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# Special cycles on orthogonal Shimura varieties

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**École doctorale de sciences mathématiques de Paris centre**

# Thèse de doctorat

Discipline : Mathématiques

présentée par

**Ruishen Zhao**

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# Special cycles on orthogonal Shimura varieties

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dirigée par Christophe Cornut

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*If people do not believe that mathematics is simple, it is only because they do not realize how complicated life is.*

John von Neumann

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# Résumé

Dans cette thése, nous développons une méthode générale pour établir les relations de distribution horizontales et verticales pour des cycles spéciaux dans les variétés de Shimura. Nous utilisons les cycles unitaires dans les variétés orthogonales impaires de Shimura comme exemple principal. Nous présentons aussi quelques autres exemples, comme les variétés de Shimura $GSp(4)$ et les variétés unitaires de Shimura. Ces cycles spéciaux avec relations de distribution sont une généralisation en dimension supérieure des points de Heegner. Ils peuvent être utilisés pour construire un système d'Euler, en vue d'applications arithmétiques.

**Mots-clés**

cycles spéciaux, relations de distribution, variétés de Shimura

## Abstract

In this thesis, we develop a general method to establish tame and norm relations for special cycles in Shimura varieties. We use unitary cycles in odd orthogonal Shimura varieties as a guiding example. We also list some other examples like $GSp(4)$ Shimura varieties and unitary Shimura varieties. Such special cycles with desired distribution relations is a higher dimensional generalization of Heegner points. It can be applied to construct an Euler system to do arithmetic applications etc.



### Keywords

special cycles, distribution relations, Shimura varieties

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# Chapitre 1

# Introduction

• **A classical example : Heegner points**

Special cycles on Shimura varieties is an important topic and has significant applications to number theory, representation theory, arithmetic geometry and so on. Among them, a classical example is Heegner points on modular curves. Before we state the main results of this thesis, we would like to illustrate this basic example as a motivation.

Let $N$ be a positive integer and $K/\mathbb{Q}$ be an imaginary quadratic field with ring of integers $\mathcal{O}_K$. Suppose that all primes of $N$ split in $K$ (Heegner hypothesis) and let $\mathcal{N}$ denote an ideal of $\mathcal{O}_K$ with norm $N$. For any positive integer $m$ prime to $N$, let $\mathcal{O}_m = \mathbb{Z} + m\mathcal{O}_K$ denote the corresponding order of $K$ and $K[m]$ denote the ring class field with conductor $m$, then there is a Heegner point $x_m$ lying in $X_0(N)(K[m])$ corresponding to the isogeny $\mathbb{C}/\mathcal{O}_m \longrightarrow \mathbb{C}/(\mathcal{N} \cap \mathcal{O}_m)^{-1}$. Define $Heeg_K = \{x_m : m \text{ prime to } N\ \}$. Among this family of special points, they satisfy the following distribution relations :

**Theorem 1.1.** *(Distribution relations)*

*Let $m$ be a positive integer and $l$ be a prime unramified in $K$. Suppose $ml$ is prime to $N$, then we have*

*(1)* ***Tame relations :*** *If $l \nmid m$, let $\lambda$ denote a prime of $K$ that lies over $l$, then*

$$Card(\frac{\mathcal{O}_m^*}{\mathcal{O}_{ml}^*}) \times Tr_{\frac{K[ml]}{K[m]}}(x_{ml}) = \begin{cases} T_l x_m, & \text{if } l \text{ is inert in } K, \\ (T_l - Frob_\lambda - Frob_\lambda^{-1})x_m, & \text{if } l \text{ splits in } K. \end{cases}$$

*Here $Frob_\lambda$ denotes the geometric Frobenius in $Gal(K[m]/K)$ corresponding to $\lambda$, $T_l$ denotes the usual Hecke operator associated to the double coset*

$$[GL_2(\mathbb{Z}_l)diag\{l, 1\}GL_2(\mathbb{Z}_l)].$$

*(2)* ***Norm relations :*** *If $l \mid m$, then*

$$Tr_{\frac{K[ml]}{K[m]}}(x_{ml}) = T_l x_m - x_{\frac{m}{l}}.$$

*Here $T_l$ is the same Hecke operator as above.*

We refer to [18] (proposition 3.10) and [29] for more details.

The distribution relations play a basic role in Kolyvagin's work on modular elliptic curves, which is an important progress towards Birch and Swinnerton-Dyer conjecture for elliptic curves. See [18] (chapter 10) and [29] for more details. Kolyvagin's work needs the modularity assumption, but due to Wiles' school's great work ([67] and [10]), this assumption holds for any elliptic curves over $\mathbb{Q}$.

The famous BSD conjecture can be generalized to the Bloch-Kato conjecture, see [5]. There is no doubt that the Bloch-Kato conjecture is one of the most fundamental problems in modern number theory and arithmetic geometry. And since the seminal work of Kolyvagin ([35]), Euler systems play an important role in the study of the Bloch-Kato conjecture. It is a powerful tool to bound Selmer groups. See Rubin's book [61] for more details.

Moreover, the Heegner points have many other important applications :

(i) Let $E$ denote an elliptic curve over $\mathbb{Q}$ with conductor $N$. By the modularity theorem, there exists a modular parametrization $X_0(N) \longrightarrow E$. For an imaginary quadratic field $K$ satisfying Heegner hypothesis (all primes of $N$ split in $K$), the Gross-Zagier formula relates the first derivative of the $L$-function for $E$ over $K$ at the central point $s = 1$, $L^{'}(E/K, 1)$, to the height of the corresponding Heegner point $P_K \in E(K)$ (see [31]), thus (together with Kolyvagin's work) proves the BSD conjecture for such elliptic curve $E$ over $K$ with analytic rank one.

(ii) A closely related problems about Galois representations is the Iwasawa theory, the study of variation of Selmer groups along $p$-adic extensions. Heegner points can help to prove (anti-cyclotomic) main conjecture, see [57], [58] and [33] for more details.

(iii) There is an important Mazur conjecture about Heegner points and it is proved in [14]. That non-triviality theorem can be used to study Selmer groups etc. See the introduction of [14] for more details.

.....

Therefore it is an important task to generalize Heegner points to higher dimensional cases, which is the main task of this thesis.

• **Main results : higher dimensional generalizations**

In this thesis, we will investigate a general method to construct special cycles satisfying certain distribution relations (tame relations and norm relations) on other Shimura varieties, which can be seen as higher dimensional generalizations of Heegner points. As we mentioned above, these distribution relations can be used to construct an Euler system, which is an important tool to study the Bloch-Kato conjecture. And in fact, the tame relations are used to construct the so called "derived classed", which form a Kolyvagin system,

see [16] and [49] for more details. The norm relations can be used to construct norm compatible systems of Galois cohomology classes (under certain ordinary conditions), which may also have applications to the related problems in the Iwasawa theory.

In this thesis, the tame relations are expressed through the Hecke polynomial (see section 2.2), which is related to the Shimura variety by the congruence conjecture in [4]. There are many results towards this conjecture already. For example, see [41] and [68]. In particular, this conjecture holds for Hodge type Shimura varieties.

For simplicity, we use orthogonal Shimura varieties as the main example.

The starting point is to consider an embedding of Shimura data $Sh_{\mathbf{H}} \longrightarrow Sh_{\mathbf{G}}$, where $\mathbf{H} \longrightarrow \mathbf{G}$ is an embedding of reductive groups over $\mathbb{Q}$ inducing compatible maps $\mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m(\mathbb{R}) \longrightarrow \mathbf{H}(\mathbb{R}) \longrightarrow \mathbf{G}(\mathbb{R})$. Our main example is as follow :

Fix a totally real number field $F$ and fix a real place $f_0 \in \mathbf{SpF} = \mathrm{Spec}(F)(\overline{\mathbb{Q}}) = \mathrm{Spec}(F)(\mathbb{R}) = \mathrm{Spec}(F)(\mathbb{C})$, here $\overline{\mathbb{Q}}$ is the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$. For a positive integer $n$, let $(V, \phi)$ denote a quadratic $F$-vector space of dimension $2n+1$. Define $(V_a, \phi_a) = (V, \phi) \otimes_{F,a} \mathbb{R}$ $(a \in \mathbf{SpF})$. Then we require the following signature condition :

$$sign(V_a, \phi_a) = \begin{cases} (2n-1, 2), & \text{if } a = f_0, \\ (2n+1, 0), & \text{if } a \neq f_0. \end{cases}$$

Now we define $\underline{\mathbf{G}} = SO(V, \phi)$, it is a reductive group over $F$. Define $\mathbf{G} = \mathrm{Res}_{F/\mathbb{Q}}\underline{\mathbf{G}}$, it is a reductive group over $\mathbb{Q}$. The signature condition determines $\mathbf{G}_{\mathbb{R}} = \prod_a \mathbf{G}_a$ with $\mathbf{G}_a = SO(V_a, \phi_a)$, so that

$$SO(V_a, \phi_a) = \begin{cases} SO(2n-1, 2) \text{ (non-compact)}, & \text{if } a = f_0, \\ SO(2n+1, 0) \text{ (compact)}, & \text{if } a \neq f_0. \end{cases}$$

Now we define the associated Hermitian symmetric domain. Let $X$ denote the space of oriented negative $\mathbb{R}$-planes in $(V_f, \phi_f)$. The action of $\mathbf{G}_{f,\mathbb{R}}(\mathbb{R}) = SO(2n-1, 2)$ on $(V_f, \phi_f)$ induces a transitive action of $\mathbf{G}(\mathbb{R})$ on $X$, and the stabilizer of any point is a maximal compact and connected subgroup. We can view $X$ as a $\mathbf{G}(\mathbb{R})$-conjugacy class of maps $h : \mathrm{Res}_{\mathbb{C}/\mathbb{R}}(\mathbb{R})(\mathbb{G}_m) \longrightarrow \mathbf{G}(\mathbb{R})$. In this way we get a Shimura variety $Sh_{\mathbf{G}}(K) = \mathbf{G}(Q)\backslash X \times \mathbf{G}(\mathbb{A}_f)/K$ (for $K$ small enough) (see [16]).

Now we define the unitary subgroup $\mathbf{H}$. Let $E$ denote a totally imaginary quadratic extension of $F$ which splits $(V, \phi)$. Then $(V, \phi)$ contains $E$-hermitian $F$-hyperplanes and all of them are conjugate under $\mathbf{G}(\mathbb{Q}) = SO(V, \phi)$ (see [16] section 9.2). We choose one such $E$-hermitian space $(W, \psi)$ and define $\underline{\mathbf{H}} = U(W, \psi)$ (a subgroup of $\underline{\mathbf{G}}$), $\mathbf{H} = \mathrm{Res}_{F/\mathbb{Q}}\underline{\mathbf{H}}$ (a subgroup of $\mathbf{G}$). We can compute its signature over $\mathbb{R}$. Define $(W_a, \psi_a) = (W, \psi) \otimes_{F,a} \mathbb{R}$ $(a \in \mathbf{Sp})$. Then $\mathbf{H}_{\mathbb{R}} = \prod_a \mathbf{H}_a$ with $\mathbf{H}_a = U(W_a, \psi_a)$, these factors are as follow :

$$U(W_a, \psi_a) = \begin{cases} U(n-1,1), & \text{if } a = f_0, \\ U(2n+1), & \text{if } a \neq f_0. \end{cases}$$

The complex embedding $f_0 : F \longrightarrow \mathbb{C}$ has two extensions to $E$, and we fix one choice $f_0^+ : E \longrightarrow \mathbb{C}$ and denote by $f_0^-$ its conjugation. Identify $E_{f_0} = E \otimes_{F,f_0} \mathbb{R}$ with $\mathbb{C}$ using $f_0^+$. Let $Y$ denote the space of negative $\mathbb{C}$-lines (via this identification) in $(W_f, \psi_f)$. Similarly, we obtain a Shimura varieties $Sh_{\mathbf{H}}(K_1) = \mathbf{H}(\mathbb{Q})\backslash Y \times \mathbf{H}(\mathbb{A}_f)/K_1$. The natural inclusion $\mathbf{H} \longrightarrow \mathbf{G}$ will induce a map between their Shimura varieties $Sh_{\mathbf{H}}(K_1) \longrightarrow Sh_{\mathbf{G}}(K)$ (if $K_1 \subset K$).

The image of a geometric connected component of the unitary Shimura varieties defines a special cycle on the orthogonal Shimura varieties (for suitable $K$, this is a closed embedding already, although we will not need this fact). Then the basic idea is to apply $\mathbf{G}(\mathbb{A}_f)$ translation to get a family of special cycles. More precisely, for any $g \in \mathbf{G}(\mathbb{A}_f)$, the $Z_{\mathbf{K}}(g) = [Y \times gK]$ defines a cycle in $Sh_{\mathbf{G}}(K)$ (here notice that $Y$ is already connected), then we get a family of special cycles $Z_K(\mathbf{G},\mathbf{H}) = \{Z_K(g)\}$. Moreover, it has the following parametrization :

$$Z_K(\mathbf{G}, \mathbf{H}) = \mathbf{H}(\mathbb{Q})\mathbf{H}^{der}(\mathbb{A}_f)\backslash \mathbf{G}(\mathbb{A}_f)/K.$$

The main object of this paper will be $R[Z_K(\mathbf{G}, \mathbf{H})]$, here $R$ is a suitable coefficient ring such as $\mathbb{Z}$ or $\mathbb{Z}_l$ etc.

The first problem is to understand the Hecke action and the Galois action on this family of special cycles. We will use Deligne's reciprocity law on the set of geometric connected components to describe the Galois action. To construct special cycles with the desired tame relations and norm relations, we will first study them locally. In the local case, we need to study the $\mathbf{H}$-action on the Bruhat-Tits buildings of $\mathbf{G}$. One important tool for both relations is the Boumasmoud's relation, see theorem 2.3. Inspired by work of Boumasmoud and Loeffler, we first obtain some purely group theoretical relations ("**the abstract relations**"), see theorem 4.3 and theorem 5.4.

The next problem is to translate these purely group theoretical relations into "true" distribution relations ("**the realization**"), which are our main results. For this step, we require more conditions.

For the tame relation, we have the following main result (theorem 4.6) unconditionally. However, to do arithmetic applications (e.g. construct an Euler system), we need the congruence conjecture in [4] to relate the Hecke polynomial with the Galois polynomial.

**Theorem 1.2.** ***tame relations***

*There exists a family of special cycles $\{z(m) | m \in \underline{\boldsymbol{N}}\}$, such that for any $m \in \underline{\boldsymbol{N}}$, the special cycle $z(m)$ is defined over $\underline{E}[m]$, and for any $v \in \underline{\boldsymbol{P}}$ with $v \nmid m$, we have $Tr_{\frac{\underline{E}[mv]}{\underline{E}[m]}}(z(mv)) = Hep_{\mu_{\boldsymbol{G},v}}(Frob_{v^+})(z(m))$.*

Here $\underline{\mathbf{P}}$ is a subset of primes ("good primes") in $F$ that split in $F \longrightarrow E$, $v^+$ is a

chosen prime of $E$ above $v$, $Frob_{v^+}$ is the corresponding geometric Frobenius, $\underline{\mathbf{N}}$ is the set of square-free products of elements in $\underline{\mathbf{P}}$, $\underline{E}[m]$ is an abelian extension of $E$, $Hep_{\mu_{\mathbf{G},v}}$ is the Hecke polynomial associated to the Shimura datum $(G, X)$ over $F_v$ (see section 2.2).

For the norm relations, we need conductor growth conditions, and ordinary conditions. The main result is theorem 5.5 :

**Theorem 1.3.** ***norm relations***

*For any positive integer $m$, there is a special cycle $z_m$ defined over the field $\widetilde{E}(m)$, such that*

$$\sum_{i=0}^{N} C_i A_i Tr_{\frac{\widetilde{E}(m+i)}{\widetilde{E}(m)}}(z_{m+i}) = 0.$$

Here $\widetilde{E}(m)$ is an abelian extension of $E$ and $\widetilde{E}(\infty) = \cup\widetilde{E}(m)$ is a nontrivial anticyclotomic $p$-adic extension of $\widetilde{E}(1)$, $C_i$ are constant integers, $A_i$ are Hecke operators determined by the related Hecke polynomial for $\mu$ (a chosen cocharacter) in section 5.1 and $N$ is the degree of that Hecke polynomial. And because we need the computation about conductors in [15], we will work over an inert prime in the norm relations.

The resulting special cycles with norm relations can be used to construct a norm compatible family (under ordinary condition) of Galois cohomology classes. Such translation technique is standard and it is stated as the lemma 5.6 in section 5.2.

• **Other examples**

Although for simplicity, we only discuss orthogonal Shimura varieties in the main part, this approach is flexible and works in other cases. We give some other examples as an application, $GSp_4$ Shimura varieties and unitary GGP pair Shimura varieties :

(i) Special cycles inside the Siegel threefold,

$$\mathbf{H} = GU(1) \times_{G_m} GL_2 \longrightarrow \mathbf{G} = GSp_4$$

(where $GU(1) = \mathrm{Res}_{E/\mathbb{Q}}\mathbb{G}_{m,E}$ and $E$ is an imaginary quadratic field).

(ii) The usual unitary GGP pair

$$U(n-1,1) \longrightarrow U(n,1) \times U(n-1,1).$$

(iii) The similitude version of this unitary GGP pair,

$$\mathbf{H} = GU(1,n-1) \times_{\mathbb{G}_m} GU(0,1) \longrightarrow \mathbf{G} = GU(1,n-1) \times GU(1,n).$$

(iv) Special cycles inside the $GU(1,2n-1)$ Shimura variety,

$$\mathbf{H} = GU(1,n-1) \times_{\mathbb{G}_m} GU(0,1)^n \longrightarrow \mathbf{G} = GU(1,2n-1).$$

Here the products are all over $\mathbb{G}_m$ through the similitude factor map.

• **comparison with previous results**

We make a comparison with previous results in the literature.

The idea of applying $\mathbf{G}(\mathbb{A}_f)$ translation is a common technique. Loeffler, Cornut and many other people have already used this idea to construct Euler systems, for example see [47], [46], [28], [16], [9], [34]. Our method is different from Loeffler's methods. He changes the level group of $\mathbf{G}(\mathbb{A}_f)$ to realize the Galois action, and for the tame relations, one of the key point is to use a multiplicity one result (branching law), which does not hold in our setting. Instead we follow the method of Cornut, use geometric connected components, and the Galois group acts on this set through Deligne's reciprocity law.

The main advantage of our method is that it works in a general setting. For both sides (tame and norm), the purely group theoretical relations are established using Boumasmoud's "Seed Relation". We then connect these abstract local relations to genuine relations. For the tame relations, this is about understanding the Galois action on $\pi_0(Sh_{\mathbf{H}})$. Boumasmoud also uses his "Seed Relation" to establish tame relations (over split primes) for the unitary GGP pair Shimura varieties in [9]. Our method can recover his results (see section 6.2). Cornut deduces tame relations for our main example over inert primes by explicit and complicated computations in [16]. His method is totally different (no "Seed Relation" etc) and use special properties of Buildings of orthogonal groups etc, while our method solves tame relations over split primes. For the norm relations, Loeffler has developed a general method to construct norm compatible systems (see [44]), our method is inspired by that paper. But we follow a similar pattern as tame relations and still use "Seed Relation", which is new. And our ordinary conditions may be different from Loeffler's ordinary conditions. And among other examples we construct, the $GSp_4$ example and the similitude version of unitary GGP pair are new. Regarding the $(GU(1,n-1)\times_{\mathbb{G}_m} GU(0,1)^n, GU(1,2n-1))$ example, Andrew Graham and Syed Waqar Ali Shah considered a similar pair $(GU(1,n-1)\times_{\mathbb{G}_m} GU(0,n), GU(1,2n-1))$ in [28]. They also construct a family of special cycles with tame relations and norm relations over split primes using the methods of Loeffler's school.

• **A byproduct : the stabilizer conjecture**

Along the study of norm relations via spherical pairs, we find some special properties about the stabilizer group associated to such pairs, and we conjecture it holds in general. This stabilizer conjecture will help us calculate conductors. For example it is used in the computation of the conductors in the norm relations for our main example in section 5.2. We put this conjecture in the appendix (section A.1) because it may have independent interest in representation theory. In that section, we prove this conjecture for our main examples, and for symmetric pairs etc.

• **Applications**

As we mentioned above, one of the main arithmetic applications of such special cycles is to construct an Euler system, then deduce results about Bloch-Kato conjecture. Regarding

the main example, Cornut has deduced rank one result for Selmer group in our setting under some technical conditions (see [16]). His result needs nontriviality conditions, which is widely open at present. We would need an explicit reciprocity law, or a Gross-Zagier formula, to relate the resulting special cycles to $L$-functions ($p$-adic or complex). Compared to his results, our tame relations are over split primes. Kolyvagin's original argument uses tame relations over inert primes, thus we can't apply his method directly. However, very recently, Jetchev, Nekovar and Skinner have managed to solve this problem by using split primes instead of inert primes in [23]. The Kolyvagin systems obtained in this way are refered to as split Kolyvagin systems. Our main example and other examples will produce such split Kolyvagin systems, and then can be used to study related Bloch-Kato conjectures.

Another important application is the Iwasawa theory. The norm compatible system will help us study related problems. Thus we may wish to use such family of special cycles to study Iwasawa Main conjectures, higher dimensional analogue of Mazur conjectures etc.

• **Outline**

The structure of this paper is as follow :

In chapter 2, we give the general setting. In its first subsection, we work globally, establish some properties of such family of special cycles and show how to translate global problems into local setting. In its second subsection, we work in a local setting, state Boumasmoud's relation (theorem 2.3), which is the key tool for both relations.

In chapter 3, we introduce some background knowledge on cohomology theory. This will be used to do arithmetic applications in chapter 6.

In chapter 4, we consider the tame relations. We first establish an abstract version of the tame relation as theorem 4.3, then we deduce the main result as theorem 4.6.

In chapter 5, we consider the norm relations and follow the same pattern as in chapter 3. The abstract version of norm relations is established as theorem 5.4 and the main result is theorem 5.5.

In chapter 6, we look at other examples and arithmetic applications. We construct other examples, for $GSp_4$ or unitary type Shimura varieties. Then we discuss some arithmetic applications, such as Euler system and related problems.

In chapter 7 (appendix), we discuss the stabilizer conjecture and prove it for some important cases, including our main example and for symmetric pairs $(H, G)$. And we also prove some miscellaneous facts to make this thesis more self-contained.

# Chapitre 2

# General Setup

## 2.1 Global setting

First we recall the definition of a Shimura datum briefly and refer to [50] for more details.

Let $\mathbb{S} = \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m$ denote the Deligne torus. A **Shimura datum** is a pair $(\mathbf{G}, X)$ consisting of a reductive group $\mathbf{G}$ over $\mathbb{Q}$ and a $\mathbf{G}(\mathbb{R})$-conjugacy class $X$ of homomorphisms $h : \mathbb{S} \longrightarrow \mathbf{G}_{\mathbb{R}}$ satisfying the following conditions :

SV1 : for all $h \in X$, the Hodge structure on $\mathrm{Lie}(\mathbf{G}_{\mathbb{R}})$ defined by $Ad \circ h$ ($Ad$ denotes the adjoint action) is of type $\{(-1,1),(0,0),(1,-1)\}$;

SV2 : for all $h \in X$, $ad(h(i))$ is a Cartan involution of the adjoint quotient group $\mathbf{G}^{ad}_{\mathbb{R}}$ ;

SV3 : the group $\mathbf{G}^{ad}$ has no $\mathbb{Q}$-factor on which the projection of $h$ is trivial.

The space $X$ is a finite disjoint union of hermitian symmetric domains, and in this thesis, it will be clear according to the text. Thus we also write a Shimura datum $(\mathbf{G}, X)$ as $Sh_{\mathbf{G}}$ or $Sh(\mathbf{G})$ for simplicity.

Some Shimura datum satisfy more conditions and there are the following additional axioms :

SV4 : the weight map $w : \mathbb{G}_{m,\mathbb{R}} \longrightarrow \mathbb{S} \longrightarrow \mathbf{G}_{\mathbb{R}}$ (this map is independent of choice of the point $h \in X$) is defined over $\mathbb{Q}$, we say that the weight is rational.

SV5 : the group $Z_{\mathbf{G}}(\mathbb{Q})$ is discrete in $Z_{\mathbf{G}}(\mathbb{A}_f)$ (here $\mathbb{A}_f$ denotes the finite adeles of $\mathbb{Q}$).

SV6 : the central central torus $Z^0_{\mathbf{G}}$ splits over a CM field.

For a compact open subgroup $K \subset \mathbf{G}(\mathbb{A}_f)$ (we will call $K$ a level subgroup) that is small enough, the double coset space $\mathbf{G}(\mathbb{Q})\backslash X \times \mathbf{G}(\mathbb{A}_f)/K$ can be seen as an algebraic variety over $\mathbb{C}$ and has a canonical model over its reflex field (a number field), we call it a **Shimura variety** and denote it as $Sh_{\mathbf{G}}(K)$.

We illustrate the following standard example :

**Example** (modular curve)

Let $\mathbf{G} = GL_2$ over $\mathbb{Q}$ and $X$ denote the $GL_2(\mathbb{R})$ conjugates of the map $h_o : \mathbb{S} \longrightarrow GL_{2,\mathbb{R}}$,

$$h_o(a + \sqrt{-1}b) = \left(\begin{smallmatrix} a & b \\ -b & a \end{smallmatrix}\right).$$

Then $(\mathbf{G}, X)$ is a Shimura datum. And there is a natural bijection between $X$ and $\mathbb{C}-\mathbb{R}$. For an integer $N \geq 6$, let $K_N$ denote the kernel of the reduction map $GL_2(\mathbb{A}_f) \longrightarrow GL_2(\mathbb{Z}/N)$, then $K_N$ is an example of level group. The resulting Shimura variety is an open modular curve, $Y(N)$.

The starting point of this thesis is **an embedding of Shimura data** ("relative version"),

$$Sh_{\mathbf{H}} \longrightarrow Sh_{\mathbf{G}},$$

where $\mathbf{H} \longrightarrow \mathbf{G}$ is an embedding of reductive groups over $\mathbb{Q}$ inducing compatible maps

$$\mathbb{S} \longrightarrow \mathbf{H}_{\mathbb{R}} \longrightarrow \mathbf{G}_{\mathbb{R}}.$$

**Example** (special points on a modular curve)

Still let $(\mathbf{G}, X)$ denote the Shimura datum of the modular curve as above. Let $\mathbf{H}$ denote $\mathrm{Res}_{\mathbb{Q}(\sqrt{-1})/\mathbb{Q}}\mathbb{G}_m$, the associated hermitian symmetric domain $Y$ is a single point, corresponding to the identity map, $\mathbb{S} \cong \mathbf{H}_{\mathbb{R}}$. We can think of $\mathbb{Q}(\sqrt{-1})$ as a two dimensional $\mathbb{Q}$-space, with an ordered basis $\{1, -\sqrt{-1}\}$. Through this basis, we can embed $\mathbf{H}$ into $\mathbf{G}$ (over $\mathbb{Q}$). And this group embedding is compatible with their associated hermitian symmetric domains. Thus we get an embedding of Shimura data. It is an example of special points on a modular curve.

Now we will state a general strategy to produce special cycles on the ambient Shimura variety.

Consider an embedding of Shimura data, $Sh_{\mathbf{H}} \longrightarrow Sh_{\mathbf{G}}$, where $Sh_{\mathbf{H}}$ is short for the Shimura datum $(H, Y)$ and $Sh_{\mathbf{G}}$ is short for the Shimura datum $(G, X)$. Let $Y^0$ denote a connected component of $Y$. For a small enough level group $K$ of $\mathbf{G}(\mathbb{A}_f)$ we obtain Shimura varieties $Sh_{\mathbf{G}}(K) = \mathbf{G}(\mathbb{Q})\backslash X \times \mathbf{G}(\mathbb{A}_f)/K$ and $Sh_{\mathbf{H}}(\mathbf{H}(\mathbb{A}_f) \bigcap K) = \mathbf{H}(\mathbb{Q})\backslash Y \times \mathbf{H}(\mathbb{A}_f)/(\mathbf{H}(\mathbb{A}_f) \bigcap K)$. For $g \in \mathbf{G}(\mathbb{A}_f)$, the image $Z_K(g) = [Y^0 \times gK]$ of $Y^0 \times gK$ in $Sh_{\mathbf{G}}(K)$ defines a **cycle** (irreducible and closed). We thus obtain **a family of special cycles**

$$Z_K(\mathbf{G}, \mathbf{H}) = \{Z_K(g) | g \in \mathbf{G}(\mathbb{A}_f)\},$$

which has the following parametrization :

**Lemma 2.1.** *The natural map $Stab_{\boldsymbol{G}(\mathbb{Q})}(Y^0)\backslash \boldsymbol{G}(\mathbb{A}_f)/K \longrightarrow Z_K(\mathbf{G}, \mathbf{H})$ is a bijection.*

*Démonstration.* It is obviously surjective, so we only need to establish injectivity.

Take $g_1, g_2 \in \mathbf{G}(\mathbb{A}_f)$ with $Z_K(g_1) = Z_K(g_2)$. Then, for any $y \in Y^0$, there exists $g \in$

$\mathbf{G}(\mathbb{Q})\bigcap g_1Kg_2^{-1}$ such that $y \in gY^0$, therefore $Y^0 = \bigcup_{g\in\mathbf{G}(\mathbb{Q})\bigcap g_1Kg_2^{-1}} Y^0\bigcap gY^0$. Notice that $Y^0$ and its translation $g(Y^0)$ are closed submanifolds of $X$, each intersection $Y^0\bigcap g(Y^0)$ is closed in $Y^0$ and this is a countable union, so by the Baire Category theorem, there exists some $g$ in $\mathbf{G}(\mathbb{Q})\bigcap g_1Kg_2^{-1}$ such that $Y^0\bigcap gY^0$ contains a non-empty open subset $U$ of $Y^0$. Take a point $y \in U$, we get equalities among tangent spaces $T_yY^0 = T_yU = T_y(gY^0)$ by dimension reason. Because both $Y^0$ and $gY^0$ are totally geodesic submanifolds, this implies $gY^0 = Y^0$. Therefore $g \in Stab_{\mathbf{G}(\mathbb{Q})}(Y^0)$. □

Now we can introduce the main object of this paper, the free $R$-module generated by $Z_K(\mathbf{G},\mathbf{H})$, $R[Z_K(\mathbf{G},\mathbf{H})]$, where $R$ is a suitable coefficient ring (such as $\mathbb{Z}$ or $\mathbb{Z}_p$). We will endow it with a Galois-Hecke action.

For the Hecke action, we can use the geometric Hecke action on cycles, or equivalently (see [16] section 5.16), think of the global Hecke algebra as $End_{\mathbf{G}(\mathbb{A}_f)}(R[G(\mathbb{A}_f)/K])$, which then has a right action on this module formally (via the parametrization). We will use this second description for the Hecke action. For the Galois action, we will use the reciprocity law for the set of geometric connected components. We have another description of $Z_K(g)$ as follow :

Consider $K_1 = gKg^{-1}\bigcap\mathbf{H}(\mathbb{A}_f)$, we have natural maps

$$Sh_{\mathbf{H}}(K_1) \longrightarrow Sh_{\mathbf{G}}(gKg^{-1}) \xrightarrow{[*g]} Sh_{\mathbf{G}}(K).$$

These maps are defined over the reflex field E for $Sh_{\mathbf{H}}$ and $Z_K(g)$ is the image of the connected component $[Y^0 \times K_1]$ of $Sh_{\mathbf{H}}(K_1)_{\overline{\mathbb{Q}}}$ in $Sh_{\mathbf{G}}(K)_{\overline{\mathbb{Q}}}$. The Galois group $Gal_E$ will act on $\pi_0(Sh_{\mathbf{H}}(K_1))$ by Deligne's reciprocity law (see [22]), then it acts on the set of special cycles $Z_K(\mathbf{G},\mathbf{H})$.

Now let's come to the main example. We will illustrate the above ideas through this example.

We first introduce the **ambient Shimura datum** $Sh_{\mathbf{G}}$.

Take a totally real number field $F$ and fix a real place $f_0$ in

$$\mathbf{SpF} = \mathrm{Spec}(F)(\overline{\mathbb{Q}}) = \mathrm{Spec}(F)(\mathbb{R}) = \mathrm{Spec}(F)(\mathbb{C}),$$

here $\overline{\mathbb{Q}}$ is the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$. Fix a positive integer $n$ and let $(V,\phi)$ denote a quadratic $F$-vector space of dimension $2n+1$. Define

$$(V_a,\phi_a) = (V,\phi)\otimes_{F,a}\mathbb{R}$$

for $a \in \mathbf{SpF}$. We require the following signature condition :

$$sign(V_a, \phi_a) = \begin{cases} (2n-1, 2), & \text{if } a = f_0, \\ (2n+1, 0), & \text{if } a \neq f_0. \end{cases}$$

Define $\underline{\mathbf{G}} = SO(V, \phi)$, it is a reductive group over $F$. Define $\mathbf{G} = \mathrm{Res}_{F/\mathbb{Q}}\underline{\mathbf{G}}$, it is a reductive group over $\mathbb{Q}$. The signature condition determines $\mathbf{G}_{\mathbb{R}} = \prod_a \mathbf{G}_a$ with $\mathbf{G}_a = SO(V_a, \phi_a)$, so that

$$\mathbf{G}_a = SO(V_a, \phi_a) = \begin{cases} SO(2n-1, 2) \text{ (non-compact)}, & \text{if } a = f_0, \\ SO(2n+1, 0) \text{ (compact)}, & \text{if } a \neq f_0. \end{cases}$$

Now we define the associated space $X$.

