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Convergence de méthodes numériques pour la mécanique des fluides : équation de Navier-Stokes stochastique et ses variantes

Jad Doghman

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Convergence of numerical methods in
fluid mechanics: The stochastic
Navier-Stokes equation and its variants
*Convergence de méthodes numériques pour la mécanique
des fluides: Équation de Navier-Stokes stochastique et ses
variantes*

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Titre: Convergence de méthodes numériques pour la mécanique des fluides: Équation de Navier-Stokes stochastique et ses variantes

Mots clés: Méthodes des éléments finis, méthode d'Euler, méthode de pénalisation, Navier-Stokes, Navier-Stokes moyennées au sens de Lagrange, bruit multiplicatif, analyse de convergence

Résumé: Bien que le développement et l'évolution des méthodes numériques pour les équations de Navier-Stokes existent depuis des décennies, elles restent, jusqu'à ce jour, un sujet ouvert pour de nouvelles recherches en raison de leurs propriétés imparfaites: allant de la complexité théorique, qui découle du comportement chaotique des solutions mathématiques, jusqu'à l'implémentation associée, qui nécessite une manipulation soignée d'algorithmes numériques ainsi que des techniques de codage efficaces permettant une compilation de code et un temps d'exécution optimaux. Les versions stochastiques des équations de Navier-Stokes ont un comportement plus complexe vis-à-vis du terme source aléatoire appliqué qui contraint les démonstrations techniques, car ses effets se propagent dans les équations et affectent éventuellement leurs solutions, entraînant souvent une perte de régularité, sans parler du temps considérable qui s'ajoute à l'exécution des codes numériques déterministes.

Cette thèse propose quelques solutions aux problèmes susmentionnés en tournant l'attention vers une variante des équations de Navier-Stokes; notamment, les équations de Navier-Stokes moyennées au sens de Lagrange (LANS- α en abrégé) qui ont de meilleures propriétés et sont paramétrées par une échelle spatiale notée α . Grosso modo, lorsque le paramètre α tend vers 0, on retrouve les équations de Navier-Stokes. Cette propriété est développée dans tout un chapitre pour détailler l'aspect théorique des solutions au modèle LANS- α stochastique lorsque α s'annule. Ainsi, en liant α avec des paramètres de discrétisation qui finissent par atteindre 0, une solution du modèle LANS- α converge vers une solution de Navier-Stokes. Par conséquent, discrétiser le problème de Navier-Stokes revient à discrétiser le modèle LANS- α , sous la condition que α s'annule en passant à la limite dans les petits paramètres de la

méthode numérique proposée.

Un autre problème considérable émergeant des équations incompressibles de Navier-Stokes, en particulier pour les schémas numériques, est la construction de sous-espaces à divergence nulle. Ces méthodes paraissent délicates à première vue puisqu'elles permettent de supprimer le champ de pression de la formulation variationnelle des équations sous-jacentes et, par conséquent, de réduire le nombre de degrés de liberté résultant en un système discret défini positif. Cependant, la gestion de leur mise en œuvre pourrait être fastidieuse et pourrait présenter des conséquences indésirables, telles que des conditionnements médiocres. Cet inconvénient peut être surmonté par une variation (appelée méthode de pénalisation ou compressibilité artificielle) de l'équation de conservation de la masse, qui implique la contrainte à divergence nulle. Cela a l'avantage de maintenir des méthodes numériques standards qui sont largement étudiées, telles que l'approximation par éléments finis, permettant d'appliquer la littérature déjà existante sans aucune restriction. La méthode de pénalisation est comparable au modèle LANS- α puisqu'elle contient un paramètre noté ε dans ce contexte, qui pourrait être considéré en fonction des paramètres de discrétisation afin de l'éliminer des équations d'intérêt lors du passage à la limite.

Concernant les méthodes numériques appliquées dans le cadre de cette thèse, la méthode des éléments finis joue un rôle essentiel dans la discrétisation spatiale du modèle stochastique LANS- α et des équations de Navier-Stokes à compressibilité artificielle. La discrétisation temporelle est assurée par la méthode d'Euler, qui peut varier d'un algorithme à l'autre en fonction des hypothèses imposées. Les schémas numériques proposés ici améliorent la vitesse de convergence des algorithmes déjà existants pour le problème de Navier-Stokes stochastique.

Title: Convergence of numerical methods in fluid mechanics: The stochastic Navier-Stokes equation and its variants

Keywords: Finite element methods, Euler method, Penalty method, Navier-Stokes, Lagrangian averaged Navier-Stokes, multiplicative noise, convergence analysis

Abstract: Although the development and evolution of numerical methods for the Navier-Stokes equations have been around for decades, they remain, until this day, an open topic for further research on account of their infamous properties: starting from the related theoretical complexity which arises from the mathematical solutions' chaotic behavior up to the associated implementation which requires a careful handling of algorithm constructions as well as efficient coding techniques permitting an optimal code compilation and execution time. The Navier-Stokes equations' stochastic versions have even worse properties regarding the applied random forcing and its restrictive effects that spread across the equations and affect their solutions eventually, often leading to loss of regularities, not to mention the significant amount of time that adds up to the code execution of the deterministic algorithm.

This thesis proposes a few solutions to the aforementioned issues by turning the attention toward a variant of the Navier-Stokes equations; namely, the Lagrangian averaged Navier-Stokes (LANS- α for short) equations which have better properties and are parametrized by a spatial scale denoted α . Loosely speaking, when the parameter α tends to 0, one retrieves the Navier-Stokes equations. This property is conducted herein within a dedicated chapter to emphasize the theoretical aspect of solutions to the stochastic LANS- α model when α vanishes. Therewith, by linking α to some discretization parameters that go to 0 eventually, a solution to the LANS- α model becomes one to the Navier-Stokes equations. That being said, discretizing the Navier-Stokes problem amounts to discretizing the LANS- α model, provided that α vanishes when passing to the limit in the proposed numerical method.

Another considerable problem emerging from the incompressible Navier-Stokes equations, especially for numerical schemes, is the construction of divergence-free subspaces. These methods appear tricky at first glance since they permit the deletion of the pressure field from the variational formulation of the underlying equations and, therefore, the reduction of the degree of freedom number resulting in a well-behaved positive definite discrete system. However, dealing with their implementation can be tedious and may present some undesirable consequences, such as lousy condition numbers. This drawback can be circumvented through a variation (called penalty method or artificial compressibility) of the mass conservation equation, which involves the null divergence constraint. This has the advantage of maintaining standard numerical methods that are extensively investigated, such as the finite element approximation, allowing for the already existing literature to be applied without any restrictions. The penalty method is comparable to the LANS- α model since it contains a parameter denoted ε herein, which might be considered in terms of the discretization parameters in order to eliminate it from the equations of interest when passing to the limit.

Speaking of the applied numerical methods within the framework of this thesis, the finite element method plays an essential role in the spatial discretization of both the stochastic LANS- α model and the Navier-Stokes equations with artificial compressibility. The time discretization is provided by the Euler method, which may vary from one algorithm to another depending on the imposed assumptions. The proposed numerical schemes herein improve the convergence rate of already existing algorithms for the stochastic Navier-Stokes problem.

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Abstract

Although the development and evolution of numerical methods for the Navier-Stokes equations have been around for decades, they remain, until this day, an open topic for further research on account of their infamous properties: starting from the related theoretical complexity which arises from the mathematical solutions' chaotic behavior up to the associated implementation which requires a careful handling of algorithm constructions as well as efficient coding techniques permitting an optimal code compilation and execution time. The Navier-Stokes equations' stochastic versions have even worse properties regarding the applied random forcing and its restrictive effects that spread across the equations and affect their solutions eventually, often leading to loss of regularities, not to mention the significant amount of time that adds up to the code execution of the deterministic algorithm.

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Résumé

Bien que le développement et l'évolution des méthodes numériques pour les équations de Navier-Stokes existent depuis des décennies, elles restent, jusqu'à ce jour, un sujet ouvert pour de nouvelles recherches en raison de leurs propriétés imparfaites: allant de la complexité théorique, qui découle du comportement chaotique des solutions mathématiques, jusqu'à l'implémentation associée, qui nécessite une manipulation soignée d'algorithmes numériques ainsi que des techniques de codage efficaces permettant une compilation de code et un temps d'exécution optimaux. Les versions stochastiques des équations de Navier-Stokes ont un comportement plus complexe vis-à-vis du terme source aléatoire appliqué qui contraint les démonstrations techniques, car ses effets se propagent dans les équations et affectent éventuellement leurs solutions, entraînant souvent une perte de régularité, sans parler du temps considérable qui s'additionne à l'exécution des codes numériques déterministes.

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List of abbreviations and symbols

a.e.	almost everywhere
a.s.	almost surely
\mathbb{R}	set of real numbers
\mathbb{N}	set of natural numbers
$:=$	equal by definition (defined on the left-hand side)
\equiv	equal by definition (defined on the right-hand side)
\lesssim	shorthand for \leq up to a universal constant
D	domain in \mathbb{R}^2 or \mathbb{R}^3
C_D	domain-dependent constant
C_P	Poincaré's constant
T	final time
$\mathbb{1}_S$	indicator function of the set S
\mathbb{P}	probability measure
$\mathbb{E}[\]$	mathematical expectation
$Var(\)$	mathematical variance
ω	realization
$(\Omega, \mathcal{F}, \mathbb{P})$	complete probability space
$(\mathcal{F}_t)_{0 \leq t \leq T}$	right-continuous filtration such that \mathcal{F}_0 contains all \mathcal{F} -null sets
\rightarrow	strong convergence (in the sense of functional analysis)
\rightharpoonup	weak convergence (in the sense of functional analysis)
$\xrightarrow{*}$	weak-* convergence (in the sense of functional analysis)
$s \wedge t$	minimum between s and t
d_t	discrete derivative with respect to time
k	time discretization step size
h	space discretization step size
I_k	a uniform partition of the interval $[0, T]$
\mathcal{T}_h	a quasi-uniform triangulation
\mathbb{H}_h	finite element space for discrete velocity vectors
L_h	finite element space for discrete pressure fields
\mathbb{V}_h	subspace of \mathbb{H}_h with weakly divergence-free functions
C^∞	set of infinitely continuously differentiable functions
$C([0, T]; V)$	space of continuous functions $f: [0, T] \rightarrow V$
L^2	Lebesgue space of square integrable functions
L^2_0	L^2 -valued functions with vanishing mean
H^m	Sobolev space of order m

H_0^1	Sobolev space involving functions with null boundary values
X_{per}	subspace of some space X endowed with periodic boundary conditions and has functions with vanishing mean.
X'	dual space of a given space X
X^*	a given space X deprived from its origin
Π_h	projection onto \mathbb{H}_h
ρ_h	projection onto L_h
\mathcal{R}_h	Ritz operator
$Tr(\cdot)$	trace of an operator
$\mathcal{L}_1(E, F)$	space of nuclear operators $\mathcal{T}: E \rightarrow F$
$\mathcal{L}_2(E, F)$	space of Hilbert-Schmidt operators $\mathcal{T}: E \rightarrow F$
$M_{\mathcal{F}_t}^p(0, T; V)$	space of \mathcal{F}_t -progressively measurable processes in $L^p(\Omega \times (0, T), d\mathbb{P} \times dt; V)$
(\cdot, \cdot)	L^2 -scalar product
$\langle \cdot, \cdot \rangle$	duality product between H_0^1 (resp. H_{per}^1) and H^{-1} .
∇	gradient operator $(\partial_x, \partial_y, \partial_z)$
Δ	Laplace operator $\partial_{xx}^2 + \partial_{yy}^2 + \partial_{zz}^2$
Δ^h	discrete Laplace operator
div	divergence operator
\mathcal{P}	Laplace projector
A	Stokes operator
A^s	s-th order of the Stokes operator
ν	kinematic viscosity
α	spatial filtering scale
f	outer force
g	diffusion coefficient
K_1, K_2	sublinearity constants of g
K_3, K_4	sublinearity constants of f
L_g	Lipschitz constant of g
L_f	Lipschitz constant of f
W	infinite-dimensional Wiener process
Q	covariance operator of the Wiener process W
K	Separable Hilbert space to which W belong
β_k	unidimensional Brownian motion of index k
ε	penalty parameter (only in Chapter 6)
\times	cross product
\otimes	dyadic product
Re	Reynolds number
NSE	Navier-Stokes equation
SNSE	stochastic Navier-Stokes equation

LANS- α Lagrangian averaged Navier-Stokes equation
LBB Ladyzhenskaya-Babuška-Brezzi

1 - Introduction (Français)

1.1 . Les équations de Navier-Stokes

Comme pour la plupart des problèmes mathématiques déconcertants, les équations de Navier-Stokes restent un dilemme pertinent car elles nécessitent une compréhension approfondie de leurs différents cadres, une gestion prudente de chaque situation étudiée tout en faisant le lien avec les circonstances réelles afin d'appliquer rigoureusement leurs résultats. Les équations de Navier-Stokes sont des lois universelles de la physique qui modélisent chaque fluide présent sur Terre, y compris les gaz, et sont considérées comme l'un des piliers de la mécanique des fluides. Elles représentent une régularisation visqueuse des équations d'Euler, qui restent également une énigme, et peuvent être utilisées pour modéliser les courants océaniques, le flux d'air autour d'une aile, le comportement météorologique, le flux sanguin, l'analyse de la pollution, la conception d'avions, de trains, de voitures et de nombreuses autres applications. Ces équations ont été développées au cours de plusieurs décennies de construction progressive à partir des théories élaborées par le physicien français Claude-Louis Navier et le mathématicien et physicien britannique Georges Gabriel Stokes.

Comprendre la turbulence était et est toujours pertinent pour tous les développements en cours, notamment à des fins industrielles et environnementales. L'apparition de la turbulence est analysée comme une perte de stabilité des solutions au sein de la classe des fonctions différentiables. Ceci explique les mauvaises propriétés habituellement fournies par les équations d'intérêt. En effet, le problème de Navier-Stokes peut être considéré comme un générateur qui reçoit une entrée u_0 ayant une certaine régularité R et est censé renvoyer une sortie u avec la même régularité donnée, c'est-à-dire R . Malheureusement, ce n'est pas tout à fait correct et c'est la raison pour laquelle ces équations sont gênantes et doivent être traitées consciencieusement.

Mathématiquement parlant, les équations tridimensionnelles de Navier-Stokes restent, à ce jour, un problème ouvert dont la résolution par une solution analytique semble impossible, et leur formulation ne peut être simplifiée davantage. Ceci se produit à cause de leur structure vectorielle qui affecte négativement les critères d'explosion associés. En d'autres termes, le seul contrôle disponible est l'inégalité d'énergie de Leray qui n'attribue qu'une majoration au champ de vitesse du fluide en moyenne quadratique mais laisse la vorticit   du fluide associ  e non trait  e. Ce probl  me provient principalement de la non-lin  arit   de ces   quations malgr   l'effet diffusif qui amortit les gradients   lev  s. En fait, la non-lin  arit  ,   galement connue sous le nom de terme convectif, transf  re les   nergies des gradients faibles aux gradients   lev  s    un rythme plus rapide que l'amortissement fourni par le terme diffusif.

De nombreux physiciens ont tent   de pallier la lacune susmentionn  e en g  n  rant des variantes aux   quations de Navier-Stokes consistant en de l  g  res modifications permettant souvent    l'effet diffusif de dominer la mal  diction du terme convectif. Ces variantes r  solvent la plupart des inconv  nients qui surgissent du mod  le original en raison de la forte r  gularit   accord  e aux solutions associ  es. Cependant, elles doivent   tre appliqu  es avec pr  caution car parfois, les effets nouvellement induits par la variantes peuvent ne pas correspondre aux circonstances qui ont men      ce choix de variante. Par exemple, certains de ces mod  les d  riv  s ont   t   cr   s pour mod  liser uniquement les fluides anisotropes tels que les cristaux liquides, contrairement    l'eau et au chloroforme, qui excluent tout ordre structurel des mol  cules. Pour n'en nommer que quelques-uns, les variantes suivantes peuvent   tre rencontr  es dans la litt  rature de la m  canique des fluides : Leray- α , Leray- α modifi  e, Clark- α , Navier-Stokes moyenn  e au sens de Lagrange. Ce dernier mod  le est expliqu   plus en d  tails dans la Section 1.2 et est choisi dans cette th  se pour   tre trait   num  riquement afin d'approcher les solutions des   quations de Navier-Stokes

d'une manière efficace. Presque toutes les variantes existantes sont paramétrées par une échelle, souvent notée α , qui, lorsqu'elle est prise égale à 0, redonne formellement le problème de Navier-Stokes.

Le mouvement des fluides visqueux incompressibles se composant d'une vitesse $u = u(t, x) \in \mathbb{R}^d$, $d \in \{2, 3\}$, et d'un champ scalaire de pression $p = p(t, x) \in \mathbb{R}$, est modélisé par les équations de Navier-Stokes incompressibles suivantes:

$$\begin{cases} \partial_t u - \nu \Delta u + [u \cdot \nabla]u + \nabla p = f, \\ \operatorname{div}(u) = 0, \end{cases} \quad (1.1)$$

dotées d'une donnée initiale $u_0 = u(0, \cdot)$, avec des conditions aux limites particulières comme Dirichlet, Neumann ou Robin. La fonction f est une force extérieure pour laquelle des expériences physiques démontrent qu'une turbulence pourrait se manifester si f présente des variations importants. La constante strictement positive ν représente la viscosité cinématique qui, lorsqu'elle est fixée à 0, transforme le système ci-dessus en un problème d'Euler. L'incompressibilité du fluide est matérialisée par la seconde identité (divergence nulle), appelée équation de conservation de masse. En termes de résolution, des solutions globales faibles au problème déterministe de Navier-Stokes incompressible dans les domaines tridimensionnels ont été illustrées par Leray (1934) et Hopf (1951). Cependant, la régularité et l'unicité des solutions de Leray-Hopf ne sont pas décrites car, comme mentionné ci-dessus, l'effet régularisant du terme visqueux ($\nu \Delta u$) n'est pas adéquat pour compenser les propriétés destructrices du terme convectif ($[u \cdot \nabla]u$). En fait,

- i. des solutions faibles globales (en temps) existent mais sans démonstration de leur unicité,
- ii. des solutions régulières locales (en temps) existent, mais leur analyse d'explosion en temps fini reste un problème ouvert.

Cette situation est beaucoup plus maîtrisée pour les domaines bidimensionnels où l'existence et l'unicité de solutions globales régulières sont démontrées.

1.1.1 . Les équations de Navier-Stokes stochastiques

La nature n'offre pas toujours des phénomènes prévisibles et dociles car plusieurs facteurs peuvent intervenir ensemble et avoir un impact sur le résultat. Les systèmes déterministes ne pourraient donc pas être l'option suprême pour tirer le meilleur parti d'un modèle proposé. Par exemple, l'étude de la turbulence autour d'une aile d'avion en hiver, où de multiples effets indésirables agissent ensemble, pourrait conduire à des résultats trop optimistes, ce qui signifie que les propriétés du résultat obtenu peuvent ne pas refléter complètement la situation qui se produit. Pour rendre ce fait plus clair, les notions d'équations différentielles stochastiques et leur théorie mathématique associée ont été introduites par le mathématicien japonais Kiyosi Itô (1940) après avoir établi le concept d'intégrales stochastiques, qui a finalement conduit à l'intégrale d'Itô, y compris la célèbre formule d'Itô qui remplace le développement de Taylor pour les cas où les fonctions sont exprimées en termes de processus stochastiques. Plus tard, le physicien russe Ruslan Stratonovich a décrit une autre approche aboutissant à une notion d'intégration stochastique dont les règles de calcul sont similaires aux intégrales ordinaires.

La forme la plus fréquente d'équations aux dérivées partielles stochastiques dans la littérature consiste à ajouter un processus stochastique; une variable aléatoire dont l'indice appartient à un ensemble non dénombrable, généralement dépendant d'un bruit blanc, en tant que force externe à une équation aux dérivées partielles donnée. Par exemple, la fonction f du système (1.1) peut contenir une somme de forces déterministes et de termes impliquant des dérivées distributionnelles d'un processus de Wiener. Définir le comportement de f est crucial pour donner un sens aux équations qui nous intéressent. Plus précisément, les bruits rajoutés sont typiquement additifs ou multiplicatifs, et selon chaque cas, les

propriétés des solutions, les hypothèses imposées ainsi que les techniques de démonstration varient considérablement. Fournir un bruit à une équation aux dérivées partielles n'améliore pas nécessairement les propriétés des solutions, comme c'est le cas avec le problème de Navier-Stokes stochastique. En effet, dans un cadre tridimensionnel, ces équations n'ont que des solutions martingales, qui est la forme la plus faible d'une solution vue dans un sens probabiliste, et l'unicité reste également un problème ouvert qui devient plus difficile à démontrer que son homologue déterministe en raison de l'interaction du forçage stochastique. D'autre part, choisir une forme particulière de bruit (par exemple, un bruit additif non dégénéré) peut fournir des propriétés absentes dans le cas déterministe, telles que l'unicité des solutions. Ceci est possible grâce aux équations de Kolmogorov et à leur relation avec les problèmes stochastiques.

Lorsqu'elles sont perturbées par un bruit, les équations de Navier-Stokes sont utilisées comme prototype complémentaire de leur version déterministe pour parvenir à une compréhension approfondie des variations mineures présentes dans les écoulements de fluides. Dans cette thèse, les équations de Navier-Stokes stochastiques sont régies par le système:

$$\begin{cases} du + (-\nu\Delta u + [u \cdot \nabla]u + \nabla p) dt = f dt + g(\cdot, u)dW, \\ \operatorname{div}(u) = 0, \end{cases} \quad (1.2)$$

et sont munies d'une donnée initiale aléatoire $u_0 = u(0, \cdot)$, sous conditions aux limites périodiques ou Dirichlet homogènes. Les quantités de l'équation ci-dessus sont similaires à celles présentées dans la Section 1.1. La fonction f étant sans bruit mais pourrait impliquer un caractère aléatoire et dépendre de la solution u . W est un processus de Wiener avec des valeurs dans un espace de Hilbert approprié. L'opérateur g est appelé coefficient de diffusion et il dépend de la solution u . Un autre type de bruit que l'on trouve dans la littérature des équations de Navier-Stokes stochastiques, et par lequel le problème (1.2) est piloté, est le processus de Lévy [25, 57, 92].

Les équations (1.2) admettent une unique solution forte (au sens probabiliste) en deux dimensions [66, 91], et des solutions martingales faibles en trois dimensions [58], ce qui signifie qu'un espace de probabilité filtré n'aurait pas pu être prescrit avant la démonstration de l'existence des solutions. Les questions tridimensionnelles connexes ne s'abordent pas de la même manière que celles du cadre bidimensionnel. Cela est dû aux inégalités de Sobolev inappropriées, y compris celle de Gagliardo-Nirenberg, qui est moins utile en trois dimensions et ne peut pas être combinée avec l'inégalité de Young pour gérer les termes supplémentaires qui surviennent généralement lors de l'identification des fonctions limites. Il convient de souligner le rôle essentiel de la propriété de monotonie locale détenue par la somme du terme visqueux et du terme convectif des équations (1.2), qui exclut toute obligation nécessitant l'utilisation des solutions faibles en deux dimensions et offre celles fortes au sens probabiliste [88, 91].

1.2 . Les équations de Navier-Stokes moyennées au sens de Lagrange (LANS- α)

Pour surmonter la malédiction des équations de Navier-Stokes, il faut passer par une autre trajectoire tout en maintenant le même environnement. Compte tenu de ses propriétés physiques et géométriques particulières, le modèle de Navier-Stokes moyenné au sens de Lagrange (LANS- α) est un compromis qui lève toutes les ambiguïtés en question et propose des fonctionnalités importantes telles que des économies de calcul analogues aux modèles de simulation des grandes structures de la turbulence (SGS) pour les écoulements turbulents forcés dans des conditions aux limites périodiques. Elles sont également un portail pour simuler la forme non modifiée des équations de Navier-Stokes incompressibles, surtout lorsque leur échelle spatiale α prend de petites valeurs. En effet, lorsque α tend vers zéro, les solutions fortes du modèle LANS- α stochastique convergent vers celles des équations de Navier-Stokes stochastiques, comme démontré au Chapitre 3. Ceci est extrêmement bénéfique car nous pourrions choisir des valeurs minimales de α lors de la simulation de turbulences sans aucune perte de propriétés associées

avec LANS- α , y compris la régularité spatiale supplémentaire de la solution, qui apporte à toute méthode numérique basée sur la discrétisation spatiale un supplément à la vitesse de convergence. Les équations de Navier-Stokes moyennées au sens de Lagrange sont également connues sous les noms suivants: modèle de Navier-Stokes- α (NS- α) ou problème visqueux de Camassa-Holm. Elles ont été développées par Shiyi Chen, Ciprian Foias, Darryl Holm, Eric Olson, Edriss Titi et Shannon Wynne en 1998-1999.

Le modèle LANS- α est attractif sur le plan informatique et pertinent sur le plan théorique. Il existe à la fois en versions anisotrope [90] et isotrope [35]. Le premier est adapté aux écoulements de cisaillement et consiste en un système couplé d'équations aux dérivées partielles, comprenant un champ moyen de vitesse de fluide et un tenseur de covariance lagrangien. Il est généralement utilisé pour les états où les propriétés d'écoulement de fluctuation dépendent des directions; notamment pour les turbulences anisotropes. Cette version ne sera pas étudiée dans cette thèse; l'attention sera plutôt tournée vers la conception isotrope dont les équations s'écrivent comme suit:

$$\begin{cases} \partial_t (u - \alpha^2 \Delta u) - \nu \Delta (u - \alpha^2 \Delta u) - u \times (\nabla \times (u - \alpha^2 \Delta u)) + \nabla p = f, \\ \operatorname{div}(u) = 0, \\ u(0, \cdot) = u_0. \end{cases} \quad (1.3)$$

Les inconnues dans le système ci-dessus sont le champ vectoriel de vitesse du fluide $u = u(t, x)$ et le champ de pression $p = p(t, x)$. Les paramètres strictement positifs ν et α représentent respectivement une viscosité cinématique du fluide et une échelle de filtrage spatial. La force extérieure f est similaire à celle des équations (1.1), et la donnée initiale u_0 peut être prise identique à celle des équations de Navier-Stokes si l'objectif principal était d'obtenir des solutions pour les équations (1.1) découlant du système ci-dessus. L'équation de conservation de masse (deuxième identité dans le système (1.3)) indique l'incompressibilité des fluides, ce qui signifie que les liquides compressibles et les gaz doivent être analysés dans des cadres différents. Contrairement à la version anisotrope, l'échantillon isotrope est dédié aux fluctuations turbulentes statistiquement uniformes dans toutes les directions. Il peut être considéré comme un écoulement turbulent éloigné des frontières et donc être au service de l'étude des propriétés fondamentales par une approche simplifiée. La dérivation du modèle isotrope LANS- α sera fournie dans la Section 1.2.1.

La non-linéarité convective apparaissant dans les équations de Navier-Stokes redirige continuellement l'énergie des grandes vers les petites échelles spatiales jusqu'à atteindre l'échelle de dissipation de Kolmogorov, où elle se transforme en chaleur. Par conséquent, pour les capturer totalement, les approches numériques associées doivent nécessiter des points de grille adéquats conduisant à un processus chronophage important, en particulier pour les écoulements complexes. Heureusement, ce terme convectif est ajusté et ne joue aucun rôle dans le modèle LANS- α , dont les solutions sont dotées d'une régularité spatiale supplémentaire grâce au terme visqueux ($\nu \Delta (I - \alpha^2 \Delta)$). Le terme non linéaire correspondant ($u \times (\nabla \times (u - \alpha^2 \Delta u))$) est responsable de l'arrêt de la perte d'énergie qui se produit dans le système de Navier-Stokes, et c'est la raison pour laquelle les équations LANS- α ne sont pas coûteuses en calcul pour les écoulements turbulents. Cette modification crée un modèle dispersif à travers le système (1.3) où une pénalité d'énergie spécifique est injectée pour empêcher de minuscules agitations en dessous d'une certaine échelle de longueur. Un avantage supplémentaire des équations (1.3) est l'existence et l'unicité de solutions régulières en deux et trois dimensions [60, 89].

De nombreuses variantes, autres que les équations de Navier-Stokes moyennées au sens de Lagrange, ont été révélées et développées pour les mêmes raisons mentionnées ci-dessus. Par exemple, nous pouvons rencontrer dans la littérature les modèles Leray- α [36], Bardina [5], Smagorinsky [104], et Leray- α modifié [77]. Néanmoins, cette thèse privilégie les équations LANS- α pour les motifs suivants:

- i. conservation de l'enstrophie bidimensionnelle, de l'hélicité et de l'énergie [59, 97],

- ii. Le théorème de circulation de Kelvin est applicable [59],
- iii. elles sont indépendantes du référentiel choisi [70],
- iv. la dissipation d'énergie et d'hélicité se produit indépendamment du nombre de Reynolds (Re), comme dans les écoulements de fluides réels [85],
- v. la cascade d'énergie à travers la région inertielle s'effectue à un rythme identique aux équations de Navier-Stokes [59],
- vi. la résolution totale est possible en $O(Re^{3/2})$ degrés de liberté, par rapport à $O(Re^{9/4})$ pour les équations de Navier-Stokes [59].

Certaines des propriétés énumérées ci-dessus ne sont pas vérifiées pour d'autres variantes; par exemple, le modèle de Bardina ne conserve pas l'énergie et l'hélicité. En outre, la variante de Leray- α ne satisfait ni le théorème de circulation de Kelvin ni la conservation de l'hélicité et est dépendante du référentiel considéré. Il en va de même pour le modèle de Smagorinsky, qui insère une hélicité non physique dans ses solutions et dissipe rapidement l'énergie via la région inertielle.

1.2.1 . Dérivation des équations LANS- α

Le modèle de Navier-Stokes moyenné au sens de Lagrange est une version visqueuse des équations d'Euler moyennées au sens de Lagrange (LAE- α), qui jouent le rôle d'une variante au problème d'Euler incompressible. Diverses techniques de dérivation du modèle LANS- α peuvent être trouvées dans la littérature, conduisant éventuellement vers les mêmes équations. L'une d'entre elles est la méthode variationnelle d'Euler-Poincaré [74, 75]. La stratégie de dérivation qui sera suivie ici provient de [59]. Elle est traitée du point de vue du théorème de circulation de Kelvin, qui est satisfait par les équations de Navier-Stokes (1.1). Pour ce faire, plusieurs étapes doivent être franchies afin que les équations LANS- α soient établies (voir figure 1.1).

