuous and such that

$$\int_{L^+}^{+\infty} |E(U(x_1, \cdot)) - E(q)| dx_1 < +\infty, \quad (3.6.17)$$

$$\lim_{x_1 \rightarrow +\infty} \|U(x_1, \cdot) - q\|_{L^2(\mathbb{R}, \mathbb{R}^k)} = 0. \quad (3.6.18)$$

Then, it holds that

$$\lim_{x_1 \rightarrow +\infty} \|U(x_1, \cdot) - q\|_{L^\infty(\mathbb{R}, \mathbb{R}^k)} = 0 \quad (3.6.19)$$

and

$$\lim_{x_2 \rightarrow \pm\infty} \|U(\cdot, x_2) - \hat{\sigma}_\pm\|_{L^\infty([L^+, +\infty), \mathbb{R}^k)} = 0. \quad (3.6.20)$$

Similarly, we have the following

Lemma 3.6.3. Assume that (H3.1), (H3.2) and (H3.3) hold. Let $(\hat{\sigma}_-, \hat{\sigma}_+) \in \Sigma^2$ (possibly equal) and $q \in X(\hat{\sigma}_-, \hat{\sigma}_+)$. Assume moreover that there exists $L^- \in \mathbb{R}$ and $U \in H_{\text{loc}}^1((-\infty, L^-] \times \mathbb{R}, \mathbb{R}^k)$ uniformly continuous and such that

$$\int_{-\infty}^{L^-} |E(U(x_1, \cdot)) - E(q)| dx_1 < +\infty, \quad (3.6.21)$$

$$\lim_{x_1 \rightarrow -\infty} \|U(x_1, \cdot) - q\|_{L^2(\mathbb{R}, \mathbb{R}^k)} = 0.$$

Then, it holds that

$$\lim_{x_1 \rightarrow -\infty} \|U(x_1, \cdot) - q\|_{L^\infty(\mathbb{R}, \mathbb{R}^k)} = 0 \quad (3.6.22)$$

and

$$\lim_{x_2 \rightarrow \pm\infty} \|U(\cdot, x_2) - \hat{\sigma}_\pm\|_{L^\infty((-\infty, L^-], \mathbb{R}^k)} = 0. \quad (3.6.23)$$

In the proof of Lemmas 3.6.2 and 3.6.3, we will need to use the following fact:

Lemma 3.6.4. Assume that (H3.1), (H3.2) and (H3.3) hold. Let $(\hat{\sigma}_-, \hat{\sigma}_+) \in \Sigma^2$ (possibly equal) and $q \in X(\hat{\sigma}_-, \hat{\sigma}_+)$. Assume that $(q_n)_{n \in \mathbb{N}}$ is a sequence in $X(\hat{\sigma}_-, \hat{\sigma}_+)$ such that

$$\lim_{n \rightarrow \infty} \|q_n - q\|_{L^2(\mathbb{R}, \mathbb{R}^k)} = 0 \quad (3.6.24)$$

and

$$\lim_{n \rightarrow \infty} E(q_n) = E(q), \quad (3.6.25)$$

then, it holds that

$$\lim_{n \rightarrow \infty} \|q_n - q\|_{H^1(\mathbb{R}, \mathbb{R}^k)} = 0. \quad (3.6.26)$$

Proof. First, notice that

$$\sup_{n \in \mathbb{N}} \|q_n\|_{L^\infty(\mathbb{R}, \mathbb{R}^k)} < +\infty. \quad (3.6.27)$$

Indeed, (3.6.25) implies that $(q'_n)_{n \in \mathbb{N}}$ is bounded in $L^2(\mathbb{R}, \mathbb{R}^k)$ which, in combination with (3.6.24)) means that $(q_n)_{n \in \mathbb{N}}$ is bounded in $H^1(\mathbb{R}, \mathbb{R}^k)$, hence in $L^\infty(\mathbb{R}, \mathbb{R}^k)$. We also have that

$$\nabla V(q) \in L^2(\mathbb{R}, \mathbb{R}^k), \quad (3.6.28)$$

which follows easily from the fact that V is smooth and quadratic near the wells. For all $n \in \mathbb{N}$, we write the following expansion

$$V(q_n) = V(q) + \langle \nabla V(q), q_n - q \rangle + \int_0^1 D^2 V(q + \lambda(q_n - q))(q_n - q)(q_n - q) d\lambda,$$

which holds pointwise in \mathbb{R} . Therefore, Cauchy-Schwartz inequality implies

$$\left(\int_{\mathbb{R}} |V(q_n) - V(q)| \right)^2 \leq \left(\int_{\mathbb{R}} |\nabla V(q)|^2 + \sup_{\substack{u \in \mathbb{R}^k \\ |u| \leq \|q_n - q\|_{L^\infty}}} |D^2 V(u)| \right) \|q_n - q\|_{L^2(\mathbb{R}, \mathbb{R}^k)}^2,$$

hence, by (3.6.27) and (3.6.28) we find a constant $C > 0$ such that for all $n \in \mathbb{N}$

$$\int_{\mathbb{R}} |V(q_n) - V(q)| \leq C \|q_n - q\|_{L^2(\mathbb{R}, \mathbb{R}^k)}$$

which by (3.6.24) means that $V(q_n) - V(q) \rightarrow 0$ in $L^1(\mathbb{R}, \mathbb{R}^k)$. As a consequence, (3.6.25) implies that

$$\lim_{n \rightarrow \infty} \|q'_n\|_{L^2(\mathbb{R}, \mathbb{R}^k)} = \|q'\|_{L^2(\mathbb{R}, \mathbb{R}^k)}. \quad (3.6.29)$$

Suppose now by contradiction that (3.6.26) does not hold. Then, we can find a subsequence $(q_{n_m})_{m \in \mathbb{N}}$ and $\hat{\delta} > 0$ such that for all $m \in \mathbb{N}$

$$\|q_{n_m} - q\|_{H^1(\mathbb{R}, \mathbb{R}^k)} \geq \hat{\delta}. \quad (3.6.30)$$

Since $(q'_{n_m})_{m \in \mathbb{N}}$ is bounded in $L^2(\mathbb{R}, \mathbb{R}^k)$, it converges weakly in L^2 up to an extraction, and the limit is q' by uniqueness of the limit in the sense of distributions. By (3.6.29), we have that such a subsequence also converges strongly in $L^2(\mathbb{R}, \mathbb{R}^k)$, which combining with (3.6.24) contradicts (3.6.30). \square

We now prove Lemma 3.6.2. The proof of Lemma 3.6.3 being analogous, we skip it.

Proof of Lemma 3.6.2. Assume by contradiction that (3.6.19) does not hold. Then, we can find a sequence $(x_{1,n})_{n \in \mathbb{N}}$ in $[L^+, +\infty) \times \mathbb{R}$ such that $x_{1,n} \rightarrow +\infty$ as $n \rightarrow \infty$ as well as $\hat{\delta} > 0$ such that for all $n \in \mathbb{N}$

$$\|U(x_{1,n}, \cdot) - q\|_{L^\infty(\mathbb{R}, \mathbb{R}^k)} \geq \hat{\delta}.$$

By uniform continuity, there exists $\nu > 0$ such that for all $n \in \mathbb{N}$ we have

$$\max_{x_1 \in [x_{1,n} - \nu, x_{1,n} + \nu]} \|U(x_{1,n}, \cdot) - q\|_{L^\infty(\mathbb{R}, \mathbb{R}^k)} \geq \frac{\hat{\delta}}{2} \quad (3.6.31)$$

Let $A := \cup_{n \in \mathbb{N}} [x_{1,n} - \nu, x_{1,n} + \nu]$. By (3.6.17) we have that

$$\int_A (E(U(x_{1,n}, \cdot)) - E(q)) dx_1 < +\infty,$$

and since A has positive measure and it is unbounded by above, we find a sequence $(y_{1,n})_{n \in \mathbb{N}}$ in A such that $y_{1,n} \rightarrow +\infty$ as $n \rightarrow \infty$ and $\lim_{n \rightarrow \infty} E(U(y_{1,n}, \cdot)) = E(q)$. Combining this fact with (3.6.18),

we have that assumptions (3.6.24) and (3.6.25) in Lemma 3.6.4 hold, which means that

$$\lim_{n \rightarrow \infty} \|U(y_{1,n}, \cdot) - q\|_{H^1(\mathbb{R}, \mathbb{R}^k)} = 0$$

which contradicts (3.6.31). Therefore, we have shown that (3.6.19) holds. In order to prove (3.6.20), we first show that there exists $\underline{L}^+ \leq L^+$ such that

$$\lim_{x_2 \rightarrow \pm\infty} \|U(\cdot, x_2) - \hat{\sigma}_{\pm}\|_{L^\infty([\underline{L}^+, +\infty), \mathbb{R}^k)} = 0. \quad (3.6.32)$$

We prove (3.6.32) by contradiction. The other case being handled in an analogous fashion, assume that there exists a sequence $(x_{2,n})_{n \in \mathbb{N}}$ in \mathbb{R} such that $x_{2,n} \rightarrow +\infty$ as $n \rightarrow \infty$, a sequence $(x_{1,n})_{n \in \mathbb{N}}$ in $[L^+, +\infty)$ tending to $+\infty$ and $\hat{\delta} > 0$ such that for all $n \in \mathbb{N}$

$$|U(x_{1,n}, x_{2,n}) - \hat{\sigma}_+| \geq \hat{\delta}. \quad (3.6.33)$$

Since we already proved that (3.6.19) holds, there exists $N_1 \in \mathbb{N}$ such that for all $n \geq N_1$ we have

$$\|U(x_{1,n}, \cdot) - q\|_{L^\infty(\mathbb{R}, \mathbb{R}^k)} \leq \frac{\hat{\delta}}{4} \quad (3.6.34)$$

and, since $q \in X(\hat{\sigma}_-, \hat{\sigma}_+)$, there exists $\hat{t} \in \mathbb{R}$ such that for all $t \geq \hat{t}$ we have

$$|q(t) - \hat{\sigma}_+| \leq \frac{\hat{\delta}}{4}. \quad (3.6.35)$$

Let $N_2 \in \mathbb{N}$ be such that for all $n \in \mathbb{N}$, $x_{2,n} \geq \hat{t}$. Taking any $n \geq \max\{N_1, N_2\}$, we obtain by (3.6.34) and (3.6.35) that

$$|U(x_{1,n}, x_{2,n}) - \hat{\sigma}_+| \leq \frac{\delta}{2},$$

which contradicts (3.6.33) and establishes (3.6.32). In order to establish (3.6.20), we handle the limit $x_2 \rightarrow +\infty$, as the other one is treated identically. Let $\rho_\Sigma^+ := \text{dist}(\sigma^+, \Sigma \setminus \{\sigma^+\}) > 0$. We claim that for every $\tilde{L} \geq L^+$ we have that if

$$\lim_{x_2 \rightarrow \pm\infty} \|U(\cdot, x_2) - \hat{\sigma}_{\pm}\|_{L^\infty([\tilde{L}, +\infty), \mathbb{R}^k)} = 0, \quad (3.6.36)$$

then

$$\lim_{x_2 \rightarrow \pm\infty} \|U(\cdot, x_2) - \hat{\sigma}_{\pm}\|_{L^\infty([\tilde{L} - \eta_\Sigma^+, +\infty), \mathbb{R}^k)} = 0, \quad (3.6.37)$$

where

$$\eta_\Sigma^+ := \min \left\{ \tilde{L} - L^+, \frac{\rho_\Sigma}{4\|D\mathfrak{U}\|_{L^\infty(\mathbb{R}^2, \mathbb{R}^k)}} \right\}. \quad (3.6.38)$$

Such a claim allows to easily complete the proof of (3.6.20) by a finite induction process, due to the fact that (3.6.32) holds. \square

It remains to establish one claim in the proof of Lemma 3.6.2.

Proof that (3.6.36) implies (3.6.37). Assume that (3.6.36) holds. We show that for every $\varepsilon \in (0, \eta_\Sigma^+)$

we have

$$\lim_{x_2 \rightarrow \pm\infty} \|U(\cdot, x_2) - \hat{\sigma}_\pm\|_{L^\infty([\tilde{L} - \eta_\Sigma^+ + \varepsilon, +\infty), \mathbb{R}^k)} = 0, \quad (3.6.39)$$

which clearly implies (3.6.37) by uniform continuity. Fix then $\varepsilon \in (0, \eta_\Sigma^+)$. By assumption, there exists $\bar{x}_2^+ \in \mathbb{R}$ such that for all $x_2 \geq \bar{x}_2^+$ we have

$$|U(\tilde{L}, x_2) - \sigma^+| \leq \frac{\rho_\Sigma^+}{4},$$

which, by (3.6.38) implies that for all $(x_1, x_2) \in [\tilde{L} - \eta_\Sigma^+ + \varepsilon, \tilde{L}] \times [\bar{x}_2, +\infty)$, it holds

$$|U(x_1, x_2) - \sigma^+| \leq \frac{\rho_\Sigma^+}{2} \quad (3.6.40)$$

and the definition of ρ_Σ^+ gives in turn that for all such (x_1, x_2) and $\sigma \in \Sigma \setminus \{\sigma^+\}$ we have

$$|U(x_1, x_2) - \sigma| \geq \frac{\rho_\Sigma^+}{2}. \quad (3.6.41)$$

Assume now that (3.6.39) does not hold. Then, inequalities (3.6.40) and (3.6.41) imply that we can find a sequence $(x_{1,n}, x_{2,n})_{n \in \mathbb{N}}$ contained in $[\tilde{L} - \eta_\Sigma^+ + \varepsilon, \tilde{L}] \times [\bar{x}_2, +\infty)$, such that $x_{2,n} \rightarrow +\infty$ as $n \rightarrow \infty$ and $\hat{\delta} > 0$ such that for all $n \in \mathbb{N}$ and $\sigma \in \Sigma$

$$|U(x_{1,n}, x_{2,n}) - \sigma| \geq \hat{\delta}.$$

By uniform continuity, we can find $\nu \in (0, \varepsilon)$ such that for all $n \in \mathbb{N}$ and

$$(x_1, x_2) \in B((x_{1,n}, x_{2,n}), \nu) \subset [\tilde{L} - \eta_\Sigma^+, \tilde{L}] \times [\bar{x}_2, +\infty)$$

we have for all $\sigma \in \Sigma$

$$|U(x_1, x_2) - \sigma| \geq \frac{\hat{\delta}}{2}$$

or, equivalently

$$V(U(x_1, x_2)) \geq V_{\hat{\delta}/2} := \min \left\{ V(u) : u \in \mathbb{R}^k, \text{dist}(u, \Sigma) \geq \frac{\hat{\delta}}{2} \right\} \quad (3.6.42)$$

which is positive by (H3.1) and (H3.3). Up to an extraction and since $x_{2,n} \rightarrow +\infty$ as $n \rightarrow \infty$, we can assume that whenever $n \neq m$ we have

$$B((x_{1,n}, x_{2,n}), \nu) \cap B((x_{1,m}, x_{2,m}), \nu) = \emptyset,$$

which, due to the definition of η_Σ^+ in (3.6.38) and (3.6.42) implies that

$$\begin{aligned} \int_{L^+}^{+\infty} |E(U(x_1)) - E(q)| dx_1 &\geq \int_{\tilde{L} - \eta_\Sigma^+}^{\tilde{L}} E(U(x_1)) dx_1 - \eta_\Sigma^+ E(q) \\ &\geq \sum_{n \in \mathbb{N}} \left(\int_{B((x_{1,n}, x_{2,n}), \nu)} V(U(x_1, x_2)) dx_1 dx_2 \right) - \eta_\Sigma^+ E(q) \\ &\geq \sum_{n \in \mathbb{N}} (\pi \nu^2 V_{\hat{\delta}/2}) - \eta_\Sigma^+ E(q) = +\infty, \end{aligned}$$

which enters in contradiction with (3.6.17). Therefore, the claim has been proven. \square

We have now all the necessary ingredients for completing the proof of Theorem 3.3:

Proof of Theorem 3.3 completed. Let (c^\star, \mathfrak{U}) be the solution given by Theorem 3.1, interpreted via the choices made in Lemma 3.6.1. We will invoke Lemma 3.6.2. The L^2 exponential convergence (3.2.12) given by Theorem 3.1 implies in particular that assumption (3.6.18) in Lemma 3.6.2 holds with $U = \mathfrak{U}$, $q = q^+(\cdot + \tau^+)$. Moreover, since $E_{2,c^\star}(\mathfrak{U}) = 0 < +\infty$, assumption (3.6.17) in Lemma 3.6.2 holds for all $L \in \mathbb{R}$ in view of the definition of E_{2,c^\star} (recall that $c^\star > 0$). Finally, we have by Theorem 3.1 that $\mathfrak{U} \in C^{2,\alpha}(\mathbb{R}^2, \mathbb{R}^k)$, $\alpha \in (0, 1)$, so that \mathfrak{U} is uniformly continuous. As a consequence, Lemma 3.6.2 applies and we have (3.6.19) and (3.6.20) for all $L \in \mathbb{R}$, and this is exactly (3.2.15) and (3.2.16) for all $L \in \mathbb{R}$.

Assume now that (H3.7) holds, so that Theorem 3.2 applies. We will show that we can invoke Lemma 3.6.3. We have that (3.2.14) in Theorem 3.2 implies that (3.6.21) in Lemma 3.6.3 holds with $U = \mathfrak{U}$ and $q = q^-(\cdot + \tau^-)$. Moreover, the abstract result Proposition 3.5.4 in combination with Lemma 3.6.1 implies in particular that for all $L \in \mathbb{R}$, (3.6.21) in Lemma 3.6.3 holds. Since \mathfrak{U} is uniformly continuous, Lemma 3.6.3 applies, which means that (3.6.22) holds, so that we have proven (3.2.17) in Theorem 3.3. Moreover, for all $L \in \mathbb{R}$ we have that (3.6.23) holds, which combined with (3.2.16) (which also holds for all $L \in \mathbb{R}$) gives (3.2.18) and completes the proof. \square

3.6.5 Proof of Theorem 3.4 completed

Assume that (H3.6) and (H3.7) hold. Arguing as in the proof of Theorem 3.1, we have that the assumptions of Theorem 3.7 are fulfilled if we choose as in Lemma 3.6.1. Notice that Theorem 3.4 is exactly Theorem 3.7 if we choose the abstract objects as in Lemma 3.6.1. Therefore, Theorem 3.4 is established. \square

Heteroclinic traveling waves of 1D parabolic systems with degenerate stable states

Outline of the current chapter

4.1 Introduction	159
4.2 Statement of the main results	161
4.2.1 Main assumptions and result	161
4.2.2 Conditions at infinity	164
4.2.3 Min-max characterization of the speed	165
4.3 Examples of potentials verifying the assumptions	166
4.4 Extensions and open problems	167
4.4.1 Other classes of potentials	167
4.4.2 The problem of traveling waves for 2D parabolic Allen-Cahn systems	168
4.4.3 Traveling waves between homoclinics	169
4.5 The abstract setting	170
4.6 Proof of the abstract results	172
4.6.1 Preliminaries	173
4.6.2 A semicontinuity result	174
4.6.3 Existence of an infimum for \mathbf{E}_c in Y_T	174
4.6.4 The comparison result	176
4.6.5 Existence of an unconstrained solution	185
4.6.6 Uniqueness of the speed	187
4.6.7 Proof of Theorem 4.4 completed	188
4.6.8 Asymptotic behavior of the constrained solutions	188
4.6.9 Proof of Theorem 4.5 completed	191
4.6.10 Proof of Theorem 4.6 completed	191
4.6.11 On the proof of Theorem 4.7	191

4.7 Proofs of the main results	191
4.7.1 Proof of Theorem 4.1 completed	192
4.7.2 Proof of Theorem 4.2 completed	192
4.7.3 Proof of Theorem 4.3 completed	192

In this chapter, we present the results contained on the preprint [129]. We have made some modifications in the presentation. In particular, we have removed some content which is already present in Chapter 3. Therefore, the present chapter is not self-contained, and the reader will be consistently referred to Chapter 3 for definitions and results.