Let $X$ denote the space of oriented negative $\mathbb{R}$-planes in $(V_{f_0}, \phi_{f_0})$ : the set of all pairs $x = (D_x, \theta_x)$ where $D_x$ is a two dimensional negative $\mathbb{R}$-plane of $(V_{f_0}, \phi_{f_0})$ and $\theta_x$ is an orientation on $D_x$. The action of $\mathbf{G}_{f_0}(\mathbb{R}) = SO(2n-1, 2)$ on $(V_{f_0}, \phi_{f_0})$ induces a transitive action of $\mathbf{G}(\mathbb{R})$ on $X$, and the stabilizer of any point is a maximal compact and connected subgroup. Moreover, we can view $X$ as a $\mathbf{G}(\mathbb{R})$-conjugacy class of maps $h : \mathrm{Res}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_m)(\mathbb{R}) \longrightarrow \mathbf{G}(\mathbb{R})$ as follow :

Recall $\mathbb{S} = \mathrm{Res}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_m)$, let $\mathbb{S}^1 \subset \mathbb{S}$ denote the kernel of the norm map $\mathcal{N} : \mathbb{S} \longrightarrow \mathbb{G}_{m,\mathbb{R}}$, and let $vm : \mathbb{S} \longrightarrow \mathbb{S}^1$ be the homomorphism which maps $z \in \mathbb{C}^* = \mathbb{S}(\mathbb{R})$ to $vm(z) = \frac{z}{\bar{z}} \in \mathbb{S}^1(\mathbb{R})$. Thinking of the orientation $\theta_x$ on the plane $D_x$ as being given by an isomorphism

$$\theta_x : \mathbb{S}^1 \xrightarrow{\cong} SO(D_x),$$

we define a morphism $h_x : \mathbb{S} \longrightarrow \mathbf{G}_{\mathbb{R}}$ by $h_x = u_x \circ vm$ where

$$u_x : \mathbb{S}^1 \xrightarrow{\theta_x} SO(D_x) \hookrightarrow SO(D_x) \times SO(D_x^{\perp}) \hookrightarrow \mathbf{G}_{f_0} \hookrightarrow \mathbf{G}_{\mathbb{R}}.$$

Therefore through the map $x \mapsto h_x$, we can identify $X$ with a $\mathbf{G}(\mathbb{R})$-conjugacy class of maps $h : \mathbb{S} \longrightarrow \mathbf{G}_{\mathbb{R}}$. And it satisfies the axioms SV1-3 for Shimura varieties in [50]. In this way we get a Shimura datum $Sh_{\mathbf{G}} = (\mathbf{G}, X)$. Moreover, this Shimura datum also satisfies the additional axioms SV4-6. For a level group $K$, we get a Shimura variety $Sh_{\mathbf{G}}(K) = \mathbf{G}(Q)\backslash X \times \mathbf{G}(\mathbb{A}_f)/K$.

Now we define the **sub-Shimura datum**.

Let $E$ denote a totally imaginary quadratic extension of $F$ which splits $(V, \phi)$, i.e. such that the quadratic space $(V, \phi) \otimes_F E$ over $E$ contains a totally isotropic $E$-subspace of dimension $n$. Then $(V, \phi)$ contains $E$-hermitian $F$-hyperplanes, and all of them are conjugate under $\mathbf{G}(\mathbb{Q}) = SO(V, \phi)$ (see [16] 9.2). We choose one such $E$-hermitian space $(W, \psi)$ and define $\underline{\mathbf{H}} = U(W, \psi)$ (a subgroup of $\underline{\mathbf{G}}$), $\mathbf{H} = \mathrm{Res}_{F/\mathbb{Q}}\underline{\mathbf{H}}$ (a subgroup of $\mathbf{G}$). We can compute its signature over $\mathbb{R}$. Define $(W_a, \psi_a) = (W, \psi) \otimes_{F,a} \mathbb{R}$ ($a \in \mathbf{SpF}$). Then

$\mathbf{H}_{\mathbb{R}} = \prod_a \mathbf{H}_a$ with $\mathbf{H}_a = U(W_a, \psi_a)$, these factors are as follow :

$$\mathbf{H}_a = U(W_a, \psi_a) = \begin{cases} U(n-1,1), & \text{if } a = f_0, \\ U(2n+1), & \text{if } a \neq f_0. \end{cases}$$

The complex embedding $f_0 : F \longrightarrow \mathbb{C}$ has two extensions to $E$, and we fix one choice $f_0^+ : E \longrightarrow \mathbb{C}$ and denote by $f_0^-$ its conjugation. Identify $E_{f_0} = E \otimes_{F,f_0} \mathbb{R}$ with $\mathbb{C}$ using $f_0^+$. Let $Y$ denote the space of negative $\mathbb{C}$-lines (via this identification) in $(W_{f_0}, \psi_{f_0})$. Similar to the ambient orthogonal case, we can view $Y$ as a $\mathbf{H}(\mathbb{R})$-conjugacy class of maps $h : \mathbb{S} \longrightarrow \mathbf{H}_{\mathbb{R}}$. In this way we get a Shimura datum $Sh_{\mathbf{H}} = (\mathbf{H}, Y)$. Moreover, the natural map

$$Sh_{\mathbf{H}} \longrightarrow Sh_{\mathbf{G}}$$

is an embedding of Shimura data. For a level group $K_1 \subset \mathbf{H}(\mathbb{A}_f)$ we obtain a Shimura variety $Sh_{\mathbf{H}}(K_1) = \mathbf{H}(\mathbb{Q})\backslash Y \times \mathbf{H}(\mathbb{A}_f)/K_1$. And the natural inclusion $\mathbf{H} \longrightarrow \mathbf{G}$ will induce a map between their Shimura varieties $Sh_{\mathbf{H}}(K_1) \longrightarrow Sh_{\mathbf{G}}(K)$ (if $K_1 \subset K$).

Compared with the general setting above, this pair has some advantages. For example, $Y^0 = Y$ ($Y$ is already connected), and the stabilizer in the parametrization of lemma 2.1 is exactly $\mathbf{H}(\mathbb{Q})$ :

Let $g \in Stab_{\mathbf{G}(\mathbb{Q})} Y$, for any negative $E_{f_0}$-line $D$ in $(W_{f_0}, \psi_{f_0})$, $g$ maps $D$ to a negative $E_{f_0}$-line $g(D)$ in $(W_{f_0}, \psi_{f_0})$. Consider the restriction of $g$ on $D$, $g : D \longrightarrow g(D)$, because the action of $g$ is an orthogonal map and it keeps the orientation, therefore this map is $E_{f_0}$-linear (this is just an identification $SO(2) \cong U(1)$ in fact). Notice that these negative lines span $W_{f_0}$, therefore $g$ preserves $W_{f_0}$ and acts on this subspace $E_{f_0}$-linearly. Therefore, $g$ lies in the intersection $\mathbf{G}(\mathbb{Q})$ with $\mathbf{H}_{f_0}(\mathbb{R}) \times \prod_{a \neq f_0} \mathbf{G}_a(\mathbb{R})$ inside $\mathbf{G}(\mathbb{R})$, which is exactly $\mathbf{H}(\mathbb{Q})$.

Next we will explicit the related Galois action on these special cycles.

We first do some preparations to compute their reflex fields, reflex norms etc.

Define $\mathbf{SpE} = \mathrm{Spec}(E)(\mathbb{C}) = \mathrm{Spec}(E)(\overline{\mathbb{Q}})$. Consider the torus

$$\mathbf{T} = \mathrm{Res}_{E/\mathbb{Q}} \mathbb{G}_m = \mathrm{Res}_{F/\mathbb{Q}} \underline{\mathbf{T}},$$

where $\underline{\mathbf{T}} = \mathrm{Res}_{E/F} \mathbb{G}_m$. It has a sub-torus

$$\mathbf{T}^1 = \ker(\mathbf{T} \xrightarrow{Norm} \mathrm{Res}_{F/\mathbb{Q}} \mathbb{G}_m) = \mathrm{Res}_{F/\mathbb{Q}} \underline{\mathbf{T}^1},$$

where $\underline{\mathbf{T}^1} = \ker(\underline{\mathbf{T}} \xrightarrow{Norm} \mathbb{G}_{m,F})$. Through the natural determinant map, $\det : \mathbf{H} \longrightarrow \mathbf{T}^1$, we can identify $\mathbf{T}^1$ with the maximal abelian quotient $\mathbf{H}^{ab}$. And det will induce a map between Shimura data $Sh_{\mathbf{H}} \longrightarrow Sh_{\mathbf{T}^1}$. This quotient map det has non-canonical sections $\mathbf{T}^1 \hookrightarrow \mathbf{H}$, thus for any $\mathbb{Q}$-algebra $R$, the map $\det : \mathbf{H}(R) \longrightarrow \mathbf{T}^1(R)$ is surjective. Moreover,

we can describe the cocharacter group of $\mathbf{T}^1$ over $\mathbb{C}$ as follow :

$$\{f : \mathbf{SpE} \longrightarrow \mathbb{Z} | f(c) + f(\overline{c}) = 0\} \cong X_*(\mathbf{T}^1_{\mathbb{C}}),$$

where $\overline{c}$ is the conjugation of $c$. This isomorphism is given by $f \mapsto \mu_f$ where for all $z \in \mathbb{C}^*$ and $c \in \mathbf{SpE}$,

$$\mu_f(z) : c \mapsto z^{f(c)} \text{ in } \mathbf{T}^1(\mathbb{C}) \subset \mathbf{T}(\mathbb{C}) = (E \otimes \mathbb{C})^* = (\prod_{c \in \mathbf{SpE}} \mathbb{C})^* = Fun(\mathbf{SpE}, \mathbb{C}^*).$$

Here $Fun(A, B)$ denotes the set of all maps from set $A$ to set $B$. And this isomorphism is $Aut(\mathbb{C})$-equivariant.

Let $\beta = (v_1, ..., v_n)$ denote an orthogonal $E$-basis for $(W, \psi)$. Then

$$\mathbf{T}(\beta) \stackrel{def}{=} \mathrm{Res}_{F/\mathbb{Q}}(U(Ev_1) \times ... \times U(Ev_n)) \subset \mathbf{H} \subset \mathbf{G}$$

is a maximal $\mathbb{Q}$-subtorus of both $\mathbf{H}$ and $\mathbf{G}$, and

$$\mathbf{T}(\beta) = \mathrm{Res}_{F/\mathbb{Q}} U(Ev_1) \times ... \times \mathrm{Res}_{F/\mathbb{Q}} U(Ev_n) \cong (\mathbf{T}^1)^n.$$

Thus similarly we have an $Aut(\mathbb{C})$-equivariant isomorphism

$$\{f : \mathbf{SpE} \longrightarrow \mathbb{Z}^n | f(c) + f(\overline{c}) = 0\} \cong X_*(\mathbf{T}(\beta)_{\mathbb{C}}).$$

Let $S_n^+ = \{\pm 1\}^n \rtimes S_n$ be the group of signed permutations, acting on $\mathbb{Z}^n$ by

$$(\epsilon \rtimes \sigma) \cdot (\lambda_1, ..., \lambda_n) = (\epsilon_1 \lambda_{\sigma^{-1}(1)}, ..., \epsilon_n \lambda_{\sigma^{-1}(n)}).$$

The Weyl groups of $\mathbf{H}$ and $\mathbf{G}$ with respect to $\mathbf{T}(\beta)$ are then respectively equal to

$$W(\mathbf{H}, \beta) = Fun(\mathbf{SpE}, S_n) \subset W(\mathbf{G}, \beta) = Fun(\mathbf{SpE}, S_n^+)$$

with actions on $X_*(\mathbf{T}(\beta)_{\mathbb{C}}) \subset Fun(\mathbf{SpE}, Z^n)$ given by $(w.f)(c) = w(c) \cdot f(c)$ $(c \in \mathbf{SpE})$.

Now we can compute their reflex fields.

Let $\beta_0 = (w_{1,0}, ..., w_{n,0})$ be the orthogonal $E_{f_0}$-basis of $(W_{f_0}, \psi_{f_0})$ obtained from $\beta$ by base change along $f_0 : F \hookrightarrow \mathbb{R}$. Because the signature of $(W_{f_0}, \psi_{f_0})$ is $(n-1, 1)$, there exists a unique $i$ in $\{1, ..., n\}$ with $\psi_{f_0}(w_{i,0}, w_{i,0}) < 0$, for simplicity we assume that $i = 1$. Then $w_{1,0}$ spans a negative $E_{f_0}$ line $y_\beta$ in $(W_{f_0}, \psi_{f_0})$, giving rise to a special point $y_\beta \in Y$ : the corresponding map $h_\beta : \mathbb{S} \longrightarrow \mathbf{H}_{\mathbb{R}}$ (or $\mathbf{G}_{\mathbb{R}}$) factors through $\mathbf{T}(\beta)_{\mathbb{R}} \hookrightarrow \mathbf{H}_{\mathbb{R}}$. The induced cocharacter $\mu_\beta : \mathbb{G}_{m,\mathbb{C}} \longrightarrow \mathbf{T}(\beta)_{\mathbb{C}}$ defined by

$$\mu_\beta = (h_\beta)_{\mathbb{C}} \circ \iota : \mathbb{G}_{m,\mathbb{C}} \longrightarrow \mathbb{G}_{m,\mathbb{C}} \times G_{m,\mathbb{C}} \cong \mathbb{S}_{\mathbb{C}} \longrightarrow \mathbf{T}(\beta)_{\mathbb{C}}$$

(where $\iota(z) = (z, 1)$) will correspond to the function $f_\beta$ as follow :

$$f_\beta : \mathbf{SpE} \longrightarrow \mathbb{Z}^n,\ c \mapsto \begin{cases} (+1, 0, ..., 0) & \text{if } c = f_0^+, \\ (-1, 0, ..., 0) & \text{if } c = f_0^-, \\ (0, ..., 0) & \text{otherwise.} \end{cases}$$

By definition, the conjugacy class of Shimura cocharacter for $Sh(\mathbf{G})$, $[\mu_{\mathbf{G}}]$, is exactly the $\mathbf{G}(\mathbb{C})$-conjugacy class of $\mu_\beta : \mathbb{G}_{m,\mathbb{C}} \longrightarrow \mathbf{G}_{\mathbb{C}}$. And the reflex field $E(\mathbf{G}, X)$ is the field of definition of this conjugacy class. By a standard argument, it is also the field of definition of the $W(\mathbf{G}, \beta)$-orbit of $\mu_\beta \in X_*(\mathbf{T}(\beta)_{\mathbb{C}})$. By the above description of the $W(\mathbf{G}, \beta) \rtimes Aut(\mathbb{C})$-module $X_*(\mathbf{T}(\beta)_{\mathbb{C}})$, we find that the reflex field is

$$E(\mathbf{G}, X) = f_0(F) \subset \mathbb{R}.$$

Similarly we find that the reflex field for $Sh_{\mathbf{H}}$ is

$$E(\mathbf{H}, Y) = f_0^+(E) \subset \mathbb{C}.$$

Now we turn to the Shimura datum for torus $\mathbf{T}^1$, which will produce zero dimensional Shimura variety that is closely related with geometric connected components of the unitary Shimura variety for $Sh_{\mathbf{H}}$. It is easy to see that the cocharacter $\mu_{\mathbf{T}^1} = \det(\mu_\beta)$ defined by

$$\mathbb{G}_{m,\mathbb{C}} \xrightarrow{\mu_\beta} \mathbf{T}(\beta)_{\mathbb{C}} \hookrightarrow \mathbf{H}_{\mathbb{C}} \xrightarrow{\det} \mathbf{T}^1_{\mathbb{C}}$$

is independent of $\beta$ and it is exactly the Shimura cocharacter for $Sh_{\mathbf{T}^1}$. This cocharacter corresponds to the function $\widetilde{f_\mu} : \mathbf{SpE} \longrightarrow \mathbb{Z}$ given by

$$c \mapsto \begin{cases} +1 & \text{if } c = f_0^+, \\ -1 & \text{if } c = f_0^-, \\ 0 & \text{otherwise.} \end{cases}$$

Then the field of definition of $\mu_{\mathbf{T}^1}$, i.e. the reflex field of $Sh_{\mathbf{T}^1}$ is also exactly $f_0^+(E)$. For simplicity, from now on, we identify the abstract number fields $F$ and $E$ with the embedded reflex fields $f_0(F)$ and $f_0^+(E)$ using respectively $f_0 : F \longrightarrow f_0(F)$ and $f_0^+ : E \longrightarrow f_0^+(E)$. By definition, the reflex norm for $Sh_{\mathbf{T}^1}$ is the following composition

$$\mathbf{T} = \mathrm{Res}_{E/\mathbb{Q}}\mathbb{G}_{m,E} \xrightarrow{\mu_{\mathbf{T}^1}} \mathrm{Res}_{E/\mathbb{Q}}(\mathbf{T}^1_E) \xrightarrow{Norm} \mathbf{T}^1.$$

By a standard argument (see section A.2), this map is given by $r = res(\underline{r})$, where

$$\underline{r} : \underline{\mathbf{T}} \longrightarrow \underline{\mathbf{T}^1}, z \mapsto \frac{z}{\overline{z}}$$

is a morphism over $F$ and recall $\mathbf{T} = \mathrm{Res}_{F/\mathbb{Q}}\underline{\mathbf{T}}$, $\mathbf{T}^1 = \mathrm{Res}_{F/\mathbb{Q}}\underline{\mathbf{T}^1}$.

Now we relate $\pi_0(Sh_{\mathbf{H}})$ with $Sh_{\mathbf{T}^1}$. It is simpler than the general case because the derived subgroup $\mathbf{H}^{der}$ is simply connected, see [50] 5.17. By the strong approximation theorem, $\mathbf{H}^{der}(\mathbb{Q})$ is dense in $\mathbf{H}^{der}(\mathbb{A}_f)$. Because $E$ is a CM field, $\mathbf{T}^1(\mathbb{Q})$ is discrete (thus closed) in $\mathbf{T}^1(\mathbb{A}_f)$. Combining this, we obtain

$$\mathbf{H}(\mathbb{Q})\mathbf{H}^{der}(\mathbb{A}_f) \subset \overline{\mathbf{H}(\mathbb{Q})} \subset (\det{}^{-1})(\mathbf{T}^1(\mathbb{Q})) = \mathbf{H}(\mathbb{Q})\mathbf{H}^{der}(\mathbb{A}_f)$$

inside $\mathbf{H}(\mathbb{A}_f)$, so

$$\overline{\mathbf{H}(\mathbb{Q})} = \mathbf{H}(\mathbb{Q})\mathbf{H}^{der}(\mathbb{A}_f).$$

For a small enough level group $K_1 \subset \mathbf{H}(\mathbb{A}_f)$, the quotient map $\det : \mathbf{H} \longrightarrow \mathbf{T}^1$ induces a natural map for Shimura varieties, $Sh_{\mathbf{H}}(K_1) \longrightarrow Sh_{\mathbf{T}^1}(\det(K_1))$. This natural map induces a natural isomorphism on the set of connected components,

$$\pi_0(Sh_{\mathbf{H}}(K_1)) = \mathbf{H}(\mathbb{Q})\backslash\mathbf{H}(\mathbb{A}_f)/K_1 = \mathbf{T}^1(\mathbb{Q})\backslash\mathbf{T}^1(\mathbb{A}_f)/\det(K_1) = \pi_0(Sh_{\mathbf{T}^1}(\det(K_1))).$$

Moreover, for a small enough level group $K \subset \mathbf{G}(\mathbb{A}_f)$, similarly we can rewrite the parametrization as follow :

$$Z_K(\mathbf{G}, \mathbf{H}) = \mathbf{H}(\mathbb{Q})\mathbf{H}^{der}(\mathbb{A}_f)\backslash\mathbf{G}(\mathbb{A}_f)/K.$$

Finally we can explicit the Galois action.

Let $F^{ab}$ and $E^{ab}$ denote the maximal abelian extensions of $F$ and $E$ in $\overline{\mathbb{Q}}$. Let $E[\infty]$ denote the subfield of $E^{ab}$ which is fixed by the image of the transfer map, or equivalently it is the union of all ring class fields of $E$. By class field theory we have the following commutative diagram with exact rows

$$\begin{tikzcd}
1 \arrow[r] & \mathrm{Res}_{F/\mathbb{Q}}\mathbb{G}_{m,F}(\mathbb{A}_f) \arrow[r] \arrow[d, "\mathrm{Art}_F"'] & \mathbf{T}(\mathbb{A}_f) \arrow[r, "r"] \arrow[d, "\mathrm{Art}_E"'] & \mathbf{T}^1(\mathbb{A}_f) \arrow[r] \arrow[d, "\mathrm{Art}_E^1"'] & 1 \\
1 \arrow[r] & \mathrm{Gal}(F^{ab}/F) \arrow[r] & \mathrm{Gal}(E^{ab}/E) \arrow[r] & \mathrm{Gal}(E[\infty]/E) \arrow[r] & 1
\end{tikzcd}$$

where $\mathrm{Art}_F$ and $\mathrm{Art}_E$ are the Artin maps sending a local uniformizer to a **geometric Frobenius**, and the last vertical map induces an isomorphism

$$\mathrm{Art}_E^1 : \mathbf{T}^1(\mathbb{A}_f)/\mathbf{T}^1(\mathbb{Q}) \cong Gal(E[\infty]/E).$$

Then the Galois group $\mathrm{Gal}_E$ acts on $Z_K(\mathbf{G}, \mathbf{H})$ as follows :

For any $\rho \in \mathrm{Gal}_E$ and $s \in \mathbf{H}(\mathbb{A}_f)$, if $\mathrm{Art}_E^1(\det(s)) = \rho|_{E[\infty]}$, then

$$\rho(Z_K(g)) = Z_K(sg) \text{ for all } g \in \mathbf{G}(\mathbb{A}_f).$$

It is also possible to describe the Galois action by $\mathrm{Gal}_F$ in a similar way. Because we won't use this fact in this thesis, we refer to [16] section 5 for more details.

We also need some preparations on ring class fields to realize abstract relations. For more explanations we refer to [16] section 7 and [9] section VI, VII.

Fix the level group $K$, choose a special cycle $\widetilde{z} = Z_K(g)$ and take a finite set $S_1^{'} = S_1^{'}(K, \widetilde{z})$ ("bad primes") of primes of $\mathbb{Q}$ large enough such that

- $S_1^{'}$ contains $\{2\}$.
- For any prime $q \notin S_1^{'}$, $\mathbb{Q} \longrightarrow E$, $\mathbf{G}$ and $\mathbf{H}$ are all unramified at $q$.
- $K = K_{S_1^{'}} \times K^{S_1^{'}}$ with $K^{S_1^{'}} = \prod_{q \notin S_1^{'}} K_q$, $K_{S_1^{'}}$ is a compact open subgroup of $\mathbf{G}(\mathbb{A}_{f,S_1})$

($\mathbb{A}_{f,S_1} = \prod_{t \in S_1} \mathbb{Q}_t$), $K_q$ is a compact open subgroup of $\mathbf{G}(\mathbb{Q}_q)$ for any $q \notin S_1^{'}$. And moreover, $g = \otimes_t g_t$ with $g_q \in K_q$ for any $q \notin S_1^{'}$.

- For any $q \notin S_1^{'}$, $K_q$ (resp $K_q \bigcap \mathbf{H}(\mathbb{Q}_q)$) is hyperspecial in $\mathbf{G}(\mathbb{Q}_q)$ (resp $\mathbf{H}(\mathbb{Q}_q)$).

Take a prime $l \notin S_1^{'}$ (we will consider $l$-adic etale cohomology in 3 to do arithmetic applications of tame relations) and define $S_1$ to be $S_1^{'} \bigcup \{l\}$. Let $S$ ("bad places") denote the finite set of places of $F$ above $S_1$. Let $\mathbb{A}_f^{S_1} = \prod_{p \notin S_1}^{'} \mathbb{Q}_p$.

By the assumption on $S_1$ and our parametrization of special cycles, the natural decomposition

$$\mathbf{G}(\mathbb{A}_{f,S_1}) \times \mathbf{G}(\mathbb{A}_f^{S_1}) \longrightarrow \mathbf{G}(\mathbb{A}_f)$$

will induce the following natural map :

$$\mathbb{Z}[\mathbf{H}^{der}(\mathbb{A}_{f,S_1}) \backslash \mathbf{G}(\mathbb{A}_{f,S_1})/K_{S_1}] \bigotimes \bigotimes_{p \notin S_1}^{'} \mathbb{Z}[\mathbf{H}^{der}(\mathbb{Q}_p) \backslash \mathbf{G}(\mathbb{Q}_p)/K_p] \longrightarrow \mathbb{Z}[Z_K(\mathbf{G}, \mathbf{H})].$$

And by our assumption on $S$, this natural map can be rewritten as

$$\mathbb{Z}[\mathbf{H}^{der}(\mathbb{A}_{f,S_1}) \backslash \mathbf{G}(\mathbb{A}_{f,S_1})/K_{S_1}] \bigotimes \bigotimes_{w \notin S}^{'} \mathbb{Z}[\underline{\mathbf{H}}^{der}(F_w) \backslash \underline{\mathbf{G}}(F_w)/K_w] \longrightarrow \mathbb{Z}[Z_K(\mathbf{G}, \mathbf{H})],$$

here $K_w$ is the corresponding component of $K$ ($K_p = \prod_{\mathfrak{p}|p} K_{\mathfrak{p}}$ for $p \notin S_1$) and we use restricted tensor products. Thus roughly speaking, we can think of $\mathbb{Z}[\underline{\mathbf{H}}^{der}(F_w) \backslash \underline{\mathbf{G}}(F_w)/K_w]$ ($w \notin S$) as a **local version** of our module for special cycles. We will use this natural map many times in this these. Roughly speaking, we usually construct an element $z_{cyc}$ in the left side, then its image in the right side ($\mathbb{Z}[Z_K(\mathbf{G}, \mathbf{H})]$) will give us a special cycle. For simplicity we will also use $z_{cyc}$ to denote this special cycle. Moreover, in application, we may need to use an extension of $\mathbb{Z}$, $\mathbb{Z} \longrightarrow R$, as our coefficient ring. It is easy to state the corresponding $R$-linear maps.

Let $\underline{\mathbf{P}}$ denote the set of primes of $\mathcal{O}_F$ that don't belong to $S$ and split in $F \longrightarrow E$, $\underline{\mathbf{N}}$ denote the set of square-free products of elements of $\underline{\mathbf{P}}$. For any ideal $m$ of $\mathcal{O}_F$ belonging

to $\underline{\mathbf{N}}$, we will associate a finite abelian extension $\underline{E}[m]$ of $E$ that lies in $E \longrightarrow E[\infty]$.

Choose a compact open subgroup $U_S^1 \subset \mathbf{T}^1(\mathbb{A}_{f,S_1})$. For other primes of $F$ ($v \notin S$), we define a filtration on $\underline{\mathbf{T}}^1(F_v)$ (by local conductor),

$$U_v^1(c) = \{\frac{a}{\bar{a}}, a \in (\mathcal{O}_{F_v} + m_{F_v}^c \mathcal{O}_{E_v})^*\},$$

where $E_v = F_v \otimes_F E$ and $\mathcal{O}_{E_v}$ is the integral closure of $\mathcal{O}_{F_v}$ inside $E_v = F_v \otimes_F E$, $c$ is a non-negative integer. Moreover, for any positive integer $c$, $U_v^1(c)$ is also equal to $\underline{r}(\ker(red_1))$ :

$$\begin{array}{ccc} \underline{\mathbf{T}}(\mathcal{O}_{F_v}) & \xrightarrow{\underline{r}} & \underline{\mathbf{T}}^1(\mathcal{O}_{F_v}) \\ {\scriptstyle red_1}\downarrow & & {\scriptstyle red_2}\downarrow \\ \underline{\mathbf{T}}(\mathcal{O}_{F_v}/\pi_v^m) & \longrightarrow & \underline{\mathbf{T}}^1(\mathcal{O}_{F_v}/\pi_v^m) \end{array}$$

here $\pi_v$ is a uniformizer of $\mathcal{O}_{F_v}$ and for simplicity we also denote by $\underline{\mathbf{T}}^1$ and $\underline{\mathbf{T}}$ the canonical Néron models of $\underline{\mathbf{T}}^1$ and $\underline{\mathbf{T}}$. In other words, the above diagram is just the same as the following diagram :

$$\begin{array}{ccc} (\mathcal{O}_{E_v})^* & \xrightarrow{\underline{r}} & \underline{\mathbf{T}}^1(\mathcal{O}_{F_v}) \subset (\mathcal{O}_{E_v})^* \\ {\scriptstyle red_1}\downarrow & & {\scriptstyle red_2}\downarrow \\ (\mathcal{O}_{E_v}/\pi_v^c)^* & \longrightarrow & \underline{\mathbf{T}}^1(\mathcal{O}_{F_v}/\pi_v^m) \subset (\mathcal{O}_{E_v}/\pi_v^c)^*. \end{array}$$

Because $\mathbf{T}^1(\mathbb{Q})$ is discrete in $\mathbf{T}^1(\mathbb{A}_f)$, we can require $U_S^1$ to be small enough such that

$$\mathbf{T}^1(\mathbb{Q}) \bigcap (U_S^1 \times \prod_{v \notin S} U_v^1(0)) = 1.$$

Moreover we can assume $U_S^1 \times \prod_{v \notin S} U_v^1(0) \subset \det(gKg^{-1} \bigcap \mathbf{H}(\mathbb{A}_f))$ due to our assumptions on $S$.

Now for an ideal $m$ as above, we define $U^1(m) = U_S^1 \times \prod_{v \notin S} U_v^1(v(m))$. We define $\underline{E}[m]$ to be the subfield of $E[\infty]$ fixed by $\mathrm{Art}_E^1(U^1(m))$. Take $m \in \underline{\mathbf{N}}$, with prime factor $h$, through direct computation, we have the following lemma :

**Lemma 2.2.** ***local-global***

*The Artin map $Art_E^1$ induces an isomorphism $\frac{U_h^1(0)}{U_h^1(1)} \cong Gal(\underline{E}[m]/\underline{E}[\frac{m}{h}])$.*

Although this lemma is simple, it is essential to translate the global distribution relation problem into local problems. By our assumption on $U^1(1)$, the cycle $\widetilde{z} = Z_K(g)$ is defined over the field $\underline{E}[1]$. In section 4.2, we will extend it into a family of special cycles defined over some $\underline{E}[m]$ with suitable tame relation.

## 2.2 Local setting (Boumasmoud's Relation)

To state Boumasmoud's Relation, we will introduce the Hecke polynomial briefly. See section A.2 for more details about related properties (background on Satake transform, rationality of the Hecke polynomial etc) and some explicit examples.

Let $F$ denote a $p$-adic field with ring of integers $\mathcal{O}_F$, a uniformizer $\pi \in \mathcal{O}_F$ and residue field $\mathcal{O}_F/\pi = \mathbb{F}_q$ with $q$ elements.

Let $\mathbf{G}$ denote a reductive group scheme over $\mathcal{O}_F$. By [27] (XXVI 7.15) and Lang's theorem [40], $\mathbf{G}$ is quasi-split over $\mathcal{O}_F$ : there is a maximal torus $\mathbf{T}$ of $\mathbf{G}$ contained in a Borel subgroup $\mathbf{B}$ with unipotent radical $\mathbf{N}$ and Levi decomposition $\mathbf{B} = \mathbf{T} \ltimes \mathbf{N}$. Let $\mathbf{S}$ denote the maximal split subtorus of $\mathbf{T}$, $X_*(\mathbf{S})$ the group of cocharacters of $\mathbf{S}$ and $X_*^+(\mathbf{S}) \subset X_*(\mathbf{S})$ the cone of $\mathbf{B}$-dominant cocharacters. Let $K$ denote the hyperspecial subgroup $\mathbf{G}(\mathcal{O}_F)$, $B$ denote $\mathbf{B}(F)$, $G$ denote $\mathbf{G}(F)$ ; we have the Iwasawa decomposition $G = BK$.

Consider the $G$ module $\mathbb{Z}[G/K]$, where $G$ acts on $G/K$ via left multiplication. We get the Hecke algebra

$$He \overset{def}{=} End_G\mathbb{Z}[G/K] = \mathbb{Z}[K\backslash G/K].$$

It has a right action on $\mathbb{Z}[G/K]$. More precisely, for a double coset $KgK = \coprod_i g_iK$, this operator $[KgK]$ will act on $\mathbb{Z}[G/K]$ as follows :

$$[KgK] : 1_{bK} \mapsto \sum_i 1_{bg_iK}.$$

Here we make a remark about the notation. Another usual method to define the Hecke algebra is to describe it as the ring of locally constant, compactly supported functions $f : G \longrightarrow \mathbb{Z}$ which are $K$-biinvariant. And the multiplication is given by the convolution product of functions (under the unique Haar measure on $G$ with $K$ having volume 1). For example we will adopt this description in the appendix A.2. The resulting ring is naturally isomorphic to $(End_G(\mathbb{Z}[G/K]))^{opp}$. However, because we are working under unramified assumptions and $K$ is a hyperspecial subgroup of $G$, the Hecke algebra is commutative. In other words, the identity map is a ring isomorphism :

$$End_G(\mathbb{Z}[G/K]) \cong End_G(\mathbb{Z}[G/K])^{opp}.$$

Therefore these two definitions are indeed the same thing.

For a cocharacter $\mu \in X_*^+(\mathbf{S})$, there is an associated $U$-operator $U_\mu \in End_B(\mathbb{Z}[G/K])$. Let $I \subset K$ denote the Iwahori subgroup determined by $K$ and $B$, with positive part $I^+ = \mathbf{N}(\mathcal{O}_F)$. In other words, $I \subset K$ is the inverse image of $\mathbf{B}(\mathcal{O}_F/\pi)$ under the reduction map $K \longrightarrow \mathbf{G}(\mathcal{O}_F/\pi)$,

By the Iwasawa decomposition, $\mathbb{Z}[G/K]$ is generated by elements of the form $1_{bK}$

$(b \in B)$, $U_\mu$ acts on these elements as follows :

$$U_\mu(1_{bK}) = \sum_{z \in \frac{I^+}{\mu(\pi)I^+\mu(\pi)^{-1}}} 1_{bz\mu(\pi)K} \tag{2.1}$$

More generally, fix an algebraic closure $\overline{F}$ of $F$ and let $F^{un}$ denote the maximal unramified extension of $F$ inside $\overline{F}$. For a conjugacy class of cocharacters $[\mu]$ for $\mathbf{G}_{\overline{F}}$, we also have an associated $U$-operator. Let $\mu \in X_*(\mathbf{T}_{\overline{F}})$ denote the unique $\mathbf{B}_{\overline{F}}$-dominant cocharacter of $\mathbf{T}_{\overline{F}}$ in this conjugacy class. Both $[\mu]$ and $\mu$ have the same field of definition, a finite extension $F(\mu) \subset \overline{F}$ of $F$. In fact, $\mathbf{G}_F$ is an unramified reductive group over $F$ by [27] XXVI 7.15, it splits over $F^{un}$, thus $F(\mu) \subset F^{un}$. Let $n(\mu) = [F(\mu) : F]$ be the degree of this extension. We still denote the descended cocharacter $\mathbb{G}_{m,F(\mu)} \longrightarrow \mathbf{T}_{F(\mu)}$ by $\mu$. Consider its norm, $\mu_0 = Norm_{F(\mu)/F}(\mu)$, it is a cocharacter for $\mathbf{T}_F$. Therefore it factors through the maximal split torus $\mathbf{S}_F$, we still denote this cocharacter $\mathbb{G}_{m,F} \longrightarrow \mathbf{S}_F$ by $\mu_0$. There is a unique cocharacter for $\mathbf{S}$ extending $\mu_0$. And it lies in $X_*^+(\mathbf{S})$. Therefore we can attach to this cocharacter a $U$-operator as above. We still denote the resulting operator by $U_\mu$.