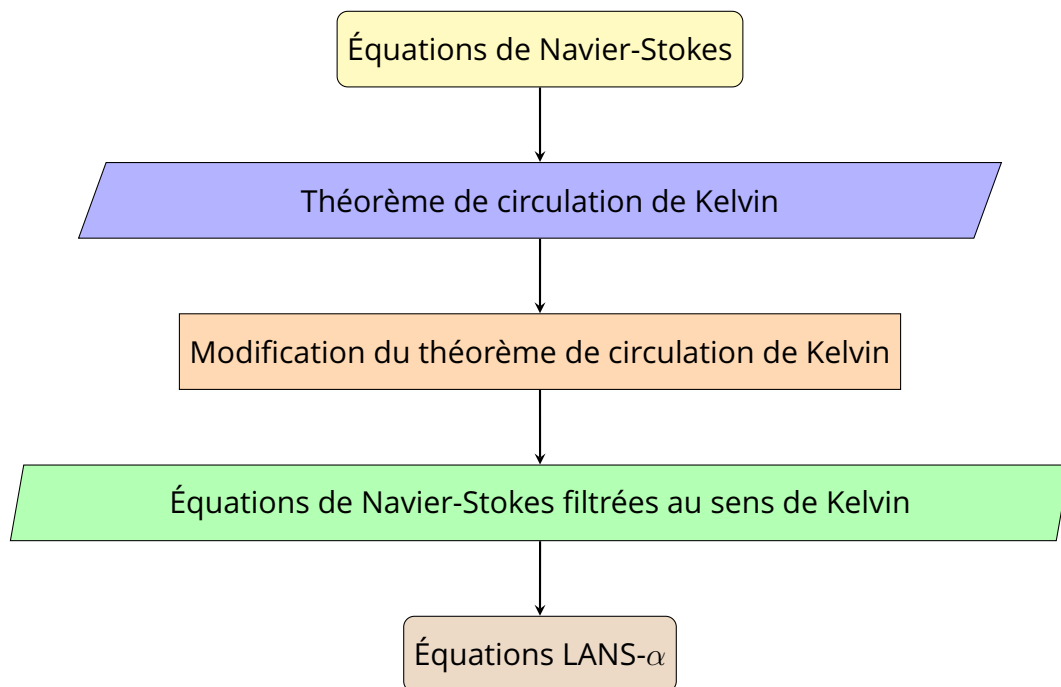


Figure 1.1 – Construction du modèle LANS- α

L'objectif principal est de démystifier le problème de Navier-Stokes filtré au sens de Kelvin, permettant

la modélisation du système (1.3). Définissons tout d'abord la circulation

$$\Gamma(t) := \oint_{C(u)} u(t, x) \cdot dx,$$

où u est la vitesse eulérienne du fluide, C représente un contour matériel fermé qui se déplace avec les particules fluides (la vitesse u dans ce cas) et $x = x(t)$ est la position relative aux particules fluides. Le théorème de circulation de Kelvin relatif au problème (1.1) se concrétise comme suit

$$\frac{d\Gamma}{dt} = \oint_{C(u)} \left(\frac{\partial u}{\partial t} + [u \cdot \nabla]u \right) \cdot dx = \oint_{C(u)} (\nu \Delta u + f) \cdot dx. \quad (1.4)$$

Pour les fluides non visqueux qui ne sont pas soumis à une force extérieure ($f = 0$), le membre de droite de l'équation (1.4) renvoie habituellement une valeur nulle, ce qui donne $\frac{d\Gamma}{dt} = 0$. Le mouvement conjoint de C avec la particule fluide est déployé dans ce cas pour accorder l'égalité $\frac{d}{dt}(dx) = du$ par laquelle le terme convectif $[u \cdot \nabla]u$ apparaît dans (1.4). Une étape cruciale pour atteindre les équations de Navier-Stokes filtrées au sens de Kelvin consiste à ajuster $C(u)$ pour qu'il permette à l'identité (1.4) de s'intégrer autour d'un contour fermé $C(\bar{u})$ qui se déplace en restant lié à une vitesse de fluide eulérienne spatialement filtrée, notée \bar{u} . Le filtre susmentionné relie à la fois u et \bar{u} à travers la relation de convolution suivante:

$$\bar{u}(t, x) = (\mathcal{F} * u)(t, x) = \int \mathcal{F}(x - y)u(t, y)d^3y.$$

Le point critique ici est de choisir \mathcal{F} de manière à autoriser l'identité $\mathcal{F} * \theta = 1$ (pourrait avoir lieu au sens distributionnel) pour un opérateur θ . Autrement dit, il faut que u soit exprimé en fonction de \bar{u} , notamment $u = \Lambda \bar{u}$, pour un certain Λ indépendant du temps, symétrique, isotrope, invariant par translation, et positif qui dépend implicitement de θ , et dont la fonction de Green est le filtre \mathcal{F} . Par conséquent, le théorème de circulation de Kelvin nouvellement modifié devient:

$$\frac{d\bar{\Gamma}}{dt} = \oint_{C(\bar{u})} (\nu \Delta u + f) \cdot dx,$$

où $\bar{\Gamma}(t) := \oint_{C(\bar{u})} u \cdot dx$ est la circulation relative au contour $C(\bar{u})$. Le membre de droite de l'égalité ci-dessus ne nécessite aucune modification, par contre la dérivée temporelle de $\bar{\Gamma}$ en a besoin. En effet, en utilisant $u = \Lambda \bar{u}$, les hypothèses imposées sur Λ ainsi que le co-mouvement de $C(\bar{u})$ avec \bar{u} , et en raisonnant par une manière similaire à la dérivation temporelle de Γ (première égalité de l'identité (1.4)), les équations de Navier-Stokes filtrées au sens de Kelvin seront les suivantes:

$$\begin{cases} \partial_t u - \nu \Delta u + [\bar{u} \cdot \nabla]u + (\nabla \bar{u})^T \cdot u + \nabla p = f, \\ u = \Lambda \bar{u}, \\ \text{div}(\bar{u}) = 0. \end{cases} \quad (1.5)$$

Observez que $[\bar{u} \cdot \nabla]u + (\nabla \bar{u})^T \cdot u = -\bar{u} \times (\nabla \times u) + \nabla(\bar{u} \cdot u)$ par une simple propriété vectorielle. La quantité $\nabla(\bar{u} \cdot u)$ peut soit être absorbée dans le terme de pression ∇p , soit annulée grâce à la divergence nulle de \bar{u} et u . L'annulation est légitime car l'analyse mathématique de ces équations les multiplie par des fonctions test à divergence nulle qui, en appliquant les formules de Green, attribuent une valeur nulle à ce membre supplémentaire. Dans cette thèse, on a préféré exprimer le terme non linéaire en fonction du produit vectoriel. Cela est dû à des objectifs de stabilité qui deviennent visibles une fois que l'analyse par éléments finis prend le relais. Finalement, le remplacement de Λ par l'opérateur de Helmholtz $I - \alpha^2 \Delta$ dans les équations (1.5) donne le modèle LANS- α (1.3). Il convient de mentionner que u et \bar{u} sont appelés fluides non filtrés et filtrés, respectivement, et que l'appellation de α comme échelle de filtrage spatiale émerge de la définition du filtre Λ .

1.2.2 . Les équations LANS- α stochastiques

Les équations de Navier-Stokes stochastiques, comme leur version déterministe, admettent des propriétés controversées (voir Section 1.1.1), et l'existence du modèle de Navier-Stokes moyenné au sens de Lagrange dans une conception sans bruit ne semble pas un substitut raisonnable en raison de l'aspect aléatoire qu'une force extérieure bruitée donne aux solutions de ces équations aux dérivées partielles. Cela étant dit, le modèle LANS- α stochastique a été introduit dans la littérature [28, 29] en complément de sa forme déterministe dans le but de donner un sens supplémentaire à la version stochastique du problème de Navier-Stokes, à condition que sa donnée initiale et la force aléatoire externe associée soient identiques à celles de Navier-Stokes. Grosso modo, annuler α dans les équations (1.3) fournirait le système déterministe de Navier-Stokes. En deux dimensions, ce fait a été démontré dans [27] en calculant une vitesse de convergence en fonction de l'échelle spatiale α de la différence entre les deux solutions correspondantes; notamment

$$\sup_{0 \leq t \leq T} \|u^\alpha(t) - u(t)\|_{L^2}^2 \leq C(\alpha),$$

où u^α et u sont des solutions aux équations (1.3) et (1.1), respectivement, et $C(\alpha)$ tend vers 0 lorsque $\alpha \rightarrow 0$. Le même concept en va de même pour les versions stochastiques bidimensionnelles associées, comme illustré au Chapitre 3 à travers une technique différente. En trois dimensions, il a été démontré dans [51] que les solutions martingales du modèle LANS- α stochastique convergent vers celles de Navier-Stokes lorsque α tend vers 0.

Tout au long de cette thèse, le modèle de LANS- α stochastique (isotrope) qui sera concerné est donné par le système d'équations suivant :

$$\begin{cases} d(\bar{u} - \alpha^2 \Delta \bar{u}) - [\nu \Delta (\bar{u} - \alpha^2 \Delta \bar{u}) + \bar{u} \times (\nabla \times (\bar{u} - \alpha^2 \Delta \bar{u})) - \nabla \bar{p}] dt \\ \quad = f(\cdot, \bar{u})dt + g(\cdot, \bar{u})dW, \\ \text{div}(\bar{u}) = 0, \\ \bar{u}(0, \cdot) = \bar{u}_0, \end{cases} \quad (1.6)$$

où la vitesse du fluide filtré \bar{u} et le champ de pression \bar{p} sont les inconnus, les constantes ν et α sont identiques à celles de l'équation (1.3), et \bar{u}_0 est la donnée initiale. Le membre de droite de la première identité est constitué d'une force extérieure $f = f(t, \bar{u})$ qui peut dépendre du caractère aléatoire, et d'un bruit aléatoire multiplicatif dépendant de l'état avec $W = \{W(t), t \in [0, T]\}$ étant un processus de Wiener de dimension infinie et $g = g(t, \bar{u})$ est un coefficient de diffusion Lipschitzien non nul. Visiblement, le système (1.6) représente un problème du quatrième ordre qui ne convient pas aux méthodes d'éléments finis largement répandues et facilement implémentables basées sur des polynômes continus par morceaux. Ce n'est en aucun cas un obstacle puisque les équations (1.6) passeront par un processus de couplage, leur permettant de s'affilier à l'ensemble des problèmes de second ordre et de lier \bar{u}_0 à la donnée initiale u_0 du système (1.2). Contrairement aux équations de Navier-Stokes stochastiques, dont l'étude a été fournie avec plusieurs options de bruit, le modèle LANS- α stochastiques a été uniquement étudié (au moment de la rédaction de cette thèse) lorsqu'il est piloté par un processus de Wiener. C'est pourquoi aucun autre processus stochastique bien connu n'a été pris en compte ici.

Pareillement au cas déterministe, le modèle de LANS- α stochastique offre de bonnes propriétés en termes de régularité et d'unicité des solutions, tant que le coefficient de diffusion g est supposé Lipschitzien. En fait, lorsque cette hypothèse est vérifiée, des solutions fortes (également appelées variationnelles) aux équations (1.6) existent et sont uniques [29] en deux et trois dimensions. Des résultats similaires sont également obtenus si les équations sous-jacentes héritent des caractéristiques [28]. En revanche, lorsque g n'est pas Lipschitzien, seules des solutions faibles (martingales) existent, et leur unicité n'est pas vérifiée [49]. Dans un tel cas, le modèle LANS- α n'apporte aucune amélioration mathématique par rapport aux équations de Navier-Stokes stochastiques tridimensionnelles. Une autre

configuration intéressante qui apparaît dans la littérature est le bruit additif, où le coefficient de diffusion g agit comme un opérateur nucléaire indépendant de la solution, résultant en l'existence et l'unicité d'une solution forte (au sens probabiliste) [68]. Le modèle de LANS- α stochastique peut également être couplé à d'autres équations afin de modéliser efficacement un phénomène naturel, comme effectuer dans [67] où le but était d'étudier la dynamique des vésicules élastiques dans un fluide visqueux incompressible en mouvement.

1.3 . Approches numériques pour les équations de Navier-Stokes stochastiques

Dans de nombreux cas, les méthodes numériques pour le problème de Navier-Stokes stochastique sont inspirées de celles proposées à la version déterministe car le processus de traitement en est rendu intuitif par l'étude déterministe préalable. L'aspect aléatoire et l'interaction avec le forçage stochastique ne facilitent pas l'étude; elles ajoutent un aspect complexe aux approximations numériques basées sur les approches déterministes, en particulier les schémas non linéaires. Par exemple, l'unicité des itérations, qui est assurée dans le cas déterministe par une contrainte de maillage et un argument de contraction ([30, Section 4]), peut devenir impossible en configurations stochastiques sauf sous une sorte de condition CFL et dans un sous-ensemble de probabilité. Un autre inconvénient qui apparaît est l'application du lemme de Grönwall qui devient beaucoup plus difficile lorsque l'espérance mathématique est appliquée à une identité, car les termes du côté droit pourront gagner des puissances supplémentaires après avoir utilisé une inégalité type Cauchy-Schwarz ou Hölder. Ceci explique pourquoi certaines propriétés déterministes ne peuvent pas être conservées lors du passage à un aspect stochastique. En général, traiter des schémas numériques linéaires corrigerait les lacunes susmentionnées, mais il y a un prix à payer pour la stabilité. Autrement dit, une régularité supplémentaire doit être déployée pour les données initiales, ce qui peut ne pas sembler naturel sauf au cas où l'étude vise la vitesse de convergence.

La convergence des schémas numériques non linéaires relatifs aux équations aux dérivées partielles stochastiques est généralement réalisée par l'une des techniques suivantes:

- i. argument de compacité impliquant l'utilisation du théorème de Skorokhod [46, Theorem 2.4], ou
- ii. une propriété de monotonie locale émergeant de certains membres d'une équation aux dérivées partielles donnée.

La technique de compacité peut être considérée comme une alternative au théorème d'Aubin-Lions-Dubinskĩ (c.f. [4, 6, 78, 103]) avec des propriétés plus délicates. En fait, s'appuyer sur un tel théorème ne préserve ni l'espace de probabilité défini à l'origine ni le processus de Wiener, ce qui signifie qu'il dévie le sens d'une solution forte s'il en existe une. Cependant, un avantage offert dont on peut tirer parti est la convergence au sens fort des suites, ce qui facilite l'analyse de la convergence d'un schéma numérique proposé. En général, l'application de l'argument de compacité est réservée aux équations qui n'ont que des solutions faibles (martingale) comme les équations tridimensionnelles de Navier-Stokes stochastiques ou le problème de Landau-Lifshitz-Gilbert stochastique [7, 24], pour en citer quelques-uns. D'autre part, une propriété de monotonie locale, lorsqu'elle est disponible, évite les inconvénients de l'argument de compacité en éliminant la complexité qui se produit (par exemple, les non-linéarités). Elle est normalement utilisée pour aborder l'identification des termes discrets avec leurs homologues dépendants de la solution. Une telle propriété est typiquement utilisée pour illustrer l'existence de solutions aux équations de Navier-Stokes stochastiques bidimensionnelles [91] et sera mise à profit lors de l'analyse de convergence des schémas numériques proposés dans cette thèse. Outre les méthodes mentionnées ci-dessus, on peut également tomber sur d'autres techniques liées à des solutions de haute régularité émergeant de problèmes du quatrième ordre comme les équations de Navier-Stokes moyennées au sens de Lagrange et le modèle de Leray- α (c.f. [29, 37]), mais peuvent ne pas être applicables directement lorsqu'il s'agit de méthodes d'éléments finis dont les fonctions de base sont uniquement continues sur tout le domaine,

non dérivables, et ayant une divergence non nulle. Ceci est à cause de la régularité maximale des fonctions de base correspondantes qui ne peut pas dépasser H^1 . D'autres modifications de l'équation de Navier-Stokes stochastique manifestent un moyen de mener des études d'approximation numérique pour les solutions aux équations de Navier-Stokes stochastiques, comme le modèle de Brinkman-Forchheimer-Navier-Stokes, qui a été étudié dans l'article [12].

Les méthodes numériques pour résoudre les équations de Navier-Stokes stochastiques ne sont pas anciennes. Leur évolution a commencé dans les années 2010, au cours desquelles l'accent a été mis sur les méthodes des éléments finis. Par exemple, l'analyse de convergence d'une discrétisation spatio-temporelle pour les équations tridimensionnelles de Navier-Stokes munies d'un bruit multiplicatif a été menée dans [24]. Cette analyse repose sur des fonctions de base d'éléments finis à divergence faiblement nulle et construit un processus de Wiener relatif à l'espace de probabilité qui ressort de l'application du théorème de Skorokhod en raison des propriétés imparfaites des solutions en trois dimensions. À cause de la mauvaise régularité de la pression [84], l'aspect point-selle encore présent dans le schéma numérique proposé n'est pas utilisé dans l'étude de la vitesse de convergence [31], qui a été illustrée dans des conditions aux limites périodiques bidimensionnelles. Son amélioration a été exécutée plus tard dans [19] grâce à une technique de décomposition qui décompose le champ de pression en un terme déterministe dépendant de la non-linéarité et une quantité stochastique exprimée en fonction du bruit multiplicatif (c.f. [18, Section 3]). Une discrétisation spatio-temporelle par éléments finis de ces équations, lorsqu'elles sont pilotées par un forçage stochastique additif et équipées de conditions aux limites à glissement nul en deux dimensions, a été réalisée dans [20] en transformant le problème de Navier-Stokes stochastique en une équation aux dérivées partielles aléatoire. Des études supplémentaires abordant le même sujet avec d'autres propriétés sont réalisées dans [11, 14]. La semi-discrétisation temporelle a également joué un rôle dans l'approximation des équations de Navier-Stokes munies de bruits multiplicatifs et additifs [9, 10, 13]. Dans la plupart des cas, la vitesse de convergence temporelle accomplie est limitée à la régularité temporelle du forçage stochastique. Par exemple, elle est presque 1/2 lorsque le bruit est un processus de Wiener dont les trajectoires sont Hölderienne d'exposant 1/2 (c.f. [40–42, 86]).

Résoudre numériquement les équations de Navier-Stokes stochastiques dans leur forme authentique n'est pas l'unique moyen et, en raison de leurs faibles propriétés mathématiques, elles pourraient ne pas offrir des fonctionnalités de haute qualité pour un schéma numérique proposé (voir Sections 1.1 et 1.1.1). Une approche améliorant une méthode numérique, dans ce contexte, consiste à choisir une variante appropriée telle que les équations de Navier-Stokes stochastiques moyennées au sens de Lagrange (voir Section 1.2.2). En fait, une solution au modèle LANS- α stochastique constitue également une solution au problème de Navier-Stokes stochastique lorsque α tend vers 0, comme illustré au chapitre 3. Cela peut être extrêmement bénéfique, en particulier lorsque l'on relie l'échelle spatiale α aux paramètres de discrétisation de manière à s'annuler lors du passage à la limite (Chapitre 4), ce qui constitue un compromis entre le système de Navier-Stokes et le modèle LANS- α ayant un α fixe, ce qui signifie que ses solutions pourraient ne pas hériter toutes les propriétés attendues. Ainsi, la haute régularité spatiale issue des équations LANS- α stochastiques intègre la vitesse de convergence spatiale en l'augmentant d'un niveau supplémentaire par rapport au problème de Navier-Stokes stochastique. Il convient de mentionner l'effet statique des méthodes numériques relatives aux équations LANS- α sur la vitesse de convergence temporelle. En effet, grâce au système (1.6), il est clair que α contribue uniquement aux régularisations spatiales à travers la quantité $\alpha\Delta$. Par conséquent, une semi-discrétisation temporelle des équations de Navier-Stokes et de LANS- α doit retourner une vitesse de convergence temporelle identique contrôlée généralement par la régularité temporelle des trajectoires du forçage stochastique appliqué, comme indiqué dans le paragraphe précédent.

Une lacune supplémentaire qui ressort des équations de Navier-Stokes stochastiques est la faible

régularité de la pression [84], qui impacte négativement la vitesse de convergence d'un schéma numérique basé sur des éléments finis traitant un aspect point-selle, comme démontré dans [31]. En fait, les fonctions de base à divergence faiblement nulle n'éliminent pas totalement le champ de pression de l'équation de quantité de mouvement approchée; c'est-à-dire qu'il faut gérer la faible régularité associée à un moment donné. D'autre part, l'approximation des équations de Navier-Stokes stochastiques par la méthode des éléments finis à divergence nulle au sens fort est largement connue comme une procédure douloureuse, en particulier en ce qui concerne les implémentations, sans parler de certains comportements étranges qui pourraient en résulter, tels que des conditionnements instables (c.f. [111]). Par suite, un compromis décent entre un problème de point-selle et les méthodes d'éléments finis à divergence nulle est une technique de compressibilité artificielle (Chapitre 6), également connue sous le nom de méthode de pénalisation, qui consiste à perturber l'équation de conservation de masse du problème de Navier-Stokes dans le but d'ajouter une quantité dépendante de la pression multipliée par un certain paramètre, généralement noté ε et considéré aussi petit que possible. Autrement dit, lorsque ε tend vers 0, le fluide retrouve son état incompressible initial. En comparaison avec l'échelle spatiale α du modèle de LANS- α et ses hypothèses associées, ε doit également subir quelques suppositions permettant à un schéma numérique suggéré de converger vers un état incompressible. Ces hypothèses sont fortement dépendantes du choix de la perturbation; notamment, ε peut être exprimé en fonction du pas de discrétisation spatiale, du pas de discrétisation temporelle ou d'une combinaison de tous les paramètres de discrétisation.

1.4 . Approche numérique pour les équations LANS- α stochastiques

L'échelle spatiale α correspondante offre deux méthodes possibles pour approcher les équations LANS- α , dont l'une est discutée dans la Section 1.3 lorsque l'échelle α est contrôlée par les paramètres de discrétisation. Une autre approche réalisable apparaît lorsque α est fixe; c'est-à-dire qu'il conserve sa valeur initiale quand tous les paramètres de discrétisation passent à 0. De cette façon, un schéma numérique fournirait une solution aux équations LANS- α au lieu du problème de Navier-Stokes. La première et la dernière approche ont également été étudiées pour des configurations déterministes (c.f. [26, 44]). Cependant, leurs manipulations diffèrent fortement par rapport aux modèles stochastiques, qui sont traités dans cette thèse.

Comme on le sait, les équations LANS- α ont été fondées pour résoudre la plupart des inconvénients de Navier-Stokes et pour constituer un substitut chaque fois que les circonstances physiques le permettent. En raison de l'échelle spatiale α , ce processus de substitution peut conduire à des résultats indésirables comme une amplitude de pression très élevée lorsque α prend des valeurs dépassant un certain seuil. En effet, les champs de pression dans les systèmes incompressibles sont généralement résolus implicitement une fois que l'existence du vecteur vitesse a été prouvée et peut être exprimée en fonction de tous les membres de l'équation de quantité de mouvement. Par conséquent, une valeur élevée de α a un effet dominant et peut être facilement observée lorsque l'analyse associée est abordée via la transformée de Fourier. Cela dit, l'approximation des équations stochastiques LANS- α avec α fixe doit être traitée avec précaution. En réalité, les valeurs de α doivent être choisies proches des minimums accessibles pour l'exécution des algorithmes.

Les équations LANS- α sont visiblement du quatrième ordre et peuvent donc être approchées par des méthodes d'éléments finis soit en divisant l'équation de quantité de mouvement en deux identités dans le but d'obtenir un problème de second ordre et d'utiliser des approximations typiques telles que des méthodes d'éléments finis basées sur des fonctions continues polynomiales par morceaux, soit en appliquant des discrétisations directement à la forme brute du modèle. Transformer les équations sous-jacentes en un problème couplé a l'avantage de s'adapter à tous les outils d'éléments finis existants dans la littérature qui offrent des techniques remarquablement simplifiées pour mettre en œuvre n'importe

quelle équation aux dérivées partielles. À première vue, cela peut sembler impeccable. Cependant, il a été montré dans [82] que la conservation de l'état d'origine d'un problème du quatrième ordre et l'approximation par des méthodes d'éléments finis ayant les fonctions de base continûment-dérivables impliquent une forme creuse plus favorable qui prend moins de temps à factoriser que les méthodes non conformes basées sur des polynômes du même ordre. Cela signifie en fait que le traitement de tels éléments implique la même précision qu'une méthode de pénalisation ou une formulation Galerkin discontinue, avec moins de dépenses. La principale cible qui doit être abordée lors de la gestion d'une méthode d'éléments finis basée sur des polynômes continûment-dérivables consiste à mapper des simplexes physiques sur des simplexes de référence, ce qui peut impliquer des transformations inhabituelles lorsqu'il s'agit d'éléments non conformes tels que les éléments d'Argyris.

1.5 . Plan du manuscrit

En excluant les sections introductives, cette thèse comprend concrètement quatre chapitres offrant un travail authentique qui n'a été considéré par aucun autre auteur. Les quatre chapitres sont liés à l'objectif principal de cette thèse qui consiste à proposer des approximations numériques aux équations de Navier-Stokes stochastiques à travers une variété de techniques.

- Le chapitre 3 est une partie préparatoire qui examine la convergence des solutions du problème de Navier-Stokes stochastique moyenné au sens de Lagrange vers les solutions des équations de Navier-Stokes stochastiques. Conceptuellement, il s'agit d'un chapitre théorique où aucun schéma numérique n'est fourni. Néanmoins, sa contribution est essentielle pour garantir que le modèle stochastique LANS- α peut se rapprocher du système de Navier-Stokes stochastique et pour clarifier les hypothèses nécessaires qui ne varient pas considérablement lors du passage à des configurations discrètes. L'ensemble de l'étude dans ce chapitre est maintenu en deux dimensions où les solutions fortes (au sens probabiliste) existent et sont plus faciles à gérer que les solutions faibles (martingales), qui nécessitent la plupart du temps une analyse minutieuse. Une méthode de Faedo-Galerkin est appliquée au modèle LANS- α stochastique, et l'échelle spatiale α correspondante est contrôlée par l'inverse d'une certaine valeur propre du Laplacien pour garantir son état de disparition à la limite. Des estimations a priori de régularités spatiales typiques et élevées sont réalisées, et une propriété de monotonie locale en rapport avec la somme du Laplacien et de la non-linéarité a été mise à profit afin de conserver l'espace de probabilité initialement introduit et le processus de Wiener. Les estimations a priori de haute régularité spatiale ne sont pas vitales pour l'objectif principal de ce chapitre mais seront bénéfiques pour étudier la vitesse de convergence d'une méthode d'éléments finis proposée. Le théorème suivant est obtenu:

Theorem 1.5.1 *L'unique solution forte (au sens probabiliste) des équations de Navier-Stokes stochastiques moyennées au sens de Lagrange (LANS- α) converge vers l'unique solution forte du problème de Navier-Stokes stochastique, lorsque l'échelle spatiale α tend vers 0 dans un domaine bidimensionnel convexe borné.*

Ce chapitre n'est ni publié ni soumis pour révision. Il représente le début d'un nouvel article dont le sujet aborde la vitesse de convergence d'une discrétisation par éléments finis des équations bidimensionnelles de Navier-Stokes stochastiques moyennées au sens de Lagrange.

- Le chapitre 4 est consacré à l'approximation des équations de Navier-Stokes stochastiques par le modèle de Navier-Stokes moyenné au sens de Lagrange stochastique dans un cadre bidimensionnel. Le contenu de cette partie est fortement inspiré des travaux théoriques du chapitre 3. Une discrétisation spatiale par éléments finis est proposée ainsi qu'une méthode d'Euler pour approcher les variables temporelles. Les équations d'intérêt sont transformées en un système couplé à travers

un filtre différentiel continu pour bénéficier des méthodes d'éléments finis ayant des fonctions de base continues sur tout le domaine et polynomiales par morceaux. Le schéma numérique suggéré a un aspect point-selle qui nécessite l'imposition d'une condition LBB discrète dans le but d'obtenir une stabilité vitesse-pression spécifique. Pour obtenir une solution aux équations de Navier-Stokes stochastiques, l'échelle spatiale α est censée être contrôlée uniquement par la taille du maillage, ce qui signifie que sa valeur est garantie de disparaître lors du passage à la limite. Cette hypothèse permet aux estimations a priori relatives aux itérés d'être uniformes en α ainsi qu'aux paramètres de discrétisation, ce qui est crucial pour acquérir des convergences faibles et faibles-*. Suivant la technique de démonstration du chapitre 3, une version discrète de la propriété de monotonie locale est prouvée et utilisée pour parvenir à la convergence du schéma numérique proposé, qui est finalement validée par une expérience numérique et évaluée contre un schéma qui a été étudié directement sur les équations de Navier-Stokes stochastiques dans un article indépendant. Ce chapitre étudie un schéma numérique spécifique (Algorithme 4.1) et propose le théorème suivant:

Theorem 1.5.2 *En deux dimensions, et sous un domaine polygonal convexe borné, une solution discrète (basée sur les méthodes d'éléments finis et d'Euler) aux équations de Navier-Stokes stochastiques moyennées au sens de Lagrange (LANS- α) provenant de l'algorithme 4.1 converge vers l'unique solution forte du système de Navier-Stokes stochastique, sous la condition que l'échelle α soit minorée et majorée par la taille du maillage.*

Le chapitre 4 fait partie d'un article intitulé *Numerical and Convergence Analysis of the Stochastic Lagrangian-Averaged Navier-Stokes Equations* et publié dans *Journal of Computational and Applied Mathematics*.

- Le chapitre 5 se concentre sur l'analyse numérique des équations de Navier-Stokes moyennées au sens de Lagrange tridimensionnelles entraînées par un bruit multiplicatif, et donne un aperçu des aspects théoriques correspondants, en particulier lorsque les équations d'intérêt sont transformées en un système couplé. Le problème de Navier-Stokes stochastique ne sera pas directement impliqué dans ce chapitre. Néanmoins, sa simulation est évoquée à titre de comparaison lorsqu'il s'agit de choisir la valeur de α lors de la partie expérimentation numérique. Cela donne un aperçu clair de la plage dans laquelle α devrait se situer car l'étude menée ne le lie à aucune hypothèse (autre que $\alpha < 1$ pour souligner sa petitesse). De manière similaire au chapitre 4, une méthode d'éléments finis est proposée pour discrétiser les variables d'espace, nécessitant la condition LBB discrète puisqu'aucune technique de stabilisation de la pression n'a été évoquée. La discrétisation temporelle repose sur la méthode d'Euler, qui fournit à la fois des schémas numériques non linéaires (Algorithme 5.1) et linéaires (Algorithme 5.2) en fonction de l'approximation de la non-linéarité et présentent des caractéristiques légèrement différentes: le schéma non linéaire offre une stabilité typique, mais ses itérés ne sont pas uniques dans tout l'espace de probabilité. Par contre, le schéma linéaire implique l'unicité presque sûre des itérés mais exige une régularité supplémentaire pour la donnée initiale. L'analyse de convergence des schémas numériques proposés est effectuée par une technique d'identification reposant sur la régularité spatiale supplémentaire offerte par les équations LANS- α et n'impliquant aucun argument de compacité qui modifie l'espace de probabilité originalement introduit. Ce chapitre peut également être considéré comme une généralisation des chapitres 4 et 6 quant au processus de Wiener utilisé où son opérateur de covariance est supposé nucléaire, mais pourrait être facilement généraliser en un processus cylindrique comme dans le chapitre 5. Puisque les équations LANS- α sont de quatrième ordre, une brève description d'une méthode d'éléments finis basée sur des polynômes par morceaux continûment dérivable sur tout le domaine est mise en évidence et appliquée à la forme brute des équations, c'est-à-dire sans subir aucun processus de couplage, et un algorithme associé est suggéré, bien que sa mise en œuvre puisse être technique et chronophage. Néanmoins, cela reste faisable car la méthode des

éléments finis proposée est conforme et la plupart des problèmes techniques sont évitables, tels que la création de transformations qui mappent des simplexes physiques à des simplexes de référence. Le chapitre 5 fournit des expériences numériques impliquant à la fois les champs de vitesse et de pression, ainsi que trois valeurs différentes de l'échelle spatiale α le long de laquelle la simulation est menée. Contrairement à la vitesse, le champ de pression donne une indication claire sur le choix de α car son amplitude varie drastiquement en conséquence. La principale contribution de ce chapitre apparaît dans le théorème énoncé grossièrement suivant:

Theorem 1.5.3 *Étant donné un $\alpha < 1$ et un domaine polyédrique convexe borné dans \mathbb{R}^3 , une approximation par éléments finis des équations de Navier-Stokes stochastiques moyennées au sens de Lagrange (LANS- α) converge vers l'unique solution forte (au sens probabiliste) du modèle stochastique LANS- α piloté par un bruit multiplicatif. L'algorithme 5.1 a une solution convergente avec une stabilité typique, et l'algorithme 5.2 approche l'unique solution du problème sous-jacent sous une hypothèse supplémentaire sur son point de départ.*

Ce chapitre fait une apparition dans l'article *Numerical and Convergence Analysis of the Stochastic Lagrangian-Averaged Navier-Stokes Equations* et a été publié dans la revue *Journal of Computational and Applied Mathematics*.

- Le chapitre 6 est consacré à l'approximation numérique des équations de Navier-Stokes stochastiques bidimensionnelles par une méthode de stabilisation de la pression (aussi appelée méthode de pénalisation ou compressibilité artificielle), qui consiste en une modification de l'équation de conservation de la masse. La technique présentée abolit l'aspect point-selle émergeant de la contrainte de divergence nulle et annule l'utilisation de conditions LBB discrètes. En vertu de cette stratégie, les méthodes d'éléments finis employées n'ont aucune restriction sur leurs degrés polynomiaux, et des estimations a priori pour la pression discrète apparaissent et semblent améliorer la vitesse de convergence des schémas impliquant un côté point-selle. Bien que l'équation de conservation de la masse ait été modifiée, l'analyse de convergence du schéma numérique proposé est effectuée de manière à obtenir les équations de Navier-Stokes incompressibles stochastiques à la limite. Ce chapitre inaugure un schéma non linéaire (Algorithme 6.1) pour lequel une propriété de monotonie locale est appliquée pour assurer sa convergence vers l'unique solution forte des équations de Navier-Stokes stochastiques. Un schéma numérique linéaire (Algorithme 6.2) est donné par la suite, dont la forme est construite en s'attaquant à l'approche implicite en temps de la non-linéarité de la méthode non linéaire. L'approche de compressibilité artificielle proposée est liée à un paramètre noté ε qui doit s'annuler à la limite afin de retrouver un état incompressible du fluide étudié. Tout au long du processus d'inspection, aucune hypothèse essentielle n'a été imposée à ε . Cependant, des nouvelles hypothèses peuvent être ajoutées si l'objectif est de comparer le schéma numérique fourni avec d'autres méthodes existantes dans la littérature. Par exemple, la comparaison de la technique de compressibilité artificielle avec un schéma basé sur le point-selle déclenche des hypothèses auxiliaires sur ε en fonction de paramètres de discrétisation. Des expériences numériques du schéma étudié sont livrées et comparées à une méthode basée sur le point-selle, où un graphique de l'erreur en fonction de ε est affiché afin d'assurer l'exactitude de l' ε choisi. Le chapitre 6 livre deux théorèmes, chacun pour un algorithme:

Theorem 1.5.4 *Dans un cadre bidimensionnel, sous un domaine polygonal convexe borné, et pour tous les paramètres de pénalisation $\varepsilon > 0$, un schéma numérique non linéaire (Algorithme 6.1) basé sur la méthode d'Euler et les éléments finis pour les équations de Navier-Stokes artificiellement compressibles entraînées par un bruit multiplicatif converge vers l'unique solution forte (au sens probabiliste) du problème de Navier-Stokes incompressible stochastique.*

Theorem 1.5.5 *En se plaçant dans un domaine polygonal convexe borné de \mathbb{R}^2 , et pour tous les paramètres de pénalité $\varepsilon > 0$, un schéma numérique linéaire (Algorithme 6.2) pour les équations de Navier-Stokes stochastiques artificiellement compressibles, consistant en une discrétisation temporelle basée sur la méthode d'Euler avec une approximation spatiale par éléments finis, converge vers l'unique solution forte (au sens probabiliste) du problème de Navier-Stokes incompressible stochastique.*

Ce chapitre a été soumis pour révision au moment de la rédaction de cette thèse.