Abstract. We study the existence of traveling waves for the parabolic system

$$\partial_t w - \partial_x^2 w = -\nabla_{\mathbf{u}} W(w) \text{ in } [0, +\infty) \times \mathbb{R},$$

where W is a potential bounded below and possessing two minima at *different levels*. We say that w is a traveling wave solution of the previous equation if there exist a *speed* $c^\star > 0$ and a *profile* $\mathbf{u} \in \mathcal{C}^2(\mathbb{R}, \mathbb{R}^k)$ such that $w(t, x) = \mathbf{u}(x - c^\star t)$. For a class of potentials W , heteroclinic traveling waves of the previous equation were shown to exist by Alikakos and Katzourakis [13]. More precisely, assuming the existence of two local minimizers of W at *different levels* which, in addition, satisfy some non-degeneracy assumptions, the authors in [13] show the existence of a speed $c^\star > 0$ and profile $\mathbf{u} \in \mathcal{C}^2(\mathbb{R}, \mathbb{R}^k)$ such that \mathbf{u} connects the two local minimizers at infinity. In this chapter, we show that the non-degeneracy assumption on the local minima can be dropped and replaced by another one which allows for potentials possessing degenerate minima. As we do in Chapter 3, our main result is in fact proven for curves which take values in a general Hilbert space and the original result is deduced as a particular case, in the spirit of the earlier works by Monteil and Santambrogio [125] and Smyrnelis [157] devoted to the existence of stationary heteroclinics for Allen-Cahn systems.

Résumé. Nous étudions l'existence des ondes progressives pour le système parabolique

$$\partial_t w - \partial_x^2 w = -\nabla_{\mathbf{u}} W(w) \text{ in } [0, +\infty) \times \mathbb{R},$$

où W est un potentiel borné inférieurement et qui possède deux minimiseurs à des *niveaux différents*. L'on dit que w est une onde progressive pour l'équation précédente s'il existe $c^\star > 0$ et $\mathbf{u} \in \mathcal{C}^2(\mathbb{R}, \mathbb{R}^k)$ tels que $w(t, x) = \mathbf{u}(x - c^\star t)$. Pour une classe de potentiels W , des ondes progressives hétéroclines pour l'équation précédente ont été établies par Alikakos et Katzourakis [13]. Plus précisément, en supposant l'existence de deux minimiseurs à des niveaux différents qui, en plus, vérifient certaines hypothèses de non dégénérescence, les auteurs de [13] démontrent l'existence d'une vitesse c^\star et un profil $\mathbf{u} \in \mathcal{C}^2(\mathbb{R}, \mathbb{R}^k)$ tels que \mathbf{u} connecte les deux minimiseurs locaux à l'infini. Dans ce chapitre, nous prouvons que l'hypothèse de non dégénérescence sur les minimiseurs locaux peut être supprimée et remplacée par une autre qui autorise des potentiels qui possèdent des minimiseurs dégénérés. Comme nous l'avons fait dans le Chapitre 3, notre résultat principal est démontré pour des courbes qui prennent ses valeurs dans un espace de Hilbert général et le résultat original est déduit comme un cas particulier, dans l'esprit des travaux précédents de Monteil et Santambrogio [125] et Smyrnelis [157] consacrés à l'existence des hétéroclines stationnaires pour des systèmes de type Allen-Cahn.

4.1 Introduction

We are concerned with the following one-dimensional parabolic system of reaction-diffusion type

$$\partial_t w - \partial_x^2 w = -\nabla_{\mathbf{u}} W(w) \text{ in } [0, +\infty) \times \mathbb{R}, \quad (4.1.1)$$

where $w : [0, +\infty) \times \mathbb{R} \rightarrow \mathbb{R}^k$ and W is an *unbalanced* double-well potential with possibly degenerate minima, meaning that we can find two minimum points of W at different levels and we do not assume that D^2W is positive definite at such points. For potentials W which possess two non-degenerate minimum points a^-, a^+ in \mathbb{R}^k such that $W(a^-) < 0 = W(a^+)$ satisfying some additional assumptions (namely, non-degeneracy and local radial monotonicity), Alikakos and Katzourakis [13] (see also Lucia, Muratov and Novaga [110] and Risler [145]) showed the existence of traveling wave solutions for (4.1.1) with heteroclinic behavior at infinity. More precisely, they showed that there exist $c^* > 0$ and $\mathbf{u} \in \mathcal{C}^2(\mathbb{R}, \mathbb{R}^k)$ such that

$$-c^* \mathbf{u}' - \mathbf{u}'' = -\nabla_{\mathbf{u}} W(\mathbf{u}) \text{ in } \mathbb{R}$$

and

$$\lim_{t \rightarrow \pm\infty} \mathbf{u}(t) = a^\pm.$$

Such a thing implies that $w : [0, +\infty) \times \mathbb{R} \rightarrow \mathbb{R}^k$ defined as

$$w(t, x) = \mathbf{u}(x - ct)$$

for $(t, x) \in [0, +\infty) \times \mathbb{R}$ is a solution to (4.1.1). More precisely, w is a traveling wave solution of (4.1.1) which propagates with speed c^* and with profile \mathbf{u} . Our main contribution here is to show that such solutions exist when we consider some class potentials W with possibly degenerate and non-isolated minima. More precisely, we assume that W takes the minimum values on sets, instead of isolated points, and that it satisfies suitable properties around such sets. Our motivation comes from Chapter 3. There, traveling waves solutions for some classes of two dimensional parabolic Allen-Cahn systems are obtained. The approach of the proof is to deduce the result from a more general one, proved in an abstract setting on Hilbert spaces, following ideas that worked for similar problems (see Monteil and Santambrogio [125] and Smyrnelis [157]). More precisely, one assumes that the potential W is defined in a Hilbert space \mathcal{H} which has no restriction in its dimension, so that in case \mathcal{H} is infinite-dimensional and W is suitably chosen, one recovers a system which has more than one dimension in space. In this chapter, we show that the abstract approach used in our previous work can be also used to recover results in the *finite-dimensional setting*. For this we mean that $\mathcal{H} = \mathbb{R}^k$, so that the resulting equation is of the type (4.1.1), which is 1D on space. These results are not included in [13]. The abstract result of Chapter 3 applies to our problem, but the main result of this chapter follows by a modified version of such abstract result, in which a key assumption in Chapter 3 (assumption (H3.7')) is replaced by another one playing the same role, (H4.1'). Such an assumption is the natural adaptation of the one used by Alikakos and Katzourakis [13] to the case of degenerate minima, and we use it following the ideas from [13] but with more involved arguments, which is not surprising since the fact that the minima might be degenerate adds extra difficulties. Nevertheless, at present we are only able to apply the abstract result of

this chapter to the finite-dimensional problem, meaning that we do not use the full strength of the abstract setting, which could be useful in other situations. We also mention that the result we obtain is weaker than that by Alikakos and Katzourakis [13] since we cannot prove convergence of the profile to the global minimum at $-\infty$, only that it stays close to it for small enough times. This might obey to merely technical limitations, but it could also be due to the fact that we allow for degenerate minima. However, we show that under an additional assumption (essentially, either an upper bound $W(a^+)$ or the set of global minimizers is a singleton, see (H4.4)) one essentially recovers the full result from [13] for our setting. We also point out that in Section 4.3, we provide some nontrivial examples of potentials for which the results of [13] do not apply but ours do.

The scheme of the proof of the abstract result of this chapter is analogous to that in Chapter 3, which is based on the variational approach of Alikakos and Katzourakis [13], inspired by that in Alikakos and Fusco [11]. While the existence of a variational structure in the context of (some) reaction-diffusion problems is known since Fife and McLeod [79, 78], it has not been widely used in the previous literature. Besides [13] and Chapter 3, other references which use such a variational structure for studying traveling waves in reaction-diffusion problems are Bouhours and Nadin [55], Lucia, Muratov and Novaga [111, 110], Muratov [128], Risler [145, 146, 144] and, more recently, Chen, Chien and Huang [68].

In [13] and Chapter 3, one considers a family of constrained minimization problems and the main difficulty of the problem can be reduced to excluding a degenerate oscillatory behavior for the minimizers of these problems, as the energy density under consideration changes its sign due to the fact that the connected minima are at different levels. In [13], this is done by imposing a non-degeneracy assumption, as well as radial monotonicity properties, on the local minima and then using the ODE system along with optimality of the minimizers. For several reasons, an assumption of this type was not available to us in Chapter 3, so that we needed to replace it by a different one which allowed an analogous type of conclusion. Such an assumption consists essentially on an upper bound on the difference between the energy of the minima. The abstract result of this chapter further replaces the previous assumption by another one which plays the same role (but which does not apply to the problem considered in Chapter 3) and still allows for potentials with degenerate minima. More precisely, it allows us to apply to the constrained minimizers an argument based on the use of the ODE system, in the spirit of [13] but more involved in some aspects. The reader is referred to Chapter 3 for a detailed scheme of proof as well as a discussion on related literature.

In order to conclude this paragraph, we recall that in the realm of *scalar* reaction-diffusion problems one can obtain important results of existence and qualitative properties of traveling waves by means of the maximum principle and comparison results. There is a large literature on the subject, the more classical papers are Fife and McLeod [79, 78], Aronson and Weinberger [20], Berestycki and Nirenberg [39], all devoted to the Fisher KPP equation (introduced by Fisher [80], Kolmogorov, Petrovsky and Piskunov [105]). In the context of the Allen-Cahn equation (which falls in a different framework than the Fisher-KPP), the stability of traveling waves was shown by Matano, Nara and Taniguchi [115]. However, the maximum and comparison principles are no longer available (in general) for vector-values problems (i. e., systems), so that one needs different tools. There is also a huge amount of research in this direction, we refer to the books by Smoller [156] as well as Volpert, Volpert and Volpert [169]. However, the use of variational methods is not

covered in the previously cited books and it has not been extensively used in the parabolic context, which is in contrast with the framework of dispersive equations.

4.2 Statement of the main results

4.2.1 Main assumptions and result

The main result of this chapter is Theorem 4.1, which establishes the existence of an heteroclinic traveling wave with the uniqueness properties on the speed already found in [13] and exponential convergence at $+\infty$. The behavior at $-\infty$ is however weaker than that proved in [13], which might be due to the fact that we allow for degenerate minima. The necessary assumptions for Theorem 4.1 are (H4.1), (H4.2) and (H4.3). In Theorem 4.2 we show that one can obtain convergence of the profile at $-\infty$ (so that one recovers the full result from [13] if one assumes either an upper bound on the difference of the energy of the minima or that the set of global minimizers is a singleton, see (H4.4). Finally, in Theorem 4.3 we show that the formula and min-max characterization of the speed given in [13] also holds here, again under assumption (H4.4).

(H4.1). $W \in \mathcal{C}^3(\mathbb{R}^k)$. There exist \mathcal{A}^- and \mathcal{A}^+ subsets of \mathbb{R}^k and $\mathfrak{h} < 0$ such that for all $a^- \in \mathcal{A}^-$ we have $W(a^-) = \mathfrak{h} < 0$, for all $a^+ \in \mathcal{A}^+$ we have $W(a^+) = 0$ and for all $\mathbf{u} \in \mathbb{R}^k$ we have $W(\mathbf{u}) \geq \mathfrak{h}$. There exist $\rho_0^\pm > 0$ such that for all $\mathbf{u} \in \mathbb{R}^k$ such that $\text{dist}(\mathbf{u}, \mathcal{A}^\pm) \leq \rho_0^\pm$ it holds

$$\text{dist}(\mathbf{u}, \mathcal{A}^\pm)^2 \leq C^\pm (W(\mathbf{u}) - \min\{\pm(-\mathfrak{h}), 0\}) \quad (4.2.1)$$

for some $C^\pm > 0$. Moreover, for every $\mathbf{u} \in \mathbb{R}^k$ there exists a unique $a^\pm(\mathbf{u}) \in \mathcal{A}^\pm$ such that

$$\text{dist}(\mathbf{u}, \mathcal{A}^\pm) = |\mathbf{u} - a^\pm(\mathbf{u})|$$

and the mappings $p^\pm : \mathbf{u} \rightarrow a^\pm(\mathbf{u})$ defined on

$$\mathcal{A}_{\rho_0^\pm}^\pm := \{\mathbf{u} \in \mathbb{R}^k : \text{dist}(\mathbf{u}, \mathcal{A}^\pm) \leq \rho_0^\pm\}$$

are \mathcal{C}^2 .

In particular, the existence of the projection mappings p^\pm holds if the sets \mathcal{A}^\pm are convex and smooth. The next assumption writes as follows:

(H4.2). One of the two follows:

1. \mathcal{A}^\pm is bounded.
2. For any $(\mathbf{u}, a^\pm) \in \mathcal{A}_{\rho_0^\pm}^\pm \times \mathcal{A}^\pm$, there exist maps $p_{(\mathbf{u}, a^\pm)}^\pm : \mathbb{R}^k \rightarrow \mathbb{R}^k$ such that $p_{(\mathbf{u}, a^\pm)}^\pm(\mathbf{u}) = a^\pm$, $W(p_{(\mathbf{u}, a^\pm)}^\pm(\mathbf{u})) = W(\mathbf{u})$ and $\text{dist}(p_{(\mathbf{u}, a^\pm)}^\pm(\mathbf{u}), \mathcal{A}^\pm) = \text{dist}(\mathbf{u}, \mathcal{A}^\pm)$. Moreover, $p_{(\mathbf{u}, a^\pm)}^\pm$ is differentiable and $|D(p_{(\mathbf{u}, a^\pm)}^\pm)(\mathbf{u}_1, \mathbf{u}_2)| = |\mathbf{u}_2|$ for any $(\mathbf{u}_1, \mathbf{u}_2) \in (\mathbb{R}^k)^2$.

In particular, if W is invariant with respect to some group action inside $\mathcal{A}_{\rho_0^\pm}^\pm$ and \mathcal{A}^\pm is a single orbit with respect to such an action, then 2. in (H4.2) is met. However, less rigid structures are also allowed by 2. Finally, for each $h \in \mathbb{R}$ define the level set

$$W^h := \{\mathbf{u} \in \mathbb{R}^k : W(\mathbf{u}) \leq h\}$$

and notice that by continuity and (H4.1) we can find $h_0 > 0$ such that

$$W^{h_0} = W^{h_0,-} \cup W^{h_0,+}$$

with $W^{h_0,\pm}$ closed and disjoint and such that $W^{h_0,+} \subset \mathcal{A}_{\rho_0^+/2}^+$. In particular

$$W^{h_0,-} \cap \mathcal{A}_{\rho_0^+/2}^+ = \emptyset.$$

For each $h \leq h_0$, we can then write

$$W^h = W^{h,-} \cup W^{h,+}$$

with $W^{h,\pm}$ closed and disjoint and such that $W^{h,\pm} \subset W^{h_0,\pm}$. It is then clear that

$$W^0 = W^{0,-} \cup \mathcal{A}^+, W^{\mathfrak{h}} = \mathcal{A}^-$$

and for all $h < 0$,

$$W^h = W^{h,-}.$$

Under these notations, our last assumption writes as follows:

(H4.3). For any $h \leq h_0$, the set $W^{h,-}$ is convex. For any $h \leq h_0$, $\mathbf{u} \in W^{h,-}$ and $a^- \in \mathcal{A}^-$ define the set

$$I(h, \mathbf{u}, a^-) := \{\lambda \in \mathbb{R} : W(a^- + \lambda(\mathbf{u} - a^-)) > h \text{ and } a^- + \lambda(\mathbf{u} - a^-) \in W^{h_0,-}\}.$$

Then, there exists $h_- \in (\mathfrak{h}, 0)$ such that

1. $W^{h_-} \subset \mathcal{A}_{\rho_0^+/2}^-$.
2. For some constant $\sigma > 0$ it holds that for $\mathbf{u} \in W^{h_0,-}$, $a^- \in \mathcal{A}^-$ and $\theta \in I((\mathfrak{h} + h_-)/2, \mathbf{u}, a^-)$ we have

$$\frac{d}{d\lambda}(W(a^- + \lambda(\mathbf{u} - a^-)))(\theta) \geq \sigma.$$

3. For all $h \in (\mathfrak{h}, h_-]$, there exists $\sigma(h) > 0$ such that for all $\mathbf{u} \in W^{h_-}$ such that $W(\mathbf{u}) \geq h$, there exists $\delta(\mathbf{u}) > 0$ such that for all $\theta \in (1 - \delta(\mathbf{u}), 1 + \delta(\mathbf{u}))$ we have

$$\frac{d}{d\lambda}(W(p^-(\mathbf{u}) + \lambda(\mathbf{u} - p^-(\mathbf{u}))))(\theta) \geq \sigma(h) \quad (4.2.2)$$

with $p(\mathbf{u})$ the projection given in (H4.1).

Essentially, assumption (H4.3) imposes some convexity on some suitable subsets of the level sets of W , as well as some uniform monotonicity on segments. Some explanatory designs are shown in Figures 4.1, 4.2, 4.3. One can see assumption (H4.2) as an adaptation of the key assumption by Alikakos and Katzourakis to the case of degenerate minima. Indeed, in case \mathcal{A}^- is reduced to a singleton, (H4.3) essentially reduces to the hypothesis formulated in [13]. Finally, recall that, as in [13], multi-well potentials can satisfy the assumptions (H4.1), (H4.2) and (H4.3) as long as they possess a local minimum at a level higher than 0 and they are modified by an additive constant. We now come back to the following equation for a pair (c, u)

$$-cu' - u'' = -\nabla_{\mathbf{u}} W(u) \text{ in } \mathbb{R} \quad (4.2.3)$$

and, assuming that (H4.3) holds, consider the conditions at infinity

$$\exists T^- \in \mathbb{R} : \forall t \leq T^-, u(t) \in W^{h-} \text{ and } \exists a^+(u) \in \mathcal{A}^+ : \lim_{t \rightarrow +\infty} |u(t) - a^+(u)| = 0. \quad (4.2.4)$$

For $c > 0$, we consider as in [13], the following weighted functional introduced in Fife and McLeod [79, 78]

$$E_c(v) := \int_{\mathbb{R}} e_c(v(t)) dt := \int_{\mathbb{R}} \left[\frac{|v'(t)|^2}{2} + W(v(t)) \right] e^{ct} dt$$

where v belongs to the class

$$\mathcal{S} := \left\{ v \in H_{\text{loc}}^1(\mathbb{R}, \mathbb{R}^k) : \exists T \geq 1 : \forall t \geq T, v(t) \in \mathcal{A}_{\rho_0^+/2}^+ \text{ and } \forall t \leq -T, v(t) \in \mathcal{A}_{\rho_0^-/2}^- \right\}$$

so that (4.2.3) is (at least formally) the Euler-Lagrange equation of E_c . As in [13], we shall find the solution profile u as a critical point (in fact, a global minimizer) of E_{c^*} in \mathcal{S} for a suitable c^* which, in addition, satisfies some uniqueness properties. In fact, assuming that (H4.3) holds we shall consider the class

$$\bar{\mathcal{S}} := \left\{ v \in H_{\text{loc}}^1(\mathbb{R}, \mathbb{R}^k) : \exists T \geq 1 : \forall t \geq T, v(t) \in \mathcal{A}_{\rho_0^+/2}^+ \text{ and } \forall t \leq -T, v(t) \in W^{h-} \right\},$$

which by 1. in (H4.3) satisfies $\bar{\mathcal{S}} \subset \mathcal{S}$. As it was pointed out in [13], for any $c > 0$, $v \in \mathcal{S}$ and $\tau \in \mathbb{R}$ we have

$$E_c(v(\cdot + \tau)) = e^{-c\tau} E_c(v)$$

which implies that for all $c > 0$

$$\inf_{v \in \mathcal{S}} E_c(v), \inf_{v \in \bar{\mathcal{S}}} E_c(v) \in \{-\infty, 0\} \quad (4.2.5)$$

and that in case the infimum above is 0, then for any $v \in \mathcal{S}$ with $E_c(v) > 0$ we have that $E_c(v(\cdot + n)) \rightarrow 0$ as $n \rightarrow +\infty$. This remark shows that one cannot expect to solve the minimization problem (4.2.5) directly, so that an indirect approach is needed. We can now state the main result of this chapter:

Theorem 4.1 (Main Theorem). *Assume that (H4.1), (H4.2) and (H4.3) hold. Then, we have that*

1. **Existence.** *There exist $c^* > 0$ and $u \in \mathcal{C}_{\text{loc}}^2(\mathbb{R}, \mathbb{R}^k) \cap \bar{\mathcal{S}}$ such that (c^*, u) fulfills (4.2.3) and (4.2.4) as well as the variational characterization*

$$E_{c^*}(u) = 0 = \inf_{v \in \bar{\mathcal{S}}} E_{c^*}(v).$$

2. **Uniqueness of the speed.** *Assume that $\bar{c}^* > 0$ is such that*

$$\inf_{v \in \bar{\mathcal{S}}} E_{\bar{c}^*}(v) = 0$$

and that $\bar{u} \in \mathcal{S}$ is such that (\bar{c}^, \bar{u}) solves (4.2.3) and $E_{\bar{c}^*}(\bar{u}) < +\infty$. Then, $\bar{c}^* = c^*$.*

3. **Exponential convergence.** *The convergence of u at $+\infty$ is exponential: There exists $\mathfrak{N}^+ > 0$ such that for all $t \in \mathbb{R}$*

$$|u(t) - a^+(u)| \leq \mathfrak{N}^+ e^{-c^* t}$$

where $a^+(u)$ is given by (4.2.4).