Here we make three remarks :

• In this section we start with a reductive group scheme $\mathbf{G}$ over the integer ring $\mathcal{O}_F$. This amounts to give an unramified reductive group $G$ over $F$ and a hyperspecial subgroup $K$ of $G(F)$.

In application, we usually starts from an unramified reductive group $G$ over $F$ and construct an integral model $\mathbf{G}$ over $\mathcal{O}_F$ for it. We state a standard strategy to construct such a integral model that will be used latter in this thesis. The idea is to **embed** $G$ into the general linear group $GL_N$ and then extend this embedding into integral level. Notice that $GL_N$ even has a model over $\mathbb{Z}$. In this thesis, we will encounter other groups like orthogonal groups and symplectic groups. These groups have standard representations. Through such a representation, we can view can such a group as the closed subgroup scheme of $GL_N$ over $F$ that fixes an associated quadratic form. When the quadratic form satisfies certain good properties, which can be detected via discriminant etc, the corresponding closed subgroup scheme of $GL_N$ over $\mathcal{O}_F$ will provide an integral model for our group.

• For simplicity, we pin down everything $(K, B, \mu...)$. In fact, this $U$-operator has a more intrinsic and geometric explanation via Bruhat-Tits building theory. We refer to [9] (section III and section V) for more details.

• For any non-negative integer $i$, we have $U_{\mu^i} = (U_\mu)^i$.

For the above conjugacy class of cocharacters $[\mu]$, we can also define a Hecke polynomial $Hep_\mu(X) \in He[X]$. There is an issue about the coefficient ring. To realize the Satake transform, we need to enlarge the coefficient ring $\mathbb{Z}$ into a larger ring $R$ such that $\mathbb{Z}[q^{\pm\frac{1}{2}}] \subset R$. We still denote the $R$-coefficient Hecke algebra $R[K\backslash G/K]$ by $He$. By scalar extension,

we get a $R[G]$-module $R[G/K]$, and an associated $U$-operator $U_\mu \in End_B R[G/K]$.

Let $\Gamma$ denote $Gal(F^{un}/F)$ with $\sigma \in \Gamma$ being the **geometric Frobenius** of $F$. Let $\rho \in X^*(\mathbf{T}_{\overline{F}}) \otimes \mathbb{Q}$ be the half-sum of all positive roots of $(\mathbf{T}_{\overline{F}}, \mathbf{B}_{\overline{F}})$.

Let $1 \longrightarrow \widehat{G} \longrightarrow {}^LG \longrightarrow \Gamma \longrightarrow 1$ be the Langlands dual of $\mathbf{G}_F$. Fix a $\Gamma$-invariant pinning $(\widehat{T}, \widehat{B}, ...)$ of $\widehat{G}$ so that ${}^LG = \widehat{G} \rtimes \Gamma$. We get a $\Gamma$-equivariant isomorphism $X_*(\mathbf{T}_{\overline{F}}) \cong X^*(\widehat{T})$. It maps $\mu$ to a $\widehat{B}$-dominant character of $\widehat{T}$ fixed by $\Gamma^{n(\mu)}$, which we also denote by $\mu$. Let $r_\mu : \widehat{G} \rtimes \Gamma^{n(\mu)} \longrightarrow GL(V_\mu)$ be the unique irreducible representation of ${}^L(\mathbf{G}_{F(\mu)})$ whose restriction to $\widehat{G}$ is the irreducible representation with highest weight $\mu$, and such that $\Gamma^{n(\mu)}$ acts trivially on the highest weight space. For $\widehat{g} \in \widehat{G}$, consider the polynomial

$$\det(X - q^{n(\mu)d(\mu)} r_\mu((\widehat{g} \rtimes \sigma^{-1})^{n(\mu)})),$$

where $d(\mu) = \langle \rho, \mu \rangle$. We denote it by $Hep_\mu$ and it is our Hecke polynomial. Its coefficients are regular functions on $\widehat{G}$ fixed by $\sigma$-conjugation. Through Satake transform, we can view these functions as elements in the Hecke algebra (with suitable coefficient ring). The Hecke polynomial only depends on the conjugacy class $[\mu]$, it does not depend on our pinning down data $(\mathbf{B}, \mathbf{T}, \mu ...)$.

**Remark :**

Here we will use the usual ("untwisted") Satake transform and we make a remark about the coefficient issue and refer to section A.2 for more details. If $\mu$ is minuscule, Wedhorn showed that the coefficients of its Hecke polynomial lie in the Hecke algebra with $\mathbb{Z}[q^{-1}]$-coefficient, see [66] section 2.8. Because the Shimura cocharacter is minuscule, in the application to tame relations, we can relax the requirement $\mathbb{Z}[q^{\pm\frac{1}{2}}] \subset R$ into $\mathbb{Z}[q^{-1}] \subset R$.

Now we can state Boumasmoud's result.

**Theorem 2.3.** *(Seed Relation)*

*The operator $U_\mu$ is a right root of the Hecke polynomial $Hep_\mu(X)$ in $End_B(R[G/K])$, in other words, in that ring, $Hep_\mu(U_\mu) = 0$.*

We refer to Boumasmoud's thesis (see [9] section IV) for its proof.

There are many occurences of the Hecke polynomial in the literature. Different papers may use different notations. To avoid such notation confusion, we conclude this section with some **remarks** :

• In this paper, the word "Frobenius" will always means geometric Frobenius except in the appendix (section A.2), while some papers are using arithmetic Frobenius to define Hecke polynomial, like [9], [16].

• We are using the usual Satake transform. In Boumasmoud's thesis, he used the usual Satake transform and a "twisted" version, but he named the latter "untwisted Satake transform", see [9] section III and section IV.

• In the definition of the Hecke polynomial, we consider the representation $V_\mu$, but some papers are using the dual space of $V_\mu$, see [41] remark 2.1.3.

• For such module $R[G/K]$, we always equip it with **left** $G$-action and **right** $He$-action, like [9]. Some papers may define the Hecke algebra as $End_G(R[G/K])$, while we're using the opposite identification $He = (End_G(R[G/K]))^{opp}$. Especially there are some notation confusion when comparing our "formal" Hecke action with geometric Hecke action. See [41] remark 6.1.7.

# Chapitre 3

# Cohomology theory

In this section, we will relate cycles to cohomology theory. We will work with global objects like Shimura varieties and use the same notations as in section 2.1. The general method is to apply $l$-adic Abel-Jacobi map. We will use the usual spectral sequence to construct it. See [54] and [51] section 23 for more details. The lecture [51] also provides some necessary background on etale cohomology.

**$l$-adic Abel-Jacobi map**

Let $X_1$ denote a smooth variety which is equidimensional of dimension $d$ over a field $k$ ($char(k) \neq l$). Let $\overline{X_1}$ denote $X_1 \times_k k^{sep}$ where $k^{sep}$ is a separable closure of $k$. For each $k \subset k^{'} \subset k^{sep}$, we have the Hochschild-Serre spectral sequence

$$E_2^{i,j} = H^i(Gal_{k'}, H^j(\overline{X_1}, \mathbb{Z}_l(n))) \Rightarrow H^{i+j}(X_1 \times_k k^{'}, \mathbb{Z}_l(n)),$$

which degenerates at $E_2$ and gives a map

$$\ker(H^{2i}(X_1 \times_k k^{'}, \mathbb{Z}_l(n)) \longrightarrow H^{2i}(\overline{X_1}, \mathbb{Z}_l(n))) \longrightarrow H^1(Gal_{k'}, H^{2i-1}(\overline{X_1}, \mathbb{Z}_l(n))).$$

Thus we get the $l$-adic Abel-Jacobi map

$$AJ_l : (\mathcal{Z}^n(X_1 \times_k k^{'}))_0 \longrightarrow H^1(Gal_{k'}, H^{2n-1}(\overline{X_1}, \mathbb{Z}_l(n))),$$

here $\mathcal{Z}^i(X_1 \times_k k^{'})$ is the Chow group of codimension $n$ cycles, $(\mathcal{Z}^n(X_1 \times_k k^{'}))_0$ is the kernel of the cycle class map $\mathcal{Z}^n(X_1 \times_k k^{'}) \longrightarrow H^{2n}(\overline{X_1}, \mathbb{Z}_l(n))$. We call elements in $(\mathcal{Z}^n(X_1 \times_k k^{'}))_0$ **cohomologically trivial** cycles. And for simplicity we will use $X_{1,k'}$ to denote $X_1 \times_k k^{'}$.

Applying this method to our main example $Sh_{\mathbf{H}} \longrightarrow Sh_{\mathbf{G}}$, the ambient Shimura variety $X_1 = Sh_{\mathbf{G}}(K)$ is $2n-1$-dimensional (suppose that the quadratic space is $2n+1$-dimensional), and our resulting special cycles (constructed through $Sh_{\mathbf{H}}$) on it are $n-1$ dimensional. Suppose we have a suitable canonical way to make these special cycles cohomologically trivial, in a Galois and Hecke-equivariant way. Then we can use them to study

$H^1(Gal_{E'}, H^{2n-1}(\overline{X_1}, \mathbb{Z}_l(n)))$ (here $E$ is the reflex field for $Sh_{\mathbf{H}}$ and $E'$ will be a suitable abelian extension of $E$). Therefore first we need some techniques to make our special cycles cohomologically trivial.

Here we make two remarks.

For our main example, the embedding $Sh_{\mathbf{H}} \longrightarrow Sh_{\mathbf{G}}$ is an example of *a special pair of Shimura data* (see [70] section 3.1). Usually the middle degree cohomology of Shimura varieties, $H^{2n-1}(\overline{X_1}, \mathbb{Z}_l(n))$, is more complicated and important than other degree. This can be seen from the following Matsushima's formula. The following localization method is a good illustration of this phenomenon. There are many problems around this middle degree cohomology. Among them, its Galois cohomology is an important topic. By **dimensional** reasons, we can exactly expect to use our special cycles to study $H^1(Gal_{E'}, H^{2n-1}(\overline{X_1}, \mathbb{Z}_l(n)))$. What's more, our other examples also have such an advantage by direct computation about dimensions.

The $l$-adic cohomology theory is most suitable for proper (or equivalently projective) Shimura varieties. Suppose that the ambient Shimura variety $X_1 = Sh_{\mathbf{G}}(K)$ is projective, let $\overline{X_1}$ denote its base change over $\mathbb{C}$. We can also view its space of complex points, $\overline{X_1}(\mathbb{C})$, as a compact complex manifold (endow it with complex topology). Pick up an isomorphism $\overline{\mathbb{Q}_l} \cong \mathbb{C}$, we have a natural comparison theorem,

$$H^*(\overline{X_1}, \mathbb{Z}_l) \otimes_{\mathbb{Z}_l} \overline{\mathbb{Q}_l} \cong H^*(\overline{X_1}(\mathbb{C}), \mathbb{C}),$$

for example see [51] section 21. In fact this comparison theorem holds more generally without properness assumption. Therefore through this comparison theorem, we can study the cohomology group on the right side to gain knowledge about $l$-adic cohomology of Shimura varieties. We can use some analytic methods (relative lie algebra cohomology, spectral decomposition etc), and we have the following Matsushima's formula,

$$H^*(\overline{X_1}(\mathbb{C}), \mathbb{C}) \cong \bigoplus_{\Pi = \Pi_f \otimes \Pi_\infty} m(\Pi)(\Pi_f^K \otimes H^*(\mathfrak{g}, K_\infty; \Pi_\infty)).$$

Here

- $\Pi$ runs through the automorphic representations of $\mathbf{G}(\mathbb{A})$, i.e. the irreducible representations of $\mathbf{G}(\mathbb{A})$ in the automorphic spectrum $L^2(\mathbf{G}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{A}), \mathbb{C})$, and $m(\Pi)$ is the corresponding multiplicity of $\Pi$;
- $\mathfrak{g}$ is the Lie algebra of $\mathbf{G}(\mathbb{R})$;
- $K_\infty$ is the maximal connected compact modulo center subgroup of $\mathbf{G}(\mathbb{R})$.

We refer to [8] section VII for its proof. This formula is quite useful to do explicit analysis and it connects $l$-adic cohomology of Shimura varieties to automorphic representations of $\mathbf{G}(\mathbb{A})$. For example, we can take of $\Pi_f$-component ($\Pi_f$ is fixed) of the whole cohomology group $H^i(\overline{X_1}, \overline{\mathbb{Q}_l})$, the resulting space is still a Galois representation. Moreover,

in many cases, cuspidal representations tend to appear in the middle degree cohomology while parabolic induced representations tend to appear in other degrees. This formula requires the properness assumption. There are some generalizations without such properness assumption, for example,

(1) Replace the left side by the $L^2$-cohomology of the Baily-Borel compactification, see [7].

(2) Consider the same cohomology group, but modify the representations that show up in the decomposition of the right side, see [26].

However, these generalizations are more complicated. Unfortunately, for our main example, the ambient Shimura variety $Sh_{\mathbf{G}}(K)$ is proper if and only if its reflex field $F$ is not $\mathbb{Q}$. Other examples in this thesis have similar non-properness issues.

**General methods to produce cohomologically trivial cycles**

Here we mention two standard methods.

One possible method is to apply a **parity projector** whose existence is the subject of the standard sign conjecture. The idea is that there exists a canonical rational Hecke correspondence

$$e \in He_{\mathbb{Q}} = End_{\mathbf{G}(\mathbb{A}_f)}\mathbb{Q}[\mathbf{G}(\mathbb{A}_f)/K]$$

such that it induces the projector to odd degree cohomology,

$$H^*(\overline{X_1}, \mathbb{Q}_l) \longrightarrow \bigoplus_{i \equiv 1 \bmod 2} H^i(\overline{X_1}, \mathbb{Q}_l).$$

We refer to [53] for more details. See also [60] (section 6.2) for unitary Shimura varieties. With the help of such projector $e$, for any special cycle $z_{cyc}$, we know that $e(z_{cyc})$ is cohomologically trial due to dimension reasons.

Another standard method is to use localization at a suitable maximal ideal of the Hecke algebra. The idea is that after such suitable localization, the cohomology will concentrate on the middle degree. And such problem is an important topic concerning cohomology of Shimura varieties. This technique is widely used when such vanishing results away from the middle degree are available. Here we will use our main example to state the rough idea. Then we list some related results.

The global Hecke algebra is not commutative and we will use a partial Hecke algebra. Let

$$He^{S_1} = \bigotimes_{p \notin S_1}{}' \mathbb{Z}[K_p \backslash \mathbf{G}(\mathbb{Q}_p)/K_p]$$

be the partial Hecke algebra. By our assumption on the finite set $S_1$, each $K_p$ is a hyperspecial subgroup of $\mathbf{G}(\mathbb{Q}_p)$ and $\mathbf{G}_{\mathbb{Q}_p}$ is unramified over $\mathbb{Q}_p$, then the local Hecke algebra $\mathbb{Z}[K_p \backslash \mathbf{G}(\mathbb{Q}_p)/K_p]$ is commutative and thus the Hecke algebra $He^{S_1}$ is commutative.

By the above Matsushima's formula, we can decompose the $l$-adic cohomology of $\overline{X_1}$

(recall that $X_1$ is the ambient Shimura variety $Sh_{\mathbf{G}}(K)$) as direct sum of automorphic representations. For each such $\Pi$ appearing in the cohomology, we can attach a homomorphism

$$\phi_\Pi : He^{S_1} \longrightarrow \mathbb{C}$$

determined by the Satake parameters of $\Pi$ at each place outside $S_1$. This is due to the decomposition theorem of Flath, $\Pi_f = \bigotimes_p^{'} \Pi_p$ ($\Pi_p$ is an irreducible representation of $\mathbf{G}(\mathbb{Q}_p)$), see [25]. And we assume that $\Pi$ is cuspidal from now on.

What's more, such $\pi$ has the so called **strong C-arithmetic** property. We refer to [62] (especially proposition 2.15) for more details. In particular, the image of $\phi_\Pi$ is contained in a number field $E_0$. We further assume that its image is contained in the ring of integers, $\mathcal{O}_{E_0}$. Thus we have a map, $\phi_\Pi : He^{S_1} \longrightarrow \mathcal{O}_{E_0}$. Let $\lambda$ be a prime of $E$ that lies above the prime $l$ and let $\mathcal{O}_\lambda$ denote the valuation ring of $E_{0,\lambda}$. Then $\phi_\Pi$ induces a morphism

$$\phi_{\Pi,\lambda} : He^{S_1} \longrightarrow \mathcal{O}_\lambda.$$

Now we introduce the following ideal of $He^{S_1}$ given by

$$\mathfrak{m} = \ker(He^{S_1} \xrightarrow{\phi_{\Pi,\lambda}} \mathcal{O}_\lambda \longrightarrow k_\lambda).$$

Here $k_\lambda$ is the residue field of $\mathcal{O}_\lambda$.

Notice that $H^*(\overline{X_1}, \mathcal{O}_\lambda)$ is a $He^{S_1}$ module and thus we can consider its localization at the ideal $\mathfrak{m}$. We expect the following vanishing result :

**Assumption** The localization $H^i(\overline{X_1}, \mathcal{O}_\lambda)_\mathfrak{m} = 0$ for $i \neq 2n-1$ (middle degree).

Moreover, usually the middle degree $H^{2n-1}(\overline{X_1}, \mathcal{O}_\lambda)_\mathfrak{m}$ is a finite free $\mathcal{O}_\lambda$ (although this is not needed for trivialization).

Under such vanishing assumption and since the localization is an exact functor, the natural map

$$H^{2n}(X_{1,E'}, \mathcal{O}_\lambda) \longrightarrow H^{2n}(X_{1,E'}, \mathcal{O}_\lambda)_\mathfrak{m} \longrightarrow H^{2n}(\overline{X_1}, \mathcal{O}_\lambda)_\mathfrak{m}$$

(here $E^{'}$ is an abelian extension of $E$) is zero because the last term is zero, and it induces the following natural map

$$H^{2n}(X_{1,E'}, \mathcal{O}_\lambda) \longrightarrow H^1(Gal_{E'}, H^{2n-1}(\overline{X_1}, \mathcal{O}_\lambda)_\mathfrak{m}).$$

Therefore we can use our special cycles to construct suitable Galois cohomology classes under such map.

Under some mild assumption on $\mathfrak{m}$ (or $\Pi$ etc), such as generic assumptions, large image assumptions etc, these vanishing assumptions have been proved for many cases. Here we list some examples :

• For Siegel modular varieties, such results have already been proved in [52]. Their work depends on the existence of a Galois representation attached to $\Pi$. Recently this technical assumption has been verified by [62]. Therefore this technique is very suitable for our $GSp_4$ example (see section 6.1).

• For some unitary Shimura varieties, Ana Caraiani and Peter Scholze made important breakthrough towards its cohomology and got such vanishing results. See [11] (compact cases) and [12] (non-compact cases). Their ideas provide a new framework towards such problems. And recently, Teruhisa Koshikawa did some generalizations and simplifications, see [36] and [37]. In particular his paper [37] throws some new insight. It is established by the great work [24], which works very generally. Therefore it maybe reasonable to expect such vanishing results for other Shimura varieties.

• For some Hilbert modular varieties, such vanishing resulting are known due to [42] section 4.

What's more, in the last part of section 5.2, lemma 5.6, we need to translate our special cycles with norm relations into a norm compatible system, this localization method is very suitable for that translation. It needs to pick up a map from local Hecke algebra to $\mathbb{C}$ first, our map $\phi_{\Pi,\lambda}$ will provide such a specialization exactly.

More precisely, let the coefficient ring $R = \mathcal{O}_\lambda$. Consider the $R$-coefficient Hecke algebra $He_R^{S_1} = He^{S_1} \otimes_{\mathbb{Z}} R$ and still denote the resulting map by $\phi_{\Pi,\lambda}$,

$$\phi_{\Pi,\lambda} : He_R^{S_1} \longrightarrow \mathcal{O}_\lambda.$$

Apply the tensor functor $- \otimes_{He_R^{S_1}} R$ (or equivalently consider the functor of modulo $\ker(\phi_{\Pi,\lambda})$) to the spectral sequence used to define Abel-Jacobi map, we get the following natural map :

$$H^{2n}(X_{1,E'}, \mathcal{O}_\lambda) \longrightarrow H^1(Gal_{E'}, H^{2n-1}(\overline{X_1}, \mathcal{O}_\lambda)/\ker(\phi_{\Pi,\lambda})),$$

and the Hecke algebra $He_R^{S_1}$ will acts on the right side via scalars. This provides the preparation to apply the translation lemma 5.6. Here we make an explanation for the above map. Usually the tensor functor $- \otimes_{He_R^{S_1}} R$ is not exact, but our above vanishing assumption helps us to avoid such problem. We may also construct this map in the following equivalent way. The map $\phi_{\Pi,\lambda}$ has the following factorization :

$$\phi_{\Pi,\lambda} : He_R^{S_1} \longrightarrow He_{R,\mathfrak{m}_R}^{S_1} \longrightarrow \mathcal{O}_\lambda$$

Here $\mathfrak{m}_R$ is the corresponding maximal ideal of $He_R^{S_1}$, $\ker(He_R^{S_1} \longrightarrow \mathcal{O}_\lambda \longrightarrow k_\lambda)$. As we explained earlier, the first map (localization) gives us the following natural map,

$$H^{2n}(X_{1,E'}, \mathcal{O}_\lambda) \longrightarrow H^1(Gal_{E'}, H^{2n-1}(\overline{X_1}, \mathcal{O}_\lambda)_{\mathfrak{m}_R}).$$

Apply the tensor functor $-\otimes_{He^{S_1}_{R,\mathfrak{m}_R}} \mathcal{O}_\lambda$ to this natural map, we get the desired map

$$H^{2n}(X_{1,E'},\mathcal{O}_\lambda) \longrightarrow H^1(Gal_{E'}, H^{2n-1}(\overline{X_1},\mathcal{O}_\lambda)/\ker(\phi_{\Pi,\lambda})).$$

**Further development**

Finally we make some remarks.

First, we only consider the simplest cohomology theory, $l$-adic cohomology with trivial coefficient (plus suitable Tate twist), as we mentioned above, this theory is most suitable for proper Shimura varieties. When the ambient Shimura variety is not proper, it is more convenient to consider some compatifications and consider cohomology theory over that compatifications. Then it is more suitable to consider intersection cohomology theory etc due to the fact that some natural compactifications (such as the Baily-Borel compactifications) are very singular. But then it will be more difficult to relate special cycles to such cohomology theory.

Another important thing is to consider non-trivial coefficients. We refer to [28] section 4 for more details. See also [1] and [64] etc.

Roughly speaking, there is a canonical additive tensor functor

$$Anc_{\mathbf{G},K} : Rep_{\mathbb{Q}_l}(\mathbf{G}_{\mathbb{Q}_l}) \longrightarrow Et(Sh_{\mathbf{G}}(K))$$

from the category of algebraic representations of $\mathbf{G}_{\mathbb{Q}_l}$ to the category of etale sheaves on $Sh_{\mathbf{G}}(K)$ with coefficients in $\mathbb{Q}_l$. And it is also possible to vary the level group $K$ to get an equivariant version,

$$Anc_{\mathbf{G}} : Rep_{\mathbb{Q}_l}(\mathbf{G}_{\mathbb{Q}_l}) \longrightarrow Et(Sh_{\mathbf{G}}).$$

Here the right side denotes the category of equivariant sheaf. It is also possible to an integral version.

The trivial coefficient system corresponds to the trivial representation. And the general coefficients cohomology will provide more applications to number theory and representation theory.

The above functors also have functoriality for the group $\mathbf{G}$, therefore very suitable to consider our relative setting, the embedding of Shimura datum, $Sh_{\mathbf{H}} \longrightarrow Sh_{\mathbf{G}}$. The pullback between sheaves will correspond to restriction of representations. Therefore, it is necessary to consider such branching law for algebraic representations,

$$Rep_{\mathbb{Q}_l}(\mathbf{G}_{\mathbb{Q}_l}) \longrightarrow Rep_{\mathbb{Q}_l}(\mathbf{H}_{\mathbb{Q}_l}).$$

Certain pair of groups $(\mathcal{G},\mathcal{H})$ have a very nice property : any irreducible algebraic representation of $\mathcal{G}$ contains the trivial representation of $\mathcal{H}$ with multiplicity one. For example, see lemma 5.1 in [28]. In that lemma they consider a special kind of algebraic

representations of $\mathcal{G}$ with such nice property and it is already enough to do arithmetic applications in their setting. Our main example also has this nice property, see [38]. That paper works for compact real groups and that proof can be adapted to algebraic setting. Therefore, it is a desirable and accessible development to consider non-trivial coefficients for our main example.

# Chapitre 4

# Tame relation

Cornut has already established the tame relation for inert places (see [16] section 7,8), which is sufficient for the main arithmetic applications, i.e. the construction of an Euler system. However, here we want to establish tame relations for split places via the seed relation, because this method can be applied to other Shimura varieties (such as $GSp_4$ lift and so on).

## 4.1 Abstract relation

In this section, we still work over a local field and use the same notations as in the previous section 2.2.

Recall $F$ is a $p$-adic field with ring of integers $\mathcal{O}_F$, a uniformizer $\pi \in \mathcal{O}_F$ and residue field $\mathcal{O}_F/\pi = \mathbb{F}_q$ with $q$ elements.

Recall $\mathbf{G}$ is the reductive group scheme over $\mathcal{O}_F$. For simplicity in this section we require it to be **split**. Then the maximal split subtorus $\mathbf{S}$ is the maximal torus $\mathbf{T}$. Notice that the natural map $X_*(\mathbf{T}) \longrightarrow X_*(\mathbf{T}_F)$ is an isomorphism. Consider a cocharacter $\mu \in X_*(\mathbf{T}_F)$, we also use $\mu$ to denote its unique extension to integral level. We require $\mu$ to be $\mathbf{B}_F$-dominant ($\mu \in X_*^+(\mathbf{T})$) and minuscule.

The quotient map $\mathcal{O}_F^* \longrightarrow \mathbb{F}_q^*$ has a natural section $\mathbb{F}_q^* \longrightarrow \mathcal{O}_F^*$. Combining it with the cocharacter $\mu$, we obtain a group map $\mathbb{F}_q^* \longrightarrow \mathcal{O}_F^* \longrightarrow \mathbf{T}(\mathcal{O}_F) \longrightarrow \mathbf{G}(\mathcal{O}_F)$. Because $\mu$ is minuscule, this map is injective. We identify $\mathbb{F}_q^*$ with $\mu(\mathbb{F}_q^*)$ inside $\mathbf{G}(\mathcal{O}_F)$. And we let $\mu(\mathbb{F}_q^*)$ act on $R[G/K]$ through our left $G$ action and denote elements $1_{bK}$ by $[b]$.

We have the following divisibility lemma :

**Lemma 4.1.** *$q-1 | Hep_\mu(\mu(\pi))([1])$ in $R[\mu(\mathbb{F}_q^*)\backslash G/K]$ .*

*Démonstration.* There is a natural $\mathbf{T}(F)$-Hecke equivariant map $pr : R[G/K] \longrightarrow R[\mu(\mathbb{F}_q^*)\backslash G/K]$. Recall that we have the seed relation (theorem 2.3), $Hep_\mu(U_\mu) = 0$ in $End_B R[G/K]$. To

prove our lemma, we need to show :

$$q-1|pr\{Hep_\mu(U_\mu)([1])-Hep_\mu(\mu(\pi))([1])\}.$$

Suppose the Hecke polynomial $Hep_\mu$ equals $\sum_i A_iX^i$ ($A_i \in He$) and recall that $U_\mu^i = U_{\mu^i}$. It is enough to show the following :

For each positive integer $i$, we have

$$q-1|pr\{(U_{\mu^i}-\mu^i(\pi))([1])\} \text{ in } R[\mu(\mathbb{F}_q^*)\backslash G/K].$$

Recall our formula for the $U_{\mu^i}$ operator (formula (2.1)), we have :

$$U_{\mu^i}([1]) = \sum_{z\in\frac{I^+}{\mu(\pi)^iI^+\mu(\pi)^{-i}}} [z\mu(\pi)^i].$$

Denote this indexing set by $In$, we need to analyze it. Recall $I^+ = \mathbf{N}(\mathcal{O}_F)$. Over $\mathrm{Spec}(\mathcal{O}_F)$, for each positive root $\alpha$, there is an associated root subgroup $\mathbf{N}_\alpha \subset \mathbf{N}$ and a $\mathbf{T}$-equivariant isomorphism $R_\alpha : \mathbb{G}_a \longrightarrow \mathbf{N}_\alpha$, here $\mathbf{T}$ acts on $\mathbb{G}_a$ through the cocharacter $\alpha : \mathbf{T} \longrightarrow \mathbb{G}_m$ and $\mathbf{T}$ acts on $\mathbf{N}_\alpha$ through conjugation. Let $\Phi^+$ denote the set of positive roots and consider the $\mathbf{T}$-equivariant product map :

$$\mathbb{G}_a^n = \prod_{\alpha\in\Phi^+}\mathbb{G}_a \cong \prod_{\alpha\in\Phi^+}\mathbf{N}_\alpha \longrightarrow \mathbf{N}.$$

This map is not compatible with the group structures, but it is an isomorphism of schemes for any ordering on the factors.

Divide these factors into two part, define $\mathbf{A} = \prod_{(\alpha,\mu)=1}\mathbf{N}_\alpha$, $\mathbf{B} = \prod_{(\beta,\mu)=0}\mathbf{N}_\beta$. The above isomorphism induces an isomorphism $\mathbf{A}(\mathcal{O}_F)\times\mathbf{B}(\mathcal{O}_F) \cong \mathbf{N}(\mathcal{O}_F)$.

Notice that for positive root $\alpha$ and $\gamma$ with $(\alpha,\mu) = 1 = (\gamma,\mu)$, there is no positive root $\xi$ in the form of $a\alpha + b\gamma$ ($a$ and $b$ are positive integers). Therefore, $\mathbf{A}$ is a commutative subgroup of $\mathbf{N}$. For any positive integer $i$, we have a natural isomorphism $\mathbf{A}(\mathcal{O}_F/\pi^i) \cong \frac{\mathbf{A}(\mathcal{O}_F)}{\mu(\pi)^i\mathbf{A}(\mathcal{O}_F)\mu(\pi)^{-i}}$. The inclusion $\mathbf{A}(\mathcal{O}_F) \hookrightarrow \mathbf{N}(\mathcal{O}_F)$ also induces a natural isomorphism $\frac{\mathbf{A}(\mathcal{O}_F)}{\mu(\pi)^i\mathbf{A}(\mathcal{O}_F)\mu(\pi)^{-i}} \cong \frac{\mathbf{N}(\mathcal{O}_F)}{\mu(\pi)^i\mathbf{N}(\mathcal{O}_F)\mu(\pi)^{-i}}$.

In summary, for the indexing set, we have found a $\mathbf{T}(\mathcal{O}_F)$-equivariant isomorphism

$$\prod_{(\alpha,\mu)=1}\mathbf{N}_\alpha(\mathcal{O}_F/\pi^i) \cong In.$$

Define $In^*$ to be $In - \{1\}$. Then $U_{\mu^i}([1]) - \mu(\pi)^i([1]) = \sum_{z\in In^*}[z\mu(\pi)^i]$.

For any $x \in In^*$, write $x = \prod_{(\alpha,\mu)=1} R_\alpha(x_\alpha)$. For any $t \in \mathbb{F}_q^*$, $\mu(t)x\mu(t)^{-1} = \prod_{(\alpha,\mu)=1}\mathbf{R}_\alpha(tx_\alpha)$.

There is at least one $\alpha$ such that $x_\alpha$ is non-zero, so the group $\mu(\mathbb{F}_q^*)$ acts on $In^*$ freely. Notice that in $\mu(\mathbb{F}_q^*)\backslash G/K$ we have

$$pr([x\mu(\pi)^i]) = pr([\mu(t)x\mu(\pi)^i\mu(t)^{-1}]) = pr([\mu(t)x\mu(t)^{-1}\mu(\pi)^i]),$$

here we use $\mu(\mathbb{F}_q^*) \subset K$. Because the cardinality of $\mu(\mathbb{F}_q^*)$ is $q-1$, we get $q-1|pr\{(U_{\mu^i} - \mu^i(\pi))([1])\}$. □

We make a remark. In fact, we can replace the element $[1]$ by any element $[t]$, where $t$ is an element in $\mathbf{T}(F)$, this lemma still holds with the same proof.

Now we will translate this lemma into the relative setting.

Suppose there exists a closed reductive subgroup $\mathbf{H} \subset \mathbf{G}_F$ and a character $v : \mathbf{H} \longrightarrow \mathbb{G}_{m,F}$ such that they satisfy the following conditions $(*)$ :

• The cocharacter $\mu : \mathbb{G}_{m,F} \longrightarrow \mathbf{G}_F$ factor through $\mathbf{H}$ and we still denote this cocharacter $\mathbb{G}_{m,F} \longrightarrow \mathbf{H}$ by $\mu$.

• $v \circ \mu$ is the identity map for $\mathbb{G}_{m,F}$.

By the second condition, the map $v : \mathbf{H} \longrightarrow \mathbb{G}_m$ is a quotient map (surjective). Moreover, taking $F$-points, we see that the induced map $\mathbf{H}(F) \longrightarrow \mathbb{G}_m(F) = F^*$ is also surjective and if we equip $\mathbf{H}(F)$ and $F^*$ with the induced $p$-adic topology, this map is an open map.

