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Nicolas Vanspranghe

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THÈSE

Pour obtenir le grade de

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**Contributions à la théorie du contrôle des systèmes non-linéaires en dimension infinie**

**Contributions to infinite-dimensional nonlinear control theory**

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## Abstract

This thesis concerns the problems of feedback stabilization and output regulation for infinite-dimensional nonlinear systems. First, we study the stability of a one-dimensional wave equation with a nonlinear velocity feedback at one extremity of the domain and a nonlinear dynamic boundary condition at the other end. This model is inspired by the behavior of torsional vibrations along drill strings, and the boundary dynamics represent a nonlinear anti-damping at the rock-bit interface that renders the uncontrolled plant unstable. Exponential stability of the set of stationary solutions is investigated by Lyapunov methods. In a second part, we consider the multi-dimensional wave equation supplied with a nonlinear and nonlocal Dirichlet feedback control acting on a part of the boundary. Well-posedness and asymptotic stability of the closed-loop dynamics are established using nonlinear contraction semigroup arguments combined with the LaSalle invariance principle and unique continuation for waves. When the feedback nonlinearity has linear growth around zero (e.g., in the case of saturating feedback), polynomial energy decay rates are derived for strong solutions. In the final part of the thesis, we are interested in constant output regulation of a class of abstract infinite-dimensional systems governed by nonlinear contraction semigroups on Hilbert spaces. The approach we propose relies on the so-called forwarding methodology, which was originally developed for the stabilization of finite-dimensional nonlinear cascade systems. We give sufficient conditions for the existence of a dynamic control law that steers the system to some equilibrium at which the output coincides with the reference. These conditions are then investigated in the particular case of semilinear systems and illustrated by examples.

**Keywords:** *Stabilization, output regulation, partial differential equations, wave equation, saturation, contraction semigroups.*

## Résumé

Dans cette thèse, nous étudions quelques problèmes de stabilisation et de régulation de sortie pour des systèmes non-linéaires en dimension infinie. Tout d'abord, nous analysons la stabilité d'une équation des ondes en dimension un d'espace, soumise à un retour stabilisant non-linéaire de type Neumann à une extrémité et à une condition au bord dynamique non-linéaire à l'autre extrémité du domaine. Ce modèle s'inspire de la propagation de vibrations de torsion le long de trains de forage, la condition au bord dynamique représentant un anti-amortissement non-linéaire à l'interface avec la roche, qui peut déstabiliser le dispositif. La stabilité exponentielle de l'ensemble des solutions stationnaires est examinée avec une approche type Lyapunov. Dans un second temps, nous considérons une équation des ondes multi-dimensionnelle soumise à un contrôle non-linéaire et non-local de type Dirichlet sur une partie de la frontière. Le caractère bien posé du problème en boucle fermée ainsi que la stabilité asymptotique sont établis à l'aide de techniques issues de la théorie des semi-groupes de contractions combinées au principe d'invariance de LaSalle et à un résultat de continuation unique pour les ondes. Lorsque la non-linéarité dans le contrôle a une croissance linéaire autour de zéro, nous démontrons que l'énergie des solutions fortes décroît de manière polynomiale. Enfin, dans une dernière partie, nous nous intéressons au problème de régulation à sortie constante pour une classe de systèmes abstraits régis par des semi-groupes de contractions sur des espaces de Hilbert. Notre approche repose sur la technique du « forwarding », originellement développée pour la stabilisation de systèmes non-linéaires de dimension finie en cascade. Nous proposons des conditions suffisantes pour l'existence d'une loi de commande dynamique menant le système à un équilibre où la sortie coïncide avec la référence. Ces conditions sont étudiées en détail dans le cas des systèmes semi-linéaires, et des exemples d'illustration sont donnés.

**Mots-clés :** *Stabilisation, régulation de sortie, équations aux dérivées partielles, équation des ondes, saturation, semi-groupes de contractions.*

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# Contents

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<b>1</b>	<b>General introduction</b>	<b>3</b>
<b>2</b>	<b>A control problem in mechanical engineering</b>	<b>9</b>
2.1	Dynamical equations and control problem . . . . .	9
2.2	Well-posedness of the feedback system . . . . .	11
2.3	Stability analysis . . . . .	19
<b>3</b>	<b>Dirichlet stabilization of the wave equation</b>	<b>29</b>
3.1	Background and closed-loop equations . . . . .	29
3.2	Well-posedness and asymptotic stability . . . . .	31
3.3	Polynomial decay rates for strong solutions . . . . .	39
<b>4</b>	<b>An output regulation problem</b>	<b>51</b>
4.1	Introduction and forwarding design . . . . .	51
4.2	Analysis of the closed-loop system . . . . .	55
4.3	The semilinear case . . . . .	63
<b>5</b>	<b>Conclusions and perspectives</b>	<b>75</b>
<b>A</b>	<b>Résumé étendu en français</b>	<b>77</b>
	<b>Bibliography</b>	<b>83</b>

# One

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## General introduction

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At the interface of applied mathematics and engineering, control theory is the study of dynamical systems in order to control them, i.e., influence their behavior with a desired goal in mind. A *control system* is a dynamical system on which one can act by choosing at each time some parameter, called the *control*, in the evolution law. The *state* of the system is the variable obeying said evolution law. Finite-dimensional control systems (i.e., the state evolves in a finite-dimensional space) are often governed by ordinary differential equations [Khalil, 2015] and have been an active area of research since the thirties. On the other hand, *infinite-dimensional* control systems include (but are not restricted to) systems modeled by partial differential equations or delay differential equations. The mathematical theory of infinite-dimensional control systems is far more recent than its finite-dimensional counterpart – see, e.g., [Lions, 1968, Russell, 1978, Coron, 2007]. It is mostly built upon the foundation of partial differential equation analysis, functional analysis and dynamical systems theory. Consider the following examples. First, given a sufficiently smooth function of two variables  $f : \mathbb{R}^d \times \mathbb{R} \rightarrow \mathbb{R}^d$ , the differential equation

$$\frac{dx}{dt} = f(x, U) \tag{1.1}$$

defines a finite-dimensional control system where  $x$  is the state and the scalar  $U$  is the control. In that case, the *state space* is the standard Euclidian space  $\mathbb{R}^d$ . Second, let  $\Omega$  be some bounded domain of  $\mathbb{R}^d$ ,  $d \geq 2$ , with boundary  $\Gamma$ ; the following wave equation with non-homogeneous Dirichlet boundary conditions

$$\frac{\partial^2 u}{\partial t^2} - \Delta u = 0 \quad \text{in } \Omega, \tag{1.2a}$$

$$u|_{\Gamma} = U \quad \text{on } \Gamma, \tag{1.2b}$$

defines (at least formally) another control system, where the state is the tuple  $[u, \partial_t u]$  and the control is the boundary data  $U$ . Following the theory of partial differential equations, both the state and the control must be chosen in some appropriate infinite-dimensional function spaces. When the control is chosen as a function of the state, we speak of *feedback* action. A *static* feedback law is a choice of control that is a function of the state at the given time only. The control of systems with static feedback laws gives rise to *dynamical systems*. In the case of the example given by (1.1), a static feedback law has the form

$$U = h(x) \tag{1.3}$$

where  $h : \mathbb{R}^d \rightarrow \mathbb{R}$  is some suitable function. The *closed-loop* system is then governed by the evolution equation

$$\frac{dx}{dt} = f(x, h(x)) \tag{1.4}$$

which, under mild assumptions, defines a dynamical system on  $\mathbb{R}^d$ .

**Control problems and orientation** Now, as mentioned above, the objective of a control law is to influence the behavior of the controlled dynamics in order to achieve a certain objective. This corresponds to a wide variety of control problems. Let us introduce the two fundamental ones we are interested in.

1. *The feedback stabilization problem.* Given an equilibrium point that is unstable for the uncontrolled dynamics, it consists in designing a feedback law such that the *closed-loop* system is steered to the equilibrium. An interesting infinite-dimensional example is [Smyshlyaev et al., 2010], where a wave equation with in-domain *anti*-damping is exponentially stabilized via a boundary controller.
2. *The output regulation problem.* Considering a control system with measured output, i.e., a function of the state that may represent some physical parameter of interest in a given application, it consists in designing a feedback law which ensures that the output of the closed-loop system, possibly subjected to external disturbance, tracks a given reference. Output regulation is one of the oldest problems in automatic control and is essential in engineering applications – see, e.g., the seminal work [Davison, 1975] for finite-dimensional linear systems.

We can now give the general direction of our work.

This thesis concerns the problems of feedback stabilization and output regulation for infinite-dimensional nonlinear systems, with a particular emphasis on those governed by partial differential equations.

We are driven by two underlying ideas that are central in control theory. *Stability* concerns the behavior of trajectories of dynamical systems under small perturbations of the initial condition and, if applicable, qualitative and quantitative properties of the convergence of trajectories to a given equilibrium. *Robustness* refers to the behavior of trajectories under *external* perturbations, such as those representing discrepancies in the model, numerical noise and disturbances, etc. This is instrumental in the successful implementation of a given controller.

A possible way to achieve stability is to ensure that the closed-loop dynamics satisfy a *contraction* property, i.e., the distance between closed-loop trajectories (typically measured in terms of *energy* in the context of partial differential equations) do not grow or even go to zero [Lohmiller and Slotine, 1998]. In our work, the property of contraction will often be the key to obtaining stability, proving the existence of suitable equilibria and also establishing existence of solutions to closed-loop equations, which is not immediate in the infinite-dimensional settings – see, e.g., [Barbu, 1976, Showalter, 2013].

**Outline of the thesis and contributions** In what follows, we briefly outline the content of the remaining chapters and indicate our contributions. Of course, in relation with the specifics of each problem, a more detailed exposition of the scientific context and literature is given in each chapter.

In Chapter 2, we investigate the stability of a one-dimensional wave equation with a nonlinear velocity feedback at one extremity of the domain and a nonlinear dynamic boundary condition at the other end. This problem is similar to the one tackled in the aforementioned work [Smyshlyaev et al., 2010] in that we analyze the action of a (boundary) feedback competing against an (dynamic boundary instead of static in-domain) *anti*-damping term. The model under consideration is inspired by the behavior of torsional vibrations along drill strings, and the boundary dynamics represent nonlinear friction at the rock-bit interface that may destabilize the plant. In that regard, such a nonlinear boundary *anti*-damping is the main novelty here. Sufficient conditions for the exponential stability are obtained after constructing a suitable Lyapunov functional.

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In Chapter 3, we study the multi-dimensional wave equation (1.2a) supplied with a nonlinear and *nonlocal* Dirichlet feedback control, which is chosen as a natural candidate dissipating the  $L^2(\Omega) \times H^{-1}(\Omega)$ -energy of the system. Unlike its well-known Neumann counterpart, the natural Dirichlet boundary feedback has been given little attention in the literature; in particular, no nonlinear treatment had been proposed so far. Well-posedness and asymptotic stability is established using nonlinear contraction semigroup arguments combined with the LaSalle invariance principle and unique continuation for waves. Under additional assumptions on the feedback nonlinearity (encompassing notably the case of *saturating* feedback), we establish a polynomial energy decay rate for strong solutions to the closed-loop equations. This is done by deriving appropriate integral energy estimates using carefully chosen differential multipliers with time-varying weights.

Chapter 4 deals with the problem of output regulation for a class of nonlinear systems that are governed by contraction semigroups. Very few output regulation results exist in the context of infinite-dimensional nonlinear systems. We propose sufficient conditions for the existence of a dynamic feedback law steering, even in presence of small constant disturbances, the closed-loop system to an equilibrium at which the reference is achieved. The construction of the control is based on the so-called *forwarding* technique originally developed in [Mazenc and Praly, 1996] for the problem of stabilizing finite-dimensional nonlinear systems in cascade. The proof of existence of a locally stable equilibrium is based on the fact that the feedback law we propose preserves, at least locally and up to a change of variable, the contraction property of the original uncontrolled system. Those sufficient conditions are then thoroughly investigated on semilinear systems.

## Publications issued from this thesis

### Journal articles

- Vanspranghe, N., Ferrante, F., and Prieur, C. (2022b). Velocity stabilization of a wave equation with a nonlinear dynamic boundary condition. To appear in the *IEEE Transactions on Automatic Control*.

Chapter 2 is partially adapted from this work.

- Vanspranghe, N., Ferrante, F., and Prieur, C. (2022a). Stabilization of the wave equation through nonlinear Dirichlet actuation. Provisionally accepted for publication in *ESAIM: Control, Optimisation and Calculus of Variations*.

Chapter 3 is adapted from this work.

- Vanspranghe, N. and Brivadis, L. (2022). Output regulation of infinite-dimensional nonlinear systems: a forwarding approach for contraction semigroups. Submitted to *SIAM Journal on Control and Optimization*.

Chapter 4 is adapted from this work. This is a collaboration with Lucas Brivadis (Laboratoire des Signaux et Systèmes, CentraleSupélec).

### Conference papers

- Vanspranghe, N., Ferrante, F., and Prieur, C. (2020). Control of a wave equation with a dynamic boundary condition. In *59th IEEE Conference on Decision and Control (CDC 2020)*, Jeju Island, South Korea.

This conference article is a preliminary work containing some results of Chapter 2.



- Vanspranghe, N., Ferrante, F., and Prieur, C. (2021). Stabilization of the wave equation by the mean of a saturating Dirichlet feedback. In *Third IFAC Conference on Modelling, Identification and Control of Nonlinear Systems*.

This conference article is a preliminary work containing some results of Chapter 3.

- Vanspranghe, N. (2022). Wave equation with hyperbolic boundary condition: a frequency domain approach. In *Fourth IFAC Workshop on Control of Systems Governed by Partial Differential Equations*.

This work is a side-project that does not pertain to the general topic of the thesis.

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## Definitions and notation used throughout the manuscript

**Linear spaces and maps** Let  $E$  be a given Banach space. The norm of  $E$  is denoted by  $\|\cdot\|_E$ . The duality bracket  $\langle \phi, x \rangle_E$  is used to write  $\phi(x)$  for any vector  $x$  in  $E$  and continuous linear form  $\phi$  in the topological dual  $E'$  of  $E$ . If  $E$  is also a Hilbert space, its scalar product is denoted by  $(\cdot, \cdot)_E$ . Given two Banach spaces  $E$  and  $F$ ,  $\mathcal{L}(E, F)$  stands for the space of bounded linear operators between  $E$  and  $F$ . It is a Banach space as well if equipped with the operator norm. If  $E$  and  $F$  are Hilbert spaces, any operator  $L$  in  $\mathcal{L}(E, F)$  possesses an adjoint  $L^*$  uniquely defined in  $\mathcal{L}(F, E)$  by  $(Lx, y)_E = (x, L^*y)_F$  for all  $x$  in  $E$  and  $y$  in  $F$ . Finally, we denote by  $\mathcal{L}(E)$  the Banach algebra  $\mathcal{L}(E, E)$ .

**Dynamical systems and semigroups** We shall describe dynamical systems using the language of evolution semigroups. Let  $X$  be a subset of the Banach space  $E$ . By a *semigroup* on  $X$ , we mean a family  $\{\mathcal{S}_t\}_{t \geq 0}$  of continuous mappings of  $X$  into itself satisfying the following properties:

- (i)  $\mathcal{S}_0 = \text{id}$ , where  $\text{id}$  denotes the identity function;
- (ii) For all  $t, s \geq 0$ ,  $\mathcal{S}_{t+s} = \mathcal{S}_t \mathcal{S}_s$ .

We use the semigroup notation and omit the parentheses even if the action of  $\{\mathcal{S}_t\}$  on  $X$  need not be linear. We say that  $\{\mathcal{S}_t\}$  is *strongly continuous* if in addition

- (iii) For all  $x \in X$ ,  $\mathcal{S}_t x \rightarrow x$  in  $E$  as  $t \rightarrow 0^+$ .

In the particular cases that  $\{\mathcal{S}_t\}$  is a semigroup of nonlinear contractions or bounded linear operators, one can infer from Items (i) to (iii) that for each  $x$  in  $X$ , the flow  $t \mapsto \mathcal{S}_t x$  belongs to  $\mathcal{C}(\mathbb{R}^+, E)$ . Given a semigroup  $\{\mathcal{S}_t\}$  on  $X$ , for any subset  $Y$  of  $X$ , we define the  $\omega$ -limit set of  $Y$  with respect to  $\{\mathcal{S}_t\}$  by

$$\omega(Y) = \bigcap_{s \geq 0} \overline{\bigcup_{t \geq s} \mathcal{S}_t Y} \quad (1.5)$$

where the closure (denoted by the bar) is taken in the topology of the ambient space  $E$ .

**Vector-valued functions and integrals** The abbreviation ‘‘a.e.’’ stands for ‘‘almost everywhere’’ or ‘‘almost every’’ in the sense of measure theory. Vector-valued integrals are intended in the sense of Bochner. For positive  $T$ , we denote by  $W^{1,p}(0, T; E)$  the subspace of  $L^p(0, T; E)$  composed of (classes of)  $E$ -valued functions  $\varphi$  such that for some  $\psi$  in  $L^p(0, T; E)$  and  $\xi$  in  $E$ ,

$$\varphi(t) = \xi + \int_0^t \psi(s) ds \quad \text{for a.e. } t \text{ in } (0, T). \quad (1.6)$$

Such a class  $\varphi$  is identified with its continuous representative and we say that  $\varphi' = \psi$  in the sense of  $E$ -valued distributions.

**Function spaces** By *domain* of  $\mathbb{R}^d$ , we mean an open and connected subset  $\Omega$  of the ambient Euclidian space  $\mathbb{R}^d$ . Given a real number  $s$ , we denote by  $H^s(\Omega)$  the (real)  $L^2(\Omega)$ -based Sobolev space of order  $s$  on  $\Omega$ . Assuming that the boundary  $\Gamma$  of  $\Omega$  is smooth, we use the notation  $H^s(\Gamma)$  for Sobolev spaces on the boundary. In that context, the notation  $dx$  indicates the Lebesgue measure on  $\mathbb{R}^d$ ; and  $d\sigma$  denotes the induced surface measure on  $\Gamma$ . We denote by  $\mathcal{D}(\Omega)$  the space of compactly supported and infinitely differentiable real-valued functions on  $\Omega$ . Then,  $\mathcal{D}'(\Omega)$  denotes the space of (real-valued) distributions on  $\Omega$ .

**Miscellaneous** If  $E$  is a metric space endowed with a distance  $d$ , the distance between an element  $x$  and a subset  $Y$  of  $E$  is defined as follows:  $\text{dist}(x, Y) \triangleq \inf_{y \in Y} d(x, y)$ . Given two normed vector spaces  $E$  and  $F$ , a map  $\Phi$  from  $E$  to  $F$  is Fréchet differentiable at a point  $x$  of  $E$  if there exists a (necessarily unique) continuous linear operator  $d\Phi(x)$  from  $E$  to  $F$  such that

$$\lim_{h \rightarrow 0} \frac{\|\Phi(x+h) - \Phi(x) - d\Phi(x)h\|_F}{\|h\|_E} = 0. \quad (1.7)$$

The Fréchet differential  $d\Phi$  of  $\Phi$ , which is defined on the set of such points  $x$ , is the map  $x \mapsto d\Phi(x)$ . Also, we say that  $\Phi$  is *locally* Lipschitz continuous if it is Lipschitz continuous on any bounded subset of  $E$ .

# Two

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## A control problem in mechanical engineering

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### Contents

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2.1	Dynamical equations and control problem . . . . .	9
2.2	Well-posedness of the feedback system . . . . .	11
2.2.1	Variational formulation and weak solutions . . . . .	11
2.2.2	Nonlinear semigroup approach and strong solutions . . . . .	13
2.2.3	Proof of the Hadamard theorem . . . . .	16
2.3	Stability analysis . . . . .	19
2.3.1	Uniform stabilization and Lyapunov methods . . . . .	19
2.3.2	The problem of saturated stabilization . . . . .	23

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### 2.1 Dynamical equations and control problem

The subject of this chapter is the stability analysis of a one-dimensional wave equation (or string equation) that is supplied with a dynamic boundary condition at one extremity and is actuated by the mean of Neumann boundary control acting on the other end. Consider a one-dimensional domain  $\Omega \triangleq (0, L)$  where  $L > 0$ . The pure wave equation posed in  $\Omega$  reads as follows:

$$\frac{\partial^2 u}{\partial t^2}(x, t) - \frac{\partial^2 u}{\partial x^2}(x, t) = 0, \quad (2.1)$$

where  $x \in \Omega$  stands for the space variable and  $t$  is the time variable. Equation (2.1) is a partial differential equation of the hyperbolic type that can model propagation of some deformation  $u$  along a one-dimensional elastic medium. We now add boundary conditions to (2.1). Assume for instance that the extremity at  $x = 0$  is attached to some tip mass and obeys a dynamical equation of the form

$$\frac{\partial^2 u}{\partial t^2}(0, t) - \frac{\partial u}{\partial x}(0, t) = F\left(\frac{\partial u}{\partial t}(0, t)\right) \quad (2.2)$$

where  $F$  is some nonlinear map representing velocity-dependent friction to which said mass is subject. By definition, dynamic (or kinetic) boundary conditions such as (2.2) involve second-order time derivative and typically arise in physical problems where the boundary carries its own momentum. Aside from the nonlinear friction term, the model is very similar to the equations in [Andrews et al., 1996]; on the other hand, such a boundary condition can also be considered in Euler-Bernoulli beam dynamics, as in [Conrad and Morgul, 1998]. Higher-dimensional examples include vibrating membranes with a given surfacic mass density that react to some in-domain deformations – see, e.g., [Gal

et al., 2003, Goldstein, 2006, Vitillaro, 2017]. As for the endpoint  $x = L$ , its momentum is neglected and we consider a standard non-homogeneous Neumann boundary condition:

$$\frac{\partial u}{\partial x}(L, t) = -g(U(t)), \quad (2.3)$$

where  $U(t)$  represents an input signal and  $g$  is a continuous monotone increasing function satisfying  $g(0) = 0$ . The scalar mapping  $g$  may represent nonlinear behavior of the actuators such as saturation or deadzone. On the other hand,  $F$  is assumed to be a globally Lipschitz continuous scalar function and represents a nonlinear boundary anti-damping term at  $x = 0$ . In the absence of feedback action and anti-damping, i.e.,  $U(t) = 0$  and  $F = 0$ , the mechanical energy of the system

$$\mathcal{E} = \frac{1}{2} \int_{\Omega} \left| \frac{\partial u}{\partial x} \right|^2 + \left| \frac{\partial u}{\partial t} \right|^2 dx + \frac{1}{2} \left| \frac{\partial u}{\partial t}(0, t) \right|^2 \quad (2.4)$$

is conserved. By setting

$$U(t) = \frac{\partial u}{\partial t}(L, t), \quad (2.5)$$

the energy is made monotone decreasing. In presence of the friction term  $F$  however, the energy need not decay. Our objective here is to investigate conditions on  $F$  and  $g$  under which the energy goes to 0; in other words, the boundary anti-damping at  $x = 0$  is compensated by the boundary dissipation at  $x = L$ , which comes from the feedback law (2.5). The goals of this chapter are twofold.

First, the coupled dynamics described by (2.1)-(2.2), or variants, have sparked interest in the control community and are very often considered in the context of minimizing torsional vibrations along drilling rods due to nonlinear friction at the rock-tip interface, where the so-called *stick-slip* phenomenon may occur and destabilize the plant [Saldivar et al., 2016, Adly and Goeleven, 2020]. System (2.1) can be seen as an infinite-dimensional model of such a plant: the rod is seen as a purely elastic medium whose angular deformation obeys the wave equation (2.6a), and the drilling tip is subject to nonlinear torsional friction, represented by  $F$  at the rock interface, which yields (2.6b). In that context, various boundary control strategies have been proposed. In [Terrand-Jeanne et al., 2020], stabilization and regulation using a proportional integral boundary controller is investigated; the elasticity of the propagation medium being possibly nonhomogenous. In [Mlayeh et al., 2018], an observer-based boundary control design is proposed. In [Roman et al., 2018], a backstepping-based controller is considered. Other related works include [Smyshlyaev and Krstic, 2009] and [Bresch-Pietri and Krstic, 2014], where first-order boundary anti-damping is considered. In all these works, the dynamical equations under study are linearized. In contrast, our analysis is valid in presence of nonlinear anti-damping (modeled by  $F$ ) and nonlinearity in the feedback loop (modeled by  $g$ ).

Second, this chapter aims at illustrating on simple equations some of the methods and operator-theoretic tools that we will use throughout the entire thesis when dealing with dynamical systems governed by partial differential equations. This includes variational framework and methods, monotonicity techniques, semigroup generation, etc. Particular emphasis is placed on the notion of *solution* to boundary value problems, which in that context is often quite weak and must sometimes be tailored to the given problem. By doing so, we hope that for the reader unfamiliar with infinite-dimensional systems, this chapter serves as suitable introductory material.

## 2.2 Well-posedness of the feedback system

Let us recall the closed-loop equations introduced in Section 2.1:

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0 \quad \text{in } \Omega \times (0, +\infty), \quad (2.6a)$$

$$\frac{\partial^2 u}{\partial t^2}(0, t) - \frac{\partial u}{\partial x}(0, t) = F\left(\frac{\partial u}{\partial t}(0, t)\right) \quad \text{for all } t \in (0, +\infty), \quad (2.6b)$$

$$\frac{\partial u}{\partial x}(L, t) = -g\left(\frac{\partial u}{\partial t}(L, t)\right) \quad \text{for all } t \in (0, +\infty). \quad (2.6c)$$

In this section, we introduce different notions of solutions to (2.6) in a unified framework and prove existence and uniqueness of such solutions.

### 2.2.1 Variational formulation and weak solutions

Let  $w$  be some smooth function on  $\Omega$ . By (formally) multiplying (2.6a) by  $w$  and integrating over  $\Omega$ , we obtain

$$\int_{\Omega} \frac{\partial^2 u}{\partial t^2} w \, dx + \frac{\partial^2 u}{\partial t^2}(0, t)w(0) + \int_{\Omega} \frac{\partial u}{\partial x} \frac{\partial w}{\partial x} \, dx + g\left(\frac{\partial u}{\partial t}(L, t)\right)w(L) = F\left(\frac{\partial u}{\partial t}(0, t)\right)w(0). \quad (2.7)$$

We introduce the pivot space  $H$  and the space of test functions  $V$ :

$$H \triangleq L^2(\Omega) \times \mathbb{R}, \quad \text{and} \quad V \triangleq \{(u, \theta) \in H^1(\Omega) \times \mathbb{R} : u(0) = \theta\}. \quad (2.8)$$

The space  $H$  is equipped with the product Hilbertian structure: for any  $[u_1, \theta_1]$  and  $[u_2, \theta_2]$  in  $H$ ,

$$([u_1, \theta_1], [u_2, \theta_2])_H \triangleq \int_{\Omega} u_1 u_2 \, dx + \theta_1 \theta_2. \quad (2.9)$$

It is not hard to see that  $V$  is a dense subset of  $H$ . On the other hand,  $V$  can be seen as the graph of the (continuous) trace operator at  $x = 0$  on  $H^1(\Omega)$ , which makes it a closed subspace of  $H^1(\Omega) \times \mathbb{R}$  by virtue of the closed graph theorem. As such, it is a Hilbert space as well if equipped with the scalar product inherited from  $H^1(\Omega) \times \mathbb{R}$ . In fact, there is an isomorphism between  $V$  and  $H^1(\Omega)$ ; thus, we will drop the tuple notation and frequently make the identification  $V \simeq H^1(\Omega)$ . We also define a continuous (nonnegative) bilinear form  $a$  on  $V \times V$ :

$$a(u_1, u_2) \triangleq \int_{\Omega} \frac{\partial u_1}{\partial x} \frac{\partial u_2}{\partial x} \, dx \quad \text{for all } u_1, u_2 \in V. \quad (2.10)$$

Then, the scalar product on  $V$  reads as follows:

$$(u_1, u_2)_V = (u_1, u_2)_H + a(u_1, u_2) \quad \text{for all } u_1, u_2 \in V. \quad (2.11)$$

Equation (2.11) shows that the embedding  $V \hookrightarrow H$  is continuous. Besides, as a Hilbert space,  $H$  can be identified to its topological dual  $H'$  by Riesz representation theorem. Therefore,  $H$  can be continuously embedded into  $V'$ . Moreover, since  $V$  is a dense subspace of  $H$ , the latter can be seen as a dense subspace of  $V'$ , which is equipped with the dual norm. This is summarized by the following classical chain of injections:

$$V \hookrightarrow H \simeq H' \hookrightarrow V', \quad (2.12)$$

where each space is dense in the following one and the inclusions are continuous. With a slight abuse of notation, we define an operator  $F : [u, \theta] \in H \mapsto [0, F(\theta)] \in H$ . That being said, we can rephrase the (formal) variational equation (2.7) as follows.

**Definition 2.2.1.** Let  $T > 0$ . A weak solution  $[u, u']$  to (2.6) on  $[0, T]$  is given by any  $u$  with regularity

$$u \in \mathcal{C}^1([0, T], H) \cap \mathcal{C}([0, T], V), \quad u(L, \cdot) \in W^{1,1}(0, T), \quad g(u'(L, \cdot)) \in L^1(0, T) \quad (2.13)$$

that satisfies the variational identity

$$\frac{d}{dt}(u', w)_H + g(u'(L, \cdot))w(L) + a(u, w) = (F(u'), w)_H \quad \text{for all } w \in V, \quad (2.14)$$

where  $d/dt$  is understood in the sense of scalar distributions on  $(0, T)$ ,  $u'$  denotes the time derivative of  $u$  in  $H$ , and  $u'(L, \cdot)$  stands for the (weak) derivative of  $u(L, \cdot)$ .

In order to define a dynamical system, we need to supplement the evolution equations (2.6) with initial conditions. Quite naturally, initial data is considered in the energy space  $\mathcal{H}$  defined by

$$\mathcal{H} \triangleq V \times H, \quad (2.15)$$

and equipped with the product Hilbertian structure. The space  $\mathcal{H}$  is also referred to as the *state* or *phase* space. Let us define a “mechanical energy” functional  $\mathcal{E} \in \mathcal{C}(\mathcal{H}, \mathbb{R})$ . Let

$$\mathcal{E}(u, v) \triangleq \frac{1}{2}\|v\|_H^2 + \frac{1}{2}a(u, u) \quad \text{for all } [u, v] \in \mathcal{H}. \quad (2.16)$$

Coming back to formal considerations, multiplying (2.6a) by  $u'$  and integrating by parts over  $\Omega$  yields the following *energy identity*:

$$\frac{d}{dt}\mathcal{E}(u, u') = -g(u'(L, \cdot))u'(L, \cdot) + (F(u'), u')_H. \quad (2.17)$$

Equation (2.17) shows that in the absence of feedback action and friction term  $F$ , the energy  $\mathcal{E}$  of the solutions is conserved. From a control perspective, what we are interested in is the competition between boundary dissipation at  $x = L$  and boundary anti-damping at  $x = 0$ . This will be investigated in Section 2.3, which is devoted to the stability analysis of the feedback system (2.6). Now, we recall that a problem is said to be well-posed in the Hadamard sense if

1. Solutions exist;
2. For each initial condition, the solution is unique;
3. The solution depends continuously on the initial data in some sense.

The next theorem shows that the initial-and-boundary problem defined by the closed-loop equations (2.6) and initial conditions in  $\mathcal{H}$  is indeed well-posed.

**Theorem 2.2.2** (Hadamard well-posedness). *For each  $[u_0, v_0] \in \mathcal{H}$ , there exists a unique weak solution  $[u, u']$  to (2.6) that satisfies the initial condition  $[u(0), u'(0)] = [u_0, v_0]$ . Furthermore, weak solutions enjoy the following properties:*

- (i) (Trace regularity.) For any  $\tau > 0$ ,  $u'(L, \cdot)$  is in fact in  $H^1(0, \tau)$ ; also,  $g(u'(L, \cdot)) \in L^2(0, \tau)$ ;
- (ii) (Energy identity.) Weak solutions satisfy the energy identity

$$\mathcal{E}(u, u')\Big|_0^\tau = - \int_0^\tau g(u'(L, t))u'(L, t) dt + \int_0^\tau (F(u'(t)), u'(t))_H dt \quad \text{for all } \tau \geq 0; \quad (2.18)$$

(iii) (*A priori estimate.*) For any  $\tau \geq 0$ , there exists a constant  $C(\tau)$  such that any given pair of weak solutions  $[u_i, u'_i]$ ,  $i \in \{1, 2\}$ , to (2.6), having let  $u \triangleq u_1 - u_2$ ,

$$\sup_{t \in [0, \tau]} \mathcal{E}(u(t), u'(t)) + \int_0^\tau |u'(L, t)|^2 dt \leq C(\tau) \mathcal{E}(u(0), u'(0)). \quad (2.19)$$

Finally, by setting  $\mathcal{S}_t : [u_0, v_0] \mapsto [u(t), u'(t)]$ , weak solutions to (2.6) describe a strongly continuous semigroup  $\{\mathcal{S}_t\}$  of Lipschitz continuous operators on  $\mathcal{H}$ .

The proof of Theorem 2.2.2 relies on tools introduced in Section 2.2.2 and is given in Section 2.2.3. We end this section by introducing some additional definitions regarding the variational formulation (2.14) of (2.6). By bilinearity and continuity of  $a$ , the mapping  $A$  defined by

$$\langle Au, w \rangle_V \triangleq a(u, w) \quad \text{for all } u, w \in V \quad (2.20)$$

belongs to  $\mathcal{L}(V, V')$ . Since  $\Omega$  is a one-dimensional domain, the Dirac distribution  $\delta_L$  at  $x = L$  is continuous with respect to the norm of  $H^1(\Omega)$ ; thus,  $\delta_L \in V'$ . This allows to reformulate (2.14) as follows:

$$\frac{d}{dt}(u', w)_H + \langle Au, w \rangle_V + g(u'(L, \cdot)) \langle \delta_L, w \rangle_V = (F(u'), w)_H \quad \text{for all } w \in H. \quad (2.21)$$

We will see later that (2.21) is equivalent to the following *functional* formulation:

$$u'' + Au + g(u'(L, \cdot))\delta_L = F(u') \quad \text{in the sense of } V'\text{-valued distributions.} \quad (2.22)$$

### 2.2.2 Nonlinear semigroup approach and strong solutions

A possible way to tackle the problem of existence of solutions to (2.6) is to recast the closed-loop equations into a first-order abstract Cauchy problem on  $\mathcal{H}$ . To do so, we shall introduce a nonlinear map  $\mathcal{A}$  defined on some subset  $\mathcal{D}(\mathcal{A})$  of  $\mathcal{H}$  such that solutions  $[u, u']$  to (2.6) satisfy, in some sense,

$$\frac{d}{dt}[u, u'] + \mathcal{A}[u, u'] = 0. \quad (2.23)$$

Bearing in mind the variational identity (2.14) and the chain of embeddings (2.12), we start by defining the subset  $\mathcal{D}(\mathcal{A})$ , referred to as the *domain* of  $\mathcal{A}$ , as follows:

$$\mathcal{D}(\mathcal{A}) \triangleq \{[u, v] \in \mathcal{H} : v \in V, Au + g(v(L))\delta_L \in H\}. \quad (2.24)$$

Next, we define  $\mathcal{A}$  by

$$\mathcal{A}[u, v] \triangleq [-v, Au + g(v(L))\delta_L - F(v)]. \quad (2.25)$$

**Lemma 2.2.3.** *The domain  $\mathcal{D}(\mathcal{A})$  can be explicitly described as follows:*

$$\mathcal{D}(\mathcal{A}) = \left\{ [u, v] \in \mathcal{H} : v \in V, u \in H^2(\Omega), \frac{\partial u}{\partial x}(L) = -g(v(L)) \right\}. \quad (2.26)$$

Furthermore,  $\mathcal{D}(\mathcal{A})$  is dense in  $\mathcal{H}$ .

*Proof.* Let  $[u, v] \in V \times V$  such that  $Au + g(v(L))\delta_L \in H$ . Recalling the embedding  $H \hookrightarrow V'$ , this means that there exists  $f = [f_1, f_2] \in H$  such that for all  $w \in V$ ,

$$\int_\Omega \frac{\partial u}{\partial x} \frac{\partial w}{\partial x} dx + g(v(L))w(L) = \int_\Omega f_1 w dx + f_2 w(0). \quad (2.27)$$



We particularize (2.27) to the case where  $w = \varphi$ , with  $\varphi$  being an arbitrary test function in  $\mathcal{D}(\Omega)$ : the boundary terms vanish and after an integration by parts, we obtain

$$-\int_{\Omega} u \frac{\partial^2 \varphi}{\partial x^2} dx = \int_{\Omega} f_1 \varphi \quad \text{for all } \varphi \in \mathcal{D}(\Omega). \quad (2.28)$$

Equation (2.28) is exactly the definition of  $-\partial_{xx}u = f_1$  in  $\mathcal{D}'(\Omega)$ , and because  $f_1 \in L^2(\Omega)$  and  $u \in H^1(\Omega)$ , this means that  $u \in H^2(\Omega)$ . Hence, we can perform a different integration by parts in (2.27) to obtain

$$\frac{\partial u}{\partial x}(L)w(L) + g(v(L))w(L) = f_2w(0). \quad (2.29)$$

Besides, there certainly exists some  $w \in V$  satisfying  $w(0) = 0$  and  $w(L) = 1$ . This leads to  $\partial_x u(L) = -g(v(L))$ . Conversely, given  $u \in H^2(\Omega)$  and  $v \in V$  with  $\partial_x u(L) = -g(v(L))$ , it is clear that

$$\int_{\Omega} \frac{\partial u}{\partial x} \frac{\partial w}{\partial x} dx + g(v(L))w(L) = -\int_{\Omega} \frac{\partial^2 u}{\partial x^2} w dx - \frac{\partial u}{\partial x}(0)w(0) \quad \text{for all } w \in V, \quad (2.30)$$

which means that  $Au + g(v(L))\delta_L = [-\partial_{xx}u, -\partial_x u(0)] \in H$ . It remains to prove that  $\mathcal{D}(\mathcal{A})$  is dense in  $\mathcal{H}$ . Let  $\varepsilon > 0$  and  $[u, v] \in \mathcal{H}$ . By denseness of  $V$  in  $H$ , we can pick  $\hat{v} \in V$  such that  $\|v - \hat{v}\|_H \leq \varepsilon$ . We let  $\theta \triangleq -g(\hat{v}(L))$ , we pick  $\varphi \in \mathcal{D}(\Omega)$  such that  $\|\varphi - \partial_x u\|_{L^2(\Omega)} \leq \varepsilon$ , and we find a piecewise affine function  $\eta$  satisfying  $\eta(L) = \theta$  and  $\|\eta\|_{L^2(\Omega)} \leq \varepsilon$ . Set

$$\hat{u}(x) \triangleq u(0) + \int_0^x \varphi(s) + \eta(s) ds. \quad (2.31)$$

Then,  $\hat{u} \in H^2(\Omega)$  and  $\partial_x \hat{u}(L) = \theta$ , so that  $[\hat{u}, \hat{v}] \in \mathcal{D}(\mathcal{A})$ . On the other hand,  $\hat{u}(0) = u(0)$  and  $\|\partial_x u - \partial_x \hat{u}\|_{L^2(\Omega)} \leq 2\varepsilon$ . By norm equivalence, it follows that  $\|u - \hat{u}\|_V \leq K\varepsilon$  for some constant  $K$  that does not depend on  $\varepsilon$  or  $[u, v]$ . In sum, we have  $\|u - \hat{u}\|_V^2 + \|v - \hat{v}\|_H^2 \leq (1 + K^2)\varepsilon^2$ , which concludes the proof.  $\square$

*Remark 2.2.4.* Here, we could have directly defined  $\mathcal{D}(\mathcal{A})$  as in (2.26), and the equivalence between (2.24) and (2.26) is quite straightforward. This is because in space dimension one,  $u \in H^1(\Omega)$  and  $\Delta u \in L^2(\Omega)$  trivially imply  $u \in H^2(\Omega)$ . In the next chapter, we will see that the matter is more complicated for higher-dimensional domains.

One can see  $\mathcal{D}(\mathcal{A})$  as the set of “smooth” initial data satisfying a *compatibility condition* with the feedback action at the boundary  $x = L$ . Our subsequent goal is to find appropriate solutions for initial data in the dense set  $\mathcal{D}(\mathcal{A})$  before passing to the limit to recover weak solutions for general initial conditions in  $\mathcal{H}$ . For that purpose, we introduce the following definition, which is standard in the context of nonlinear semigroups and abstract first-order Cauchy problems.