Remark 4.2.1. Notice that the conditions at infinity (4.2.4) show that u is not constant, since we clearly have that $\mathcal{A}_{\rho_0^-/2}^- \cap \mathcal{A}_{\rho_0^+/2}^+ = \emptyset$. Regarding the uniqueness of the speed, in particular it holds that c^* is unique among the class of global minimizers.

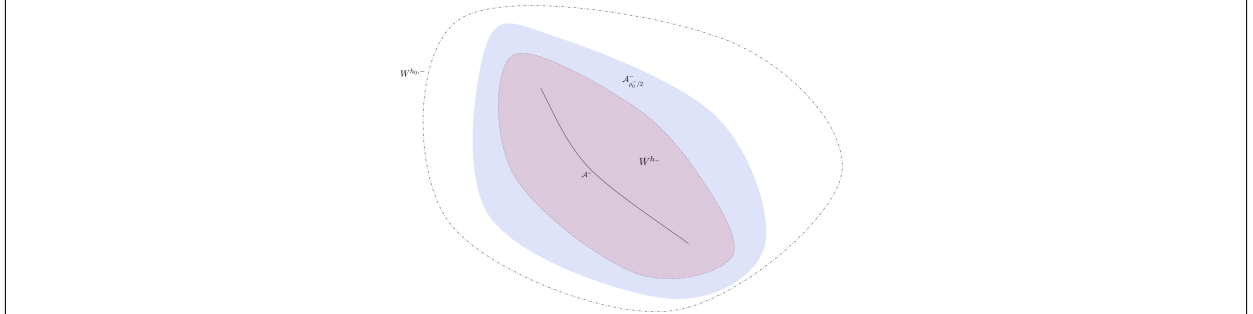


Figure 4.1: Representation of 1. in (H4.3). The curve represents the set \mathcal{A}^- , while the inner convex shadowed region corresponds to the level set $W^{h,-}$, which is contained in $\mathcal{A}_{\rho_0^-/2}^-$, the larger shadowed region. Finally, the outer punctured line contains the convex region representing $W^{h_0,-}$.

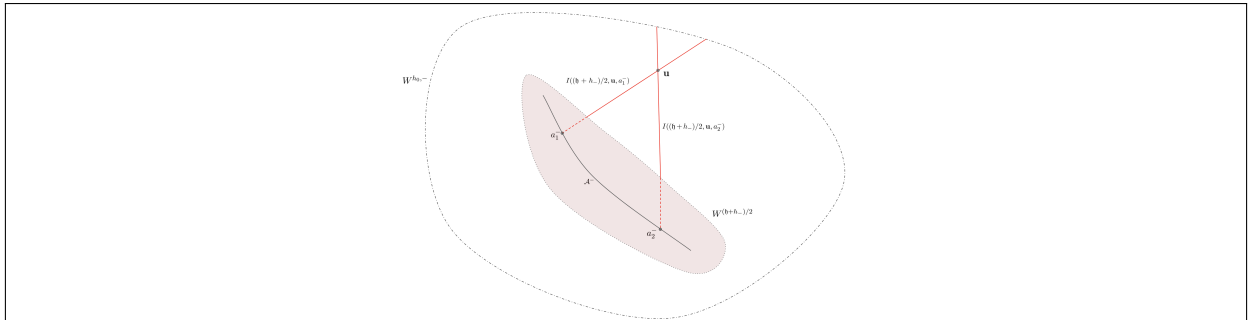


Figure 4.2: Representation of 2. in (H4.3). We have u , which is contained in $W^{h_0,-}$ but not in $W^{(h+h_-)/2,-}$. The full lines represent the segments $I((h+h_-)/2, u, a_1^-)$ and $I((h+h_-)/2, u, a_2^-)$ for a_1^- and a_2^- in \mathcal{A}^- . The discontinuous lines complete the previous segments into the segment starting in a_1^- and a_2^- .

Once the main result has been stated, we give some properties on the solution (c^*, u) , which are essentially the adaptation from chapter 3 to the current setting.

4.2.2 Conditions at infinity

We now explain how the condition at infinity (4.2.4) can be upgraded. In particular, we would like conditions at infinity of the type

$$\exists a^\pm(u) \in \mathcal{A}^\pm : \lim_{t \rightarrow \pm\infty} |u(t) - a^\pm(u)| = 0. \quad (4.2.6)$$

Under the previous assumptions, we have no proof that (c, u) satisfies (4.2.6). However, we can establish such a behavior adding the following assumption:

(H4.4). We have one of the two following situations:

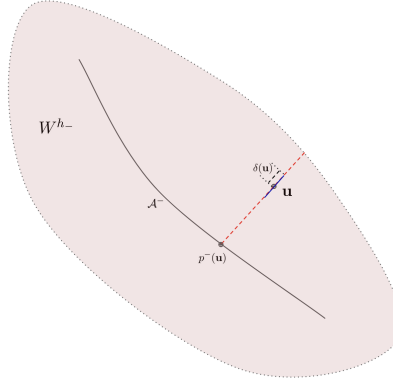


Figure 4.3: Representation of 3. in (H4.3). We take $\mathbf{u} \in W^{h-}$ not contained in \mathcal{A}^- . The discontinuous red line is a segment starting in $p^-(\mathbf{u})$ and containing \mathbf{u} . The blue line represents the small portion of the segment centered in \mathbf{u} contained in the line going through $p^-(\mathbf{u})$ and with length $2\delta(\mathbf{u})$, $\delta(\mathbf{u}) > 0$. We ask uniform strict monotonicity property (4.2.2) to hold in this segment.

1. $\mathcal{A}^- := \{a^-\}$.
2. We have that

$$-\mathfrak{h} < \frac{(d\eta)^2}{2}$$

where

$$d := \text{dist}_{\mathbb{R}^k}(\mathcal{A}_{\rho_0^+/2}^+, W^{h_0,-}) \quad (4.2.7)$$

which is positive by the definition of h_0 and

$$\eta := \frac{1}{\frac{1}{2}(C^-)^2((C^-)^2 + (C^- + 1)^2) + C^-} > 0, \quad (4.2.8)$$

where C^- is the constant from (H4.1).

Essentially, (H4.4) requires that either \mathcal{A}^- is a singleton (while \mathcal{A}^+ does not need to be) or that the value $-\mathfrak{h}$ is not too large. Then we have the following:

Theorem 4.2. *Assume that (H4.1), (H4.2), (H4.3) and (H4.4) hold. Let (c^*, \mathbf{u}) be a solution given by Theorem 4.1. Then, \mathbf{u} also satisfies (4.2.6). Moreover, the convergence is exponential in the following way: We have that $c^* < \eta$, η as in (4.2.8), and there exists $\mathfrak{N}^- > 0$ such that for all $t \in \mathbb{R}$*

$$|\mathbf{u}(t) - a^-(\mathbf{u})| \leq \mathfrak{N}^- e^{(\eta - c^*)t}$$

4.2.3 Min-max characterization of the speed

We now give some results which characterize the speed c^* . As before, such results are very close to the ones obtained in [13] and Chapter 3. More specifically, in case (H4.4) holds, the speed c^* satisfies the following additional properties, which include a min-max characterization as in Heinze [93], Heinze, Papanicolau and Stevens [94]:

Theorem 4.3. Assume that (H4.1), (H4.2), (H4.3) and (H4.4) hold. Let (c^\star, u) be a solution given by Theorem 4.1. Then, for any $\tilde{u} \in \bar{\mathcal{S}}$ such that $E_{c^\star}(\tilde{u}) = 0$ we have that (c^\star, \tilde{u}) solves (4.2.3) and

$$c^\star = \frac{-\mathfrak{h}}{\int_{\mathbb{R}} |\tilde{u}'|^2}.$$

In particular, the quantity $\int_{\mathbb{R}} |\tilde{u}'|^2$ is well-defined and constant among the class of global minimizers of E_{c^\star} in $\bar{\mathcal{S}}$. Moreover, it holds

$$c^\star \leq \frac{\sqrt{-2\mathfrak{h}}}{d} < \eta,$$

with d as in (4.2.7), where we have also used (H4.4). Finally, we have the variational characterization

$$c^\star = \sup\{c > 0 : \inf_{v \in \bar{\mathcal{S}}} E_c(v) = -\infty\} = \inf\{c > 0 : \inf_{v \in \bar{\mathcal{S}}} E_c(v) = 0\}.$$

Remark 4.2.2. Combining the uniqueness part in Theorem 4.1 and Theorem 4.3, we obtain that for any $c > c^\star$, the infimum

$$\inf_{v \in \bar{\mathcal{S}}} E_c(v) = 0$$

is never attained.

4.3 Examples of potentials verifying the assumptions

We now sketch some examples of potentials on the plane which verify the assumptions introduced in Section 4.2 and for which the results of the previous works by Alikakos and Katzourakis [13] and Risler [145] do not apply.

Consider a smooth function $f : \mathbb{R} \rightarrow \mathbb{R}$ possessing $a < b < c$ three critical points such that $f(a) < f(c) < f(b)$, a and c are local minima and b is a local maximum. Moreover, we assume that for some $d \in (a, b)$ with $f(d) > f(c)$ the function f is strictly convex in (a, d) . See Figure 4.4. It is not difficult to see that one could give such an f explicitly. Our first example is obtained by considering the potential $W_1 : (u_1, u_2) \in \mathbb{R}^2 \rightarrow f(u_1) - f(c) \in \mathbb{R}$, see Figure 4.5. Then, we have that

1. The potential W_1 fulfills (H4.1) with $\mathfrak{h} = f(a) - f(c) < 0$, $\mathcal{A}^- = \{a\} \times \mathbb{R}$ and $\mathcal{A}^+ = \{c\} \times \mathbb{R}$. The existence and smoothness of the projection maps p^\pm follow from the fact that \mathcal{A}^- and \mathcal{A}^+ are straight lines.
2. It is clear that (H4.2) holds due to 2. as W_1 does not depend on the variable x_2 .
3. The fact that f is strictly convex in (c, d) and that $f(d) > f(b)$ implies that (H4.3) holds.
4. If we also assume that $f(b)$ close enough to $f(a)$, then we have that (H4.4) holds due to 2.

Notice that W_1 does not fit into the framework of [13, 145] due to the fact that the set \mathcal{A}^+ is a straight line and not an isolated equilibrium. For our second example, we chose $(a, c) = (0, 1)$ and $W_2 : u = (u_1, u_2) \in \mathbb{R}^2 \rightarrow f(|u|^2) - f(c) \in \mathbb{R}$, so that W_2 is radially symmetric. We see that $\mathcal{A}^- = \{0\}$, so that (H4.4) holds due to 1. We also see that $\mathcal{A}^+ = \mathbb{S}^1$, the unit circle. As before, it is clear that (H4.1) holds and the same goes for (H4.2) as \mathcal{A}^\pm is bounded. Regarding (H4.3), this is proven as

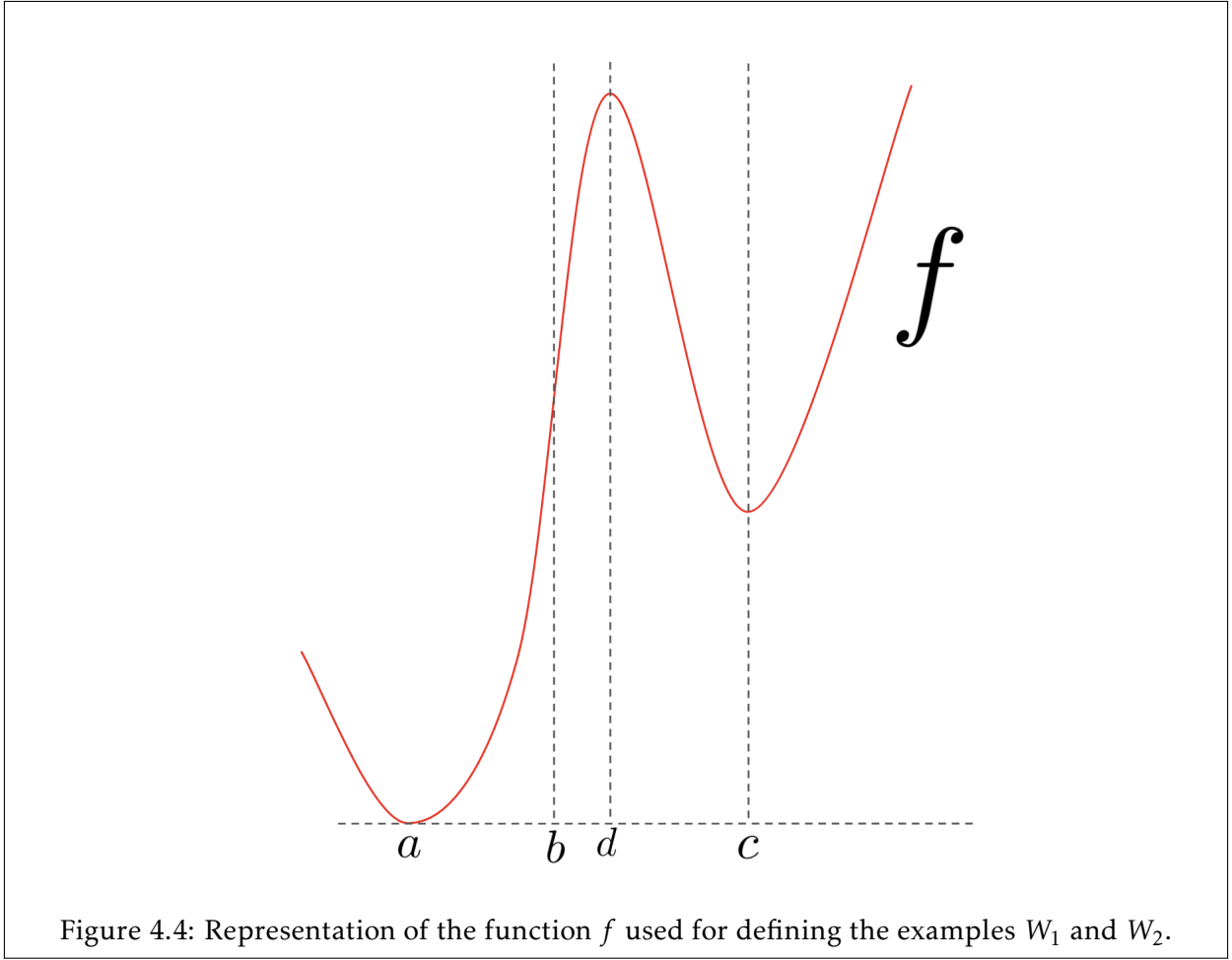


Figure 4.4: Representation of the function f used for defining the examples W_1 and W_2 .

for W_1 . Again that the potential W_2 does not fit into the framework of [13, 145]. Moreover, we see that W_2 is reminiscent of the Ginzburg-Landau potential

$$V_{GL} : u = (u_1, u_2) \in \mathbb{R}^2 \rightarrow \frac{(1 - |u|^2)^2}{4} \in \mathbb{R}$$

which has attracted a significant amount of attention since the seminal work of Bethuel, Brezis and Helein [42]. Our potential W_2 vanishes as V_{GL} on the unit circle \mathbb{S}^1 . However, the fact that W_2 changes sign makes it not to fit into the class of potentials which can be considered in the Ginzburg-Landau theory of [42] and its posterior developments.

As a final remark (which we owe to Emmanuel Risler), we point out that if W verifies the assumption (H4.1) (so in particular W is C^2), then we must have by (4.2.1) that $\text{int}(\mathcal{A}^-) = \text{int}(\mathcal{A}^+) = \emptyset$, as otherwise we would have a discontinuity on D^2W .

4.4 Extensions and open problems

4.4.1 Other classes of potentials

Applying directly the abstract result from Chapter 3, it is possible to replace (H4.3) by an assumption which consists on an upper bound on \mathfrak{h} . Following the idea in Chapter 3, a particular family of potentials satisfying such an assumption can be obtained by suitably perturbing a given multi-well

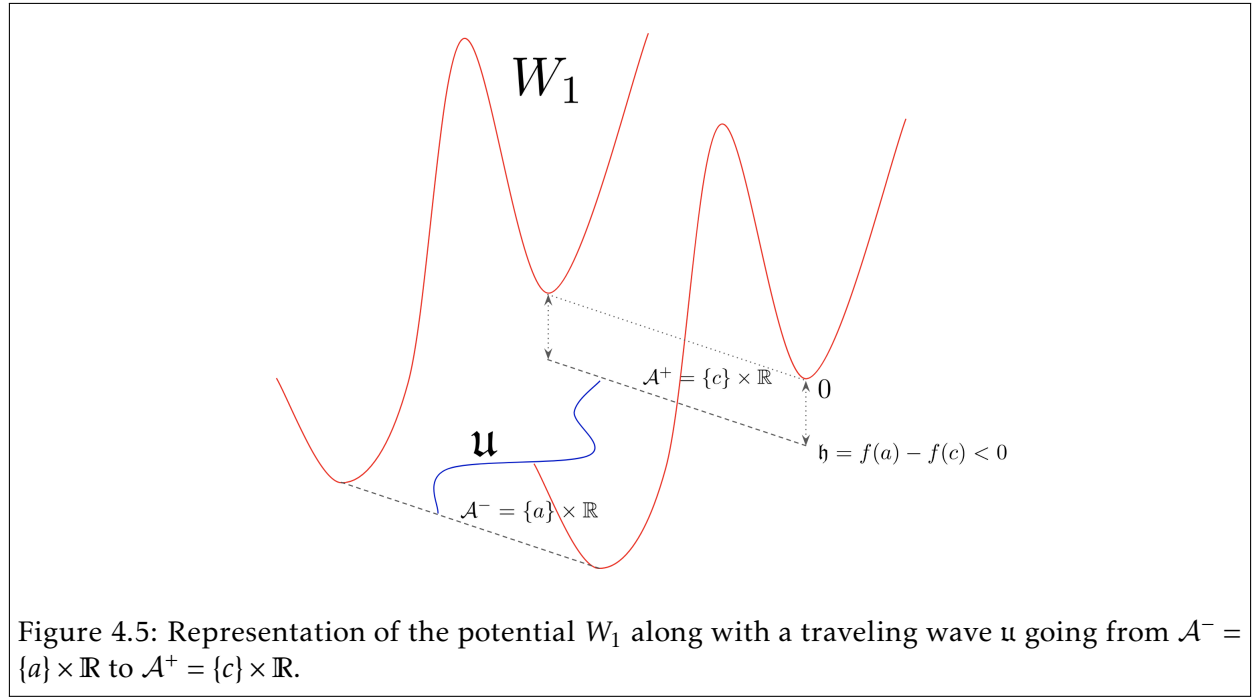


Figure 4.5: Representation of the potential W_1 along with a traveling wave u going from $\mathcal{A}^- = \{a\} \times \mathbb{R}$ to $\mathcal{A}^+ = \{c\} \times \mathbb{R}$.

potential (or even a more general potential). More precisely, one can consider a non-negative smooth potential V vanishing in two sets \mathcal{A}_1 and \mathcal{A}_2 in \mathbb{R}^k and with local properties as in (H4.1) and (H4.2), with the obvious modifications. A suitable perturbation around \mathcal{A}_2 gives a family of unbalanced potentials $(W_\delta)_{\delta \geq 0}$ such that $W_0 = V$ and for $\delta \in (0, \delta_0]$ (for some $\delta_0 > 0$), the abstract framework from Chapter 3 applies to W_δ (up to an additive constant). In particular, one has results as the ones of this chapter for W_δ , which was a type of potential not allowed in [13].

