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CY Cergy Paris University - École doctorale n° 405 Économie, management,  
mathématiques, physique et sciences informatiques (EM2PSI)

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# Strategic Compression for Multi-User Bayesian Persuasion

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by

**Rony Bou Roupael**

This dissertation is submitted for the degree of

*Doctor of Philosophy from CY Cergy Paris University*

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Submitted: October, 5<sup>th</sup>, 2022



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---

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# DEDICATION

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*To my father and the loving memory of my mother.*

*To Johnny, Tony, and Benny.*

# ABSTRACT

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In decentralized, decision-oriented communication networks, information compression policies are designed to satisfy a long-term strategic goal. The encoders select signals in order to trigger preferred actions from the decoders by controlling their local information environments. Henceforth, strategic communication can be naturally studied at the intersection of two disciplines: Information theory, which analyzes the optimal performance of communication systems, and game theory, which describes the behavior of agents in strategic interactions. Originally formulated with no restrictions on the amount of transmitted information, the Bayesian persuasion game models a strategic communication between an informed encoder observing a source variable, and transmitting signals to one or many decoders who are supposed to take actions that affect the encoder's performance. Before observing the source, the encoder commits to, and announces the encoding strategy to be implemented. All communicating agents are assumed to be rational, and endowed with mismatched objectives captured by distinct and arbitrary cost functions. In this thesis, we consider a large number of independent and identically distributed (i.i.d.) copies of the one-shot Bayesian persuasion game, subject to information constraints. Because of the restrictions imposed on the communication, not enough messages are available to transmit the whole source. We need to consider block-coding instead of one-shot coding. This setting generalizes the lossy source coding problem of Shannon, where agents fully cooperate to achieve truthful and reliable information transmission that minimizes their aligned distortion measures, to the non-cooperative scenario in which players do not necessarily share a common objective, but select the coding strategies that minimize their respective non-aligned cost functions. We investigate the strategic communication problem in three different settings. First, we consider the successive refinement coding setup in which a single encoder communicates a public message to two decoders, and a private message to only one of them. Upon reception of the message from the encoder, each decoder draws the action sequence that minimizes its respective long-run cost functions. Second, we consider the Gray-Wyner communication network, with a single encoder and two decoders, each observing a public and a private signal from the encoder, and are endowed with cost functions that depend on the actions of both decoders and the state. In this setting, each commitment of the encoder induces a Bayesian game among the decoders which admits perfect Bayes-Nash equilibria. Third, we combine the Bayesian persuasion model with the cascade multiple description network, where information is transmitted from the encoder to the decoder through an interested relay. In each of these settings, we study the information-theoretic limits of strategic communication and describe the asymptotic behavior of the encoder's optimal long-run cost function. Using auxiliary random variables, we characterize the encoder's minimal single-letter cost function subject to the optimal compression scheme that satisfies the constraints imposed on the amount of information transmitted in each model, as well as the incentive constraints of the decoders.

## Keywords:

Information Theory; Game Theory; Strategic Communication; Bayesian Persuasion; Lossy Source Coding; Multi-User Channel; Successive Refinement; Gray-Wyner Network, Cascade Multiple Description Network

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# RÉSUMÉ

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Dans les réseaux de communication décentralisés et orientés décision, les politiques de compression de l'information sont conçues pour satisfaire un objectif stratégique à long terme. Les codeurs sélectionnent des signaux afin de déclencher des actions préférées chez les décodeurs en contrôlant leurs environnements d'informations locaux. Désormais, la communication stratégique peut être naturellement étudiée à l'intersection de deux disciplines : la théorie de l'information, qui analyse les performances optimales des systèmes de communication, et la théorie des jeux, qui décrit le comportement des agents dans les interactions stratégiques. Formulé à l'origine sans restriction sur la quantité d'informations transmises, le jeu de persuasion Bayésienne modélise une communication stratégique entre un encodeur informé observant une variable de source et transmettant des signaux à un ou plusieurs décodeurs censés prendre des actions affectant la performance de l'encodeur. Avant d'observer la source, l'encodeur s'engage et annonce la stratégie d'encodage à mettre en œuvre. Tous les agents communicants sont supposés rationnels, et dotés d'objectifs variés capturés par des fonctions de coût distinctes et arbitraires. Dans cette thèse, nous considérons un grand nombre de copies indépendantes et identiquement distribuées (i.i.d.) du jeu de persuasion Bayésienne, sous contraintes d'information. En raison des restrictions imposées à la communication, il n'y a pas assez de messages disponibles pour transmettre la source. Nous devons donc envisager le codage par blocs au lieu du codage ponctuel. Ce modèle généralise le problème de codage source avec perte de Shannon, où les agents coopèrent pleinement pour obtenir une transmission d'informations véridique et fiable qui minimise leurs mesures de distorsion alignées, au scénario non-coopératif dans lequel les joueurs ne partagent pas nécessairement un objectif commun, mais sélectionnent les stratégies de codage qui minimisent leurs fonctions de coût respectives et non alignées. Nous étudions le problème de communication stratégique dans trois scénarios différents. Premièrement, nous considérons la configuration de codage par raffinement successif dans laquelle un seul encodeur communique un message public à deux décodeurs, et un message privé à un seul décodeur d'entre eux. Lors de la réception du message, chaque décodeur dessine la suite d'action qui minimise sa fonction de coût à long terme. Deuxièmement, nous considérons le réseau de communication Gray-Wyner, avec un seul encodeur et deux décodeurs, dont le coût de l'un dépend de l'action de l'autre, et chacun observant un signal public et un signal privé. Dans ce cadre, chaque engagement du codeur induit un jeu Bayésien entre les décodeurs qui admet des équilibres Bayes-Nash parfaits. Troisièmement, nous combinons le modèle de persuasion Bayésienne avec le réseau de description multiple en cascade, où l'information est transmise de l'encodeur au décodeur via un relais. Dans chacun de ces cas, nous étudions les limites théoriques de l'information de la communication stratégique et décrivons le comportement asymptotique de la fonction de coût à long terme optimale de l'encodeur. À l'aide de variables aléatoires auxiliaires, nous caractérisons la fonction de coût minimal du codeur soumise au schéma de compression optimal qui satisfait les contraintes imposées sur la quantité d'informations transmises, ainsi que les contraintes d'incitation des décodeurs.

## Mots-Clés:

Théorie de l'information; Théorie des jeux; Communication Stratégique; Persuasion Bayésienne; Codage Source; Canal Multi-utilisateur; Raffinement Successif; Réseau Gray-Wyner; Réseau de Description Multiple en Cascade

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# MY PUBLICATIONS

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## International journal:

- **Rony Bou Rouphael** and Maël Le Treust. ‘*Strategic Successive Refinement Coding with Interdependent Decoders Cost Functions*’. In international IEEE journal of selected areas in Information Theory, April. 2022 (Second round revision).

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- **Rony Bou Rouphael** and Maël Le Treust. ‘*Strategic Communication via the Gray-Wyner Network with Cost-Dependent Decoders*’. In IEEE: Information Theory Workshop 2022, Mumbai, India. Nov. 2022.
- **Rony Bou Rouphael** and Maël Le Treust. ‘*Strategic Communication via Cascade Multiple-Description Network*’. In IEEE: 58th Allerton Conference on Control Communication and Computing, Monticello, Illinois. Nov. 2022.
- **Rony Bou Rouphael** and Maël Le Treust. ‘*Strategic Successive Refinement Coding for Bayesian Persuasion with Two Decoders*’. In IEEE: Information Theory Workshop 2021, Kanazawa, Japan. Oct. 2021.
- **Rony Bou Rouphael** and Maël Le Treust. ‘*Impact of Private Observation in Bayesian Persuasion*’. In International Conference on NETWORK Games CONTROL and OPTimization, NETGCOOP2020. Carsège, France. Mar. 2020.

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- **Rony Bou Rouphael** and Maël Le Treust. ‘*Strategic Compression for Bayesian Persuasion via Gray-Wyner Network*’. In Colloque Francophone de Traitement du signal et des images, GRETSI’22. Nancy, France. Sept. 2022.

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- **Rony Bou Rouphael** and Maël Le Treust. *'Impact of Private Observation in Bayesian Persuasion'*. In European Network for Game Theory Conference, GAMENET2019. Institute of Information Theory and Automation, gamenet19.utia.cas.cz. Prague, Czech Republic. Nov. 2019.

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- **Rony Bou Rouphael**. *'Multi-User Bayesian Persuasion with Side Information'*. In Panthéon-Sorbonne University, Master's thesis (M2). July 2019.
- **Rony Bou Rouphael**. *'Classification of Finite Reflection Groups Generated by Reflections'*. In Sorbonne Paris Nord University, Master's thesis (M1). July 2018.

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# LIST OF SYMBOLS

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The list below describes several symbols that will be used throughout this document.

## General Notation

$\mathcal{A}$  Calligraphic letters for sets of alphabets

$\mathcal{A}^n$  Cartesian product  $\prod_{t=1}^n \mathcal{A}$  of the set  $\mathcal{A}$

$A$  Uppercase letters for random variables

$a$  Lowercase letters for symbol realizations

$A^n$  Sequence of random variables of length  $n$

$a^n$  Sequence of symbols of length  $n$

$\|x^n\|$   $L^1$ - norm  $\sum_{t=1}^n |x_t|$  of  $x^n \in \mathbb{R}^n$

$|x|$  Absolute value of  $x$

$a_i$   $i$ -th component of the sequence  $a^n$

$\lfloor x \rfloor$   $= \max\{m \in \mathbb{Z}, m \leq x\}$

$[i : 2^x]$  The set  $\{i, i + 1, \dots, 2^{\lfloor x \rfloor}\}$

$\log$  Logarithm of base 2

$\text{conv}\mathcal{A}$  The convex envelop of set  $\mathcal{A}$

## Blackboard Bold

$\mathbb{N}$  Set of natural numbers  $\{0, 1, \dots\}$

$\mathbb{R}$  Real number field

$\mathbb{R}_+$  Positive real numbers

$\mathbb{N}^*$  Set  $\mathbb{N} \setminus \{0\}$

$\mathbb{R}^*$  Set  $\mathbb{R} \setminus \{0\}$

$\mathbb{R}^n$   $n$ -dimensional real Euclidean space

$\mathbb{R}^{n \times m}$  Set of matrices with  $n$  rows and  $m$  columns and real values

## Set Theory

$a \in \mathcal{A}$   $a$  is an element in  $\mathcal{A}$

$|\mathcal{A}|$  Cardinality of the finite set  $\mathcal{A}$

$\mathcal{A} \cup \mathcal{B}$  Union of sets  $\mathcal{A}$  and  $\mathcal{B}$

$\mathcal{A} \cap \mathcal{B}$  Intersection of sets  $\mathcal{A}$  and  $\mathcal{B}$

$\mathcal{A} \subseteq \mathcal{B}$   $\mathcal{A}$  subset of  $\mathcal{B}$

$\mathcal{A} \setminus \{a\}$  Set  $\mathcal{A}$  without the element  $a$

## Probability Theory

$\mathcal{P}_X(x)$  Probability mass function  $\mathbb{P}\{X = x\}$

$\Delta(\mathcal{X})$  Probability simplex over  $\mathcal{X}$

$\prod_{t=1}^n \Delta(\mathcal{X}) \subset \Delta(\mathcal{X}^n)$  The  $n$ -th Cartesian product of the probability simplex  $\Delta(\mathcal{X})$  and a proper subset of  $\Delta(\mathcal{X}^n)$

$\mathbb{E}_{\mathcal{P}}[X]$  Expected value of  $X$  drawn from distribution  $\mathcal{P}$

$\mathcal{P}_X^{\otimes n} \in \prod_{t=1}^n \Delta(\mathcal{X})$  Probability distribution over  $\mathcal{X}^n$  defined as the  $n$ -th product of distributions  $\mathcal{P}_X$  over  $\mathcal{X}$

## Information Theory

$\mathcal{E}$  The encoder

$\mathcal{D}, \mathcal{D}_i$  The decoder or the  $i$ -th decoder

$R$  The rate of symbols per transmission

$H(X)$  Shannon's entropy of a random variable  $X$

$H_b(X)$  Binary entropy of a binary random variable  $X$

$I(X; Y)$  Shannon's mutual information between two random variables  $X$  and  $Y$

$\mathcal{T}_\delta^n(\mathcal{P}_X)$  The set of all  $\delta$ -typical sequences  $x^n \in \mathcal{X}^n$  corresponding to  $\mathcal{P}_X$ : Let  $\mathcal{X}$  be a finite alphabet and  $x^n$  a sequence in  $\mathcal{X}^n$ , and let  $\pi_{x^n}$  the empirical probability mass function (pmf) over  $\mathcal{X}$  corresponding to the relative frequency of symbols in  $x^n$ , i.e.  $\pi_{x^n}(x) = \frac{|t: x_t=x|}{n}$  for  $x \in \mathcal{X}$ .

The sequence  $x^n$  is said to be  $\delta$ -typical with respect to a probability distribution  $\mathcal{P}_X$  on  $\mathcal{X}$  if

$$\sum_{x \in \mathcal{X}} |\pi_{x^n}(x) - \mathcal{P}_X(x)| \leq \delta \mathcal{P}_X(x).$$

$\mathcal{T}_\delta^n(\mathcal{P}_{X_1 X_2 \dots X_k})$  The set of all  $k$ -tuples  $(x_1^n, x_2^n, \dots, x_k^n) \in \mathcal{X}_1^n \times \mathcal{X}_2^n \dots \times \mathcal{X}_k^n$  that are jointly  $\delta$ -typical with respect to the joint distribution  $\mathcal{P}_{X_1 X_2 \dots X_k}$

$D(\mathcal{P} \parallel \mathcal{Q})$  *Kullback-Leiber* (KL) divergence between two probability distributions  $\mathcal{P}$  and  $\mathcal{Q}$

## Game Theory

$G = (\mathcal{N}, (\mathcal{V}_i)_{i \in \mathcal{N}}, (\mathcal{T}_i)_{i \in \mathcal{N}}, (\mathcal{P}^i)_{i \in \mathcal{N}}, (C_i)_{i \in \mathcal{N}})$  A finite Bayesian game, often denoted by  $G = ((\mathcal{T}_i)_{i \in \mathcal{N}}, (\mathcal{V}_i)_{i \in \mathcal{N}})$ , where:

- $\mathcal{N} = \{1, 2, \dots, n\}$  is the set of players
- $\forall i \in \mathcal{N}$ ,  $\mathcal{V}_i$  is the set of actions of Player  $i$
- $\forall i \in \mathcal{N}$ ,  $\mathcal{T}_i$  is the set of types of Player  $i$ ,
- $\forall i \in \mathcal{N}$ ,  $\mathcal{P}^i : \mathcal{T}_i \mapsto \Delta(\mathcal{T}_1^{i-1} \times \mathcal{T}_{i+1}^n)$  associates to each type of Player  $i$  a belief about the types of other players
- $\forall i \in \mathcal{N}$ ,  $C_i : \prod_{i=1}^n \mathcal{V}_i \times \prod_{i=1}^n \mathcal{T}_i \mapsto \mathbb{R}$  gives the cost of Player  $i$  as a function of profiles of actions and types

# 1

## INTRODUCTION

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### 1.1 Strategic communication

When communicating information about themselves in order to reach a certain goal, individuals can attempt to obtain a more favorable outcome by manipulating, selecting or disclosing parts of the available information to influence the decision of a decision-maker. Such strategic individuals exploit the information they possess about the decision-maker, like the decision-maker's objective function, while designing their information disclosure policy. Situations like these are increasingly being observed in Machine Learning (ML), for instance in classification problems [Brückner et al. \(2012\)](#). As ML is being used to make important decisions about individuals in areas such as education, health, employment, and commerce among others, a strategic person can try to get a better classification result by selectively revealing information to the classifier, especially if the transmitter is in possession of information about the classifier like its objective function. Another interesting situation observed in ML and involving strategic communication between multiple agents is negotiation. ML is used to train agents in complex multiple-agent contexts featuring private information. For instance, [Lewis et al. \(2017\)](#) found out that an agent trained on negotiations under incomplete information with reinforcement learning, fails to reach an agreement with humans more often than a human-imitative model did.

As [Kamenica and Gentzkow \(2011\)](#) described it, the problem of strategic communication comes about each time the question of *who gets to know what* arises. They described a setting from court where a prosecutor (the encoder) tries to persuade a judge (the decoder)

that a certain defendant is guilty. The prosecutor is better off if the judge chooses to convict regardless of whether the defendant is innocent or guilty. However, the judge prefers to convict a guilty defendant and acquit an innocent one.

Other real-world examples involving persuasion arise from a wide range of domains including economics, finance, marketing, politics, education, and security among others. When a seller wants to advertise for a product so a potential client would buy it, or a political candidate devises an electoral campaign trying to persuade people to vote for him/her, or a student fills in an application for a university scholarship program, or a potential employee presents to a job interview, (the list goes on and on), the transmitted information in each of these different cases is strategically selected in a way that best achieves the objective of the interested encoder. The decoder on the other end, updates its Bayesian beliefs about the state based on the strategically selected information by the encoder, and takes an action accordingly. In other words, the encoder's goal consists of designing an information disclosure policy specifying what bits of information should be revealed in order to attain a specific objective subject to the challenges imposed by the channel's bandwidth.

Communication systems involving strategic agents are not exclusive to human interactions, or human-machine interactions but can also be found in the interactions of connected autonomous devices. The Internet of Things (IoT) has enabled seamless communication between connected devices with minimal human intervention. In such settings, autonomous devices use collected data to make informed decisions, according to the goals that have been ingrained as their purpose. For example, autonomous vehicles perceive their surroundings information from a combined variety of sensors which makes them capable of moving safely with little or no human input. As artificial intelligence (AI) systems are deployed to act on behalf of humans in more real-world circumstances, from commercial negotiations in the nearer term [Chakraborty et al. \(2020\)](#), to high-stakes strategic decision-making in the longer term [Geist and Lohn \(2018\)](#), they will need to be able to act effectively in environments where agents can have common or conflicting interests. Such contexts of multi-agent communication and interaction often exhibit opportunities for cooperation (when a joint action leads to mutual benefit) or conflict (when interests of different agents are misaligned). In fact, these advanced computer agents don't just think and act like humans, but they think

and act rationally, and rationality is a descriptive feature of game-theoretical behaviors.

This brings us to the essence of this thesis, which explores an emerging line of interdisciplinary research, between information theory and game theory. Information Theory is a mathematical science concerned with the representation, communication, processing and utilization of Information using optimal methods and algorithms. While Game Theory is a branch of applied Mathematics, that provides a theoretical framework for the analysis of strategic interactions among rational agents. As this work addresses the problem of strategic information selection and transmission through decentralized, decision-oriented, multi-user communication networks, it can be naturally studied at the intersection of these two disciplines.

## 1.2 Position of the problem

Shannon's theory of communication is mainly descriptive of settings involving cooperating agents. But what if agents are self-interested, and the communication of information involves incentive constraints? Such non-cooperative communication scenarios which frequently arise in real-world circumstances, require new multi-disciplinary approaches and techniques to achieve optimal outcomes. In this thesis, we are interested in scenarios where an informed and interested encoder aims to "persuade" one or two decoders to choose a certain action from a finite set of actions by controlling their information environment. In the persuasion problem discussed above, between a classifier and an individual generating data, the individual interested in improving their classification result can be viewed as the encoder and the classifier making the utmost decision that affects the encoder is the decoder.

We study a strategic communication game through a decentralized network with restricted communication. A decentralized network is one where information-processing workload is distributed across multiple separate devices instead of relying on a single central server. We consider several models of such decentralized networks, where communication happens between one informed encoder, observing an independent and identically distributed (i.i.d) source sequence, and one or two decoders making the ultimate decisions that

affect the encoder, based on locally available information. All communicating agents are endowed with non-aligned objectives captured by distinct and arbitrary cost functions. We assume that each player is aware of the objectives of other players, i.e. cost functions are known by all players and thus encoding and decoding functions are selected accordingly. The strategic communication game is assumed to be played by blocks of  $n$  symbols, and the Bayesian beliefs of the decoders about the state are updated once the transmission is over. The state is an  $n$ -sequence of random variables generated by a memoryless source. Before the game starts, the encoder commits to and reveals its encoding function to the decoders. Because of the restrictions imposed on communication, not enough messages are available to transmit the whole source. We need to consider block-coding instead of one-shot coding. This setting generalizes the lossy source coding problem of Shannon, where agents fully cooperate to achieve truthful and reliable information transmission that minimizes their aligned distortion measures, to the non-cooperative scenario in which players do not necessarily share a common objective, but select the coding strategies that minimize their respective non-aligned cost functions. This setting, also referred to as the Bayesian persuasion game, enables the use of entropy constraints in order to derive the fundamental information-theoretic limits of compression and communication.

The timeline of the game is as follows: The encoder commits to an encoding scheme and reveals its commitment to the decoder(s). Then, the source sequence of independent and identically distributed random variables is drawn and observed by the encoder who then transmits one or many signals to the decoder(s), according to its encoding function. In situations involving two decoders, our models assume that at least one decoder has access to a private signal transmitted by the encoder. Thereafter, each decoder observes its received message, and draws a sequence of actions according to its respective decoding function.

When communication involves two decoders, we model two distinct scenarios: In one case, we consider communication involving independent decoders, i.e. the cost function of one decoder depends on its choice of action and on the state only and does not depend on the action taken by the other decoder. In the other, communication involves cost-dependent decoders, i.e. the cost function of one decoder depends on the action taken by the other decoder in addition to its own action and the state. The latter case turned out to be quite

more challenging as each commitment of the encoder induces a Bayesian game among the decoders. This Bayesian game admits Bayes-Nash equilibria at which the pair of decoding functions will be played.

We are interested in the asymptotic behavior of the long-run cost function of the encoder in each of the considered scenarios. The goal is to design an achievable coding scheme that minimizes the encoder's long-run cost function subject to the challenges imposed by the channel, and satisfying the incentive constraints of the decoders.

### 1.3 Literature Review

The information-theoretic approach to Bayesian persuasion, which takes into account the limits of the communication capacity, is an emerging field of research. Strategic compression in the context of strategic information transmission was addressed in [Akyol et al. \(2017\)](#). The problem is modeled as a Stackelberg game with mismatched objectives, and asymptotic limits of equilibrium strategies are characterized along with their associated costs for three different problem settings. In several recent contributions, Vora and Kulkarni investigate the region of achievable rates of strategic communication between agents with distinct utility functions. In [Vora and Kulkarni \(2020d\)](#), [Vora and Kulkarni \(2020c\)](#), and [Vora and Kulkarni \(2021\)](#), authors address the problem of extracting truthful information from a strategic sender with an incentive to misreport information. Modeled as a Stackelberg game in which the decoder is the Stackelberg leader, authors investigate the fundamental limits to the amount of truthful information that can be perfectly recovered by the receiver. They introduce the notion of information extraction capacity which quantifies the rate at which the number of perfectly recovered source sequences grows with the sequence block-length in the presence of a noiseless channel. They compute upper and lower bounds on the maximal rate of information extraction from the strategic sender. [Vora and Kulkarni \(2020b\)](#), [Vora and Kulkarni \(2020a\)](#), study a related information extraction problem and show that there exists a strategy which allows the receiver to recover a large number of source sequences while achieving an asymptotically vanishing error probability. The computational aspects

of the persuasion game are considered in [Dughmi et al. \(2016\)](#), where the impact of the channel's capacity on the optimal utility is investigated.

When objective functions of the players are perfectly aligned, the common goal of the encoder and the decoder is to reliably and truthfully transmit information over the communication channel. In this case, the communication problem under study, boils down to the lossy source coding problem of [Shannon \(1959\)](#). [Kamenica and Gentzkow \(2011\)](#) consider mismatched utility functions, and assume that the communication channel is perfect. However; if the channel is imperfect and the players have common objectives, the problem becomes the joint source-channel coding problem of [Shannon \(1959\)](#). In fact, in his theory of information [Shannon \(1948\)](#), [Shannon \(1959\)](#) analyzes the rate at which a source of information can be reliably transmitted over a channel, i.e with an arbitrarily small probability of error. [Lapidoth \(1997\)](#) studies the source coding problem with mismatched distortions between the encoder and the decoder. The author analyzes the minimum distortion at which the source sequence can be encoded knowing that the reconstruction of the source sequence is done to minimize the decoder's distortion. Using a random coding argument, Lapidoth derives an upper bound on the resulting distortion. Coding for multiple distortion functions is also addressed in [Wolf et al. \(1980\)](#), [El Gamal and Cover \(1982\)](#), multiple description coding where several descriptions are produced from the same source, and the distortion of the reconstruction depends on which subset of descriptions is available to the decoder.

The problem of strategic communication was formulated in the game theory literature, and originally referred to as the sender-receiver game, with no communication constraints. The Nash equilibrium solution of the cheap talk game was investigated in the seminal paper of [Crawford and Sobel \(1982\)](#), in which the encoder and the decoder are endowed with distinct objectives and choose their coding strategies simultaneously. The message sent by the sender is assumed to be costless and unverifiable and the communication happens in a single transmission. Crawford and Sobel's solution is based on the quantization of a continuous state. In our approach, the source is discrete and the solution relies on the quantization of the source using an auxiliary random variable which is correlated to the source according to a conditional distribution over the auxiliary random variable given the source. In the continuous case, a different kind of quantization based on intervals is required.

Blume et al. (2007) study the cheap talk through a noisy channel. They show that the presence of noise could possibly improve the sender’s welfare. Kamenica and Gentzkow (2011) study strategic communication with sufficiently large message sets. The encoder commits to an encoding strategy before observing the source. In this thesis, we generalize their model, which is presented in section 2.2, to limited communication scenarios with two decoders.

In our approach, we investigate the Bayesian persuasion game that is played by blocks of source sequences and action sequences which allows us to use coding techniques. We combine the game-theoretic mismatched motives, with the information-theoretic bounds on communication. The mismatched objectives of the players are captured by cost functions, defined as the distortion functions in the lossy source coding problem. We use the bounds on the regions of achievable rates (i.e the rates that achieve truthful information transmission) derived in information theory, to restrict the sets of target distributions of the encoder, and we analyze the asymptotic behavior of the encoder’s optimal cost. We follow the approach of Le Treust and Tomala (2019a) who study the fundamental limits of the one-to-one Bayesian persuasion game through the lens of Information Theory. This model will be further explained in section 2.3.

Our approach to Bayesian persuasion under information constraints, is closely related to the rate-distortion theory which describes the trade-offs between lossy compression rate and the resulting distortion. We model Bayesian persuasion as a similar trade-off problem between the rate of symbols per transmission, and the cost function of the encoder subject to the incentive constraints of the decoders. For a given compression rate, we analyze the minimal cost of the encoder for sufficiently large blocklengths. Rate-distortion theory has been used in wide range of applications including secrecy systems Schieler and Cuff (2014), cellular signaling Andrews and Iglesias (2007), image caching Weidmann et al. (1997) among other. Dumas et al. (2017) study the problem of image compression in which learning is performed on whole images under a global rate-distortion constraint. Image patches compete together when computing their sparse representation, enabling variable rate image compression for images of any size after a single training. Nikbakht et al. (2022) surveyed the mixed-delay traffic in 5G and 6G wireless networks from an information-theoretic perspec-

tive. Applications of these modern networks differ in reliability and latency requirements. Authors focus on the set of simultaneously achievable rate pairs for Ultra-Reliable Low-Latency Communication and enhanced Mobile Broadband messages. [Saeidian et al. \(2021\)](#) investigate membership privacy in machine learning models via an operationally meaningful privacy metric: maximal leakage. Authors propose a pointwise membership privacy approach for studying information leakage of individual data entries in a database.

Bayesian persuasion has been studied in contexts where communication is restricted by either exogenous constraints imposed by the adopted model, or endogenous constraints that eventually emerge. Our work considers Bayesian persuasion under exogenous constraints imposed by the communication channel’s capacity which is determined by a fixed and limited rate of transmission and hence imposes constraints on the cardinality of the message sets. Bayesian persuasion under exogenous constraints has been the subject of interest of several papers in the literature. In such settings, the encoder’s set of signals is restricted, which results in a smaller set of feasible posterior beliefs of the decoder. This could result from constraints on the amount of information, [Le Treust and Tomala \(2019a\)](#), [Dughmi et al. \(2016\)](#), or communication being coarse [Gradwohl et al. \(2021\)](#), or the presence of other considerations like privacy or discrimination [Babichenko et al. \(2021\)](#). [Dughmi et al.](#), used Bayesian persuasion under exogenous restrictions to model auction settings [S. Dughmi and Roth \(2014\)](#) and bilateral trade [Dughmi et al. \(2016\)](#). Recently, [Babichenko et al. \(2021\)](#), consider the problem of strategic signaling in online ad auctions in a way that prevents discrimination and protects the privacy of the user when sending information to an advertiser. Authors classify exogenous constraints into two families: ex-ante and ex-post. Posteriors in both cases are mapped via a continuous function to a real value. Ex-ante constraints impose restrictions in expectation, while ex-post constraints limit each instance of Sender-Receiver communication to be below a certain value (refer to [Volund \(2018\)](#)). In our model, the encoder commits to an encoding strategy and reveals it to the decoder(s) before observing the source which induces ex-ante restrictions.

Endogenously emerging constraints on Bayesian persuasion have also been studied in several models. In such setups, the encoder’s commitment to truthfully reveal information is weakened, but there are no restrictions on the set of feasible signals that are available

for the encoder. In [Nguyen and Tan \(2021\)](#), the commitment assumption of the encoder is weakened in the persuasion problem and quantified as a communication cost. The encoder observes the signal realization in private and transmits any message to the decoder. In a similar model, [Guo and Shmaya \(2021\)](#) study a sender-receiver game between a platform and its customers, while assuming that customers trust the platform's forecasts in an equilibrium even though they do not observe the algorithm used by the platform to generate information. The platform, which is the encoder in that case, does not reveal ex-ante the experiment to the customer or the decoder. As in the cheap-talk literature [Crawford and Sobel \(1982\)](#), the decoder observes only the message, but not the strategy of the encoder.

## 1.4 Contributions

It is only recently that the problem of strategic communication started to gain attention from the information-theoretic community in response to the arising challenges in communication settings involving strategic agents. We are proud to be among the earliest to put together the rate-distortion theory with strategic, non-cooperative communication settings involving agents with mismatched objectives, and study the fundamental limits of communication in multi-user networks under incentive constraints.

**Chapter 2:** Our first contribution to this work was to analyze the impact of the side information on the encoder's optimal cost via a binary example. Using the convex characterizations of the optimal costs derived in [Le Treust and Tomala \(2021\)](#), with and without side information, we run numerical simulations to get the favorable settings under which the encoder can benefit from persuasion. We identify the optimal splitting of the decoder's beliefs satisfying the information constraint imposed by the restricted communication channel, and we compute the encoder's optimal cost value, with and without private signal. Varying the parameters such as the prior belief, the precision of the private signal and the channel capacity, we propose the settings that are more favorable to the encoder.

**Chapter 3:** We are the first to model the Bayesian persuasion game with two decoders in a successive refinement coding setup. We establish a model for strategic successive refinement coding setup, in which each decoder is endowed with a non-aligned cost function that depends on the state, and its own action, whereas the encoder is endowed with a cost function that depends on the actions of both decoders and the state. This setting differs from the standard successive refinement coding setup where only decoders are endowed with distortion measures and the encoder's goal is to transmit information subject to a fidelity criterion. We define the encoder's optimal signal as a conditional distribution on the message pairs given the source under the constraints imposed on the achievable region of rates in the successive refinement setup. We characterize the single-letter best responses of the decoders that satisfy their respective incentives, and we derive the expected value of the encoder's optimal single-letter cost as a function of the rate pairs. We fully describe the limiting behavior of the encoder's long run expected cost as well as its lower bound for a sufficiently large block-length  $n$ .

We provide a technical novelty in proving the achievability statement of our strategic successive refinement: We reformulate the single-letter solution as an optimization over a dense subset of target distributions in which every commitment of the encoder induces Bayesian beliefs according to which the decoders are not indifferent between two actions. Moreover, as we shift from the game by blocks of length  $n$ , to the one-shot game, we characterize the single-letter Bayesian beliefs using auxiliary random variables, and we show that those beliefs are close on average, under the kl-divergence, to the beliefs induced by the coding functions. which relies on the standard arguments of codebook generation and analysis of the probability of the error events, was to investigate the beliefs of the decoders about the state. These arguments are crucial in order to guarantee the desired convergence. In the converse proof, we identify our auxiliary random variables and show that the single-letter characterization is a lower bound for the expected value of the encoder's long-run optimal cost.

**Chapter 4:** We are the first to model the Bayesian persuasion game with two decoders under the information constraints imposed by the Gray-Wyner network. We establish a

model for strategic lossy compression via a Gray-Wyner network. Each decoder is endowed with a non-aligned cost function that depends on the state, and on actions of both decoders, and the encoder is endowed with a cost function that depends on the actions of both decoders and the state. We define the encoder's optimal signal as a conditional distribution on the message pairs given the source under the constraints imposed on the achievable region of rates in the successive refinement setup. Every commitment of the encoder in this setting induces a finite Bayesian game among the decoders which admits perfect Bayes-Nash equilibria. Using two auxiliary random variables, we characterize the single-letter Bayesian game and its set of Bayes-Nash equilibria. We characterize the encoder's optimal signal as a conditional distribution on the auxiliary random variables given the source subject to the achievable rate region of the Gray-Wyner network. We group them in two sets: one with all conditional distributions that satisfy a certain Markov chain, and one with the general distributions. We derive an upper bound on the long run cost of the encoder in the achievability statement of our main result which depends on the set of target distributions that satisfies a Markov chains, and a lower bound in the converse statement, which depends on the set of target distributions that does not satisfy any Markov chains. Besides from the standard arguments of codebook generation and analysis of the probabilities of error events, our achievability proof presents a novel technique which allows the passage from the long run cost of the encoder in the game by blocks of length  $n$ , to the single-letter optimal cost in the one shot game. We characterize each of the intermediate games and we control the beliefs of each decoder about the state and about the type of the other decoder. In our converse proof we identify the auxiliary random variables and show that single-letter characterizations induce decoding strategies that form Bayes-Nash equilibria.

**Chapter 5:** We model the sequential Bayesian persuasion game through a relay using a Cascade multiple description network with an encoder, a decoder and a relay. We modify the standard cascade multiple description network by assuming that all three players have distinct cost functions which depend on the state and on the decoder's action. We study several cases which either relate to other works in the information theory literature or game theory literature: 1) the cooperative case with equal cost functions 2) unrestricted communication 3) locally restricted communication 4) local cooperation. We also study a

binary example under which communication is not subject to any information constraints. Following the convexification method of Kamenica and Gentzkow, we use the convex closure of the relay's expected cost in order to characterize the optimal solution for the encoder.

## 1.5 Thesis organization

This thesis is organized as follows. In [chapter 2](#), we start with the point-to-point communication model in which a single encoder communicates to persuade one decoder to change action. In [section 2.1](#), we review some fundamental concepts of point-to-point communication: the lossy source coding setup in [subsection 2.1.1](#), the joint source-channel coding and the separation theorem in [subsection 2.1.2](#). Wyner-Ziv's communication model with side Information at the decoder in [subsection 2.1.3](#). In [section 2.2](#), we start with the game theoretic approach to Bayesian persuasion with no restrictions on the amount of transmitted information. In [subsection 2.2.1](#), we introduce the base model of [Kamenica and Gentzkow \(2011\)](#) and in [subsection 2.2.2](#), we present their convexification method. In [section 2.3](#), we study Bayesian persuasion under information constraints in two different models: In [subsection 2.3.1](#), we review the model of [Le Treust and Tomala \(2019a\)](#), which considers noisy Bayesian persuasion by combining both the model of [Kamenica and Gentzkow \(2011\)](#) with Shannon's lossy source coding. In [subsection 2.3.2](#), we present [Le Treust and Tomala \(2021\)](#) model of noisy Bayesian persuasion with side information at the decoder. In [section 2.4](#), we analyze a binary example [subsection 2.4.1](#), and study the impact of side information on the encoder's optimal cost in [subsection 2.4.2](#). We conclude the chapter in [section 2.5](#).

In [chapter 3](#), we study the multi-user Bayesian persuasion between one encoder and two decoders via a successive refinement coding setup. In [section 3.1](#) we summarize our problem and contributions. In [section 3.2](#), we review the successive refinement coding setup. In [section 3.3](#), we present the Bayesian persuasion model with two independent decoders endowed with distinct cost functions ([subsection 3.3.1](#)), and we characterize the optimal encoding subject to the successive refinement information constraints, as well as the optimal single letter expected cost of the encoder ([subsection 3.3.2](#)). Then we present our main result which describes the asymptotic behavior of the encoder's optimal long run expected cost

and determines its lower bound. In [section 3.4](#), we present the proof of achievability which consists of four main parts: first, in [subsection 3.4.3](#), we provide an alternative formulation to our solution concept by restricting the set of target distributions to a dense subset of distributions inducing beliefs based on which the decoders are not indifferent between two or more actions. Second, in [subsection 3.4.2](#), we generate the codebook and analyze the error probability. Third in [subsection 3.4.1](#), we restrict our optimization to a dense subset of target distributions in order to eliminate decoder's indifferences between two pairs of actions.. Finally in [subsection 3.4.4](#), we control the Bayesian beliefs of each decoder about the state concluding by that our proof of achievability. In [section 3.5](#), we present the proof of the converse statement of our main result. In [section 3.6](#), we wrap up the chapter with some concluding remarks.

In [chapter 4](#), we investigate Bayesian persuasion over the Gray-Wyner network. In [section 4.1](#), we summarize our contributions. In [section 4.2](#) we introduce the model of Gray-Wyner and the region of achievable rates. In [section 4.3](#), we study the persuasion game between one encoder and two cost-dependent decoders, i.e the cost function of one decoder depends on the action of the other. In [subsection 4.3.1](#), we present the model via a Gray-Wyner network. In [subsection 4.3.2](#), we introduce auxiliary random variables and characterize the sets of optimal distributions subject to the Gray-Wyner information constraints. We characterize the one-shot Bayesian game, and two formulations of the optimal single-letter cost of the encoder: one which corresponds to the case where auxiliary random variables satisfy a certain Markov chain and another formulation in which auxiliary random variables do not satisfy a Markov chain. Our main result consists of two statements: an achievability statement and a converse statement. In [section 4.4](#), we study a particular case with  $R_2 = 0$ . This corresponds to the strategic successive refinement network with interdependent decoders' cost functions. In [section 4.5](#), we present the achievability proof of our main result. This proof consists of four main parts: First in [subsection 4.5.2](#), we present the Gray-Wyner codebook generation and analyze the error events. Second in [subsection 4.5.3](#), we reformulate our solution using a dense subset of target distributions which induce essential equilibria only for decoder's subsequent Bayesian game. Third in [subsection 4.5.4](#), we control the Bayesian beliefs of each decoder about the state and about the type of the other

decoder. Finally in [subsection 4.5.1](#), we define and analyze the different Bayesian games at each stage  $t \in \{1, 2, \dots, n\}$  which concludes our proof of achievability. In [4.5](#), we identify the auxiliary random variables and prove the converse statement of our main result. In [4.6](#), we sum up the chapter with some concluding remarks and possible extensions.

In [chapter 5](#), we study the problem of persuasion in which information is cascaded from an encoder to a relay, and from the relay to the decoder. In [section 5.1](#) we briefly summarize the chapter. In [section 5.2](#), we present the cascade multiple description coding setup. In [section 5.3](#), we present the Bayesian persuasion game via a Cascade network. In [section 5.4](#), we consider the unrestricted communication with  $R_1 = R_2 = \log |\mathcal{U}|$ . Second in [section 5.5](#), we consider the cooperative scenario with  $c_1 = c_2 = c_3$ . In [section 5.6](#), we consider the case where either the encoding is unrestricted i.e  $R_1 = \log |\mathcal{U}|$  or relaying information is not subject to any information constraint with  $R_2 = \log |\mathcal{U}|$ . We also present the covering lemma with a slight modification. In [section 5.7](#), we consider local cooperation between either the relay and the encoder with  $c_1 = c_2$  or the relay shares the same cost function with the decoder  $c_2 = c_3$ . In [section 5.8](#), we illustrate the problem with a binary example. We apply the convexification method of Kamenica-Gentzkow, and we illustrate the optimal settings for the encoder. In [section 5.9](#), we conclude this section. Finally, in [chapter 6](#), we wrap-up this thesis with some concluding remarks and perspectives.

# 2

## POINT-TO-POINT STRATEGIC COMMUNICATION

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*In this chapter, we present the base models of strategic communication between one encoder and one decoder. We start by reviewing some preliminary concepts of point-to-point communication theory. Obviously, section 2.1 can be omitted by the familiar reader.*

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## 2.1 Introduction to point-to-point communication

A point-to-point communication scheme is the simplest form of communication networks. It is represented by a directed graph  $G = (V, e)$ , where  $V = \{\mathcal{E}, \mathcal{D}\}$  is the set of vertices or nodes in the communication network consisting of an encoder  $\mathcal{E}$  and a decoder  $\mathcal{D}$ , and  $e = \{(\mathcal{E}, \mathcal{D})\}$  is the set of edges which contains a single edge  $(\mathcal{E}, \mathcal{D})$  representing a wireless communication link from the encoder to the decoder. In the context of telecommunication, point-to-point communication without any relay or compression of data is a rare sight in modern networks. A simple phone call is the most prominent example of point-to-point communication between two connected nodes sending and receiving audio. However; a call via Whatsapp or Zoom cannot be considered as a point-to-point communication as data input is first received and encrypted by a cloud or gateway, before it is broadcasted to the terminal node. In the following, we will briefly review the results of point-to-point information theory. We present the models of lossy source coding in [subsection 2.1.1](#), joint source-channel separation result in [subsection 2.1.2](#), and the communication model of Wyner-Ziv with decoder's side information in [subsection 2.1.3](#). These concepts are fundamental for the study of the Bayesian persuasion game under information constraints which will be developed in [section 2.3](#) and [section 2.4](#), whereas [section 2.2](#) will be devoted for the model of Kamenica and Gentzkow where information is unconstrained.

### 2.1.1 Lossy source coding

[Shannon \(1959\)](#) introduced the concept of specifying the rate  $R > 0$ , i.e the exponential growth rate of the number of available messages, required to represent a discrete source under a certain fidelity criterion. The communication system in Shannon's theory represents a cooperative scenario in which the transmitter and the receiver share the same objective of transmitting reliable and truthful information. Let  $\mathcal{U}$  denote a discrete alphabet representing the source, and  $\mathcal{P}_{\mathcal{U}} \in \Delta(\mathcal{U})$  a probability distribution over  $\mathcal{U}$  that is assumed to be known by the encoder and the decoder. The lossy source coding setup, depicted in [Figure 2.1](#), models communication between an encoder  $\mathcal{E}$  observing an  $n$ -sequence,  $n \in \mathbb{N}^*$ , of discrete, independent and identically distributed (i.i.d) random variables  $U^n$ , distributed according

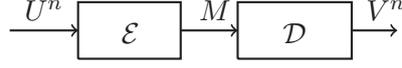


Figure 2.1: Source coding setup

to the i.i.d distribution  $\mathcal{P}_{U^n} = \prod_{t=1}^n \mathcal{P}_{U_t}$ , and transmitting to decoder  $\mathcal{D}$  a message  $M$  taking values from a set of size  $2^{\lfloor nR \rfloor}$ . The decoder observes the encoded message and draws an  $n$ -sequence  $V^n$  from a discrete alphabet  $\mathcal{V}$  that is different from  $\mathcal{U}$ . The encoder and the decoder share a cost function  $c : \mathcal{U} \times \mathcal{V} \mapsto \mathbb{R}$ , and a common goal of keeping the expected average distortion between  $U^n$  and  $V^n$  small. In the following, we define an  $(R, n)$  lossy source code and present the lossy source coding theorem of Shannon. More details about lossy source coding can be found in (El Gamal and Kim, 2011, section 3.6).

**Definition 2.1.1.** *An  $(R, n)$  lossy source code consists of the following:*

- *An index set  $\{1, 2, \dots, 2^{\lfloor nR \rfloor}\}$ .*
- *An encoding function  $\sigma : \mathcal{U}^n \rightarrow \{1, 2, \dots, 2^{\lfloor nR \rfloor}\}$  which assigns an index  $m$  to each sequence  $u^n$ .*
- *A decoding function  $\tau : \{1, 2, \dots, 2^{\lfloor nR \rfloor}\} \rightarrow \mathcal{V}^n$  which assigns an estimate  $v^n \in \mathcal{V}^n$  to each index  $m$ .*
- *A joint distribution defined by*

$$\mathcal{P}^{\sigma, \tau} = \left( \prod_{t=1}^n \mathcal{P}_{U_t} \right) \mathcal{P}_{U^n|M}^{\sigma} \mathcal{P}_{V^n|M}^{\tau}. \quad (2.1)$$

We denote by  $\mathcal{C}(R, n)$  the set of pairs  $(\sigma, \tau)$  of an  $(R, n)$  code.

**Definition 2.1.2.** *Let  $c : \mathcal{U} \times \mathcal{V} \rightarrow \mathbb{R}$  a cost function. We define the long run distortion function  $c^n(\sigma, \tau)$  as follows:*

$$c^n(\sigma, \tau) = \sum_{u^n, v^n} \mathcal{P}^{\sigma, \tau}(u^n, v^n) \frac{1}{n} \sum_{t=1}^n c(u_t, v_t). \quad (2.2)$$

In the following, we present the analogous problem of minimizing the cost for a given rate  $R$  as it is more relevant to the models we present in the next chapters.

**Definition 2.1.3.** The long-run cost-rate function  $\Gamma^n(R)$  gives, for a rate  $R$ , the minimal long-run cost that could be achieved by an  $(R, n)$  lossy source code

$$\Gamma^n(R) = \min_{(\sigma, \tau) \in \mathcal{C}(R, n)} c^n(\sigma, \tau). \quad (2.3)$$

**Definition 2.1.4.** We define the single-letter cost-rate function  $\Gamma(R)$  that gives the lowest cost provided  $R \geq 0$  by

$$\Gamma(R) = \min_{\mathcal{P}_{V|U}: I(U; V) \leq R} \mathbb{E}(c(U, V)). \quad (2.4)$$

**Theorem 2.1.1** (Lossy Source Coding for a DMS [Shannon \(1959\)](#)). For a rate  $R \in \mathbb{R}^+$ , we have

$$\lim_{n \rightarrow \infty} \Gamma^n(R) = \inf_{n \in \mathbb{N}^*} \Gamma^n(R) = \Gamma(R). \quad (2.5)$$

The achievability proof of [Theorem 2.1.1](#) consists of showing that  $\lim_{n \rightarrow \infty} \Gamma^n(R) \leq \Gamma(R)$  and the converse proof shows that for all  $n$ ,  $\inf_{n \in \mathbb{N}^*} \Gamma^n(R) \geq \Gamma(R)$ . Along with Fekete's lemma we get  $\lim_{n \rightarrow \infty} \Gamma^n(R) = \inf_{n \in \mathbb{N}^*} \Gamma^n(R)$ .

### 2.1.2 Joint source-channel coding

Consider the case where the encoder wishes to communicate  $n$  symbols of an uncompressed discrete memoryless source  $U$  with entropy  $H(U)$  in  $n$  transmissions through the discrete memoryless channel  $(\mathcal{X}, \mathcal{Y}, \mathcal{P}_{Y|X})$  with capacity  $C$ . Let  $c : \mathcal{U} \times \mathcal{V} \mapsto \mathbb{R}$  a cost function,  $C(R)$  a cost-rate function, a DMS source  $U$  and a DMC  $\mathcal{P}_{Y|X}$  of capacity  $C$ .



Figure 2.2: Lossy joint source-channel coding setup

**Definition 2.1.5.** A lossy joint source-channel  $n$ -code consists of the following:

- An encoding function  $\sigma : \mathcal{U}^n \rightarrow \mathcal{X}^n$  which assigns a codeword  $x^n \in \mathcal{X}^n$  to each sequence  $u^n \in \mathcal{U}^n$ .
- A decoding function  $\tau : \mathcal{Y}^n \rightarrow \mathcal{V}^n$  which assigns an estimate  $v^n \in \mathcal{V}^n$  to each sequence  $y^n \in \mathcal{Y}^n$ .
- A joint distribution defined by

$$\mathcal{P}^{\sigma, \tau} = \left( \prod_{t=1}^n \mathcal{P}_{U_t} \right) \mathcal{P}_{X^n|U^n}^{\sigma} \left( \prod_{t=1}^n \mathcal{P}_{Y|X} \right) \mathcal{P}_{V^n|Y^n}^{\tau} \quad (2.6)$$

The long-run distortion  $c^n(\sigma, \tau)$  of the decoder is defined as in Def. 2.1.2. We are interested in the asymptotic behavior of the optimal long-run cost i.e. the limit as  $n$  goes to infinity of  $\Gamma^n = \min_{\sigma, \tau} c^n(\sigma, \tau)$ . The single-letter cost-rate function  $\Gamma(R)$  is given as in Def. 2.1.4

Note that the channel's capacity  $C$  (Shannon (1948)) is given by

$$C = \max_{\mathcal{P}_X} I(X; Y) = \max_{\mathcal{P}_X} \sum_{(x, y) \in \text{supp} \mathcal{P}_X \mathcal{P}_{Y|X}} \mathcal{P}_X(x) \mathcal{P}_{Y|X}(y|x) \log_2 \frac{\mathcal{P}_{Y|X}(y|x)}{\sum_{x'} \mathcal{P}_X(x') \mathcal{P}_{Y|X}(y|x')}. \quad (2.7)$$

The maximum in (2.7) is taken over the set of probability distributions  $\mathcal{P}_X \in \Delta(\mathcal{X})$ . When the channel is perfect, i.e.  $\mathcal{P}_{Y|X}(y|x) = \mathbb{1}_{\{y=x\}}$ , the channel capacity  $C$  satisfies

$$C = \max_{\mathcal{P}_X} I(X; Y) = \max_{\mathcal{P}_X} H(X) = \log_2 |\mathcal{X}|. \quad (2.8)$$

**Theorem 2.1.2** (Lossy Source-Channel Separation). *Given a DMC channel  $(\mathcal{X}, \mathcal{P}_{Y|X}, \mathcal{Y})$  of capacity  $C$ , the following holds:*

$$\lim_{n \rightarrow \infty} \Gamma^n = \inf_{n \in \mathbb{N}^*} \Gamma^n = \Gamma(C). \quad (2.9)$$

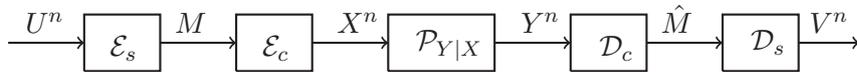


Figure 2.3: Source-channel separation

### 2.1.3 Communication with side information at the decoder

Wyner and Ziv (1976) extended the work of Slepian and Wolf (1973), to establish information-theoretic bounds for lossy compression with side information at the decoder. Let  $U$  and  $Z$  represent two i.i.d. random variables, distributed according to  $\mathcal{P}_{UZ}^{\otimes n}$ , of possibly infinite alphabets  $\mathcal{U}$  and  $\mathcal{Z}$ , modeling source data and side information respectively. We assume that the source values are encoded without access to the side information as depicted in Figure 2.4. Unlike the encoder, the decoder has access to the side information  $Z$ , and obtains a reconstruction of the source values in alphabet  $\mathcal{V}$ .

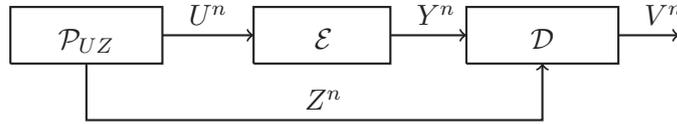


Figure 2.4: Lossy compression with side information at the decoder

**Definition 2.1.6.** A lossy source  $n$ -code with side information at the decoder consists of the following:

- An encoding function  $\sigma : \mathcal{U}^n \rightarrow \mathcal{Y}^n$  which assigns a codeword  $y^n \in \mathcal{Y}^n$  to each sequence  $u^n \in \mathcal{U}^n$ .
- A decoding function  $\tau : \mathcal{Y}^n \times \mathcal{Z}^n \rightarrow \mathcal{V}^n$  which assigns an estimate  $v^n \in \mathcal{V}^n$  to each pair of sequences  $(y^n, z^n) \in \mathcal{Y}^n \times \mathcal{Z}^n$ .
- A joint distribution defined by

$$\mathcal{P}^{\sigma, \tau} = \left( \prod_{t=1}^n \mathcal{P}_{U_t Z_t} \right) \mathcal{P}_{Y^n | U^n}^{\sigma} \mathcal{P}_{V^n | Y^n Z^n}^{\tau} \quad (2.10)$$

Given  $\mathcal{P}_{UZ}$ , the long run cost function  $c^n(\sigma, \tau)$  is given as in Def. 2.1.2. Let  $W \in \mathcal{W}$  denote an auxiliary random variable such that  $|\mathcal{W}| \leq |\mathcal{U}| + 1$ .

**Definition 2.1.7.** Let  $R > 0$  be given. We define the Wyner-Ziv's cost-rate function  $\Gamma_{U|Z}(R)$  with side information at the decoder by

$$\Gamma_{U|Z}(R) = \min_{\mathcal{P}_{W|U} \mathcal{P}_{V|WZ}: R \geq I(U; W|Z)} \mathbb{E} \left( \frac{1}{n} \sum_{t=1}^n c(U_t, V_t) \right). \quad (2.11)$$

In the following, we present [Wyner and Ziv \(1976\)](#)'s Theorem for lossy compression with side information at the decoder.

**Theorem 2.1.3** (Lossy Compression with Side Information at the Decoder). *Given a rate  $R \geq 0$ , a cost function  $c : \mathcal{U} \times \mathcal{V} \rightarrow \mathbb{R}$ , a DMS source  $U$  and a side information  $Z$ , distributed according to  $\mathcal{P}_{UZ}^{\otimes n}$ , the following holds:*

$$\lim_{n \rightarrow \infty} \Gamma^n(R) = \inf_n \Gamma^n(R) = \Gamma_{U|Z}(R). \quad (2.12)$$

## 2.2 Bayesian persuasion with unlimited communication capacity

Bayesian persuasion analyzes situations in which one party, the encoder, transmits information optimally to the other party, the decoder, who has to make a decision based on the provided information. Optimality means either maximizing the encoder's payoff, or minimizing its cost. In the game theory literature, the Bayesian persuasion game was introduced as a point-to-point communication model with no limits on the amount of information transmitted. Each player is endowed with a distinct, non-aligned cost function which is known by the other player. The informed encoder communicates the i.i.d. source, with a sufficiently large number of bits, to a decoder who updates its Bayesian beliefs based on the signal received from the encoder, and draws an action sequence that affects the encoder's cost.