Consider the following conductor filtration on $\mathbb{G}_m(F)$ : For any non-negative integer $m$, define

$$\mathbb{G}_m(m) = \begin{cases} \mathcal{O}_F^* & \text{if } m = 0 \\ 1 + \pi^m \mathcal{O}_F & \text{if } m > 0 \end{cases}.$$

We will use $v$ to define a conductor filtration on $\mathbf{H}(F)$, just define $H(m) = v^{-1}(\mathbb{G}_m(m))$. By the second condition, we get $\mu(\mathbb{F}_q^*) \subset H(0)$. Now we can translate the above lemma as this theorem :

**Theorem 4.2.** ***divisibility***

*$q-1|\ Hep_\mu(\mu(\pi))([1])$ in $R[H(0)\backslash G/K]$.*

This divisibility can be translated as a kind of abstract relation. And it will also explain another reason behind the appearance of $q-1$.

Define $H^d = \ker(\mathbf{H}(F) \xrightarrow{v} \mathbb{G}_m(F))$ and consider the module $R[H^d\backslash G/K]$, which will be a local analogue of our module of special cycles (see next section 4.2). For an element $x \in H^d\backslash G/K$, we denote its corresponding element in $R[H^d\backslash G/K]$ by $[x]$. We have a distinguished element $[1]$. Since $H^d$ is a normal subgroup of $\mathbf{H}(F)$, the latter group acts on $H^d\backslash G/K$, and this action factors through $v$. We thus obtain an $\mathbf{H}(F)$-action on $R[H^d\backslash G/K]$ which factors through $v$.

**Theorem 4.3.** ***abstract relation***

*The element* $Hep_\mu(\mu(\pi))([1]) \in R[H^d\backslash G/K]$ *lies in the image of the trace map*

$$Tr_{1,0} \overset{def}{=} Tr_{\frac{H(0)}{H(1)}} : R[H^d\backslash G/K]^{H(1)} \longrightarrow R[H^d\backslash G/K]^{H(0)}.$$

*Démonstration.* Because $\mu(\mathbb{G}_m(0)) \subset K$, so $H(0)$ fixes $[1]$. And the $H(0)$-action commutes with the operator $Hep_\mu(\mu(\pi))$, thus the element $Hep_\mu(\mu(\pi))([1])$ lies in $R[H^d\backslash G/K]^{H(0)}$.

Notice that each $H(0)$-orbit in $R[H^d\backslash G/K]$ is finite, we obtain an $R$-linear isomorphism

$$R[H(0)\backslash G/K] \cong R[H^d\backslash G/K]^{H(0)}.$$

Denote the projection $H^d\backslash G/K \longrightarrow H(0)\backslash G/K$ by $pr_0$. For any element $C \in H(0)\backslash G/K$, the above map is given by sending $[C]$ to $\sum_{x\in pr_0^{-1}(C)} [x]$.

Similarly, denote the projection $H^d\backslash G/K \longrightarrow H(1)\backslash G/K$ by $pr_1$, denote $H(1)\backslash G/K \longrightarrow H(0)\backslash G/K$ by $pr$. We get a $R$-linear isomorphism $R[H(1)\backslash G/K] \cong R[H^d\backslash G/K]^{H(1)}$.

Consider the following commutative diagram :

$$\begin{array}{ccc} R[H(1)\backslash G/K] & \xrightarrow{\cong} & R[H^d\backslash G/K]^{H(1)} \\ \downarrow{\scriptstyle Tr} & & \downarrow{\scriptstyle Tr_{1,0}} \\ R[H(0)\backslash G/K] & \xrightarrow{\cong} & R[H^d\backslash G/K]^{H(0)} \end{array}$$

Here $Tr$ is the map induced by $Tr_{1,0}$, **not** the map induced by $pr$. Now let's explicit this map $Tr$ :

For any $C \in H(1)\backslash G/K$, let $S(C)$ denote the cardinality of its stabilizer group in $\frac{H(0)}{H(1)}$. Then we have $S(C)|q-1$, because $q-1$ is the cardinality of $\frac{H(0)}{H(1)}$. And $Tr([C]) = S(C)[pr(C)]$.

Now in $R[H(0)\backslash G/K]$, write $Hep_\mu(\mu(\pi))([1]) = \sum_D a_D[D]$, where $D$ runs over elements in $H(0)\backslash G/K$ with non-zero $a_D$. For each $D$, choose an element $C_D \in H(1)\backslash G/K$ such that $pr(C_D) = D$. By our divisibility theorem 4.2, for each $D$, $q-1|a_D$, thus there exists $b_D \in R$ such that $a_D = (q-1)b_D$. Define $S_1 = \sum_D b_D[C_D]$, then $Tr(S_1) = Hep_\mu(\mu(\pi))([1])$. We're done. □

As in our previous lemma, we can also replace $[1]$ by $[t]$ ($t \in \mathbf{T}(F)$) with the same proof.

Finally we propose a variant. This theorem is already enough for our main example in next section 4.2. For more general cases, we only need a slight change, see section 6.1 and section 6.2.

Suppose we have the following integral version conditions (♠) of conditions (*) :

♠ Suppose there exists a closed reductive group $\mathbf{H} \hookrightarrow \mathbf{G}$ over $\mathrm{Spec}(\mathcal{O}_F)$ and a character $v : \mathbf{H} \longrightarrow \mathbb{G}_m$ over $\mathrm{Spec}(\mathcal{O}_F)$ such that

• The cocharacter $\mu : \mathbb{G}_m \longrightarrow \mathbf{G}$ factor through $\mathbf{H}$ and we still denote this cocharacter $\mathbb{G}_m \longrightarrow \mathbf{H}$ by $\mu$.

• $v \circ \mu$ is the identity map for $\mathbb{G}_m$.

And we will define the conductor filtration on $\mathbf{H}(\mathcal{O}_F)$ instead of the whole group $\mathbf{H}(F)$. For any non-negative integer, define $H(m) = v^{-1}(\mathbb{G}_m(m))$. Next we replace the group $H^d$ by $H^{der} = \mathbf{H}^{der}(F)$. Other things remain the same, then we get the following version of abstract tame relation :

**Theorem 4.4.** ***variant abstract relation***

*The element $Hep_\mu(\mu(\pi))([1]) \in R[H^{der}\backslash G/K]$ lies in the image of the trace map $Tr_{1,0} \overset{def}{=} Tr_{\frac{H(0)}{H(1)}} : R[H^{der}\backslash G/K]^{H(1)} \longrightarrow R[H^{der}\backslash G/K]^{H(0)}$.*

## 4.2 Realization

In this section, we will show that the abstract relation can be translated into "real" tame relations for special cycles. This section is in global setting, we will use notations in section 2.1 again.

First we enlarge the coefficient ring $\mathbb{Z}$ into $R$. Here we take $R$ to be an $l$-adic integer ring. Recall we have taken a finite set $S_1$ ("bad primes") containing this prime $l$. Then for any prime $p \notin S_1$, we have $p^{-1} \in R$, so by the discussion about coefficients for Hecke algebra in section 2.2 (or see section A.2), $R$ is large enough, i.e. the coefficients of the Hecke polynomial belong to the Hecke algebra with coefficients in $R$.

Next we will use the previous section's theorems to deduce a local result for our main example.

Recall that $F$ is a totally real field with a quadratic CM extension $F \longrightarrow E$. And $V$ is a $2n+1$-dimensional $F$-space with a quadratic form $\phi$. Inside $(V, \phi)$, we have a $n$-dimensional $E$-hermitian space $(W, \psi)$. We have a pair of $F$-reductive groups $(\underline{\mathbf{G}}, \underline{\mathbf{H}}) = (SO(V), U(W))$ and our main example is

$$(\mathbf{G}, \mathbf{H}) = (\mathrm{Res}_{F/\mathbb{Q}}\underline{\mathbf{G}}, \mathrm{Res}_{F/\mathbb{Q}}\underline{\mathbf{H}}).$$

In the previous section, we work with split groups. Thus we can't apply the results in that section to our main example directly. We will do a two step reduction to overcome this problem.

Let $v$ be a finite place of $F$ which splits in $F \longrightarrow E$ and does not belong to the set $S$ ("bad places"). Let $p$ be the underlying prime of $\mathbb{Q}$, so that $p \notin S_1$. Fix an isomorphism

$$F \otimes \mathbb{Q}_p \cong \prod_{\mathfrak{p}|p} F_{\mathfrak{p}},$$

for simplicity, we require the first factor to be $F_v$.

Consider the base change of our pair $(\mathbf{G}, \mathbf{H})$ to $\mathbb{Q}_p$. The above isomorphism $F \otimes \mathbb{Q}_p \cong \prod_{\mathfrak{p}|p} F_{\mathfrak{p}}$ will induce an isomorphism

$$\mathbf{G}_{\mathbb{Q}_p} \cong \prod_{\mathfrak{p}|p} \mathrm{Res}_{F_{\mathfrak{p}}/\mathbb{Q}_p} \underline{\mathbf{G}}_{F_{\mathfrak{p}}},$$

and for simplicity we define $\mathbf{G}_p = \mathbf{G}_{\mathbb{Q}_p}$, $\mathbf{G}_{\mathfrak{p}} = \mathrm{Res}_{F_{\mathfrak{p}}/\mathbb{Q}_p} \underline{\mathbf{G}}_{F_{\mathfrak{p}}}$. Similarly we have

$$\mathbf{H}_{\mathbb{Q}_p} \cong \prod_{\mathfrak{p}|p} \mathrm{Res}_{F_{\mathfrak{p}}/\mathbb{Q}_p} \underline{\mathbf{H}}_{F_{\mathfrak{p}}}$$

and define $\mathbf{H}_p = \mathbf{H}_{\mathbb{Q}_p}$, $\mathbf{H}_{\mathfrak{p}} = \mathrm{Res}_{F_{\mathfrak{p}}/\mathbb{Q}_p} \underline{\mathbf{H}}_{F_{\mathfrak{p}}}$.

Extend the embedding $F \longrightarrow F_v$ into an embedding $\overline{\mathbb{Q}} \longrightarrow \overline{F_v} = \overline{\mathbb{Q}_p}$. Along this way, we pick up a prime $v^+$ of $E$ lying over $v$ and identify $E_{v^+}$ with $F_v$. Denote the conjugation of $v^+$ by $v^-$ and denote $E_{v^-}$ by $F_{\overline{v}}$. Under this isomorphism $E \otimes_F F_v \cong F_v \times F_{\overline{v}}$, the conjugation on the left side will correspond to the involution of swapping factors on the right side.

Recall our Shimura cocharacter conjugacy class $[\mu_{\mathbf{G}}]$ and now transfer it into the $p$-adic setting. The embeddings between fields $\mathbb{C} \hookleftarrow \overline{\mathbb{Q}} \hookrightarrow \overline{F_v}$ induce isomorphisms

$$\mathrm{Hom}_{\mathbb{Q}}(F, \mathbb{C}) \cong \mathrm{Hom}_{\mathbb{Q}}(F, \overline{\mathbb{Q}}) \cong \mathrm{Hom}_{\mathbb{Q}}(F, \overline{F_v}),$$

and conjugacy classes of cocharacters

$$C_*(\mathbf{G}_{\mathbb{C}}) \cong C_*(\mathbf{G}_{\overline{\mathbb{Q}}}) \cong C_*(\mathbf{G}_{\overline{F_v}}).$$

The isomorphism $\mathbf{G}_p \cong \prod_{\mathfrak{p}} \mathbf{G}_{\mathfrak{p}}$ induces an isomorphism $C_*(\mathbf{G}_{p,\overline{F_v}}) \cong \prod_{\mathfrak{p}} C_*(\mathbf{G}_{\mathfrak{p},\overline{F_v}})$, and our conjugacy class of Shimura cocharacter $[\mu_{\mathbf{G}}]$ will correspond to an element

$$[\mu_{\mathbf{G},v}] = ([\mu_v], 1, ..., 1).$$

Denote $\mathbf{G}_p(\mathbb{Q}_p)$ by $G_p$ with associated Hecke algebra $He_p$ and $\mathbf{G}_{\mathfrak{p}}(\mathbb{Q}_p)$ by $G_{\mathfrak{p}}$ with associated Hecke algebra $He_{\mathfrak{p}}$. Through the natural isomorphism $He_p \cong \bigotimes_{\mathfrak{p}|p} He_{\mathfrak{p}}$, we have an identity for their Hecke polynomials

$$Hep_{\mu_{\mathbf{G},v}} = Hep_{\mu_v}$$

(also see [16] section 10.2). This finishes the first step reduction.

To apply previous section's theorems, we next analyze our subgroup $\mathbf{H}_v$.

Recall $\mathbf{H}_v = \mathrm{Res}_{F_v/\mathbb{Q}_p} \underline{\mathbf{H}}_{F_v}$. Because $v$ is a split prime and we have fixed $E_{v^+} \cong F_v$, the

unitary group $\underline{\mathbf{H}}_{F_v}$ splits. More precisely, through $E \otimes_F F_v \cong F_v \times F_{\overline{v}}$, we have $W_{F_v} = W \otimes_F F_v \cong W_v \oplus W_{\overline{v}}$, where $W_v = W \otimes_E F_v$ and $W_{\overline{v}} = W \otimes_E F_{\overline{v}}$. Then we get

$$\underline{\mathbf{H}}_{F_v} \cong GL(W_v)_{F_v}.$$

Similarly we have

$$\underline{\mathbf{T}}_{F_v} \cong \mathbb{G}_{m,F_v} \times \mathbb{G}_{m,F_{\overline{v}}}$$

and

$$\underline{\mathbf{T}}^1{}_{F_v} \cong \mathbb{G}_{m,F_v}.$$

Under these isomorphisms, the determinant map $\det : \underline{\mathbf{H}}_{F_v} \longrightarrow \underline{\mathbf{T}}^1{}_{F_v}$ corresponds to the determinant map for general linear groups

$$\det : GL(W_v) \longrightarrow \mathbb{G}_{m,F_v}.$$

And for any $F_v$-algebra $Al$, $(x, y) \in Al^* \times Al^*$, the map $\underline{r}$ is as follow

$$\underline{r} : (x, y) \longrightarrow \frac{x}{y},$$

and the inclusion $\underline{\mathbf{T}}^1{}_{F_v}(Al) \hookrightarrow \underline{\mathbf{T}}_{F_v}(Al)$ becomes

$$x \longrightarrow (x, x^{-1}).$$

Now let $\underline{\mathcal{G}}_v$ be the reductive integral model over $\mathrm{Spec}(\mathcal{O}_{F_v})$ for $\underline{\mathbf{G}}_{F_v}$ such that $\underline{\mathcal{G}}_v(\mathcal{O}_{F_v})$ equals $K_v$. By our assumption, $K_v \bigcap \underline{\mathbf{H}}_{F_v}(F_v)$ is a hyperspecial subgroup of $\underline{\mathbf{H}}_{F_v}(F_v)$, we can take a Borel pair $(\underline{\mathcal{B}}_v, \underline{\mathcal{T}}_v)$ for $\underline{\mathcal{G}}_v$ such that $(\underline{\mathcal{B}}_{v,F_v} \bigcap \underline{\mathbf{H}}_{F_v}, \underline{\mathcal{T}}_{v,F_v} \bigcap \underline{\mathbf{H}}_{F_v})$ is a Borel pair for $\underline{\mathbf{H}}_{F_v}$. This Borel pair for $\underline{\mathbf{H}}_{F_v}$ will give us an ordered $F_v$-basis $(x_1, ..., x_n)$ for $W_v$ such that $\underline{\mathcal{B}}_{v,F_v} \bigcap \underline{\mathbf{H}}_{F_v}$ corresponds to upper triangular matrices and $\underline{\mathcal{T}}_{v,F_v} \bigcap \underline{\mathbf{H}}_{F_v}$ corresponds to diagonal matrices. Now define a cocharacter $\mu_{v,0} : \mathbb{G}_{m,F_v} \longrightarrow \underline{\mathbf{H}}_{F_v}$ by sending $t$ to $(t, 1, ..., 1)$ under this ordered basis. Composing it with the embedding $\underline{\mathbf{H}}_{F_v} \hookrightarrow \underline{\mathbf{G}}_{F_v}$, we get a cocharacter for $\underline{\mathbf{G}}_{F_v}$ factoring through its maximal torus $\underline{\mathcal{T}}_{v,F_v}$ and this cocharacter has a unique extension to $\underline{\mathcal{T}}_v$ over $\mathrm{Spec}(\mathcal{O}_{F_v})$, for simplicity we still denote them by $\mu_{v,0}$.

Now we relate $\mu_{0,v}$ to $[\mu_v]$. Recall $\mathbf{G}_v = \mathrm{Res}_{F_v/\mathbb{Q}_p}\underline{\mathbf{G}}_{F_v}$, then over $\overline{F_v} = \overline{\mathbb{Q}_p}$, we have an isomorphism for conjugacy classes of cocharacters,

$$C_*(\mathbf{G}_{v,\overline{F_v}}) \cong \prod_{\delta \in Hom_{\mathbb{Q}_p}(F_v, \overline{F_v})} C_*(\underline{\mathbf{G}}_{v,\delta,\overline{F_v}}),$$

here $\underline{\mathbf{G}}_{v,\delta,\overline{F_v}} = \underline{\mathbf{G}}_{F_v} \otimes_{F_v,\delta} \overline{F_v}$ and for simplicity we assume the first factor corresponds to

our fixed inclusion $F_v \hookrightarrow \overline{F_v}$. Then

$$[\mu_v] = ([\mu_{v,0}], 1, ..., 1).$$

For $(\underline{\mathbf{G}}_{F_v}, [\mu_{v,0}])$ we have the Hecke polynomial $Hep_{\mu_{v,0}} \in \underline{He}_v[X]$, where $\underline{He}_v$ is the Hecke algebra $R[K_v \backslash \underline{\mathbf{G}}_{F_v}(F_v)/K_v] = He_v$, then

$$Hep_{\mu_{v,0}} = Hep_{\mu_v}$$

by [16] section 10.2.

Therefore we have finished the second reduction and we can now apply the results of the previous section.

Because $\det(\mu_{v,0})$ is the identity map for $\mathbb{G}_{m,F_v}$, the pair $(\underline{\mathbf{H}}_{F_v}, \det)$ satisfies conditions (*) in previous section. The map det induces a conductor filtration $\{\underline{\mathbf{H}}_v(m) | m \geq 0\}$ on $\underline{\mathbf{H}}_{F_v}(F_v) = \mathbf{H}_v(\mathbb{Q}_p)$ and its kernel is exactly $\underline{\mathbf{H}}^{der}_{F_v}(F_v)$. Denote this kernel by $H^d_v$ and define $H_v(m) = \underline{\mathbf{H}}_v(m)$. Take a uniformizer $\pi_v \in \mathcal{O}_{F_v}$ and apply theorem 4.3 to this pair $(\underline{\mathbf{H}}_{F_v}, \underline{\mathbf{G}}_v)$, we get the following local result :

**Theorem 4.5.** ***local tame relation***

*The element $Hep_{\mu_{v,0}}(\mu_{v,0}(\pi_v))([1]) \in R[H^d_v \backslash G_v/K_v]$ lies in the image of trace map $Tr_{v,1,0} \overset{def}{=} Tr_{\frac{H_v(0)}{H_v(1)}} : R[H^d_v \backslash G_v/K_v]^{H_v(1)} \longrightarrow R[H^d_v \backslash G_v/K_v]^{H_v(0)}$.*

Here $R[H^d_v \backslash G_v/K_v]$ is exactly the local version of our module for special cycles and $\mu_{v,0}(\pi_v)$ can be seen as an analogue of the **Frobenius** :

Still denote the image of $\mu_{v,0}(\pi_v)$ (resp. $\pi_v$) under the inclusion $\underline{\mathbf{H}}(F_v) \hookrightarrow \mathbf{H}(\mathbb{A}_f)$ (resp. $\underline{\mathbf{T}}^1(F_v) \hookrightarrow \mathbf{T}^1(\mathbb{A}_f)$) by $\mu_{v,0}(\pi_v)$ (resp. $\pi_v$). Here recall we have identified $\underline{\mathbf{T}}^1_{F_v}$ with $\mathbb{G}_{m,F_v}$ and $\pi_v \in \mathbb{G}_{m,F_v}(F_v)$. Denote the class of $\pi_v$ in $\mathbf{T}^1(\mathbb{A}_f)/\mathbf{T}^1(\mathbb{Q})$ by $[\pi_v]$. We have

$$\det(\mu_{v,0}(\pi_v)) = \pi_v = \underline{r}((..., 1, \pi_v, 1, ...))$$

and

$$\mathrm{Art}^1_E([\pi_v]) = Frob_{v^+}|_{E[\infty]}.$$

According to the reciprocity law, for any special cycle $Z_K(\widetilde{g})$, we have

$$Frob_{v^+}(Z_K(\widetilde{g})) = Z_K(\mu_{v,0}(\pi_v)\widetilde{g}).$$

We also notice that the two conductor filtrations for torus (defined in section 2.1 and section 4.1) are indeed the same. For any positive integer $m$, we have the following exact sequences :

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathbb{G}_m(\mathcal{O}_{F_v}) & \longrightarrow & \underline{\mathbf{T}}(\mathcal{O}_{F_v}) & \xrightarrow{\underline{r}} & \underline{\mathbf{T}}^1(\mathcal{O}_{F_v}) & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle red_0} & & \downarrow{\scriptstyle red_1} & & \downarrow{\scriptstyle red_2} & & \\
1 & \longrightarrow & \mathbb{G}_m(\mathcal{O}_{F_v}/\pi_v^m) & \longrightarrow & \underline{\mathbf{T}}(\mathcal{O}_{F_v}/\pi_v^m) & \longrightarrow & \underline{\mathbf{T}}^1(\mathcal{O}_{F_v}/\pi_v^m) & \longrightarrow & 1
\end{array}$$

These three vertical reduction maps ($red_0$, $red_1$, $red_2$) are surjective, then the snake lemma implies $\underline{r}(\ker(red_1)) = \ker(red_2)$. And this equality holds for any prime of $F$. The zero step in both filtrations are $\underline{\mathbf{T}}^1(\mathcal{O}_{F_v})$. In summary these two filtrations are the same.

Now recall our base cycle $\widetilde{z} = Z_K(g)$ with $g = g_{S_1} \times g^{S_1}$, by our assumption for $S_1$ and $S$, we can assume $\widetilde{z} = \widetilde{z}_S \otimes 1^S$ (here $1^S = \bigotimes_{w \notin S} 1$) through the following natural map :

$$R[\mathbf{H}^{der}(\mathbb{A}_{f,S_1})\backslash\mathbf{G}(\mathbb{A}_{f,S_1})/K_{S_1}] \bigotimes \bigotimes_{w\notin S}{}' R[\underline{\mathbf{H}}^{der}(F_w)\backslash\underline{\mathbf{G}}(F_w)/K_w] \longrightarrow R[Z_K(\mathbf{G},\mathbf{H})].$$

For each prime $v \in \underline{\mathbf{P}}$, by the above theorem, there exists an element $\widehat{z_v} \in R[\underline{\mathbf{H}}^{der}(F_v)\backslash\underline{\mathbf{G}}(F_v)/K_v]^{H_v(1)}$ such that

$$Tr_{v,1,0}(\widehat{z_v}) = Hep_{\mu_{v,0}}(Frob_{v^+})([1]).$$

For any ideal $m \in \underline{\mathbf{N}}$, we define an element $z(m)$ in the left side of the above natural map,

$$z(m) = \widetilde{z}_S \otimes (\bigotimes_{v|m} \widehat{z_v}) \otimes (\bigotimes_{v\nmid n, v\notin S} 1)$$

and we also use $z(m)$ to denote the resulting cycle in $R[Z_K(\mathbf{G},\mathbf{H})]$. Putting everything together, we finally obtain the following result :

**Theorem 4.6.** ***tame relation***

*We have $z(1) = \widetilde{z}$. For any $m \in \underline{\mathbf{N}}$, the special cycle $z(m)$ is defined over $\underline{E}[m]$, and for any $v \in \underline{\mathbf{P}}$ that does not divide $m$, we have $Tr_{\frac{\underline{E}[mv]}{\underline{E}[m]}}(z(mv)) = Hep_{\mu_{\mathbf{G},v}}(Frob_{v^+})(z(m))$.*

*Démonstration.* Inside $Z_K(\mathbf{G},\mathbf{H})$, the special cycle $z(m)$ is fixed by $U^1(m)$, thus it is defined over the field $\underline{E}[m]$ and by definition $z(1) = \widetilde{z}$.

Notice that for any prime $w \notin S\bigcup\{v\}$, we have $z(m)_w = z(mv)_w$ ($w$-"component"). Applying the previous results, we get

$$Tr_{\frac{\underline{E}[mv]}{\underline{E}[m]}}(z(mv)) \overset{lemma\ 2.2}{=\!=\!=} Tr_{\frac{U_v^1(0)}{U_v^1(1)}}(z(mv))$$

$$= \widetilde{z}_S \otimes (Tr_{v,1,0}(\widehat{z_v})) \otimes (\bigotimes_{w\notin S\bigcup\{v\}} z(m)_w)$$

$$\overset{theorem\ 4.5}{=\!=\!=} \widetilde{z}_S \otimes (Hep_{\mu_{\mathbf{G},v}}(Frob_{v^+})([1])) \otimes (\bigotimes_{w\notin S\bigcup\{v\}} z(m)_w)$$

$$= Hep_{\mu_{\mathbf{G},v}}(Frob_{v^+})(z(m)).$$

□

Finally we make two remarks :

• For simplicity, we have taken $\widetilde{z} = Z_K(g)$ as our base cycle. The result easily generalize to any base cycle $z^{'} = \sum_i Z_K(g_i)$.

• In this paper, we always use double coset description for Hecke algebra and use "formal" Hecke action. The geometric Hecke action is through the Geometric Hecke correspondence. These two actions are the same, we refer to Cornut's [16] section 5.16 for details.

# Chapitre 5

# Norm relation

## 5.1 Abstract relation

In this section, we come back to the local setting again.

Let $F$ denote a $p$-adic local field with a uniformizer $\pi$. Denote the cardinality of the residue field $\mathcal{O}_F/\pi$ by $q$. Let $\mathbf{G}$ denote a reductive group scheme over $\mathrm{Spec}(\mathcal{O}_F)$ and $\mathbf{H}$ a closed subgroup scheme of $\mathbf{G}$. We assume this pair $(\mathbf{G}, \mathbf{H})$ to be spherical over $\mathrm{Spec}(\mathcal{O}_F)$ : there exists a Borel subgroup scheme $\overline{\mathbf{B}}$ of $\mathbf{G}$ such that the $\mathbf{H}$-orbit of $[1]$ in $\mathbf{G}/\overline{\mathbf{B}}$ is open, equivalently, $Lie(\mathbf{H}) + Lie(\overline{\mathbf{B}}) = Lie(\mathbf{G})$. Take a maximal torus $\mathbf{T}$ inside $\overline{\mathbf{B}}$ and denote by $\mathbf{B}$ the Borel subgroup of $\mathbf{G}$ opposed to $\overline{\mathbf{B}}$ with respect to $\mathbf{T}$. Here we use notations compatible with those of Loeffler's [44] section 4. Loeffler considers a more general situation where $\mathbf{B}$ is replaced by a parabolic subgroup $Q_{\mathbf{G}}$ and $[1]$ is replaced by $[u]$ $(u \in \mathbf{G}(\mathcal{O}_F))$. Our method can be generalized similarly but here this simplification is sufficient for our application.

**Example** (spherical pair)

There are many spherical pairs. Here we mention a standard example : $(GL_n \times GL_n, GL_n)$.

Consider the diagonal embedding $GL_n \longrightarrow GL_n \times GL_n$. We take $\mathbf{G} = GL_n \times GL_n$, $\mathbf{H} = GL_n$. Take $\mathbf{T}_1 \subset \mathbf{H}$ to be the maximal torus corresponding to the set of diagonal matrix, $\mathbf{B}_1 \subset \mathbf{H}$ to be the Borel subgroup of $\mathbf{H}$ that corresponds to the set of upper triangular matrix, $\mathbf{B}_2 \subset \mathbf{H}$ to be the Borel subgroup of $\mathbf{H}$ that corresponds to the set of lower triangular matrix. Then $(\mathbf{T}_1 \times \mathbf{T}_1, \mathbf{B}_1 \times \mathbf{B}_2)$ is a Borel pair for $\mathbf{G}$, and the related opposite Borel subgroup $\overline{\mathbf{B}_1 \times \mathbf{B}_2}$ is $\mathbf{B}_2 \times \mathbf{B}_1$. It is easy to verify that $Lie(\mathbf{H}) + Lie(\mathbf{B}_2 \times \mathbf{B}_1) = Lie(\mathbf{G})$, thus $(GL_n \times GL_n, GL_n)$ is a spherical pair. In fact, it is a kind of symmetric pair, in the appendix A.1, we will encounter this example again.

Now we come back to our topic.

Denote $G = \mathbf{G}(F)$, $H = \mathbf{H}(F)$, $K = \mathbf{G}(\mathcal{O}_F)$ and $K_H = \mathbf{H}(\mathcal{O}_F) = H \bigcap K$. Denote the Hecke algebra for $G$ by $He = R[K\backslash G/K]$, here $R$ is a suitable coefficient ring to realize Hecke polynomial (i.e. contains $\mathbb{Z}[q^{\pm\frac{1}{2}}]$). Now we will define a filtration on $G$ and $H$ ("level

group filtration").

Denote the unipotent radical of $\mathbf{B}$ and $\overline{\mathbf{B}}$ by $\mathbf{N}$ and $\overline{\mathbf{N}}$, so we have $\mathbf{B} = \mathbf{T} \ltimes \mathbf{N}$ and $\overline{\mathbf{B}} = \mathbf{T} \ltimes \overline{\mathbf{N}}$. Take a strict $\mathbf{B}$-dominant cocharacter $\mu \in X^+(\mathbf{T})$, here strict means the associated parabolic group $\mathbf{P}_\mu$ for $\mu$ is exactly $\mathbf{B}$. Set $\tau = \mu(\pi)$ and thus we have $\tau\mathbf{N}(O_F)\tau^{-1} \subset \mathbf{N}(O_F)$, $\tau^{-1}\overline{\mathbf{N}}(O_F)\tau \subset \overline{\mathbf{N}}(O_F)$. Let $Hep_\mu \in He[X]$ and $U_\mu \in End_{\mathbf{B}(F)}R[G/K]$ be the associated Hecke polynomial and $U$-operator. Now for any integer $m \geq 0$, define

$$G_m = K \bigcap \tau^m K \tau^{-m},\ H_m = G_m \bigcap \mathbf{H}(F).$$

So $G_0 = K$, $H_0 = K_H$. For a positive integer $m$, $G_m$ lies in the $\mathcal{O}_F$-points of the "Big Bruhat cell" $\mathbf{N} \times \mathbf{T} \times \overline{\mathbf{N}}$ and decomposes as

$$G_m = \tau^m \mathbf{N}(\mathcal{O}_F)\tau^{-m} \times \mathbf{T}(\mathcal{O}_F) \times \overline{\mathbf{N}}(\mathcal{O}_F).$$

It follows that $G_m \supset G_{m+1}$ and $H_m \supset H_{m+1}$ for every $m$. Moreover, define

$$N_m = \mathbf{N}(F) \bigcap G_m,$$

then $N_m = \tau^m \mathbf{N}(\mathcal{O}_F)\tau^{-m}$ and $N_m \hookrightarrow G_m$ induces a bijection

$$N_m/N_{m+i} = G_m/G_{m+i}$$

for every positive integer $m$ and $i$.

Regarding these level group filtrations, we have the following comparison :

**Lemma 5.1.** ***comparison*** *1*

*For $m, i \geq 1$, the natural inclusion $H_m \subset G_m$ induces an isomorphism $H_m/H_{m+i} = G_m/G_{m+i}$.*

*Démonstration.* This map is obviously injective and we only need to establish surjectivity.

Take an element $x \in G_m$ and denote its class in $(\mathbf{G}/\overline{\mathbf{B}})(\mathcal{O}_F)$ by $[x]$. Consider the reduction (or specialization) map $red : \mathbf{G}(\mathcal{O}_F) \longrightarrow \mathbf{G}(\mathcal{O}_F/\pi)$. Then $red(x) \in \overline{\mathbf{B}}(\mathcal{O}_F/\pi)$. Therefore $red([x]) = red([1])$, here $red$ also denotes the reduction map for our flag scheme $(\mathbf{G}/\overline{\mathbf{B}})(\mathcal{O}_F) \longrightarrow (\mathbf{G}/\overline{\mathbf{B}})(\mathcal{O}_F/\pi)$.

Let $\Psi : \mathbf{H} \longrightarrow \mathbf{G}/\overline{\mathbf{B}}$ be the $\mathbf{H}$-orbit map of [1]. By our spherical assumption ($Lie(\mathbf{H}) + Lie(\overline{\mathbf{B}}) = Lie(\mathbf{G})$), this map is a smooth map. Because $red([x]) = red([1]) = \Psi(1)$ and $\mathcal{O}_F$ is Henselian, there exists an element $y \in \mathbf{H}(\mathcal{O}_F)$ such that $\Psi(y) = [x]$ and $red(y) = 1$ (here $red$ is for $\mathbf{H}$). Then $x = y \cdot c$, where $c \in \overline{\mathbf{B}}(\mathcal{O}_F)$. Since $c \in G_{m+i}$, it follows that $y$ belongs to $H_m$, and we have $[y] = [x]$ in $G_m/G_{m+i}$. □

Therefore we can identify $H_m/H_{m+i}$ with $N_m/N_{m+i}$ through these isomorphisms.

Now consider our module $R[G/K]$. Denote $x_0 = [1] = 1_K$ and define $x_m = [\tau^m] = 1_{\tau^m K}$ ($m$ is any positive integer). Through $H \hookrightarrow G$, the group $H$ acts on the set $G/K$ by left multiplication, and for each non-negative integer $m$, $H_m$ fixes the element $x_m$ inside $G/K$.