4.4.2 The problem of traveling waves for 2D parabolic Allen-Cahn systems

Our initial motivation comes from the study of the parabolic Allen-Cahn system

$$\partial_t w - \Delta w = -\nabla_{\mathbf{u}} V(w) \text{ in } [0, +\infty) \times \mathbb{R}^2, \quad (4.4.1)$$

where V is a standard, smooth, multi-well potential. In Chapter 3, we show by an abstract approach that there exist traveling wave solutions $w : (t, x_1, x_2) \in [0, +\infty) \times \mathbb{R}^2 \rightarrow \mathcal{U}(x_1 - c^*t, x_2) \in \mathbb{R}^k$ of (4.4.1) such that

$$\lim_{x_1 \rightarrow \pm\infty} \|\mathcal{U}(x_1, \cdot) - q^\pm\|_{H^1(\mathbb{R}, \mathbb{R}^k)} = 0$$

where q^\pm solve

$$(q^\pm)'' = \nabla_{\mathbf{u}} V(q^\pm) \text{ in } \mathbb{R}$$

and satisfy

$$\lim_{t \rightarrow \pm\infty} q^\pm(t) = \sigma^\pm$$

with $\sigma^\pm \in \Sigma := \{\mathbf{u} \in \mathbb{R}^k : V(\mathbf{u}) = 0\}$. Moreover, q^- and q^+ are (in some sense) *local minimizers* of the energy

$$E : q \in H_{\text{loc}}^1(\mathbb{R}, \mathbb{R}^k) \rightarrow E(q) := \int_{\mathbb{R}} \left[\frac{|q'(t)|^2}{2} + V(q(t)) \right] dt$$

with *different energy levels*. That is

$$E(q^\pm) = m^\pm$$

with $m^+ > m^-$ and m^- the global minima. The main point is that $m^+ > m^-$, as the case $m^+ = m^-$ had already been treated by several authors and it is by now well understood. See Fusco [83], Monteil and Santambrogio [125], Schatzman [153], Smyrnelis [157]. As already explained, in Chapter 3 we have to assume that $0 < m^+ - m^-$ is bounded above by a given constant. As in here, the approach of the proof is to deduce the result from an abstract setting. In an attempt to extend our result to other situations, we were able to replace the perturbation assumption by another one in the abstract setting. The other assumption is the abstract version of (H4.3) and the one we use here (see (H4.1')). However, the inconvenient is that such condition does not seem to apply when dealing with traveling waves of the system (4.4.1). Even for simple examples of the potential V , we cannot prove that such condition is satisfied, which does not allow us to apply the abstract result. The reason is that properties such as convexity of level sets and uniform strict monotonicity along segments are harder to verify in infinite dimensions. However, if an example of potential V such that (H4.1') is satisfied could be found, then we would have the existence of a pair (c^*, \mathfrak{U}) satisfying the same properties than the solution obtained in Chapter 3.

4.4.3 Traveling waves between homoclinics

One considers again the Allen-Cahn system

$$\partial_t w - \Delta w = -\nabla_{\mathbf{u}} V(w) \text{ in } [0, +\infty) \times \mathbb{R}^2,$$

with V as in the previous paragraph. We use the same notation as above. This time, we consider the conditions at infinity

$$\lim_{x_1 \rightarrow \pm\infty} \|\mathfrak{U}(x_1, \cdot) - \mathbf{u}^\pm\|_{H^1(\mathbb{R}, \mathbb{R}^k)} = 0$$

for the associated profile \mathfrak{U} . Here \mathbf{u}^- is the constant homoclinic solution equal to $\sigma \in \Sigma$ and \mathbf{u}^+ is such that $E(\mathbf{u}^+) > 0$ and it solves

$$(\mathbf{u}^+)'' = \nabla_{\mathbf{u}} V(\mathbf{u}^+) \text{ in } \mathbb{R}$$

satisfying

$$\lim_{t \rightarrow \pm\infty} \mathbf{u}^+(t) = \sigma.$$

That is, \mathbf{u}^+ is a non-constant homoclinic emanating from σ . If \mathbf{u}^+ is locally minimizing and, moreover, it satisfies the assumptions given by some of our abstract results¹, then the existence of a solution (c^*, \mathfrak{U}) is guaranteed. The fact that $\mathbf{u}^+ \neq \mathbf{u}^-$ implies that \mathfrak{U} is not constant, and since \mathbf{u}^- and \mathbf{u}^+ are homoclinic, then \mathfrak{U} can be thought as a heteroclinic between homoclinics. Moreover, notice that such type of solution cannot exist in the stationary case, as stationary waves join two different 1D solutions which are global minimizers, which is not possible in the homoclinic case². Even though for the moment we have no explicit example of potential V such that the necessary assumptions are fulfilled, the main result of Chapter 2 suggests that homoclinic solutions might exist in some situations.

¹That is, either the one that we use in Chapter 3 or the one that we give here.

²Clearly, the only globally minimizing homoclinic is the constant one.

4.5 The abstract setting

As we advanced in the introduction, the main results of this chapter are obtained from exploiting the abstract setting introduced in Chapter 4. Essentially, we replace (H3.7') by a new assumption, (H4.1'). We will consistently make references to the definitions and results from Chapter 4. Let us just briefly recall that we consider \mathcal{L} be a Hilbert space with inner product $\langle \cdot, \cdot \rangle_{\mathcal{L}}$ and induced norm $\|\cdot\|_{\mathcal{L}}$. We also consider $\mathcal{H} \subset \mathcal{L}$ a Hilbert space with inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ such that $\mathcal{H} \subset \mathcal{L}$ and a functional $\mathcal{E} : \mathcal{L} \rightarrow (-\infty, +\infty]$. We will invoke the hypotheses (H3.1'), (H3.2'), (H3.3'), (H3.4'), (H3.5'), (H3.6') from Chapter 4 and replace (H3.7') by

(H4.1'). Assume that (H3.1') and (H3.2') hold. We suppose that for all $\alpha \in [a, \alpha_0]$, we have that $\mathcal{E}^{\alpha,-}$ is convex. For all $\mathbf{v}^- \in \mathcal{F}^-$, $\alpha \in [a, \alpha_0]$ and $v \in \mathcal{E}^{\alpha_0,-}$ define

$$A(\alpha, \mathbf{v}^-, v) := \{\lambda \in \mathbb{R} : \mathcal{E}(\mathbf{v}^- + \lambda(v - \mathbf{v}^-)) > \alpha \text{ and } \mathbf{v}^- + \lambda(v - \mathbf{v}^-) \in \mathcal{E}^{\alpha_0,-}\}. \quad (4.5.1)$$

Then, there exists $\alpha_- \in (a, 0)$ such that

$$1. \quad \mathcal{E}^{\alpha_-} \subset \mathcal{F}_{r_0^-/2}^-.$$

2. There exists $\omega > 0$ such that

$$\forall v \in \mathcal{E}^{\alpha_0,-}, \forall \mathbf{v}^- \in \mathcal{F}^-, \forall \theta \in A\left(\frac{\alpha_- + a}{2}, \mathbf{v}^-, v\right), \quad \frac{d}{d\lambda}(\mathcal{E}(\mathbf{v}^- + \lambda(v - \mathbf{v}^-)))(\theta) \geq \omega. \quad (4.5.2)$$

3. We have that for all $\alpha \in (a, \alpha_-]$, there exists $\omega(\alpha) > 0$ such that, for all $v \in \mathcal{E}^{\alpha_0,-}$ such that $\mathcal{E}(v) \geq \alpha$, it holds

$$\exists \delta(v) > 0, \forall \theta \in (1 - \delta(v), 1 + \delta(v)), \quad \frac{d}{d\lambda}(\mathcal{E}(P^-(v) + \lambda(v - P^-(v))))(\theta) \geq \omega(\alpha), \quad (4.5.3)$$

where P^- is the projection map defined in (H3.1'), which is well defined in \mathcal{E}^{α_-} because $\mathcal{E}^{\alpha_-} \subset \mathcal{F}_{r_0^-/2}^-$.

Furthermore, we have the following global bound property: There exists $R > 0$ such that for any $v \in \mathcal{E}^{\alpha_0,-}$ there exists $\mathbf{v}^-(v) \in \mathcal{F}^-$ such that $\|v - \mathbf{v}^-(v)\|_{\mathcal{H}} \leq R$. Moreover we have that:

$$\forall R > 0, \exists C_{\mathcal{E}}(R) > 0, \forall (v, w) \in \mathcal{H}^2 : \|v\|_{\mathcal{H}} \leq R, \quad |D\mathcal{E}(v)(w)| \leq C_{\mathcal{E}}(R)\|w\|_{\mathcal{H}}. \quad (4.5.4)$$

Finally, it holds

$$\forall v \in \mathcal{H}, \quad \|v\|_{\mathcal{H}}^2 - \|v\|_{\mathcal{L}}^2 \leq C_c(\mathcal{E}(v) - a). \quad (4.5.5)$$

Let us next define the space

$$X := \left\{ U \in H_{\text{loc}}^1(\mathbb{R}, \mathcal{L}) : \exists T \geq 1, \forall t \geq T, \quad \text{dist}_{\mathcal{L}}(U(t), \mathcal{F}^+) \leq \frac{r_0^+}{2}, \right. \\ \left. \forall t \leq -T, \quad \text{dist}_{\mathcal{L}}(U(t), \mathcal{F}^-) \leq \frac{r_0^-}{2} \right\}, \quad (4.5.6)$$

already introduced in Chapter 3. If (H4.1') holds, we define the class

$$Y := \left\{ U \in H_{\text{loc}}^1(\mathbb{R}, \mathcal{L}) : \exists T \geq 1, \forall t \geq T, \text{dist}_{\mathcal{L}}(U(t), \mathcal{F}^+) \leq \frac{r_0^+}{2}, \right. \\ \left. \forall t \leq -T, \mathcal{E}(U(t)) \leq \alpha_- \right\} \quad (4.5.7)$$

so that $Y \subset X$ by (H4.1'). For working under assumption (H4.1'), the space Y becomes more suitable in certain aspects. We can now finally state the abstract result, which shows the existence of a solution to

$$\mathbf{U}'' - D_{\mathcal{L}} \mathcal{E}(\mathbf{U}) = -c\mathbf{U}' \text{ in } \mathbb{R} \quad (4.5.8)$$

satisfying the conditions at infinity

$$\exists T \geq 1 : \forall t \leq -T, \mathbf{U}(t) \in \mathcal{F}_{r_0/2}^-, \\ \forall t \geq T, \mathbf{U}(t) \in \mathcal{F}_{r_0/2}^+. \quad (4.5.9)$$

Theorem 4.4 (Main abstract result). *Assume that (H3.3'), (H3.4'), (H3.5'), (H3.6') and (H4.1') hold. Then, it holds*

1. **Existence.** *There exists $c^* > 0$ and $\mathbf{U} \in \mathcal{A}(\mathbb{R}) \cap Y$, $\mathcal{A}(\mathbb{R})$ as in (3.4.23), such that (c^*, \mathbf{U}) solves (4.5.8) with conditions at infinity (4.5.9) and \mathbf{U} is a global minimizer of \mathbf{E}_c in Y (that is, $\mathbf{E}_c(\mathbf{U}) = 0$).*
2. **Uniqueness of the speed.** *The speed c^* is unique in the following sense: if $\overline{c^*} > 0$ is such that*

$$\inf_{U \in Y} \mathbf{E}_{\overline{c^*}}(U) = 0$$

and there exists $\overline{\mathbf{U}} \in \mathcal{A}(\mathbb{R}) \cap X$ such that $(\overline{c^}, \overline{\mathbf{U}})$ solves (4.5.8) and $\mathbf{E}_{\overline{c^*}}(\overline{\mathbf{U}}) < +\infty$, then $\overline{c^*} = c^*$.*

3. **Exponential convergence.** *We have that for some $M^+ > 0$ it holds for all $t \in \mathbb{R}$*

$$\|\mathbf{U}(t) - \mathbf{v}^+(U)\|_{\mathcal{L}} \leq M^+ e^{-c^* t} \quad (4.5.10)$$

where $\mathbf{v}^+(U)$ is given by (4.5.9).

As we can see, the statement of Theorem 4.4 is very similar to that of Theorem 3.5 in Chapter 3. If \mathcal{F}^- is reduced to a simple point and (H4.1') holds, we can shown convergence with respect to the \mathcal{L} -norm as $t \rightarrow -\infty$:

Theorem 4.5. *Assume that (H3.3'), (H3.4'), (H3.5'), (H3.6') and (H4.1') hold. Assume moreover that $\mathcal{F}^- = \{\mathbf{v}_i^-\}$. Then, if (c^*, \mathbf{U}) is the solution given by Theorem 4.4, \mathbf{U} satisfies in addition*

$$\lim_{t \rightarrow -\infty} \|\mathbf{U}(t) - \mathbf{v}_i^-\|_{\mathcal{L}} = 0. \quad (4.5.11)$$

Remark 4.5.1. Notice that (4.5.11) does not imply that $\mathcal{E}(\mathbf{U}(t)) \rightarrow a$ as $t \rightarrow -\infty$, since \mathcal{E} is only supposed to be lower-semicontinuous with respect to \mathcal{L} -convergence.

We can impose an additional assumption in order to obtain stronger conditions at $-\infty$ on the solution, which is similar to (H3.8') in Chapter 3

(H4.2'). Hypothesis (H4.1') is fulfilled and, additionally:

$$-a < \frac{(d_{\alpha_0} \gamma^-)^2}{2} \quad (4.5.12)$$

where γ^- is as in (3.4.28) and

$$d_{\alpha_0} := \text{dist}_{\mathcal{L}}(\mathcal{E}^{\alpha_0, -}, \mathcal{F}_{r_0^+/2}^+) \quad (4.5.13)$$

which is positive by (H4.1') and Lemma 3.4.2.

If (H4.2') holds, we can show the following:

Theorem 4.6. *Assume that (H3.3'), (H3.4'), (H3.5'), (H3.6') and (H4.2') hold. Then, if (c^*, \mathbf{U}) is the solution given by Theorem 4.4, it holds that $c^* < \gamma^-$, γ^- as in (3.4.28) and there exists $M^- > 0$ such that for all $t \in \mathbb{R}$*

$$\|\mathbf{U}(t) - \mathbf{v}^-(\mathbf{U})\|_{\mathcal{L}} \leq M^- e^{(\gamma^- - c^*)t}$$

for some $\mathbf{v}^-(U) \in \mathcal{F}^-$.

As we see Theorem 4.6 is analogous to Theorem 3.6 in Chapter 3. Finally, we have the characterization of the speed, analogous to Theorem 3.7 in Chapter 3

Theorem 4.7. *Assume that (H3.3'), (H3.4'), (H3.5'), (H3.6') and (H4.2') hold. Let (c^*, \mathbf{U}) be the solution given by Theorem 4.4. Then, if $\tilde{\mathbf{U}} \in \mathcal{A}(\mathbb{R}) \cap Y$ is such that*

$$\mathbf{E}_{c^*}(\tilde{\mathbf{U}}) = 0$$

then we have that $(c^*, \tilde{\mathbf{U}})$ solves (4.5.8) and

$$c^* = \frac{-a}{\int_{\mathbb{R}} \|\tilde{\mathbf{U}}'(t)\|_{\mathcal{L}}^2 dt}.$$

In particular, the quantity $\int_{\mathbb{R}} \|\tilde{\mathbf{U}}'(t)\|_{\mathcal{L}}^2 dt$ is finite. Moreover, we have that

$$c^* \leq \frac{\sqrt{-2a}}{d_{\alpha_0}} < \gamma^-,$$

where we have used (H4.2') and d_{α_0} , γ^- are as in (4.5.13) and (3.4.28) respectively. Finally, we have

$$c^* = \sup\{c > 0 : \inf_{U \in Y} \mathbf{E}_c(U) = -\infty\} = \inf\{c > 0 : \inf_{U \in Y} \mathbf{E}_c(U) = 0\}.$$

4.6 Proof of the abstract results

The strategy is essentially the same that the one used in Chapter 3, which is inspired by [13]. See also the book by Alikakos, Fusco and Smyrnelis [12]. We reproduce the main steps and provide the proof of the new results, which are essentially those in Section 3.5.6, for which we use (H4.1') instead of (H3.7'). Therefore, each result here has its equivalent one in Chapter 3 and in some cases the arguments adapt in a straightforward way or the result can even be applied directly.

4.6.1 Preliminaries

Let r_0^- and r_0^+ be the constants introduced before and $\mathcal{F}_{r_0^\pm/2}^\pm$ be the corresponding closed balls as in (3.4.4). Assume that (H3.1') holds. For $T \geq 1$, we define the sets

$$X_T^- := \left\{ U \in H_{\text{loc}}^1(\mathbb{R}, \mathcal{L}) : \forall t \leq -T, U(t) \in \mathcal{F}_{r_0^-/2}^- \right\},$$

$$X_T^+ := \left\{ U \in H_{\text{loc}}^1(\mathbb{R}, \mathcal{L}) : \forall t \geq T, U(t) \in \mathcal{F}_{r_0^+/2}^+ \right\},$$

where the constant . Subsequently, we set

$$X_T := X_T^- \cap X_T^+. \quad (4.6.1)$$

In case (H4.1') holds, we can define for $T \geq 1$

$$Y_T^- := \left\{ U \in H_{\text{loc}}^1(\mathbb{R}, \mathcal{L}) : \forall t \leq -T, \mathcal{E}(U(t)) \leq \alpha_- \right\},$$

where α_- is the constant introduced in assumption (H4.1'). Subsequently, we set

$$Y_T := Y_T^- \cap X_T^+. \quad (4.6.2)$$

The spaces Y_T will play here the role that the spaces X_T played in Chapter 3. We have that

$$X = \bigcup_{T \geq 1} X_T$$

and

$$Y = \bigcup_{T \geq 1} Y_T.$$

The following property is immediate:

Lemma 4.6.1. *Assume that (H4.1') holds. Then, we have that*

$$Y_T \subset X_T \quad (4.6.3)$$

Proof. From (H4.1') it follows that whenever $v \in \mathcal{H}$ is such that $\mathcal{E}(v) \leq \alpha_-$, it then holds that $v \in \mathcal{F}_{r_0^-/2}^-$. The inclusion (4.6.3) then follows by the definitions (4.6.1) and (4.6.2). \square

Lemma 4.6.1 shows that whenever (H4.1') holds, Y_T is a subspace of X_T . Therefore, properties that hold for X_T (and that we proved in Chapter 3) will apply to Y_T . Define now the infimum value

$$\mathbf{b}_{c,T} := \inf_{U \in Y_T} \mathbf{E}_c(U). \quad (4.6.4)$$

It will be later shown that $\mathbf{b}_{c,T}$ is always attained. We have the following immediate property:

Lemma 4.6.2. *Assume that (H3.1') and (H3.2') hold. Fix $\hat{\mathbf{v}}^\pm \in \mathcal{F}^\pm$. Let $c > 0$ and $T \geq 1$. For all $T \geq 1$*

the function

$$\Psi(t) := \begin{cases} \hat{\mathbf{v}}^- & \text{if } t \leq -1, \\ \frac{1-t}{2}\hat{\mathbf{v}}^- + \frac{t+1}{2}\hat{\mathbf{v}}^+ & \text{if } -1 \leq t \leq 1, \\ \hat{\mathbf{v}}^+ & \text{if } t \geq 1, \end{cases} \quad (4.6.5)$$

belongs to X_T . If (H4.1') holds, we also have $\Psi \in Y_T$. Moreover, for all $c > 0$

$$\mathbf{E}_c(\Psi) < +\infty.$$

Furthermore, we have

$$-\infty < \mathbf{b}_{c,T} < +\infty. \quad (4.6.6)$$

The proof of Lemma 4.6.2 follows the same lines that Lemma 3.5.4 in Chapter 3, so we skip it.