[Kamenica and Gentzkow \(2011\)](#) formulate the Stackelberg version of the one-shot strategic communication game, in which the encoder (the Stackelberg leader) commits to and reveals an encoding strategy before observing the state, and the decoder (the Stackelberg follower) chooses its strategies as a best-response to the encoder's strategy. The authors study the ex ante solution of a belief-based model of persuasion, in which a strategic transmitter wants to persuade a decoder to change its action, which affects the welfare of both agents, by influencing its Bayesian beliefs about the state. This model considers the encoder to be an information designer ([Bergemann and Morris \(2019\)](#)) who commits to and reveals

the signaling scheme, without knowledge of the state. A signal in this model is assumed to be costless and can be viewed as a map from the true state of the world to a distribution over some signal realization space. The utility functions in this game are known by all players. The encoder's goal is to select the signal that maximizes its own payoff while satisfying the incentive constraints of the decoder. Kamenica and Gentzkow characterize the signals that are optimal from the encoder's perspective and derive necessary and sufficient conditions for the existence of such signals. They show that the problem of choosing an optimal signal can be reexpressed as a problem of choosing a distribution of posterior beliefs such that the average of posteriors equal the prior belief. This allows the use of a geometric approach (the concavification/convexification method) in order to find the optimal signals.

Information design with multiple designers interacting with a set of agents is studied in [Koessler et al. \(2021\)](#). Authors establish the existence of equilibria, characterize mixed or pure equilibria in special situations where interested parties design information to influence the behavior of decision-makers, and show when it is without loss of generality to restrict designers to the use of simple information policies. In [Sarıtaş et al. \(2017\)](#), [Sarıtaş et al. \(2019\)](#), the Nash equilibrium solution is investigated for multi-dimensional sources and quadratic cost functions, whereas the Stackelberg solution is studied in [Sarıtaş et al. \(2020\)](#). The one shot noisy Bayesian persuasion game was also studied in [Tsakas and Tsakas \(2021\)](#) where they show the existence of an optimal signaling structure and establish necessary and sufficient conditions to weakly increase the sender's payoff with respect to the Blackwell-informativeness of the noisy channel when the message space and the channel are binary.

### 2.2.1 The model of Kamenica-Gentzkow with distinct objectives

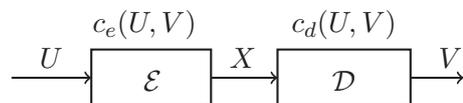


Figure 2.5: Kamenica-Gentzkow's one-shot Bayesian persuasion

As depicted in Figure 2.5, the encoder  $\mathcal{E}$  observes a state variable  $U$  drawn by nature according to  $\mathcal{P}_U$  and taking its values from a finite alphabet  $\mathcal{U}$ , and sends a symbol  $x \in \mathcal{X}$ , where  $\mathcal{X}$  is a finite alphabet such that  $|\mathcal{X}| \geq |\mathcal{U}|$ , to decoder  $\mathcal{D}$  who then selects an action  $v$  from a finite action set  $\mathcal{V}$ . Players in this game are endowed with distinct costs defined as follows.

**Definition 2.2.1.** *The encoder and decoder's cost functions  $c_e$  and  $c_d$  respectively are given by*

$$c_e : \mathcal{U} \times \mathcal{V} \mapsto \mathbb{R}, \quad (2.13)$$

$$c_d : \mathcal{U} \times \mathcal{V} \mapsto \mathbb{R}. \quad (2.14)$$

We assume that cost functions and alphabets are known by all players. The sender and the receiver share a prior belief  $\mathcal{P}_U \in \Delta(\mathcal{U})$  about the state. The signal designed and revealed by the encoder before observing the source, is a profile of conditional distributions over the symbols  $x \in \mathcal{X}$  for each given source symbol  $u \in \mathcal{U}$ . We denote the encoder's signal by  $\sigma$  and the decoder's strategy by  $\tau$ .

**Definition 2.2.2.** *The encoding  $\sigma$  and decoding  $\tau$  of the encoder and the decoder are given as follows:*

$$\sigma : \mathcal{U} \longrightarrow \Delta(\mathcal{X}), \quad (2.15)$$

$$\tau : \mathcal{X} \longrightarrow \Delta(\mathcal{V}). \quad (2.16)$$

Each pair of stochastic strategies  $(\sigma, \tau)$  induces a joint distribution  $\mathcal{P}_{UXV}^{\sigma, \tau}$  over  $\mathcal{U} \times \mathcal{X} \times \mathcal{V}$  defined by

$$\mathcal{P}_{UXV}^{\sigma, \tau} = \mathcal{P}_U \mathcal{P}_{X|U}^{\sigma} \mathcal{P}_{V|X}^{\tau} \quad (2.17)$$

For every given signal  $\sigma$  of the encoder, the decoder's set  $BR(\sigma)$  of best-response

strategies  $\tau$  is defined as follows

$$BR(\sigma) = \left\{ \tau, \mathbb{E}_{\mathcal{P}^{\sigma, \tau}}[c_d(U, V)] \leq \mathbb{E}_{\mathcal{P}^{\sigma, \tilde{\tau}}}[c_d(U, V)], \forall \tilde{\tau} \neq \tau \right\}. \quad (2.18)$$

The encoder's optimal signal is the one that minimizes its expected cost and satisfies the incentives of the decoder. Therefore, the encoder has to solve the following problem:

$$\Gamma_e = \min_{\sigma} \min_{\tau \in BR(\sigma)} \mathbb{E}_{\mathcal{P}^{\sigma, \tau}}[c_e(U, V)]. \quad (2.19)$$

Below is the ordered timeline of the Bayesian persuasion game:

- The encoder  $\mathcal{E}$  chooses, announces the signal  $\sigma$ .
- The state  $U$  is drawn according to  $\mathcal{P}_U$ .
- The symbol  $x \in \mathcal{X}$  is transmitted by the encoder according to  $\mathcal{P}_{X|U}^{\sigma}$ .
- Knowing  $\sigma$ , decoder  $\mathcal{D}$  observes  $x$  and selects an action  $V$  according to the behavior<sup>1</sup> strategy  $\tau \in BR(\sigma)$ .
- The cost values are given by  $c_e(U, V)$  and  $c_d(U, V)$

### 2.2.2 Kamenica-Gentzkow's convexification method

Kamenica and Gentzkow's solution concept is an encoder-preferred subgame perfect equilibrium, i.e for a given encoding strategy  $\sigma$ , and a signal realization  $x$ , the decoder updates its posterior belief  $\mathcal{P}_U(\cdot|x)$  about the state using Bayes's rule, and accordingly selects a best-response action  $v$ . Let  $p \in \Delta(\mathcal{U})$  denote the posterior belief of the decoder about the state.

**Definition 2.2.3.** For a given posterior belief  $p \in \Delta(\mathcal{U})$ , we define the decoder's set of best-response actions  $V^*(p) \subset \mathcal{V}$  as follows

$$V^*(p) = \arg \min_{v \in \mathcal{V}} \mathbb{E}_p[c_d(U, v)]. \quad (2.20)$$

<sup>1</sup>A behavior strategy  $\tau_i$  of player  $i$  in a Bayesian game is a mapping  $\mathcal{T}_i \mapsto \Delta(\mathcal{V}_i)$ , where  $\mathcal{T}_i$  is the set of types of player  $i$  and  $\mathcal{V}_i$  is the set of actions.

If at a given belief, the decoder is indifferent between two or more actions, the model assumes it takes the one that minimizes the encoder's expected cost.

For a given posterior belief  $p \in \Delta(\mathcal{U})$ , let  $v^*(p)$  denote a best-response action in  $V^*(p)$ . The sender's expected cost value  $\Psi_e(p)$  is given by

$$\Psi_e(p) = \mathbb{E}_p[c_e(U, v^*(p))] \quad (2.21)$$

The optimal cost of the encoder is the convex envelop of the encoder's expected cost function evaluated at the prior  $\mathcal{P}_U$  as follows

$$\Gamma_e = \inf_{(\lambda_x, p_x)_{x \in \mathcal{X}}} \left\{ \sum_{x \in \mathcal{X}} \lambda_x \cdot \Psi_e(p_x), \quad \sum_{x \in \mathcal{X}} \lambda_x = 1, \quad \sum_{x \in \mathcal{X}} \lambda_x \cdot p_x = \mathcal{P}_U \right\} \quad (2.22)$$

The infimum is taken over  $(\lambda_x, p_x)_{x \in \mathcal{X}}$  which is an optimal splitting of the prior belief  $\mathcal{P}_U$  of the decoder. This splitting needs to be Bayes plausible ([Kamenica and Gentzkow \(2011\)](#)), or in other words it needs to satisfy the splitting lemma ([Aumann and Maschler \(1995\)](#)) which says that on average the posterior beliefs are equal to the prior belief. This convex formulation provides an interesting geometric approach to compute the encoder's optimal cost for the one-shot scenario.

## 2.3 Bayesian Persuasion under information constraint

The model of Le Treust and Tomala [Le Treust and Tomala \(2019a\)](#) considers that the communication channel is subject to exogenous noise, and assumes that the encoder and the decoder are engaged in a large number of i.i.d. copies of the same base game. They formulate the strategic communication problem as a joint source-channel coding problem, in which one informed encoder transmits to a decoder who's suppose to choose an action that affects the encoder. The players are endowed with mismatched objectives captured by distinct utility functions. Each player aims to maximize its respective payoff. They investigate the effect of the noise on the sender's expected utility as the number of problems and reported messages increases, and formulate their solution as a function of the channel's

capacity. They show that the optimal strategy of the sender correlates all the messages with the state parameters of all the problems, and hence the sender can achieve a better payoff by linking the independent problems together. The main result describes two features of the asymptotic behavior of the long-run optimal utility of the encoder: 1) An achievability statement showing that the encoder's optimal long-run expected utility converges to its single letter expected utility. 2) A converse statement showing that the expected single letter utility of the encoder forms an upper bound to its optimal long run expected utility. Using separation source-channel coding, the authors prove the existence of an achievable coding scheme that is optimal for the encoder while satisfying the incentive constraints of the decoder. Then they show that the beliefs of the decoder that are induced by the coding strategies and the single-letter beliefs that correspond to the target distribution that satisfies the information constraints imposed by the channel are close on average under the  $KL$ -divergence. Using Lagrangian methods, the authors characterize the optimal payoff of the encoder as the concave closure of its expected payoff function evaluated at the prior belief of the decoder. The presence of noise reduces the set of feasible posterior beliefs of the decoders, which harms the encoder's payoff compared to what it could have achieved in persuasion through a perfect communication channel. The result of [Le Treust and Tomala \(2019a\)](#) is reviewed and generalized by [Doval and Skreta \(2018\)](#) to multiple constraints, providing by that a general toolbox for studying constrained Bayesian persuasion. In the same spirit, [Jackson and Sonnenschein \(2007\)](#), found that in an incentive compatible mechanism design problem, the designer can achieve a better outcome when the problem is linked with a large number of independent copies of itself. We generalize the model in [Le Treust and Tomala \(2019a\)](#) to strategic communication involving multiple agents. In [Matysková and Montes \(2021\)](#), authors extend the Bayesian persuasion game to the case where decision makers may have access to some additional information at a costly effort. Unlike in the model of [Kamenica and Gentzkow \(2011\)](#), the receiver chooses her own signal at a cost to acquire more information before taking an action. In this setting, such additional information can decrease the encoder's payoff. Therefore, the encoder has to solve the Bayesian persuasion game under an additional constraint: the receiver never learns. Authors suggest a new solution method that does not rely directly on the concavification of the encoder's expected payoff, and which is also applicable to the standard Bayesian persuasion model.

The case where the decoder privately observes a signal correlated to the state, also referred to as the [Wyner and Ziv \(1976\)](#) setting, is studied in [Akyol et al. \(2017\)](#), [Bou Rouphael and Le Treust \(2020a\)](#) and [Le Treust and Tomala \(2021\)](#). Wyner and Ziv establish a lower bound for the achievable rates under a distortion constraint when a single source is encoded independently but encoded with the presence of a side information at the decoder [Wyner and Ziv \(1976\)](#). The main result of Wyner and Ziv shows that in presence of decoder's side information, encoding the source independently can result in a rate loss compared to the setting where the side information is also accessible by the encoder.

In [Le Treust and Tomala \(2021\)](#), authors extend the model in [Le Treust and Tomala \(2019a\)](#) by assuming that the decision maker or the decoder has access to a side information correlated to the state. They assume an analogous scenario in which mismatched utilities are replaced by misaligned distortion functions and that the strategic goal of each player is to minimize its respective distortion function. They showed that the main result of [Le Treust and Tomala \(2019a\)](#) can be extended to the noisy communication involving side information at the decoder. The solution has also been characterized as the convex envelop of the encoder's optimal expected distortion evaluated at the prior. Our first contribution to this work was to investigate the impact of the decoder's side information on the outcome that could be achieved by the encoder [Bou Rouphael and Le Treust \(2020a\)](#). The work by [Akyol et al. \(2017\)](#), [Akyol et al. \(2016\)](#), and [Akyol et al. \(2015\)](#) considers Gaussian source and channels with the Crawford-Sobel's quadratic cost functions. Their result suggests that the optimal solution for the one-shot communication problem remains optimal when considering several strategic communication problems. This is not the case when the source and the channel are general and discrete source, channel and general mismatched cost functions. In [Nadendla et al. \(2018\)](#), authors further expanded these results for cases where the prior beliefs about the source and the channel are heterogeneous and non identical. Bayesian persuasion involving rational individuals with different prior beliefs was also studied in [Alonso and Câmara \(2016\)](#). Authors provide necessary and sufficient conditions under which a sender benefits from persuading decoders with distinct prior beliefs. Quadratic cost measures that depend on the state in the context of Bayesian persuasion were also addressed in [Sayin and Başar \(2021\)](#). In our model, the cost functions are general,

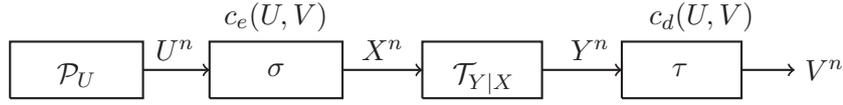


Figure 2.6: Noisy Bayesian persuasion

and the source is discrete. When two decoders are considered, we assume they both share the same prior belief about the state. The case of general cost functions with a Gaussian source could be very interesting to tackle.

### 2.3.1 Le Treust and Tomala's approach to strategic lossy source-channel coding

The encoder  $\mathcal{E}$  observes an i.i.d. source  $U^n \sim \mathcal{P}_U^{\otimes n}$  taking its values from a finite alphabet  $\mathcal{U}$ , and inputs a sequence  $X^n$ , where  $\mathcal{X}$  is a finite alphabet into the channel.  $Y^n$  is the channel output drawn according to the memoryless channel  $\mathcal{T}_{Y|X}$  and observed by decoder  $\mathcal{D}$  who then selects an action sequence  $V^n$  with  $\mathcal{V}$  a finite alphabet as in Figure 2.6.

**Definition 2.3.1.** *Let  $n \in \mathbb{N}^*$ . The encoding  $\sigma$  and decoding  $\tau$  of the encoder and the decoder respectively are given as follows:*

$$\sigma : \mathcal{U}^n \longrightarrow \Delta(\mathcal{X}^n), \quad (2.23)$$

$$\tau : \mathcal{Y}^n \longrightarrow \Delta(\mathcal{V}^n). \quad (2.24)$$

Every coding pair  $(\sigma, \tau)$  induces a joint probability distribution  $\mathcal{P}^{\sigma, \tau}$  over  $\mathcal{U}^n \times \mathcal{X}^n \times \mathcal{Y}^n \times \mathcal{V}^n$  given by

$$\mathcal{P}^{\sigma, \tau} = \left( \prod_{t=1}^n \mathcal{P}_{U_t} \right) \times \mathcal{P}_{X^n|U^n}^\sigma \times \left( \prod_{t=1}^n \mathcal{T}_{Y_t|X_t} \right) \times \mathcal{P}_{V^n|Y^n}^\tau. \quad (2.25)$$

**Definition 2.3.2.** *The long run cost functions of the encoder and the decoder are defined by*

$$c_e^n(\sigma, \tau) = \mathbb{E}_{\sigma, \tau} \left[ \frac{1}{n} \sum_{t=1}^n c_e(U_t, V_t) \right] = \sum_{u^n, v^n} \mathcal{P}_{U^n V^n}^{\sigma, \tau}(u^n, v^n) \cdot \left[ \frac{1}{n} \sum_{t=1}^n c_e(u_t, v_t) \right], \quad (2.26)$$

$$c_d^n(\sigma, \tau) = \mathbb{E}_{\sigma, \tau} \left[ \frac{1}{n} \sum_{t=1}^n c_d(U_t, V_t) \right] = \sum_{u^n, v^n} \mathcal{P}_{U^n V^n}^{\sigma, \tau}(u^n, v^n) \cdot \left[ \frac{1}{n} \sum_{t=1}^n c_d(u_t, v_t) \right]. \quad (2.27)$$

where single-letter costs  $c_e$  and  $c_d$  are the same as in Definition 2.2.1.

**Definition 2.3.3.** For a given encoding  $\sigma$ , the set of decoder's best-response strategies  $BR(\sigma)$  is given by

$$BR(\sigma) = \left\{ \tau, \quad c_d^n(\sigma, \tau) \leq c_d^n(\sigma, \tilde{\tau}), \forall \tilde{\tau} \neq \tau \right\}. \quad (2.28)$$

If for a given  $\sigma$ , there exists more than one strategy  $\tau \in BR(\sigma)$ , we assume that the decoder will select the one that maximizes the encoder's long-run cost.

**Definition 2.3.4.** The encoding problem for the noisy Bayesian persuasion is given by

$$\Gamma_e^n = \inf_{\sigma} \max_{\tau \in BR(\sigma)} c_e^n(\sigma, \tau). \quad (2.29)$$

### 2.3.1.1 Solution Formulation

Let  $W$  be an auxiliary random variable taking values from  $\mathcal{W}$  such that  $|\mathcal{W}| = \min(|\mathcal{U}| + 1, |\mathcal{V}|)$ . The authors characterize the optimal signal of the encoder as a conditional distribution that satisfies the information constraints imposed by the noisy channel, and formulate the optimal single-letter cost of the encoder accordingly.

**Definition 2.3.5.** The encoder's set of target distributions  $\mathbb{Q}_0$ , and the decoder's set of best-responses  $\mathbb{Q}_2(\mathcal{Q}_{W|U})$  for every conditional distribution  $\mathcal{Q}_{W|U}$  are respectively characterized as follows

$$\mathbb{Q}_0 = \left\{ \mathcal{Q}_{W|U}, \quad \max_{\mathcal{P}_X} I(X; Y) - I(U; W) \geq 0 \right\}, \quad (2.30)$$

$$\mathbb{Q}_2(\mathcal{Q}_{W|U}) = \arg \min_{\mathcal{Q}_{V|W}} \mathbb{E}_{\mathcal{P}_U \mathcal{Q}_{W|U} \mathcal{Q}_{V|W}} [c_d(U, V)]. \quad (2.31)$$

Therefore, the linear program formulation of the encoder's optimal single-letter cost is given by

$$\Gamma_e^* = \inf_{\mathcal{Q}_{W|U} \in \mathcal{Q}_0} \max_{\mathcal{Q}_{V|W} \in \mathcal{Q}_2(\mathcal{Q}_{W|U})} \mathbb{E}_{\mathcal{P}_U \mathcal{Q}_{W|U} \mathcal{Q}_{V|W}} [c_e(U, V)]. \quad (2.32)$$

The main result of [Le Treust and Tomala \(2019a\)](#), fully describes the limiting behavior of the encoder's optimal long-run expected cost  $\Gamma_e^n$ , and describes its lower bound.

**Theorem 2.3.1** ([Le Treust and Tomala \(2019a\)](#)).

$$\lim_{n \rightarrow \infty} \Gamma_e^n = \inf_{n \in \mathbb{N}^*} \Gamma_e^n = \Gamma_e^*. \quad (2.33)$$

The proof of [Theorem 2.3.1](#) can be found in ([Le Treust and Tomala, 2019a](#), Subsection 3.1.1) and consists of proving the achievability of the optimal single-letter cost for sufficiently large blocklength  $n$ , and conversely, the proof of the lower bound. Note that the problem in [Le Treust and Tomala \(2019a\)](#) was formulated with utility functions instead of costs where the player's goal is to maximize their respective utility functions and the converse statement of the main result provides an upper bound for the encoder's optimal long-run expected cost. In the following, we provide the convex formulation of the solution for the Bayesian persuasion game over a noisy channel.

Let  $C > 0$  denote the channel's capacity and  $p \in \Delta(\mathcal{U})$  the belief parameter of the decoder about the state. Assuming that if two or more actions minimize the decoder's cost, the latter will select the action that maximizes the encoder's cost, the best-response action of the decoder for a given belief  $p$  is given by

$$v^*(p) \in \arg \max_{v \in \arg \min \mathbb{E}_p [c_d(U, v)]} \mathbb{E}_p [c_e(U, v)]. \quad (2.34)$$

The encoder's expected cost for a belief  $p$  of the decoder is given by

$$\Psi_e(p) = \mathbb{E}_p [c_e(U, v^*(p))]. \quad (2.35)$$

Therefore, the optimal single-letter cost that can be achieved by the encoder in the noisy Bayesian persuasion game is characterized as the convex envelop of its expected cost formulated as a function of the decoder's beliefs as follows

$$\Gamma_e^* = \inf_{(\lambda_w, p_w)_{w \in \mathcal{W}}} \left\{ \sum_{w \in \mathcal{W}} \lambda_w \cdot \Psi_e(p_w), \quad \sum_{w \in \mathcal{W}} \lambda_w = 1, \quad \sum_{w \in \mathcal{W}} \lambda_w \cdot p_w = \mathcal{P}_U, \right. \\ \left. \sum_w \lambda_w \cdot H(p_w) \geq H(U) - C, |\mathcal{W}| = \min(|\mathcal{U}| + 1, |\mathcal{V}|) \right\}. \quad (2.36)$$

If we remove the information constraint imposed by the noisy channel from (2.36), and the cardinality bound of  $\mathcal{W}$ , we revert back to convex formulation (2.22) of [Kamenica and Gentzkow \(2011\)](#).

### 2.3.2 Persuasion with decoder's side information

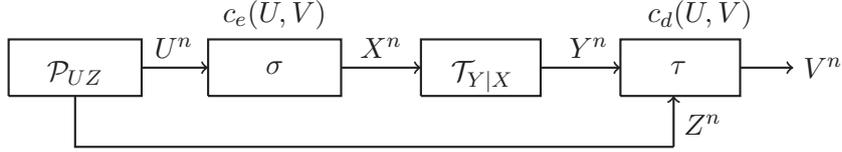


Figure 2.7: Noisy Bayesian persuasion with decoder's side information

Now we study the Bayesian persuasion problem over a noisy channel with the additional assumption that the decoder privately observes a side information  $Z$  correlated to the state  $U$  according to the conditional probability distribution  $\mathcal{P}_{Z|U}$  as in [Figure 2.7](#). In the following, we present the system model and the solution formulation for the noisy Bayesian persuasion with decoder's side information. Finally we derive the convex formulation of the solution.

**Definition 2.3.6.** *The encoding  $\sigma$  and decoding  $\tau$  of the encoder and the decoder respectively are given as follows:*

$$\sigma : \mathcal{U}^n \longrightarrow \Delta(\mathcal{X}^n), \quad (2.37)$$

$$\tau : \mathcal{Y}^n \times \mathcal{Z}^n \longrightarrow \Delta(\mathcal{V}^n). \quad (2.38)$$

Every coding pair  $(\sigma, \tau)$  induces a joint probability distribution  $\mathcal{P}^{\sigma, \tau}$  over  $\mathcal{U}^n \times \mathcal{Z}^n \times \mathcal{X}^n \times \mathcal{Y}^n \times \mathcal{V}^n$  given by

$$\mathcal{P}^{\sigma, \tau} = \left( \prod_{t=1}^n \mathcal{P}_{U_t Z_t} \right) \times \mathcal{P}_{X^n | U^n}^\sigma \times \left( \prod_{t=1}^n \mathcal{T}_{Y_t | X_t} \right) \times \mathcal{P}_{V^n | Y^n Z^n}^\tau. \quad (2.39)$$

**Definition 2.3.7.** *The encoding problem for the noisy Bayesian persuasion with side information at the decoder is given by*

$$\Gamma_s^n = \inf_{\sigma} \max_{\tau \in BR(\sigma)} c_e^n(\sigma, \tau),$$

where  $BR(\sigma) = \left\{ \tau, c_d^n(\sigma, \tau) \leq c_d^n(\sigma, \tilde{\tau}), \forall \tilde{\tau} \neq \tau \right\}$  is the set of best-response strategies  $\tau$  of the decoder for a given encoding  $\sigma$ .

### 2.3.2.1 Solution Formulation

Let  $W$  be an auxiliary random variable such that  $|\mathcal{W}| = \min(|\mathcal{U}| + 1, |\mathcal{V}|^{|\mathcal{Z}|})$ . [Le Treust and Tomala \(2021\)](#) characterize the single letter solution and derive the convex formulation for Bayesian persuasion over a noisy channel with decoder's side information.

**Definition 2.3.8.** *The encoder's set of target distributions  $\mathbb{Q}_0$ , and the decoder's set of best-responses  $\mathbb{Q}_2(\mathcal{Q}_{W|U})$  for every conditional distribution  $\mathcal{Q}_{W|U}$  are characterized as follows*

$$\mathbb{Q}_0 = \left\{ \mathcal{Q}_{W|U}, \max_{\mathcal{P}_X} I(X; Y) - I(U; W|Z) \geq 0 \right\}, \quad (2.40)$$

$$\mathbb{Q}_2(\mathcal{Q}_{W|U}) = \arg \min_{\mathcal{Q}_{V|WZ}} \mathbb{E}_{\mathcal{P}_{UZ} \mathcal{Q}_{W|U} \mathcal{Q}_{V|WZ}} [c_d(U, V)]. \quad (2.41)$$

**Definition 2.3.9.** *The optimal single-letter cost of the encoder  $\Gamma_s$  for the noisy Bayesian persuasion with decoder's side information is characterized as follows*

$$\Gamma_s = \inf_{\mathcal{Q}_{W|U} \in \mathbb{Q}_0} \max_{\mathcal{Q}_{V|WZ} \in \mathbb{Q}_2(\mathcal{Q}_{W|U})} \mathbb{E}_{\mathcal{P}_{UZ} \mathcal{Q}_{W|U} \mathcal{Q}_{V|WZ}} [c_e(U, V)]. \quad (2.42)$$

**Theorem 2.3.2** (Le Treust and Tomala (2021)).

$$\lim_{n \rightarrow \infty} \Gamma_s^n = \inf_{n \in \mathbb{N}^*} \Gamma_s^n = \Gamma_s. \quad (2.43)$$

The proof of [Theorem 2.3.2](#) can be found in (Le Treust and Tomala, 2018, App. B and C).

Let  $p \in \Delta(\mathcal{U})$  denote the decoder's belief about the state. The encoder's expected cost for belief  $p$  is given by

$$\psi_e(p) = \mathbb{E}_p[c_e(U, v^*(p))], \quad (2.44)$$

where  $v^*(p) \in \arg \max_{v \in \arg \min \mathbb{E}[c_d(U, v)]} \mathbb{E}_p[c_e(U, v)]$  the decoder's best-response action for belief  $p$ .

Because of the presence of side information at the decoder, we define the encoder's cost function for a given belief of the decoder about the state averaged over  $Z$ . Similarly, we define the average entropy which is the conditional entropy  $H(U|Z)$  evaluated with respect to  $p \cdot \mathcal{P}_{Z|U}$  instead of  $\mathcal{P}_{UZ}$ .

**Definition 2.3.10.** *The encoder's average utility function  $\Psi_e(p)$  and the average entropy function  $h(p)$  are given as follows:*

$$\Psi_e(p) = \sum_{u,z} p(u) \cdot \mathcal{P}(z|u) \cdot \psi_e\left(\frac{p(\cdot)\mathcal{P}(z|\cdot)}{\sum_{u'} p(u') \cdot \mathcal{P}(z|u')}\right), \quad (2.45)$$

$$h(p) = \sum_{u,z} \mathcal{P}(z) \frac{p(u)\mathcal{P}(z|u)}{\sum_{u'} p(u')\mathcal{P}(z|u')} \cdot \log_2 \frac{\sum_{u'} p(u') \cdot \mathcal{P}(z|u')}{p(u) \cdot \mathcal{P}(z|u)}. \quad (2.46)$$

Therefore, the convex formulation of the optimal single-letter cost for channel's capacity  $C = \max_{\mathcal{P}_X} I(X; Y)$  is given as follows

$$\Gamma_s = \inf_{(\lambda_w, p_w)_{w \in \mathcal{W}}} \left\{ \sum_{w \in \mathcal{W}} \lambda_w \cdot \delta_e^z(p_w), \quad \sum_{w \in \mathcal{W}} \lambda_w = 1, \quad \sum_{w \in \mathcal{W}} \lambda_w \cdot p_w = \mathcal{P}_U, \right.$$

$$\left. \sum_w \lambda_w \cdot h^z(p_w) \geq H(U|Z) - C, |\mathcal{W}| = \min(|\mathcal{U}| + 1, |\mathcal{V}|^{|Z|}) \right\}. \quad (2.47)$$

When removing the information constraints imposed by the noisy channel from formulas (2.36) and (2.47) we revert back to the persuasion game (2.22) of [Kamenica and Gentzkow \(2011\)](#). When the cost functions  $c_e = c_d$ , we are in the [Wyner and Ziv \(1976\)](#) setting and when  $c_e \neq c_d$ , the problem is the one in [Le Treust and Tomala \(2019a\)](#). If the state variable  $U$  and the side information of the decoder  $Z$  are two independent random variables, then formulas (2.36) and (2.47) are equal.

## 2.4 Impact of side observation on Bayesian persuasion

The convex formulations that were presented in the previous sections, are particularly interesting for the one-shot scenario as they allow a geometric representation of the solution as the convex envelop of the encoder's expected cost evaluated at the prior belief of the decoder. In the following, we illustrate the problem using an example with a binary state  $U$ , binary action  $V$ , and binary side observation  $Z$ . We analyze the impact of side information on the encoder's cost by comparing the convex formulations for noisy Bayesian persuasion with (2.47) and without (2.36) decoder's side information.

### 2.4.1 Binary example

Let  $\mathcal{U} = \{u_0, u_1\}$  the binary state space,  $\mathcal{V} = \{v_0, v_1\}$  the decoder's action space, and  $p_0 = \mathbb{P}(U = u_1) \in [0, 1]$  the decoder's prior belief parameter. We consider a binary symmetric noisy channel where  $\mathcal{X} = \{x_0, x_1\}$  denotes the set of channel inputs,  $\mathcal{Y} = \{y_0, y_1\}$  denotes the set of channel outputs. The binary symmetric channel's capacity is given by  $C = \max_{\mathcal{P}_X} (H_b(X) - H_b(X|Y)) = \max_{\mathcal{P}_X} (H_b(X) - \sum_x \mathcal{P}(x) H_b(\epsilon)) = 1 - H_b(\epsilon)$  bits per transmission, where  $\epsilon \in [0, \frac{1}{2}]$  is the noise level such that  $\epsilon = \mathbb{P}(y_j|x_i), i \neq j$  as in [Figure 2.8](#) and  $H_b(p)$  denotes the binary entropy. Let  $\delta \in [0, \frac{1}{2}]$  denote the correlation parameter between the state  $U$  and the side information  $Z$  such that  $\delta = \mathcal{P}_{U|Z}(u_0|z_1) = \mathcal{P}_{U|Z}(u_1|z_0)$ . Note that if

$\delta = 0.5$ , then the state  $U$  and the side  $Z$  are independent, and the persuasion game is as in subsection 2.3.1.

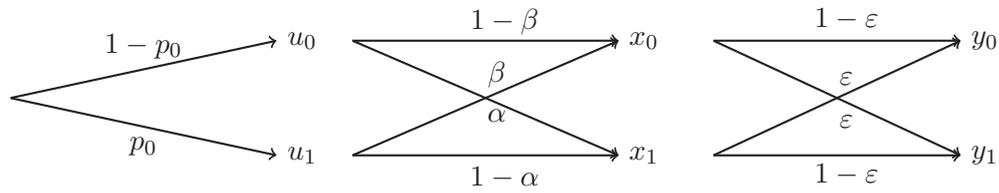


Figure 2.8: Encoder's joint strategy  $\sigma$  over the binary symmetric channel

The cost functions of the decoder and the encoder are given in Table 2.2 and Table 2.1 respectively and the expected costs as functions of the belief  $\mathcal{P}_U(u_1)$  of the decoder are given in Figure 2.9. As the decoder's goal is to minimize its cost, the red lines show the decoder's expected cost for the best-response actions that depend on its belief about the state. The parameter  $\gamma \in [0, 1]$  denotes the threshold at which the decoder changes action from  $v_0$ , the default action that corresponds to belief  $p_0$ , to action  $v_1$  which is preferred by the encoder. In this example, we consider the prior  $p_0 = 0.8$  and the utility threshold  $\gamma = 0.6$ .

Table 2.1: Decoder's cost

	$v_0$	$v_1$
$u_0$	9	0
$u_1$	4	10

Table 2.2: Encoder's cost

	$v_0$	$v_1$
$u_0$	1	0
$u_1$	1	0

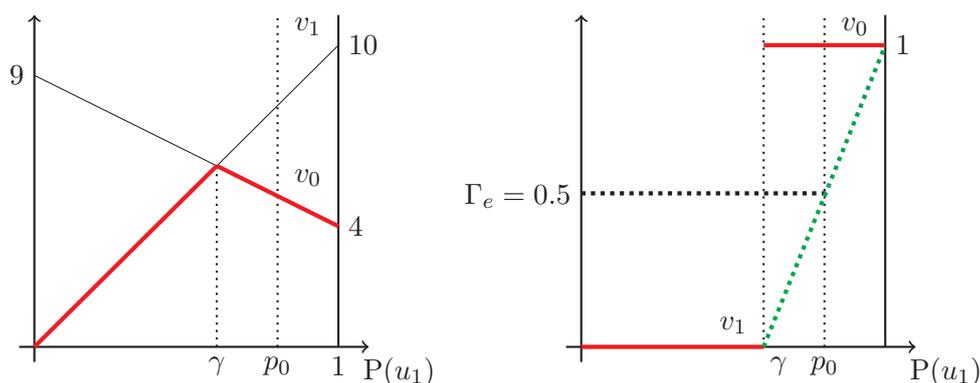


Figure 2.9: Expected cost functions of decoder and encoder with  $p_0 = 0.8$ ,  $\delta = 0.5$ , and  $\gamma = 0.6$ .

The encoder's optimal cost value  $\Gamma_e$  given in (2.22), which corresponds to the Bayesian persuasion game of Kamenica and Gentzkow (2011) in which the side and the state are independent, i.e  $\delta = 0.5$ , and the channel is perfect, i.e  $\epsilon = 0$ , is computed by taking the

convex envelop of the encoder's expected utility represented by the green dotted line in [Figure 2.9](#). The optimal splitting of the decoder's prior belief in this case corresponds to posterior beliefs  $p_1 = \gamma$  and  $p_2 = 1$ . When evaluated at the prior  $p_0$  so as to satisfy the splitting lemma, we get the optimal cost value  $\Gamma_e = 0.5$ .

Using Bayes-rule we compute the decoder's beliefs as follows

$$p_1 = \mathcal{P}(u_1|x_0) = \frac{p_0\alpha}{p_0\alpha + (1-p_0)(1-\beta)}, \quad (2.48)$$

$$p_2 = \mathcal{P}(u_1|x_1) = \frac{p_0(1-\alpha)}{p_0(1-\alpha) + (1-p_0)\beta}. \quad (2.49)$$

Therefore, the encoder's strategy can be derived as follows

$$\alpha = \frac{p_1(p_2 - p_0)}{p_0(p_2 - p_1)}, \quad (2.50)$$

$$\beta = \frac{(1-p_2)(p_0 - p_1)}{(1-p_0)(p_2 - p_1)}. \quad (2.51)$$

Thus, the posterior belief that are feasible need to satisfy either  $0 \leq p_1 \leq p_2 \leq 1$  or  $0 \leq p_2 \leq p_1 \leq 1$ . By substituting  $p_1$  and  $p_2$  with their optimal values 1 and  $\gamma$  as in [Figure 2.9](#), we get the optimal strategy of the encoder for  $\alpha = 0.625$  and  $\beta = 1$ .

### Noisy Bayesian Persuasion without Side Information (Equation for $\Gamma_e^*$ (2.36))

Now assume that the communication is imperfect i.e  $\epsilon > 0$ , and we still assume that the decoder's threshold  $\gamma < p_0$ . In fact, if  $\gamma \geq p_0$ , then the decoder will play  $v_1$  by default as shown in [Figure 2.9](#), and therefore the encoder has no interest in persuasion. The optimal number of posterior beliefs when no side information is available at the decoder is two ([Le Treust and Tomala, 2019a](#), lemma 6.1). Let  $\lambda \in [0, 1]$  and posteriors  $p_1, p_2 \in \Delta(\mathcal{U})$ . These posterior beliefs of the decoder need to satisfy the splitting condition and information constraint imposed by the noisy channel.

$$\left\{ \begin{array}{l} \lambda p_1 + (1-\lambda)p_2 = p_0 \iff \lambda = \frac{p_0 - p_2}{p_1 - p_2} \iff 1 - \lambda = \frac{p_1 - p_0}{p_1 - p_2}, \\ \lambda H_b(p_1) + (1-\lambda)H_b(p_2) \geq H_b(p_0) - C. \end{array} \right. \quad (2.52)$$

Assuming that the information constraint is binding at the optimal, we get

$$\lambda H_b(p_1) + (1 - \lambda)H_1(p_2) = H_1(p_0) - C \quad (2.53)$$

$$\iff H_b(p_1) = \frac{p_0 H_b(p_2) - p_2 (H_b(p_0) - C)}{(p_0 - p_2)} + p_1 \frac{(-H_b(p_2) + H_b(p_0) - C)}{(p_0 - p_2)} \quad (2.54)$$

Let  $p \in [0, 1]$  denote a belief parameter. The encoder's expected cost function depicted in Figure 2.9 is given by  $\Psi_e(p) = \mathbb{1}_{p \in [\gamma, 1]}$ . For each  $p_2 \in [p_0, 1]$ , we denote by  $p_1(p_2)$  the unique solution of (2.54) for a given pair  $(p_0, C)$ . Since the decoder's threshold  $\gamma > p_0$ , hence at the optimum  $p_2 = \gamma$ , thus

$$\Gamma_e^* = \inf_{p_2 \in [0, 1]} \left( \lambda \Psi_e(p_1(p_2)) + (1 - \lambda) \Psi_e(p_2) \right) = \frac{p_1(\gamma) - p_0}{p_1(\gamma) - \gamma}. \quad (2.55)$$

As depicted in Figure 2.10, the orange curve is the binary entropy  $H_b(p)$  and the red lines represent the encoder's expected cost as a function of the decoder's belief about the state. The green dotted line corresponds to the convexification of (2.22) for prior  $p_0 = 0.7$  and threshold  $\gamma = 0.6$ . Assuming that the channel's capacity is  $C = 0.1$ , the optimal splitting of the prior has to satisfy the information constraint (2.53). Therefore, the optimal posteriors are  $\gamma$  and  $p_1 = 0.8$ . The blue dotted line represents the convexification of the expected cost and it intersects the prior at the optimal cost  $\Gamma_e^*$  which could be achieved in this noisy persuasion game without decoder's side information.

### Persuasion with Side Information (Equation for $\Gamma_s$ (2.47))

Assume  $\delta \neq 0.5$ , i.e. the decoder observes a side information  $\mathcal{Z}$  correlated to the state  $U$ . In this case, (Le Treust and Tomala, 2019a, Lemma 6.3) ensures that the optimal number of posterior beliefs is three. The posterior distributions  $(p_1, p_2, p_3)$  from observing the message delivered by the encoder, must satisfy the information constraint given by

$$\lambda_1 \cdot h(p_1) + \lambda_2 \cdot h(p_2) + \lambda_3 \cdot h(p_3) \geq H(U|Z) - \max_{\mathcal{P}(x)} I(X; Y) =: IC. \quad (2.56)$$

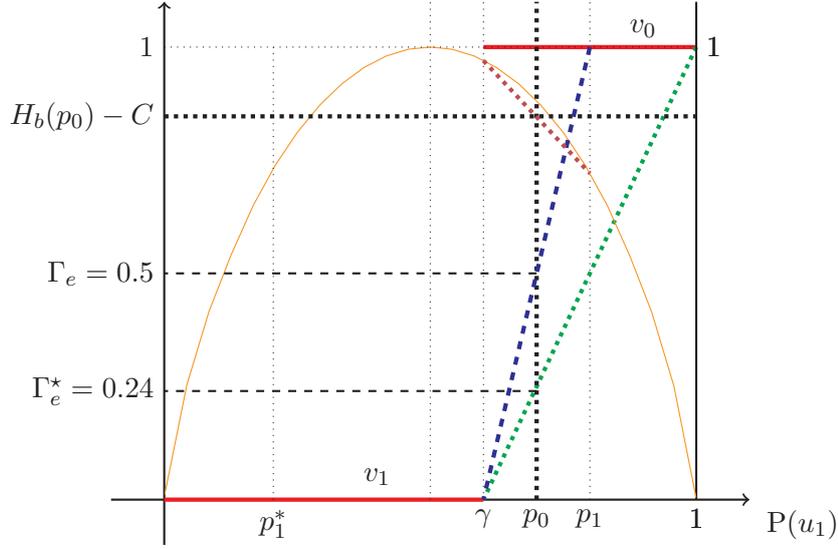


Figure 2.10: Optimal splitting for noisy persuasion without side information for  $C = 0.1$ ,  $\gamma = 0.6$  and  $p_0 = 0.7$

Thus  $(\lambda_1, \lambda_2, \lambda_3)$  can be computed from the above information constraint (2.56), and the splitting lemma  $\lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 q_3 = p_0$  and the fact that  $\lambda_1 + \lambda_2 + \lambda_3 = 1$ . We assume that the information constraint is binding. Using (Le Treust and Tomala, 2019b, Eq. (57)-(59)),

$$\lambda_1 = \frac{IC \cdot (p_2 - q_3) + h(p_2) \cdot (q_3 - p_0) + h(q_3) \cdot (p_0 - p_2)}{h(p_1) \cdot (p_2 - q_3) + h(p_2) \cdot (q_3 - p_1) + h(q_3) \cdot (p_1 - p_2)}, \quad (2.57)$$

$$\lambda_2 = \frac{IC \cdot (q_3 - p_1) + h(q_3) \cdot (p_1 - p_0) + h(p_1) \cdot (p_0 - q_3)}{h(p_1) \cdot (p_2 - q_3) + h(p_2) \cdot (q_3 - p_1) + h(q_3) \cdot (p_1 - p_2)}, \quad (2.58)$$

$$\lambda_3 = \frac{IC \cdot (p_1 - p_2) + h(p_1) \cdot (p_2 - p_0) + h(p_2) \cdot (p_0 - p_1)}{h(p_1) \cdot (p_2 - q_3) + h(p_2) \cdot (q_3 - p_1) + h(q_3) \cdot (p_1 - p_2)}. \quad (2.59)$$

Given a *interim* belief parameter  $q \in [0, 1]$ , the decoder's side information might be  $z_0$  or  $z_1$ , thus inducing the two following posterior beliefs

$$p_1(q) = \frac{q \cdot \delta}{(1 - q) \cdot (1 - \delta) + q \cdot \delta}, \quad (2.60)$$

$$p_2(q) = \frac{q \cdot (1 - \delta)}{(1 - q) \cdot \delta + q \cdot (1 - \delta)}. \quad (2.61)$$

The decoder's threshold  $\gamma$  induces the two corresponding thresholds  $\nu_1$  and  $\nu_2$  for the *interim* belief parameter  $q \in [0, 1]$  such that

$$\nu_1 = \frac{\gamma \cdot (1 - \delta)}{\delta \cdot (1 - \gamma) + \gamma \cdot (1 - \delta)}, \quad (2.62)$$

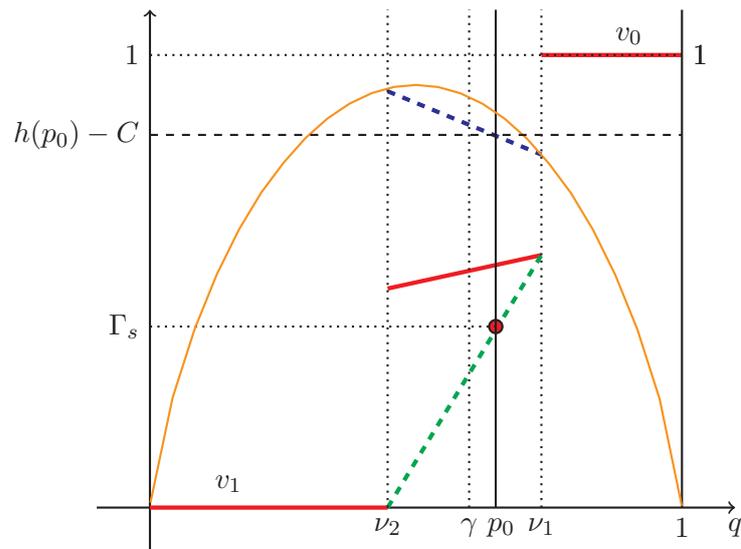


Figure 2.11: Splitting over 2 posteriors ( $q_1 = \nu_2; q_2 = \nu_1$ ) with  $C = 0.25$ ,  $p_0 = 0.65$ ,  $\delta = 0.35$ ,  $\gamma = 0.6$ .

$$\nu_2 = \frac{\gamma \cdot \delta}{\gamma \cdot \delta + (1 - \delta) \cdot (1 - \gamma)}. \quad (2.63)$$

Thus the encoder's utility function  $\Psi_e(q)$  represented by the red lines in Fig. 2.11 and the conditional entropy  $h(q)$  reformulate as

$$\Psi_e(q) = 0 \cdot \mathbb{1}_{\{q \in [0, \nu_2]\}} + ((1 - q) \cdot \delta + q \cdot (1 - \delta)) \cdot \mathbb{1}_{\{q \in [\nu_2, \nu_1]\}} + 1 \cdot \mathbb{1}_{\{q \in [\nu_1, 1]\}}, \quad (2.64)$$

$$h(q) = ((1 - q) \cdot (1 - \delta) + q \cdot \delta) \cdot H_b(p_1(q)) + ((1 - q) \cdot \delta + q \cdot (1 - \delta)) H_b(p_2(q)). \quad (2.65)$$

The encoder's optimal utility value is given by

$$\begin{aligned} \Gamma_s &= \inf_{\substack{p_1 \in [0, \nu_2], p_2 \in [\nu_2, \nu_1], \\ q_3 \in [\nu_1, 1]}} \left( \lambda_1 \cdot \Psi_e(p_1) + \lambda_2 \cdot \Psi_e(p_2) + \lambda_3 \cdot \Psi_e(q_3) \right) \quad (2.66) \\ &= \inf_{\substack{p_1 \in [0, \nu_2], p_2 \in [\nu_2, \nu_1], \\ q_3 \in [\nu_1, 1]}} \left( \frac{((h(p_0) - C)((q_3 - p_1) \cdot (p_2 \cdot (1 - 2\delta) + \delta) + (p_1 - p_2))}{h(p_1) \cdot (p_2 - q_3) + h(p_2) \cdot (q_3 - p_1) + h(q_3) \cdot (p_1 - p_2)} \right. \\ &\quad + \frac{(h(q_3) \cdot (p_1 - p_0) + h(p_1) \cdot (p_0 - q_3)) \cdot (p_2 \cdot (1 - 2\delta) + \delta)}{h(p_1) \cdot (p_2 - q_3) + h(p_2) \cdot (q_3 - p_1) + h(q_3) \cdot (p_1 - p_2)} \\ &\quad \left. + \frac{h(p_1) \cdot (p_2 - p_0) + h(p_2) \cdot (p_0 - p_1)}{h(p_1) \cdot (p_2 - q_3) + h(p_2) \cdot (q_3 - p_1) + h(q_3) \cdot (p_1 - p_2)} \right) \quad (2.67) \end{aligned}$$

As depicted in Figure 2.12, the average entropy is the orange curve and the red lines are the encoder's expected cost function averaged over  $Z$ . The green triangle represents the

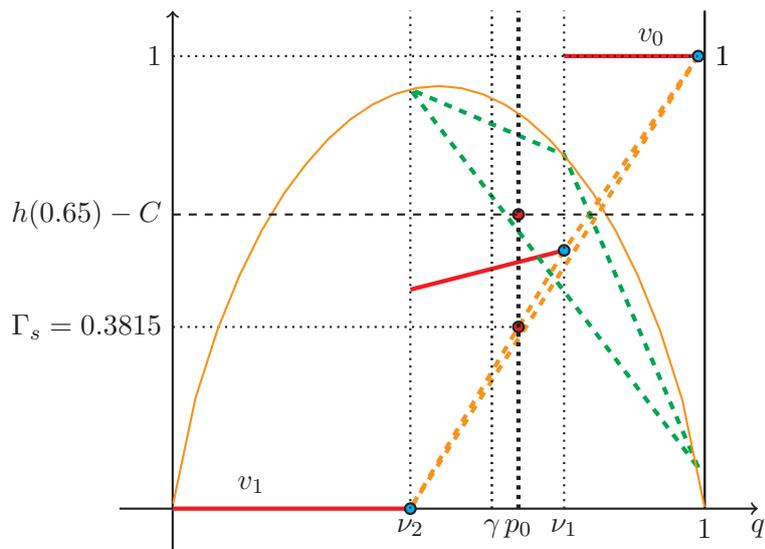


Figure 2.12: Optimal splittings over 3 posteriors with  $C = 0.25$ ,  $p_0 = 0.65$ ,  $\delta = 0.35$ ,  $\gamma = 0.6$ .

information constraint, where the edges are the entropy values of the optimal posterior beliefs of the decoder. The edges of the orange triangle are the expected cost that could be achieved by the encoder with respect to the optimal splitting.

#### 2.4.2 When is side information beneficial for the encoder?

In the following, we investigate the impact of the private observation on the encoder's optimal cost. We perform numerical simulations over the  $(C, \delta)$  region for both convexification formulas  $\Gamma_e^*$  given in (2.36) and  $\Gamma_s$  given in (2.47), revealing the encoder's optimal cost values with and without decoder's private observation respectively. The optimal splitting of the prior over 3 posterior beliefs results in the encoder's optimal payoff values shown in Fig.2.13 with respect to the  $(C, \delta)$  regions.

As the channel's capacity increases, the encoder's cost decreases without decoder's side information. This is due to the fact that more capacity allows the transmission of more information and hence information can be optimally disclosed. However; with low capacity, the decoder's side observation can be beneficial for the encoder until the encoder has no capacity at all, it becomes optimal to have private information up to some threshold  $\delta^*$  evaluated in Proposition 2.4.1 below.

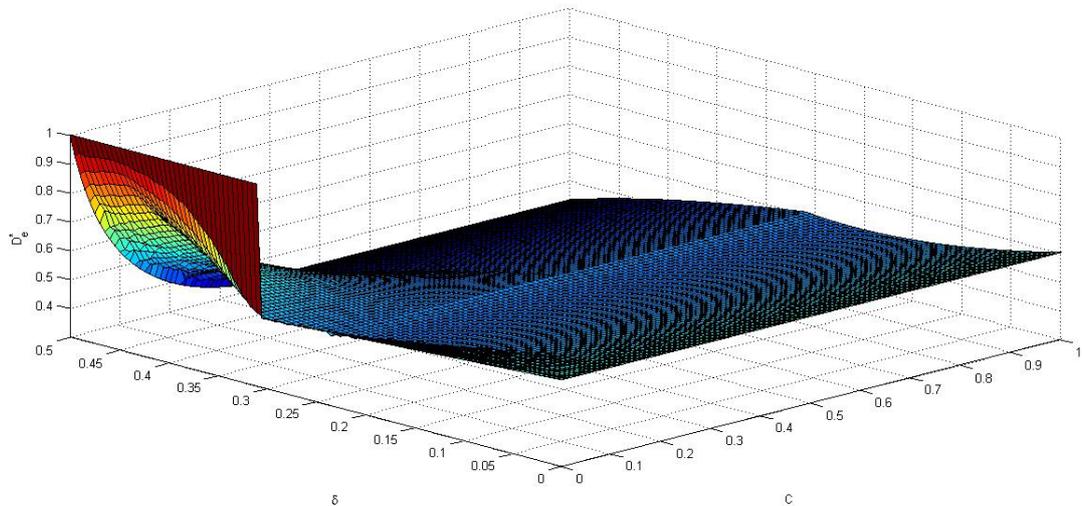


Figure 2.13: Encoder's optimal cost evaluated with three posteriors w.r.t.  $\delta$  and  $C$  for  $p_0 = 0.4$  and  $\gamma = 0.6$ .

**Proposition 2.4.1.** *Let  $C = 0$ .*

- *If  $p_0 < \gamma$  and  $\delta \in [0, \frac{p_0 \cdot (\gamma - 1)}{p_0 \cdot (-1 + 2\gamma) - \gamma}] \cup [\frac{\gamma \cdot (1 - p_0)}{p_0 \cdot (1 - 2\gamma) + \gamma}, 1]$ , then  $\Gamma_s > \Gamma_e^*$ .*
- *If  $p_0 \geq \gamma$  then  $\Gamma_e^* \geq \Gamma_s$ .*

The proof of Proposition 2.4.1 can be found in [Appendix A](#).

Comparing the encoder's cost values for three posterior splittings versus 2 posterior splittings, [Figure 2.16](#) confirms the result of [Le Treust and Tomala \(2021\)](#) that the optimal number of posteriors for the noisy Bayesian persuasion game with decoder's side information potentially achieve a lower cost than splitting the prior over two posteriors only.

## 2.5 Conclusion

Point-to-point communication theory provides a rigorous mathematical framework to study the fundamental information-theoretic limits of communication between a single encoder and a single decoder. Our approach is closely related to the rate-distortion theory ([Shannon \(1959\)](#)) in which the rate region which achieves a small distortion up to a fidelity

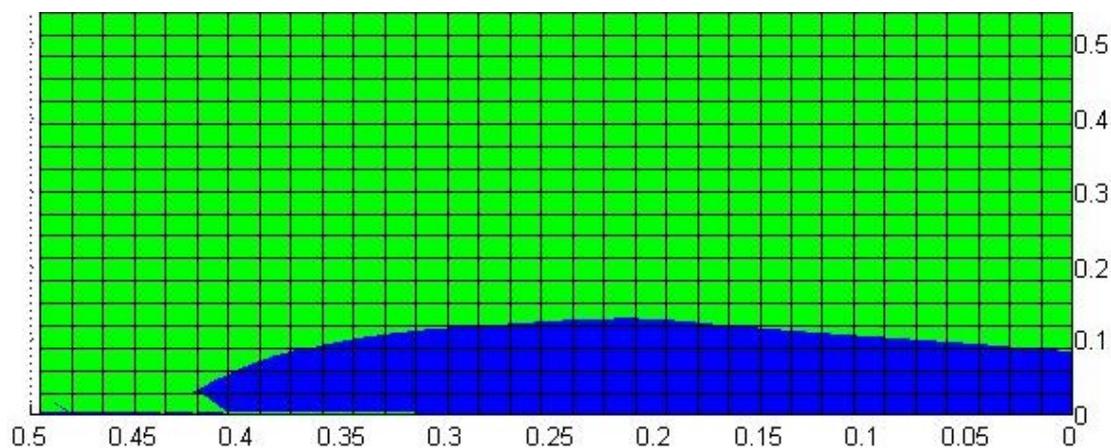


Figure 2.14:  $(\delta, C)$  regions for encoder's optimal cost with (blue) and without (green) decoder's private observation for  $p_0 = 0.4$  and  $\gamma = 0.6$ .