For any integer $m > 0, i \geq 0$, we have the following lemma :

**Lemma 5.2.** ***comparison 2***
*$Tr_{\frac{H_m}{H_{m+i}}}(x_{m+i}) = U_{\mu^i}(x_m)$ in $R[G/K]$, here $Tr_{\frac{H_m}{H_{m+i}}}(x_{m+i}) \overset{def}{=} \sum_{\delta \in H_m/H_{m+i}} \delta(x_{m+i})$.*

*Démonstration.* If $i = 0$, both operator are the identity. For positive integer $i$, it is a corollary of the above lemma 5.1 :

Through the identifications $H_m/H_{m+i} = N_m/N_{m+i} = \tau^m(N_0/N_i)\tau^{-m}$, we find that

$$Tr_{\frac{H_m}{H_{m+i}}}(x_{m+i}) = \sum_{\delta \in N_m/N_{m+i}} \delta(x_{m+i}) = \sum_{z \in \frac{\mathbf{N}(\mathcal{O}_F)}{\tau^i \mathbf{N}(\mathcal{O}_F)\tau^{-i}}} [\tau^m z \tau^i] = U_{\mu^i}(x_m),$$

where the last equality comes from the explicit formula (2.1) for $U$-operator, there $I^+ = \mathbf{N}(\mathcal{O}_F)$. □

Now suppose the Hecke polynomial $Hep_\mu = \sum_i A_i X^i$, where $A_i \in R[K\backslash G/K]$ (Hecke algebra). Since $U_{\mu^i} = (U_\mu)^i$, applying Boumasmoud's relation (theorem 2.3), we have the following relation :

**Lemma 5.3.** *$\sum_i A_i Tr_{\frac{H_m}{H_{m+i}}}(x_{m+i}) = 0$ in $R[G/K]$.*

Now recall the local analogue of our modules for special cycles is $R[H^{der}\backslash G/K]$ where $H^{der} = \mathbf{H}^{der}(F)$. Through the natural projection $G/K \longrightarrow H^{der}\backslash G/K$, the above relation also holds in the second module. More precisely, define

$$H^{ab} = \frac{H}{H^{der}}, H_m^{ab} = Im(H_m).$$

All fibers of the map $H_m/H_{m+i} \longrightarrow H_m^{ab}/H_{m+i}^{ab}$ have the same cardinality, which we denote by $c(m,i)$. Then we get :

**Theorem 5.4.** ***abstract norm relation***
*$\sum_i c(m,i) A_i Tr_{\frac{H_m^{ab}}{H_{m+i}^{ab}}}(x_{m+i}) = 0$ in $R[H^{der}\backslash G/K]$.*

Finally we make a remark. For many spherical pairs $(\mathbf{G}, \mathbf{H})$, we find that $H_m = H \bigcap \tau^m K \tau^{-m}$. In other words, the $H$-stabilizer of $x_m \in G/K$ also stabilizes $x_0 \in G/K$. We conjecture that this holds in the general setup of this section (**stabilizer conjecture**) and verify it in many cases in appendix A.1. In particular, it holds for our main example. This observation will be used in next section 5.2 for the calculation of $c(m,i)$.

## 5.2 Realization

We return to the global setting again and to the notations in section 2.1.

Recall that we have chosen a special cycle $z = Z_K(g) \in R[Z_K(\mathbf{G}, \mathbf{H})]$ and $z$ is defined over the field $\underline{E}[1]$, which is the fixed field by $U^1(1)$. Choose a prime $v$ of $F$ that doesn't belong to $S$ and $v \nmid disc(W), v \nmid disc(V)$ (discriminant of $W$ and $V$).

First we verify the spherical condition over $v$ to apply previous section's theorems.

Denote $\mathcal{O}_{E_v} = \mathcal{O}_E \otimes_{\mathcal{O}_F} \mathcal{O}_{F_v}$. Because $v$ is an unramified prime, there exists an element $\eta \in \mathcal{O}^*_{E_v}$ such that $Tr(\eta) = \eta + \overline{\eta} = 0$. Denote $(V_v, \phi_v) = (V, \phi) \otimes_F F_v$, $(W_v, \psi_v) = (W, \psi) \otimes_F F_v$. Because $v \nmid disc(W)$ and the unitary group $\underline{\mathbf{H}}_{F_v}$ is unramified, there exists an orthogonal $E_v$-basis $\{w_1, ..., w_n\}$ for $W_{F_v} = W \otimes_F F_v$, such that $\phi_v(w_i, w_i) \in \mathcal{O}^*_{F_v}$. Suppose $V_v = W_v \perp F_v v_n$, we can assume $\phi_v(v_n, v_n) \in \mathcal{O}^*_{F_v}$ due to $v \nmid disc(V)$. Because $Norm(\mathcal{O}^*_{E_v}) = \mathcal{O}^*_{F_v}$, we can rescale $\{w_1, ..., w_n\}$ so that $\phi(w_n, w_n) + \phi(v_n, v_n) = 0$, $\phi(w_i, w_i) + \phi(\eta w_{i+1}, \eta w_{i+1}) = 0$ $(1 \leq i \leq n-1)$. Define $v_{i-1} = \eta w_i$ $(1 \leq i \leq n)$, we get an orthogonal $F_v$-basis for $V_v$

$$\mathcal{B}_v = (v_0, w_1, v_1, w_2, ..., w_n, v_n).$$

Such a basis is called a special basis in Cornut's [15] section 5.1.5. It defines two orthogonal decomposition of $V_v$,

$$V_v = E_v w_1 \perp ... \perp E w_n \perp F_v v_n = F_v v_0 \perp H_1 \perp ... \perp H_n,$$

where $H_i = F_v w_i \perp F_v v_i$ is an hyperbolic $F_v$-plane whose isotropic $F_v$-lines are spanned by $e_{\pm i} = \frac{1}{2}(v_i \pm w_i)$. Consider the ordered basis $(e_n, ..., e_1, v_0, e_{-1}, ..., e_{-n})$ of $V_v$, it defines an embedding

$$\underline{\mathbf{G}}_{F_v} \hookrightarrow GL(V_v)_{F_v} = GL(2n+1)_{F_v}$$

and defines a Borel pair $(\underline{\mathbf{Ts}}_v, \underline{\mathbf{B}}_v)$ for $\underline{\mathbf{G}}_{F_v}$, here $\underline{\mathbf{Ts}}_v$ corresponds to diagonal matrices and $\underline{\mathbf{B}}_v$ corresponds to upper triangular matrices. Consider the $\mathcal{O}_{F_v}$-lattice $L_v$ spanned by the above basis, it is a self-dual lattice and extends our embedding over $\mathrm{Spec}(\mathcal{O}_{F_v})$,

$$SO(L_v) \hookrightarrow GL(L_v).$$

Through this embedding, we get a reductive integral model $\underline{\mathcal{G}}_v$ for $\underline{\mathbf{G}}_{F_v}$ with the associated Borel pair $(\underline{\mathfrak{Ts}}_v, \underline{\mathfrak{B}}_v)$. Consider the $\mathcal{O}_{E_v}$-lattice $\widetilde{L}_w$ spanned by $(w_1, ..., w_n)$, this is a self-dual lattice in $W_v$ and it extends the embedding $\underline{\mathbf{H}}_{F_v} \hookrightarrow \underline{\mathbf{G}}_{F_v}$ over $\mathrm{Spec}(\mathcal{O}_{F_v})$,

$$U(\widetilde{L}_w) \hookrightarrow SO(L_v),$$

and we get a reductive integral model $\underline{\mathcal{H}}_v$ for $\underline{\mathbf{H}}_{F_v}$. It is easy to check that

$$\underline{\mathcal{H}}_v \bigcap \overline{\underline{\mathfrak{B}}_v} = 1$$

($\overline{\underline{\mathfrak{B}}_v}$ is the Borel group opposite to $\underline{\mathfrak{B}}_v$ with respect to $\underline{\mathfrak{T}\mathfrak{s}}_v$) and then $(\underline{\mathcal{H}}_v, \underline{\mathcal{G}}_v, \overline{\underline{\mathfrak{B}}_v})$ satisfies the spherical condition by dimension reason.

Here we make some remarks. In fact, since the intersection of $\underline{\mathcal{H}}_v$ and $\overline{\underline{\mathfrak{B}}_v}$ is trivial, the action of $\underline{\mathcal{H}}_v(\mathcal{O}_{F_v})$ on the open orbit of [1] in $\underline{\mathcal{G}}_v/\overline{\underline{\mathfrak{B}}_v}(\mathcal{O}_{F_v})$ is transitive. In other words, any Borel group $\widetilde{B}_v \subset \underline{\mathcal{G}}_v$ with $Lie(\underline{\mathcal{H}}_v) + Lie(\widetilde{B}_v) = Lie(\underline{\mathcal{G}}_v)$ can be constructed as above.

Now we can apply previous section's results. Take a uniformizer $\pi_v$ for $\mathcal{O}_{F_v}$ and choose a strict $\underline{\mathfrak{B}}_v$-dominant cocharacter $\mu_v \in X_*^+(\underline{\mathfrak{T}\mathfrak{s}}_v)$ in the form of $(s_n, ..., s_1, 0, -s_1, ..., -s_n)$ where $s_i$ are integers with $0 < s_1 < ... < s_n$. In other words, for any $\mathcal{O}_{F_v}$ algebra $Al$ and any $t \in Al^*$, the map $\mu_v$ is given by

$$t \mapsto (t^{s_n}, ..., t^{s_1}, 1, t^{-s_1}, ..., t^{-s_n}) \in \underline{\mathfrak{T}\mathfrak{s}}_v(Al) \subset \underline{\mathcal{G}}_v(Al) \subset GL(L_v)(Al).$$

Still define $G_v = \underline{\mathbf{G}}(F_v)$ with the hyperspecial group $K_v = \underline{\mathcal{G}}(\mathcal{O}_{F_v})$. Notice that we already chose a global level group $K$ with $v$-component "$K_v$". We can require these two notations defines the same group to avoid notation confusion. This can be done either by modifying the $v$-component of the global level group $K$ or by changing our local integral model. Such slight change influences nothing. Let $\tau = \mu_v(\pi_v)$ and define $x_m = [\tau^m] = 1_{\tau^m K_v} \in G_v/K_v$ for any non-negative integer $m$. Applying theorem 5.4, we get the following relation (for positive integer $m$) :

$$\sum_i c(m,i) A_i Tr_{\frac{T^1_{v,m}}{T^1_{v,m+i}}}(x_{m+i}) = 0 \in R[H_v^{der} \backslash G_v / K_v], \tag{5.1}$$

here $H_v = \underline{\mathbf{H}}_v(F_v)$, $H_v^{der} = \underline{\mathbf{H}}_v^{der}(F_v)$ and $T^1_{v,m} = H^{ab}_{v,m}$ with other notations defined in the previous section.

Now we describe these constants $c(m,i)$. We will need Cornut's explicit computation in [15]. That paper works under inert assumption therefore from now on we assume $v$ to be inert in $E$. By Cornut's computation in [15] section 5.1.14, for any non-negative integer $m$, we have

$$T^1_{v,m} = \det(H_{v,m}) = U^1_v(ms_1),$$

here $U^1_v(ms_1)$ is the conductor filtration defined in global setting 2.1. For the benefit of readers, we make some remarks about this computation. The case $m = 0$ is trivial, we only need to care about $m > 0$. We use similar notation (special basis etc) as Cornut's [15]. The related computation can be found in [15] section 5.1.5-5.1.14. He dealt with more general cases there but we only need to use a very good case. The elements $x_m$ correspond to hyperspecial vertices in the Bruhat-Tits building of $G_v = \underline{\mathbf{G}}_{F_v}(F_v)$. Cornut implicitly

described this building via self-dual norms on the orthogonal space $(V_v, \phi_v)$. There he studied the conductor of the stabilizer of $x_m$ in $H_v = \underline{\mathbf{H}}_{F_v}(F_v)$. By the stabilizer conjecture (see appendix A.1), this group is exactly the level group for $H_v$ that we defined in previous section. In his notations, our element $x_m$ will correspond to a norm $\alpha$ determined by the $n$-tuple $(ms_1, ..., ms_n)$. Because $0 < ms_1 < ... < ms_n$, this $n$-tuple satisfies his condition (**SP**) (see [15] section 5.1.6). And this $n$-tuple determines another $n$-tuple $(c_1, ..., c_n) \in \mathbb{Z}^n$, $c_1 = ms_1$, $c_i = ms_i + ms_{i-1}$ $(2 \leq i \leq n)$. By [15] section 5.1.9, we have $\omega(\alpha) = [ms_1, ..., ms_n]_D$. Then the lemma in [15] section 5.1.14 tells us $\det U(\alpha) = U_r$ (in his notation) with $r = c_1 = ms_1$. In our notations, this group is exactly $U_v^1(ms_1)$.

For any positive integers $m$ and $i$, recall we have a natural identification $H_{v,m}/H_{v,m+i} = N_{v,m}/N_{v,m+i}$, here $N_{v,m} = \tau^m \mathcal{N}_v(\mathcal{O}_{F_v})\tau^{-m}$ ($\mathcal{N}_v$ is the unipotent radical of $\underline{\mathfrak{B}}_v$). Denote its cardinality by $y(m,i)$, then $y(m,i)$ only depends on $i$. Moreover, using the root group map for $\mathcal{N}_v$, we can also compute it explicitly. We have

$$y(m,i) = q_{F_v}^{\langle i\mu_v, 2\rho_v\rangle},$$

here $q_{F_v}$ is the cardinality for the residue field $\mathcal{O}_{F_v}/\pi_v$ and $\rho_v$ is the half sum of positive roots of $\underline{\mathcal{G}}_v$. Now for each $m$, we introduce the following definition of conductor :

$$con(m) = \min\{cc | cc \in \mathbb{N}, H_{v,m}^{ab} = \det(H_{v,m}) \supset U_v^1(cc)\}$$

Here by the above computation, we already know $con(m) = ms_1$. Moreover, we have an equality in fact :

$$\det(H_{v,m}) = U_v^1(ms_1).$$

Thus in this situation, we obtain the following explicit formula for $c(m,i)$,

$$c(m,i) = Card(H_{v,m}/H_{v,m+i})/Card(H_{v,m}^{ab}/H_{v,m+i}^{ab})$$

$$= \frac{y(m,i)}{q_{F_v}^{con(m+i)-con(m)}} = q_{F_v}^{i(\langle\mu_v, 2\rho_v\rangle - s_1)} = q_{F_v}^{i(\sum_{j=1}^{n}(2n-2j+1)s_{n+1-j} - s_1)}.$$

Here we make an important **remark** :

In fact, the key outcome of the above computations is that our main example satisfies the following **conditions** :

♣ For large enough integer $m$, the numbers $c(m,1)$ are constant.

♡ The conductor $con(m)$ grows to infinity.

We make some comments on these two conditions. The condition ♡ will cut out a non-trivial $p$-adic (suppose $v|p$) extension for our resulting special cycles. We have a general strategy to verify this condition, avoiding explicit calculation. For positive integer $m$, the image of $G_{v,m}$ in $\underline{\mathcal{G}}_v(\mathcal{O}_{F_v}/\pi_v^m)$ lies in $\overline{\underline{\mathfrak{B}}_v}(\mathcal{O}_{F_v}/\pi_v^m)$. Because we know $\underline{\mathcal{H}}_v \bigcap \overline{\underline{\mathfrak{B}}_v} = 1$ ("small

intersection"), it follows that the image of $H_{v,m}$ in $\underline{\mathcal{G}}_v(\mathcal{O}_{F_v}/\pi_v^m)$ is trivial. Then its abelian quotient lies in $U_v^1(m)$, so $con(m) \geq m$ and the condition $\heartsuit$ is satisfied. For general Shimura pair $Sh_{\widetilde{H}} \longrightarrow Sh_{\widetilde{G}}$ we may not have such trivial intersection property, but we can use a similar argument, see $GSp_4$ example in section 6.1 and the similitude version unitary GGP pair in section 6.2. The condition $\clubsuit$ is needed to turn abstract norm relations into a norm compatible family (under ordinary condition). At present the author doesn't know how to verify it formally without explicit computation. Here the large enough assumption on $m$ is a very mild assumption in the study of norm relations. It will simplify the study about $c(m,1)$ under condition $\heartsuit$.

Now we continue the discussion of our main example.

Due to condition $\clubsuit$, we define $C_1 = c(m,1)$ and we have $C_1^i = c(m,i)$ and denote it by $C_i$. Now we construct a family of special cycles. Recall we have the following natural map

$$R[\mathbf{H}^{der}(\mathbb{A}_{f,S_1})\backslash\mathbf{G}(\mathbb{A}_{f,S_1})/K_{S_1}]\bigotimes\bigotimes_{p\notin S_1}{}' R[\mathbf{H}^{der}(\mathbb{Q}_p)\backslash\mathbf{G}(\mathbb{Q}_p)/K_p] \longrightarrow R[Z_K(\mathbf{G},\mathbf{H})],$$

through this map, we write the base cycle $\widetilde{z} = z^v \otimes 1_v$. For any non-negative integer $m$, we define

$$z_{cy(m)} = z^v \otimes [\tau^m], U^1(cg(m)) = U^{1,v} \times T_{v,m}^1 = U^{1,v} \times U_v^1(ms_1) = U^1((v)^{ms_1})$$

and denote the corresponding fixed subfield inside $E[\infty]$ by $\widetilde{E}(cg(m))$. We have the following realization :

**Theorem 5.5.** ***norm relation***

*We have $z_{cy(0)} = \widetilde{z}$. And for $m > 0$, the cycle $z_{cy(m)}$ is defined over the field $\widetilde{E}(cg(m))$, they satisfy*

$$\sum_i C_i A_i Tr_{\frac{\widetilde{E}(cg(m+i))}{\widetilde{E}(cg(m))}}(z_{cy(m+i)}) = 0.$$

*Démonstration.* This is a translation of the above results.

The first two statements are obvious. Notice that $U^1(1)\bigcap\mathbf{T}^1(\mathbb{Q}) = 1$, similar to tame relation case (lemma 2.2), we also have a local-global connection in norm case,

$$Gal(\widetilde{E}(cg(m+i))/\widetilde{E}(cg(m))) = \frac{T_{v,m}^1}{T_{v,m+i}^1}.$$

Now apply the above relation 5.1, we're done. □

This norm relation can be used to get a norm compatible family (under ordinary conditions). There is a standard technique. For example, in Heegner point case, see [55] section 1.5. We will state the general form as a lemma.

**The general form**

In view of the norm relation 5.5, our special cycles $\{z_{cy(m)}\}$ are related by a polynomial

$$\widetilde{Pol} = \sum_i C_i A_i X^i \in R[K_v \backslash G_v / K_v][X]$$

with coefficients in the Hecke algebra $R[K_v \backslash G_v / K_v]$. We wish to produce a norm compatible family of special cycles which are linear combination of these cycles. To do so, we usually map $\widetilde{Pol}$ to a numerical polynomial (e.g. polynomial with complex coefficients) $Pol$ first, this can be done (assume $R \subset \mathbb{C}$) by taking an $R$-algebra homomorphism

$$R[K_v \backslash G_v / K_v] \longrightarrow \mathbb{C},$$

such specialization map is determined by the associated Satake parameters. Under such a map, we get $Pol \in \mathbb{C}[X]$ and suppose it has the following decomposition :

$$Pol = \sum_{i=0}^{k} e_i X^i = (X - b)(\sum_{i=0}^{k-1} p_i X^i),$$

where $e_k$, $p_{k-1}$, $b$ are nonzero complex numbers. It is possible to turn elements related by $Pol$ into a norm compatible family.

We state a simple analogue in the appendix A.2. The reader may also see that analogue to understand the key point.

Fix an embedding $\iota_p : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}_p}$. Suppose we have a polynomial $Pol \in \overline{\mathbb{Q}}[X]$ and denote the completion of its splitting field through $\iota_p$ by $PF$. Denote the integer ring of $PF$ by $\mathcal{O}$ with a uniformizer $\pi$ and suppose $Pol$ has a root $b \in \mathcal{O}^*$ (**ordinary condition**). Suppose it decomposes as

$$Pol = \sum_{i=0}^{k} e_i X^i = (X - b)(\sum_{i=0}^{k-1} p_i X^i).$$

Now suppose the Galois group $Gal_E$ acts on a $PF$-space $V_{La}$ continuously ($p$-adic Galois representation) with a Galois stable $\mathcal{O}$-lattice $La$. Consider another Galois lattice $La^{'} = \frac{1}{\pi^{ck}} La$, here $ck$ is a suitable integer such that $p_i(La) \subset La^{'}$ and $e_i(La) \subset La^{'}$. Now for simplicity we denote by $Tr_{l,r}$ the corestriction map in Galois cohomology from $Gal_{\widetilde{E}(cg(l))}$ to $Gal_{\widetilde{E}(cg(r))}$, then we have the following lemma :

**Lemma 5.6.** ***translation***

*Suppose that for each integer $m \gg 0$, we have an element $\mathcal{Y}_m \in H^1(Gal_{\widetilde{E}(cg(m))}, La)$ such that*

$$\sum_{i=0}^{k} e_i Tr_{m+i,m}(\mathcal{Y}_{m+i}) = 0$$

*in* $H^1(Gal_{\widetilde{E}(cg(m))}, La')$. *Then define*

$$\mathcal{X}_m = b^{-m} \sum_{i=0}^{k-1} p_i Tr_{m+i,m}(\mathcal{Y}_{m+i})$$

*(for* $m \gg 0$*) in the latter group. We have*

$$Tr_{m+1,m}(\mathcal{X}_{m+1}) = \mathcal{X}_m.$$

*Démonstration.* This is a routine check. Notice

$$Tr_{m+1,m}(b^{m+1}\mathcal{X}_{m+1}) - b(b^m \mathcal{X}_m) = \sum_{i=0}^{k-1} p_i Tr_{m+i+1,m}(\mathcal{Y}_{m+1+i}) - b\sum_{i=0}^{k-1} p_i Tr_{m+i,m}(\mathcal{Y}_{m+i})$$

$$= \sum_{i=0}^{k} e_i Tr_{m+i,m}(\mathcal{Y}_{m+i}) = 0.$$

Because $b \in \mathcal{O}^*$, we're done. □

Usually the main obstruction to produce norm compatible family is whether $Pol$ has a root $b$ that is invertible in $\mathcal{O}$. This is exactly our **ordinary condition** :

$$v_p(b) = 0.$$

Here $v_p$ is the normalized valuation of $\overline{\mathbb{Q}_p}$ such that $v_p(p) = 1$.

Now we show how these things, $V_{La}$, $Pol$ etc come naturally.

In practice, we need to construct the $p$-adic Galois representation $V_{La}$ with local Hecke action via scalars. This preparation can be done through considering $p$-adic cohomology (recall $p$ is the underlying prime of the place $v$) of the related Shimura varieties. See section 3 (in particular the localization method) for related knowledge.

Here we use our main example to show how to relate our special cycles to Galois cohomology classes in the lemma 5.6.

Again we use similar notations as in section 3. And for simplicity we ignore the problem of adding suitable Tate twist to cohomology.

Denote the ambient Shimura variety $Sh_{\mathbf{G}}(K)$ ($2n-1$ dimensional) as $X_1$ (over its reflex field $F$), and use $\overline{X_1}$ to denote the resulting variety over $\overline{\mathbb{Q}}$.

Pick up an isomorphism $\overline{\mathbb{Q}_p} \cong \mathbb{C}$, we have the following Matsushima's formula :

$$H^*(\overline{X_1}, \overline{\mathbb{Q}_p}) \cong \bigoplus_{\Pi = \Pi_f \otimes \Pi_\infty} m(\Pi)(\Pi_f^K \otimes H^*(\mathfrak{g}, K_\infty; \Pi_\infty)).$$

In particular, we chose a cuspidal representation $\Pi$ appearing in the middle cohomology

group $H^{2n-1}(\overline{X_1}, \overline{\mathbb{Q}_p})$ and the $\Pi_f$-component of the whole middle cohomology group is still a $p$-adic Galois representation. By the so called **strong C-arithmetic** property of $\Pi$, the defining field of $\Pi$ is a number field $E_0$. Take a finite place $\lambda$ of $E_0$ that lies above the prime $p$, and denote the corresponding valuation ring of $E_{0,\lambda}$ by $\mathcal{O}_\lambda$. Then we can consider the $p$-adic cohomology group $H^{2n-1}(\overline{X_1}, E_{0,\lambda})$ and our $p$-adic representation will be a suitable Galois-Hecke quotient of its $\Pi_f$-component :

$$H^{2n-1}(\overline{X_1}, E_{0,\lambda})[\Pi_f] \longrightarrow V_{La},$$

and the Galois lattice $La$ will the corresponding image inside $V_{La}$ of integral cohomology group $H^{2n-1}(\overline{X_1}, \mathcal{O}_\lambda)$. In general, the natural map $H^{2n-1}(\overline{X_1}, \mathcal{O}_\lambda) \longrightarrow H^{2n-1}(\overline{X_1}, E_{0,\lambda})$ may not be injective (torsion parts exists for integral cohomology group), this will influences nothing.

In particular, the representation $\Pi$ will determine an unramified representation of $G_v$, $\Pi_v$, and the local Hecke algebra acts on $V_{La}$ via scalars determined by $\Pi_v$,

$$\phi_{\Pi,\lambda,v} : \mathbb{Z}[K_v \backslash G_v / K_v] \longrightarrow E_{0,\lambda}.$$

Moreover, we usually have further integrality result, its image lies in the valuation ring $\mathcal{O}_\lambda$. Then let the coefficient ring $R = \mathcal{O}_\lambda$, we still denote the resulting map by $\phi_{\Pi,\lambda,v}$,

$$\phi_{\Pi,\lambda,v} : R[K_v \backslash G_v / K_v] \longrightarrow \mathcal{O}_\lambda.$$

Then we can take the second lattice $La^{'} = La$ due to such integrality.

Through the cycle map, our special cycle $z_{cy(m)}$ will be mapped into an element in the cohomology group $H^{2n}(X_{1,\widetilde{E}(cg(m))}, \mathcal{O}_\lambda)$. Therefore we need to consider some canonical methods to make cycles cohomologically trivial. See section 3. In particular, the localization at some Hecke ideal is very suitable for our such translation. Take that localization as example, it provides a natural map

$$H^{2n}(X_{1,\widetilde{E}(cg(m))}, \mathcal{O}_\lambda) \longrightarrow H^1(Gal_{\widetilde{E}(cg(m))}, H^{2n-1}(\overline{X_1}, \mathcal{O}_\lambda)/\ker(\phi_{\Pi,\lambda})).$$

Therefore, we can take $V_{La}$ as suitable Galois-Hecke quotient of $H^{2n-1}(\overline{X_1}, E_{0,\lambda}/\ker(\phi_{\Pi,\lambda}))$ and take $La$ as the corresponding lattice inside it. And we can use our special cycles to get Galois cohomology classes and apply the above lemma now.

**computation of the "universal" ordinary conditions**

Now we will figure out an explicit ordinary condition for our main example. And after this computation, we will see that there exists a "universal" ordinary condition on Satake parameters, such that this condition does not depend on the choice of the strict dominant cocharacter.

The original polynomial $\widetilde{Pol}$ only differs from the related Hecke polynomial by a

constant. More precisely, we have

$$\widetilde{Pol} = Hep_{\mu_v}(c(m,1)X) = Hep_{\mu_v}(C_1 X).$$

As explained above, we want to find a root of the polynomial $Pol$ ($Pol$ will be a suitable specialization of $\widetilde{Pol}$) which is a $p$-adic unit. The idea is to first relate $Pol$ with Weyl character formula (just differ by some normalization constants), and we will choose a "canonical" factor to get the ordinary condition.

Recall our notation, over the local field $F_v$, $\underline{\mathbf{G}}_{F_v}$ is split. The Galois action on the dual group $\widehat{\underline{\mathbf{G}}_{F_v}}$ is trivial and thus we can ignore it. Then the representation $V_{\mu_v}$ that is used to define the Hecke polynomial is just the highest weight representation of $\widehat{\underline{\mathbf{G}}_{F_v}} \cong Sp(2n)$ with highest weight $\mu_v$, here we think of $\mu_v$ as a character of the dual torus $\widehat{\underline{\mathbf{Ts}}_v}$.

A Satake parameter is an element in $\widehat{\underline{\mathbf{Ts}}_v}(\mathbb{C}) \cong (\mathbb{C}^*)^n$. Here we choose an isomorphism compatible with our previous pinning data. More precisely, take any Satake parameter $sa$ associated to the $n$-tuple $(st_1, ...st_n) \in \widehat{\underline{\mathbf{Ts}}_v}(\mathbb{C})$, under the natural pairing

$$X^*(\widehat{\underline{\mathbf{Ts}}_v}) \times \widehat{\underline{\mathbf{Ts}}_v}(\mathbb{C}) \longrightarrow \mathbb{C}^*,$$

we have

$$(\mu_v, sa) = \prod_{i=1}^{n} st_i^{s_i},$$

here recall that $\mu_v$ is associated to the $n$-tuple $(s_1, ..., s_n)$. We make a remark on the word "Satake parameter". In strict sense, a Satake parameters is an element in $\widehat{\underline{\mathbf{Ts}}_v}(\mathbb{C})/W(\widehat{\underline{\mathbf{G}}_{F_v}}, \widehat{\underline{\mathbf{Ts}}_v})$ (equivalent classes under the action of the Weyl group), here for simplicity we also use a Satake parameter to denote an element in $\widehat{\underline{\mathbf{Ts}}_v}(\mathbb{C})$ (its lift).

Let $p$ denote the prime of $\mathbb{Q}$ that underlies the place $v$ of $F$. Fix an isomorphism $\overline{\mathbb{Q}_p} \cong \mathbb{C}$ to calculate $p$-adic valuations of the Satake parameters etc. We assume that the valuation $v_p$ is normalized so that $v_p(p) = 1$ and set $k_v = v_p(q_{F_v})$.

Denote the Satake map $He(\underline{\mathbf{G}}_{F_v}) \longrightarrow He(\underline{\mathbf{Ts}}_v)$ as $S$, here both sides are complex-valued Hecke algebras. Denote the unramified principal series associated to the Satake parameter $sa$ as $\pi_{sa}$. Then for any element $f$ in the Hecke algebra, we have

$$(Sf, sa) = (f, \pi_{sa}^{K_v}).$$

Here the pairing on the left side is determined by the above natural pairing, the pairing on the right side denotes the action of the Hecke algebra on the one-dimensional space $\pi_{sa}^{K_v}$.

Therefore, through this equality, we can first apply Satake map to translate our Hecke polynomial $Hep_{\mu_v}$ as an element in $He(\underline{\mathbf{Ts}}_v)[X]$, then evaluate it with $sa$ to get a polynomial in $\mathbb{C}[X]$. However, recall our definition of Hecke polynomial, after such Satake map,

it is exactly

$$\prod_{\lambda}(X - q_{F_v}^{\langle \mu_v, \rho_v \rangle}\lambda)^{m(\lambda)}.$$

Here $q_{F_v}^{\langle \mu_v, \rho_v \rangle}$ is the normalization factor, $\lambda$ runs over the weights of $\widehat{\underline{\mathbf{Ts}}_v}$ in the highest weight representation $V_{\mu_v}$ and $m(\lambda)$ is the corresponding multiplicity. Thus we can write down this polynomial with the help of Weyl character formula. In fact, we don't need to write down all factors because one root is enough. Obviously, this polynomial will have a "canonical" factor

$$X - q_{F_v}^{\langle \mu_v, \rho_v \rangle}\mu_v$$

(the weight $\mu_v$ will always appear with multiplicity one).

Combining all things together, the polynomial $Pol$ will have a factor

$$C_1 X - q_{F_v}^{\langle \mu_v, \rho_v \rangle}\prod_{i=1}^{n} st_i^{s_i}.$$

This linear factor will contribute to a root $b$ for $Pol$. Denote the evaluation

$$v_p(sa) = (v_p(st_1), ..., v_p(st_n)).$$

Then the **ordinary condition**

$$v_p(b) = 0$$

is exactly

$$v_p(C_1) = \langle \mu_v, v_p(sa) \rangle + k_v \langle \mu_v, \rho_v \rangle.$$

Now recall the formula

$$C_1 = q_{F_v}^{(\langle \mu_v, 2\rho_v \rangle - s_1)} = q_{F_v}^{(\sum_{j=1}^{n}(2n-2j+1)s_{n+1-j} - s_1)},$$

we thus obtain

$$\langle \mu_v, v_p(sa) \rangle = \langle \mu_v, k_v\rho_v - k_v(1, 0, ..., 0) \rangle.$$

In particular, we get a "universal" ordinary condition, which holds for all $\mu_v$, namely :

$$v_p(sa) = k_v(\rho_v - (1, 0, ..., 0)) = \begin{cases} 0, & \text{if } n = 1, \\ k_v(0, 3, ..., 2n-1), & \text{if } n > 1. \end{cases}$$

Finally we make some general remarks.

The above computation of ordinary condition doesn't require any special properties of orthogonal groups etc, the only requirement is the splitting condition. Thus this arguments holds for any split cases. In particular, we can work out ordinary conditions similarly in other examples (such as examples in section 6.1 and section 6.2).

It is also possible to express the ordinary condition through some Hecke operators. For example, see the assumption 9.1 and remark 9.2 in [28]. Their ordinary condition is stated via some Hecke operators and their remark explains the connection between those Hecke operators and Satake parameters. This relations relies on the work [56], which holds in very general situations (including for all split reductive groups). Such translation is off the topic of this thesis, therefore we leave this work for interested readers.

# Chapitre 6

# Other examples and arithmetic applications

## 6.1 $GSp_4$ example

We will consider the following embedding :

$$\mathbf{H} = GU(1) \times_{G_m} GL_2 \longrightarrow \mathbf{G} = GSp_4$$

(where $GU(1) = \mathrm{Res}_{E/\mathbb{Q}}\mathbb{G}_{m,E}$ and $E$ is an imaginary quadratic field).

Consider the 4-dim $\mathbb{Q}$-space $V$ with a symplectic form $J$, we choose a suitable ordered basis $(e_1, e_2, e_3, e_4)$ for $V$ so that $J$ corresponds to the matrix

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}.$$

For simplicity, we will use $E = \mathbb{Q}(\sqrt{-1})$ as an example, the general case is similar. Then $GU(1)$ can be embedded into $GL_2$, sending $a + b\sqrt{-1}$ to the matrix $\left(\begin{smallmatrix} a & b \\ -b & a \end{smallmatrix}\right)$.