4.6.2 A semicontinuity result

We extend Lemma 3.5.6 in Chapter 3.

Lemma 4.6.3. Assume that (H3.1'), (H3.3') and (H3.4') hold. Let $T \geq 1$ be fixed. Let $(U_n^i)_{n \in \mathbb{N}}$ be a sequence in X_T and $(c_n)_{n \in \mathbb{N}}$ a convergent sequence of positive real numbers such that

$$\sup_{n \in \mathbb{N}} \mathbf{E}_{c_n}(U_n^i) < +\infty.$$

Then, there exists a sequence (U_n) in X_T and $U_\infty \in X_T$ such that up to extracting a subsequence in $(U_n, c_n)_{n \in \mathbb{N}}$ it holds

$$\begin{aligned} \forall n \in \mathbb{N}, \quad \mathbf{E}_c(U_n) &= \mathbf{E}_c(U_n^i), \\ \forall t \in \mathbb{R}, \quad U_n(t) &\rightharpoonup U_\infty(t) \text{ weakly in } \mathcal{L} \\ U_n' h_{c_n} &\rightharpoonup U_\infty' h_{c_\infty} \text{ weakly in } L^2(\mathbb{R}, \mathcal{L}) \end{aligned} \quad (4.6.7)$$

and

$$\mathbf{E}_{c_\infty}(U_\infty) \leq \liminf_{n \rightarrow \infty} \mathbf{E}_{c_n}(U_n),$$

where, for $k \in \mathbb{R}$, we set $h_k : t \in \mathbb{R} \rightarrow e^{kt/2} \in \mathbb{R}$, $c_\infty := \lim_{n \rightarrow \infty} c_n$. Moreover, if (H4.1') holds and the sequence $(U_n^i)_{n \in \mathbb{N}}$ is contained in Y_T , then $U_\infty \in Y_T$.

Proof. Except for the last part, the result is exactly the same as Lemma 3.5.6 in Chapter 2. Notice that if $(U_n^i)_{n \in \mathbb{N}}$ is contained in Y_T , then by (4.6.7) and the lower-semicontinuity of \mathcal{E} we have that for all $t \leq -T$ it holds

$$\mathcal{E}(U_\infty(t)) \leq \liminf_{n \rightarrow \infty} \mathcal{E}(U_n(t)) \leq \alpha_-$$

which shows that $U_\infty \in Y_T$ and concludes the proof. \square

4.6.3 Existence of an infimum for \mathbf{E}_c in Y_T

We now show that for any $c > 0$ and $T \geq 1$, the infimum value $\mathbf{b}_{c,T}$ defined in (4.6.4) is attained. The result is as follows:

Lemma 4.6.4. *Assume that (H3.3'), (H3.4') and (H4.1') hold. Let $c > 0$, $T \geq 1$ and $\mathbf{b}_{c,T}$ be as in (4.6.4). Then, $\mathbf{b}_{c,T}$ is attained for some $\mathbf{V}_{c,T} \in Y_T$.*

Proof. By (4.6.6) in Lemma 4.6.2, we have that there exists a minimizing sequence $(U_n)_{n \in \mathbb{N}}$ in Y_T . Therefore, Lemma 4.6.3 implies the existence of $\mathbf{V}_{c,T} \in Y_T$. \square

We next show that constrained minimizers are solutions of the equation (4.5.8) on a (possibly proper) subset of \mathbb{R} .

Lemma 4.6.5. *Assume that (H3.6') and (H4.1') hold. Let $c > 0$, $T \geq 1$ and $\mathbf{b}_{c,T}$ be as in (4.6.4). Let $\mathbf{V}_{c,T} \in Y_T$ be such that $\mathbf{E}_c(\mathbf{V}_{c,T}) = \mathbf{b}_{c,T}$. Then, $\mathbf{V}_{c,T} \in \mathcal{A}((-T, T))$ and*

$$\mathbf{V}_{c,T}'' - D\mathcal{L}\mathcal{E}(\mathbf{V}_{c,T}) = -c\mathbf{V}_{c,T}' \text{ in } (-T, T). \quad (4.6.8)$$

Moreover, if $t \geq T$ is such that

$$\text{dist}_{\mathcal{L}}(\mathbf{V}_{c,T}(t), \mathcal{F}^+) < \frac{r_0^+}{2}, \quad (4.6.9)$$

then, there exists $\delta_Y^+(t) > 0$ such that $\mathbf{V}_{c,T} \in \mathcal{A}((t - \delta_Y^+(t), t + \delta_Y^+(t)))$ and

$$\mathbf{V}_{c,T}'' - D\mathcal{L}\mathcal{E}(\mathbf{V}_{c,T}) = -c\mathbf{V}_{c,T}' \text{ in } (t - \delta_Y^+(t), t + \delta_Y^+(t)). \quad (4.6.10)$$

Similarly, if $t \leq -T$ is such that there exist $\varepsilon_Y^-(t) > 0$ and $\delta_Y^-(t) > 0$ such that

$$\forall s \in [t - \delta_Y^-(t), t + \delta_Y^-(t)], \quad \mathcal{E}(\mathbf{V}_{c,T}(s)) \leq \alpha_- - \varepsilon_Y^-(t) \quad (4.6.11)$$

then, $\mathbf{V}_{c,T} \in \mathcal{A}((t - \delta_Y^-(t), t + \delta_Y^-(t)))$ and

$$\mathbf{V}_{c,T}'' - D\mathcal{L}\mathcal{E}(\mathbf{V}_{c,T}) = -c\mathbf{V}_{c,T}' \text{ in } (t - \delta_Y^-(t), t + \delta_Y^-(t)). \quad (4.6.12)$$

Proof. We begin by showing that

$$\forall t \in \mathbb{R}, \quad \mathbf{V}_{c,T}(t) = \mathbb{P}(\mathbf{V}_{c,T}(t)). \quad (4.6.13)$$

Arguing as in the proof of Lemma 3.5.8 in Chapter 3, to prove (4.6.13) it suffices to show that

$$\mathbf{V}_{c,T}^{\mathbb{P}} : t \in \mathbb{R} \rightarrow \mathbf{V}_{c,T}^{\mathbb{P}}(t)$$

belongs to Y_T . Indeed, the proof given in Chapter 3 implies that $\mathbf{V}_{c,T}^{\mathbb{P}} \in X_T$. Thus, the fact that $\mathbf{V}_{c,T} \in \mathcal{A}((-T, T))$ and $\mathbf{V}_{c,T}$ solves (4.6.8) is proven as in Chapter 3. Next, notice that (3.4.20) implies that for all $t \leq -T$, we have $\mathcal{E}(\mathbf{V}_{c,T}^{\mathbb{P}}(t)) \leq \mathcal{E}(\mathbf{V}_{c,T}(t)) \leq \alpha_-$. Therefore, $\mathbf{V}_{c,T}^{\mathbb{P}} \in Y_T$, which shows (4.6.13). Let now $t \geq T$ such that (4.6.9) holds. Again, arguing as in Lemma 3.5.8, we obtain that there exists $\delta_Y^+(t) > 0$ such that $\mathbf{V}_{c,T} \in \mathcal{A}((t - \delta_Y^+(t), t + \delta_Y^+(t)))$ and (4.6.10) holds. To conclude the proof, let $t \leq -T$ be such that (4.6.11) holds. Notice that $\mathcal{E}(\mathbf{V}_{c,T})$ is finite on $(-\infty, T]$, which by assumption (H3.2') implies that $\mathbf{V}_{c,T}$ takes values in \mathcal{H} on $(-\infty, T]$. Moreover $\mathbf{V}_{c,T}$ is \mathcal{L} -continuous. Therefore, using (4.5.5) we get

$$\forall s \in [t - \delta_Y^-(t), t + \delta_Y^-(t)], \quad \|\mathbf{V}_{c,T}(s)\|_{\mathcal{H}} \leq R_1, \quad (4.6.14)$$

with

$$0 < R_1 := \left(C_c(\alpha_- - a) + \max_{s \in [t - \delta_Y^-(t), t + \delta_Y^-(t)]} \|\mathbf{V}_{c,T}(s)\|_{\mathcal{L}} \right)^{\frac{1}{2}} < +\infty.$$

Let $\phi \in \mathcal{C}_c^1((t - \delta_Y^-(t), t + \delta_Y^-(t)), (\mathcal{H}, \|\cdot\|_{\mathcal{H}}))$ be such that

$$\max_{s \in [t - \delta_Y^-(t), t + \delta_Y^-(t)]} \|\phi(s)\|_{\mathcal{H}} \leq R_1, \quad (4.6.15)$$

by (4.5.4), we have that if $v \in \mathcal{H}$ is such that $\|v\|_{\mathcal{H}} \leq 3R_1$, it holds

$$\forall s \in [t - \delta_Y^-(t), t + \delta_Y^-(t)], \quad |D\mathcal{E}(v)(\phi(s))| \leq C_{\mathcal{E}}(3R_1)\|\phi(s)\|_{\mathcal{H}} \quad (4.6.16)$$

with $C_{\mathcal{E}}(3R_1)$ independent on v . We have by (4.6.11) that

$$\forall s \in [t - \delta_Y^-(t), t + \delta_Y^-(t)], \quad \mathcal{E}(\mathbf{V}_{c,T}(s) + \phi(s)) \leq \alpha_- - \varepsilon_Y^-(t) + D\mathcal{E}(h(s))(\phi(s)), \quad (4.6.17)$$

where, for all $s \in [t - \delta_Y^-(t), t + \delta_Y^-(t)]$, $h(s) \in \mathcal{H}$ lies on the segment joining $\mathbf{V}_{c,T}(s) + \phi(s)$ and $\mathbf{V}_{c,T}(s)$, so that $\|h(s)\|_{\mathcal{H}} \leq 3R_1$ by (4.6.14) and (4.6.15). Therefore, we can plug (4.6.16) into (4.6.17) to obtain

$$\forall s \in [t - \delta_Y^-(t), t + \delta_Y^-(t)], \quad \mathcal{E}(\mathbf{V}_{c,T}(s) + \phi(s)) \leq \alpha_- - \varepsilon_Y^-(t) + C_{\mathcal{E}}(3R_1)\|\phi(s)\|_{\mathcal{H}}, \quad (4.6.18)$$

meaning that if we choose any $\phi \in \mathcal{C}_c^1((t - \delta_Y^-(t), t + \delta_Y^-(t)), (\mathcal{H}, \|\cdot\|_{\mathcal{H}}))$ such that

$$\max_{s \in [t - \delta_Y^-(t), t + \delta_Y^-(t)]} \|\phi(s)\|_{\mathcal{H}} \leq \min \left\{ R_1, \frac{\varepsilon_Y^-(t)}{2C_{\mathcal{E}}(3R_1)} \right\}$$

we get by (4.6.18) that

$$\forall s \in [t - \delta_Y^-(t), t + \delta_Y^-(t)], \quad \mathcal{E}(\mathbf{V}_{c,T}(s) + \phi(s)) \leq \alpha_- - \frac{\varepsilon_Y^-(t)}{2} < \alpha_-$$

so that $\mathbf{V}_{c,T} + \phi \in Y_T$. Therefore, we have that $\mathbf{V}_{c,T}$ is a local minimizer of $\mathbf{E}_c(\cdot; [t - \delta_Y^-(t), t + \delta_Y^-(t)])$ in the sense of Definition 3.4.1. Since, in addition, $\mathbf{V}_{c,T}$ satisfies (4.6.13), we can apply (H3.6') and we obtain that $\mathbf{V}_{c,T} \in \mathcal{A}((t - \delta_Y^-(t), t + \delta_Y^-(t)))$ and (4.6.12) holds. As a consequence, the proof is finished. \square

4.6.4 The comparison result

As in Chapter 3, we establish the necessary properties regarding the behavior of the constrained minimizers. This is done by using the fact that the constrained minimizers solve the ODE system along with comparison arguments. Essentially, we adapt the proof of the analogous result from [13, 12] in the framework of (H4.3), which makes it more involved but uses the same ideas. We will use the constants defined in Section 3.5.6. We begin by proving two preliminary comparison results:

Lemma 4.6.6. *Assume that (H4.1') holds. Let $t_1 < t_2$ and $U \in H^1([t_1, t_2], \mathcal{L})$. Assume that*

$$U(t_1), U(t_2) \in \mathcal{E}^{\alpha, -} \text{ with } \alpha \leq 0 \quad (4.6.19)$$

and

$$\exists t_3 \in (t_1, t_2) : \mathcal{E}(U(t_3)) > \alpha. \quad (4.6.20)$$

Then, there exists $\tilde{U} \in H^1([t_1, t_2], \mathcal{L})$ such that

$$\forall t \in (t_1, t_2), \forall i \in \{1, 2\}, \quad U(t) \in \mathcal{E}^{\alpha, -}, \quad \tilde{U}(t_i) = U(t_i) \quad (4.6.21)$$

and

$$\forall c > 0, \quad \mathbf{E}_c(\tilde{U}) < \mathbf{E}_c(U). \quad (4.6.22)$$

The following is inspired by similar results from [13], although our version is weaker:

Lemma 4.6.7. Assume that (H4.1') holds. Let $t_1 < t_2$, $\alpha \in \left(\frac{\alpha_- + a}{2}, \alpha_0\right)$ and U_α be the constant function

$$U_\alpha : t \in [t_1, t_2] \rightarrow v_\alpha \in \mathcal{E}^{\alpha, -}, \quad \mathcal{E}(v_\alpha) = \alpha.$$

Then, there exists $\tilde{U} \in H_{\text{loc}}^1([t_1, t_2], \mathcal{L})$ such that

$$\forall t \in (t_1, t_2), \forall i \in \{1, 2\}, \quad \mathcal{E}(\tilde{U}(t)) < \alpha, \quad \tilde{U}(t_i) = v_\alpha \quad (4.6.23)$$

and

$$\forall c > 0, \quad \mathbf{E}_c(\tilde{U}) < \mathbf{E}_c(U_\alpha). \quad (4.6.24)$$

Proof of Lemma 4.6.6. From (H4.1'), we have that $\mathcal{E}^{\alpha, -}$ is closed and convex in \mathcal{L} . Therefore, we can consider the corresponding orthogonal projection $P^{\alpha, -} : \mathcal{L} \rightarrow \mathcal{E}^{\alpha, -}$. Define now

$$\forall t \in [t_1, t_2], \quad \tilde{U}(t) := P^{\alpha, -}(U(t))$$

which clearly belongs to $H^1([t_1, t_2], \mathcal{L})$. Moreover,

$$\forall t \in [t_1, t_2], \quad \|\tilde{U}'(t)\|_{\mathcal{L}} \leq \|U'(t)\|_{\mathcal{L}}. \quad (4.6.25)$$

Applying (4.6.19), we see that \tilde{U} verifies (4.6.21). Now, notice that by (H3.1') implies that $t \in [t_1, t_2] \rightarrow \mathcal{E}(U(t))$ is lower-semicontinuous. Therefore, $\{t \in [t_1, t_2] : \mathcal{E}(U(t)) > \alpha\}$ is open, and by (4.6.20) we have that it is also non-empty. As a consequence, there exists $I \subset [t_1, t_2]$ non-empty an open such that

$$\forall t \in I, \quad \mathcal{E}(U(t)) > \alpha$$

which implies that

$$\int_I \mathcal{E}(\tilde{U}(t)) e^{ct} dt < \int_I \mathcal{E}(U(t)) e^{ct} dt$$

and, since for all $v \in \mathcal{H}$ we have $\mathcal{E}(P^{\alpha, -}(v)) \leq \mathcal{E}(v)$ by definition of the orthogonal projection and because $\alpha \leq 0$, we conclude

$$\int_{t_1}^{t_2} \mathcal{E}(\tilde{U}(t)) e^{ct} dt < \int_{t_1}^{t_2} \mathcal{E}(U(t)) e^{ct} dt,$$

which in combination with (4.6.25) gives (4.6.22) and concludes the proof. \square

Proof of Lemma 4.6.7. Let $\varepsilon > 0$, $\mathbf{v}^- \in \mathcal{F}^-$ and $\phi \in \mathcal{C}_c^1([t_1, t_2])$ such that $\phi > 0$ in (t_1, t_2) . Define the

function

$$\forall t \in [t_1, t_2], \quad U_\varepsilon(t) := \mathbf{v}^- + (1 - \varepsilon\phi(t))(v_\alpha - \mathbf{v}^-)$$

so that $U_0 = U_\alpha$. It is clear that for all $\varepsilon > 0$ we have $U_\varepsilon \in H^1([t_1, t_2], \mathcal{L})$. Moreover, we also have that for $i \in \{1, 2\}$, it holds $U_\varepsilon(t_i) = v_\alpha$. For all $t \in [t_1, t_2]$, consider $A\left(\frac{\alpha_- + a}{2}, \mathbf{v}^-, U(t)\right)$ as in (4.5.1). Since $\alpha \in \left(\frac{\alpha_- + a}{2}, \alpha_0^-\right)$, $v_\alpha \in \mathcal{E}^{\alpha, -}$ and $\|\phi\|_{C^1([t_1, t_2])} < +\infty$, there exists $\varepsilon_1 > 0$ independent on t such that

$$(1 - \varepsilon_1\phi(t), 1 + \varepsilon_1\phi(t)) \subset A\left(\frac{\alpha_- + a}{2}, \mathbf{v}^-, U(t)\right).$$

Therefore, using (4.5.2) we have that

$$\begin{aligned} \forall \varepsilon \in (0, \varepsilon_1), \forall t \in [t_1, t_2], \quad \mathcal{E}(U_\varepsilon(t)) - \mathcal{E}(U_0(t)) &= - \int_{1-\varepsilon\phi(t)}^1 \frac{d}{d\lambda} (\mathcal{E}(\mathbf{v}^- + \lambda(v_\alpha - \mathbf{v}^-)))(\theta) d\theta \\ &\leq -\varepsilon\phi(t)\omega, \end{aligned} \quad (4.6.26)$$

where $\omega > 0$. In particular, we have that for all $\varepsilon \in (0, \varepsilon_1)$, U_ε fulfills (4.6.23). Now, notice that

$$\forall \varepsilon > 0, \text{ for a. e. } t \in [t_1, t_2], \quad U'_\varepsilon(t) = -\varepsilon\phi'(t)(v_\alpha - \mathbf{v}^-)$$

which means that, using also (4.6.26), we obtain

$$\begin{aligned} \forall c > 0, \forall \varepsilon \in (0, \varepsilon_1), \\ \mathbf{E}_c(U_\varepsilon) - \mathbf{E}_c(U_0) &\leq (-\varepsilon\|\phi\|_{L^\infty([t_1, t_2])}\omega + \frac{1}{2}\varepsilon^2\|\phi'\|_{L^\infty([t_1, t_2])}^2\|v_\alpha - \mathbf{v}^-\|_{\mathcal{L}}^2) \frac{e^{ct_2} - e^{ct_1}}{c} \end{aligned}$$

so that if $\varepsilon \in (0, \varepsilon_2)$, where

$$\varepsilon_2 := \frac{2\|\phi\|_{L^\infty([t_1, t_2])}\omega}{\|\phi'\|_{L^\infty([t_1, t_2])}^2\|v_\alpha - \mathbf{v}^-\|_{\mathcal{L}}^2} > 0$$

then (4.6.24) holds for U_ε . Hence, we have shown that if $\varepsilon \in (0, \varepsilon_3)$ with $\varepsilon_3 := \min\{\varepsilon_1, \varepsilon_2\} > 0$, then U_ε fulfills (4.6.23) and (4.6.24), which completes the proof. \square