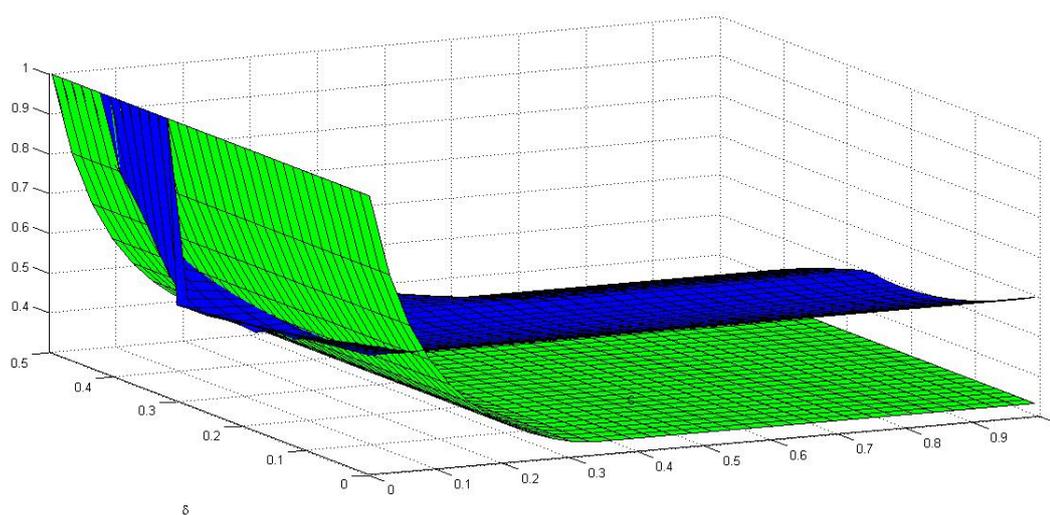


Figure 2.15: Encoder's optimal cost with three posterior splittings for  $(\delta, C)$  with (blue) and without (green) decoder's side information for  $p_0 = 0.4$  and  $\gamma = 0.6$ .

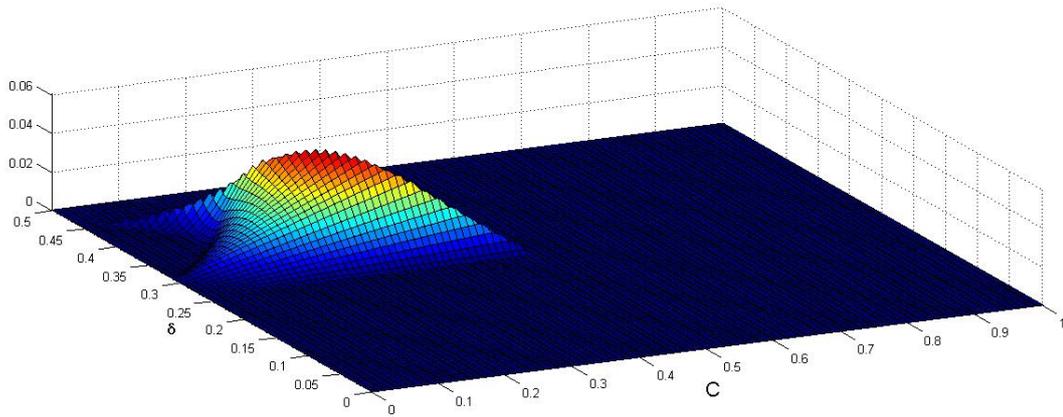


Figure 2.16: Difference between optimal cost values obtained by splitting with three posteriors and two posteriors.

criterion is derived. We consider distinct cost functions instead of distortions and we minimize the encoder's cost function subject to the optimal signal that satisfies the bounds of the achievable rate region. Bayesian persuasion in game theory does not take into consideration information constraints. Adding information constraints to the Bayesian persuasion game is an emerging research field. The cases where the channel is noisy and the decoder either has or not access to a side observation have been fully studied. The impact of the side information on the encoder's cost function was also investigated through a binary example. We illustrate the unrestricted communication, the noisy communication and the noisy communication with decoder's side information. We compute the optimal splittings in each case and derive the encoder's optimal settings.

In the following chapter, we extend the analysis to multi-user communication models with two decoders. We study strategic lossy source coding for the successive refinement setup.

# 3

## STRATEGIC SUCCESSIVE REFINEMENT CODING FOR BAYESIAN PERSUASION

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*In this chapter, we propose a new model to study strategic communication via a successive refinement coding setup. We fully solve this case and prove achievability and converse.*

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## 3.1 Introduction

From the point-to-point communication paradigms, we shift our attention to multi-user communication networks. We study a Bayesian persuasion game modeled by the successive refinement source coding setup. This setting puts together an encoder and two decoders, one of which has access to the observation of the other. The importance of successive refinement coding is that it can be applied to cases that go beyond the traditional rate-distortion coding to different settings including multimedia systems [Verbist et al. \(2013\)](#), hypothesis testing [Tian and Chen \(2008\)](#), pattern recognition [O’Sullivan et al. \(2006\)](#), and privacy [Girgis et al. \(2020\)](#) among others. In the standard successive refinement source coding scheme, formulated as a rate-distortion problem, a source stream is encoded into more than one description in a progressive order, which allows later descriptions to be used in order to refine the early ones. This improves the quality of the reconstructions progressively. The early works of [Koshelev \(1980\)](#) and [Rimoldi \(1994\)](#), focused on the fundamental problem of characterizing the optimal rate-distortion region, in which such a progressive coding does not result in any performance loss, compared to single stage coding systems. Subsequently, the problem has been thoroughly researched with various extensions including the notable works by [Tuncel and Rose \(2003a\)](#), [Tuncel and Rose \(2003b\)](#), [Effros \(1999\)](#), and [Effros \(2001\)](#).

We propose a non-cooperative setup with a strategic encoder linked to two decoders via two perfect links: one to the first decoder only, and the other to both decoders. We consider that the encoder and both decoders are endowed with distinct and arbitrary cost functions, unlike the standard model in which only decoders are endowed with distortion measures. We investigate the strategic source coding problem in which the encoder commits to an encoding while the decoders select the sequences of symbols that minimize their long-run respective cost functions. We fully characterize the optimal single-letter cost of the encoder using successive refinement coding with respect to a specific probability distribution which involves two auxiliary random variables, and captures the incentive constraints of both decoders. Our achievability proof presents a technical novelty which consists of characterizing the beliefs of the decoders that are induced by the successive refinement coding, and we show that these

beliefs are close on average to the posterior beliefs induced by block coding.

### 3.2 Successive refinement coding setup

Successive refinement is a special case of multiple description code for a discrete memoryless source  $U$  with two descriptions one of which is a refinement of the other. We review a setup with one encoder and two decoders as depicted in Figure 3.1. Decoder  $\mathcal{D}_1$  receives two descriptions  $M_0$  at rate  $R_0$  and  $M_1$  at rate  $R_1$ , and decoder  $\mathcal{D}_2$  receives only one description  $M_0$ . Both decoders  $\mathcal{D}_1$  and  $\mathcal{D}_2$  are endowed with cost functions to be minimized. More details about successive refinement coding can be found in (El Gamal and Kim, 2011, Chapter 13.5). Obviously, this section can be omitted by the familiar reader.

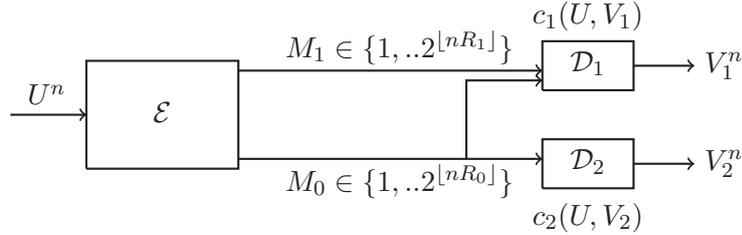


Figure 3.1: Successive refinement coding setup

**Definition 3.2.1.** Let  $(R_0, R_1) \in \mathbb{R}_+^2$ . An  $(R_1, R_0, n)$  successive refinement multiple description code consists of the following:

- Two index sets  $\{1, 2, \dots, 2^{[nR_0]}\}$  and  $\{1, 2, \dots, 2^{[nR_1]}\}$ .
- An encoding function  $\sigma : \mathcal{U}^n \rightarrow \{1, 2, \dots, 2^{[nR_0]}\} \times \{1, 2, \dots, 2^{[nR_1]}\}$  which assigns a pair  $(m_1, m_0)$  to each sequence  $u^n \in \mathcal{U}^n$ .
- Two decoding functions  $\tau_1 : \{1, 2, \dots, 2^{[nR_0]}\} \times \{1, 2, \dots, 2^{[nR_1]}\} \rightarrow \mathcal{V}_1^n$  and  $\tau_2 : \{1, 2, \dots, 2^{[nR_0]}\} \rightarrow \mathcal{V}_2^n$ .

**Definition 3.2.2.** Let  $c_1 : \mathcal{U} \times \mathcal{V}_1 \rightarrow \mathbb{R}$  and  $c_2 : \mathcal{U} \times \mathcal{V}_2 \rightarrow \mathbb{R}$  two cost functions. We define the long run distortion functions  $c_1^n(\sigma, \tau_1)$  and  $c_2^n(\sigma, \tau_1)$  as follows:

$$c_i^n(\sigma, \tau_i) = \sum_{u^n, v^n} \mathcal{P}^{\sigma, \tau_i}(u^n, v_i^n) \frac{1}{n} \sum_{t=1}^n c_i(u_t, v_{i,t}), \quad i \in \{1, 2\}. \quad (3.1)$$

**Definition 3.2.3.** A rate-cost quadruple  $(R_0, R_1, C_1, C_2) \in \mathbb{R}_+^4$  is said to be achievable if there exists a sequence of  $(R_0, R_1, n)$  successive refinement multiple description codes such that  $\limsup_{n \rightarrow \infty} c_i^n(U^n, V_i^n) \leq C_i$ , for  $i \in \{1, 2\}$ .

For a given rate pair  $(R_0, R_1)$ , we denote by  $\mathcal{B}(R_0, R_1)$  the closure of the set of pairs  $(C_1, C_2)$  such that the quadruples  $(R_0, R_1, C_1, C_2)$  are achievable.

**Theorem 3.2.1** (Successive Refinement Multiple Description Coding for a DMS). *Let  $U$  be a discrete memoryless source, distributed according to a probability distribution  $\mathcal{P}_U$  over  $\mathcal{U}$ . Let  $\mathcal{V}_1$  and  $\mathcal{V}_2$  be two discrete alphabets and  $c_i : \mathcal{U} \times \mathcal{V}_i \mapsto \mathbb{R}^+$  for  $i \in \{1, 2\}$  be two cost functions. Then, given a rate pair  $(R_0, R_1)$ , we have*

$$\mathcal{B}(R_0, R_1) = \left\{ \left( \mathbb{E}_{\mathcal{Q}}(c_1(U, V_1)), \mathbb{E}_{\mathcal{Q}}(c_2(U, V_2)) \right), \quad \mathcal{Q} \in \mathcal{Q}(R_0, R_1) \right\} \quad (3.2)$$

where

$$\mathcal{Q}(R_0, R_1) = \{ \mathcal{Q}_{V_1 V_2 | U}, \quad I(U; V_2) \leq R_0, \quad I(U; V_1, V_2) \leq R_0 + R_1 \}. \quad (3.3)$$

### 3.3 Bayesian persuasion with two decoders

We investigate the Bayesian persuasion game between an encoder and two decoders under restricted communication. We assume that the encoder is also endowed with a cost function that depends on the state and actions of both decoders. The observation of the first decoder contains the observation of the second decoder, as depicted in [Figure 3.2](#). The source sequence is drawn according to the i.i.d. distribution  $\mathcal{P}_U \in \Delta(\mathcal{U})$ , and the encoder  $\mathcal{E}$  selects and announces beforehand the compression scheme to be implemented. All three players are endowed with distinct and arbitrary cost functions capturing their mismatched objectives. The encoder's cost depends on the state and on the action of both decoders. However; the cost function of each decoder depends on the state and on its own action. Upon reception of the indices, the decoders  $\mathcal{D}_1$  and  $\mathcal{D}_2$  update their Bayesian beliefs about the source sequence and select the action sequences that minimizes their respective cost

functions. In the following, we present our model of strategic successive refinement coding with mismatched costs at all players, and we characterize the optimal cost of the encoder subject to the information constraint imposed by the successive refinement network and the incentive constraints of the decoders.

### 3.3.1 New model for strategic successive refinement

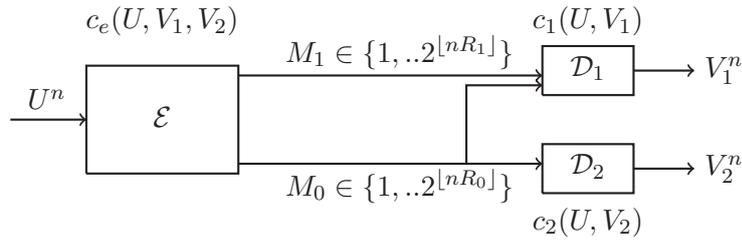


Figure 3.2: Strategic successive refinement source coding

**Definition 3.3.1.** Let  $R_0, R_1 \in \mathbb{R}_+^2 = [0, +\infty]^2$ , and  $n \in \mathbb{N}^* = \mathbb{N} \setminus \{0\}$ . The encoding  $\sigma$  and decoding  $\tau_i$  strategies of the encoder  $\mathcal{E}$  and decoders  $\mathcal{D}_i$ ,  $i \in \{1, 2\}$  are defined by

$$\sigma : U^n \longrightarrow \Delta(\{1, 2, \dots, 2^{\lfloor nR_0 \rfloor}\} \times \{1, 2, \dots, 2^{\lfloor nR_1 \rfloor}\}), \quad (3.4)$$

$$\tau_1 : \{1, 2, \dots, 2^{\lfloor nR_0 \rfloor}\} \times \{1, 2, \dots, 2^{\lfloor nR_1 \rfloor}\} \longrightarrow \Delta(\mathcal{V}_1^n), \quad (3.5)$$

$$\tau_2 : \{1, 2, \dots, 2^{\lfloor nR_0 \rfloor}\} \longrightarrow \Delta(\mathcal{V}_2^n), \quad (3.6)$$

We denote by  $\mathcal{S}(n, R_1, R_0)$  the set of coding triplets  $(\sigma, \tau_1, \tau_2)$ .

The stochastic coding strategies  $(\sigma, \tau_1, \tau_2) \in \mathcal{S}(n, R_1, R_0)$  induce a joint probability distribution  $\mathcal{P}^{\sigma\tau_1\tau_2} \in \Delta(\mathcal{U}^n \times \{1, 2, \dots, 2^{\lfloor nR_1 \rfloor}\} \times \{1, 2, \dots, 2^{\lfloor nR_0 \rfloor}\} \times \mathcal{V}_1^n \times \mathcal{V}_2^n)$  defined for all  $(u^n, m_1, m_0, v_1^n, v_2^n)$  by

$$\mathcal{P}^{\sigma\tau_1\tau_2}(u^n, m_1, m_0, v_1^n, v_2^n) = \left( \prod_{t=1}^n \mathcal{P}_U(u_t) \right) \sigma(m_1, m_0 | u^n) \tau_1(v_1^n | m_1, m_0) \tau_2(v_2^n | m_0). \quad (3.7)$$

**Definition 3.3.2.** We consider arbitrary single-letter cost functions  $c_e : \mathcal{U} \times \mathcal{V}_1 \times \mathcal{V}_2 \longrightarrow \mathbb{R}$  for the encoder  $\mathcal{E}$ ,  $c_1 : \mathcal{U} \times \mathcal{V}_1 \longrightarrow \mathbb{R}$  for the decoder  $\mathcal{D}_1$  and  $c_2 : \mathcal{U} \times \mathcal{V}_2 \longrightarrow \mathbb{R}$  for the decoder

$\mathcal{D}_2$ . The long-run cost functions are defined by

$$c_e^n(\sigma, \tau_1, \tau_2) = \mathbb{E}_{\sigma, \tau_1, \tau_2} \left[ \frac{1}{n} \sum_{t=1}^n c_e(U_t, V_{1,t}, V_{2,t}) \right] \quad (3.8)$$

$$= \sum_{u^n, v_1^n, v_2^n} \mathcal{P}_{U^n V_1^n V_2^n}^{\sigma, \tau_1, \tau_2}(u^n, v_1^n, v_2^n) \cdot \left[ \frac{1}{n} \sum_{t=1}^n c_e(u_t, v_{1,t}, v_{2,t}) \right], \quad (3.9)$$

$$c_1^n(\sigma, \tau_1) = \sum_{u^n, v_1^n} \mathcal{P}_{U^n V_1^n}^{\sigma, \tau_1}(u^n, v_1^n) \cdot \left[ \frac{1}{n} \sum_{t=1}^n c_1(u_t, v_{1,t}) \right], \quad (3.10)$$

$$c_2^n(\sigma, \tau_2) = \sum_{u^n, v_2^n} \mathcal{P}_{U^n V_2^n}^{\sigma, \tau_2}(u^n, v_2^n) \cdot \left[ \frac{1}{n} \sum_{t=1}^n c_2(u_t, v_{2,t}) \right]. \quad (3.11)$$

In the above equations,  $\mathcal{P}_{U^n V_1^n V_2^n}^{\sigma, \tau_1, \tau_2}$ ,  $\mathcal{P}_{U^n V_1^n}^{\sigma, \tau_1}$  and  $\mathcal{P}_{U^n V_2^n}^{\sigma, \tau_2}$  denote the marginal distributions of  $\mathcal{P}^{\sigma, \tau_1, \tau_2}$  defined in (3.7) over  $(U^n, V_1^n, V_2^n)$ ,  $(U^n, V_1^n)$ , and  $(U^n, V_2^n)$  respectively.

**Definition 3.3.3.** For any encoding strategy  $\sigma$ , the set of best-response strategies of decoder  $\mathcal{D}_i$  for  $i \in \{1, 2\}$  is defined by

$$BR_i(\sigma) = \left\{ \tau_i, c_i^n(\sigma, \tau_i) \leq c_i^n(\sigma, \tilde{\tau}_i), \forall \tilde{\tau}_i \right\}. \quad (3.12)$$

If several pairs of best-response strategies  $(\tau_1, \tau_2) \in BR_1(\sigma) \times BR_2(\sigma)$  are available, we assume that the pair  $(\tau_1, \tau_2)$  that maximizes the encoder's cost is selected. This assumption is motivated by our attempt to obtain a robust solution concept by solving for the worst case scenario. Our solution is robust to the exact specification of the decoding strategies. For  $(R_0, R_1) \in \mathbb{R}_+^2$  and  $n \in \mathbb{N}^*$ , the strategic successive refinement coding problem under study is given by

$$\Gamma_{sr}^n(R_0, R_1) = \inf_{\sigma} \max_{\substack{\tau_1 \in BR_1(\sigma), \\ \tau_2 \in BR_2(\sigma)}} c_e^n(\sigma, \tau_1, \tau_2). \quad (3.13)$$

**Remark 3.3.1.** Suppose that the decoders choose, among their best-response strategies, the pair that also minimizes the encoder distortion. This ‘‘optimistic’’ coding problem writes

$$\Gamma_o^n(R_0, R_1) = \min_{\sigma} \min_{\substack{\tau_1 \in BR_1(\sigma), \\ \tau_2 \in BR_2(\sigma)}} c_e^n(\sigma, \tau_1, \tau_2). \quad (3.14)$$

For generic problems, i.e. on a dense subset of the game space,  $\Gamma_o^n(R_0, R_1) = \Gamma_{sr}^n(R_0, R_1)$  (Le Treust and Tomala, 2019a, pp. 8).

The operational significance of (3.13) corresponds to the persuasion game that is played in the following steps:

- Encoder  $\mathcal{E}$  chooses, announces the encoding  $\sigma$ .
- Sequence  $U^n$  is drawn i.i.d with distribution  $\mathcal{P}_U$ .
- Messages  $(M_1, M_0)$  are encoded according to  $\mathcal{P}_{M_1 M_0 | U^n}^\sigma$ .
- Knowing  $\sigma$ , decoder  $\mathcal{D}_1$  observes  $(M_1, M_0)$  and draws  $V_1^n$  according to  $\tau_1 \in BR_1(\sigma)$ , and decoder  $\mathcal{D}_2$  observes  $M_0$  and draws  $V_2^n$  according to  $\tau_2 \in BR_2(\sigma)$ .
- Distortion values are  $c_e^n(\sigma, \tau_1, \tau_2)$ ,  $c_1^n(\sigma, \tau_1)$ ,  $c_2^n(\sigma, \tau_2)$ .

**Lemma 3.3.1.** *The sequence  $(n\Gamma_{sr}^n(R_0, R_1))_{n \in \mathbb{N}^*}$  is sub-additive.*

The proof of Lemma 3.3.1 is stated in Appendix B.

### 3.3.2 Proposed single-letter characterizations and fundamental limit of the encoder's optimal long-run cost

In the following, we characterize the asymptotic behaviour of  $\Gamma_{sr}^n(R_0, R_1)$ . Our main theorem consists of an achievability statement describing the limiting behaviour of the encoder's long-run cost function and a converse statement determining its lower bound. Our solution combines the decoders incentive constraints with the information constraints of the successive refinement source coding.

**Definition 3.3.4.** *We consider two auxiliary random variables  $W_1 \in \mathcal{W}_1$  and  $W_0 \in \mathcal{W}_0$  with  $|\mathcal{W}_1| = |\mathcal{V}_1|$ , and  $|\mathcal{W}_0| = |\mathcal{V}_2|$ . For  $(R_0, R_1) \in \mathbb{R}_+^2$ , we define*

$$\bar{\mathcal{Q}}_0(R_0, R_1) = \{\mathcal{Q}_{W_1 W_0 | U}, \quad R_0 \geq I(U; W_0), R_1 + R_0 \geq I(U; W_1, W_0)\}. \quad (3.15)$$

For every distribution  $\mathcal{Q}_{W_1 W_0|U} \in \Delta(\mathcal{W}_1 \times \mathcal{W}_0)^{|\mathcal{U}|}$ , we define

$$\mathbb{Q}_1(\mathcal{Q}_{W_1 W_0|U}) = \arg \min_{\substack{\mathcal{Q}_{V_1|W_1 W_0} \\ \mathcal{Q}_{V_1|W_1 W_0}}} \mathbb{E}_{\mathcal{Q}_{W_1 W_0|U}} [c_1(U, V_1)], \quad (3.16)$$

$$\mathbb{Q}_2(\mathcal{Q}_{W_0|U}) = \arg \min_{\substack{\mathcal{Q}_{V_2|W_0} \\ \mathcal{Q}_{V_2|W_0}}} \mathbb{E}_{\mathcal{Q}_{W_0|U}} [c_2(U, V_2)]. \quad (3.17)$$

Note that  $\mathbb{Q}_1(\mathcal{Q}_{W_1 W_0|U}) \in \Delta(\mathcal{V}_1)^{|\mathcal{W}_1 \times \mathcal{W}_0|}$  and  $\mathbb{Q}_2(\mathcal{Q}_{W_0|U}) \in \Delta(\mathcal{V}_2)^{|\mathcal{W}_0|}$ . The encoder's optimal cost is defined by

$$\Gamma_{sr}^*(R_0, R_1) = \inf_{\substack{\mathcal{Q}_{W_1 W_0|U} \\ \in \mathcal{Q}_0(R_0, R_1)}} \max_{\substack{\mathcal{Q}_{V_1|W_1 W_0} \in \mathbb{Q}_1(\mathcal{Q}_{W_1 W_0|U}) \\ \mathcal{Q}_{V_2|W_0} \in \mathbb{Q}_2(\mathcal{Q}_{W_0|U})}} \mathbb{E}[c_\varepsilon(U, V_1, V_2)], \quad (3.18)$$

where the expectation in (3.18) is evaluated with respect to  $\mathcal{P}_U \mathcal{Q}_{W_1 W_0|U} \mathcal{Q}_{V_1|W_1 W_0} \mathcal{Q}_{V_2|W_0}$ .

**Remark 3.3.2.** *The random variables  $U, W_1, W_0, V_1, V_2$  satisfy*

$$(U, V_2) \text{ --- } (W_1, W_0) \text{ --- } V_1, \quad (U, W_1, V_1) \text{ --- } W_0 \text{ --- } V_2.$$

Given  $\mathcal{Q}_{W_1 W_0|U}$ , we denote by  $\mathcal{Q}_{U|W_1 W_0} \in \Delta(\mathcal{U})^{|\mathcal{W}_1 \times \mathcal{W}_0|}$  and  $\mathcal{Q}_{U|W_0} \in \Delta(\mathcal{U})^{|\mathcal{W}_0|}$  the posterior beliefs of decoders  $\mathcal{D}_1$  and  $\mathcal{D}_2$ . Moreover, for  $(w_1, w_0) \in \mathcal{W}_1 \times \mathcal{W}_0$ , we introduce the notations  $\mathcal{Q}_U^{w_1 w_0} = \mathcal{Q}_{U|W_1 W_0}(\cdot | w_1, w_0) \in \Delta(\mathcal{U})$  and  $\mathcal{Q}_U^{w_0} = \mathcal{Q}_{U|W_0}(\cdot | w_0) \in \Delta(\mathcal{U})$ .

**Theorem 3.3.1.** *Let  $(R_0, R_1) \in \mathbb{R}_+^2$ , we have*

- a)  $\forall \varepsilon > 0, \exists \hat{n} \in \mathbb{N}, \forall n \geq \hat{n}, \Gamma_{sr}^n(R_0, R_1) \leq \Gamma_{sr}^*(R_0, R_1) + \varepsilon,$
- b)  $\forall n \in \mathbb{N}, \Gamma_{sr}^n(R_0, R_1) \geq \Gamma_{sr}^*(R_0, R_1).$

The proof of Theorem 3.3.1 is stated in [section 3.4](#) and [section 3.5](#). Together with Fekete's Lemma for the sub-additive sequence  $(n\Gamma_{sr}^n(R_0, R_1))_{n \in \mathbb{N}^*}$  (see [Lemma 3.3.1](#)), we obtain

$$\lim_{n \rightarrow \infty} \Gamma_{sr}^n(R_0, R_1) = \inf_{n \in \mathbb{N}^*} \Gamma_{sr}^n(R_0, R_1) = \Gamma_{sr}^*(R_0, R_1). \quad (3.19)$$

### 3.4 Proof of achievability for strategic successive refinement

The proof of achievability of Theorem 3.3.1 consists of four main parts. In subsection 3.4.3, we provide an alternative formulation to our solution concept by restricting the set of target distributions to a dense subset of distributions inducing beliefs based on which the decoders are not indifferent between two or more actions. In subsection 3.4.2, we generate the codebook and analyze the error probability. In subsection 3.4.1, we restrict our optimization to a dense subset of target distributions in order to eliminate decoder's indifferences between two pairs of actions.. Finally in subsection 3.4.4, we control the Bayesian beliefs of each decoder about the state concluding by that our proof of achievability.

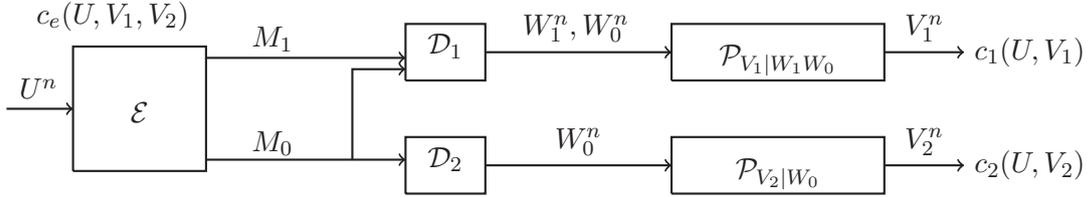


Figure 3.3: Achievability of successive refinement source coding.

#### 3.4.1 Alternative formulation

In the following, we reformulate the optimal cost of the encoder  $\Gamma_{sr}^*(R_0, R_1)$  by restricting the set of target distribution  $Q_0(R_0, R_1)$  to distributions that induce beliefs according to which the decoders are not indifferent between two actions. We show that both problems since the restricted subset is dense in  $Q_0(R_0, R_1)$ , and optimizing over a dense subset is equal to optimizing over the full set.

**Definition 3.4.1.** We denote by  $V_1^*(q_1)$  and  $V_2^*(q_2)$ , the respective action sets of decoders  $\mathcal{D}_1$  and  $\mathcal{D}_2$  for belief parameters  $q_1 \in \Delta(\mathcal{U})$  and  $q_2 \in \Delta(\mathcal{U})$ .

$$V_1^*(q_1) = \arg \min_{v_1 \in V_1} \sum_u q_1(u) c_1(u, v_1), \quad (3.20)$$

$$V_2^*(q_2) = \arg \min_{v_2 \in V_2} \sum_u q_2(u) c_2(u, v_2). \quad (3.21)$$

**Definition 3.4.2.** Fix a strategy  $\mathcal{Q}_{W_1W_0|U}$ . Let  $\tilde{A}(\mathcal{Q}_{W_1W_0|U}, w_1, w_0)$  denote the set of action pairs  $(v_1, v_2)$  that are optimal for the decoders and worst for the encoder. This set is given by:

$$\tilde{A}(\mathcal{Q}_{W_1W_0|U}, w_1, w_0) = \arg \max_{\substack{(v_1, v_2) \in V_1^*(\mathcal{Q}_{W_1W_0|U}^{w_1, w_0}) \times \\ V_2^*(\mathcal{Q}_{W_0|U}^{w_0})}} \left\{ \sum_u \mathcal{Q}^{w_1, w_0}(u) c_e(u, v_1, v_2) \right\} \subset \mathcal{V}_1 \times \mathcal{V}_2. \quad (3.22)$$

The set  $\tilde{\mathcal{Q}}_0(R_0, R_1)$  of target probability distributions for  $(R_0, R_1) \in \mathbb{R}_+^2$  is given by:

$$\tilde{\mathcal{Q}}_0(R_0, R_1) = \left\{ \mathcal{Q}_{W_1W_0|U}, R_0 > I(U; W_0), R_1 + R_0 > I(U; W_1, W_0), \max_{w_1, w_0} |\tilde{A}(\mathcal{Q}_{W_1W_0|U}, w_1, w_0)| = 1 \right\}. \quad (3.23)$$

**Definition 3.4.3.** Consider the following program:

$$\tilde{\Gamma}_{sr}(R_0, R_1) = \inf_{\mathcal{Q}_{W_1W_0|U} \in \tilde{\mathcal{Q}}_0(R_0, R_1)} \max_{\substack{\mathcal{Q}_{V_1|W_1W_0} \in \mathcal{Q}_1(\mathcal{Q}_{W_1W_0|U}) \\ \mathcal{Q}_{V_2|W_0} \in \mathcal{Q}_2(\mathcal{Q}_{W_0|U})}} \mathbb{E}_{\mathcal{P}_U \mathcal{Q}_{W_1W_0|U}} \left[ c_e(U, V_1, V_2) \right]. \quad (3.24)$$

**Lemma 3.4.1.** For  $(R_0, R_1) \in \mathbb{R}_+^2$ , we have

$$\Gamma_{sr}^*(R_0, R_1) = \tilde{\Gamma}_{sr}(R_0, R_1) \quad (3.25)$$

*Proof.* of lemma 3.4.1 Consider the following sets:

$$\bar{\mathcal{Q}}_0(R_0, R_1) = \{ \mathcal{Q}_{W_1W_0|U} \text{ s.t. } R_0 \geq I(U; W_0), R_1 + R_0 \geq I(U; W_1, W_0) \}, \quad (3.26)$$

$$\mathcal{Q}_{01} = \{ \mathcal{Q}_{W_1W_0|U} \text{ s.t. } \max_{w_1, w_0} |\tilde{A}(\mathcal{Q}_{W_1W_0|U}, w_1, w_0)| = 1 \}, \quad (3.27)$$

$$\mathcal{Q}_{02}(R_0, R_1) = \{ \mathcal{Q}_{W_1W_0|U} \text{ s.t. } R_0 > I(U; W_0), R_1 + R_0 > I(U; W_1, W_0) \}. \quad (3.28)$$

We will show that  $\mathcal{Q}_{01} \cap \mathcal{Q}_{02}(R_0, R_1) = \tilde{\mathcal{Q}}_0(R_0, R_1)$  is dense in  $\bar{\mathcal{Q}}_0(R_0, R_1)$ . We first show that  $\mathcal{Q}_{01} \cap \bar{\mathcal{Q}}_0(R_0, R_1)$  is open and dense in  $\bar{\mathcal{Q}}_0(R_0, R_1)$ .

**Definition 3.4.4** (Equivalent actions). *Two action  $v_i$  and  $\tilde{v}_i$  for decoder  $\mathcal{D}_i$ ,  $i \in \{1, 2\}$  are said to be equivalent if:  $c_i(u, v_i) = c_i(u, \tilde{v}_i)$  for all  $u \in \mathcal{U}$ ,  $i \in \{1, 2\}$ . We denote this equivalence relation by  $\sim_i$ . We use  $\not\sim_i$  for non equivalent actions  $v_i$  and  $\tilde{v}_i$ , i.e. there exists  $u \in \mathcal{U}$ , such that  $c_i(u, v_i) \neq c_i(u, \tilde{v}_i)$  for  $i \in \{1, 2\}$ .*

*Two action pairs  $(v_1, v_2)$  and  $(\tilde{v}_1, \tilde{v}_2)$  are equivalent for the encoder  $\mathcal{E}$  if :  $c_e(u, v_1, v_2) = c_e(u, \tilde{v}_1, \tilde{v}_2)$  for all  $u \in \mathcal{U}$ . We denote this equivalence relation by  $\sim_e$ . We use  $\not\sim_e$  for non equivalent action pairs  $(v_1, v_2)$  and  $(\tilde{v}_1, \tilde{v}_2)$  i.e. there exists  $u \in \mathcal{U}$ , such that  $c_e(u, v_1, v_2) \neq c_e(u, \tilde{v}_1, \tilde{v}_2)$ . We say that two pairs of actions  $(v_1, v_2)$  and  $(\tilde{v}_1, \tilde{v}_2)$  are completely equivalent if:*

1.  $(v_1, v_2) \sim_e (\tilde{v}_1, \tilde{v}_2)$ ,
2.  $v_1 \sim_1 \tilde{v}_1$ ,
3.  $v_2 \sim_2 \tilde{v}_2$ .

*Without loss of generality we can assume that no pairs of actions are completely equivalent, otherwise we can merge them into one action and reduce the set of actions.*

**Definition 3.4.5.** *For a fixed i.i.d distribution  $\mathcal{P}_U \in \Delta(\mathcal{U})$ , we denote by  $\mathbb{Q}_i$  for  $i \in \{1, 2\}$ , the set of distributions  $\mathcal{Q}_{W_1 W_0 | U} \in \Delta(\mathcal{W}_1 \times \mathcal{W}_0)^{|\mathcal{U}|}$  for which decoder  $\mathcal{D}_i$  is indifferent between two actions  $v_i$  and  $\tilde{v}_i$  that are not equivalent,*

$$\mathbb{Q}_1 = \left\{ \mathcal{Q}_{W_1 W_0 | U}, \exists v_1 \not\sim_1 \tilde{v}_1, \exists w_1, w_0, \right. \\ \left. \mathbb{E}_{\mathcal{Q}_U^{w_1 w_0}} [c_1(U, v_1)] = \mathbb{E}_{\mathcal{Q}_U^{w_1 w_0}} [c_1(U, \tilde{v}_1)] \right\}, \quad (3.29)$$

$$\mathbb{Q}_2 = \left\{ \mathcal{Q}_{W_1 W_0 | U}, \exists v_2 \not\sim_2 \tilde{v}_2, \exists w_0, \right. \\ \left. \mathbb{E}_{\mathcal{Q}_U^{w_0}} [c_2(U, v_2)] = \mathbb{E}_{\mathcal{Q}_U^{w_0}} [c_2(U, \tilde{v}_2)] \right\}, \quad (3.30)$$

*and by  $\mathbb{Q}_e$ , the set of distributions  $\mathcal{Q}_{W_1 W_0 | U} \in \Delta(\mathcal{W}_1 \times \mathcal{W}_0)^{|\mathcal{U}|}$  for which the encoder  $\mathcal{E}$  is indifferent between two action pairs  $(v_1, v_2)$  and  $(\tilde{v}_1, \tilde{v}_2)$  that are not equivalent:*

$$\mathbb{Q}_e = \left\{ \mathcal{Q}_{W_1 W_0 | U}, \exists (v_1, v_2) \not\sim_e (\tilde{v}_1, \tilde{v}_2), \exists w_1, w_0 \right.$$

$$\left. \mathbb{E}_{\mathcal{Q}_U^{w_1 w_0}} [c_e(U, v_1, v_2)] = \mathbb{E}_{\mathcal{Q}_U^{w_1 w_0}} [c_e(U, \tilde{v}_1, \tilde{v}_2)] \right\}. \quad (3.31)$$

Let  $\mathbb{Q}^c = \Delta(\mathcal{W}_1 \times \mathcal{W}_0)^{|\mathcal{U}|} \setminus (\mathbb{Q}_e \cup \mathbb{Q}_1 \cup \mathbb{Q}_2)$  the set of distributions  $\mathcal{Q}_{W_1 W_0 | U}$  where for all  $w_1, w_0$ , at least one of the following statements hold: i) The encoder is not indifferent between any two pairs of actions, ii) At least one of the decoders is not indifferent between any two actions.  $\square$

**Lemma 3.4.2.** For each distribution  $\mathcal{Q}_{W_1 W_0 | U}$  in  $\mathbb{Q}^c$ , the set  $\tilde{A}(\mathcal{Q}_{W_1 W_0 | U}, w_{1,t}, w_{0,t})$  is a singleton.

*Proof.* of lemma 3.4.2 We proceed by contradiction. Let  $\mathcal{Q}_{W_1 W_0 | U} \in \mathbb{Q}^c$  and suppose there exists  $(w_1, w_0) \in \mathcal{W}_1 \times \mathcal{W}_0$  such that  $|\tilde{A}(\mathcal{Q}_{W_1 W_0 | U}, w_1, w_0)| = 2$ . This means there exists two distinct action pairs  $(v_1, v_2) \neq (\tilde{v}_1, \tilde{v}_2)$  with  $v_1, \tilde{v}_1 \in V^*(\mathcal{Q}_U^{w_1 w_0})$  and  $v_2, \tilde{v}_2 \in V^*(\mathcal{Q}_U^{w_0})$  such that:

$$\mathbb{E}_{\mathcal{Q}_U^{w_1 w_0}} [c_e(U, v_1, v_2)] = \mathbb{E}_{\mathcal{Q}_U^{w_1 w_0}} [c_e(U, \tilde{v}_1, \tilde{v}_2)], \quad (3.32)$$

$$\mathbb{E}_{\mathcal{Q}_U^{w_1 w_0}} [c_1(U, v_1)] = \mathbb{E}_{\mathcal{Q}_U^{w_1 w_0}} [c_1(U, \tilde{v}_1)], \quad (3.33)$$

$$\mathbb{E}_{\mathcal{Q}_U^{w_0}} [c_2(U, v_2)] = \mathbb{E}_{\mathcal{Q}_U^{w_0}} [c_2(U, \tilde{v}_2)]. \quad (3.34)$$

By hypothesis,  $(v_1, v_2)$  and  $(\tilde{v}_1, \tilde{v}_2)$  are not completely equivalent. Therefore, we must have either  $(v_1, v_2) \simeq_e (\tilde{v}_1, \tilde{v}_2)$ , or  $v_1 \simeq_1 \tilde{v}_1$ , or  $v_2 \simeq_0 \tilde{v}_2$ , which imply that  $\mathcal{Q}_{W_1 W_0 | U} \in \mathbb{Q}_e \cup \mathbb{Q}_1 \cup \mathbb{Q}_2$ .

This contradicts the hypothesis  $\mathcal{Q}_{W_1 W_0 | U} \in \mathbb{Q}^c$ . Thus,  $\tilde{A}(\mathcal{Q}_{W_1 W_0 | U}, w_{1,t}, w_{0,t})$  is a singleton.

$\square$

**Lemma 3.4.3.** The set  $\mathbb{Q}^c$  is open and dense in  $\Delta(\mathcal{W}_1 \times \mathcal{W}_2)^{|\mathcal{U}|}$ .

*Proof.* of lemma 3.4.3 For each  $v_i \simeq_i \tilde{v}_i$ ,  $i \in \{1, 2\}$ , and pairs  $(v_1, v_2) \simeq_e (\tilde{v}_1, \tilde{v}_2)$  each set

$$\mathbb{Q}(v_i, \tilde{v}_i) = \left\{ \mathcal{Q}_U \in \Delta(\mathcal{U}), \mathbb{E}_{\mathcal{Q}_U} [c_i(U, v_i)] = \mathbb{E}_{\mathcal{Q}_U} [c_i(U, \tilde{v}_i)] \right\}, \quad i \in \{1, 2\}, \quad (3.35)$$

$$\mathbb{Q}(v_1, v_2, \tilde{v}_1, \tilde{v}_2) = \left\{ \mathcal{Q}_U \in \Delta(\mathcal{U}), \mathbb{E}_{\mathcal{Q}_U} [(c_e(U, v_1, v_2))] = \mathbb{E}_{\mathcal{Q}_U} [(c_e(U, \tilde{v}_1, \tilde{v}_2))] \right\}, \quad (3.36)$$

is a closed hyperplane of dimension  $\dim(\mathbb{Q}(v_i, \tilde{v}_i) = \dim\mathbb{Q}(v_1, v_2, \tilde{v}_1, \tilde{v}_2) = |\mathcal{U}| - 2$ . Consider the set  $B = \left( \bigcup_{v_1, \tilde{v}_1} \mathbb{Q}(v_1, \tilde{v}_1) \right) \cup \left( \bigcup_{v_2, \tilde{v}_2} \mathbb{Q}(v_2, \tilde{v}_2) \right) \cup \left( \bigcup_{v_1, v_2, \tilde{v}_1, \tilde{v}_2} \mathbb{Q}(v_1, v_2, \tilde{v}_1, \tilde{v}_2) \right)$ . The set  $B$  is a finite union of hyperplanes of dimension at most  $|\mathcal{U}| - 2$ . Hence,  $\Delta(\mathcal{U}) \setminus B$  is dense in  $\Delta(\mathcal{U})$ . If we consider the set  $A_0 := ([0, 1] \times \Delta(\mathcal{U}))^{|\mathcal{W}_1 \times \mathcal{W}_0|}$ , it follows that the set  $A := ([0, 1] \times (\Delta(\mathcal{U}) \setminus B))^{|\mathcal{W}_1 \times \mathcal{W}_0|}$  is a dense subset of  $A_0$ .

Let  $\Psi : A_0 \mapsto \Delta(\mathcal{W}_1 \times \mathcal{W}_0)^{|\mathcal{U}|}$  a continuous and onto function such that  $\Psi((\lambda_{w_1 w_0}, \mathcal{Q}_U^{w_1 w_0})_{w_1, w_0}) = \frac{\lambda_{w_1 w_0} \mathcal{Q}_U^{w_1 w_0}}{\mathcal{P}_U}$ . Let  $\Psi(A)$  denote the image of  $A$  under  $\Psi$ . We show that  $\Psi(A)$  is dense in  $\Delta(\mathcal{W}_1 \times \mathcal{W}_0)^{|\mathcal{U}|}$ . Take a distribution  $\mathcal{Q}_{\mathcal{W}_1 \mathcal{W}_0 | U} \in \Delta(\mathcal{W}_1 \times \mathcal{W}_0)^{|\mathcal{U}|}$ . Since  $A$  is dense in  $A_0$ , for each distribution  $\mathcal{Q}_U^{w_1 w_0} \in \Delta(U)$ , there exists a sequence  $(\mathcal{Q}_U^{w_1 w_0})_{(w_1, w_0)} \in \Delta(U \setminus B)$  that converges to it under the KL-divergence. By the continuity of  $\Psi$ , the image  $\Psi((\mathcal{Q}_U^{w_1 w_0})_{(w_1, w_0)}) \in \Psi(A)$  of  $(\mathcal{Q}_U^{w_1 w_0})_{(w_1, w_0)}$ , is a sequence that converges to  $\Psi(\mathcal{Q}_{\mathcal{W}_1 \mathcal{W}_0 | U}) \in \Delta(\mathcal{W}_1 \times \mathcal{W}_0)^{|\mathcal{U}|}$ . Therefore,  $\Psi(A)$  is dense in  $\Delta(\mathcal{W}_1 \times \mathcal{W}_0)^{|\mathcal{U}|}$ .

It follows that  $\mathbb{Q}^c \cap \mathbb{Q}_0(R_0, R_1) = \mathbb{Q}_{01} \cap \mathbb{Q}_0(R_0, R_1)$  is open and dense in  $\bar{\mathbb{Q}}_0(R_0, R_1) \cap \Delta(\mathcal{W}_1 \times \mathcal{W}_0)^{|\mathcal{U}|} = \bar{\mathbb{Q}}_0(R_0, R_1)$  as desired.  $\square$

**Lemma 3.4.4.** *If  $(R_0, R_1) \in \mathbb{R}_+^2$ , the set  $\mathbb{Q}_{02}(R_0, R_1)$  is nonempty, open and dense in  $\bar{\mathbb{Q}}_0(R_0, R_1)$ .*

*Proof.* of lemma 3.4.4 For  $(R_0, R_1) \in \mathbb{R}_+^2$ , the sets  $\bar{\mathbb{Q}}_0(R_0, R_1)$  and  $\mathbb{Q}_{02}(R_0, R_1)$  are non-empty. Moreover, the set  $\mathbb{Q}_{02}(R_0, R_1)$  is open being defined with strict inequalities on the continuous mutual information function, which means its complement  $\mathbb{Q}_{02}^c(R_0, R_1) = \{\mathcal{Q}_{\mathcal{W}_1 \mathcal{W}_0 | U} \text{ s.t. } R_0 \leq I(U; W_0), R_1 + R_0 \leq I(U; W_1, W_0)\}$  is closed. Take a feasible distribution  $\mathcal{Q}_{\mathcal{W}_1 \mathcal{W}_0 | U} \in \bar{\mathbb{Q}}_0(R_0, R_1)$  such that  $R_0 \geq I(U; W_0)$  and  $R_1 + R_0 \geq I(U; W_1, W_0)$ . Consider the distributions  $\mathcal{P}_{\mathcal{W}_1 \mathcal{W}_0}(w_1, w_0) = \sum_u \mathcal{P}(u) \mathcal{Q}(w_1, w_0 | u) \forall (w_1, w_0) \in \mathcal{W}_1 \times \mathcal{W}_2$  and  $\mathcal{P}_{\mathcal{W}_0}(w_0) = \sum_u \mathcal{P}(u) \mathcal{Q}(w_0 | u) \forall w_0 \in \mathcal{W}_2$ . For  $\varepsilon > 0$ , consider the perturbed distributions  $\mathcal{Q}_{\mathcal{W}_1 \mathcal{W}_0 | U}^\varepsilon = (1 - \varepsilon) \mathcal{Q}_{\mathcal{W}_1 \mathcal{W}_0 | U} + \varepsilon \mathcal{P}_{\mathcal{W}_1 \mathcal{W}_0}$ , and  $\mathcal{Q}_{\mathcal{W}_0 | U}^\varepsilon = (1 - \varepsilon) \mathcal{Q}_{\mathcal{W}_0 | U} + \varepsilon \mathcal{P}_{\mathcal{W}_0}$ . As  $\varepsilon \rightarrow 0$ , we have  $\mathcal{Q}_{\mathcal{W}_1 \mathcal{W}_0 | U}^\varepsilon \rightarrow \mathcal{Q}_{\mathcal{W}_1 \mathcal{W}_0 | U}$ , and  $\mathcal{Q}_{\mathcal{W}_0 | U}^\varepsilon \rightarrow \mathcal{Q}_{\mathcal{W}_0 | U}$ . Therefore,

$$I_{\mathcal{Q}_{\mathcal{W}_1 \mathcal{W}_0 | U}^\varepsilon}(U; W_1 W_0) \leq (1 - \varepsilon) \cdot I_{\mathcal{Q}_{\mathcal{W}_1 \mathcal{W}_0 | U}}(U; W_1 W_0) + \varepsilon \cdot I_{\mathcal{P}_{\mathcal{W}_1 \mathcal{W}_0}}(U; W_1 W_0) \quad (3.37)$$

$$< I_{\mathcal{Q}_{\mathcal{W}_1 \mathcal{W}_0 | U}}(U; W_1 W_0) \quad (3.38)$$

$$\leq R_1 + R_0. \quad (3.39)$$

Similarly,

$$I_{\mathcal{Q}^\varepsilon_{W_0|U}}(U; W_0) \leq (1 - \varepsilon) \cdot I_{\mathcal{Q}_{W_0|U}}(U; W_0) + \varepsilon \cdot I_{\mathcal{P}_{W_0}}(U; W_0) \quad (3.40)$$

$$< I_{\mathcal{Q}_{W_0|U}}(U; W_0) \quad (3.41)$$

$$\leq R_0. \quad (3.42)$$

Equations (3.37) and (3.40) follow from the convexity of the mutual information with respect to  $\mathcal{Q}_{W_1W_0|U}$  and  $\mathcal{Q}_{W_0|U}$  respectively for fixed  $\mathcal{P}_U$ . The strict inequalities in (3.38) and (3.41) follow since  $I_{\mathcal{P}_{W_1W_0}}(U; W_1W_0) = 0$  and  $I_{\mathcal{P}_{W_0}}(U; W_0) = 0$  and  $\varepsilon > 0$ , and last inequalities in equations (3.39) and (3.42) come from the definition of the set  $\bar{\mathbb{Q}}_0(R_0, R_1)$ . This means that both distributions  $\mathcal{Q}^\varepsilon_{W_1W_0|U}$  and  $\mathcal{Q}^\varepsilon_{W_0|U}$  belong to the set  $\bar{\mathbb{Q}}_0(R_0, R_1)$ . Hence, the set  $\mathbb{Q}_{02}(R_0, R_1)$  is dense in  $\bar{\mathbb{Q}}_0(R_0, R_1)$  which concludes the proof of lemma 3.4.4.  $\square$

Since  $\mathbb{Q}_{01}$  and  $\mathbb{Q}_{02}(R_0, R_1)$  are open and dense,  $\mathbb{Q}_{01} \cap \mathbb{Q}_{02}(R_0, R_1)$  is also open and dense in  $\bar{\mathbb{Q}}_0(R_0, R_1)$ . We now show that  $\Gamma_{sr}^*(R_0, R_1) = \tilde{\Gamma}_e(R_0, R_1)$ . In fact, the function

$$\mathcal{Q}_{W_1W_0|U} \mapsto \max_{\substack{\mathcal{Q}_{V_1|W_1W_0} \in \mathbb{Q}_1(\mathcal{Q}_{W_1W_0|U}) \\ \mathcal{Q}_{V_2|W_0} \in \mathbb{Q}_2(\mathcal{Q}_{W_0|U})}} \mathbb{E}_{\mathcal{P}_U \mathcal{Q}_{W_1W_0|U} \mathcal{Q}_{V_1|W_1W_0} \mathcal{Q}_{V_2|W_0}} \left[ c_e(U, V_1, V_2) \right]$$

is upper semi-continuous (u.s.c) and the infimum of an u.s.c function over a dense set is the infimum over the full set.

In this part of the proof, the assumption that each decoder chooses the optimal action that is worst for the encoder plays an important role. In fact, if decoders were to choose the pair of actions that is best for the encoder's cost, our function becomes

$$\mathcal{Q}_{W_1W_0|U} \mapsto \min_{\substack{\mathcal{Q}_{V_1|W_1W_0} \in \mathbb{Q}_1(\mathcal{Q}_{W_1W_0|U}) \\ \mathcal{Q}_{V_2|W_0} \in \mathbb{Q}_2(\mathcal{Q}_{W_0|U})}} \mathbb{E}_{\mathcal{P}_U \mathcal{Q}_{W_1W_0|U} \mathcal{Q}_{V_1|W_1W_0} \mathcal{Q}_{V_2|W_0}} \left[ c_e(U, V_1, V_2) \right]$$

which is lower semi continuous. The infimum of a lower semi continuous (l.s.c) function over a dense subset  $\mathbb{Q}_{01} \cap \mathbb{Q}_{02}(R_0, R_1)$  might be greater than the infimum over the whole set  $\bar{\mathbb{Q}}_0(R_0, R_1)$ . However, this is only the case whenever the information is constrained, and the information constraint is binding at optimum and all posterior beliefs of each decoder induce actions between which decoder is indifferent. This case is nongeneric in our class of persuasion games: if we slightly perturb the cost functions of our decoders, we perturb the

points of indifference for each decoder, and thus the points of discontinuity in our l.s.c. or u.s.c. This ends the proof of lemma 3.4.1.  $\square$

**Remark 3.4.1.** Our achievability proof assumes that both rates  $R_0$  and  $R_1$  are strictly positive. Some particular cases with at least one of the rates equal to zero are also interesting:

- If  $R_1 = R_0 = 0$ . The auxiliary random variables  $(W_1, W_0)$  are independent of  $U$ . The message sets are singletons, and the only possible encoding strategy  $\sigma_0$  is given by  $\sigma_0 : \mathcal{U}^n \rightarrow \{1\} \times \{1\}$ . The codebook consists of two sequences  $W_0^n(1)$  and  $W_1^n(1, 1)$  only. Therefore,  $\forall n \in \mathbb{N}^*$ ,  $\Gamma_{sr}^*(0, 0) = \Gamma_{sr}^n(0, 0)$ .
- If  $R_1 > 0$  and  $R_0 = 0$ . Random variables  $W_0$  and  $U$  are independent for  $R_1 > 0$  and  $R_2 = 0$ , i.e.  $\mathcal{Q}_{W_1 W_0 | U} = \mathcal{Q}_{W_0} \mathcal{Q}_{W_1 | W_0 U}$ . This means that decoder  $\mathcal{D}_2$  will repeatedly chose the action  $v_{2,0} \in V^*(\mathcal{P}_U)$  that corresponds to its prior belief  $\mathcal{P}_U$  and maximizes the encoder's distortion. The persuasion game is thus reduced to the point-to-point problem with one decoder  $\mathcal{D}_1$ , as in [Le Treust and Tomala \(2019a\)](#).
- If  $R_1 = 0$  and  $R_2 > 0$ . The auxiliary random variable  $W_1$  is independent of  $U$ . Hence, the encoder transmits the same index to both decoders. Therefore, both decoders will have the same posterior belief  $\mathcal{Q}_U^{w_0} \in \Delta(\mathcal{U})$ ,  $\forall w_0 \in \mathcal{W}_0$ . In that case, the optimal distortion can be reformulated in terms of a convexification of its expected distortion as in [Le Treust and Tomala \(2019a\)](#),  $\Gamma_{sr}^*(0, R_0) = \inf_{(\lambda_{w_0}, \mathcal{Q}_U^{w_0})_{w_0 \in \mathcal{W}_0}} \sum_{w_0 \in \mathcal{W}_0} \lambda_{w_0} \Psi_\epsilon(\mathcal{Q}_U^{w_0})$  where  $\Psi_\epsilon(q) = \max_{\substack{(v_1, v_2) \in \\ V_1^*(\bar{q}) \times V_2^*(q)}} \mathbb{E}_q[c_\epsilon(U, v_1, v_2)]$ .