**Definition 2.2.5.** Let  $T > 0$ . A *strong* solution  $[u, u']$  to (2.6) on  $[0, T]$  is any  $[u, u']$  with regularity

$$[u, u'] \in W^{1,1}(0, T; \mathcal{H}) \quad (2.32)$$

that takes values in  $\mathcal{D}(\mathcal{A})$  and satisfies

$$\frac{d}{dt}[u, u'] + \mathcal{A}[u, u'] = 0 \quad \text{a.e. in } (0, T) \quad (2.33)$$

in the sense of strong differentiation in  $\mathcal{H}$ .

*Remark 2.2.6.* For a strong solution  $[u, u']$ , (2.33) implies that, as expected, the trace of  $u'$  at  $x = L$  coincides a.e. with the weak derivative of the trace of  $u$  at  $x = L$ .

**Proposition 2.2.7** (Existence of strong solutions). *For any  $[u_0, v_0] \in \mathcal{D}(\mathcal{A})$ , there exists a unique strong solution  $[u, u']$  to (2.6) that satisfies  $[u(0), u'(0)] = [u_0, v_0]$ . In addition,  $[u, u']$  is differentiable from the right and for any  $\tau > 0$ ,*

$$[u, u'] \in W^{1,\infty}(0, \tau; \mathcal{H}). \quad (2.34)$$

*Proof.* We shall express  $\mathcal{A}$  as the perturbation of a maximal monotone operator by some globally Lipschitz mapping. First, we define a mapping  $\mathcal{F}$  on  $\mathcal{H}$  as follows:

$$\mathcal{F}[u, v] = [0, F(v)] \quad \text{for all } [u, v] \in \mathcal{H}. \quad (2.35)$$

Clearly,  $\mathcal{F}$  is globally Lipschitz continuous due to  $F$  being globally Lipschitz continuous. We let

$$\mathcal{A}_0 \triangleq \mathcal{A} + \frac{1}{2} \text{id} + \mathcal{F}. \quad (2.36)$$

Let us now prove that  $\mathcal{A}_0$  is maximal monotone; once this is established, the existence of a unique strong solution  $[u, u']$  to (2.6) with  $[u, u'] \in W^{1,\infty}(0, \tau; \mathcal{H})$ ,  $\tau > 0$ , and  $[u, u']$  right-differentiable in  $\mathcal{H}$  will follow from [Showalter, 2013, Chapter IV, Corollary 4.1]. Starting with the monotonicity condition, we pick a pair of elements  $[u_i, v_i] \in \mathcal{D}(\mathcal{A})$  and write

$$\begin{aligned} & (\mathcal{A}_0[u_1, v_1] - \mathcal{A}_0[u_2, v_2], [u, v])_{\mathcal{H}} \\ &= -(u, v)_H - a(u, v) + (Au + \{g(v_1(L)) - g(v_2(L))\}\delta_L, v)_H + \frac{1}{2}\|u\|_V^2 + \frac{1}{2}\|v\|_H^2. \end{aligned} \quad (2.37)$$

where  $u \triangleq u_1 - u_2$  and  $v \triangleq v_1 - v_2$ . First, we have

$$(Au + \{g(v_1(L)) - g(v_2(L))\}\delta_L, v)_H = a(u, v) + (g(v_1(L)) - g(v_2(L)))(v_1(L) - v_2(L)) \geq a(u, v) \quad (2.38)$$

because  $g$  is monotone increasing. Besides,  $\|u\|_V^2 + \|v\|_H^2 \geq \|u\|_H^2 + \|v\|_H^2 \geq 2|(u, v)_H|$ . It follows that

$$(\mathcal{A}_0[u_1, v_1] - \mathcal{A}_0[u_2, v_2], [u, v])_{\mathcal{H}} \geq 0, \quad (2.39)$$

which proves that  $\mathcal{A}_0$  is monotone. Next, to obtain the range condition, it suffices to prove that  $\mathcal{A}_0 + (1/2)\text{id}$  is onto. Let  $[f, g] \in \mathcal{H}$ . We must find  $[u, v] \in \mathcal{D}(\mathcal{A})$  such that

$$-v + u = f_1 \quad \text{in } V, \quad (2.40a)$$

$$Au + g(v(L))\delta_L + v = f_2 \quad \text{in } H. \quad (2.40b)$$

Substituting (2.40a) into (2.40b) shows that for any solution  $[u, v]$  to (2.40),  $v$  must satisfy

$$Av + v + g(v(L))\delta_L = Af_1 + f_2 \quad \text{in } V'. \quad (2.41)$$

To solve the variational problem (2.41), we introduce a nonlinear operator  $\Phi : V \rightarrow V'$  given by

$$\Phi(v) \triangleq Av + v + g(v(L))\delta_L, \quad (2.42)$$

so that (2.41) rewrites as  $\Phi(v) = Af_1 + f_2$ . We note that

$$\langle \Phi(v), v \rangle_V = a(v, v) + \|v\|_H^2 + g(v(L))v(L) \geq \|v\|_V^2 \quad \text{for all } v \in V. \quad (2.43)$$

Let us prove that  $\Phi$  is onto. To do so, we use [Showalter, 2013, Chapter II, Lemma 2.1 and Theorem 2.1]. Due to (2.43), we only have to check that

1.  $\Phi$  is monotone, i.e.,  $\langle \Phi(v_1) - \Phi(v_2), v_1 - v_2 \rangle_V \geq 0$  for any  $v_1, v_2 \in V$ ;
2.  $\Phi$  maps bounded sets of  $V$  into bounded sets of  $V'$ ;
3. For any  $v_1, v_2 \in V$ , the scalar function  $t \mapsto \langle \Phi(v_1 + tv_2), v_2 \rangle_V$  is continuous.

This is done by using that  $g$  is continuous and monotone increasing; besides,  $g$  being a scalar continuous function,  $g(v(L))$  remains bounded when  $v$  describes bounded sets of  $V$ . Therefore, there exists  $v \in V$  satisfying (2.41). By letting  $u = f_1 + v$ , we readily get (2.40b), which also means that  $[u, v] \in \mathcal{D}(\mathcal{A})$ . The proof is now complete.  $\square$

**Proposition 2.2.8.** *Let  $[u_0, v_0] \in \mathcal{D}(\mathcal{A})$ . The strong solution  $[u, u']$  to (2.6) originating from  $[u_0, v_0]$  is also a weak solution to (2.6).*

*Proof.* Let  $[u_0, v_0] \in \mathcal{D}(\mathcal{A})$  and  $[u, u']$  the associated strong solution to (2.6) as provided by Proposition 2.2.7. Let  $\tau \geq 0$ . Then,  $u' \in L^\infty(0, \tau; V) \cap W^{1, \infty}(0, \tau; H)$ . In particular, following Remark 2.2.6,  $u(L, \cdot)$  belongs to  $W^{1, \infty}(0, \tau)$ . Also,

$$u'' + Au + g(u'(L, \cdot))\delta_L = F(u') \quad \text{a.e. on } (0, \tau) \quad (2.44)$$

in the sense of strong differentiation in  $H$ . We recall that  $H$  is continuously embedded into  $V'$ , so that  $u' \in W^{1, \infty}(0, \tau; V')$  and (2.44) holds in the sense of strong differentiation in  $V'$  as well. Then, we can use general results on vector-valued distributions [Temam, 1997, Lemma 3.1, Chapter II] to deduce from (2.44) that

$$\frac{d}{dt}(u', w) + a(u, w) + g(u'(L, \cdot))w(L) = (F(u'), w) \quad \text{for all } w \in V \quad (2.45)$$

in the sense of scalar distributions on  $(0, \tau)$ .  $\square$

### 2.2.3 Proof of the Hadamard theorem

In this section, we bridge the gap between the variational problem (2.14) and the first-order abstract Cauchy problem (2.33) by showing that limits of strong solutions are in fact weak solutions to (2.6). This motivates the following definition, which provides an intermediate notion of solution.

**Definition 2.2.9.** Let  $T > 0$ . A *generalized* solution  $[u, u']$  to (2.6) is any limit of strong solutions in  $\mathcal{C}([0, T], \mathcal{H})$  with respect to the topology of uniform convergence, i.e., there exists a sequence of strong solutions  $[u_n, u'_n]$  such that

$$\sup_{t \in [0, T]} \|[u_n, u'_n] - [u, u']\|_{\mathcal{H}} \rightarrow 0 \quad \text{as } n \rightarrow +\infty. \quad (2.46)$$

This definition naturally arises in the context of evolution equations involving maximal monotone operators [Barbu, 1976, Showalter, 2013], where the contraction property of solutions provides Cauchy sequences in  $\mathcal{C}([0, T], \mathcal{H})$ .

**Lemma 2.2.10** (Existence of generalized solutions). *Let  $T > 0$ . For any  $[u_0, v_0] \in \mathcal{H}$ , there exists a unique generalized solution  $[u, u'] \in \mathcal{C}([0, T], \mathcal{H})$  to (2.6) satisfying  $[u(0), u'(0)] = [u_0, v_0]$ .*

*Proof.* We recall that  $\mathcal{D}(\mathcal{A})$  is dense in  $\mathcal{H}$  and apply [Showalter, 2013, Corollary 4.1A] to  $\mathcal{A}$  written as a globally Lipschitz perturbation of the maximal monotone operator  $\mathcal{A}_0$ .  $\square$

We are finally in position to prove Theorem 2.2.2. We need to show that strong solutions satisfy the various additional properties stated in the theorem, and then pass to the limit in suitable spaces.

*Proof of Theorem 2.2.2.* We start by proving that strong solutions satisfy the energy identity (2.18) and the estimate (2.19). To do so, given initial data in  $\mathcal{D}(\mathcal{A})$ , we consider a pair  $[u_i, u'_i]$  of strong solutions to (2.6). As usual, we let  $u \triangleq u_1 - u_2$ . Let  $\tau \geq 0$ . Then,  $u' \in W^{1,\infty}(0, \tau; H)$  and  $u'$  solves

$$u'' + Au + \{g(u_1(L, \cdot)) - g(u_2(L, \cdot))\} \delta_L = F(u'_1) - F(u'_2) \quad \text{in } H. \quad (2.47)$$

We take the scalar product of (2.47) with  $u'$  and integrate over  $(0, \tau)$  to obtain

$$\begin{aligned} & \int_0^\tau (u'', u')_H + a(u, u') dt \\ &= - \int_0^\tau \{g(u'_1(L, t)) - g(u'_2(L, t))\} u'(L, t) dt + \int_0^\tau (F(u'_1) - F(u'_2), u')_H dt \end{aligned} \quad (2.48)$$

Because  $u' \in W^{1,\infty}(0, \tau; H)$ ,  $(u'', u')_H = (d/dt) \|u'\|_H^2 / 2$ , and because  $u' \in W^{1,\infty}(0, \tau; V)$  and  $a$  is a continuous and symmetric bilinear form,  $a(u, u') = (d/dt) a(u, u) / 2$ , so that

$$\mathcal{E}(u, u') \Big|_0^\tau = - \int_0^\tau \{g(u'_1(L, t)) - g(u'_2(L, t))\} u'(L, t) dt + \int_0^\tau (F(u'_1) - F(u'_2), u')_H dt. \quad (2.49)$$

Equation (2.49) in the particular case where  $[u_2(0), u'_2(0)] = 0$  yields the desired energy identity (2.18). Bearing in mind that  $g$  is monotone increasing and  $F$  is globally  $q$ -Lipschitz continuous for some constant positive  $q$ , we deduce from (2.49) that

$$\mathcal{E}(u, u') \Big|_0^\tau \leq q \int_0^\tau \|u'\|_H^2 dt \leq 2q \int_0^\tau \mathcal{E}(u, u') dt. \quad (2.50)$$

The parameter  $\tau$  being arbitrary, (2.50) holds with  $\tau$  replaced by any  $t \leq \tau$ . By using Grönwall's inequality, we infer from (2.50) that

$$\mathcal{E}(u(t), u'(t)) \leq \exp(2qt) \mathcal{E}(u(0), u'(0)) \leq \exp(2q\tau) \mathcal{E}(u(0), u'(0)) \quad \text{for all } t \leq \tau. \quad (2.51)$$

Next, detailing (2.47), we get

$$u'' - \frac{\partial^2 u}{\partial x^2} = 0 \quad \text{a.e. in } \Omega \times (0, \tau), \quad (2.52a)$$

$$u''(0, t) - \frac{\partial u}{\partial x}(0, t) = F(u'_1(0, t)) - F(u'_2(0, t)) \quad \text{for a.e. } t \in (0, \tau), \quad (2.52b)$$

where  $u''(0, \cdot)$  is the weak derivative of  $u'(0, \cdot)$ . Let us define a differential multiplier  $\mathcal{M}$  as follows:

$$\mathcal{M}u(x, t) \triangleq x \frac{\partial u}{\partial x}(x, t). \quad (2.53)$$

Then,  $\mathcal{M}u \in W^{1,\infty}(0, \tau; L^2(\Omega))$ , which allows to take the scalar product of (2.52a) in  $L^2(\Omega)$  and integrate by parts with respect to the time variable: having let  $Q_\tau \triangleq \Omega \times (0, \tau)$ ,

$$\int_\Omega u' \mathcal{M}u dx \Big|_0^\tau - \iint_{Q_\tau} u' \mathcal{M}u' dx dt - \iint_{Q_\tau} \frac{\partial^2 u}{\partial x^2} \mathcal{M}u dx dt = 0. \quad (2.54)$$

Integrating by parts with respect to the space variable leads to following standard trace identity:

$$\frac{L}{2} \int_0^\tau \left| \frac{\partial u}{\partial x}(L, t) \right|^2 + |u'(L, t)|^2 dt = \int_\Omega u' \mathcal{M}u dx \Big|_0^\tau + \frac{1}{2} \iint_{Q_\tau} \left| \frac{\partial u}{\partial x} \right|^2 + |u'|^2 dx dt. \quad (2.55)$$

First, by using Cauchy-Swcharz and Young inequalities, we obtain

$$\int_{\Omega} u' \mathcal{M} u dx \Big|_0^{\tau} \leq L \mathcal{E}(u(0), u'(0)) + L \mathcal{E}(u(\tau), u'(\tau)). \quad (2.56)$$

In particular, it follows from (2.51) and (2.55) that

$$\int_0^{\tau} |u'(L, t)|^2 dt \leq 2\{1 + (1 + \tau/L) \exp(2q\tau)\} \mathcal{E}(u(0), u'(0)). \quad (2.57)$$

Combining (2.51) and (2.57) yields (2.19). Now we consider general initial data  $[u_0, v_0] \in \mathcal{H}$ . By Lemma 2.2.10, there exists a unique generalized solution  $[u, u'] \in \mathcal{C}(\mathbb{R}^+, \mathcal{H})$  to (2.6). Fix  $\tau > 0$ . We can pick a sequence of strong solutions  $[u_n, u'_n]$  such that  $[u_n, u'_n] \rightarrow [u, u']$  in  $\mathcal{C}([0, \tau], \mathcal{H})$ . Let us first prove that

$$u(L, \cdot) \in H^1(0, \tau), \quad \text{and} \quad u'_n(L, \cdot) \rightarrow u'(L, \cdot) \quad \text{in } L^2(0, \tau). \quad (2.58)$$

By continuity of the trace at  $x = L$  with respect to the  $H^1(\Omega)$ -norm,  $\{u_n(L, \cdot)\}$  converges to  $u(\cdot, L)$  in  $\mathcal{C}([0, \tau])$ . Furthermore, we deduce from (2.19) that  $\{u_n(L, \cdot)\}$  is a Cauchy sequence in  $H^1(0, \tau)$ . Uniqueness of the limit implies (2.58). Next, we prove that

$$g(u'(L, \cdot)) \in L^2(\Omega), \quad \text{and} \quad g(u'_n(L, \cdot)) \rightarrow g(u'(L, \cdot)) \quad \text{in } L^2(0, \tau). \quad (2.59)$$

We infer from (2.55) that

$$\left\{ \frac{\partial u_n}{\partial x}(L, \cdot) \right\} = \{g(u'_n(L, \cdot))\} \text{ is a Cauchy sequence in } L^2(0, \tau) \quad (2.60)$$

and converges to some function  $U \in L^2(0, \tau)$ . Let us now prove that  $U = g(u'(L, \cdot))$ . Recall from the theory of  $L^p$ -spaces that there exists a negligible subset  $\mathcal{N}_1$  of  $(0, \tau)$  such that, up to a subsequence,

$$u'_n(L, t) \rightarrow u'(L, t) \quad \text{for all } t \in (0, \tau) \setminus \mathcal{N}_1. \quad (2.61)$$

For such  $t$ , by continuity of  $g$ ,  $g(u'_n(L, t)) \rightarrow g(u'(L, t))$ . On the other hand, there exists a negligible set  $\mathcal{N}_2 \subset (0, \tau)$  such that, up to another subsequence,  $u'_n(L, t) \rightarrow U(t)$  for all  $t \in (0, \tau) \setminus \mathcal{N}_2$ . Now,  $\mathcal{N}_1 \cup \mathcal{N}_2$  is a set of measure 0 as well, so that by uniqueness of the limit,  $U(t) = g(u'(L, t))$  a.e. in  $(0, \tau)$ . It remains to prove that  $[u, u']$  satisfies the variational identity (2.14) and the energy identity (2.18). To see this, we pick  $\varphi \in \mathcal{D}(0, \tau)$  and must prove that for any  $w \in V$ ,

$$-\int_0^{\tau} (u', v)_H \varphi' dt + \int_0^{\tau} a(u, v) \varphi dt = -\int_0^{\tau} g(u(L, t)) w(L) \varphi(t) dt + \int_0^{\tau} (F(u'), w) \varphi dt. \quad (2.62)$$

Equation (2.62) is satisfied by the  $[u_n, u'_n]$  because those are strong solutions to (2.6); then, one can pass to the limit using the convergence in  $\mathcal{C}([0, \tau], \mathcal{H})$  together with (2.59). The same argument shows that (2.18) holds for  $[u, u']$  as well. The last statement of Theorem 2.2.2 concerns the semigroup  $\{\mathcal{S}_t\}$  associated with (2.6). That  $\{\mathcal{S}_t\}$  is well-defined and enjoys the semigroup properties readily follows from existence and uniqueness of solutions to the (autonomous) equations (2.6). Then, each  $\mathcal{S}_t$  is Lipschitz continuous because of the incremental estimate (2.19) combined with

$$\|u_1(t) - u_2(t)\|_H \leq \|u_1(0) - u_2(0)\|_H + \int_0^t \|u'_1(s) - u'_2(s)\|_H ds \quad (2.63)$$

for any pair  $[u_i, u'_i]$  of solutions to (2.6), which completes the proof.  $\square$

## 2.3 Stability analysis

This section is devoted to the stability properties of the feedback system (2.6). Our objective is to derive sufficient conditions under which the mechanical energy  $\mathcal{E}$  of solutions to (2.6) decays exponentially. For that purpose, we need to investigate the competition between the nonlinear boundary anti-damping at  $x = 0$  and the dissipative feedback action at  $x = L$ .

### 2.3.1 Uniform stabilization and Lyapunov methods

The first theorem states that under some (global) conditions on the anti-damping nonlinearity  $F$  and the feedback function  $g$ , the energy of solutions to (2.6) decays exponentially and uniformly.

**Theorem 2.3.1** (Exponential stabilization). *Assume that  $F$  is globally  $q$ -Lipschitz continuous for some  $q < 1/2$ . Suppose also that the feedback function  $g$  satisfies the following sector condition:*

$$\alpha_1 |s| \leq |g(s)| \leq \alpha_2 |s| \quad \text{for all } s \in \mathbb{R}, \quad (2.64)$$

where  $\alpha_1$  and  $\alpha_2$  are positive constants satisfying

$$\frac{\alpha_1}{1 + \alpha_2^2} > q. \quad (2.65)$$

Then, there exist positive constants  $\mu$  and  $M$  such that for all solutions  $[u, u']$  to (2.6),

$$\mathcal{E}(u(t), u'(t)) \leq M \exp(-\mu t) \mathcal{E}(u(0), u'(0)) \quad \text{for all } t \geq 0. \quad (2.66)$$

Theorem 2.3.1 is proved and discussed below. Let us state a consequence of the energy decay property (2.66) regarding the stability and attractivity of the set  $\mathfrak{A} \subset \mathcal{H}$  of stationary solutions to (2.6). The set  $\mathfrak{A}$  is given by

$$\mathfrak{A} = \text{span}(\mathbb{1}) \times \{0\} \quad (2.67)$$

where  $\mathbb{1}$  denotes the constant function whose value is 1. Note that

$$\mathfrak{A} = \{[u, v] \in \mathcal{H} : \mathcal{E}(u, v) = 0\}. \quad (2.68)$$

In fact, the (squared) distance between  $\mathfrak{A}$  and a given point  $[u, v]$  of the state space  $\mathcal{H}$  can be upper- and lower-bounded by its energy  $\mathcal{E}(u, v)$  – see (2.92) below. This leads us to the following corollary.

**Corollary 2.3.2** (Stability of the set of stationary solutions). *Under the hypotheses of Theorem 2.3.1, the set  $\mathfrak{A}$  of stationary solutions is pointwise asymptotically stable, i.e.,*

- Each point  $[u, 0] \in \mathfrak{A}$  is Lyapunov stable;
- Every solution  $[u, u']$  to (2.6) converges in  $\mathcal{H}$  to some limit  $[u^*, 0] \in \mathfrak{A}$ .

Furthermore, the following exponential stability property holds: there exists  $K > 0$  such that all solutions  $[u, u']$  to (2.6) satisfy

$$\text{dist}([u(t), u'(t)], \mathfrak{A})^2 \leq K \exp(-\mu t) \text{dist}([u(0), u'(0)], \mathfrak{A})^2 \quad \text{for all } t \geq 0. \quad (2.69)$$

Coming back to Theorem 2.3.1, a simple examination of the energy identity (2.17) is not enough to draw any conclusion regarding the decay (or lack thereof) of  $\mathcal{E}$  along solutions to (2.6). In order to overcome this issue and exhibit the boundary coupling between the two extremities of  $\Omega$ , we

introduce a class of Lyapunov candidates that are perturbations of the energy functional  $\mathcal{E}$ . Given a continuous scalar function  $\rho$  on  $[0, L]$ , we define  $\mathcal{E}_\rho$  as follows:

$$\mathcal{E}_\rho(u, v) \triangleq \mathcal{E}(u, v) + \int_{\Omega} \rho(x) \frac{\partial u}{\partial x}(x) v(x) dx \quad \text{for all } [u, v] \in \mathcal{H}. \quad (2.70)$$

Then,  $\mathcal{E}_\rho$  is continuous and there exists  $M > 0$  such that  $\mathcal{E}_\rho(u, v) \leq M\mathcal{E}(u, v)$  for all  $[u, v]$  in  $\mathcal{H}$ . Furthermore, if  $\max_{x \in [0, L]} |\rho(x)| < 1$ , then there also exists  $m > 0$  such that

$$m\mathcal{E}_\rho(u, v) \leq \mathcal{E}(u, v) \leq M\mathcal{E}_\rho(u, v) \quad \text{for all } [u, v] \in \mathcal{H}. \quad (2.71)$$

The constants  $m$  and  $M$  in the sandwich inequality (2.71) depend on  $\rho$  only. The idea behind the cross term in  $\mathcal{E}_\rho$  is to supplement (2.17) with trace identities that are similar to those used in the proof of Theorem 2.2.2 and will help us quantifying the behavior of the energy of the system.

**Proposition 2.3.3** (Lyapunov analysis). *Under the assumptions of Theorem 2.3.1, there exists an affine weight function  $\rho$  taking values in  $[0, 1]$  and a positive constant  $\mu$  such that for any solution  $[u, u']$  to (2.6),*

$$\mathcal{E}_\rho(u, u') \Big|_0^\tau \leq -\mu \int_0^\tau \mathcal{E}_\rho(u, u') dt \quad \text{for all } \tau \geq 0. \quad (2.72)$$

Furthermore, if the incremental version of (2.64) is satisfied, i.e.,

$$\alpha_1 |r_1 - r_2| \leq |g(r_1) - g(r_2)| \leq \alpha_2 |r_1 - r_2| \quad \text{for all } r_1, r_2 \in \mathbb{R}, \quad (2.73)$$

then, for any pair  $[u_1, u'_1], [u_2, u'_2]$  of solutions to (2.6), the incremental version of (2.72) holds:

$$\mathcal{E}_\rho(u_1 - u_2, u'_1 - u'_2) \Big|_0^\tau \leq -\mu \int_0^\tau \mathcal{E}_\rho(u_1 - u_2, u'_1 - u'_2) dt \quad \text{for all } \tau \geq 0. \quad (2.74)$$

Once Proposition 2.3.3 is established, Theorem 2.3.1 becomes a simple consequence of the integral inequality (2.72), as shown below. The incremental inequality (2.74), which is stronger than (2.72) in that it concerns pairs of solutions, will be used for technical reasons when dealing with the problem of saturated stabilization in Section 2.3.2.

*Proof of Proposition 2.3.3.* Pick two strong solutions  $[u_1, u'_1]$  and  $[u_2, u'_2]$  to (2.6). We assume that the incremental sector condition (2.73) is satisfied; if only (2.64) holds, then all the computations below are valid in the case  $u_2 = 0$ . Again, we write  $u \triangleq u_1 - u_2$ . Let  $\rho$  be affine, positive and monotone increasing. Fix  $\tau \geq 0$ . Similarly as in the proof of Theorem 2.2.2, by multiplying (2.52a) by  $\rho \partial_x u$ , integrating over  $Q_\tau = \Omega \times (0, \tau)$  and performing some integrations by parts, we obtain a trace identity that generalizes (2.55):

$$2 \int_{\Omega} \rho u' \frac{\partial u}{\partial x} dx \Big|_0^\tau + \rho' \iint_{Q_\tau} \left| \frac{\partial u}{\partial x} \right|^2 + |u'|^2 dx dt = \left[ \rho \int_0^\tau \left| \frac{\partial u}{\partial x} \right|^2 + |u'|^2 dt \right]_{x=0}^{x=L} \quad (2.75)$$

where we recall the notation  $Q_\tau = \Omega \times (0, T)$ . By summing the incremental energy identity (2.49) and one half of (2.75), we obtain an identity giving the variation of  $\mathcal{E}_\rho(u, u')$ :

$$\mathcal{E}_\rho(u, u') \Big|_0^\tau = \mathcal{E}(u, u') \Big|_0^\tau - \frac{\rho'}{2} \iint_{Q_\tau} \left| \frac{\partial u}{\partial x} \right|^2 + |u'|^2 dx dt + \left[ \frac{\rho}{2} \int_0^\tau \left| \frac{\partial u}{\partial x} \right|^2 + |u'|^2 dt \right]_{x=0}^{x=L} \quad (2.76)$$

Since  $g$  is monotone increasing and the sector condition (2.64) holds, we have

$$-\int_0^\tau \{g(u'_1(L, t)) - g(u'_2(L, t))\} u'(L, t) dt \leq -\alpha_1 \int_0^\tau |u'(L, t)|^2 dt. \quad (2.77)$$

Using the  $q$ -Lipschitz continuity of  $F$ , we deduce from (2.49), (2.76) and (2.77) that

$$\begin{aligned} \mathcal{E}_\rho(u, u')|_0^\tau \leq & -\frac{\rho'}{2} \iint_{Q_\tau} \left| \frac{\partial u}{\partial x} \right|^2 + |u'|^2 dx dt + \left\{ q - \frac{\rho(0)}{2} \right\} \int_0^\tau |u'(0, t)|^2 dt \\ & + \int_0^\tau \frac{\rho(L)}{2} \left| \frac{\partial u}{\partial x}(L, t) \right|^2 + \left\{ \frac{\rho(L)}{2} - \alpha_1 \right\} |u'(L, t)|^2 dt. \end{aligned} \quad (2.78)$$

Next, we use (2.64) again and write

$$\int_0^\tau \left| \frac{\partial u}{\partial x}(L, t) \right|^2 dt = \int_0^\tau |g(u'_1(L, t)) - g(u'_2(L, t))|^2 dt \leq \alpha_2^2 \int_0^\tau |u'(L, t)|^2 dt. \quad (2.79)$$

Plugging (2.79) into (2.78) leads to

$$\begin{aligned} \mathcal{E}_\rho(u, u')|_0^\tau \leq & -\frac{\rho'}{2} \iint_{Q_\tau} \left| \frac{\partial u}{\partial x} \right|^2 + |u'|^2 dx dt + \left\{ q - \frac{\rho(0)}{2} \right\} \int_0^\tau |u'(0, t)|^2 dt \\ & + \int_0^\tau \left\{ \frac{\rho(L)}{2} (1 + \alpha_2^2) - \alpha_1 \right\} |u'(L, t)|^2 dt. \end{aligned} \quad (2.80)$$

It remains to tune the affine weight  $\rho$  appropriately. Let  $\varepsilon > 0$ . Recall from the hypothesis (2.65) that

$$\frac{\alpha_1}{1 + \alpha_2^2} > q, \quad \text{hence} \quad \frac{\alpha_1 - \varepsilon}{1 + \alpha_2^2} > q + \varepsilon \quad (2.81)$$

provided that  $\varepsilon$  is sufficiently small. The choice  $\rho(0) = 2q + 2\varepsilon$  and  $\rho(L) = 2(\alpha_1 - \varepsilon)(1 + \alpha_2^2)^{-1}$  defines a unique affine function  $\rho$ , which is strictly increasing by (2.81) (meaning that  $\rho' > 0$ ) and positive as  $q > 0$ . Then,  $q - \rho(0)/2 = -\varepsilon$ ,  $\rho(L)(1 + \alpha_2^2)/2 - \alpha_1 = -\varepsilon$  and (2.80) yields

$$\mathcal{E}_\rho(u, u')|_0^\tau \leq -\frac{\rho'}{2} \iint_{Q_\tau} \left| \frac{\partial u}{\partial x} \right|^2 + |u'|^2 dx dt - \varepsilon \int_0^\tau |u'(0, t)|^2 dt. \quad (2.82)$$

Furthermore, since  $\alpha_1 \leq \alpha_2$ , we have  $\alpha_1(1 + \alpha_2^2)^{-1} \leq 1/2$ , hence  $\rho(L) < 1$ . As a result, the sandwich inequality (2.71) is valid for some positive constants  $m$  and  $M$  depending on  $\rho$ . Having let  $\mu \triangleq m^{-1} \min(\rho', 2\varepsilon) > 0$ , we finally obtain (2.74) holding for arbitrary  $\tau \geq 0$ . This concludes the proof.  $\square$

We end this section by completing the proofs of Theorem 2.3.1 and Corollary 2.3.2.

*Proof of Theorem 2.3.1.* Let  $[u, u']$  be a solution to (2.6). Since  $\mathcal{E}_\rho(u, u')$  is continuous and satisfies the integral inequality (2.72), Grönwall's lemma yields

$$\mathcal{E}_\rho(u(t), u'(t)) \leq \exp(-\mu t) \mathcal{E}_\rho(u(0), u'(0)) \quad \text{for all } t \geq 0. \quad (2.83)$$

The conclusion follows from the sandwich inequality (2.71).  $\square$



*Proof of Corollary 2.3.2.* By equivalence between the norms of  $V$  and  $H^1(\Omega)$ , there exist positive constants  $c$  and  $C$  such that

$$c\|w\|_{H^1(\Omega)}^2 \leq \|w\|_V^2 \leq C\|w\|_{H^1(\Omega)}^2 \quad \text{for all } w \in V. \quad (2.84)$$

Recall that  $\mathfrak{A} = \text{span}(\mathbf{1}) \times \{0\}$ . Given  $[u, v] \in \mathcal{H}$ , we have

$$\text{dist}([u, v], \mathfrak{A})^2 = \inf_{[w, 0] \in \mathfrak{A}} \|u - w\|_V^2 + \|v\|_H^2. \quad (2.85)$$

Let  $u_\Omega$  be the constant function given by the mean value of  $u$  over  $\Omega$ :

$$u_\Omega \triangleq \frac{1}{L} \int_{\Omega} u \, dx. \quad (2.86)$$

Then,  $u_\Omega$  is the orthogonal projection in  $H^1(\Omega)$  of  $u$  onto the vector line of constant functions. Thus, it follows from (2.84) that

$$\|u - w\|_V^2 \geq c\|u - u_\Omega\|_{H^1(\Omega)}^2 = c \int_{\Omega} \left| \frac{\partial u}{\partial x} \right|^2 + |u - u_\Omega|^2 \, dx \quad \text{for all } [w, 0] \in \mathfrak{A}. \quad (2.87)$$

We deduce from (2.85) and (2.87) that

$$\text{dist}([u, v], \mathfrak{A})^2 \geq c \int_{\Omega} \left| \frac{\partial u}{\partial x} \right|^2 + |u - u_\Omega|^2 \, dx + c\|v\|_H^2 \geq 2c\mathcal{E}(u, v). \quad (2.88)$$

On the other hand,  $[u_\Omega, 0] \in \mathfrak{A}$ , which implies that

$$\text{dist}([u, v], \mathfrak{A})^2 \leq \|u - u_\Omega\|_V^2 + \|v\|_H^2 \leq C\|u - u_\Omega\|_{H^1(\Omega)}^2 + \|v\|_H^2 = C \int_{\Omega} \left| \frac{\partial u}{\partial x} \right|^2 + |u - u_\Omega|^2 \, dx + \|v\|_H^2, \quad (2.89)$$

where we used (2.84) again. Now, the Poincaré-Wirtinger inequality states that for some constant  $K > 0$  and arbitrary  $u \in H^1(\Omega)$ ,

$$\int_{\Omega} |u - u_\Omega|^2 \, dx \leq K \int_{\Omega} \left| \frac{\partial u}{\partial x} \right|^2 \, dx. \quad (2.90)$$

Plugging (2.90) into (2.89) yields

$$\text{dist}([u, v], \mathfrak{A})^2 \leq C(1 + K) \int_{\Omega} \left| \frac{\partial u}{\partial x} \right|^2 \, dx + \|v\|_H^2 \leq 2 \max\{C(1 + K), 1\} \mathcal{E}(u, v). \quad (2.91)$$

Renaming the constants if needed, we infer from (2.88) and (2.91) that for some positive  $c$  and  $C$ ,

$$c\mathcal{E}(u, v) \leq \text{dist}([u, v], \mathfrak{A})^2 \leq C\mathcal{E}(u, v) \quad \text{for all } [u, v] \in \mathcal{H}. \quad (2.92)$$

Using (2.66), we apply (2.92) to a solution  $[u, u']$  to (2.6) and obtain that for all  $t \geq 0$ ,

$$\begin{aligned} \text{dist}([u(t), u'(t)], \mathfrak{A})^2 &\leq C\mathcal{E}(u(t), u'(t)) \\ &\leq CM \exp(-\mu t) \mathcal{E}(u(0), u'(0)) \\ &\leq c^{-1} CM \exp(-\mu t) \text{dist}([u(0), u'(0)], \mathfrak{A})^2, \end{aligned} \quad (2.93)$$

which is the desired inequality (2.69). Next, we show that each point of  $\mathfrak{A}$  is Lyapunov stable. Let  $[w, 0] \in \mathfrak{A}$  and  $\varepsilon > 0$ . We must find  $\delta > 0$  such that for any  $[u_0, v_0] \in \mathcal{B}_{\mathcal{H}}([w, 0], \delta)$ ,  $\mathcal{S}_t[u_0, v_0] \in \mathcal{B}_{\mathcal{H}}([w, 0], \varepsilon)$  for all  $t \geq 0$ . Given  $[u_0, v_0] \in \mathcal{H}$ , we denote the associated solution to (2.6) by  $[u(t), u'(t)] = \mathcal{S}_t[u_0, v_0]$ . First,  $u \in W^{1,1}(0, +\infty; H)$  and

$$\|u(t) - u_0\|_H \leq \int_0^{+\infty} \|u'(r)\|_H dr \quad \text{for all } t \geq 0. \quad (2.94)$$

The right-hand side of (2.94) is finite because of the exponential decay of  $\mathcal{E}(u, u')$ . In fact, we have

$$\|u(t) - u_0\|_H \leq \frac{2M^{1/2}}{\mu} \mathcal{E}(u_0, v_0)^{1/2} \quad \text{for all } t \geq 0. \quad (2.95)$$

On the other hand, since  $[w, 0] \in \mathfrak{A}$ ,  $w$  is a constant, and we can write

$$\|[u(t) - w, u'(t)]\|_{\mathcal{H}}^2 = \|u'(t)\|_H^2 + a(u(t), u(t)) + \|u(t) - w\|_H^2 \leq 2\mathcal{E}(u(t), u'(t)) + \|u(t) - w\|_H^2. \quad (2.96)$$

We also have  $\|u(t) - w\|_H^2 \leq 2\|u(t) - u_0\|_H^2 + 2\|w - u_0\|_H^2$  for all  $t \geq 0$ . Thus, combining (2.66), (2.95) and (2.96), we obtain that

$$\|[u(t) - w, u'(t)]\|_{\mathcal{H}}^2 \leq K\{\mathcal{E}(u_0, v_0) + \|w - u_0\|_H^2\} \quad \text{for all } t \geq 0, \quad (2.97)$$

for some positive constant  $K$ . Because  $[w, 0] \in \mathfrak{A}$ ,  $\mathcal{E}(w, 0) = 0$ , and by continuity of  $\mathcal{E}$  with respect to the norm of  $\mathcal{H}$ , there exists  $\delta > 0$  such that any  $[u_0, v_0] \in \mathcal{B}_{\mathcal{H}}(Y, \delta)$ ,  $\mathcal{E}(u_0, v_0) \leq \varepsilon/2K$ . Taking  $\delta$  smaller than  $\varepsilon/2K$  as well if needed, since  $\|w - u_0\|_H^2 \leq \|w - u_0\|_V^2 \leq \|[u_0 - w, v_0]\|_{\mathcal{H}}^2$ , we obtain the desired Lyapunov stability property. It remains to prove that for each  $[u_0, v_0] \in \mathcal{H}$ ,  $[u(t), u'(t)]$  goes to some limit  $[u^*, 0] \in \mathfrak{A}$  as  $t$  goes to  $+\infty$ . We already know that  $u'$  goes to 0 in  $H$ . Equation (2.95) combined with the semigroup properties of  $\{\mathcal{S}_t\}$  and  $a(u(t), u(t)) \rightarrow 0$  implies that for each increasing sequence of nonnegative real numbers  $\{t_n\}$ ,  $\{u_{t_n}\}$  is a Cauchy sequence in  $V$  and therefore converges to some  $u^*$ , which must then satisfies  $a(u^*, u^*) = 0$ , i.e.,  $[u^*, 0] \in \mathfrak{A}$ . By using that  $[u^*, 0]$  is Lyapunov stable, one can check that  $u^*$  does not depend on the particular choice of sequence  $\{t_n\}$ .  $\square$

### 2.3.2 The problem of saturated stabilization

It is clear that Theorem 2.3.1 does not apply for bounded feedback functions  $g$  such as ‘‘hard’’ saturation maps  $\text{sat}_S$  with threshold  $S > 0$  defined by

$$\text{sat}_S(s) \triangleq \begin{cases} s & \text{if } |s| \leq S, \\ S \frac{s}{|s|} & \text{otherwise,} \end{cases} \quad (2.98)$$

which only satisfy the sector condition (2.64) around zero. Nevertheless, one may expect that a local version of the result holds in that case. We shall prove the following theorem.

**Theorem 2.3.4** (Saturated stabilization). *Instead of (2.64), assume there exist positive constants  $S$ ,  $\alpha_1$  and  $\alpha_2$ , with  $\alpha_1$  and  $\alpha_2$  satisfying (2.65), such that*

$$\alpha_1|s_1 - s_2| \leq |g(s_1) - g(s_2)| \leq \alpha_2|s_1 - s_2| \quad \text{for all } s_1, s_2 \in \mathbb{R} \text{ with } |s_i| \leq S. \quad (2.99)$$

*Then, there exists positive constants  $K$ ,  $M$  and  $\mu$  such that (2.66) holds for all strong solutions  $[u, v]$  with initial data  $[u_0, v_0] \in \mathcal{D}(\mathcal{A})$  satisfying*

$$\|u_0\|_{H^1(\Omega)}^2 + \left\| \frac{\partial v_0}{\partial x} \right\|_{H^1(\Omega)}^2 \leq K. \quad (2.100)$$

In other words, smooth solutions originating from some *small* ball around the origin still have exponentially decaying energy; however, this ball is defined with respect to a norm that is stronger than that of  $\mathcal{H}$ . Now, establishing Theorem 2.3.4 essentially amounts to proving that sufficiently small balls in the  $H^2(\Omega) \times H^1(\Omega)$ -norm are contained in some subset of  $\mathcal{H}$  that is invariant with respect to the closed-loop dynamics. Indeed, roughly speaking, if the  $H^1(\Omega)$ -norm of  $u'$  is sufficiently small, then  $u(L, t) \leq S$  for all  $t \geq 0$ , and the analysis of Proposition 2.3.3 is valid. On the other hand, if  $[u_0, v_0]$  is taken small in some ball of the energy space  $\mathcal{H}$ , using (2.19), we only get  $L^2$ -estimates of  $u'(L, \cdot)$ , as opposed to pointwise estimates. Thus, we need to bound some higher-degree norm of strong solutions. While the general theory of contraction semigroups directly provides such results in problems where the original (uncontrolled) dynamics are non-expansive, in our case the original system can be unstable due to the anti-damping term; and, in the example of saturated proportional control, only when the feedback effectively operates does the contraction property of the closed-loop equations (2.6) holds. That is, modulo constant solutions, which is another fact that requires some workaround. To overcome these issues, we shall prove a series of technical lemmas that are given in the sequel.