We will now define for $U \in Y_T$

$$t_2^-(U) := \inf\{t \in \mathbb{R} : \mathcal{E}(U(t)) = \alpha_0, U(t) \in \mathcal{E}^{\alpha_0, -}\} \quad (4.6.27)$$

and

$$t_1^-(U) := \sup\{t \leq t_2^-(U) : \mathcal{E}(U(t)) \leq \alpha_-\} \quad (4.6.28)$$

where the constants $a < \alpha_- < 0 < \alpha_0$ are those from (H4.1'). The following preliminary property is straightforward:

Lemma 4.6.8. *Let $U \in Y_T$ and $t_2^-(U)$, $t_1^-(U)$ be as in (4.6.27) and (4.6.28) respectively. Then, it holds*

$$-T \leq t_1^-(U) \leq t_2^-(U)$$

Proof. We have that $\mathcal{E}(U(-T)) \leq \alpha_-$, which implies that $-T \leq t_1^-(U)$. The inequality $t_1^-(U) \leq t_2^-(U)$ follows from the definition. \square

Lemmas 4.6.6 and 4.6.7 apply mainly to obtain the following information on constrained minimizers in Y_T :

Proposition 4.6.1. *Assume that (H3.3'), (H3.4'), (H3.6') and (H4.1') hold. Let $c > 0$ and $T \geq 1$. Let $\mathbf{V}_{c,T}$ be a constrained minimizer of \mathbf{E}_c in Y_T given by Lemma 4.6.4. Let $t_1^- := t_1^-(\mathbf{V}_{c,T})$ and $t_2^- := t_2^-(\mathbf{V}_{c,T})$, where $t_1^-(\mathbf{V}_{c,T})$ and $t_2^-(\mathbf{V}_{c,T})$ are as in (4.6.28), (4.6.27) respectively. Then, it holds that $\mathcal{E}(\mathbf{V}_{c,T}(t_2^-)) = \alpha_0$ and*

$$t_2^- - t_1^- \leq T_1(c), \quad (4.6.29)$$

where

$$T_1(c) := \frac{2Rc + 2\sqrt{R^2c^2 + 2R\omega}}{\omega}. \quad (4.6.30)$$

Moreover, we have that

$$\forall t \leq t_1^-, \quad \mathcal{E}(\mathbf{V}_{c,T}(t)) \leq \alpha_- \quad (4.6.31)$$

and there exist $\delta_0^- > 0$ and $t_0^- \leq t_1^-$ such that

$$\begin{cases} \mathcal{E}(\mathbf{V}_{c,T}(\cdot)) \text{ is monotone in } (-\infty, t_0^- + \delta_0^-] \text{ and} \\ \forall t \leq t_0^- + \delta_0^-, \quad \mathcal{E}(\mathbf{V}_{c,T}(t)) < \alpha_-. \end{cases} \quad (4.6.32)$$

Proof. Notice that due to the definition of t_2^- in (4.6.27) for all $t \leq t_2^-$ we have $\mathbf{V}_{c,T}(t) \in \mathcal{E}^{\alpha_0,-}$. Therefore, $t_2^- < T$. Notice now that if $t_1^- = t_2^-$, then (4.6.30) holds trivially. Assume then that $t_1^- < t_2^-$. By Lemma 4.6.8, we have that $(t_1^-, t_2^-) \subset (-T, T)$ with $t_2^- < T$. Therefore, since we assume that (H3.6') holds we have that $\mathbf{V}_{c,T} \in \mathcal{A}((-T, T))$ fulfills (4.6.8) by Lemma 4.6.5. The definition of $\mathcal{A}((-T, T))$ in (3.4.23) implies that $t \in [t_1^-, t_2^-] \rightarrow \mathcal{E}(\mathbf{V}_{c,T}(t))$ is continuous, which means that $\mathcal{E}(\mathbf{V}_{c,T}(t_2^-)) = \alpha_0$. Due to the definitions of t_1^- and t_2^- in (4.6.28) and (4.6.27) respectively, we have that

$$\forall t \in (t_1^-, t_2^-), \quad \mathcal{E}(\mathbf{V}(t)) \in (\alpha_-, \alpha_0), \quad \mathbf{V}(t) \in \mathcal{E}^{\alpha_0,-}$$

which means that we can use (4.5.2) to obtain

$$\forall \mathbf{v}^- \in \mathcal{F}^-, \forall t \in (t_1^-, t_2^-), \quad \frac{d}{d\lambda}(\mathcal{E}(\mathbf{v}^- + \lambda(\mathbf{V}_{c,T}(t) - \mathbf{v}^-)))(1) \geq \omega > 0,$$

which means, using (3.4.10) in (H3.2')

$$\forall \mathbf{v}^- \in \mathcal{F}^-, \forall t \in (t_1^-, t_2^-), \quad \langle D_{\mathcal{L}}(\mathbf{V}_{c,T}), \mathbf{V}_{c,T}(t) - \mathbf{v}^- \rangle_{\mathcal{L}} \geq \omega. \quad (4.6.33)$$

By (H4.1'), there exists $\mathbf{v}_{c,T}^- \in \mathcal{F}^-$ such that $\|\mathbf{V}_{c,T}(t_2^-) - \mathbf{v}_{c,T}^-\|_{\mathcal{L}} \leq R$. Define now the function

$$\rho : t \in [t_1^-, t_2^-] \rightarrow \|\mathbf{V}_{c,T}(t) - \mathbf{v}_{c,T}^-\|_{\mathcal{L}}^2 \in \mathbb{R}$$

so that

$$\rho(t_2^-) \leq R. \quad (4.6.34)$$

By (4.6.8) and (H3.6'), we have that $\rho \in \mathcal{C}^2((t_1^-, t_2^-))$. We have

$$\forall t \in (t_1^-, t_2^-), \quad \rho'(t) = \langle \mathbf{V}'_{c,T}(t), \mathbf{V}_{c,T}(t) - \mathbf{v}_{c,T}^- \rangle_{\mathcal{L}}$$

and, subsequently, using (4.6.8)

$$\forall t \in (t_1^-, t_2^-), \quad \rho''(t) = \|\mathbf{V}_{c,T}\|_{\mathcal{L}}^2 + \langle D_{\mathcal{L}}(\mathbf{V}_{c,T}(t)), \mathbf{V}_{c,T}(t) - \mathbf{v}_{c,T}^- \rangle_{\mathcal{L}} - c\rho'(t),$$

which, by (4.6.33) means that

$$\forall t \in (t_1^-, t_2^-), \quad \rho''(t) + c\rho'(t) \geq \omega. \quad (4.6.35)$$

Notice that (4.6.35) implies that ρ does not possess any local maximum in $[t_1^-, t_2^-]$, as we have $\omega > 0$. That is, either

1. The function ρ is non-decreasing in $[t_1^-, t_2^-]$.
2. The function ρ possesses a unique local minimum at $t_3^- \in (t_1^-, t_2^-)$.

Assume that 1. holds. Then for each $t \in (t_1^-, t_2^-)$ we can integrate (4.6.35) in $[t_1^-, t]$ to obtain

$$c(\rho(t) - \rho(t_1^-)) + \rho'(t) - \rho'(t_1^-) \geq \omega(t - t_1^-)$$

so that, since we assume that $\rho' \geq 0$ in (t_1^-, t_2^-)

$$c\rho(t) + \rho'(t) \geq \omega(t - t_1^-) \quad (4.6.36)$$

which, integrating now in $[t_1^-, t_2^-]$ gives

$$c \int_{t_1^-}^{t_2^-} \rho(t) dt + \rho(t_2^-) \geq \frac{\omega}{2}(t_2^- - t_1^-)^2.$$

By (4.6.34), and since ρ does not possess any local maximum, the previous becomes,

$$Rc(t_2^- - t_1^-) + R \geq \frac{\omega}{2}(t_2^- - t_1^-)^2. \quad (4.6.37)$$

The roots of the polynomial

$$\frac{\omega}{2}x^2 - Rcx - R$$

are

$$x_1 = \frac{Rc + \sqrt{R^2c^2 + 2R\omega}}{\omega} > 0 \text{ and } x_2 = \frac{Rc - \sqrt{R^2c^2 + 2R\omega}}{\omega} < 0, \quad (4.6.38)$$

therefore, since (4.6.37) holds, we must have

$$t_2^- - t_1^- \leq x_1. \quad (4.6.39)$$

Assume now that 1. does not hold. Then 2. holds. Notice that since t_3^- is a local minimum of ρ in (t_1^-, t_2^-) we have $\rho'(t_3^-) = 0$. We first integrate (4.6.35) in $[t_3^-, t]$ for $t \in (t_3^-, t_2^-)$ to get

$$c(\rho(t) - \rho(t_3^-)) + \rho'(t) \geq \omega(t - t_3^-)$$

which means that

$$\forall t \in (t_3^-, t_2^-), \quad c\rho(t) + \rho'(t) \geq \omega(t - t_3^-),$$

which is exactly (4.6.36) with t_1^- replaced by t_3^- . The same reasoning as above gives

$$t_2^- - t_3^- \leq x_1 \quad (4.6.40)$$

with x_1 as in (4.6.38). We now take $t \in (t_1^-, t_3^-)$ and integrate (4.6.35) in $[t, t_3^-]$. We obtain

$$-\rho'(t) + c(\rho(t_3^-) - \rho(t)) \geq \omega(t_3^- - t)$$

so that

$$\forall t \in [t_1^-, t_3^-], \quad -\rho'(t) + c\rho(t_3) \geq \omega(t_3^- - t)$$

which, by integrating in $[t_1^-, t_3^-]$ and using (4.6.34) along with the fact that ρ has no local maximum in $[t_1^-, t_2^-]$, gives

$$R + Rc(t_3^- - t_1^-) \geq \frac{\omega}{2}(t_3^- - t_1^-)^2$$

which is exactly (4.6.37) but with t_2^- replaced by t_3^- . Therefore, we get

$$t_3^- - t_1^- \leq x_1$$

which in combination with (4.6.40) gives

$$t_2^- - t_1^- \leq 2x_1. \quad (4.6.41)$$

As a consequence, we have that either (4.6.39) or (4.6.41) holds. That is, we have shown that (4.6.29), with $T_1(c)$ as in (4.6.30), holds. We now show that (4.6.31) holds. Otherwise, we would have $-T < t_1^-$ and some $\tilde{t} \in (-T, t_1^-)$ such that

$$\mathcal{E}(\mathbf{V}_{c,T}(\tilde{t})) > \alpha_-$$

and, since we have that $\mathcal{E}(\mathbf{V}_{c,T}(-T))$ and $\mathcal{E}(\mathbf{V}_{c,T}(t_1^-))$ are smaller than α_- , we obtain a contradiction by Lemma 4.6.6. Therefore, (4.6.31) has been established. We now show the existence $t_0^- \leq t_1^-$ and $\delta_0^- > 0$ such that (4.6.32) holds. We begin by showing the existence of $\tilde{t}_0^- \leq t_1^-$ such that

$$\forall t \leq \tilde{t}_0^-, \quad \mathcal{E}(\mathbf{V}_{c,T}(t)) < \alpha_-. \quad (4.6.42)$$

By contradiction, assume that there exists a sequence $(t_n)_{n \in \mathbb{N}}$ in $(-\infty, t_1^-]$ such that $t_n \rightarrow -\infty$ as $n \rightarrow \infty$ and

$$\forall n \in \mathbb{N}, \quad \mathcal{E}(\mathbf{V}_{c,T}(t_n)) = \alpha_-. \quad (4.6.43)$$

Assume by contradiction that there exists a sequence $(\tilde{t}_n)_{n \in \mathbb{N}}$ in $(-\infty, t_1^-]$ such that $\tilde{t}_n \rightarrow -\infty$ and

$$\forall n \in \mathbb{N}, \quad \mathcal{E}(\mathbf{V}_{c,T}(\tilde{t}_n)) < \alpha_-$$

which, in combination with (4.6.43), shows the existence of $t_a < t_b < t_c \leq t_1^-$ and $\tilde{\alpha}_- < \alpha_- < 0$ such that

$$\mathbf{V}_{c,T}(t_a), \mathbf{V}_{c,T}(t_c) \in \mathcal{E}^{\tilde{\alpha}_-} = \mathcal{E}^{\tilde{\alpha}_-, -} \text{ and } \mathcal{E}(\mathbf{V}_{c,T}(t_b)) = \alpha_- > \tilde{\alpha}_-$$

which, by Lemma 4.6.6, gives a contradiction with the minimality of $\mathbf{V}_{c,T}$. As a consequence, we

have that there exists $\tilde{t} \leq t_1^-$ such that

$$\forall t \leq \tilde{t}, \quad \mathcal{E}(\mathbf{V}_{c,T}(t)) = \alpha_-. \quad (4.6.44)$$

By Lemma 4.6.7, we have that $\mathbf{V}_{c,T}$ cannot be constant in any non-empty interval $I \subset (-\infty, \tilde{t}]$. Therefore, we have in particular that

$$\int_{-\infty}^{\tilde{t}} \frac{\|\mathbf{V}'_{c,T}(t)\|_{\mathcal{L}}^2}{2} e^{ct} dt > 0. \quad (4.6.45)$$

Let \tilde{U} be the function defined as

$$\tilde{U}(t) = \begin{cases} \mathbf{V}_{c,T}\left(\frac{1}{2}t + \frac{1}{2}\tilde{t}\right) & \text{if } t \leq \tilde{t} \\ \mathbf{V}_{c,T}(t) & \text{if } t \geq \tilde{t} \end{cases}$$

which is well defined and belongs to Y_T . Notice that for all $t \leq \tilde{t}$, we have that $\frac{1}{2}t + \frac{1}{2}\tilde{t} \leq \tilde{t}$, so that $\mathcal{E}(\tilde{U}(t)) = \alpha_-$ by (4.6.44). Therefore,

$$\int_{-\infty}^{\tilde{t}} \mathcal{E}(\tilde{U}(t)) e^{ct} dt = \int_{-\infty}^{\tilde{t}} \mathcal{E}(\mathbf{V}_{c,T}(t)) e^{ct} dt,$$

meaning that

$$\int_{\mathbb{R}} \mathcal{E}(\tilde{U}(t)) e^{ct} dt = \int_{\mathbb{R}} \mathcal{E}(\mathbf{V}_{c,T}(t)) e^{ct} dt. \quad (4.6.46)$$

Now, notice that for a. e. $t \leq \tilde{t}$ it holds $U'(t) = \frac{1}{2}\mathbf{V}_{c,T}\left(\frac{1}{2}t + \frac{1}{2}\tilde{t}\right)$. Hence, we have that

$$\begin{aligned} \int_{-\infty}^{\tilde{t}} \frac{\|\tilde{U}'(t)\|_{\mathcal{L}}^2}{2} e^{ct} dt &= \frac{1}{4} \int_{-\infty}^{\tilde{t}} \frac{\|\mathbf{V}_{c,T}\left(\frac{1}{2}t + \frac{1}{2}\tilde{t}\right)\|_{\mathcal{L}}^2}{2} e^{ct} dt \\ &\leq \frac{1}{4} \int_{-\infty}^{\tilde{t}} \frac{\|\mathbf{V}_{c,T}\left(\frac{1}{2}t + \frac{1}{2}\tilde{t}\right)\|_{\mathcal{L}}^2}{2} e^{c(\frac{1}{2}t + \frac{1}{2}\tilde{t})} dt \\ &= \frac{1}{2} \int_{-\infty}^{\tilde{t}} \frac{\|\mathbf{V}'_{c,T}(t)\|_{\mathcal{L}}^2}{2} e^{ct} dt, \end{aligned}$$

which, by (4.6.45), means that

$$\int_{\mathbb{R}} \frac{\|\tilde{U}'(t)\|_{\mathcal{L}}^2}{2} e^{ct} dt < \int_{\mathbb{R}} \frac{\|\mathbf{V}'_{c,T}(t)\|_{\mathcal{L}}^2}{2} e^{ct} dt,$$

so that, taking also into account (4.6.46), we have obtained

$$\mathbf{E}_c(\tilde{U}) < \mathbf{E}_c(\mathbf{V}_{c,T}),$$

which is a contradiction, because $\mathbf{V}_{c,T}$ minimizes \mathbf{E}_c in Y_T . As a consequence, (4.6.43) cannot hold, which means that there exists $\tilde{t}_0^- \leq t_1^-$ such that (4.6.42) holds. We now establish the existence of $t_0^- \leq t_1^-$ and $\delta_0^- > 0$ such that (4.6.32) holds. Applying Lemma 4.6.6, we have that $\mathcal{E}(\mathbf{V}_{c,T}(\cdot))$ does not possess any strict local maximum, and at most one strict local minimum, in $(-\infty, t_1^-]$. Therefore, there exist $t_0^- \leq \tilde{t}_0^-$ and $\delta_0^- > 0$ such that $\mathcal{E}(\mathbf{V}_{c,T}(\cdot))$ is monotone in $(-\infty, t_0^- + \delta_0^-]$ and $t_0^- + \delta_0^- \leq \tilde{t}_0^-$.

Therefore, (4.6.32) follows by (4.6.42). \square

For $U \in Y_T$, recall the definition of $t^+(U, \varepsilon_0^+)$ in (3.5.40). The goal now will be to provide a uniform bound on $t^+(U, \varepsilon_0^+) - t_2^-(U)$. The result is very close to the second part of the comparison result in Chapter 3, Proposition 3.5.1:

Proposition 4.6.2. *Assume that (H3.3'), (H3.4'), (H3.5'), (H3.6') and (H4.1') hold. Let $c > 0$ and $T \geq 1$. Consider $\mathbf{V}_{c,T} \in Y_T$ a minimizer of \mathbf{E}_c in Y_T , which exists by Lemma 4.6.4. Let $t_2^- := t_2^-(\mathbf{V}_{c,T})$ as in (4.6.27) and $t^+ := t^+(\mathbf{V}_{c,T}, \varepsilon_0^+)$ as in (3.5.40). Then, we have that*

$$\forall t \geq t^+, \quad \text{dist}_{\mathcal{L}}(\mathbf{V}_{c,T}(t), \mathcal{F}^+) < \frac{r_0^+}{2} \quad (4.6.47)$$

Furthermore, we have that

$$0 < t^+ - t_2^- \leq T_2(c), \quad (4.6.48)$$

where

$$T_2(c) := \frac{1}{c} \ln \left(\frac{-a}{\alpha_{\star\star}} + 1 \right) \quad (4.6.49)$$

with $\alpha_{\star\star} > 0$ a constant independent on c , T and U .