### 3.4.2 Successive refinement codebook generation

Let  $(R_0, R_1) \in \mathbb{R}_+^2$ . Fix a conditional probability distribution  $\mathcal{Q}_{W_1 W_0 | U} \in \tilde{\mathcal{Q}}_0(R_0, R_1)$  given in (3.23). There exists  $\eta > 0$  such that

$$R_0 = I(U; W_0) + \eta, \quad (3.43)$$

$$R_1 = I(U; W_1 | W_0) + \eta. \quad (3.44)$$

Randomly and independently generate  $2^{\lfloor nR_0 \rfloor}$  sequences  $w_0^n(m_0)$  for  $m_0 \in [1 : 2^{\lfloor nR_0 \rfloor}]$ , according to the i.i.d distribution  $\mathcal{Q}_{W_0^n} = \prod_{t=1}^n \mathcal{Q}_{W_0}(w_{0t})$ . For each  $(m_1, m_0) \in [1 : 2^{\lfloor nR_1 \rfloor}] \times [1 : 2^{\lfloor nR_0 \rfloor}]$  generate a sequence  $w_1^n(m_1, m_0)$  randomly and conditionally independently according to the i.i.d conditional distribution  $\mathcal{Q}_{W_1^n|W_0^n} = \prod_{t=1}^n \mathcal{Q}_{W_1|W_0}(w_{1t}|w_{0t}(m_0))$ . Encoder  $\mathcal{E}$  observes  $u^n$  and chooses  $(m_1, m_0)$  such that  $(u^n, w_1^n(m_1, m_0), w_0^n(m_0)) \in \mathcal{T}_\delta^n(\mathcal{P}_U \mathcal{Q}_{W_1 W_0|U})$ , i.e. the sequences are jointly typical with tolerance parameter  $\delta > 0$ . If such a jointly typical tuple doesn't exist, the source encoder sets  $(m_1, m_0)$  to  $(1, 1)$ . Then, it sends  $m_0$  to decoder  $\mathcal{D}_2$ , and  $(m_1, m_0)$  to decoder  $\mathcal{D}_1$ .

Here comes the main difference with the successive refinement coding, which is due to the strategic nature of the problem. Instead of declaring  $w_1^n(m_1, m_0)$  and  $w_0^n(m_0)$  and selecting  $V_1^n$  and  $V_2^n$  i.i.d. with respect to  $\mathcal{Q}_{V_1|W_1 W_0} \in \mathbb{Q}_1(\mathcal{Q}_{W_1 W_0|U})$  and  $\mathcal{Q}_{V_2|W_0} \in \mathbb{Q}_2(\mathcal{Q}_{W_0|U})$ , at each stage  $t \in \{1, \dots, n\}$  the decoders  $\mathcal{D}_1$  and  $\mathcal{D}_2$  compute their Bayesian posterior beliefs  $\mathcal{P}_{U_t|M_1 M_0}^\sigma(\cdot|m_1, m_0)$  and  $\mathcal{P}_{U_t|M_0}^\sigma(\cdot|m_0)$  and select the actions  $v_{1,t} \in V_1^*(\mathcal{P}_{U_t|M_1 M_0}^\sigma)$  and  $v_{2,t} \in V_2^*(\mathcal{P}_{U_t|M_0}^\sigma)$  that minimize their own cost functions. If several pairs are available, they select the worst one for the encoder's cost.

### 3.4.3 Analysis of the error probability

Given a tolerance  $\delta > 0$ , the error event is given by

$$\mathcal{F} = \{(U^n, W_0^n(m_0), W_1^n(m_0, m_1)) \notin \mathcal{T}_\delta^n\}. \quad (3.45)$$

We have by the union of events bound

$$\mathcal{P}(\mathcal{F}) \leq \mathcal{P}(\mathcal{F}_1) + \mathcal{P}(\mathcal{F}_2(M_0) \cap \mathcal{F}_1^c), \quad (3.46)$$

where

$$\mathcal{F}_1 = \{(U^n, W_0^n(m_0)) \notin \mathcal{T}_\delta^n \forall m_0\}, \quad (3.47)$$

$$\forall m_0, \mathcal{F}_2(m_0) = \{(U^n, W_0^n(m_0), W_1^n(m_0, m_1)) \notin \mathcal{T}_\delta^n \forall m_1\}. \quad (3.48)$$

By (El Gamal and Kim, 2011, Lemma 3.3, pp. 62),  $\mathcal{P}(\mathcal{F}_1)$  tends to zero as  $n \rightarrow \infty$  if  $R_0 > I(U; W_0) + \eta$ . By (El Gamal and Kim, 2011, Lemma 3.3, pp. 62),  $\mathcal{P}(\mathcal{F}_1^c \cap \mathcal{F}_2(M_0))$  goes to zero if  $R_1 + R_0 > I(U; W_1, W_0) + \eta$ .

Since the expected error probability evaluated with respect to the random codebook is small, we have that for all  $\varepsilon_2 > 0$ , for all  $\eta > 0$ , there exists  $\bar{\delta} > 0$ , for all  $\delta \leq \bar{\delta}$ , there exists  $\bar{n} \in \mathbb{N}$  such that for all  $n \geq \bar{n}$ , we have

$$\mathbb{E}[\mathcal{P}(\mathcal{F}_1)] \leq \varepsilon_2, \quad (3.49)$$

$$\forall m_0, \mathbb{E}[\mathcal{P}(\mathcal{F}_2(m_0))] \leq \varepsilon_2. \quad (3.50)$$

The proof of achievability consists of four main parts: first, in subsection 3.4.2, we generate the codebook and analyze the error probability. Second in subsection 3.4.3, we provide an alternative formulation to our solution concept by restricting the set of target distributions to a dense subset of distributions inducing beliefs based on which the decoders are not indifferent between two or more actions. third in subsection 3.4.1, we restrict our optimization to a dense subset of target distributions in order to eliminate decoder's indifferences between two pairs of actions.. Finally in subsection 3.4.4, we control the Bayesian beliefs of each decoder about the state concluding by that our proof of achievability.

### 3.4.4 Control of the Bayesian beliefs

We introduce the indicators of error events  $E_\delta^1 \in \{0, 1\}$  for decoder  $\mathcal{D}_1$ , and  $E_\delta^2 \in \{0, 1\}$  for decoder  $\mathcal{D}_2$  defined as follows

$$E_\delta^1 = \begin{cases} 1, & \text{if } (u^n, w_1^n, w_0^n) \notin \mathcal{T}_\delta^n(\mathcal{P}_U \mathcal{Q}_{W_1 W_0 | U}). \\ 0, & \text{otherwise.} \end{cases} \quad (3.51)$$

$$E_\delta^2 = \begin{cases} 1, & \text{if } (u^n, w_0^n) \notin \mathcal{T}_\delta^n(\mathcal{P}_U \mathcal{Q}_{W_0 | U}). \\ 0, & \text{otherwise.} \end{cases} \quad (3.52)$$

**Remark 3.4.2.** Note that  $E_\delta^1 = 0 \iff (u^n, w_1^n, w_0^n) \in \mathcal{T}_\delta^n(\mathcal{P}_U \mathcal{Q}_{W_1 W_0 | U}) \implies (u^n, w_0^n) \in \mathcal{T}_\delta^n(\mathcal{P}_U \mathcal{Q}_{W_0 | U}) \iff E_\delta^2 = 0$ . Conversely,  $E_\delta^2 = 1 \iff (u^n, w_0^n) \notin \mathcal{T}_\delta^n(\mathcal{P}_U \mathcal{Q}_{W_0 | U}) \implies (u^n, w_1^n, w_0^n) \notin \mathcal{T}_\delta^n(\mathcal{P}_U \mathcal{Q}_{W_1 W_0 | U}) \iff E_\delta^1 = 1$ . Moreover,  $\mathcal{P}(E_\delta^1 = 0) \leq \mathcal{P}(E_\delta^2 = 0)$  and  $\mathcal{P}(E_\delta^1 = 1) \geq \mathcal{P}(E_\delta^2 = 1)$ . Assuming the distribution  $\mathcal{P}_{U|W_1 W_0}$  is fully supported, the beliefs of decoder  $\mathcal{D}_1$  are controlled as follows

$$\mathbb{E}\left[\frac{1}{n} \sum_{t=1}^n D(\mathcal{P}_t^{m_1, m_0} \| \mathcal{P}_{U|W_1 W_2}(\cdot | W_{1t}, W_{0t})) \middle| E_\delta^1 = 0\right] \quad (3.53)$$

$$= \sum_{m_1, m_0, w_1^n, w_0^n} \mathcal{P}^{\sigma\tau_1\tau_2}(m_1, m_0, w_1^n, w_0^n | E_\delta^1 = 0) \cdot \frac{1}{n} \sum_{t=1}^n D(\mathcal{P}_t^{m_1, m_0} \| \mathcal{P}_{U|W_1 W_0}(\cdot | W_{1t}, W_{0t})) \quad (3.54)$$

$$= \sum_{m_1, m_0, w_1^n, w_0^n} \mathcal{P}^{\sigma\tau_1\tau_2}(m_1, m_0, w_1^n, w_0^n | E_\delta^1 = 0) \cdot \frac{1}{n} \sum_{t=1}^n \sum_u \mathcal{P}_t^{m_1 m_0}(u) \log_2 \frac{\mathcal{P}_t^{m_1 m_0}(u)}{\mathcal{P}_{U|W_1 W_2}(u | w_{1t}, w_{0t})} \quad (3.55)$$

$$= \sum_{m_1, m_0, w_1^n, w_0^n} \mathcal{P}^{\sigma\tau_1\tau_2}(m_1, m_0, w_1^n, w_0^n | E_\delta^1 = 0) \cdot \frac{1}{n} \sum_{t=1}^n \sum_u \mathcal{P}_t^{m_1 m_0}(u) \log_2 \frac{1}{\mathcal{P}_{U|W_1 W_0}(u | w_{1t}, w_{0t})} \\ - \sum_{m_1, m_0, w_1^n, w_0^n} \mathcal{P}^{\sigma\tau_1\tau_2}(m_1, m_0, w_1^n, w_0^n | E_\delta^1 = 0) \cdot \frac{1}{n} \sum_{t=1}^n \sum_u \mathcal{P}_t^{m_1 m_0}(u) \log_2 \frac{1}{\mathcal{P}_t^{m_1 m_0}(u)} \quad (3.56)$$

$$= \frac{1}{n} \sum_{m_1, m_0, w_1^n, w_0^n} \mathcal{P}^{\sigma\tau_1\tau_2}(m_1, m_0, w_1^n, w_0^n | E_\delta^1 = 0) \cdot \frac{1}{n} \sum_{t=1}^n \sum_u \mathcal{P}_t^{m_1 m_0}(u) \log_2 \frac{1}{\mathcal{P}_{U|W_1 W_0}(u | w_{1t}, w_{0t})} \\ - \frac{1}{n} \sum_{t=1}^n H(U_t | M_1, M_0, E_\delta^1 = 0) \quad (3.57)$$

$$= \frac{1}{n} \sum_{u^n, w_1^n, w_0^n} \mathcal{P}^{\sigma\tau_1\tau_2}(u^n, w_1^n, w_0^n | E_\delta^1 = 0) \cdot \log_2 \frac{1}{\prod_{t=1}^n \mathcal{P}_{U|W_1 W_0}(u_t | w_{1t}, w_{0t})} \\ - \frac{1}{n} \sum_{t=1}^n H(U_t | M_1, M_0, E_\delta^1 = 0) \quad (3.58)$$

$$= \frac{1}{n} \sum_{u^n, w_1^n, w_0^n \in \mathcal{T}_\delta^n} \mathcal{P}^{\sigma\tau_1\tau_2}(u^n, w_1^n, w_0^n | E_\delta^1 = 0) \cdot \log_2 \frac{1}{\prod_{t=1}^n \mathcal{P}_{U|W_1 W_0}(u_t | w_{1t}, w_{0t})} \\ - \frac{1}{n} \sum_{t=1}^n H(U_t | M_1, M_0, E_\delta^1 = 0) \quad (3.59)$$

$$\leq \frac{1}{n} \sum_{u^n, w_1^n, w_0^n \in \mathcal{T}_\delta^n} \mathcal{P}^{\sigma\tau_1\tau_2}(u^n, w_1^n, w_0^n | E_\delta^1 = 0) \cdot n \cdot (H(U | W_1, W_0) + \delta) - \frac{1}{n} H(U^n | M_1, M_0, E_\delta^1 = 0) \quad (3.60)$$

$$\leq \frac{1}{n} I(U^n; M_1, M_0 | E_\delta^1 = 0) - I(U; W_1, W_0) + \delta + \frac{1}{n} + \log_2 |\mathcal{U}| \cdot \mathcal{P}^{\sigma\tau_1\tau_2}(E_\delta^1 = 1) \quad (3.61)$$

$$\leq \eta + \delta + \frac{1}{n} + \log_2 |\mathcal{U}| \cdot \mathcal{P}^{\sigma\tau_1\tau_2}(E_\delta^1 = 1). \quad (3.62)$$

- Equation (3.54) comes from the definition of expected K-L divergence.
- Equation (3.55) comes from the definition of K-L divergence.
- Equation (3.56) comes from splitting the logarithm.
- Equation (3.57) follows since:

$$\sum_{m_1, m_0, w_1^n, w_0^n} \mathcal{P}^{\sigma\tau_1\tau_2}(m_1, m_0, w_1^n, w_0^n | E_\delta^1 = 0) \cdot \frac{1}{n} \sum_{t=1}^n \sum_u \mathcal{P}_t^{m_1 m_0}(u) \log_2 \frac{1}{\mathcal{P}_t^{m_1 m_0}(u)} \quad (3.63)$$

$$= \sum_{m_1, m_0, w_1^n, w_0^n} \mathcal{P}^{\sigma\tau_1\tau_2}(m_1, m_0, w_1^n, w_0^n | E_\delta^1 = 0) \cdot \frac{1}{n} \sum_{t=1}^n H(U_t | M_1 = m_1, M_0 = m_0) \quad (3.64)$$

$$= \frac{1}{n} \sum_{t=1}^n \sum_{m_1, m_0, w_1^n, w_0^n} \mathcal{P}^{\sigma\tau_1\tau_2}(m_1, m_0, w_1^n, w_0^n | E_\delta^1 = 0) \cdot H(U_t | M_1 = m_1, M_0 = m_0) \quad (3.65)$$

$$= \frac{1}{n} \sum_{t=1}^n \sum_{m_1, m_0} \mathcal{P}^{\sigma\tau_1\tau_2}(m_1, m_0 | E_\delta^1 = 0) \cdot H(U_t | M_1 = m_1, M_0 = m_0) \quad (3.66)$$

$$= \frac{1}{n} \sum_{t=1}^n H(U_t | M_1, M_0, E_\delta^1 = 0). \quad (3.67)$$

- Equation (3.58) follows since:

$$\sum_{m_1, m_0, w_1^n, w_0^n} \mathcal{P}^{\sigma\tau_1\tau_2}(m_1, m_0, w_1^n, w_0^n | E_\delta^1 = 0) \cdot \frac{1}{n} \sum_{t=1}^n \sum_u \mathcal{P}_t^{m_1 m_0}(u) \log_2 \frac{1}{\mathcal{P}_{U|W_1 W_0}(u | w_{1t}, w_{0t})} \quad (3.68)$$

$$= \frac{1}{n} \sum_{t=1}^n \sum_{u_t, m_1, m_0, w_1^n, w_0^n} \mathcal{P}^{\sigma\tau_1\tau_2}(u_t, m_1, m_0, w_1^n, w_0^n | E_\delta^1 = 0) \cdot \log_2 \frac{1}{\mathcal{P}_{U|W_1 W_0}(u_t | w_{1t}, w_{0t})} \quad (3.69)$$

$$= \frac{1}{n} \sum_{t=1}^n \sum_{u^n, m_1, m_0, w_1^n, w_0^n} \mathcal{P}^{\sigma\tau_1\tau_2}(u^n, m_1, m_0, w_1^n, w_0^n | E_\delta^1 = 0) \cdot \log_2 \frac{1}{\mathcal{P}_{U|W_1 W_0}(u_t | w_{1t}, w_{0t})} \quad (3.70)$$

$$= \frac{1}{n} \sum_{u^n, m_1, m_0, w_1^n, w_0^n} \mathcal{P}^{\sigma\tau_1\tau_2}(u^n, m_1, m_0, w_1^n, w_0^n | E_\delta^1 = 0) \cdot \log_2 \frac{1}{\prod_{t=1}^n \mathcal{P}_{U|W_1W_0}(u_t | w_{1t}, w_{0t})} \quad (3.71)$$

$$= \frac{1}{n} \sum_{u^n, w_1^n, w_0^n} \mathcal{P}^{\sigma\tau_1\tau_2}(u^n, w_1^n, w_0^n | E_\delta^1 = 0) \cdot \log_2 \frac{1}{\prod_{t=1}^n \mathcal{P}_{U|W_1W_0}(u_t | w_{1t}, w_{0t})}. \quad (3.72)$$

- Equation (3.59) follows since the support of  $\mathcal{P}^{\sigma\tau_1\tau_2}(u^n, w_1^n, w_0^n | E_\delta^1) = \mathbb{P}\{(u^n, w_1^n, w_0^n) \in \mathcal{T}_\delta^n\}$  is included in  $\mathcal{T}_\delta^n$ .
- Equation (3.60) follows from the typical average lemma property (Property 1 pp.26 in *El Gamal and Kim (2011)*) given in lemma G.0.1, and the chain rule of entropy:  $H(U^n | M_1, M_0, W_1^n, W_0^n) \leq \sum_{t=1}^n H(U_t | M_1, M_0, W_1, W_0)$ .
- Equation (3.61) comes from the conditional entropy property and the fact that  $H(U^n) = nH(U)$  for an i.i.d random variable  $U$  and lemma G.0.2.
- Equation (3.62) follows since  $I(U^n; M_1, M_0) \leq H(M_1, M_0) \leq \log_2 |\mathcal{C}| = n \cdot (R_1 + R_0) = n \cdot (I(U; W_1, W_0) + \eta)$  and lemma G.0.2.

Similarly for decoder  $\mathcal{D}_2$  we have

$$\begin{aligned} & \mathbb{E} \left[ \frac{1}{n} \sum_{t=1}^n D(\mathcal{P}_t^{m_0} | \prod_{t=1}^n \mathcal{P}_{U_t|W_{0t}}) | E_\delta^2 = 0 \right] \quad (3.73) \\ &= \frac{1}{n} \sum_{(u^n, m_0, w_0^n) \in \mathcal{T}_\delta^n} \mathcal{P}^{\sigma, \tau_2}(m_0, w_0^n | E_\delta^2 = 0) \cdot \log_2 \frac{1}{\prod_{t=1}^n \mathcal{P}_{U_t|W_{0t}}} - \frac{1}{n} \sum_{t=1}^n H(U_t | M_0, E_\delta^2 = 0) \end{aligned} \quad (3.74)$$

$$\leq \frac{1}{n} \sum_{(u^n, m_0, w_0^n) \in \mathcal{T}_\delta^n} \mathcal{P}^{\sigma, \tau_2}(u^n, m_0, w_0^n | E_\delta^2 = 0) \cdot n \cdot (H(U|W_0) + \delta) - \frac{1}{n} H(U^n | M_0, E_\delta^2 = 0) \quad (3.75)$$

$$\leq \frac{1}{n} I(U^n; M_0, E_\delta^2 = 0) - I(U; W_0) + \delta + \frac{1}{n} + \log_2 |\mathcal{U}| \cdot \mathcal{P}^{\sigma, \tau_2}(E_\delta^2 = 1) \quad (3.76)$$

$$\leq \eta + \delta + \frac{1}{n} + \log_2 |\mathcal{U}| \cdot \mathcal{P}^{\sigma\tau_1\tau_2}(E_\delta^1 = 1). \quad (3.77)$$

By combining equations (3.49), (3.50), (3.62), (3.77) with (Le Treust and Tomala, 2019a, Lemma A.21, equations (40)-(46), Lemma A.8 ), we obtain  $\forall \varepsilon > 0, \exists \hat{n}, \forall n \geq \hat{n}, \Gamma_\varepsilon^n(R_0, R_1) \leq \Gamma_\varepsilon^*(R_0, R_1) + \varepsilon$ . This completes the proof of achievability.  $\square$

### 3.5 Converse proof for strategic successive refinement

In the following, we prove the converse statement of our main result [Theorem 3.3.1](#). We provide identifications for our auxiliary random variables satisfying Markov chain properties, and we show they satisfy the information constraints of the successive refinement coding setup for any given rate pair.

Let  $(R_0, R_1) \in \mathbb{R}_+^2$  and  $n \in \mathbb{N}^*$ . We consider  $(\sigma, \tau_1, \tau_2) \in \mathcal{S}(n, R_1, R_0)$  and a random variable  $T$  uniformly distributed over  $\{1, 2, \dots, n\}$  and independent of  $(U^n, M_1, M_0, V_1^n, V_2^n)$ . We introduce the auxiliary random variables  $W_1 = (M_1, T)$ ,  $W_0 = (M_0, T)$ ,  $(U, V_1, V_2) = (U_T, V_{1,T}, V_{2,T})$ <sup>1</sup>, distributed according to  $\mathcal{P}_{UW_1W_0V_1V_2}^{\sigma\tau_1\tau_2}$  defined for all  $(u, w_1, w_0, v_1, v_2) = (u_t, m_1, m_0, t, v_{1,t}, v_{2,t})$  by

$$\mathcal{P}_{UW_1W_0V_1V_2}^{\sigma\tau_1\tau_2}(u, w_1, w_0, v_1, v_2) = \mathcal{P}_{UM_1M_0TV_1TV_2T}^{\sigma\tau_1\tau_2}(u_t, m_1, m_0, t, v_{1,t}, v_{2,t}) \quad (3.78)$$

$$= \frac{1}{n} \sum_{\substack{u^{t-1} \\ u^{t+1}}} \sum_{\substack{v_1^{t-1}, v_{1,t+1}^n \\ v_2^{t-1}, v_{2,t+1}^n}} \left( \prod_{t=1}^n \mathcal{P}_U(u_t) \right) \sigma(m_1, m_0 | u^n) \tau_1(v_1^n | m_1, m_0) \tau_2(v_2^n | m_0). \quad (3.79)$$

**Lemma 3.5.1.** *The distribution  $\mathcal{P}_{UW_1W_0V_1V_2}^{\sigma\tau_1\tau_2}$  has marginal on  $\Delta(\mathcal{U})$  given by  $\mathcal{P}_U$  and satisfies the Markov chain properties*

$$(U, V_2) \text{---} (W_1, W_0) \text{---} V_1, \quad (U, W_1, V_1) \text{---} W_0 \text{---} V_2.$$

*Proof.* [[Lemma 3.5.1](#)] The i.i.d. property of the source ensures that the marginal distribution

<sup>1</sup>We denote by  $U_T, V_{1,T}, V_{2,T}$  components of sequences  $U^n, V_1^n, V_2^n$  respectively, selected uniformly at random.

is  $\mathcal{P}_U$ . By the definition of the decoding functions  $\tau_1$  and  $\tau_2$  we have

$$(U_T, V_{2,T}) \text{---} (M_1, M_0, T) \text{---} V_{1,T},$$

$$(U_T, M_1, V_{1,T}) \text{---} (M_0, T) \text{---} V_{2,T}.$$

Therefore  $\mathcal{P}_{UW_1W_0V_1V_2}^{\sigma\tau_1\tau_2} = \mathcal{P}_U \mathcal{P}_{W_1W_0|U}^\sigma \mathcal{P}_{V_1|W_1W_0}^{\tau_1} \mathcal{P}_{V_2|W_0}^{\tau_2}$ .  $\square$

**Lemma 3.5.2.** *For all  $\sigma$ , the distribution  $\mathcal{P}_{W_1W_0|U}^\sigma \in \bar{\mathbb{Q}}_0(R_0, R_1)$ .*

*Proof.* [Lemma 3.5.2] We consider an encoding strategy  $\sigma$ , then

$$\lfloor nR_0 \rfloor \geq H(M_0) \geq I(M_0; U^n) \tag{3.80}$$

$$= \sum_{t=1}^n I(U_t; M_0 | U^{t-1}) \tag{3.81}$$

$$= nI(U_T; M_0 | U^{T-1}, T) \tag{3.82}$$

$$= nI(U_T; M_0, U^{T-1}, T) \tag{3.83}$$

$$\geq nI(U_T; M_0, T) \tag{3.84}$$

$$= nI(U; W_0). \tag{3.85}$$

In fact, (3.82) follows from the introduction of the uniform random variable  $T \in \{1, \dots, n\}$ , (3.83) comes from the i.i.d. property of the source, and (3.85) follows from the identification of the auxiliary random variables  $(U, W_0)$  and the independence between  $T$  and  $U_T$ . Similarly,

$$\lfloor n(R_1 + R_0) \rfloor \geq H(M_1, M_0) \geq I(U^n; M_1, M_0) \tag{3.86}$$

$$= \sum_{t=1}^n I(U_t; M_1, M_0 | U^{t-1}) \tag{3.87}$$

$$= nI(U_T; M_1, M_0 | U^{T-1}, T) \tag{3.88}$$

$$\geq nI(U_T; M_1, M_0, T) \tag{3.89}$$

$$= nI(U; W_1, W_0). \tag{3.90}$$

$\square$

**Lemma 3.5.3.** For all  $(\sigma, \tau_1, \tau_2)$  and  $i \in \{1, 2\}$ , we have

$$c_e^n(\sigma, \tau_1, \tau_2) = \mathbb{E}[c_e(U, V_1, V_2)] \quad (3.91)$$

$$c_i^n(\sigma, \tau_i) = \mathbb{E}[c_i(U, V_i)] \quad (3.92)$$

evaluated with respect to  $\mathcal{P}_U \mathcal{P}_{W_1 W_0 | U}^\sigma \mathcal{P}_{V_1 | W_1 W_0}^{\tau_1} \mathcal{P}_{V_2 | W_0}^{\tau_2}$ . Moreover, for all  $\sigma$ , we have

$$\mathcal{Q}_1(\mathcal{P}_{W_1 W_0 | U}^\sigma) = \left\{ \mathcal{Q}_{V_1 | W_1 W_0}, \quad \exists \tau_1 \in BR_1(\sigma), \mathcal{Q}_{V_1 | W_1 W_0} = \mathcal{P}_{V_1 | W_1 W_0}^{\tau_1} \right\}, \quad (3.93)$$

$$\mathcal{Q}_2(\mathcal{P}_{W_0 | U}^\sigma) = \left\{ \mathcal{Q}_{V_2 | W_0}, \quad \exists \tau_2 \in BR_2(\sigma), \mathcal{Q}_{V_2 | W_0} = \mathcal{P}_{V_2 | W_0}^{\tau_2} \right\}. \quad (3.94)$$

*Proof.* [Lemma 3.5.3] By Definition 3.3.2 and (3.7), (3.79), we have

$$\begin{aligned} c_e^n(\sigma, \tau_1, \tau_2) &= \sum_{\substack{u^n, m_1, m_0, \\ v_1^n, v_2^n}} \left( \prod_{t=1}^n \mathcal{P}_U(u_t) \right) \sigma(m_1, m_0 | u^n) \times \tau_1(v_1^n | m_1, m_0) \tau_2(v_2^n | m_0) \left[ \frac{1}{n} \sum_{t=1}^n c_e(u_t, v_{1,t}, v_{2,t}) \right] \\ &= \sum_{t=1}^n \sum_{\substack{u_t, m_1, m_0, \\ v_{1,t}, v_{2,t}}} \mathcal{P}^{\sigma \tau_1 \tau_2}(u_t, m_1, m_0, t, v_{1,t}, v_{2,t}) \times c_e(u_t, v_{1,t}, v_{2,t}) \\ &= \mathbb{E}[c_e(U_T, V_{1,T}, V_{2,T})] = \mathbb{E}[c_e(U, V_1, V_2)]. \end{aligned} \quad (3.95)$$

Now we prove the second part of lemma 3.5.3. For any  $\sigma$  and any  $\mathcal{Q}_{V_1 | W_1 W_0} \in \mathcal{Q}_1(\mathcal{P}_{W_1 W_0 | U}^\sigma)$ , we define  $\tilde{\tau}_1$  by

$$\tilde{\tau}_1(v_1^n | m_1, m_0) = \prod_{s=1}^n \mathcal{Q}_{V_1 | W_1 W_0}(v_{1,s} | m_1, m_0, s), \quad \forall (m_1, m_0, v_1^n). \quad (3.96)$$

Then for all  $(w_1, w_0, v_1) = (m_1, m_0, t, v_{1,t})$

$$\mathcal{P}_{V_1 | W_1 W_0}^{\tilde{\tau}_1}(v_1 | w_1, w_0) = \mathcal{P}_{V_1 | W_1 W_0}^{\tilde{\tau}_1}(v_{1,t} | m_1, m_0, t) \quad (3.97)$$

$$= \sum_{v_1^{t-1}, v_{1,t+1}^n} \tilde{\tau}_1(v_1^n | m_1, m_0) \quad (3.98)$$

$$= \sum_{v_1^{t-1}, v_{1,t+1}^n} \prod_{s=1}^n \mathcal{Q}_{V_1 | W_1 W_0}(v_{1,s} | m_1, m_0, s) \quad (3.99)$$

$$= \mathcal{Q}_{V_1 | W_1 W_0}(v_{1,t} | m_1, m_0, t) \sum_{v_1^{t-1}, v_{1,t+1}^n} \prod_{s \neq t} \mathcal{Q}_{V_1 | W_1 W_0}(v_{1,s} | m_1, m_0, s) \quad (3.100)$$

$$= \mathcal{Q}_{V_1|W_1W_0}(v_{1,t}|m_1, m_0, t) = \mathcal{Q}_{V_1|W_1W_0}(v_1|w_1, w_0). \quad (3.101)$$

Moreover assume that  $\tilde{\tau}_1 \notin BR_1(\sigma)$ , then there exists  $\bar{\tau}_1 \neq \tilde{\tau}_1$  such that

$$\mathbb{E}_{\substack{\mathcal{P}_{W_1W_0|U}^\sigma \\ \mathcal{P}_{V_1|W_1W_0}^{\tilde{\tau}_1}}} [c_1(U, V_1)] = c_1^n(\sigma, \bar{\tau}_1) < c_1^n(\sigma, \tilde{\tau}_1) \quad (3.102)$$

$$= \mathbb{E}_{\substack{\mathcal{P}_{W_1W_0|U}^\sigma \\ \mathcal{P}_{V_1|W_1W_0}^{\tilde{\tau}_1}}} [c_1(U, V_1)] = \mathbb{E}_{\substack{\mathcal{P}_{W_1W_0|U}^\sigma \\ \mathcal{Q}_{V_1|W_1W_0}}} [c_1(U, V_1)], \quad (3.103)$$

which contradicts  $\mathcal{Q}_{V_1|W_1W_0} \in \mathbb{Q}_1(\mathcal{P}_{W_1W_0|U}^\sigma)$ . Therefore,  $\tilde{\tau}_1 \in BR_1(\sigma)$  and thus  $\mathcal{Q}_{V_1|W_1W_0}$  belongs to the right-hand side of (3.93). For the other inclusion, we assume that  $\mathcal{Q}_{V_1|W_1W_0}$  belongs to the right-hand side of (3.93) and does not belong to  $\mathbb{Q}_1(\mathcal{P}_{W_1W_0|U}^\sigma)$ , then we show that it leads to a contradiction. Similar arguments imply (3.94).  $\square$

For any strategy  $\sigma$ , we have

$$\max_{\substack{\tau_1 \in BR_1(\sigma), \\ \tau_2 \in BR_2(\sigma)}} c_e^n(\sigma, \tau_1, \tau_2) = \max_{\substack{\tau_1 \in BR_1(\sigma), \\ \tau_2 \in BR_2(\sigma)}} \mathbb{E}_{\substack{\mathcal{P}_{W_1W_0|U}^\sigma \\ \mathcal{P}_{V_1|W_1W_0}^{\tau_1} \mathcal{P}_{V_2|W_0}^{\tau_2}}} [c_e(U, V_1, V_2)] \quad (3.104)$$

$$= \max_{\substack{\mathcal{Q}_{V_1|W_1W_0} \in \mathbb{Q}_1(\mathcal{P}_{W_1W_0|U}^\sigma) \\ \mathcal{Q}_{V_2|W_0} \in \mathbb{Q}_2(\mathcal{P}_{W_0|U}^\sigma)}} \mathbb{E}_{\substack{\mathcal{P}_{W_1W_0|U}^\sigma \\ \mathcal{Q}_{V_1|W_1W_0} \mathcal{Q}_{V_2|W_0}}} [c_e(U, V_1, V_2)] \quad (3.105)$$

$$\geq \inf_{\substack{\mathcal{Q}_{W_1W_0|U} \in \bar{\mathbb{Q}}_0(R_0, R_1) \\ \mathcal{Q}_{V_1|W_1W_0} \in \mathbb{Q}_1(\mathcal{Q}_{W_1W_0|U}) \\ \mathcal{Q}_{V_2|W_0} \in \mathbb{Q}_2(\mathcal{Q}_{W_0|U})}} \max_{\substack{\mathcal{Q}_{V_1|W_1W_0} \in \mathbb{Q}_1(\mathcal{Q}_{W_1W_0|U}) \\ \mathcal{Q}_{V_2|W_0} \in \mathbb{Q}_2(\mathcal{Q}_{W_0|U})}} \mathbb{E} [c_e(U, V_1, V_2)] \quad (3.106)$$

$$= \Gamma_{sr}^*(R_0, R_1). \quad (3.107)$$

Equations (3.104) and (3.105) come from Lemma 3.5.3, whereas (3.106) comes from Lemma 3.5.2 and (3.107) follows from (3.18). Therefore,  $\Gamma_{sr}^*(R_0, R_1) \leq \inf_{\sigma} \max_{\substack{\tau_1 \in BR_1(\sigma), \\ \tau_2 \in BR_2(\sigma)}} c_e^n(\sigma, \tau_1, \tau_2) = \Gamma_{sr}^n(R_0, R_1)$ , for all  $n$ .

## 3.6 Conclusion

We study strategic lossy source coding through a modified model of the successive refinement network where one decoder observes a public message from the encoder, and the other decoder observes a private message in addition to the common public message.

In addition to the distinct and arbitrary cost functions of both decoders, we assume that the encoder is endowed with a cost function that depends on the state and on the actions from both decoders. We characterize the single-letter signal of the encoder as a conditional distribution over a pair of auxiliary random variables satisfying the information constraints imposed by the successive refinement coding setup. The encoder's optimal single-letter cost is characterized using the set of optimal encoding signals which satisfy the incentive constraints of both decoders. We fully describe the limiting behaviour of the long-run optimal cost function of the encoder and we determine its lower bound for any blocklength  $n \in \mathbb{N}^*$ . Our achievability proof consists of 1) reformulating the problem by restricting the set of optimal signals to a dense subset of signals that induce beliefs according to which the decoders are not indifferent between any pairs of actions so as to ensure convergence, 2) generating the codebook and analyzing the probability of the error events according to the successive refinement coding setup, 3) showing that the Bayesian beliefs defined using auxiliary random variables are close under the KL-divergence to the beliefs induced by the coding strategies. Our converse proof provides an identification of the auxiliary random and shows that the long-run cost of the encoder does not go below its single-letter expected cost. A natural extension to this model is to assume that the links are not perfect and add one or two noisy channels on each link. This corresponds to considering the degraded broadcast channel model, whose capacity region was conjectured by [Cover \(1972\)](#) and proven by [Bergmans \(1973\)](#) and [Gallager \(1974\)](#), with a strategic encoder. The information constraints imposed by the noisy channel are then added to the encoder's set of target distributions.

# 4

## STRATEGIC COMMUNICATION VIA THE GRAY-WYNER NETWORK

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*In this chapter, we propose a strategic Gray-Wyner network coding for Bayesian persuasion. Our solution determines upper and lower bounds on the encoder's long-run cost.*

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## 4.1 Introduction

Persuading two decoders endowed with distinct objectives is a challenging problem that can be studied in various settings. When both decoders have access to a public and a private information from a single encoder, the setting can be modeled using a Gray-Wyner network. The simple Gray-Wyner network was formulated in the seminal work [Gray and Wyner \(1974\)](#) modeling a broadcast communication scenario composed of an encoder, encoding a correlated source into a common or public message and two private messages, and two decoders which aim to recover the source based on their respective observations of the received messages. [Gray and Wyner \(1974\)](#) fully characterized the region of attainable rates, both in the lossless and lossy cases. The optimal region of second-order coding for the lossy Gray-Wyner network was derived in [Zhou et al. \(2016\)](#). The rate region for jointly Gaussian sources, has been partially computed in [Xu et al. \(2011\)](#), [Viswanatha et al. \(2014\)](#) before being fully computed in a recent contribution [Sula and Gastpar \(2022\)](#). However; the question of optimal rate region for any other sources remains open. We propose a novel communication setting à la Gray-Wyner, in which the encoder is also endowed with a cost function that depends on the state and the actions of both decoders, and we assume that the cost of each decoder depends on the state and on the actions of both decoders. Under these assumptions, every agent is strategic and has a goal which consists of minimizing its cost. We still assume the encoder commits to and reveals an encoding before observing the source. However; in this setting, every commitment of the encoder induces a finite Bayesian game among the decoders.

## 4.2 Gray-Wyner’s model

The Gray–Wyner system depicted in [Figure 4.1](#) is a lossy source coding setup in which a single encoder transmits three descriptions: one public and two private descriptions. Decoder  $\mathcal{D}_i$  for  $i \in \{1, 2\}$ , receives the common description  $M_0$  transmitted at rate  $R_0$ , and a private description  $M_i$  transmitted at rate  $R_i$ . Then each decoder draws a reconstruction

sequence  $V_i^n$  that minimizes its cost  $c_i(U, V_i)$ . More details can be found in (El Gamal and Kim, 2011, chapter 14.2.1).

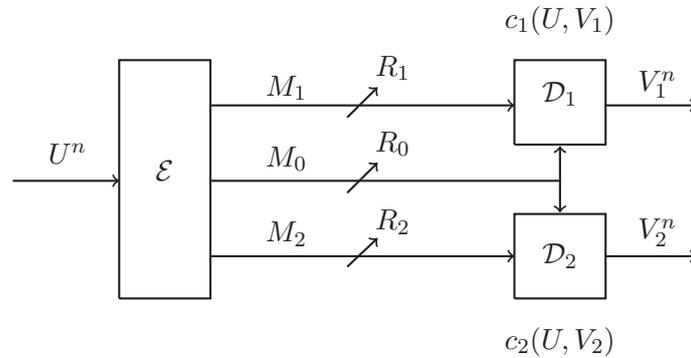


Figure 4.1: Gray-Wyner's lossy source coding setup

**Definition 4.2.1.** Let  $(R_0, R_1, R_2) \in \mathbb{R}_+^3$ . An  $(R_0, R_1, R_2, n)$  code for the Gray-Wyner system consists of the following:

- Three index sets  $\{1, 2, \dots, 2^{\lfloor nR_0 \rfloor}\}$ ,  $\{1, 2, \dots, 2^{\lfloor nR_1 \rfloor}\}$ , and  $\{1, 2, \dots, 2^{\lfloor nR_2 \rfloor}\}$ .
- An encoding function  $\sigma : \mathcal{U}^n \rightarrow \{1, 2, \dots, 2^{\lfloor nR_0 \rfloor}\} \times \{1, 2, \dots, 2^{\lfloor nR_1 \rfloor}\} \times \{1, 2, \dots, 2^{\lfloor nR_2 \rfloor}\}$  which assigns a triple  $(m_0, m_1, m_2)$  to each sequence  $u^n \in \mathcal{U}^n$ .
- Two decoding functions  $\tau_1 : \{1, 2, \dots, 2^{\lfloor nR_0 \rfloor}\} \times \{1, 2, \dots, 2^{\lfloor nR_1 \rfloor}\} \rightarrow \mathcal{V}_1^n$  and  $\tau_2 : \{1, 2, \dots, 2^{\lfloor nR_0 \rfloor}\} \times \{1, 2, \dots, 2^{\lfloor nR_1 \rfloor}\} \rightarrow \mathcal{V}_2^n$ .

Long-run cost functions are given as in definition 3.2.2.

**Definition 4.2.2.** Let  $c_1 : \mathcal{U} \times \mathcal{V}_1 \rightarrow \mathbb{R}$  and  $c_2 : \mathcal{U} \times \mathcal{V}_2 \rightarrow \mathbb{R}$  two cost functions. We define the long run distortion functions  $c_1^n(\sigma, \tau_1)$  and  $c_2^n(\sigma, \tau_1)$  as follows:

$$c_i^n(\sigma, \tau_i) = \sum_{u^n, v_i^n} \mathcal{P}^{\sigma, \tau_i}(u^n, v_i^n) \frac{1}{n} \sum_{t=1}^n c_i(u_t, v_{i,t}), \quad i \in \{1, 2\}. \quad (4.1)$$

**Definition 4.2.3.** A rate-cost quintuple  $(R_0, R_1, R_2, C_1, C_2) \in \mathbb{R}_+^5$  is said to be achievable if there exists a sequence of  $(R_0, R_1, R_2, n)$  codes for the Gray-Wyner system such that  $\limsup_{n \rightarrow \infty} \mathbb{E}(c_i^n(U^n, V_i^n)) \leq C_i$ , for  $i \in \{1, 2\}$ .

For a rate triple  $(R_0, R_1, R_2)$ , we denote by  $\mathcal{J}(R_0, R_1, R_2)$  the closure of the set of pairs  $(C_1, C_2)$  such that quintuples  $(R_0, R_1, R_2, C_1, C_2)$  are achievable. Let  $W_0 \in \mathcal{W}_0$ ,  $W_1 \in \mathcal{W}_1$ , and  $W_2 \in \mathcal{W}_2$  three auxiliary random variables such that  $|\mathcal{W}_0| = |\mathcal{U}|$ ,  $|\mathcal{W}_1| = |\mathcal{V}_1|$  and  $|\mathcal{W}_2| = |\mathcal{V}_2|$ .

**Theorem 4.2.1** (Gray-Wyner Lossy source coding). *Let  $U$  be a discrete memoryless source, distributed according to a probability distribution  $\mathcal{P}_U$  over  $\mathcal{U}$ . Let  $\mathcal{V}_1$  and  $\mathcal{V}_2$  be two discrete alphabets and  $c_i : \mathcal{U} \times \mathcal{V}_i \mapsto \mathbb{R}$  for  $i \in \{1, 2\}$  be two cost functions. Then, given a rate triple  $(R_0, R_1, R_2)$ , we have*

$$\mathcal{J}(R_0, R_1, R_2) = \left\{ \left( \mathbb{E}_{\mathcal{Q}}(c_1(U, V_1)), \mathbb{E}_{\mathcal{Q}}(c_2(U, V_2)) \right), \quad \mathcal{Q} \in \mathcal{Q}(R_0, R_1, R_2) \right\} \quad (4.2)$$

where

$$\mathcal{Q}(R_0, R_1, R_2) = \left\{ \mathcal{Q}_{W_0 W_1 | U} \mathcal{Q}_{W_2 | U W_0}, \quad I(U; W_0) \leq R_0, I(U; W_1 | W_0) \leq R_1, \right. \\ \left. I(U; W_2 | W_0) \leq R_2 \right\} \quad (4.3)$$

### 4.3 Gray-Wyner network for Bayesian persuasion with cost-dependent Decoders

Our strategic communication model via the Gray-Wyner network considers an informed encoder endowed with a cost function that depends on the (i.i.d.) source and on the actions of both decoder as in [Figure 4.2](#). We also assume that decoders are cost-dependent, i.e. the cost of each decoder depends on the action taken by the other decoder. We still assume that the encoder commits to an encoding strategy and announces it before the game starts. This setting is particularly challenging since each commitment of the encoder induces a finite Bayesian game among the decoders which admits perfect Bayes-Nash equilibria. In the following, we introduce the strategic Gray-Wyner setup and the coding problem to be solved by the encoder. We formally define coding strategies, cost functions and formulate the Bayesian game induced by each encoding function.

## 4.3.1 Novel model of strategic Gray-Wyner network

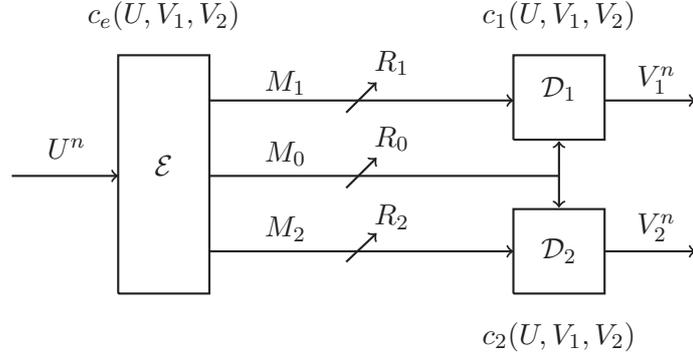


Figure 4.2: Strategic Gray-Wyner network with cost-dependent decoders.

**Definition 4.3.1.** Let  $R_0, R_1, R_2 \in \mathbb{R}_+^3 = [0, +\infty[^3$ , and  $n \in \mathbb{N}^* = \mathbb{N} \setminus \{0\}$ . The encoding function  $\sigma$  and the decoding functions  $\tau_i$  of the encoder  $\mathcal{E}$  and decoders  $\mathcal{D}_i$  for  $i \in \{1, 2\}$  respectively, are given by

$$\sigma : \mathcal{U}^n \mapsto \Delta(\{1, \dots, 2^{\lfloor nR_0 \rfloor}\} \times \{1, \dots, 2^{\lfloor nR_1 \rfloor}\} \times \{1, \dots, 2^{\lfloor nR_2 \rfloor}\}), \quad (4.4)$$

$$\tau_1 : (\{1, 2, \dots, 2^{\lfloor nR_0 \rfloor}\} \times \{1, 2, \dots, 2^{\lfloor nR_1 \rfloor}\}) \mapsto \Delta(\mathcal{V}_1^n), \quad (4.5)$$

$$\tau_2 : (\{1, 2, \dots, 2^{\lfloor nR_0 \rfloor}\} \times \{1, 2, \dots, 2^{\lfloor nR_2 \rfloor}\}) \mapsto \Delta(\mathcal{V}_2^n). \quad (4.6)$$

The coding triplets  $(\sigma, \tau_1, \tau_2)$  are stochastic and induce a joint probability distribution  $\mathcal{P}^{\sigma, \tau_1, \tau_2} \in \Delta(\mathcal{U}^n \times \{1, 2, \dots, 2^{\lfloor nR_0 \rfloor}\} \times \{1, 2, \dots, 2^{\lfloor nR_1 \rfloor}\} \times \{1, 2, \dots, 2^{\lfloor nR_2 \rfloor}\} \times \mathcal{V}_1^n \times \mathcal{V}_2^n)$  defined by

$$\mathcal{P}_{U^n M_0 M_1 M_2 V_1^n V_2^n}^{\sigma, \tau_1, \tau_2} = \left( \prod_{t=1}^n \mathcal{P}_{U_t} \right) \mathcal{P}_{M_0 M_1 M_2 | U^n}^{\sigma} \mathcal{P}_{V_1^n | M_0 M_1}^{\tau_1} \mathcal{P}_{V_2^n | M_0 M_2}^{\tau_2}.$$

**Definition 4.3.2.** Single-letter cost functions  $c_e : \mathcal{U} \times \mathcal{V}_1 \times \mathcal{V}_2 \mapsto \mathbb{R}$  of the encoder and  $c_i : \mathcal{U} \times \mathcal{V}_1 \times \mathcal{V}_2 \mapsto \mathbb{R}$  of decoder  $\mathcal{D}_i$  for  $i \in \{1, 2\}$  induce long-run cost functions  $c_e^n(\sigma, \tau_1, \tau_2)$  and  $c_i^n(\sigma, \tau_1, \tau_2)$  for  $i \in \{1, 2\}$  as follows

$$c_e^n(\sigma, \tau_1, \tau_2) = \mathbb{E}_{\sigma, \tau_1, \tau_2} \left[ \frac{1}{n} \sum_{t=1}^n c_e(U_t, V_{1,t}, V_{2,t}) \right] \quad (4.7)$$

$$= \sum_{u^n, v_1^n, v_2^n} \mathcal{P}_{U^n V_1^n V_2^n}^{\sigma, \tau_1, \tau_2}(u^n, v_1^n, v_2^n) \cdot \left[ \frac{1}{n} \sum_{t=1}^n c_e(u_t, v_{1,t}, v_{2,t}) \right], \quad (4.8)$$

$$c_i^n(\sigma, \tau_1, \tau_2) = \mathbb{E}_{\sigma, \tau_1, \tau_2} \left[ \frac{1}{n} \sum_{t=1}^n c_i(U_t, V_{1,t}, V_{2,t}) \right]. \quad (4.9)$$

where  $\mathcal{P}_{U^n V_1^n V_2^n}^{\sigma \tau_1 \tau_2}$  denotes the marginal distributions of  $\mathcal{P}^{\sigma \tau_1 \tau_2}$  over the  $n$ -sequences  $(U^n, V_1^n, V_2^n)$ .

We consider the strategic communication game in which each player aims to minimize its long-run cost function. Since the decoders' cost functions are interdependent, each encoding function  $\sigma$  induces a Bayesian game  $G^\sigma(M_0, M_1, M_2, V_1^n, V_2^n)$  among the decoders which is defined below.

**Definition 4.3.3.** For each encoding  $\sigma$ , the finite Bayesian game  $G^\sigma(M_0, M_1, M_2, V_1^n, V_2^n)$  consists of:

- The decoders  $\mathcal{D}_i, i \in \{1, 2\}$  as the players of the game,
- $\mathcal{V}_i^n$  is the set of action sequences of  $\mathcal{D}_i$ ,
- $(M_0, M_i)$  is the type<sup>1</sup> of decoder  $\mathcal{D}_i$ ,
- $\tau_i$  is a behavior<sup>2</sup> strategy of decoder  $\mathcal{D}_i$ ,
- The belief of decoder  $\mathcal{D}_1$  (resp.  $\mathcal{D}_2$ ) over the type of decoder  $\mathcal{D}_2$  (resp.  $\mathcal{D}_1$ ) is given by  $\mathcal{P}_{M_2|M_0M_1}^\sigma$  (resp.  $\mathcal{P}_{M_1|M_0M_2}^\sigma$ ).
- $C_i^\sigma : \{1, 2, \dots, 2^{\lfloor nR_0 \rfloor}\} \times \{1, 2, \dots, 2^{\lfloor nR_1 \rfloor}\} \times \{1, 2, \dots, 2^{\lfloor nR_2 \rfloor}\} \times \mathcal{V}_1^n \times \mathcal{V}_2^n \mapsto \mathbb{R}$  is the  $\sigma$ -cost function of  $\mathcal{D}_i$  such that  $\forall v_1^n, v_2^n, m_0, m_1, m_2$ ,

$$C_i^\sigma(m_0, m_1, m_2, v_1^n, v_2^n) = \sum_{u^n} \mathcal{P}^\sigma(u^n | m_0, m_1, m_2) \left[ \frac{1}{n} \sum_{t=1}^n c_i(u_t, v_{1,t}, v_{2,t}) \right].$$

- For a fixed strategy profile  $(\tau_1, \tau_2)$ , the expected  $\sigma$ -costs  $\Psi_1^\sigma(\tau_1, \tau_2, m_0, m_1)$  of  $\mathcal{D}_1$  with type  $(m_0, m_1)$  is given by

$$\begin{aligned} \Psi_1^\sigma(\tau_1, \tau_2, m_0, m_1) &= \sum_{m_2} \mathcal{P}^\sigma(m_2 | m_0, m_1) \sum_{v_1^n, v_2^n} \mathcal{P}^{\tau_1}(v_1^n | m_0, m_1) \mathcal{P}^{\tau_2}(v_2^n | m_0, m_2) \\ &\quad \times C_1^\sigma(v_1^n, v_2^n, m_0, m_1, m_2). \end{aligned} \quad (4.10)$$

<sup>1</sup>Note that we use the term "type" to refer to the observation of a decoder. The term type in this manuscript does not refer to the empirical distribution of a "typical" sequence.

<sup>2</sup>A behavior strategy  $\tau_i$  of player  $i$  in a Bayesian game is a mapping  $\mathcal{T}_i \mapsto \Delta(\mathcal{V}_i)$ , where  $\mathcal{T}_i$  is the set of types of player  $i$  and  $\mathcal{V}_i$  is the set of actions.

Similarly,  $\Psi_2^\sigma(\tau_1, \tau_2, m_0, m_2)$  can be defined.

**Definition 4.3.4.** Given  $\sigma$ , for each behavior strategy  $\tau_2$ , decoder  $\mathcal{D}_1$ , computes the set  $BR_1^\sigma(\tau_2)$  of best-response strategies

$$BR_1^\sigma(\tau_2) = \{\tau_1, \Psi_1^\sigma(\tau_1, \tau_2, m_0, m_1) \leq \Psi_1^\sigma(\tilde{\tau}_1, \tau_2, m_0, m_1), \forall \tilde{\tau}_1, m_0, m_1\}.$$

Similarly,  $\mathcal{D}_2$  computes  $BR_2^\sigma(\tau_1)$ .

This Bayesian game  $G^\sigma(M_0, M_1, M_2, V_1^n, V_2^n)$  is finite, the players use behavioral strategies and the Theorem of Nash (1951) ensures the existence of at least one Bayes-Nash equilibrium. In the following, we define the set of such equilibria.