We start by introducing the quotient space  $\tilde{\mathcal{H}} \triangleq \mathcal{H}/\mathfrak{A}$ , i.e.,  $\tilde{\mathcal{H}}$  is the vector space of equivalence classes modulo stationary solutions. Since  $\mathfrak{A}$  is a closed subspace of the Banach space  $\mathcal{H}$ ,  $\tilde{\mathcal{H}}$  endowed with the norm

$$\|[u, v]\|_{\tilde{\mathcal{H}}} \triangleq \inf_{[w, 0] \in \mathfrak{A}} \|[u - w, v]\|_{\mathcal{H}} \quad (2.101)$$

is also a Banach space. The first lemma shows that the semigroup  $\{\mathcal{S}_t\}$  is compatible with the quotient by  $\mathfrak{A}$ , i.e., the image by each  $\mathcal{S}_t$  of an equivalence class modulo  $\mathfrak{A}$  is a single equivalence class.

**Lemma 2.3.5.** *Let  $[u_1, v_1]$  and  $[u_2, v_2]$  in  $\mathcal{H}$ . If  $[u_1 - u_2, v_1 - v_2] \in \mathfrak{A}$ , then  $\mathcal{S}_t[u_1, v_1] - \mathcal{S}_t[u_2, v_2] \in \mathfrak{A}$ .*

*Proof.* Let  $t \geq 0$ . A consequence of the *a priori* estimate (2.19) is that

$$\mathcal{E}(\mathcal{S}_t[u_1, v_1] - \mathcal{S}_t[u_2, v_2]) \leq C(t)\mathcal{E}(u_1 - u_2, v_1 - v_2) = 0, \quad (2.102)$$

where we used that  $[u_1 - u_2, v_1 - v_2] \in \mathfrak{A}$ . The conclusion follows from (2.68).  $\square$

This allows us to define a semigroup  $\{\tilde{\mathcal{S}}_t\}$  that operates on the quotient space  $\tilde{\mathcal{H}}$ : an equivalence class in  $\tilde{\mathcal{H}}$  containing some element  $[u, v]$  is mapped by  $\tilde{\mathcal{S}}_t$  to the equivalence class of  $\mathcal{S}_t[u, v]$ , and by Lemma 2.3.5 the result does not depend on the representative  $[u, v]$ . The semigroup property is easily verified as well. Next, we introduce some additional notation. Given a continuous  $\rho$  satisfying  $\max_{x \in [0, L]} |\rho(x)| < 1$ , we let

$$\|[u, v]\|_{\tilde{\mathcal{H}}, \rho} \triangleq \mathcal{E}_\rho(u, v)^{1/2} \quad \text{for all } [u, v] \in \mathcal{H}. \quad (2.103)$$

This definition is valid for either single points in  $\mathcal{H}$  or equivalence classes in  $\tilde{\mathcal{H}}$  since

$$\mathcal{E}_\rho(u + w, v) = \mathcal{E}_\rho(u, v) \quad \text{for all } [w, 0] \in \mathfrak{A}. \quad (2.104)$$

**Lemma 2.3.6.** *The function  $\|\cdot\|_{\tilde{\mathcal{H}}, \rho}$  is an Hilbert norm on  $\tilde{\mathcal{H}}$  that is equivalent to  $\|\cdot\|_{\tilde{\mathcal{H}}}$ .*

*Proof.* Triangular inequality and absolute homogeneity are easily verified. Positive-definiteness follows from (2.68). By the sandwich inequality (2.71), it is sufficient to prove that  $\|\cdot\|_{\tilde{\mathcal{H}}, 0}$  is equivalent to  $\|\cdot\|_{\tilde{\mathcal{H}}}$ , i.e., there exist some positive constants  $c$  and  $C$  such that for all  $[u, v] \in \mathcal{H}$ ,

$$c\mathcal{E}(u, v) \leq \inf_{[w, 0] \in \mathfrak{A}} \|[u - w, v]\|_{\mathcal{H}}^2 \leq C\mathcal{E}(u, v). \quad (2.105)$$

Equation (2.105) is exactly (2.92) and has already been established in the proof of Corollary 2.3.2. To see that  $\|\cdot\|_{\tilde{\mathcal{H}},\rho}$  is induced by a scalar product, we can check for instance that the following parallelogram identity is valid:

$$2\mathcal{E}_\rho(u_1, v_1) + 2\mathcal{E}_\rho(u_2, v_2) = \mathcal{E}_\rho(u_1 - u_2, v_1 - v_2) + \mathcal{E}_\rho(u_1 + u_2, v_1 + v_2) \quad (2.106)$$

for all pairs  $[u_i, v_i] \in \mathcal{H}$ .  $\square$

As a consequence of Lemma 2.3.6, the quotient space  $\tilde{\mathcal{H}}$  equipped with any of the (equivalent) norms  $\|\cdot\|_{\tilde{\mathcal{H}},\rho}$  is a Hilbert space.

Our next objective is to use contraction properties of the “quotient semigroup” in order to prove that for strong solutions, the  $H^1(\Omega)$ -norm of the velocity remains bounded. Key to the proof of Theorem 2.3.4 is the following lemma.

**Lemma 2.3.7.** *Let  $r > 0$ . Under the full hypotheses of Proposition 2.3.3, consider the (nonempty) set*

$$\mathcal{K}_{\rho,r} \triangleq \{[u, v] \in \mathcal{D}(\mathcal{A}) : \|\mathcal{A}[u, v]\|_{\tilde{\mathcal{H}},\rho} \leq r\}. \quad (2.107)$$

where  $\rho$  is the weight given by the proposition. Then,  $\mathcal{K}_{\rho,r}$  is positively invariant with respect to  $\{\mathcal{S}_t\}$ , i.e.,

$$\mathcal{S}_t \mathcal{K}_r \subset \mathcal{K}_r \quad \text{for all } t \geq 0. \quad (2.108)$$

*Proof.* Let  $[u_0, v_0] \in \mathcal{K}_r$ . We write  $[u(t), u'(t)] = \mathcal{S}_t[u_0, v_0]$ . Then, since  $[u_0, v_0] \in \mathcal{D}(\mathcal{A})$ ,  $[u, u']$  is a strong solution to (2.6) and by virtue of Proposition 2.2.7,  $[u, u']$  is differentiable from the right and

$$\lim_{s \rightarrow 0^+} \left\| \frac{\mathcal{S}_{s+t}[u_0, v_0] - \mathcal{S}_t[u_0, v_0]}{s} + \mathcal{A} \mathcal{S}_t[u_0, v_0] \right\|_{\mathcal{H}} = 0 \quad \text{for all } t \geq 0. \quad (2.109)$$

We now make a detour by the quotient space  $\tilde{\mathcal{H}}$ . Since  $\mathcal{E} \leq (1/2)\|\cdot\|_{\tilde{\mathcal{H}}}^2$ , the linear map from  $\mathcal{H}$  to  $\tilde{\mathcal{H}}$  sending an element  $[u, v]$  to its equivalence class modulo  $\mathfrak{A}$  is continuous. It follows that the semigroup  $\{\tilde{\mathcal{S}}_t\}$  is strongly continuous as well; in addition, (2.109) means that for each  $t \geq 0$ , (the class of)  $-\mathcal{A}[u(t), u'(t)]$  is the right-derivative at  $t$  of  $s \mapsto \tilde{\mathcal{S}}_s[u_0, v_0]$  with respect to the norm of  $\tilde{\mathcal{H}}$ . Furthermore, we infer from (2.74) in Proposition 2.3.3 that it is in fact a contraction semigroup. In sum,  $\{\tilde{\mathcal{S}}_t\}$  is a strongly continuous semigroup of contractions on (the Hilbert space)  $\tilde{\mathcal{H}}$ , and its trajectory originating from  $[u_0, v_0]$  is right-differentiable; therefore, by virtue of [Crandall and Pazy, 1969, Theorem 1.4], the norm of the right-derivative is a monotone decreasing function of the time variable. Putting aside the matter of equivalence classes, this means that

$$\|\mathcal{A}[u(t), u'(t)]\|_{\tilde{\mathcal{H}},\rho} \leq \|\mathcal{A}[u(0), u'(0)]\|_{\tilde{\mathcal{H}},\rho} \quad \text{for all } t \geq 0. \quad (2.110)$$

The result is now proved.  $\square$

**Lemma 2.3.8.** *There exist positive constants  $C_1$  and  $C_2$  such that for all  $[u, v] \in \mathcal{D}(\mathcal{A})$ ,*

$$C_1 \left\{ \|v\|_{H^1(\Omega)}^2 + \left\| \frac{\partial u}{\partial x} \right\|_{H^1(\Omega)}^2 \right\} \leq \|\mathcal{A}[u, v]\|_{\tilde{\mathcal{H}},\rho}^2 + \|[u, v]\|_{\tilde{\mathcal{H}},\rho}^2 \leq C_2 \left\{ \|v\|_{H^1(\Omega)} + \left\| \frac{\partial u}{\partial x} \right\|_{H^1(\Omega)} \right\}. \quad (2.111)$$

*Proof.* By (2.71), it suffices to prove the case  $\rho = 0$ . We have

$$\|\mathcal{A}[u, v]\|_{\tilde{\mathcal{H}},0}^2 = \mathcal{E}(-v, Au + g(v(L))\delta_L) = \frac{1}{2} \int_{\Omega} \left| \frac{\partial v}{\partial x} \right|^2 + \left| \frac{\partial^2 u}{\partial x^2} \right|^2 dx + \frac{1}{2} \left| \frac{\partial u}{\partial x}(0) \right|^2. \quad (2.112)$$

Therefore,

$$\|\mathcal{A}[u, v]\|_{\mathcal{H},0}^2 + \|[u, v]\|_{\mathcal{H},0}^2 = \frac{1}{2} \int_{\Omega} \left| \frac{\partial^2 u}{\partial x^2} \right|^2 + \left| \frac{\partial u}{\partial x} \right|^2 + \left| \frac{\partial v}{\partial x} \right|^2 + |v|^2 dx + \frac{1}{2} \left| \frac{\partial u}{\partial x}(0) \right|^2 + \frac{1}{2} |v(0)|^2. \quad (2.113)$$

Since the trace is continuous with respect to the  $H^1(\Omega)$ -norm, the conclusion readily follows.  $\square$

We are finally in position to prove Theorem 2.3.4. Note that the lemmas established so far require that the feedback linearity  $g$  satisfies the global sector condition of Proposition 2.3.3. Nonetheless, by proving that well-chosen sets  $\mathcal{K}_{\rho,r}$  are invariant for the saturated dynamics as well, we can obtain the desired result.

*Proof of Theorem 2.3.4.* First, we claim that there exists a continuous monotone increasing function  $f$  satisfying the global incremental sector condition (2.64) and such that

$$g(s) = f(s) \quad \text{for all } |s| \leq S. \quad (2.114)$$

For instance, we can explicitly build one as follows:

$$f(s) \triangleq g(s) \quad \text{for } s \in (-S, S), \quad (2.115a)$$

$$f(s) \triangleq g(S) + \frac{\alpha_1 + \alpha_2}{2}(s - S) \quad \text{for } s \geq S, \quad (2.115b)$$

$$f(s) \triangleq g(-S) + \frac{\alpha_1 + \alpha_2}{2}(s + S) \quad \text{for } s \leq -S. \quad (2.115c)$$

The closed-loop equations (2.6) with  $g$  replaced by  $f$  generate another semigroup of nonlinear operators on  $\mathcal{H}$ , which we denote by  $\{\mathcal{T}_t\}$ . All results in Proposition 2.3.3 and Theorem 2.3.1 apply to  $\{\mathcal{T}_t\}$ . Let  $\rho$  be the weight function given by Proposition 2.3.3. By Lemma 2.3.7, all sets  $\mathcal{K}_{\rho,r}$  are positively invariant for  $\{\mathcal{T}_t\}$ . On the other hand, by Lemma 2.3.8, contraction property of  $\{\mathcal{T}_t\}$  with respect to the  $\|\cdot\|_{\mathcal{H},\rho}$ -norm and continuity of the trace, one can choose  $r > 0$  such that

$$|v(L)| \leq S \quad \text{for all } [u, v] \in \mathcal{K}_r. \quad (2.116)$$

Let us prove that  $\mathcal{K}_{\rho,r}$  is positively invariant for the original semigroup  $\{\mathcal{S}_t\}$ . Pick  $[u_0, v_0] \in \mathcal{K}_{\rho,r}$ . Let  $[u(t), u'(t)] \triangleq \mathcal{S}_t[u_0, v_0]$  and  $[z(t), z'(t)] \triangleq \mathcal{T}_t[u_0, v_0]$ . Then,  $[z, z']$  satisfies

$$z'' + Az + f(z'(L, \cdot))\delta_L = F(z') \quad \text{in } V' \text{ (or even } H) \quad (2.117)$$

a.e. in  $(0, +\infty)$ . On the other hand, by positive invariance of  $\mathcal{K}_r$ ,  $|z'(L, t)| \leq S$  for all  $t \geq 0$ , which means that  $f(z'(L, t)) = g(z'(L, t))$  for all  $t \geq 0$ . In other words,  $[u, u']$  and  $[z, z']$  satisfy the same variational identity (2.14) and the same initial condition. By uniqueness of weak solutions to (2.6) in Theorem 2.2.2, this implies  $[u, u'] = [z, z']$ . We have proved that for some positive constants  $\mu$  and  $M$  given by Theorem 2.3.1 applied to  $\{\mathcal{T}_t\}$ , all solutions  $[u, u'] = \mathcal{S}_t[u_0, v_0]$  originating from  $\mathcal{K}_r$  satisfy the exponential energy decay property (2.66). Using Lemma 2.3.8 again, there exists  $K > 0$  such that

$$\text{If } \|v_0\|_{H^1(\Omega)}^2 + \left\| \frac{\partial u_0}{\partial x} \right\|_{H^1(\Omega)}^2 \leq K, \quad \text{then } [u_0, v_0] \in \mathcal{K}_r, \quad (2.118)$$

which concludes the proof.  $\square$

## Comments on Chapter 2

**One-dimensional hyperbolic systems** Using the Riemannian invariants  $\partial_t u - \partial_x u$  and  $\partial_t u + \partial_x u$ , the one-dimensional wave equation (2.1) can be recast into the framework of first-order one-dimensional hyperbolic equations, for which a wide variety of general results exist – see, e.g., [Coron et al., 2008]. We do not follow this approach here for two reasons:

- Due to the nonlinear aspect of the boundary conditions (2.6b) and (2.6c), they do not translate conveniently in the new Riemann coordinates;
- The formalism we use in Chapter 2 is a starting point for analyzing more general second-order evolution equations, including Euler-Bernoulli beam equations or multi-dimensional wave equations as in Chapter 3; in particular, it should be noted that in dimension higher than one, the “wave” differential operator (also known as d’Alembertian)  $\partial/\partial t - \Delta$  no longer factorizes into a product of transport operators.

**Lagrangian mechanics and variational identity** The mechanical energy functional  $\mathcal{E}$  belongs to  $\mathcal{C}^1(\mathcal{H}, \mathbb{R})$  and its Fréchet differential is given by

$$d\mathcal{E}(u, v)[w, z] = a(u, w) + (v, z)_H \quad \text{for all } [u, v], [w, z] \in \mathcal{H}. \quad (2.119)$$

Using the notation of physics, the variational identity (2.14), which characterizes solutions  $[u, \dot{u}]$ , can be rewritten as a balance of energy. For all virtual displacement in the test function space  $V$ , the following Euler-Lagrange equation is satisfied:

$$\frac{d}{dt} \left\{ \frac{\partial \mathcal{E}}{\partial \dot{u}}(u, \dot{u}) \delta \right\} - \frac{\partial \mathcal{E}}{\partial u}(u, \dot{u}) \delta = \text{Virtual power of external forces}, \quad (2.120)$$

where  $\partial/\partial u$  and  $\partial/\partial \dot{u}$  denote partial Fréchet differentials with respect to the position and velocity variable respectively. Such an energy-based formulation highlights the close relation between variational methods and the *principle of least action* in mechanics.



# Three

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## Dirichlet stabilization of the wave equation

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### Contents

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3.1	Background and closed-loop equations . . . . .	29
3.2	Well-posedness and asymptotic stability . . . . .	31
3.2.1	Functional preliminaries . . . . .	31
3.2.2	Well-posedness and additional properties of the semigroup . . . . .	33
3.2.3	Asymptotic stability . . . . .	38
3.3	Polynomial decay rates for strong solutions . . . . .	39
3.3.1	Statement of the result and outline of the proof . . . . .	39
3.3.2	The multiplier identity . . . . .	41
3.3.3	Integral estimates and conclusion . . . . .	44

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### 3.1 Background and closed-loop equations

In this chapter, we study the multi-dimensional wave equation with Dirichlet boundary control subject to a nonlinearity  $g$ . Let  $\Omega$  be a bounded domain of  $\mathbb{R}^d$ ,  $d \geq 2$ , with smooth boundary  $\Gamma$ . Given a relatively open nonempty subset  $\Gamma_0$  of  $\Gamma$ , we consider the following control system:

$$\frac{\partial^2 u}{\partial t^2} - \Delta u = 0 \quad \text{in } \Omega \times (0, +\infty), \quad (3.1a)$$

$$u|_{\Gamma} = -g(U(t)) \quad \text{on } \Gamma_0 \times (0, +\infty), \quad (3.1b)$$

$$u|_{\Gamma} = 0 \quad \text{on } \Gamma_1 \times (0, +\infty), \quad (3.1c)$$

where  $U$  represents the control input and  $\Gamma_1 \triangleq \Gamma \setminus \Gamma_0$  is the uncontrolled part of the boundary. It was proved in [Lasiecka et al., 1986] that, roughly speaking, the largest state space  $\mathcal{H}$  for which, with reference to the wave equation (3.1a), the property

$$[u(0), u'(0)] \in \mathcal{H}, \quad u|_{\Gamma} \in L^2(0, T; L^2(\Gamma)) \quad \text{implies} \quad [u, u'] \in \mathcal{C}([0, T], \mathcal{H}), \quad (3.2)$$

or, in other words, the optimal energy space  $\mathcal{H}$  for the waves with  $L^2(0, T; L^2(\Gamma))$ -Dirichlet control is in fact

$$\mathcal{H} \triangleq L^2(\Omega) \times H^{-1}(\Omega). \quad (3.3)$$

where we recall that the Sobolev space of negative order  $H^{-1}(\Omega)$  is the topological dual of  $H_0^1(\Omega)$ , which is the closure of  $\mathcal{D}(\Omega)$  in  $H^1(\Omega)$ . Note that  $H^{-1}(\Omega)$  is a space of distributions. It is a Hilbert



space if equipped with the following scalar product:

$$(u_1, u_2)_{H^{-1}(\Omega)} \triangleq (A^{-1/2}u_1, A^{-1/2}u_2)_{L^2(\Omega)} \quad \text{for all } u_1, u_2 \in H^{-1}(\Omega), \quad (3.4)$$

where the “ $-\Delta$  with homogeneous Dirichlet boundary conditions” operator  $A$  and its fractional powers are properly introduced in Section 3.2.1. The aforementioned optimal regularity result suggests working in  $\mathcal{H}$  for the feedback stabilization problem as well. As usual, we define an energy functional  $\mathcal{E}$  on  $\mathcal{H}$  by letting

$$\mathcal{E}(u, v) \triangleq \frac{1}{2}\|u\|_{L^2(\Omega)}^2 + \frac{1}{2}\|v\|_{H^{-1}(\Omega)}^2 \quad \text{for all } [u, v] \in \mathcal{H}. \quad (3.5)$$

Formally differentiating  $\mathcal{E}$  along “trajectories” of the open-loop system (3.1) yields the energy identity

$$\frac{d}{dt}\mathcal{E}(u, u') = \int_{\Gamma_0} g(U(t)) \frac{\partial[A^{-1}u']}{\partial\nu} d\sigma \quad (3.6)$$

where  $\partial/\partial\nu$  denotes the outward normal derivative. Assuming that  $g(s)s \geq 0$  for all real  $s$ , (3.5) suggests the following choice of (velocity) feedback:

$$U(t) = -\frac{\partial[A^{-1}u']}{\partial\nu}. \quad (3.7)$$

The feedback law (3.7) was introduced in [Lasićka and Triggiani, 1987] for the linear problem (i.e.,  $g = \text{id}$ ). The authors then proved that the linear version of (3.1)-(3.7) gives rise to an exponentially stable semigroup of operators on  $\mathcal{H}$  under the assumption that the whole boundary is actuated (i.e.,  $\Gamma = \Gamma_0$ ) and that  $\Omega$  satisfies suitable geometric conditions. The proof relies on the analysis of a new variable  $p$  defined as

$$p \triangleq A^{-1}u' \quad (3.8)$$

which is smoother and solves a wave-type equation as well. The result was later refined by the same authors in [Lasićka and Triggiani, 1992b] where feedback acting only on a subset of the boundary is allowed and, most importantly, specific geometric conditions related to the analysis of the  $p$ -variable by multipliers are relaxed. This was achieved by the mean of another change of variable operating at the level of pseudodifferential calculus. In short, after transposing problem (3.1)-(3.7) to the half-space via partition of unity and truncating the solution with respect to the time variable, one defines a new variable  $w$  by

$$\mathcal{F}[w](\xi, \omega; x) = \lambda(\xi, \omega)\mathcal{F}[u](\xi, \omega; x), \quad \xi \in \mathbb{R}^{d-1}, \quad \omega \in \mathbb{R}, \quad x \geq 0, \quad (3.9)$$

where  $\mathcal{F}$  denotes the Fourier transform in both tangential and time variables and  $\lambda$  is a carefully constructed symbol. While transformations (3.8) and (3.9) are quite different in nature, both enable computations on variables with  $H^1(\Omega) \times L^2(\Omega)$ -regularity.

The closed-loop equations (3.1)-(3.7) constitute a natural Dirichlet counterpart to the wave equation with nonlinear Neumann boundary dissipation

$$\frac{\partial^2 u}{\partial t^2} - \Delta u = 0 \quad \text{in } \Omega \times (0, +\infty) \quad (3.10a)$$

$$\frac{\partial u}{\partial \nu} = -g\left(\frac{\partial u}{\partial t}\right) \quad \text{on } \Gamma_0 \times (0, +\infty), \quad (3.10b)$$

$$u|_{\Gamma} = 0 \quad \text{on } \Gamma_1 \times (0, +\infty), \quad (3.10c)$$

which, in contrast, have been extensively studied in the literature. To cite only a few, when the nonlinearity  $g$  has linear growth at infinity, *uniform* decay of the  $H^1(\Omega) \times L^2(\Omega)$ -energy of solutions to

(3.10) can be achieved, as in [Zuazua, 1990] or [Lasićka and Tataru, 1993] – see also [Komornik, 1994a] and the references therein, or more recently [Daoulatli et al., 2009]. In the one-dimensional settings, arguments based on Riemann invariants are available, and the decay of the energy can be analyzed via appropriate iterated sequences. See for instance [Chitour et al., 2021], where  $g$  is allowed to be a multivalued monotone mapping, or [Vancostenoble and Martinez, 2000], where it is proved, in particular, that exponential or polynomial uniform decay cannot be achieved when  $g$  represents a pointwise saturation mapping – see also [Prieur et al., 2016, Xu and Xu, 2019] for a stability analysis in the saturated case. Our objective in this chapter is to bridge the gap between Neumann and Dirichlet boundary conditions and extend the stability analysis of the closed-loop system (3.1)-(3.7) to the nonlinear case. As in Chapter 2, our main motivation behind the term  $g$  in the feedback loop is the problem of saturated stabilization, and one can think of the maps  $\text{sat}_s$  defined by (2.98) as prototype nonlinearities satisfying all the assumptions given below. Let us introduce a first set of hypotheses satisfied by the scalar mapping  $g$ .

*Hypothesis 3.1.1.* The (real) scalar mapping  $g$  satisfies the following properties:

- (i)  $g$  is globally Lipschitz continuous and monotone increasing;
- (ii)  $g(s) = 0$  if and only if  $s = 0$ .

## 3.2 Well-posedness and asymptotic stability

Our objective in this section is to establish well-posedness of the closed-loop equations (3.1)-(3.7) with initial data in  $\mathcal{H} = L^2(\Omega) \times H^{-1}(\Omega)$  as well as asymptotic stability of the zero equilibrium. Now, in comparison with Chapter 2 where initial data had  $H^1(\Omega) \times L^2(\Omega)$ -regularity, we have to deal with *very weak* solutions, and the operator-theoretic formulation of the evolution problem is more involved. In particular, we need fractional powers of the “positive Laplacian with homogeneous Dirichlet boundary conditions”  $A$ , which we introduce in Section 3.2.1. Well-posedness is then stated and proved in Section 3.2.2. Finally, asymptotic stability is established in Section 3.2.3.

### 3.2.1 Functional preliminaries

We start by defining a (symmetric) continuous bilinear form  $a$  on  $H_0^1(\Omega) \times H_0^1(\Omega)$ :

$$a(u_1, u_2) \triangleq \int_{\Omega} \nabla u_1 \cdot \nabla u_2 \, dx \quad \text{for all } u_1, u_2 \in H_0^1(\Omega). \quad (3.11)$$

The bilinear form  $a$  is coercive since  $\|\cdot\|_{H_0^1(\Omega)} = a(\cdot, \cdot)$ . By letting

$$\langle Au, w \rangle_{H_0^1(\Omega)} \triangleq a(u, w) \quad \text{for all } u, w \in H_0^1(\Omega), \quad (3.12)$$

we define a duality mapping  $A \in \mathcal{L}(H_0^1(\Omega), H^{-1}(\Omega))$ . By symmetry of  $a$ ,  $A$  is self-adjoint in the sense that

$$\langle Au_1, u_2 \rangle_{H_0^1(\Omega)} = \langle Au_2, u_1 \rangle_{H_0^1(\Omega)} \quad \text{for all } u_1, u_2 \in H_0^1(\Omega). \quad (3.13)$$

Note that because  $a$  is coercive, by virtue of the Lax-Milgram lemma,  $A^{-1}$  exists in  $\mathcal{L}(H^{-1}(\Omega), H_0^1(\Omega))$  (and *a fortiori* in  $\mathcal{L}(L^2(\Omega))$ ). At this point, following the ideas presented in Chapter 2, we could easily define  $H^1(\Omega) \times L^2(\Omega)$ -solutions to the wave equation with Neumann boundary control; for the Dirichlet problem on  $\mathcal{H}$ , however, we need to go further in the analysis. It is possible to see  $A$  as a linear unbounded operator on  $L^2(\Omega)$ . To do so, bearing in mind the embedding chain  $H_0^1(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow H^{-1}(\Omega)$ , we let

$$\mathcal{D}(A) \triangleq \{u \in H_0^1(\Omega) : Au \in L^2(\Omega)\}. \quad (3.14)$$

The domain  $\mathcal{D}(A)$  is equipped with the norm  $\|\cdot\|_{\mathcal{D}(A)}$  given by

$$\|u\|_{\mathcal{D}(A)} \triangleq \|Au\|_{L^2(\Omega)} \quad \text{for all } u \in \mathcal{D}(A). \quad (3.15)$$

Then,  $A$  is an isomorphism from  $\mathcal{D}(A)$  onto  $L^2(\Omega)$ . Besides, by using the coercivity of  $a$ , the Poincaré inequality, and Cauchy-Schwarz and Young inequalities, one obtains that  $\mathcal{D}(A)$  is continuously embedded into  $H_0^1(\Omega)$  and  $L^2(\Omega)$ . Furthermore, the domain  $\mathcal{D}(A)$  of  $A$  is dense in  $L^2(\Omega)$  – for instance, one can see that  $\mathcal{D}(\Omega) \subset \mathcal{D}(\mathcal{A})$  and  $\mathcal{D}(\Omega)$  is dense in  $L^2(\Omega)$ . Then, by symmetry of  $a$ , it is easy to see that  $A$  considered as an unbounded operator is self-adjoint, i.e.,  $\mathcal{D}(A^*) = \mathcal{D}(A)$  and  $A = A^*$ . Finally,

$$(Au, u)_{L^2(\Omega)} \geq \lambda \|u\|_{L^2(\Omega)}^2 \quad \text{for all } u \in \mathcal{D}(A), \quad (3.16)$$

where  $\lambda > 0$  comes from the Poincaré inequality, meaning that  $A$  is *strictly* positive.

**Elliptic regularity** By using test functions in  $\mathcal{D}(\Omega) \subset H_0^1(\Omega)$ , we see that  $\mathcal{D}(A)$  is the set of elements  $u$  of  $H_0^1(\Omega)$  such that the (distributional) Laplacian  $\Delta u$  is in fact a function in  $L^2(\Omega)$ . Then, recalling that  $\Omega$  is assumed to be smooth, elliptic regularity theory [Lions and Magenes, 1968] gives the following characterization:

$$\mathcal{D}(A) = H^2(\Omega) \cap H_0^1(\Omega), \quad (3.17)$$

with equivalence between the norm  $\|\cdot\|_{\mathcal{D}(A)}$  and the norm induced by  $H^2(\Omega)$ .

**Fractional powers** As a closed positive self-adjoint operator,  $A$  possesses fractional power  $A^s$ ,  $s \in \mathbb{R}$  [Yosida, 2012]. More precisely, there exists a family of unbounded operators  $A^s$  with dense domains  $\mathcal{D}(A^s) \subset L^2(\Omega)$ ,  $s \geq 0$ , such that, given  $s_1, s_2 \in \mathbb{R}$ ,  $A^{s_1+s_2}u = A^{s_1}A^{s_2}u = A^{s_2}A^{s_1}u$  for any  $u \in \mathcal{D}(A^s)$  where  $s = \max\{s_1, s_2, s_1 + s_2\}$ . Those are strictly positive and self-adjoint as well. Each  $\mathcal{D}(A^s)$  is a Hilbert space if equipped with the scalar product

$$(u_1, u_2)_{\mathcal{D}(A^s)} \triangleq (A^s u_1, A^s u_2)_{L^2(\Omega)} \quad \text{for all } u_1, u_2 \in \mathcal{D}(A^s). \quad (3.18)$$

Then, for  $s \geq 0$ ,  $A^s$  is an isomorphism from  $\mathcal{D}(A^s)$  onto  $L^2(\Omega)$ . For  $s > 0$ , after letting  $\mathcal{D}(A^{-s}) \triangleq \mathcal{D}(A^s)'$ , one proves that  $A^s$  can be extended by continuity as an isomorphism from  $L^2(\Omega)$  onto  $\mathcal{D}(A^{-s})$ , and the extension is denoted with the same symbol. Then, an equivalent norm on  $\mathcal{D}(A^{-s})$  is induced by the scalar product (3.18) where  $s$  is replaced by  $-s$ . The set family  $\{\mathcal{D}(A^s)\}_{s \in \mathbb{R}}$  is increasing (with respect to the inclusion) and for each  $s_1 \in \mathbb{R}$ ,  $\mathcal{D}(A^{s_1})$  is continuously embedded into every  $\mathcal{D}(A^{s_2})$ ,  $s_2 \leq s_1$ . Finally, for all  $s_1 \geq s_2$ ,  $A^{s_1-s_2}$  is an isomorphism from  $\mathcal{D}(A^{s_1})$  onto  $\mathcal{D}(A^{s_2})$ . We also recover

$$\mathcal{D}(A^{1/2}) = H_0^1(\Omega), \quad \text{and thus} \quad \mathcal{D}(A^{-1/2}) = H^{-1}(\Omega), \quad (3.19)$$

where the norms are the same in both equalities. For additional details and references on that matter, the reader is referred to [Temam, 1997, Section II.2.1], where inspiration for the presentation of this paragraph was taken from. The main properties that we shall use in the sequel can be summarized with the continuous embedding chain

$$\mathcal{D}(A) \hookrightarrow H_0^1(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow H^{-1}(\Omega) \hookrightarrow \mathcal{D}(A^{-1}) \quad (3.20)$$

and the isomorphism property

$$A \in \mathcal{L}(\mathcal{D}(A^s), \mathcal{D}(A^{s-1})), \quad A^{-1} \in \mathcal{L}(\mathcal{D}(A^s), \mathcal{D}(A^{s+1})), \quad s \in \mathbb{R}. \quad (3.21)$$

**Harmonic extension operator** Next, we introduce the Dirichlet map  $D$ . Let  $f \in H^{1/2}(\Gamma)$ ; there exists a unique  $Df \triangleq u \in H^1(\Omega)$  such that  $u|_{\Gamma} = f$  and  $\Delta u = 0$ . The linear operator  $D$  defined that way enjoys the continuity property

$$D \in \mathcal{L}(H^{1/2}(\Gamma), H^1(\Omega)). \quad (3.22)$$

In other words,  $D$  is a continuous right-inverse for the trace. In fact,  $D$  possesses extensions on fractional Sobolev spaces [Lions and Magenes, 1968], which we denote with the same symbol:

$$D \in \mathcal{L}(H^s(\Gamma), H^{s+1/2}(\Omega)) \quad \text{for all } s \in \mathbb{R}. \quad (3.23)$$

In particular,  $D$  continuously maps  $L^2(\Gamma)$  into  $L^2(\Omega)$ , allowing us to define the adjoint  $D^*$  with respect to these spaces:

$$(D^*u, f)_{L^2(\Gamma)} = (u, Df)_{L^2(\Omega)} \quad \text{for all } u \in L^2(\Omega) \text{ and } f \in L^2(\Gamma). \quad (3.24)$$

The adjoint  $D^*$  admits continuous extensions as well:

$$D^* \in \mathcal{L}(H^s(\Omega), H^{s+1/2}(\Gamma)) \quad \text{for all } s \in \mathbb{R}. \quad (3.25)$$

**Lemma 3.2.1.** *We have the following identity:*

$$-D^*Au = \frac{\partial u}{\partial \nu} \quad \text{for all } u \in \mathcal{D}(A). \quad (3.26)$$

*Proof.* This follows from the Green formula. Let  $u \in \mathcal{D}(A)$ . Let  $\varphi \in H^{1/2}(\Gamma)$  (say). Then,

$$\begin{aligned} (-D^*Au, \varphi)_{L^2(\Gamma)} &= (-Au, D\varphi)_{L^2(\Omega)} = \int_{\Omega} \Delta u D\varphi \, dx = - \int_{\Omega} \nabla u \cdot \nabla [D\varphi] \, dx + \int_{\Gamma} \frac{\partial u}{\partial \nu} [D\varphi]_{|\Gamma} \, d\sigma \\ &= \int_{\Gamma} \frac{\partial u}{\partial \nu} \varphi \, d\sigma, \end{aligned} \quad (3.27)$$

where we used that  $[D\varphi]_{|\Gamma} = \varphi$  and  $\Delta [D\varphi] = 0$  by definition of  $D$ . Since  $\varphi$  is chosen arbitrarily in  $H^{1/2}(\Gamma)$ , which is dense in  $L^2(\Gamma)$ , the result is proved.  $\square$

### 3.2.2 Well-posedness and additional properties of the semigroup

With a little abuse of notation, we denote by  $g$  the Lipschitz mapping on  $L^2(\Gamma)$  defined by

$$g(f)(\cdot) \triangleq g(f(\cdot)) \quad \text{for all } f \in L^2(\Gamma) \quad (3.28)$$

We also define a projection operator  $P \in L^2(\Gamma)$  by

$$[Pf](\cdot) = \mathbb{1}_{\Gamma_0}(\cdot) f(\cdot) \quad \text{for all } f \in L^2(\Gamma). \quad (3.29)$$

Equation (3.26) in Lemma 3.2.1 shows that the boundary conditions given by (3.1b) and (3.1c) together with the feedback law (3.7) can be rewritten as follows:

$$u|_{\Gamma} = -Pg(D^*u'). \quad (3.30)$$

Next, we introduce the nonlinear operator  $\mathcal{A}$  associated with the closed-loop system (3.1)-(3.7). Bearing in mind the chain of embeddings (3.20) and that  $A$  maps  $L^2(\Omega)$  onto  $\mathcal{D}(A^{-1})$ , we define  $\mathcal{D}(\mathcal{A})$  by

$$\begin{aligned} \mathcal{D}(\mathcal{A}) &\triangleq \{[u, v] \in \mathcal{H} : v \in L^2(\Omega), A[u + DPg(D^*v)] \in H^{-1}(\Omega)\} \\ &= \{[u, v] \in \mathcal{H} : v \in L^2(\Omega), u + DPg(D^*v) \in H_0^1(\Omega)\}. \end{aligned} \quad (3.31)$$

Next, we let

$$\mathcal{A}[u, v] \triangleq [-v, Au + ADPg(D^*v)] \quad (3.32)$$

In the sequel, we employ the standard nonlinear semigroup terminology that has been introduced in Chapter 2: by a *strong* solution to (3.1)-(3.7), we mean any  $[u, u'] \in W^{1,1}(0, T; \mathcal{H})$ ,  $T > 0$ , that satisfies  $[u(t), u'(t)] \in \mathcal{D}(\mathcal{A})$  for all  $t \geq 0$  and

$$\frac{d}{dt}[u, u'] + \mathcal{A}[u, u'] = 0 \quad \text{a.e.} \quad (3.33)$$

in the sense of strong differentiation in  $\mathcal{H}$ ; by a *generalized* solution to (3.1)-(3.7), we mean a continuous  $\mathcal{H}$ -valued function  $[u, v]$  that is, on each interval  $[0, T]$ , the uniform limit of some sequence of strong solutions.

**Theorem 3.2.2** (Hadamard well-posedness). *The nonlinear operator  $\mathcal{A}$  is densely defined and maximal monotone. Thus,  $-\mathcal{A}$  is the infinitesimal generator of a strongly continuous semigroup  $\{\mathcal{S}_t\}$  of contractions on the energy space  $\mathcal{H}$ . For all initial data  $[u_0, v_0]$  in  $\mathcal{H}$ , there exists a unique generalized solution  $[u, u'] \in \mathcal{C}(\mathbb{R}^+, \mathcal{H})$  to (3.1)-(3.7). If  $[u_0, v_0]$  belongs to  $\mathcal{D}(\mathcal{A})$ , then  $[u, u']$  is a strong solution to (3.1)-(3.7). Furthermore,*

(i) *Strong solutions satisfy the inequality*

$$\|\mathcal{A}[u(t), u'(t)]\|_{\mathcal{H}} \leq \|\mathcal{A}[u_0, v_0]\|_{\mathcal{H}} \quad \text{for all } t \geq 0; \quad (3.34)$$

(ii) *Strong solutions satisfy the energy identity*

$$\frac{d}{dt}\mathcal{E}(u, u') = - \int_{\Gamma_0} g(D^*u')D^*u' d\sigma = \int_{\Gamma_0} g\left(-\frac{\partial[A^{-1}u']}{\partial\nu}\right) \frac{\partial[A^{-1}u']}{\partial\nu} d\sigma \quad (3.35)$$

*in the scalar distribution sense on  $(0, +\infty)$ .*

*Remark 3.2.3.* If we also assume that  $|g(s)| \geq \alpha|s|$  for all  $s \in \mathbb{R}$  and some  $\alpha > 0$ , the energy identity (3.35) provides a uniform estimate of the  $L^2(0, +\infty; L^2(\Gamma_0))$ -norm of  $\partial_\nu[A^{-1}u']$  for strong solutions. From there, one can prove that (3.35) holds for generalized solution as well by passing to the limit and recovering the traces  $u|_\Gamma$  and  $\partial_\nu[A^{-1}u']$  in  $L^2(0, +\infty; L^2(\Gamma_0))$  – see for instance [Chueshov et al., 2002] for similar arguments in the Neumann case.

*Proof of Theorem 3.2.2.* Once proven that  $\mathcal{A}$  is maximal monotone, existence and uniqueness of strong and generalized solutions to (3.1)-(3.7), together with the appropriate semigroup properties, follow from Kato's theorem and nonlinear semigroup theory – see, e.g., [Showalter, 2013, Chapter IV].