Now we can embed $\mathbf{H}$ into $\mathbf{G}$, it will correspond to the following matrices (where $a^2 + b^2 = xw - zy$) inside $GSp_4$ :

$$\begin{pmatrix} a & 0 & 0 & b \\ 0 & x & y & 0 \\ 0 & z & w & 0 \\ -b & 0 & 0 & a \end{pmatrix}.$$

In fact, this group embedding has a factorization $\mathbf{H} \longrightarrow \mathbf{M} = GL_2 \times_{\mathbb{G}_m} GL_2 \longrightarrow \mathbf{G}$. The embedding $\mathbf{M} \longrightarrow \mathbf{G}$ is obtained by replacing $\left(\begin{smallmatrix} a & b \\ -b & a \end{smallmatrix}\right)$ with $\left(\begin{smallmatrix} l & p \\ q & r \end{smallmatrix}\right)$ in the above expression.

These group maps will induce maps between Shimura varieties. Recall the torus $GU(1)$ defines a Shimura variety, the map from Deligne torus $\mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m(\mathbb{R})$ to $GU(1)(\mathbb{R})$ is an identification, sending $x+y\sqrt{-1}$ to $x+y\sqrt{-1}$ $(x, y \in \mathbb{R})$. And the inclusion $GU(1) \longrightarrow GL_2$ will induce a map between their Shimura varieties. This is the usual map defining a CM point inside the modular curve. Similarly, we have maps $Sh_{\mathbf{H}} \longrightarrow Sh_{\mathbf{M}} \longrightarrow Sh_{\mathbf{G}}$.

Loeffler, Skinner and Zerbes have used the embedding $GL_2 \times_{\mathbb{G}_m} GL_2 \longrightarrow GSp_4$ to construct an Euler system (see [47]). The first embedding $\mathbf{H} \longrightarrow \mathbf{M}$ can be seen as a kind of "base change" of Heegner points. Roughly speaking, we can think of Shimura varieties for$Sh_{\mathbf{H}}$ as a family of Heegner points or product of Heegner points with the modular curve.

What's more, for $E = \mathbb{Q}(i)$, this pair can be seen as a kind of lift for our pair $(U(1,1), SO(3,2))$. Consider the standard representation $V$ for $GSp_4$ and let $\rho$ be the induced representation of $GSp_4$ on the 6-dimensional space $\wedge^2 V$. Twist $\rho$ by $v^{-1}$, where $v$ is the standard 1-dimensional character defining $GSp_4$, $J(g(x), g(y)) = v(g)J(x, y)$. There is a quadratic form on $\wedge^2 V$ defined by the wedge product

$$\wedge^2 V \times \wedge^2 V \longrightarrow \wedge^4 V \cong \mathbb{Q}.$$

Then the representation $\rho v^{-1}$ is an orthogonal representation for $GSp_4$. And it fixes the line generated by $e_1 \wedge e_4 + e_2 \wedge e_3$. Then the orthogonal complement $L$ for this line will provide a 5-dimensional orthogonal representation of $GSp_4$. In this way, we obtain a morphism $GSp_4 \longrightarrow SO(3,2)$ whose kernel is the center of $GSp_4$, $Z(GSp_4) = \mathbb{G}_m$.

Then we need to construct a 2-dimensional $E$-hermitian space inside this 5-dimensional quadratic $\mathbb{Q}$-space. This is equivalent to define a "complex" structure (linear map $S$ with $S^2 = -1$) compatible with the quadratic form ($S$ is orthogonal) on a suitable 4-dimensional subspace $W$. We choose $W$ to be the space generated by $\{e_1 \wedge e_2, e_2 \wedge e_4, e_3 \wedge e_4, e_1 \wedge e_3\}$.

Consider the following central element $S$ of $\mathbf{H} \subset \mathbf{G}$ :

$$\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Then $S$ also acts on $\wedge^2 V$ and we can check that $S$ stabilizes $W$ and induces the desired hermitian structure.

Consider the conjugation of $S$ on $GSp_4$ (this is a four-order automorphism), its fixed subgroup is exactly $\mathbf{H} = GU(1) \times_{\mathbb{G}_m} GL_2$. Moreover, here is an additional interesting fact. The fixed subgroup for conjugation by $S^2$ is exactly $\mathbf{M} = GL_2 \times_{\mathbb{G}_m} GL_2$. Therefore we get two symmetric pairs, $\mathbf{M} \longrightarrow \mathbf{G}$ and $\mathbf{H} \longrightarrow \mathbf{M}$. Because $\mathbf{H}$ commutes with $S$, its action on $\wedge^2 V$ also commutes with $S$. Then $W$ is an $\mathbf{H}$ representation, and we get the map $\mathbf{H} \longrightarrow U(W)$. This is a surjection with kernel $\mathbb{G}_m$. We have the following commutative

diagram :

$$\begin{array}{ccc} \mathbf{H} = GU(1) \times_{\mathbb{G}_m} GL_2 & \longrightarrow & \mathbf{G} = GSp_4 \\ \downarrow & & \downarrow \\ U(1,1) & \longrightarrow & SO(3,2) \end{array}$$

Moreover, it induces maps between Shimura datum, thus we can think of $Sh_{\mathbf{H}} \longrightarrow Sh_{\mathbf{G}}$ as a kind of lift for $Sh_{U(1,1)} \longrightarrow Sh_{SO(3,2)}$. This lift allows us to replace the abelian type Shimura variety for $SO(3,2)$ by the much more familiar Siegel threefold, and thus opens the way to a study of these special cycles using PEL type moduli spaces.

Now we study this pair in the framework of this paper.

We first compute the stabilizer involved in the parametrization lemma 2.1. It is well known that the associated Hermitian symmetric domain for the Siegel modular variety is the Siegel half-space. More precisely, we can rearrange the chosen ordered basis $(e_1, e_2, e_3, e_4)$ into $(e_1, e_2, e_4, e_3)$ so that the symplectic form $J$ is $\left(\begin{smallmatrix} 0 & I_2 \\ -I_2 & 0 \end{smallmatrix}\right)$. Then $GSp_4(\mathbb{R})$ will act on the Siegel half-space

$$\mathbb{H}_2 = \{Z \in M_2(\mathbb{C}), Z^T = Z, Im(Z) > 0\},$$

the matrix $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ sends $Z$ into $(AZ+B)(CZ+D)^{-1}$. This is the hermitian symmetric domain $X$ for $Sh(\mathbf{G})$. The hermitian symmetric domain $Y$ for $Sh(\mathbf{H})$ is the positive half plane, it is already connected (unlike in the $GL_2$ case). Inside $X$, $Y$ will correspond to $\left(\begin{smallmatrix} \sqrt{-1} & 0 \\ 0 & z \end{smallmatrix}\right)$, where $z = a + b\sqrt{-1}(b > 0)$ lies in the positive half plane. A direct argument shows that $Stab_{\mathbf{G}(\mathbb{Q})}(Y) = \mathbf{H}(\mathbb{Q})$. Choosing a suitable neat level group $K \subset \mathbf{G}(\mathbb{A}_f)$, we obtain a similar parametrization :

$$Z_K(\mathbf{G}, \mathbf{H}) = \mathbf{H}(\mathbb{Q}) \backslash \mathbf{G}(\mathbb{A}_f)/K.$$

Now let's consider the reciprocity law for $\pi_0(Sh_{\mathbf{H}})$. The derived subgroup $\mathbf{H}^{der} = SL_2$ is simply connected, and the abelian quotient map is the projection to the first factor

$$pr : \mathbf{H} = GU(1) \times_{\mathbb{G}_m} GL_2 \longrightarrow \mathbf{H}^{ab} = GU(1).$$

Then $E$ is the common reflex field for $Sh_{\mathbf{H}}$ and $Sh_{\mathbf{H}^{ab}}$ and the reciprocity law for $Sh_{\mathbf{H}^{ab}}$ is the identity map,

$$\mathrm{Res}_{E/\mathbb{Q}}\mathbb{G}_m \xrightarrow{=} GU(1).$$

As in our main example, we also have $\overline{\mathbf{H}(\mathbb{Q})} = \mathbf{H}(\mathbb{Q})\mathbf{H}^{der}(\mathbb{A}_f)$ inside $\mathbf{H}(\mathbb{A}_f)$. Here we use a special property about imaginary quadratic field, $\mathbf{H}^{ab}(\mathbb{Q})$ is closed (in fact discrete) in $\mathbf{H}^{ab}(\mathbb{A}_f)$, this is not true for more general CM fields. And we also have the following

natural map induced by $pr$ :

$$\pi_0(Sh_{\mathbf{H}}(K_1)) \cong \pi_0(Sh_{\mathbf{H}^{ab}}(pr(K_1))).$$

Now everything is similar to our main example, we have the Hecke action on the right, and the Galois action through left multiplication by $\mathbf{H}(\mathbb{A}_f)$ via reciprocity law. And we can rewrite the parametrization as follows :

$$Z_K(\mathbf{G},\mathbf{H}) = \mathbf{H}(\mathbb{Q})\mathbf{H}^{der}(\mathbb{A}_f)\backslash\mathbf{G}(\mathbb{A}_f)/K.$$

Choose a special cycle $\widetilde{z} = Z_K(g)$ and take a finite subset $S = S(K,z)$ ("bad primes") of primes of $\mathbb{Q}$ large enough such that

- $S$ contains $\{2\}$.
- For any prime $q \notin S$, $\mathbb{Q} \longrightarrow E$, $\mathbf{H}$, $\mathbf{G}$ are all unramified at $q$.
- $K = K_S \times K^S$ with $K^S = \prod_{q\notin S} K_q$, $K_S$ is a compact open subgroup of $\mathbf{G}(\mathbb{A}_{f,S})$ and $K_q$ is a compact open subgroup of $\mathbf{G}(\mathbb{Q}_q)$ for any $q \notin S$. Moreover, $g = \otimes_t g_t$ with $g_q \in K_q$ for any $q \notin S$.
- For any $q \notin S$, $K_q$ (resp $K_q \bigcap \mathbf{H}(\mathbb{Q}_q)$) is hyperspecial in $\mathbf{G}(\mathbb{Q}_q)$ (resp $\mathbf{H}(\mathbb{Q}_q)$).

Then we also have the following natural map :

$$\mathbb{Z}[\mathbf{H}^{der}(\mathbb{A}_{f,S})\backslash\mathbf{G}(\mathbb{A}_{f,S})/K_S] \bigotimes \bigotimes_{p\notin S}{}' \mathbb{Z}[\mathbf{H}^{der}(\mathbb{Q}_p)\backslash\mathbf{G}(\mathbb{Q}_p)/K_p] \longrightarrow \mathbb{Z}[Z_K(\mathbf{G},\mathbf{H})].$$

Therefore we can apply our general strategy to this pair now.

We first introduce a one dimensional torus quotient of $\mathbf{H}^{ab}$ to define conductor filtrations. Define $\mathbf{T}^1 = U(1) = \ker(GU(1) \overset{Norm}{\longrightarrow} \mathbb{G}_{m,\mathbb{Q}})$ and consider the map $r : GU(1) \longrightarrow \mathbf{T}^1$ over $\mathbb{Q}$ by sending $z$ to $\frac{z}{\bar{z}}$. Then for tame relations, over a split prime $p$, we will use this map $r(pr) : \mathbf{H} \longrightarrow \mathbf{T}^1$ to define conductor filtrations and apply theorem 4.4 to get abstract tame relations. Following the arguments in section 4.2 (realization of tame relations), we can produce special cycles for $Sh_{\mathbf{G}}(K)$ with the desired tame relations.

For norm relations, the first task is to construct a suitable Borel subgroup to establish the spherical condition for this pair. Suppose the base local field is $\mathbb{Q}_p$ with $p \notin S$.

Here we can work over the global field $\mathbb{Q}$ directly in fact, the construction is uniform.

Recall we have chosen a suitable ordered basis $(e_1, e_2, e_3, e_4)$ so that J corresponds to

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}.$$

Then consider a new ordered basis $(\gamma_4, \gamma_3, \gamma_2, \gamma_1) = (e_1 + e_2, e_3 - e_4, e_1 - e_2, e_3 + e_4)$. This

ordered basis defines a Borel pair $(\mathbf{T}, \overline{\mathbf{B}})$ for $\mathbf{G}$ over $\mathbb{Q}$. And we will use $\mathbf{B}$ to denote the Borel subgroup of $\mathbf{G}$ opposite to $\overline{\mathbf{B}}$ with respect to $\mathbf{T}$. We have $\mathbf{H} \bigcap \overline{\mathbf{B}} = \mathbb{G}_{m,\mathbb{Q}} = Z(\mathbf{G})$. For all but finitely many primes $p$ for $\mathbb{Q}$, we can extend these groups over $\mathrm{Spec}(\mathbb{Z}_p)$, and denote $(\mathcal{H}_p, \mathcal{G}_p, \overline{\mathcal{B}}_p, \mathcal{T}_p, \mathcal{B}_p)$ for the integral models of $(\mathbf{H}_{\mathbb{Q}_p}, \mathbf{G}_{\mathbb{Q}_p}, \overline{\mathbf{B}}_{\mathbb{Q}_p}, \mathbf{T}_{\mathbb{Q}_p}, \mathbf{B}_{Q_p})$. Then $(\mathcal{H}_p, \mathcal{G}_p)$ is a spherical pair (with respect to $\overline{\mathcal{B}}_p$) by dimension reason.

Recall our torus $\mathbf{T}^1 = U(1)$ over $\mathbb{Q}$, we will still use it to define conductors for norm relations. Take a good unramified odd prime $p$ so that we have a reductive integral model $\mathcal{T}_p^1$ for $\mathbf{T}^1_{Q_p}$ and extend the quotient maps $\mathbf{H} \xrightarrow{pr} GU(1) \xrightarrow{r} \mathbf{T}^1$ over $\mathrm{Spec}(\mathbb{Z}_p)$ (still denote these maps by $pr, r$). From now on, we will work over $\mathbb{Q}_p$. We will do some explicit matrix calculations first.

Under this ordered basis $(\gamma_1, \gamma_2, \gamma_3, \gamma_4)$, the Borel subgroup $\mathbf{B}_p$ will correspond to upper triangular matrices. Take a strictly $\mathcal{B}_p$-dominant cocharacter $\mu = (a_1, a_2, a_3, a_4) \in X_*^+(\mathcal{T}_p)$(where $a_1 > a_2 > a_3 > a_4$ and $a_1 + a_4 = a_2 + a_3$). Take $\tau = \mu(\pi)$, where $\pi$ is a uniformizer for $\mathbb{Q}_p$, for example $\pi = p$). The elements of $\mathbf{H}(\mathbb{Q}_p)$ given by

$$\begin{pmatrix} a & 0 & 0 & b \\ 0 & x & y & 0 \\ 0 & z & w & 0 \\ -b & 0 & 0 & a \end{pmatrix}$$

under the ordered basis $(e_1, e_2, e_3, e_4)$ correspond to the elements of $\mathbf{G}(\mathbb{Q}_p)$ given by

$$\frac{1}{2}\begin{pmatrix} w+a & -b-z & w-a & z-b \\ b-y & a+x & -y-b & a-x \\ w-a & b-z & w+a & b+z \\ b+y & a-x & y-b & a+x \end{pmatrix}$$

under the ordered basis $(\gamma_1, \gamma_2, \gamma_3, \gamma_4)$. Then it is easy to see that $\mathbf{H}(\mathbb{Q}_p) \bigcap \tau^m \mathcal{G}_p(\mathbb{Z}_p)\tau^{-m}$ lies in $\mathcal{H}_p(\mathbb{Z}_p)$, in other words, the stabilizer conjecture holds. Then for any non-negative integer $m$, we define $H_{p,m} = \mathbf{H}(\mathbb{Q}_p) \bigcap \tau^m \mathcal{G}_p(\mathbb{Z}_p)\tau^{-m}$ as in section 5.1. The next task is to compute the numbers $c(m, i)$ involved in the abstract norm relations (theorem 5.4).

Consider the conductor filtration on $\mathcal{T}_p^1(\mathbb{Z}_p)$. As usual, $T_p^1(m) = \{\frac{z}{\bar{z}}, z \in \mathbb{Z}_p + \pi^m \mathcal{O}_{E_p}\}$. And we similarly define conductors for $H_{p,m}$, $con(m)$ is defined to be the minimal non-negative integer $cc$ with $r(pr)(H_{p,m}) \supset T_p^1(cc)$.

We first make a formal argument. Because the intersection $\mathbf{H} \bigcap \overline{\mathbf{B}} = \mathbb{G}_m$ lies in the kernel of $r(pr)$, the conductors $con(m)$ grows to infinity, just like in our main example. Thus condition $\heartsuit$ (in section 5.2) is satisfied.

To check condition $\clubsuit$ (in section 5.2), we will explicit these numbers $c(m, i)$. As in our main example, this involves computing conductors.

For any positive integer $m$, consider the elements in $H_{p,m}$, through their matrices

under the ordered basis $(\gamma_1, \gamma_2, \gamma_3, \gamma_4)$. We find $v_p(a) = 0$ and $v_p(b) \geq m(a_1 - a_2)$. Thus $pr(H_{p,m}) \subset W_m = \mathbb{Z}_p^* + \sqrt{-1}\pi^{m(a_1-a_2)}\mathbb{Z}_p$. Let us now see that this is actually a surjection, i.e. $pr(H_{p,m}) = W_m$.

For any $a \in \mathbb{Z}_p^*$ and $b \in p^m\mathbb{Z}_p$, consider this element $x_{a,b} \in \mathbf{H}(\mathbb{Q}_p)$ given by

$$\begin{pmatrix} a & 0 & 0 & b \\ 0 & a & -b & 0 \\ 0 & b & a & 0 \\ -b & 0 & 0 & a \end{pmatrix}$$

under the ordered basis $(e_1, e_2, e_3, e_4)$. Computing $x_{a,b}$ in the ordered basis $(\gamma_1, \gamma_2, \gamma_3, \gamma_4)$, we find that $x_{a,b} \in H_{p,m}$. Notice that $pr(x_{a,b}) = a + b\sqrt{-1}$, thus $pr(H_{p,m}) = W_m$.

Then the conductor $con(m)$ equals $m(a_1 - a_2)$ and condition ♣ holds. And we have the following explicit formula for $c(m, i)$ :

$$c(m, i) = p^{i(2a_1+a_2-2a_3-a_4)}.$$

Thus we can apply the same idea as in section 5.2 to construct special cycles with norm relations along the anticyclotomic extension.

The explicit computation of ordinary condition is similar to our main example and there also exists a "universal" ordinary condition (independent of the choice of the strict dominant cocharacter).

Fix an isomorphism $\overline{\mathbb{Q}_p} \cong \mathbb{C}$ and use the normalized valuation $v_p$ so that $v_p(p) = 1$. Suppose the Satake parameter $sa \in \widehat{\mathbf{T}_{\mathbb{Q}_p}}(\mathbb{C})$ is associated to the tuple $(s_1, s_2, s_3, s_4) \in (\mathbb{C}^*)^4$ with $s_1s_2 = s_3s_4$, here we use the fact that the dual group $\widehat{GSp_4}$ is isomorphic to $GSp_4$ and we embed it into $GL_4$ so that we can view the dual torus $\widehat{\mathbf{T}_{\mathbb{Q}_p}}$ as a subtorus of $\mathbb{G}_m^4$ (diagonal torus in $GL_4$). Denote the $p$-adic valuation as $v_p(sa) = (vs_1, vs_2, vs_3, vs_4)$. Then the universal ordinary condition is :

$$(vs_1, vs_2, vs_3, vs_4) = (\frac{1}{2}, 1, -1, -\frac{1}{2}).$$

Finally we make a remark. We have another "dual" version embedding. We can swap factors to get $\overline{\mathbf{H}} = GL_2 \times_{\mathbb{G}_m} GU(1) \longrightarrow GSp_4$. Everything is similar. Moreover, the resulting family of special cycles is the same in fact. These two embedding are conjugated by an element $g \in GSp_4(\mathbb{Q})$, where $g$ is

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Therefore $Z_K(\mathbf{G},\mathbf{H})$ is the same as $Z_K(\mathbf{G},\overline{\mathbf{H}})$. But we can't do such automorphic translation inside $Sh(\mathbf{M})$ ($g$ doesn't belong to $\mathbf{M}(\mathbb{Q})$). In this viewpoint, $Sh(GSp_4)$ is a better ambient Shimura variety and we can expect some interesting results about these special cycles.

## 6.2 Unitary GGP pair and some variants

The usual unitary GGP pair $U(n) \longrightarrow U(n+1) \times U(n)$ has already been studied by Boumasmoud in his thesis (especially tame relations, see [9]). Therefore we won't repeat details and refer to his thesis.

The construction of embedding is standard. And the unitary group is similar to our unitary subgroup :

Fix a totally real field $F$, and an imaginary quadratic extension $F \longrightarrow E$. Fix a real place $f$ of $F$. For a positive integer $n$, let $W$ denote an $n$-dimensional $E$-hermitian space with quadratic form $\psi$ and consider the $n+1$-dimensional $E$-hermitian space $V{=}W \perp Ee_{n+1}$ and denote its quadratic form as $\phi$. Similar to our main example, we define $\underline{\mathbf{H}} = U(W,\psi)$, $\mathbf{H} = \mathrm{Res}_{F/\mathbb{Q}}\underline{\mathbf{H}}$. Define $(W_a,\psi_a) = (W,\psi)\otimes_{F,a}\mathbb{R}$ ($a$ is any real place of $F$). Then we put the following signature condition, notice that $\mathbf{H}_{\mathbb{R}} = \prod_a \mathbf{H}_a$ with $\mathbf{H}_a = U(W_a,\psi_a)$, these factors are as follow :

$$U(W_a,\psi_a) = \begin{cases} U(n-1,1), & \text{if } a = f, \\ U(2n+1), & \text{if } a \neq f. \end{cases}$$

We also require the signature for $Ee_{n+1}$ to be (1,0) at any real place of $F$.

Now we have a natural embedding $U(W) \longrightarrow U(V)\times U(W)$ (the second factor map is identity). Taking Weil restriction, we get $\mathbf{H} = \mathrm{Res}_{F/\mathbb{Q}}U(W) \longrightarrow \mathbf{G} = \mathrm{Res}_{F/\mathbb{Q}}(U(V)\times U(W))$. The construction of Hermitian symmetric domain is similar to our main example. And the inclusion will induce a morphism between Shimura datum $Sh(\mathbf{H}) \longrightarrow Sh(\mathbf{G})$. Then we can apply our general method.

The parametrization (here the involved stabilizer is $\mathbf{N_G}(\mathbf{H})(\mathbb{Q})$, larger than $\mathbf{H}(\mathbb{Q})$) and reciprocity law for geometric connected components can be found in Boumasmoud's thesis [9]. Over a split prime, similarly to our main example, we can reduce to the split reductive case, which corresponds to $GL_n \longrightarrow GL_{n+1}\times GL_n$. We just point out that the split assumption is enough to apply the general argument via root groups. We don't need to analyse $H\backslash G/K$ explicitly as in Boumasmoud's thesis.

For norm relations, the first thing is still to construct a suitable Borel subgroup. We will work over a split prime $v$ of $F$ and the involved embedding becomes $(H_v,G_v) = (GL_n, GL_{n+1}\times GL_n)$, after fixing isomorphisms $E\otimes_F F_v \cong F_v\times F_{\overline{v}}, U(W_{F_v}) \cong GL(W_{F_v}), U(V_{F_v}) \cong GL(V_{F_v})$ etc like in our main example (see section 4.2 for these isomorphisms). Here $H_v$

(resp. $G_v$) is the $v$-component of $\mathbf{H}_{\mathbb{Q}_p}$ ($p$ is the underlying prime) (resp. $\mathbf{G}_{\mathbb{Q}_p}$).

Choose an ordered basis $(e_1, ..., e_n)$ for $W_{F_v}$ and $(v_1, ..., v_n, v_{n+1})$ ($v_i = e_i, 1 \leq i \leq n$) for $V_{F_v}$. Then the embedding $GL_n \longrightarrow GL_{n+1} \times GL_n$ is given by sending the matrix $A$ to $\begin{pmatrix} A & \\ & 1 \end{pmatrix} \times A$.

The ordered basis $(e_n, ..., e_1)$ for $W_{F_v}$ defines a Borel pair $(T_2, B_2)$ for $GL(n)$ and we use $\overline{B_2}$ to denote the Borel subgroup of $GL(n)$ that is opposed to $B_2$ with respect to $T_2$. Consider the ordered basis $(v_1, ...v_n, v_1 + v_2 + ... + v_{n+1})$ for $V_{F_v}$, similarly we get a Borel pair $(T_1, B_1)$ for $GL(n+1)$ and the opposite Borel subgroup $\overline{B_1}$.

Consider the Borel pair $(T, B)$=$(T_1 \times T_2,\ B_1 \times B_2)$ for $GL(n+1) \times GL(n)$ and the opposite Borel subgroup $\overline{B} = \overline{B_1} \times \overline{B_2}$. Then the subgroup $H_v$ has trivial intersection with $\overline{B}$. Extending them to integral level, we can verify the spherical condition for $(\mathcal{H}_v, \mathcal{G}_v)$ (integral models of $(H_v, G_v)$) by a dimension argument.

As in the $GSp_4$ example, we will use explicit matrix computations to calculate conductors.

Set the vector $ve = (1, ..., 1)^T$($n$ elements), the elements of $H_v(F_v)$ given by the matrix $A$ under the ordered basis $(e_1, .., e_n)$ for $W_{F_v}$ corresponds to the elements of $G_v(F_v)$ given by

$$\begin{pmatrix} A & (A - I_n)ve \\ 0 & 1 \end{pmatrix} \times A$$

under the ordered basis $(v_1, ..., v_n, v_1 + ... + v_{n+1}); (e_1, ..., e_n)$ for $V_{F_v} \times W_{F_v}$.

Take a strictly $B_1$-dominant cocharacter $\mu_1 = (b_1, ..., b_{n+1}) \in X_*(T_1)$ and a strictly $B_2$-dominant cocharacter $\mu_2 = (a_n, ..., a_1) \in X_*(T_2)$, where $b_1 > b_2 > ... > b_{n+1}$ and $a_1 < a_2 < ... < a_n$. Then $\mu = (\mu_1, \mu_2) \in X_*(T_1 \times T_2)$ is a strictly $B$-dominant cocharacter.

Set $\tau = \mu(\pi)$ (again $\pi$ is a uniformizer of $F_v$), then the stabilizer conjecture holds by the following direct argument :

For any non-negative integer $m$, define the subgroup $H_{v,m} = H_v(F_v) \bigcap \tau^m \mathcal{G}_v(\mathcal{O}_{F_v})\tau^{-m}$. The stabilizer conjecture says that $H_{v,m} \subset \mathcal{H}_v(\mathcal{O}_{F_v})$. The case $m = 0$ is trivial, thus we only need to deal with $m > 0$.

Now for any element $x \in H_{v,m}$, suppose it corresponds to a matrix $A$ under the ordered basis $(e_1, .., e_n)$ for $W_{F_v}$. Let $x_{i,j}$ denote the $(i, j)$ entry of the matrix $A$. By the above computation about changing basis and $x \in \tau^m \mathcal{G}_v(\mathcal{O}_{F_v})\tau^{-m}$, we get

$$v_{F_v}(x_{i,j}) \geq 0$$

for $i \neq j$. And because each entry of the vector $(A - I_n)ve$ also belongs to $\mathcal{O}_{F_v}$, we get

$$v_{F_v}(\sum_{j=1}^{n} x_{i,j}) \geq 0$$

for any $1 \leq i \leq n$, therefore each entry $x_{i,j} \in \mathcal{O}_{F_v}$, $A$ is an integral matrix, $x \in \mathcal{H}_v(\mathcal{O}_{F_v})$.

As in our main example, we have a formal argument for condition ♡ (section 5.2), the trivial intersection property implies that the conductor grows to infinity. Now we explicit the numbers $c(m,i)$ involved in the abstract norm relation (theorem 5.4).

The main task is still computing conductors of $H_{v,m}$. Suppose $m > 0$. Set $w = \min\{a_{i+1} - a_i, b_j - b_{j+1}\}$. Then we have $\det(H_{v,m}) = 1 + \pi^{mw}\mathcal{O}_{F_v}$, so the conductor is exactly $mw$ and the condition ♣ is also satisfied. Denote by $q$ the cardinality of the residue field. Then the explicit formula for $c(m,i)$ is

$$c(m,i) = q^{i(\sum_{k=1}^{n+1}(n+2-2k)b_k + \sum_{k=1}^{n}(n+1-2k)a_{n+1-k} - w)}.$$

Then we can apply the general method in section 5.2 to construct special cycles with norm relations.

Here is the detailed computation for the conductor :

First, we will bound the conductor $\det(H_{v,m}) \subset 1 + \pi^{mw}\mathcal{O}_{F_v}$.

Denote the matrix $A \in H_{v,m}$ by

$$\begin{pmatrix} x_{1,1} & \dots & x_{1,n} \\ \dots & \dots & \dots \\ x_{n,1} & \dots & x_{n,n} \end{pmatrix}.$$

By the definition of $H_{v,m}$, we get

$$v_{F_v}(x_{i,j}) \geq \max\{m(a_i - a_j), m(b_i - b_j), 0\}.$$

Thus under the reduction map $red : GL_n(\mathcal{O}_{F_v}) \longrightarrow GL_n(\mathcal{O}_{F_v}/\pi)$, $red(A)$ is a diagonal matrix (lies in the diagonal torus). Therefore $A$ lies in the "Big Bruhat cell" for $GL_n(\mathcal{O}_{F_v})$. Inspired by this fact, we divide $\det(A) - 1$ into two parts

$$\det(A) - 1 = Pa1 + Pa2,$$

where

$$Pa1 = \prod_{i=1}^{n} x_{i,i} - 1, Pa2 = \sum_{\lambda \in S_n, \lambda \neq Id} sgn(\lambda) \prod_{i=1}^{n} x_{i,\lambda(i)}.$$

For the second part, apply the fact $v_{F_v}(x_{i,j}) \geq \max\{m(a_i - a_j), m(b_i - b_j), 0\}$, we get $v_{F_v}(Pa2) \geq mw$.

For the first part, use the definition of $H_{v,m}$ again, we get

$$v_{F_v}(x_{i,i} - 1 + \sum_{j \neq i} x_{i,j}) \geq m(b_{n+1} - b_i).$$

Therefore we rewrite

$$Pa1 = \prod_{i=1}^{n} x_{i,i} - 1 = \prod_{i=1}^{n}(x_{i,i} - 1 + 1) - 1.$$

Then we get $v_{F_v} \geq mw$. Combine them together, $v_{F_v}(det(A) - 1) \geq mw$.

Second, we next establish the equality by constructing suitable elements.

To do this, we will specify "regions" for $w$. For any $1 \leq i \leq n-1$, we call the set of parameters $\{(b_1, ..., b_{n+1}); (a_1, ..., a_n) | w = a_{i+1} - a_i\}$ a "region". Similarly for any $1 \leq j \leq n$, we also have a "region" for $w = b_j - b_{j+1}$. Different "regions" are not disjoint, but any parameter will lie in some "regions".

For each $1 \leq i \leq n-1$ and any parameter in the "region" for "$w = a_{i+1} - a_i$", and any $y_{i+1,i} \in \pi^{mw}\mathcal{O}_{F_v}$, we can define the following matrix $A$ : set $x_{i+1,i} = -y_{i+1,i}$, $x_{i+1,i+1} = 1 + y_{i+1,i}$, $x_{dia,dia} = 1$ (here $1 \leq dia \leq n$, $dia \neq i+1$) and set other entries $x_{l,r}$ to be 0 :

$$\begin{pmatrix} 1 & 0 & ... & ... & 0 \\ ... & ... & ... & ... & ... \\ 0 & ... & -y_{i+1,i} & 1 + y_{i+1,i} & ... \\ ... & ... & ... & ... & ... \end{pmatrix}.$$

Then we can verify that $A \in H_{v,m}$ and $det(A) = 1 + y_{i+1,i}$.

For each $1 \leq j \leq n-1$ and any parameter in the "region" for $w = b_j - b_{j+1}$, we can construct the desired element similarly to reach the bound of conductor.

For any parameter in the "region" for $w = b_n - b_{n+1}$, the idea is similar and the resulting matrix is in fact simpler. Still take any $y_{n,n+1} \in \pi^{mw}\mathcal{O}_{F_v}$, set the matrix $A$ to be the diagonal matrix : set $x_{n,n} = 1 + y_{n,n+1}$ and set other diagonal elements to be 1,

$$\begin{pmatrix} 1 & ... & 0 \\ ... & ... & ... \\ 0 & ... & 1 + y_{n,n+1} \end{pmatrix}.$$

In conclusion, the bound can always be reached, thus $\det(H_{v,m}) = 1 + \pi^{mw}\mathcal{O}_{F_v}$.

Finally, we also have a similar method as in our previous examples to do explicit computations about the ordinary condition. However, the result is not "universal". The problem is due to the fact that $w = \min\{a_{i+1} - a_i, b_j - b_{j+1}\}$ is not a "linear" function of the parameter $(b_1, ..., b_{n+1}); (a_n, ..., a_1)$. The resulting ordinary condition is constant on each "region" for $w$.

Next we consider the similitude version. For simplicity we work over $\mathbb{Q}$ and suppose

the quadratic imaginary field is $E$. The similitude version is

$$\mathbf{H} = GU(1, n-1) \times_{\mathbb{G}_m} GU(0,1) \longrightarrow \mathbf{G} = GU(1, n-1) \times GU(1, n).$$

This pair is similar to the usual unitary GGP pair. The advantage is that they are the usual PEL type $GU$ Shimura varieties (the $U(n)$ Shimura variety is only an abelian type Shimura variety).

As in the $GSp_4$ example (section 6.1), we introduce

$$\mathbf{T}^1 = U(1) = \ker(GU(1) \overset{Norm}{\longrightarrow} G_{m,\mathbb{Q}})$$

and consider the quotient map

$$q : \mathbf{H} \longrightarrow \mathbf{T}^1, (A, z) \mapsto \frac{det(A)}{z^n}$$

to get a one dimensional torus quotient.