**Definition 4.3.5.** For each encoding strategy  $\sigma$ , we define the set  $BNE(\sigma)$  of Bayes-Nash equilibria  $(\tau_1, \tau_2)$  of  $G^\sigma(M_0, M_1, M_2, V_1^n, V_2^n)$  as follows

$$BNE(\sigma) = \{(\tau_1, \tau_2), \tau_1 \in BR_1^\sigma(\tau_2) \text{ and } \tau_2 \in BR_2^\sigma(\tau_1)\}.$$

The communication game goes in the following order:

- The encoder  $\mathcal{E}$  chooses, announces the encoding  $\sigma$ .
- The sequence  $U^n$  is drawn i.i.d with distribution  $\mathcal{P}_U$ , and the game  $G^\sigma(M_0, M_1, M_2, V_1^n, V_2^n)$  begins.
- The messages  $(M_0, M_1, M_2)$  are encoded according to  $\mathcal{P}_{M_0 M_1 M_2 | U^n}^\sigma$ .
- Knowing  $\sigma$ , decoder  $\mathcal{D}_1$  observes  $(M_0, M_1)$  and draws a sequence  $V_1^n$  according to the behavior strategy  $\tau_1 \in BR_1^\sigma(\tau_2)$ , such that  $(\tau_1, \tau_2) \in BNE(\sigma)$ . Similarly, decoder  $\mathcal{D}_2$  observes  $(M_0, M_2)$  and draws a sequence  $V_2^n$  according to the strategy  $\tau_2 \in BR_2^\sigma(\tau_1)$  such that  $(\tau_1, \tau_2) \in BNE(\sigma)$ .
- The cost values are given by  $c_e^n(\sigma, \tau_1, \tau_2)$ ,  $\Psi_1^\sigma(\tau_1, \tau_2, m_0, m_1)$ ,  $\Psi_2^\sigma(\tau_1, \tau_2, m_0, m_2)$ .

For  $(R_0, R_1, R_2) \in \mathbb{R}_+^3$  and  $n \in \mathbb{N}^*$ , the encoder has to solve the following coding problem.

$$\Gamma_{GW}^n(R_0, R_1, R_2) = \inf_{\sigma} \max_{(\tau_1, \tau_2) \in BNE(\sigma)} c_e^n(\sigma, \tau_1, \tau_2). \quad (4.11)$$

### 4.3.2 Proposed bounds on encoder's optimal cost

We consider three auxiliary random variables  $W_0 \in \mathcal{W}_0$ ,  $W_1 \in \mathcal{W}_1$  and  $W_2 \in \mathcal{W}_2$  with  $|\mathcal{W}_0| = |\mathcal{V}_1| \times |\mathcal{V}_2| + 1$ , and  $|\mathcal{W}_i| = |\mathcal{V}_i|$ , for  $i \in \{1, 2\}$ .

**Definition 4.3.6.** For  $(R_0, R_1, R_2) \in \mathbb{R}_+^3$ , we define

$$\mathbb{Q}_0(R_0, R_1, R_2) = \left\{ \mathcal{Q}_{W_0|U} \mathcal{Q}_{W_1|UW_0} \mathcal{Q}_{W_2|UW_0}, \quad R_0 \geq I(U; W_0), \quad R_1 \geq I(U; W_1|W_0), \right. \\ \left. R_2 \geq I(U; W_2|W_0) \right\}, \quad (4.12)$$

$$\hat{\mathbb{Q}}_0(R_0, R_1, R_2) = \left\{ \mathcal{Q}_{W_0W_1W_2|U}, \quad R_0 \geq I(U; W_0), \quad R_1 \geq I(U; W_1|W_0), \right. \\ \left. R_2 \geq I(U; W_2|W_0) \right\}. \quad (4.13)$$

In the following, we define the Bayesian game played among the decoders in the one-shot scenario where the types are expressed using our three auxiliary random variables.

**Definition 4.3.7.** For each distribution  $\mathcal{Q}_{W_0W_1W_2|U} \in \Delta(\mathcal{W}_0 \times \mathcal{W}_1 \times \mathcal{W}_2)^{|\mathcal{U}|}$ , the auxiliary single-letter Bayesian game  $G^w(W_0, W_1, W_2, V_1, V_2)$  is given as follows:

- $(w_0, w_i)$  is the type of decoder  $\mathcal{D}_i$ ,  $i \in \{1, 2\}$ ,
- The belief of decoder  $\mathcal{D}_1$  (resp.  $\mathcal{D}_2$ ) over the type of decoder  $\mathcal{D}_2$  (resp.  $\mathcal{D}_1$ ) is given by  $\mathcal{Q}_{W_2|W_0W_1}$  (resp.  $\mathcal{Q}_{W_1|W_0W_2}$ ).
- $C_i^* : \mathcal{V}_1 \times \mathcal{V}_2 \times \mathcal{W}_0 \times \mathcal{W}_1 \times \mathcal{W}_2 \mapsto \mathbb{R}$  is the single-letter cost of  $\mathcal{D}_i$  such that  $\forall v_1, v_2, w_0, w_1, w_2$

$$C_i^*(v_1, v_2, w_0, w_1, w_2) = \sum_u \mathcal{Q}(u|w_0, w_1, w_2) c_i(u, v_1, v_2),$$

- For each pair  $(\mathcal{Q}_{V_1|W_0W_1}, \mathcal{Q}_{V_2|W_0W_2})$  and profile  $(w_0, w_i)$ , the single-letter expected costs  $\Psi_i^*(\mathcal{Q}_{V_1|W_0W_1}, \mathcal{Q}_{V_2|W_0W_2}, w_0, w_i)$  of  $\mathcal{D}_i$  are given by

$$\Psi_1^*(\mathcal{Q}_{V_1|W_0W_1}, \mathcal{Q}_{V_2|W_0W_2}, w_0, w_1) = \sum_{w_2} \mathcal{Q}(w_2|w_0, w_1) \sum_{v_1, v_2} \mathcal{Q}(v_1|w_0, w_1) \times \\ \mathcal{Q}(v_2|w_0, w_2) C_1^*(v_1, v_2, w_0, w_1, w_2), \quad (4.14)$$

$$\begin{aligned} \Psi_2^*(\mathcal{Q}_{V_1|W_0W_1}, \mathcal{Q}_{V_2|W_0W_2}, w_0, w_2) &= \sum_{w_1} \mathcal{Q}(w_1|w_0, w_2) \sum_{v_1, v_2} \mathcal{Q}(v_1|w_0, w_1) \times \\ &\quad \mathcal{Q}(v_2|w_0, w_2) C_2^*(v_1, v_2, w_0, w_1, w_2). \end{aligned} \quad (4.15)$$

For each distribution  $\mathcal{Q}_{W_0W_1W_2|U} \in \Delta(\mathcal{W}_0 \times \mathcal{W}_1 \times \mathcal{W}_2)^{|\mathcal{U}|}$ , the auxiliary set of Bayesian Nash equilibria is given by

$$\begin{aligned} \mathbb{BNE}(\mathcal{Q}_{W_0W_1W_2|U}) &= \{(\mathcal{Q}_{V_1|W_0W_1}, \mathcal{Q}_{V_2|W_0W_2}), \\ \Psi_1^*(\mathcal{Q}_{V_1|W_0W_1}, \mathcal{Q}_{V_2|W_0W_2}, w_0, w_1) &\leq \Psi_1^*(\tilde{\mathcal{Q}}_{V_1|W_0W_1}, \mathcal{Q}_{V_2|W_0W_2}, w_0, w_1) \vee \tilde{\mathcal{Q}}_{V_1|W_0W_1}, w_0, w_1, \\ \Psi_2^*(\mathcal{Q}_{V_1|W_0W_1}, \mathcal{Q}_{V_2|W_0W_2}, w_0, w_2) &\leq \Psi_2^*(\mathcal{Q}_{V_1|W_0W_1}, \tilde{\mathcal{Q}}_{V_2|W_0W_2}, w_0, w_2) \vee \tilde{\mathcal{Q}}_{V_2|W_0W_2}, w_0, w_2\}. \end{aligned} \quad (4.16)$$

The encoder's optimal cost is defined with respect to  $\mathbb{Q}_0(R_0, R_1, R_2)$  and  $\hat{\mathbb{Q}}_0(R_0, R_1, R_2)$  respectively as follows

$$\Gamma_{GW}^*(R_0, R_1, R_2) = \inf_{\substack{\mathcal{Q}_{W_0|U} \mathcal{Q}_{W_1|W_0U} \\ \mathcal{Q}_{W_2|W_0U} \in \mathbb{Q}_0(R_0, R_1, R_2)}} \max_{(\mathcal{Q}_{V_1|W_0W_1}, \mathcal{Q}_{V_2|W_0W_2}) \in \mathbb{BNE}(\mathcal{Q}_{W_0|U} \mathcal{Q}_{W_1|W_0U} \mathcal{Q}_{W_2|W_0U})} \mathbb{E}[c_e(U, V_1, V_2)], \quad (4.17)$$

$$\hat{\Gamma}_{GW}(R_0, R_1, R_2) = \inf_{\substack{\mathcal{Q}_{W_0W_1W_2|U} \\ \in \hat{\mathbb{Q}}_0(R_0, R_1, R_2)}} \max_{(\mathcal{Q}_{V_1|W_0W_1}, \mathcal{Q}_{V_2|W_0W_2}) \in \mathbb{BNE}(\mathcal{Q}_{W_0W_1W_2|U})} \mathbb{E}[c_e(U, V_1, V_2)], \quad (4.18)$$

where in (4.17), the *inf.* is taken over the set  $\mathbb{Q}_0(R_0, R_1, R_2)$ , and the expectation is evaluated with respect to  $\mathcal{P}_U \mathcal{Q}_{W_0|U} \mathcal{Q}_{W_1|W_0U} \mathcal{Q}_{W_2|UW_0} \mathcal{Q}_{V_1|W_0W_1} \mathcal{Q}_{V_2|W_0W_2}$ . In (4.18), the *inf.* is taken over  $\hat{\mathbb{Q}}_0(R_0, R_1, R_2)$  and the expectation is evaluated with respect to  $\mathcal{P}_U \mathcal{Q}_{W_0W_1W_2|U} \mathcal{Q}_{V_1|W_0W_1} \mathcal{Q}_{V_2|W_0W_2}$ .

**Remark 4.3.1.** The random variables  $U, W_0, W_1, W_2, V_1$  and  $V_2$  satisfy the following Markov chains

$$\begin{aligned} (U, W_2, V_2) &\text{---} (W_0, W_1) \text{---} V_1, \\ (U, W_1, V_1) &\text{---} (W_0, W_2) \text{---} V_2. \end{aligned}$$

For each distribution of  $\mathbb{Q}_0(R_0, R_1, R_2)$ , the following Markov chain is satisfied

$$W_1 \text{---} (U, W_0) \text{---} W_2.$$

Our main result consists of an achievability statement determining an upper bound on the encoder's long-run optimal cost  $\Gamma_{GW}^n(R_0, R_1, R_2)$ , and a converse statement determining its lower bound.

**Theorem 4.3.1.** *Let  $(R_0, R_1, R_2) \in \mathbb{R}_+^3$ , we have*

$$\forall \varepsilon > 0, \exists \hat{n} \in \mathbb{N}, \forall n \geq \hat{n}, \quad \Gamma_{GW}^n(R_0, R_1, R_2) \leq \Gamma_{GW}^*(R_0, R_1, R_2) + \varepsilon, \quad (4.19)$$

$$\forall n \in \mathbb{N}, \quad \Gamma_{GW}^n(R_0, R_1, R_2) \geq \hat{\Gamma}_{GW}(R_0, R_1, R_2). \quad (4.20)$$

The proof of Theorem 4.3.1 is provided in [section 4.5](#) and [section 4.6](#).

**Lemma 4.3.1.** *Let  $(R_0, R_1, R_2) \in \mathbb{R}_+^3$ , and consider  $c_{e1} : \mathcal{U} \times \mathcal{V}_1 \mapsto \mathbb{R}$ , and  $c_{e2} : \mathcal{U} \times \mathcal{V}_2 \mapsto \mathbb{R}$ .*

*If for all  $(u, v_1, v_2)$ ,  $c_e(u, v_1, v_2) = c_{e1}(u, v_1) + c_{e2}(u, v_2)$ , then*

$$\Gamma_{GW}^*(R_0, R_1, R_2) = \hat{\Gamma}_{GW}(R_0, R_1, R_2) \quad (4.21)$$

Using Fekete's Lemma for the sub-additive sequence  $(n\Gamma_{GW}^n(R_0, R_1, R_2))_{n \in \mathbb{N}^*}$  ([App. 3.3.1](#)), we get

$$\lim_{n \rightarrow \infty} \Gamma_{GW}^n(R_0, R_1, R_2) = \inf_n \Gamma_{GW}^n(R_0, R_1, R_2) = \Gamma_{GW}^*(R_0, R_1, R_2). \quad (4.22)$$

In the following, we consider a special case with  $R_2 = 0$  and the cost functions  $c_1$  and  $c_2$  of decoders  $\mathcal{D}_1$  and  $\mathcal{D}_2$  respectively are interdependent.

#### 4.4 Special case: Strategic successive refinement with interdependent decoders

Consider the Gray-Wyner network with  $R_2 = 0$  (or analogously  $R_1 = 0$ ). In other words, the message  $M_2$  is no longer informative and takes values from the singleton  $\{1\}$ . This particular case corresponds to the strategic successive refinement model with interdependent

decoders' cost functions as depicted in Figure 4.3. Therefore, it combines the encoding and decoding strategies of subsection 3.3.1 given in Definition 3.3.1 with the cost functions of subsection 4.3.1 given in Definition 4.3.2.

**Definition 4.4.1.** *Let  $(R_0, R_1) \in \mathbb{R}_+^2$  be given. We define the encoder's optimal cost for the successive refinement network with interdependent decoders as follows*

$$\Gamma_{SRID}^*(R_0, R_1) = \inf_{\substack{\mathcal{Q}_{W_0W_1|U} \\ \in \bar{\mathcal{Q}}_0(R_0, R_1)}} \max_{\substack{(\mathcal{Q}_{V_1|W_0W_1}, \mathcal{Q}_{V_2|W_0}) \in \\ \mathbb{BNE}(\mathcal{Q}_{W_0W_1|U})}} \mathbb{E} \left[ c_e(U, V_1, V_2) \right], \quad (4.23)$$

where the expectation in (4.23) is evaluated with respect to the distribution  $\mathcal{P}_U \mathcal{Q}_{W_0W_1|U} \mathcal{Q}_{V_1|W_0W_1} \mathcal{Q}_{V_2|W_0}$ , the set  $\bar{\mathcal{Q}}_0(R_0, R_1)$  is given in Equation 3.15 and for each  $\mathcal{Q}_{W_0W_1|U} \in \Delta(\mathcal{W}_0 \times \mathcal{W}_1)^{|\mathcal{U}|}$ , the set  $\mathbb{BNE}(\mathcal{Q}_{W_0W_1|U})$  is given as follows

$$\mathbb{BNE}(\mathcal{Q}_{W_0W_1|U}) = \left\{ (\mathcal{Q}_{V_1|W_0W_1}, \mathcal{Q}_{V_2|W_0}), \Psi_1^*(\mathcal{Q}_{V_1|W_0W_1}, \mathcal{Q}_{V_2|W_0}, w_0, w_1) \leq \Psi_1^*(\tilde{\mathcal{Q}}_{V_1|W_0W_1}, \mathcal{Q}_{V_2|W_0}, w_0, w_1) \right. \\ \left. \forall \tilde{\mathcal{Q}}_{V_1|W_0W_1}, w_0, w_1, \Psi_2^o(\mathcal{Q}_{V_1|W_0W_1}, \mathcal{Q}_{V_2|W_0}, w_0) \leq \Psi_2^o(\mathcal{Q}_{V_1|W_0W_1}, \tilde{\mathcal{Q}}_{V_2|W_0}, w_0) \forall \tilde{\mathcal{Q}}_{V_2|W_0}, w_0 \right\}.$$

with  $\Psi_1^*(\mathcal{Q}_{V_1|W_0W_1}, \mathcal{Q}_{V_2|W_0}, w_0, w_1)$  as given in Equation 4.14, and

$$\Psi_2^o(\mathcal{Q}_{V_1|W_0W_1}, \mathcal{Q}_{V_2|W_0}, w_0) = \sum_{w_1} \mathcal{Q}(w_1|w_0) \sum_{v_1, v_2} \mathcal{Q}(v_1|w_0, w_1) \mathcal{Q}(v_2|w_0) C_2^o(v_1, v_2, w_0, w_1). \quad (4.24)$$

and  $C_2^o : \mathcal{V}_1 \times \mathcal{V}_2 \times \mathcal{W}_0 \times \mathcal{W}_1 \mapsto \mathbb{R}$  such that for all  $v_1, v_2, w_0, w_1$ ,

$$C_i^o(v_1, v_2, w_0, w_1) = \sum_u \mathcal{Q}(u|w_0, w_1) c_2(u, v_1, v_2). \quad (4.25)$$

**Lemma 4.4.1.** *Let  $(R_0, R_1) \in \mathbb{R}_+^2$  be given. The following holds*

$$\Gamma_{SRID}^*(R_0, R_1) = \Gamma_{GW}^*(R_0, R_1, 0). \quad (4.26)$$

*Proof.* We have  $\Gamma_{SRID}^*(R_0, R_1) \geq \Gamma_{GW}^*(R_0, R_1, 0)$  by identifying the auxiliary random variable  $W_2$  to a constant. The set  $\mathcal{Q}_0(R_0, R_1, 0)$  is now restricted to distributions  $\mathcal{Q}_{W_0|U} \mathcal{Q}_{W_1|UW_0}$

which coincides with  $\bar{\mathbb{Q}}_0(R_0, R_1)$  and gives the desired inequality. For the second inequality  $\Gamma_{SRID}^*(R_0, R_1) \leq \Gamma_{GW}^*(R_0, R_1, 0)$ , we have by the achievability of [Theorem 4.3.1](#),

$$\inf_n \Gamma_{GW}^n(R_0, R_1, R_2) = \lim_{n \rightarrow \infty} \Gamma_{GW}^n(R_0, R_1, R_2) \leq \Gamma_{GW}^*(R_0, R_1, R_2). \quad (4.27)$$

By the converse statement of [Bou Rouphael and Le Treust \(2022\)](#), we have

$$\Gamma_{SRID}^*(R_0, R_1) \leq \inf_n \Gamma_{GW}^n(R_0, R_1, R_2). \quad (4.28)$$

This can be shown following similar arguments as in the converse proof of [Theorem 4.3.1](#) given below in [section 4.6](#) with the identification of the auxiliary random variable  $W_2 = M_2$  which is a constant since  $R_2 = 0$

$$\max_{\tau_1, \tau_2} c_e^n(\sigma, \tau_1, \tau_2) = \max_{\tau_1, \tau_2} \mathbb{E}_{\substack{\mathcal{P}_{W_0 W_1 W_2 | U}^\sigma \\ \mathcal{P}_{V_1 | W_0 W_1}^{\tau_1} \mathcal{P}_{V_2 | W_0 W_2}^{\tau_2}}} [c_e(U, V_1, V_2)] \quad (4.29)$$

$$= \max_{\substack{(\mathcal{Q}_{V_1 | W_0 W_1}, \mathcal{Q}_{V_2 | W_0}) \in \\ \text{BNE}(\mathcal{P}_{W_0 W_1 | U}^\sigma)}} \mathbb{E}_{\substack{\mathcal{P}_{W_0 W_1 | U}^\sigma \\ \mathcal{Q}_{V_1 | W_0 W_1} \mathcal{Q}_{V_2 | W_0}}} [c_e(U, V_1, V_2)] \quad (4.30)$$

$$\geq \inf_{\substack{\mathcal{Q}_{W_0 W_1 | U} \\ \in \bar{\mathbb{Q}}_0(R_0, R_1)}} \max_{\substack{(\mathcal{Q}_{V_1 | W_0 W_1}, \mathcal{Q}_{V_2 | W_0}) \in \\ \text{BNE}(\mathcal{P}_{W_0 W_1 | U}^\sigma)}} \mathbb{E} [c_e(U, V_1, V_2)] \quad (4.31)$$

$$= \Gamma_{SRID}^*(R_0, R_1). \quad (4.32)$$

□

We have solved the problem of strategic communication with interdependent decoders' cost functions via a successive refinement network in [Bou Rouphael and Le Treust \(2022\)](#) where we have shown that the optimal long cost of the encoder converges to  $\Gamma_{SRID}^*(R_0, R_1)$ . In [Bou Rouphael and Le Treust \(2022\)](#), the solution is characterized with only two auxiliary random variables  $W_0$  and  $W_1$ . However, for the Gray-Wyner model with  $R_2 = 0$ , whether  $\hat{\Gamma}_{GW}(R_0, R_1, 0) = \Gamma_{GW}^*(R_0, R_1, 0)$ , or  $\hat{\Gamma}_{GW}(R_0, R_1, 0) < \Gamma_{GW}^*(R_0, R_1, 0)$  is still unsolved.

In the following section, we prove the achievability statement [\(4.19\)](#) of our main result [Theorem 4.3.1](#).

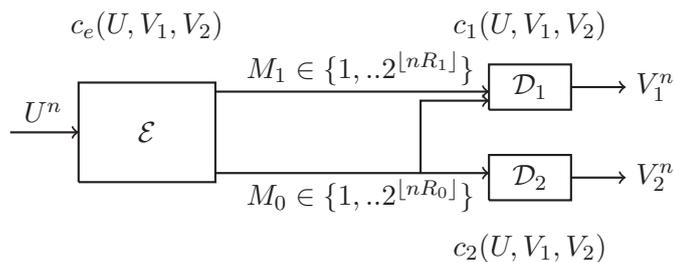


Figure 4.3: Strategic successive refinement coding setup with cost-interdependent decoders

## 4.5 Proof of achievability for the strategic Gray-Wyner communication network

Our proof consists of three main parts: Firstly, we restrict the optimization to a dense subset of distributions inducing essential equilibria only. Secondly, we generate the codebook and show that the probability of error over the codebook is small. Thirdly, we show that the beliefs induced by the coding functions are close under the KL-distance to the single-letter beliefs described using auxiliary random variables.

### 4.5.1 Essential equilibria

The random coding scheme may induce some perturbations in the probability distribution  $\mathcal{Q}_{W_0|U} \mathcal{Q}_{W_1|W_0U} \mathcal{Q}_{W_2|W_0U}$  of the Bayesian game of Definition 4.3.7. A non-cooperative game is essential (Yu, 1999, Definition 4.1) if small perturbations of the probability distributions may induce small perturbations of the set of Bayes-Nash equilibria. According to (Yu, 1999, Theorem 4.2), the set of essential games is a dense subset of the set of non-cooperative games. In the following we formally develop those ideas and derive an alternative formulation to our encoding problem.

**Definition 4.5.1.** *Given  $\mathcal{Q}_{W_0W_1W_2|U} \in \Delta(\mathcal{W}_0 \times \mathcal{W}_1 \times \mathcal{W}_2)^{|\mathcal{U}|}$ , an equilibrium  $(\mathcal{Q}_{V_1|W_1, W_0}, \mathcal{Q}_{V_2|W_2, W_0}) \in \mathbb{BNE}(\mathcal{Q}_{W_0W_1W_2|U})$  is said to be essential if for all  $\varepsilon > 0$ , there exists an open neighborhood*

$\Omega$  of  $\mathcal{Q}_{W_0W_1W_2|U}$  such that for all  $\hat{\mathcal{Q}}_{W_0W_1W_2|U} \in \Omega$ ,

$$\begin{aligned} (\hat{\mathcal{Q}}_{V_1|W_1,W_0}, \hat{\mathcal{Q}}_{V_2|W_2,W_0}) &\in \mathbb{BNE}(\hat{\mathcal{Q}}_{W_0W_1W_2|U}) \implies \\ \|\mathcal{Q}_{V_1|W_0,W_1} - \hat{\mathcal{Q}}_{V_1|W_0,W_1}\| + \|\mathcal{Q}_{V_2|W_0,W_2} - \hat{\mathcal{Q}}_{V_2|W_0,W_2}\| &\leq \varepsilon. \end{aligned} \quad (4.33)$$

We denote by  $\mathbb{EBNE}(\mathcal{Q}_{W_0W_1W_2|U})$  the set of essential Bayes-Nash equilibria.

**Definition 4.5.2.** For  $(R_0, R_1, R_2) \in \mathbb{R}_+^3$ , we define the set

$$\begin{aligned} \tilde{\mathcal{Q}}_0(R_0, R_1, R_2) &= \left\{ \mathcal{Q}_{W_0|U} \mathcal{Q}_{W_1|W_0U} \mathcal{Q}_{W_2|W_0U}, \min_{u, w_0, w_1, w_2} \mathcal{Q}(w_0|u) \mathcal{Q}(w_1|w_0, u) \mathcal{Q}(w_2|w_0, u) > 0, \right. \\ &\quad R_0 > I(U; W_0), R_1 > I(U; W_1|W_0), R_2 > I(U; W_2|W_0), \\ &\quad \left. \mathbb{BNE}(\mathcal{Q}_{W_0|U} \mathcal{Q}_{W_1|W_0U} \mathcal{Q}_{W_2|W_0U}) = \mathbb{EBNE}(\mathcal{Q}_{W_0|U} \mathcal{Q}_{W_1|W_0U} \mathcal{Q}_{W_2|W_0U}) \right\}. \end{aligned}$$

In the following, we show that optimizing over the full set of target distributions results in the same cost as when the optimization is taken over the set of target distributions restricted to unique Nash Equilibrium.

**Definition 4.5.3.** We replace the set  $\mathcal{Q}_0(R_0, R_1, R_2)$  by the set  $\tilde{\mathcal{Q}}_0(R_0, R_1, R_2)$  and we define the following program:

$$\tilde{\Gamma}_{GW}(R_0, R_1, R_2) = \inf_{\substack{\mathcal{Q}_{W_0|U} \mathcal{Q}_{W_1|W_0U} \mathcal{Q}_{W_2|W_0U} \\ \in \tilde{\mathcal{Q}}_0(R_0, R_1, R_2)}} \max_{\substack{(\mathcal{Q}_{V_1|W_0W_1}, \mathcal{Q}_{V_2|W_0W_2}) \in \\ \mathbb{EBNE}(\mathcal{Q}_{W_0|U} \mathcal{Q}_{W_1|W_0U} \mathcal{Q}_{W_2|W_0U})}} \mathbb{E} \left[ c_e(U, V_1, V_2) \right]. \quad (4.34)$$

**Lemma 4.5.1.** For  $(R_0, R_1, R_2) \in \mathbb{R}_+^3$ , we have

$$\Gamma_{GW}^*(R_0, R_1, R_2) = \tilde{\Gamma}_{GW}(R_0, R_1, R_2). \quad (4.35)$$

*Proof.* The proof of Lemma 4.5.1 follows from [Theorem 4.2, Yu (1999)] and Lemmas 4.5.2, and 4.5.3 given below. For a given  $(R_0, R_1, R_2)$ , the set  $\tilde{\mathcal{Q}}_0(R_0, R_1, R_2)$  is dense in the set

$\mathcal{Q}_0(R_0, R_1, R_2)$  of target distributions  $\mathcal{Q}_{W_0W_1W_2|U}$ . The function

$$\mathcal{Q}_{W_0W_1W_2|U} \mapsto \max_{\substack{(\mathcal{Q}_{V_1|W_0W_1}, \mathcal{Q}_{V_2|W_0W_2}) \in \mathbb{E} \\ \in \mathbb{BNE}(\mathcal{Q}_{W_0|U}, \mathcal{Q}_{W_1|W_0U}, \mathcal{Q}_{W_2|W_0U})}} \mathbb{E}[c_e(U, V_1, V_2)] = \max_{\substack{(\mathcal{Q}_{V_1|W_0W_1}, \mathcal{Q}_{V_2|W_0W_2}) \in \mathbb{E} \\ \in \mathbb{BNE}(\mathcal{Q}_{W_0|U}, \mathcal{Q}_{W_1|W_0U}, \mathcal{Q}_{W_2|W_0U})}} \mathbb{E}[c_e(U, V_1, V_2)]$$

is upper semi-continuous (u.s.c) and the infimum of an u.s.c function over a dense set is the infimum over the full set.  $\square$

**Lemma 4.5.2.** *Given a distribution  $\mathcal{Q}_{W_0W_1W_2|U} \in \Delta(\mathcal{W}_0 \times \mathcal{W}_1 \times \mathcal{W}_2)^{|\mathcal{U}|}$ , we have*

$$\max_{\substack{(\mathcal{Q}_{V_1|W_0W_1}, \mathcal{Q}_{V_2|W_0W_2}) \in \mathbb{E} \\ \in \mathbb{BNE}(\mathcal{Q}_{W_0W_1W_2|U})}} \mathbb{E}[c_e(U, V_1, V_2)] = \max_{\substack{(\mathcal{Q}_{V_1|W_0W_1}, \mathcal{Q}_{V_2|W_0W_2}) \in \mathbb{E} \\ \in \mathbb{BNE}(\mathcal{Q}_{W_0W_1W_2|U})}} \mathbb{E}[c_e(U, V_1, V_2)]. \quad (4.36)$$

*Proof.* [Lemma 4.5.2] Consider the upper semi-continuous [Theorem 3.3, Yu (1999)] correspondence

$$\mathcal{Q}_{W_0W_1W_2|U} \rightrightarrows \mathbb{BNE}(\mathcal{Q}_{W_0W_1W_2|U}).$$

Using (Fort, 1951, Theorem 2), the correspondence is lower semi-continuous, therefore continuous. Using Def. 4.5.1 and Berge's Maximum Theorem Berge (1963), the result follows.  $\square$

**Lemma 4.5.3.** *Let  $(R_0, R_1, R_2) \in \mathbb{R}_+^3$ , the set  $\tilde{\mathcal{Q}}_0(R_0, R_1, R_2)$  is a dense subset of  $\mathcal{Q}_0(R_0, R_1, R_2)$ .*

*Proof.* [Lemma 4.5.3] Fix a distribution  $\mathcal{Q}_{W_0W_1W_2|U} \in \Delta(\mathcal{W}_0 \times \mathcal{W}_1 \times \mathcal{W}_2)^{|\mathcal{U}|}$ . The Bayes-Nash equilibrium correspondence  $\mathcal{Q}_{W_0W_1W_2|U} \rightrightarrows \mathbb{BNE}(\mathcal{Q}_{W_0W_1W_2|U})$  is non-empty for finite games, compact valued and upper semi-continuous (Yu, 1999, Theorems 3.2, 3.3). Therefore, every probability distribution in the set  $\tilde{\mathcal{Q}}_0(R_0, R_1, R_2)$  induces an essential equilibria. Using (Yu, 1999, Theorem 4.2), the result follows.  $\square$

### 4.5.2 Codebook Generation for the Gray-Wyner Network

Fix a conditional probability distribution  $\mathcal{Q}_{W_0|U} \mathcal{Q}_{W_1|UW_0} \mathcal{Q}_{W_2|UW_0} \in \tilde{\mathcal{Q}}_0(R_0, R_1, R_2)$ .

There exists  $\eta > 0$  such that

$$R_0 = I(U; W_0) + \eta, \quad (4.37)$$

$$R_1 = I(U; W_1|W_0) + \eta, \quad (4.38)$$

$$R_2 = I(U; W_2|W_0) + \eta. \quad (4.39)$$

Randomly and independently generate  $2^{\lfloor nR_0 \rfloor}$  sequences  $w_0^n(m_0)$  for  $m_0 \in [1 : 2^{\lfloor nR_0 \rfloor}]$ , according to the i.i.d distribution  $\mathcal{Q}_{W_0^n} = \prod_{t=1}^n \mathcal{Q}_{W_0}(w_{0t})$ . For each  $(m_1, m_0) \in [1 : 2^{\lfloor nR_1 \rfloor}] \times [1 : 2^{\lfloor nR_0 \rfloor}]$  generate a sequence  $w_1^n(m_1, m_0)$  randomly and conditionally independently according to the i.i.d conditional distribution  $\mathcal{Q}_{W_1^n|W_0^n} = \prod_{t=1}^n \mathcal{Q}_{W_1|W_0}(w_{1t}|w_{0t}(m_0))$ . For each  $(m_2, m_0) \in [1 : 2^{\lfloor nR_2 \rfloor}] \times [1 : 2^{\lfloor nR_0 \rfloor}]$  generate a sequence  $w_2^n(m_2, m_0)$  randomly and conditionally independently according to the i.i.d conditional distribution  $\mathcal{Q}_{W_2^n|W_0^n} = \prod_{t=1}^n \mathcal{Q}_{W_2|W_0}(w_{2t}|w_{0t}(m_0))$ .

Coding algorithm: Encoder  $\mathcal{E}$  observes  $u^n$  and determines  $m_0$  such that  $(U^n, W_0^n(m_0)) \in \mathcal{T}_\delta^n(\mathcal{P}_U \mathcal{Q}_{W_0|U})$ ,  $m_1$  such that  $(U^n, W_0^n(m_0), W_1^n(m_1, m_0)) \in \mathcal{T}_\delta^n(\mathcal{P}_U \mathcal{Q}_{W_0W_1|U})$ , and  $m_2$  such that  $(U^n, W_0^n(m_0), W_2^n(m_2, m_0)) \in \mathcal{T}_\delta^n(\mathcal{P}_U \mathcal{Q}_{W_0W_2|U})$ . If such a jointly typical tuple doesn't exist, the source encoder sets  $(m_0, m_1, m_2)$  to  $(1, 1, 1)$ . Then, it sends  $(m_0, m_1)$  to decoder  $\mathcal{D}_1$ , and  $(m_0, m_2)$  to decoder  $\mathcal{D}_2$ . Decoder  $\mathcal{D}_1$  declares  $v_1^n$  and decoder  $\mathcal{D}_2$  declares  $v_2^n$  according to  $\tau_1$  and  $\tau_2$ .

### 4.5.3 Analysis of the Error Probability

We define the following error events

$$\begin{aligned} \mathcal{F}_0 &= \{(U^n, W_0^n(m_0)) \notin \mathcal{T}_\delta^n, \forall m_0\}, \\ \forall m_0, \quad \mathcal{F}_1(m_0) &= \{(U^n, W_0^n(m_0), W_1^n(m_1, m_0)) \notin \mathcal{T}_\delta^n, \forall m_1\}, \\ \forall m_0, \quad \mathcal{F}_2(m_0) &= \{(U^n, W_0^n(m_0), W_2^n(m_2, m_0)) \notin \mathcal{T}_\delta^n, \forall m_2\}. \end{aligned}$$

By the covering lemma (El Gamal and Kim, 2011, Lemma 3.3) given in Lemma G.0.3,  $\mathcal{P}(\mathcal{F}_0)$  tends to zero as  $n \rightarrow \infty$  if  $R_0 \geq I(U; W_0) + \eta$ ,  $\mathcal{P}(\mathcal{F}_1(M_0)|\mathcal{F}_0^c)$  goes to zero by the covering lemma if  $R_1 \geq I(U; W_1|W_0) + \eta$ , and  $\mathcal{P}(\mathcal{F}_2(M_0)|\mathcal{F}_0^c)$  goes to zero by the covering lemma if  $R_2 \geq I(U; W_2|W_0) + \eta$ .

The expected probability of error over the codebook being small means that for all  $\varepsilon_2 > 0$ , for all  $\eta > 0$ , there exists  $\bar{\delta} > 0$ , for all  $\delta \leq \bar{\delta}$ , there exists  $\bar{n} \in \mathbb{N}$  such that for all  $n \geq \bar{n}$  we have:

$$\mathbb{E}[\mathcal{P}(\mathcal{F}_0)] \leq \varepsilon_2, \tag{4.40}$$

$$\mathbb{E}[\mathcal{P}(\mathcal{F}_1(m_0)|\mathcal{F}_0^c)] \leq \varepsilon_2, \tag{4.41}$$

$$\mathbb{E}[\mathcal{P}(\mathcal{F}_2(m_0)|\mathcal{F}_0^c)] \leq \varepsilon_2, \tag{4.42}$$

#### 4.5.4 Analysis of the Bayes-Nash equilibria

In this section, we break down the Bayesian game  $G^\sigma(M_0, M_1, M_2, V_1^n, V_2^n)$  played by blocks of  $n$ -sequences, into several games of stage  $t$ , for  $t \in \{1, 2, \dots, n\}$  selected uniformly at random, that differ in the decoders' types and actions until we reach the single-letter game characterized with auxiliary random variables as illustrated in Fig.4.4. We analyze the different Bayes-Nash equilibria of these Bayesian games. In order to do so, we need to control the beliefs of each decoder about the state and about the type of the other decoder.

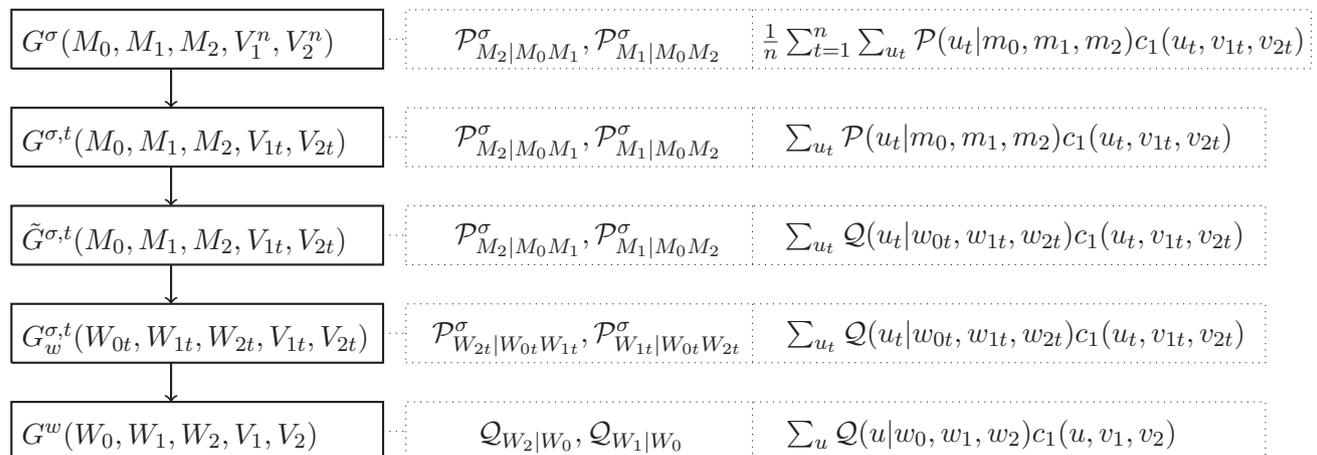


Figure 4.4: Chain of the Bayesian games played among the decoders for achievability.

**Definition 4.5.4.** For each encoding  $\sigma$ , and  $t \in \{1, 2, \dots, n\}$ , the finite Bayesian game

$G^{\sigma,t}(M_0, M_1, M_2, V_{1,t}, V_{2,t})$  at stage  $t$  consists of:

- The decoders  $\mathcal{D}_i, i \in \{1, 2\}$  as the players of the game,
- $\mathcal{V}_{i,t}$  is the set of action sequences of  $\mathcal{D}_i$ ,
- $(M_0, M_i)$  is the type of decoder  $\mathcal{D}_i$ ,
- $\tau_{i,t} : (\{1, 2, \dots, 2^{\lfloor nR_0 \rfloor}\} \times \{1, 2, \dots, 2^{\lfloor nR_i \rfloor}\}) \mapsto \Delta(\mathcal{V}_{i,t})$  is a behavior strategy of decoder  $\mathcal{D}_i$ ,
- The belief of decoder  $\mathcal{D}_1$  (resp.  $\mathcal{D}_2$ ) over the type of decoder  $\mathcal{D}_2$  (resp.  $\mathcal{D}_1$ ) is given by  $\mathcal{P}_{M_2|M_0M_1}^\sigma$  (resp.  $\mathcal{P}_{M_1|M_0M_2}^\sigma$ ).
- $C_i^{\sigma,t} : \{1, 2, \dots, 2^{\lfloor nR_0 \rfloor}\} \times \{1, 2, \dots, 2^{\lfloor nR_1 \rfloor}\} \times \{1, 2, \dots, 2^{\lfloor nR_2 \rfloor}\} \times \mathcal{V}_{1,t} \times \mathcal{V}_{2,t} \mapsto \mathbb{R}$  is the  $\sigma, t$ -cost function of  $\mathcal{D}_i$  at stage  $t$  such that

$$C_i^{\sigma,t}(m_0, m_1, m_2, v_{1,t}, v_{2,t}) = \sum_{u_t} \mathcal{P}^\sigma(u_t | m_0, m_1, m_2) c_i(u_t, v_{1,t}, v_{2,t}). \quad \forall m_0, m_1, m_2, v_{1,t}, v_{2,t}.$$

- For a fixed strategy profile  $(\tau_{1,t}, \tau_{2,t})$ , the expected  $\sigma, t$ -costs  $\Psi_1^{\sigma,t}(\tau_{1,t}, \tau_{2,t}, m_0, m_1)$  of  $\mathcal{D}_1$  at stage  $t$  with type  $(m_0, m_1)$  is given by

$$\begin{aligned} \Psi_1^{\sigma,t}(\tau_{1,t}, \tau_{2,t}, m_0, m_1) &= \sum_{m_2} \mathcal{P}^\sigma(m_2 | m_0, m_1) \sum_{v_{1,t}, v_{2,t}} \mathcal{P}^{\tau_{1,t}}(v_{1,t} | m_0, m_1) \mathcal{P}^{\tau_{2,t}}(v_{2,t} | m_0, m_2) \times \\ &C_1^{\sigma,t}(m_0, m_1, m_2, v_{1,t}, v_{2,t}). \end{aligned} \quad (4.43)$$

Similarly,  $\Psi_2^{\sigma,t}(\tau_{1,t}, \tau_{2,t}, m_0, m_2)$  can be defined.

For each encoding strategy  $\sigma$  and stage  $t$ , we define the set  $BNE(\sigma, t)$  of Bayes-Nash equilibria  $(\tau_{1,t}, \tau_{2,t})$  of  $G^{\sigma,t}(M_0, M_1, M_2, V_{1,t}, V_{2,t})$  as follows

$$\begin{aligned} BNE(\sigma, t) &= \left\{ (\tau_{1,t}, \tau_{2,t}), \Psi_1^{\sigma,t}(\tau_{1,t}, \tau_{2,t}, m_0, m_1) \leq \Psi_1^{\sigma,t}(\tilde{\tau}_{1,t}, \tau_{2,t}, m_0, m_1) \forall \tilde{\tau}_{1,t}, m_0, m_1 \right. \\ &\quad \left. \Psi_2^{\sigma,t}(\tau_{1,t}, \tau_{2,t}, m_0, m_2) \leq \Psi_2^{\sigma,t}(\tau_{1,t}, \tilde{\tau}_{2,t}, m_0, m_2) \forall \tilde{\tau}_{2,t}, m_0, m_2 \right\}. \end{aligned} \quad (4.44)$$

The following lemma shows that every Bayes-Nash equilibrium of the game  $G^\sigma(M_0, M_1, M_2, V_1^n, V_2^n)$  played by blocks of  $n$ -sequences induces an equilibrium of the game  $G^{\sigma,t}(M_0, M_1, M_2, V_{1,t}, V_{2,t})$  at stage  $t$ , for  $t \in \{1, 2, \dots, n\}$ . The converse is also true.

**Lemma 4.5.4.** 1. If  $(\tau_1, \tau_2) \in BNE(\sigma)$ , then  $(\tau_{1,t}, \tau_{2,t}) \in BNE(\sigma, t)$  for all  $t \in \{1, 2, \dots, n\}$ .  
 2. If  $(\tau_{1,t}, \tau_{2,t}) \in BNE(\sigma, t)$  for all  $t \in \{1, 2, \dots, n\}$ , then  $(\prod_{t=1}^n \tau_{1,t}, \prod_{t=1}^n \tau_{2,t}) \in BNE(\sigma)$ .

*Proof.* Given  $\sigma$ , let  $(\tau_1, \tau_2) \in BNE(\sigma)$ . Assume that there exists  $t$  such that  $(\tau_{1,t}, \tau_{2,t}) \notin BNE(\sigma, t)$ . This means that at  $t$ , at least one of the decoders is better off if it deviates from its strategy. Without loss of generality, assume  $\mathcal{D}_1$  deviates to  $\tilde{\tau}_{1,t}$  and selects  $\tilde{v}_{1,t}$  accordingly. This shifts the action sequence  $v_1^n$  that corresponds to  $\tau_1$ , to  $\tilde{v}_1^n = (v_{1,1}, v_{1,2}, \dots, \tilde{v}_{1,t}, \dots, v_{1,n})$ . Thus  $\tau_1 \notin BR_1^\sigma(\tau_2)$ , and  $(\tau_1, \tau_2) \notin BNE(\sigma)$ .

Conversely, if  $(\tau_{1,t}, \tau_{2,t}) \in BNE(\sigma, t)$  for all  $t \in \{1, 2, \dots, n\}$ , we define  $(\tau_1, \tau_2)$  such that

$$\mathcal{P}^{\tau_1}(v_1^n | m_0, m_1) = \prod_{t=1}^n \mathcal{P}^{\tau_{1,t}}(v_{1,t} | m_0, m_1), \quad \forall v_1^n, m_0, m_1 \quad (4.45)$$

$$\mathcal{P}^{\tau_2}(v_2^n | m_0, m_2) = \prod_{t=1}^n \mathcal{P}^{\tau_{2,t}}(v_{2,t} | m_0, m_2), \quad \forall v_2^n, m_0, m_2. \quad (4.46)$$

Suppose that  $(\tau_1, \tau_2) \notin BNE(\sigma)$ . Without loss of generality, assume  $\tau_1 \notin BR_1^\sigma(\tau_2)$ , i.e there exists  $\tilde{\tau}_1 \in BR_1(\sigma)$  such that  $\Psi_1^\sigma(\tau_1, \tau_2, m_0, m_1) \geq \Psi_1^\sigma(\tilde{\tau}_1, \tau_2, m_0, m_1) \forall m_0, m_1$ . Therefore, there exists  $t \in \{1, \dots, n\}$  such that  $\Psi_1^{\sigma,t}(\tau_{1,t}, \tau_{2,t}, m_0, m_1) \geq \Psi_1^{\sigma,t}(\tilde{\tau}_{1,t}, \tau_{2,t}, m_0, m_1) \forall m_0, m_1$ . Thus,  $(\tau_{1,t}, \tau_{2,t}) \notin BNE(\sigma, t)$  which leads to the desired contradiction.  $\square$

**Lemma 4.5.5.** For all  $\sigma$ , we have

$$\max_{\substack{(\tau_1, \tau_2) \\ \in BNE(\sigma)}} \mathbb{E} \left[ \sum_{t=1}^n \frac{1}{n} c_e(U_t, V_{1,t}, V_{2,t}) \right] = \frac{1}{n} \sum_{t=1}^n \max_{\substack{(\tau_{1,t}, \tau_{2,t}) \\ \in BNE(\sigma, t)}} \mathbb{E} \left[ c_e(U_t, V_{1,t}, V_{2,t}) \right]. \quad (4.47)$$

*Proof.* Let  $\sigma$  be given. We will show the equality by showing double inequalities. Let  $(\tau_1, \tau_2) \in BNE(\sigma)$  be arbitrarily chosen, and for all  $t \in \{1, 2, \dots, n\}$ , let  $(\tau_{1,t}, \tau_{2,t}) \in BNE(\sigma, t)$  the corresponding equilibrium strategy pair for the game of stage  $t$ . Therefore,

$$\mathbb{E}_{\sigma, \tau_1, \tau_2} \left[ \sum_{t=1}^n \frac{1}{n} c_e(U_t, V_{1,t}, V_{2,t}) \right] = \frac{1}{n} \sum_{t=1}^n \mathbb{E}_{\sigma, \tau_1, \tau_2} \left[ c_e(U_t, V_{1,t}, V_{2,t}) \right] \quad (4.48)$$

$$\leq \frac{1}{n} \sum_{t=1}^n \mathbb{E}_{\sigma, \tau_{1,t}, \tau_{2,t}} \left[ c_e(U_t, V_{1,t}, V_{2,t}) \right] \quad (4.49)$$

$$\leq \frac{1}{n} \sum_{t=1}^n \max_{\substack{(\tau_{1,t}, \tau_{2,t}) \\ \in BNE(\sigma, t)}} \mathbb{E} \left[ c_e(U_t, V_{1,t}, V_{2,t}) \right]. \quad (4.50)$$

where (4.50) follows since  $(\tau_1, \tau_2) \in BNE(\sigma)$ . Since this is true for all  $(\tau_1, \tau_2) \in BNE(\sigma)$ , then

$$\max_{\substack{(\tau_1, \tau_2) \\ \in BNE(\sigma)}} \mathbb{E} \left[ \sum_{t=1}^n \frac{1}{n} c_e(U_t, V_{1,t}, V_{2,t}) \right] \leq \frac{1}{n} \sum_{t=1}^n \max_{\substack{(\tau_1, \tau_2, \tau_{2,t}) \\ \in BNE(\sigma, t)}} \mathbb{E} \left[ c_e(U_t, V_{1,t}, V_{2,t}) \right]. \quad (4.51)$$

Similarly, let  $(\tau_{1,t}, \tau_{2,t}) \in BNE(\sigma, t)$  be arbitrarily chosen for each  $t \in \{1, 2, \dots, n\}$ . We have,

$$\frac{1}{n} \cdot \sum_{t=1}^n \mathbb{E}_{\sigma, \tau_{1,t}, \tau_{2,t}} \left[ c_e(U_t, V_{1,t}, V_{2,t}) \right] = \frac{1}{n} \cdot \mathbb{E}_{\sigma, \tau_{1,t}, \tau_{2,t}} \left[ \sum_{t=1}^n c_e(U_t, V_{1,t}, V_{2,t}) \right] \quad (4.52)$$

$$\leq \frac{1}{n} \mathbb{E}_{\sigma, \tau_1, \tau_2} \left[ \sum_{t=1}^n c_e(U_t, V_{1,t}, V_{2,t}) \right] \quad (4.53)$$

$$\leq \max_{\substack{(\tau_1, \tau_2) \\ \in BNE(\sigma)}} \mathbb{E} \left[ \sum_{t=1}^n \frac{1}{n} c_e(U_t, V_{1,t}, V_{2,t}) \right]. \quad (4.54)$$

where (4.53) follows since  $(\tau_{1,t}, \tau_{2,t}) \in BNE(\sigma, t)$  for all  $t \in \{1, 2, \dots, n\}$ . Since this is true for all  $(\tau_{1,t}, \tau_{2,t}) \in BNE(\sigma, t)$ , then

$$\max_{\substack{(\tau_1, \tau_2) \\ \in BNE(\sigma)}} \mathbb{E} \left[ \sum_{t=1}^n \frac{1}{n} c_e(U_t, V_{1,t}, V_{2,t}) \right] \geq \frac{1}{n} \sum_{t=1}^n \max_{\substack{(\tau_1, \tau_2, \tau_{2,t}) \\ \in BNE(\sigma, t)}} \mathbb{E} \left[ c_e(U_t, V_{1,t}, V_{2,t}) \right]. \quad (4.55)$$

This concludes the proof.  $\square$

We introduce the indicator of error events  $E_\delta \in \{0, 1\}$  defined as follows

$$E_\delta = \begin{cases} 1, & \text{if } (u^n, w_1^n, w_2^n, w_0^n) \notin \mathcal{T}_\delta^n. \\ 0, & \text{otherwise.} \end{cases} \quad (4.56)$$

We control the Bayesian belief of decoder  $\mathcal{D}_1$  (resp.  $\mathcal{D}_2$ ) about the type of  $\mathcal{D}_2$  (resp.  $\mathcal{D}_1$ ). Let  $\mathcal{P}_{W_{2,t}}^{w_0^n, w_1^n} \in \Delta(\mathcal{W}_2)$  denote  $\mathcal{P}_{W_{2,t}|W_0^n, W_1^n}(\cdot | w_0^n, w_1^n)$  and  $\mathcal{P}_{W_{1,t}}^{w_0^n, w_2^n} \in \Delta(\mathcal{W}_1)$  denote  $\mathcal{P}_{W_{1,t}|W_0^n, W_2^n}(\cdot | w_0^n, w_2^n)$ .

**Lemma 4.5.6.** *For all  $w_0^n, w_1^n, w_2^n, w_0, w_1, w_2$ , we have*

$$\lim_{n \rightarrow \infty} \mathbb{E} \left[ \frac{1}{n} \sum_{t=1}^n D(\mathcal{P}_{W_{2,t}}^{w_0^n, w_1^n} \| \mathcal{Q}_{W_2}^{w_0, w_1}) \middle| E_\delta = 0 \right] = 0, \quad (4.57)$$

$$\lim_{n \rightarrow \infty} \mathbb{E} \left[ \frac{1}{n} \sum_{t=1}^n D(\mathcal{P}_{W_{1,t}}^{w_0^n, w_2^n} \| \mathcal{Q}_{W_1}^{w_0, w_2}) \middle| E_\delta = 0 \right] = 0. \quad (4.58)$$

*Proof.* The proof of Lemma 4.5.6 is stated in Appendix C.  $\square$

We denote the Bayesian posterior beliefs  $\mathcal{P}_{U_t|M_1M_2M_0}^\sigma(\cdot|m_1, m_2, m_0) \in \Delta(\mathcal{U})$  by  $\mathcal{P}_{U_t}^{m_1m_2m_0}$  and by  $\mathcal{Q}_U^{w_1w_2w_0}$  the single-letter belief  $\mathcal{Q}_{U|W_1W_2W_0}(\cdot|w_1, w_2, w_0)$ .

**Lemma 4.5.7.** *For all  $m_0, m_1, m_2, w_0, w_1, w_2$ , we have*

$$\lim_{n \rightarrow \infty} \mathbb{E} \left[ \frac{1}{n} \sum_{t=1}^n D(\mathcal{P}_{U_t}^{m_1m_2m_0} \| \mathcal{Q}_U^{w_1w_2w_0}) \middle| E_\delta = 0 \right] = 0. \quad (4.59)$$

*Proof.* The proof of Lemma 4.5.7 is stated in Appendix D.  $\square$  In a similar fashion, we denote by  $\mathcal{Q}_{W_2}^{w_0, w_1}$  and  $\mathcal{Q}_{W_1}^{w_0, w_2}$  the distributions  $\mathcal{Q}_{W_2|W_0, W_1}(\cdot|w_0, w_1)$  and  $\mathcal{Q}_{W_1|W_0, W_2}(\cdot|w_0, w_2)$  respectively.

**Lemma 4.5.8.** *For all  $w_0^n, w_1^n, w_2^n$ , we have*

$$\lim_{n \rightarrow \infty} \mathbb{E} \left[ \frac{1}{n} \sum_{t=1}^n D(\mathcal{P}_{W_{1,t}|W_{0,t}W_{2,t}}^\sigma \| \mathcal{Q}_{W_1}^{w_0, w_2}) \middle| E_\delta = 0 \right] = 0, \quad (4.60)$$

$$\lim_{n \rightarrow \infty} \mathbb{E} \left[ \frac{1}{n} \sum_{t=1}^n D(\mathcal{P}_{W_{2,t}|W_{0,t}W_{1,t}}^\sigma \| \mathcal{Q}_{W_2}^{w_0, w_1}) \middle| E_\delta = 0 \right] = 0. \quad (4.61)$$

*Proof.* The proof of lemma 4.5.8 is stated in Appendix E.  $\square$

In the following we define the game  $\tilde{G}^{\sigma, t}(M_0, M_1, M_2, V_{1,t}, V_{2,t})$  of stage  $t$ , for  $t \in \{1, 2, \dots, n\}$ , in which the costs are computed using the single-letter beliefs that correspond to essential equilibria.