*Step 1: Monotonicity.* Let  $[u_1, v_1]$  and  $[u_2, v_2]$  in  $\mathcal{D}(\mathcal{A})$ . Then,

$$\begin{aligned} & (\mathcal{A}[u_1, v_1] - \mathcal{A}[u_2, v_2], [u_1, v_1] - [u_2, v_2])_{\mathcal{H}} \\ &= -(v_1 - v_2, u_1 - u_2)_{L^2(\Omega)} + (A^{1/2}[u_1 - u_2 + DPg(D^*v_1) - DPg(D^*v_2)], A^{-1/2}[v_1 - v_2])_{L^2(\Omega)}. \end{aligned} \quad (3.36)$$

Now we use that  $A^{-1/2}[v_1 - v_2]$  belongs to  $\mathcal{D}(A^{1/2})$  and that  $A^{1/2}$  is self-adjoint to obtain

$$\begin{aligned} & (\mathcal{A}[u_1, v_1] - \mathcal{A}[u_2, v_2], [u_1, v_1] - [u_2, v_2])_{\mathcal{H}} = (DPg(D^*v_1) - DPg(D^*v_2), v_1 - v_2)_{L^2(\Omega)} \\ &= (Pg(D^*v_1) - Pg(D^*v_2), D^*v_1 - D^*v_2)_{L^2(\Gamma)} \\ &= (g(D^*v_1) - g(D^*v_2), D^*v_1 - D^*v_2)_{L^2(\Gamma_0)} \geq 0, \end{aligned} \quad (3.37)$$

the right-hand side being nonnegative by  $g$  being monotone increasing, which proves that  $\mathcal{A}$  is monotone.

*Step 2: Range condition.* Let  $\lambda > 0$  and  $[f_1, f_2] \in \mathcal{H}$ . To solve the equation  $\mathcal{A}[u, v] + \lambda[u, v] = [f_1, f_2]$ , it suffices to find  $v \in L^2(\Omega)$  such that

$$\lambda^{-1}v + DPg(D^*v) + \lambda A^{-1}v = A^{-1}f_2 - \lambda^{-1}f_1. \quad (3.38)$$

This is seen by substituting  $-v + \lambda u = f_1$  into the second coordinate of the equation and applying  $A^{-1}$  to the result. If such an element  $v \in L^2(\Omega)$  is found, then  $[u, v]$  belongs to  $\mathcal{D}(\mathcal{A})$  (and solves the desired equation). Indeed, we then have  $u + DPg(D^*v) + \lambda A^{-1}v = A^{-1}f_2$ , which implies that  $u + DPg(D^*v) \in H_0^1(\Omega)$  since  $A^{-1}f_2$  and  $\lambda A^{-1}v$  both belong to  $H_0^1(\Omega)$ .

The method is the same as in Chapter 2. We define a nonlinear operator  $\Theta$  on  $L^2(\Omega)$  by  $\Theta(v) \triangleq \lambda^{-1}v + DPg(D^*v) + \lambda A^{-1}v$  for all  $v \in L^2(\Omega)$ . Then,  $\Theta$  enjoys the following properties:

- (i)  $\Theta$  maps bounded sets into bounded sets;
- (ii)  $(\Theta(v_1) - \Theta(v_2), v_1 - v_2)_{L^2(\Omega)} \geq 0$  for all  $v_1$  and  $v_2$  in  $L^2(\Omega)$ ;
- (iii) The scalar function  $t \mapsto (\Theta(v_1 + tv_2), v_2)_{L^2(\Omega)}$  is continuous for all  $v_1$  and  $v_2$  in  $L^2(\Omega)$ .

Also, we have

$$(\Theta(v), v)_{L^2(\Omega)} \geq \lambda^{-1} \|v\|_{L^2(\Omega)}^2 \quad \text{for all } v \in L^2(\Omega). \quad (3.39)$$

Thus, it follows from [Showalter, 2013, Chapter II, Lemma 2.1 and Theorem 2.1] that  $\Theta$  is onto. Consequently, the equation  $\mathcal{A}[u, v] + \lambda[u, v] = [f_1, f_2]$  has a solution in  $\mathcal{D}(\mathcal{A})$ .

*Step 3: Denseness of the domain.* Let  $[u, v] \in \mathcal{H}$  and  $\varepsilon > 0$ . Since  $A^{-1}v \in H_0^1(\Omega)$  and  $\mathcal{D}(\Omega)$  is dense in  $H_0^1(\Omega)$ , we can pick  $\phi \in \mathcal{D}(\Omega)$  such that

$$\|A^{-1}v - \phi\|_{H_0^1(\Omega)}^2 \leq \varepsilon, \quad \text{and thus } \|v - A\phi\|_{H^{-1}(\Omega)}^2 \leq C\varepsilon, \quad (3.40)$$

where  $C > 0$  comes from  $A \in \mathcal{L}(H_0^1(\Omega), H^{-1}(\Omega))$ . Besides, there exists  $\psi \in \mathcal{D}(\Omega)$  such that  $\|u - \psi\|_{L^2(\Omega)}^2 \leq \varepsilon$ . Since  $\phi \in \mathcal{D}(\Omega) \subset \mathcal{D}(A)$ , we have  $A\phi \in L^2(\Omega)$  and also, using (3.26),

$$g(D^*A\phi) = g\left(-\frac{\partial\phi}{\partial\nu}\right) = 0. \quad (3.41)$$

Thus,  $[\psi, A\phi] \in \mathcal{D}(\mathcal{A})$ ; also, we have  $\|[u, v] - [\psi, A\phi]\|_{\mathcal{H}}^2 \leq (1 + C)\varepsilon$ . It is now proved that  $\mathcal{D}(\mathcal{A})$  is dense in  $\mathcal{H}$ .

*Step 4: Energy identity.* Let  $[u, v]$  be a strong solution to (3.1)-(3.7). We recall that  $\mathcal{E}(u, u') = (1/2)\|[u, u']\|_{\mathcal{H}}^2$ . Consequently, by the chain rule,  $\mathcal{E}(u, u')$  belongs to  $W^{1,\infty}(0, +\infty)$  and

$$\frac{d}{dt}\mathcal{E}(u, u') = (-\mathcal{A}[u, u'], [u, u'])_{\mathcal{H}} \quad \text{a.e.} \quad (3.42)$$

Thus, the desired identity (3.35) follows from (3.37).  $\square$

Now, we establish some compactness and regularity properties that are useful in the proof of the stability results presented in Sections 3.2.3 and 3.3.1. We start by introducing the following proposition, which enables us to prove asymptotic stability of the feedback system (3.1)-(3.7) using LaSalle's invariance principle.

**Proposition 3.2.4** (Compactness). *For any  $\lambda > 0$ , the (nonlinear) resolvent operator  $(\mathcal{A} + \lambda \text{id})^{-1}$  is well-defined on  $\mathcal{H}$  and compact. In particular, for all initial data  $[u_0, v_0] \in \mathcal{H}$ , the (semi)trajectory  $\{\mathcal{S}_t[u_0, v_0]\}_{t \geq 0}$  is relatively compact in  $\mathcal{H}$ .*

*Proof.* Assume for a moment that  $(\mathcal{A} + \lambda \text{id})^{-1}$  is well-defined and compact for some  $\lambda > 0$ . Then, since  $\mathcal{A}(0) = 0$ , relative compactness of the trajectories follows from [Dafermos and Slemrod, 1973, Theorem 3].

Let  $\lambda > 0$ . We already know from the proof of Theorem 3.2.2 that the equation

$$\mathcal{A}[u, v] = [f_1, f_2] \quad (3.43)$$

has a solution in  $\mathcal{D}(\mathcal{A})$  for all  $[f_1, f_2] \in \mathcal{H}$ .

*Step 1: Uniqueness.* Consider two solutions  $[u_1, v_1]$  and  $[u_2, v_2]$  to (3.43). Then, we recall from (3.38) in the proof of Theorem 3.2.2 that

$$\lambda^{-1}[v_1 - v_2] + DP[g(D^*v_1) - g(D^*v_2)] + \lambda A^{-1}[v_1 - v_2] = 0. \quad (3.44)$$

Taking the scalar product in  $L^2(\Omega)$  of (3.44) with  $v_1 - v_2$  yields

$$\lambda^{-1}\|v_1 - v_2\|_{L^2(\Omega)}^2 + (g(D^*v_1) - g(D^*v_2), D^*v_1 - D^*v_2)_{L^2(\Gamma_0)} + \lambda\|v_1 - v_2\|_{H^{-1}(\Omega)}^2 = 0. \quad (3.45)$$

In particular, since  $g$  is monotone increasing, we infer from (3.45) that  $v_1 = v_2$ ; thus,  $[u_1, v_1] = [u_2, v_2]$  and  $(\mathcal{A} + \lambda \text{id})^{-1}$  is well-defined.

*Step 2: Compactness of the resolvent operator:* In what follows, we let  $[u, v] \triangleq (\mathcal{A} + \lambda \text{id})^{-1}[f_1, f_2]$  and we look for estimates of  $[u, v] \in \mathcal{D}(\mathcal{A})$  in stronger norms. First, as in the previous step, we obtain

$$\lambda^{-1}\|v\|_{L^2(\Omega)}^2 + (g(D^*v), D^*v)_{L^2(\Gamma_0)} + \lambda\|v\|_{H^{-1}(\Omega)}^2 = (A^{-1/2}f_2, A^{-1/2}v)_{L^2(\Omega)} - \lambda^{-1}(f_1, v)_{L^2(\Omega)}, \quad (3.46)$$

where it is used that  $A^{-1/2}$  is self-adjoint. Recalling that  $(g(D^*v), D^*v)_{L^2(\Gamma_0)} \geq 0$ , we use Cauchy-Schwarz and Young inequalities with positive parameters  $\varepsilon_1$  and  $\varepsilon_2$  to infer from (3.46) that

$$\frac{1}{\lambda}\|v\|_{L^2(\Omega)}^2 + \lambda\|v\|_{H^{-1}(\Omega)}^2 \leq \frac{1}{2\varepsilon_1}\|f_2\|_{H^{-1}(\Omega)}^2 + \frac{\varepsilon_1}{2}\|v\|_{H^{-1}(\Omega)}^2 + \frac{1}{2\varepsilon_2\lambda}\|f_1\|_{L^2(\Omega)}^2 + \frac{\varepsilon_2}{2\lambda}\|v\|_{L^2(\Omega)}^2. \quad (3.47)$$

Letting  $\varepsilon_1 = 2\lambda$  and  $\varepsilon_2 = 1$  in (3.47) yields

$$\|v\|_{L^2(\Omega)}^2 \leq \frac{1}{2}\|f_2\|_{H^{-1}(\Omega)}^2 + \|f_1\|_{L^2(\Omega)}^2. \quad (3.48)$$

The remainder of the proof relies on elliptic regularity theory and in particular [Lions and Magenes, 1968, Théorème 10.1]. Since  $[u, v] \in \mathcal{D}(\mathcal{A})$ , we know that  $u + DPg(D^*v)$  belongs to  $H_0^1(\Omega)$  and

$$u + DPg(D^*v) = A^{-1}f_2 - \lambda A^{-1}v. \quad (3.49)$$

Picking an arbitrary test function  $\phi$  in  $\mathcal{D}(\Omega) \subset \mathcal{D}(A)$ , taking the scalar product in  $L^2(\Omega)$  of (3.49) with  $A\phi$  and using again that  $D^*A\phi = -\partial_\nu\phi = 0$  leads to

$$-\Delta u = f_2 - \lambda v \quad \text{in } \mathcal{D}'(\Omega). \quad (3.50)$$

Besides, since  $-\Delta u \in H^{-1}(\Omega)$ ,  $u|_\Gamma$  is well-defined in  $H^{-1/2}(\Gamma)$ ; then, we infer from  $u + DPg(D^*v) \in H_0^1(\Omega)$  that  $u|_\Gamma = -Pg(D^*v) \in L^2(\Gamma)$ . Applying the aforementioned theorem, we obtain  $u \in H^{1/2}(\Omega)$  along with the estimate

$$\|u\|_{H^{1/2}(\Omega)}^2 \leq C_1 \left\{ \|Pg(D^*v)\|_{L^2(\Gamma)}^2 + \|f_2 - \lambda v\|_{H^{-1}(\Omega)}^2 \right\} \quad (3.51)$$

where  $C_1 > 0$  is solution independent. Since  $P$  and  $g$  are Lipschitz continuous on  $L^2(\Gamma)$ ,  $g(0) = 0$  and  $D^*$  is linear continuous from  $L^2(\Omega)$  into  $L^2(\Gamma)$ , plugging (3.48) into (3.51) yields

$$\|u\|_{H^{1/2}(\Omega)}^2 \leq C_1 C_2 \|v\|_{L^2(\Omega)}^2 + 2C_1 \lambda^2 \|v\|_{H^{-1}(\Omega)}^2 + 2C_1 \|f_2\|_{H^{-1}(\Omega)}^2, \quad (3.52)$$

where  $C_2 > 0$  is some other constant.

Combining (3.48) and (3.52), we see that  $(\mathcal{A} + \lambda \text{id})^{-1}$  maps bounded sets of  $\mathcal{H} = L^2(\Omega) \times H^{-1}(\Omega)$  into bounded sets of  $H^{1/2}(\Omega) \times L^2(\Omega)$ , the latter being compactly embedded into  $\mathcal{H}$ . Thus, the result is proved.  $\square$

The next proposition is meant for use in Section 3.3, where we work under additional assumptions on  $\Omega$ ; however, since it is a direct continuation of the proof of Proposition 3.2.4, we introduce it here.

**Proposition 3.2.5** (Regularity). *Suppose that  $\overline{\Gamma_0} \cap \overline{\Gamma_1} = \emptyset$ . Then, the following explicit characterization of  $\mathcal{D}(\mathcal{A})$  holds:*

$$\mathcal{D}(\mathcal{A}) = \{[u, v] \in \mathcal{H} : v \in L^2(\Omega), u \in H^1(\Omega), u|_{\Gamma} = -\mathbb{1}_{\Gamma_0} g(D^*v)\}. \quad (3.53)$$

Thus, strong solutions  $[u, u']$  take values in  $H^1(\Omega) \times L^2(\Omega)$ . Furthermore, there exists a constant  $K > 0$  such that any strong solution to (3.1)-(3.7) satisfies

$$\|[u(t), u'(t)]\|_{H^1(\Omega) \times L^2(\Omega)} \leq K \|[u(0), u'(0)]\|_{\mathcal{H}} \quad \text{for all } t \geq 0. \quad (3.54)$$

*Remark 3.2.6.* Since  $H^1(\Omega) \times L^2(\Omega)$  is continuously embedded into  $\mathcal{H}$ , it follows from (3.54) evaluated at  $t = 0$  that for some constant  $K' > 0$ ,

$$\mathcal{E}(u_0, v_0) \leq K' \|[u_0, v_0]\|_{\mathcal{H}}^2 \quad \text{for all } [u_0, v_0] \in \mathcal{D}(\mathcal{A}). \quad (3.55)$$

*Proof of Proposition 3.2.5.* Let  $u \in H^1(\Omega)$  and  $v \in L^2(\Omega)$  such that

$$u|_{\Gamma} = -\mathbb{1}_{\Gamma_0} g(D^*v) = -Pg(D^*v). \quad (3.56)$$

By trace regularity,  $Pg(D^*v) \in H^{1/2}(\Gamma)$ , and by (3.23),  $DPg(D^*v) \in H^1(\Omega)$ . It follows that  $u + DPG(D^*v) \in H_0^1(\Omega)$ , i.e.,  $[u, v] \in \mathcal{D}(\mathcal{A})$ .

Conversely, let  $[u, v] \in \mathcal{D}(\mathcal{A})$ . Recalling calculations made in Proposition 3.2.4, we already know that  $[u, v]$  must satisfy  $-\Delta u \in H^{-1}(\Omega)$  and  $u|_{\Gamma} = -Pg(D^*v)$ . Therefore, in comparison with the proof of Proposition 3.2.4, it suffices to show that  $u|_{\Gamma}$  belongs to  $H^{1/2}(\Gamma)$  instead of  $L^2(\Gamma)$  and apply the elliptic regularity theorem to gain the desired extra half-unit of regularity. By virtue of (3.23), we have  $D^*v \in H^{1/2}(\Gamma)$ .

First, recall<sup>1</sup> that pointwise Lipschitz nonlinearities such as  $g$  map bounded sets of  $H^{1/2}(\Gamma)$  into bounded sets of  $H^{1/2}(\Gamma)$ . Indeed, using the definition of Sobolev spaces on manifold by local charts and the Sobolev-Slobodeckij characterization of the fractional spaces  $H^s(\mathbb{R}^{d-1})$  (see [Di Nezza et al., 2012]), we know that for a given  $f$  in  $H^{1/2}(\Gamma)$ ,  $g \in H^{1/2}(\Gamma)$  if and only if

$$\iint_{\mathbb{R}^{d-1} \times \mathbb{R}^{d-1}} \frac{|\phi_i(x_1)g([f \circ \psi_i](x_1)) - \phi_i(x_2)g([f \circ \psi_i](x_2))|^2}{\|x_1 - x_2\|^d} dx_1 dx_2 < +\infty, \quad (3.57)$$

for all suitable  $(\phi_i, \psi_i)$ , where the functions  $\phi_i \in \mathcal{D}(\mathbb{R}^{d-1})$  are chosen from a partition of unity subordinate to some (finite) covering of  $\Gamma$  and the functions  $\psi_i$  are corresponding local representations of

<sup>1</sup> This fact seems to be commonly used in the literature; however we did not find a proof, so we give one here for the reader's convenience.



the surface. The integral term in (3.57) is finite because  $g$  and  $\phi_i$  are globally Lipschitz continuous; hence,  $g(f) \in H^{1/2}(\Omega)$ . Furthermore, taking the integral term in (3.57) plus some appropriate lower-order  $L^2$ -term defines a norm on  $H^{1/2}(\mathbb{R}^{d-1})$  equivalent to the one given by interpolation. Thus, after coming back to functions on  $\Gamma$ , it follows from (3.57) that

$$\|g(f)\|_{H^{1/2}(\Gamma)} \leq K \|f\|_{H^{1/2}(\Gamma)} \quad \text{for all } f \in H^{1/2}(\Gamma), \quad (3.58)$$

where  $K$  is some positive constant coming from the Lipschitz continuity of  $g$  and norm equivalence.

Next we have to check that  $P \in \mathcal{L}(H^{1/2}(\Gamma))$ . Again, this is a consequence of (3.57): we observe that since  $\bar{\Gamma}_0 \cap \bar{\Gamma}_1 = \emptyset$ , there exists  $m > 0$  such that  $\|x_1 - x_2\| > m$  whenever  $(\psi_i(x_1), \psi_i(x_2)) \in [\Gamma_0 \times \Gamma_1] \cup [\Gamma_1 \times \Gamma_0]$ .

Finally, combining (3.23) for  $s = 1/2$ , the estimate (3.58), the fact that  $P \in \mathcal{L}(H^{1/2}(\Gamma))$  together with the elliptic regularity theorem, we obtain  $u \in H^1(\Omega)$  and the stronger estimate

$$\begin{aligned} \|u\|_{H^1(\Omega)} &\leq C \{ \|\Delta u\|_{H^{-1}(\Omega)} + \|v\|_{L^2(\Omega)} \} \\ &= C \{ \|A[u + DPg(D^*v)]\|_{H^{-1}(\Omega)} + \|v\|_{L^2(\Omega)} \} \leq C' \|\mathcal{A}[u, v]\|_{\mathcal{H}} \end{aligned} \quad (3.59)$$

where  $C$  and  $C'$  are some positive constants that do not depend on  $[u, v]$ . The set equality in (3.53) is now proved and the property (3.54) readily follows from (3.59) and (3.34).  $\square$

### 3.2.3 Asymptotic stability

Next, we state the second main result of the section, which asserts that the zero equilibrium of the closed-loop system (3.1)-(3.7) is globally asymptotically stable.

**Theorem 3.2.7** (Asymptotic stability of the closed-loop system). *Let  $[u_0, v_0] \in \mathcal{H}$ . Then,*

$$\|\mathcal{S}_t[u_0, v_0]\|_{\mathcal{H}} \rightarrow 0 \quad \text{as } t \rightarrow +\infty. \quad (3.60)$$

*Together with the contraction property of  $\{\mathcal{S}_t\}$ , (3.60) implies that 0 is a globally asymptotically stable equilibrium point for the feedback system (3.1)-(3.7).*

*Proof.* By the contraction property of the semigroup  $\{\mathcal{S}_t\}$  and denseness of  $\mathcal{D}(\mathcal{A})$  in  $\mathcal{H}$ , it suffices to prove (3.60) for initial data  $[u_0, v_0]$  in  $\mathcal{D}(\mathcal{A})$ .

To do so, we use a Lasalle-type invariance approach. Let us recall the classical line of arguments. We consider the  $\omega$ -limit set  $\omega([u_0, v_0])$  of  $[u_0, v_0]$ , which can be characterized as follows:  $[w_0, z_0] \in \mathcal{H}$  belongs to  $\omega([u_0, v_0])$  if there exists an increasing sequence  $\{t_n\} \in \mathbb{R}^{\mathbb{N}}$  such that  $t_n \rightarrow +\infty$  and

$$\mathcal{S}_{t_n}[u_0, v_0] \rightarrow [w_0, z_0] \quad \text{in } \mathcal{H} \text{ as } n \rightarrow +\infty. \quad (3.61)$$

Recall that  $\{\mathcal{S}_t[u_0, v_0]\}_{t \geq 0}$  is relatively compact in  $\mathcal{H}$ . Therefore,  $\omega([u_0, v_0])$  is a nonempty (positively) invariant compact set, and  $\text{dist}(\mathcal{S}_t[u_0, v_0], \omega([u_0, v_0])) \rightarrow 0$  as  $t \rightarrow +\infty$  – see [Haraux, 1991, Théorème 1.1.8]. Moreover, since  $t \mapsto \|\mathcal{A}(\mathcal{S}_t[u_0, v_0])\|_{\mathcal{H}}$  is bounded, it follows from [Crandall and Pazy, 1969, Lemma 2.3] and (3.61) that  $\omega([u_0, v_0]) \subset \mathcal{D}(\mathcal{A})$ . Besides, since  $\mathcal{E}(\mathcal{S}_t[u_0, v_0])$  is bounded and monotone decreasing with respect to  $t$ , it must converge to some  $\mathcal{E}_\infty \geq 0$  as  $t$  goes to  $+\infty$ . By (3.61) and continuity of  $\mathcal{E}$ , we have  $\mathcal{E}(w_0, z_0) = \mathcal{E}_\infty$  for any  $[w_0, z_0] \in \omega([u_0, v_0])$ .

The remainder consists in proving that  $\omega([u_0, v_0])$  is reduced to  $\{0\}$ . Let  $[w_0, z_0] \in \omega([u_0, v_0])$ ; we write  $[w(t), w'(t)] = \mathcal{S}_t[w_0, z_0]$  and we notice that  $\mathcal{E}(w(t), w'(t)) = \mathcal{E}_\infty$  for all  $t \geq 0$ . Furthermore,  $[w, w']$  is a strong solution to (3.1)-(3.7) and we infer from the energy identity (3.35) that

$$\int_0^\tau \int_{\Gamma_0} g \left( -\frac{\partial[A^{-1}w']}{\partial \nu} \right) \frac{\partial[A^{-1}w']}{\partial \nu} d\sigma dt = 0 \quad \text{for all } \tau \geq 0. \quad (3.62)$$

It follows from Hypothesis 3.1.1 that  $g(s)s > 0$  for every nonzero  $s$ . Thus, letting  $p \triangleq A^{-1}w' \in \mathcal{C}(\mathbb{R}^+, H_0^1(\Omega)) \cap L^\infty(0, +\infty; \mathcal{D}(A))$ , (3.62) leads to

$$\frac{\partial p}{\partial \nu} = 0 \quad \text{a.e. on } \Gamma_0 \times (0, +\infty). \quad (3.63)$$

Next, we recall that  $w' \in W^{1,\infty}(0, +\infty; H^{-1}(\Omega))$  and, using (3.63) together with the operator-theoretic formulation of (3.1)-(3.7), we obtain  $w'' + Aw = 0$ . Hence,  $p \in W^{1,\infty}(0, +\infty; H_0^1(\Omega))$  satisfies  $p' + w = 0$ , which in turn implies that  $p \in W^{2,\infty}(0, +\infty; H^{-1}(\Omega))$  and solves  $p'' + Ap = 0$  in  $H^{-1}(\Omega)$ , i.e., the standard variational formulation of the wave equation with homogeneous Dirichlet boundary conditions. In particular,  $p \in \mathcal{C}(\mathbb{R}^+, H_0^1(\Omega)) \cap \mathcal{C}^1(\mathbb{R}^+, L^2(\Omega))$  and solves the following boundary value problem:

$$\frac{\partial^2 p}{\partial t^2} - \Delta p = 0 \quad \text{in } \Omega \times (0, +\infty), \quad (3.64a)$$

$$\frac{\partial p}{\partial \nu} = 0 \quad \text{on } \Gamma_0 \times (0, +\infty), \quad (3.64b)$$

$$p|_\Gamma = 0 \quad \text{on } \Gamma \times (0, +\infty). \quad (3.64c)$$

We see that  $p$  has vanishing Cauchy data on  $\Gamma_0$ . The subset  $\Gamma_0$  being relatively open in  $\Gamma$ , a unique continuation argument for waves yields  $p = 0$  – for instance, one can directly apply [Robbiano, 1991, Théorème 2]. Therefore,  $w' = 0$ ,  $Aw = 0$  and finally  $w = 0$ , which concludes the proof.  $\square$

### 3.3 Polynomial decay rates for strong solutions

This section is dedicated to the analysis of the decay rate of strong solutions under additional assumptions on the feedback nonlinearity and the geometry of the problem.

#### 3.3.1 Statement of the result and outline of the proof

In what follows, we work under stronger assumptions that are given next.

*Hypothesis 3.3.1.* There exist positive constants  $S$ ,  $\alpha_1$  and  $\alpha_2$  such that

$$\alpha_1 |s| \leq |g(s)| \leq \alpha_2 |s| \quad \text{for all } |s| \leq S. \quad (3.65)$$

*Hypothesis 3.3.2.* The domain  $\Omega \subset \mathbb{R}^d$  with smooth boundary  $\Gamma = \Gamma_0 \cup \Gamma_1$  satisfies the following conditions:

1. The boundary is such that

$$\overline{\Gamma_0} \cap \overline{\Gamma_1} = \emptyset; \quad (3.66)$$

2. There exists a point  $x_0 \in \mathbb{R}^d$  such that, setting  $h(x) \triangleq x - x_0$ ,

$$h \cdot \nu \leq 0 \quad \text{on } \Gamma_1. \quad (3.67)$$

As mentioned earlier, the class of nonlinearities  $g$  satisfying (3.65) includes the saturation maps  $\text{sat}_S$  introduced in (2.98). Then, we can estimate the decay rate of each strong solution.

**Theorem 3.3.3** (Non-uniform polynomial decay rate). *Let  $r \geq \max\{d-1, 2\}$ . Under Hypotheses 3.3.1 and 3.3.2, strong solutions  $[u, u']$  to (3.1)-(3.7) satisfy*

$$\mathcal{E}(u(t), u'(t)) \leq C_u t^{-2/(r-1)} \quad \text{for all } t \geq 0, \quad (3.68)$$

where  $C_u$  is a positive constant depending only on  $\mathcal{E}(u_0, v_0)$  and  $\|\mathcal{A}[u_0, v_0]\|_{\mathcal{H}}$ .

Theorem 3.3.3 is a Dirichlet counterpart to non-uniform polynomial decay results that are well-known in the case of Neumann boundary conditions – see, e.g., [Komornik, 1994a, Theorem 9.10]. Let us introduce the following notation: if  $[u, u']$  is a given solution to (3.1)-(3.7), we define a (continuous) function  $\mathcal{E}_u$  over  $\mathbb{R}^+$  by

$$\mathcal{E}_u(t) \triangleq \mathcal{E}(u(t), u'(t)). \quad (3.69)$$

Here, polynomial decay rate is obtained by applying the following classical lemma to the (monotone decreasing) energy  $\mathcal{E}_u$  of each solution.

**Lemma 3.3.4.** *Let  $E : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  be a monotone decreasing function. Assume that there exist two positive constants  $\gamma$  and  $T$  such that*

$$\int_{\tau}^{+\infty} E^{\gamma+1} dt \leq TE(0)^{\gamma}E(\tau) \quad \text{for all } \tau \geq 0. \quad (3.70)$$

Then,

$$E(t) \leq E(0) \left( \frac{T + \gamma t}{T + \gamma T} \right)^{-1/\gamma} \quad \text{for all } t \geq T. \quad (3.71)$$

The reader is referred to [Komornik, 1994a, Theorem 9.1] for a proof of the lemma. We already know from Section 3.2 that  $\mathcal{E}_u(t)$  converges to 0 as  $t$  goes to  $+\infty$ . Our subsequent efforts focus on estimating

$$\int_{\tau_1}^{\tau_2} \mathcal{E}_u^{(r+1)/2} dt \quad \text{for arbitrary } 0 \leq \tau_1 \leq \tau_2, \quad (3.72)$$

where we recall that

$$\mathcal{E}_u(t) = \frac{1}{2} \|u\|_{L^2(\Omega)}^2 + \frac{1}{2} \|u'\|_{H^{-1}(\Omega)}^2. \quad (3.73)$$

As mentioned earlier, the proof is based on an analysis of the variable  $p$  defined by

$$p = A^{-1}u' \quad (3.74)$$

which solves, at least formally, the following boundary-value problem:

$$\frac{\partial^2 p}{\partial t^2} - \Delta p = -\frac{\partial}{\partial t} \left\{ DPg \left( -\frac{\partial p}{\partial \nu} \right) \right\} \quad \text{in } \Omega \times (0, +\infty), \quad (3.75a)$$

$$p|_{\Gamma} = 0 \quad \text{on } \Gamma \times (0, +\infty). \quad (3.75b)$$

If  $u'$  takes values in  $L^2(\Omega)$ , recalling the formula (3.26), we have

$$-\frac{\partial p}{\partial \nu} = -\frac{\partial[A^{-1}u']}{\partial \nu} = D^*Ap = D^*u'. \quad (3.76)$$

To alleviate notation, in the sequel we denote by  $\Phi$  the term

$$\Phi(t) \triangleq DPg(D^*u'(t)). \quad (3.77)$$

In regards to (3.30), we see that  $-\Phi$  is the harmonic extension of the trace  $u|_{\Gamma}$ . As mentioned earlier, the  $p$ -variable is smoother, which permits, in regards to the wave-type equation (3.75a) satisfied by  $p$ , the use of a differential multiplier technique to obtain estimates of the integral over time of

$$\frac{1}{2} \int_{\Omega} \|\nabla p\|^2 + |p'|^2 dx \quad (3.78)$$

premultiplied by an appropriate power of  $\mathcal{E}_u$ . The quantity (3.78) is the natural energy of  $[p, p']$  at the  $H^1(\Omega) \times L^2(\Omega)$ -level (i.e., the standard variational framework); from there, we will be able to deduce a suitable integral estimate of the energy  $\mathcal{E}_u$  associated with the less regular  $u$ -variable.

*Remark 3.3.5.* In fact, since we want to avoid differentiating terms involving  $g$  so that our results remain valid when the nonlinearity is only continuous, we will rather multiply an integrated version of (3.75a), namely the formula  $u = -[p' + \Phi]$ , by the time derivative of the multiplier – see Lemma 3.3.6 below. In particular,  $p'$  need not be continuous.

### 3.3.2 The multiplier identity

In this section, we give more precise properties of the  $p$ -variable and derive an expression of (3.72) in the form of a identity obtained by applying an appropriate multiplier to (3.75a).

**Lemma 3.3.6.** *Let  $[u, u']$  be a strong solution. The corresponding functions  $p$  and  $\Phi$  enjoy the regularity*

$$p \in L^\infty(0, +\infty; \mathcal{D}(A)) \cap W^{1,\infty}(0, +\infty; H_0^1(\Omega)), \quad \Phi \in L^\infty(0, +\infty; H^1(\Omega)), \quad (3.79)$$

Also, the following identity holds:

$$u = -[p' + \Phi] \in \mathcal{C}(\mathbb{R}, L^2(\Omega)). \quad (3.80)$$

*Proof.* We infer from Proposition 3.2.5 that  $[u, u'] \in L^\infty(0, +\infty; H^1(\Omega) \times L^2(\Omega))$ . As a consequence, since  $A^{-1}$  is continuous from  $L^2(\Omega)$  into  $\mathcal{D}(A)$ , we get

$$p = A^{-1}u' \in L^\infty(0, +\infty; \mathcal{D}(A)). \quad (3.81)$$

Besides,  $u' \in W^{1,\infty}(0, +\infty; H^{-1}(\Omega))$  and

$$u'' + Au = -ADPg(D^*u') \quad \text{a.e.} \quad (3.82)$$

Thus, applying  $A^{-1}$  to (3.82) yields

$$p' = A^{-1}u'' \in L^\infty(0, +\infty; H_0^1(\Omega)), \quad (3.83)$$

and also

$$u = -[p' + \Phi], \quad \text{and } \Phi \in L^\infty(0, +\infty; H^1(\Omega)), \quad (3.84)$$

which concludes the proof.  $\square$

Define the usual wave multiplier as follows:

$$\mathcal{M}p \triangleq 2h \cdot \nabla p + (d-1)p, \quad (3.85)$$

where  $h(x) = x - x_0$  as defined in Hypothesis 3.3.2 and  $d$  is the space dimension. Since  $p$  satisfies a wave equation, we know that the integral of  $\int_\Omega \|\nabla p\|^2 + |p'|^2 dx dt$  over  $(\tau_1, \tau_2)$  can be estimated by multiplying (3.75a) by  $\mathcal{M}p$  and integrating over  $\Omega \times (\tau_1, \tau_2)$ . Since we are looking for estimates of  $\mathcal{E}_u$  at the power  $(r+1)/2$ , we premultiply  $\mathcal{M}p$  by  $\mathcal{E}_u$  at the power  $(r-1)/2$ . Thus, we shall multiply (3.75a) by

$$\mathcal{E}_u^{(r-1)/2}(t) \mathcal{M}p(x, t). \quad (3.86)$$

The resulting identity is given in the next lemma.

**Lemma 3.3.7** (Multiplier identity). *The following equality holds for any  $0 \leq \tau_1 \leq \tau_2$ :*

$$\begin{aligned} 2 \int_{\tau_1}^{\tau_2} \mathcal{E}_u^{(r+1)/2} dt &= \mathcal{E}_u^{(r-1)/2} \int_{\Omega} u \mathcal{M} p dx \Big|_{\tau_1}^{\tau_2} + \int_{\tau_1}^{\tau_2} \mathcal{E}_u^{(r-1)/2} \int_{\Gamma} (h \cdot \nu) \left| \frac{\partial p}{\partial \nu} \right|^2 d\sigma \\ &\quad - \int_{\Gamma_0} (h \cdot \nu) |g(D^* u')|^2 d\sigma dt - \int_{\tau_1}^{\tau_2} \mathcal{E}_u^{(r-1)/2} \int_{\Omega} (d+1)\Phi u + \Phi[2h \cdot \nabla u] dx dt \\ &\quad - \frac{(r-1)}{2} \int_{\tau_1}^{\tau_2} \mathcal{E}'_u \mathcal{E}_u^{(r-3)/2} \int_{\Omega} u \mathcal{M} p dx dt. \end{aligned} \quad (3.87)$$

*Proof.* The proof is split into four steps.