2 - Introduction

2.1 . The Navier-Stokes equations

As with most baffling mathematical problems, the Navier-Stokes equations remain a relevant dilemma as they require a thorough understanding of their various frameworks, careful handling of each occurring situation, and a connection with actual circumstances allowing their results to be applied rigorously. The Navier-Stokes equations are universal laws of physics that model every fluid one has on the globe, including gases, and are considered one of the pillars of fluid mechanics. They represent a viscous regularization of the Euler equations, which remain an enigma, and may be employed to model ocean currents, airflow around a wing, weather behavior, blood flow, pollution analysis, designing aircraft, trains, cars, and many other applications. These equations were developed over multiple decades of gradually constructing the theories by the French physicist Claude-Louis Navier and the British mathematician and physicist Georges Gabriel Stokes.

Understanding turbulence was and is still pertinent to all ongoing developments, especially for industrial and environmental purposes. The onset of turbulence is analyzed as a loss of solutions' stability within the class of differentiable functions. This explains the poor properties usually yielded by the equations of interest. Indeed, the Navier-Stokes problem can be thought of as a generator that gets an input u_0 of certain regularity R and is expected to return an output u with the same given regularity i.e. R . Unfortunately this is false and is the reason why those equations are troublesome and need to be dealt with conscientiously.

Mathematically speaking, the three-dimensional Navier-Stokes equations remain, until this day, an open problem whose resolution via an analytical solution seems impossible, and their formulation cannot be further simplified. This is on account of their vectorial structure, which does not obstruct blow-ups. In other words, the only available control is Leray's energy inequality which only assigns a bound to the fluid velocity field in L^2 -norm but leaves the associated fluid vorticity untreated. This issue arises mainly from the nonlinearity of these equations despite the diffusive effect that dampens the high gradients. In fact, the nonlinearity, also known as the convective term, transfers energies from low to high gradients at a rate swifter than the damping provided by the diffusive term.

Many physicists attempted to circumvent the aforementioned shortcoming by introducing variants to the Navier-Stokes equations consisting of slight modifications, often permitting the diffusive effect to dominate the convective term. These variants solve most drawbacks that jump up from the original model due to the granted high regularity of the associated solutions. However, they may be applied with care as sometimes, the occurring conditions might not fit the opted variant, and therefore divergent misleading outcomes may pop out. For example, some of these derived models were created to solely model anisotropic fluids such as liquid crystals, unlike water and chloroform, which exclude any structural ordering of molecules. To name a few, the following variants can be found in the fluid mechanics literature: Leray- α , modified Leray- α , Clark- α , Lagrangian averaged Navier-Stokes. The latter model is explained further in Section 2.2 and is chosen in this thesis to be numerically treated for the sake of approximating solutions to the Navier-Stokes equations in an efficient fashion. Almost all of the existing variants are parametrized by a scale, often denoted α , which when taken equal to 0, renders roughly the Navier-Stokes problem.

Viscous incompressible fluids' motion with velocity vector field $u = u(t, x) \in \mathbb{R}^d$, $d \in \{2, 3\}$, and

pressure field $p = p(t, x) \in \mathbb{R}$, is modeled via the incompressible Navier-Stokes equations:

$$\begin{cases} \partial_t u - \nu \Delta u + [u \cdot \nabla]u + \nabla p = f, \\ \operatorname{div}(u) = 0, \end{cases} \quad (2.1)$$

equipped with an initial datum $u_0 = u(0, \cdot)$, and specific Dirichlet, Neumann, or Robin boundary conditions. The function f is an outer force for which physical experiments demonstrate that turbulence occurs if it has significant variations. The positive constant ν represents the kinematic viscosity which, when set to 0, transforms the above system into an Euler problem. The fluid's incompressibility is embodied through the second identity (null divergence), called the mass conservation equation. In terms of resolution, weak global solutions to the deterministic incompressible Navier-Stokes problem in three-dimensional domains were illustrated by Leray (1934) and Hopf (1951). However, the smoothness and uniqueness of Leray-Hopf solutions remain undisclosed because, as mentioned above, the smoothing effect of the viscous term ($\nu \Delta u$) is not adequate to cease the convective term's ($[u \cdot \nabla]u$) destructive properties. In point of fact,

- i. global (in time) weak solutions exist except for their uniqueness,
- ii. local (in time) solutions exist and are smooth, but their blow-up analysis is still an open problem.

This situation is much more relaxed for two-dimensional domains where the existence and uniqueness of global smooth solutions are gained.

2.1.1 . The stochastic Navier-Stokes equations

Nature does not always offer predictable well-behaved phenomena as several factors can intervene together and impact an outcome of an underlying subject. Deterministic systems can therefore not be the supreme option for getting the most out of a proposed model. For example, investigating turbulence around an airplane wing during winter, where multiple undesirable effects might add up, can lead to very optimistic results, meaning that the obtained theoretical outcome's properties may not thoroughly reflect the occurring situation. To render this fact clearer, notions of stochastic differential equations and their related mathematical theory were introduced by the Japanese mathematician Kiyosi Itô (1940) after establishing the concept of stochastic integrals, which eventually led to the Itô calculus, including the famous Itô formula that replaces the Taylor expansion for cases where functions are expressed in terms of stochastic processes. Later, the Russian physicist Ruslan Stratonovich initiated another approach resulting in calculus similar to ordinary calculus.

The most frequent form of stochastic partial differential equations in the literature consists in adding a stochastic process; a random variable whose index belongs to an uncountable set, usually dependent on a white noise, as an external force to a given partial differential equation. For example, the function f in system (2.1) can contain a sum of deterministic forces and terms involving distributional derivatives of a Wiener process. Defining the behavior of f is crucial in order to give the equations of interest a sense. More precisely, appended noises can be additive or multiplicative, and according to each case, properties of solutions, imposed assumptions along with demonstration techniques may vary dramatically. Supplying a partial differential equation with a noise does not necessarily improve solutions' properties, as is the case with the stochastic Navier-Stokes problem. Indeed, within a three-dimensional framework, these equations have only martingale solutions, which is the weakest form of a solution seen within a probabilistic sense, and uniqueness also remains an open problem that becomes harder to demonstrate than its deterministic counterpart due to the additional interplay of the stochastic forcing. On the other hand, opting for a particular form of a noise (for instance, nondegenerate additive noise) may provide properties absent in the deterministic case, such as the uniqueness of solutions. This is possible thanks to the Kolmogorov equations and their relation to stochastic problems.

When perturbed with a noise, the Navier-Stokes equations are employed as a complementary prototype of their deterministic version for achieving a deep understanding of minor variations present in fluid flows. In this thesis, the stochastic Navier-Stokes equations read

$$\begin{cases} du + (-\nu\Delta u + [u \cdot \nabla]u + \nabla p) dt = f dt + g(\cdot, u)dW, \\ \operatorname{div}(u) = 0, \end{cases} \quad (2.2)$$

endowed with a random initial datum $u_0 = u(0, \cdot)$, subject to homogeneous Dirichlet or periodic boundary conditions. Quantities of the above system are similar to those presented in Section 2.1. The function f is noise-free but can depend on the randomness and the solution u , and W is a Wiener process with values in an appropriate Hilbert space. Operator g is called the diffusion coefficient and depends on the solution u . Another type of noise that is found in the stochastic Navier-Stokes equations' literature and by which equations (2.2) may be driven is the Lévy process [25, 57, 92].

Equations (2.2) have unique strong (probabilistic) solution in two dimensions [66, 91], and weak martingale solutions in three dimensions [58] meaning that a filtered probability space cannot be prescribed prior to the solutions' existence. The related three-dimensional issues cannot be tackled in a similar way to that of the two-dimensional framework. This is on account of the inappropriate Sobolev inequalities, including the Gagliardo-Nirenberg estimate, which is less helpful in three dimensions, and cannot be combined with Young's inequality to handle the additional terms that usually arise while carrying out identifications of limiting functions. It is worth highlighting the essential role of the local monotonicity property that is owned by the sum of the viscous term and the convective term of equations (2.2), which renders weak (martingale) solutions obsolete in two dimensions and puzzles out strong probabilistic ones [88, 91].

2.2 . The Lagrangian averaged Navier-Stokes equations (LANS- α)

To overcome the curse of the Navier-Stokes equations, one must seek a bargain. Given its particular physical and geometrical properties, the Lagrangian averaged Navier-Stokes (LANS- α) model is a compromise that clears up all ambiguities in question and proposes significant features such as analogous computational savings as Large Eddy Simulation (LES) models for forced turbulent flows within periodic boundary conditions. They are also a portal to simulate the unamended form of the incompressible Navier-Stokes equations, especially when their spatial scale α takes on small values. In point of fact, when α tends to zero, strong solutions to the stochastic LANS- α model converge toward the stochastic Navier-Stokes solutions, as demonstrated in Chapter 3. This is tremendously beneficial as one can choose minimal values of α when simulating turbulences without any loss of LANS- α properties, including the additional solution's space-regularity, which supplies any utilized space discretization-based numerical method one extra rate of convergence. The Lagrangian averaged Navier-Stokes equations are also known as the Navier-Stokes- α (NS- α) model or the viscous Camassa-Holm problem. They were developed by Shiyi Chen, Ciprian Foias, Darryl Holm, Eric Olson, Edriss Titi, and Shannon Wynne during 1998-1999.

The LANS- α model is computationally attractive and theoretically relevant. It comes in both anisotropic [90] and isotropic [35] versions. The former is suitable for shear flows and consists of a coupled system of partial differential equations, including a mean fluid velocity field and a Lagrangian covariance tensor. It is typically employed for states where the fluctuation flow properties depend upon directions, namely, anisotropic turbulences. This version will not be investigated in this thesis; the focus

will be turned instead toward the isotropic design whose equations read:

$$\begin{cases} \partial_t (u - \alpha^2 \Delta u) - \nu \Delta (u - \alpha^2 \Delta u) - u \times (\nabla \times (u - \alpha^2 \Delta u)) + \nabla p = f, \\ \operatorname{div}(u) = 0, \\ u(0, \cdot) = u_0. \end{cases} \quad (2.3)$$

The unknowns in the above system are the fluid velocity vector field $u = u(t, x)$, and the pressure field $p = p(t, x)$. The positive parameters ν and α represent a kinematic fluid viscosity and a spatial filtering scale, respectively. The outer force f is similar to that of equations (2.1), and the initial datum u_0 may be taken identical to that of the Navier-Stokes equations if the main target was to obtain solutions streaming from the above system to equations (2.1). The mass conservation equation (second identity in system (2.3)) indicates fluids' incompressibility, meaning that compressible liquids and gases need to be analyzed within different frameworks. Unlike the anisotropic version, the isotropic sample is dedicated to turbulent fluctuations that are statistically uniform in all directions. It can be thought of as a turbulent flow distanced from boundaries and therefore be in the service of studying fundamental properties via a simplified approach. Derivation of the isotropic LANS- α model will be provided in Section 2.2.1.

The convective nonlinearity appearing in the Navier-Stokes equations continuously redirects the energy from large to small spatial scales until reaching the Kolmogorov dissipation scale, where it can be turned into heat. Consequently, to capture them all, associated numerical approaches must require adequate grid points leading to a significant time-demanding process, especially for complex flows. Fortunately, this convective term is adjusted and does not play a role in the LANS- α model, whose solutions are supplied with additional space-regularity through the viscous term ($\nu \Delta (I - \alpha^2 \Delta)$). The corresponding nonlinear term ($u \times (\nabla \times (u - \alpha^2 \Delta u))$) is responsible for ceasing the energy loss that occurs within the Navier-Stokes system, and it is the reason why the LANS- α equations are not computationally expensive for turbulent flows. This modification makes a dispersive model from system (2.3) where a specific energy penalty is inserted to prevent tiny excitations below some length scale. One extra advantage of equations (2.3) is the existence and uniqueness of regular solutions in two and three dimensions [60, 89].

Many variants, other than the Lagrangian averaged Navier-Stokes equations, were revealed and developed for the same above-mentioned reasons. For example, one may encounter in the literature the Leray- α [36], Bardina [5], Smagorinsky [104], and modified Leray- α [77] models. Nevertheless, this thesis favors the LANS- α equations for the following motives:

- i. conservation of two-dimensional enstrophy, helicity, and energy [59, 97],
- ii. Kelvin's circulation theorem is applicable [59],
- iii. they are frame invariant [70],
- iv. energy and helicity dissipation occurs independently of the Reynolds number (Re), as in genuine fluid flows [85],
- v. energy cascading via the inertial range at an identical rate as the Navier-Stokes equations [59],
- vi. full resolution is possible within $O(Re^{3/2})$ degree of freedom, compared to $O(Re^{9/4})$ for the Navier-Stokes equations [59].

Some of the above-enumerated properties do not populate other variants; for instance, the Bardina model does not conserve energy and helicity. On the other hand, the Leray- α variant fulfills neither Kelvin's circulation theorem nor helicity conservation and is not frame invariant. The same goes for Smagorinsky's model, which inserts nonphysically helicity into its solutions and swiftly dissipates energy via the inertial range.

2.2.1 . Derivation of the LANS- α equations

The Lagrangian averaged Navier-Stokes model is a viscous version of the Lagrangian averaged Euler (LAE- α) equations, which serve as a variant to the incompressible Euler problem. Various derivation techniques for the LANS- α model might be found in the literature, eventually conducting the same equations. One amongst them is Euler-Poincaré's variational fashion [74, 75]. The derivation strategy that will be followed herein [59] is seen from the viewpoint of Kelvin's circulation theorem, which is satisfied by the Navier-Stokes equations (2.1). For this purpose, several steps must be passed through in order for the LANS- α equations to be established (see figure 2.1). The primary target is to demystify the Kelvin-filtered Navier-Stokes problem, allowing system (2.3) to be puzzled out. Defining first the circulation

$$\Gamma(t) := \oint_{C(u)} u(t, x) \cdot dx,$$

where u is the Eulerian fluid velocity, and C represents a closed material contour that moves with the fluid particles (the velocity u in this case). The Kelvin circulation theorem relative to problem (2.1) reads

$$\frac{d\Gamma}{dt} = \oint_{C(u)} \left(\frac{\partial u}{\partial t} + [u \cdot \nabla]u \right) \cdot dx = \oint_{C(u)} (\nu \Delta u + f) \cdot dx. \quad (2.4)$$

For inviscid fluids that are not subject to an external force ($f = 0$), the right-hand side of equation (2.4) returns habitually a null value, resulting in $\frac{d\Gamma}{dt} = 0$. The co-movement of C with the fluid particle is deployed in this case to grant the equality $\frac{d}{dt}(dx) = du$ from which the convective term $[u \cdot \nabla]u$ appears in (2.4).

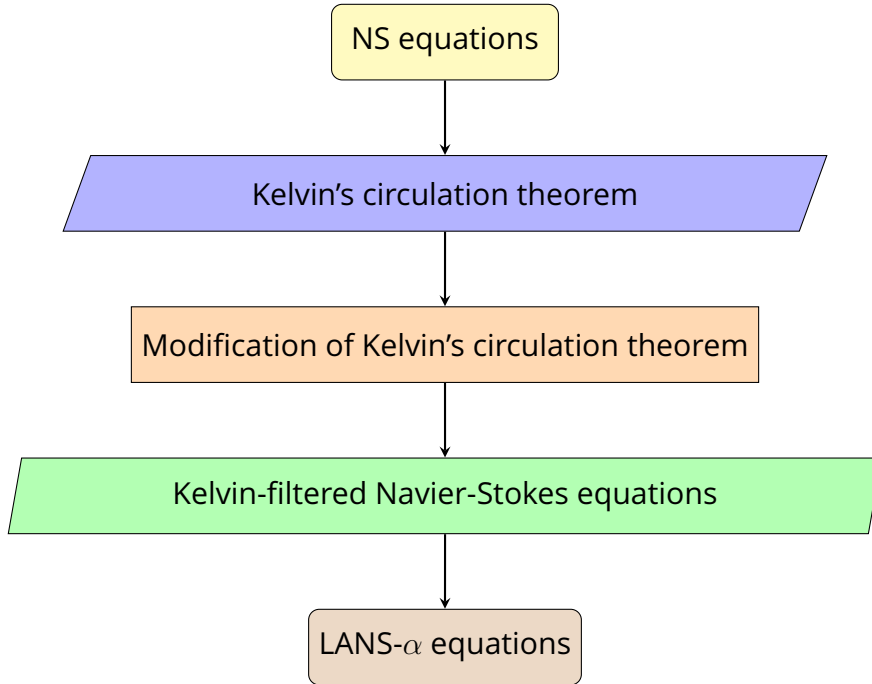


Figure 2.1 – The LANS- α construction

Adjusting $C(u)$ so that it authorizes identity (2.4) to integrate around a closed contour $C(\bar{u})$ that moves correlatively with a spatially filtered Eulerian fluid velocity, denoted \bar{u} , is one crucial step to come up with the Kelvin-filtered Navier-Stokes equations. The aforesaid filter links both u and \bar{u} via the following convolution relationship

$$\bar{u}(t, x) = (\mathcal{F} * u)(t, x) = \int \mathcal{F}(x - y)u(t, y)d^3y.$$

The critical point here is to choose \mathcal{F} in a way that permits the identity $\mathcal{F} * \theta = 1$ to hold (could be in a distributional sense) for some operator θ . In other words, one needs u to be expressed in terms of \bar{u} , such as $u = \Lambda \bar{u}$, for some time-independent, symmetric, isotropic, translation-invariant, and positive Λ that depends implicitly on θ , and whose Green's function is the filter \mathcal{F} . That said, the newly modified Kelvin's circulation theorem becomes

$$\frac{d\bar{\Gamma}}{dt} = \oint_{C(\bar{u})} (\nu \Delta u + f) \cdot dx,$$

where $\bar{\Gamma} := \oint_{C(\bar{u})} u \cdot dx$ is the circulation relative to the contour $C(\bar{u})$. The right-hand side of the above equality does not require any modification; the time-derivative of $\bar{\Gamma}$ does. Indeed, making use of $u = \Lambda \bar{u}$, the assumed properties on Λ together with the co-movement of $C(\bar{u})$ with \bar{u} , and arguing in a similar way to the time-derivation of Γ (first equality in identity (2.4)), the Kelvin-filtered Navier-Stokes equations follows:

$$\begin{cases} \partial_t u - \nu \Delta u + [\bar{u} \cdot \nabla] u + (\nabla \bar{u})^T \cdot u + \nabla p = f, \\ u = \Lambda \bar{u}, \\ \operatorname{div}(\bar{u}) = 0. \end{cases} \quad (2.5)$$

Note that $[\bar{u} \cdot \nabla] u + (\nabla \bar{u})^T \cdot u = -\bar{u} \times (\nabla \times u) + \nabla(\bar{u} \cdot u)$ by a simple vectorial property. The quantity $\nabla(\bar{u} \cdot u)$ can either be absorbed in the pressure term ∇p or canceled out on account of the null divergence of both \bar{u} and u . The cancellation makes sense because the mathematical analysis of these equations multiplies them by divergence-free test functions which, by applying Green's formulas, returns a null value of this supplementary item. The preferred form of the nonlinear term will be expressed in terms of the cross product in the present thesis. This is due to stability purposes that become visible once the finite element analysis takes over. Eventually, replacing Λ with the Helmholtz operator $I - \alpha^2 \Delta$ in equations (2.5) yields the LANS- α model (2.3). It is worth mentioning that u and \bar{u} are called unfiltered and filtered fluids, respectively, and that the nomenclature of α (spatial filtering scale) emerges from the filter Λ itself.

2.2.2 . The stochastic LANS- α equations

The stochastic Navier-Stokes equations, as their deterministic version, admit controversial properties (see Section 2.1.1), and the existence of the Lagrangian averaged Navier-Stokes model in only noise-free design does not seem a reasonable substitute due to the random aspect that a noise gives to solutions to partial differential equations. That said, the stochastic LANS- α model was introduced to the literature [28, 29] as a complement to its deterministic form for the sake of giving additional sense to the stochastic version of the Navier-Stokes problem, provided that the equipped initial datum and the external random force are identical to Navier-Stokes'. Roughly speaking, setting α to zero in equations (2.3) yields the deterministic Navier-Stokes system. This fact was demonstrated in [27] within two dimensions by calculating a convergence rate in terms of the spatial scale α of the difference between both corresponding solutions; namely

$$\sup_{0 \leq t \leq T} \|u^\alpha(t) - u(t)\|_{L^2}^2 \leq C(\alpha),$$

where u^α and u are solutions to equations (2.3) and (2.1), respectively, and $C(\alpha)$ goes to 0 when $\alpha \rightarrow 0$. The same concept goes for the associated two-dimensional stochastic versions, as illustrated in Chapter 3 through a different technique. In three dimensions, it was shown [51] that martingale solutions to the stochastic LANS- α model converge toward the Navier-Stokes martingale solutions as α tends to 0.

Throughout this thesis, the stochastic (isotropic) LANS- α model that will be concerned is given by

the following system of equations:

$$\begin{cases} d\left(\bar{u} - \alpha^2 \Delta \bar{u}\right) - \left[\nu \Delta \left(\bar{u} - \alpha^2 \Delta \bar{u}\right) + \bar{u} \times \left(\nabla \times \left(\bar{u} - \alpha^2 \Delta \bar{u}\right)\right) - \nabla p\right] dt \\ \quad = f(\cdot, \bar{u})dt + g(\cdot, \bar{u})dW, \\ \operatorname{div}(\bar{u}) = 0, \\ \bar{u}(0, \cdot) = \bar{u}_0, \end{cases} \quad (2.6)$$

where the filtered fluid velocity \bar{u} and the pressure field p are the unknowns, ν and α are identical to those of equations (2.3), and \bar{u}_0 is the initial datum. The right-hand side of the first identity consists of an outer force $f = f(t, \bar{u})$ that can depend on the randomness, and a state-dependent multiplicative random noise with $W = \{W(t), t \in [0, T]\}$ being an infinite-dimensional Wiener process and $g = g(t, \bar{u})$ is a non-null Lipschitz-continuous diffusion coefficient. Visibly, system (2.6) represents a fourth-order problem that is unsuitable for the widely spread and easily implementable finite element methods that are based on piecewise continuous polynomials. This is by no means an obstacle since equations (2.6) will go through a coupling process, allowing them to affiliate to second-order problems' set and to link \bar{u}_0 to the initial datum u_0 of system (2.2). Unlike the stochastic Navier-Stokes equations, whose study has been provided with several noise options, the stochastic LANS- α model was uniquely investigated (at the time of writing this thesis) when it is driven by a Wiener process, which is why no other well-known stochastic processes were considered herein.

As with the deterministic case, the stochastic LANS- α model offers good properties in terms of solutions' regularity and uniqueness, as long as the diffusion coefficient g is assumed to be Lipschitz-continuous. In point of fact, when this hypothesis holds, strong (also called variational) solutions to equations (2.6) exist and are unique [29] in two and three dimensions. Similar results are also obtained if the underlying equations are subject to hereditary characteristics [28]. In contrast, when g is not Lipschitz-continuous, only weak (martingale) solutions exist, and their uniqueness does not hold [49]. In such a case, the LANS- α model does not bring any mathematical improvement regarding the stochastic three-dimensional Navier-Stokes equations. Another interesting configuration that comes into view in the literature is the additive noise, where the diffusion coefficient g acts as a solution-independent nuclear operator, resulting in the existence and uniqueness of a strong (probabilistic) solution [68]. The stochastic LANS- α model can also be coupled with other equations in order for a natural phenomenon to be effectively modeled, as conducted in [67] where the aim is to investigate the elastic vesicle dynamics in a moving incompressible viscous fluid.

2.3 . Numerical approximations for the stochastic Navier-Stokes equations

In many instances, numerical methods for the stochastic Navier-Stokes problem are inspired by those proposed for the deterministic version because the handling process could be intuitive. The randomness intervention and the stochastic forcing interplay do not facilitate the study; they add a complex aspect to the deterministic-based numerical approximations, especially to nonlinear schemes. For instance, iterates' uniqueness, which is ensured in the deterministic case through a mesh constraint and a contraction argument ([30, Section 4]), may become impossible in stochastic configurations except under some sort of CFL condition and within a probability subset. Another drawback that comes into view is the Grönwall lemma's application which becomes challenging after taking the mathematical expectation of an equation, as terms on the right-hand side may gain additional power exponents after employing the Cauchy-Schwarz or Hölder inequality. This explains why some deterministic properties cannot be maintained when passing to a stochastic aspect. In general, dealing with linear numerical schemes fixes the aforementioned shortcomings, but there is a price to pay for stability. In other words, extra regularity must be deployed for the initial data, which may not seem natural unless the convergence rate is

conducted.

Convergence of nonlinear numerical schemes relative to stochastic partial differential equations is usually carried out through either of the following techniques:

- i. compactness argument involving the use of Skorokhod's theorem [46, Theorem 2.4], or
- ii. a local monotonicity property of some members of a given partial differential equation.

The compactness technique can be thought of as an alternative to the Aubin-Lions-Dubinskiĭ theorem (c.f. [4, 6, 78, 103]) with more delicate properties. In point of fact, relying on such a theorem preserves neither the originally defined probability space nor the Wiener process, meaning that it deviates the sense of a strong solution if one exists. However, an offered benefit that can be taken advantage of is the strong convergence of sequences, making the convergence analysis of a proposed numerical scheme easier. In general, the compactness argument's application is reserved for equations that only have weak (martingale) solutions as the three-dimensional stochastic Navier-Stokes equations or the stochastic Landau-Lifshitz-Gilbert problem [7, 24], to state a few. On the other hand, a local monotonicity property, when it is available, avoids the compactness argument's drawbacks by eliminating the occurring complexity (e.g., nonlinearities). It is normally employed to tackle the identification of discrete terms with their solution-dependent counterparts. Such a property is typically used to illustrate the solutions' existence to the two-dimensional stochastic Navier-Stokes equations [91] and will be taken advantage of during the convergence analysis of the proposed numerical schemes in the present thesis. Besides the above-mentioned methods, one might also stumble across other techniques involved with high regularity solutions emerging from fourth-order problems as the Lagrangian averaged Navier-Stokes equations and the Leray- α model (c.f. [29, 37]), but may not be applicable straightforwardly when dealing with non exactly divergence-free C^0 piecewise polynomials-based finite element methods, as the corresponding basis functions' maximal regularity cannot exceed H^1 . Other stochastic Navier-Stokes modifications manifest a way to conduct numerical approximation studies for solutions to the stochastic Navier-Stokes equations, such as the Brinkman-Forchheimer-Navier-Stokes model, which was studied in article [12].

Numerical methods to solve the stochastic Navier-Stokes equations are not ancient. Their evolution began in the 2010s, during which the focus was directed toward the finite element methods. For instance, convergence analysis of a time-space discretization for the three-dimensional Navier-Stokes equations driven by a multiplicative noise was conducted in [24]. The analysis therein relies on weakly divergence-free finite element basis functions and constructs a Wiener process relative to the probability space that pops out from the application of Skorokhod's theorem due to the lousy solutions' properties in three dimensions. On account of the pressure's poor regularity [84], the saddle point aspect of the proposed numerical scheme therein did not help with the convergence rate study [31], which was illustrated within two-dimensional periodic boundary conditions. Its improvement was executed later on in [19], thanks to a decomposition technique that splits the pressure field into a deterministic term depending on the nonlinearity and a stochastic quantity expressed in terms of the multiplicative noise (c.f. [18, Section 3]). A finite element-based space-time discretization of these equations, when driven by an additive stochastic forcing and equipped with no-slip boundary conditions in two dimensions, was achieved in [20] by transforming the stochastic Navier-Stokes problem into a random partial differential equation. Additional studies covering the same topic with further properties are performed in [11, 14]. Semi-discretization in time has also played a role in approximating the Navier-Stokes equations driven by both multiplicative and additive noises [9, 10, 13]. On most occasions, the accomplished time convergence rate is limited to the time-regularity of the stochastic forcing. For example, it is almost $1/2$ when the equipped noise embodies the Wiener process whose paths are Hölder-continuous with exponent $1/2$ (c.f. [40–42, 86]).

Solving the stochastic Navier-Stokes equations numerically in their genuine form is not the only way and might not offer high-quality properties relative to a proposed numerical scheme due to their poor features (see Sections 2.1 and 2.1.1). A decent routine that enhances a numerical method, in this context, is choosing an appropriate variant such as the stochastic Lagrangian averaged Navier-Stokes equations (see Section 2.2.2). In fact, a solution to the stochastic LANS- α model also constitutes a solution to the stochastic Navier-Stokes problem when α tends to 0, as illustrated in Chapter 3. This can be tremendously beneficial, especially when linking the spatial scale α to the discretization parameters in a way to vanish when passing to the limit (Chapter 4), which makes up a compromise between the infamous Navier-Stokes system and the LANS- α model with fixed α , meaning that its solutions may not have thorough properties. Therewith, the high-space regularity emerging from the stochastic LANS- α equations integrates the space convergence rate by increasing it one extra level in comparison with the stochastic Navier-Stokes problem. It is worth mentioning the static effect of numerical methods to the LANS- α equations on the time convergence rate. Indeed, owing to system (2.6), it is straightforward that α contributes solely to space regularizations through the quantity $\alpha\Delta$. Consequently, a semi-discretization in time of both Navier-Stokes and LANS- α equations must return an identical time convergence rate controlled generally by the paths' time-regularity of the applied stochastic forcing, as indicated in the previous paragraph.

One additional shortcoming that pops out from the stochastic Navier-Stokes equations is the pressure's low regularity [84], which negatively impacts the convergence rate of a finite element-based numerical scheme that deals with a saddle point aspect, as demonstrated in [31]. In point of fact, weakly divergence-free basis functions do not totally eliminate the pressure field from the approximate momentum equation; that is, one has to handle the associated low regularity at some point. On the other hand, approximating the stochastic Navier-Stokes equations through exactly divergence-free finite element method is widely known as a painful procedure, particularly when it comes to implementations, not to mention some uncanny behaviors that might result such as unstable condition numbers (c.f. [111]). Subsequently, a decent compromise between a saddle point problem and exactly divergence-free finite element methods is an artificial compressibility technique (Chapter 6), also known as the penalty method, that consists in perturbing the mass conservation equation of the Navier-Stokes problem to add a pressure-dependent quantity multiplied by a certain parameter, usually denoted ε and considered as small as possible. In other words, when ε tends to 0, the fluid recovers its initial incompressible state. In comparison with the spatial scale α of the LANS- α model and its related assumptions, ε may also undergo a few hypotheses allowing a suggested numerical scheme to convergence toward an incompressible state. These hypotheses are strongly dependent on the perturbation choice; namely, ε can be expressed in terms of the space discretization step size, the time discretization step size, or a combination of all the discretization parameters.

2.4 . Numerical approximation for the stochastic LANS- α equations

The corresponding spatial scale α hands out two possible fashions to approximate the LANS- α equations, one of which is discussed in Section 2.3 when α is controlled by the discretization parameters. Another attainable approach comes to light when α is fixed; that is, it maintains its initial value when all discretization parameters go to 0. This way, a numerical scheme would deliver a solution to the LANS- α equations instead of the Navier-Stokes problem. The former and the latter approaches were also investigated for deterministic configurations (c.f. [26, 44]). However, their handling remains remarkably distinguishable compared to the stochastic models, which are dealt with in this thesis.

As widely known, the LANS- α equations were founded to solve most of the Navier-Stokes drawbacks and to make up a substitute whenever the physical circumstances permit. On account of the spatial scale

α , this substitution process might lead to undesirable results such as a very high pressure amplitude when α takes on values exceeding a certain threshold. This is because pressure fields in incompressible systems are usually solved implicitly once the velocity vector's existence has been proven and can be expressed in terms of all members of the momentum equation. As a result, a high value of α has a dominant effect and can be readily seen when the associated analysis is carried out through the Fourier transform. That said, approximating the stochastic LANS- α equations with fixed α shall be treated carefully. In other words, values of α should be kept to a minimum.