**Definition 4.5.5.** *For each encoding  $\sigma$ , the Bayesian game  $\tilde{G}^{\sigma, t}(M_0, M_1, M_2, V_{1,t}, V_{2,t})$  at stage  $t$ , for  $t \in \{1, 2, \dots, n\}$  consists of:*

- The decoders  $\mathcal{D}_i, i \in \{1, 2\}$  as the players of the game,
- $\mathcal{V}_{i,t}$  is the set of action sequences of  $\mathcal{D}_i$ ,
- $(M_0, M_i)$  is the type of decoder  $\mathcal{D}_i$ ,
- $\tau_{i,t} : (\{1, 2, \dots, 2^{\lfloor nR_0 \rfloor}\} \times \{1, 2, \dots, 2^{\lfloor nR_i \rfloor}\}) \mapsto \Delta(\mathcal{V}_{i,t})$  is a behavior strategy of decoder  $\mathcal{D}_i$ ,

- The belief of decoder  $\mathcal{D}_1$  (resp.  $\mathcal{D}_2$ ) over the type of decoder  $\mathcal{D}_2$  (resp.  $\mathcal{D}_1$ ) is given by  $\mathcal{P}_{M_2|M_0M_1}^\sigma$  (resp.  $\mathcal{P}_{M_1|M_0M_2}^\sigma$ ).
- $\tilde{C}_i^{\sigma,t} : \{1, 2, \dots, 2^{\lfloor nR_i \rfloor}\} \times \{1, 2, \dots, 2^{\lfloor nR_1 \rfloor}\} \times \{1, 2, \dots, 2^{\lfloor nR_2 \rfloor}\} \times \mathcal{V}_{1,t} \times \mathcal{V}_{2,t} \mapsto \mathbb{R}$  is the  $\sigma$ -cost function of  $\mathcal{D}_i$  at stage  $t$  such that

$$\tilde{C}_i^{\sigma,t}(m_0, m_1, m_2, v_{1,t}, v_{2,t}) = \sum_{u_t} \mathcal{Q}(u_t | w_{0,t}(m_0), w_{1,t}(m_0, m_1), w_{2,t}(m_0, m_2)) \times c_i(u_t, v_{1,t}, v_{2,t}), \quad \forall v_{1,t}, v_{2,t}, m_0, m_1, m_2.$$

- For a fixed strategy profile  $(\tau_{1,t}, \tau_{2,t})$ , the expected  $\sigma$ -costs  $\Psi_1^\sigma(\tau_{1,t}, \tau_{2,t}, m_0, m_1)$  of  $\mathcal{D}_1$  at stage  $t$  with type  $(m_0, m_1)$  is given by

$$\begin{aligned} \tilde{\Psi}_1^{\sigma,t}(\tau_{1,t}, \tau_{2,t}, m_0, m_1) &= \sum_{m_2} \mathcal{P}^\sigma(m_2 | m_0, m_1) \sum_{v_{1,t}, v_{2,t}} \mathcal{P}^{\tau_{1,t}}(v_{1,t} | m_0, m_1) \mathcal{P}^{\tau_{2,t}}(v_{2,t} | m_0, m_2) \times \\ &\quad \tilde{C}_1^{\sigma,t}(v_{1,t}, v_{2,t}, m_0, m_1, m_2). \end{aligned} \quad (4.62)$$

Similarly,  $\tilde{\Psi}_2^{\sigma,t}(\tau_{1,t}, \tau_{2,t}, m_0, m_2)$  can be defined.

For each distribution  $\mathcal{P}_{W_{0,t}W_{1,t}W_{2,t}|U}^\sigma \in \Delta(\mathcal{W}_{0,t} \times \mathcal{W}_{1,t} \times \mathcal{W}_{2,t})^{|\mathcal{U}_t|}$ , the set of Bayes-Nash equilibria of stage  $t$  is given by

$$\begin{aligned} B\tilde{N}E(\sigma, t) &= \left\{ (\tau_{1,t}, \tau_{2,t}), \tilde{\Psi}_1^{\sigma,t}(\tau_{1,t}, \tau_{2,t}, m_0, m_1) \leq \tilde{\Psi}_1^{\sigma,t}(\tilde{\tau}_{1,t}, \tau_{2,t}, m_0, m_1) \forall \tilde{\tau}_{1,t}, m_0, m_1 \right. \\ &\quad \left. \tilde{\Psi}_2^{\sigma,t}(\tau_{1,t}, \tau_{2,t}, m_0, m_2) \leq \tilde{\Psi}_2^{\sigma,t}(\tau_{1,t}, \tilde{\tau}_{2,t}, m_0, m_2) \forall \tilde{\tau}_{2,t}, m_0, m_2 \right\}. \end{aligned} \quad (4.63)$$

**Lemma 4.5.9.** For all  $\sigma$ , and  $\varepsilon > 0$ , we have

$$\lim_{n \rightarrow \infty} \sum_{t=1}^n \left| \max_{\substack{(\tau_{1,t}, \tau_{2,t}) \\ \in B\tilde{N}E(\sigma, t)}} \mathbb{E}[c_e(U_t, V_{1,t}, V_{2,t})] - \max_{\substack{(\tau_{1,t}, \tau_{2,t}) \\ \in B\tilde{N}E(\sigma, t)}} \mathbb{E}[c_e(U_t, V_{1,t}, V_{2,t})] \right| \leq \varepsilon. \quad (4.64)$$

*Proof.* Consider the following correspondence

$$\mathcal{P}_{U_t|M_0M_1M_2}^\sigma \Rightarrow \left\{ (\mathcal{P}_{V_{1,t}|M_0M_1}^{\tau_{1,t}}, \mathcal{P}_{V_{2,t}|M_0M_2}^{\tau_{2,t}}), \tilde{\Psi}_1^{\sigma,t}(\tau_{1,t}, \tau_{2,t}, m_0, m_1) \leq \tilde{\Psi}_1^{\sigma,t}(\tilde{\tau}_{1,t}, \tau_{2,t}, m_0, m_1) \forall \tilde{\tau}_{1,t}, m_0, m_1 \right.$$

$$\tilde{\Psi}_2^{\sigma,t}(\tau_{1,t}, \tau_{2,t}, m_0, m_2) \leq \tilde{\Psi}_2^{\sigma,t}(\tau_{1,t}, \tilde{\tau}_{2,t}, m_0, m_2) \forall \tilde{\tau}_{2,t}, m_0, m_2 \}. \quad (4.65)$$

Denote by  $N(\mathcal{P}_{U_t|M_0M_1M_2}^\sigma)$  the RHS of equation (4.65). It follows from lemma 4.5.7 that for a given  $\varepsilon > 0$ , we have,

$$\begin{aligned} & \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{t=1}^n \left| C_i^{\sigma,t}(m_0, m_1, m_2, v_{1,t}, v_{2,t}) - \tilde{C}_i^{\sigma,t}(m_0, m_1, m_2, v_{1,t}, v_{2,t}) \right| = \\ & \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{t=1}^n \left| \sum_{u_t} \mathcal{P}^\sigma(u_t|m_0, m_1, m_2) c_i(u_t, v_{1,t}, v_{2,t}) - \sum_{u_t} \mathcal{Q}(u_t|w_{0,t}(m_0), w_{1,t}(m_0, m_1), w_{2,t}(m_0, m_2)) \times \right. \\ & \left. c_i(u_t, v_{1,t}, v_{2,t}) \right| \leq \varepsilon, \quad \forall m_0, m_1, m_2, v_{1,t}, v_{2,t}. \end{aligned} \quad (4.66)$$

Consequently, for all  $m_0, m_1, m_2, v_{1,t}, v_{2,t}$ ,

$$\begin{aligned} & \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{t=1}^n \left| \Psi_1^{\sigma,t}(\tau_{1,t}, \tau_{2,t}, m_0, m_1) - \tilde{\Psi}_1^{\sigma,t}(\tau_{1,t}, \tau_{2,t}, m_0, m_1) \right| \\ & = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{t=1}^n \left| \sum_{m_2} \mathcal{P}^\sigma(m_2|m_0, m_1) \sum_{v_{1,t}, v_{2,t}} \mathcal{P}^{\tau_{1,t}}(v_{1,t}|m_0, m_1) \mathcal{P}^{\tau_{2,t}}(v_{2,t}|m_0, m_2) C_1^\sigma(v_{1,t}, v_{2,t}, m_0, m_1, m_2) \right. \\ & \left. - \sum_{m_2} \mathcal{P}^\sigma(m_2|m_0, m_1) \sum_{v_{1,t}, v_{2,t}} \mathcal{P}^{\tau_{1,t}}(v_{1,t}|m_0, m_1) \mathcal{P}^{\tau_{2,t}}(v_{2,t}|m_0, m_2) \tilde{C}_1^\sigma(v_{1,t}, v_{2,t}, m_0, m_1, m_2) \right| \leq \varepsilon. \end{aligned} \quad (4.67)$$

Therefore, using (Fort, 1951, Theorem 2), the correspondence in (4.65) is continuous. Using Berge's Maximum Theorem (Berge (1963)), the function

$$\mathcal{P}_{U_t|M_0M_1M_2}^\sigma \mapsto \max_{\substack{(\tau_{1,t}, \tau_{2,t}) \\ \in N(\mathcal{P}_{U_t|M_0M_1M_2}^\sigma)}} \mathbb{E}[c_e(U_t, V_{1,t}, V_{2,t})] \quad (4.68)$$

is well-defined and continuous. Hence,  $(\tau_{1,t}, \tau_{2,t}) \in B\tilde{N}E(\sigma, t)$ . Therefore, varying  $\sigma$  in a small neighborhood, slightly perturbs the expected cost functions resulting in a slightly perturbed set of Bayes-Nash equilibria. By the continuity of the max-value function in (4.68), we get the desired inequality.  $\square$

Similarly, denote by  $G_w^{\sigma,t}(W_{0,t}, W_{1,t}, W_{2,t}, V_{1,t}, V_{2,t})$  the Bayesian game restricted to types  $W_{0,t}, W_{1,t}, W_{2,t}$  at stage  $t$ , for  $t \in \{1, 2, \dots, n\}$  defined as follows.

**Definition 4.5.6.** For each encoding  $\sigma$ , the finite Bayesian game  $G_w^{\sigma,t}(W_{0,t}, W_{1,t}, W_{2,t}, V_{1,t}, V_{2,t})$  at stage  $t$ , for  $t \in \{1, 2, \dots, n\}$  consists of:

- The decoders  $\mathcal{D}_i, i \in \{1, 2\}$  as the players of the game,
- $\mathcal{V}_{i,t}$  is the set of action sequences of  $\mathcal{D}_i$ ,
- $(W_{0,t}, W_{i,t})$  is the type of decoder  $\mathcal{D}_i$ ,
- $\tau_{i,t} : (\{1, 2, \dots, 2^{\lfloor nR_0 \rfloor}\} \times \{1, 2, \dots, 2^{\lfloor nR_i \rfloor}\}) \mapsto \Delta(\mathcal{V}_{i,t})$  is a behavior strategy of decoder  $\mathcal{D}_i$ ,
- The belief of decoder  $\mathcal{D}_1$  (resp.  $\mathcal{D}_2$ ) over the type of decoder  $\mathcal{D}_2$  (resp.  $\mathcal{D}_1$ ) is given by  $\mathcal{P}_{W_{2,t}|W_{0,t}W_{1,t}}^\sigma$  (resp.  $\mathcal{P}_{W_{1,t}|W_{0,t}W_{2,t}}^\sigma$ ).
- $C_i^{\sigma,w,t} : \mathcal{W}_0 \times \mathcal{W}_1 \times \mathcal{W}_2 \times \mathcal{V}_{1,t} \times \mathcal{V}_{2,t} \mapsto \mathbb{R}$  is the  $\sigma$ -cost function of  $\mathcal{D}_i$  at stage  $t$  such that  $\forall v_{1,t}, v_{2,t}, w_{0,t}, w_{1,t}, w_{2,t}$ ,

$$C_i^{\sigma,w,t}(w_{0,t}, w_{1,t}, w_{2,t}, v_{1,t}, v_{2,t}) = \sum_{u_t} \mathcal{Q}(u_t|w_{0,t}, w_{1,t}, w_{2,t}) c_i(u_t, v_{1,t}, v_{2,t}). \quad (4.69)$$

- For a fixed strategy profile  $(\tau_{1,t}, \tau_{2,t})$ , the expected  $\sigma$ -costs  $\Psi_1^{\sigma,w,t}(\tau_{1,t}, \tau_{2,t}, w_{0,t}, w_{1,t})$  of  $\mathcal{D}_1$  at stage  $t$  with type  $(w_{0,t}, w_{1,t})$  is given by

$$\begin{aligned} \Psi_1^{\sigma,w,t}(\tau_{1,t}, \tau_{2,t}, w_{0,t}, w_{1,t}) &= \sum_{w_{2,t}} \mathcal{P}^\sigma(w_{2,t}|w_{0,t}, w_{1,t}) \sum_{v_{1,t}, v_{2,t}} \mathcal{P}^{\tau_{1,t}}(v_{1,t}|w_{0,t}, w_{1,t}) \mathcal{P}^{\tau_{2,t}}(v_{2,t}|w_{0,t}, w_{2,t}) \times \\ &C_1^{\sigma,w,t}(v_{1,t}, v_{2,t}, w_{0,t}, w_{1,t}, w_{2,t}). \end{aligned} \quad (4.70)$$

Similarly,  $\Psi_2^{\sigma,w,t}(\tau_{1,t}, \tau_{2,t}, w_{0,t}, w_{2,t})$  can be defined.

For each encoding strategy  $\sigma$  and stage  $t$ , we define the set  $BNE^w(\sigma, t)$  of Bayes-Nash equilibria  $(\tau_{1,t}, \tau_{2,t})$  of  $G_w^{\sigma,t}(W_{0,t}, W_{1,t}, W_{2,t}, V_{1,t}, V_{2,t})$  as follows

$$\begin{aligned} BNE^w(\sigma, t) &= \left\{ (\tau_{1,t}, \tau_{2,t}), \Psi_1^{\sigma,w,t}(\tau_{1,t}, \tau_{2,t}, w_0, w_1) \leq \Psi_1^{\sigma,w,t}(\tilde{\tau}_{1,t}, \tau_{2,t}, w_0, w_1) \forall \tilde{\tau}_{1,t}, w_0, w_1 \right. \\ &\quad \left. \Psi_2^{\sigma,w,t}(\tau_{1,t}, \tau_{2,t}, w_0, w_2) \leq \Psi_2^{\sigma,w,t}(\tau_{1,t}, \tilde{\tau}_{2,t}, w_0, w_2) \forall \tilde{\tau}_{2,t}, w_0, w_2 \right\}. \end{aligned}$$

The game  $G_w^{\sigma,t}(W_{0,t}, W_{1,t}, W_{2,t}, V_{1,t}, V_{2,t})$  of stage  $t$  directly derives from  $\tilde{G}^{\sigma,t}(M_0, M_1, M_2, V_{1,t}, V_{2,t})$  by marginalizing with respect to components  $W_{0,t}, W_{1,t}$  and  $W_{2,t}$  of  $W_0^n(M_0), W_1^n(M_0, M_1)$  and  $W_2^n(M_0, M_2)$  respectively according to the injections in Fig. 4.5.

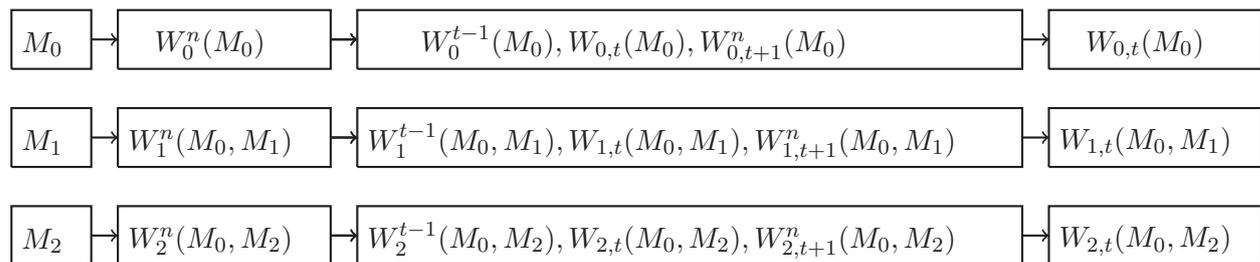


Figure 4.5: Marginalization scheme of the decoders' types

**Lemma 4.5.10.** *For all  $\sigma$ , we have*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{t=1}^n \max_{\substack{(\tau_{1,t}, \tau_{2,t}) \\ \in B\tilde{N}E(\sigma, t)}} \mathbb{E}[c_e(U_t, V_{1,t}, V_{2,t})] = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{t=1}^n \max_{\substack{(\tau_{1,t}, \tau_{2,t}) \\ \in BNE^w(\sigma, t)}} \mathbb{E}[c_e(U_t, V_{1,t}, V_{2,t})] \quad (4.71)$$

*Proof.* We will proceed by showing that every equilibrium  $(\tau_{1,t}, \tau_{2,t}) \in B\tilde{N}E(\sigma, t)$  induces an equilibrium in  $BNE^w(\sigma, t)$ . For all  $\sigma, \tau_{1,t}, \tau_{2,t}, w_{0,t}, w_{1,t}$ , we have,

$$\Psi_1^{\sigma, w, t}(\tau_{1,t}, \tau_{2,t}, w_{0,t}, w_{1,t}) \quad (4.72)$$

$$= \sum_{w_{2,t}} \mathcal{P}^\sigma(w_{2,t} | w_{0,t}, w_{1,t}) \sum_{v_{1,t}, v_{2,t}} \mathcal{P}^{\tau_{1,t}}(v_{1,t} | w_{0,t}, w_{1,t}) \mathcal{P}^{\tau_{2,t}}(v_{2,t} | w_{0,t}, w_{2,t}) \\ \times \sum_{u_t} \mathcal{Q}(u_t | w_{0,t}, w_{1,t}, w_{2,t}) c_i(u_t, v_{1,t}, v_{2,t}) \quad (4.73)$$

$$= \sum_{\substack{w_2^n, w_{0,t+1}^n, w_0^{t-1} \\ w_1^{t-1}, w_{1,t+1}^n}} \mathcal{P}^\sigma(w_2^n | w_0^n, w_1^n) \mathcal{P}^\sigma(w_{0,t+1}^n, w_0^{t-1}, w_1^{t-1}, w_{1,t+1}^n | w_{0,t}, w_{1,t}) \\ \times \sum_{v_{1,t}, v_{2,t}} \sum_{\substack{w_{0,t+1}^n, w_0^{t-1} \\ w_1^{t-1}, w_{1,t+1}^n}} \mathcal{P}^{\tau_{1,t}}(v_{1,t} | w_0^n, w_1^n) \mathcal{P}(w_{0,t+1}^n, w_0^{t-1}, w_1^{t-1}, w_{1,t+1}^n | w_{0,t}, w_{1,t}) \\ \times \sum_{\substack{w_{0,t+1}^n, w_0^{t-1} \\ w_2^{t-1}, w_{2,t+1}^n}} \mathcal{P}^{\tau_{2,t}}(v_{2,t} | w_0^n, w_2^n) \mathcal{P}(w_{0,t+1}^n, w_0^{t-1}, w_2^{t-1}, w_{2,t+1}^n | w_{0,t}, w_{2,t}) \\ \times \sum_{\substack{u_t, w_{0,t+1}^n, w_0^{t-1} \\ w_1^{t-1}, w_{1,t+1}^n, w_2^{t-1}, w_{2,t+1}^n}} \mathcal{Q}(u_t | w_0^n, w_1^n, w_2^n) \mathcal{Q}(w_{0,t+1}^n, w_0^{t-1}, w_1^{t-1}, w_{1,t+1}^n, w_2^{t-1}, w_{2,t+1}^n | w_{0,t}, w_{1,t}, w_{2,t}) \\ \times c_1(u_t, v_{1,t}, v_{2,t}). \quad (4.74)$$

Thus, if  $(\tau_{1,t}, \tau_{2,t}) \in BNE(\sigma)$ , then  $(\tau_{1,t}, \tau_{2,t}) \in BNE^w(\sigma, t)$ .  $\square$

Finally, we define the essential game  $\tilde{G}^{w,t}(W_{0,t}, W_{1,t}, W_{2,t}, V_{1,t}, V_{2,t})$  of stage  $t$ , for  $t \in \{1, 2, \dots, n\}$ .

**Definition 4.5.7.** For each encoding  $\sigma$ , the essential Bayesian game  $\tilde{G}^{w,t}(W_{0,t}, W_{1,t}, W_{2,t}, V_{1,t}, V_{2,t})$  at stage  $t$ , for  $t \in \{1, 2, \dots, n\}$  consists of:

- The decoders  $\mathcal{D}_i, i \in \{1, 2\}$  as the players of the game,
- $\mathcal{V}_{i,t}$  is the set of action sequences of  $\mathcal{D}_i$ ,
- $(W_{0,t}, W_{i,t})$  is the type of decoder  $\mathcal{D}_i$ ,
- $\tau_{i,t} : \mathcal{W}_{0,t} \times \mathcal{W}_{i,t} \mapsto \Delta(\mathcal{V}_{i,t})$  is a behavior strategy of decoder  $\mathcal{D}_i$ ,
- The belief of decoder  $\mathcal{D}_1$  (resp.  $\mathcal{D}_2$ ) over the type of decoder  $\mathcal{D}_2$  (resp.  $\mathcal{D}_1$ ) is given by  $\mathcal{Q}_{W_{2,t}|W_{0,t}}$  (resp.  $\mathcal{Q}_{W_{1,t}|W_{0,t}}$ ).
- $\tilde{C}_i^{w,t} : \mathcal{W}_0 \times \mathcal{W}_1 \times \mathcal{W}_2 \times \mathcal{V}_{1,t} \times \mathcal{V}_{2,t} \mapsto \mathbb{R}$  is the  $\sigma$ -cost function of  $\mathcal{D}_i$  at stage  $t$  such that  $\forall v_{1,t}, v_{2,t}, w_{0,t}, w_{1,t}, w_{2,t}$ ,

$$\tilde{C}_i^\sigma(w_{0,t}, w_{1,t}, w_{2,t}, v_{1,t}, v_{2,t}) = \sum_{u_t} \mathcal{Q}(u_t|w_{0,t}, w_{1,t}, w_{2,t}) c_i(u_t, v_{1,t}, v_{2,t}).$$

- For a fixed strategy profile  $(\tau_{1,t}, \tau_{2,t})$ , the expected  $\sigma$ -costs  $\tilde{\Psi}_1^{\sigma,t}(\tau_{1,t}, \tau_{2,t}, w_{0,t}, w_{1,t})$  of  $\mathcal{D}_1$  at stage  $t$  with type  $(w_{0,t}, w_{1,t})$  is given by

$$\begin{aligned} \tilde{\Psi}_1^{w,t}(\tau_{1,t}, \tau_{2,t}, w_{0,t}, w_{1,t}) &= \sum_{w_{2,t}} \mathcal{Q}(w_{2,t}|w_{0,t}) \sum_{v_{1,t}, v_{2,t}} \mathcal{P}^{\tau_{1,t}}(v_{1,t}|w_{0,t}, w_{1,t}) \mathcal{P}^{\tau_{2,t}}(v_{2,t}|w_{0,t}, w_{2,t}) \times \\ &\quad \tilde{C}_1^{w,t}(v_{1,t}, v_{2,t}, w_{0,t}, w_{1,t}, w_{2,t}). \end{aligned} \quad (4.75)$$

Similarly,  $\tilde{\Psi}_2^{w,t}(\tau_{1,t}, \tau_{2,t}, w_{0,t}, w_{2,t})$  can be defined.

For each distribution  $\mathcal{Q}_{W_{0,t}W_{1,t}W_{2,t}|U_t} \in \Delta(\mathcal{W}_{0,t} \times \mathcal{W}_{1,t} \times \mathcal{W}_{2,t})^{|\mathcal{U}_t|}$ , the set of essential Bayes-Nash equilibria of stage  $t$  is given by

$$\begin{aligned} \mathbb{BNE}(\mathcal{Q}_{W_{0,t}W_{1,t}W_{2,t}|U_t}) &= \left\{ (\mathcal{Q}_{V_{1,t}|W_{0,t}W_{1,t}}, \mathcal{Q}_{V_{2,t}|W_{0,t}W_{2,t}}), \right. \\ &\quad \tilde{\Psi}_1^{w,t}(\mathcal{Q}_{V_{1,t}|W_{0,t}W_{1,t}}, \mathcal{Q}_{V_{2,t}|W_{0,t}W_{2,t}}, w_{0,t}, w_{1,t}) \leq \tilde{\Psi}_1^{w,t}(\tilde{\mathcal{Q}}_{V_{1,t}|W_{0,t}W_{1,t}}, \mathcal{Q}_{V_{2,t}|W_{0,t}W_{2,t}}, w_{0,t}, w_{1,t}) \\ &\quad \forall \tilde{\mathcal{Q}}_{V_{1,t}|W_{0,t}W_{1,t}}, w_{0,t}, w_{1,t}, \\ &\quad \left. \tilde{\Psi}_2^{w,t}(\mathcal{Q}_{V_{1,t}|W_{0,t}W_{1,t}}, \mathcal{Q}_{V_{2,t}|W_{0,t}W_{2,t}}, w_{0,t}, w_{2,t}) \leq \tilde{\Psi}_2^{w,t}(\mathcal{Q}_{V_{1,t}|W_{0,t}W_{1,t}}, \tilde{\mathcal{Q}}_{V_{2,t}|W_{0,t}W_{2,t}}, w_{0,t}, w_{2,t}) \right\} \end{aligned}$$

$$\forall \tilde{\mathcal{Q}}_{V_{2,t}|W_{0,t}W_{2,t}}, w_{0,t}, w_{2,t}\}. \quad (4.76)$$

**Lemma 4.5.11.** For all  $t, \sigma, \mathcal{Q}_{W_{0,t}|U} \mathcal{Q}_{W_{1,t}|W_{0,t}U} \mathcal{Q}_{W_{2,t}|W_{0,t}U} \in \mathbb{Q}_0(R_0, R_1, R_2)$  and  $\varepsilon > 0$ , we have

$$\left| \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{t=1}^n \max_{(\tau_{1,t}, \tau_{2,t}) \in \text{BNE}^w(\sigma, t)} \mathbb{E}[c_e(U_t, V_{1,t}, V_{2,t})] - \max_{(\mathcal{Q}_{V_{1,t}|W_{0,t}W_{1,t}}, \mathcal{Q}_{V_{2,t}|W_{0,t}W_{2,t}}) \in \text{BNE}(\mathcal{Q}_{W_{0,t}|U_t} \mathcal{Q}_{W_{1,t}|W_{0,t}U_t} \mathcal{Q}_{W_{2,t}|W_{0,t}U_t})} \mathbb{E}[c_e(U_t, V_{1,t}, V_{2,t})] \right| \leq \varepsilon. \quad (4.77)$$

*Proof.* Consider the following correspondence

$$\begin{aligned} \mathcal{P}_{U_t|W_{0,t}W_{1,t}W_{2,t}}^\sigma &\Rightarrow \left\{ (\mathcal{P}_{V_{1,t}|W_{0,t}W_{1,t}}^{\tau_{1,t}}, \mathcal{P}_{V_{2,t}|W_{0,t}W_{2,t}}^{\tau_{2,t}}), \right. \\ &\quad \tilde{\Psi}_1^{w,t}(\tau_{1,t}, \tau_{2,t}, w_{0,t}, w_{1,t}) \leq \tilde{\Psi}_1^{w,t}(\tilde{\tau}_{1,t}, \tau_{2,t}, w_{0,t}, w_{1,t}), \forall \tilde{\tau}_{1,t}, w_{0,t}, w_{1,t}, \\ &\quad \left. \tilde{\Psi}_2^{w,t}(\tau_{1,t}, \tau_{2,t}, w_{0,t}, w_{2,t}) \leq \tilde{\Psi}_2^{w,t}(\tau_{1,t}, \tilde{\tau}_{2,t}, w_{0,t}, w_{2,t}), \forall \tilde{\tau}_{2,t}, w_{0,t}, w_{2,t} \right\}. \end{aligned} \quad (4.78)$$

Denote by  $N^w(\mathcal{P}_{U_t|W_{0,t}W_{1,t}W_{2,t}}^\sigma)$  the RHS of equation (4.65). It follows from lemma 4.5.8 that for a given  $\varepsilon > 0$ , and for all  $w_{0,t}, w_{1,t}, w_{2,t}, v_{1,t}, v_{2,t}$ ,

$$\begin{aligned} &\left| \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{t=1}^n \Psi_1^{w,t}(\tau_{1,t}, \tau_{2,t}, w_{0,t}, w_{1,t}) - \tilde{\Psi}_1^{w,t}(\tau_{1,t}, \tau_{2,t}, w_{0,t}, w_{1,t}) \right| \\ &= \left| \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{t=1}^n \sum_{w_{2,t}} \mathcal{P}^\sigma(w_{2,t}|w_{0,t}, w_{1,t}) \sum_{v_{1,t}, v_{2,t}} \mathcal{P}^{\tau_{1,t}}(v_{1,t}|w_{0,t}, w_{1,t}) \times \right. \\ &\quad \mathcal{P}^{\tau_{2,t}}(v_{2,t}|w_{0,t}, w_{2,t}) C_1^{w,t}(v_{1,t}, v_{2,t}, w_{0,t}, w_{1,t}, w_{2,t}) - \\ &\quad \sum_{w_{2,t}} \mathcal{Q}(w_{2,t}|w_{0,t}) \sum_{v_{1,t}, v_{2,t}} \mathcal{Q}^{\tau_{1,t}}(v_{1,t}|w_{0,t}, w_{1,t}) \times \\ &\quad \left. \mathcal{Q}^{\tau_{2,t}}(v_{2,t}|w_{0,t}, w_{2,t}) \tilde{C}_1^{w,t}(v_{1,t}, v_{2,t}, w_{0,t}, w_{1,t}, w_{2,t}) \right| \leq \varepsilon. \end{aligned} \quad (4.79)$$

Therefore, using (Fort, 1951, Theorem 2), the correspondence in (4.78) is continuous. Using Berge's Maximum Theorem Berge (1963), the function

$$\mathcal{P}_{U_t|W_{0,t}W_{1,t}W_{2,t}}^\sigma \mapsto \max_{(\tau_{1,t}, \tau_{2,t}) \in N^w(\mathcal{P}_{U_t|W_{0,t}W_{1,t}W_{2,t}}^\sigma)} \mathbb{E}[c_e(U_t, V_{1,t}, V_{2,t})] \quad (4.80)$$

is well-defined and continuous. Hence,  $(\tau_{1,t}, \tau_{2,t}) \in B\tilde{N}E(\sigma, t)$ . Therefore, varying  $\sigma$  in a small neighborhood, slightly perturbs the expected cost functions resulting in a slightly perturbed set of Bayes-Nash equilibria. By the continuity of the max-value function in (4.80), we get the desired inequality.  $\square$

It follows from lemmas 4.5.5, 4.5.9, 4.5.10, and 4.5.11 that for all  $\varepsilon > 0$ , there exists  $\hat{n}$  such that for all  $n \geq \hat{n}$ ,

$$\max_{\substack{(\tau_1, \tau_2) \\ \in BNE(\sigma)}} \mathbb{E}\left[\sum_{t=1}^n \frac{1}{n} c_e(U_t, V_{1,t}, V_{2,t})\right] = \frac{1}{n} \sum_{t=1}^n \max_{\substack{(\tau_{1,t}, \tau_{2,t}) \\ \in BNE(\sigma, t)}} \mathbb{E}[c_e(U_t, V_{1,t}, V_{2,t})] \quad (4.81)$$

$$\leq \frac{1}{n} \sum_{t=1}^n \max_{\substack{(\tau_{1,t}, \tau_{2,t}) \\ \in B\tilde{N}E(\sigma, t)}} \mathbb{E}[c_e(U_t, V_{1,t}, V_{2,t})] \quad (4.82)$$

$$\leq \frac{1}{n} \sum_{t=1}^n \max_{\substack{(\tau_{1,t}, \tau_{2,t}) \\ \in B\tilde{N}E(\sigma, t)}} \mathbb{E}[c_e(U_t, V_{1,t}, V_{2,t})] \quad (4.83)$$

$$= \frac{1}{n} \sum_{t=1}^n \max_{\substack{(\tau_{1,t}, \tau_{2,t}) \\ \in BNE^w(\sigma, t)}} \mathbb{E}[c_e(U_t, V_{1,t}, V_{2,t})] \quad (4.84)$$

$$\leq \max_{\substack{(\mathcal{Q}_{V_{1,t}|W_{0,t}W_{1,t}}, \mathcal{Q}_{V_{2,t}|W_{0,t}W_{2,t}}) \in \\ BNE(\mathcal{Q}_{W_{0,t}|U_t}, \mathcal{Q}_{W_{1,t}|W_{0,t}U_t}, \mathcal{Q}_{W_{2,t}|W_{0,t}U_t})}} \mathbb{E}[c_e(U_t, V_{1,t}, V_{2,t})] + \varepsilon. \quad (4.85)$$

This concludes the proof of the achievability statement of Theorem 4.3.1.

## 4.6 Converse proof for the strategic Gray-Wyner communication network

Let  $(R_0, R_1, R_2) \in \mathbb{R}_+^3$  and  $n \in \mathbb{N}^*$ . Fix  $(\sigma, \tau_1, \tau_2)$ , and consider a random variable  $T$  uniformly distributed over  $\{1, 2, \dots, n\}$  and independent of  $(U^n, M_0, M_1, M_2, V_1^n, V_2^n)$ . We introduce the auxiliary random variables  $W_0 = (M_0, T)$ ,  $W_1 = M_1$ ,  $W_2 = M_2$ ,  $(U, V_1, V_2) = (U_T, V_{1,T}, V_{2,T})$ <sup>3</sup>, distributed according to  $\mathcal{P}_{UW_0W_1W_2V_1V_2}^{\sigma\tau_1\tau_2}$  defined for all  $(u, w_0, w_1, w_2, v_1, v_2) =$

<sup>3</sup>We denote by  $U_T, V_{1,T}, V_{2,T}$  components of sequences  $U^n, V_1^n, V_2^n$  respectively, selected uniformly at random.

$(u_t, m_0, m_1, m_2, t, v_{1,t}, v_{2,t})$  by

$$\begin{aligned} \mathcal{P}_{UW_0W_1W_2V_1V_2}^{\sigma\tau_1\tau_2}(u, w_0, w_1, w_2, v_1, v_2) &= \mathcal{P}_{UM_0M_1M_2TV_1TV_2T}^{\sigma\tau_1\tau_2}(u_t, m_0, m_1, m_2, t, v_{1,t}, v_{2,t}) \\ &= \frac{1}{n} \sum_{\substack{u^{t-1} \\ u^{t+1}}} \sum_{\substack{v_1^{t-1}, v_{1,t+1}^n \\ v_2^{t-1}, v_{2,t+1}^n}} \left( \prod_{t=1}^n \mathcal{P}_U(u_t) \right) \mathcal{P}_{M_0M_1M_2|U^n}^\sigma(m_0, m_1, m_2|u^n) \\ &\quad \times \mathcal{P}_{V_1^n|M_0M_1}^{\tau_1}(v_1^n|m_0, m_1) \mathcal{P}_{V_2^n|M_0M_2}^{\tau_2}(v_2^n|m_0, m_2). \end{aligned}$$

**Lemma 4.6.1.** *The distribution  $\mathcal{P}_{UW_0W_1W_2V_1V_2}^{\sigma\tau_1\tau_2}$  has marginal on  $\Delta(U)$  given by  $\mathcal{P}_U$  and satisfies the Markov chain properties*

$$\begin{aligned} (U, V_2) &\text{---} (W_0, W_1) \text{---} V_1; \\ (U, W_1, V_1) &\text{---} (W_0, W_2) \text{---} V_2. \end{aligned}$$

*Proof.* [Lemma 4.6.1] The i.i.d. property of the source ensures that the marginal distribution is  $\mathcal{P}_U$ . By the definition of the decoding functions  $\tau_1$  and  $\tau_2$  we have

$$\begin{aligned} (U_T, V_{2,T}) &\text{---} (M_1, M_0, T) \text{---} V_{1,T}, \\ (U_T, M_1, V_{1,T}) &\text{---} (M_2, M_0, T) \text{---} V_{2,T}. \end{aligned}$$

□ Therefore  $\mathcal{P}_{UW_0W_1W_2V_1V_2}^{\sigma\tau_1\tau_2} = \mathcal{P}_U \mathcal{P}_{W_0|U}^\sigma \mathcal{P}_{W_1|W_0U}^\sigma \mathcal{P}_{W_2|W_0U}^\sigma \mathcal{P}_{V_1|W_0W_1}^{\tau_1} \mathcal{P}_{V_2|W_0W_2}^{\tau_2}$ .

**Lemma 4.6.2.** *For all  $\sigma$ , the distribution  $\mathcal{P}_{W_0W_1W_2|U}^\sigma \in \mathbb{Q}_0(R_0, R_1, R_2)$ .*

*Proof.* [Lemma 4.6.2] We consider an encoding strategy  $\sigma$ , then

$$\lfloor nR_0 \rfloor \geq H(M_0) \geq I(M_0; U^n) \tag{4.86}$$

$$= \sum_{t=1}^n I(U_t; M_0|U^{t-1}) \tag{4.87}$$

$$= nI(U_T; M_0|U^{T-1}, T) \tag{4.88}$$

$$= nI(U_T; M_0, U^{T-1}, T) \tag{4.89}$$

$$\geq nI(U_T; M_0, T) \tag{4.90}$$

$$=nI(U; W_0). \quad (4.91)$$

In fact, (4.88) follows from the introduction of the uniform random variable  $T \in \{1, \dots, n\}$ , (4.89) comes from the i.i.d. property of the source and (4.91) follows from the identification of the auxiliary random variables  $(U, W_2)$ . Similarly,

$$\lfloor nR_1 \rfloor \geq H(M_1) \geq I(U^n; M_1 | M_0) \quad (4.92)$$

$$= \sum_{t=1}^n I(U_t; M_1, M_0 | U^{t-1}) \quad (4.93)$$

$$=nI(U_T; M_1, M_0 | U^{T-1}, T) \quad (4.94)$$

$$\geq nI(U_T; M_1, M_0, T) \quad (4.95)$$

$$\geq nI(U_T; M_1 | M_0, T) \quad (4.96)$$

$$=nI(U; W_1 | W_0). \quad (4.97)$$

Similarly,  $nR_2 \geq nI(U; W_2 | W_0)$ .  $\square$

**Lemma 4.6.3.** *For all  $(\sigma, \tau_1, \tau_2)$  and  $i \in \{1, 2\}$ , we have*

$$c_e^n(\sigma, \tau_1, \tau_2) = \mathbb{E}[c_e(U, V_1, V_2)], \quad (4.98)$$

$$c_i^n(\sigma, \tau_1, \tau_2) = \mathbb{E}[c_i(U, V_1, V_2)]. \quad (4.99)$$

evaluated with respect to  $\mathcal{P}_U \mathcal{P}_{W_0 W_1 W_2 | U}^\sigma \mathcal{P}_{V_1 | W_0 W_1}^{\tau_1} \mathcal{P}_{V_2 | W_0 W_2}^{\tau_2}$ .

Moreover, for each  $(m_0, m_1, m_2, v_1^n, v_2^n)$ , we have

$$C_i^\sigma(m_0, m_1, m_2, v_1^n, v_2^n) = C_i^*(w_0, w_1, w_2, v_1, v_2), \quad (4.100)$$

$$\Psi_i^\sigma(\tau_1, \tau_2, m_0, m_i) = \mathbb{E}_{\mathcal{P}_U} [\Psi_i^*(\mathcal{P}_{V_1 | W_0 W_1}^{\tau_1}, \mathcal{P}_{V_2 | W_0 W_2}^{\tau_2}, w_0, w_i)].$$

*Proof.* [Lemma 4.6.3] By Definition 4.3.2 we have

$$c_e^n(\sigma, \tau_1, \tau_2) = \sum_{\substack{u^n, m_0, m_1, m_2, \\ v_1^n, v_2^n}} \left( \prod_{t=1}^n \mathcal{P}_U(u_t) \right) \mathcal{P}_{M_0 M_1 M_2 | U^n}^\sigma(m_0, m_1, m_2 | u^n) \mathcal{P}_{V_1^n | M_0 M_1}^{\tau_1}(v_1^n | m_0, m_1)$$

$$\times \mathcal{P}_{V_2^n | M_0 M_2}^{\tau_2}(v_2^n | m_0, m_2) \cdot \left[ \frac{1}{n} \sum_{t=1}^n c_e(u_t, v_{1,t}, v_{2,t}) \right] \quad (4.101)$$

$$= \sum_{t=1}^n \sum_{\substack{u_t, m_0, m_1, m_2, \\ v_{1,t}, v_{2,t}}} \mathcal{P}^{\sigma \tau_1 \tau_2}(u_t, m_0, m_1, m_2, t, v_{1,t}, v_{2,t}) c_e(u_t, v_{1,t}, v_{2,t}) \quad (4.102)$$

$$= \mathbb{E}[c_e(U, V_1, V_2)]. \quad (4.103)$$

For all  $(m_0, m_1, m_2, v_1^n, v_2^n)$  and  $i \in \{1, 2\}$  we have

$$C_i^\sigma(m_0, m_1, m_2, v_1^n, v_2^n) = \sum_{u^n} \mathcal{P}_{U^n | M_0 M_1 M_2}^\sigma(u^n | m_0, m_1, m_2) \left[ \frac{1}{n} \sum_{t=1}^n c_i(u_t, v_{1,t}, v_{2,t}) \right] \quad (4.104)$$

$$= \sum_{t=1}^n \sum_{u_t} \mathcal{P}^\sigma(u_t | m_0, m_1, m_2, t) c_i(u_t, v_{1,t}, v_{2,t}) \quad (4.105)$$

$$= \sum_u \mathcal{P}^\sigma(u | w_0, w_1, w_2) c_i(u, v_1, v_2) \quad (4.106)$$

$$= C_i^*(v_1, v_2, w_0, w_1, w_2). \quad (4.107)$$

Moreover,

$$\begin{aligned} \Psi_1^\sigma(\tau_1, \tau_2, m_0, m_1) &= \\ & \sum_{u^n} \left( \prod_{t=1}^n \mathcal{P}_U(u_t) \right) \sum_{m_2} \mathcal{P}^\sigma(m_2 | m_0, m_1) \sum_{v_1^n, v_2^n} \mathcal{P}^{\tau_1}(v_1^n | m_0, m_1) \mathcal{P}^{\tau_2}(v_2^n | m_0, m_2) \\ & \times \left[ \frac{1}{n} \sum_{t=1}^n c_1(u_t, v_{1,t}, v_{2,t}) \right] \end{aligned} \quad (4.108)$$

$$\begin{aligned} &= \sum_{t=1}^n \sum_{m_2, u_t} \mathcal{P}_U(u_t) \mathcal{P}^\sigma(m_0, m_2, u_t, t | m_0, m_1, t) \sum_{v_{1,t}, v_{2,t}} \mathcal{P}^{\tau_1, t}(v_{1,t} | m_0, m_1, t) \mathcal{P}^{\tau_2, t}(v_{2,t} | m_0, m_2, t) \\ & \times [c_1(u_t, v_{1,t}, v_{2,t})]. \end{aligned} \quad (4.109)$$

$$\begin{aligned} &= \sum_{m_2, u_t, t} \mathcal{P}_U(u_t) \mathcal{P}^\sigma(m_2, t | m_0, m_1, t) \sum_{v_{1,t}, v_{2,t}} \mathcal{P}^{\tau_1, t}(v_{1,t} | m_0, m_1, t) \mathcal{P}^{\tau_2, t}(v_{2,t} | m_0, m_2, t) \times \\ & C_1^*(v_{1,t}, v_{2,t}, m_0, m_1, m_2, t) \end{aligned} \quad (4.110)$$

$$\begin{aligned} &= \sum_{w_2, u} \mathcal{P}_U(u) \mathcal{P}^\sigma(w_0, w_2 | w_0, w_1) \sum_{v_1, v_2} \mathcal{P}^{\tau_1, t}(v_1 | w_0, w_1) \mathcal{P}^{\tau_2, t}(v_2 | w_0, w_2) C_1^*(v_1, v_2, w_0, w_1, w_2) \end{aligned} \quad (4.111)$$

$$= \mathbb{E}_{\mathcal{P}_U} [\Psi_1^*(\mathcal{P}_{V_1 | W_0 W_1}^{\tau_1, t}, \mathcal{P}_{V_2 | W_0 W_2}^{\tau_2, t}, w_0, w_1)]. \quad (4.112)$$

□

**Lemma 4.6.4.** *For all  $\sigma$ , we have*

$$\begin{aligned} \mathbb{BNE}(\mathcal{P}_{W_0W_1W_2|U}^\sigma) = & \left\{ (\mathcal{Q}_{V_1|W_0W_1}, \mathcal{Q}_{V_2|W_0W_2}), \exists(\tau_1, \tau_2), \tau_1 \in BR_1^\sigma(\tau_2), \tau_2 \in BR_2^\sigma(\tau_1), \right. \\ & \left. \mathcal{Q}_{V_1|W_0W_1} = \mathcal{P}_{V_1|W_0W_1}^{\tau_1}, \mathcal{Q}_{V_2|W_0W_2} = \mathcal{P}_{V_2|W_0W_2}^{\tau_2} \right\}. \end{aligned} \quad (4.113)$$

*Proof.* [Lemma 4.6.4] Fix  $\sigma$  and let  $(\mathcal{Q}_{V_1|W_0W_1}, \mathcal{Q}_{V_2|W_0W_2}) \in \mathbb{BNE}(\mathcal{P}_{W_0W_1W_2|U}^\sigma)$ . We consider  $(\tau_1, \tau_2)$  such that

$$\begin{aligned} \mathcal{P}_{V_1^n|M_0M_1}^{\tau_1}(v_1^n|m_0, m_1) &= \prod_{t=1}^n \mathcal{Q}_{V_1|W_0W_1}(v_{1,t}|m_0, m_1, t), \\ \mathcal{P}_{V_2^n|M_0M_2}^{\tau_2}(v_2^n|m_0, m_2) &= \prod_{t=1}^n \mathcal{Q}_{V_2|W_0W_2}(v_{2,t}|m_0, m_2, t). \end{aligned}$$

Then  $\forall(w_0, w_1, v_1) = (m_0, m_1, t, v_{1,t})$ ,

$$\mathcal{P}_{V_1|W_0W_1}^{\tau_1}(v_1|w_0, w_1) = \mathcal{P}_{V_1|W_0W_1}^{\tau_1}(v_{1,t}|m_0, m_1, t) \quad (4.114)$$

$$= \sum_{v_1^{t-1}, v_{1,t+1}^n} \prod_{s=1}^n \mathcal{Q}_{V_1|W_0W_1}(v_{1,s}|m_0, m_1, s) \quad (4.115)$$

$$= \mathcal{Q}_{V_1|W_0W_1}(v_{1,t}|m_0, m_1, t) \sum_{v_1^{t-1}, v_{1,t+1}^n} \prod_{s \neq t} \mathcal{Q}_{V_1|W_0W_1}(v_{1,s}|m_0, m_1, s) \quad (4.116)$$

$$= \mathcal{Q}_{V_1|W_0W_1}(v_{1,t}|m_0, m_1, t) = \mathcal{Q}_{V_1|W_0W_1}(v_1|w_0, w_1). \quad (4.117)$$

Assume that  $\forall \tau_2, \tau_1 \notin BR_1^\sigma(\tau_2)$ , then there exists  $\bar{\tau}_1 \neq \tau_1$  such that

$$\begin{aligned} \mathbb{E}_{\mathcal{P}_U^\sigma} [\Psi_1^*(\mathcal{P}_{V_1|W_0W_1}^{\bar{\tau}_1}, \mathcal{P}_{V_2|W_0W_2}^{\tau_2}, w_0, w_1)] &= \Psi_1^\sigma(\bar{\tau}_1, \tau_2, m_0, m_1) < \Psi_1^\sigma(\tau_1, \tau_2, m_0, m_1) \\ &= \mathbb{E}_{\mathcal{P}_{U|W_0W_1W_2}^\sigma} [\Psi_1^*(\mathcal{P}_{V_1|W_0W_1}^{\bar{\tau}_1}, \mathcal{P}_{V_2|W_0W_2}^{\tau_2}, w_0, w_1)], \forall \tau_2, \end{aligned}$$

which contradicts  $(\mathcal{Q}_{V_1|W_0W_1}, \mathcal{Q}_{V_2|W_0W_2}) \in \mathbb{BNE}(\mathcal{P}_{W_0W_1W_2|U}^\sigma)$ . Therefore,  $\tau_1 \in BR_1^\sigma(\tau_2)$  and thus  $\mathcal{Q}_{V_1|W_0W_1}$  belongs to the right-hand side of (4.113).

Therefore for all  $m_0, m_1$ ,

$$\begin{aligned}
\Psi_1^\sigma(\tau_1, \tau_2, m_0, m_1) &= \mathbb{E}_{\mathcal{P}_U^\sigma} [\Psi_1^*(\mathcal{P}_{V_1|W_0W_1}^{\tau_1}, \mathcal{P}_{V_2|W_0W_2}^{\tau_2}, w_0, w_1)] \\
&= \min_{\mathcal{Q}_{V_1|W_0W_1}} \mathbb{E}_{\mathcal{P}_U^\sigma} [\Psi_1^*(\mathcal{Q}_{V_1|W_0W_1}, \mathcal{Q}_{V_2|W_0W_2}, w_0, w_1)] \\
&\leq \min_{\tilde{\tau}_1} \mathbb{E}_{\mathcal{P}_U^\sigma} [\Psi_1^*(\mathcal{P}_{V_1|W_0W_1}^{\tilde{\tau}_1}, \mathcal{P}_{V_2|W_0W_2}^{\tau_2}, w_0, w_1)] \\
&= \min_{\tilde{\tau}_1} \Psi_1^\sigma(\tilde{\tau}_1, \tau_2, m_0, m_1).
\end{aligned}$$

Hence  $\tau_1 \in BR_1^\sigma(\tau_2)$ . Similarly,  $\tau_2 \in BR_2^\sigma(\tau_1)$ . The other inclusion is direct.  $\square$

For any strategy  $\sigma$ , we have

$$\max_{\tau_1, \tau_2} c_e^n(\sigma, \tau_1, \tau_2) = \max_{\tau_1, \tau_2} \mathbb{E}_{\substack{\mathcal{P}_{W_0W_1W_2|U}^\sigma \\ \mathcal{P}_{V_1|W_0W_1}^{\tau_1} \mathcal{P}_{V_2|W_0W_2}^{\tau_2}}} [c_e(U, V_1, V_2)] \quad (4.118)$$

$$= \max_{\substack{(\mathcal{Q}_{V_1|W_0W_1}, \mathcal{Q}_{V_2|W_0W_2}) \in \\ \text{BNE}(\mathcal{P}_{W_0W_1W_2|U}^\sigma)}} \mathbb{E}_{\substack{\mathcal{P}_{W_0W_1W_2|U}^\sigma \\ \mathcal{Q}_{V_1|W_0W_1} \mathcal{Q}_{V_2|W_0W_2}}} [c_e(U, V_1, V_2)] \quad (4.119)$$

$$\geq \inf_{\substack{\mathcal{Q}_{W_0W_1W_2|U} \\ \in \hat{\mathcal{Q}}_0(R_0, R_1, R_2)}} \max_{\substack{(\mathcal{Q}_{V_1|W_0W_1}, \mathcal{Q}_{V_2|W_0W_2}) \in \\ \text{BNE}(\mathcal{Q}_{W_0W_1W_2|U})}} \mathbb{E} [c_e(U, V_1, V_2)] \quad (4.120)$$

$$= \hat{\Gamma}_{GW}(R_0, R_1, R_2). \quad (4.121)$$

Equations (4.118) and (4.119) follow from Lemma 4.6.3, whereas (4.120) comes from Lemma 4.6.2. This concludes the converse proof of Theorem 4.3.1.

## 4.7 Conclusion

Strategic communication between one encoder and two decoders is under study. We propose a non-cooperative Gray-Wyner setting, in which the encoder is endowed with a cost function that depends on the actions of both decoders and on the state, and the decoders are cost-dependent, i.e. the cost function of each decoder depends on the action of the other decoder. The encoder commits to an encoding before observing the source. Upon revelation of the encoder's commitment, a Bayesian game is played among the decoders. This game admits perfect Bayes-Nash equilibria. Henceforth, the strategic goal of the encoder consists

of minimizing its long-run cost subject to the information constraints imposed by the Gray-Wyner network, and the equilibrium strategies of the decoders in the subsequent Bayesian game. Using the Gray-Wyner lossy source coding that satisfies the incentive constraints of both decoders, we derive two single-letter characterizations of the encoder's optimal cost, one with an optimal signal that satisfies a Markov chain and forms an upper bound for the encoder's optimal long-run cost in our achievability statement, and one with a general joint distribution that serves as a lower bound to the long-run cost of the encoder in the converse statement. Our achievability proof provides a technical novelty which consists of identifying a series of intermediate Bayesian games starting from the block-game until reaching the one-shot game. We characterize each game, and we analyze the encoder's expected cost that corresponds to the set of Bayes-Nash equilibria of each game. This analysis requires controlling the Bayesian beliefs of the decoders which can be done under specific considerations in selecting the codebook sequences.

# 5

## CASCADE MULTIPLE DESCRIPTION NETWORK FOR STRATEGIC COMMUNICATION

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*In this chapter, we examine particular cases of Strategic Communication between an encoder and a decoder through a strategic relay.*

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## 5.1 Introduction

We consider a three-node cascade network with an encoder, a relay and a decoder, having distinct objectives captured by cost functions. In such a cascade network, agents choose their respective strategies sequentially, as a response to the former agent's strategy and in a way to influence the decision of the latter agent in the network. We assume the encoder commits to a strategy before the communication takes place. Upon revelation of the encoding strategy, the relay commits to a strategy and reveals it. The communication starts, the source sequence is drawn and processed by the encoder and relay. Then, the decoder observes a sequences of symbols, updates its Bayesian posterior beliefs accordingly, and takes the optimal action. This is an extension of the Bayesian persuasion problem in the Game Theory literature. In this work, we provide an information-theoretic approach to study the fundamental limit of the strategic communication via three-node cascade network. Our goal is to characterize the optimal strategies of the encoder, the relay and the decoder, and study the asymptotic behavior of the encoder's minimal long-run cost function.

Cascade source coding consists of compressing a source sequence through an intermediate or relay node which then reconstructs the source and transmits it to the next node. Yamamoto (1981) considered the source coding problem for cascade and branching communication systems, and established the region of achievable rates for cascade systems and bounds for the branching systems. Lossy source coding for cascade communication systems was also considered in Vasudevan et al. (2006) where both the relay and the terminal node have access to side information and wish to reconstruct the source with certain fidelities. The cascade source coding framework is a building block for various compression and communication scenarios. It captures major aspects of multihop coding for wireless communication networks Broch et al. (1998), including cellular communication Lin and Hsu (2000), ad hoc networks Johnson et al. (2002), and sensor networks Woo et al. (2003).