The argument for tame relations is the same as in our previous examples. For norm relations, like in the usual unitary case, we also work over a split prime $p$. This pair will split as

$$(GL(n) \times GL(1)) \times \mathbb{G}_m \longrightarrow GL(n) \times \mathbb{G}_m \times GL(n+1) \times \mathbb{G}_m,$$

where the three copies of $\mathbb{G}_m$ correspond to the similitude factor. Such similitude factor will influence nothing and we can ignore it. Then we will only need to consider the embedding

$$GL(n) \times GL(1) \longrightarrow GL(n) \times GL(n+1).$$

Then it is very similar to the above usual unitary GGP example. The construction of maximal torus, Borel subgroups etc are the same. Although in this case, we don't have trivial intersection property, the intersection is $GL(1)$ diagonally embedded in the center of $GL(n) \times GL(1)$, but this intersection lies in the kernel of our map $q$, thus just as in our $GSp_4$ example, the conductor also grows to infinity (condition $\heartsuit$ in section 5.2 holds). Moreover, the computation of conductors are also the same. Then condition $\clubsuit$ in section 5.2 also holds. Thus we can apply the same argument to produce special cycles with norm relations.

There is another possible variant. We can also consider the embedding

$$\mathbf{H} = GU(1, n-1) \times_{\mathbb{G}_m} GU(0,1)^n \longrightarrow \mathbf{G} = GU(1, 2n-1).$$

Here the product are all over $\mathbb{G}_m$ through the similitude map. And we will use the map

$$q : \mathbf{H} \longrightarrow U(1), (A, z_1, ..., z_n) \mapsto \frac{det(A)}{\prod_{i=1}^n z_i}$$

to cut out a one dimensional torus quotient. Similar argument will produce special cycles with tame relations and norm relations over split primes. Here for norm relations, there is a slight change. To obtain the spherical condition, we can not use Borel subgroups (too small), so we will instead work with a suitable parabolic subgroup of $GL(2n)$ corresponding to the partition $(n, n)$ of $2n$. Direct matrix computations show that the conditions ♣ and ♡ of section 5.2 hold. Then the general method still works. And finally we make a remark. This pair is very similar to the pair $(GU(1, n-1) \times_{\mathbb{G}_m} GU(0, n), GU(1, 2n-1))$, which is already studied by Andrew Graham and Syed Waqar Ali Shah in [28]. They also construct cycles with tame relations and norm relations over split primes using the methods of Loeffler's school.

## 6.3 Arithmetic applications

The special cycles can be used to construct Galois cohomology classes and then study related Selmer groups.

In the section 3, we introduce cohomology theory with trivial coefficients and show how to relate special cycles to cohomology classes (Abel-Jacobi map, trivialization methods etc). With the help of tame relations in section 4.2 and norm relations in section 5.2, we hope to construct an Euler system through our special cycles. Our tame relations are over split primes and thus need to use the machine of split Kolyvagin systems, see [23]. The standard application of Euler systems is to give upper bound for the related Selmer groups. For example, Christophe Cornut has deduced a rank one result in [16] under some assumptions. As we mentioned earlier, his tame relations are over inert primes.

Our tame relations involve the Hecke polynomial. Under the congruence conjecture of Blasius-Rogawski (see [4]), this polynomial is related with the characteristic polynomial of the Frobenius, which corresponds to the requirement of tame relation in Euler systems. This conjecture has now been established in many cases. For example, Si Ying Lee proved this conjecture for Hodge type Shimura varieties under some assumptions in [41]. More recently, Zhiyou Wu proved this conjecture for all Hodge type Shimura varieties in [68] as a corollary of the $S = T$ conjecture (see [69]). This covers the $GSp_4$ example, and the $GU$ analogues of the GGP-pair examples, but it seems that for our main example, where $Sh_{\mathbf{G}}(K)$ is an abelian type Shimura variety, the congruence conjecture remains unknown.

The norm relations can also be used to construct a norm compatible family under ordinary conditions, which may have potential other applications to Selmer groups and related Iwasawa theory. For example, regard Heegner points, see [55] for such an application towards parity of Selmer group and see also [2] for applications to Iwasawa theory.

For any applications of Euler system, we first have to show that the resulting Euler system is nontrivial. Such thing is very difficult to verify. The main idea is to connect special elements to $L$-function. Such connection is already an important topic and has

independent interest. We list four usual ideas.

• One method is complex Gross-Zagier formula. For example in the Heegner points case, we have Gross-Zagier formulae relating heights of Heegner points with derivatives of $L$-functions. Regarding the examples in this paper, unfortunately the Gross-Zagier formula for them is still widely open. The GGP program can be seen as an attempt to generalize the Gross-Zagier formula. For unitary GGP pair, there are many developments now, such as the arithmetic fundamental lemma program, the relative trace formula method etc. We refer to [70] for more details.

• Another strategy to study special cycles is to consider $p$-adic Gross-Zagier formula. For example, Henri Darmon and Victor Rotger developed a $p$-adic Gross-Zagier formula to study the generalised Gross-Kudla-Schoen diagonal cycles (see [19],[20]).

• What's more, we can also investigate explicit reciprocity laws. For example, Loeffler and Zerbes established an explicit reciprocity law for their Euler system of $GSp_4$ (not constructed by cycle class from special cycles), and they proved new cases of the Bloch-Kato conjecture (see [48]). They used higher Hida theory to study $p$-adic $L$-functions (see [45]).

• There is another different routine to get explicit reciprocity laws, it is (arithmetic) level raising. The work [3] is a classical example. We also refer to [65] for more details about such method for the triple product of Shimura curves. See [42] for Hilbert modular varieties. See also the recent breakthrough, [43]. They consider unitary Shimura varieties.

# Annexe A

# Appendix

## A.1 The stabilizer conjecture

In this section we will work in a local setting. First let's recall our setting in section 5.1.

Let $F$ be a $p$-adic field ($p$ is an odd prime) with a uniformizer $\pi$ and denote its residue field $\mathcal{O}_F/\pi$ by $k$. Let $\mathbf{G}$ denote a reductive group scheme over $\mathrm{Spec}(\mathcal{O}_F)$ and let $\mathbf{H}$ be a closed reductive subgroup scheme of $\mathbf{G}$. We assume this pair $(\mathbf{G}, \mathbf{H})$ to be spherical over $\mathrm{Spec}(\mathcal{O}_F)$ : there exists a Borel subgroup scheme $\overline{\mathbf{B}}$ of $\mathbf{G}$ such that the $\mathbf{H}$-orbit of $[1]$ in $\mathbf{G}/\overline{\mathbf{B}}$ is open, equivalently, $Lie(\mathbf{H}) + Lie(\overline{\mathbf{B}}) = Lie(\mathbf{G})$. Take a maximal torus $\mathbf{T}$ inside $\overline{\mathbf{B}}$ and denote by $\mathbf{B}$ the Borel subgroup of $\mathbf{G}$ opposed to $\overline{\mathbf{B}}$ with respect to $\mathbf{T}$. Denote $G = \mathbf{G}(F)$, $H = \mathbf{H}(F)$, $K = \mathbf{G}(\mathcal{O}_F)$ and $K_H = \mathbf{H}(\mathcal{O}_F) = H \bigcap K$. Choose a strict $\mathbf{B}$-dominant cocharacter $\mu$ of $\mathbf{T}$ defined over $\mathcal{O}_F$ and set $\tau = \mu(\pi)$. For any non-negative integer $m$, define $\overline{H_m} = \tau^m K \tau^{-m} \bigcap \mathbf{H}(F)$, $H_m = \overline{H_m} \bigcap K$. Denote their images in $H^{ab} = \frac{H}{H^{der}}$ by $H_m^{ab}$ and $\overline{H_m}^{ab}$. And define $x_m = [\tau^m] \in G/K$. We consider the following two properties :

(1) For any $m$, we have $H_m = \overline{H_m}$.

(2) For any $m$, we have $H_m^{ab} = \overline{H_m}^{ab}$.

Obviously we know $H_m \subset \overline{H_m}$, $H_m^{ab} \subset \overline{H_m}^{ab}$ and property (1) implies property (2). Property (1) is equivalent to say that the $H$-stabilizer of $x_m \in G/K$ also stabilizes $x_0 \in G/K$. We propose the following conjectures :

**Stabilizer Conjecture :** Property (1) always holds.

**Weak Stabilizer Conjecture :** Property (2) always holds.

In section 5.2, we needed to compute conductors and constants $c(m, i)$. The weak version is sufficient for our application to norm relations. At present we don't know how to deal with this weaker conjecture in general directly. We will propose some methods for the stabilizer conjecture. The first observation is that we may assume that $\mathbf{G}$ is split. Because we can take a finite unramified extension $F \longrightarrow F_0$ to split $\mathbf{G}$, and consider the spherical

pair $(\mathbf{H}_{\mathcal{O}_{F_0}}, \mathbf{G}_{\mathcal{O}_{F_0}})$. If $(\mathbf{H}_{\mathcal{O}_{F_0}}, \mathbf{G}_{\mathcal{O}_{F_0}})$ satisfies the stabilizer conjecture, then $(\mathbf{H}, \mathbf{G})$ also satisfies the stabilizer conjecture.

In fact, our main example has a stronger property than property (1). To state it, we first need some preparation about filtrations and Bruhat-Tits building theory. We refer to [21] for filtrations, [63] for buildings, [17] for some generalization and relations between buildings and filtrations.

Let $\mathbf{B}(\mathbf{G}_F)$ denote the extended Bruhat-Tits building of $\mathbf{G}_F$. Assume $\mathbf{G}_F$ is semisimple, then there is a $\mathbf{G}(F)$ invariant metric (unique up to scalar) on $\mathbf{B}(\mathbf{G}_F)$,

$$d : \mathbf{B}(\mathbf{G}_F) \times \mathbf{B}(\mathbf{G}_F) \longrightarrow \mathbb{R}_{\geq 0}.$$

And $(\mathbf{B}(\mathbf{G}_F), d)$ is a complete CAT(0)-space. For general reductive groups, there also exists such invariant metric but it may not be unique up to scalar. See [17] section 6.2 for more details. Now we fix such a metric. In our situation, we can identify $G/K$ with a $G$-orbit of hyperspecial vertices in $\mathbf{B}(\mathbf{G}_F)$ and thus view $x_m$ as an hyperspecial vertex in $\mathbf{B}(\mathbf{G}_F)$. Through $\mathbf{H}(\mathcal{O}_F) \subset \mathbf{G}(\mathcal{O}_F)$, we can embed the building $\mathbf{B}(\mathbf{H}_F)$ for $\mathbf{H}_F$ into a $H$-stable closed convex subset of $\mathbf{B}(\mathbf{G}_F)$ and $x_0 = [1]$ becomes a common origin. See [39] for more details (functoriality property). Because $\mathbf{B}(\mathbf{H}_F)$ is a closed convex set, there is a convex projection

$$pr : \mathbf{B}(\mathbf{G}_F) \longrightarrow \mathbf{B}(\mathbf{H}_F),$$

sending $x$ to its closest point in $\mathbf{B}(\mathbf{H}_F)$. Now we consider the following property :

(0) We have $pr(x_m) = x_0$.

Because the projection $pr$ is $H$-invariant, property (0) implies property (1). We will verify property (0) for our main example. Then it satisfies the stabilizer conjecture. We will use tools about buildings and filtrations from [17]. First we will translate property (0) into a property about filtrations.

Following [17], let $\Gamma = (\Gamma, +, \leq)$ denote a totally ordered commutative group, $\mathbb{D}_{\widetilde{S}}(\Gamma)$ denote the diagonalized multiplicative group over a base scheme $\widetilde{S}$ with character group $\Gamma$. For a reductive group $\widetilde{G}$ over $\widetilde{S}$, Cornut defined and studied the following sequence between $\widetilde{S}$-schemes in [17] section 2

$$\mathbb{G}^{\Gamma}(\widetilde{G}) \xrightarrow{Fil} \mathbb{F}^{\Gamma}(\widetilde{G}) \xrightarrow{t} \mathbb{C}^{\Gamma}(\widetilde{G}),$$

here $\mathbb{G}^{\Gamma}(\widetilde{G}) = \underline{\mathrm{Hom}}(\mathbb{D}_S(\Gamma), \widetilde{G})$. Also see [17] section 1 for some motivations. Here we will only work over $S = \mathrm{Spec}(R)$ with $R$ equal to one of the local rings $F$, $\mathcal{O}_F$ or $k$ and $\Gamma = (\mathbb{R}, +, \leq)$. For simplicity we denote $\mathbf{F}(\widetilde{G}) = \mathbb{F}^{\mathbb{R}}(\widetilde{G})(R)$ (filtrations). According to [17] section 4.1.15, there is an additive structure on $\mathbf{F}(\widetilde{G})$,

$$\mathbf{F}(\widetilde{G}) \times \mathbf{F}(\widetilde{G}) \xrightarrow{+} \mathbf{F}(\widetilde{G}).$$

The choice of a faithful finite dimensional representation $\rho$ induces a $\widetilde{G}(R)$-invariant scalar product

$$\langle -,- \rangle_\rho : \mathbf{F}(\widetilde{G}) \times \mathbf{F}(\widetilde{G}) \longrightarrow \mathbb{R}.$$

See [17] section 4.2 (especially section 4.2.2 and section 4.2.10) for more details. Here in application we will always fix such a representation first and simplify $\langle -,- \rangle_\rho$ as $\langle -,- \rangle$. Moreover, according to corollary 87 in [17] (see section 4.2.10), any embedding between groups $\widetilde{H} \hookrightarrow \widetilde{G}$ will induce an embedding of filtrations $\mathbf{F}(\widetilde{H}) \hookrightarrow \mathbf{F}(\widetilde{G})$. Especially for classical groups, we can use their standard embedding to $GL(V)$ to view their filtrations as a (closed convex) subspace of $\mathbf{F}(GL(V))$.

Now we connect filtrations and buildings. There is an action of $\mathbf{F}(\mathbf{G}_F)$ on $\mathbf{B}(\mathbf{G}_F)$

$$\mathbf{B}(\mathbf{G}_F) \times \mathbf{F}(\mathbf{G}_F) \xrightarrow{+} \mathbf{B}(\mathbf{G}_F).$$

And $\mathbf{B}(\mathbf{G}_F)$ becomes an affine $\mathbf{F}(\mathbf{G}_F)$-space. However, this action is not compatible with the addition map on $\mathbf{F}(\mathbf{G}_F)$. See [17] section 6.2 for more details. Be careful about the following **sign issue** : for $x_0$ and $\tau = \mu(\pi)$ as above, we have $\tau(x_0) = x_0 + \mathcal{F}_{\mu^{-1}}$, where $\mathcal{F}_{\mu^{-1}}$ is the filtration determined by $\mu^{-1}$ ($\mathcal{F}_{\mu^{-1}} = Fil(\mu^{-1})$). This can be traced back to the definition of buildings, see [63], section 1. By [17] section 6.4.8, there is a reduction map

$$\mathbf{F}(\mathbf{G}_F) \xleftarrow{\cong} \mathbf{F}(\mathbf{G}) \xrightarrow{red} \mathbf{F}(\mathbf{G}_k).$$

Through this reduction map, according to [17] section 6.4.13, section 5.5.2 and section 5.5.12, the property (0) is equivalent to the following property :

(A) Over the residue field $k$, for any filtration $\mathcal{F} \in \mathbf{F}(\mathbf{H}_k) \subset \mathbf{F}(\mathbf{G}_k)$, we have $\langle \mathcal{F}, red(\mathcal{F}_{\mu^{-1}}) \rangle \leq 0$.

We will prove that our main example satisfies property (A) in the following. Before that, we first briefly take $GL(m)$ as an example for the above concepts. This example is already widely explained, such as in [17] section 6.1. Here we illustrate it for the benefit of readers.

**Example** ($GL(m)$)

Consider a $m$-dimensional space $V$ over $F$ and the general linear group $\widetilde{G} = GL(V)$. And we will always fix its standard representation to compute scalar products of filtrations.

Its building can be identified with the space of $F$-norms on $V$. And the subset of hyperspecial vertices $hyp(\widetilde{G})$ can be identified with the set of $\mathcal{O}_F$-lattices in $V$.

A filtration on $V$ is a decreasing map with special properties

$$\mathcal{F} : \mathbb{R} \longrightarrow \{F\text{-subspaces of X}\}, \quad x \longrightarrow \mathcal{F}^x.$$

Here the set of subspace of $V$ is partially ordered by inclusion and we require $\mathcal{F}$ to satisfy the following two conditions :

(i) There exists $x \in \mathbb{R}$ such that $\mathcal{F}^x = (0)$ and $\mathcal{F}^{-x} = V$, i.e. the filtration is exhaustive and separating.

(ii) Let $\mathcal{F}^{x^-} = \bigcap_{y<x} \mathcal{F}^y$, then $\mathcal{F}^x = \mathcal{F}^{x^-}$ (left continuous).

Denote $\mathcal{F}^{>x} = \mathcal{F}^{x^+} = \bigcup_{y>x} \mathcal{F}_y$ , $\mathcal{F}^{\geq x} = \bigcup_{y\geq x} \mathcal{F}_y = \mathcal{F}^x$ and $gr^x_{\mathcal{F}}(V) = \mathcal{F}^{\geq x}/\mathcal{F}^{>x}$. And we refer to elements in $\{x|gr^x_{\mathcal{F}}(V) \neq 0\}$ as the breaks (or jumps) of $\mathcal{F}$. Suppose these elements are $y_l > y_{l-1} > ... > y_1$, we may also denote $\mathcal{F}$ as a tuple $(y_l, F^{y_l}, y_{l-1}, F^{y_{l-1}}, ..., y_1)$. For example, (0) is the filtration with a single jump at 0. If all these breaks are integers, then we call $\mathcal{F}$ a $\mathbb{Z}$-filtration, and the subset $\mathbf{F}^{\mathbb{Z}}(\widetilde{G}) = \mathbb{F}^{\mathbb{Z}}(\widetilde{G})(F)$ consists of $\mathbb{Z}$-filtrations. For a subspace $\widetilde{V} \subset V$, we can restrict $\mathcal{F}$ to $\widetilde{V}$ and denote it by $\mathcal{F}|_{\widetilde{V}}$, with $\mathcal{F}|^x_{\widetilde{V}} = \mathcal{F}^x \bigcap \widetilde{V}$. For a quotient $\Delta : V \twoheadrightarrow \overline{V}$ we can also define $\mathcal{F}|_{\overline{V}}$, $x \longrightarrow \Delta(\mathcal{F}^x)$. Combining these, we can define a filtration $\mathcal{F}|_{V'}$ for any sub-quotient of $V$. Now we describe the map $\mathbb{G}^{\mathbb{Z}}(\widetilde{G})(F) \xrightarrow{Fil} \mathbb{F}^{\mathbb{Z}}(\widetilde{G})(F)$. The elements in the left side correspond to cocharacters $\alpha : \mathbb{G}_m \longrightarrow GL(V)$. Through such $\alpha$, $V$ has a weight decomposition, $V = \oplus_i V(i)$, where $V(i) = \{v \in V, \alpha(t)(v) = t^i(v)\}$. For any $x \in \mathbb{R}$, define $V^x = \bigoplus_{y\geq x, y\in\mathbb{Z}} V(y)$, then the map $\mathcal{F} : x \longrightarrow \mathcal{F}^x = V^x$ is a $\mathbb{Z}$-filtration and it is exactly $Fil(\alpha)$. If we take a Borel pair $(\widetilde{B}, \widetilde{T})$, then the map $Fil : X^+(\widetilde{T}) \longrightarrow \mathbf{F}(\widetilde{G})$ is compatible with the addition maps. Now we describe the scalar product. For a filtration $\mathcal{F}$, we define $\deg(\mathcal{F}|V) = \sum_x x \dim(gr^x_{\mathcal{F}}(V))$. For two filtrations $\mathcal{F}_1$, $\mathcal{F}_2$, their scalar product is as follow

$$\langle \mathcal{F}_1, \mathcal{F}_2 \rangle = \sum_{x,y} xy \dim \frac{\mathcal{F}_1^x \bigcap \mathcal{F}_2^y}{\mathcal{F}_1^{>x} \bigcap \mathcal{F}_2^y + \mathcal{F}_1^x \bigcap \mathcal{F}_2^{>y}}$$

$$= \sum_x x \deg(\mathcal{F}_2|_{gr^x_{\mathcal{F}_1}(V)}) = \sum_x x \deg(\mathcal{F}_1|_{gr^x_{\mathcal{F}_2}(V)}).$$

Notice that we can rewrite $\deg(\mathcal{F}|V) = \langle \mathcal{F}, (0) \rangle$.

Now we illustrate the relation between filtrations and buildings. The action of $\mathbf{F}^{\mathbb{Z}}(\widetilde{G}) = \mathbb{F}^{\mathbb{Z}}(\widetilde{G})(F)$ on $Hyp(\widetilde{G})$ is as follow

$$L + \mathcal{F} = \sum_{i\in\mathbb{Z}} \frac{1}{\pi^i} L \bigcap \mathcal{F}^i.$$

Take a lattice $L_0$, it will give a reductive integral model $\mathcal{G} = GL(L_0)$ over $\mathrm{Spec}(\mathcal{O}_F)$. The reduction map for filtrations is as follows

$$red(\mathcal{F}) : x \longrightarrow (\mathcal{F}^x \bigcap L_0)/\pi.$$

Now let's come back to the proof of (A) for our main example. First let's recall its construction in section 5.2. Let $F \longrightarrow E$ denote a quadratic unramified extension ($E$ is a field or split as $F \times F$) and let $\mathcal{O}_E$ denote the integral closure of $\mathcal{O}_F$ in $E$. Let $(V, \phi)$

denote a $2n+1$-dimensional quadratic $F$-space with an $n$-dimensional hermitian $E$-space $(W,\psi)$. Our main example (over generic fiber) is $(SO(V),U(W))$. Now recall the notation of special basis (see section 5.2). Take an element $\eta \in \mathcal{O}_E^*$ with $\eta+\overline{\eta}=0$, we have the following orthogonal $F$-basis for $V$,

$$\beta = (v_0, w_1, v_1, w_2, ..., w_n, v_n),$$

where $\{w_1, ..., w_n\}$ is an orthogonal $E$-basis for $W$, $V = W \perp Fv_n$, $v_{i-1} = \eta w_i$ $(1 \le i \le n)$ and $\phi(w_i, w_i) + \phi(v_i, v_i) = 0$. And we assume that $\psi(v_i, v_i)$, $\psi(w_j, w_j)$ all belong to $\mathcal{O}_F^*$. This special basis defines two orthogonal decomposition of $V$,

$$V = Ew_1 \perp ... \perp Ew_n \perp Fv_n = Fv_0 \perp H_1 \perp ... \perp H_n,$$

where $H_i = Fw_i \perp Fv_i$ is an hyperbolic $F$-plane with isotropic lines spanned by $e_{\pm i} = \frac{1}{2}(v_i \pm w_i)$. Define $e_0 = v_0$ and consider the ordered basis $(e_n, .., e_{-n})$. Consider the $\mathcal{O}_F$-lattice $L_0$ spanned by this basis, it is a self-dual lattice and gives us a reductive integral model $\mathbf{G} = SO(L_0)$ over $\mathrm{Spec}(\mathcal{O}_F)$. Through $SO(L_0) \hookrightarrow GL(L_0)$ we get a Borel pair $(\mathbf{T}, \mathbf{B})$ for $\mathbf{G}$ ($\mathbf{B}$ corresponds to upper triangular matrices under this ordered basis). Consider the $\mathcal{O}_E$-lattice $\widetilde{L}$ spanned by $(w_1, ..., w_n)$, it defines a reductive integral model $\mathbf{H} = U(\widetilde{L})$. It is easy to check $\mathbf{H} \bigcap \overline{\mathbf{B}} = 1$ ($\overline{\mathbf{B}}$ is the Borel group opposite to $\mathbf{B}$ with respect to $\mathbf{T}$) and then $(\mathbf{H}, \mathbf{G}, \overline{\mathbf{B}})$ satisfies the spherical condition by dimension reason.

Fix the standard representation for the orthogonal group to compute scalar products of filtrations. Take a strict $\mathbf{B}$-dominant cocharacter $\mu \in X_*^+(\mathbf{T})$ associated to $(s_n, ..., s_1, 0, -s_1, ..., -s_n)$, where $s_i$ are integers with $0 < s_1 < ... < s_n$, we will verify property (A) for $red(\mathcal{F}_{\mu^{-1}})$. We first make some simplifications to our notations. Let $V_k$ denote the $2n+1$-dimensional quadratic $k$-space $L_0/\pi$, $h$ denote $\mathcal{O}_E/\pi$, $\theta$ denote $\eta/\pi \in h^*$, $W_k$ denote the $n$-dimensional hermitian $h$-space $\widetilde{L}/\pi$, $G_k$ denote the orthogonal group $\mathbf{G}_k = SO(V_k)$ and $H_k$ denote the unitary group $\mathbf{H}_k = U(W_k)$. Define the ordered basis $(f_n, ..., f_{-n}) = (\overline{e_{-n}}, ..., \overline{e_n})$ for $V_k$ and define a cocharacter $\zeta = (s_n, ..., s_1, 0, -s_1, ..., -s_n)$ under this ordered basis. Then $red(\mathcal{F}_{\mu^{-1}}) = \mathcal{F}_\zeta$.

Through the natural embedding $SO(V_k) \hookrightarrow GL(V_k)$, we embed $\mathbf{F}(G_k)$ into $\mathbf{F}(GL(V_k))$.

Denote $s_0 = 0$, $s_{-i} = -s_i$ $(1 \le i \le n)$. For each $-n \le i \le n$, define $V_i = \oplus_{j \ge i} k(f_j)$. The filtration $\mathcal{F}_\zeta$ corresponds to the tuple $(s_n, V_n, ..., s_{-n})$. It satisfies a self-dual property, i.e. $V_{-i} = V_{i+1}^\perp$ $(0 \le i \le n-1)$ and $s_{-i} = -s_i$ $(0 \le i \le n)$. In fact, any filtration in $\mathbf{F}(G_k)$ has this property.

The following observation shows that the filtration $\mathcal{F}_\zeta$ is "orthogonal" to the $h$-linear structure.

**Lemma A.1.** *For any vector $v = \sum_{i=1}^m c_i f_i$ with $1 \le m \le n-1$ and $c_m \ne 0$, we have $\theta(v) \in V_{-m-1} - V_{-m}$.*

*Démonstration.* For any $0 \leq m \leq n-1$, we have $V_{-m} = \oplus_{j=m+1}^{n} k(f_j) \oplus k(\theta(\overline{w_{m+1}})) \oplus_{j=1}^{m} h(\overline{w_j})$ : The right side is a subspace of the left side and they have the same dimension ($n+m+1$ dimensional), thus they are equal.

Now for such vector $v$, we have

$$\theta(v) = \sum_{i=1}^{m-1} c_i \theta(f_i) + c_m(\theta^2(\overline{w_{m+1}}) - \theta(\overline{w_m})).$$

Then it lies in $V_{-m-1}$. Because $c_m\theta^2$ is nonzero elements in $k$ and $\overline{w_{m+1}} \notin V_{-m}$, thus

$$\theta(v) \in V_{-m-1} - V_{-m}.$$

□

Now we first verify property (A) for filtrations of minimal type, namely those with at most three jumps. For any isotropic $h$-subspace $X \subset W_k$ and a positive real number $x$, we have a filtration $\mathcal{F}_{X,x} \in \mathbf{F}(H_k)$, $\mathcal{F}_{X,x} = (x, X, 0, X^{\perp}, -x)$. We have the following lemma :

**Lemma A.2.** *We have* $\deg(\mathcal{F}_{\zeta}|X) \leq 0$.

*Démonstration.* Denote the breaks set of $\mathcal{F}_{\zeta}|X$ as $Se$. Then $Se \subset \{s_n, ..., s_{-n}\}$. And we divide it into two parts, $Se = Se^+ \coprod Se^-$, where $Se^+ = \{s_i \in Se, 1 \leq i \leq n\}$, $Se^- = \{s_i, -n \leq i \leq 0\}$. Denote the degree $\deg(\mathcal{F}_{\zeta}|X)$ as $S$, so that $S = \sum_{s_i \in Se} s_i = S^+ + S^-$ with $S^+ = \sum_{s_i \in Se^+} s_i$, $S^- = \sum_{s_i \in Se^-} s_i$. Then $S^+ \geq 0$ and $S^- \leq 0$.

If $S^+ = 0$, we're done. If $S^+ > 0$, then $Se^+ \neq \emptyset$ and we will cancel its contribution through $S^-$.

Now define a function $Ma : V - 0 \longrightarrow \mathbb{Z}$.

For any nonzero vector $v$, denote $v = \sum_{l=-n}^{i} c_l f_l$ with $c_i \neq 0$, then define $Ma(v) = i$.

We next define a "**reverse**" function $r : Se^+ \longrightarrow \mathbb{Z}$.

Take any $s_i \in Se^+$, then $i \leq n-1$, and we define $r(s_i) = \min\limits_{v \in (V_i \bigcap X - V_{i+1} \bigcap X)} Ma(v)$.

By definition we know $r(s_i) \geq i$ and it is injective :

Suppose there exists $i < j$, with $r(s_i) = r(s_j)$. By definition, there exists $v_i \in V_i \bigcap X - V_{i+1} \bigcap X$ (resp $v_j \in V_j \bigcap X - V_{j+1} \bigcap X$) such that $v_i = \sum_{l=i}^{r(s_i)} c_l f_l$ (resp $v_j = \sum_{l=j}^{r(s_j)} d_l f_l$) with $c_{r(s_i)} \neq 0$ (resp $d_{r(s_j)} \neq 0$). Then consider the vector $v = d_{r(s_i)} v_j - c_{r(s_j)} v_i \in V_i \bigcap X - V_{i+1} \bigcap X$, but $Ma(v) < Ma(v_i)$, this contradicts the definition of $r$.

For each $s_i \in Se^+$, there exists $v_i \in V_i \bigcap X - V_{i+1} \bigcap X$ such that $v_i = \sum_{l=1}^{r(s_i)} c_l f_l$ with $c_{r(s_i)} \neq 0$. By the above lemma A.2, $\theta(v_i) \in V_{-r(s_i)-1} - V_{-r(s_i)}$. Then $s_{-r(s_i)-1} \in Se^-$. Consider the sum

$$S' = \sum_{s_i \in Se^+} s_i - s_{r(s_i)+1}.$$

Then $S = S' + S''$ with $S'' \leq 0$, $S' < 0$ (due to $r(s_i) \geq i$). Thus the degree is non-positive. □

By self-dual properties, we have $\langle \mathcal{F}_{X,x}, \mathcal{F}_\zeta \rangle = 2x(\deg(\mathcal{F}_\zeta|X))$. Thus this lemma verifies property (A) for minimal type filtrations. Moreover, it also helps us to reduce the general case into the minimal case.

**Theorem A.3.** *For any $\mathcal{F}_1 \in \boldsymbol{F}(H_k)$, we have $\langle F_1, F_\zeta \rangle \leq 0$.*

*Démonstration.* The case $\mathcal{F}_1 = (0)$ is trivial. For other cases, the tuple corresponding to $\mathcal{F}_1$ will be of the form $(a_m, X_m, ..., a_{-m})$, where $a_m > ... > a_{-m}$, $a_i + a_{-i} = 0$, $X_{-i} = X_{i+1}^\perp$ $(0 \leq i \leq m-1)$, $X_i$ $(1 \leq i \leq m)$ are isotropic $h$-spaces. We have shown the case $m = 1$. Now we do induction for $m \geq 2$.

Consider another filtration $\mathcal{F}_2 \in \mathbf{F}(H_k)$, defined by the tuple $(a_{m-1}, X_{m-1}, ..., a_{-(m-1)})$. By self-duality properties, we have $\langle \mathcal{F}_1, \mathcal{F}_\zeta \rangle = \langle \mathcal{F}_2, \mathcal{F}_\zeta \rangle + 2(a_m - a_{m-1}) \deg(\mathcal{F}_\zeta|X_m)$. Because $a_m - a_{m-1} > 0$, by lemma A.2 and induction argument, we're done. □

We make some remarks.

• We can relax the condition (for cocharacter $\zeta$) $s_1 < ... < s_m$ into $s_1 \leq ... \leq s_m$, the above theorem still holds by an analogous proof.

• Property (A) holds for any $\mu$ which is stricly dominant with respect to a Borel subgroup in the open $\mathbf{H}$-orbit. Indeed, property (A) only depens upon the filtration $\mathcal{F}_{\mu^{-1}}$, and any such filtration is split by some special basis $\beta$ as above. In other words, property (A) is an intrinsical property of our spherical pair $(\mathbf{H}, \mathbf{G})$.

Now we discuss other cases, especially the eight infinite families of indecomposable pairs $(\mathbf{H}, \mathbf{G})$ in [44] section 6. Using similar methods, we can show that the following four kinds of families also have property (A) :

•$(GL(n), Sp(2n))$

•$(SO(n), GL(n))$

•$(SO(n) \times SO(n+1), SO(2n+1))$

•$(SO(n) \times SO(n), SO(2n))$

But in general, we shouldn't expect property (A) for spherical pairs. Below is a classical counterexample.

Suppose the pair is $(G_0, G_0 \times G_0)$ over $\mathrm{Spec}(\mathcal{O}_F)$ with $G_0$ being semi-simple. Take a Borel pair $(T, B_1)$ for $G_0$ and let $B_2$ be the Borel subgroup of $G_0$ opposed to $B_1$ with respect to $T$. Then $B_1 \times B_2$ is a Borel subgroup of $G_0 \times G_0$ and $(G_0, G_0 \times G_0)$ is a spherical pair. This is a standard example. Consider a strict $B_1$-dominant cocharacter $\mu_1 \in X_*(T)$ and a strict $B_2$-dominant cocharacter $\mu_2 \in X_*(T)$ ; then for any positive integer $N$, we have a strict $B_1 \times B_2$-dominant cocharacter $\mu_N = ((\mu_1)^N, \mu_2) \in X_*(T \times T)$. The corresponding filtration involved in property (0) is $\mathcal{F}_{\mu_N} = (N(\mathcal{F}_{\mu_1^{-1}}), \mathcal{F}_{\mu_2^{-1}}) \in \mathbf{F}(G_{0,F} \times G_{0,F}) = \mathbf{F}(G_{0,F}) \times \mathbf{F}(G_{0,F})$. To do computation about filtrations, choose an embedding $G_{0,F} \hookrightarrow GL_N$ and the induced embedding $G_{0,F} \times G_{0,F} \hookrightarrow GL_N \times GL_N \hookrightarrow GL_{2N}$. Consider the filtration $\mathcal{F}_N = N(\mathcal{F}_{\mu_1^{-1}}) \in \mathbf{F}(G_{0,F})$, then

$$\langle \mathcal{F}_N, \mathcal{F}_{\mu_N} \rangle = N^2 \langle \mathcal{F}_{\mu_1^{-1}}, \mathcal{F}_{\mu_1^{-1}} \rangle + N \langle \mathcal{F}_{\mu_1^{-1}}, \mathcal{F}_{\mu_2^{-1}} \rangle;$$

since $\langle \mathcal{F}_{\mu_1^{-1}}, \mathcal{F}_{\mu_1^{-1}} \rangle > 0$, this scalar product will be positive for $N \gg 0$. Similar argument over the residue field $k$ will produce a counterexample to property (A).