The LANS- α equations are visibly of fourth-order and can therefore be approximated through finite element methods either by splitting the momentum equation into two identities for the sake of obtaining a second-order problem and employing typical approximations such as continuous piecewise polynomials-based finite element methods, or by applying discretizations directly to the raw form of the problem. Turning the underlying equations into a coupled problem has the advantage of acquiring all finite element existing tools in the literature that offer remarkably simplified techniques to implement any partial differential equation. At first glance, this might sound impeccable. However, it has been shown [82] that conserving the original state of a fourth-order problem and approximating it through continuously differentiable finite element methods has a more favorable sparsity pattern that consumes less time to factor than non-conforming methods based on polynomials of the same order. This actually means that dealing with such elements implies the same accuracy of a penalty method or a discontinuous Galerkin formulation, with fewer expenses. The principle target that needs to be tackled while managing a continuously differentiable polynomials-based finite element method is mapping physical simplexes to reference simplexes, which may involve unusual transformations when coping with non-conforming elements such as the Argyris elements.

2.5 . Outline

By excluding the introductory sections, this thesis contains concretely four chapters offering genuine work that has not been considered by any other authors. All four chapters are hooked to the primary objective of this thesis which consists in proposing numerical approximations to the stochastic Navier-Stokes equations through various techniques.

- Chapter 3 is a preparatory part that inspects the convergence of solutions to the stochastic Lagrangian averaged Navier-Stokes problem toward solutions to the stochastic Navier-Stokes equations. Conceptually, this is a theoretical chapter where no numerical schemes are provided. Still its contribution is essential to ensure that the stochastic LANS- α model can approximate the stochastic Navier-Stokes system and clarify the needed assumptions that do not dramatically vary when moving into discrete settings. The whole study within this chapter is maintained in two dimensions where strong (probabilistic) solutions exist and are easier to manage than weak (martingale) solutions, which require most of the time a careful analysis. A Faedo-Galerkin method is applied to the stochastic LANS- α model, and the corresponding spatial scale α is controlled by the inverse of a certain Laplace operator's eigenvalue to guarantee its vanishing state at the limit. Typical and high spatial regularity a priori estimates are carried out, and a local monotonicity property related to the sum of the Laplace operator and the nonlinearity has been taken advantage of in order to retain the initially introduced probability space and the Wiener process. The high spatial regularity a priori estimates are not vital for the main purpose of this chapter but can be beneficial to perform the convergence rate study of a proposed finite element method. The following theorem is achieved:

Theorem 2.5.1 *The unique strong (probabilistic) solution of the stochastic Lagrangian averaged Navier-Stokes (LANS- α) equations converges toward the unique strong solution of the stochastic*

Navier-Stokes problem, as the spatial scale α goes to 0, in a bounded convex two-dimensional domain.

This chapter is neither published nor submitted for review. It represents the beginning of a new article whose topic covers the convergence rate of a finite element-based discretization of the two-dimensional stochastic Lagrangian averaged Navier-Stokes equations.

- Chapter 4 is devoted to approximating the stochastic Navier-Stokes equations through the stochastic Lagrangian averaged Navier-Stokes model in a two-dimensional framework. The content in this part is heavily inspired by the theoretical work of Chapter 3. A finite element-based space-discretization is proposed together with an Euler method to approximate the time variables. The equations of interest are transformed into a coupled system through a continuous differential filter to benefit from continuous piecewise-based finite element methods. The suggested numerical scheme has a saddle point aspect that requires a discrete LBB condition to be imposed for the sake of obtaining specific velocity-pressure stability. To gain a solution to the stochastic Navier-Stokes equations, the spatial scale α is assumed to be solely controlled by the mesh size, meaning that its value is guaranteed to vanish when passing to the limit. This assumption allows the a priori estimates relative to the iterates to be uniform in α as well as the discretization parameters, which is crucial to accomplish weak and weak-* convergences. Following the demonstration technique of Chapter 3, a discrete version of the local monotonicity property is proven and employed to come up with the convergence of the proposed numerical scheme, which is eventually validated through a numerical experiment and evaluated against a scheme that was investigated directly on the stochastic Navier-Stokes equations in an independent paper. This chapter investigates a specific numerical scheme (see Algorithm 4.1) and offers the following theorem:

Theorem 2.5.2 *In two dimensions, and under a bounded convex polygonal domain, a discrete solution (based on the finite element and Euler methods) to the stochastic Lagrangian averaged Navier-Stokes (LANS- α) equations emerging from Algorithm 4.1 converges toward the unique strong solution of the stochastic Navier-Stokes system, provided the spatial scale α is bounded from above and below by the mesh size.*

Chapter 4 is part of an article entitled *Numerical and Convergence Analysis of the Stochastic Lagrangian-Averaged Navier-Stokes Equations* that was published in the *Journal of Computational and Applied Mathematics*.

- Chapter 5 focuses on the numerical analysis of the three-dimensional Lagrangian averaged Navier-Stokes equations driven by a multiplicative noise and gives an insight into the corresponding theoretical aspects, especially when the equations of interest are turned into a coupled system. The stochastic Navier-Stokes equations will not be directly involved in this chapter. Nevertheless, their simulation is evoked for the sake of comparison when it comes to choosing the value of α during the numerical experiment part. This gives a clear insight into the range in which α should lie because the conducted study does not link it to any assumption (other than $\alpha < 1$ to emphasize its smallness). In a similar fashion to Chapter 4, a finite element method is proposed to discretize the space variables, requiring the discrete LBB condition since no pressure stabilization techniques were evoked. Temporal discretization relies on Euler's method, which provides both nonlinear (Algorithm 5.1) and linear (Algorithm 5.2) numerical schemes, depending on the nonlinearity's approximation, and has slightly different features: the nonlinear scheme provides typical stability, but its iterates are not unique in the whole probability space. Whereas the linear scheme implies the iterates' uniqueness almost surely but requires an additional regularity for the initial datum. The convergence analysis of the proposed numerical schemes is carried out through an identification technique relying on the extra space-regularity that the LANS- α equations offer and not involving

any compactness argument that modifies the originally introduced probability space. This chapter can also be regarded as a generalization of Chapters 4 and 6 in terms of the utilized Wiener process, which is assumed to have a nuclear covariance operator therein but is cylindrical within Chapter 5. Due to the fourth-order aspect of the LANS- α equations, a short description of a continuously differentiable piecewise polynomials-based finite element method is highlighted and applied to the equations' raw form i.e. without undergoing any coupling process, and an associated algorithm is suggested, though its implementation might be technical and time-demanding. Nevertheless, it remains feasible because the proposed finite element method is conforming, and most technical issues can be avoided, such as creating transformations that map physical simplexes to reference simplexes. Chapter 5 provides numerical experiments involving both the velocity and pressure fields, together with three different values of the spatial scale α along which the simulation is conducted. Unlike the velocity, the pressure field gives a clear hint about the choice of α because its amplitude varies accordingly. The main contribution of this chapter appears in the following roughly stated theorem:

Theorem 2.5.3 *Given a fixed $\alpha < 1$ and a bounded convex polyhedral domain in \mathbb{R}^3 , a finite element-based approximation of the stochastic Lagrangian averaged Navier-Stokes (LANS- α) equations converges toward the unique strong (probabilistic) solution of the LANS- α model driven by a multiplicative noise. Algorithm 5.1 has a convergent solution with typical stability and Algorithm 5.2 approaches the unique solution of the underlying problem under an additional assumption on its starting point.*

This chapter makes an appearance in the article *Numerical and Convergence Analysis of the Stochastic Lagrangian-Averaged Navier-Stokes Equations*, and was published in the *Journal of Computational and Applied Mathematics*.

- Chapter 6 is dedicated to the numerical approximation of the two-dimensional stochastic Navier-Stokes equations through a pressure stabilization method (also known as the penalty method or the artificial compressibility), which consists of a modification of the mass conservation equation. The presented technique abolishes the saddle point aspect that emerges from the null divergence constraint and cancels out the use of discrete LBB conditions. By virtue of this strategy, the employed finite element methods have no restrictions upon their polynomial degrees, and a priori estimates for the discrete pressure come into view and seem to improve the convergence rate of schemes involving a saddle point side. Although the mass conservation equation has been modified, the convergence analysis of the proposed numerical scheme is carried out in a way to obtain the stochastic incompressible Navier-Stokes equations at the limit. This chapter inaugurates a nonlinear scheme (Algorithm 6.1) for which a local monotonicity property is applied to ensure its convergence toward the unique strong solution of the stochastic Navier-Stokes equations. A linear numerical scheme (Algorithm 6.2) is given afterward, whose form is built through tackling the nonlinearity's time-implicit design of the nonlinear method. The suggested artificial compressibility approach is linked to a parameter denoted ε that should vanish at the limit in order to recover an incompressible state of the studied fluid. Throughout the inspection process, no essential assumptions have been imposed on ε . However, supplementary hypotheses can be added out if the aim is to compare the provided numerical scheme with other existing methods in the literature. For instance, comparing the artificial compressibility technique with a saddle point-based scheme triggers new assumptions on ε in terms of the discretization parameters. Numerical experiments of the investigated scheme are delivered and compared to a saddle point-based method, where a graph of the error in terms of ε is displayed in order to ensure the accuracy of the chosen ε . Chapter 6 delivers two theorems, each for an algorithm:

Theorem 2.5.4 *Within a two-dimensional framework, in a bounded convex polygonal domain, and for all penalty parameters $\varepsilon > 0$, an Euler and finite element-based nonlinear numerical scheme (Algorithm 6.1) of the artificially compressible Navier-Stokes equations driven by a multiplicative noise converges toward the unique strong (probabilistic) solution of the stochastic incompressible Navier-Stokes problem.*

Theorem 2.5.5 *For a bounded convex polygonal domain of \mathbb{R}^2 , and for all penalty parameters $\varepsilon > 0$, a linear numerical scheme (Algorithm 6.2) of the artificially compressible stochastic Navier-Stokes equations, consisting of an Euler method-based time-discretization along with a finite element approximation of the space variables, converges toward the unique strong (probabilistic) solution of the stochastic incompressible Navier-Stokes problem.*

This chapter has been submitted for review at the time of writing this thesis.

3 - Convergence of the stochastic LANS- α solutions toward the stochastic Navier-Stokes solutions

3.1 . Introduction

Given a solution to the stochastic Lagrangian averaged Navier-Stokes (LANS- α) model, which reads,

$$\begin{cases} \frac{\partial}{\partial t} (\bar{u} - \alpha^2 \Delta \bar{u}) - \nu \Delta (\bar{u} - \alpha^2 \Delta \bar{u}) - \bar{u} \times (\nabla \times (\bar{u} - \alpha^2 \Delta \bar{u})) + \nabla \bar{p} = g(\cdot, \bar{u}) \frac{\partial W}{\partial t}, \\ \operatorname{div}(\bar{u}) = 0, \\ \bar{u}(0, \cdot) = \bar{u}_0, \end{cases} \quad (3.1)$$

the main interest in this chapter is to check whether or not it converges toward a solution of the stochastic Navier-Stokes equations (NSEs)

$$\begin{cases} \frac{\partial u}{\partial t} - \nu \Delta u + [u \cdot \nabla] u + \nabla p = g(\cdot, u) \frac{\partial W}{\partial t}, \\ \operatorname{div}(u) = 0, \\ u(0, \cdot) = \bar{u}_0, \end{cases} \quad (3.2)$$

when the spatial scale α tends to 0. Both equations are equipped with the same configurations, including the initial datum \bar{u}_0 to guarantee a similar fluid state at time $t = 0$. \bar{u} and u denote the fluid velocities, p and \bar{p} represent the pressure fields, ν symbolizes the kinematic fluid viscosity, α is a small spatial scale, g is a diffusion coefficient, and W is an infinite-dimensional (cylindrical) Wiener process. On account of the poor uniqueness properties of three-dimensional solutions to the stochastic NSEs, the conducted study herein will be limited to two dimensions to guarantee that the unique solution of the stochastic LANS- α equations converges toward a sole one as α goes to 0.

In this chapter, the study is accomplished through periodic boundary conditions for the sake of investigating the effect of α on the space regularity of a solution and to take advantage of the nonlinearity's properties that occur within this framework. It could have been carried out with Dirichlet boundary conditions if only the typical solution's space regularity was intended. Observe that α is always multiplied by $\Delta \bar{u}$ in equations (3.1), meaning that the extra granted regularity that does not figure in problem (3.2) can be loosened through a particular assumption on α when dealing with a finite-dimensional system. The pressure field will be eliminated from the corresponding weak formulation throughout this work through the null divergence criterion, and the focus will be turned toward the velocity vector.

Investigating the convergence of equations (3.1) toward system (3.2) is beneficial because the principal reason for which the LANS- α model was introduced is to overcome most of the Navier-Stokes shortcomings. If the converse scenario took place, equations (3.1) would have become obsolete, but fortunately, it is not. This convergence was also conducted for the deterministic settings (i.e. when $g = 0$) in [27] where the convergence rate in terms of α is revealed. The theoretical study herein has the advantage of building efficient numerical schemes for the stochastic Navier-Stokes problem while considering minimal assumptions on the spatial scale α (Chapter 4). Since α is solely involved with solutions' space regularity, any time discretization should not come into play in any further hypotheses upon α .

Equations (3.1) were first introduced in [28, 29], where the existence of a unique variational solution was proven. It is worth highlighting one drawback of this model relative to the pressure's regularity that appears after applying a generalization of the De Rham theorem [84], which links the velocity's smoothness to that of the pressure. In point of fact, it was shown (c.f. [29, Theorem 3.3]) that \bar{p} is

H^{-1} -valued, meaning that it is lower than that of p , which is L^2 -valued. This inconvenience originates from the biharmonic operator that appears in the first identity of system (3.1) and might have an uncooperative effect on convergence rates of numerical schemes concerned with a non-null divergence of velocities. The same goes for other stochastic Navier-Stokes variants, such as the Leray- α model [50]. Further examinations of equations (3.1) were performed in [49, 51], including a splitting-up scheme in [48].

This chapter is organized as follows: all preliminaries, assumptions and configurations are presented in Section 3.2. The main theorem of this chapter is then presented in Section 3.3, followed by the Faedo-Galerkin approximation where equations (3.1) are projected onto a finite-dimensional subspace, and a priori estimates are carried out. Section 3.5 provides additional properties to a certain limiting function, which will turn out to be a solution to equations (3.2). Finally, a conclusion concerning the relationship between an achieved equation and the Navier-Stokes problem along with a few outlines are given in Section 3.6.

3.2 . Configuration and materials

Given a positive number L , the domain D represents a two-dimensional torus $(0, L)^2$, and for a given $T > 0$, the time interval reads $[0, T]$. Throughout this chapter, the Lebesgue and Sobolev spaces are denoted L^p and H^m (or $W^{m,p}$) respectively, and for an arbitrary normed vector space X , its associated norm will be symbolized by $\|\cdot\|_X$. The notation X_{per} signifies that all its members are periodic functions whose mean is null. Regarding the small spatial scale α that is present in equations (3.1), a special norm $\|\cdot\|_\alpha$ is associated with it and defined by $\|\cdot\|_\alpha^2 := \|\cdot\|_{L^2}^2 + \alpha^2 \|\nabla \cdot\|_{L^2}^2$. The notation $\mathcal{L}_2(E, F)$ is the space of all Hilbert-Schmidt operators; with E and F being two given Banach spaces, \lesssim embodies a shorthand for the less or equal symbol \leq up to a universal non-negative constant, and C_D will denote throughout this chapter a positive constant depending only on the domain D . The solely employed Gelfand triple in this chapter is $(H_{per}^1(D), L_{per}^2(D), H_{per}^{-1}(D))$, where $H_{per}^{-1}(D)$ is the dual space of $H_{per}^1(D)$. The $L^2(D)$ space will be endowed with its standard inner product (\cdot, \cdot) , and the duality brackets $\langle \cdot, \cdot \rangle$ will represent the duality product between $H_{per}^1(D)$ and $H_{per}^{-1}(D)$. Following the mathematical notations for the Navier-Stokes framework, the function spaces that will be frequently encountered herein are

$$\begin{aligned} \mathcal{V} &:= \left\{ u \in [C_{per}^\infty(D)]^2 \mid \operatorname{div}(u) = 0 \right\}, \\ \mathbb{H} &:= \left\{ u \in [L_{per}^2(D)]^2 \mid \operatorname{div}(u) = 0 \text{ a.e. in } D \right\}, \\ \mathbb{V} &:= \left\{ u \in [H_{per}^1(D)]^2 \mid \operatorname{div}(u) = 0 \text{ a.e. in } D \right\}. \end{aligned}$$

Let A be the Stokes operator defined from $D(A) := [H^2(D)]^2 \cap \mathbb{V}$ into \mathbb{H} by $A := -\mathcal{P}\Delta$, where $\mathcal{P}: [L_{per}^2(D)]^2 \rightarrow \mathbb{H}$ is the Leray Projector. In two-dimensional domains and with imposed periodic boundary conditions, it is well-known that the commutator $[\mathcal{P}, \Delta]$ vanishes; namely $\mathcal{P}\Delta = \Delta\mathcal{P}$. Recall that operator A is self-adjoint whose inverse is compact. From now on, all Cartesian products of a sole linear space will be symbolized by blackboard bold letters with the domain D being omitted. For instance, the Sobolev space $[H_{per}^1(D)]^2$ will become \mathbb{H}_{per}^1 .

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P})$ be a complete probability space whose filtration $(\mathcal{F}_t)_{0 \leq t \leq T}$ is right-continuous. Given a separable Hilbert space K equipped with a complete orthonormal basis $\{w_k, k \geq 1\}$, the K -valued cylindrical Wiener process $W(t), t \in [0, T]$ reads $W(t) := \sum_{k \geq 1} \beta_k(t) w_k$, for all $t \in [0, T]$, where $\{\beta_k, k \geq 1\}$ is a family of independent and identically distributed \mathbb{R} -valued Brownian motions on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P})$. For clarity's sake, the nonlinear term in equations (3.1) will be denoted \tilde{b} ; that is $\tilde{b}(u, v, w) = -\left(u \times (\nabla \times v), w\right)$ for appropriate vector fields u, v and w , where $v = u - \alpha^2 \Delta u$ in the

equations of interest. The bilinear operator that can be derived from \tilde{b} will be denoted \tilde{B} and it reads:

$$\tilde{B}(u, v) := -u \times (\nabla \times v), \quad \text{for all } u, v \in \mathbb{V}.$$

The below proposition lists a few useful properties of the bilinear operator \tilde{B} .

Proposition 3.2.1 (i) For all $u, v, w \in \mathbb{H}^1$, $\langle \tilde{B}(u, v), w \rangle = -\langle \tilde{B}(w, v), u \rangle$. In particular, identity $\langle \tilde{B}(u, v), u \rangle = 0$ holds true for all $u, v \in \mathbb{H}^1$.

(ii) $\langle \tilde{B}(u, v), w \rangle = ([u \cdot \nabla]v, w) - ([w \cdot \nabla]v, u)$, for all $u, v, w \in \mathbb{H}_{per}^1$. If additionally, u and v are divergence-free then, $\langle \tilde{B}(u, v), v \rangle = -([v \cdot \nabla]v, u)$.

(iii) $|\langle \tilde{B}(u, v), w \rangle| \leq C_D \|u\|_{\mathbb{L}^4} \|\nabla v\|_{\mathbb{L}^2} \|w\|_{\mathbb{L}^2}^{\frac{1}{2}} \|\nabla w\|_{\mathbb{L}^2}^{\frac{1}{2}}$, for all $u, v, w \in \mathbb{H}_{per}^1$ (only in 2D).

Proof: Assertion (i) can be proven by a simple application of the identity $(u \times v) \cdot w = -(w \times v) \cdot u$. To demonstrate equality (ii), we need to employ the following property:

$$\langle \tilde{B}(u, v), w \rangle = ([u \cdot \nabla]v, w) + ((\nabla u)^T \cdot v, w) - (\nabla(u \cdot v), w), \quad (3.3)$$

which may be straightforwardly proven via the identity

$$[u \cdot \nabla]v + (\nabla u)^T \cdot v - \nabla(u \cdot v) = -u \times (\nabla \times v).$$

Indeed, the quantity $((\nabla u)^T \cdot v, w)$ of equation (3.3) turns into $-([w \cdot \nabla]v, u) + (\nabla(u \cdot v), w)$ after applying two integration by parts. Plugging it back in equation (3.3) completes the proof of (ii). Finally, the Hölder and Gagliardo-Nirenberg inequalities applied to assertion (ii) yield estimate (iii). \square

The operator \tilde{b} can be readily expressed via the trilinear form associated with the Navier-Stokes equations, as mentioned in Proposition 3.2.1-(ii). For brevity's sake, we deploy the next proposition to grant a few corresponding properties. The reader may refer to [106, Remark 2.2] for further information.

Proposition 3.2.2 (i) $([u \cdot \nabla]v, v) = 0$ for all $u, v \in \mathbb{V}$.

(ii) $|([u \cdot \nabla]v, w)| \leq C_D \|u\|_{\mathbb{L}^2} \|\nabla v\|_{\mathbb{L}^2} \|w\|_{\mathbb{L}^2}^{\frac{1}{2}} \|Aw\|_{\mathbb{L}^2}^{\frac{1}{2}}$, for all $u \in \mathbb{H}$, $v \in \mathbb{V}$ and $w \in D(A)$.

Assumptions

(S₁) $\mathbb{E} \left[\|\bar{u}_0\|_{\mathbb{H}^1}^{2p} \right] < +\infty$, for some $p \in [1, +\infty)$,

(S₂) $g: \Omega \times (0, T) \times \mathbb{V} \rightarrow \mathcal{L}_2(K, \mathbb{L}^2)$ satisfies: for all $u \in \mathbb{V}$, $g(\cdot, u)$ is \mathcal{F}_t -progressively measurable, and a.e. in $\Omega \times (0, T)$, it holds that

$$\begin{aligned} \|g(\cdot, u) - g(\cdot, v)\|_{\mathcal{L}_2(K, \mathbb{L}^2)} &\leq L_g \|u - v\|_{\alpha}, \quad \forall u, v \in \mathbb{V}, \\ \|g(\cdot, u)\|_{\mathcal{L}_2(K, \mathbb{H}^1)} &\leq K_1 + K_2 \|u\|_{\alpha}, \quad \forall u \in \mathbb{V}. \end{aligned}$$

for some real, nonnegative, time-independent constants L_g, K_1, K_2 such that $L_g \leq \frac{\sqrt{p}}{C_P \sqrt{2}}$, where C_P is the Poincaré constant.

The Lipschitz-continuity of the diffusion coefficient g must hold to ensure the existence of a unique strong solution for both equations (3.1) and (3.2). If such a condition were not imposed, only a non-unique weak (martingale) solution would have existed (c.f. [49]).

Remark 3.2.1 Inequality $\|g(\cdot, u)\|_{\mathcal{L}_2(K, \mathbb{H}^1)} \leq K_1 + K_2 \|u\|_{\alpha}$ of assumption (S₂) is imposed in \mathbb{H}^1 instead of \mathbb{L}^2 to be able to execute high space-regularity estimates for the velocity field.

To reduce repetitions, the below proposition gathers a few properties that will be employed throughout this chapter.

Proposition 3.2.3 (i) $x^p \leq 1 + x^q$ for all $x \geq 0$, and $1 \leq p \leq q < +\infty$.

(ii) $2(a, b) = \|a\|_{\mathbb{L}^2}^2 - \|b\|_{\mathbb{L}^2}^2 + \|a - b\|_{\mathbb{L}^2}^2$, for all $a, b \in \mathbb{L}^2$.

(iii) $|a + b|^p \leq 2^{p-1}(|a|^p + |b|^p)$, for all $a, b \in \mathbb{R}$ and $p \geq 1$.

3.2.1 . Concept of solutions

The underlying equations consist of a fourth-order problem which is not appropriate for C^0 piecewise polynomials-based finite element methods. Therefore, a continuous differential filter shall be introduced allowing equations (3.1) to turn into a second-order coupled problem.

Definition 3.2.1 (Continuous differential filter) Let $v \in \mathbb{L}^2$ be a given vector field. A continuous differential filter \bar{u} of v is defined as part of the unique solution $(\bar{u}, \bar{p}) \in \mathbb{V} \times L_0^2(D)$ to the problem:

$$\begin{cases} -\alpha^2 \Delta \bar{u} + \bar{u} + \nabla \bar{p} = v, & \text{in } D, \\ \operatorname{div}(\bar{u}) = 0, & \text{in } D. \end{cases} \quad (3.4)$$

The notation \bar{v} (instead of \bar{u}) is widely spread in the literature of differential filters. However, to maintain a visible relationship between equations (3.1) and (3.4), \bar{v} will be substituted by the notation \bar{u} . Observe that system (3.4) represents a deterministic steady Stokes problem and that v plays the role of an outer force. Additionally, projecting system (3.4) using the Leray projector \mathcal{P} yields

$$\alpha^2 A \bar{u} + \bar{u} = \mathcal{P}v, \text{ in } D.$$

which has a unique solution \bar{u} according to [69, Subsection 8.2]. Thereby, when it comes to the process $\{\bar{u}(t), t \in [0, T]\}$ of problem (3.1), the multiplication in L^2 of the above equation by $\varphi \in \mathbb{V}$ returns for all $t \in [0, T]$,

$$(v(t), \varphi) = (\bar{u}(t), \varphi) + \alpha^2 (\nabla \bar{u}(t), \nabla \varphi).$$

Based on the above identity, we define v_0 as the solution of $(v_0, \varphi) = (\bar{u}_0, \varphi) + \alpha^2 (\nabla \bar{u}_0, \nabla \varphi)$, for all $\varphi \in \mathbb{V}$. Since \bar{u}_0 belongs to \mathbb{V} , it is straightforward that $\alpha^2 \mathbb{E}[(\nabla \bar{u}_0, \nabla \varphi)] \rightarrow 0$ as $\alpha \rightarrow 0$. Subsequently, $\mathbb{E}[(v_0, \varphi)] = \mathbb{E}[(\bar{u}_0, \varphi)]$ for all $\varphi \in \mathbb{V}$ as $\alpha \rightarrow 0$. As a result, $v_0 = \bar{u}_0$ \mathbb{P} -a.s. and a.e. in D when α vanishes. The next definition states the compound of a solution to equations (3.1) whose existence and uniqueness are illustrated in [29].

Definition 3.2.2 Let $T > 0$ and assume (S_1) – (S_2) . A $\mathbb{V} \times \mathbb{H}$ -valued stochastic process $(\bar{u}(t), v(t))$, $t \in [0, T]$ is said to be a variational solution to problem (3.1) if it fulfills the following conditions:

(i) $\bar{u} \in L^2(\Omega; L^2(0, T; \mathbb{H}^2 \cap \mathbb{V})) \cap L^2(\Omega; L^\infty(0, T; \mathbb{V}))$,

(ii) $v \in L^2(\Omega; L^2(0, T; \mathbb{V})) \cap L^2(\Omega; L^\infty(0, T; \mathbb{H}))$,

(iii) \mathbb{P} -almost surely, \bar{u} is weakly continuous with values in \mathbb{V} , and v is continuous with values in \mathbb{H} ,

(iv) for all $t \in [0, T]$, (\bar{u}, v) satisfies the following equation \mathbb{P} -almost surely

$$\begin{cases} (v(t), \varphi) + \nu \int_0^t (\nabla v(s), \nabla \varphi) ds + \int_0^t \tilde{b}(\bar{u}(s), v(s), \varphi) ds \\ = (v_0, \varphi) + \left(\int_0^t g(s, \bar{u}(s)) dW(s), \varphi \right), \quad \forall \varphi \in \mathbb{V}, \\ (v(t), \psi) = (\bar{u}(t), \psi) + \alpha^2 (\nabla \bar{u}(t), \nabla \psi), \quad \forall \psi \in \mathbb{V}. \end{cases} \quad (3.5)$$

The pressure field \bar{p} does not appear in equations (3.5) because the test function φ is divergence-free. One way to recover it is through De Rham's theorem. Another possible way that can help further in the convergence rate study of a numerical scheme not relying on strongly divergence-free elements is through [18, Theorem 5] where the pressure is decomposed into a deterministic part and a stochastic part. For the current case, considering $\phi \in [C_{per}^2(D)]^2$ as a test function, and taking into account that $\phi - \nabla \Delta^{-1} \operatorname{div}(\phi)$ is divergence-free, we assign it to φ of equations (3.5) to obtain:

$$\begin{aligned} & (v(t), \phi) + \nu \int_0^t (\nabla v(s), \nabla \phi) ds + \int_0^t \tilde{b}(\bar{u}(s), v(s), \phi) ds \\ &= (v_0, \phi) + \int_0^t (p^{det}(s), \operatorname{div}(\phi)) ds + \left(\int_0^t p^{sto}(s) dW(s), \phi \right) + \left(\int_0^t g(s, \bar{u}(s)) dW(s), \phi \right), \end{aligned}$$

where $p^{det} := \Delta^{-1} \operatorname{div}[\bar{u} \times (\nabla \times v)]$, and $p^{sto} := -\nabla \Delta^{-1} \operatorname{div}(g(\cdot, \bar{u}))$.

A solution's sense will be required hereafter to check whether the obtained function makes up a solution to equations (3.2) or not when α goes to 0.

Definition 3.2.3 *Let $T > 0$ be fixed and assumptions (S_1) - (S_2) be fulfilled. A process $u(t)$, $t \in [0, T]$ on a stochastic filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, T]}, \mathbb{P})$ is said to be a strong solution to equations (3.2) if it belongs to $L^2(\Omega; C([0, T]; \mathbb{H}) \cap L^2(0, T; \mathbb{V}))$, and it satisfies \mathbb{P} -a.s. for all $t \in [0, T]$, the weak formulation*

$$\begin{aligned} & (u(t), \varphi) + \nu \int_0^t (\nabla u(s), \nabla \varphi) ds + \int_0^t ([u(s) \cdot \nabla] u(s), \varphi) ds \\ &= (\bar{u}_0, \varphi) + \left(\int_0^t g(s, u(s)) dW(s), \varphi \right), \quad \forall \varphi \in \mathbb{V}. \end{aligned}$$

3.3 . Main result

Theorem 3.3.1 *Let $T > 0$, $L > 0$, $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P})$ be a filtered probability space, $D = (0, L)^2$ be a two-dimensional torus subject to periodic boundary conditions, and $1 \leq p < +\infty$ be given. Let $\{e_k, k \geq 1\}$ be a complete orthonormal basis of \mathbb{H} consisting of eigenfunctions of the Stokes operator A , and $\{\mu_k, k \geq 1\}$ be the associated eigenvalues whose values diverge when $k \rightarrow +\infty$. Assume that hypotheses (S_1) - (S_2) are fulfilled, and that for all $N \in \mathbb{N} \setminus \{0\}$, the spatial scale follows the decreasing rate $C_{min} \mu_N^{-3/4} \leq \alpha := \alpha_N \leq C_{max} \mu_N^{-3/4}$, for some constants $C_{min}, C_{max} > 0$ independent of N . Then, a solution $(\bar{u}, v) := (\bar{u}_{\alpha_N}, v_{\alpha_N})$ to equations (3.1) in the sense of Definition 3.2.2, for a given α , converges, according to equations (3.15)-(3.18), toward the unique strong solution (v_{NS}, v_{NS}) of equations (3.2) in the sense of Definition 3.2.3 when $N \rightarrow +\infty$, and it satisfies:*

$$(i) \quad \mathbb{E} \left[\sup_{t \in [0, T]} \|v_{NS}(t)\|_{\mathbb{L}^2}^{2p} + 2p\nu \int_0^T \|v_{NS}(t)\|_{\mathbb{L}^2}^{2(p-1)} \|\nabla v_{NS}(t)\|_{\mathbb{L}^2}^2 dt \right] \leq C_2,$$

$$(ii) \quad \mathbb{E} \left[\sup_{t \in [0, T]} \|\nabla v_{NS}(t)\|_{\mathbb{L}^2}^{2p} + \left(\nu \int_0^T \|Av_{NS}(t)\|_{\mathbb{L}^2}^2 \right)^p \right] \leq C_4,$$

where $C_2 > 0$ depends on constants C_{max}, C_1 of Lemma 3.4.1 and its parameters, and $C_4 > 0$ depends on $C_1, \|\bar{u}_0\|_{L^{6p}(\Omega; \mathbb{V})}$ and C_{max} .