*Step 1: Integration by parts with respect to time.* First, by linearity and continuity of  $\mathcal{M}$ ,  $\mathcal{M}p$  belongs to  $W^{1,\infty}(0, +\infty; L^2(\Omega))$ . On the other hand, recall that  $\mathcal{E}_u$  is bounded and absolutely continuous with

$$\mathcal{E}'_u = - \int_{\Gamma_0} g(D^* u') D^* u' d\sigma \quad \text{a.e.,} \quad \mathcal{E}'_u \in L^\infty(0, +\infty), \quad (3.88)$$

because strong solutions are Lipschitz continuous with respect to time. Thus,  $\mathcal{E}_u^{(r-1)/2}$  belongs to  $W^{1,\infty}(0, +\infty)$  and  $\mathcal{E}_u^{(r-1)/2} \mathcal{M}p$  belongs to  $W^{1,\infty}(0, +\infty; L^2(\Omega))$  with

$$[\mathcal{E}_u^{(r-1)/2} \mathcal{M}p]' = \mathcal{E}_u^{(r-1)/2} \mathcal{M}p' + \frac{(r-1)}{2} \mathcal{E}'_u \mathcal{E}_u^{(r-3)/2} \mathcal{M}p \quad \text{a.e. in } L^2(\Omega). \quad (3.89)$$

Now, it follows from (3.80) that  $p' + \Phi$  belongs to  $W^{1,\infty}(0, +\infty; L^2(\Omega))$  and

$$-[p' + \Phi]' = u' = Ap. \quad (3.90)$$

Let  $0 \leq \tau_1 \leq \tau_2$ . Recall that, since  $p \in \mathcal{D}(A)$ ,  $Ap = -\Delta p \in L^2(\Omega)$ . Thus, taking the scalar product of (3.90) with  $\mathcal{E}_u^{(r-1)/2} \mathcal{M}p$  in  $L^2(\tau_1, \tau_2; L^2(\Omega))$  and using the integration by parts formula in  $W^{1,2}(\tau_1, \tau_2; L^2(\Omega))$  leads to

$$\begin{aligned} - \int_{\tau_1}^{\tau_2} \mathcal{E}_u^{(r-1)/2} \int_{\Omega} \Delta p \mathcal{M} p dx dt &= \mathcal{E}_u^{(r-1)/2} \int_{\Omega} u \mathcal{M} p dx \Big|_{\tau_1}^{\tau_2} - \int_{\tau_1}^{\tau_2} \mathcal{E}_u^{(r-1)/2} \int_{\Omega} u \mathcal{M} p' dx dt \\ &\quad - \frac{(r-1)}{2} \int_{\tau_1}^{\tau_2} \mathcal{E}'_u \mathcal{E}_u^{(r-3)/2} \int_{\Omega} u \mathcal{M} p dx dt. \end{aligned} \quad (3.91)$$

*Step 2: Multiplier technique for the wave equation.* In what follows, we apply classical vector calculus identities to recover the  $H_0^1(\Omega) \times L^2(\Omega)$ -energy of  $[p, p']$ . Since  $p$  takes values in  $H^2(\Omega)$ , Rellich's identity yields

$$\int_{\Omega} \Delta p [2h \cdot \nabla p] dx = (d-2) \int_{\Omega} \|\nabla p\|^2 dx + \int_{\Gamma} \frac{\partial p}{\partial \nu} [2h \cdot \nabla p] d\sigma - \int_{\Gamma} (h \cdot \nu) \|\nabla p\|^2 d\sigma. \quad (3.92)$$

Furthermore,  $p|_{\Gamma} = 0$ ; thus

$$\nabla p = \left( \frac{\partial p}{\partial \nu} \right) \nu \quad \text{on } \Gamma. \quad (3.93)$$

Combining (3.92) and (3.93), we obtain

$$\int_{\Omega} \Delta p [2h \cdot \nabla p] = (d-2) \int_{\Omega} \|\nabla p\|^2 dx + \int_{\Gamma} (h \cdot \nu) |\partial_{\nu} p|^2 d\sigma. \quad (3.94)$$

On the other hand,

$$\int_{\Omega} \Delta p (d-1)p \, dx = -(d-1) \int_{\Omega} \|\nabla p\|^2 \, dx, \quad (3.95)$$

where we use again that  $p$  vanishes on the boundary. Summing (3.94) and (3.95) yields

$$\int_{\Omega} \Delta p \mathcal{M} p \, dx = - \int_{\Omega} \|\nabla p\|^2 \, dx + \int_{\Gamma} (h \cdot \nu) |\partial_{\nu} p|^2 \, dx. \quad (3.96)$$

Coming back to (3.91) and recalling (3.80), let us write

$$\begin{aligned} \int_{\Omega} u \mathcal{M} p' &= - \int_{\Omega} p' \mathcal{M} p' \, dx - \int_{\Omega} \Phi \mathcal{M} p' \, dx \\ &= - \int_{\Omega} p' [2h \cdot \nabla p'] + (d-1) |p'|^2 \, dx - \int_{\Omega} \Phi \mathcal{M} p' \, dx \\ &= \int_{\Omega} |p'|^2 \, dx - \int_{\Omega} \Phi \mathcal{M} p' \, dx, \end{aligned} \quad (3.97)$$

where we use the identity

$$\int_{\Omega} \phi [2h \cdot \nabla \phi] \, dx = \int_{\Gamma} (h \cdot \nu) (\phi|_{\Gamma})^2 \, d\sigma - \int_{\Omega} (\phi)^2 \operatorname{div} h \, dx \quad \text{for all } \phi \in H^1(\Omega) \quad (3.98)$$

together with  $\operatorname{div} h = d$  and  $p|_{\Gamma} = 0$ . Therefore, combining (3.91) with (3.96) and (3.97) leads to

$$\begin{aligned} \int_{\tau_1}^{\tau_2} \mathcal{E}_u^{(r-1)/2} \int_{\Omega} \|\nabla p\|^2 + |p'|^2 \, dx \, dt &= \mathcal{E}_u^{(r-1)/2} \int_{\Omega} u \mathcal{M} p \, dx \Big|_{\tau_1}^{\tau_2} + \int_{\tau_1}^{\tau_2} \mathcal{E}_u^{(r-1)/2} \int_{\Omega} \Phi \mathcal{M} p' \, dx \, dt \\ &\quad + \int_{\tau_1}^{\tau_2} \mathcal{E}_u^{(r-1)/2} \int_{\Gamma} (h \cdot \nu) \left| \frac{\partial p}{\partial \nu} \right|^2 \, d\sigma \, dt - \frac{(r-1)}{2} \int_{\tau_1}^{\tau_2} \mathcal{E}'_u \mathcal{E}_u^{(r-3)/2} \int_{\Omega} u \mathcal{M} p \, dx \, dt. \end{aligned} \quad (3.99)$$

*Step 3: Additional terms.* Here, we put the terms involving  $\Phi$  into a form suitable for further estimation. It follows from (3.80) that

$$\int_{\Omega} \Phi \mathcal{M} p' \, dx = - \int_{\Omega} \Phi \mathcal{M} u \, dx - \int_{\Omega} \Phi \mathcal{M} \Phi \, dx. \quad (3.100)$$

Applying (3.98) to  $\Phi$ , similarly to (3.97), we obtain

$$\begin{aligned} - \int_{\Omega} \Phi \mathcal{M} \Phi \, dx &= \int_{\Omega} |\Phi|^2 \, dx - \int_{\Gamma} (h \cdot \nu) (\Phi|_{\Gamma})^2 \\ &= \int_{\Omega} |\Phi|^2 \, dx - \int_{\Gamma_0} (h \cdot \nu) |g(D^* u')|^2 \, d\sigma, \end{aligned} \quad (3.101)$$

where we use that, by definition,

$$\Phi|_{\Gamma} = g(D^* A p) = g(D^* u') \text{ on } \Gamma_0, \quad \Phi|_{\Gamma} = 0 \text{ on } \Gamma_1. \quad (3.102)$$

On the other hand, we recall that

$$- \int_{\Omega} \Phi \mathcal{M} u \, dx = - \int_{\Omega} \Phi [2h \cdot \nabla u] + (d-1) \Phi u \, dx. \quad (3.103)$$

Plugging (3.101) and (3.103) into (3.99) leads to

$$\begin{aligned}
 & \int_{\tau_1}^{\tau_2} \mathcal{E}_u^{(r-1)/2} \int_{\Omega} \|\nabla p\|^2 + |p'|^2 \, dx \, dt \\
 & \quad = \mathcal{E}_u^{(r-1)/2} \int_{\Omega} u \mathcal{M} p \, dx \Big|_{\tau_1}^{\tau_2} - \frac{(r-1)}{2} \int_{\tau_1}^{\tau_2} \mathcal{E}'_u \mathcal{E}_u^{(r-3)/2} \int_{\Omega} u \mathcal{M} p \, dx \, dt \\
 & + \int_{\tau_1}^{\tau_2} \mathcal{E}_u^{(r-1)/2} \int_{\Gamma} (h \cdot \nu) \left| \frac{\partial p}{\partial \nu} \right|^2 \, d\sigma - \int_{\Gamma_0} (h \cdot \nu) |g(D^* A p)|^2 \, d\sigma - \int_{\Omega} \Phi [2h \cdot \nabla u] + (d-1)\Phi u - |\Phi|^2 \, dx \, dt.
 \end{aligned} \tag{3.104}$$

*Step 4: Conclusion.* We finish the proof by rewriting (3.104) as an estimate of  $\int_{\tau_1}^{\tau_2} \mathcal{E}_u^{(r+1)/2} \, dt$ . This is done as follows. First, by definition of the  $p$ -variable,

$$\int_{\Omega} \|\nabla p\|^2 \, dx = \|A^{1/2} p\|_{L^2(\Omega)}^2 = \|A^{-1/2} u'\|_{L^2(\Omega)}^2 = \|u'\|_{H^{-1}(\Omega)}^2. \tag{3.105}$$

On the other hand, it immediately follows from  $u = -[p' + \Phi]$  that

$$\int_{\Omega} |p'|^2 \, dx = \int_{\Omega} |u|^2 \, dx + \int_{\Omega} 2\Phi u + |\Phi|^2 \, dx \quad \text{a.e.} \tag{3.106}$$

Summing (3.105) and (3.106) yields

$$\int_{\Omega} \|\nabla p\|^2 + |p'|^2 \, dx = 2\mathcal{E}_u + \int_{\Omega} 2\Phi u + |\Phi|^2 \, dx \quad \text{a.e.} \tag{3.107}$$

Plugging (3.107) into (3.104), we get the desired identity  $\square$

### 3.3.3 Integral estimates and conclusion

Our goal in this section is to establish an integral inequality in the form of

$$(1 - \mu) \int_{\tau_1}^{\tau_2} \mathcal{E}_u^{(r+1)/2} \, dt \leq K_u \{ \mathcal{E}_u(\tau_1) + \mathcal{E}_u(\tau_2) \} \quad \text{for all } 0 \leq \tau_1 \leq \tau_2, \tag{3.108}$$

where  $K_u$  is a constant that may depend on the initial data and  $\mu$  is a sufficiently small constant that may depend on  $u$  as well. Bearing in mind the full statement of Theorem 3.3.3, we aim at finding such constants that depend on  $\|\mathcal{A}[u_0, v_0]\|$  and  $\mathcal{E}_u(0) = \mathcal{E}(u_0, v_0)$  only.

Assuming that (3.108) holds, we let  $\tau_2$  go to  $+\infty$  to obtain

$$\int_{\tau}^{+\infty} \mathcal{E}_u^{(r+1)/2} \, dt \leq \frac{K_u}{1 - \mu} \mathcal{E}_u(\tau) \quad \text{for all } \tau \geq 0. \tag{3.109}$$

Then, Theorem 3.3.3 follows readily from Lemma 3.3.4 if we choose

$$\gamma = \frac{(r-1)}{2} \quad \text{and} \quad T = \frac{K_u}{1 - \mu} \mathcal{E}_u(0)^{-(r-1)/2}. \tag{3.110}$$

To prove (3.108), we shall examine each term in the multiplier identity (3.87) and derive estimates in terms of

- Either directly  $\mathcal{E}_u(\tau_1)$  or  $\mathcal{E}_u(\tau_2)$ ;
- Or the boundary dissipation term  $\int_{\tau_1}^{\tau_2} \int_{\Gamma_0} g(D^*u')D^*u' d\sigma$  which is nonnegative and can be integrated, since

$$-\mathcal{E}'_u = \int_{\Gamma_0} g(D^*u')D^*u' d\sigma = - \int_{\Gamma_0} g\left(-\frac{\partial p}{\partial \nu}\right) \frac{\partial p}{\partial \nu} d\sigma; \quad (3.111)$$

- And also  $\int_{\tau_1}^{\tau_2} \mathcal{E}_u^{(r+1)/2} dt$  premultiplied by small  $\mu$  so that it can be absorbed in the left-hand side.

*Remark 3.3.8.* In what follows, we shall denote by  $K, K'$ , etc. generic constants that *do not* depend on the initial data.

We can write estimates in terms of  $\int_{\tau_1}^{\tau_2} \int_{\Gamma_0} |g(D^*u')|^2 d\sigma dt$  instead of  $\int_{\tau_1}^{\tau_2} \int_{\Gamma_0} g(D^*u')D^*u' d\sigma dt$  since

$$0 \leq \int_{\Gamma_0} |g(D^*u')|^2 d\sigma \leq K \int_{\Gamma_0} |g(D^*u')D^*u'| d\sigma = K \int_{\Gamma_0} g(D^*u')D^*u' d\sigma \quad (3.112)$$

by Lipschitz continuity and monotonicity of  $g$ , together with  $g(0) = 0$ .

Moreover, recalling (3.67) in Hypothesis 3.3.2 and looking at the sign of each term, we observe that  $|\partial_\nu p|$  in (3.87) need not be estimated on the uncontrolled boundary  $\Gamma_1$ .

That being said, let us start by estimating the term involving  $2h \cdot \nabla u$  in (3.87). This is done in the following lemma.

**Lemma 3.3.9.** *Suppose that  $r \geq 2$ . Then, there exists a positive constant  $K$  such that*

$$\left| \int_{\tau_1}^{\tau_2} \mathcal{E}_u^{(r-1)/2} \int_{\Omega} \Phi[2h \cdot \nabla u] dx dt \right| \leq K \|\mathcal{A}[u_0, v_0]\|_{\mathcal{H}}^{1/2} \left\{ \int_{\tau_1}^{\tau_2} \frac{1}{\mu} \int_{\Gamma_0} |g(D^*u')|^2 d\sigma + \mu \mathcal{E}_u^{(r-2)/2}(0) \mathcal{E}_u^{(r+1)/2} dt \right\} \quad (3.113)$$

for all  $\tau_2 \geq \tau_1 \geq 0$  and  $\mu > 0$ .

*Proof.* We start by writing

$$(\Phi, 2h \cdot \nabla u)_{L^2(\Omega)} = (DPg(D^*u'), 2h \cdot \nabla u)_{L^2(\Omega)} = (g(D^*u'), D^*[2h \cdot \nabla u])_{L^2(\Gamma_0)}. \quad (3.114)$$

Thus, applying the Cauchy-Schwarz inequality yields

$$\left| \int_{\Omega} \Phi[2h \cdot \nabla u] dx \right| \leq \|g(D^*u')\|_{L^2(\Gamma_0)} \|D^*[2h \cdot \nabla u]\|_{L^2(\Gamma_0)} \leq \|g(D^*u')\|_{L^2(\Gamma_0)} \|D^*[2h \cdot \nabla u]\|_{L^2(\Gamma)}. \quad (3.115)$$

Next, recall from (3.25) that

$$D^* \in \mathcal{L}(H^{-1/2}(\Omega), L^2(\Gamma)). \quad (3.116)$$

Therefore, it follows from (3.115) that

$$\left| \int_{\Omega} \Phi[2h \cdot \nabla u] dx \right| \leq K \|g(D^*Ap)\|_{L^2(\Gamma_0)} \|2h \cdot \nabla u\|_{H^{-1/2}(\Omega)}. \quad (3.117)$$

Linear interpolation between the Sobolev spaces  $L^2(\Omega)$  and  $H^{-1}(\Omega)$  leads to

$$\|2h \cdot \nabla u\|_{H^{-1/2}(\Omega)} \leq K \|2h \cdot \nabla u\|_{L^2(\Omega)}^{1/2} \|2h \cdot \nabla u\|_{H^{-1}(\Omega)}^{1/2} \quad (3.118)$$



First, by Proposition 3.2.5,

$$\|2h \cdot \nabla u\|_{L^2(\Omega)}^{1/2} \leq K \|h\|_{L^\infty(\Omega)^d}^{1/2} \|\nabla u\|_{L^2(\Omega)^d}^{1/2} \leq K' \|\mathcal{A}[u_0, v_0]\|_{\mathcal{H}}^{1/2}. \quad (3.119)$$

Besides, since  $2h \cdot \nabla u$  belongs to  $L^2(\Omega)$ , we have

$$\langle 2h \cdot \nabla u, w \rangle_{H_0^1(\Omega)} = (2h \cdot \nabla u, w)_{L^2(\Omega)} \quad \text{for all } w \in H_0^1(\Omega). \quad (3.120)$$

Let us write

$$\begin{aligned} \int_{\Omega} [2h \cdot \nabla u] w \, dx &= 2 \sum_{i=1}^d \int_{\Omega} h_i \frac{\partial u}{\partial x_i} w \, dx \\ &= -2 \sum_{i=1}^d \int_{\Omega} u \left[ w \frac{\partial h_i}{\partial x_i} + h_i \frac{\partial w}{\partial x_i} \right] dx + \sum_{i=1}^d \int_{\Gamma} \nu_i h_i u w \, d\sigma \\ &= -2 \int_{\Omega} u [w \operatorname{div} h + h \cdot \nabla w] \, dx, \end{aligned} \quad (3.121)$$

where  $h_i$  (resp.  $\nu_i$ ) denotes the  $i$ -th coordinate of the vector field  $h$  (resp. the outward normal vector  $\nu$ ). Recall that  $h \in \mathcal{C}^1(\overline{\Omega})$ . Then, using the Poincaré inequality on  $w$ , we obtain that for some  $K > 0$ ,

$$|(2h \cdot \nabla u, w)_{L^2(\Omega)}| \leq K \|u\|_{L^2(\Omega)} \|w\|_{H_0^1(\Omega)}. \quad (3.122)$$

By norm equivalence between  $H^{-1}(\Omega)$  and  $H_0^1(\Omega)'$ , we infer from (3.122) that

$$\|2h \cdot \nabla u\|_{H^{-1}(\Omega)} \leq K \|u\|_{L^2(\Omega)}. \quad (3.123)$$

Coming back to (3.117), combining (3.119) and (3.123) yields

$$\begin{aligned} \left| \int_{\Omega} Dg(D^*u') [2h \cdot \nabla u] \, dx \right| &\leq K \|\mathcal{A}[u_0, v_0]\|_{\mathcal{H}}^{1/2} \|g(D^*u')\|_{L^2(\Gamma_0)} \|u\|_{L^2(\Omega)}^{1/2} \\ &\leq K' \|\mathcal{A}[u_0, v_0]\|_{\mathcal{H}}^{1/2} \|g(D^*u')\|_{L^2(\Gamma_0)} \mathcal{E}_u^{1/4}. \end{aligned} \quad (3.124)$$

Therefore, since  $\mathcal{E}_u \geq 0$ , we have

$$\left| \mathcal{E}_u^{(r-1)/2} \int_{\Omega} Dg(D^*u') 2h \cdot \nabla u \, dx \right| \leq K \|\mathcal{A}[u_0, v_0]\|_{\mathcal{H}}^{1/2} \|g(D^*u')\|_{L^2(\Gamma_0)} \mathcal{E}_u^{(r-1)/2+1/4}. \quad (3.125)$$

Applying Young's inequality with a parameter  $\mu > 0$ , we obtain

$$\|g(D^*u')\|_{L^2(\Gamma_0)} \mathcal{E}_u^{(r-1)/2+1/4} \leq \frac{1}{2\mu} \|g(D^*u')\|_{L^2(\Gamma_0)}^2 + \frac{\mu}{2} \mathcal{E}_u^{r-1/2}. \quad (3.126)$$

It is assumed that  $r \geq 2$ . Thus, letting  $\eta \triangleq (r-1)/2 - (r+1)/2 \geq 0$ , by nonincreasingness of  $\mathcal{E}_u$ , we have

$$\mathcal{E}_u^{r-1/2} = \mathcal{E}_u^\eta \mathcal{E}_u^{(r+1)/2} \leq \mathcal{E}_u^\eta(0) \mathcal{E}_u^{(r+1)/2}. \quad (3.127)$$

Plugging (3.127) into (3.126) and integrating over  $(\tau_1, \tau_2)$  yields the desired result.  $\square$

Next, we deal with the term involving  $|\partial_\nu p| = |D^*u'|$  on the controlled boundary  $\Gamma_0$ . Here, the arguments are very similar to those employed in the case of saturated Neumann feedback – see, e.g., [Komornik, 1994a, Theorem 9.10].

**Lemma 3.3.10.** *Suppose that  $r \geq d - 1$ . Then, there exists  $K > 0$  and  $\eta \in (0, 1)$  such that*

$$\begin{aligned} \left| \int_{\tau_1}^{\tau_2} \mathcal{E}_u^{(r-1)/2} \int_{\Gamma_0} \left| \frac{\partial p}{\partial \nu} \right|^2 d\sigma dt \right| &\leq K \int_{\tau_1}^{\tau_2} \int_{\Gamma_0} |g(D^*u')|^2 d\sigma dt \\ &+ K \|\mathcal{A}[u_0, v_0]\|_{\mathcal{H}}^{2-\eta} \left\{ \mu^{1/\eta} \int_{\tau_1}^{\tau_2} \int_{\Gamma_0} g(D^*u') D^*u' d\sigma dt + \mu^{-1/(1-\eta)} \int_{\tau_1}^{\tau_2} \mathcal{E}_u^{(r+1)/2} dt \right\} \end{aligned} \quad (3.128)$$

for all  $0 \leq \tau_1 \leq \tau_2$  and  $\mu > 0$ .

*Proof.* For each  $t \geq 0$ , we set

$$\Gamma_t^0 \triangleq \left\{ \sigma \in \Gamma_0 : \left| \frac{\partial p}{\partial \nu}(\sigma, t) \right| \leq S \right\} \quad \text{and} \quad \Gamma_t^1 \triangleq \Gamma_0 \setminus \Gamma_t^0. \quad (3.129)$$

where we recall that the constant  $S$  is introduced in Hypothesis 3.3.1. Then,

$$\left| \int_{\Gamma_0} (h \cdot \nu) \left| \frac{\partial p}{\partial \nu} \right|^2 d\sigma \right| \leq K \int_{\Gamma_t^0} \left| \frac{\partial p}{\partial \nu} \right|^2 d\sigma + K \int_{\Gamma_t^1} \left| \frac{\partial p}{\partial \nu} \right|^2 d\sigma. \quad (3.130)$$

Using (3.65) in Hypothesis 3.3.1, we estimate the first term in (3.130) as follows:

$$\int_{\Gamma_t^0} \left| \frac{\partial p}{\partial \nu} \right|^2 d\sigma \leq \alpha_1^{-2} \int_{\Gamma_t^0} \left| g \left( -\frac{\partial p}{\partial \nu} \right) \right|^2 d\sigma \leq \alpha_1^{-2} \int_{\Gamma_0} \left| g \left( -\frac{\partial p}{\partial \nu} \right) \right|^2 d\sigma = \alpha_1^{-2} \int_{\Gamma_0} |g(D^*u')|^2 d\sigma. \quad (3.131)$$

Let us examine the second term in (3.130). Setting a parameter  $\eta \in (0, 1)$  to be tuned later on, we have

$$\int_{\Gamma_t^1} \left| \frac{\partial p}{\partial \nu} \right|^2 d\sigma = \int_{\Gamma_t^1} |D^*u'|^2 d\sigma = \int_{\Gamma_t^1} |D^*u'|^{2-\eta} \frac{|g(D^*u') D^*u'|^\eta}{|g(D^*u')|^\eta} d\sigma. \quad (3.132)$$

Equation (3.132) makes sense since  $|g(D^*u')| \geq \min\{g(S), -g(-S)\} > 0$  on  $\Gamma_t^1$ . In fact, we have

$$\begin{aligned} \int_{\Gamma_t^1} |D^*u'|^2 d\sigma &\leq \min\{g(S), -g(-S)\}^{-\eta} \int_{\Gamma_t^1} |D^*u'|^{2-\eta} |g(D^*u') D^*u'|^\eta d\sigma \\ &\leq K \int_{\Gamma_0} |D^*u'|^{2-\eta} |g(D^*u') D^*u'|^\eta d\sigma. \end{aligned} \quad (3.133)$$

Using Hölder's inequality with conjugates  $1/\eta$  and  $1/(1-\eta)$ , we infer from (3.133) that

$$\int_{\Gamma_t^1} |D^*u'|^2 d\sigma \leq K \left( \int_{\Gamma_0} |D^*u'|^{\frac{2-\eta}{1-\eta}} d\sigma \right)^{1-\eta} \left( \int_{\Gamma_0} |g(D^*u') D^*u'|^\eta d\sigma \right)^\eta. \quad (3.134)$$

Now,  $[u, u']$  being a strong solution to (3.1)-(3.7), we recall from Proposition 3.2.5 that  $u'$  takes values in  $L^2(\Omega)$  and  $\|u'(t)\|_{L^2(\Omega)} \leq K \|\mathcal{A}[u_0, v_0]\|_{\mathcal{H}}$  for all  $t \geq 0$ . The continuity property (3.23) yields

$$\|D^*u'(t)\|_{H^{1/2}(\Gamma)} \leq K \|\mathcal{A}[u_0, v_0]\|_{\mathcal{H}} \quad \text{for all } t \geq 0. \quad (3.135)$$

In what follows, we rely on (fractional) Sobolev inequalities – see [Di Nezza et al., 2012, Theorems 6.5 and 6.9]. First, we consider the case  $d \geq 3$ , where we recall that  $d$  denotes the space dimension. We have the continuous embedding

$$H^{1/2}(\Gamma_0) \hookrightarrow L^q(\Gamma_0) \quad \text{for all } q \in \left[ 2, \frac{2(d-1)}{d-2} \right] \triangleq I_d. \quad (3.136)$$

Furthermore, since  $r+1 \geq d$ , if we choose  $\eta = 2/(r+1)$ , some computations yield  $(2-\eta)/(1-\eta) \in I_d$ ; hence

$$H^{1/2}(\Gamma_0) \hookrightarrow L^{\frac{2-\eta}{1-\eta}}(\Gamma_0). \quad (3.137)$$

If  $d = 2$ , then the embedding (3.136) holds in fact for any  $q \in [2, +\infty)$ ; therefore, (3.137) is valid as well. Coming back to (3.134), combining (3.137) with (3.135) yields

$$\mathcal{E}_u^{(r-1)/2} \int_{\Gamma_1^1} |D^* u'|^2 d\sigma \leq K \|\mathcal{A}[u_0, v_0]\|_{\mathcal{H}}^{2-\eta} \mathcal{E}_u^{(r-1)/2} \left( \int_{\Gamma_0} g(D^* u') D^* u' d\sigma \right)^\eta. \quad (3.138)$$

Applying the Young inequality with conjugates  $1/\eta$  and  $1/(1-\eta)$ , we get that for all  $\mu > 0$ ,

$$\mathcal{E}_u^{(r-1)/2} \int_{\Gamma_1^1} |D^* u'|^2 d\sigma \leq K \|\mathcal{A}[u_0, v_0]\|_{\mathcal{H}}^{2-\eta} \left\{ \mu^{-\frac{1}{1-\eta}} \mathcal{E}_u^{\frac{r-1}{2(1-\eta)}} + \mu^{\frac{1}{\eta}} \int_{\Gamma_0} g(D^* u') D^* u' d\sigma \right\}. \quad (3.139)$$

Since  $(r-1)/2(1-\eta) = (r+1)/2$ , we conclude the proof by combining (3.139) and (3.131) together with (3.130).  $\square$

At this point, the proof of Theorem 3.3.3 is almost complete. Estimates of the remaining terms in (3.87) are given in the next lemmas. Following our remarks at the beginning of the section, we claim that Theorem 3.3.3 is proved once those are established.

**Lemma 3.3.11.** *There exists a positive constant  $K$  such that*

$$\left| \left[ \mathcal{E}_u^{(r-1)/2} \int_{\Omega} u \mathcal{M} p dx \right]_{\tau_1}^{\tau_2} \right| \leq K \mathcal{E}_u^{(r-1)/2}(0) \{ \mathcal{E}_u(\tau_1) + \mathcal{E}_u(\tau_2) \} \quad \text{for all } 0 \leq \tau_1 \leq \tau_2. \quad (3.140)$$

*Proof.* Let  $\tau \geq 0$ . Then,

$$\begin{aligned} \left| \mathcal{E}_u^{(r-1)/2}(\tau) \int_{\Omega} u(\tau) \mathcal{M} p(\tau) dx \right| &\leq \mathcal{E}_u^{(r-1)/2}(\tau) \|u(\tau)\|_{L^2(\Omega)} \|\mathcal{M} p(\tau)\|_{L^2(\Omega)} \\ &\leq K \mathcal{E}_u^{(r-1)/2}(\tau) \|u(\tau)\|_{L^2(\Omega)} \|p(\tau)\|_{H_0^1(\Omega)} \\ &\leq K \mathcal{E}_u^{(r-1)/2}(\tau) \|u(\tau)\|_{L^2(\Omega)} \|u'(\tau)\|_{H^{-1}(\Omega)} \\ &\leq K \mathcal{E}_u^{(r-1)/2}(0) \mathcal{E}_u(\tau), \end{aligned} \quad (3.141)$$

where it used that  $\mathcal{E}_u$  is monotone decreasing. Equation (3.140) readily follows from the triangular inequality.  $\square$

**Lemma 3.3.12.** *There exists a positive constant  $K$  such that*

$$\left| \int_{\tau_1}^{\tau_2} \mathcal{E}_u' \mathcal{E}_u^{(r-3)/2} \int_{\Omega} u \mathcal{M} p dx dt \right| \leq K \mathcal{E}_u^{(r-1)/2}(0) \{ \mathcal{E}_u(\tau_1) + \mathcal{E}_u(\tau_2) \} \quad \text{for all } 0 \leq \tau_1 \leq \tau_2. \quad (3.142)$$

*Proof.* Again, we write

$$\left| \int_{\Omega} u \mathcal{M} p dx \right| \leq K \mathcal{E}_u. \quad (3.143)$$

Therefore,

$$\begin{aligned} \left| \int_{\tau_1}^{\tau_2} \mathcal{E}_u' \mathcal{E}_u^{(r-3)/2} \int_{\Omega} u \mathcal{M} p \, dx \, dt \right| &\leq K \int_{\tau_1}^{\tau_2} (-\mathcal{E}_u') \mathcal{E}_u^{(r-1)/2} \, dt = -K \int_{\tau_1}^{\tau_2} \left[ \frac{2}{r+1} \mathcal{E}_u^{(r+1)/2} \right]' \, dt \\ &= \frac{2K}{r+1} \{ \mathcal{E}_u^{(r+1)/2}(\tau_1) - \mathcal{E}_u^{(r+1)/2}(\tau_2) \}. \end{aligned} \quad (3.144)$$

The desired inequality follows from the fact that  $\mathcal{E}_u$  is monotone decreasing and (3.144).  $\square$

**Lemma 3.3.13.** *There exists a positive constant  $K$  such that*

$$\left| \int_{\tau_1}^{\tau_2} \mathcal{E}_u^{(r-1)/2} \int_{\Omega} \Phi u \, dx \, dt \right| \leq K \left\{ \frac{1}{\mu} \int_{\tau_1}^{\tau_2} \mathcal{E}_u^{(r+1)/2} \, dt + \mu \mathcal{E}_u^{(r-1)/2}(0) \int_{\tau_1}^{\tau_2} \int_{\Gamma_0} |g(D^*u')|^2 \, d\sigma \, dt \right\} \quad (3.145)$$

for all  $0 \leq \tau_1 \leq \tau_2$  and  $\mu > 0$ .

*Proof.* First, using Cauchy-Schwarz and Young inequalities, we obtain

$$\begin{aligned} \left| \int_{\tau_1}^{\tau_2} \mathcal{E}_u^{(r-1)/2} \int_{\Omega} \Phi u \, dx \, dt \right| &\leq \int_{\tau_1}^{\tau_2} \mathcal{E}_u^{(r-1)/2} \|\Phi\|_{L^2(\Omega)} \|u\|_{L^2(\Omega)} \, dt \\ &\leq \frac{1}{2\mu} \int_{\tau_1}^{\tau_2} \mathcal{E}_u^{(r-1)/2} \|\Phi\|_{L^2(\Omega)}^2 \, dt + \mu \int_{\tau_1}^{\tau_2} \mathcal{E}_u^{(r+1)/2} \, dt. \end{aligned} \quad (3.146)$$

Next, recall that  $\Phi = DPg(D^*u')$  and that  $D$  continuously maps  $L^2(\Gamma)$  into  $L^2(\Omega)$ . Therefore,

$$\|\Phi\|_{L^2(\Omega)}^2 \leq K \|Pg(D^*u')\|_{L^2(\Gamma)}^2 = K \|g(D^*u')\|_{L^2(\Gamma_0)}^2. \quad (3.147)$$

We conclude the proof by plugging (3.147) into (3.146) and using that  $\mathcal{E}_u$  is monotone decreasing.  $\square$

## Comments on Chapter 3

**The multiplier method** The use of differential multipliers in order to derive appropriate integral energy estimates, which we illustrated in Section 3.3, has been widely developed in the literature – see, e.g., [Komornik, 1994a] or [Alabau-Boussouira, 2012]. To mention only a few, this strategy has been successfully applied to the Schrödinger equation [Lasiecka and Triggiani, 1992a] or Kirchhoff plates [Komornik, 1994b] in the context of exact observability and/or feedback stabilization. Coming back to the wave equation, an alternative approach based on Carleman estimates has been proposed in [Baudouin et al., 2013], with the goal of establishing exact controllability of the waves in presence of a potential term together with a robust reconstruction method of said potential – see also the recent work [Baudouin et al., 2022]. However, both methods lead to quite strong geometric assumptions, such as “star-shaped” conditions (as in our Hypothesis 3.3.2) or other more general convexity-based hypotheses. Those are known to be non-optimal (at least as long as the pure wave equation is concerned) since the seminal work [Bardos et al., 1992] where the celebrated *geometric control conditions*, which involve the technical apparatus of *microlocal analysis*, were proved to be necessary and sufficient for the problems of exact observability and exponential feedback stabilization. For further discussions on that matter, the reader is referred to [Miller, 2002] or [Laurent and Léautaud, 2016].

**Optimality of the non-uniform decay rate** Theorem 3.3.3 deals with the decay rate of strong solutions to (3.1)-(3.7), which remain bounded in a stronger norm (here, in  $H^1(\Omega) \times L^2(\Omega)$ ). In particular, this enables the use of Sobolev embeddings to obtain estimates of the boundary term  $\partial_\nu[A^{-1}u']$  in  $L^\infty(0, +\infty; L^q(\Gamma))$  for some appropriate  $q$ . In view of the energy identity (3.35), and as done in Lemma 3.3.10, we can then derive an estimate involving only the “dissipation term”  $g(-\partial_\nu[A^{-1}u'])$  and lower-order energy terms, even though no lower bound on nonlinearity  $g$  is prescribed at infinity. Here, using the terminology of [Vancostenoble and Martinez, 2000], the feedback is allowed to be *weak*, i.e.,  $g(s)/s$  can go to 0 as  $|s|$  goes to infinity, as it is the case when  $g$  represents a saturation mapping; then, loss of uniformity is to be expected. More precisely, coming back to the Neumann problem, the one-dimensional version of (3.10) with  $g$  being a saturation mapping as given by (2.98) is known to possess weak solutions that decay to zero (in the natural energy space  $H^1(\Omega) \times L^2(\Omega)$ ) slower than any exponential or polynomial, whereas strong solutions decay exponentially to zero but in a non-uniform way – see [Vancostenoble and Martinez, 2000, Theorem 4.1] or also [Chitour et al., 2021, Theorem 4.33]. Proving a similar result in our Dirichlet case would be interesting.

**Uniform stabilization** Putting aside the matter of saturated feedback and assuming if needed that  $g$  has linear growth at infinity, we see that, unfortunately, the strategy followed here is not sufficient to prove *uniform* decay of solutions to (3.1)-(3.7). Indeed, while estimating the term  $(\Phi, 2h \cdot \nabla u)_{L^2(\Omega)}$  as in Lemma 3.3.9 is good enough for the purpose of proving Theorem 3.3.3, it requires, again, that solutions remain bounded in a norm stronger than that of the energy space  $\mathcal{H}$ . If, instead of (3.113), one manages to prove something in the likes of

$$\int_0^\tau \int_\Omega \Phi[2h \cdot \nabla u] dx dt \leq K(\tau) \int_0^\tau |g(D^*u')|^2 d\sigma dt + K' \{ \mathcal{E}_u(0) + \mathcal{E}_u(\tau) \} + \varepsilon \int_0^\tau \mathcal{E}_u dt \quad (3.148)$$

for some  $\tau > 0$ , where  $K(\tau)$  is allowed to depend on  $\tau$  and  $\varepsilon$  can be chosen sufficiently small, then, by remarking that the multiplier identity (3.87) is still valid with the time-varying weight  $\mathcal{E}(u, u')^{(r-1)/2}$  replaced by the constant 1, one could easily adapt the rest of our proof to obtain exponential uniform stability. By following the proof of [Lasićka and Triggiani, 1987, Lemma 3.3], we can prove such an estimate when  $g$  is the identity, at least under some specific geometric conditions; however, the argument breaks down in the nonlinear case. Therefore, the problem of uniform stability is still open.

# Four

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## An output regulation problem

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### Contents

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4.1	Introduction and forwarding design . . . . .	51
4.1.1	Background and control problem . . . . .	51
4.1.2	Forwarding technique and feedback law . . . . .	52
4.1.3	System under consideration . . . . .	54
4.2	Analysis of the closed-loop system . . . . .	55
4.2.1	Sufficient conditions for output regulation . . . . .	55
4.2.2	Change of coordinates and proof of well-posedness . . . . .	56
4.2.3	Proof of the output regulation theorem . . . . .	58
4.3	The semilinear case . . . . .	63
4.3.1	Existence of an invariant graph for the cascade dynamics . . . . .	63
4.3.2	Additional properties in the semilinear settings and proof . . . . .	65
4.3.3	Applications . . . . .	70

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### 4.1 Introduction and forwarding design

This chapter is an attempt at tackling the problem of output regulation for a class of infinite-dimensional nonlinear systems governed by dynamics of contraction.

#### 4.1.1 Background and control problem

Consider a control system with measured output governed by an abstract evolution equation of the form

$$\frac{dw}{dt} + \mathcal{A}(w) = BU(t), \quad (4.1a)$$

$$y = Cw. \quad (4.1b)$$

The problem of robust output regulation consists in designing a feedback law which ensures that the output  $y$  of the closed-loop system tracks a given reference  $y_{\text{ref}}$  even in presence of external (constant here) disturbances  $d$  acting on the original dynamics (4.1a).

The idea of *integral action*, which derives from the more general *internal model principle* [Isidori and Byrnes, 1990], is instrumental in solving that problem. Let us add an output integrator to the

control loop, i.e., extend the original  $w$ -dynamics with a new variable  $z$  solving

$$\frac{dz}{dt} = Cw - y_{\text{ref}}. \quad (4.2)$$

Assume that one manages to build a feedback law  $U(t) = \mathcal{K}(w, z)$  of the extended state variable  $[w, z]$  such that trajectories of the system (4.1)-(4.2) are steered to some equilibrium  $[w^*, z^*]$ ; then, one immediately infer from (4.2) that  $Cw^* = y_{\text{ref}}$ , i.e., the output  $y$  is at the reference  $y_{\text{ref}}$  when the full state is at the equilibrium. From here, establishing that the output  $y$  tracks the reference  $y_{\text{ref}}$  amounts to proving suitable attractivity properties of said equilibrium.

The theory of output regulation for linear finite-dimensional systems has been well understood for several decades – see, e.g., [Davison, 1975, Davison, 1976, Francis, 1977]. However, nonlinear (finite-dimensional) output regulation theory is still an active research area. The matter is especially delicate when dealing with more general (time-varying) disturbances, as illustrated for instance in [Natarajan and Weiss, 2019]. Another interesting example is [Astolfi et al., 2021] where an infinite-dimensional controller is designed in order to regulate a finite-dimensional nonlinear system and reject periodic disturbances. Extensions of the linear output regulation theory to the infinite-dimensional settings were first proposed in [Pohjolainen, 1982, Pohjolainen, 1985], where an output integrator was considered. This research effort was pursued in [Paunonen and Pohjolainen, 2010, Paunonen, 2019, Phan and Paunonen, 2021], where internal models were implemented in order to reject wider classes of disturbances. Other works dealing with the problem of output regulation for particular linear partial differential equations include [Guo and Meng, 2021, Guo and Zhao, 2022] or [Guo et al., 2018].

On the other hand, very few papers in the literature deal with output regulation problems in the infinite-dimensional nonlinear settings. Most of them are very “transfer-function oriented”, and either rely on linearizing the plant around the origin, as in [Natarajan and Bentsman, 2016], where an approximate linear controller is proposed with the goal of regulating a finite-number of harmonics of the output with reference to a periodic signal, or deal with dynamics that are linear without control but are subject to some appropriate nonlinearity applied to the input, as in [Logemann and Ryan, 2000]. Also, in all of them, the output is finite-dimensional. The approach we propose in this chapter is quite different: we take the point of view of dynamical systems theory and are driven by Lyapunov calculus (more precisely, our control design relies on the forwarding methodology, which we introduce in the next section), hopefully paving the way to the output regulation of more general nonlinear dynamics.

Before further investigating the control strategy, let us introduce some notation and specify the formal control system (4.1). The original state variable  $w$ , the output  $y$  (together with the state  $z$  of the integrator) and the control signal  $U(t)$  take values in some Hilbert spaces  $H$ ,  $Z$  and  $E$  respectively. The input and output maps  $B$  and  $C$  are linear, with  $B : E \rightarrow H$  bounded and  $C : \mathcal{D}(C) \rightarrow Z$ ,  $\mathcal{D}(C) \subset H$ , possibly unbounded. The properties of the nonlinear map  $\mathcal{A} : \mathcal{D}(\mathcal{A}) \rightarrow H$  will be precised later on, for the moment we simply assume that  $\mathcal{D}(\mathcal{A})$  is included in  $\mathcal{D}(C)$ .

#### 4.1.2 Forwarding technique and feedback law

Our control design relies on the forwarding technique [Mazenc and Praly, 1996] for the stabilization of (finite-dimensional) nonlinear systems in cascade. This methodology has been developed with different purposes in, e.g., [Kaliora and Astolfi, 2005, Benachour et al., 2013, Astolfi and Praly, 2017]. It has also been used for nonlinear output regulation purpose in the recent work [Giaccagli et al., 2021]. Note that while extensions of this approach to the infinite-dimensional settings have been proposed [Terrand-Jeanne et al., 2019, Marx et al., 2021a, Marx et al., 2021b, Balogoun et al., 2021], in all of them the uncontrolled dynamics are linear.

In what follows, we introduce our choice of control law for the extended system composed of the original  $w$ -equation and the output integrator. For a moment, let us forget about the reference  $y_{\text{ref}}$

and the disturbance  $d$ , and consider the following cascade:

$$\frac{dw}{dt} + \mathcal{A}(w) = BU(t), \quad (4.3a)$$

$$\frac{dz}{dt} = Cw, \quad (4.3b)$$

Let us make two assumptions (rigorous formulations will be given later on) on the system:

1. (*Input-to-state stability of the  $w$ -subsystem.*) There exists a Lyapunov functional  $\mathcal{V}$  and positive  $\alpha$  and  $\beta$  such that

$$\frac{d}{dt} \mathcal{V}(w) \leq -\alpha \mathcal{V}(w) + \beta \|U(t)\|_E^2 \quad (4.4)$$

along solutions to the  $w$ -equation (4.3a) (in other words,  $\mathcal{V}$  is a *control-Lyapunov function*);

2. (*Invariant graph.*) There exists a smooth mapping  $\mathcal{M} : H \rightarrow Z$  such that  $\mathcal{M}(0) = 0$  and the graph of  $\mathcal{M}$  is invariant with respect to the uncontrolled cascade dynamics (4.3), i.e.,<sup>1</sup>

$$d\mathcal{M}(w) \cdot \mathcal{A}(w) + Cw = 0 \quad \text{for all suitable } w \in H. \quad (4.5)$$

Then, consider the Lyapunov candidate

$$\mathcal{W}(w, z) \triangleq \mathcal{V}(w) + \frac{\rho}{2} \|z - \mathcal{M}(w)\|_Z^2, \quad (4.6)$$

where  $\rho > 0$  is a parameter to be tuned. The functional  $\mathcal{W}$  is the “energy”  $\mathcal{V}$  of the  $w$ -subsystem plus an additional term penalizing the distance to the invariant graph. By the chain rule, we obtain

$$\begin{aligned} \frac{d}{dt} \mathcal{W}(w, z) &= \frac{d}{dt} \mathcal{V}(w) + \rho(Cw - d\mathcal{M}(w)[- \mathcal{A}(w) + BU(t)], z - \mathcal{M}(w))_Z \\ &= \frac{d}{dt} \mathcal{V}(w) - \rho(U(t), B^* d\mathcal{M}(w)^*[z - \mathcal{M}(w)])_E, \end{aligned} \quad (4.7)$$

where we plugged (4.5) into (4.7). Equation (4.7) suggests setting the feedback law as

$$U(t) = B^* d\mathcal{M}(w)^*[z - \mathcal{M}(w)]. \quad (4.8)$$

Indeed, by choosing  $\rho \geq 2\beta$  and using (4.4) together with Young’s inequality, we obtain

$$\frac{d}{dt} \mathcal{W}(w, z) \leq -\alpha \mathcal{V}(w) - \beta \|B^* d\mathcal{M}(w)^*[z - \mathcal{M}(w)]\|_Z^2. \quad (4.9)$$

Thus,  $\mathcal{W}$  is rendered monotone decreasing along trajectories of (4.3) in closed loop with (4.8). Applying (formally!) a LaSalle invariance argument and assuming in addition that  $d\mathcal{M}(0)^*B^*z = 0$  implies  $z = 0$ , since  $\mathcal{V}$  is definite positive and  $\mathcal{M}(0) = 0$ , we expect that solutions  $[w, z]$  to (4.3)-(4.8) go to 0 in  $H \times Z$ .