Moreover, we can produce counterexamples to property (A) for the three remaining families (the idea is similar except for the last family) :

•$(GL(n), GL(n) \times GL(n+1))$

•$(SO(n), SO(n) \times SO(n+1))$

•$(Sp(2n), SL(2n+1))$.

Now we turn to our original conjecture concerning property (1). We have a general strategy to verify it for symmetric pairs.

The starting input is a reductive group scheme $\mathbf{G}$ over $\mathrm{Spec}(\mathcal{O}_F)$ with a nontrivial involution $\theta$. Let $\mathbf{H}$ denote the $\theta$ fixed subgroup. Such a pair $(\mathbf{H}, \mathbf{G})$ is called a symmetric pair. We assume that there exists a Borel subgroup $\mathbf{B}$ of $\mathbf{G}$ such that $\theta(\mathbf{B}) = \overline{\mathbf{B}}$ is opposed to $\mathbf{B}$. Then $\mathbf{T} = \theta(\mathbf{B}) \bigcap \mathbf{B}$ is a $\theta$-stable maximal torus and $(\mathbf{G}, \mathbf{H})$ is a spherical pair (see [32]).

Here we make some remarks :

• A parabolic group $\mathbf{P}$ such that $\theta(\mathbf{P})$ is opposed to $\mathbf{P}$ is called a $\theta$-split parabolic subgroup. In general, we can replace $\mathbf{B}$ by a minimal $\theta$-split parabolic subgroup, our argument below still works.

• Over a field $F$, there are many studies about symmetric pairs. For example, there always exists a $\theta$-stable maximal torus and under suitable conditions, there exists a non-trivial minimal $\theta$-split parabolic subgroup, we refer to [32] for more details. Moreover, the connected component of the fixed subgroup is a reductive group by [59].

Now take a strictly $\mathbf{B}$-dominant cocharacter $\mu$ of $\mathbf{T}$ and define $\tau = \mu(\pi)$, we will show that property (1) holds.

Because $\mu$ is strictly $\mathbf{B}$-dominant, the cocharacters $\eta = \theta(\mu)$ and $\mu^{-1}\eta$ are all strict $\overline{\mathbf{B}}$-dominant. Define $\xi = (\mu^{-1}\eta)(\pi)$. Let $\mathbf{N}$ be the unipotent radical of $\mathbf{B}$ and $\overline{\mathbf{N}}$ be the unipotent radical of $\overline{\mathbf{B}}$.

For any positive integer $m$, take an element $x \in \overline{H_m} = \mathbf{H}(F) \bigcap \tau^m \mathbf{G}(\mathcal{O}_F) \tau^{-m}$ and write $x = \tau^m y \tau^{-m}$ with $y \in \mathbf{G}(\mathcal{O}_F)$.

Then we know $x = \theta(x)$, thus $\tau^m y \tau^{-m} = \theta(\tau)^m \theta(y) \theta(\tau)^{-m}$.

Therefore $y = \xi^m \theta(y) \xi^{-m}$. Since $y$ and $\theta(y)$ belong to $\mathbf{G}(\mathcal{O}_F)$, this implies that $y$ lies in the big Bruhat cell $\overline{\mathbf{N}} \times \mathbf{T} \times \mathbf{N}$, and moreover $y$ decomposes as $\xi^m \overline{n_1} \xi^{-m} \times t \times n_2$ with $\overline{n_1} \in \overline{\mathbf{N}}(\mathcal{O}_F)$ and $n_2 \in \mathbf{N}(\mathcal{O}_F)$.

Thus $x = \tau^m y \tau^{-m} = \eta(\pi)^m \overline{n_1} \eta(\pi)^{-m} \times t \times \tau^m n_2 \tau^{-m}$. Then property (1) holds since $\eta$ is $\overline{\mathbf{B}}$-dominant and $\mu$ is $\mathbf{B}$-dominant.

Now we list some examples to which this method applies. Consider the diagonal embedding mentioned above, $G_0 \longrightarrow G_0 \times G_0$. Here we don't need $G_0$ to be semisimple. Consider the involution $\theta$ on $G_0 \times G_0$ defined by swapping factors, $(x, y) \longrightarrow (y, x)$. Then the fixed subgroup is exactly $G_0$ and the Borel subgroup $B \times \overline{B}$ is a $\theta$-split minimal parabolic sub-

group. Thus we can apply the above method to show property (1). Another example is $SO(n) \longrightarrow GL(n)$ and the involution $\theta$ is given by $A \longrightarrow A^{-T}$.

Involutions can also be used to study property (A). If there exists an involution $\theta$ of $\mathbf{F}(\mathbf{G}_k)$ that preserves the scalar product, fixes $\mathbf{F}(\mathbf{H}_k)$ and such that $\theta(red(\mathcal{F}_{\mu^{-1}})) + red(\mathcal{F}_{\mu^{-1}}) = 0$, then property (A) holds : According to [17] corollary 92, we have

$$2\langle \mathcal{F}_1, red(\mathcal{F}_{\mu^{-1}})\rangle = \langle \mathcal{F}_1, red(\mathcal{F}_{\mu^{-1}})\rangle + \langle \mathcal{F}_1, \theta(red(\mathcal{F}_{\mu^{-1}}))\rangle \leq \langle \mathcal{F}_1, red(\mathcal{F}_{\mu^{-1}}) + \theta(red(\mathcal{F}_{\mu^{-1}}))\rangle = 0.$$

Finally, we mention that in some concrete cases beyond symmetric pairs, we can also use explicit calculations to show property (1). See the example $(GU(1) \times_{\mathbb{G}_m} GL_2, GSp_4)$ in section 6.1 and the unitary GGP pair in section 6.2. In particular, we have the following example that satisfies property (1) but doesn't satisfy property (A) :

•$(GL(n), GL(n) \times GL(n+1))$

It is just the spherical pair appearing in section 6.2.

## A.2 Miscellaneous facts

To make this thesis more self-contained, in this section we prove some facts used in the main part.

**1. computation of the reflex norm**

In this section, we explain the reflex norm used in our main example.

This is a standard exercise with tori.

Let $F_1$ denote a field and fix a separate closure $F_1^s$. Let $F_2 \subset F_1^s$ denote a finite sparable extension of $F_1$ and denote the set $Gal_{F_1}/Gal_{F_2}$ by $Se$. The set $Se$ has a distinguished element [1].

Consider a torus $T_2$ over $F_2$ and another torus $T_1 = Res_{F_2/F_1} T_2$ over $F_1$. Over $F_1^s$, we have the following natural isomorphism

$$X_*(T_{1,F_1^s}) \cong \mathbb{Z}[Gal_{F_1}] \otimes_{\mathbb{Z}[Gal_{F_2}]} X_*(T_{2,F_1^s}).$$

Through this isomorphism we get a $Gal_{F_2}$-equivariant map $i : X_*(T_{2,F_1^s}) \longrightarrow X_*(T_{1,F_1^s})$, sending $x$ to $[1] \otimes x$. This map will induce a map between torus over $F_2$, $i : T_2 \longrightarrow T_{1,F_2}$.

Now suppose there is a cocharacter $\mu_2$ for $T_2$ over $F_2$, composing it with $i$, we get a cocharacter $\mu_1 = i(\mu_2)$ for $T_{1,F_2}$. We have the following lemma :

**Lemma A.4.** ***norm***

*The following diagram is commutative :*

$$\begin{tikzcd} Res_{F_2/F_1}\mathbb{G}_m \arrow[r, "res(\mu_1)"] \arrow[d, "res(\mu_2)"] & Res_{F_2/F_1}T_{1,F_2} \arrow[d, "Norm_{F_2/F_1}"] \\ Res_{F_2/F_1}T_2 \arrow[r, "="] & T_1 \end{tikzcd}$$

*Démonstration.* It is equivalent to check commutativity of this diagram over $F_1^s$ for their corresponding cocharacter groups.

Under the following identifications,

$$X_*((Res_{F_2/F_1}\mathbb{G}_m)_{F_1^s}) \cong \mathbb{Z}[Gal_{F_1}] \otimes_{\mathbb{Z}[Gal_{F_2}]} \mathbb{Z},$$

$$X_*((Res_{F_2/F_1}T_{1,F_2})_{F_1^s}) \cong \mathbb{Z}[Gal_{F_1}] \otimes_{\mathbb{Z}[Gal_{F_2}]} X_*(T_{1,F_1^s}),$$

$$X_*(T_{1,F_1^s}) \cong \mathbb{Z}[Gal_{F_1}] \otimes_{\mathbb{Z}[Gal_{F_2}]} X_*(T_{2,F_1^s}),$$

we have

$$res(\mu_1)(\sum_{\rho\in Se} a_\rho([\rho]\otimes 1)) = \sum_{\rho\in Se} a_\rho([\rho]\otimes \mu_1),$$

$$Norm_{F_2/F_1}(\sum_{\rho\in Se} a_\rho([\rho]\otimes \mu_1)) = \sum_{\rho\in Se} a_\rho\rho(\mu_1),$$

$$res(\mu_2)(\sum_{\rho\in Se} a_\rho([\rho]\otimes 1)) = \sum_{\rho\in Se} a_\rho([\rho]\otimes \mu_2).$$

Since $\rho(\mu_1) = [\rho]\otimes mu_2$, our diagram is indeed commutative. □

Applying this lemma to our main example, we obtain the reflex norm.

**2. Some facts about the Hecke polynomial**

This section is mainly a complement for section 2.2 thus we will use similar notations. However, for simplicity in this section we will use **arithmetic Frobenius**.

From now on let $F$ denote a $p$-adic field with ring of integers $\mathcal{O}_F$, choose a uniformizer $\pi \in \mathcal{O}_F$ and denote the cardinality of the residue field $\mathcal{O}_F/\pi$ by $q$. Fix an algebraic closure $\overline{F}$ of $F$ and let $F^{un}$ denote the maximal unramified extension of $F$ inside $\overline{F}$.

Let $\mathbf{G}$ denote a reductive group scheme over $\mathcal{O}_F$, then $\mathbf{G}$ is quasi-split over $\mathcal{O}_F$. Thus we can choose a Borel pair $(\mathbf{T}, \mathbf{B})$ for $\mathbf{G}$, where $\mathbf{T}$ is a maximal torus and the Borel group $\mathbf{B}$ contains $\mathbf{T}$. Let the unipotent radical of the Borel subgroup $\mathbf{B}$ be $\mathbf{N}$, we have a Levi decomposition $\mathbf{B} = \mathbf{T} \ltimes \mathbf{N}$. Let $\mathbf{S}$ denote the maximal split subtorus of $\mathbf{T}$, $X_*(\mathbf{S})$ the group of cocharacters of $\mathbf{S}$ and $X_*^+(\mathbf{S}) \subset X_*(\mathbf{S})$ the cone of $\mathbf{B}$-dominant cocharacters.

Let $K$ denote the hyperspecial subgroup $\mathbf{G}(\mathcal{O}_F)$, $B$ denote $\mathbf{B}(F)$, $G$ denote $\mathbf{G}(F)$, $T$ denote $\mathbf{T}(F)$, $T^0$ denote $\mathbf{T}(\mathcal{O}_F) = T \cap K$, $N$ denote $\mathbf{N}(F)$. Then $T^0$ is a maximal compact subgroup of $T$, the natural inclusion $\mathbf{S}_F \hookrightarrow \mathbf{T}_F$ will identify $X_*(\mathbf{S}) = X_*(\mathbf{S}_F)$ with $X_{F,*}(\mathbf{T}_{\overline{F}})$, the $F$-rational cocharacter for $\mathbf{T}_F$, or equivalently the subgroup of $X_*(\mathbf{T}_{\overline{F}})$ fixed

by $Gal(\overline{F}/F)$. Moreover, we can identify $X_*(\mathbf{S})$ with $T/T^0$ by sending $\lambda$ (a cocharacter) to the coset of $\lambda(\pi)$.

Now we introduce the Hecke algebra. Let $dg$ denote the unique Haar measure on $G$ with $K$ having volume 1. The Hecke algebra $He$ is the ring of locally constant, compactly supported functions $f : G \longrightarrow \mathbb{Z}$ which are $K$-biinvariant, $f(kg) = f(gk) = f(g)$ for any $k$ in $K$. The multiplication in $He$ is given by the convolution product :

$$f_1 \cdot f_2(z) = \int_G f_1(g) \cdot f_2(g^{-1}z)dg.$$

The characteristic function of $K$ is the unit element.

Notice that the characteristic functions $char(KgK)$ give a $\mathbb{Z}$-basis for $He$, and it is easy to check that this definition of the Hecke algebra is equivalent to the previous definition via endomorphism algebra. Moreover, we have the following Cartan decomposition :

$$G = \coprod_{\lambda \in X_*^+(\mathbf{S})} K\lambda(\pi)K.$$

For each such $\lambda$, let $c_\lambda$ denote the corresponding characteristic function $char(K\lambda(\pi)K)$.

For any ring $R$, we will denote the $R$-valued Hecke algebra $He \otimes R$ as $He(G, K, R)$. Equivalently, we can also think of it as the ring of locally constant, compactly supported $R$-valued functions on $G$.

For the torus $\mathbf{T}$, obviously its Hecke algebra $He(T, T^0, R)$ is commutative and we can identify it with the group algebra $R[X_*(\mathbf{S})]$ due to $X_*(\mathbf{S}) \cong T/T^0$ as we mentioned above. The following Satake transform will identify the Hecke algebra for $\mathbf{G}$ to a subalgebra of the Hecke algebra of $\mathbf{T}$. In particular, it is also a commutative ring.

**Satake transform**

Although we work with unramified groups, the Satake transform can be defined more generally for any reductive group. In that setting the idea is similar. We refer to Cartier's paper [13] for such generalization. Also see [30] for some concrete examples etc.

The usual Satake transform is an isomorphism

$$S : He(G, K, R_0) \longrightarrow He(T, T^0, R_0)^{\Omega_F},$$

here $R_0$ is a ring containing $\mathbb{Z}[q^{\pm\frac{1}{2}}]$ and $\Omega_F$ is the relative Weyl group of $\mathbf{T}$ in $\mathbf{G}$.

Let $dn$ be the unique Haar measure on the group $N$ with $N(\mathcal{O}_F) = N \cap K$ having volume 1. Let $\delta : B \longrightarrow \mathbb{R}_+^*$ be the modulus factor for $B$, defined by the formula

$$d(bnb^{-1}) = \delta(b) \cdot dn.$$

Obviously $\delta$ is trivial on $N$, thus defines a character $T \longrightarrow \mathbb{R}_+^*$. Let $\delta^{\frac{1}{2}}$ be the positive square-root of this cocharacter.

Let $\rho \in X^*(\mathbf{T}_{\overline{F}}) \otimes \mathbb{Q}$ denote the half sum of all positive roots of $(\mathbf{T}_{\overline{F}}, \mathbf{B}_{\overline{F}})$, $\langle -, - \rangle$ denote the natural pairing

$$X_*(\mathbf{T}_{\overline{F}}) \otimes \mathbb{Q} \times X^*(\mathbf{T}_{\overline{F}}) \otimes \mathbb{Q} \longrightarrow \mathbb{Q}.$$

For the element $t = \lambda(\pi)$, where $\lambda \in X_*(\mathbf{S})$, we have the following formula :

$$\begin{aligned} \delta^{\frac{1}{2}}(t) &= |det(adt|Lie(N))|^{\frac{1}{2}} \\ &= |\pi^{\langle \lambda, 2\rho \rangle}|^{\frac{1}{2}} = q^{-\langle \lambda, \rho \rangle}, \end{aligned}$$

here $adt$ denotes the adjoint action. In particular, its value lies in the subgroup $q^{\frac{1}{2}\mathbb{Z}}$.

For any element $f \in He(G, K, R_0)$, the Satake transform will define $S(f)$ as a function on $T$ by the following integral

$$Sf(t) = \delta(t)^{\frac{1}{2}} \int_N f(tn)dn.$$

Then obviously $Sf$ is a function on $T/T^0 = X_*(\mathbf{S})$ with values in $R_0$ (recall $R_0$ contains $q^{\pm\frac{1}{2}}$). Therefore $Sf$ lies in $He(T, T^0, R_0) = R_0[X_*(\mathbf{S})]$. Moreover, this transform is an injective ring map whose image is the $\Omega_F$-invariant submodule of $He(T, T^0, R_0)$. We refer to Cartier's paper [13] for the proof of these facts.

**Twisted version**

From the integral formula, we see that the usual Satake transform will involve issues about the coefficient ring, this is due to the factor $\delta^{\frac{1}{2}}$. In many papers (especially those about the Hecke polynomial), it is convenient to introduce the **twisted version** Satake transform $S^\bullet$. It will normalize the Satake transform by $\delta^{\frac{1}{2}}$ to keep "rationality". We will follow Wedhorn's notation in [66]. The twisted Satake transform is a ring map

$$S^\bullet : He(G, K, \mathbb{Z}) \longrightarrow He(T, T^0, \mathbb{Z}),$$

sending any element $f$ to a function $S^\bullet(f)$ on $T$ by the following integral

$$S^\bullet f(t) = \int_N f(nt)dn = \delta(t) \int_N f(tn)dn.$$

The advantage is that this twisted Satake transform can be defined over any coefficient ring. This map is also an injection and if we require that the coefficient ring $R_0$ contains $\mathbb{Z}[q^{-1}]$ then its image is the submodule of "$\Omega_F$" invariants, here we have to use a "dot-action" of $\Omega_F$ on the Hecke algebra, which is defined as follow (also see [66] section 1.8) :

For $w \in \Omega_F$, $\lambda \in X_*(\mathbf{S})$ and view $\lambda$ as an element in $R_0[X_*(\mathbf{S})]$, then we define

$$w \bullet \lambda = q^{\langle \lambda - w(\lambda), \rho \rangle} w(\lambda).$$

Notice that $\langle \lambda - w(\lambda), \rho \rangle \in \mathbb{Z}$, so this action is well-defined due to $R_0$ contains $\mathbb{Z}[q^{-1}]$.

Consider the following $\mathbb{C}$-linear map

$$\alpha : \mathbb{C}[X_*(\mathbf{S})] \longrightarrow \mathbb{C}[X_*(\mathbf{S})],$$

$$\lambda \mapsto q^{-\langle \lambda, \rho \rangle} \lambda.$$

It is an isomorphism between these two group algebras and we have the following relation between our two Satake maps :

$$\alpha \circ S = S^{\bullet}.$$

Now we introduce the **dual viewpoint** for the Satake transform.

Let $\Gamma$ denote $Gal(F^{un}/F)$ with $\sigma \in \Gamma$ being the **arithmetic Frobenius** of $F$. By our assumption for $\mathbf{G}$, the group $\mathbf{G}_{\overline{F}}$ will split over $F^{un}$. Therefore we can consider the unramified version of Langlands dual group :

$$1 \longrightarrow \widehat{G} \longrightarrow {}^L G \longrightarrow \Gamma \longrightarrow 1,$$

here $\widehat{G}$ is the dual group, a reductive group over $\mathbb{C}$ with root datum dual to the root datum defined by $(\mathbf{T}_{\overline{F}}, \mathbf{B}_{\overline{F}}, ...)$ of $\mathbf{G}_{\overline{F}}$.

Fix a $\Gamma$-invariant pinning $(\widehat{T}, \widehat{B}, ...)$ of $\widehat{G}$ so that ${}^L G = \widehat{G} \rtimes \Gamma$. There is a $\Gamma$-equivariant isomorphism $X_*(\mathbf{T}_{\overline{F}}) \cong X^*(\widehat{T})$. Then we can identify $\mathbb{C}[X_*(\mathbf{S})]^{\Omega_F}$ as the algebra of regular functions on $\widehat{G}$ which are invariant under $\sigma$-conjugation, see Borel's paper [6] section 6. In particular, through the Satake transform

$$S : He(G, K, \mathbb{C}) \longrightarrow He(T, T^0, \mathbb{C})^{\Omega_F},$$

we can view such a function as an element of the complex Hecke algebra.

In the **split case**, we have $\Omega_F = \Omega$ (absolute Weyl group), $\mathbf{S} = \mathbf{T}$, and we can also view the Hecke algebra as the representation ring of the dual group $\widehat{G}$,

$$\mathbb{C}[X_*(\mathbf{S})]^{\Omega_F} \cong Rep(\widehat{G}, \mathbb{C}).$$

The irreducible representation of $\widehat{G}$ is a highest weight representation $V_\lambda$ determined by the highest weight $\lambda \in X^*(\widehat{T})$. Let $\chi_\lambda = Trace(V_\lambda)$ be the corresponding element in $\mathbb{C}[X^*(\widehat{T})]^{\Omega}$. Then $\{\chi_\lambda\}$ is a natural basis for the representation ring. Therefore it is natural to explicit the Satake transform under these two basis $\{c_\lambda\}$ and $\{\chi_\lambda\}$. We have the following result :

$$S(c_\lambda) = q^{\langle \lambda, \rho \rangle} \chi_\lambda + \sum_{\eta < \lambda} a_\lambda(\eta) \chi_\eta.$$

We refer to [13] and [30] for its proof. In particular if $\lambda$ is a minuscule (minimal) weight

for $\widehat{G}$, then we have

$$q^{\langle\lambda,\rho\rangle}\chi_\lambda = S(c_\lambda).$$

In the general case, these coefficients $a_\lambda(\eta)$ are related to Kazhdan-Lusztig polynomials, we refer to [30] for more details.

Now we have the following explicit example for $\mathbf{G} = GL(n)$.

**Example**

For $\mathbf{G} = GL(n)$, take $\mathbf{T}$ to be the diagonal torus, $\mathbf{B}$ to be the group of upper triangular matrices.

For each $1 \leq i \leq n$, let $\mu_i$ denote the cocharacter of $\mathbf{T}$ given by

$$t \mapsto (t, .., t, 1, ..., 1) \text{ (the first } i \text{ terms are } t).$$

It corresponds to a minuscule weight of the dual group $\widehat{G} \cong GL(n)$, and the corresponding highest weight representation is $\bigwedge^i \mathbb{C}^n$, where $\mathbb{C}^n$ denotes the standard representation. Thus we have the following formula :

$$S(c_{\mu_i}) = q^{\frac{i(n-i)}{2}} Trace(\bigwedge^i \mathbb{C}^n).$$

**Hecke polynomial**

Now we recall the Hecke polynomial.

Let $\mu$ be a conjugacy class of cocharacters of $\mathbf{G}_{\overline{F}}$ and we also use $\mu$ to denote the unique $\mathbf{B}_{\overline{F}}$-dominant cocharacter of $\mathbf{T}_{\overline{F}}$. Both variants of $\mu$ have the same field of definition, a finite unramified extension $F(\mu) \subset F^{un}$ of $F$. Let $n(\mu) = [F(\mu) : F]$ be the degree of this extension. Through our identification $X_*(\mathbf{T}_{\overline{F}}) \cong X^*(\widehat{T})$, we can also view $\mu$ as a $\widehat{B}$-dominant character of $\widehat{T}$ that is fixed by $\Gamma^{n(\mu)}$. Consider the following representation :

$$r_\mu :^L (\mathbf{G}_{F(\mu)}) = \widehat{G} \rtimes \Gamma^{n(\mu)} \longrightarrow GL(V_\mu),$$

where the restriction to $\widehat{G}$ is the highest weight representation with weight $\mu$ and $\Gamma^{n(\mu)}$ acts trivially on the highest weight space. Now for any $\widehat{g} \in \widehat{G}$, consider the characteristic polynomial

$$\det(X - q^{n(\mu)d(\mu)} r_\mu((\widehat{g} \rtimes \sigma)^{n(\mu)})),$$

where $d(\mu) = \langle\mu, \rho\rangle)$. Its coefficients, viewed as functions on $\widehat{G}$, are regular functions on $\widehat{G}$ which are invariant under $\sigma$-conjugation. Thus through the Satake transform, there exists a polynomial $Hep_\mu \in He(G, K, \mathbb{C})[X]$ such that $S(Hep_\mu)$ corresponds to this characteristic polynomial.

Notice that for a linear transform $\phi$ on a $n$-dimensional vector space $V$, we can expand

the characteristic polynomial as follow :

$$det(X-\phi)=\sum_{i=0}^{n}X^{n-i}(-1)^{i}Trace(\phi|\wedge^{i}V).$$

Therefore we have an explicit example for $GL(n)$ : Because the group splits, the Galois action is trivial. Recall our above example for $GL(n)$, for $\mu=\mu_1$, the corresponding representation is exactly $\mathbb{C}^n$ (standard representation), then through the above computation of Satake transforms, we have

$$Hep_{\mu}=\sum_{i=0}^{n}(-1)^{i}q^{\frac{i^2-i}{2}}c_{\mu_i}X^{n-i}.$$

**Rationality property of the Hecke polynomial**

As we mentioned earlier, the Hecke polynomial has certain rationality properties. For $\mu$ minuscule, in [66] Wedhorn showed that the Hecke polynomial in fact lies in $He(G,K,\mathbb{Z}[q^{-1}])[X]$. In fact we have a similar result in the general case, where $\mu$ is not necessarily minuscule.

Suppose $\mathbf{G}_F$ splits over $F_1$, and let $d$ denote the degree $[F_1:F]$, $R$ denote the ring $\mathbb{Z}[q^{-1},\zeta_d]$ ($\zeta_d$ is a primitive $d$-th root of unity). Then we have the following theorem :

**Theorem A.5.** *(rationality)*

*The Hecke polynomial lies in $He(G,K,R)$.*

*Démonstration.* Through the Satake transform $S$, we can view $Hep_{\mu}$ as an element in $\mathbb{C}[X_*(\mathbf{S})][X]$. Now we restrict the representation $r_{\mu}$ as a representation of the dual torus $\widehat{T}$ and consider its weight decomposition and the Galois action, $\sigma^{n(\mu)}$ acts on the set of weights.

Notice that $(1\rtimes\sigma^{-n(\mu)})(\widehat{t}\rtimes 1)(1\times\sigma^{n(\mu)})=(\sigma^{-n(\mu)}(\widehat{t})\rtimes 1)$, for each weight $\lambda\in X^*(\widehat{T})$ that appears in $V_{\mu}$, we have an isomorphism

$$V_{\mu}(\lambda)\cong V_{\mu}(\sigma^{n(\mu)}(\lambda))$$

($V_{\mu}(\lambda)$ is the corresponding weight space) given by

$$v\mapsto r_{\mu}(1\rtimes\sigma^{n(\mu)})(v).$$

Suppose the size of the orbit $\{\sigma^{n(\mu)k}(\lambda)|k\in\mathbb{Z}\}$ is $m$, in particular $\sigma^{n(\mu)m}(\lambda)=\lambda$. Then the linear operator $r_{\mu}(1\rtimes\sigma^{n(\mu)m})$ acts on $V_{\mu}(\lambda)$ and it will generate a cyclic subgroup of the general linear group with order $d_1$ dividing $d$. In particular this linear operator is diagonalizable. Take a basis $\{v_1,...v_k\}$ for $V_{\mu}(\lambda)$ consisting of eigenvectors of $r_{\mu}(1\rtimes\sigma^{n(\mu)m})$.

Now suppose $r_{\mu}(1\rtimes\sigma^{n(\mu)m})v_1=\zeta v_1$, where $\zeta$ is a $d_1$-th root of unity. Then consider the space $W$ generated by the ordered basis $\{v_1,r_{\mu}(1\rtimes\sigma^{n(\mu)})(v_1),...,r_{\mu}(1\rtimes\sigma^{n(\mu)(m-1)})(v_1)\}$.

Then $r_\mu(1 \rtimes \sigma^{n(\mu)})$ acts on $W$, and the corresponding matrix is

$$\begin{pmatrix} 0 & 0 & ... & \zeta \\ 1 & 0 & ... & 0 \\ 0 & 1 & ... & 0 \\ ... & ... & ... & ... \\ ... & ... & ... & 0 \end{pmatrix}.$$

Notice that $(\widehat{t} \rtimes \sigma)^{n(\mu)} = (\prod_{i=0}^{n(\mu)-1} \sigma^i(\widehat{t})) \rtimes \sigma^{n(\mu)}$, the corresponding operator $r_\mu((\widehat{t} \rtimes \sigma)^{n(\mu)})$ will acts on $W$, in particular its determinant polynomial on $W$ (denote it as $P_{\lambda,v_1}$) will contribute to a factor of the Hecke polynomial, and this factor lies in $\mathbb{C}[X^*(\widehat{T})][X]$, it is

$$X^m - q^{n(\mu)d(\mu)m}\zeta(\prod_{i=1}^{n(\mu)m}(\sigma^{i-n(\mu)}(\lambda))).$$

Notice that the character $\prod_{i=1}^{n(\mu)m}(\sigma^{i-n(\mu)}(\lambda)) \in X^*(\widehat{T})$ is fixed by $\sigma$, thus through the identification $X^*(\widehat{T}) \cong X_*(\mathbf{T}_{\overline{F}})$, we can think of it as an element in $X_*(\mathbf{S})$. Now apply the normalization map $\alpha$,

$$\alpha(\prod_{i=1}^{n(\mu)m}(\sigma^{i-n(\mu)}(\lambda))) = q^{-\langle \sum_{i=1}^{n(\mu)m}(\sigma^{i-n(\mu)}(\lambda)),\rho\rangle} \prod_{i=1}^{n(\mu)m}(\sigma^{i-n(\mu)}(\lambda)).$$

Notice that $q^{-\langle \sum_{i=1}^{n(\mu)m}(\sigma^{i-n(\mu)}(\lambda)),\rho\rangle} = q^{-n(\mu)m\langle\lambda,\rho\rangle}$, combine with the factor $q^{n(\mu)d(\mu)m}$, and apply the highest weight property of $V_\mu$, we know that $\mu - \lambda = \sum_{\alpha^\vee} c(\alpha^\vee)\alpha^\vee$, where $\alpha^\vee$ is positive root of $\widehat{T}$, $c(\alpha^\vee)$ is an integer, and $\langle\alpha^\vee,\rho\rangle$ is also an integer. Therefore we know that after the normalization map $\alpha$, this polynomial lies in $R[X_*(\mathbf{S})]$.

By definition of the determinant polynomial, we have a decomposition

$$Hep_\mu = \prod_{\lambda,v} P_{\lambda,v}.$$

Finally recall that $S^\bullet = \alpha \circ S$, and the twisted Satake transform $S^\bullet$ keeps the coefficient ring, therefore we're done.

□

If $\mu$ is minuscule, Wedhorn claimed that each root of unity appearing in the above argument is always 1 in [66]. Therefore his result is a little stronger : the coefficient $R$ is only require to contain $\mathbb{Z}[q^{-1}]$. This happens in many cases. We list some examples :

• If each weight space $V_\mu(\lambda)$ is one dimensional (this includes minuscule cases), then each related root of unity belongs to $\{\pm 1\} \subset \mathbb{Z}$, so we get the stronger result. The reason is that the complex representation $V_\mu$ has a model over $\mathbb{Q}$. Working over $\mathbb{Q}$, we still have

weight decompositions etc. And for a one dimensional $\mathbb{Q}$-space, a cyclic subgroup of the automorphism group $GL_1(\mathbb{Q}) = \mathbb{Q}^*$ can only be $\{\pm 1\}$ or $\{1\}$.

• For groups of the form $Res_{F_1/F}G_1$, where $G_1$ is a split reductive groups over $F_1$. In particular this includes split reductive groups ;

• If the degree $d$ of the splitting field is 2, then the corresponding roots of unity $\pm 1$ already lie in the integer ring $\mathbb{Z}$, in particular unitary groups satisfy such condition.

**3. A simple analogue**

In the last part of section 5.2, lemma 5.6, we explained how to modify our special cycles with norm relations into a norm compatible family. For the readers who are not familiar with this idea, we describe here a simple analogue : how to obtain a constant sequence from a sequence satisfying a certain type of linear relations.

Let $R$ denote a ring, $M$ be a $R$-module, $Pol \in R[X]$ denote a polynomial with degree larger than one. Suppose $Pol = \sum_{i=0}^{k} e_i X^i$ with $e_k \neq 0$. We also assume that $Pol$ has a root $b \in R^*$. In other words, we also have the following expansion :

$$Pol = (X - b)(\sum_{i=0}^{k-1} p_i X^i).$$

Now let $\{x_i | i \in \mathbb{N}\} \subset \widetilde{M}$ denote a sequence of elements that are related by $Pol$, in other words, for each integer $m > 0$, they satisfy

$$\sum_{i=0}^{k} e_i x_{m+i} = 0.$$

Now define a sequence of elements inside $M$ (for $m > 0$) :

$$\widehat{x_m} = b^{-m} \sum_{i=0}^{k-1} p_i x_{m+i},$$

then this sequence is constant :

$$\widehat{x_{m+1}} = \widehat{x_m}.$$

The proof is straightforward :

$$\sum_{i=0}^{k-1} p_i x_{m+i+1} - b(\sum_{i=0}^{k-1} p_i x_{m+i}) = 0$$

$$b^{m+1}\widehat{x_{m+1}} - b(b^m \widehat{x_m}) = 0$$

$$b^{m+1}(\widehat{x_{m+1}} - \widehat{x_m}) = 0,$$

because $b^{-1}$ lies in $R$, we have $\widehat{x_{m+1}} = \widehat{x_m}$.

Formally, we can think of the element $Tr_{m+i,m}(\mathcal{Y}_{m+i})$ in the lemma 5.6 as an analogue of $x_{m+i}$. Then the key idea hidden in the lemma 5.6 is exactly the same as the above idea.

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