Remark 3.3.1 *Throughout this chapter, there will only be a single limit concept parameterized by N ; no successive double limits are intended within this context. In a more accurate way, we will neither treat the case $\alpha \rightarrow 0$ while fixing N nor the independent convergences of α and N . The whole study revolves around the convergence of N to $+\infty$, which leads α to vanish.*

3.4 . Faedo-Galerkin technique

It is well-known that the trilinear term of the Navier-Stokes equations $\int_D [z \cdot \nabla] z \Delta z dx$ vanishes if the configurations were set to two-dimensional domain with periodic boundary conditions. This property is unfortunately inapplicable to $\tilde{b}(z, z - \alpha^2 \Delta z, \Delta z)$. Therefore, we must find a way to achieve high spatial regularity estimates. To this purpose, let $N \in \mathbb{N} \setminus \{0\}$ be a large integer, $\{e_k, k \geq 1\}$ be a complete orthonormal basis of \mathbb{H} consisting of eigenfunctions of the Stokes operator A whose domain is $\mathbb{H}^2 \cap \mathbb{V}$, and $(\mu_k)_{k \geq 1}$ be the associated eigenvalues. Denote by $V_N := \text{span}\{e_1, \dots, e_N\}$ the finite-dimensional vector subspace of \mathbb{H} , $P_N: \mathbb{H} \rightarrow \mathbb{H}$ the projection operator of \mathbb{H} onto V_N , and $A_N := P_N \circ A$ the projected version of the Stokes operator onto V_N . We will assume from now on that $C_{\min} \mu_N^{-3/4} \leq \alpha \leq C_{\max} \mu_N^{-3/4}$, for some constants $C_{\min}, C_{\max} > 0$ independent of N . That way, when N tends to $+\infty$, the spatial scale α goes to 0, thanks to the property $\mu_1 < \mu_2 < \dots < \mu_N \rightarrow +\infty$ as $N \rightarrow \infty$. N is opted to be significant to ensure that $1/\mu_N \leq 1$. Consequently, we introduce the following Faedo-Galerkin approximate system:

$$\begin{cases} (v_N(t), e_k) + \nu \int_0^t (\nabla v_N(s), \nabla e_k) ds + \int_0^t \tilde{b}(\bar{u}_N(s), v_N(s), e_k) ds \\ = (v_0, e_k) + \left(\int_0^t g(s, \bar{u}_N(s)) dW(s), e_k \right), \\ (v_N(t), e_k) = (\bar{u}_N(t), e_k) + \alpha^2 (\nabla \bar{u}_N(t), \nabla e_k), \end{cases} \quad (3.6)$$

for all $t \in [0, T]$, $k \in \{1, \dots, N\}$ and \mathbb{P} -almost surely, with initial datum $\bar{u}_N(0) = P_N \bar{u}_0$ i.e. $v_N(0) = P_N v_0 = (P_N + \alpha^2 A_N) \bar{u}_0$. System (3.6) converges to the unique strong solution of the stochastic Navier-Stokes equations when N tends to $+\infty$, as demonstrated in the sequel of this chapter.

Remark 3.4.1 Assumption $\alpha \leq C_{\max} \mu_N^{-3/4}$ could have been $\alpha \leq C_{\max} \mu_N^{-1/2}$ if only the convergence of solutions to equations (3.1) toward solutions to problem (3.2) was intended. The additional negative exponent on μ_N is exclusively required in this context to obtain high spatial regularity for the velocities v and \bar{u} .

3.4.1 . A priori estimates

This section is devoted to providing a priori estimates to the projected process (\bar{u}_N, v_N) .

Lemma 3.4.1 Let $T > 0$, $N \in \mathbb{N} \setminus \{0\}$, $p \geq 1$, and assumptions (S_1) - (S_2) be valid. Then, the finite-dimensional system (3.6) has a $\mathbb{V} \times \mathbb{H}$ -valued solution (\bar{u}_N, v_N) that satisfies the following estimates:

$$(i) \sup_{0 \leq t \leq T} \mathbb{E} \left[\|\bar{u}_N(t)\|_{\alpha}^{2p} \right] + 2p\nu \mathbb{E} \left[\int_0^T \|\bar{u}_N(t)\|_{\alpha}^{2(p-1)} \|\nabla \bar{u}_N(t)\|_{\alpha}^2 dt \right] \leq C_1,$$

$$(ii) \mathbb{E} \left[\sup_{0 \leq t \leq T} \|\bar{u}_N(t)\|_{\alpha}^{2p} \right] \leq C_1,$$

for a certain constant $C_1 > 0$ depending only on $\mathbb{E} \left[\|\bar{u}_0\|_{\mathbb{H}^1}^{2p} \right]$, p, D, K_1, K_2 , and T . Moreover, if one assumes $\alpha \leq \mu_N^{-1/2}$ then, it holds that

$$(iii) \mathbb{E} \left[\sup_{0 \leq t \leq T} \|v_N(t)\|_{\mathbb{L}^2}^{2p} \right] + 2p\nu \mathbb{E} \left[\int_0^T \|v_N(t)\|_{\mathbb{L}^2}^{2(p-1)} \|\nabla v_N(t)\|_{\mathbb{L}^2}^2 dt \right] \leq C_2,$$

where C_2 is a positive constant depending only on C_1 .

Proof: Problem (3.6) is a finite-dimensional system of ordinary differential equations subject to a polynomial nonlinearity. Therefore, it has a local solution (\bar{u}_N, v_N) . In order to apply Itô's formula, we need to define, for $n \in \mathbb{N} \setminus \{0\}$, the following stopping time:

$$\tau_N^n := \begin{cases} \inf \{t \in [0, T] : \|(I + \alpha^2 A)^{-1/2} v_N(t)\|_{\mathbb{L}^2} > n\} & \text{if the set is non-empty,} \\ +\infty & \text{otherwise.} \end{cases}$$

For $p \geq 1$, and $t \in [0, T]$, we define the process

$$F(v_N(t)) := \|(I + \alpha^2 A)^{-1/2} v_N(t)\|_{\mathbb{L}^2}^{2p}.$$

From equation (3.6)₂, and taking into account that $I + \alpha^2 A$ is self-adjoint and bijective from $D(A)$ to \mathbb{H} , it is straightforward that $F(v_N) = \|\bar{u}_N\|_{\alpha}^{2p}$. Moreover,

$$\begin{aligned} DF(v_N) &= 2p \|(I + \alpha^2 A)^{-1/2} v_N\|_{\mathbb{L}^2}^{2(p-1)} (I + \alpha^2 A)^{-1} v_N = 2p \|\bar{u}_N\|_{\alpha}^{2(p-1)} \bar{u}_N, \text{ and,} \\ D^2 F(v_N) &= 4p(p-1) \|\bar{u}_N\|_{\alpha}^{2p-4} \bar{u}_N \otimes \bar{u}_N + 2p \|\bar{u}_N\|_{\alpha}^{2p-2} (I + \alpha^2 A)^{-1}, \end{aligned}$$

where the symbol \otimes denotes the usual dyadic product. Apply now Itô's formula to the stochastic process $F(v_N(t \wedge \tau_N^n))$:

$$\begin{aligned} \|\bar{u}_N(t \wedge \tau_N^n)\|_{\alpha}^{2p} &= \|\bar{u}_N(0)\|_{\alpha}^{2p} + 2p \int_0^{t \wedge \tau_N^n} \|\bar{u}_N(s)\|_{\alpha}^{2(p-1)} (\bar{u}_N(s), g(s, \bar{u}_N(s))) dW(s) \\ &+ 2p(p-1) \int_0^{t \wedge \tau_N^n} \|\bar{u}_N(s)\|_{\alpha}^{2p-4} \|(\bar{u}_N(s))^* g(s, \bar{u}_N(s))\|_K^2 ds \\ &+ p \int_0^{t \wedge \tau_N^n} \|\bar{u}_N(s)\|_{\alpha}^{2(p-1)} \|(I + \alpha^2 A)^{-1/2} g(s, \bar{u}_N(s))\|_{\mathcal{L}_2(K, \mathbb{L}^2)}^2 ds \\ &+ 2p \int_0^{t \wedge \tau_N^n} \|\bar{u}_N(s)\|_{\alpha}^{2(p-1)} \langle \bar{u}_N(s), -\nu A v_N(s) - \tilde{B}(\bar{u}_N(s), v_N(s)) \rangle ds. \end{aligned}$$

We have $\langle \bar{u}_N(s), A v_N(s) \rangle = \langle \nabla \bar{u}_N(s), \nabla (I + \alpha^2 A) \bar{u}_N(s) \rangle = \|\nabla \bar{u}_N(s)\|_{\alpha}^2$, and by Proposition 3.2.1-(i), the nonlinear operator \tilde{B} in the last term on the right-hand side of the above equation vanishes so that

$$\begin{aligned} \|\bar{u}_N(t \wedge \tau_N^n)\|_{\alpha}^{2p} &+ 2p\nu \int_0^{t \wedge \tau_N^n} \|\bar{u}_N(s)\|_{\alpha}^{2p-2} \|\nabla \bar{u}_N(s)\|_{\alpha}^2 ds \leq \|\bar{u}_N(0)\|_{\alpha}^{2p} \\ &+ 2p \int_0^{t \wedge \tau_N^n} \|\bar{u}_N(s)\|_{\alpha}^{2p-2} (\bar{u}_N(s), g(s, \bar{u}_N(s))) dW(s) \\ &+ 2p(p-1) \int_0^{t \wedge \tau_N^n} \|\bar{u}_N(s)\|_{\alpha}^{2p-4} \|\bar{u}_N(s)\|_{\mathbb{L}^2}^2 \|g(s, \bar{u}_N(s))\|_{\mathcal{L}_2(K, \mathbb{L}^2)}^2 ds \\ &+ p \int_0^{t \wedge \tau_N^n} \|\bar{u}_N(s)\|_{\alpha}^{2p-2} \|(I + \alpha^2 A)^{-1/2} g(s, \bar{u}_N(s))\|_{\mathcal{L}_2(K, \mathbb{L}^2)}^2 ds \\ &= \|\bar{u}_N(0)\|_{\alpha}^{2p} + I_1 + I_2 + I_3. \end{aligned} \tag{3.7}$$

Assumption (S_2) together with the stopping time τ_N^n yield $\mathbb{E}[I_1] = 0$. On the other hand, by virtue of Proposition 3.2.3-(i), assumption (S_2) , and estimate $\|(I + \alpha^2 A)^{-1/2} z\|_{\mathbb{L}^2} \leq \|z\|_{\mathbb{L}^2}$, it holds that

$$\begin{aligned} I_2 + I_3 &\leq 2p(p-1) \int_0^{t \wedge \tau_N^n} \|\bar{u}_N(s)\|_{\alpha}^{2p-4} \|\bar{u}_N(s)\|_{\mathbb{L}^2}^2 (K_1 + K_2 \|\bar{u}_N(s)\|_{\alpha})^2 ds \\ &+ p \int_0^{t \wedge \tau_N^n} \|\bar{u}_N(s)\|_{\alpha}^{2p-2} (K_1 + K_2 \|\bar{u}_N(s)\|_{\alpha})^2 ds \\ &\leq p(2p-1) \int_0^{t \wedge \tau_N^n} \|\bar{u}_N(s)\|_{\alpha}^{2p-2} (K_1 + K_2 \|\bar{u}_N(s)\|_{\alpha})^2 ds \\ &\leq 2p(2p-1) K_1^2 t \wedge \tau_N^n + 2p(2p-1) (K_1^2 + K_2^2) \int_0^{t \wedge \tau_N^n} \|\bar{u}_N(s)\|_{\alpha}^{2p} ds. \end{aligned}$$

Putting it all together and applying the mathematical expectation to equation (3.7) return

$$\begin{aligned} & \mathbb{E} \left[\|\bar{u}_N(t \wedge \tau_N^n)\|_\alpha^{2p} \right] + 2p\nu \mathbb{E} \left[\int_0^{t \wedge \tau_N^n} \|\bar{u}_N(s)\|_\alpha^{2p-2} \|\nabla \bar{u}_N(s)\|_\alpha^2 ds \right] \leq \mathbb{E} \left[\|\bar{u}_N(0)\|_\alpha^{2p} \right] \\ & + 2p(2p-1)K_1^2 \mathbb{E} [t \wedge \tau_N^n] + 2p(2p-1)(K_1^2 + K_2^2) \int_0^{t \wedge \tau_N^n} \mathbb{E} \left[\|\bar{u}_N(s)\|_\alpha^{2p} \right] ds. \end{aligned}$$

The Grönwall lemma finally implies

$$\begin{aligned} & \sup_{0 \leq t \leq T} \mathbb{E} \left[\|\bar{u}_N(t \wedge \tau_N^n)\|_\alpha^{2p} \right] + 2p\nu \mathbb{E} \left[\int_0^{t \wedge \tau_N^n} \|\bar{u}_N(s)\|_\alpha^{2p-2} \|\nabla \bar{u}_N(s)\|_\alpha^2 ds \right] \\ & \leq \left(\mathbb{E} \left[\|\bar{u}_N(0)\|_\alpha^{2p} \right] + 2p(2p-1)K_1^2 \mathbb{E} [t \wedge \tau_N^n] \right) \exp \left(2p(2p-1)(K_1^2 + K_2^2)t \wedge \tau_N^n \right). \end{aligned} \quad (3.8)$$

Taking into account that $\mathbb{E} [\|\bar{u}_N(0)\|_\alpha^{2p}] \leq \mathbb{E} [\|\bar{u}_0\|_\alpha^{2p}]$, and letting $n \rightarrow +\infty$ in equation (3.8) complete the proof of estimate (i). Now that we have illustrated that $\|\bar{u}_N\|_\alpha$ has finite moments, we can drop the stopping time in equation (3.7). whose supremum in time returns

$$\begin{aligned} & \mathbb{E} \left[\sup_{0 \leq t \leq T} \|\bar{u}_N(t)\|_\alpha^{2p} \right] \leq \mathbb{E} \left[\|\bar{u}_N(0)\|_\alpha^{2p} \right] \\ & + 2p \mathbb{E} \left[\sup_{0 \leq t \leq T} \left| \int_0^t \|\bar{u}_N(s)\|_\alpha^{2p-2} (\bar{u}_N(s), g(s, \bar{u}_N(s)) dW(s)) \right| \right] \\ & + 2p(2p-1)K_1^2 T + 2p(2p-1)(K_1^2 + K_2^2) T \sup_{0 \leq t \leq T} \mathbb{E} \left[\|\bar{u}_N(t)\|_\alpha^{2p} \right]. \end{aligned} \quad (3.9)$$

By virtue of Proposition 3.2.3-(i), assumption (S_2) , the Burkholder-Davis-Gundy and Young inequalities, the second term on the right-hand side can be bounded by

$$\begin{aligned} & \lesssim \mathbb{E} \left[\left(\int_0^T \|\bar{u}_N(t)\|_\alpha^{4p-2} \|g(t, \bar{u}_N(t))\|_{\mathcal{L}_2(K, \mathbb{L}^2)}^2 dt \right)^{1/2} \right] \\ & \lesssim \mathbb{E} \left[\sup_{0 \leq t \leq T} \left(\|\bar{u}_N(t)\|_\alpha^{\frac{2p-1}{2}} \|g(t, \bar{u}_N(t))\|_{\mathcal{L}_2(K, \mathbb{L}^2)}^{1/2} \right) \left(\int_0^T \|\bar{u}_N(t)\|_\alpha^{2p-1} \|g(t, \bar{u}_N(t))\|_{\mathcal{L}_2(K, \mathbb{L}^2)} dt \right)^{1/2} \right] \\ & \leq \frac{\varepsilon}{2} \mathbb{E} \left[K_1 + (K_1 + K_2) \sup_{0 \leq t \leq T} \|\bar{u}_N(t)\|_\alpha^{2p} \right] + \frac{1}{2\varepsilon} \left(K_1 T + (K_1 + K_2) T \sup_{0 \leq t \leq T} \mathbb{E} \left[\|\bar{u}_N(t)\|_\alpha^{2p} \right] \right), \end{aligned}$$

for some constant $\varepsilon > 0$ emerging from the Young inequality. Taking $\varepsilon = \frac{1}{K_1 + K_2}$, merging the above result into equation (3.9), and employing assertion (i) complete the proof of estimate (ii). Moving on to the inequality (iii), we have $(v_N, \psi) = (\bar{u}_N, \psi) + \alpha^2 (\nabla \bar{u}_N, \nabla \psi)$ \mathbb{P} -a.s. for all $\psi \in V_N$, thanks to equation (3.6)₂. Therefore, substituting ψ by $v_N(t)$ and employing the Cauchy-Schwarz inequality to get:

$$\|v_N(t)\|_{\mathbb{L}^2}^2 \leq \|\bar{u}_N(t)\|_{\mathbb{L}^2} \|v_N(t)\|_{\mathbb{L}^2} + \alpha^2 \|\nabla \bar{u}_N(t)\|_{\mathbb{L}^2} \|\nabla v_N(t)\|_{\mathbb{L}^2}.$$

On the other hand, the estimate $\|\nabla v_N(t)\|_{\mathbb{L}^2} \leq \sqrt{\mu_N} \|v_N(t)\|_{\mathbb{L}^2}$ together with the hypothesis $\alpha \leq \mu_N^{-1/2}$ and the Young inequality lead to

$$\|v_N(t)\|_{\mathbb{L}^2} \leq \sqrt{2} \|\bar{u}_N(t)\|_\alpha. \quad (3.10)$$

Following the same technique, but this time replacing ψ by $A v_N(t) \in V_N$, one obtains

$$\|\nabla v_N(t)\|_{\mathbb{L}^2} \leq \sqrt{2} \|\nabla \bar{u}_N(t)\|_\alpha. \quad (3.11)$$

It suffices now to raise inequality (3.10) to the power $2p$, take the supremum over $t \in [0, T]$, apply to it the mathematical expectation, and employ estimate (ii) to get $\mathbb{E} \left[\sup_{0 \leq t \leq T} \|v_N(t)\|_{\mathbb{L}^2}^{2p} \right] \lesssim C_1$. Similarly,

$\|v_N(t)\|_{\mathbb{L}^2}^{2(p-1)} \|\nabla v_N(t)\|_{\mathbb{L}^2}^2 \lesssim \|\bar{u}_N(t)\|_{\alpha}^{2(p-1)} \|\nabla \bar{u}_N(t)\|_{\alpha}^2$, thanks to (3.10) and (3.11). Integrating over $[0, T]$, applying the mathematical expectation and employing estimate (i) terminate the proof. \square

Notice that assumption (S_1) requires \bar{u}_0 to be \mathbb{H}^1 -valued independently of the spatial scale α . In other words, we do not recommend the condition $\mathbb{E} \left[\|\bar{u}_0\|_{\alpha}^{2p} \right] < +\infty$ because, when α tends to 0, we will still necessitate \bar{u}_0 to be in \mathbb{H}^1 for the sake of acquiring high spatial regularity for \bar{u} and v . The next lemma exhibits the smoothness of v_0 with respect to that of \bar{u}_0 .

Lemma 3.4.2 *Let $1 \leq p < +\infty$, and assume (S_1) . If $C_{\min} \mu_N^{-1/2} \leq \alpha \leq C_{\max} \mu_N^{-1/2}$ for some constants $C_{\min}, C_{\max} > 0$, then $v_0 \in L^{2p}(\Omega; \mathbb{V})$, and*

$$\|\nabla v_0\|_{L^{2p}(\Omega; \mathbb{L}^2)} \leq \|\bar{u}_0\|_{L^{2p}(\Omega; \mathbb{V})}.$$

Proof: By definition, we have $P_N v_0 = P_N \bar{u}_0 + \alpha^2 A_N \bar{u}_0$, \mathbb{P} -a.s and a.e. in D . Particularly,

$$(P_N v_0, \varphi_N) = (P_N \bar{u}_0, \varphi_N) + \alpha^2 (A_N \bar{u}_0, \varphi_N), \text{ for all } \varphi_N \in V_N.$$

We point out that $(A_N \bar{u}_0, \varphi_N) = (A \bar{u}_0, \varphi_N) = (\bar{u}_0, A \varphi_N) = (P_N \bar{u}_0, A \varphi_N) = (A P_N \bar{u}_0, \varphi_N)$, for all $\varphi_N \in V_N$. Therefore, replacing φ_N by $A P_N v_0 \in V_N$ and employing the Cauchy-Schwarz inequality yield

$$\|A^{1/2} P_N v_0\|_{\mathbb{L}^2}^2 \leq \|A^{1/2} P_N \bar{u}_0\|_{\mathbb{L}^2} \|A^{1/2} P_N v_0\|_{\mathbb{L}^2} + \alpha^2 \|A P_N \bar{u}_0\|_{\mathbb{L}^2} \|A P_N v_0\|_{\mathbb{L}^2}.$$

Taking into account the estimate $\|Az\|_{\mathbb{L}^2} \leq \sqrt{\mu_N} \|A^{1/2} z\|_{\mathbb{L}^2}$ for all $z \in V_N$, we apply it to $\|A P_N \bar{u}_0\|_{\mathbb{L}^2}$ and $\|A P_N v_0\|_{\mathbb{L}^2}$ to obtain $\|A^{1/2} P_N v_0\|_{\mathbb{L}^2} \leq (1 + C_{\max}^2) \|A^{1/2} P_N \bar{u}_0\|_{\mathbb{L}^2}$. Subsequently,

$$\mathbb{E} \left[\|A^{1/2} P_N v_0\|_{\mathbb{L}^2}^{2p} \right] \leq (1 + C_{\max}^2)^{2p} \mathbb{E} \left[\|A^{1/2} P_N \bar{u}_0\|_{\mathbb{L}^2}^{2p} \right] \leq (1 + C_{\max}^2)^{2p} \mathbb{E} \left[\|A^{1/2} \bar{u}_0\|_{\mathbb{L}^2}^{2p} \right] =: M,$$

which implies that $(P_N v_0)_N$ is bounded in the reflexive Banach space $L^{2p}(\Omega; \mathbb{H}^1)$. Thus, there exists a subsequence $(P_{N_k} v_0)_k$ that converges weakly in $L^{2p}(\Omega; \mathbb{H}^1)$ toward some limit ξ , and one gets $\mathbb{E} \left[\|\xi\|_{\mathbb{H}^1}^{2p} \right] \leq \liminf \mathbb{E} \left[\|P_{N_k} v_0\|_{\mathbb{H}^1}^{2p} \right] \leq C_D M$, thanks to the Poincaré inequality. It remains to identify ξ with v_0 . Indeed, since $L^{2p}(\Omega; \mathbb{H}^1) \hookrightarrow L^{2p}(\Omega; \mathbb{L}^2)$, the weak convergence of $(P_{N_k} v_0)_k$ also takes place in $L^{2p}(\Omega; \mathbb{L}^2)$. Observe that $P_N v_0$ converges strongly (and therefore weakly) toward v_0 in $L^{2p}(\Omega; \mathbb{L}^2)$ as $N \rightarrow +\infty$, thanks to the properties of the projector P_N . Consequently, by the weak limit uniqueness, $\xi = v_0$ \mathbb{P} -a.s. and a.e. in D , and the result follows. \square

Owing to Lemma 3.4.2, high space-regularity estimates are illustrated below for the process (\bar{u}_N, v_N) .

Lemma 3.4.3 *Let $N \in \mathbb{N} \setminus \{0\}$, and $p \in [1, +\infty)$. Assume that (S_1) - (S_2) are valid and that $\alpha \leq C_{\max} \mu_N^{-3/4}$, for some constant $C_{\max} > 0$ independent of N . Then, the solution (\bar{u}_N, v_N) of equation (3.6) satisfies*

$$(i) \mathbb{E} \left[\sup_{t \in [0, T]} \|\nabla \bar{u}_N(t)\|_{\alpha}^{2p} + \left(\nu \int_0^T \|A_N \bar{u}_N(t)\|_{\alpha}^2 dt \right)^p \right] \leq C_3,$$

$$(ii) \mathbb{E} \left[\sup_{t \in [0, T]} \|\nabla v_N(t)\|_{\mathbb{L}^2}^{2p} + \left(\nu \int_0^T \|A_N v_N(t)\|_{\mathbb{L}^2}^2 dt \right)^p \right] \leq C_4,$$

where $C_3 > 0$ depends on C_1 and $\|\bar{u}_0\|_{L^{6p}(\Omega; \mathbb{V})}$, and C_4 depends only on C_3 and C_{\max} .

Proof: Define the stopping time

$$\tau_N^n := \begin{cases} \inf \{t \in [0, T] : \|A^{1/2}(I + \alpha^2 A)^{-1/2} v_N(t)\|_{\mathbb{L}^2} > n\} & \text{if the set is non-empty,} \\ +\infty & \text{otherwise,} \end{cases}$$

and the process $F(v_N) := \|A^{1/2}(I + \alpha^2 A)^{-1/2} v_N\|_{\mathbb{L}^2}^2$. By equation (3.6)₂, one gets $F(v_N(t \wedge \tau_N^n)) = \|\nabla \bar{u}_N(t \wedge \tau_N^n)\|_{\alpha}^2$. Moreover, $DF(x) = 2A(I + \alpha^2 A)^{-1}x$, and $D^2F(x) = 2A(I + \alpha^2 A)^{-1}$. In particular, $DF(v_N) = 2A\bar{u}_N$, thanks to equation (3.6)₂. By applying Itô's formula to the process $F(v_N(t \wedge \tau_N^n))$, it follows that

$$\begin{aligned} & \|\nabla \bar{u}_N(t \wedge \tau_N^n)\|_{\alpha}^2 + 2\nu \int_0^{t \wedge \tau_N^n} \|A\bar{u}_N(s)\|_{\alpha}^2 ds = \|\nabla \bar{u}_N(0)\|_{\alpha}^2 + 2 \int_0^{t \wedge \tau_N^n} (A\bar{u}_N(s), g(s, \bar{u}_N(s))) dW(s) \\ & + \int_0^{t \wedge \tau_N^n} \|A^{1/2}(I + \alpha^2 A)^{-1/2} g(s, \bar{u}_N(s))\|_{\mathcal{L}_2(K, \mathbb{L}^2)}^2 ds - 2 \int_0^{t \wedge \tau_N^n} \langle \tilde{B}(\bar{u}_N(s), v_N(s)), A\bar{u}_N(s) \rangle ds \\ & = \|\nabla \bar{u}_N(0)\|_{\alpha}^2 + I_1 + I_2 - I_3. \end{aligned}$$

On account of assumption (S₂) and the measurability of \bar{u}_N , we have $\mathbb{E}[I_1] = 0$. Now, the fact that for any $z \in \mathbb{L}^2$, the quantity $\|A^{1/2}(I + \alpha^2 A)^{-1/2} z\|_{\mathbb{L}^2}$ is optimally bounded by $\frac{1}{\alpha} \|z\|_{\mathbb{L}^2}$ justifies the opted assumption on $\|g(\cdot, z)\|_{\mathcal{L}_2(K, \mathbb{H}^1)}$. Therewith, (S₂) leads to

$$I_2 \leq \int_0^{t \wedge \tau_N^n} \|g(s, \bar{u}_N(s))\|_{\mathcal{L}_2(K, \mathbb{H}^1)}^2 ds \leq 2K_1^2 t \wedge \tau_N^n + 2K_2^2 \int_0^{t \wedge \tau_N^n} \|\bar{u}_N(s)\|_{\alpha}^2 ds.$$

Moreover, by virtue of equation (3.6)₂, the identity $v_N = \bar{u}_N + \alpha^2 A\bar{u}_N$ holds \mathbb{P} -a.s. and a.e. in $(0, T) \times D$. Thus, the integrand of I_3 can be amended to the following form

$$\left(\tilde{B}(\bar{u}_N(s), \bar{u}_N(s)), A\bar{u}_N(s) \right) + \alpha^2 \left(\tilde{B}(\bar{u}_N(s), A\bar{u}_N(s)), A\bar{u}_N(s) \right) =: B_1 + B_2.$$

Proposition 3.2.1-(ii) yields $B_1 = ([\bar{u}_N(s) \cdot \nabla] \bar{u}_N(s), A\bar{u}_N(s)) - ([A\bar{u}_N(s) \cdot \nabla] \bar{u}_N(s), \bar{u}_N(s))$. The first term vanishes thanks to [106, Lemma 3.1], as well as the second term (see Proposition 3.2.2-(i)). Thereby, $B_1 = 0$. On the other hand, by Proposition 3.2.1-(ii), it follows that

$$B_2 = -\alpha^2 ([A\bar{u}_N(s) \cdot \nabla] A\bar{u}_N(s), \bar{u}_N(s)).$$

Hence,

$$\begin{aligned} |I_3| & \leq 2\alpha^2 C_D \int_0^{t \wedge \tau_N^n} \|A\bar{u}_N(s)\|_{\mathbb{L}^2} \|A^{3/2} \bar{u}_N(s)\|_{\mathbb{L}^2} \|\bar{u}_N(s)\|_{\mathbb{L}^2}^{1/2} \|A\bar{u}_N(s)\|_{\mathbb{L}^2}^{1/2} ds \\ & \leq 2\alpha^2 C_D \mu_N^{3/2} \int_0^{t \wedge \tau_N^n} \|A\bar{u}_N(s)\|_{\mathbb{L}^2}^{3/2} \|\bar{u}_N(s)\|_{\mathbb{L}^2}^{3/2} ds \\ & \leq \frac{4C_{max}^8 C_D^4}{\nu^3} \int_0^{t \wedge \tau_N^n} \|\bar{u}_N(s)\|_{\mathbb{L}^2}^6 ds + \frac{3\nu}{4} \int_0^{t \wedge \tau_N^n} \|A\bar{u}_N(s)\|_{\mathbb{L}^2}^2 ds. \end{aligned}$$

where Proposition 3.2.2-(ii), estimate $\|A^{3/2} z\|_{\mathbb{L}^2} \leq \mu_N^{3/2} \|z\|_{\mathbb{L}^2}$, for all $z \in V_N$, condition $\alpha \leq C_{max} \mu_N^{-3/4}$ together with the Young inequality with conjugate exponents 1/4 and 3/4 were taken advantage of. Observe that

$$\begin{aligned} \|\nabla \bar{u}_N(0)\|_{\alpha}^2 & = \|\nabla \bar{u}_N(0)\|_{\mathbb{L}^2}^2 + \alpha^2 \|A\bar{u}_N(0)\|_{\mathbb{L}^2}^2 \leq \|\nabla \bar{u}_N(0)\|_{\mathbb{L}^2}^2 + C_{max}^2 \|\nabla \bar{u}_0\|_{\mathbb{L}^2}^2 \\ & \leq (1 + C_{max}^2) \|\nabla \bar{u}_0\|_{\mathbb{L}^2}^2, \end{aligned}$$

thanks to $\alpha \leq C_{max} / \mu_N^{3/4} \leq C_{max}$, and estimate $\|Az\|_{\mathbb{L}^2} \leq \sqrt{\mu_N} \|\nabla z\|_{\mathbb{L}^2}$ for all $z \in V_N$. Taking into account that $\sup_{0 \leq t \leq T} \|\bar{u}_N(t)\|_{\alpha}^{2q}$ is almost surely finite for all $q \geq 2$ on account of Lemma 3.4.1, the

stopping time of last and first terms on the right-hand side of I_2 and I_3 can be omitted. Thereby,

$$\begin{aligned} & \|\nabla \bar{u}_N(t \wedge \tau_N^n)\|_\alpha^2 + \frac{5\nu}{4} \int_0^{t \wedge \tau_N^n} \|A\bar{u}_N(s)\|_\alpha^2 ds \leq (1 + \mathcal{C}_{max}^2) \|\nabla \bar{u}_0\|_{\mathbb{L}^2}^2 + 2K_1^2 t \wedge \tau_N^n \\ & + 2 \int_0^{t \wedge \tau_N^n} (A\bar{u}_N(s), g(s, \bar{u}_N(s))) dW(s) + 2K_2^2 \int_0^t \|\bar{u}_N(s)\|_\alpha^2 ds \\ & + \frac{4\mathcal{C}_{max}^8 C_D^4}{\nu^3} \int_0^t \|\bar{u}_N(s)\|_{\mathbb{L}^2}^6 ds. \end{aligned} \quad (3.12)$$

Subsequently, taking the mathematical expectation, employing Lemma 3.4.1 and letting $n \rightarrow +\infty$ imply

$$\begin{aligned} & \mathbb{E} \left[\|\nabla \bar{u}_N(t)\|_\alpha^2 \right] + \frac{5\nu}{4} \mathbb{E} \left[\int_0^t \|A\bar{u}_N(s)\|_\alpha^2 ds \right] \\ & \leq (1 + \mathcal{C}_{max}^2) \mathbb{E} \left[\|\nabla \bar{u}_0\|_{\mathbb{L}^2}^2 \right] + 2K_1^2 T + (2K_2^2 + \frac{4\mathcal{C}_{max}^8 C_D^4}{\nu^3}) T C_1. \end{aligned} \quad (3.13)$$

We now raise equation (3.12) to the power p , use Proposition 3.2.3-(iii), and drop the stopping time τ_N^n , thanks to estimate (3.13). We obtain

$$\begin{aligned} & \sup_{0 \leq t \leq T} \|\nabla \bar{u}_N(t)\|_\alpha^{2p} + \left(\frac{5\nu}{4} \int_0^T \|A\bar{u}_N(t)\|_\alpha^2 dt \right)^p \lesssim \|\nabla \bar{u}_0\|_{\mathbb{L}^2}^{2p} + (K_1^2 T)^p \\ & + \left(\sup_{0 \leq t \leq T} \int_0^t (\nabla \bar{u}_N(s), \nabla g(s, \bar{u}_N(s))) dW(s) \right)^p + (K_2^2 T)^p \sup_{0 \leq t \leq T} \|\bar{u}_N(t)\|_\alpha^{2p} \\ & + (\mathcal{C}_{max}^8 C_D^4 T / \nu^3)^p \sup_{0 \leq t \leq T} \|\bar{u}_N(t)\|_{\mathbb{L}^2}^{6p}. \end{aligned} \quad (3.14)$$

We bound the third term on the right-hand side using the Burkholder-Davis-Gundy and Young inequalities, assumption (S_2) , and Proposition 3.2.3-(iii):

$$\begin{aligned} & \mathbb{E} \left[\left(\sup_{0 \leq t \leq T} \int_0^t (\nabla \bar{u}_N(s), \nabla g(s, \bar{u}_N(s))) dW(s) \right)^p \right] \\ & \lesssim \mathbb{E} \left[\left(\int_0^T \|\nabla \bar{u}_N(t)\|_{\mathbb{L}^2}^2 \|\nabla g(t, \bar{u}_N(t))\|_{\mathcal{L}_2(K; \mathbb{L}^2)}^2 dt \right)^{p/2} \right] \\ & \leq \frac{1}{2} \mathbb{E} \left[\sup_{0 \leq t \leq T} \|\nabla \bar{u}_N(t)\|_\alpha^{2p} \right] + 2^{2p-2} T^p \mathbb{E} \left[K_1^{2p} + K_2^{2p} \sup_{0 \leq t \leq T} \|\bar{u}_N(t)\|_\alpha^{2p} \right], \end{aligned}$$

Taking afterwards the mathematical expectation of equation (3.14) and employing Lemma 3.4.1 complete the proof of estimate (i). On the other hand, $\|\nabla v_N(t)\|_{\mathbb{L}^2}^2 \leq 2 \max(1, \mathcal{C}_{max}^2) \|\nabla \bar{u}_N(t)\|_\alpha^2$ holds for all $t \in [0, T]$, thanks to equation (3.11) which is slightly amended here to fit the case $\alpha \leq \mathcal{C}_{max} \mu_N^{-3/4}$. Furthermore, multiplying in \mathbb{L}^2 the identity $v_N(t) = \bar{u}_N(t) + \alpha^2 A \bar{u}_N(t)$ by $A^2 v_N$ and making use of Cauchy-Schwarz inequality give

$$\|Av_N(t)\|_{\mathbb{L}^2}^2 \leq \|A\bar{u}_N(t)\|_{\mathbb{L}^2} \|Av_N(t)\|_{\mathbb{L}^2} + \alpha^2 \|A^{3/2} \bar{u}_N(t)\|_{\mathbb{L}^2} \|A^{3/2} v_N(t)\|_{\mathbb{L}^2}.$$

We use $\alpha \leq \mathcal{C}_{max} \mu_N^{-3/4}$, $\|A^{3/2} v_N\|_{\mathbb{L}^2} \leq \sqrt{\mu_N} \|Av_N\|_{\mathbb{L}^2}$, and simplify by $\|Av_N(t)\|_{\mathbb{L}^2}$ to obtain eventually

$$\|Av_N(t)\|_{\mathbb{L}^2} \leq \max(1, \mathcal{C}_{max}) \left(\|A\bar{u}_N(t)\|_{\mathbb{L}^2} + \alpha \|A^{3/2} \bar{u}_N(t)\|_{\mathbb{L}^2} \right).$$

Squaring both sides implies $\|A_N v_N(t)\|_{\mathbb{L}^2}^2 \leq 2 \max(1, \mathcal{C}_{max}^2) \|A_N \bar{u}_N(t)\|_\alpha^2$. The proof of inequality (ii) follows after applying estimate (i). \square

3.4.2 . Convergence

Now that all the necessary a priori estimates are set, the next step will consist in extracting a convergent subsequence from $\{v_N\}_N$ and $\{\bar{u}_N\}_N$. The proof hereafter is broken down into multiple steps.