Proof. We claim that for all $t \geq t^+$, it holds that $\mathcal{E}(\mathbf{V}_{c,T}(t)) \geq 0$. Indeed, otherwise we could find $\tilde{t} \geq t^+$ such that $\mathcal{E}(\mathbf{V}_{c,T}(\tilde{t})) = \alpha < 0$. Since we necessarily have $t^+ > -T$, and $\mathbf{V}_{c,T}(-T) \in \mathcal{E}^{\max\{\alpha_-, \alpha\}}$, by Lemma 4.6.6, we can find $\tilde{U} \in Y_T$ such that $\mathbf{E}_c(\tilde{U}) < \mathbf{E}_c(\mathbf{V}_{c,T})$, a contradiction. Therefore, arguing as in the second part of Proposition 3.5.1 in Chapter 3, we obtain that (4.6.47) must hold. In order to prove (4.6.48), we first show that $t^+ > t_2^-$. This follows from the fact that for all $t \leq t_2^-$ we must have $\mathbf{V}_{c,T}(t) \in \mathcal{E}^{\alpha_0, -}$, $\mathbf{V}_{c,T}(t^+) \in \mathcal{F}_{r_0^+/2}^+$ and $\mathcal{F}_{r_0^+/2}^+ \cap \mathcal{E}^{\alpha_0, -} = \emptyset$. See (H4.1'). Let $\alpha_{\star\star} := \min\{\alpha_0, \varepsilon_0^+\}$. We claim that

$$\forall t \in [t_2^-, t^+], \quad \mathcal{E}(\mathbf{V}_{c,T}(t)) \geq \alpha_{\star\star}. \quad (4.6.50)$$

Indeed, assume by contradiction that for some $t \in (t_2^-, t^+)$ we have $\mathcal{E}(\mathbf{V}_{c,T}(t)) = \alpha < \alpha_{\star\star}$. If $\mathbf{V}_{c,T}(t) \in \mathcal{F}_{r_0^+/2}^+$, we have a contradiction with the definition of t^+ since $\alpha < \varepsilon_0^+$. Assume then $\mathbf{V}_{c,T}(t) \notin \mathcal{F}_{r_0^+/2}^+$. Since $\mathcal{E}^{\alpha, +} \subset \mathcal{E}^{\alpha_0, +} \subset \mathcal{F}_{r_0^+/2}^+$ by (H4.1'), we have that $\mathbf{V}_{c,T}(t) \in \mathcal{E}^{\alpha, -}$. We have that $\mathbf{V}_{c,T}(-T) \in \mathcal{E}^{\alpha_-} \subset \mathcal{E}^{\alpha, -}$ and $\mathbf{V}_{c,T}(t_2^-) = \alpha_0 > \alpha$ by Proposition 4.6.1. Therefore, Lemma 4.6.6 leads to a contradiction. As a consequence, we have shown that (4.6.50) holds. Arguing as in the last part of the corresponding result in Chapter 3, we obtain that (4.6.48), with $T_2(c)$ as in (4.6.49) holds. \square

Propositions 4.6.1 and 4.6.2 can be summarized as follows:

Corollary 4.6.1. *Assume that (H3.3'), (H3.4'), (H3.5'), (H3.6') and (H4.1') hold. Let $c > 0$ and $T \geq 1$. Consider $\mathbf{V}_{c,T} \in Y_T$ a minimizer of \mathbf{E}_c in Y_T , which exists by Lemma 4.6.4. Let $t_1^- := t_1^-(\mathbf{V}_{c,T})$ as in (4.6.28) and $t^+ := t^+(\mathbf{V}_{c,T}, \varepsilon_0^+)$ as in (3.5.40). Then it holds that*

$$\forall t \geq t^+, \quad \text{dist}_{\mathcal{L}}(\mathbf{V}_{c,T}(t), \mathcal{F}^+) < \frac{r_0^+}{2}, \quad (4.6.51)$$

$$\forall t \leq t_1^-, \quad \mathcal{E}(\mathbf{V}_{c,T}(t)) \leq \alpha_- \quad (4.6.52)$$

and there exist $\delta_0^- > 0$ and $t_0^- \leq t_1^-$ such that

$$\begin{cases} \mathcal{E}(\mathbf{V}_{c,T}(\cdot)) \text{ is monotone in } (-\infty, t_0^- + \delta_0^-] \text{ and} \\ \forall t \leq t_0^- + \delta_0^-, \mathcal{E}(\mathbf{V}_{c,T}(t)) < \alpha_-. \end{cases} \quad (4.6.53)$$

Moreover, we have

$$0 < t^+ - t_1^- \leq T_{\star\star}(c) \quad (4.6.54)$$

where

$$T_{\star\star}(c) := T_1(c) + T_2(c)$$

with $T_1(c)$ as in (4.6.30) and $T_2(c)$ as in (4.6.49). That is,

$$c \in (0, +\infty) \rightarrow T_{\star\star}(c) \in (0, +\infty)$$

is continuous and independent on T . Finally, if $t_2^- := t_2^-(\mathbf{V}_{c,T})$ as in (4.6.27), we have $t_2^- \in (t_1^-, t^+)$, $\mathbf{V}_{c,T}(t_2^-) \in \mathcal{E}^{\alpha_0, -}$ and

$$\forall t \geq t_2^-, \mathcal{E}(\mathbf{V}_{c,T}(t)) \geq 0. \quad (4.6.55)$$

Finally, we have the following property on constrained solutions:

Corollary 4.6.2. Assume that (H3.3'), (H3.4'), (H3.5'), (H3.6') and (H4.1') hold. Let $c > 0$ and $T \geq 1$. Consider $\mathbf{V}_{c,T} \in Y_T$ a minimizer of \mathbf{E}_c in Y_T , which exists by Lemma 4.6.4. Let $t_1^- := t_1^-(\mathbf{V}_{c,T})$ as in (4.6.28) and $t^+ := t^+(\mathbf{V}_{c,T}, \varepsilon_0^+)$ as in (3.5.40) and $t_0^- \leq t_1^-$, also given by Corollary 4.6.1. Then, there exists $\delta_{Y,c,T} > 0$ such that the set

$$S_{Y,c,T} := (-\infty, t_0^- + \delta_{Y,c,T}) \cup (-T, T) \cup (t^+ - \delta_{Y,c,T}, +\infty)$$

is such that $\mathbf{V}_{c,T} \in \mathcal{A}(S_{Y,c,T})$ (see (3.4.23)) and

$$\mathbf{V}_{c,T}'' - D_{\mathcal{L}}\mathcal{E}(\mathbf{V}_{c,T}) = -c\mathbf{V}_{c,T}' \text{ in } S_{Y,c,T}. \quad (4.6.56)$$

Proof. Using (4.6.53) in Corollary 4.6.1, we have that for each $\tilde{T} > 0$ it holds

$$\forall t \leq [t_0^- - \tilde{T}, t_0^- + \delta_0^-], \mathcal{E}(\mathbf{V}_{c,T}(t)) \leq \tilde{\alpha} < \alpha_-,$$

so that (4.6.11) in Lemma 4.6.5 is satisfied for any closed interval contained in $(-\infty, t_0^-]$. As a consequence, (4.6.12) follows in such interval. That means that for $t_0^- \leq t_1^-$ and $\delta_0^- > 0$ as in Corollary 4.6.1, we have that $\mathbf{V}_{c,T} \in \mathcal{A}((-\infty, t_0^- + \delta_0^-))$ and

$$\mathbf{V}_{c,T}'' - D_{\mathcal{L}}\mathcal{E}(\mathbf{V}_{c,T}) = -c\mathbf{V}_{c,T}'' \text{ in } (-\infty, t_0^- + \delta_0^-).$$

Using again Lemma 4.6.5 in combination with (4.6.51) in Corollary 4.6.1, we obtain for some $\delta_0^+ > 0$ that

$$\mathbf{V}_{c,T} \in \mathcal{A}((-T, T) \cup (t^+ - \delta_0^+, +\infty))$$

and

$$\mathbf{V}_{c,T}'' - D_{\mathcal{L}}\mathcal{E}(\mathbf{V}_{c,T}) = -c\mathbf{V}_{c,T}'' \text{ in } (-T, T) \cup (t^+ - \delta_0^+, +\infty).$$

As a consequence, (4.6.56) follows by taking $\delta_{Y,c,T} := \min\{\delta_0^-, \delta_0^+\} > 0$. \square

4.6.5 Existence of an unconstrained solution

We can now establish the existence of an unconstrained solution for the proper speed. Let

$$\mathcal{D} := \{c > 0 : \exists T \geq 1 \text{ and } U \in Y_T \text{ such that } \mathbf{E}_c(U) < 0\}. \quad (4.6.57)$$

We have:

Lemma 4.6.9. *Assume that (H3.3'), (H3.4'), (H3.5') and (H4.1') hold. Let \mathcal{D} be the set defined in (4.6.57). Then, \mathcal{D} is open and non-empty and bounded such that*

$$\sup \mathcal{D} \leq \frac{\sqrt{-2a}}{d_{\alpha_0}}, \quad (4.6.58)$$

where $d_{\alpha_0} > 0$ is as in (4.5.13)

Proof. Let Ψ be the function defined in (4.6.5). Since (H4.1') holds we have by Lemma 4.6.2 that $\Psi \in Y_T$. Therefore, the argument from Lemma 3.5.11 in Chapter 3 applies here to show that $\mathcal{D} \neq \emptyset$. Subsequently, let $c \in \mathcal{D}$. By definition, there exists $T \geq 1$ such that $\mathbf{E}_c(\mathbf{V}_{c,T}) < 0$, where $\mathbf{V}_{c,T} \in Y_T$ is a minimizer of \mathbf{E}_c in Y_T , which exists by Lemma 4.6.4. Let $\mathbf{v}_{c,T}^+$ be such that $\|\mathbf{V}_{c,T}(t) - \mathbf{v}_{c,T}^+\|_{\mathcal{H}} \rightarrow 0$ as $t \rightarrow +\infty$, which occurs due to Lemma 3.5.3 in Chapter 3. Define for $t \geq T$

$$\mathbf{V}_{c,T}^t(s) := \begin{cases} \mathbf{V}_{c,T}(s) & \text{if } s \leq t, \\ (1+t-s)\mathbf{V}_{c,T}(t) + (s-t)\mathbf{v}_{c,T}^+ & \text{if } t \leq s \leq t+1, \\ \mathbf{v}_{c,T}^+ & \text{if } t+1 \leq s. \end{cases}$$

which belongs to Y_T . Arguing as in Lemma 3.5.11, we can show that for $t \geq T$ large enough and some $\delta > 0$ we have for all $\tilde{c} \in (c - \delta, c + \delta)$ that $\mathbf{E}_{\tilde{c}}(\mathbf{V}_{c,T}^t(s)) < 0$. This shows that \mathcal{D} is open. In order to establish the bound (4.6.58), we use Corollary 4.6.1. Let $c \in \mathcal{D}$, there exists then $T \geq 1$ such that $\mathbf{E}_c(\mathbf{V}_{c,T}) < 0$, with $\mathbf{V}_{c,T}$ a minimizer of \mathbf{E}_c in Y_T . Let $t_2^- := t_2^-(\mathbf{V}_{c,T})$ as in (4.6.27) and $t^+ := t^+(\mathbf{V}_{c,T}, \varepsilon_0^+)$ as in (3.5.40). Corollary 4.6.1 implies that $\mathbf{V}_{c,T}(t_2^-) \in \mathcal{E}^{\alpha_0-}$ and $\mathbf{V}_{c,T}(t^+) \in \mathcal{F}_{r_0^+/2}^+$. Therefore, the definition of d_{α_0} in (4.5.13) implies

$$(d_{\alpha_0})^2 \leq \|\mathbf{V}_{c,T}(t^+) - \mathbf{V}_{c,T}(t_2^-)\|_{\mathcal{H}}^2$$

so that

$$(d_{\alpha_0})^2 \leq 2 \int_{\mathbb{R}} \frac{\|\mathbf{V}_{c,T}'(t)\|_{\mathcal{H}}^2}{2} e^{ct} dt \left(\frac{e^{-ct_2^-} - e^{-ct^+}}{c} \right).$$

Using (4.6.55) in Corollary 4.6.1, the inequality above becomes

$$(d_{\alpha_0})^2 \leq 2 \left(\mathbf{E}_c(\mathbf{V}_{c,T}) - \frac{e^{ct_2^-}}{c} \right) \left(\frac{e^{-ct_2^-} - e^{-ct^+}}{c} \right)$$

which, using that $\mathbf{E}_c(\mathbf{V}_{c,T}) < 0$ and $t^+ > t_2^-$, gives (4.6.58) and concludes the proof. \square

We can now establish the existence of an unconstrained solution:

Proposition 4.6.3. *Assume that (H3.3'), (H3.4'), (H3.5'), (H3.6') and (H4.1') hold. Let $\bar{c} \in \partial(\mathcal{D}) \cap (0, +\infty)$, where $\partial(\mathcal{D})$ stands for the boundary of the set \mathcal{D} defined in (4.6.57). Then, there exists $\bar{T} \geq 1$ such that $\mathbf{b}_{\bar{c}, \bar{T}} = 0$ ($\mathbf{b}_{\bar{c}, \bar{T}}$ as in (4.6.4)) and $\bar{\mathbf{V}} \in Y_{\bar{T}}$ an associated minimizer of $\mathbf{E}_{\bar{c}}$ which does not saturate the constraints, i. e.,*

$$\forall t \geq \bar{T}, \quad \text{dist}_{\mathcal{L}}(\bar{\mathbf{V}}(t), \mathcal{F}^+) < \frac{r_0^+}{2} \quad (4.6.59)$$

and

$$\forall t \leq -\bar{T}, \quad \mathcal{E}(\bar{\mathbf{V}}(t)) < \alpha_-. \quad (4.6.60)$$

In particular, the pair $(\bar{c}, \bar{\mathbf{V}})$ solves (4.5.8).

Remark 4.6.1. Notice that Lemma 4.6.9 implies that (under the necessary assumptions) the set \mathcal{D} is bounded, meaning that the set $\partial(\mathcal{D}) \cap (0, +\infty)$ is not empty. Such a fact, in combination with Proposition 4.6.3, shows the existence of the unconstrained solution.

Proof of Proposition 4.6.3. We essentially mimic the proof from Proposition 3.5.2 in Chapter 3, with the obvious modifications. Lemma 4.6.9 implies that $\partial(\mathcal{D}) \subset \mathbb{R} \setminus \mathcal{D}$, so that $\bar{c} \notin \mathcal{D}$. The definition of \mathcal{D} in (4.6.57) implies then that

$$\forall T \geq 1, \quad \mathbf{b}_{\bar{c}, T} \geq 0. \quad (4.6.61)$$

Let $(c_n)_{n \in \mathbb{N}}$ be a sequence in \mathcal{D} such that $c_n \rightarrow \bar{c}$ as $n \rightarrow \infty$. By definition, for each $n \in \mathbb{N}$ there exists $T_n \geq 1$ such that $\mathbf{E}_{c_n}(\mathbf{V}_{c_n, T_n}) < 0$ where, for each $n \in \mathbb{N}$, \mathbf{V}_{c_n, T_n} is a minimizer of \mathbf{E}_{c_n} in Y_{T_n} . For each $n \in \mathbb{N}$, set $t_{1,n}^- := t_1^-(\mathbf{V}_{c_n, T_n})$ as in (4.6.28) and $t_n^+ := t^+(\mathbf{V}_{c_n, T_n}, \varepsilon_0^+)$ as in (3.5.40). Using (4.6.54) in Corollary 4.6.1 we have

$$\forall n \in \mathbb{N}, \quad 0 < t_n^+ - t_{1,n}^+ \leq T_{\star\star}(c_n)$$

with

$$c \in (0, +\infty) \rightarrow T_{\star\star}(c) \in (0, +\infty)$$

a continuous function. Since the sequence $(c_n)_{n \in \mathbb{N}}$ is bounded, we have that

$$T_{\star\star} := \max \left\{ 1, \sup_{n \in \mathbb{N}} T_{\star\star}(c_n) \right\} < +\infty$$

and

$$\forall n \in \mathbb{N}, \quad 0 < t_n^+ - t_{1,n}^- \leq T_{\star\star}, \quad (4.6.62)$$

so that we have a bound on $(t_n^+ - t_{1,n}^-)_{n \in \mathbb{N}}$. Moreover, (4.6.51) and (4.6.52) in Corollary 4.6.1 imply

$$\forall n \in \mathbb{N}, \forall t \geq t_{1,n}^+, \quad \text{dist}_{\mathcal{L}}(\mathbf{V}_{c_n, T_n}(t), \mathcal{F}^+) < \frac{r_0^+}{2} \quad (4.6.63)$$

and

$$\forall n \in \mathbb{N}, \forall t \leq t_n^-, \quad \mathcal{E}(\mathbf{V}_{c_n, T_n}(t)) \leq \alpha_-. \quad (4.6.64)$$

For each $n \in \mathbb{N}$, define the function $\mathbf{V}_{c_n, T_n}^{t_n^+} := \mathbf{V}_{c_n, T_n}(\cdot + t_n^+)$. Then, (4.6.62) implies that (4.6.63) and (4.6.64) write as

$$\forall n \in \mathbb{N}, \forall t \geq 0, \quad \text{dist}_{\mathcal{L}}(\mathbf{V}_{c_n, T_n}^{t_n^+}(t), \mathcal{F}^-) < \frac{r_0^-}{2}.$$

and

$$\forall n \in \mathbb{N}, \forall t \leq -T_{\star\star}, \quad \mathcal{E}(\mathbf{V}_{c_n, T_n}^{t_n^+}(t)) \leq \alpha_-$$

so that for all $n \in \mathbb{N}$ we have $\mathbf{V}_{c_n, T_n}^{t_n^+} \in Y_{T_{**}}$. Moreover, a computation shows

$$\forall n \in \mathbb{N}, \quad \mathbf{E}_{c_n}(\mathbf{V}_{c_n, T_n}^{t_n^+}) = e^{-c_n t_n^+} \mathbf{E}_{c_n}(\mathbf{V}_{c_n, T_n}) < 0.$$

Therefore, if we apply Lemma 4.6.3 with sequence of speeds $(c_n)_{n \in \mathbb{N}}$ and the sequence $(\mathbf{V}_{c_n, T_n}^{t_n^+})_{n \in \mathbb{N}}$ in $Y_{T_{**}}$, we obtain $\bar{\mathbf{V}} \in Y_{T_{**}}$ such that

$$\mathbf{E}_{\bar{c}}(\bar{\mathbf{V}}) \leq \liminf_{n \rightarrow \infty} \mathbf{E}_{c_n}(\mathbf{V}_{c_n, T_n}^{t_n^+}) \leq 0,$$

which in combination with (4.6.61) implies that $\mathbf{b}_{\bar{c}, T_{**}} = 0$. Therefore, we have $\mathbf{E}_{\bar{c}}(\bar{\mathbf{V}}) = 0$, so that $\bar{\mathbf{V}}$ is a minimizer of $\mathbf{E}_{\bar{c}}$ in $Y_{T_{**}}$. Set $t_{1, **}^- := t_1^-(\bar{\mathbf{V}})$ as in (4.6.28) and $t_{**}^+ := t^+(\bar{\mathbf{V}}, \varepsilon_0^\pm)$ as in (3.5.40). Invoking (4.6.51) and (4.6.53) in Corollary 4.6.1, we obtain as before that for some $t_0^- \leq t_1^-$ and $\delta_0^- > 0$

$$\forall t \geq t_{**}^+, \quad \text{dist}_{\mathcal{L}}(\bar{\mathbf{V}}(t), \mathcal{F}^+) < \frac{r_0^+}{2}, \quad (4.6.65)$$

$$\begin{cases} \mathcal{E}(\bar{\mathbf{V}}(\cdot)) \text{ is monotone in } (-\infty, t_0^- + \delta_0^-] \text{ and} \\ \forall t \leq t_0^- + \delta_0^-, \quad \mathcal{E}(\bar{\mathbf{V}}(t)) < \alpha_- \end{cases} \quad (4.6.66)$$

If we set $\bar{T} = \max\{1, t_{**}^+, -t_0^-\}$, then (4.6.65) and (4.6.66) imply that $\bar{\mathbf{V}} \in Y_{\bar{T}}$ and that (4.6.59), (4.6.60) hold. Moreover, we have that $\mathbf{E}_{\bar{c}}(\bar{\mathbf{V}}) = 0$, so that $\bar{\mathbf{V}}$ is a minimizer of $\mathbf{E}_{\bar{c}}$ in $Y_{\bar{T}}$ due to (4.6.61). We will now apply Corollary 4.6.2 with \bar{c} , \bar{T} as constants and $\bar{\mathbf{V}}$ as constrained minimizer. By (4.6.65) and (4.6.66), we have that $\bar{\mathbf{V}} \in \mathcal{A}(S_{Y, \bar{c}, \bar{T}})$ and

$$\bar{\mathbf{V}}'' - D_{\mathcal{L}} \mathcal{E}(\bar{\mathbf{V}}) = -\bar{c} \bar{\mathbf{V}}' \text{ in } S_{Y, \bar{c}, \bar{T}},$$

with

$$S_{Y, \bar{c}, \bar{T}} := (-\infty, t_0^- + \delta_{Y, \bar{c}, \bar{T}}) \cup (-\bar{T}, \bar{T}) \cup (t_{**}^+ - \delta_{Y, \bar{c}, \bar{T}}, +\infty),$$

for some $\delta_{Y, \bar{c}, \bar{T}} > 0$. The choice of \bar{T} implies that $S_{Y, \bar{c}, \bar{T}} = \mathbb{R}$, which concludes the proof. \square

4.6.6 Uniqueness of the speed

Recall that Proposition 3.5.3 in Chapter 3 gives a uniqueness statement on the speed. This allows to show the following:

Corollary 4.6.3. *Assume that (H3.3'), (H3.4'), (H3.5'), (H3.6') and (H4.1') hold. Let*

$$c(\mathcal{D}) := \sup \mathcal{D}.$$

Then, we have $\mathcal{D} = (0, c(\mathcal{D}))$.