Information transmission via relays can also involve endogenous constraints. In Kosenko (2020), a strategic designer transmits state-dependent signals to a decoder through a relay who can garble these signals. Comparing outcomes with and without a relay, the encoder seems to never benefit from mediation, while the decoder might. Arieli et al. (2022) studied

a similar model of communication between an informed encoder and a decoder through a sequence of relays. Authors characterize the encoder's optimal value for any number of relays and show that the presence of a relay is never beneficial for the sender. In this regard, the last model investigated in this thesis, considers a Cascade channel where information travels from a strategic encoder to a decoder through a relay. However, we assume the encoder commits to an encoding and announces its commitment. Then, the relay commits to a strategy accordingly. Henceforth, our model takes the relay as well as the amount of information transmitted as exogenous parameters. Note here that the order of commitment between the encoder and the relay is crucial. In our setting, the encoder is the Stackleberg leader, who commits first and declares its commitment to the other players. If the relay was assumed to commit to a strategy before the encoder, then the problem boils down to a strategic joint source-channel coding of Shannon, like the one investigated in [Le Treust and Tomala \(2019a\)](#).

## 5.2 Cascade multiple description coding

The three-node Cascade communication system depicted in [Figure 5.1](#) is a lossy source coding setup in which an encoder transmits a description  $M_1$  at rate  $R_1$  to a relay which reconstructs the source with  $V_1^n$  and relays a description  $M_2$  at rate  $R_2$  to the decoder which reconstructs the source with  $V_2^n$ . All alphabets  $\mathcal{U}$ ,  $\mathcal{V}_1$  and  $\mathcal{V}_2$  are discrete and the source  $U$  is memoryless. More details can be found in ([El Gamal and Kim, 2011](#), chapter 20.2.1).

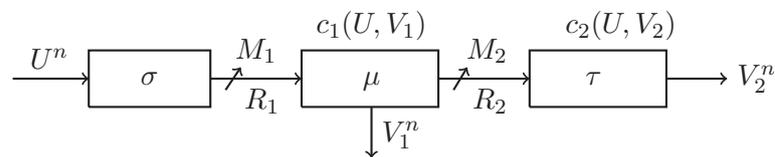


Figure 5.1: Cascade multiple description coding.

**Definition 5.2.1.** Let  $(R_1, R_2) \in \mathbb{R}_+^2$ . An  $(R_1, R_2, n)$  code for the three-node Cascade multiple description network consists of the following:

- Two index sets  $\{1, 2, \dots, 2^{\lfloor nR_1 \rfloor}\}$ , and  $\{1, 2, \dots, 2^{\lfloor nR_2 \rfloor}\}$ .

- An encoding function  $\sigma : \mathcal{U}^n \rightarrow \{1, 2, \dots, 2^{\lfloor nR_1 \rfloor}\}$  which assigns a message  $m_1 \in \{1, 2, \dots, 2^{\lfloor nR_1 \rfloor}\}$  to each sequence  $u^n \in \mathcal{U}^n$ .
- A relay function  $\mu : \{1, 2, \dots, 2^{\lfloor nR_1 \rfloor}\} \rightarrow \{1, 2, \dots, 2^{\lfloor nR_2 \rfloor}\} \times \mathcal{V}_1^n$
- A decoding function  $\tau : \{1, 2, \dots, 2^{\lfloor nR_2 \rfloor}\} \rightarrow \mathcal{V}_2^n$ .

**Definition 5.2.2.** Let  $c_1 : \mathcal{U} \times \mathcal{V}_1 \rightarrow \mathbb{R}$  and  $c_2 : \mathcal{U} \times \mathcal{V}_2 \rightarrow \mathbb{R}$  two single-letter cost functions of the relay and the decoder respectively. We define the long-run cost functions  $c_1^n(\sigma, \mu)$  and  $c_2^n(\sigma, \tau)$  as follows:

$$c_1^n(\sigma, \mu) = \sum_{u^n, v_1^n} \mathcal{P}^{\sigma, \mu}(u^n, v_1^n) \frac{1}{n} \sum_{t=1}^n c_1(u_t, v_{1,t}), \quad (5.1)$$

$$c_2^n(\sigma, \tau) = \sum_{u^n, v_2^n} \mathcal{P}^{\sigma, \tau}(u^n, v_2^n) \frac{1}{n} \sum_{t=1}^n c_2(u_t, v_{2,t}). \quad (5.2)$$

**Definition 5.2.3.** A rate-cost quadruple  $(R_1, R_2, C_1, C_2) \in \mathbb{R}_+^4$  is said to be achievable if there exists a sequence of  $(R_1, R_2, n)$  codes for the three-node Cascade network such that  $\limsup_{n \rightarrow \infty} \mathbb{E}(c_i^n(U^n, V_i^n)) \leq C_i$ , for  $i \in \{1, 2\}$ .

For a rate pair  $(R_1, R_2)$ , we denote by  $\mathcal{K}(R_1, R_2)$  the closure of the set of pairs  $(C_1, C_2)$  such that quadruples  $(R_1, R_2, C_1, C_2)$  are achievable.

Let  $W_1 \in \mathcal{W}_1$ , and  $W_2 \in \mathcal{W}_2$  two auxiliary random variables such that  $|\mathcal{W}_1| = |\mathcal{V}_1|$  and  $|\mathcal{W}_2| = |\mathcal{V}_2|$ .

**Theorem 5.2.1** (Three-node Cascade multiple description coding). Let  $U$  be a discrete memoryless source, distributed according to a probability distribution  $\mathcal{P}_U$  over  $\mathcal{U}$ . Let  $\mathcal{V}_1$  and  $\mathcal{V}_2$  be two discrete alphabets and  $c_i : \mathcal{U} \times \mathcal{V}_i \mapsto \mathbb{R}$  for  $i \in \{1, 2\}$  be two cost functions. Then, given a rate pair  $(R_1, R_2)$  we have

$$\mathcal{K}(R_1, R_2) = \left\{ \left( \mathbb{E}_{\mathcal{Q}}(c_1(U, V_1)), \mathbb{E}_{\mathcal{Q}}(c_2(U, V_2)) \right), \quad \mathcal{Q} \in \mathcal{Q}(R_1, R_2) \right\} \quad (5.3)$$

where

$$\mathcal{Q}(R_1, R_2) = \left\{ \mathcal{Q}_{W_1 W_2 | U}, \quad I(U; W_2) \leq R_2, I(U; W_1, W_2) \leq R_1 \right\} \quad (5.4)$$

### 5.3 Modified model for strategic information cascade

We propose a strategic model for the Bayesian persuasion game via the three-node cascade multiple description network. We modify the original model by assuming that the encoder is endowed with a cost function which depends on the state  $U$  and the decoder's action  $V$ . We also assume that the relay, unlike the standard model presented in [section 5.2](#), does not draw a sequence  $V_1^n$  upon receiving the message from the encoder, but only relays the message  $M_1$  of the encoder. The cost function of the relay in our setting depends on the action  $V$  taken by the decoder and on the state  $U$ . We still assume that the encoder commits to an encoding function  $\sigma$  before observing the source. Upon the announcement of the encoder's commitment, the relay commits to a strategy  $\mu$  and announces it as well. In the following, we formulate the Strategic cascade lossy source coding problem.

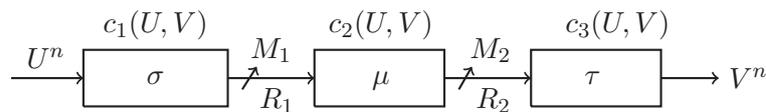


Figure 5.2: Strategic source coding for cascade channel with successive commitment.

Let  $n \in \mathbb{N}^*$ , and  $(R_1, R_2) \in \mathbb{R}_+^2$  denote the rate pair. We assume that the information source  $U$  follows the i.i.d probability distribution  $\mathcal{P}_U \in \Delta(\mathcal{U})$ .

**Definition 5.3.1.** *The coding strategies  $\sigma$ ,  $\mu$  and  $\tau$  of the encoder, relay, and decoder respectively are defined by*

$$\sigma : \mathcal{U}^n \longrightarrow \Delta(\{1, \dots, 2^{\lfloor nR_1 \rfloor}\}), \quad (5.5)$$

$$\mu : \{1, \dots, 2^{\lfloor nR_1 \rfloor}\} \longrightarrow \Delta(\{1, \dots, 2^{\lfloor nR_2 \rfloor}\}), \quad (5.6)$$

$$\tau : \{1, \dots, 2^{\lfloor nR_2 \rfloor}\} \longrightarrow \Delta(\mathcal{V}^n). \quad (5.7)$$

The stochastic coding strategies  $(\sigma, \mu, \tau)$  induce a joint probability distribution  $\mathcal{P}^{\sigma\mu\tau} \in$

$\Delta(\mathcal{U}^n \times \{1, 2, \dots, 2^{\lfloor nR_1 \rfloor}\} \times \{1, 2, \dots, 2^{\lfloor nR_2 \rfloor}\} \times \mathcal{V}^n)$  defined for all  $(u^n, m_1, m_2, v^n)$  by

$$\mathcal{P}^{\sigma\mu\tau}(u^n, m_1, m_2, v^n) = \left( \prod_{t=1}^n \mathcal{P}_U(u_t) \right) \sigma(m_1|u^n) \mu(m_2|m_1) \tau(v^n|m_2). \quad (5.8)$$

**Definition 5.3.2.** We consider arbitrary single-letter cost functions  $c_1 : \mathcal{U} \times \mathcal{V} \rightarrow \mathbb{R}$  for the encoder  $\mathcal{E}$ ,  $c_2 : \mathcal{U} \times \mathcal{V} \rightarrow \mathbb{R}$  for the relay, and  $c_3 : \mathcal{U} \times \mathcal{V} \rightarrow \mathbb{R}$  for the decoder. The long-run cost functions are defined for  $i \in \{1, 2, 3\}$  by

$$c_i^n(\sigma, \mu, \tau) = \mathbb{E}_{\sigma, \mu, \tau} \left[ \frac{1}{n} \sum_{t=1}^n c_i(U_t, V_t) \right] \quad (5.9)$$

$$= \sum_{u^n, v^n} \mathcal{P}_{U^n V^n}^{\sigma, \mu, \tau}(u^n, v^n) \cdot \left[ \frac{1}{n} \sum_{t=1}^n c_i(u_t, v_t) \right]. \quad (5.10)$$

In the above equations,  $\mathcal{P}_{U^n V^n}^{\sigma\mu\tau}$  denote the marginal distributions over the sequences  $(U^n, V^n)$  of  $\mathcal{P}^{\sigma\mu\tau}$  defined in (5.8) over the  $n$ -sequences  $(U^n, M_1, M_2, V^n)$ .

**Definition 5.3.3.** For any strategy pair  $(\sigma, \mu)$ , of the encoder and the relay, the set of best-response strategies  $\tau$  of the decoder is defined by

$$\mathbb{A}_3(\sigma, \mu) = \arg \min_{\tau} c_3^n(\sigma, \mu, \tau). \quad (5.11)$$

For any strategy  $\sigma$ , the set of pairs  $(\mu, \tau)$  which are best-responses for the relay and the decoder is defined by

$$\mathbb{A}_2(\sigma) = \arg \min_{\substack{(\mu, \tau) \text{ s.t.} \\ \tau \in \mathbb{A}_3(\sigma, \mu)}} c_2^n(\sigma, \mu, \tau). \quad (5.12)$$

Therefore, the encoder has to solve the following coding problem,

$$\Gamma_{cas}^n(R_1, R_2) = \inf_{\sigma} \max_{\substack{(\mu, \tau) \in \\ \mathbb{A}_2(\sigma)}} c_1^n(\sigma, \mu, \tau). \quad (5.13)$$

**Remark 5.3.1.** In order to get a robust solution concept, we assume that the encoder solves the problem for the worst case scenario, i.e. if more than one pair of strategies are available in  $\mathbb{A}_2(\sigma)$ , we consider the one that maximizes the encoder's cost.

The operational significance of (5.13) corresponds to the persuasion game that is played in the following steps:

- The encoder chooses, announces the encoding  $\sigma$ .
- knowing  $\sigma$ , the relay chooses, announces the encoding  $\mu$ .
- Knowing  $(\sigma, \mu)$ , the decoder compute its best-response strategy  $\tau$ .
- Sequences  $U^n$  are drawn i.i.d with distribution  $\mathcal{P}_U$ .
- Message sequence  $M_1$  are encoded according to  $\sigma_{M_1|U^n}$ .
- Message sequence  $M_2$  are encoded according to  $\mu_{M_2|M_1}$ .
- The decoder observes  $M_2$  and draws  $V^n$  according to  $\tau_{V^n|M_2}$ .
- Cost functions  $c_1^n(\sigma, \mu, \tau)$ ,  $c_2^n(\sigma, \mu, \tau)$ ,  $c_3^n(\sigma, \mu, \tau)$  are computed.

## 5.4 Cooperative scenario

Consider the cooperative communication scenario where  $c_1 = c_2 = c_3$ , and all three agents share the objective of minimizing the same cost function. This setting corresponds to the standard coding setup of a cascade multiple description network [El Gamal and Kim \(2011\)](#), under the assumption that the relay does not reconstruct the source, but only relays a message  $M_2$ , and the cost functions of the three players depend on the source and the decoder's action.

Consider an auxiliary random variables  $W \in \mathcal{W}$  such that  $|\mathcal{W}| = |\mathcal{U}|$ . The set  $\mathbb{Q}_0^c(R_1, R_2)$  of target distributions is defined by:

$$\mathbb{Q}_0^c(R_1, R_2) = \{\mathcal{Q}_{W_2|U}; \min(R_1, R_2) \geq I(U; W_2)\}. \quad (5.14)$$

The single-letter best-response of the decoder is defined by:

$$\mathbb{Q}_3^c(\mathcal{Q}_{W_2|U}) = \arg \min_{\mathcal{Q}_{V|W_2}} \mathbb{E}[c_3(U, V)]. \quad (5.15)$$

The single-letter optimal cost  $\Gamma_{cas}^c(R_1, R_2)$  of the encoder is given by

$$\Gamma_{cas}^c(R_1, R_2) = \inf_{\mathcal{Q}_{W_2|U} \in \mathcal{Q}_0^c(R_1, R_2)} \max_{\mathcal{Q}_{V|W_2} \in \mathcal{Q}_3^c(\mathcal{Q}_{W_2|U})} \mathbb{E}[c_3(U, V)]. \quad (5.16)$$

**Theorem 5.4.1.** *Let  $(R_1, R_2) \in \mathbb{R}_+^2$ . If  $c_1 = c_2 = c_3$ , then*

$$\lim_{n \rightarrow \infty} \Gamma_{cas}^n(R_1, R_2) = \inf_{n \in \mathbb{N}^*} \Gamma_{cas}^n(R_1, R_2) = \Gamma_{cas}^c(R_1, R_2). \quad (5.17)$$

### Achievability of Theorem 5.4.1

The proof of Theorem 5.4.1 can be directly derived from the proof of (El Gamal and Kim, 2011, Theorem 20.4) by considering the relay's estimate to be a constant and its role is to only transition the message received from the encoder.

Fix probability distributions  $\mathcal{Q}_{W_2|U} \in \mathcal{Q}_0^c(R_1, R_2)$ . There exists  $\eta > 0$  such that

$$\min(R_1, R_2) = I(U; W_2) + \eta, \quad (5.18)$$

Let  $(R_1, R_2) \in \mathbb{R}_+^2$ . Randomly and independently generate  $2^{nR_2}$  sequences  $w_2^n(m_2)$  for each  $m_2 \in \{1, \dots, 2^{\lfloor nR_2 \rfloor}\}$ , according to the i.i.d distribution  $\mathcal{Q}_{W_2^n} = \prod_{t=1}^n \mathcal{Q}_{W_2}(w_{2t})$ .

Encoder  $\mathcal{E}$  observes  $u^n$  and looks in the codebook for a sequence  $w_2^n(m_2)$  such that  $(u^n, w_2^n(m_2)) \in \mathcal{T}_\delta^n(\mathcal{P}_U \mathcal{Q}_{W_2|U})$ . If such a jointly typical tuple doesn't exist, the source encoder sets  $w_2^n$  to  $(1, 1, \dots, 1)$ . Then, it sends  $m_2$  to the relay. The relay observes  $m_2$  and sends it to the decoder. Then, the decoder  $\mathcal{D}$  observes  $m_2$  and declares  $v^n$  according to  $\tau$ .

The error event is given by  $\mathcal{F} = \{(U^n, W_2^n(M_2)) \notin \mathcal{T}_\delta^n(\mathcal{P}_U \mathcal{Q}_{W_2|U})\}$ .

By the covering lemma, we get  $\mathcal{P}(\mathcal{F})$  tends to zero as  $n \rightarrow \infty$  since

$$R_2 \geq I(U; W_2) + \eta. \quad (5.19)$$

### Converse of [Theorem 5.4.1](#)

The converse proof of [Theorem 5.4.1](#) follows the same line of arguments as in the previous chapters. The full proof can be found in [Appendix F](#).

## 5.5 Bayesian persuasion with unrestricted communication

We assume that the communication is perfect and unrestricted, i.e.  $R_1 = R_2 = \log |\mathcal{U}|$ . Fix  $\mathcal{Q}_{W_1|U} \mathcal{Q}_{W_2|W_1}$ . Consider two auxiliary random variables  $W_1 \in \mathcal{W}_1$  and  $W_2 \in \mathcal{W}_2$  such that  $|\mathcal{W}_1| = |\mathcal{W}_2| = |\mathcal{U}|$  and

$$U \text{---} W_1 \text{---} W_2, \quad W_1 \text{---} W_2 \text{---} V.$$

The single-letter best-responses are defined by:

$$\begin{aligned} \mathbb{Q}_3(\mathcal{Q}_{W_1|U}, \mathcal{Q}_{W_2|W_1}) &= \arg \min_{\mathcal{Q}_{V|W_2}} \mathbb{E}[c_3(U, V)], \\ \mathbb{Q}_2(\mathcal{Q}_{W_1|U}) &= \arg \min_{\substack{(\mathcal{Q}_{W_2|W_1}, \mathcal{Q}_{V|W_2}), \\ \mathcal{Q}_{V|W_2} \in \mathbb{Q}_3(\mathcal{Q}_{W_1|U}, \mathcal{Q}_{W_2|W_1})}} \mathbb{E}[c_2(U, V)], \end{aligned}$$

The single-letter optimal cost  $\Gamma_e$  of the encoder is given by

$$\Gamma_e = \inf_{\mathcal{Q}_{W_1|U}} \max_{\substack{\mathcal{Q}_{W_2|W_1}, \mathcal{Q}_{V|W_2} \\ \in \mathbb{Q}_2(\mathcal{Q}_{W_1|U})}} \mathbb{E}_{\mathcal{P}_U \mathcal{Q}_{W_1|U}} [c_1(U, V)].$$

**Theorem 5.5.1.** *If  $R_1 = R_2 = \log |\mathcal{U}|$ , then*

$$\lim_{n \rightarrow \infty} \Gamma_e^n = \inf_{n \in \mathbb{N}^*} \Gamma_e^n = \Gamma_e \tag{5.20}$$

### Achievability of Theorem 5.5.1

Let  $R_1 = R_2 = \log |\mathcal{U}|$ , and fix a joint probability distribution  $\mathcal{Q}_{W_1|U} \mathcal{Q}_{W_2|W_1}$ . The sequences  $U^n$  are drawn according to the i.i.d. distribution  $\mathcal{P}_{U^n}$ . Randomly and independently generate  $2^{nR_1}$  sequences  $w_1^n(m_1)$  for each  $m_1 \in \{1, \dots, 2^{\lfloor nR_1 \rfloor}\}$ , according to the i.i.d. distribution  $\mathcal{Q}_{W_1^n|U^n} = \prod_{t=1}^n \mathcal{Q}_{W_1|U}(w_{1t}|u_t)$ . Similarly, generate  $2^{nR_2}$  sequences  $w_2^n(m_2)$  for  $m_2 \in \{1, \dots, 2^{\lfloor nR_2 \rfloor}\}$  randomly and independently according to the i.i.d. distribution  $\mathcal{Q}_{W_2^n|W_1^n} = \prod_{t=1}^n \mathcal{Q}_{W_2|W_1}(w_{2t}|w_{1t})$ .

Since  $R_1 = \log |\mathcal{U}| = \log |\mathcal{W}_1|$  and  $R_2 = \log |\mathcal{U}| = \log |\mathcal{W}_2|$ , encoder  $\mathcal{E}$  observes  $u^n$  and looks in the codebook for the corresponding sequences  $w_1^n(m_1)$  and sends  $m_1$  to the relay. The relay observes  $m_1$  and sends  $m_2$  to the decoder. Then, the decoder  $\mathcal{D}$  observes  $m_2$  and declares  $v^n$  according to  $\tau$ .

### Converse Proof of Theorem 5.5.1

Given a triple  $(\sigma, \mu, \tau)$  and a random variable  $T$  uniformly distributed over  $\{1, 2, \dots, n\}$  and independent of  $(U^n, M_1, M_2, V^n)$ . We identify the auxiliary random variables  $W_1 = (M_1, T)$ ,  $W_2 = M_2$ ,  $(U, V) = (U_T, V_T)$ , distributed according to  $\mathcal{P}_{UW_1W_2V}^{\sigma\mu\tau}$  defined for all  $(u, w_1, w_2, v) = (u_t, x_1, x_2, t, v_t)$  by

$$\begin{aligned} \mathcal{P}_{UW_1W_2V}^{\sigma\mu\tau}(u, w_1, w_2, v) &= \mathcal{P}_{U_T W_1 W_2 T V_T}^{\sigma\mu\tau}(u_t, x_1, x_2, t, v_t) \\ &= \frac{1}{n} \sum_{\substack{u^{t-1} \\ u^{t+1}}} \sum_{\substack{x_1^{t-1}, x_1^n \\ x_1^{t+1}, x_2^{t+1}}} \sum_{\substack{v^{t-1}, v^{t+1} \\ v^n}} \left( \prod_{t=1}^n \mathcal{P}_U(u_t) \right) \mathcal{P}_{M_1|U^n}^\sigma(m_1|u^n) \\ &\quad \times \mathcal{P}_{M_2|M_1}^\mu(m_2|m_1) \mathcal{P}_{V^n|M_2}^\tau(v^n|m_2). \end{aligned}$$

**Lemma 5.5.1.** *The distribution  $\mathcal{P}_{UW_1W_2V}^{\sigma\mu\tau}$  has marginal on  $\Delta(\mathcal{U})$  given by  $\mathcal{P}_U$  and satisfies the following Markov chain property*

$$U \text{ --- } W_1 \text{ --- } W_2, \quad W_1 \text{ --- } W_2 \text{ --- } V.$$

*Proof.* [Lemma 5.5.1] The i.i.d. property of the source ensures that the marginal distribution is  $\mathcal{P}_U$ . By the definition of the coding functions  $\sigma$ ,  $\mu$  and  $\tau$  we have

$$\begin{aligned} (U_T) \text{---} (M_1, T) \text{---} M_2, \\ (M_1, T) \text{---} M_2 \text{---} V_T. \end{aligned}$$

□ Therefore  $\mathcal{P}_{UW_1W_2V}^{\sigma\mu\tau} = \mathcal{P}_U \mathcal{P}_{W_1|U}^\sigma \mathcal{P}_{W_2|W_1}^\mu \mathcal{P}_{V|W_2}^\tau$ .

**Lemma 5.5.2.** For all  $(\sigma, \tau_1, \tau_2)$  and  $i \in \{1, 2, 3\}$ , we have

$$c_i^n(\sigma, \mu, \tau) = \mathbb{E}[c_i(U, V)], \quad (5.21)$$

evaluated with respect to  $\mathcal{P}_U \mathcal{P}_{W_1|U}^\sigma \mathcal{P}_{W_2|W_1}^\mu \mathcal{P}_{V|W_2}^\tau$ . Moreover for all  $\sigma, \mu$  we have

$$\mathbb{Q}_3(\mathcal{P}_{W_1|U}^\sigma, \mathcal{P}_{W_2|W_1}^\mu) = \left\{ \mathcal{Q}_{V|W_2}, \exists \tau \in \mathbb{A}_3(\sigma, \mu), \mathcal{Q}_{V|W_2} = \mathcal{P}_{V|W_2}^\tau \right\}, \quad (5.22)$$

$$\begin{aligned} \mathbb{Q}_2(\mathcal{P}_{W_1|U}^\sigma) = \left\{ (\mathcal{Q}_{W_2|W_1}, \mathcal{Q}_{V|W_2}), \exists (\mu, \tau) \in \mathbb{A}_2(\sigma), \mathcal{Q}_{W_2|W_1} = \mathcal{P}_{W_2|W_1}^\mu, \right. \\ \left. \mathcal{Q}_{V|W_2} = \mathcal{P}_{V|W_2}^\tau \right\}. \quad (5.23) \end{aligned}$$

*Proof.* [Lemma 5.5.2] By Definition 5.3.2 we have for  $i \in \{1, 2, 3\}$

$$\begin{aligned} c_i^n(\sigma, \mu, \tau) &= \sum_{\substack{u^n, m_1, \\ m_2, v^n}} \left( \prod_{t=1}^n \mathcal{P}_U(u_t) \right) \mathcal{P}_{M_1|U^n}^\sigma(m_1|u^n) \mathcal{P}_{M_2|M_1}^\mu(m_2|m_1) \\ &\quad \times \mathcal{P}_{V^n|M_2}^\tau(v^n|m_2) \cdot \left[ \frac{1}{n} \sum_{t=1}^n c_i(u_t, v_t) \right] \quad (5.24) \end{aligned}$$

$$= \sum_{t=1}^n \sum_{\substack{u_t, x_1, \\ x_2, t, v_t}} \mathcal{P}^{\sigma, \mu, \tau}(u_t, x_1, x_2, t, v_t) \times c_i(u_t, v_t) = \mathbb{E}[c_i(U, V)]. \quad (5.25)$$

Given  $\mathcal{Q}_{V|W_2} \in \mathbb{Q}_3(\mathcal{P}_{W_1|U}^\sigma, \mathcal{P}_{W_2|W_1}^\mu)$ , we consider  $\tau$  such that

$$\mathcal{P}_{V^n|M_2}^\tau(v^n|m_2) = \prod_{t=1}^n \mathcal{Q}_{V|W_2}(v_{1,t}|m_2). \quad (5.26)$$

Given  $(\mathcal{Q}_{W_2|W_1}, \mathcal{Q}_{V|W_2}) \in \mathbb{Q}_2(\mathcal{P}_{W_1|U}^\sigma)$ , we consider  $(\mu, \tau)$  such that

$$\mathcal{P}_{M_2|M_1}^\mu(m_2|m_1) = \prod_{t=1}^n \mathcal{Q}_{W_2|W_1}(m_2|m_1, t), \quad (5.27)$$

$$\mathcal{P}_{V^n|M_2}^\tau(v^n|m_2) = \prod_{t=1}^n \mathcal{Q}_{V|W_2}(v_{1,t}|m_2). \quad (5.28)$$

Therefore

$$c_3^n(\sigma, \mu, \tau) = \mathbb{E}_{\substack{\mathcal{P}_{W_1|U}^\sigma \\ \mathcal{P}_{W_2|W_1}^\mu, \mathcal{Q}_{V|W_2}}} [c_3(U, V)] \quad (5.29)$$

$$= \min_{\mathcal{P}_{V|W_2}} \mathbb{E}_{\substack{\mathcal{P}_{W_1|U}^\sigma \\ \mathcal{P}_{W_2|W_1}^\mu, \mathcal{P}_{V|W_2}}} [c_3(U, V)] \quad (5.30)$$

$$\leq \min_{\tilde{\tau}} \mathbb{E}_{\substack{\mathcal{P}_{W_1|U}^\sigma \\ \mathcal{P}_{W_2|W_1}^\mu, \mathcal{P}_{V|W_2}^{\tilde{\tau}}}} [c_3(U, V)] \quad (5.31)$$

$$= \min_{\tilde{\tau}} c_3^n(\sigma, \mu, \tilde{\tau}), \quad (5.32)$$

hence  $\tau \in \mathbb{A}_3(\sigma, \mu)$ . Similarly,

$$c_2^n(\sigma, \mu, \tau) = \mathbb{E}_{\substack{\mathcal{P}_{W_1|U}^\sigma \\ \mathcal{Q}_{W_2|W_1}, \mathcal{Q}_{V|W_2}}} [c_2(U, V)] \quad (5.33)$$

$$= \min_{(\mathcal{P}_{W_2|W_1}, \mathcal{P}_{V|W_2})} \mathbb{E}_{\substack{\mathcal{P}_{W_1|U}^\sigma \\ \mathcal{P}_{W_2|W_1}, \mathcal{P}_{V|W_2}}} [c_2(U, V)] \quad (5.34)$$

$$\leq \min_{(\tilde{\mu}, \tilde{\tau})} \mathbb{E}_{\substack{\mathcal{P}_{W_1|U}^\sigma \\ \mathcal{P}_{W_2|W_1}^{\tilde{\mu}}, \mathcal{P}_{V|W_2}^{\tilde{\tau}}}} [c_2(U, V)] \quad (5.35)$$

$$= \min_{(\tilde{\mu}, \tilde{\tau})} c_2^n(\sigma, \mu, \tilde{\tau}), \quad (5.36)$$

and thus  $(\mu, \tau) \in \mathbb{A}_2(\sigma)$ . The other inclusions are direct and the same arguments imply (5.23) and (5.22).  $\square$

For any strategy  $\sigma$ , we have

$$\max_{\mu, \tau} c_1^n(\sigma, \mu, \tau) = \max_{\mu, \tau} \mathbb{E}_{\substack{\mathcal{P}_{W_1|U}^\sigma \\ \mathcal{P}_{W_2|W_1}^\mu, \mathcal{P}_{V|W_2}^\tau}} [c_1(U, V)] \quad (5.37)$$

$$\geq \max_{\substack{\mathcal{Q}_{W_2|W_1}, \mathcal{Q}_{V|W_2} \\ \in \mathbb{Q}_2(\mathcal{Q}_{W_1|U})}} \mathbb{E}_{\substack{\mathcal{P}_{W_1|U}^\sigma \\ \mathcal{Q}_{W_2|W_1}, \mathcal{Q}_{V|W_2}}} [c_1(U, V)] \quad (5.38)$$

$$\geq \inf_{\mathcal{Q}_{W_1|U}} \max_{\substack{\mathcal{Q}_{W_2|W_1}, \mathcal{Q}_{V|W_2} \\ \in \mathcal{Q}_2(\mathcal{Q}_{W_1|U})}} \mathbb{E}[c_1(U, V)] \quad (5.39)$$

$$= \Gamma_e(R_1, R_2). \quad (5.40)$$

Equations (5.37) and (5.38) comes from Lemma 5.5.2, whereas (5.39) comes from taking the infimum over  $\mathcal{Q}_{W_1|U}$ . This concludes the converse proof of Theorem 5.5.1.

## 5.6 Locally restricted communication

### Relay's Restriction

Assume that the encoder can send messages at large enough rate  $R_1 = \log |\mathcal{U}|$ , but the relay sends at a fixed smaller rate  $R_2$ . Fix  $\mathcal{Q}_{W_1|U}$ . In this setting, the single-letter best-responses are defined by:

$$\mathcal{Q}_3(\mathcal{Q}_{W_1|U}, \mathcal{Q}_{W_2|W_1}) = \arg \min_{\mathcal{Q}_{V|W_2}} \mathbb{E}[c_3(U, V)], \quad (5.41)$$

$$\mathcal{Q}_2^r(\mathcal{Q}_{W_1|U}) = \arg \min_{\substack{(\mathcal{Q}_{W_2|W_1}, \mathcal{Q}_{V|W_2}) \text{ s.t. } R_2 \geq I(W_1; W_2), \\ \mathcal{Q}_{V|W_2} \in \mathcal{Q}_3(\mathcal{Q}_{W_1|U}, \mathcal{Q}_{W_2|W_1})}} \mathbb{E}[c_2(U, V)], \quad (5.42)$$

The single-letter optimal cost  $\Gamma_{cas}^r(R_2)$  of the encoder is given by

$$\Gamma_{cas}^r(R_2) = \inf_{\mathcal{Q}_{W_1|U}} \max_{\substack{\mathcal{Q}_{W_2|W_1}, \mathcal{Q}_{V|W_2} \\ \in \mathcal{Q}_2(\mathcal{Q}_{W_1|U})}} \mathbb{E}[c_1(U, V)]. \quad (5.43)$$

**Theorem 5.6.1.** *Let  $R_2 \in \mathbb{R}^+$ . If  $R_1 = \log |\mathcal{U}|$ , then*

$$\lim_{n \rightarrow \infty} \Gamma_{cas}^n(R_2) = \inf_{n \in \mathbb{N}^*} \Gamma_{cas}^n(R_2) = \Gamma_{cas}^r(R_2). \quad (5.44)$$

The proof of Theorem 5.6.1 relies on the lossy source coding at the relay by considering the source to be the observed message which is uniformly drawn from the codebook of size

$2^{nR_1}$ . This slight modification does not affect the condition on the covering lemma as the coding will only depend on the size  $2^{nR_2}$  of the message set of the relay.

### Encoder's Restriction

Now assume that  $R_2 = \log |\mathcal{U}|$ , i.e. the encoder is restricted to a limited amount of bits per transmission, but the relay can transmit with no information constraints. Therefore, the set  $\mathcal{Q}_0^e(R_1)$  of the encoder's target distributions is given by

$$\mathcal{Q}_0^e(R_1) = \{\mathcal{Q}_{W_1|U}; R_1 \geq I(U; W_1)\}. \quad (5.45)$$

Single-letter best-responses are defined by:

$$\mathcal{Q}_3(\mathcal{Q}_{W_1|U}, \mathcal{Q}_{W_2|W_1}) = \arg \min_{\mathcal{Q}_{V|W_2}} \mathbb{E}[c_3(U, V)], \quad (5.46)$$

$$\mathcal{Q}_2^e(\mathcal{Q}_{W_1|U}) = \arg \min_{\substack{(\mathcal{Q}_{W_2|W_1}, \mathcal{Q}_{V|W_2}), \\ \mathcal{Q}_{V|W_2} \in \mathcal{Q}_3(\mathcal{Q}_{W_1|U}, \mathcal{Q}_{W_2|W_1})}} \mathbb{E}[c_2(U, V)], \quad (5.47)$$

The single-letter optimal cost  $\Gamma_{cas}^*(R_1)$  of the encoder is given by

$$\tilde{\Gamma}_{cas}(R_1) = \inf_{\mathcal{Q}_{W_1|U} \in \mathcal{Q}_0^e(R_1)} \max_{\substack{\mathcal{Q}_{W_2|W_1}, \mathcal{Q}_{V|W_2} \\ \in \mathcal{Q}_2^e(\mathcal{Q}_{W_1|U})}} \mathbb{E}[c_1(U, V)]. \quad (5.48)$$

**Theorem 5.6.2.** Let  $R_1 \in \mathbb{R}^+$ . If  $R_2 = \log |\mathcal{U}|$ , then

$$\lim_{n \rightarrow \infty} \Gamma_{cas}^n(R_2) = \inf_{n \in \mathbb{N}^*} \Gamma_{cas}^n(R_2) = \tilde{\Gamma}_{cas}(R_2). \quad (5.49)$$

## 5.7 Locally cooperating agents

Consider now that either the relay and the encoder or the relay and the decoder are cooperating. In other words, we assume that either  $c_1 = c_2$  or  $c_2 = c_3$  holds.

### Encoder-Relay Cooperation

Assume  $c_1 = c_2$  the encoder and the relay are cooperating. The encoder will reveal information using the maximal rate  $R_1$ . The set  $\mathbb{Q}_0^s(R_1, R_2)$  of target distributions:

$$\mathbb{Q}_0^s(R_1, R_2) = \{\mathcal{Q}_{W_1|U}; R_1 \geq I(U; W_1)\}. \quad (5.50)$$

Single-letter best-responses are defined by:

$$\mathbb{Q}_3(\mathcal{Q}_{W_1|U}, \mathcal{Q}_{W_2|W_1}) = \arg \min_{\mathcal{Q}_{V|W_2}} \mathbb{E}[c_3(U, V)], \quad (5.51)$$

$$\mathbb{Q}_2^s(\mathcal{Q}_{W_1|U}) = \arg \min_{\substack{(\mathcal{Q}_{W_2|W_1}, \mathcal{Q}_{V|W_2}), R_2 \geq I(W_1; W_2) \\ \mathcal{Q}_{V|W_2} \in \mathbb{Q}_3(\mathcal{Q}_{W_1|U}, \mathcal{Q}_{W_2|W_1})}} \mathbb{E}[c_2(U, V)], \quad (5.52)$$

The single-letter optimal cost  $\Gamma_{cas}^s(R_1, R_2)$  of the encoder is given by

$$\Gamma_{cas}^s(R_1, R_2) = \inf_{\mathcal{Q}_{W_1|U} \in \mathbb{Q}_0^s(R_1, R_2)} \max_{\substack{\mathcal{Q}_{W_2|W_1}, \mathcal{Q}_{V|W_2} \\ \in \mathbb{Q}_2^s(\mathcal{Q}_{W_1|U})}} \mathbb{E}[c_1(U, V)]. \quad (5.53)$$

**Theorem 5.7.1.** Let  $(R_1, R_2) \in \mathbb{R}_+^2$ . If  $c_1 = c_2$ , then

$$\lim_{n \rightarrow \infty} \Gamma_{cas}^n(R_1, R_2) = \inf_{n \in \mathbb{N}^*} \Gamma_{cas}^n(R_1, R_2) = \Gamma_{cas}^s(R_1, R_2). \quad (5.54)$$

The proof relies on considering that the relay observes the source as the encoder can fully reveal it, and the Bayesian persuasion setting between the relay and the decoder.

### Relay-Decoder Cooperation

Assume now that the decoder cooperates with the relay because  $c_2 = c_3$ . The set  $\mathbb{Q}_0^d(R_1, R_2)$  of target distributions:

$$\mathbb{Q}_0^d(R_1, R_2) = \{\mathcal{Q}_{W_1|U}; R_1 \geq I(U; W_1)\}. \quad (5.55)$$

Single-letter best-responses are defined by:

$$\mathcal{Q}_3(\mathcal{Q}_{W_1|U}, \mathcal{Q}_{W_2|W_1}) = \arg \min_{\mathcal{Q}_{V|W_2}} \mathbb{E}[c_3(U, V)], \quad (5.56)$$

$$\mathcal{Q}_2^d(\mathcal{Q}_{W_1|U}) = \arg \min_{\substack{(\mathcal{Q}_{W_2|W_1}, \mathcal{Q}_{V|W_2}), R_2 \geq I(W_1; W_2) \\ \mathcal{Q}_{V|W_2} \in \mathcal{Q}_3(\mathcal{Q}_{W_1|U}, \mathcal{Q}_{W_2|W_1})}} \mathbb{E}[c_2(U, V)]. \quad (5.57)$$

The single-letter optimal cost  $\Gamma_{cas}^d(R_1, R_2)$  of the encoder is given by

$$\Gamma_{cas}^d(R_1, R_2) = \inf_{\mathcal{Q}_{W_1|U} \in \mathcal{Q}_0^d(R_1, R_2)} \max_{\substack{\mathcal{Q}_{W_2|W_1}, \mathcal{Q}_{V|W_2} \\ \in \mathcal{Q}_2^d(\mathcal{Q}_{W_1|U})}} \mathbb{E}[c_1(U, V)]. \quad (5.58)$$

**Theorem 5.7.2.**

$$\lim_{n \rightarrow \infty} \Gamma_{cas}^n(R_1, R_2) = \inf_{n \in \mathbb{N}^*} \Gamma_{cas}^n(R_1, R_2) = \Gamma_{cas}^d(R_1, R_2). \quad (5.59)$$

The proof follows by considering the relay and the decoder as one party, and lossy source coding at the encoder.

## 5.8 Illustrating example

Assume that  $R_1 = R_2 = \log |\mathcal{U}|$ , and  $c_1 = c_3$ . We illustrate the problem using a binary source information  $\mathcal{U} = \{u_0, u_1\}$ , binary channel inputs  $\mathcal{X}_2 = \{x_{2,0}, x_{2,1}\}$ ,  $\mathcal{X}_1 = \{x_{1,0}, x_{1,1}\}$  and binary action set  $\mathcal{V} = \{v_0, v_1\}$ . We denote by the parameter  $p_0 \in [0, 1]$ , the prior belief  $p_0 = \mathcal{P}_U(u_1)$  of the decoder about the state  $U$ . Single-letter cost functions of the encoder, the relay and the decoder are given in the tables below.

Table 5.1:  $c_1(u, v)$

	$v_0$	$v_1$
$u_0$	9	0
$u_1$	4	10

Table 5.2:  $c_2(u, v)$

	$v_0$	$v_1$
$u_0$	1	0
$u_1$	1	0

Table 5.3:  $c_3(u, v)$

	$v_0$	$v_1$
$u_0$	9	0
$u_1$	4	10

Let  $(\alpha, \beta), (\gamma, \delta),$  and  $(\varepsilon, \eta) \in [0, 1]^2$ . The expected cost functions of the encoder, the relay and the decoder are respectively given in Figure 5.4 for  $p_0 = 0.4$ . The threshold  $\gamma$  corresponds to the belief  $\mathcal{P}_U(u_1)$  about the state at which the decoder changes action from

its default action  $v_1$  (which corresponds to its prior belief  $p_0$ ), to action  $v_0$ . In the absence of the relay, i.e. when the relaying strategy is to fully reveal the encoder's message ( $\delta = \gamma = 0$ , or  $\delta = \gamma = 1$ ), the encoder's optimal cost  $\Gamma_e$  can be obtained using the convexification method represented by the green dashed line in Figure 5.4, evaluated at the prior  $p_0$ . This case corresponds to the setting of [Kamenica and Gentzkow \(2011\)](#).

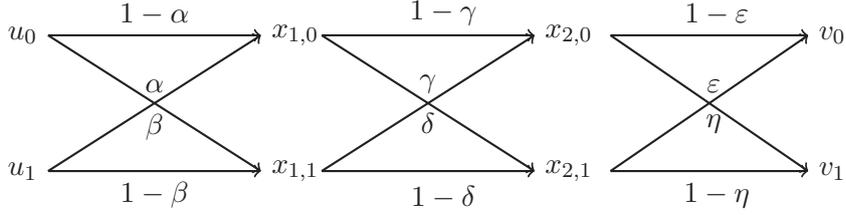


Figure 5.3: Encoders' joint strategies  $\sigma_1$  and  $\sigma_2$  and decoder's strategy  $\sigma_3$ .

Using Baye's rule, we compute the posterior beliefs of the decoder about the state, and about the observation of the relay updated after observing the symbol realization  $x_{2,0}$  and  $x_{2,1}$  from the relay as follows,

$$q_0^1 = \mathcal{P}(u_1|x_{2,0}) = \frac{\mathcal{P}(u_1, x_{2,0})}{\mathcal{P}(x_{2,0})} = \frac{(\beta(1-\gamma) + (1-\beta)\delta) \cdot p_0}{(\beta(1-\gamma) + (1-\beta)\delta) \cdot p_0 + ((1-\alpha)(1-\gamma) + \alpha\delta) \cdot (1-p_0)}, \quad (5.60)$$

$$q_1^1 = \mathcal{P}(u_1|x_{2,1}) = \frac{\mathcal{P}(u_1, x_{2,1})}{\mathcal{P}(x_{2,1})} = \frac{(\beta\gamma + (1-\beta)(1-\delta)) \cdot p_0}{(\beta\gamma + (1-\beta)(1-\delta)) \cdot p_0 + ((1-\alpha)\gamma + \alpha(1-\delta)) \cdot (1-p_0)}. \quad (5.61)$$

Given an encoding  $(\alpha, \beta)$  of the encoder, a Bayesian persuasion game  $G(\alpha, \beta)$  takes place between the relay and the decoder. We denote by  $p_1(\alpha, \beta) = \mathcal{P}_{X_1}(x_{1,1})$ , the belief of the decoder about  $X_1$  before receiving the relay's message. Thus,

$$p_1(\alpha, \beta) = \mathcal{P}_{X_1}(x_{1,1}) = \sum_u \mathcal{P}_U(u) \cdot \mathcal{P}_{X_1|U}(x_{1,1}|u) = (1-p_0)\alpha + p_0(1-\beta). \quad (5.62)$$

We define the costs  $c_i^x(x_1, v)$ ,  $i \in \{1, 2, 3\}$  of all players as functions of the channel input  $X_1$  and the decoder's action  $V$  as follows

$$c_1^x(x_1, v) = \sum_u \mathcal{P}_{U|X_1}(u|x_1) c_1(u, v), \quad \forall x_1, v \quad (5.63)$$

$$c_2^x(x_1, v) = \sum_u \mathcal{P}_{U|X_1}(u|x_1) c_2(u, v), \quad \forall x_1, v \quad (5.64)$$

$$c_3^x(x_1, v) = \sum_u \mathcal{P}_{U|X_1}(u|x_1) c_3(u, v). \quad \forall x_1, v, \quad (5.65)$$

where the distributions  $\mathcal{P}_{U|X_1}(u_0|x_{1,0})$  and  $\mathcal{P}_{U|X_1}(u_1|x_{1,1})$  are computed as follows

$$\mathcal{P}(u_0|x_{1,0}) = \frac{\mathcal{P}(u_0, x_{1,0})}{\mathcal{P}(x_{1,0})} = \frac{(1 - \alpha) \cdot (1 - p_0)}{\beta \cdot p_0 + (1 - \alpha) \cdot (1 - p_0)}, \quad (5.66)$$

$$\mathcal{P}(u_1|x_{1,1}) = \frac{\mathcal{P}(u_1, x_{1,1})}{\mathcal{P}(x_{1,1})} = \frac{(1 - \beta) \cdot p_0}{(1 - \beta) \cdot p_0 + \alpha \cdot (1 - p_0)}. \quad (5.67)$$

By setting  $\mathbb{E}_{\mathcal{P}_{X_1}}[c_3^x(x_1, v_0)] = \mathbb{E}_{\mathcal{P}_{X_1}}[c_3^x(x_1, v_1)]$ , the threshold  $g(\alpha, \beta) = \mathcal{P}_{X_1}(x_{1,1})$  at which the decoder changes action is computed as follows

$$g(\alpha, \beta) = \frac{2 - \mathcal{P}(u_0|x_{1,0}) \cdot 5}{5 \cdot (1 - \mathcal{P}(u_1|x_{1,1}) - \mathcal{P}(u_0|x_{1,0}))} \quad (5.68)$$

**Remark 5.8.1.** *The order of commitment is crucial in this setting. If the relay commits to a strategy  $(\gamma, \delta)$  and announces it before the encoder commits to and announces a strategy, thus the problem boils down to the one tackled in [Le Treust and Tomala \(2019a\)](#).*

**Remark 5.8.2.** • *If  $\alpha + \beta = 1$ , then the source  $U$  and channel's input  $X_1$  are independent. In that case, the decoder will stick to its prior belief  $p_0$  disregarding any information received from the relay, and play its default action  $v_1$ . The corresponding costs are  $10 \times 0.4 = 4$  for the encoder, and 1 for the relay.*

- *If  $\alpha = \beta = 1$  or  $\alpha = \beta = 0$ , the encoder is fully revealing the source to the relay. In this case, the communication can be considered as a one-to-one Bayesian persuasion game between the relay and the decoder. The optimal cost of the relay is given by  $C_2^*$  in [Figure 5.4](#).*

We define the single-letter cost of the encoder and the relay as a function of the belief parameter  $q \in [0, 1]$  about  $X_1$  as follows:

$$c_1^x(q) = \sum_{x_1} q(x_1) c_1^x(x_1, v^*(q(x_1))), \quad (5.69)$$

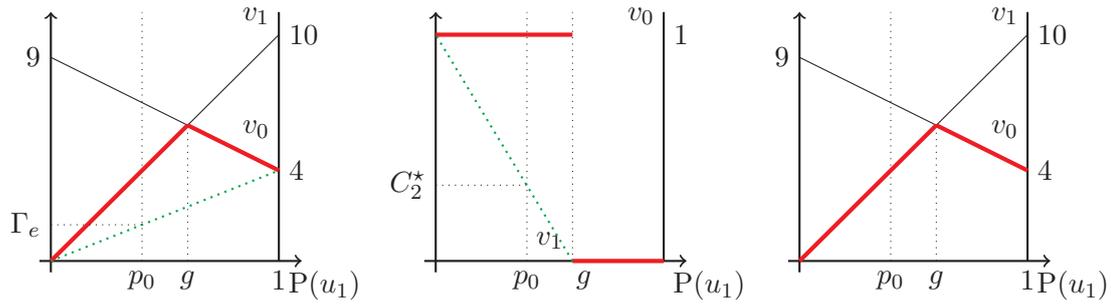


Figure 5.4: Expected cost functions with  $p_0 = 0.4$ ,  $g = 0.6$ ,  $C_2^* = 0.33$  and  $\Gamma_e = 1.6$  for large enough rates  $R_1, R_2 \geq \log |\mathcal{U}| = 1$ .

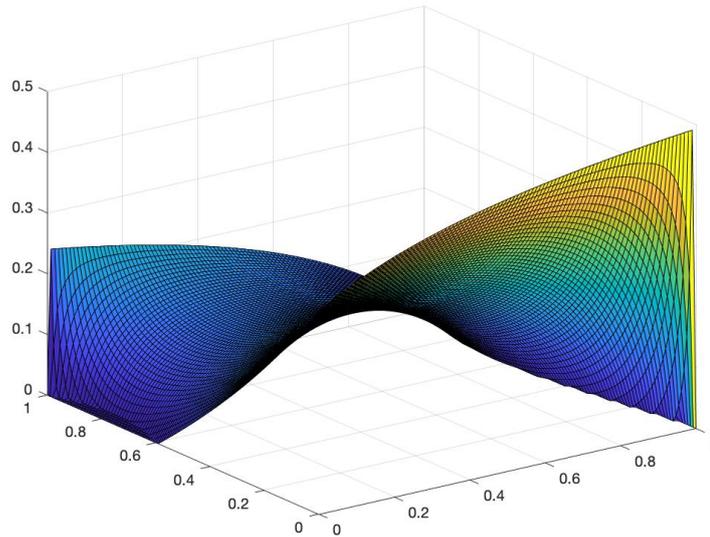


Figure 5.5: The relay's optimal cost  $C_2^*(\alpha, \beta)$  for  $p_0 = 0.4$  and  $\alpha, \beta \in [0, 1]$ .

$$c_2^x(q) = \sum_{x_1} q(x_1) c_2^x(x_1, v^*(q(x_1))), \quad (5.70)$$

where

$$v^*(q(x_1)) = \arg \min_v \sum_{x_1} q(x_1) c_3^x(x_1, v). \quad (5.71)$$

For a given  $(\alpha, \beta)$ , the optimal cost of the relay can be computed using the convexification method as follows:

$$C_2^*(\alpha, \beta) = \inf_{(\lambda, q)_{x_2}} \left\{ \sum_{x_2} \lambda_{x_2} c_2^x(q_{x_2}), \sum_k \lambda_k^2 = 1, \sum_{x_2} \lambda_{x_2} q_{x_2} = p_1(\alpha, \beta) \right\}. \quad (5.72)$$

The optimal single-letter cost of the encoder is therefore given by

$$\Gamma_{cas}^*(R_1, R_2) = \inf_{\alpha, \beta} \left\{ \sum_{x_2} \lambda_{x_2} c_1^x(q_{x_2}), (\lambda_{x_2}, q_{x_2})_{x_2} \in \arg \min_{(\lambda_{x_2}, q_{x_2})_{x_2}} C_2^*(\alpha, \beta) \right\}. \quad (5.73)$$

## 5.9 Conclusion

We modify the base model of a three-node cascade by assuming that the encoder, the relay and the decoder are endowed with distinct and arbitrary cost functions that depend on the state and on the decoder's action. We assume that the encoder commits to and announces an encoding before observing the i.i.d. source. Then, the relay commits to and announces a relaying strategy. This setting differs from the original cooperative setting where only the relay and the decoder are endowed with distortions that depend on the source and their respective reconstructions. We study the fundamental limit of the encoder's long run cost subject to the information constraints and the incentives of the relay and the decoder. We study particular cases by relating them to existing models in the literature. When communication is unrestricted, the problem boils down to two persuasion games of [Kamenica and Gentzkow \(2011\)](#), repeated in  $n$  independent copies. The fully cooperative scenario where players share the same cost functions can be approached using standard cascade multiple description coding. Partial cooperation or partial communication restrictions require a combination of information-theoretic tools with game-theoretic behavior strategies as in [Le Treust and Tomala \(2019a\)](#).