Boundedness and convergence: We begin by bounding each term of equations (3.6) in a suitable reflexive Banach space. The bilinear operator $\{\tilde{B}(\bar{u}_N, v_N)\}_N$ is bounded in $L^2(\Omega; L^2(0, T; \mathbb{V}'))$. Indeed, Proposition 3.2.1-(iii), the embedding $\mathbb{H}^1 \hookrightarrow \mathbb{L}^4$, the Cauchy-Schwarz inequality, and Lemma 3.4.3 yield

$$\mathbb{E} \left[\int_0^T \left\| \tilde{B}(\bar{u}_N(t), v_N(t)) \right\|_{\mathbb{V}'}^2 dt \right] \leq C_D \mathbb{E} \left[\sup_{t \in [0, T]} \|\nabla \bar{u}_N(t)\|_{\mathbb{L}^2}^2 \int_0^T \|\nabla v_N(t)\|_{\mathbb{L}^2}^2 dt \right] \leq C_D C_3 C_4.$$

Therefore, setting $R(\bar{u}_N) := -\nu \Delta v_N + \tilde{B}(\bar{u}_N, v_N)$, we conclude from Lemma 3.4.1 that $\{R(\bar{u}_N)\}_N$ is bounded in $L^2(\Omega; L^2(0, T; \mathbb{V}'))$. Moreover, by virtue of Lemma 3.4.1 and assumption (S_2) , $\{v_N\}_N$, $\{\bar{u}_N\}_N$ are bounded in $L^2(\Omega; L^\infty(0, T; \mathbb{H}) \cap L^2(0, T; \mathbb{V}))$, and $\{g(\cdot, \bar{u}_N)\}_N$ too in the Hilbert space $L^2(\Omega; L^2(0, T; \mathcal{L}_2(K, \mathbb{L}^2)))$. This implies the existence of two subsequences $\{v_{N_k}\}_k$, $\{\bar{u}_{N_k}\}_k$ of $\{v_N\}_N$, $\{\bar{u}_N\}_N$ respectively, and four limiting functions $v_{NS}, u_{NS} \in L^2(\Omega; L^\infty(0, T; \mathbb{H}) \cap L^2(0, T; \mathbb{V}))$, $R_0 \in L^2(\Omega; L^2(0, T; \mathbb{V}'))$, and $g_0 \in L^2(\Omega; L^2(0, T; \mathcal{L}_2(K, \mathbb{L}^2)))$ such that

$$v_{N_k} \rightharpoonup v_{NS} \quad \& \quad \bar{u}_{N_k} \rightharpoonup u_{NS} \quad \text{in } L^2(\Omega; L^2(0, T; \mathbb{V})), \quad (3.15)$$

$$v_{N_k} \overset{*}{\rightharpoonup} v_{NS} \quad \& \quad \bar{u}_{N_k} \overset{*}{\rightharpoonup} u_{NS} \quad \text{in } L^2(\Omega; L^\infty(0, T; \mathbb{H})), \quad (3.16)$$

$$R(\bar{u}_{N_k}) \rightharpoonup R_0 \quad \text{in } L^2(\Omega; L^2(0, T; \mathbb{V}')), \quad (3.17)$$

$$g(\cdot, \bar{u}_{N_k}) \rightharpoonup g_0 \quad \text{in } L^2(\Omega; L^2(0, T; \mathcal{L}_2(K, \mathbb{L}^2))). \quad (3.18)$$

As a result, the limiting function v_{NS} satisfies \mathbb{P} -a.s. and for all $t \in [0, T]$ the equation:

$$(v_{NS}(t), \varphi) + \int_0^t \langle R_0(s), \varphi \rangle ds = (v_0, \varphi) + \left(\int_0^t g_0(s) dW(s), \varphi \right), \quad \forall \varphi \in \mathbb{V}, \quad (3.19)$$

where we recall the v_0 is the limit of $P_N v_0$ as $N \rightarrow +\infty$ in $L^4(\Omega; \mathbb{H})$. Making use of the classical approach in [95], and taking into account equation (3.19) which is fulfilled by v_{NS} , it is straightforward to show that $v_{NS} \in L^2(\Omega; C([0, T]; \mathbb{H}))$. Besides, identity $v_{N_k} = \bar{u}_{N_k} + \alpha^2 A \bar{u}_{N_k}$ grants equality between processes u_{NS} and v_{NS} . Indeed, for all $\varphi \in \mathbb{H}$, it holds that

$$\left| \alpha^2 \mathbb{E} \left[\int_0^T (A \bar{u}_{N_k}(t), \varphi) dt \right] \right| \leq \alpha \|\varphi\|_{\mathbb{L}^2} \mathbb{E} \left[\int_0^T \alpha^2 \|A \bar{u}_{N_k}(t)\|_{\mathbb{L}^2}^2 dt \right]^{1/2} \leq \alpha \|\varphi\|_{\mathbb{L}^2} C_1 \rightarrow 0$$

as $k \rightarrow +\infty$, thanks to the hypothesis $\alpha \leq C_{max} \mu_N^{-3/4}$. Subsequently, $\{\alpha^2 A \bar{u}_{N_k}\}_k$ converges weakly in $L^2(\Omega; L^2(0, T; \mathbb{L}^2))$ to 0, which offers, by the use of the aforementioned identity together with (3.15), the equality $u_{NS} = v_{NS}$ \mathbb{P} -a.s. and a.e. in $[0, T] \times D$. The only remaining task in this section consists in identifying R_0 and g_0 with their solution-dependent counterparts. To this purpose, we must first state one essential property that enables such an identification.

Proposition 3.4.1 *For $N \in \mathbb{N} \setminus \{0\}$, assume that $\alpha \leq C_{max} \mu_N^{-3/4}$. Let v_N^1, v_N^2 be two vector fields in V_N such that $v_N^1 = \bar{u}_N^1 + \alpha^2 A \bar{u}_N^1$ and $v_N^2 = \bar{u}_N^2 + \alpha^2 A \bar{u}_N^2$. If $L_g \leq \frac{\sqrt{\nu}}{C_P \sqrt{2}}$ then, there exists a constant $\mathcal{K} > 0$ depending only on D and C_{max} such that*

$$\begin{aligned} & \left\langle -\nu \Delta (v_N^1 - v_N^2) + \tilde{B}(\bar{u}_N^1, v_N^1) - \tilde{B}(\bar{u}_N^2, v_N^2) + \frac{\mathcal{K}}{\nu^3} \left\| \bar{u}_N^2 \right\|_{\mathbb{L}^4}^4 (\bar{u}_N^1 - \bar{u}_N^2), \bar{u}_N^1 - \bar{u}_N^2 \right\rangle \\ & - \left\| g(\cdot, \bar{u}_N^1) - g(\cdot, \bar{u}_N^2) \right\|_{\mathcal{L}_2(K, \mathbb{L}^2)}^2 \geq 0, \end{aligned}$$

where $C_P > 0$ is the Poincaré constant.

Proof: Let $w_N := \bar{u}_N^1 - \bar{u}_N^2$. We have

$$\langle -\nu\Delta(v_N^1 - v_N^2), w_N \rangle = \nu \left(A^{1/2}(I + \alpha^2 A)w_N, A^{1/2}w_N \right) = \nu \|\nabla w_N\|_{\alpha}^2.$$

Additionally, Proposition 3.2.1-(i) and (iii) yield

$$\begin{aligned} \left| \langle \tilde{B}(\bar{u}_N^1, v_N^1) - \tilde{B}(\bar{u}_N^2, v_N^2), w_N \rangle \right| &= \left| \langle \tilde{B}(\bar{u}_N^2, v_N^1 - v_N^2), w_N \rangle \right| \\ &\leq C_D \|\bar{u}_N^2\|_{\mathbb{L}^4} \|\nabla(v_N^1 - v_N^2)\|_{\mathbb{L}^2} \|w_N\|_{\mathbb{L}^2}^{\frac{1}{2}} \|\nabla w_N\|_{\mathbb{L}^2}^{\frac{1}{2}}, \end{aligned} \quad (3.20)$$

where identity $v_N^1 - v_N^2 = w_N + \alpha^2 A w_N$ implies $\nabla(v_N^1 - v_N^2) = \nabla w_N + \alpha^2 \nabla A w_N$ and therefore, it follows that $\|\nabla(v_N^1 - v_N^2)\|_{\mathbb{L}^2} \leq (1 + C_{max}) \|\nabla w_N\|_{\mathbb{L}^2}$, thanks to the condition $\alpha \leq C_{max} \mu_N^{-3/4}$. Plugging this result back into equation (3.20) and applying the Young inequality to get

$$\left| \langle \tilde{B}(\bar{u}_N^1, v_N^1) - \tilde{B}(\bar{u}_N^2, v_N^2), w_N \rangle \right| \leq \frac{\nu}{4} \|\nabla w_N\|_{\mathbb{L}^2}^2 + \frac{\mathcal{K}}{\nu^3} \|\bar{u}_N^2\|_{\mathbb{L}^4}^4 \|w_N\|_{\mathbb{L}^2}^2,$$

where $\mathcal{K} > 0$ depends only on C_D and C_{max} . Assumption (S_2) implies

$$-\|g(\cdot, \bar{u}_N^1) - g(\cdot, \bar{u}_N^2)\|_{\mathcal{L}_2(K, \mathbb{L}^2)}^2 \geq -L_g^2 \|w_N\|_{\alpha}^2$$

in addition. Putting it all together and employing the Poincaré inequality, we obtain

$$\begin{aligned} &\langle -\nu\Delta(v_N^1 - v_N^2) + \tilde{B}(\bar{u}_N^1, v_N^1) - \tilde{B}(\bar{u}_N^2, v_N^2) + \frac{\mathcal{K}}{\nu^3} \|\bar{u}_N^2\|_{\mathbb{L}^4}^4 w_N, w_N \rangle - \|g(\cdot, \bar{u}_N^1) - g(\cdot, \bar{u}_N^2)\|_{\mathcal{L}_2(K, \mathbb{L}^2)}^2 \\ &\geq \left(\frac{\nu}{2} - L_g^2 C_P^2\right) \|\nabla w_N\|_{\mathbb{L}^2}^2 + \alpha^2 (\nu - L_g^2 C_P^2) \|A_N w_N\|_{\mathbb{L}^2}^2 \end{aligned}$$

which is nonnegative when $L_g \leq \frac{\sqrt{\nu}}{C_P \sqrt{2}}$. □

Remark 3.4.2 The quantities \bar{u}_N^1 and \bar{u}_N^2 in the statement of Proposition 3.4.1 exist and are unique, thanks to the bijectivity of operator $I + \alpha^2 A$ from $D(A)$ to \mathbb{H} .

Limits identification: For clarity's sake, the subsequences' subscript N_k will be henceforth denoted N . Let $0 < \ell < N$ be a fixed integer, and $z, \bar{z} \in L^\infty(\Omega \times (0, T); V_\ell)$ be such that $z = \bar{z} + \alpha^2 A \bar{z}$. For $t \in [0, T]$, define the real valued process $\rho(\omega, t) := \frac{2\mathcal{K}}{\nu^3} \int_0^t \|z(\omega, s)\|_{\mathbb{L}^4}^4 ds$, where the constant \mathcal{K} is that of Proposition 3.4.1. Due to the properties of z , the process ρ is clearly time-continuous and adapted. By application of Itô's formula to the process $t \mapsto e^{-\rho(t)} \|v_N(t)\|_{\mathbb{L}^2}^2$, it follows that

$$\begin{aligned} e^{-\rho(t)} \|v_N(t)\|_{\mathbb{L}^2}^2 &= \|v_N(0)\|_{\mathbb{L}^2}^2 + 2 \int_0^t e^{-\rho(s)} (v_N(s), g(s, \bar{u}_N(s)) dW(s)) \\ &\quad - \frac{2\mathcal{K}}{\nu^3} \int_0^t e^{-\rho(s)} \|z(s)\|_{\mathbb{L}^4}^4 \|v_N(s)\|_{\mathbb{L}^2}^2 ds - 2 \int_0^t e^{-\rho(s)} (v_N(s), R(\bar{u}_N(s))) ds \\ &\quad + \int_0^t e^{-\rho(s)} \|P_N g(s, \bar{u}_N(s))\|_{\mathcal{L}_2(K, \mathbb{L}^2)}^2 ds, \end{aligned}$$

where we recall that $R(\bar{u}_N) = -\nu\Delta v_N + \tilde{B}(\bar{u}_N, v_N)$. The mathematical expectation of the second term on the right-hand side is null, thanks to assumption (S_2) and the measurability of v_N . Therefore, the

above equation transforms into

$$\begin{aligned}
& \mathbb{E} \left[e^{-\rho(T)} \|v_N(T)\|_{\mathbb{L}^2}^2 - \|v_N(0)\|_{\mathbb{L}^2}^2 \right] \\
&= \frac{2\mathcal{K}}{\nu^3} \mathbb{E} \left[\int_0^T e^{-\rho(t)} \|z(t)\|_{\mathbb{L}^4}^4 \left\{ \|z(t)\|_{\mathbb{L}^2}^2 - 2(v_N(t), z(t)) \right\} dt \right] \\
&- 2\mathbb{E} \left[\int_0^T e^{-\rho(t)} \left(R(\bar{u}_N(t)) - R(\bar{z}(t)) + \frac{\mathcal{K}}{\nu^3} \|z(t)\|_{\mathbb{L}^4}^4 (v_N(t) - z(t), v_N(t) - z(t)) \right) dt \right] \\
&- 2\mathbb{E} \left[\int_0^T e^{-\rho(t)} (R(\bar{u}_N(t)) - R(\bar{z}(t)), z(t)) dt \right] - 2\mathbb{E} \left[\int_0^T e^{-\rho(t)} (R(\bar{z}(t)), v_N(t)) dt \right] \quad (3.21) \\
&+ \mathbb{E} \left[\int_0^T e^{-\rho(t)} \|P_N g(t, \bar{u}_N(t)) - P_N g(t, \bar{z}(t))\|_{\mathcal{L}_2(K, \mathbb{L}^2)}^2 dt \right] \\
&+ 2\mathbb{E} \left[\int_0^T e^{-\rho(t)} (P_N g(t, \bar{u}_N(t)), P_N g(t, \bar{z}(t)))_{\mathcal{L}_2(K, \mathbb{L}^2)} dt \right] \\
&- \mathbb{E} \left[\int_0^T e^{-\rho(t)} \|P_N g(t, \bar{z}(t))\|_{\mathcal{L}_2(K, \mathbb{L}^2)}^2 dt \right] =: I_1 + \dots + I_7,
\end{aligned}$$

where the notation $(\cdot, \cdot)_{\mathcal{L}_2(K, \mathbb{L}^2)}$ represents the $\mathcal{L}_2(K, \mathbb{L}^2)$ -scalar product. By convergence 3.15, I_1 converges toward $\frac{2\mathcal{K}}{\nu^3} \mathbb{E} \left[\int_0^T e^{-\rho(t)} \|z(t)\|_{\mathbb{L}^4}^4 \left\{ \|z(t)\|_{\mathbb{L}^2}^2 - 2(v_{NS}(t), z(t)) \right\} dt \right]$ as $N \rightarrow +\infty$. Moreover,

$$\begin{aligned}
I_2 &= -2\mathbb{E} \left[\int_0^T e^{-\rho(t)} \left(R(\bar{u}_N(t)) - R(\bar{z}(t)) + \frac{\mathcal{K}}{\nu^3} \|z(t)\|_{\mathbb{L}^4}^4 (\bar{u}_N(t) - \bar{z}(t), \bar{u}_N(t) - \bar{z}(t)) \right) dt \right] \\
&- 2\alpha^2 \mathbb{E} \left[\int_0^T e^{-\rho(t)} \left(R(\bar{u}_N(t)) - R(\bar{z}(t)) + \frac{\mathcal{K}}{\nu^3} \|z(t)\|_{\mathbb{L}^4}^4 (\bar{u}_N(t) - \bar{z}(t)), A\bar{u}_N(t) - A\bar{z}(t) \right) dt \right] \\
&- \frac{2\mathcal{K}\alpha^2}{\nu^3} \mathbb{E} \left[\int_0^T e^{-\rho(t)} \|z(t)\|_{\mathbb{L}^4}^4 (A\bar{u}_N(t) - A\bar{z}(t), v_N(t) - z(t)) dt \right] =: I_{2,1} + I_{2,2} + I_{2,3}.
\end{aligned}$$

Proposition 3.4.1 implies that $I_{2,1} + I_5 \leq 0$. Additionally, by turning (\cdot, \cdot) into $\langle \cdot, \cdot \rangle$, it follows that

$$|I_{2,2}| \leq 2\alpha^2 \mathbb{E} \left[\int_0^T \left(\|R(\bar{u}_N)\|_{\mathbb{H}^{-1}} + \|R(\bar{z})\|_{\mathbb{H}^{-1}} + \frac{\mathcal{K}}{\nu^3} \|z\|_{\mathbb{L}^4}^4 \|\bar{u}_N - \bar{z}\|_{\mathbb{H}^{-1}} \right) \|A\bar{u}_N - A\bar{z}\|_{\mathbb{H}^1} dt \right].$$

By the definition of operator R , one gets

$$\begin{aligned}
\|R(\bar{u}_N(t))\|_{\mathbb{H}^{-1}} &\leq \nu \|\Delta \bar{u}_N(t)\|_{\mathbb{H}^{-1}} + \|\tilde{B}(\bar{u}_N(t), v_N(t))\|_{\mathbb{H}^{-1}} \\
&\leq \nu \|\nabla \bar{u}_N(t)\|_{\mathbb{L}^2} + C_D \|\bar{u}_N(t)\|_{\mathbb{L}^2}^{\frac{1}{2}} \|\nabla \bar{u}_N(t)\|_{\mathbb{L}^2}^{\frac{1}{2}} \|\nabla v_N(t)\|_{\mathbb{L}^2},
\end{aligned}$$

thanks to Proposition 3.2.1-(iii) and the Gagliardo-Nirenberg inequality. Therefore,

$$\begin{aligned}
& 2\alpha^2 \mathbb{E} \left[\int_0^T \|R(\bar{u}_N(t))\|_{\mathbb{H}^{-1}} \|A\bar{u}_N(t) - A\bar{z}(t)\|_{\mathbb{H}^1} dt \right] \\
&\leq 2\alpha\nu \mathbb{E} \left[\sup_{t \in [0, T]} \|\nabla \bar{u}_N(t)\|_{\mathbb{L}^2}^2 \right]^{\frac{1}{2}} \mathbb{E} \left[\int_0^T \alpha^2 \|A\bar{u}_N(t) - A\bar{z}(t)\|_{\mathbb{H}^1}^2 dt \right]^{\frac{1}{2}} \\
&\quad + 2\alpha C_D \mathbb{E} \left[\sup_{t \in [0, T]} \|\bar{u}_N(t)\|_{\mathbb{L}^2} \|\nabla \bar{u}_N(t)\|_{\mathbb{L}^2} \|\nabla v_N(t)\|_{\mathbb{L}^2}^2 \right]^{\frac{1}{2}} \mathbb{E} \left[\int_0^T \alpha^2 \|A\bar{u}_N(t) - A\bar{z}(t)\|_{\mathbb{H}^1}^2 dt \right]^{\frac{1}{2}} \\
&\lesssim 2\alpha\nu C_3 + 2\alpha C_D C_3 C_4 \rightarrow 0 \text{ as } N \rightarrow +\infty,
\end{aligned}$$

thanks to Lemma 3.4.3 and the assumption $\alpha \leq C_{max}\mu_N^{-3/4}$. The same goes for the remaining terms of $I_{2,2}$, which are easier to handle. Thus, $I_{2,2} \rightarrow 0$ as $N \rightarrow +\infty$. Moving on to $I_{2,3}$, we have

$$\begin{aligned} |I_{2,3}| &\leq \frac{2\mathcal{K}\alpha}{\nu^3} \mathbb{E} \left[\sup_{t \in [0, T]} \|z(t)\|_{\mathbb{L}^4}^8 \|v_N(t) - z(t)\|_{\mathbb{L}^2}^2 \right]^{\frac{1}{2}} \mathbb{E} \left[\int_0^T \alpha^2 \|A\bar{u}_N(t) - A\bar{z}(t)\|_{\mathbb{L}^2}^2 dt \right]^{\frac{1}{2}} \\ &\lesssim \frac{2\mathcal{K}\alpha}{\nu^3} C_2 C_1 \rightarrow 0 \text{ as } N \rightarrow +\infty, \end{aligned}$$

by virtue of Lemma 3.4.1 and $\alpha \leq C_{max}\mu_N^{-3/4}$. It is straightforward to show that when $N \rightarrow +\infty$, z and \bar{z} become equal \mathbb{P} -a.s. and a.e. in $[0, T] \times D$. We exploit this fact and convergence 3.17 to obtain $I_3 \rightarrow -2\mathbb{E} \left[\int_0^T e^{-\rho(t)} \langle R_0(t) - R(z(t)), z(t) \rangle dt \right]$ as $N \rightarrow +\infty$, and convergence (3.15) to accomplish $I_4 \rightarrow -2\mathbb{E} \left[\int_0^T e^{-\rho(t)} \langle R(z(t)), v_{NS}(t) \rangle dt \right]$. Similarly, $I_6 \rightarrow 2\mathbb{E} \left[\int_0^T e^{-\rho(t)} (g_0(t), g(t, z(t)))_{\mathcal{L}_2(K, \mathbb{L}^2)} dt \right]$, by virtue of result (3.18), the continuity of g with respect to its second variable, and the properties of projector P_N which also grant the convergence of I_7 i.e. $I_7 \rightarrow -\mathbb{E} \left[\int_0^T e^{-\rho(t)} \|g(t, z(t))\|_{\mathcal{L}_2(K, \mathbb{L}^2)}^2 dt \right]$. Consequently, we pass to the limit in equation (3.21) while taking advantage of all generated results to achieve eventually:

$$\begin{aligned} \mathbb{E} \left[e^{-\rho(T)} \|v_{NS}(T)\|_{\mathbb{L}^2}^2 - \|v_{NS}(0)\|_{\mathbb{L}^2}^2 \right] &\leq \liminf_{N \rightarrow +\infty} \mathbb{E} \left[e^{-\rho(T)} \|v_N(T)\|_{\mathbb{L}^2}^2 - \|v_N(0)\|_{\mathbb{L}^2}^2 \right] \\ &\leq \frac{2\mathcal{K}}{\nu^3} \mathbb{E} \left[\int_0^T e^{-\rho(t)} \|z(t)\|_{\mathbb{L}^4}^4 \left\{ \|z(t)\|_{\mathbb{L}^2}^2 - 2(v_{NS}(t), z(t)) \right\} dt \right] \\ &\quad - 2\mathbb{E} \left[\int_0^T e^{-\rho(t)} \langle R_0(t) - R(z(t)), z(t) \rangle dt \right] - 2\mathbb{E} \left[\int_0^T e^{-\rho(t)} \langle R(z(t)), v_{NS}(t) \rangle dt \right] \\ &\quad + 2\mathbb{E} \left[\int_0^T e^{-\rho(t)} (g_0(t), g(t, z(t)))_{\mathcal{L}_2(K, \mathbb{L}^2)} dt \right] - \mathbb{E} \left[\int_0^T e^{-\rho(t)} \|g(t, z(t))\|_{\mathcal{L}_2(K, \mathbb{L}^2)}^2 dt \right]. \end{aligned} \tag{3.22}$$

Next, we apply Itô's formula to the process $t \mapsto e^{-\rho(t)} \|v_{NS}(t)\|_{\mathbb{L}^2}^2$, where we recall that v_{NS} satisfies equation (3.19). It holds that

$$\begin{aligned} \mathbb{E} \left[e^{-\rho(T)} \|v_{NS}(T)\|_{\mathbb{L}^2}^2 - \|v_{NS}(0)\|_{\mathbb{L}^2}^2 \right] &= -\frac{2\mathcal{K}}{\nu^3} \mathbb{E} \left[\int_0^T e^{-\rho(t)} \|z(t)\|_{\mathbb{L}^4}^4 \|v_{NS}(t)\|_{\mathbb{L}^2}^2 dt \right] \\ &\quad - 2\mathbb{E} \left[\int_0^T e^{-\rho(t)} \langle R_0(t), v_{NS}(t) \rangle dt \right] + \mathbb{E} \left[\int_0^T e^{-\rho(t)} \|g_0(t)\|_{\mathcal{L}_2(K, \mathbb{L}^2)}^2 dt \right]. \end{aligned} \tag{3.23}$$

Plugging result (3.23) in equation (3.22) grants:

$$\begin{aligned} &\frac{2\mathcal{K}}{\nu^3} \mathbb{E} \left[\int_0^T e^{-\rho(t)} \|z(t)\|_{\mathbb{L}^4}^4 \|v_{NS}(t) - z(t)\|_{\mathbb{L}^2}^2 dt \right] \\ &\quad + 2\mathbb{E} \left[\int_0^T e^{-\rho(t)} \langle R_0(t) - R(z(t)), v_{NS}(t) - z(t) \rangle dt \right] \\ &\geq \mathbb{E} \left[\int_0^T e^{-\rho(t)} \|g_0(t) - g(t, z(t))\|_{\mathcal{L}_2(K, \mathbb{L}^2)}^2 dt \right], \quad \forall z \in L^\infty(\Omega \times (0, T); V_\ell). \end{aligned} \tag{3.24}$$

Arguing by density, the above inequality holds for all $z \in L^4(\Omega; L^\infty(0, T; \mathbb{H})) \cap L^2(\Omega; L^2(0, T; \mathbb{V}))$. Setting $z = v_{NS}$ in equation (3.24) yields $g(\cdot, v) = g_0$ \mathbb{P} -a.s. and a.e. in $(0, T) \times D$. Furthermore, for an arbitrary $w \in L^4(\Omega; L^\infty(0, T; \mathbb{H})) \cap L^2(\Omega; L^2(0, T; \mathbb{V}))$ and $\theta \in \mathbb{R}_+^*$, we set $z = v_{NS} + \theta w$, and make

use of equation (3.24) once again to obtain:

$$\begin{aligned} & \frac{\mathcal{K}\theta}{\nu^3} \mathbb{E} \left[\int_0^T e^{-\rho(t)} \|v_{NS}(t) + \theta w(t)\|_{\mathbb{L}^4}^4 \|w(t)\|_{\mathbb{L}^2}^2 dt \right] \\ & - \mathbb{E} \left[\int_0^T e^{-\rho(t)} \langle R_0(t) - R(v_{NS}(t) + \theta w(t)), w(t) \rangle dt \right] \geq 0. \end{aligned}$$

Letting θ go to 0 and using the hemi-continuity of the operator R lead to

$$\mathbb{E} \left[e^{-\rho(t)} \langle R_0(t) - R(v_{NS}(t)), w(t) \rangle dt \right] \leq 0, \quad \forall w \in L^4(\Omega; L^\infty(0, T; \mathbb{H})) \cap L^2(\Omega; L^2(0, T; \mathbb{V})),$$

which eventually implies $R_0 = R(v_{NS})$ in $L^2(\Omega; L^2(0, T; \mathbb{H}^{-1}))$.

3.5 . Further properties

This part gathers complementary properties of function v_{NS} that was obtained in Section 3.4. As intended, standard and high spatial regularities for v_{NS} are achieved in the below lemma.

Lemma 3.5.1 *Let $T > 0$, $1 \leq p < +\infty$, and $N \in \mathbb{N} \setminus \{0\}$ be given. Suppose that assumptions (S_1) - (S_2) are met, and that for some constant $C_{max} > 0$ independent of N , the spatial scale $\alpha \leq C_{max} \mu_N^{-3/4}$. Then, the stochastic process $\{v_{NS}(t), t \in [0, T]\}$ that satisfies equation (3.19) has the following boundedness properties:*

$$\begin{aligned} (i) \quad & \mathbb{E} \left[\sup_{t \in [0, T]} \|v_{NS}(t)\|_{\mathbb{L}^2}^{2p} + 2p\nu \int_0^T \|v_{NS}(t)\|_{\mathbb{L}^2}^{2(p-1)} \|\nabla v_{NS}(t)\|_{\mathbb{L}^2}^2 dt \right] \leq C_2, \\ (ii) \quad & \mathbb{E} \left[\sup_{t \in [0, T]} \|\nabla v_{NS}(t)\|_{\mathbb{L}^2}^{2p} + \left(\nu \int_0^T \|Av_{NS}(t)\|_{\mathbb{L}^2}^2 dt \right)^p \right] \leq C_4, \end{aligned}$$

where $C_2 > 0$ depends on constants C_{max} , C_1 and its parameters, and $C_4 > 0$ depends on C_1 , $\|\bar{u}_0\|_{L^{6p}(\Omega; \mathbb{V})}$ and C_{max} .