<sup>1</sup>Equation (4.5) is obtained by formally differentiating  $\mathcal{M}(w) = z$  along solutions  $[w, z]$  to (4.3) with  $U(t) = 0$ .



### 4.1.3 System under consideration

Coming back to the problem of (local) output regulation, our motivation is that for small reference  $y_{\text{ref}}$  and disturbance  $d$ , the forwarding-based feedback law (4.8) still steers the system to a new equilibrium  $[w^*, z^*]$  near the origin, at which the output  $y = Cw$  is regulated at  $y_{\text{ref}}$ . Proving this reasonably requires to focus on a specific class of nonlinear dynamics. Inspired by previous work in the theory of output regulation for finite-dimensional nonlinear systems [Pavlov et al., 2006, Giaccagli et al., 2021], and following the general spirit of this thesis, we shall concentrate our attention on systems of contractions.

*Hypothesis 4.1.1* (Monotonicity). The nonlinear mapping  $\mathcal{A} : \mathcal{D}(\mathcal{A}) \rightarrow H$  is a (single-valued) densely defined maximal monotone operator satisfying  $\mathcal{A}(0) = 0$ . Furthermore,  $\mathcal{A}$  is *strongly* monotone, i.e., there exists  $\alpha > 0$  such that

$$(\mathcal{A}(w_1) - \mathcal{A}(w_2), w_1 - w_2)_H \geq \alpha \|w_1 - w_2\|_H^2 \quad \text{for all } w_1, w_2 \in \mathcal{D}(\mathcal{A}). \quad (4.10)$$

Then, according to [Showalter, 2013, Chapter IV, Proposition 3.1],  $-\mathcal{A}$  is the generator of a strongly continuous contraction semigroup on  $H$ , which is denoted by  $\{\mathcal{T}_t\}$ . Equation (4.10) implies that the semigroup  $\{\mathcal{T}_t\}$  satisfies

$$\|\mathcal{T}_t w_1 - \mathcal{T}_t w_2\|_H \leq \exp(-\alpha t) \|w_1 - w_2\|_H \quad \text{for all } w_1, w_2 \in H \text{ and } t \geq 0. \quad (4.11)$$

In other words, the original  $w$ -equation exponentially contracts distances in  $H$ . With reference to our preliminary Lyapunov analysis in Section 4.1.2, we will see later on that the additional strong monotonicity property (4.10) yields an incremental version of (4.4) if we let

$$\mathcal{V}(w) \triangleq \frac{1}{2} \|w\|_H^2 \quad \text{for all } w \in H. \quad (4.12)$$

Next, we give the precise formulation of the “invariance equation” (4.5) satisfied by the mapping  $\mathcal{M}$  upon which we wish to build the controller (4.8).

*Hypothesis 4.1.2* (Praly-Mazenc equation). There exists a continuously Fréchet differentiable solution  $\mathcal{M} \in \mathcal{C}^1(H, Z)$  to

$$d\mathcal{M}(w) \cdot \mathcal{A}(w) + Cw = 0 \quad \text{for all } w \in \mathcal{D}(\mathcal{A}) \quad (4.13)$$

that satisfies  $\mathcal{M}(0) = 0$ .

Note that a formal candidate to (4.13) is always given by

$$\mathcal{M}(w) = -C \int_0^{+\infty} \mathcal{T}_t w \, dt; \quad (4.14)$$

however, what is difficult to check for general  $\mathcal{A}$  is that the map  $\mathcal{M}$  given by (4.14) is well-defined for all  $w$  in  $H$  (even if the integral converges in  $H$ , it need not take values in  $\mathcal{D}(C)$ ) and sufficiently smooth. This is all the more true since both  $\mathcal{M}$  and  $d\mathcal{M}$  appear in the feedback law (4.8). Nevertheless, in the case that  $\mathcal{A}$  has a *semilinear* structure, we are able to work around these difficulties – see Section 4.3 below. Note also that uniqueness of solutions to (4.13) is not investigated here. In the sequel, when working under Hypothesis 4.1.2, we implicitly fix a particular choice of  $\mathcal{M}$ .

Let  $[d, y_{\text{ref}}]$  in  $H \times Z$ . We now end this section by recalling the full extended control system with output integrator relative to the reference  $y_{\text{ref}}$  and unknown disturbance  $d$ :

$$\frac{dw}{dt} + \mathcal{A}(w) = BU(t) + d, \quad (4.15a)$$

$$\frac{dz}{dt} = Cw - y_{\text{ref}}, \quad (4.15b)$$

together with the (full-state) feedback law, which is well-defined under Hypothesis 4.1.2,

$$U(t) = B^* d \mathcal{M}(w)^* [z - \mathcal{M}(w)]. \quad (4.16)$$

Again, when referring to the closed-loop equations (4.15)-(4.16), there is an implicit dependence on the pair  $[d, y_{\text{ref}}]$  and the choice of  $\mathcal{M}$ .

## 4.2 Analysis of the closed-loop system

Our aim in this section is to derive sufficient conditions for the feedback law (4.16) (which we assume exists) to steer the system (4.15) to some equilibrium at the prescribed output reference. Due to the presence of the output integrator (4.15b) in the feedback loop, this amounts to proving the existence of a (locally or globally) attractive equilibrium for the closed-loop equations (4.15)-(4.16). The precise statements are given in Section 4.2.1. The main idea here is that, up to a certain change of variable, which we introduce in Section 4.2.2, the feedback law (4.16) preserves at least locally the contraction property of the original  $w$ -dynamics. This reasoning is made precise in Section 4.2.3, where the main result is proved.

### 4.2.1 Sufficient conditions for output regulation

The first thing to do is to establish existence and uniqueness of solutions to (4.15)-(4.16). Let us slightly adapt the definitions of solutions given in Chapter 2.

**Definition 4.2.1** (Strong and generalized solutions). Let  $T \in (0, +\infty]$ . We say that a  $H \times Z$ -valued function  $[w, z]$  defined on  $[0, T)$  is

- (i) A *strong* solution to (4.15) – (4.16) on  $[0, T)$  if  $[w, z]$  belongs to  $W^{1,1}(0, \tau; H \times Z)$  for any  $\tau \in (0, T)$ , takes values in  $\mathcal{D}(\mathcal{A}) \times Z$  and satisfies (4.15)-(4.16) a.e. on  $(0, T)$ ;
- (ii) A *generalized* solution to (4.15)-(4.16) if  $[w, z]$  is a limit of strong solutions in  $\mathcal{C}([0, \tau], H \times Z)$  for any fixed  $\tau \in (0, T)$ .

**Theorem 4.2.2** (Well-posedness). Assume Hypotheses 4.1.1 and 4.1.2 hold and  $d \mathcal{M}$  is locally Lipschitz continuous. Then,

- (i) For all  $[d, y_{\text{ref}}] \in H \times Z$  and all  $[w_0, z_0] \in H \times Z$ , the closed-loop system (4.15)-(4.16) admits a unique maximal<sup>2</sup> satisfying the initial condition  $[w(0), z(0)] = [w_0, z_0]$  and defined on  $[0, T_{\text{max}})$ , where  $T_{\text{max}} \in (0, +\infty]$  depends on  $[w_0, z_0]$ ;
- (ii) Moreover, if  $w_0 \in \mathcal{D}(\mathcal{A})$ , then  $[w, z]$  is a strong solution to (4.15)-(4.16) and satisfies  $[w, z] \in W^{1,\infty}(0, \tau; H \times Z)$  for any  $\tau \in (0, T_{\text{max}})$ ;
- (iii) Furthermore, if  $d = 0$  or  $d \mathcal{M}$  is globally Lipschitz continuous, then (4.15)-(4.16) is forward complete, i.e.,  $T_{\text{max}} = +\infty$  for any initial condition.

Theorem 4.2.2 is proved in Section 4.2.2. Then, in the next theorem, we establish sufficient conditions for achieving output regulation in terms of the map  $\mathcal{M}$  (which makes the link between the operator  $\mathcal{A}$  governing the  $w$ -dynamics and the output operator  $C$ ), the input operator  $B$  and the pair of disturbance-reference  $[d, y_{\text{ref}}]$ .

<sup>2</sup>This means that there is no solution defined on an interval  $[0, T)$  where  $T > T_{\text{max}}$ .

**Theorem 4.2.3** (Sufficient conditions for output regulation). *Assume Hypotheses 4.1.1 and 4.1.2 are satisfied. Assume that*

$$\text{Range } d\mathcal{M}(0)B = Z, \quad \text{i.e.,} \quad \exists \lambda > 0 \mid \forall z \in Z, \|B^* d\mathcal{M}(0)^* z\|_Z^2 \geq \lambda \|z\|_Z^2. \quad (4.17)$$

If  $d\mathcal{M}$  is globally (resp. locally) Lipschitz continuous, then there exist positive constants  $M$ ,  $\kappa$  and  $r$  and a neighborhood  $\mathcal{N}$  of the origin in  $H \times Z$  such that for any  $[d, y_{\text{ref}}]$  in  $\mathcal{B}_{H \times Z}(0, r)$  (resp. in  $\{0\} \times \mathcal{B}_Z(0, r)$ ), the following results hold.

- (i) *There exists an equilibrium point  $[w^*, z^*] \in \mathcal{D}(\mathcal{A}) \times Z$ , which lies in  $\mathcal{N}$ , of the closed-loop system (4.15)-(4.16) such that  $Cw^* = y_{\text{ref}}$  and for all initial data  $[w_0, z_0]$  in  $\mathcal{N}$ , the corresponding solution  $[w, z]$  to (4.15)-(4.16) satisfies, for all  $t \geq 0$ ,*

$$\|[w(t) - w^*, z(t) - z^*]\|_{H \times Z} \leq M \exp(-\kappa t) \|[w_0 - w^*, z_0 - z^*]\|_{H \times Z}. \quad (4.18)$$

- (ii) *Moreover, if for all  $z \in Z$  and  $w \in H$ ,*

$$\|B^* d\mathcal{M}(w)^* z\|_Z^2 \geq \lambda \|z\|_Z^2 \quad (4.19)$$

*then for any initial data  $[w_0, z_0]$  in  $H \times Z$ , the corresponding solution  $[w, z]$  to (4.15)-(4.16) satisfies*

$$[w(t), z(t)] \rightarrow [w^*, z^*] \quad \text{in } H \times Z \quad \text{as } t \rightarrow +\infty. \quad (4.20)$$

The proof of Theorem 4.2.3 is given in Section 4.2.3, together with an outline of the strategy we followed and some appropriate auxiliary results. In terms of stability,

- Equation (4.18) means that the equilibrium  $[w^*, z^*]$  is locally exponentially stable, and its basin of attraction includes the origin;
- Under the uniform coercivity condition (4.19), the equilibrium  $[w^*, z^*]$  is also globally asymptotically stable by (4.20).

Note that (4.19) is stronger than a global version of the range condition (4.17): if  $\text{Range } d\mathcal{M}(w)B = Z$  for all  $w$  in  $H$ , then the adjoint argument provides a family of positive  $\lambda_w$  such that  $\|B^* d\mathcal{M}(w)^* z\|_Z^2 \geq \lambda_w \|z\|_Z^2$  for all  $w$  in  $H$  and  $z$  in  $Z$ ; however, the infimum of the  $\lambda_w$  might very well be 0.

*Remark 4.2.4* (Bassin of attraction). For the sake of clarity, we chose to establish, on one hand, a quite general local result (Item (i) of Theorem 4.2.3) under the relatively weak range condition (4.17), and on the other hand, a global result (Item (ii)) under the very strong condition (4.19). However, it will appear clear that the basin of attraction of the equilibrium  $[w^*, z^*]$  is, roughly speaking, as large as the set where (4.19) holds. This can be inferred from the proof of Lemma 4.2.7 below.

Let us conclude the section with an important point that is specific to the infinite-dimensional settings. It is clear that if the operator  $C$  is continuous, i.e.,  $C$  belongs to  $\mathcal{L}(H, Z)$ , then the output  $Cw(t)$  goes to  $y_{\text{ref}}$  in  $Z$  whenever  $w(t)$  goes to  $w^*$  in  $H$ . If  $C$  is unbounded, then  $Cw$  need not be defined for all closed-loop solutions, and even for *strong* solutions, it need not be continuous. This is further discussed at the end of chapter.

## 4.2.2 Change of coordinates and proof of well-posedness

In what follows, we will investigate the properties of the closed-loop system (4.15)-(4.16) in the new coordinates  $[w, \eta]$  where  $\eta$  is given by

$$\eta \triangleq z - \mathcal{M}(w). \quad (4.21)$$

In  $[w, \eta]$ -coordinates, (4.15)-(4.16) can be equivalently rewritten as

$$\frac{dw}{dt} + \mathcal{A}(w) = BB^* d\mathcal{M}(w)^* \eta + d, \quad (4.22a)$$

$$\frac{d\eta}{dt} + d\mathcal{M}(w)BB^* d\mathcal{M}(w)^* \eta = -y_{\text{ref}} - d\mathcal{M}(w)d. \quad (4.22b)$$

Equation (4.22b) is obtained by differentiating (4.21) and using (4.13) combined with (4.22a). By continuity of  $\mathcal{M}$ , the change of variables is continuous. In particular, given  $[w^*, z^*] \in H \times Z$ , any solution  $[w, z]$  to (4.15)-(4.16) converges to  $[w^*, z^*]$  if and only if  $[w, \eta]$  converges to  $[w^*, \eta^*]$ , where  $\eta = z - \mathcal{M}(z)$  and  $\eta^* = z^* - \mathcal{M}(w^*)$ . Moreover, given a ball of  $H$  containing  $w^*$ , by local Lipschitz continuity of  $\mathcal{M}$ , there exist positive constants  $K_1$  and  $K_2$  (depending on said ball) such that

$$K_1 \| [w - w^*, z - z^*] \|_{H \times Z} \leq \| [w - w^*, \eta - \eta^*] \|_{H \times Z} \leq K_2 \| [w - w^*, z - z^*] \|_{H \times Z} \quad (4.23)$$

whenever  $w$  lies in that ball. Bearing in mind the statements of Theorems 4.2.2 and 4.2.3, this means that well-posedness and existence of a locally exponentially stable and/or global attractive equilibrium can be investigated in the new  $[w, \eta]$ -coordinates, i.e., by considering (4.22).

Before proving Theorem 4.2.2, in order to alleviate notation when needed, let us define

$$\mathcal{K}(w) \triangleq d\mathcal{M}(w)B \quad \text{for all } w \in H. \quad (4.24)$$

*Proof of Theorem 4.2.2.* We start by proving that solutions to (4.22) exist at least on a finite time interval (Items (i) and (ii) of Theorem 4.2.2), and then we investigate forward completeness (Item (iii)).

*Step 1: Local well-posedness.* First, we observe that (4.22) represents a locally Lipschitz perturbation of the following maximal monotone problem:

$$\frac{d}{dt} [w, \eta] + [\mathcal{A}(w), \eta] = 0. \quad (4.25)$$

Indeed, by letting

$$\mathcal{F}_{d, y_{\text{ref}}}[w, \eta] \triangleq \begin{bmatrix} -B\mathcal{K}(w)^* \eta - d \\ -\eta + \mathcal{K}(w)\mathcal{K}(w)^* \eta + y_{\text{ref}} + d\mathcal{M}(w)d \end{bmatrix} \quad \text{for all } [w, \eta] \in H \times Z, \quad (4.26)$$

we can rewrite (4.22) as follows:

$$\frac{d}{dt} [w, \eta] + [\mathcal{A}(w), \eta] + \mathcal{F}_{d, y_{\text{ref}}}[w, \eta] = 0. \quad (4.27)$$

The nonlinear map  $\mathcal{F}_{d, y_{\text{ref}}}$  is locally Lipschitz continuous on  $H \times Z$ . Besides, since  $\mathcal{A}$  is maximal monotone on  $H$ , so is the mapping  $[w, \eta] \mapsto [\mathcal{A}(w), \eta]$  on  $H \times Z$  (with dense domain  $\mathcal{D}(\mathcal{A}) \times Z$ ). Furthermore,  $[\mathcal{A}(0), 0] = 0$ . Thus, it follows from [Chueshov et al., 2002, Theorem 7.2] that for each initial condition  $[w_0, \eta_0] \in H \times Z$ , there exists a unique maximal generalized solution  $[w, \eta]$  to (4.22) defined on  $[0, T_{\text{max}})$ , with  $T_{\text{max}} \in (0, +\infty]$ . Moreover, if  $w_0 \in \mathcal{D}(\mathcal{A})$ ,  $[w, \eta]$  is in fact a strong solution. Recalling that  $z = \eta + \mathcal{M}(w)$  and  $\mathcal{M} \in \mathcal{C}^1(H, Z)$ , we obtain Items (i) and (ii). Finally, the aforementioned theorem also states that if  $T_{\text{max}}$  is finite, then the norm of  $[w(t), \eta(t)]$  must go to  $+\infty$  as  $t$  approaches  $T_{\text{max}}$ .

*Step 2: Sufficient conditions for forward completeness.* Let us prove that if  $d = 0$  or  $d\mathcal{M}$  is globally Lipschitz continuous, then  $T_{\text{max}} = +\infty$  for any initial condition  $[w_0, \eta_0] \in H \times Z$ . Let  $\rho > 0$  to be fixed (large enough) later on. Given a (strong) solution  $[w, \eta]$  with initial data  $[w_0, \eta_0] \in \mathcal{D}(\mathcal{A}) \times Z$ , we have

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \{ \|w\|_H^2 + \rho \|\eta\|_Z^2 \} &= -(\mathcal{A}(w), w)_H + (B\mathcal{K}(w)^* \eta, w)_H + (d, w)_H \\ &\quad - \rho \|\mathcal{K}(w)^* \eta\|_E^2 - \rho (y_{\text{ref}}, \eta)_Z - \rho (d\mathcal{M}(w)d, \eta)_Z \end{aligned} \quad (4.28)$$

holding a.e. on  $(0, T_{\max})$ . Similarly as in our preliminary Lyapunov analysis in Section 4.1.2, in order to deal with the term  $B\mathcal{K}(w)^*\eta$ , we use (4.10) together with Young's inequality and choose  $2\rho \geq \|B\|_{\mathcal{L}(E,H)}^2$  to obtain

$$\frac{d}{dt} \{ \|w\|_H^2 + \rho \|\eta\|_Z^2 \} \leq 2\|w\|_H^2 + 2\rho \|\eta\|_Z^2 + \rho \|y_{\text{ref}}\|_Z^2 + \|d\|_H^2 + \rho \|\mathcal{d}\mathcal{M}(w)d\|_Z^2 \quad (4.29)$$

If  $d = 0$  or  $\mathcal{d}\mathcal{M}$  is globally Lipschitz continuous, the following inequality holds:

$$\|\mathcal{d}\mathcal{M}(w)d\|_Z^2 \leq K(1 + \|w\|_H^2)\|d\|_H^2 \quad (4.30)$$

for some  $K > 0$  independent of  $w$  or  $d$ . Combining (4.29) and (4.30) yields

$$\frac{d}{dt} \{ \|w\|_H^2 + \rho \|\eta\|_Z^2 \} \leq K' \{ \|w\|_H^2 + \rho \|\eta\|_Z^2 \} + K' \quad \text{a.e.} \quad (4.31)$$

for some  $K' > 0$  independent of the initial data. As a strong solution to (4.22),  $[w, \eta]$  is absolutely continuous in  $H \times Z$ ; therefore, we can deduce from (4.31) and Grönwall's inequality the following uniform estimate: for all  $t \in [0, T_{\max})$ ,

$$\|w(t)\|_H^2 + \rho \|\eta(t)\|_Z^2 \leq \exp(K't) \{ \|w_0\|_H^2 + \rho \|\eta_0\|_Z^2 + 1 \}. \quad (4.32)$$

We infer from (4.32) that the norm of  $[w, \eta]$  cannot blow up in finite time; thus,  $T_{\max} = +\infty$ . Furthermore, by passing to the limit, we see that (4.32) is satisfied for generalized solutions as well, which means that the same conclusion holds for any initial data in  $H \times Z$ . This proves Item (iii) and ends the proof of Theorem 4.2.2.  $\square$

### 4.2.3 Proof of the output regulation theorem

The proof of Theorem 4.2.3 relies on a series of lemmas that are given below. Let us first give some insight on the main strategy.

1. Under the range condition (4.17), the operators  $\mathcal{K}(w)^* = B^*\mathcal{d}\mathcal{M}(w)^*$  involved in the  $\eta$ -equation (4.22b) enjoy a coercivity property that is uniform with respect to  $w$ , provided that  $w$  remains in some ball around the origin. This is shown in Lemma 4.2.5.
2. This allows to prove that there exists a (not necessarily invariant) region around the origin of  $H \times Z$  where the dynamics generated by (4.22) are strictly contractive. This is stated in Lemma 4.2.6.
3. Another consequence of the local coercivity property is the existence of suitable attracting sets for the closed-loop dynamics (4.22), as demonstrated in Lemma 4.2.7. Furthermore, if the reference  $y_{\text{ref}}$  and the perturbation  $d$  are sufficiently small, then those sets are contained in the contraction region of the previous step.
4. Finally, existence of a locally exponentially stable equilibrium for (4.22) is shown by using Banach fixed point arguments and properties of  $\omega$ -limit sets associated with contraction semi-groups.
5. Additionally, under the global coercivity condition (4.19), the attracting sets from Lemma 4.2.7 are in fact globally attractive, hence the equilibrium is also globally asymptotically stable.

Coming back to the  $[w, z]$ -coordinates and thus to the original closed-loop system (4.15)-(4.16) is then straightforward. In what follows, we shall be careful regarding the dependence of the various neighborhoods involved in the analysis with respect to each parameter. Before proceeding further, we introduce some additional notation. Recalling (4.26) and (4.27), given  $[d, y_{\text{ref}}]$ , we denote by  $\mathcal{A}_{d, y_{\text{ref}}}$  the nonlinear operator

$$\mathcal{A}_{d, y_{\text{ref}}} \triangleq [\mathcal{A}, \text{id}] + \mathcal{F}_{d, y_{\text{ref}}} \quad (4.33)$$

defined on  $\mathcal{D}(\mathcal{A}) \times Z$  and associated with the closed-loop system (4.22) in the  $[w, \eta]$ -coordinates. Also, for  $\rho > 0$ , we denote by  $\|\cdot\|_{H \times Z, \rho}$  the Hilbert norm on  $H \times Z$  given by

$$\|[w, \eta]\|_{H \times Z, \rho}^2 \triangleq \|w\|_H^2 + \rho \|\eta\|_Z^2 \quad \text{for all } [w, \eta] \in H \times Z. \quad (4.34)$$

We denote by  $(\cdot, \cdot)_{H \times Z, \rho}$  the associated scalar product. All these norms are equivalent. Bearing in mind our Lyapunov analysis in the  $[w, z]$ -coordinates in Section 4.1.2, the  $\rho$ -norm is connected to the Lyapunov functional  $\mathcal{W}$  by

$$\mathcal{W}(w, z) = \frac{1}{2} \|[w, z - \mathcal{M}(w)]\|_{H \times Z, \rho}^2. \quad (4.35)$$

In the sequel, we either take  $d = 0$  or assume that  $d\mathcal{M}$  is globally Lipschitz continuous. Therefore, by virtue of Theorem 4.2.2, solutions to the closed-loop equations (4.22) are well-defined for all positive time and any initial condition in  $H \times Z$ . Recall from (4.17) in the statement of Theorem 4.2.3 that the following coercivity assumption is in force:

$$\|\mathcal{K}(0)^* z\|_E^2 \geq \lambda \|z\|_Z^2 \quad \text{for all } z \in Z. \quad (4.36)$$

The first lemma is a simple consequence of (4.36) and continuity of  $d\mathcal{M}$ .

**Lemma 4.2.5** (Local coercivity). *There exist positive constants  $\tilde{\lambda}$  and  $\tilde{r}$  such that*

$$\|\mathcal{K}(w)^* z\|_E^2 \geq \tilde{\lambda} \|z\|_Z^2 \quad \text{for all } z \in Z \text{ and } w \in \mathcal{B}_H(0, \tilde{r}). \quad (4.37)$$

*Proof.* Let  $w \in H$  and  $z \in Z$ . Recall that  $\mathcal{K}(w)^* z = B^* d\mathcal{M}(0)^* z + B^* [d\mathcal{M}(w) - d\mathcal{M}(0)]^* z$ . Using Cauchy-Schwarz and Young inequalities together with (4.36), we get

$$\|\mathcal{K}(w)^* z\|_E^2 \geq \frac{2\lambda}{3} \|z\|_H^2 - 2\|B\|_{\mathcal{L}(E, H)}^2 \|[d\mathcal{M}(w) - d\mathcal{M}(0)]^*\|_{\mathcal{L}(Z, E)}^2 \|z\|_Z^2. \quad (4.38)$$

By continuity of  $d\mathcal{M}$  at 0, we can choose  $\tilde{r} > 0$  such that

$$\|d\mathcal{M}(w) - d\mathcal{M}(0)\|_{\mathcal{L}(H, Z)}^2 \leq \frac{\lambda}{6\|B\|_{\mathcal{L}(E, H)}^2} \quad \text{for all } w \in \mathcal{B}_H(0, \tilde{r}). \quad (4.39)$$

Thus, by letting  $\tilde{\lambda} \triangleq \lambda/3 > 0$ , we obtain the desired inequality (4.37).  $\square$

The following constant appears in the next two lemmas:

$$\kappa \triangleq \min\{\alpha/4, \tilde{\lambda}/4\}. \quad (4.40)$$

Those concern the contraction property of the dynamics governed by (4.22) around the origin and the existence of attractive sets depending on  $[d, y_{\text{ref}}]$ .

**Lemma 4.2.6** (Local strong monotonicity). *There exists  $\rho_0 > 0$  such that the following property holds: for all  $\rho \geq \rho_0$ , there exists a positive  $r_{0,\rho} \leq \tilde{r}$  such that*

$$(\tilde{\mathcal{A}}_{d,y_{\text{ref}}}[w_1, \eta_1] - \tilde{\mathcal{A}}_{d,y_{\text{ref}}}[w_2, \eta_2], [w_1, \eta_1] - [w_2, \eta_2])_{H \times Z, \rho} \geq \kappa \| [w_1, \eta_1] - [w_2, \eta_2] \|_{H \times Z, \rho}^2 \quad (4.41)$$

for all  $[w_1, \eta_1], [w_2, \eta_2] \in \mathcal{B}_{H \times Z}(0, r_{0,\rho}) \cap \mathcal{D}(\mathcal{A}) \times Z$ ,  $d \in \mathcal{B}_H(0, r_{0,\rho})$ , and  $y_{\text{ref}} \in Z$ .

*Proof.* Let  $[w_1, \eta_1]$  and  $[w_2, \eta_2]$  in  $\mathcal{D}(\mathcal{A}) \times Z$ . We write  $\tilde{w} \triangleq w_1 - w_2$  and  $\tilde{\eta} \triangleq \eta_1 - \eta_2$ . Then, for any  $\rho > 0$ ,

$$\begin{aligned} (\tilde{\mathcal{A}}_{d,y_{\text{ref}}}[w_1, \eta_1] - \tilde{\mathcal{A}}_{d,y_{\text{ref}}}[w_2, \eta_2], [\tilde{w}, \tilde{\eta}])_{H \times Z, \rho} &= (\mathcal{A}(w_1) - \mathcal{A}(w_2) - B[\mathcal{K}(w_1)^* \eta_1 - \mathcal{K}(w_2)^* \eta_2], \tilde{w})_H \\ &\quad + \rho(\mathcal{K}(w_1)\mathcal{K}(w_1)^* \eta_1 - \mathcal{K}(w_2)\mathcal{K}(w_2)^* \eta_2 + [d\mathcal{M}(w_1) - d\mathcal{M}(w_2)]d, \tilde{\eta})_Z. \end{aligned} \quad (4.42)$$

By adding and removing some terms, (4.42) can be rewritten as follows:

$$\begin{aligned} (\tilde{\mathcal{A}}_{d,y_{\text{ref}}}[w_1, \eta_1] - \tilde{\mathcal{A}}_{d,y_{\text{ref}}}[w_2, \eta_2], [\tilde{w}, \tilde{\eta}])_{H \times Z, \rho} &= (\mathcal{A}(w_1) - \mathcal{A}(w_2), \tilde{w})_H \\ &\quad + \rho(\mathcal{K}(w_1)\mathcal{K}(w_1)^* \tilde{\eta}, \tilde{\eta})_Z + \rho([\mathcal{K}(w_1)\mathcal{K}(w_1)^* - \mathcal{K}(w_2)\mathcal{K}(w_2)^*] \eta_2, \tilde{\eta})_Z \\ &\quad - (B\mathcal{K}(w_1)^* \tilde{\eta}, \tilde{w})_H - (B[\mathcal{K}(w_1) - \mathcal{K}(w_2)]^* \eta_2, \tilde{w})_H + \rho([d\mathcal{M}(w_1) - d\mathcal{M}(w_2)]d, \tilde{\eta})_Z. \end{aligned} \quad (4.43)$$

Assume for a moment that  $\|w_i\|_H \leq \tilde{r}/2$ , where  $\tilde{r}$  is given by Lemma 4.2.5. Therefore,  $\|\tilde{w}\|_H \leq \tilde{r}$ ; and using (4.10) and (4.37) together with the Lipschitz continuity of  $\mathcal{K}$  and  $\mathcal{K}(\cdot)\mathcal{K}(\cdot)^*$  on  $\mathcal{B}_H(0, \tilde{r})$ , we infer from (4.43) that

$$\begin{aligned} (\tilde{\mathcal{A}}_{d,y_{\text{ref}}}[w_1, \eta_1] - \tilde{\mathcal{A}}_{d,y_{\text{ref}}}[w_2, \eta_2], [\tilde{w}, \tilde{\eta}])_{H \times Z, \rho} &\geq \alpha \|\tilde{w}\|_H^2 + \rho \tilde{\lambda} \|\tilde{\eta}\|_Z^2 \\ &\quad - \rho K_1 \|\tilde{w}\|_H \|\eta_2\|_Z \|\tilde{\eta}\|_Z - K_2 \|\tilde{\eta}\|_Z \|\tilde{w}\|_H - K_3 \|\eta_2\|_Z \|\tilde{w}\|_H^2 - \rho K_4 \|\tilde{w}\|_H \|d\|_H \|\tilde{\eta}\|_Z. \end{aligned} \quad (4.44)$$

where the  $K_i$  are some positive constants independent of  $[w_i, \eta_i]$ ,  $[d, y_{\text{ref}}]$ , and  $\rho$ . Given  $\varepsilon > 0$ , we employ Cauchy-Schwarz and Young inequalities to obtain

$$\begin{aligned} (\tilde{\mathcal{A}}_{d,y_{\text{ref}}}[w_1, \eta_1] - \tilde{\mathcal{A}}_{d,y_{\text{ref}}}[w_2, \eta_2], [\tilde{w}, \tilde{\eta}])_{H \times Z, \rho} &\geq \alpha \|\tilde{w}\|_H^2 + \rho \tilde{\lambda} \|\tilde{\eta}\|_Z^2 \\ &\quad - \frac{\rho}{2} \{K_1 \|\eta_2\|_Z + K_4 \|d\|_H\} \{\|\tilde{w}\|_H^2 + \|\tilde{\eta}\|_Z^2\} - \frac{K_2}{2\varepsilon} \|\tilde{\eta}\|_Z^2 - \frac{\varepsilon K_2}{2} \|\tilde{w}\|_H^2 - K_3 \|\eta_2\|_Z \|\tilde{w}\|_H^2. \end{aligned} \quad (4.45)$$

Let  $\varepsilon = \alpha/(2K_2)$  in (4.45) and define  $\rho_0 \triangleq 2K_2/(\tilde{\lambda}\varepsilon)$ . For all  $\rho \geq \rho_0$ ,

$$\begin{aligned} (\tilde{\mathcal{A}}_{d,y_{\text{ref}}}[w_1, \eta_1] - \tilde{\mathcal{A}}_{d,y_{\text{ref}}}[w_2, \eta_2], [\tilde{w}, \tilde{\eta}])_{H \times Z, \rho} &\geq \frac{\alpha}{2} \|\tilde{w}\|_H^2 + \frac{\rho \tilde{\lambda}}{2} \|\tilde{\eta}\|_Z^2 \\ &\quad - \frac{\rho}{2} \{K_1 \|\eta_2\|_Z + K_4 \|d\|_H\} \{\|\tilde{w}\|_H^2 + \|\tilde{\eta}\|_Z^2\} - K_3 \|\eta_2\|_Z \|\tilde{w}\|_H^2. \end{aligned} \quad (4.46)$$

Let  $\rho \geq \rho_0$ . We infer from (4.46) that there exists  $r_{0,\rho} \leq \tilde{r}/2$  such that

$$(\tilde{\mathcal{A}}_{d,y_{\text{ref}}}[w_1, \eta_1] - \tilde{\mathcal{A}}_{d,y_{\text{ref}}}[w_2, \eta_2], [\tilde{w}, \tilde{\eta}])_{H \times Z, \rho} \geq \frac{\alpha}{4} \|\tilde{w}\|_H^2 + \frac{\rho \tilde{\lambda}}{4} \|\tilde{\eta}\|_Z^2 \quad (4.47)$$

as long as  $[w_i, \eta_i] \in \mathcal{B}_{H \times Z}(0, r_{0,\rho})$  and  $d \in \mathcal{B}_H(0, r_{0,\rho})$ , which completes the proof.  $\square$

**Lemma 4.2.7** (Absorbing balls). *There exists  $\rho_1 > 0$  such that the following property holds: for any  $\rho \geq \rho_1$ , there exist positive  $r_{1,\rho}$  and  $K_\rho$  such that, if  $[d, y_{\text{ref}}] \in \mathcal{B}_{H \times Z}(0, r_{1,\rho})$ , then the estimate*

$$\|[w(t), \eta(t)]\|_{H \times Z, \rho}^2 \leq \exp(-\kappa t) \|[w_0, \eta_0]\|_{H \times Z, \rho}^2 + K_\rho \{1 - \exp(-\kappa t)\} \|[d, y_{\text{ref}}]\|_{H \times Z}^2 \quad (4.48)$$

holds for any solution  $[w, \eta]$  to (4.22) with initial data  $[w_0, \eta_0]$  in  $\mathcal{B}_{H \times Z}(0, r_{1,\rho})$ . Furthermore, if (4.37) holds globally, then (4.48) is true for all initial data in  $H \times Z$ .

*Proof.* All formal computations performed below are justified by considering appropriate sequence of strong solutions as provided by Theorem 4.2.2 and then passing to the limit at the very end. First, let

$$\rho_1 \triangleq \|B\|_{\mathcal{L}(E,H)}^2 \max\{1, 2\alpha^{-1}\}. \quad (4.49)$$

Let  $\rho \geq \rho_1$  be fixed and let  $[w, \eta]$  be a solution to (4.22) with initial condition  $[w_0, \eta_0]$ . As in the proof of Theorem 4.2.2, using (4.28) we obtain

$$\frac{d}{dt} \|[w, \eta]\|_{H \times Z, \rho}^2 \leq k_\rho \{ \|[w, \eta]\|_{H \times Z, \rho}^2 + \|y_{\text{ref}}\|_Z^2 + \|d\|_H^2 \} \quad (4.50)$$

for some  $k_\rho > 0$  independent of  $[w_0, \eta_0]$  and  $[d, y_{\text{ref}}]$ . It follows from (4.50) that

$$\|[w(t), \eta(t)]\|_{H \times Z, \rho}^2 \leq \exp(k_\rho t) \|[w_0, \eta_0]\|_{H \times Z, \rho}^2 + \exp(k_\rho t) \{ \|y_{\text{ref}}\|_Z^2 + \|d\|_H^2 \} \quad (4.51)$$

for all  $t \in [0, 1]$ . As a consequence of (4.51), there exists a positive constant  $r_\rho$  such that

$$\max\{\|w_0\|_H, \|\eta_0\|_Z, \|d\|_H, \|y_{\text{ref}}\|_Z\} \leq r_\rho \quad \text{implies} \quad \|w(t)\|_H \leq \tilde{r} \quad \text{for all } t \in [0, 1], \quad (4.52)$$

where  $\tilde{r}$  is defined in Lemma 4.2.5. Therefore, for such initial data, we can use (4.37) in (4.28) to refine our previous estimate: for all positive  $\varepsilon$  and  $\mu$ , on  $(0, 1)$  we have

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|[w, \eta]\|_{H \times Z, \rho}^2 &\leq -\alpha \|w\|_H^2 + \frac{1}{2\varepsilon} \|B\|_{\mathcal{L}(E,H)}^2 \|\mathcal{X}(w)^* \eta\|_U^2 + \varepsilon \|w\|_H^2 \\ &\quad + \frac{1}{2\varepsilon} \|d\|_H^2 - \rho \|\mathcal{X}(w)^* \eta\|_U^2 + \frac{\rho}{2\mu} \|y_{\text{ref}}\|_Z^2 + \frac{\rho}{2\mu} \|d\mathcal{M}(w)d\|_Z^2 + \rho\mu \|\eta\|_Z^2. \end{aligned} \quad (4.53)$$

First, recall that we have either  $d\mathcal{M}$  globally Lipschitz continuous or  $d = 0$ . Hence, there exists  $k > 0$  independent of  $[w_0, \eta_0]$  and  $[d, y_{\text{ref}}]$  such that  $\|d\mathcal{M}(w)d\|_Z^2 \leq k \|d\|_H^2$ . By choosing  $\mu = \tilde{\lambda}/4$  and  $\varepsilon = \alpha/2$ , we deduce from (4.53) the following differential inequality, valid on  $(0, 1)$ :

$$\frac{1}{2} \frac{d}{dt} \|[w, \eta]\|_{H \times Z, \rho}^2 \leq -\frac{\alpha}{2} \|w\|_H^2 - \frac{\rho \tilde{\lambda}}{4} \|\eta\|_Z^2 + \frac{\rho}{2\mu} \|y_{\text{ref}}\|_Z^2 + \left\{ \frac{\rho k}{2\mu} + \frac{1}{2\varepsilon} \right\} \|d\|_H^2 \quad (4.54)$$

Applying Grönwall's inequality to (4.54) yields

$$\|[w(t), \eta(t)]\|_{H \times Z, \rho}^2 \leq \exp(-\kappa t) \|[w_0, \eta_0]\|_{H \times Z, \rho}^2 + K_\rho (1 - \exp(-\kappa t)) \|[d, y_{\text{ref}}]\|_{H \times Z}^2 \quad (4.55)$$

for all  $t \in [0, 1]$ , where  $\kappa$  is defined in (4.40) and  $K_\rho \geq 1$  is some constant independent of  $[w_0, \eta_0]$  and  $[d, y_{\text{ref}}]$ . Next, by norm equivalence, there exists  $r_{1,\rho} > 0$  such that the following implication holds: if  $[w_0, \eta_0]$  and  $[d, y_{\text{ref}}]$  are in  $\mathcal{B}_{H \times Z}(0, r_{1,\rho})$ , then

$$\|[w_0, \eta_0]\|_{H \times Z, \rho} \leq 2^{-1/2} r_\rho, \quad \text{and} \quad \|[d, y_{\text{ref}}]\|_{H \times Z} \leq (2K_\rho)^{-1/2} r_\rho. \quad (4.56)$$

Now, we claim that the estimate (4.55) remains valid for all  $t \geq 0$ . Indeed, (4.55) shows that  $\|[w(1), \eta(1)]\|_{H \times Z, \rho} \leq r_\rho$ . Therefore, by definition of  $r_\rho$ , we infer from the estimate (4.51) applied to the initial data  $[w(1), \eta(1)]$  that  $\|w(t)\|_H \leq \tilde{r}$  for all  $t \in [1, 2]$ . As a consequence, the differential inequality (4.54) is valid on  $(0, 2)$ ; hence, (4.55) holds on  $[0, 2]$ , with in particular  $\|[w(2), \eta(2)]\|_{H \times Z, \rho} \leq r_\rho$ , and so on. The conclusion readily follows by induction.

Moreover, if it is assumed that (4.37) holds for all  $w \in H$ , then (4.54) is valid on  $(0, +\infty)$  whatever the initial condition, so that (4.51) immediately holds for all  $t \geq 0$ . In this case, no additional condition on  $[d, y_{\text{ref}}]$  is required.  $\square$



We are finally ready to prove Theorem 4.2.3 by following the strategy we outlined above.