Proof of Corollary 4.6.3. The idea is the same than in Corollary 3.5.3, Chapter 3. As before, we need to show that

$$\partial(\mathcal{D}) \cap (0, +\infty) = \{c(\mathcal{D})\}.$$

Let $\bar{c} \in \partial(\mathcal{D}) \cap (0, +\infty)$, we need to show that $\bar{c} = c(\mathcal{D})$ in order to finish the proof. Using Proposition

4.6.3 we obtain $\mathbf{V}^{\mathcal{D}}$ and $\bar{\mathbf{V}}$ in $Y \subset X$ such that $(c(\mathcal{D}), \mathbf{V}(\mathcal{D}))$ and $(\bar{c}, \bar{\mathbf{V}})$ solve (4.5.8). Moreover,

$$\inf_{U \in Y} \mathbf{E}_{c(\mathcal{D})}(U) = \mathbf{E}_{c(\mathcal{D})}(\mathbf{V}^{\mathcal{D}}) = 0 = \mathbf{E}_{\bar{c}}(\bar{\mathbf{V}}) = \inf_{U \in Y} \mathbf{E}_{\bar{c}}(U)$$

so that we also have

$$\mathbf{E}_{c(\mathcal{D})}(\bar{\mathbf{V}}) \geq 0 \text{ and } \mathbf{E}_{\bar{c}}(\mathbf{V}^{\mathcal{D}}) \geq 0.$$

Therefore, the requirements of Proposition 3.5.3 are met, which means that $\bar{c} = c(\mathcal{D})$. \square

4.6.7 Proof of Theorem 4.4 completed

If (H4.1') holds, using Proposition 4.6.3 and Corollary 4.6.3, we can argue as in Chapter 3 to establish the existence of a solution solving (4.5.8) with condition (4.5.9) and speed $c^{\star} = c(\mathcal{D})$. The statement regarding the uniqueness of the speed c^{\star} follows from Proposition 3.5.3. Finally, we have that the exponential convergence (4.5.10) follows from (3.5.7) in Lemma 3.5.3, Chapter 3. Thus, the proof of Theorem 4.4 is completed. \square

4.6.8 Asymptotic behavior of the constrained solutions

We essentially adapt the corresponding results of Chapter 3 to this setting. We have the following result:

Lemma 4.6.10. *Assume that (H3.3'), (H3.4'), (H3.5'), (H3.6') and (H4.1') hold. Let $\mathbf{U}_{c,T}$ be a constrained solution given by Lemma 4.6.4 and $t^{-} := t^{-}(\mathbf{U}_{c,T}, \mathcal{E}_{\max}^{-})$ be as in (3.5.39). Then for all $t < t^{-}$ we have the inequality*

$$\frac{\|\mathbf{U}'_{c,T}(t)\|_{\mathcal{F}}^2}{2} \leq \mathcal{E}(\mathbf{U}_{c,T}(t)) - a.$$

Similarly, it holds that for all $t > t^{+}$

$$\mathcal{E}(\mathbf{U}_{c,T}(t)) \leq \frac{\|\mathbf{U}'_{c,T}(t)\|_{\mathcal{F}}^2}{2},$$

where $t^{+} := t^{+}(\mathbf{U}_{c,T}, \mathcal{E}_{\max}^{+})$ is as in (3.5.40).

The proof goes exactly as Lemma 3.5.13 in Chapter 3, with the obvious minor modifications. From Lemma 4.6.10 it follows a convergence result at $-\infty$, which one also might prove by modifying from Proposition 3.5.4, Chapter 3:

Proposition 4.6.4. *Assume that (H3.3'), (H3.4'), (H3.5'), (H3.6') and (H4.1') hold. Let $c > 0$ and $T \geq 1$. Assume moreover that $c < \gamma^{-}$, where γ^{-} is defined in (3.4.28). Let $\mathbf{U}_{c,T}$ be a constrained solution given by Lemma 4.6.4. Then, there exists $\bar{M} > 0$ such that for all $\varepsilon \in (0, \gamma^{-} - c)$ and $t \in \mathbb{R}$ it holds*

$$\int_{-\infty}^t (\mathcal{E}(\mathbf{U}_{c,T}(s)) - a) e^{-\varepsilon s} ds \leq \bar{M} e^{(\gamma^{-} - c - \varepsilon)t}.$$

Furthermore, there exist $M^{-} > 0$ and $\mathbf{v}_{c,T}^{-} \in \mathcal{F}^{-}$ such that for all $t \in \mathbb{R}$

$$\|\mathbf{U}_{c,T}(t) - \mathbf{v}_{c,T}^{-}\|_{\mathcal{F}}^2 \leq M^{-} e^{(\gamma^{-} - c)t}. \quad (4.6.67)$$

In order to close this section, we prove the following result (which in fact generalizes Theorem 4.5) for the case in which (H4.1') holds and \mathcal{F}^- is a singleton:

Lemma 4.6.11. *Assume that (H3.3'), (H3.4'), (H3.5'), (H3.6') and (H4.1') hold. Let $c > 0$ and $T \geq 1$. Assume moreover that $\mathcal{F}^- = \{\mathbf{v}^-\}$. Let $\mathbf{V}_{c,T}$ be a constrained solution given by Lemma 4.6.5. Then, it holds*

$$\lim_{t \rightarrow -\infty} \|\mathbf{V}_{c,T}(t) - \mathbf{v}^-\|_{\mathcal{Z}} = 0.$$

Proof. We essentially argue similarly to [12] and [13], using the fact that $\mathcal{F}^- = \{\mathbf{v}^-\}$. In such a case, (4.5.3) in assumption (H4.1') implies that

$$\forall v \in \mathcal{E}^{\alpha_0^-}, \exists \lambda(v) > 1, \forall \theta \in (0, \lambda(v)), \frac{d}{d\lambda}(\mathcal{E}(\mathbf{v}^- + \lambda(v - \mathbf{v}^-)))(\theta) \geq 0. \quad (4.6.68)$$

Corollary 4.6.2 implies in particular the existence of $t_0^- \in \mathbb{R}$ such that

$$\forall t \leq t_0^-, \mathcal{E}(\mathbf{V}_{c,T}(t)) < \alpha_-, \quad (4.6.69)$$

$$\mathbf{V} \in \mathcal{A}((-\infty, t_0^-))$$

and

$$\mathbf{V}_{c,T}'' - D_{\mathcal{Z}}\mathcal{E}(\mathbf{V}_{c,T}) = -c\mathbf{V}_{c,T}' \text{ in } (-\infty, t_0^-). \quad (4.6.70)$$

The combination of (4.6.68) and (4.6.69) gives that

$$\forall t \leq t_0^-, \exists \lambda(t) > 1, \forall \theta \in (0, \lambda(t)), \frac{d}{d\lambda}(\mathcal{E}(\mathbf{v}^- + \lambda(\mathbf{V}_{c,T}(t) - \mathbf{v}^-)))(\theta) \geq 0$$

which, using that $\mathbf{V}_{c,T} \in \mathcal{A}((-\infty, t_0^-))$ and (3.4.10) in (H3.2'), implies

$$\forall t < t_0^-, \langle D_{\mathcal{Z}}\mathcal{E}(\mathbf{V}_{c,T}(t)), \mathbf{V}_{c,T}(t) - \mathbf{v}^- \rangle_{\mathcal{Z}} \geq 0. \quad (4.6.71)$$

Define now the function

$$\rho_{c,T} : t \in (-\infty, t_0^-) \rightarrow \|\mathbf{V}_{c,T}(t) - \mathbf{v}^-\|_{\mathcal{Z}}^2 \in \mathbb{R}.$$

Notice that (4.6.69) along with (H4.1') implies that

$$\forall t \leq t_0^-, \rho_{c,T}(t) \leq \frac{(r_0^-)^2}{2}. \quad (4.6.72)$$

The fact that $\mathbf{V}_{c,T} \in \mathcal{A}((-\infty, t_0^-))$ implies that $\rho \in \mathcal{C}^2((-\infty, t_0^-))$. We have that

$$\forall t < t_0^-, \rho'_{c,T}(t) = 2\langle \mathbf{V}_{c,T}(t) - \mathbf{v}^-, \mathbf{V}_{c,T}'(t) \rangle_{\mathcal{Z}} \quad (4.6.73)$$

and

$$\forall t < t_0^-, \rho''_{c,T}(t) = 2\|\mathbf{V}_{c,T}'(t)\|_{\mathcal{Z}}^2 + 2\langle \mathbf{V}_{c,T}(t) - \mathbf{v}^-, \mathbf{V}_{c,T}''(t) \rangle_{\mathcal{Z}} \quad (4.6.74)$$

so that, using (4.6.70)

$$\begin{aligned} \forall t < t_0^-, \quad \rho_{c,T}''(t) &= 2\|\mathbf{V}_{c,T}'(t)\|_{\mathcal{L}}^2 + 2\langle D_{\mathcal{L}}\mathcal{E}(\mathbf{V}_{c,T}(t)), \mathbf{V}_{c,T}(t) - \mathbf{v}^- \rangle_{\mathcal{L}} \\ &\quad - 2c\langle \mathbf{V}_{c,T}(t) - \mathbf{v}^-, \mathbf{V}_{c,T}'(t) \rangle_{\mathcal{L}}, \end{aligned}$$

which, by (4.6.71) and (4.6.73) gives the inequality

$$\forall t < t_0^-, \quad \rho_{c,T}''(t) + c\rho_{c,T}'(t) \geq 0. \quad (4.6.75)$$

Inequality (4.6.75) implies that $\rho_{c,T}$ does not possess any strict local maximum. Therefore, we can find $\tilde{t}_1 < t_0^-$ such that $\rho_{c,T}$ is monotone in $(-\infty, \tilde{t}_1)$. In combination with the bound (4.6.72), monotony implies the existence of $l \in [0, (r_0^-)^2/2]$ such that

$$\lim_{t \rightarrow -\infty} \rho_{c,T}(t) = l. \quad (4.6.76)$$

In order to conclude the proof, we need to show that $l = 0$. Assume by contradiction that $l \neq 0$. Decreasing the value of \tilde{t}_1 if necessary, we find $\tilde{l} > 0$ such that for all $t \leq \tilde{t}_1$ it holds $\mathcal{E}(\mathbf{V}_{c,T}(t)) \geq \tilde{l}$. Therefore, using (4.5.3) in (H4.1'), we have that

$$\forall t \leq \tilde{t}_1, \exists \delta(t) > 0, \forall \theta \in (1 - \delta(t), 1 + \delta(t)), \quad \frac{d}{d\lambda}(\mathcal{E}(\mathbf{v}^- + \lambda(\mathbf{V}_{c,T}(t) - \mathbf{v}^-)))(\theta) \geq \omega(\tilde{l}),$$

with $\omega(\tilde{l}) > 0$. In particular, it holds

$$\forall t \leq \tilde{t}_1, \quad \langle D_{\mathcal{L}}\mathcal{E}(\mathbf{V}_{c,T}(t)), \mathbf{V}_{c,T}(t) - \mathbf{v}^- \rangle_{\mathcal{L}} \geq \omega(\tilde{l}),$$

which by (4.6.73) and (4.6.74) gives

$$\forall t \leq \tilde{t}_1, \quad \rho_{c,T}''(t) + c\rho_{c,T}'(t) \geq 2\omega(\tilde{l}). \quad (4.6.77)$$

Suppose first that $\rho_{c,T}$ is non increasing in $(-\infty, \tilde{t}_1)$, meaning that for all $t \leq \tilde{t}_1$, $\rho_{c,T}(\tilde{t}_1) \leq \rho_{c,T}(t)$ and $\rho_{c,T}'(\tilde{t}_1) \leq 0$. Moreover, integrating (4.6.77) in (s, \tilde{t}_1) for $s < \tilde{t}_1$, we obtain

$$\forall s < \tilde{t}_1, \quad \rho_{c,T}'(\tilde{t}_1) - \rho_{c,T}'(s) + c(\rho_{c,T}(\tilde{t}_1) - \rho_{c,T}(s)) \geq 2\omega(\tilde{l})(\tilde{t}_1 - s).$$

which becomes

$$\forall s < \tilde{t}_1, \quad 2\omega(\tilde{l})(s - \tilde{t}_1) \geq \rho_{c,T}'(s)$$

meaning that $\lim_{s \rightarrow -\infty} \rho_{c,T}'(s) = -\infty$, which contradicts (4.6.76). Therefore, $\rho_{c,T}$ must be non decreasing in $(-\infty, \tilde{t}_1]$, which means that for all $t \leq \tilde{t}_1$, $\rho_{c,T}'(t) \geq 0$. We now fix $s < t < \tilde{t}_1$ and integrate (4.6.77) in $[s, t]$. We obtain

$$\rho_{c,T}'(t) - \rho_{c,T}'(s) + c(\rho_{c,T}(t) - \rho_{c,T}(s)) \geq 2\omega(\tilde{l})(t - s),$$

which, since $\rho_{c,T}$ is non decreasing and we have the bound (4.6.72), gives

$$\rho_{c,T}'(t) \geq 2\omega(\tilde{l})(t - s) - c\frac{(r_0^-)^2}{2}$$

so that, by integrating again

$$\forall s < \tilde{t}_2, \quad \rho_{c,T}(\tilde{t}_1) - \rho_{c,T}(s) \geq \omega(\tilde{t})(\tilde{t}_1 - s)^2 - c \frac{(r_0^-)^2}{2} (\tilde{t}_1 - s)$$

which gives a contradiction by passing to the limit $s \rightarrow -\infty$, since the right-hand side term tends in that case to $+\infty$. Therefore, we must have that the limit l defined in (4.6.76) equals 0, which concludes the proof. \square

Remark 4.6.2. The arguments in Lemma 4.6.11 do not seem to apply to the case in which \mathcal{F}^- is no longer a singleton. Indeed, in such an event, the function $\rho_{c,T}$ would have to be defined as

$$\rho_{c,T} : t \in (-\infty, t_0^-) \rightarrow \|\mathbf{V}_{c,T}(t) - P^-(\mathbf{V}_{c,T}(t))\|_{\mathcal{L}}^2$$

so that, when computing its derivatives, the (possibly nonzero) differential of DP^- appears and with it some terms which do not seem to have necessarily the right sign. Recall also that the limit property given by Lemma 4.6.11 does not allow, by itself, to obtain information on $\mathcal{E}(\mathbf{V}_{c,T}(\cdot))$ at $-\infty$, since \mathcal{E} is only lower semicontinuous with respect to \mathcal{L} -convergence.

4.6.9 Proof of Theorem 4.5 completed

It is a particular case of Lemma 4.6.11. Indeed, the solution (c^\star, \mathbf{U}) given by Theorem 4.4 is given by Proposition 4.6.3, so in particular is a constrained solution in Y_T for suitable $T \geq 1$ and Lemma 4.6.11 applies. \square

4.6.10 Proof of Theorem 4.6 completed

If (H4.1') holds, then using Proposition 4.6.3, inequality (4.6.58) in Lemma 4.6.9 implies that $c^\star < \gamma^-$, where we have also used (4.5.12) in (H4.2'). The proof then follows by using identity (4.6.67) in Proposition 4.6.4. \square

4.6.11 On the proof of Theorem 4.7

The proof works exactly as the proof of Theorem 3.7 in Chapter 3, so we skip it.

4.7 Proofs of the main results

The main results of this chapter are obtained by the following, which links the main setting with the abstract setting:

Lemma 4.7.1. *Assume that (H4.1), (H4.2) and (H4.3) hold. Let*

$$\mathcal{L} = \mathcal{H} = \tilde{\mathcal{H}} = \mathbb{R}^k$$

and $\mathcal{E} = W$. Then, (H3.1'), (H3.2'), (H3.3'), (H3.4'), (H3.5'), (H3.6') and (H4.1') hold. Moreover, if (H4.4) holds, then (H4.2') holds.

Proof. The proof is straightforward. Indeed, assumption (H3.1') follows from (H4.1). Assumption (H3.2') becomes tautological because $\mathcal{L} = \mathcal{H} = \tilde{\mathcal{H}} = \mathbb{R}^k$. Assumption (H3.3') is straightforward. Assumption (H3.4') corresponds directly to (H4.2). Assumption (H3.5') is also tautological. Regarding (H3.6'), the existence of the map \mathfrak{P} follows simply by taking $\mathfrak{P} = \text{Id}$. The rest of the statement follows from elliptic regularity results or it is tautological. Finally, we clearly have that (H4.2') holds if (H4.4) holds. \square

Notice that most of the strength of the abstract results will not be used as we choose all the Hilbert spaces equal to \mathbb{R}^k . This also makes all the

4.7.1 Proof of Theorem 4.1 completed

We apply Theorem 4.4 with the choice of objects given by Lemma 4.7.1. It is clear that (4.5.8) reads now as (4.2.3) and (4.5.9) reads as (4.2.4), which establishes the existence part as we also have that $\mathcal{A}(\mathbb{R})$ becomes a subset of $\mathcal{C}_{\text{loc}}^2(\mathbb{R}, \mathbb{R}^k)$ by its definition in (3.4.23). Due to the fact that $X = \mathcal{S}$, $Y = \overline{\mathcal{S}}$ (with X as in (4.5.6) and Y as in (4.5.7)) the uniqueness statement of Theorem 4.1 follows directly from that in Theorem 4.4. Finally, it is clear that the exponential convergence in Theorem 4.4 writes as that in Theorem 4.1, which concludes the proof. \square

4.7.2 Proof of Theorem 4.2 completed

We apply Theorems 4.5 and 4.6 with the choice of objects given by Lemma 4.7.1. Then, we have that the case 1. in (H4.4) corresponds to the assumptions of Theorem 4.5 and case 2. in (H4.4) is the abstract assumption (H4.2'), required in Theorem 4.6. Therefore, Theorem 4.2 follows. \square

4.7.3 Proof of Theorem 4.3 completed

We apply Theorem 4.7 with the choice of objects given by Lemma 4.7.1. It is then clear that Theorem 4.3 holds, as recall that $Y = \overline{\mathcal{S}}$. \square

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