Proof: We only illustrate here the proof of estimate (ii) as (i) can be concluded from (ii). Let $p \geq 1$. On account of Lemma 3.4.3-(ii), the sequence $(v_N)_N$ is bounded in $L^{2p}(\Omega; L^\infty(0, T; \mathbb{V}))$ which implies the existence of a function $\xi \in L^{2p}(\Omega; L^\infty(0, T; \mathbb{V}))$ such that for some subsequence $(v_{N_k})_k$, it holds that $v_{N_k} \xrightarrow{*} \xi$ in $L^{2p}(\Omega; L^\infty(0, T; \mathbb{V})) \hookrightarrow L^2(\Omega; L^\infty(0, T; \mathbb{H}))$, and

$$\mathbb{E} \left[\sup_{t \in [0, T]} \|\xi(t)\|_{\mathbb{V}}^{2p} \right] \leq \liminf \mathbb{E} \left[\sup_{t \in [0, T]} \|v_{N_k}(t)\|_{\mathbb{V}}^{2p} \right] \leq C_4,$$

thanks to Lemma 3.4.3-(ii). By convergence (3.16) and the weak limit uniqueness, we infer that $\xi = v_{NS}$ \mathbb{P} -a.s. and a.e. in $(0, T) \times D$. This is valid because v_{NS} is the unique solution to equations (3.2) which means that the whole sequence $(v_N)_N$ is convergent. Arguing in a similar fashion, and owing to Lemma 3.4.3-(ii), $(v_N)_N$ is bounded in the reflexive Banach space $L^{2p}(\Omega; L^2(0, T; D(A)))$, which signifies that for some $(v_{N_k})_k$ and $\eta \in L^{2p}(\Omega; L^2(0, T; D(A)))$, we have $v_{N_k} \rightharpoonup \eta$ in $L^{2p}(\Omega; L^2(0, T; D(A)))$ and

$$\mathbb{E} \left[\left(\nu \int_0^T \|A\eta(t)\|_{\mathbb{L}^2}^2 dt \right)^p \right] \leq \liminf \mathbb{E} \left[\left(\nu \int_0^T \|Av_{N_k}\|_{\mathbb{L}^2}^2 dt \right)^p \right] \leq C_4.$$

As done earlier in this proof, one obtains $\eta = v_{NS}$ \mathbb{P} -a.s. and a.e. in $(0, T) \times D$. \square

We recall that estimate (i) of Lemma 3.5.1 required the assumption $\alpha \lesssim \mu_N^{-1/2}$, which represents the optimal upper bound when it comes to obtaining a solution to the stochastic Navier-Stokes equations

through the LANS- α model. Whereas, assertion (ii) of Lemma 3.5.1 necessitates the hypothesis $\alpha \lesssim \mu_N^{-3/4}$ together with the two-dimensional periodic boundary conditions, which means that the vanishing rate of α must be swifter if a high spatial regularity of a solution is needed.

Remark 3.5.1 *The acquired estimates in Lemma 3.5.1 are not applicable to solutions of equations (3.1) because the spatial scale α has henceforth disappeared. As shown in Section 3.6, the stochastic process $\{v_{NS}(t), t \in [0, T]\}$ will become a solution to equations (3.2). Subsequently, the estimates of Lemma 3.5.1 are dedicated to the stochastic Navier-Stokes problem.*

3.6 . Conclusion

Owing to Section 3.4, the limiting function v_{NS} satisfies for all $t \in [0, T]$, and \mathbb{P} -a.s. the following equation:

$$\begin{aligned} & (v_{NS}(t), \varphi) + \nu \int_0^t (\nabla v_{NS}(s), \nabla \varphi) + \int_0^t \tilde{b}(v_{NS}(s), v_{NS}(s), \varphi) ds \\ &= (v_0, \varphi) + \left(\int_0^t g(s, v_{NS}(s)) dW(s), \varphi \right), \quad \forall \varphi \in \mathbb{V}. \end{aligned}$$

By virtue of Proposition 3.2.1, one gets

$$\tilde{b}(v_{NS}(s), v_{NS}(s), \varphi) = -\tilde{b}(\varphi, v_{NS}(s), v_{NS}(s)) = ([v_{NS}(s) \cdot \nabla] v_{NS}(s), \varphi).$$

Moreover, as mentioned in Section 3.4.2, v_{NS} belongs to $L^2(\Omega; C([0, T]; \mathbb{H}))$. Besides the latter fact, Lemma 3.5.1 guarantees that $v_{NS} \in L^2(\Omega; L^2(0, T; \mathbb{V}))$. Consequently, collecting all results and comparing them with Definition 3.2.3, it follows that v_{NS} is the unique solution to equations (3.2) in the sense of Definition 3.2.3.

The convergence analysis followed in this chapter could have been carried out differently. For instance, instead of controlling the spatial scale α with a quantity that vanishes at the limit, a convergence rate of the difference $\|v_{NS} - \bar{u}\|$ in terms of α could have made up an alternative approach, as conducted in [15] for the stochastic Leray- α equations. We emphasize the uselessness of the imposed periodic boundary conditions if high spatial regularities of the solution were not utilized. In this case, Dirichlet boundary conditions are required.

The demonstration techniques employed in this chapter are only functional for two-dimensional domains. In three dimensions, another approach must be applied to acquire a solution to the stochastic Navier-Stokes problem from the stochastic LANS- α model, as performed in article [51].

4 - Numerical and convergence analysis of the stochastic Navier-Stokes equations through the LANS- α model

4.1 . Introduction

Over the last few decades, many regularization models of the Navier-Stokes equations (NSEs) have arisen, especially the α -regularizations, for the sake of better understanding the closure problem of averaged quantities in turbulent flows. Such turbulent modeling schemes (e.g. Leray- α , Navier-Stokes- α , Clark- α , Modified Leray- α) were introduced as effective subgrid-scale models of the NSEs which require massive grid points or Fourier modes, allowing for approximation to capture all the spatial scales down to the Kolmogorov scale (see for instance [27] and the references therein), as well as their suitability with the empirical and experimental data for a thorough range of Reynolds numbers.

In the present chapter, we consider the stochastic version of the LANS- α equations [89] (also known as the viscous Camassa-Holm equations [16], or the Navier-Stokes- α model [34, 75])

$$\begin{cases} \partial_t (\bar{u} - \alpha^2 \Delta \bar{u}) - \nu \Delta (\bar{u} - \alpha^2 \Delta \bar{u}) - \bar{u} \times (\nabla \times (\bar{u} - \alpha^2 \Delta \bar{u})) + \nabla \bar{p} \\ \quad = f(\cdot, \bar{u}) + g(\cdot, \bar{u}) \dot{W}, \\ \text{div}(\bar{u}) = 0, \\ \bar{u}(0, \cdot) = \bar{u}_0, \end{cases} \quad (4.1)$$

for internal flow i.e. for a bounded domain in \mathbb{R}^2 . The unknown vector field \bar{u} is called the filtered fluid velocity, and it depends on time and space variables, ν is the fluid kinematic viscosity, and α is a small spatial scale at which fluid motion is filtered. Note that both ν and α are positive constants. $f = f(t, \bar{u})$ is an external force, the scalar quantity $\bar{p} = \bar{p}(t, x)$ represents the pressure and \bar{u}_0 is the corresponding initial datum. The last term of equations (4.1)₁ describes a state-dependent random noise, and it is defined by $g(\cdot, \bar{u}) \dot{W} := g(t, \bar{u}) \partial_t W(t, x)$, where g is a diffusion coefficient and W is a Wiener process whose covariance operator is of trace class. It is worth highlighting the fact that W could be considered cylindrical in this chapter, resulting in a simplification of the needed assumptions. The aim herein is to approach the two-dimensional solutions of the stochastic NSEs via the LANS- α model, numerically. Whence the need to evoke the former equations with similar configurations:

$$\begin{cases} \partial_t v - \nu \Delta v + [v \cdot \nabla] v + \nabla p = f(\cdot, v) + g(\cdot, v) \dot{W}, \\ \text{div}(v) = 0, \\ v(0, \cdot) = v_0, \end{cases} \quad (4.2)$$

where v (resp. p) is the corresponding fluid velocity (resp. pressure), and v_0 embodies its initial datum.

Equations (4.1) and (4.2) are usually employed as a complementary model to their deterministic versions to better understand the situation of tiny variations or perturbations present in fluid flows. The former represents a modification of the latter by performing Lagrangian means, asymptotic expansions, and an assumption of isotropy of fluctuations in the Hamilton principle, which grant further physical properties (e.g. conservation laws for energy and momentum). More specifically, the convective nonlinearity $[v \cdot \nabla] v$ in the NSEs is adjusted so that the cascading of turbulence at scales under specific length stops. The latter adjustment is called a nonlinearly dispersive modification.

The existence and uniqueness of a variational solution to the problem (4.1) were investigated in [29] under Lipschitz-continuous conditions in a three-dimensional bounded domain. A similar study is proposed in [49], but this time with a genuine finite-dimensional Wiener process depending only on time.

LANS- α model driven by an additive space-time noise of trace class was considered in [68], where the authors proved the existence and uniqueness of an invariant measure, and a probabilistic strong solution.

Speaking of the numerical approach, the convergence analysis of suitable numerical methods for the stochastic LANS- α equations is less well developed. In connection with the deterministic version, both convergence rate and convergence analysis of an algorithm consisting of a finite element method were investigated in [44] where the spatial scale α is considered in terms of the space discretization's step. The author in [26] conducted a similar study, with α being independent of the discretization parameters. On the other hand, numerical schemes for stochastic nonlinear equations admitting local Lipschitz nonlinearities related to the Navier-Stokes systems had been already investigated. For instance, authors in [24] studied a finite element-based space-time discretization of the incompressible NSEs driven by a multiplicative noise. An enhancement of [24] in dimension 2 was carried out in [11].

This chapter aims to provide a fully discrete finite element-based discretization of equations (4.2) through system (4.1) in a two-dimensional bounded convex polygonal domain. Notice that the underlying model consists of a fourth-order problem, nevertheless we avoid the use of C^1 piecewise polynomials-based finite element methods by introducing a notion of differential filters that transform equations (4.1) into a coupled problem of second-order. The employed time-discretization herein is an Euler scheme. One highly valued characteristic of the finite element method is the prospect of meticulous interpretation provided by the functional analysis framework. In contrast to the linear stochastic partial differential equations, since we are dealing here with a nonlinear model, one cannot make use of the semigroup method or Green's function. Those techniques are effectively replaced by monotone or Lipschitz-continuous drift functions. It is worth highlighting the importance of constructing practical numerical schemes provided with exact divergence-free finite element functions (e.g. see [17, 54, 93, 94]). However, due to their computational complexity, one may notice the usage of a weak divergence-free condition that compensates for the strong sense's absence.

The associated spatial scale α will be considered hereafter in terms of the space discretization's step size. Therefore, the main result consists of the convergence in 2D of Algorithm 4.1 toward the continuous solution of the 2D stochastic NSEs. Speaking of the followed approach, we begin by performing a priori estimates characterized by their uniformity in α and the discretizations' parameters, allowing us to extract convergent subsequences of the approximate solution. It is worth mentioning that Skorokhod's theorem is not employed to achieve solutions' existence, which means that the probability basis that is defined in Section 4.2 is maintained throughout the demonstration. This avoidance is possible due to a specific identification method that exploits a local monotonicity property attributed to the nonlinear term of equations (4.1). This approach was first introduced in [91] to illustrate solutions' existence through an abstract Faedo-Galerkin method. However, since we are dealing with discrete settings herein, Itô's formula is no longer the go-to tool. Fortunately, discrete derivations solve this issue despite the appearance of a few associated extra terms.

This chapter is organized as follows. We introduce in Section 4.2 a few notions and preliminaries, including the spatial framework, the needed assumptions, the time and space discretizations alongside their properties, definition of solutions to problems (4.1) and (4.2), the definition of continuous and discrete differential filters along with the investigated algorithm. Section 4.3 is tailored for the main results of this chapter. We dedicate Section 4.4 to all possible a priori estimates within standard regularities, together with the local monotonicity property related to the nonlinear term. In Section 4.5, we study the convergence analysis of the proposed numerical scheme. Accordingly, we identify both deterministic and stochastic integrals, as the discretization steps tend to 0, with their corresponding solution-dependent counterparts. We terminate this chapter (Section 4.6) with a conclusion concerning the obtained limiting

functions and how one can relate them to the stochastic NSEs. We equip this section with a computational experiment to visualize the outcomes and to evaluate the performance of the proposed numerical scheme.

4.2 . Notations and preliminaries

We state, in this section, preliminary background material following the usual notation employed in the context of the mathematical theory of Navier-Stokes equations.

Given $T > 0$, we denote by $D \subset \mathbb{R}^2$ a bounded convex polygonal domain with boundary ∂D , in which we seek a solution, namely a stochastic process $(\bar{u}(t), p(t)), t \in [0, T]$ satisfying equations (4.1) in a certain sense. We define almost everywhere on ∂D the unit outward normal vector field $\vec{n}: \partial D \rightarrow \mathbb{R}^2$. The following function spaces are required hereafter:

$$\begin{aligned} \mathbb{H} &:= \left\{ z \in [L^2(D)]^2 \mid \operatorname{div}(z) = 0 \text{ a.e. in } D, z \cdot \vec{n} = 0 \text{ a.e. on } \partial D \right\}, \\ \mathbb{V} &:= \left\{ z \in [H_0^1(D)]^2 \mid \operatorname{div}(z) = 0 \text{ a.e. in } D \right\}. \end{aligned}$$

From now on, the spaces of vector valued functions will be indicated with blackboard bold letters, for instance $\mathbb{L}^2 := (L^2(D))^2$ denotes the Lebesgue space of vector valued functions defined on D . Denote by $\mathcal{P}: \mathbb{L}^2 \rightarrow \mathbb{H}$ the Leray projector, and by $A: D(A) \rightarrow \mathbb{H}$ the Stokes operator defined by $A := -\mathcal{P}\Delta$ with domain $D(A) = \mathbb{H}^2 \cap \mathbb{V}$. A is a self-adjoint positive operator, and has a compact inverse, see for instance [32]. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, T]}, \mathbb{P})$ be a complete probability space, Q be a nuclear operator, and K be a separable Hilbert space on which we define the Q -Wiener process $W(t)$, $t \in [0, T]$ such that

$$W(t) = \sum_{k \geq 1} \sqrt{q_k} \beta^k(t) w_k, \quad \forall t \in [0, T], \quad (4.3)$$

where $\{\beta^k(\cdot), k \geq 1\}$ is a sequence of independent and identically distributed \mathbb{R} -valued Brownian motions on the probability basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, T]}, \mathbb{P})$, $\{w_k, k \geq 1\}$ is a complete orthonormal basis of the Hilbert space K consisting of the eigenfunctions of Q , with eigenvalues $\{q_k, k \geq 1\}$. The following estimate will play a role in the sequel, cf. [76].

$$\mathbb{E} \left[\|W(t) - W(s)\|_K^{2r} \right] \leq (2r - 1)!! (t - s)^r (\operatorname{Tr}(Q))^r, \quad \forall r \in \mathbb{N}, \quad (4.4)$$

where $(2r - 1)!! := (2r - 1)(2r - 3) \dots \times 5 \times 3 \times 1$, and $\operatorname{Tr}(Q)$ denotes the trace of Q .

For any arbitrary Hilbert spaces X, Y , the sets $\mathcal{L}_1(X, Y)$ and $\mathcal{L}_2(X, Y)$ denote the nuclear, and Hilbert-Schmidt operators from X to Y , respectively. For brevity's sake, if $X = Y$, we set $\mathcal{L}_i(X, X) = \mathcal{L}_i(X)$, $i \in \{1, 2\}$. Hereafter, $M_{\mathcal{F}_t}^p(0, T; X)$ denotes the space of all \mathcal{F}_t -progressively measurable processes belonging to $L^p(\Omega \times (0, T), d\mathbb{P} \times dt; X)$, for any Banach space X .

Throughout this chapter, the nonnegative constant C_D depends only on the domain D , the symbols (\cdot, \cdot) and $\langle \cdot, \cdot \rangle$ stand for the inner product in \mathbb{L}^2 and the duality product between \mathbb{H}^{-1} and \mathbb{H}^1 , respectively. Recall that α is a small spatial scale, thereby we assume that $\alpha \leq 1$. The latter leads to the following norm equivalence

$$\alpha \|\cdot\|_{\mathbb{H}^1} \leq \|\cdot\|_{\alpha} \leq \|\cdot\|_{\mathbb{H}^1}, \quad (4.5)$$

where we define $\|\cdot\|_{\alpha}$ by

$$\|\cdot\|_{\alpha}^2 := \|\cdot\|_{\mathbb{L}^2}^2 + \alpha^2 \|\nabla \cdot\|_{\mathbb{L}^2}^2.$$

We point out that the whole study herein maintains all the stated properties if one chooses $\alpha \leq \tilde{\alpha}$, for some $\tilde{\alpha} \in \mathbb{R}_+^*$. Additionally, the symbol \lesssim will denote all this chapter long a shorthand to the inequality \leq up to a universal constant. We list below the needed assumptions on the data \bar{u}_0, g, Q , and f .

Assumptions

(S₁) $Q \in \mathcal{L}_1(K)$ is a symmetric, positive definite operator.

(S₂) $f \in L^2(\Omega; C([0, T]; \mathbb{H}^{-1}))$ and $g \in L^2(\Omega; C([0, T]; \mathcal{L}_2(K, \mathbb{L}^2)))$ are sublinear Lipschitz-continuous mappings, i.e. for all $z_1, z_2 \in \mathbb{V}$, $g(\cdot, z_1)$ and $f(\cdot, z_2)$ are \mathcal{F}_t -progressively measurable, and $d\mathbb{P} \times dt$ -a.e. in $\Omega \times (0, T)$,

$$\begin{aligned} \|g(\cdot, z_1) - g(\cdot, z_2)\|_{\mathcal{L}_2(K, \mathbb{L}^2)} &\leq L_g \|z_1 - z_2\|_\alpha, \quad \forall z_1, z_2 \in \mathbb{V}, \\ \|g(\cdot, z)\|_{\mathcal{L}_2(K, \mathbb{L}^2)} &\leq K_1 + K_2 \|z\|_\alpha, \quad \forall z \in \mathbb{V}, \\ \|f(\cdot, z_1) - f(\cdot, z_2)\|_{\mathbb{H}^{-1}} &\leq L_f \|z_1 - z_2\|_\alpha, \quad \forall z_1, z_2 \in \mathbb{V}, \\ \|f(\cdot, z)\|_{\mathbb{H}^{-1}} &\leq K_3 + K_4 \|z\|_\alpha, \quad \forall z \in \mathbb{V}, \end{aligned}$$

for some time-independent nonnegative constants $K_1, K_2, K_3, K_4, L_f, L_g$ such that $L_f \leq \nu/\sqrt{2}C_P^2$ and $L_g \leq \text{Tr}(Q)^{-1/2} \left(\nu^2 - 2L_f^2 C_P^4 \right)^{1/2} / C_P \sqrt{2\nu}$, with $C_P > 0$ being the Poincaré constant.

(S₃) $\bar{u}_0 \in L^{2p}(\Omega, \mathcal{F}_0, \mathbb{P}; \mathbb{V})$, for some $p \in [1, +\infty)$.

Remark 4.2.1 *The time-continuity of both f and g in assumption (S₂) can be readily relaxed and turned into $L^2(0, T)$. This emerges from the approximation choice of f and g occurring in Algorithm 4.1. For instance, $f(t_{m-1}, \cdot)$ could have been approximated by $f^m(\cdot) := \frac{1}{k} \int_{(m-1)k}^{mk} f(t, \cdot) dt$, where k and m are introduced in subsection 4.2.2.*

Remark 4.2.2 *Assumption (S₁) can be readily eliminated, which means that the considered Wiener process W can be cylindrical. The reader may refer to Chapter 5 to check the utilized demonstration technique that avoids the use of such an assumption. This fact can also bring a few simplifications on the imposed assumptions on L_g (see Section 4.3).*

To avoid repetitions later on, we state the following identity

$$(a - b, a) = \frac{1}{2} \left(\|a\|_{\mathbb{L}^2}^2 - \|b\|_{\mathbb{L}^2}^2 + \|a - b\|_{\mathbb{L}^2}^2 \right) \quad \text{for all } a, b \in \mathbb{L}^2. \quad (4.6)$$

The trilinear form

We define the trilinear form \tilde{b} , associated with the LANS- α equations, by

$$\tilde{b}(z_1, z_2, w) = -\langle z_1 \times (\nabla \times z_2), w \rangle, \quad \forall z_1, z_2, w \in \mathbb{H}_0^1.$$

\tilde{b} can also be associated with a bilinear operator \tilde{B} as defined in Chapter 3. The following proposition contains a few corresponding properties.

Proposition 4.2.1

- (i) $(z_1 \times (\nabla \times z_2), w) = -(w \times (\nabla \times z_2), z_1)$ for all $z_1, z_2, w \in \mathbb{H}^1$. Particularly, $(z_1 \times (\nabla \times z_2), z_1) = 0$.
- (ii) $-(z_1 \times (\nabla \times z_2), w) = ([z_1 \cdot \nabla] z_2, w) + ((\nabla z_1)^T \cdot z_2, w) - (\nabla(z_1 \cdot z_2), w)$, for all $z_1, z_2, w \in \mathbb{H}_0^1$. In particular, $\tilde{b}(z_1, z_2, w) = ([z_1 \cdot \nabla] z_2, w) + \langle (\nabla z_1)^T \cdot z_2, w \rangle$ if $z_1, z_2 \in \mathbb{H}_0^1$ and $w \in \mathbb{V}$.
- (iii) $-(z_1 \times (\nabla \times z_2), w) = ([z_1 \cdot \nabla] z_2, w) - ([w \cdot \nabla] z_2, z_1)$, for all $z_1, z_2, w \in \mathbb{H}_0^1$.
- (iv) $\forall z_1, z_2, w \in \mathbb{H}_0^1$, $|(z_1 \times (\nabla \times z_2), w)| \leq C_D \|z_1\|_{\mathbb{L}^4} \|\nabla z_2\|_{\mathbb{L}^2} \|w\|_{\mathbb{L}^2}^{1/2} \|\nabla w\|_{\mathbb{L}^2}^{1/2}$ (only valid in 2D).

Proof: Identities (i) and (ii) follow straightforwardly from the triple product property

$$(a \times b) \cdot c = -(c \times b) \cdot a,$$

and the equality

$$[z_1 \cdot \nabla]z_2 + (\nabla z_1)^T \cdot z_2 - \nabla(z_1 \cdot z_2) = -z_1 \times (\nabla \times z_2),$$

respectively. To justify assertion (iii), we make use of (ii); more precisely we apply two integrations by parts to $((\nabla z_1)^T \cdot z_2, w) = \sum_{i,j=1}^d \int_D \partial_i z_1^j z_2^j w^i dx$ to obtain $-([w \cdot \nabla]z_2, z_1) + (\nabla(z_1 \cdot z_2), w)$. Plugging it back in (ii) yields the result. Estimate (iv) can therefore be concluded from assertion (iii) after employing the Hölder and Gagliardo-Nirenberg inequalities. \square

It is well-known that finite element methods based on C^1 piecewise polynomials are not easily implementable. Therefore, our fourth-order partial differential equation (4.1) must undergo a modification so that it turns into a second-order problem. To this end, we shall propose a differential filter that deals with a Stokes problem. Such an idea emerges from [62] within a slight adjustment for the sake of fitting the current framework.

Definition 4.2.1 (Continuous differential filter)

Given a vector field $v \in \mathbb{L}^2$, its continuous differential filter, denoted \bar{u} , is part of the unique solution $(\bar{u}, \bar{p}) \in \mathbb{V} \times L_0^2(D)$ to

$$\begin{cases} -\alpha^2 \Delta \bar{u} + \bar{u} + \nabla \bar{p} = v, & \text{in } D, \\ \operatorname{div}(\bar{u}) = 0, & \text{in } D, \\ \bar{u} = 0, & \text{on } \partial D. \end{cases} \quad (4.7)$$

Note that the differential filter of a function v is usually denoted by \bar{v} . Nevertheless, the employed notation herein will be \bar{u} to obtain a clear vision of the relationship between the differential filter and equations (4.1). For a given $v \in \mathbb{L}^2$, problem (4.7) yields a unique $\bar{u} \in \mathbb{H}^2 \cap \mathbb{V}$ provided that $D \subset \mathbb{R}^d$, is a bounded convex two-dimensional polygonal domain. Moreover, the solution \bar{u} satisfies $\|\bar{u}\|_{\mathbb{H}^2} \leq C_D \alpha^{-2} \|v\|_{\mathbb{L}^2}$. The former and the latter properties are provided in [69, Subsection 8.2].

4.2.1 . Definition of solutions

Relying on paper [29], a solution to equations (4.1) can be defined as follows:

Definition 4.2.2 Let $T > 0$ and assume that (S_1) - (S_3) are valid. A \mathbb{V} -valued stochastic process \bar{u} is said to be a variational solution to problem (4.1) if it belongs to $M_{\mathcal{F}_t}^2(0, T; D(A)) \cap L^2(\Omega; L^\infty(0, T; \mathbb{V}))$, is weakly continuous with values in \mathbb{V} , and it satisfies \mathbb{P} -a.s. and for all $t \in [0, T]$ the following:

$$\begin{aligned} & (\bar{u}(t), \phi) + \alpha^2 (\nabla \bar{u}(t), \nabla \phi) + \nu \int_0^t (\bar{u}(s) + \alpha^2 A \bar{u}(s), A \phi) ds \\ & + \int_0^t \tilde{b}(\bar{u}(s), \bar{u}(s) - \alpha^2 \Delta \bar{u}(s), \phi) ds = (\bar{u}_0, \phi) + \alpha^2 (\nabla \bar{u}_0, \nabla \phi) \\ & + \int_0^t \langle f(s, \bar{u}(s)), \phi \rangle ds + \left(\int_0^t g(s, \bar{u}(s)) dW(s), \phi \right), \quad \forall \phi \in D(A). \end{aligned} \quad (4.8)$$

If \bar{u} is a solution to problem (4.1) in the sense of Definition 4.2.2, then it has the sufficient regularity to define the right-hand side v of system (4.7). Therefore, considering $v = v(t)$ as in problem (4.7) grants a new (equivalent) formula for equation (4.8), namely for all $t \in [0, T]$, it holds \mathbb{P} -almost surely

$$\begin{aligned} & (v(t), \phi) + \nu \int_0^t (\nabla v(s), \nabla \phi) ds + \int_0^t \tilde{b}(\bar{u}(s), v(s), \phi) ds = (v_0, \phi) \\ & + \int_0^t \langle f(s, \bar{u}(s)), \phi \rangle ds + \left(\int_0^t g(s, \bar{u}(s)) dW(s), \phi \right), \quad \forall \phi \in \mathbb{V}, \end{aligned} \quad (4.9)$$

where $v_0 \in \mathbb{L}^2$ is given by equation (4.7) when $\bar{u} = \bar{u}_0$. The trilinear term involving the pressure $\int_0^t \tilde{b}(\bar{u}(s), \nabla \bar{p}(s), \phi) ds$ does not appear in equation (4.9) because

$$\tilde{b}(\bar{u}, \nabla \bar{p}, \phi) = \sum_{i,j=1}^d \int_D \bar{u}^i \partial_i \partial_j \bar{p} \phi^j dx + \sum_{i,j=1}^d \int_D \partial_i \bar{u}^j \partial_j \bar{p} \phi^i dx.$$

The first term on the right-hand side turns into $-\int_D [\phi \cdot \nabla] \bar{u} \nabla \bar{p} dx$ after performing an integration by parts, and the second term can be rewritten as $\int_D [\phi \cdot \nabla] \bar{u} \nabla \bar{p} dx$. We highlight the absence of $(\nabla(\bar{u} \cdot \nabla \bar{p}), \phi)$ in the above calculation which results from Proposition 4.2.1-(ii). It is worth mentioning that (4.9), coupled with the weak formulation of (4.7), establishes a well-posed problem whose solution satisfies equations (4.1) in the sense of Definition 4.2.2.

According to Definition 4.2.1, it is straightforward to infer that assumption (S_3) can be substituted by

$$(S_3^{alt}) \quad v_0 \in L^{2p}(\Omega, \mathcal{F}_0, \mathbb{P}; \mathbb{H}), \text{ for some } p \in [1, +\infty).$$

Indeed, the \mathbb{V} regularity of \bar{u}_0 arises from the variational formulation study of system (4.7) which implies that $\|\bar{u}_0\|_\alpha \leq \|v_0\|_{\mathbb{L}^2}$.

Remark 4.2.3 Both functions v and \bar{u} of equation (4.9) must vanish at the boundary ∂D , which includes the condition $A\bar{u} = 0$ on ∂D . This goes back to the way in which the LANS- α model is constructed. For additional interpretations, the reader may refer to article [89].

Next, we give a definition of strong solutions to problem (4.2) in 2D.

Definition 4.2.3 Given $T > 0$, let assumptions (S_1) , (S_2) and (S_3^{alt}) be fulfilled. An \mathbb{H} -valued stochastic process $v(t), t \in [0, T]$ is said to be a strong solution to equations (4.2) if it belongs to $M_{\mathcal{F}_t}^2(0, T; \mathbb{V}) \cap L^2(\Omega; C([0, T]; \mathbb{H}))$ and it satisfies \mathbb{P} -a.s., for all $t \in [0, T]$:

$$\begin{aligned} (v(t), \varphi) + \nu \int_0^t (\nabla v(s), \nabla \varphi) ds + \int_0^t \langle [v(s) \cdot \nabla] v(s), \varphi \rangle ds &= (v_0, \varphi) \\ + \int_0^t \langle f(s, v(s)), \varphi \rangle ds + \left(\int_0^t g(s, v(s)) dW(s), \varphi \right), \quad \forall \varphi \in \mathbb{V}. \end{aligned} \quad (4.10)$$

4.2.2 . Discretizations and algorithm

Time Discretization

Let $M \in \mathbb{N}^*$ be given, and $I_k = \{t_\ell\}_{\ell=0}^M$ be an equidistant partition of the interval $[0, T]$, where $t_0 := 0$, $t_M := T$ and $k := T/M$ is the time-step size. The nodes' equidistance is not mandatory in the sequel; it is imposed, however, for simplicity. One can generalize the presented method by associating a time-step k_m with each sub-interval $[t_{m-1}, t_m]$, for all $m \in \{1, \dots, M\}$.

Space discretization For simplicity's sake, we let \mathcal{T}_h be a quasi-uniform triangulation of the domain $D \subset \mathbb{R}^2$ into simplexes of maximal diameter $h > 0$, and $\bar{D} = \bigcup_{K \in \mathcal{T}_h} \bar{K}$. The space of polynomial vector

fields on an arbitrary set O with degree less than or equal to $n \in \mathbb{N}$ is denoted by $\mathbb{P}_n(O) := (P_n(O))^2$. For $n_1, n_2 \in \mathbb{N} \setminus \{0\}$, we let

$$\begin{aligned} \mathbb{H}_h &:= \left\{ z_h \in \mathbb{H}_0^1 \cap [C^0(\bar{D})]^2 \mid z_h|_K \in \mathbb{P}_{n_1}(K), \quad \forall K \in \mathcal{T}_h \right\}, \\ L_h &:= \left\{ q_h \in L_0^2(D) \mid q_h|_K \in P_{n_2}(K), \quad \forall K \in \mathcal{T}_h \right\}, \\ \mathbb{V}_h &:= \left\{ z_h \in \mathbb{H}_h \mid (\operatorname{div}(z_h), q_h) = 0, \quad \forall q_h \in L_h \right\}, \end{aligned}$$

be the finite element function spaces. For fixed $n_1, n_2 \in \mathbb{N} \setminus \{0\}$, we assume that (\mathbb{H}_h, L_h) satisfies the discrete inf-sup condition; namely there is a constant $\beta > 0$ independent of the mesh size h such that

$$\sup_{z_h \in \mathbb{H}_h \setminus \{0\}} \frac{(\operatorname{div}(z_h), q_h)}{\|\nabla z_h\|_{\mathbb{L}^2}} \geq \beta \|q_h\|_{L^2}, \quad \forall q_h \in L_h. \quad (4.11)$$

Examples of finite-dimensional spaces that satisfy inequality (4.11) are provided in [1, 64, 71].

Given $z \in \mathbb{L}^2$, we denote by $\Pi_h : \mathbb{L}^2 \rightarrow \mathbb{V}_h$ the \mathbb{L}^2 -orthogonal projections, defined as the unique solution of the identity

$$(z - \Pi_h z, \varphi_h) = 0, \quad \forall \varphi_h \in \mathbb{V}_h. \quad (4.12)$$

For $z \in \mathbb{H}_0^1$, $\Delta^h : \mathbb{H}_0^1 \rightarrow \mathbb{V}_h$ denotes the discrete Laplace operator, defined as the unique solution of

$$\left(\Delta^h z, \varphi_h \right) = -(\nabla z, \nabla \varphi_h), \quad \forall \varphi_h \in \mathbb{V}_h. \quad (4.13)$$

Operator Δ^h can also be seen as a discrete Stokes operator since its image is \mathbb{V}_h . We point out that one must not blindly interchange \mathbb{H}_h and \mathbb{V}_h in the definition of Δ^h because a discrete pressure shall be eliminated from Algorithm 4.1 later on. This also appears in Lemma 4.2.1.