*Proof of Theorem 4.2.3.* The proof is split into four steps for the reader's convenience.

*Step 1: Setting all neighborhoods.* Pick  $\rho$  such that  $\rho \geq \max\{\rho_0, \rho_1\}$  as in Lemmas 4.2.6 and 4.2.7. In the sequel,  $[d, y_{\text{ref}}]$  is assumed to lie in the intersection of  $\mathcal{B}_H(0, r_{0,\rho}) \times Z$  and  $\mathcal{B}_{H \times Z}(0, r_{1,\rho})$ , so that the lemmas apply. Now that  $\rho$  is fixed, we will omit the dependence on  $\rho$  in further notation. Lemma 4.2.6 provides a neighborhood  $\mathfrak{K}$  of the origin in  $H \times Z$ , which we will refer to as the contraction region, where (4.41) holds. On the other hand, according to (4.48) in Lemma 4.2.7, the set

$$\mathfrak{V}_{d,y_{\text{ref}}} \triangleq \mathcal{B}_{H \times Z, \rho}(0, K^{1/2} \| [d, y_{\text{ref}}] \|_{H \times Z}) \quad (4.57)$$

attracts all solutions to (4.22) originating from  $\mathcal{B}_{H \times Z}(0, r_{1,\rho})$ . By norm equivalence, any sufficiently small ball for the  $\rho$ -norm that is centered at the origin is contained in both  $\mathfrak{K}$  and  $\mathcal{B}_{H \times Z}(0, r_{1,\rho})$ . That being said, as a consequence of (4.48) and (4.57), there exists positive numbers  $\delta$  and  $r$  such that, having let  $\mathcal{B} \triangleq \mathcal{B}_{H \times Z, \rho}(0, \delta)$ , the following properties hold for any  $[d, y_{\text{ref}}] \in \mathcal{B}_{H \times Z}(0, r)$ :

- The closure of the corresponding attracting set  $\mathfrak{V}_{d,y_{\text{ref}}}$  is contained in  $\mathcal{B}$ ;
- Solutions to (4.22) with initial data in  $\mathcal{B}$  remain in  $\mathfrak{K}$ .

If in addition we assume that (4.19) holds, i.e., (4.37) holds globally, then by Lemma 4.2.7,

- The set  $\mathfrak{V}_{d,y_{\text{ref}}}$  attracts all solutions to (4.22), whatever the initial data.

In what follows, we omit the dependence on  $[d, y_{\text{ref}}]$  in the notation and we denote by  $\{\tilde{\mathcal{T}}_t\}$  the strongly continuous semigroup associated with (4.22). Then, since solutions originating from (the nonempty open set)  $\mathcal{B}$  remain in the contraction region  $\mathfrak{K}$ , we infer from (4.41) together with a density argument that

$$\|\tilde{\mathcal{T}}_t[w_1, \eta_1] - \tilde{\mathcal{T}}_t[w_2, \eta_2]\|_{H \times Z, \rho} \leq \exp(-\kappa t) \| [w_1, \eta_1] - [w_2, \eta_2] \|_{H \times Z, \rho} \quad (4.58)$$

for all  $t \geq 0$  and  $[w_i, \eta_i] \in \mathcal{B}$ ,  $i \in \{1, 2\}$ .

*Step 2: Existence of a fixed point.* Pick an arbitrary  $[w_0, \eta_0] \in \mathcal{B}$ . By (4.58) and a usual contraction argument, we see that

$$\{\tilde{\mathcal{T}}_n[w_0, \eta_0]\}_{n \geq 0} \text{ is a Cauchy sequence in } H \times Z \quad (4.59)$$

and converges to a fixed point  $[w_0^*, \eta_0^*]$  of the nonlinear operator  $\tilde{\mathcal{T}}_1$ . Now, consider the  $\omega$ -limit set  $\omega([w_0, \eta_0])$  of  $[w_0, \eta_0]$  with respect to the evolution semigroup  $\{\tilde{\mathcal{T}}_t\}$ . By the sequential characterization of  $\omega$ -limit sets (see [Chueshov, 2002, Lemma 2.1, p.19]), we observe that  $[w_0^*, \eta_0^*] \in \omega([w_0, \eta_0])$ , which means that  $\omega([w_0, \eta_0])$  is nonempty. Moreover, it is positively invariant by definition. Finally, since  $\mathfrak{V}_{d,y_{\text{ref}}}$  attracts all solutions originating from  $\mathcal{B}$ , we must have

$$\omega([w_0, \eta_0]) \subset \overline{\mathfrak{V}_{d,y_{\text{ref}}}} \subset \mathcal{B}. \quad (4.60)$$

By following *verbatim*<sup>3</sup> the proof of [Dafermos and Slemrod, 1973, Theorem 1], we obtain that for each  $t \geq 0$ ,  $\tilde{\mathcal{T}}_t$  is an isometry on  $\omega([w_0, \eta_0])$ . On the other hand, for positive  $t$ ,  $\tilde{\mathcal{T}}_t$  is a strict contraction on  $\omega([w_0, \eta_0])$ ; thus,  $\omega([w_0, \eta_0])$  must be reduced to the singleton  $\{[w_0^*, \eta_0^*]\}$ . By invariance of the  $\omega$ -limit set,  $[w_0^*, \eta_0^*]$  is fixed by the semigroup  $\{\tilde{\mathcal{T}}_t\}$ . Moreover, it follows from (4.58) that

<sup>3</sup> The only difference with [Dafermos and Slemrod, 1973, Theorem 1] is that  $\{\tilde{\mathcal{T}}_t\}$  is a contraction only on a region (containing the  $\omega$ -limit set) that is not *a priori* positively invariant. However, in order to obtain the isometry property, contraction is only needed on the points of the  $\omega$ -limit set.

$[w_0^*, \eta_0^*]$  is the unique fixed point of  $\{\tilde{\mathcal{T}}_t\}$  in  $\mathcal{B}$  and is exponentially attractive in  $\mathcal{B}$ . Thus, we write  $[w^*, \eta^*] \triangleq [w_0^*, \eta_0^*]$ .

*Step 3: The fixed point lies in the domain.* We now prove that  $w^* \in \mathcal{D}(\mathcal{A})$ . Let  $\varepsilon > 0$  sufficiently small and consider the ball

$$\mathcal{C} \triangleq \mathcal{B}_{H \times Z, \rho}([w^*, \eta^*], \varepsilon) \subset \mathcal{R}. \quad (4.61)$$

Since  $\mathcal{D}(\mathcal{A})$  is dense in  $H$  and  $\mathcal{C}$  has nonempty interior, we can pick some  $[w_0, \eta_0] \in [\mathcal{D}(\mathcal{A}) \times Z] \cap \mathcal{C}$ . Let  $[w(t), \eta(t)] \triangleq \tilde{\mathcal{T}}_t[w_0, \eta_0]$ . As a strong solution to (4.22),  $[w, \eta]$  is differentiable a.e. in  $H \times Z$  and

$$\frac{d}{dt}[w, \eta] + \tilde{\mathcal{A}}_{d, y_{\text{ref}}}[w, \eta] = 0 \quad \text{a.e.} \quad (4.62)$$

Besides, we infer from (4.58) that  $\mathcal{C}$  is positively invariant so that  $\{\tilde{\mathcal{T}}_t\}$  restricted to  $\mathcal{C}$  is still a well-defined contraction semigroup. Thus, we can apply [Crandall and Pazy, 1969, Theorem 1.4] to obtain

$$\|\tilde{\mathcal{A}}_{d, y_{\text{ref}}}[w, \eta]\|_{H \times Z, \rho} \leq \|\tilde{\mathcal{A}}_{d, y_{\text{ref}}}[w(t_0), \eta(t_0)]\|_{H \times Z, \rho} \quad \text{a.e. on } (t_0, +\infty) \quad (4.63)$$

for some  $t_0 \geq 0$ . In particular,  $\|\mathcal{A}(w)\|_H$  is bounded a.e. on  $(t_0, +\infty)$ . On the other hand,  $w(t)$  converges to  $w^*$  in  $H$  when  $t$  goes to  $+\infty$ . Therefore, it follows from  $\mathcal{A}$  being maximal monotone and [Crandall and Pazy, 1969, Lemma 2.3] that  $w^* \in \mathcal{D}(\mathcal{A})$ .

*Step 4: Conclusion.* Since  $[w^*, \eta^*]$  belongs to  $\mathcal{D}(\mathcal{A}) \times Z$  and is fixed by  $\{\tilde{\mathcal{T}}_t\}$ ,  $\tilde{\mathcal{A}}_{d, y_{\text{ref}}}[w^*, \eta^*] = 0$ . We come back to the original  $[w, z]$  coordinates by letting  $z^* \triangleq \eta^* + \mathcal{M}(w^*)$  and  $\mathcal{N} \triangleq \{[w, \eta + \mathcal{M}(w)], [w, \eta] \in \mathcal{B}\}$ . Then,  $[w^*, z^*]$  belongs to  $[\mathcal{D}(\mathcal{A}) \times Z] \cap \mathcal{N}$  and is an equilibrium for (4.15)-(3.7); hence,  $Cw^* = y_{\text{ref}}$ . Because  $\mathcal{M}$  vanishes at 0 and is continuous,  $\mathcal{N}$  is indeed a neighborhood of 0. Local exponential stability of  $[w^*, z^*]$  with decay rate  $\kappa$  and bassin of attraction containing  $\mathcal{N}$  follows from (4.58) and our prior remarks regarding the change of coordinates; the constant  $M$  in (4.11) comes from (4.23) and equivalence with the  $\rho$ -norm. Additionally, under the stronger condition (4.19),  $[w^*, z^*]$  is globally asymptotically stable. The proof is now complete.  $\square$

### 4.3 The semilinear case

Our objective in this section is to prove the existence (in fact, build “explicitly”) a solution  $\mathcal{M}$  to (4.13) in the semilinear settings, thereby guaranteeing the existence of the control law (4.16) for a wide class of systems. We will see that for such a map, the requirements of Theorem 4.2.3 translate into (mostly) easy to verify conditions on the original  $w$ -dynamics, the input operator  $B$  and the output operator  $C$ . This is demonstrated on examples in Section 4.3.3.

#### 4.3.1 Existence of an invariant graph for the cascade dynamics

Consider the case where  $\mathcal{A} = A + F$  with  $\mathcal{D}(\mathcal{A}) = \mathcal{D}(A)$ , where  $-A$  is the infinitesimal generator of a strongly continuous semigroup  $\{\mathcal{S}_t\}$  of linear operators on  $H$ , and  $F$  is a nonlinear mapping satisfying

$$F \in \mathcal{C}^1(H), \quad dF \text{ locally Lipschitz continuous}, \quad (4.64)$$

and without loss of generality,  $dF(0) = 0$ . Following [Pazy, 2012], we shall say that  $\mathcal{A}$  is semilinear. In that context, let us introduce the following set of assumptions.

*Hypothesis 4.3.1.* The operator  $\mathcal{A} = A + F$  is semilinear and satisfies the following properties:

- (i)  $C$  is  $A$ -bounded, i.e., there exist positive constants  $a$  and  $b$  such that

$$\|Cw\|_Z \leq a\|Aw\|_H + b\|w\|_H \quad \text{for all } w \in \mathcal{D}(A); \quad (4.65)$$

- (ii) 0 is in the resolvent set of  $A$ , i.e.,  $A^{-1} \in \mathcal{L}(H)$ ;
- (iii) There exists  $\alpha > 0$  such that

$$(Ah + dF(w)h, h)_H \geq \alpha \|h\|_H^2 \quad \text{for all } w \in H \text{ and } h \in \mathcal{D}(A). \quad (4.66)$$

We claim that Hypothesis 4.3.1 implies Hypothesis 4.1.1. Indeed, writing  $F(w_1) - F(w_2)$  as an integral of  $dF$  along the line segment joining  $w_1$  to  $w_2$ , one can show that (4.66) implies (4.10). Then, standard results on Lipschitz perturbations of linear systems (see [Pazy, 2012, Section 6.1]) together with (4.10) yield that  $-\mathcal{A}$  generates a strongly continuous semigroup of contractions (denoted by  $\{\mathcal{T}_t\}$ ) on  $H$ , which in turn implies that  $\mathcal{A}$  is maximal monotone by virtue of [Komura, 1969, Theorem 4]. Under Hypothesis 4.3.1, the functional equation (4.13) admits a solution  $\mathcal{M}$ , which is a function of  $C, A, F$  and the original semigroup  $\{\mathcal{T}_t\}$  – meaning that Hypothesis 4.3.1 also implies Hypothesis 4.1.2. This is shown in the next theorem.

**Theorem 4.3.2** (Existence of  $\mathcal{M}$  in the semilinear case). *Assume Hypothesis 4.3.1 is satisfied. Then there exists a map  $\mathcal{M} \in \mathcal{C}^1(H, Z)$  satisfying (4.13) with  $d\mathcal{M}$  locally Lipschitz continuous and given by*

$$\mathcal{M}(w) \triangleq -C \left\{ \lim_{\tau \rightarrow +\infty} \int_0^\tau \mathcal{T}_t w \, dt \right\} = -CA^{-1}w + CA^{-1} \int_0^{+\infty} F(\mathcal{T}_t w) \, dt \quad \text{for all } w \in H, \quad (4.67)$$

where the limit on the left-hand side of (4.67) is taken in  $H$  and the integrand on the right-hand is absolutely integrable in  $H$ . The differential  $d\mathcal{M}$  is given by

$$d\mathcal{M}(w)h = -CA^{-1}h + CA^{-1} \int_0^{+\infty} dF(\mathcal{T}_t w) d\mathcal{T}_t(w)h \, dt \quad \text{for all } w, h \in H, \quad (4.68)$$

where  $d\mathcal{T}_t(w)$  denotes the differential of  $\mathcal{T}_t$  at  $w$ . Moreover, if  $F$  and  $dF$  are globally Lipschitz continuous, then so is  $d\mathcal{M}$ .

The proof of Theorem 4.3.2, including the limit equality in (4.67) and the differentiability of the semigroup  $\{\mathcal{T}_t\}$ , is given below in Section 4.3.2. A notable consequence of (4.68) is that the coercivity condition (4.17) in Theorem 4.2.3 simply reads as

$$\text{Range } CA^{-1}B = Z, \quad (4.69)$$

which, in the context of output regulation of finite-dimensional linear systems, corresponds to a *non-resonance* condition between  $A$  and the zero dynamics of the integrator via the Schur complement – see, e.g., [Isidori et al., 2003]. On the other hand, Theorem 4.3.2 also states that  $d\mathcal{M}$  inherits the Lipschitz properties of  $F$  and  $dF$ .

Now, regarding Hypothesis 4.3.1, (4.66) is a sufficient condition under which solutions to the uncontrolled  $w$ -equation linearized around a given trajectory (also called *first variation equation*, see (4.86) below) uniformly converge to that trajectory. Equation (4.66) is easily verified in (at least) two situations of interest.

- The nonlinearity  $F$  contributes to the contraction behavior of the  $w$ -dynamics. This is the case for example if  $A$  is coercive and  $F$  is monotone (i.e. such that  $(dF(w)h, h)_H$  is nonnegative for all  $w, h$  in  $H$ ). More generally, the reader may refer to [Lohmiller and Slotine, 1998] in finite dimension or [Temam, 1997, Chapter V] for the contraction analysis of the system by means of its first variation equation.

- Variations of  $F$  are small with respect to linear dissipation brought by  $A$ . More precisely, if  $F$  is  $K$ -Lipschitz continuous, then  $\|dF(\cdot)\|_{\mathcal{L}(H)}$  is bounded by  $K$ , hence (4.66) is satisfied if  $(Ah, h)_H \geq \beta \|h\|_H^2$  for some constant  $\beta > K$ . In the same spirit, under (4.66) and (4.69), by observing<sup>4</sup> that for each  $w$  the integral map in (4.68) has operator norm bounded by  $K/\alpha$ , one deduces from (4.68) together with a Neumann series argument that *global* uniform coercivity (4.19) holds whenever  $K < \alpha$ .

*Remark 4.3.3* (Linear case). If  $\mathcal{A}$  is linear, i.e.,  $F = 0$ , and 0 lies in the resolvent set of  $A$ , then the solution  $\mathcal{M}$  to (4.13) is unique and given by  $\mathcal{M} = -CA^{-1}$ .

### 4.3.2 Additional properties in the semilinear settings and proof

We start by establishing some auxiliary results valid under the hypotheses of Section 4.3. In what follows,  $\mathcal{D}(A)$  is endowed with the graph norm:

$$\|w\|_{\mathcal{D}(A)}^2 \triangleq \|w\|_H^2 + \|Aw\|_H^2 \quad \text{for all } w \in \mathcal{D}(A). \quad (4.70)$$

Of course, in the semilinear case, the uncontrolled  $w$ -equation

$$\frac{dw}{dt} + \mathcal{A}(w) = 0 \quad (4.71)$$

rewrites as the following perturbed linear equation:

$$\frac{dw}{dt} + Aw + F(w) = 0. \quad (4.72)$$

The next lemmas provide some additional properties of solutions to the uncontrolled  $w$ -equation (4.71) that stem from its semilinear structure.

**Lemma 4.3.4.** *Generalized solutions  $w$  to (4.71) with initial data  $w_0 \in H$  are characterized by*

$$w(t) = \mathcal{S}_t w_0 - \int_0^t \mathcal{S}_{t-s} F(w(s)) ds \quad \text{for all } t \geq 0, \quad (4.73)$$

where we recall that  $\{\mathcal{S}_t\}$  is the semigroup of linear operators generated by  $-A$ . Furthermore, strong solutions  $w$  to (4.71) enjoy the additional regularity

$$w \in \mathcal{C}(\mathbb{R}^+, \mathcal{D}(A)) \cap \mathcal{C}^1(\mathbb{R}^+, H). \quad (4.74)$$

*Proof.* Let  $T > 0$  and  $w^0 \in H$ . First, let us prove that if  $w \in \mathcal{C}([0, T], H)$  satisfies (4.73), then it is a limit of strong solutions to (4.71). Pick a sequence  $\{w_n^0\} \subset \mathcal{D}(\mathcal{A})$  such that  $w_n^0 \rightarrow w^0$  in  $H$ . Then, by the contraction property of  $\{\mathcal{S}_t\}$ , letting  $w_n(t) \triangleq \mathcal{S}_t w_n^0$  and  $\tilde{w}(t) \triangleq \mathcal{S}_t w^0$ , we have  $w_n \rightarrow \tilde{w}$  in  $\mathcal{C}([0, T], H)$ , and it suffices to prove that  $w = \tilde{w}$ . Now, each  $w_n$  belongs to  $W^{1,\infty}(0, T; H)$ , takes values in  $\mathcal{D}(\mathcal{A}) = \mathcal{D}(A)$ , and solves

$$\frac{dw_n}{dt} + Aw_n + F(w_n) = 0 \quad \text{a.e. in } (0, T). \quad (4.75)$$

Applying variation of constants, we get

$$w_n(t) = \mathcal{S}_t w_n^0 - \int_0^t \mathcal{S}_{t-s} F(w_n(s)) ds \quad \text{for all } t \in [0, T]. \quad (4.76)$$

<sup>4</sup> See Lemma 4.3.8 below for the bound  $K/\alpha$ .

Then, taking the difference between (4.73) and (4.76) yields

$$w_n(t) - w(t) = \mathcal{S}_t[w_n^0 - w^0] - \int_0^t \mathcal{S}_{t-s}[F(w_n(s)) - F(w(s))] ds \quad \text{for all } t \in [0, T]. \quad (4.77)$$

Since  $\{w_n^0\}$  is a bounded sequence in  $H$ , the functions  $w_n$  are bounded in  $\mathcal{C}([0, T], H)$ , and for a suitable local Lipschitz constant  $K$ , we infer from (4.77) together with  $\|\mathcal{S}_t\|_{\mathcal{L}(H)} \leq 1$  that

$$\|w_n(t) - w(t)\|_H \leq \|w_n^0 - w^0\|_H + K \int_0^t \|w_n(s) - w(s)\|_H ds \quad \text{for all } t \in [0, T]. \quad (4.78)$$

Passing to the limit after an application of the Grönwall's lemma yields  $\tilde{w} = w$ . Conversely, let  $w$  be a generalized solution to (4.71). By passing to the limit in (4.76), where the  $w_n$  as a suitable sequence of strong solutions to (4.71), we see that  $w$  satisfies (4.73) and is indeed a mild solution to (4.72).

Finally, since  $F$  is continuously differentiable, the regularity property (4.74) follows from [Pazy, 2012, Theorem 1.5, p.187].  $\square$

*Remark 4.3.5.* Lemma 4.3.4 means that, as expected, *generalized* solutions to (4.71) (in the sense of nonlinear semigroup theory) coincide with *mild* solutions to (4.72) (in the sense of perturbed linear problems).

Next, Lemma 4.3.6 given below allows us to circumvent the possible unboundedness of the output operator  $C$  and together with the  $A$ -boundeness of  $C$  guarantees that  $\mathcal{M}$  given by the limit in (4.67) is a well-defined mapping on the whole space  $H$ .

**Lemma 4.3.6** (Integral formula). *For any  $w_0 \in H$ ,*

$$\lim_{\tau \rightarrow +\infty} \int_0^\tau \mathcal{T}_t w_0 dt = A^{-1} w_0 - A^{-1} \int_0^{+\infty} F(\mathcal{T}_t w_0) dt. \quad (4.79)$$

*In particular,  $\lim_{\tau \rightarrow +\infty} \int_0^\tau \mathcal{T}_t w_0 dt$  belongs to  $\mathcal{D}(A)$ .*

*Proof.* Let  $w_0 \in H$ . Recall from standard linear semigroup theory that

$$\int_0^\tau \mathcal{T}_t w_0 dt \in \mathcal{D}(A), \quad \text{and} \quad A \int_0^\tau \mathcal{T}_t w_0 dt = w_0 - \mathcal{T}_\tau w_0. \quad (4.80)$$

Thus, by integrating the variation of the constant formula (4.73) over  $(0, \tau)$ ,  $\tau \geq 0$ , we obtain

$$\int_0^\tau \mathcal{T}_t w_0 dt = A^{-1} w_0 - A^{-1} \mathcal{T}_\tau w_0 - \int_0^\tau \int_0^t \mathcal{S}_{t-s} F(\mathcal{T}_s w_0) ds dt. \quad (4.81)$$

Let  $\tau \geq 0$ . Then, by letting  $I_\tau \triangleq \{(s, t) \in \mathbb{R}^2, 0 \leq s \leq t \leq \tau\}$ , we can write the last term in (4.81) as an absolutely convergent integral over  $\mathbb{R}^2$ :

$$\int_0^\tau \int_0^t \mathcal{S}_{t-s} F(\mathcal{T}_s w_0) ds dt = \iint_{\mathbb{R}^2} \mathbb{1}_{I_\tau}(s, t) \mathcal{S}_{t-s} F(\mathcal{T}_s w_0) ds dt, \quad (4.82)$$

where  $\mathbb{1}_{I_\tau}$  denotes the indicator function of the set  $I_\tau$ . Switching the order of integration in the right-hand side of (4.82) leads to

$$\int_0^\tau \int_0^t \mathcal{S}_{t-s} F(\mathcal{T}_s w_0) ds dt = \int_0^{+\infty} \left\{ \mathbb{1}_{I_\tau}(s, s) \int_0^{\tau-s} \mathcal{S}_t F(\mathcal{T}_s w_0) dt \right\} ds. \quad (4.83)$$

We shall apply Lebesgue's dominated convergence theorem to conclude, the parameter being  $\tau$ . For fixed  $s \geq 0$ ,  $\lim_{\tau \rightarrow +\infty} \mathbb{1}_{I_\tau} = 1$ , and  $\lim_{\tau \rightarrow +\infty} \mathcal{S}_\tau F(\mathcal{T}_s w_0) = 0$  in  $H$ ; hence,

$$\lim_{\tau \rightarrow +\infty} \mathbb{1}_{I_\tau}(s, s) \int_0^{\tau-s} \mathcal{S}_t F(\mathcal{T}_s w_0) dt = A^{-1} F(\mathcal{T}_s w_0). \quad (4.84)$$

On the other hand, by  $\alpha$ -exponential stability of  $\{\mathcal{S}_t\}$ , for all  $\tau \geq 0$  and  $s \geq 0$ ,

$$\left\| \mathbb{1}_{I_\tau}(s, s) \int_0^{\tau-s} \mathcal{S}_t F(\mathcal{T}_s w_0) dt \right\|_H \leq \frac{1}{\alpha} \|F(\mathcal{T}_s w_0)\|_H, \quad (4.85)$$

Equation (4.85) provides an integrable dominating function, which ends the proof.  $\square$

Now, let us come back to the uncontrolled  $w$ -equation (4.71), which we linearize around a given trajectory  $\{\mathcal{T}_t w_0, t \geq 0\}$ :

$$\frac{dv}{dt} + Av + dF(\mathcal{T}_t w_0)v = 0. \quad (4.86)$$

Equation (4.86) is linear but non-autonomous in general. Given  $h \in H$ , (4.86) possesses a unique mild solution  $t \mapsto v(t) = v(t; w_0, h)$  satisfying  $v(0) = h$  and  $v \in \mathcal{C}(\mathbb{R}^+, H)$  (see [Pazy, 2012, Theorem 1.2, p.184]). Given  $w_0$  in  $H$ , the next lemma states that the nonlinear operators  $\mathcal{T}_t$  are all differentiable at  $w_0$  and their differentials coincide with the evolution family associated with (4.86). This is a classical fact for sufficiently smooth nonlinear dynamics; here, we give a proof that matches our particular set of hypotheses.

**Lemma 4.3.7** (Differentiability of the semigroup). *Each operator  $\mathcal{T}_t$  is Fréchet differentiable. Furthermore, for any  $t \geq 0$  and  $w_0 \in H$ , the differential  $d\mathcal{T}_t(w_0)$  is given by*

$$d\mathcal{T}_t(w_0)h = v(t; w_0, h) \quad \text{for all } h \in H, \quad (4.87)$$

where  $t \mapsto v(t; w_0, h)$  is the unique mild solution to (4.86) with initial data  $h$ .

*Proof.* Let  $\tau \geq 0$  and  $w_0 \in H$ . It is clear that the mapping  $h \mapsto v(\tau; w_0, h)$  is linear; it is also continuous by [Pazy, 2012, Theorem 1.2, p.184]. First, since  $F$  is differentiable, the following Taylor formula holds:

$$F(a + b) - F(a) = dF(a)b + R(a, b) \quad \text{for all } a, b \in H, \quad (4.88)$$

with  $R(a, b) = o(\|b\|_H)$  when  $\|b\|_H \rightarrow 0$  for fixed  $a \in H$ . Now, take a nonzero  $h \in H$ . Combining the variation of the constant formula (4.73) with (4.88) leads to

$$\begin{aligned} \mathcal{T}_t(w_0 + h) - \mathcal{T}_t w_0 &= \mathcal{S}_t h - \int_0^t \mathcal{S}_{t-s} dF(\mathcal{T}_s w_0) \{ \mathcal{T}_s(w_0 + h) - \mathcal{T}_s w_0 \} ds \\ &\quad - \int_0^t \mathcal{S}_{t-s} R(\mathcal{T}_s w_0, \mathcal{T}_s(w_0 + h) - \mathcal{T}_s w_0) ds. \end{aligned} \quad (4.89)$$

Omitting the dependence on  $w_0$  and  $h$  for the moment, we write  $v(t) \triangleq v(t; w_0, h)$  and  $\mathcal{R}(t) \triangleq \mathcal{T}_t(w_0 + h) - \mathcal{T}_t w_0 - v(t)$ . Now, taking the difference between (4.89) and the integral identity satisfied by  $v$  as a mild solution to (4.86), one obtains

$$\mathcal{R}(t) = \int_0^t \mathcal{S}_{t-s} dF(\mathcal{T}_s w_0) \mathcal{R}(s) ds - \int_0^t \mathcal{S}_{t-s} R(\mathcal{T}_s w_0, \mathcal{T}_s(w_0 + h) - \mathcal{T}_s w_0) ds \quad (4.90)$$

holding for all  $0 \leq t \leq \tau$ . Next, because  $dF$  is continuous on the compact set  $\{\mathcal{T}_s w_0, 0 \leq s \leq \tau\}$ ,  $\|dF(\mathcal{T}_s w_0)\|_{\mathcal{L}(H)}$  is bounded by some  $m$  independent of  $h$ . Besides,  $\|\mathcal{T}_s\|_{\mathcal{L}(H)} \leq 1$  for all  $s \geq 0$ . Therefore, it follows from (4.90) that for all  $0 \leq t \leq \tau$ ,

$$\|\mathcal{R}(t)\|_H \leq m \int_0^t \|\mathcal{R}(s)\|_H ds + \int_0^t \|R(\mathcal{T}_s w_0, \mathcal{T}_s(w_0 + h) - \mathcal{T}_s w_0)\|_H ds. \quad (4.91)$$

Using Grönwall's inequality in its integral form, we deduce from (4.91) that

$$\frac{\|\mathcal{R}(\tau)\|_H}{\|h\|_H} \leq \exp(m\tau) \int_0^\tau \frac{\|R(\mathcal{T}_s w_0, \mathcal{T}_s(w_0 + h) - \mathcal{T}_s w_0)\|_H}{\|h\|_H} ds. \quad (4.92)$$

To obtain the desired differentiability property, it suffices to show that the right-hand side of (4.92) converges to 0 as  $\|h\|_H$  goes to 0. This is done using Lebesgue's dominated convergence theorem. Let  $s \in [0, \tau]$ . By the contraction property of  $\{\mathcal{T}_t\}$ , for any nonzero  $h \in H$  such that  $\mathcal{T}_s(w_0 + h) - \mathcal{T}_s w_0$  is nonzero, we have

$$\frac{\|R(\mathcal{T}_s w_0, \mathcal{T}_s(w_0 + h) - \mathcal{T}_s w_0)\|_H}{\|h\|_H} \leq \frac{\|R(\mathcal{T}_s w_0, \mathcal{T}_s(w_0 + h) - \mathcal{T}_s w_0)\|_H}{\|\mathcal{T}_s(w_0 + h) - \mathcal{T}_s w_0\|_H}, \quad (4.93)$$

and if  $\mathcal{T}_s(w_0 + h) - \mathcal{T}_s w_0 = 0$ , then  $R(\mathcal{T}_s w_0, \mathcal{T}_s(w_0 + h) - \mathcal{T}_s w_0) = 0$ . Either way, when  $\|h\|_H \rightarrow 0$ ,  $\|\mathcal{T}_s(w_0 + h) - \mathcal{T}_s w_0\|_H \rightarrow 0$ , and by definition of the residual term  $R$ , the right-hand side of (4.93) must converge to 0 as well. To conclude the proof, let us estimate the left-hand side of (4.93) uniformly with respect to  $h$ . We observe that the set of all points  $\mathcal{T}_s(w_0 + h)$ ,  $0 \leq s \leq \tau$ ,  $\|h\|_H \leq 1$ , is contained in some open ball, on which  $F$  is  $K$ -Lipschitz continuous for some  $K > 0$ . Thus, using (4.88), one obtains that the left-hand side of (4.93) is smaller than  $2K$ .  $\square$

We continue by establishing exponential decay of solutions to (4.86).

**Lemma 4.3.8** (Stability of the linearized equation). *Let  $w_0 \in H$ . For any  $h \in H$ , the solution  $t \mapsto v(t; w_0, h)$  to (4.86) with initial data  $h$  satisfies*

$$\|v(t; w_0, h)\|_H \leq \exp(-\alpha t) \|h\|_H \quad \text{for all } t \geq 0. \quad (4.94)$$

*Proof.* Let  $w_0$  and  $h$  in  $H$ . Let  $\tau \geq 0$  and  $f \triangleq t \mapsto dF(\mathcal{T}_t w_0)v(t) \in \mathcal{C}([0, \tau], H)$ . Then, pick sequences  $\{h_n\} \subset \mathcal{D}(A)$  and  $\{f_n\} \subset W^{1,2}(0, \tau; H)$  such that  $h_n \rightarrow h$  in  $H$  and  $f_n \rightarrow f$  in  $L^2(0, \tau; H)$ . For each  $n$ , there exists a unique strong solution  $v_n$  to  $dv_n/dt + Av_n + f_n(t) = 0$  satisfying the initial condition  $v_n(0) = h_n$ . Furthermore,

$$\frac{1}{2} \frac{d}{dt} \|v_n\|_H^2 = -(Av_n, v_n)_H - (f_n, h)_H \quad \text{a.e. on } (0, \tau), \quad (4.95)$$

and  $v_n$  converges to  $v$  in  $\mathcal{C}([0, \tau], H)$ . Plugging (4.66) into (4.95) yields

$$\frac{1}{2} \frac{d}{dt} \|v_n\|_H^2 \leq -\alpha \|v_n\|_H^2 + (dF(\mathcal{T}_t w_0)v_n - f_n, v_n)_H \quad \text{a.e. on } (0, \tau). \quad (4.96)$$

We deduce from (4.96) that for all  $0 \leq t \leq \tau$ ,

$$\|v_n(t)\|_H^2 \leq \exp(-2\alpha t) \|h_n\|_H^2 + \frac{1}{2\alpha} \int_0^t |(dF(\mathcal{T}_s w_0)v_n(s) - f_n(s), v_n(s))_H| ds. \quad (4.97)$$

As  $n$  goes to  $+\infty$ , the integral term in (4.97) tends to 0 and we obtain the desired result by passing to the limit.  $\square$

Now that we have established that all objects in the statement of Theorem 4.3.2 are well-defined, we can conclude this section by proving that  $\mathcal{M}$  as given by (4.67) enjoys the required regularity properties, namely  $\mathcal{M} \in \mathcal{C}^1(H, Z)$  with locally (or globally) Lipschitz continuous differential, and solves the functional equation (4.13).

*Proof of Theorem 4.3.2.* Recall that  $CA^{-1}$  is a bounded linear operator. In view of the formula (4.79), the desired properties of  $\mathcal{M}$  and  $d\mathcal{M}$  readily follow from those of the mappings

$$w \mapsto \int_0^{+\infty} F(\mathcal{T}_t w) dt \quad \text{and} \quad w \mapsto \int_0^{+\infty} dF(\mathcal{T}_t w) d\mathcal{T}_t(w) dt, \quad (4.98)$$

which we investigate next.

*Step 1: Differentiability.* Let  $w_0 \in H$ . It suffices to prove that

$$\int_0^{+\infty} \frac{\|F(\mathcal{T}_t(w_0 + h)) - F(\mathcal{T}_t w_0) - dF(\mathcal{T}_t w_0) d\mathcal{T}_t(w_0)h\|_H}{\|h\|_H} dt \rightarrow 0 \quad (4.99)$$

when  $\|h\|_H$  goes to 0. We use Lebesgue's dominated convergence theorem. Since  $F$  and  $\mathcal{T}_t$  are differentiable, by the chain rule, the integrand in (4.99) converges to 0 pointwise. Let us now find some integrable dominating function. As in the proof of Lemma 4.3.7, we can find some open ball where  $F$  is  $K$ -Lipschitz and which contains the set of all  $\mathcal{T}_t(w_0 + h)$ ,  $\|h\|_H \leq 1$ ,  $t \geq 0$ . Thus, it follows from (4.11) and (4.94) that the integrand in (4.99) is dominated by  $t \mapsto 2K \exp(-\alpha t)$ , which is integrable.

*Step 2: Lipschitz continuity of the differential.* Pick two elements  $w_1$  and  $w_2$  in  $H$ . We write  $R \triangleq \max\{\|w_1\|_H, \|w_2\|_H\}$ . Subsequent estimates are motivated by the following decomposition:

$$\begin{aligned} dF(\mathcal{T}_t w_1) d\mathcal{T}_t(w_1) - dF(\mathcal{T}_t w_2) d\mathcal{T}_t(w_2) \\ = dF(\mathcal{T}_t w_1) [d\mathcal{T}_t(w_1) - d\mathcal{T}_t(w_2)] + [dF(\mathcal{T}_t w_1) - dF(\mathcal{T}_t w_2)] d\mathcal{T}_t(w_2). \end{aligned} \quad (4.100)$$

First,  $\mathcal{T}_t w_1$  and  $\mathcal{T}_t w_2$  must remain in  $\mathcal{B}_H(0, R)$ , where  $dF$  is, say,  $K_R$ -Lipschitz continuous. Since  $dF(0) = 0$ , for all  $t \geq 0$  we have

$$\|dF(\mathcal{T}_t w_1)\|_{\mathcal{L}(H)} \leq K_R \|\mathcal{T}_t w_1\|_H \leq K_R \|w_1\|_H. \quad (4.101)$$

Now, let us estimate  $d\mathcal{T}_t(w_1) - d\mathcal{T}_t(w_2)$  in operator norm. Pick  $h \in H$ . In what follows, we denote by  $v(t)$  the difference  $v(t) \triangleq d\mathcal{T}_t(w_1)h - d\mathcal{T}_t(w_2)h$ . Then,  $v(0) = 0$  and  $v$  is a (mild) solution to the following non-autonomous equation:

$$\frac{dv}{dt} + Av + [dF(\mathcal{T}_t w_1) d\mathcal{T}_t(w_1) - dF(\mathcal{T}_t w_2) d\mathcal{T}_t(w_2)]h = 0, \quad (4.102)$$

which can be rewritten as

$$\frac{dv}{dt} + Av + dF(\mathcal{T}_t w_1)v + [dF(\mathcal{T}_t w_1) - dF(\mathcal{T}_t w_2)]d\mathcal{T}_t(w_2)h = 0. \quad (4.103)$$

Justifications for the formal computations performed below are similar to those in the proof of Lemma 4.3.8; they are omitted here. Taking the scalar product in  $H$  of (4.103) with  $v$  and using (4.66) along with Cauchy-Schwarz and Young inequalities leads to

$$\frac{1}{2} \frac{d}{dt} \|v\|_H^2 \leq -\frac{\alpha}{2} \|v\|_H^2 + \frac{1}{2\alpha} \|[dF(\mathcal{T}_t w_1) - dF(\mathcal{T}_t w_2)]d\mathcal{T}_t(w_2)\|_{\mathcal{L}(H)}^2 \|h\|_H^2. \quad (4.104)$$



Since  $v(0) = 0$ , we deduce from (4.104) multiplied by  $\exp(\alpha t)$ , (4.94) and (4.11) that

$$\|v(t)\|_H^2 \leq \frac{K_R^2 \exp(-\alpha t)}{3\alpha^2} \|w_1 - w_2\|_H^2 \|h\|_H^2 \quad \text{for all } t \geq 0. \quad (4.105)$$

We infer from (4.101) and (4.105) that

$$\|dF(\mathcal{T}_t w_1)[d\mathcal{T}_t(w_1) - d\mathcal{T}_t(w_2)]\|_{\mathcal{L}(H)} \leq mRK_R^2 \exp(-\alpha t/2) \|w_1 - w_2\|_H \quad (4.106)$$

for all  $t \geq 0$ , where  $m$  is some constant independent of  $R$ ; on the other hand, coming back to the second term of (4.100), we also have

$$\|[dF(\mathcal{T}_t w_1) - dF(\mathcal{T}_t w_2)]d\mathcal{T}_t(w_2)\|_{\mathcal{L}(H)} \leq K_R \exp(-2\alpha t) \|w_1 - w_2\|_H. \quad (4.107)$$

Thus, the desired local Lipschitz continuity is obtained by applying the triangular inequality to (4.100) and integrating (4.106) and (4.107) over  $(0, +\infty)$ . Furthermore, if we assume that both  $F$  and  $dF$  are globally Lipschitz continuous, then for some  $K_F$  we can choose  $K_R = K_F$  independent of  $R$  and replace (4.101) with  $\|dF(\mathcal{T}_t w_1)\|_{\mathcal{L}(H)} \leq K_F$ , thereby proving global Lipschitz continuity of the differential  $d\mathcal{M}$ .