# 6

## CONCLUSION AND PERSPECTIVES

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*Communication systems involving strategic agents with misaligned objectives are being increasingly observed in modern wireless networks. Coding the source in these systems is not only subjected to the information constraints imposed by the channel, but also to the incentive constraints of the rational decoders. We propose a multidisciplinary approach that studies the information-theoretic limits of strategic communication in a game-theoretic framework. The Bayesian persuasion of [Kamenica and Gentzkow \(2011\)](#), investigate the Stackleberg version of the one-shot strategic communication game between a single encoder (the Stackleberg leader) and a decoder (the Stackleberg follower) with no limits on the amount of transmitted information. The encoder commits to an encoding before observing the source and reveals its commitment to the decoders. We consider a large number of identical and independent Bayesian persuasion games under information constraints. We modify the standard lossy source coding setup of [Shannon \(1959\)](#), by assuming that the encoder is endowed with a cost function that depends on the source and on the actions taken by the decoders upon reception of the encoder's signal. The informed encoder transmits description(s) of the source to the decoder(s) in order to minimize its own long-run cost function. We follow the approach of [Le Treust and Tomala \(2019a\)](#) which studies noisy point-to-point Bayesian persuasion, to the multi-user strategic communication in three different settings. First, we consider the Successive Refinement coding setup in which a single encoder communicates a public message to two decoders, and a private message to only one of them. Upon reception of the message from the encoder, each decoder draws the action sequence that minimizes its respective long-run cost functions. Second, we consider the Gray-Wyner communication Network, with a single encoder and two decoders, each observing a public and a private signal from the encoder, and are endowed with cost functions that depend on the actions of both decoders and*

*the state. In this setting, each commitment of the encoder induces a Bayesian game among the decoders which admits perfect Bayes-Nash equilibria. Third, we combine the Bayesian Persuasion model with the Cascade Multiple Description Network, where information is transmitted from the encoder to the decoder through an interested relay. In each of these settings, we study the information-theoretic limits of strategic communication and describe the asymptotic behavior of the encoder's optimal long-run cost function. Using auxiliary random variables, we characterize the encoder's minimal single-letter cost function subject to the optimal compression scheme that satisfies the constraints imposed on the amount of information transmitted in each model, as well as the incentive constraints of the decoders.*

*Further extensions of the models considered in this manuscript could be interesting to investigate. In our communication systems, the encoder is linked to the decoders through perfect links with a fixed and limited rate of information transmission. A noisy channel can be added on each of these links with a conditional distribution that is known by all players. As source-channel separation does not hold in general for networks, optimal compression schemes which capture the incentive constraints of the decoders need to be derived. When the cost of each decoder depends on the action of the other decoder, each commitment of the encoder induces a Bayesian game among the decoders. Since the first decoder in the successive refinement setting has access to the observation of the second decoder, assuming that the cost function of the first decoder depends on the action of the second decoder won't modify our solution. Another interesting extension to the models is to assume that the decoders have access to side information correlated to the state. Under such assumption, the joint distributions would include the side variable and the encoder's expected optimal cost and entropy constraints will be averaged over the side variable. Considering particular distributions of the source, like the Gaussian distribution, could also be interesting to study.*

# 7

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# A

## PROOF OF PROPOSITION 2.4.1

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**Proposition.** *Let  $C = 0$ .*

- *If  $p_0 < \gamma$  and  $\delta \in [0, \frac{p_0 \cdot (\gamma - 1)}{p_0 \cdot (-1 + 2\gamma) - \gamma}] \cup [\frac{\gamma \cdot (1 - p_0)}{p_0 \cdot (1 - 2\gamma) + \gamma}, 1]$ , then  $\Gamma_s > \Gamma_e^*$ .*
- *If  $p_0 \geq \gamma$  then  $\Gamma_e^* \geq \Gamma_s$ .*

*Proof.* Assume  $p_0 < \gamma$  By the feasibility condition (Le Treust and Tomala, 2018, Lemma 2),  $\delta$  belongs to the interval  $[0, 0.5]$  whenever either one of the following conditions is satisfied:

$$p_1 \leq p_0 \leq p_2 \quad \text{or} \quad p_2 \leq p_0 \leq p_1$$

Since the decoder changes action at threshold  $\gamma$  the encoder's utility increases with either  $p_1 > \gamma$  or  $p_2 > \gamma$ . Otherwise, if both  $p_1$  and  $p_2$  are smaller than  $\gamma$  the receiver will not change action and thus the sender will get a utility of 0 Fig. 3(b).

$$p_2 > \gamma \iff \frac{p_0 \cdot (1 - \delta)}{p_0 \cdot (1 - \delta) + (1 - p_0) \cdot \delta} > \gamma \tag{A.1}$$

$$\iff p_0 \cdot (1 - \delta) > \gamma(p_0 \cdot (1 - \delta) + (1 - p_0) \cdot \delta) \tag{A.2}$$

$$\iff \delta(-p_0 + \gamma \cdot p_0 - \gamma + \gamma \cdot p_0) > \gamma \cdot p_0 - p_0 \tag{A.3}$$

$$\iff \delta < \frac{p_0 \cdot (\gamma - 1)}{p_0 \cdot (-1 + 2\gamma) - \gamma} =: \delta^*. \tag{A.4}$$

Symmetrically, one could also tackle the problem as follows:

$$p_1 > \gamma \iff \frac{p_0 \cdot \delta}{p_0 \cdot \delta + (1 - p_0) \cdot (1 - \delta)} > \gamma \tag{A.5}$$

$$\iff p_0 \cdot \gamma > \gamma \cdot \delta + \gamma \cdot (1 - p_0) \cdot (1 - \delta) \tag{A.6}$$

$$\iff \delta \cdot (p_0 - 2 \cdot \gamma \cdot 0 + \gamma) > \gamma - \gamma \cdot 0 \tag{A.7}$$

$$\iff \delta > \frac{\gamma \cdot (1 - p_0)}{p_0 \cdot (1 - 2\gamma) + \gamma} =: \delta^{**}. \quad (\text{A.8})$$

Since the channel's capacity is  $C = 0$ , we evaluate the utility of the encoder as a function of  $\delta$  by applying the splitting lemma to  $p_1$  and  $p_2$ . Let  $\lambda = \mathbb{P}(z_{01}) \in [0, 1]$  hence  $1 - \lambda = \mathbb{P}(z_{02})$ . Applying the splitting lemma, we have

$$\lambda \cdot p_1 + (1 - \lambda) \cdot p_2 = p_0 \iff \lambda \cdot (p_1 - p_2) = p_0 - p_2 \quad (\text{A.9})$$

$$\iff \lambda = \frac{p_0 - p_2}{p_1 - p_2} \quad (\text{A.10})$$

$$\iff 1 - \lambda = \frac{p_1 - p_0}{p_1 - p_2} \quad (\text{A.11})$$

The encoder's expected utility  $\Psi_e$  is defined as follows:

$$\Psi_e = [\lambda \cdot 0 + (1 - \lambda) \cdot 1] \cdot \mathbb{1}_{\{p_2 > \gamma\}} \cdot \mathbb{1}_{\{p_1 \leq \gamma\}} + [\lambda \cdot 0 + (1 - \lambda) \cdot 0] \quad (\text{A.12})$$

$$\times \mathbb{1}_{\{p_2 < \gamma\}} \cdot \mathbb{1}_{\{p_1 < \gamma\}} + [\lambda \cdot 1 + (1 - \lambda) \cdot 0] \cdot \mathbb{1}_{\{p_2 \leq \gamma\}} \cdot \mathbb{1}_{\{p_1 > \gamma\}} \quad (\text{A.13})$$

$$= [\lambda \cdot 0 + (1 - \lambda) \cdot 1] \cdot \mathbb{1}_{\{0 \leq \delta < \delta^*\}} + 0 \cdot \mathbb{1}_{\{\delta^* < \delta < \delta^{**}\}} \quad (\text{A.14})$$

$$+ [\lambda \cdot 1 + (1 - \lambda) \cdot 0] \cdot \mathbb{1}_{\{\delta^{**} < \delta \leq 1\}}. \quad (\text{A.15})$$

Hence,  $\Gamma_s = \Gamma_e^* = 0$  for  $\delta \in [\delta^*, \delta^{**}]$  and otherwise  $\Gamma_s > \Gamma_e^* = 0$ . Now assume  $p_0 \geq \gamma$ . The sender's expected utility  $U_s$  becomes:

$$\begin{aligned} \Psi_e &= [\lambda \cdot 0 + (1 - \lambda) \cdot 1] \cdot \mathbb{1}_{\{p_2 \geq \gamma\}} \cdot \mathbb{1}_{\{p_1 < \gamma\}} + [\lambda \cdot 0 + (1 - \lambda) \cdot 0] \cdot \mathbb{1}_{\{p_2 < \gamma\}} \cdot \mathbb{1}_{\{p_1 < \gamma\}} \\ &\quad + [\lambda \cdot 1 + (1 - \lambda) \cdot 0] \cdot \mathbb{1}_{\{p_2 < \gamma\}} \cdot \mathbb{1}_{\{p_1 \geq \gamma\}} + [\lambda \cdot 1 + (1 - \lambda) \cdot 1] \cdot \mathbb{1}_{\{p_2 \geq \gamma\}} \cdot \mathbb{1}_{\{p_1 \geq \gamma\}} \end{aligned} \quad (\text{A.16})$$

$$= [\lambda \cdot 0 + (1 - \lambda) \cdot 1] \cdot \mathbb{1}_{\{0 \leq \delta \leq \delta^*\}} + [\lambda \cdot 1 + (1 - \lambda) \cdot 0] \cdot \mathbb{1}_{\{\delta^* \leq \delta \leq 1\}} + 1 \cdot \mathbb{1}_{\{\delta^* \leq \delta \leq \delta^{**}\}}. \quad (\text{A.17})$$

Therefore,  $\Gamma_e^* = 1 \geq \Gamma_s \quad \forall \delta \in [0, 0.5]$ .

□

# B

## PROOF OF LEMMA 3.3.1

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**Lemma.** *The sequence  $(n\Gamma_{sr}^n(R_0, R_1))_{n \in \mathbb{N}^*}$  is sub-additive.*

*Proof. [Lemma 3.3.1] Let  $n, m \in \mathbb{Z}$ . We denote by  $\sigma_c^{n+m}$ , the concatenation of the strategies  $\sigma^n, \sigma^m$  where  $\sigma^n$  is implemented over the first  $n$  stages and  $\sigma^m$  is implemented over the last  $m$  stages. For decoder  $i \in \{1, 2\}$ , consider the best responses  $\tau_i^n \in BR_i(\sigma^n)$  and  $\tau_i^m \in BR_i(\sigma^m)$ . Then, the concatenation  $\tau_{i,c}^{n+m}$  of  $\tau_i^n$  and  $\tau_i^m$  is also a best response  $\tau_{i,c}^{n+m} \in BR_i(\sigma_c^{n+m})$ . Therefore, we have the inequality*

$$n\Gamma_{sr}^n(R_0, R_1) + m\Gamma_{sr}^m(R_0, R_1) = \inf_{\sigma^n} \max_{\substack{\tau_1^n \in BR_1(\sigma^n), \\ \tau_2^n \in BR_2(\sigma^n)}} \mathbb{E} \left[ \sum_{t=1}^n c_e(U_t, V_{1,t}, V_{2,t}) \right] \quad (\text{B.1})$$

$$+ \inf_{\sigma^m} \max_{\substack{\tau_1^m \in BR_1(\sigma^m), \\ \tau_2^m \in BR_2(\sigma^m)}} \mathbb{E} \left[ \sum_{t=1}^m c_e(U_t, V_{1,t}, V_{2,t}) \right] \quad (\text{B.2})$$

$$= \inf_{\sigma_c^{n+m}} \max_{\substack{\tau_1^{n+m} \in BR_1(\sigma_c^{n+m}), \\ \tau_2^{n+m} \in BR_2(\sigma_c^{n+m})}} \mathbb{E} \left[ \sum_{t=1}^{n+m} c_e(U_t, V_{1,t}, V_{2,t}) \right] \quad (\text{B.3})$$

$$\geq \inf_{\sigma^{n+m}} \max_{\substack{\tau_1^{n+m} \in BR_1(\sigma^{n+m}), \\ \tau_2^{n+m} \in BR_2(\sigma^{n+m})}} \mathbb{E} \left[ \sum_{t=1}^{n+m} c_e(U_t, V_{1,t}, V_{2,t}) \right] \quad (\text{B.4})$$

$$= (n+m)\Gamma_{sr}^{n+m}(R_0, R_1), \quad (\text{B.5})$$

where the notation  $\sigma_c^{n+m}$  stands for the encoding strategies obtained by concatenation.  $\square$

# C

## PROOF OF LEMMA 4.5.6

**Lemma.** For all  $w_0^n, w_1^n, w_2^n, w_0, w_1, w_2$ , we have

$$\lim_{n \rightarrow \infty} \mathbb{E} \left[ \frac{1}{n} \sum_{t=1}^n D(\mathcal{P}_{W_{2,t}}^{w_0^n, w_1^n} \| \mathcal{Q}_{W_2}^{w_0, w_1}) \mid E_\delta = 0 \right] = 0, \quad (\text{C.1})$$

$$\lim_{n \rightarrow \infty} \mathbb{E} \left[ \frac{1}{n} \sum_{t=1}^n D(\mathcal{P}_{W_{1,t}}^{w_0^n, w_2^n} \| \mathcal{Q}_{W_1}^{w_0, w_2}) \mid E_\delta = 0 \right] = 0. \quad (\text{C.2})$$

*Proof.* In the following, we control the beliefs of decoder  $\mathcal{D}_2$  about the type of decoder  $\mathcal{D}_1$ . The analogous case can be proven following similar arguments. We denote the codebook by  $\mathcal{C}$ , and for each  $m_0$ , we denote the inner codebook by  $\mathcal{C}^1(m_0) = \{w_1^n(m_0, m_1) \in \mathcal{C}, m_1 \in \{1, \dots, 2^{\lfloor nR_1 \rfloor}\}\}$ , and  $\mathcal{C}^2(m_0) = \{w_2^n(m_0, m_2) \in \mathcal{C}, m_2 \in \{1, \dots, 2^{\lfloor nR_2 \rfloor}\}\}$ . For each  $m_0 \in \{1, \dots, 2^{\lfloor nR_0 \rfloor}\}$ ,  $w_1 \in \mathcal{W}_1$ ,  $w_2 \in \mathcal{W}_2$ ,  $t \in \{1, \dots, n\}$  and  $\delta > 0$ , we introduce the following sets

$$\mathcal{A}_1^t(w_1 | m_0) = \{w_1^n \in \mathcal{C}(m_0), w_{1,t} = w_1\}, \quad (\text{C.3})$$

$$\mathcal{A}_2^t(w_2 | m_0) = \{w_2^n \in \mathcal{C}(m_0), w_{2,t} = w_2\}, \quad (\text{C.4})$$

$$\mathcal{J}(m_0) = \left\{ t \in \{1, \dots, n\}, \quad \left\| \mathcal{Q}_{W_1|W_0}(\cdot | w_{0t}) - \frac{|\mathcal{A}_1^t(\cdot | m_0)|}{|\mathcal{C}^1(m_0)|} \right\| > \delta, \right. \\ \left. \left\| \mathcal{Q}_{W_2|W_0}(\cdot | w_{0t}) - \frac{|\mathcal{A}_2^t(\cdot | m_0)|}{|\mathcal{C}^2(m_0)|} \right\| > \delta \right\}, \quad (\text{C.5})$$

where  $\|\cdot\|$  denote the  $L_1$ -norm. We have by the law of large numbers,  $\mathcal{P}(|\mathcal{J}(m_0)| > \delta) \xrightarrow[n \rightarrow \infty]{} 0$ .

For any  $w_1^n, w_0^n$ , and sufficiently small  $\delta > 0$ , and sufficiently large  $n$  we have

$$\mathcal{P}(w_1^n | w_0^n) = \sum_{u^n \in \mathcal{T}_\delta^n(\mathcal{P}_{U|W_0}, w_0^n, w_1^n)} \mathcal{P}(w_1^n, u^n | w_0^n) \quad (\text{C.6})$$

$$= \sum_{u^n \in \mathcal{T}_\delta^n(\mathcal{P}_{U|W_0W_1}, w_0^n, w_1^n)} \mathcal{P}(u^n|w_0^n) \mathcal{P}(w_1^n|u^n, w_0^n) \quad (\text{C.7})$$

$$= \sum_{u^n \in \mathcal{T}_\delta^n(\mathcal{P}_{U|W_0W_1}, w_0^n, w_1^n)} 2^{-nH(U|W_0)} \mathbb{1}(E_\delta = 0) \quad (\text{C.8})$$

$$= |\mathcal{T}_\delta^n(\mathcal{P}_{U|W_0W_1}, w_0^n, w_1^n)| 2^{-nH(U|W_0)} \quad (\text{C.9})$$

$$\simeq 2^{nH(U|W_0W_1)} 2^{-nH(U|W_0)} \quad (\text{C.10})$$

$$= 2^{-nI(U;W_1|W_0)} \quad (\text{C.11})$$

$$= 2^{-n(R_1 - \eta)}. \quad (\text{C.12})$$

For any  $t, w_{1t}, w_0^n$ , and sufficiently small  $\delta > 0$ , we have

$$\mathcal{P}_{W_{1t}|W_0^n}(w_{1t}|w_0^n) = \sum_{w_1^n \in \mathcal{T}_\delta^n(\mathcal{P}_{W_1|W_0}, w_0^n)} \mathcal{P}(w_1^n|w_0^n) \quad (\text{C.13})$$

$$= \sum_{w_1^n \in \mathcal{A}_1^t(w_{1t}|w_0^n)} 2^{-n(R_1 - \eta)} \quad (\text{C.14})$$

$$= |\mathcal{A}_1^t(w_{1t}|w_0^n)| 2^{-n(R_1 - \eta)} \quad (\text{C.15})$$

$$= \frac{|\mathcal{A}_1^t(w_{1t}|w_0^n)|}{|\mathcal{C}^1(w_0^n)|} \quad (\text{C.16})$$

$$= \mathcal{Q}_{W_{1t}|W_{0t}}(w_{1t}|w_{0t}) \quad (\text{C.17})$$

where (C.14) follows from (C.12), and (C.17) follows from (C.5). Therefore, we get

$$\lim_{n \rightarrow \infty} \mathbb{E} \left[ \frac{1}{n} \sum_{t=1}^n D(\mathcal{P}_{W_{1t}|W_0^n}(\cdot|W_0^n) \| \mathcal{Q}_{W_{1t}|W_{0t}}(\cdot|W_{0t})) \middle| E_\delta = 0 \right] \quad (\text{C.18})$$

$$= \lim_{n \rightarrow \infty} \sum_{w_1^n, w_0^n} \mathcal{P}^\sigma(w_1^n, w_0^n | E_\delta = 0) \frac{1}{n} \sum_{t=1}^n \sum_{w_1} \mathcal{P}(w_1|w_0^n) \log_2 \frac{\mathcal{P}(w_1|w_0^n)}{\mathcal{Q}(w_{1t}|w_{0t})} \quad (\text{C.19})$$

$$= 0. \quad (\text{C.20})$$

where (C.20) follows since  $\lim_{n \rightarrow \infty} \log \frac{\mathcal{P}_{W_{1t}|W_0^n}(w_{1t}|w_0^n)}{\mathcal{Q}_{W_{1t}|W_{0t}}(w_{1t}|w_{0t})} = 0$ .  $\square$

# D

## PROOF OF LEMMA 4.5.7

---

**Lemma.** For all  $m_0, m_1, m_2, w_0, w_1, w_2$ , we have

$$\lim_{n \rightarrow \infty} \mathbb{E} \left[ \frac{1}{n} \sum_{t=1}^n D(\mathcal{P}_{U_t}^{m_1 m_2 m_0} \| \mathcal{Q}_U^{w_1 w_2 w_0}) \middle| E_\delta = 0 \right] = 0. \quad (\text{D.1})$$

*Proof.* We denote the Bayesian posterior belief about the state  $\mathcal{P}_{U_t | M_1 M_2 M_0}^\sigma(\cdot | m_1, m_2, m_0) \in \Delta(\mathcal{U})$  by  $\mathcal{P}_{U_t}^{m_1, m_2, m_0}$ . We show that on average, the Bayesian belief is close in KL distance to the target belief  $\mathcal{Q}_{U | W_0 W_1 W_2}$  induced by the single-letter distribution  $\mathcal{Q}_{W_0 W_1 W_2 | U}$ . The indicator of error event  $E_\delta \in \{0, 1\}$  is as given in (4.56). Assuming the distribution  $\mathcal{Q}_{U | W_1 W_2 W_0}$  is fully supported, the beliefs about the state are controlled as follows

$$\mathbb{E} \left[ \frac{1}{n} \sum_{t=1}^n D(\mathcal{P}_{U_t}^{m_1, m_2, m_0} \| \mathcal{Q}_U(\cdot | W_{1t}, W_{2t}, W_{0t})) \middle| E_\delta = 0 \right] \quad (\text{D.2})$$

$$= \sum_{\substack{m_1, m_2, m_0, \\ w_2^n, w_1^n, w_0^n}} \mathcal{P}^{\sigma \tau_1 \tau_2}(m_1, m_2, m_0, w_1^n, w_2^n, w_0^n | E_\delta = 0) \frac{1}{n} \sum_{t=1}^n D(\mathcal{P}_{U_t}^{m_1, m_2, m_0} \| \mathcal{Q}_{U | W_0 W_1 W_2}(\cdot | W_{1t}, W_{2t}, W_{0t})) \quad (\text{D.3})$$

$$= \sum_{\substack{m_1, m_2, m_0, \\ w_2^n, w_1^n, w_0^n}} \mathcal{P}^{\sigma \tau_1 \tau_2}(m_1, m_2, m_0, w_1^n, w_2^n, w_0^n | E_\delta = 0) \frac{1}{n} \sum_{t=1}^n \sum_u \mathcal{P}_{U_t}^{m_1, m_2, m_0}(u) \log_2 \frac{\mathcal{P}_{U_t}^{m_1, m_2, m_0}(u)}{\mathcal{Q}_{U | W_0 W_1 W_2}(u | w_{1t}, w_{2t}, w_{0t})} \quad (\text{D.4})$$

$$= \sum_{\substack{m_1, m_2, m_0, \\ w_2^n, w_1^n, w_0^n}} \mathcal{P}^{\sigma \tau_1 \tau_2}(m_1, m_2, m_0, w_1^n, w_2^n, w_0^n | E_\delta = 0) \frac{1}{n} \sum_{t=1}^n \sum_u \mathcal{P}_{U_t}^{m_1, m_2, m_0}(u) \log_2 \frac{1}{\mathcal{Q}_{U | W_0 W_1 W_2}(u | w_{1t}, w_{2t}, w_{0t})} \\ - \sum_{\substack{m_1, m_2, m_0, \\ w_2^n, w_1^n, w_0^n}} \mathcal{P}^{\sigma \tau_1 \tau_2}(m_1, m_2, m_0, w_1^n, w_2^n, w_0^n | E_\delta = 0) \frac{1}{n} \sum_{t=1}^n \sum_u \mathcal{P}_{U_t}^{m_1, m_2, m_0}(u) \log_2 \frac{1}{\mathcal{P}_{U_t}^{m_1, m_2, m_0}(u)} \quad (\text{D.5})$$

$$\begin{aligned}
&= \frac{1}{n} \sum_{\substack{m_1, m_2, m_0, \\ w_2^n, w_1^n, w_0^n}} \mathcal{P}^{\sigma\tau_1\tau_2}(m_1, m_2, m_0, w_1^n, w_2^n, w_0^n | E_\delta = 0) \sum_{t=1}^n \sum_u \mathcal{P}_{U_t}^{m_1 m_2 m_0}(u) \log_2 \frac{1}{\mathcal{Q}_{U|W_0 W_1 W_2}(u | w_{1t}, w_{2t}, w_{0t})} \\
&\quad - \frac{1}{n} \sum_{t=1}^n H(U_t | M_1, M_2, M_0, E_\delta = 0) \tag{D.6}
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{n} \sum_{u^n, w_1^n, w_2^n, w_0^n} \mathcal{P}^{\sigma\tau_1\tau_2}(u^n, w_1^n, w_2^n, w_0^n | E_\delta = 0) \log_2 \frac{1}{\prod_{t=1}^n \mathcal{Q}_{U|W_0 W_1 W_2}(u_t | w_{1t}, w_{2t}, w_{0t})} \\
&\quad - \frac{1}{n} \sum_{t=1}^n H(U_t | M_1, M_2, M_0, E_\delta = 0) \tag{D.7}
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{n} \sum_{u^n, w_1^n, w_2^n, w_0^n \in \mathcal{T}_\delta^n} \mathcal{P}^{\sigma\tau_1\tau_2}(u^n, w_1^n, w_2^n, w_0^n | E_\delta = 0) \log_2 \frac{1}{\prod_{t=1}^n \mathcal{Q}_{U|W_0 W_1 W_2}(u_t | w_{1t}, w_{2t}, w_{0t})} \\
&\quad - \frac{1}{n} \sum_{t=1}^n H(U_t | M_1, M_2, M_0, E_\delta = 0) \tag{D.8}
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{n} \sum_{\substack{u^n, w_1^n, \\ w_2^n, w_0^n \in \mathcal{T}_\delta^n}} \mathcal{P}^{\sigma\tau_1\tau_2}(u^n, w_1^n, w_2^n, w_0^n | E_\delta = 0) \cdot n(H(U | W_1, W_2, W_0) + \delta) - \frac{1}{n} H(U^n | M_1, M_2, M_0, E_\delta = 0) \\
&\tag{D.9}
\end{aligned}$$

$$\leq \frac{1}{n} I(U^n; M_1, M_2, M_0 | E_\delta = 0) - I(U; W_1, W_2, W_0) + \delta + \frac{1}{n} + \log_2 |\mathcal{U}| \cdot \mathcal{P}^{\sigma\tau_1\tau_2}(E_\delta = 1) \tag{D.10}$$

$$\leq \eta + \delta + \frac{1}{n} + \log_2 |\mathcal{U}| \cdot \mathcal{P}^{\sigma\tau_1\tau_2}(E_\delta = 1). \tag{D.11}$$

- Equation (D.3) comes from the definition of expected K-L divergence.
- Equation (D.4) comes from the definition of K-L divergence.
- Equation (D.5) comes from splitting the logarithm.
- Equation (D.6) follows since:

$$\begin{aligned}
&\sum_{\substack{m_1, m_2, m_0, \\ w_2^n, w_1^n, w_0^n}} \mathcal{P}^{\sigma\tau_1\tau_2}(m_1, m_2, m_0, w_1^n, w_2^n, w_0^n | E_\delta = 0) \frac{1}{n} \sum_{t=1}^n \sum_u \mathcal{P}_{U_t}^{m_1 m_2 m_0}(u) \log_2 \frac{1}{\mathcal{P}_{U_t}^{m_1 m_2 m_0}(u)} \\
&\tag{D.12}
\end{aligned}$$

$$\begin{aligned}
&= \sum_{\substack{m_1, m_2, m_0, \\ w_2^n, w_1^n, w_0^n}} \mathcal{P}^{\sigma\tau_1\tau_2}(m_1, m_2, m_0, w_1^n, w_2^n, w_0^n | E_\delta = 0) \frac{1}{n} \sum_{t=1}^n H(U_t | M_1 = m_1, M_2 = m_2, M_0 = m_0) \\
&\tag{D.13}
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{n} \sum_{t=1}^n \sum_{\substack{m_1, m_2, m_0, \\ w_2^n, w_1^n, w_0^n}} \mathcal{P}(m_1, m_2, m_0, w_1^n, w_2^n, w_0^n | E_\delta = 0) H(U_t | M_1 = m_1, M_2 = m_2, M_0 = m_0) \\
&\tag{D.14}
\end{aligned}$$

$$= \frac{1}{n} \sum_{t=1}^n \sum_{m_1, m_2, m_0} \mathcal{P}^{\sigma\tau_1\tau_2}(m_1, m_2, m_0 | E_\delta = 0) H(U_t | M_1 = m_1, M_2 = m_2, M_0 = m_0) \quad (\text{D.15})$$

$$= \frac{1}{n} \sum_{t=1}^n H(U_t | M_1, M_2, M_0, E_\delta = 0). \quad (\text{D.16})$$

- Equation (D.7) follows since:

$$\sum_{\substack{m_1, m_2, m_0, \\ w_2, w_1^n, w_0^n}} \mathcal{P}^{\sigma\tau_1\tau_2}(m_1, m_2, m_0, w_1^n, w_0^n | E_\delta = 0) \frac{1}{n} \sum_{t=1}^n \sum_u \mathcal{P}_{U_t}^{m_1 m_2 m_0}(u) \log_2 \frac{1}{\mathcal{Q}_{U|W_0 W_1 W_2}(u | w_{1t}, w_{2t}, w_{0t})} \quad (\text{D.17})$$

$$= \frac{1}{n} \sum_{t=1}^n \sum_{\substack{u_t, m_1, m_2, \\ m_0, w_1^n, w_0^n}} \mathcal{P}^{\sigma\tau_1\tau_2}(u_t, m_1, m_2, m_0, w_1^n, w_0^n | E_\delta = 0) \log_2 \frac{1}{\mathcal{Q}_{U|W_0 W_1 W_2}(u_t | w_{1t}, w_{2t}, w_{0t})} \quad (\text{D.18})$$

$$= \frac{1}{n} \sum_{t=1}^n \sum_{\substack{u^n, m_1, m_2, \\ m_0, w_1^n, w_0^n}} \mathcal{P}^{\sigma, \tau_1, \tau_2}(u^n, m_1, m_2, m_0, w_1^n, w_0^n | E_\delta = 0) \log_2 \frac{1}{\mathcal{Q}_{U|W_0 W_1 W_2}(u_t | w_{1t}, w_{2t}, w_{0t})} \quad (\text{D.19})$$

$$= \frac{1}{n} \sum_{\substack{u^n, m_1, m_2, \\ m_0, w_1^n, w_0^n}} \mathcal{P}^{\sigma\tau_1\tau_2}(u^n, m_1, m_2, m_0, w_1^n, w_2^n, w_0^n | E_\delta = 0) \log_2 \frac{1}{\prod_{t=1}^n \mathcal{Q}_{U|W_0 W_1 W_2}(u_t | w_{1t}, w_{2t}, w_{0t})} \quad (\text{D.20})$$

$$= \frac{1}{n} \sum_{u^n, w_1^n, w_2^n, w_0^n} \mathcal{P}^{\sigma\tau_1\tau_2}(u^n, w_1^n, w_2^n, w_0^n | E_\delta = 0) \log_2 \frac{1}{\prod_{t=1}^n \mathcal{Q}_{U|W_0 W_1 W_2}(u_t | w_{1t}, w_{2t}, w_{0t})}. \quad (\text{D.21})$$

- Equation (D.8) follows since the support of  $\mathcal{P}^{\sigma\tau_1\tau_2}(u^n, w_1^n, w_2^n, w_0^n | E_\delta) = \mathbb{P}\{(u^n, w_1^n, w_2^n, w_0^n) \in \mathcal{T}_\delta^n\}$  is included in  $\mathcal{T}_\delta^n$ .
- Equation (D.9) follows from the typical average lemma property (Property 1 pp.26 in *El Gamal and Kim (2011)*) given in lemma G.0.1, and the chain rule of entropy:  $H(U^n | M_1, M_2, M_0, W_1^n, W_2^n, W_0^n) \leq \sum_{t=1}^n H(U_t | M_1, M_2, M_0, W_1, W_2, W_0)$ .
- Equation (D.10) comes from the conditional entropy property and the fact that  $H(U^n) = nH(U)$  for an i.i.d random variable  $U$  and lemma G.0.2.

- Equation (D.11) follows since  $I(U^n; M_1, M_2, M_0) \leq H(M_1, M_2, M_0) \leq \log_2 |J| = n \cdot (R_1 + R_2 + R_0) = n \cdot (I(U; W_1, W_2, W_0) + \eta)$  and lemma G.0.2.

If the expected probability of error is small over the codebooks, then it has to be small over at least one codebook. Therefore, equations (4.40) and (4.41) imply that:

$$\begin{aligned} \forall \varepsilon_2 > 0, \forall \eta > 0, \exists \bar{\delta} > 0, \forall \delta \leq \bar{\delta}, \exists \bar{n} \in \mathbb{N}, \forall n \geq \bar{n}, \exists b^*, \\ \text{such that } \mathcal{P}_{b^*}(E_\delta^2 = 1) \leq \varepsilon_2. \end{aligned} \quad (\text{D.22})$$

The strategy  $\sigma$  of the encoder consists of using  $b^*$  in order to transmit the pair  $(m_1, m_2, m_0)$  such that

$(U^n, W_0^n(m_0), W_1^n(m_0, m_1))$  is a jointly typical sequence. By construction, this satisfies equation (D.22).

**Lemma D.0.1.** Let  $\mathcal{Q}_{W_0 W_1 W_2 | U} \in \tilde{\mathcal{Q}}_0(R_1, R_2, R_0)$ , then  $\forall \varepsilon > 0, \forall \alpha > 0,$

$\text{Gamma}_s > 0$ , there exists  $\bar{\delta}, \forall \delta \leq \bar{\delta}, \exists \bar{n}, \forall n \geq \bar{n}, \exists \sigma$ , such that  $1 - \mathcal{P}^\sigma(B_{\alpha, \gamma, \delta}) \leq \varepsilon$ .

*Proof.* of lemma D.0.1 We have:

$$1 - \mathcal{P}^\sigma(B_{\alpha, \gamma, \delta}) := \mathcal{P}^\sigma(B_{\alpha, \gamma, \delta}^c) \quad (\text{D.23})$$

$$= \mathcal{P}^\sigma(E_\delta = 0) \mathcal{P}^\sigma(B_{\alpha, \gamma, \delta}^c | E_\delta = 0) \quad (\text{D.24})$$

$$\leq \mathcal{P}^\sigma(B_{\alpha, \gamma, \delta}^c | E_\delta = 0) \quad (\text{D.25})$$

$$\leq \varepsilon_2 + \mathcal{P}^\sigma(B_{\alpha, \gamma, \delta}^c | E_\delta = 1). \quad (\text{D.26})$$

Moreover,

$$\mathcal{P}^\sigma(B_{\alpha, \gamma, \delta}^c | E_\delta = 0) = \sum_{\substack{w_1^n, w_2^n, w_0^n, \\ m_1, m_2, m_0}} \mathcal{P}^\sigma \left( (w_1^n, w_0^n, m_1, m_2, m_0) \in B_{\alpha, \gamma, \delta}^c \middle| E_\delta = 0 \right) \quad (\text{D.27})$$

$$= \sum_{\substack{w_1^n, w_2^n, w_0^n, \\ m_1, m_2, m_0}} \mathcal{P}^\sigma \left( (w_1^n, w_0^n, m_1, m_2, m_0), \frac{|T_\alpha(w_1^n, w_0^n, m_1, m_2, m_0)|}{n} \leq 1 - \gamma \middle| E_\delta = 0 \right) \quad (\text{D.28})$$

$$= \mathcal{P}^\sigma \left( \frac{\#}{n} \left\{ t, D \left( \mathcal{P}_{U^t}^{m_1, m_2, m_0} \middle| \mathcal{Q}_{U | W_0 W_1 W_2}(\cdot | W_{1t}, W_{2t}, W_{0t}) \right) \leq \frac{\alpha^2}{2 \ln 2} < 1 - \gamma \middle| E_\delta = 0 \right\} \right) \quad (\text{D.29})$$

$$= \mathcal{P}^\sigma \left( \frac{\#}{n} \left\{ t, D \left( \mathcal{P}_{U_t}^{m_1, m_2, m_0} \middle\| \mathcal{Q}_{U|W_0 W_1 W_2}(\cdot | W_{1t}, W_{2t}, W_{0t}) \right) > \frac{\alpha^2}{2 \ln 2} \geq \gamma \middle| E_\delta = 0 \right\} \right) \quad (\text{D.30})$$

$$\leq \frac{2 \ln 2}{\alpha^2 \gamma} \cdot \mathbb{E}_\sigma \left[ \frac{1}{n} \sum_{t=1}^n D \left( \mathcal{P}_t^{m_1, m_2, m_0} \middle\| \mathcal{Q}_U(\cdot | W_{1t}, W_{2t}, W_{0t}) \right) \right] \quad (\text{D.31})$$

$$\leq \frac{2 \ln 2}{\alpha^2 \gamma} \cdot \left( \eta + \delta + \frac{2}{n} + 2 \log_2 |\mathcal{U}| \cdot \mathcal{P}^\sigma(E_\delta^2 = 1), \right) \quad (\text{D.32})$$

- Equations (D.27) to (D.30) are simple reformulations.
- Equation (D.31) comes from using Markov's inequality given in lemma D.0.2.
- Equation (D.32) comes from equation (D.11).

**Lemma D.0.2.** (Markov's Inequality). For all  $\varepsilon_1 > 0$ ,  $\varepsilon_2 > 0$  we have:

$$\begin{aligned} \mathbb{E}_\sigma \left[ \frac{1}{n} \sum_{t=1}^n D \left( \mathcal{P}_{U_t}^{m_1, m_2, m_0} \middle\| \mathcal{Q}_{U|W_0 W_1 W_2}(\cdot | W_{1t}, W_{2t}, W_{0t}) \right) \right] \leq \varepsilon_0 &\implies \\ \mathcal{P} \left( \frac{\#}{n} \left\{ t, D \left( \mathcal{P}_{U_t}^{m_1, m_2, m_0} \middle\| \mathcal{Q}(\cdot | W_{1t}, W_{2t}, W_{0t}) \right) > \varepsilon_1 \right\} > \varepsilon_2 \right) &\leq \frac{\varepsilon_0}{\varepsilon_1 \cdot \varepsilon_2}. \end{aligned} \quad (\text{D.33})$$

*Proof.* (of lemma D.0.2) We denote by  $D_t = D(\mathcal{P}_{U_t}^{m_1, m_2, m_0} \middle\| \mathcal{Q}_{U|W_0 W_1 W_2}(\cdot | W_{1t}, W_{2t}, W_{0t}))$  and  $D^n = \{D_t\}_t$  the K-L divergence. We have that:

$$\mathcal{P} \left( \frac{\#}{n} \left\{ t, \text{s.t. } D_t > \varepsilon_1 \right\} > \varepsilon_2 \right) = \mathcal{P} \left( \frac{1}{n} \cdot \sum_{t=1}^n \mathbb{1} \left\{ D_t > \varepsilon_1 \right\} > \varepsilon_2 \right) \quad (\text{D.34})$$

$$\leq \frac{\mathbb{E} \left[ \frac{1}{n} \cdot \sum_{t=1}^n \mathbb{1} \left\{ D_t > \varepsilon_1 \right\} \right]}{\varepsilon_2} \quad (\text{D.35})$$

$$= \frac{\frac{1}{n} \sum_{t=1}^n \mathbb{E} \left[ \mathbb{1} \left\{ D_t > \varepsilon_1 \right\} \right]}{\varepsilon_2} \quad (\text{D.36})$$

$$= \frac{\frac{1}{n} \sum_{t=1}^n \mathcal{P} \left( D_t > \varepsilon_1 \right)}{\varepsilon_2} \quad (\text{D.37})$$

$$\leq \frac{\frac{1}{n} \sum_{t=1}^n \frac{\mathbb{E} \left[ D_t \right]}{\varepsilon_1}}{\varepsilon_2} \quad (\text{D.38})$$

$$= \frac{1}{\varepsilon_1 \cdot \varepsilon_2} \cdot \mathbb{E} \left[ \frac{1}{n} \sum_{t=1}^n D_t \right] \leq \frac{\varepsilon_0}{\varepsilon_1 \cdot \varepsilon_2}. \quad (\text{D.39})$$

- Equations (D.34), (D.36), (D.37) and (D.39) are reformulations of probabilities and expectations.
- Equations (D.35) and (D.38), come from Markov's inequality  $\mathcal{P}(X \geq \alpha) \leq \frac{\mathbb{E}[X]}{\alpha}$ ,  $\forall \alpha > 0$ .

□

# E

## PROOF OF LEMMA 4.5.8

**Lemma.** For all  $w_0^n, w_1^n, w_2^n$ , we have

$$\lim_{n \rightarrow \infty} \mathbb{E} \left[ \frac{1}{n} \sum_{t=1}^n D(\mathcal{P}_{W_{1,t}|W_{0,t}W_{2,t}}^\sigma \| \mathcal{Q}_{W_1}^{w_0, w_2}) \middle| E_\delta = 0 \right] = 0, \quad (\text{E.1})$$

$$\lim_{n \rightarrow \infty} \mathbb{E} \left[ \frac{1}{n} \sum_{t=1}^n D(\mathcal{P}_{W_{2,t}|W_{0,t}W_{1,t}}^\sigma \| \mathcal{Q}_{W_2}^{w_0, w_1}) \middle| E_\delta = 0 \right] = 0. \quad (\text{E.2})$$

*Proof.* Given a codebook  $\mathcal{C}$ , a stage  $t \in \{1, 2, \dots, n\}$  and a symbol  $m_0$ , we denote the  $t$ -codebook by  $\mathcal{C}^{1,t}(m_0) = \{w_{1,t}(m_0, m_1), w_1^n(m_0, m_1) \in \mathcal{C}, m_1 \in \{1, \dots, 2^{\lfloor nR_1 \rfloor}\}\}$  and  $\mathcal{C}^{2,t}(m_0) = \{w_{2,t}(m_0, m_2), w_2^n(m_0, m_2) \in \mathcal{C}, m_2 \in \{1, \dots, 2^{\lfloor nR_2 \rfloor}\}\}$ . For each  $m_0 \in \{1, \dots, 2^{\lfloor nR_0 \rfloor}\}$ ,  $w_1 \in \mathcal{W}_1$ ,  $w_2 \in \mathcal{W}_2$ ,  $t \in \{1, \dots, n\}$  and  $\delta > 0$ , we introduce the following sets

$$\mathcal{B}_1^t(w_1|m_0) = \{w_{1,t} \in \mathcal{C}^t(m_0), w_{1,t} = w_1\}, \quad (\text{E.3})$$

$$\mathcal{B}_2^t(w_2|m_0) = \{w_{2,t} \in \mathcal{C}^t(m_0), w_{2,t} = w_2\}, \quad (\text{E.4})$$

$$\mathcal{T}(m_0) = \left\{ t \in \{1, \dots, n\}, \quad \left\| \mathcal{Q}_{W_1|W_0}(\cdot|w_{0t}) - \frac{|\mathcal{B}_1^t(\cdot|m_0)|}{|\mathcal{C}^t(m_0)|} \right\| > \delta, \right. \\ \left. \left\| \mathcal{Q}_{W_2|W_0}(\cdot|w_{0t}) - \frac{|\mathcal{B}_2^t(\cdot|m_0)|}{|\mathcal{C}^t(m_0)|} \right\| > \delta \right\}, \quad (\text{E.5})$$

where  $\|\cdot\|$  denote the  $L_1$ -norm. We have by the law of large numbers,  $\mathcal{P}(|\mathcal{T}(m_0)| > \delta) \xrightarrow{n \rightarrow \infty} 0$ .

For all  $w_0^n, w_1^n, w_2^n, t$ , we have

$$\mathcal{P}^\sigma(w_{1,t}|w_{0,t}, w_{2,t}) = \sum_{\substack{w_0^{t-1}, w_{0,t+1}^n \\ w_2^{t-1}, w_{2,t+1}^n}} \mathcal{P}^\sigma(w_0^{t-1}, w_{0,t+1}^n, w_2^{t-1}, w_{2,t+1}^n | w_{0,t}, w_{2,t}) \cdot \mathcal{P}^\sigma(w_{1,t} | w_0^n, w_2^n). \quad (\text{E.6})$$

Similarly,

$$\mathcal{P}^\sigma(w_{2,t}|w_{0,t}, w_{1,t}) = \sum_{\substack{w_0^{t-1}, w_{0,t+1}^n \\ w_1^{t-1}, w_{1,t+1}^n}} \mathcal{P}^\sigma(w_0^{t-1}, w_{0,t+1}^n, w_1^{t-1}, w_{1,t+1}^n | w_{0,t}, w_{1,t}) \cdot \mathcal{P}^\sigma(w_{2,t} | w_0^n, w_1^n). \quad (\text{E.7})$$

Therefore, using Lemma 4.5.6 the result follows.  $\square$

# F

## CONVERSE PROOF OF THEOREM 5.4.1

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**Theorem.** Let  $(R_1, R_2) \in \mathbb{R}_+^2$ . If  $c_1 = c_2 = c_3$ , then

$$\lim_{n \rightarrow \infty} \Gamma_{cas}^n(R_1, R_2) = \inf_{n \in \mathbb{N}^*} \Gamma_{cas}^n(R_1, R_2) = \Gamma_{cas}^c(R_1, R_2). \quad (\text{F.1})$$

*Proof.* Given a triple  $(\sigma, \mu, \tau)$  and a random variable  $T$  uniformly distributed over  $\{1, 2, \dots, n\}$  and independent of  $(U^n, M_1, M_2, V^n)$ . We identify the auxiliary random variables  $W_2 = (M_2, T)$ ,  $(U, V) = (U_T, V_T)$ , distributed according to  $\mathcal{P}_{UW_2V}^{\sigma\mu\tau}$  defined for all  $(u, w_2, v) = (u_t, x_1, x_2, t, v_t)$  by

$$\begin{aligned} \mathcal{P}_{UW_2V}^{\sigma\mu\tau}(u, w_2, v) &= \mathcal{P}_{U_T W_2 T V_T}^{\sigma\mu\tau}(u_t, x_1, x_2, t, v_t) \\ &= \frac{1}{n} \sum_{\substack{u^{t-1} \\ u_{t+1}^n}} \sum_{\substack{x_1^{t-1}, x_{1,t+1}^n \\ x_2^{t-1}, x_{2,t+1}^n}} \sum_{v^{t-1}, v_{t+1}^n} \left( \prod_{t=1}^n \mathcal{P}_U(u_t) \right) \mathcal{P}_{M_1|U^n}^\sigma(m_1|u^n) \mathcal{P}_{M_2|M_1}^\mu(m_2|m_1) \mathcal{P}_{V^n|M_2}^\tau(v^n|m_2). \end{aligned}$$

Therefore  $\mathcal{P}_{UW_2V}^{\sigma\mu\tau} = \mathcal{P}_U \mathcal{P}_{W_2|U}^\sigma \mathcal{P}_{V|W_2}^\tau$ .

**Lemma F.0.1.** For all  $\sigma$ , the distribution  $\mathcal{P}_{W_2|U}^\sigma \in \mathbb{Q}_0^c(R_1, R_2)$ .

*Proof.* [Lemma F.0.1] Given an encoding strategy  $\sigma$ , we have

$$[nR_1] \geq H(M_1) \geq I(M_1; U^n) \quad (\text{F.2})$$

$$= \sum_{t=1}^n I(U_t; M_1 | U^{t-1}) \quad (\text{F.3})$$

$$= nI(U_T; M_1 | U^{T-1}, T) \quad (\text{F.4})$$

$$= nI(U_T; M_1, U^{T-1}, T) \quad (\text{F.5})$$

$$\geq nI(U_T; M_1, T) \quad (\text{F.6})$$

$$\geq nI(U_T; M_2, T) \quad (\text{F.7})$$

$$= nI(U; W_2) \quad (\text{F.8})$$

In fact, (F.4) follows from the introduction of the uniform random variable  $T \in \{1, \dots, n\}$ , (F.5) comes from the i.i.d. property of the source and (F.7) comes from the data processing inequality, and equation (F.8) follows from the identification of the auxiliary random variable  $W_2$ . Similarly, one can show that

$$nR_2 \geq nI(U; W_2), \quad (\text{F.9})$$

which concludes the proof of Lemma F.0.1.  $\square$

**Lemma F.0.2.** For all  $(\sigma, \tau_1, \tau_2)$  and  $i \in \{1, 2, 3\}$ , we have

$$c_i^n(\sigma, \mu, \tau) = \mathbb{E}[c_i(U, V)], \quad (\text{F.10})$$

evaluated with respect to  $\mathcal{P}_U \mathcal{P}_{W_1|U}^\sigma \mathcal{P}_{W_2|W_1}^\mu \mathcal{P}_{V|W_2}^\tau$ . Moreover for all  $\sigma, \mu$  we have

$$\mathbb{Q}_3^c(\mathcal{P}_{W_2|U}^\sigma) = \left\{ \mathcal{Q}_{V|W_2}, \exists \tau \in \mathbb{A}_3(\sigma, \mu), \mathcal{Q}_{V|W_2} = \mathcal{P}_{V|W_2}^\tau \right\}. \quad (\text{F.11})$$

*Proof.* [Lemma F.0.2] By Definition 5.3.2 we have for  $i \in \{1, 2, 3\}$

$$\begin{aligned} c_i^n(\sigma, \mu, \tau) &= \sum_{\substack{u^n, m_1, \\ m_2, v^n}} \left( \prod_{t=1}^n \mathcal{P}_U(u_t) \right) \mathcal{P}_{M_1|U^n}^\sigma(m_1|u^n) \mathcal{P}_{M_2|M_1}^\mu(m_2|m_1) \\ &\quad \times \mathcal{P}_{V^n|M_2}^\tau(v^n|m_2) \cdot \left[ \frac{1}{n} \sum_{t=1}^n c_i(u_t, v_t) \right] \\ &= \sum_{t=1}^n \sum_{\substack{u_t, x_1, \\ x_2, t, v_t}} \mathcal{P}^{\sigma, \mu, \tau}(u_t, x_1, x_2, t, v_t) \times c_i(u_t, v_t) = \mathbb{E}[c_i(U, V)]. \end{aligned} \quad (\text{F.12})$$

Given  $\mathcal{Q}_{V|W_2} \in \mathbb{Q}_3^c(\mathcal{P}_{W_2|U}^\sigma)$ , we consider  $\tau$  such that

$$\mathcal{P}_{V^n|M_2}^\tau(v^n|m_2) = \prod_{t=1}^n \mathcal{Q}_{V|W_2}(v_{1,t}|m_2).$$

Therefore

$$c_3^n(\sigma, \mu, \tau) = \mathbb{E}_{\mathcal{P}_{W_2|U}^\sigma \mathcal{Q}_{V|W_2}} [c_3(U, V)] \quad (\text{F.13})$$

$$= \min_{\mathcal{P}_{V|W_2}} \mathbb{E}_{\mathcal{P}_{W_2|U}^\sigma \mathcal{P}_{V|W_2}} [c_3(U, V)] \quad (\text{F.14})$$

$$\leq \min_{\tilde{\tau}} \mathbb{E}_{\mathcal{P}_{W_2|U}^\sigma \mathcal{P}_{V|W_2}^{\tilde{\tau}}} [c_3(U, V)] \quad (\text{F.15})$$

$$= \min_{\tilde{\tau}} c_3^n(\sigma, \mu, \tilde{\tau}), \quad (\text{F.16})$$

hence  $\tau \in \mathbb{A}_3(\sigma, \mu)$ . Similarly,

$$c_2^n(\sigma, \mu, \tau) = \mathbb{E}_{\substack{\mathcal{P}_{W_1|U}^\sigma \\ \mathcal{Q}_{W_2|W_1} \mathcal{Q}_{V|W_2}}} [c_2(U, V)] \quad (\text{F.17})$$

$$= \min_{(\mathcal{P}_{W_2|W_1}, \mathcal{P}_{V|W_2})} \mathbb{E}_{\substack{\mathcal{P}_{W_1|U}^\sigma \\ \mathcal{P}_{W_2|W_1}, \mathcal{P}_{V|W_2}}} [c_2(U, V)] \quad (\text{F.18})$$

$$\leq \min_{(\tilde{\mu}, \tilde{\tau})} \mathbb{E}_{\substack{\mathcal{P}_{W_1|U}^\sigma \\ \mathcal{P}_{W_2|W_1}^{\tilde{\mu}}, \mathcal{P}_{V|W_2}^{\tilde{\tau}}}} [c_2(U, V)] \quad (\text{F.19})$$

$$= \min_{(\tilde{\mu}, \tilde{\tau})} c_2^n(\sigma, \mu, \tilde{\tau}), \quad (\text{F.20})$$

and thus  $(\mu, \tau) \in \mathbb{A}_2(\sigma)$ . The other inclusions are direct and the same arguments imply (F.11).  $\square$

For any strategy  $\sigma$ , we have

$$\max_{\mu, \tau} c_1^n(\sigma, \mu, \tau) = \max_{\mu, \tau} \mathbb{E}_{\mathcal{P}_{W_2|U}^\sigma \mathcal{P}_{V|W_2}^\tau} [c_1(U, V)] \quad (\text{F.21})$$

$$\geq \max_{\mathcal{Q}_{V|W_2} \in \mathbb{Q}_3^c(\mathcal{Q}_{W_2|U})} \mathbb{E}_{\mathcal{P}_{W_2|U}^\sigma \mathcal{Q}_{V|W_2}} [c_1(U, V)] \quad (\text{F.22})$$

$$\geq \inf_{\substack{\mathcal{Q}_{W_2|U} \in \\ \mathbb{Q}_0^c(R_1, R_2)}} \max_{\mathcal{Q}_{V|W_2} \in \mathbb{Q}_3^c(\mathcal{Q}_{W_2|U})} \mathbb{E} [c_1(U, V)] \quad (\text{F.23})$$

$$= \Gamma_{cas}^c(R_1, R_2). \quad (\text{F.24})$$

Equations (F.21) and (F.22) comes from Lemma F.0.2, whereas (F.23) comes from Lemma F.0.1. This concludes the converse proof of Theorem 5.4.1.  $\square$

# G

## ADDITIONAL LEMMAS

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**Lemma G.0.1.** (*Typical Sequences Property 1, pp.26 in El Gamal and Kim (2011)*). The typical sequences  $(u^n, w_1^n, w_0^n) \in \mathcal{T}_\delta^n$  satisfy:

$$\forall \varepsilon > 0, \exists \bar{\delta} > 0, \forall \delta \leq \bar{\delta}, \forall n, \forall (u^n, w_1^n, w_0^n) \in \mathcal{T}_\delta^n, \left| \frac{1}{n} \cdot \log_2 \frac{1}{\prod_{t=1}^n \mathcal{P}(u|w_{1t}, w_{2t})} - H(U|W_1, W_0) \right| \leq \varepsilon, \quad (\text{G.1})$$

where  $\bar{\delta} = \varepsilon \cdot H(U|W_1, W_0)$ .

**Lemma G.0.2.** Let  $U^n$  an i.i.d random variable and  $M$  a random variable. For all  $\varepsilon > 0$ , there exists  $\bar{n} \in \mathbb{N}$ , such that for all  $n \geq \bar{n}$ , we have

$$H(U^n|E_\delta = 0) \geq n \cdot (H(U) - \varepsilon). \quad (\text{G.2})$$

*Proof.*

$$H(U^n|E_\delta = 0) = \frac{1}{\mathcal{P}(E_\delta = 0)} \cdot \left( H(U^n|E_\delta = 1) - \mathcal{P}(E_\delta = 1) \cdot H(U^n|E_\delta = 1) \right) \quad (\text{G.3})$$

$$\geq H(U^n|E_\delta) - \mathcal{P}(E_\delta = 1) \cdot H(U^n|E_\delta = 1) \quad (\text{G.4})$$

$$\geq H(U^n) - H(E_\delta) - \mathcal{P}(E_\delta = 1) \cdot H(U^n|E_\delta = 1) \quad (\text{G.5})$$

$$\geq H(U^n) - n \cdot \varepsilon. \quad (\text{G.6})$$

□

- Equation (G.3) follows from the conditional entropy definition.

- Equation (G.4) follows since  $\mathcal{P}(E_\delta = 0) \leq 1$ .
- Equation (G.5) comes from the property  $H(U^n|M, E_\delta) = H(U^n, M, E_\delta) - H(M) - H(E_\delta) \geq H(U^n) - H(M) - H(E_\delta)$ .
- Equation (G.6) follows since  $U$  is i.i.d and the definition of  $E_\delta = 1$ . Hence, for all  $\varepsilon$ , there exists an  $\bar{n} \in \mathbb{N}$  such that for all  $n \geq \bar{n}$  we have  $H(\mathcal{P}(E_\delta = 1)) + H(M) + \mathcal{P}(E_\delta = 1) \cdot \log_2 |\mathcal{U}| \leq \varepsilon$ .

**Lemma G.0.3.** (Covering Lemma (El Gamal and Kim, 2011, Lemma 3.3)). Let  $\delta > 0$ ,  $R > 0$ , a block length  $n$ , and let  $X^n \in \mathcal{T}_\delta^n(P_X)$  where  $P_X$  is an i.i.d distribution. Let  $\{Y^n(m)\}^{2^{nR}}$  be a set of  $P_Y$ -i.i.d. sequences for  $m \in [1 : 2^{nR}]$ . Assume  $\{X^n, \{Y^n(m)\}^{2^{nR}}$  are mutually independent. For any joint distribution  $\mathcal{P}_{XY}$  with marginals  $P_X$  and  $P_Y$ , if  $R > I(X; Y) + \eta$ , for all  $\varepsilon > 0$ , there exists  $\bar{\delta} > 0$ ,  $\exists \bar{n} \in \mathbb{N}$ ,  $\exists \eta \rightarrow 0$  such that  $\forall \delta \leq \bar{\delta}, \forall n \geq \bar{n}$  such that

$$\mathcal{P}\left(\bigcap_{m \in [1:2^{nR}]} (X^n, Y^n(m)) \notin \mathcal{T}_\delta^n(\mathcal{P}_{XY})\right) \leq \varepsilon. \quad (\text{G.7})$$