*Step 3: Conclusion.* At this point, it remains to check that our candidate  $\mathcal{M}$  is a solution to (4.13). It is clear that  $\mathcal{M}(0) = 0$ . Take  $w_0$  in  $\mathcal{D}(\mathcal{A})$  and consider the associated strong solution  $w \triangleq t \mapsto \mathcal{T}_t w_0 \in \mathcal{C}(\mathbb{R}^+, \mathcal{D}(A)) \cap \mathcal{C}^1(\mathbb{R}^+, H)$  to (4.71). Then,

$$\frac{\mathcal{M}(\mathcal{T}_t w_0) - \mathcal{M}(w_0)}{t} = C \left[ \frac{1}{t} \int_0^t \mathcal{T}_s w_0 \, ds \right]. \quad (4.108)$$

As  $w$  is continuous in  $\mathcal{D}(A)$ , the term  $t^{-1} \int_0^t \mathcal{T}_s w_0 \, ds$  converges to  $w_0$  in  $\mathcal{D}(A)$  when  $t$  approaches 0. Thus,  $C$  being  $A$ -bounded, the difference quotient in (4.108) converges to  $Cw_0$  in  $Z$ . On the other hand, since  $w \in \mathcal{C}^1(\mathbb{R}^+, H)$  and  $\mathcal{M} \in \mathcal{C}^1(H, Z)$ , by the chain rule, we have

$$\frac{d}{dt} [\mathcal{M}(\mathcal{T}_t w_0)] = -d\mathcal{M}(\mathcal{T}_t w_0) \cdot \mathcal{A}(\mathcal{T}_t w_0) \quad \text{for all } t \geq 0. \quad (4.109)$$

Evaluating (4.109) at  $t = 0$  yields  $d\mathcal{M}(w_0) \cdot \mathcal{A}(w_0) + Cw_0 = 0$  by uniqueness of the limit.  $\square$

### 4.3.3 Applications

In this section, we provide two semilinear examples for which all our results apply.

**Sine-Gordon equation** Let  $\xi$ ,  $\gamma$ , and  $L$  be positive constants. Let  $\mathcal{O}$  be a nonempty open subset of  $\Omega \triangleq (0, L)$ . Consider the following damped sine-Gordon equation with control acting on  $\mathcal{O}$  and homogeneous Dirichlet boundary conditions:

$$\frac{\partial^2 \theta}{\partial t^2} + \xi \frac{\partial \theta}{\partial t} - \frac{\partial^2 \theta}{\partial x^2} + \gamma \sin(\theta) = \mathbf{1}_{\mathcal{O}}(x)U(x, t) \quad \text{in } (0, L) \times (0, +\infty), \quad (4.110a)$$

$$\theta(0, t) = \theta(L, t) = 0 \quad \text{for all } t \geq 0. \quad (4.110b)$$

In this example inspired by [Temam, 1997, Chapter IV], (4.110) may represent the voltage dynamics of the continuous limit case for coupled Josephson junctions, with the control  $U$  being proportional to the applied current. The uncontrolled dynamics generated by (4.110) are well-posed on the energy space  $H \triangleq H_0^1(\Omega) \times L^2(\Omega)$ . Using the state variable  $w = [\theta, \theta']$ , we can recast (4.110) into a semilinear

evolution problem on  $H$  as in Section 4.3. Letting  $\mathcal{D}(A) \triangleq [H^2(\Omega) \times H_0^1(\Omega)] \cap H$  and  $U \triangleq L^2(\mathcal{O})$ , we define the unbounded linear operator  $A$ , the input operator  $B$ , and the nonlinear mapping  $F$  by

$$A[\theta, \zeta] \triangleq \left[ -\zeta, -\frac{\partial^2 \theta}{\partial x^2} + \xi \zeta + \gamma \theta \right], \quad [\theta, \zeta] \in \mathcal{D}(A), \quad (4.111a)$$

$$BU \triangleq [0, \mathbb{1}_{\mathcal{O}} U], \quad U \in E, \quad (4.111b)$$

$$F[\theta, \zeta] \triangleq [0, \gamma \sin(\theta) - \gamma \theta], \quad [\theta, \zeta] \in H. \quad (4.111c)$$

As an output, consider the Neumann trace at, say,  $x = 0$ :

$$y(t) = Cw(t) = \frac{\partial \theta}{\partial x}(0, t), \quad (4.112)$$

which is modeled by an unbounded (but  $A$ -bounded) scalar-valued operator. That  $A^{-1}$  exists in  $\mathcal{L}(H)$  can be proved using Riesz representation theorem in  $H_0^1(\Omega)$ . Besides,  $F$  is in  $\mathcal{C}^1(H)$  and both  $F$  and  $dF$  are globally Lipschitz continuous,  $dF$  being given by

$$dF(\theta, \zeta)[h_1, h_2] = [0, \gamma \cos(\theta)h_1 - \gamma h_1] \quad \text{for all } [\theta, \zeta], [h_1, h_2] \in H. \quad (4.113)$$

We equip  $H$  with a scalar product that is equivalent to the usual one:

$$([\theta_1, \zeta_1], [\theta_2, \zeta_2])_{H, \varepsilon} \triangleq \int_{\Omega} \frac{\partial \theta_1}{\partial x} \frac{\partial \theta_2}{\partial x} dx + \int_{\Omega} (\zeta_1 + \varepsilon \theta_1)(\zeta_2 + \varepsilon \theta_2) dx \quad (4.114)$$

where  $\varepsilon \triangleq \min\{\xi/4, \lambda_1/(2\xi)\}$ , with  $\lambda_1$  being the optimal Poincaré inequality constant. Then, after some computations similar to [Temam, 1997, Section IV.1.2], we obtain

$$(Ah + dF(\theta, \eta)h, h)_{H, \varepsilon} \geq \frac{\varepsilon}{2} \|h\|_{H, \varepsilon}^2 - \gamma \lambda_1 \|h\|_{H, \varepsilon}^2 \quad \text{for all } h \in \mathcal{D}(A) \text{ and } [\theta, \zeta] \in H. \quad (4.115)$$

Therefore, Hypothesis 4.3.1 is satisfied as long as  $\gamma < \varepsilon/(2\lambda_1)$ . In that case, Theorem 4.3.2 provides a suitable solution  $\mathcal{M}$  to (4.13) with  $d\mathcal{M}$  globally Lipschitz continuous, upon which a forwarding control law can be built for the output regulation problem. Since the range of  $CA^{-1}B$  is non-zero<sup>5</sup> and hence  $\mathbb{R}$ , Theorem 4.2.3 guarantees the existence of a locally exponentially stable equilibrium for the closed-loop system with small reference and disturbance. Furthermore, following the discussion subsequent to Theorem 4.2.3, the global coercivity condition (4.19) holds whenever  $\varepsilon/2(1 + \lambda_1) > \gamma$ , in which case the theorem provides a globally asymptotically stable equilibrium.

**A pre-stabilized Wilson-Cowan equation** The following example is inspired by the study of neural fields – see for instance [Boscain et al., 2021]. Let  $\Omega$  be a bounded domain in  $\mathbb{R}^n$  and  $\mathcal{O}$  be an open subset of  $\Omega$ . Given a positive gain  $\alpha$ , a kernel  $k \in L^\infty(\Omega \times \Omega)$ , and a smooth scalar nonlinearity  $s$  that has bounded derivative and vanishes at 0, consider the following non-local evolution equation:

$$\frac{\partial w}{\partial t}(x, t) + \alpha w(x, t) + \int_{\Omega} k(x, v)s(w(v, t)) dv = \mathbb{1}_{\mathcal{O}}(x)U(x, t) \quad \text{in } \Omega \times (0, +\infty). \quad (4.116)$$

We look at the (vector-valued) output given by

$$y(t) = Cw(t) = w|_{\mathcal{O}}(t). \quad (4.117)$$

<sup>5</sup> Consider for instance the image by  $CA^{-1}B$  of  $\mathbb{1}_I$  where  $I$  is some segment contained in  $\mathcal{O}$ .

Having set  $H \triangleq L^2(\Omega)$ , we define two mappings  $K$  and  $F$  on  $H$  by

$$[Kw](x) \triangleq s'(0) \int_{\Omega} k(x, \nu) w(\nu) d\nu, \text{ and } [F(w)](x) \triangleq \int_{\Omega} k(x, \nu) \{s(w(\nu)) - s'(0)w(\nu)\} d\nu \quad (4.118)$$

for all  $w \in H$ . Then,  $K$  is a bounded linear operator, and  $F$  is continuously differentiable with  $F$  and  $dF$  globally Lipschitz continuous. We also let  $A \triangleq \alpha \text{id} + K$  and  $BU \triangleq \mathbb{1}_{\mathcal{O}}U$  with  $E \triangleq L^2(\mathcal{O})$ . As an integral operator,  $K$  is compact; thus,  $0$  lies in the resolvent set of  $A$  except for a bounded and countable set of values for  $\alpha$ . Here,  $A$  is continuous; therefore, we can choose  $Z \triangleq \text{Range}A^{-1}B$ , which as a closed subspace of  $H$  is a Hilbert space as well. By letting  $C \triangleq \text{id}$ , the condition (4.69) is automatically satisfied. Furthermore,  $Z = H$  if and only if  $\mathcal{O} = \Omega$ . As for condition (4.66), we have

$$(Ah + dF(w)h, h)_H \geq (\alpha - M_{k,s}) \|h\|_H^2 \quad \text{for all } w, h \in H, \quad (4.119)$$

where  $M_{k,s} \triangleq \iint_{\Omega \times \Omega} |k(x, \nu) s'(\nu)|^2 dx d\nu$ . Hypothesis 4.3.1 is satisfied whenever  $\alpha > M_{k,s}$ , while global uniform coercivity (4.19) holds provided that  $\alpha > 2M_{k,s}$ .

## Comments on Chapter 4

**The case of unbounded output** Let us hold an informal discussion regarding the convergence of the output  $Cw(t)$  towards the reference when the operator  $C$  is unbounded. To get some insight on the situation, assume for a moment that the original  $w$ -system is linear. Then, the solution  $\mathcal{M}$  to (4.13) provided by Theorem 4.3.2 is a bounded linear operator, and the closed-loop dynamics around the equilibrium are governed by a strongly continuous linear semigroup, which commutes with its generator. Recalling the notation from Section 4.3, it follows that for a *strong* solution  $[w, z]$  to (4.15)-(4.16),  $w(t) - w^*$  goes to  $0$  in  $\mathcal{D}(A)$  endowed the graph norm, hence  $Cw(t)$  converges to  $y_{\text{ref}}$  provided that  $C$  is  $A$ -bounded. In the nonlinear case, the argument breaks down.

Alternatively, one may look for weaker notions of convergence. In many applications, unbounded output operators of interest enjoy an *admissibility* property with respect to the uncontrolled dynamics. In the linear theory,  $C$  is said to be  $A$ -admissible if  $C$  is  $A$ -bounded and there exist positive constants  $K$  and  $T$  such that

$$\int_0^T \|C \mathcal{S}_t w_0\|_Z^2 dt \leq K \|w_0\|_H^2 \quad \text{for all } w_0 \in \mathcal{D}(A). \quad (4.120)$$

In the semilinear case, one can deduce from (4.120) that, first of all, the output  $Cw$  is well-defined in  $L^2_{\text{loc}}(0, +\infty; Z)$  even for generalized solutions  $[w, z]$  to the closed-loop equations (4.15)-(4.16), and secondly, that the output converges “in average” to the reference:

$$\lim_{\tau \rightarrow +\infty} \int_{\tau}^{T+\tau} \|Cw(t) - y_{\text{ref}}\|_Z^2 dt = 0 \quad (4.121)$$

for any (including generalized) solution  $[w, z]$  that converges to the equilibrium  $[w^*, z^*]$  in  $H \times Z$ . When no semilinear structure is prescribed for the maximal monotone operator  $\mathcal{A}$  governing the  $w$ -dynamics, we generalize (4.120) by assuming that for any  $w_i^0$  in  $\mathcal{D}(\mathcal{A})$  and  $f_i$  in  $W^{1,2}(0, T; H)$ , the solution<sup>6</sup>  $w_i$  to  $dw_i/dt + \mathcal{A}(w_i) = f_i$  with initial condition  $w_i(0) = w_i^0$ ,  $i \in \{1, 2\}$ , satisfies

$$\int_0^T \|Cw_1(t) - Cw_2(t)\|_Z^2 dt \leq K \|w_1(0) - w_2(0)\|_H^2 + K \int_0^T \|f_1 - f_2\|_H^2 dt. \quad (4.122)$$

In that case, the same conclusions hold for closed-loop solutions.

<sup>6</sup>Existence and uniqueness of a strong solution is guaranteed by [Showalter, 2013, Chapter IV, Theorem 4.1]. In particular, each  $w_i$  is absolutely continuous; hence  $\mathcal{A}(w_i)$  is measurable, and so is  $Cw_i$  by (4.13).

**On robustness and bounded control** In order to take into account nonlinear behavior of actuators (e.g., saturation) in the feedback loop, one may investigate stability properties of (4.15)-(4.16) when a nonlinearity  $g$  is applied to the input  $U$ . This is also relevant in applications where the control signal must satisfy some prescribed bound in norm (see e.g. [Marx et al., 2021b]). Consider a Lipschitz continuous map  $g$  that vanishes at 0 and is *strongly* monotone in some neighborhood of 0. Then, the local (strict) contraction property of the  $[w, \eta]$ -dynamics is preserved. This leaves room for a possible adaptation of Theorem 4.2.3 in the case of saturated or *a priori* bounded control.

We also believe that our framework provides tools to analyse the behavior of the closed-loop (4.15)-(4.16) under certain time-varying disturbances. Indeed, given  $[d_0, y_{\text{ref}}]$  as in Theorem 4.2.3, to which we associate an equilibrium  $[w^*, z^*]$ , consider a disturbance of the form  $d(t) = d_0 + d_1(t)$  with  $d_1$  small in  $L^2(0, +\infty; H)$ . It can then be deduced from (4.41) that the system (4.15)-(4.16) is *incrementally* input-to-state stable in a neighborhood of  $[w^*, z^*]$ , allowing us to quantify the deviation from equilibrium due to the exogenous signal  $d_1$  in terms of its  $L^2$ -norm.

**Stabilization of cascade systems** Putting aside the problem of output regulation and following [Marx et al., 2021b, Marx et al., 2021a], we might be interested in stabilizing the cascade composed of the (actuated)  $w$ -subsystem and a more general  $z$ -subsystem governed by  $dz/dt = Sz + Cw$ , where  $S$  is a skew-adjoint operator on  $Z$ . This would require to investigate a nonlinear Sylvester equation of the form

$$d\mathcal{M}(w)\mathcal{A}(w) + S\mathcal{M}(w) + Cw = 0, \quad w \in \mathcal{D}(\mathcal{A}). \quad (4.123)$$

Under the condition that a sufficiently regular solution  $\mathcal{M}$  to (4.123) exists, the Lyapunov analysis performed in Section 4.1.2 remains valid, which is a good starting point for analysing stability of the new closed-loop with control law given by (4.16).



## Five

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# Conclusions and perspectives

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In this thesis, several problems related to the control of infinite-dimensional nonlinear systems were studied. Our results can be summarized as follows.

Chapter 2 deals with a one-dimensional wave equation supplied with nonlinear unstable boundary dynamics and anticollocated nonlinear velocity feedback. Existence, uniqueness and regularity of (variational and strong) solutions to the closed-loop equations were investigated. A Lyapunov-based analysis showed that solutions converge exponentially to the set of stationary points, provided that the Lipschitz constant of the anti-damping term is small and the feedback nonlinearity  $g$  satisfies a suitable global sector condition. Furthermore, an incremental Lyapunov analysis combined with fine properties of nonlinear contraction semigroups and a certain invariant set argument allowed to prove a local (in the sense of a stronger norm, however) version of the exponential stability result which holds in the case of saturating control.

In Chapter 3, we considered the problem of nonlinear stabilization of the multi-dimensional wave equation with Dirichlet boundary conditions. It was shown that the wave dynamics subject to a specific nonlocal feedback generate a strongly continuous semigroup of contractions on the optimal energy space  $L^2(\Omega) \times H^{-1}(\Omega)$ . Using a standard LaSalle invariance argument combined with unique continuation, we proved that any solution to the closed-loop equations converges to zero in the aforementioned topology. Secondly, under the condition that the feedback nonlinearity has linear growth around zero, polynomial energy decay rates were established for strong solutions. This result is, in particular, valid in the case that the feedback nonlinearity represents a (nonsmooth) pointwise saturation mapping. This constitutes new Dirichlet counterparts to well-known results pertaining to nonlinear stabilization in  $H^1(\Omega) \times L^2(\Omega)$  of the wave equation with Neumann boundary conditions.

In Chapter 4, we tackled the problem of (set-point) output regulation of a class of abstract systems governed by nonlinear contraction semigroups. After having extended the original plant with an output integrator, we designed a feedback law based on the so-called forwarding approach. The proposed controller exists provided that a well-behaved solution to a certain functional equation exists, which, geometrically speaking, amounts to finding a smooth invariant graph for the cascade dynamics composed of the original system and the output integrator. We then gave sufficient conditions for the existence of a locally exponentially stable (and under additional hypotheses, globally asymptotically stable) equilibrium at which the output of the closed-loop system coincides with the reference. All these conditions were investigated in the particular case of semilinear systems and illustrated with examples.

We believe that our work opens up new research perspectives, some of which are given next.

**Uniform Dirichlet stabilization of the waves and nonlocal terms** With reference to the closed-loop equations (3.1)-(3.7), the question of whether *uniform* stability holds under appropriate global assumptions on the feedback nonlinearity is still open. As discussed at the end of Chapter 3, we

fear that this problem is beyond the reach of the methods developed in this thesis. Regardless, we believe that the difficulties we encounter illustrate a more general and interesting matter, namely the mix of *nonlocal* terms (involving for instance the operator  $A^{-1}$ , harmonic extensions, etc.) on one hand and pointwise, local nonlinearities (such as  $g$ ) on the other hand. The same goes for Wilson-Cowan equations, such as the simplified version (4.116), where the nonlinearity  $s$  typically has some monotonicity that is however “mixed up” by the convolution kernel, rendering the analysis more involved.

**Nonlinear output regulation** While the literature on feedback stabilization of infinite-dimensional systems has become quite mature over the last decades, the problem of output regulation of infinite-dimensional nonlinear systems, despite its importance in engineering applications, is quite overlooked. For that reason, we believe that our related research effort, which we describe in Chapter 4, is worth pursuing. More specifically, we find that the following points constitute interesting perspectives for future works.

- Functional equations of the form of (4.13) are worth investigating *per se*. As mentioned at the end of Chapter 4, we remark that (4.13) is a (somehow degenerate) nonlinear Sylvester-type equation. This indicates that our approach somehow generalizes the linear theory. Even in the semilinear case, establishing existence of solutions to more involved version of (4.13), such as (4.123), would be an interesting challenge.
- On a similar note, for general nonlinear  $\mathcal{A}$ , formally speaking, a candidate solution  $\mathcal{M}$  to (4.13) is given by (4.14). which, at least in the case where the output  $C$  is bounded, is well-posed provided that solutions to the uncontrolled dynamics go to zero sufficiently fast. The difficulty lies in proving that the candidate is sufficiently regular, namely continuously differentiable with locally Lipschitz differential. We believe that the arguments used in the semilinear case partially transpose to more general nonlinear systems, provided that those enjoy some smoothing properties. For that reason, we think that nonlinear parabolic equations of reaction-diffusion type constitute an interesting class of “non-semilinear” systems for which existence of a solution to (4.13) could be investigated.
- The feedback law (4.16) requires, in addition to the integrated tracking error  $z$ , the knowledge of the full state  $w$  of the system. In control, a challenging problem is to design a *output feedback* controller, i.e., one that would only require the knowledge of the output  $y = Cw$  and the integrator state  $z$ . A natural strategy to do so is to estimate the state  $w$  via an *observer* in the control scheme. In that context, an interesting approach to consider is the Lyapunov *strictification* via observer developed in [Praly, 2019] and applied in, e.g., [Marx et al., 2021a].
- Finally, if one desires to implement the controller given by (4.16), rigorously analysing the performance of an approximate linearized version would be interesting. To that end, the methodology developed for tackling the semilinear case in Section 4.3 seems to be a good starting basis.

# A

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## Résumé étendu en français

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### Introduction générale

A l'interface des mathématiques appliquées et des sciences de l'ingénieur, la théorie du contrôle est l'étude des systèmes dynamiques dans le but de les contrôler, c'est-à-dire, leur donner un comportement souhaité. Un *système contrôlé* est un système dynamique dont un paramètre, le *contrôle*, est choisi à chaque instant. L'*état* du système est la variable régie par cette dynamique. Un système contrôlé de dimension finie est souvent gouverné par des équations différentielles ordinaires, tandis qu'un système modélisé par des *équations aux dérivées partielles* constitue un système de *dimension infinie*. La théorie mathématique de tels systèmes est plus récente qu'en dimension finie ; elle repose notamment sur l'analyse des équations aux dérivées partielles, l'analyse fonctionnelle et la théorie des systèmes dynamiques. Donnons quelques exemples. Tout d'abord, supposons que  $f : \mathbb{R}^d \times \mathbb{R} \rightarrow \mathbb{R}^d$  est une fonction suffisamment régulière, alors l'équation différentielle

$$\frac{dx}{dt} = f(x, U) \quad (\text{A.1})$$

définit un système contrôlé de dimension finie où  $x$  est l'état et le scalaire  $U$  est le contrôle. Dans ce cas, l'*espace d'état* est l'espace euclidien  $\mathbb{R}^d$ . Ensuite, considérons un domaine borné  $\Omega$  de  $\mathbb{R}^d$ ,  $d \geq 2$ , dont la frontière est  $\Gamma$ . Alors, l'équation des ondes avec donnée au bord de Dirichlet non-homogène

$$\frac{\partial^2 u}{\partial t^2} - \Delta u = 0 \quad \text{dans } \Omega, \quad (\text{A.2a})$$

$$u|_{\Gamma} = U \quad \text{sur } \Gamma, \quad (\text{A.2b})$$

définit (au moins formellement) un autre système contrôlé dont l'état est le couple  $[u, \partial_t u]$ . La théorie des équations aux dérivées partielles nous indique que l'état et le contrôle doivent être choisis dans des espaces fonctionnels adaptés. Lorsque le contrôle est choisi comme une fonction de l'état, on parle d'un système commandé par rétroaction. Cette dernière est dite *statique* si elle dépend uniquement de l'état du système à l'instant présent. L'étude des systèmes commandés par un retour d'état statique mène à celle des *systèmes dynamiques*. Dans le cas de l'exemple donné par (A.1), un retour statique a la forme

$$U = h(x) \quad (\text{A.3})$$

où  $h : \mathbb{R}^d \rightarrow \mathbb{R}$  est une fonction bien choisie. Le système en *boucle fermée* est alors régi par l'équation d'évolution

$$\frac{dx}{dt} = f(x, h(x)), \quad (\text{A.4})$$

laquelle définit, sous certaines hypothèses, un système dynamique sur  $\mathbb{R}^d$ .



Les problèmes de contrôle sur lesquels nous nous concentrons tout particulièrement sont les suivants.

1. *Le problème de stabilisation par rétroaction.* Etant donné un point d'équilibre instable pour le système sans contrôle, il s'agit de trouver une loi de commande par rétroaction telle que la *boucle fermée* converge vers l'équilibre.
2. *Le problème de régulation de sortie.* Considérons un système dont on mesure une sortie, c'est-à-dire une fonction de l'état représentant par exemple une grandeur physique que l'on souhaite contrôler. On cherche alors à concevoir une loi de commande garantissant que la sortie du système en boucle fermée est à une référence donnée même si ce dernier est soumis à une perturbation externe inconnue.

Voici l'orientation générale de ce travail :

Dans ce mémoire, nous nous intéressons à des problèmes de stabilisation et de régulation de sortie pour des systèmes non-linéaires en dimension infinie.

Gardons à l'esprit deux idées fondamentales en automatique. Premièrement, la notion de *stabilité*, qui est liée au comportement des trajectoires de systèmes dynamiques lorsque la donnée initiale est perturbée, et également aux propriétés qualitatives et quantitatives de la convergence des trajectoires vers un équilibre donné, lorsqu'il en existe. Deuxièmement, la notion de *robustesse*, renvoyant à l'analyse de l'influence de perturbations *exogènes* sur le système, lesquelles représentent par exemple un écart du modèle à la réalité, un bruit numérique, etc. Leur prise en compte est cruciale pour implémenter un contrôleur.

Une façon d'obtenir de la stabilité est d'imposer une propriété de *contraction* à la dynamique en boucle fermée, c'est-à-dire que la distance entre deux trajectoires (typiquement mesurée en termes d'*énergie* dans le contexte des équations aux dérivées partielles) soit décroissante voire tende vers zéro avec le temps. Dans notre travail, l'idée de contraction intervient à la fois dans la stabilité, l'existence d'équilibres et l'existence de solutions.

## Résumé du Chapitre 2

Ce chapitre, motivé par des problèmes non-linéaires de propagation de vibrations mécaniques au sein de dispositifs de forage, porte sur le système dynamique défini par les équations suivantes :

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0 \quad \text{dans } (0, L) \times (0, +\infty), \quad (\text{A.5a})$$

$$\frac{\partial^2 u}{\partial t^2}(0, t) - \frac{\partial u}{\partial x}(0, t) = F \left( \frac{\partial u}{\partial t}(0, t) \right) \quad \text{pour tout } t \in (0, +\infty), \quad (\text{A.5b})$$

$$\frac{\partial u}{\partial x}(L, t) = -g \left( \frac{\partial u}{\partial t}(L, t) \right) \quad \text{pour tout } t \in (0, +\infty), \quad (\text{A.5c})$$

où  $g$  et  $F$  sont deux fonctions non-linéaires. L'équation (A.5b) est une condition au bord cinétique représentant une dynamique de second ordre à l'extrémité  $x = 0$ ,  $F$  modélisant un terme de friction non-linéaire. En supposant que  $g$  est une fonction croissante avec  $g(0) = 0$ , l'équation (A.5c) définit un retour en vitesse dissipatif au bord en condition au bord de Neumann. La question est de savoir si l'action au bord  $x = L$  permet de compenser l'éventuel *anti-amortissement* lié à la dynamique au bord  $x = 0$  et de stabiliser l'ensemble du système. Nous donnons des conditions sur  $g$  et  $F$  sous lesquelles

la réponse est positive. L'analyse repose sur la construction d'une fonctionnelle de Lyapunov exhibant le couplage entre les deux bords, de la forme

$$\mathcal{E}_\rho = \frac{1}{2} \left| \frac{\partial u(0, t)}{\partial t} \right|^2 + \frac{1}{2} \int_0^L \left| \frac{\partial u}{\partial x}(x, t) \right|^2 + \left| \frac{\partial u}{\partial t}(x, t) \right|^2 + 2\rho(x) \frac{\partial u}{\partial x}(x, t) \frac{\partial u}{\partial t}(x, t) dx, \quad (\text{A.6})$$

où  $\rho$  est un poids variable dans l'espace à choisir de manière appropriée. Notons la présence dans l'énergie de la vitesse en  $x = 0$  considérée ponctuellement du fait de la condition au bord dynamique (A.5b).

Il est démontré que lorsque  $g$  vérifie une certaine condition de secteur dépendant de la constante de Lipschitz de  $F$ , alors l'ensemble des solutions stationnaires est globalement exponentiellement stable. Plus précisément, en notant  $q$  la constante de Lipschitz de  $F$ , alors on supposera qu'il existe des constantes strictement positives  $\alpha_1$  et  $\alpha_2$  telles que

$$\alpha_1 |s| \leq |g(s)| \leq \alpha_2 |s| \quad \text{pour tout } s \in \mathbb{R}, \quad (\text{A.7})$$

avec

$$\frac{\alpha_1}{1 + \alpha_2^2} > q. \quad (\text{A.8})$$

Cependant, cette condition exclut le cas intéressant du contrôle saturé, puisqu'elle implique que  $g$  est approximativement linéaire à l'infini. Dans cette situation, nous démontrons un résultat alternatif, valable sous une hypothèse « locale » sur  $g$  seulement, portant sur la décroissance de l'énergie des solutions *fortes* dont la donnée initiale est bornée dans une norme d'ordre plus élevé par rapport à l'espace d'énergie naturel du système. La démonstration de ce résultat repose notamment sur une analyse de contraction où l'on applique la fonctionnelle définie par (A.6) à la différence de deux solutions de (A.5). Quelques outils issus de la théorie des semi-groupes de contractions non-linéaires interviennent également.

Ce chapitre est également l'occasion d'introduire un certain nombre d'outils théoriques relatifs à l'étude des systèmes dynamiques en dimension infinie et des équations aux dérivées partielles. En particulier, les différentes notions de solution utilisées dans le manuscrit, ainsi que des méthodes permettant d'obtenir leur existence, sont présentées en détail.

### Résumé du Chapitre 3

On se donne un domaine borné et régulier  $\Omega$  de  $\mathbb{R}^d$ ,  $d \geq 2$ , et on considère le système suivant :

$$\frac{\partial^2 u}{\partial t^2} - \Delta u = 0 \quad \text{dans } \Omega \times (0, +\infty), \quad (\text{A.9a})$$

$$u|_\Gamma = -g(U(t)) \quad \text{sur } \Gamma_0 \times (0, +\infty), \quad (\text{A.9b})$$

$$u|_\Gamma = 0, \quad \text{sur } \Gamma_1 \times (0, +\infty), \quad (\text{A.9c})$$

où  $\Delta$  est l'opérateur Laplacien,  $U(t)$  représente un contrôle exercé sur un sous-ensemble  $\Gamma_0$  de la frontière, et  $g$  est une non-linéarité scalaire supposée croissante, avec  $g(s) = 0$  si et seulement si  $s = 0$ . Le reste de la frontière, noté  $\Gamma_1 \triangleq \Gamma \setminus \Gamma_0$  et éventuellement vide, est laissé « au repos ». Typiquement,  $g$  peut représenter une fonction de saturation.

Par rapport au problème précédent, nous notons deux difficultés supplémentaires.

- D'une part, l'équation des ondes est désormais posée en dimension d'espace plus grande que 1, et l'entrée est désormais à valeurs vectorielles et non scalaires. En particulier, l'analyse de régularité des solutions est plus délicate.

- D'autre part, l'espace d'état adapté au contrôle des solutions de l'équation des ondes par la condition au bord de Dirichlet est en fait

$$\mathcal{H} \triangleq L^2(\Omega) \times H^{-1}(\Omega) \quad (\text{A.10})$$

et non un espace du type  $H^1(\Omega) \times L^2(\Omega)$  comme dans le cas Neumann. L'espace  $\mathcal{H}$ , plus grand, est davantage rencontré dans les problèmes de *contrôlabilité* où l'on se ramène par transposition à une variable possédant une régularité  $H^1(\Omega) \times L^2(\Omega)$ . Rappelons que  $H^{-1}(\Omega)$  est le dual topologique de  $H_0^1(\Omega)$  muni d'un produit scalaire, et qu'il s'agit d'un espace de distributions.

En dérivant une certaine fonctionnelle d'énergie (correspondant à la norme au carré dans  $\mathcal{H}$ ) le long des « trajectoires » du système en boucle ouverte (A.9), on obtient un choix naturel de retour en vitesse :

$$U(t) = -\frac{\partial[A^{-1}u']}{\partial \nu} \quad (\text{A.11})$$

où  $\partial/\partial \nu$  désigne la dérivée normale orientée vers l'extérieur et  $A^{-1}$  est l'inverse de l'opérateur « moins Laplacien avec condition au bord de Dirichlet homogène ». Autrement dit, on choisit un retour dissipatif consistant à résoudre à chaque instant

$$-\Delta p = u' \quad \text{dans } \Omega, \quad (\text{A.12a})$$

$$p|_{\Gamma} = 0 \quad \text{sur } \Gamma, \quad (\text{A.12b})$$

puis prendre (moins) la dérivée normale du résultat  $p \triangleq A^{-1}u'$  en guise de contrôle.

En ce qui concerne le cas linéaire, il a été démontré dans [Lasiecka and Triggiani, 1992b] que la loi de commande (A.11) stabilise exponentiellement le système. L'analyse est plus délicate que celle du problème équivalent en condition au bord de Neumann et, à notre connaissance, n'a pas été étendue au cas où une non-linéarité statique est présente dans la boucle.

Dans un premier temps, nous démontrons un résultat de stabilité asymptotique globale. Dans un second temps, et afin d'obtenir un résultat plus précis mais valable lorsque  $g$  est une fonction de saturation, nous faisons uniquement l'hypothèse que  $g$  est approximativement linéaire dans un voisinage de 0, et nous démontrons que les solutions *fortes* de (A.9), c'est-à-dire les solutions régulières dont les données initiales vérifient certaines conditions de compatibilité, voient leur énergie décroître comme un polynôme  $1/t^{-r}$ , où  $r$  dépend de la dimension d'espace  $d$  du problème – sous l'hypothèse que certaines conditions géométriques soient vérifiées si jamais le contrôle n'est pas exercé sur toute la frontière. La constante devant le taux de décroissance, en revanche, dépend de la condition initiale de manière *a priori* non-continue – plus précisément, elle dépend aussi de la norme de la condition initiale dans  $H^1(\Omega) \times L^2(\Omega)$ , laquelle est définie uniquement pour les solutions fortes. En cela, il s'agit d'un résultat de décroissance *non-uniforme*. Mais en l'absence de conditions sur le comportement de  $g$  à l'infini, on s'attend à ce type de résultat : dans le cas Neumann, on sait que le système équivalent en boucle fermée avec un contrôle saturé *n'est pas* uniformément exponentiellement stable – voire par exemple [Vancostenoble and Martinez, 2000] ; et on peut conjecturer qu'il en est de même pour le cas Dirichlet.

De la même façon, par analogie avec le cas Neumann, on s'attend à ce que le système soit exponentiellement (et uniformément) stable si la non-linéarité  $g$  est également approximativement linéaire à l'infini ; mais ce problème reste ouvert et semble hors d'atteinte pour le moment.

## Résumé du Chapitre 4

Dans ce chapitre, nous nous intéressons au problème de régulation de sortie. Il s'agit de concevoir une commande permettant à la sortie du système étudié de suivre une référence donnée, et ce, même en

présence d'une perturbation externe inconnue. Nous considérons des systèmes non-linéaires abstraits de la forme

$$\frac{dw}{dt} + \mathcal{A}(w) = BU(t), \quad (\text{A.13a})$$

$$y = Cw, \quad (\text{A.13b})$$

où l'état  $w$ , le contrôle  $U(t)$  et la sortie  $y$  vivent respectivement dans des espaces de Hilbert  $H$ ,  $E$  et  $Z$ , où  $\mathcal{A}$  est un opérateur *maximal monotone*, et  $B$  et  $C$  sont des opérateurs d'entrée et de sortie linéaires.

L'objectif est de trouver une loi de commande, possiblement dynamique, telle que, pour une sortie de référence  $y_{\text{ref}}$  donnée, les sorties du système en boucle fermée convergent vers  $y_{\text{ref}}$ . Nous supposons au préalable que  $\mathcal{A}$  est en fait *strictement* monotone, garantissant une propriété de stabilité de type « input-to-state » : il existe une fonctionnelle de Lyapunov  $\mathcal{V} \in \mathcal{C}^1(H)$  et deux constantes strictement positives  $\alpha$  et  $\beta$  telles que

$$\frac{d}{dt} \mathcal{V}(w) \leq -\alpha \mathcal{V}(w) + \beta \|U(t)\|_E^2 \quad (\text{A.14})$$

pour chaque solution de (A.13a). En fait,  $\mathcal{V}$  sera simplement une norme au carré, et une telle inégalité est également vérifiée par la différence entre deux solutions.

Revenant aux fondamentaux de l'automatique, on utilise une action intégrale en ajoutant au système une nouvelle variable

$$\frac{dz}{dt} = Cw - y_{\text{ref}}. \quad (\text{A.15})$$

La loi de commande souhaitée sera une fonction de l'état étendu  $[w, z]$ ; si un équilibre pour le système contrôlé existe, la sortie correspondante sera nécessairement  $y_{\text{ref}}$ .

Avant toute chose, on considère le cas  $y_{\text{ref}} = 0$  et on cherche à stabiliser la cascade « sous-système  $w$  » et intégrateur. L'approche adoptée ici est celle du *forwarding* [Mazenc and Praly, 1996] : supposons qu'il existe une fonction  $\mathcal{M} \in \mathcal{C}^1(H, Z)$  vérifiant  $\mathcal{M}(0) = 0$  et l'équation fonctionnelle

$$d\mathcal{M}(w) \cdot \mathcal{A}(w) + Cw = 0 \quad (\text{A.16})$$

où  $d\mathcal{M}(w)$  est la différentielle (au sens de Fréchet) de  $\mathcal{M}$  au point  $w$ . Alors, le graphe de  $\mathcal{M}$  est une variété stable pour la dynamique donnée par (A.13a)-(A.15), et en définissant une fonctionnelle de Lyapunov  $\mathcal{W}$  sur  $H \times Z$  par

$$\mathcal{W}(w, z) \triangleq \mathcal{V}(w) + \frac{\rho}{2} \|z - \mathcal{M}(w)\|_Z^2, \quad (\text{A.17})$$

on obtient, au moins formellement, le long des solutions de (A.13a)-(A.15)

$$\frac{d}{dt} \mathcal{W}(w, z) \leq -\alpha \mathcal{V}(w) + \beta \|U(t)\|^2 - \rho (U(t), B^* d\mathcal{M}(w)^* [z - \mathcal{M}(w)])_Z, \quad (\text{A.18})$$

où  $(\cdot, \cdot)_Z$  est le produit scalaire dans  $Z$  et  $*$  désigne l'adjoint. L'équation (A.18) suggère, en choisissant bien  $\rho$ , la loi de commande (non-linéaire)

$$U(t) = B^* d\mathcal{M}(w)^* [z - \mathcal{M}(w)], \quad (\text{A.19})$$

qui rend ainsi  $\mathcal{W}$  décroissante le long des trajectoires du système en boucle fermée.

On peut expliciter une solution formelle à (A.16) :

$$\mathcal{M}(w_0) = - \int_0^{+\infty} C \mathcal{T}_s w_0 ds, \quad (\text{A.20})$$

où  $\{\mathcal{T}_s\}$  est le semi-groupe continu engendré par  $-\mathcal{A}$ , i.e.,  $s \mapsto \mathcal{T}_s w_0 \in \mathcal{C}(\mathbb{R}^+, H)$  est la solution de (A.13a) sans second membre. En dimension finie, sous l'hypothèse de stabilité (A.14),  $\mathcal{M}$  est nécessairement bien défini. En dimension infinie, l'observation peut être non-bornée – c'est souvent le cas pour une observation au bord par exemple, et en général,  $C\mathcal{T}_s w_0$  n'est défini « ponctuellement » que pour une donnée initiale  $w_0$  régulière. Cependant, dans le cas *semi-linéaire*, ou, plus précisément, quand  $\mathcal{A}$  est une perturbation lipschitzienne d'un générateur infinitésimal de semi-groupe linéaire, on démontre une formule alternative pour (A.20) permettant d'écrire  $\mathcal{M}$  comme un opérateur non-linéaire bien défini sur  $H$  tout entier.

Sous l'hypothèse de l'existence d'une solution  $\mathcal{M}$  suffisamment régulière, et lorsque la référence  $y_{\text{ref}}$  et la perturbation extérieure (constante) sont suffisamment petites, on démontre l'existence d'un équilibre local exponentiellement attractif pour la dynamique donnée par (A.13a) en boucle fermée avec (A.19), pour peu que  $\mathcal{M}$  vérifie une condition de surjectivité de type

$$\text{Image } d\mathcal{M}(0)B = Z. \quad (\text{A.21})$$

Dans le cas de  $\mathcal{M}$  construit dans le cas semi-linéaire, on fait le lien avec la théorie linéaire en remarquant que cette condition s'écrit comme une condition de *non-résonance* :

$$\text{Image } CA^{-1}B = Z. \quad (\text{A.22})$$

Nous appliquons ensuite ces résultats à quelques exemples. Tout d'abord, l'équation de Sine-Gordon amortie, qui est une équation d'onde semi-linéaire :

$$\frac{\partial^2 \theta}{\partial t^2} + \xi \frac{\partial \theta}{\partial t} - \frac{\partial^2 \theta}{\partial x^2} + \gamma \sin(\theta) = \mathbb{1}_{\mathcal{O}}(x)U(x, t) \quad \text{dans } (0, L) \times (0, +\infty), \quad (\text{A.23a})$$

$$\theta(0, t) = \theta(L, t) = 0 \quad \text{pour tout } t \in (0, +\infty), \quad (\text{A.23b})$$

où  $\xi$  et  $\gamma$  sont des paramètres strictement positifs, et  $\mathcal{O}$  est un sous-ensemble ouvert de  $(0, L)$ . Enfin, une équation de Wilson-Cowan pré-stabilisée issue des neurosciences :

$$\frac{\partial w}{\partial t}(x, t) + \alpha w(x, t) + \int_{\Omega} k(x, \nu) s(w(\nu, t)) d\nu = \mathbb{1}_{\mathcal{O}}(x)U(x, t) \quad \text{dans } \Omega \times (0, +\infty), \quad (\text{A.24})$$

où  $\Omega$  est un domaine borné de  $\mathbb{R}^n$ ,  $\mathcal{O}$  est un sous-ensemble ouvert de  $\Omega$ ,  $\alpha$  est un paramètre strictement positif,  $k$  est un noyau intégral borné et  $s$  est une non-linéarité scalaire suffisamment régulière.

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