Estimate (4.14) and the inverse inequality (4.15) below need to be satisfied by the recently defined approximate function spaces. Let \mathbb{S}_h be a finite-dimensional subspace of \mathbb{H}_0^1 equipped with an \mathbb{L}^2 -projector $\Pi_{\mathbb{S}_h} : \mathbb{L}^2 \rightarrow \mathbb{S}_h$, satisfying the following property:

Let $n \in \mathbb{N}$ be the polynomials' degree in \mathbb{S}_h . For all $s \in [2, n+1]$, and $z \in \mathbb{H}^s \cap \mathbb{H}_0^1$, there is a positive constant C independent of h such that

$$\sum_{j=0}^1 h^j \left\| D^j (z - \Pi_{\mathbb{S}_h} z) \right\|_{\mathbb{L}^2} \leq Ch^s \|z\|_{\mathbb{H}^s}. \quad (4.14)$$

Furthermore, assume that \mathbb{S}_h fulfills the following inverse inequality:

For $\ell \in \mathbb{N}$, $1 \leq p, q \leq +\infty$ and $0 \leq m \leq \ell$, there exists a constant C independent of h such that

$$\|z_h\|_{\mathbb{W}^{\ell,p}} \leq Ch^{m-\ell+2\min(\frac{1}{p}-\frac{1}{q}, 0)} \|z_h\|_{\mathbb{W}^{m,q}}, \quad \forall z_h \in \mathbb{S}_h. \quad (4.15)$$

Provided the triangulation of the domain D is quasi-uniform, one can easily check that the space \mathbb{H}_h satisfies both estimates (4.14) and (4.15). The reader may refer to [21] for adequate proofs. Subsequently, we take $\mathbb{S}_h = \mathbb{H}_h$.

Identity (4.13) together with the inverse inequality (4.15) ensure the following estimate:

$$\left\| \Delta^h z_h \right\|_{\mathbb{L}^2} \leq Ch^{-1} \|\nabla z_h\|_{\mathbb{L}^2}, \quad \forall z_h \in \mathbb{V}_h. \quad (4.16)$$

The discrete differential filter is somewhat defined as its continuous counterpart, but this time by involving the weak formulation of problem (4.7).

Definition 4.2.4 (Discrete differential filter)

Let v be the vector field of Definition 4.2.1. Its discrete differential filter, denoted by $\bar{u}_h \in \mathbb{V}_h$, is given by the unique solution of

$$\alpha^2 (\nabla \bar{u}_h, \nabla \varphi_h) + (\bar{u}_h, \varphi_h) = (v, \varphi_h), \quad \forall \varphi_h \in \mathbb{V}_h.$$

Additional information are stated in article [81, Section 4]. We list some of its properties in the following lemma.

Lemma 4.2.1 *Let $v = v_h \in \mathbb{V}_h$ and $\bar{u}_h \in \mathbb{V}_h$ be its discrete differential filter. Then,*

(i) $v_h = \bar{u}_h - \alpha^2 \Delta^h \bar{u}_h$ and $\nabla v_h = \nabla \bar{u}_h - \alpha^2 \nabla \Delta^h \bar{u}_h$ a.e. in D .

(ii) $(\nabla v_h, \nabla \bar{u}_h) = \|\nabla \bar{u}_h\|_{\mathbb{L}^2}^2 + \alpha^2 \left\| \Delta^h \bar{u}_h \right\|_{\mathbb{L}^2}^2$.

Proof: Assertions (i) and (ii) are covered in [44, Lemma 2.1]. \square

Before exhibiting the algorithm, we will define new notations for the approximate functions. The subscript h of the utilized test functions will be dropped throughout the rest of this chapter for the sake of clarity. For $t \in [0, T]$, we set $V(t) := v_h(t)$ for $v_h \in \mathbb{V}_h$, and denote by $U(t)$ its discrete differential filter, i.e. $U(t) := \bar{u}_h(t)$. Besides, let $\Pi(t) := p_h(t)$ and $\tilde{\Pi}(t) := \bar{p}_h(t)$ be the (space) approximate pressures. We point out that Algorithm 4.1 is derived from equation (4.9), which contains both terms \bar{u} and v .

Algorithm 4.1

Let $U^0 \in \mathbb{H}_h$ be given.

Project v_0 through $(V^0, \psi) = (U^0, \psi) + \alpha^2 (\nabla U^0, \nabla \psi)$, for all $\psi \in \mathbb{H}_h$.

If for $m \in \{1, \dots, M\}$, iterates $U^{m-1}, V^{m-1}, \Pi^{m-1}, \tilde{\Pi}^{m-1}$ are known, find a 4-tuple stochastic process $(U^m, V^m, \Pi^m, \tilde{\Pi}^m) \in \mathbb{H}_h \times \mathbb{H}_h \times L_h \times L_h$ such that for all $(\varphi, \psi, \Lambda_1, \Lambda_2) \in \mathbb{H}_h \times \mathbb{H}_h \times L_h \times L_h$, there holds \mathbb{P} -a.s.

$$\left\{ \begin{array}{l} (V^m - V^{m-1}, \varphi) + k\nu(\nabla V^m, \nabla \varphi) - k(U^m \times (\nabla \times V^m), \varphi) - k(\Pi^m, \operatorname{div}(\varphi)) \\ \quad = k\langle f(t_{m-1}, U^{m-1}), \varphi \rangle + (g(t_{m-1}, U^{m-1})\Delta_m W, \varphi), \\ (V^m, \psi) = (U^m, \psi) + \alpha^2 (\nabla U^m, \nabla \psi) - (\tilde{\Pi}^m, \operatorname{div}(\psi)), \\ (\operatorname{div}(U^m), \Lambda_1) = (\operatorname{div}(V^m), \Lambda_2) = 0, \end{array} \right.$$

where $\Delta_m W = W(t_m) - W(t_{m-1})$ for all $m \in \{1, \dots, M\}$.

The choice of the starting point U^0 of Algorithm 4.1 is provided in Section 4.3. Although the trilinear term's second variable could have been chosen to be explicit i.e. $\nabla \times V^{m-1}$, this choice may force an extra smoothness assumption on v_0 when illustrating the a priori estimates of $\{V^m\}_{m=1}^M$; namely one needs v_0 to be in \mathbb{H}_0^1 , which is not really practical as v_0 is just a transfer tool that should not play an important role within the study. For each $m \in \{0, \dots, M\}$, we may conclude from the second and third equations of Algorithm 4.1 along with Definition 4.2.4 two facts:

- (i) U^m is the discrete differential filter of V^m and thereby, all the associated properties are valid.
- (ii) The Algorithm's starting point U^0 could be exchanged with V^0 .

4.3 . Main result

All this chapter long, the spatial scale α will only vanish when the space discretization step size h goes to 0. We will neither consider a vanishing α while fixing h nor a double limit. In other words, α and h will be proportional.

In the light of the preceding preliminaries, we are now able to state the main results of this chapter.

Theorem 4.3.1 Let $T > 0$, $D \subset \mathbb{R}^2$ be a bounded convex polygonal domain and $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, T]}, \mathbb{P})$ be a filtered probability space. Assume assumptions (S_1) , (S_2) and (S_3^{alt}) . Given an integer $M \in \mathbb{N} \setminus \{0\}$, define the time discretization step size $k := \frac{T}{M}$ such that $0 < k < 1$, and I_k forms a uniform partition of the time interval $[0, T]$. Let $h \in (0, 1)$ be the space discretization step size such that the triangulation \mathcal{T}_h is quasi-uniform. Define the finite element triple $(\mathbb{H}_h, L_h, U_h^0)$ such that the pair (\mathbb{H}_h, L_h) satisfies the LBB condition (4.11) and U_h^0 ($:= U^0$) belongs to \mathbb{H}_h . The following results hold:

- For a given $(k, h, \alpha) \in (0, 1)^3$, there exists a solution $\left\{ (U^m, V^m, \Pi^m, \tilde{\Pi}^m) \right\}_{m=1}^M$ to Algorithm 4.1 that satisfies Lemma 4.4.1.

- For a family $\{k, h, \alpha\}$ of parameters satisfying $k, h, \alpha \rightarrow 0$ instantly such that the following supplementary assumptions are imposed:

★ the spatial scale α lies in the range $[\alpha_{min}h, \alpha_{max}h]$ for some $\alpha_{min}, \alpha_{max} > 0$ independent of k and h ,

★ $U^0 \rightarrow \bar{u}_0$ in the space $L^4(\Omega; \mathbb{H}^1)$ (particularly, $\|U^0\|_{L^4(\Omega; \mathbb{H}^1)}$ is uniformly bounded in h),

the solution $\left\{ \left(U^m, V^m, \Pi^m, \tilde{\Pi}^m \right) \right\}_{m=1}^M$ of Algorithm 4.1 fulfills Lemma 4.4.5 and it converges toward the unique solution of the stochastic Navier-Stokes equations (4.2) in the sense of Definition 4.2.3.

The initial datum U^0 of Algorithm 4.1 can be prescribed through the Ritz operator which grants a uniform bound in \mathbb{H}^1 with respect to h . That is, we evoke the Ritz operator \mathcal{R}_h which is stable in \mathbb{H}^1 i.e. there is a positive non-decreasing function ζ , uniform in h such that $\|\mathcal{R}_h v\|_{\mathbb{H}^1} \leq \zeta \|v\|_{\mathbb{H}^1}$ for all $v \in \mathbb{H}^1$, such that for a given $v \in \mathbb{H}^1$, $\mathcal{R}_h: \mathbb{H}^1 \rightarrow \mathbb{V}_h$ is defined as the unique solution of

$$(\nabla \mathcal{R}_h v, \nabla v_h) = (\nabla v, \nabla v_h), \quad \forall v_h \in \mathbb{V}_h.$$

Therefore, we define U^0 by $U^0 = \mathcal{R}_h \bar{u}_0$ where \bar{u}_0 is the initial datum of equations (4.1), which also represents the continuous differential filter of v_0 . Besides, the second equation in Algorithm 4.1 together with Lemma 4.2.1-(i), inequality (4.16), $\alpha \leq 1$ and $\alpha \leq \alpha_{max}h$ lead to $\|V^0\|_{\mathbb{L}^2} \leq \sqrt{2} \max(1, C\alpha_{max}) \|U^0\|_{\mathbb{H}^1}$ which means that $\|V^0\|_{\mathbb{L}^2}$ is also uniformly bounded in h .

4.4 . Solvability, stability and a priori estimates

Notice that the system of equations proposed in Algorithm 4.1 can be reformulated after taking the test functions φ and ψ in \mathbb{V}_h :

$$\begin{cases} (V^m - V^{m-1}, \varphi) + k\nu(\nabla V^m, \nabla \varphi) - k(U^m \times (\nabla \times V^m), \varphi) \\ = k\langle f(t_{m-1}, U^{m-1}), \varphi \rangle + (g(t_{m-1}, U^{m-1})\Delta_m W, \varphi), \quad \forall \varphi \in \mathbb{V}_h. \\ (V^m, \psi) = (U^m, \psi) + \alpha^2(\nabla U^m, \nabla \psi), \quad \forall \psi \in \mathbb{V}_h. \end{cases} \quad (4.17)$$

In the lemma down below, we illustrate the solvability of Algorithm 4.1, the iterates' measurability, and some a priori estimates whose role is to afford the proposed numerical scheme with stability.

Lemma 4.4.1 *Assume that assumptions (S_1) - (S_3) are valid and let $p \in [2, \infty) \cap \mathbb{N}$. Then, there exists a $\mathbb{V}_h \times \mathbb{V}_h \times L_h \times L_h$ -valued sequence of random variables $\{(U^m, V^m, \Pi^m, \tilde{\Pi}^m)\}_{m=1}^M$ that solves \mathbb{P} -a.s. Algorithm 4.1, and fulfills the following assertions:*

(i) for any $m \in \{1, \dots, M\}$, the maps $U^m, V^m: \Omega \rightarrow \mathbb{H}_h$ are \mathcal{F}_{t_m} -measurable.

$$(ii) \mathbb{E} \left[\max_{1 \leq m \leq M} \|U^m\|_\alpha^2 + \frac{k\nu}{2} \sum_{m=1}^M \left(\|\nabla U^m\|_{\mathbb{L}^2}^2 + \alpha^2 \|\Delta^h U^m\|_{\mathbb{L}^2}^2 \right) + \frac{1}{4} \sum_{m=1}^M \|U^m - U^{m-1}\|_\alpha^2 \right] \leq C_T,$$

$$(iii) \mathbb{E} \left[\max_{1 \leq m \leq M} \|U^m\|_\alpha^{2p} + \sum_{m=1}^M \|U^m\|_\alpha^{2p-1} \|U^m - U^{m-1}\|_\alpha^2 + k\nu \sum_{m=1}^M \|U^m\|_\alpha^{2p-1} \left(\|\nabla U^m\|_{\mathbb{L}^2}^2 + \alpha^2 \|\Delta^h U^m\|_{\mathbb{L}^2}^2 \right) \right] \leq C_{T,p},$$

where $C_{T,p} > 0$ depends on $\|U^0\|_{L^{2p}(\Omega; \mathbb{H}^1)}$, T , $\{K_i\}_{i=1}^4$, $Tr(Q)$, ν , and D , and is independent of α , k and h . Note that $C_T := C_{T,1}$.

Proof: *Solvability*

To prove the Algorithm's solvability, we will follow a technique similar to that in [7, Lemma 4.1] while relying on equations (4.17). Since $V^m \in \mathbb{V}_h$ for all $m \in \{1, \dots, M\}$ then, by Lemma 4.2.1-(i), we get $V^m = U^m - \alpha^2 \Delta^h U^m$, \mathbb{P} -a.s. and a.e. in D . This means that the existence of U^m implies that of V^m . Assume that, for some $2 \leq \ell \leq M$ and for almost every $\omega \in \Omega$, a sequence $\{(U^m(\omega), V^m(\omega))\}_{m=1}^{\ell-1}$ has been found by induction. For $\omega \in \Omega$, define \mathbb{P} -a.s. the mapping $\mathcal{F}_{\ell-1}^\omega: \mathbb{V}_h \rightarrow \mathbb{V}'_h$ by

$$\begin{aligned} \mathcal{F}_{\ell-1}^\omega(\varphi) := & \varphi - \alpha^2 \Delta^h \varphi - V^{\ell-1}(\omega) - k\nu \Delta \left(\varphi - \alpha^2 \Delta^h \varphi \right) - k\varphi \times \left(\nabla \times (\varphi - \alpha^2 \Delta^h \varphi) \right) \\ & - kf(t_{\ell-1}, U^{\ell-1}(\omega)) - g(t_{\ell-1}, U^{\ell-1}(\omega)) \Delta_\ell W(\omega), \end{aligned}$$

for all $\varphi \in \mathbb{V}_h$. The continuity of $\mathcal{F}_{\ell-1}^\omega$ can be shown by a straightforward argument. Since, \mathbb{V}_h equipped with the inner product (\cdot, \cdot) , is a Hilbert space, then by Riesz representation theorem, functional $\mathcal{F}_{\ell-1}^\omega$ can be defined through the L^2 -inner product, namely for $\varphi \in \mathbb{V}_h$,

$$(\mathcal{F}_{\ell-1}^\omega(\varphi))(\psi) = (\mathcal{F}_{\ell-1}^\omega(\varphi), \psi), \quad \text{for all } \psi \in \mathbb{V}_h.$$

Therefore, for $\psi = \varphi \in \mathbb{V}_h$ and by Proposition 4.2.1-(i), the discrete Laplace operator (4.13), assumption (S_2) , the Cauchy-Schwarz and Young inequalities,

$$\begin{aligned} (\mathcal{F}_{\ell-1}^\omega(\varphi), \varphi) & \geq \|\varphi\|_{\mathbb{L}^2}^2 + (\alpha^2 + k\nu) \|\nabla \varphi\|_{\mathbb{L}^2}^2 - \|V^{\ell-1}(\omega)\|_{\mathbb{L}^2} \|\varphi\|_{\mathbb{L}^2} + k\nu \alpha^2 \|\Delta^h \varphi\|_{\mathbb{L}^2}^2 \\ & - k \left(K_3 + K_4 \|U^{\ell-1}(\omega)\|_\alpha \right) \|\varphi\|_{\mathbb{H}^1} - \left(K_1 + K_2 \|U^{\ell-1}(\omega)\|_\alpha \right) \|\Delta_\ell W(\omega)\|_K \|\varphi\|_{\mathbb{L}^2} \\ & \geq \frac{1}{2} \|\varphi\|_{\mathbb{L}^2}^2 + \left(\alpha^2 + \frac{k\nu}{2} \right) \|\nabla \varphi\|_{\mathbb{L}^2}^2 - \|V^{\ell-1}(\omega)\|_{\mathbb{L}^2}^2 - \frac{kC_D^2}{2\nu} \left(K_3 + K_4 \|U^{\ell-1}(\omega)\|_\alpha \right)^2 \\ & - \left(K_1 + K_2 \|U^{\ell-1}(\omega)\|_\alpha \right)^2 \|\Delta_\ell W(\omega)\|_K^2 \geq \frac{1}{2} \|\varphi\|_{\mathbb{L}^2}^2 - L_{\ell-1}(\omega), \end{aligned}$$

where

$$L_{\ell-1} := 2K_1^2 \|\Delta_\ell W\|_K^2 + \frac{kC_D^2 K_3^2}{\nu} + \|V^{\ell-1}\|_{\mathbb{L}^2}^2 + \left(\frac{kC_D^2 K_4^2}{\nu} + 2K_2^2 \|\Delta_\ell W\|_K^2 \right) \|U^{\ell-1}\|_\alpha^2.$$

By inequality (4.4) and the induction's hypothesis, it holds \mathbb{P} -a.s. $L_{\ell-1}(\omega) < +\infty$. Therefore, taking $\varphi \in \mathbb{V}_h$ such that $\|\varphi\|_{\mathbb{L}^2} = \sqrt{2L_{\ell-1}(\omega)}$ yields $(\mathcal{F}_{\ell-1}^\omega(\varphi), \varphi) \geq 0$. Subsequently, Brouwer's fixed point theorem (see [65, Corollary 1.1, p. 279]) ensures the existence (but not uniqueness) of a function $\phi = \phi(\omega) \in \mathbb{V}_h$ such that $\mathcal{F}_{\ell-1}^\omega(\phi) = 0$. Hence, $(U^\ell, V^\ell) \in \mathbb{V}_h \times \mathbb{V}_h$ exists \mathbb{P} -a.s. . The discrete LBB condition (4.11) yields the existence of an $L_h \times L_h$ -valued process $\{(\Pi^m, \tilde{\Pi}^m)\}_{m=1}^M$ satisfying Algorithm 4.1.

Measurability

After proving the algorithm's solvability through the functional $\mathcal{F}_{\ell-1}^\omega$, the measurability of iterates U^m , $m \in \{1, \dots, M\}$ follows by induction (see [7, Lemma 4.1]). Moreover, by Lemma 4.2.1-(i), one infers the measurability of $\{V^m\}_{m=1}^M$.

A priori energy estimate

Let us denote by $\|\cdot\|_{h,\alpha}^2$ the quantity $\|\nabla \cdot\|_{\mathbb{L}^2}^2 + \alpha^2 \|\Delta^h \cdot\|_{\mathbb{L}^2}^2$. In equation (4.17), we take $\varphi = \psi = U^m$ and employ identity (4.6), Lemma 4.2.1-(ii) together with Proposition 4.2.1-(i):

$$\begin{aligned} \frac{1}{2} \left(\|U^m\|_\alpha^2 - \|U^{m-1}\|_\alpha^2 + \|U^m - U^{m-1}\|_\alpha^2 \right) + k\nu \|U^m\|_{h,\alpha}^2 & = k \langle f(t_{m-1}, U^{m-1}), U^m \rangle \\ + \left(g(t_{m-1}, U^{m-1}) \Delta_m W, U^m - U^{m-1} \right) + \left(g(t_{m-1}, U^{m-1}) \Delta_m W, U^{m-1} \right). \end{aligned} \quad (4.18)$$

After employing the Cauchy-Schwarz and Young inequalities along with assumption (S_2) , we take the sum over m from 1 to M :

$$\begin{aligned} & \frac{1}{2} \|U^M\|_\alpha^2 - \frac{1}{2} \|U^0\|_\alpha^2 + \frac{1}{4} \sum_{m=1}^M \|U^m - U^{m-1}\|_\alpha^2 + \frac{k\nu}{2} \sum_{m=1}^M \|U^m\|_{h,\alpha}^2 \\ & \leq \frac{C_D^2 T K_3^2}{\nu} + \frac{C_D^2 K_4^2}{\nu} k \sum_{m=1}^M \|U^{m-1}\|_\alpha^2 + \sum_{m=1}^M \|g(t_{m-1}, U^{m-1}) \Delta_m W\|_{\mathbb{L}^2}^2 \\ & \quad + \sum_{m=1}^M (g(t_{m-1}, U^{m-1}) \Delta_m W, U^{m-1}). \end{aligned} \quad (4.19)$$

Due to the measurability of U^m , the last term on the right-hand side vanishes when taking its expectation. The penultimate term is controlled as follows:

$$\begin{aligned} & \mathbb{E} \left[\|g(t_{m-1}, U^{m-1})\|_{\mathcal{L}_2(K, \mathbb{L}^2)}^2 \|\Delta_m W\|_K^2 \right] \\ & = \mathbb{E} \left[\|g(t_{m-1}, U^{m-1})\|_{\mathcal{L}_2(K, \mathbb{L}^2)}^2 \right] \mathbb{E} \left[\|\Delta_m W\|_K^2 \middle| \mathcal{F}_{t_{m-1}} \right] \\ & \leq 2Tr(Q) K_1^2 k + 2K_2^2 k Tr(Q) \mathbb{E} \left[\|U^{m-1}\|_\alpha^2 \right], \end{aligned} \quad (4.20)$$

thanks to the tower property of the conditional expectation, the increments independence of the Wiener process, property (4.4), and assumption (S_2) . Plugging estimate (4.20) in equation (4.19) returns

$$\begin{aligned} & \frac{1}{2} \mathbb{E} \left[\|U^M\|_\alpha^2 \right] + \frac{1}{4} \sum_{m=1}^M \mathbb{E} \left[\|U^m - U^{m-1}\|_\alpha^2 \right] + \frac{k\nu}{2} \sum_{m=1}^M \mathbb{E} \left[\|U^m\|_{h,\alpha}^2 \right] \leq \frac{1}{2} \mathbb{E} \left[\|U^0\|_\alpha^2 \right] \\ & \quad + \left(\frac{C_D^2 K_3^2}{\nu} + 2Tr(Q) K_1^2 \right) T + \left(\frac{C_D^2 K_4^2}{\nu} + 2K_2^2 Tr(Q) \right) k \sum_{m=0}^{M-1} \mathbb{E} \left[\|U^m\|_\alpha^2 \right]. \end{aligned} \quad (4.21)$$

Now, we employ the discrete Grönwall inequality (see for instance [109, Lemma 10.5]) in order to prove the sought estimate. We replace M in equation (4.21) by any other index $\ell \geq 1$. We get

$$\mathbb{E} \left[\|U^\ell\|_\alpha^2 \right] \leq \left[\mathbb{E} \left[\|U^0\|_{\mathbb{H}^1}^2 \right] + 2 \left(\frac{C_D^2 K_3^2}{\nu} + 2Tr(Q) K_1^2 \right) T \right] e^{T \left(\frac{C_D^2 K_4^2}{\nu} + 2K_2^2 Tr(Q) \right)} =: K_T$$

for all $\ell \in \{1, \dots, M\}$, where $\|U^0\|_\alpha \leq \|U^0\|_{\mathbb{H}^1}$ thanks to (4.5). Consequently,

$$\max_{1 \leq m \leq M} \mathbb{E} \left[\|U^m\|_\alpha^2 \right] \leq K_T. \quad (4.22)$$

By virtue of estimate (4.21) and the discrete Grönwall lemma, one also obtains the following two estimates:

$$\frac{k\nu}{2} \sum_{m=1}^M \mathbb{E} \left[\|U^m\|_{h,\alpha}^2 \right] \leq K_T \quad \text{and} \quad \frac{1}{4} \sum_{m=1}^M \mathbb{E} \left[\|U^m - U^{m-1}\|_\alpha^2 \right] \leq K_T.$$

We still need to prove $\mathbb{E} \left[\max_{1 \leq m \leq M} \|U^m\|_\alpha^2 \right] \leq C_T$, for a certain positive constant C_T independent of α , k and h . To this end, we make use of estimate (4.19), but this time by summing from $m = 1$ to $m = \ell$ where $\ell \geq 1$ is an integer. Then, we take the maximum over ℓ and apply the mathematical expectation on both sides to get

$$\begin{aligned} & \frac{1}{2} \mathbb{E} \left[\max_{1 \leq \ell \leq M} \|U^\ell\|_\alpha^2 \right] \leq \frac{1}{2} \mathbb{E} \left[\|U^0\|_\alpha^2 \right] + \frac{C_D^2 T K_3^2}{\nu} + \frac{C_D^2 K_4^2}{\nu} k \sum_{m=1}^M \mathbb{E} \left[\|U^{m-1}\|_\alpha^2 \right] \\ & \quad + \sum_{m=1}^M \mathbb{E} \left[\|g(t_{m-1}, U^{m-1}) \Delta_m W\|_{\mathbb{L}^2}^2 \right] + \mathbb{E} \left[\max_{1 \leq \ell \leq M} \sum_{m=1}^{\ell} (g(t_{m-1}, U^{m-1}) \Delta_m W, U^{m-1}) \right]. \end{aligned} \quad (4.23)$$

To bound the last term on the right-hand side, we use assumption (S_2) , the Burkholder-Davis-Gundy and Young inequalities, after considering the sum as the stochastic integral of a piecewise constant integrand:

$$\begin{aligned}
& \mathbb{E} \left[\max_{1 \leq \ell \leq M} \sum_{m=1}^{\ell} \left(g(t_{m-1}, U^{m-1}) \Delta_m W, U^{m-1} \right) \right] \\
& \lesssim \mathbb{E} \left[\left(k \sum_{m=1}^M \left\| g(t_{m-1}, U^{m-1}) \right\|_{\mathcal{L}_2(K, \mathbb{L}^2)}^2 \|U^{m-1}\|_{\mathbb{L}^2}^2 \right)^{1/2} \right] \\
& \leq \frac{1}{4} \mathbb{E} \left[\|U^0\|_{\mathbb{L}^2}^2 \right] + 2K_1^2 T + \mathbb{E} \left[\frac{1}{4} \max_{1 \leq \ell \leq M} \|U^\ell\|_{\mathbb{L}^2}^2 + 2K_2^2 k \sum_{m=1}^M \|U^{m-1}\|_{\alpha}^2 \right].
\end{aligned} \tag{4.24}$$

Returning to estimate (4.23), we avail ourselves of (4.20), (4.22) and (4.24) to conclude

$$\mathbb{E} \left[\max_{1 \leq m \leq M} \|U^m\|_{\alpha}^2 \right] \leq C_T,$$

where $C_T > 0$ depends only on the parameters of K_T .

Bounds for higher velocity moments

We will demonstrate below the case $p = 2$. The reader may refer to [24] for additional hints. We start by multiplying equation (4.18) by the norm $\|U^m\|_{\alpha}^2$.

$$\begin{aligned}
& \frac{1}{2} \|U^m\|_{\alpha}^4 - \frac{1}{2} \|U^{m-1}\|_{\alpha}^2 \|U^m\|_{\alpha}^2 + \frac{1}{2} \|U^m - U^{m-1}\|_{\alpha}^2 \|U^m\|_{\alpha}^2 + k\nu \|U^m\|_{h, \alpha}^2 \|U^m\|_{\alpha}^2 \\
& = k \langle f(t_{m-1}, U^{m-1}), U^m \rangle \|U^m\|_{\alpha}^2 + \left(g(t_{m-1}, U^{m-1}) \Delta_m W, U^m - U^{m-1} \right) \|U^m\|_{\alpha}^2 \\
& \quad + \left(g(t_{m-1}, U^{m-1}) \Delta_m W, U^{m-1} \right) \|U^m\|_{\alpha}^2 = I + II + III.
\end{aligned} \tag{4.25}$$

For I , we apply the norm equivalence (4.5), the Young inequality and estimate $|a+b|^p \leq 2^{p-1}(|a|^p + |b|^p)$ for $p = 4$:

$$\begin{aligned}
I & \leq kC_D \left(K_3 + K_4 \|U^{m-1}\|_{\alpha} \right) \|\nabla U^m\|_{\mathbb{L}^2}^{\frac{3}{2}} \|U^m\|_{\alpha}^{\frac{3}{2}} \leq \frac{kC_D^4}{4\nu^3} \left(K_3 + K_4 \|U^{m-1}\|_{\alpha} \right)^4 \\
& \quad + \frac{3k\nu}{4} \|U^m\|_{h, \alpha}^2 \|U^m\|_{\alpha}^2 \leq \frac{2kC_D^4 K_3^4}{\nu^3} + \frac{2kC_D^4 K_4^4}{\nu^3} \|U^{m-1}\|_{\alpha}^4 + \frac{3k\nu}{4} \|U^m\|_{h, \alpha}^2 \|U^m\|_{\alpha}^2.
\end{aligned}$$

For II ,

$$\begin{aligned}
II & \leq \|g(t_{m-1}, U^{m-1})\|_{\mathcal{L}_2(K, \mathbb{L}^2)}^2 \|\Delta_m W\|_K^2 \left(\|U^m\|_{\alpha}^2 - \|U^{m-1}\|_{\alpha}^2 + \|U^{m-1}\|_{\alpha}^2 \right) \\
& \quad + \frac{1}{4} \|U^m - U^{m-1}\|_{\mathbb{L}^2}^2 \|U^m\|_{\alpha}^2 \\
& \leq \|g(t_{m-1}, U^{m-1})\|_{\mathcal{L}_2(K, \mathbb{L}^2)}^2 \|\Delta_m W\|_K^2 \|U^{m-1}\|_{\alpha}^2 + \frac{1}{16} \left| \|U^m\|_{\alpha}^2 - \|U^{m-1}\|_{\alpha}^2 \right|^2 \\
& \quad + 4 \|g(t_{m-1}, U^{m-1})\|_{\mathcal{L}_2(K, \mathbb{L}^2)}^4 \|\Delta_m W\|_K^4 + \frac{1}{4} \|U^m - U^{m-1}\|_{\mathbb{L}^2}^2 \|U^m\|_{\alpha}^2.
\end{aligned}$$

For III ,

$$\begin{aligned}
III & := \left(g(t_{m-1}, U^{m-1}) \Delta_m W, U^{m-1} \right) \left(\|U^m\|_{\alpha}^2 - \|U^{m-1}\|_{\alpha}^2 + \|U^{m-1}\|_{\alpha}^2 \right) \\
& \leq \left(g(t_{m-1}, U^{m-1}) \Delta_m W, U^{m-1} \right) \|U^{m-1}\|_{\alpha}^2 + \frac{1}{16} \left| \|U^m\|_{\alpha}^2 - \|U^{m-1}\|_{\alpha}^2 \right|^2 \\
& \quad + 4 \|g(t_{m-1}, U^{m-1})\|_{\mathcal{L}_2(K, \mathbb{L}^2)}^2 \|\Delta_m W\|_K^2 \|U^{m-1}\|_{\alpha}^2.
\end{aligned}$$

Equation (4.25) becomes

$$\frac{1}{2} \|U^m\|_{\alpha}^4 - \frac{1}{2} \|U^{m-1}\|_{\alpha}^2 \|U^m\|_{\alpha}^2 + \frac{1}{4} \|U^m - U^{m-1}\|_{\alpha}^2 \|U^m\|_{\alpha}^2 + \frac{k\nu}{4} \|U^m\|_{h, \alpha}^2 \|U^m\|_{\alpha}^2$$

$$\begin{aligned}
&\leq \frac{2kC_D^4 K_3^4}{\nu^3} + \frac{2kC_D^4 K_4^4}{\nu^3} \|U^{m-1}\|_\alpha^4 + \frac{1}{8} \left| \|U^m\|_\alpha^2 - \|U^{m-1}\|_\alpha^2 \right|^2 \\
&\quad + \left(g(t_{m-1}, U^{m-1}) \Delta_m W, U^{m-1} \right) \|U^{m-1}\|_\alpha^2 + 4 \|g(t_{m-1}, U^{m-1})\|_{\mathcal{L}_2(K, \mathbb{L}^2)}^4 \|\Delta_m W\|_K^4 \\
&\quad + 5 \|g(t_{m-1}, U^{m-1})\|_{\mathcal{L}_2(K, \mathbb{L}^2)}^2 \|\Delta_m W\|_K^2 \|U^{m-1}\|_\alpha^2.
\end{aligned}$$

Note that $\|U^m\|_\alpha^4 - \|U^{m-1}\|_\alpha^2 \|U^m\|_\alpha^2 = \frac{1}{2} (\|U^m\|_\alpha^4 - \|U^{m-1}\|_\alpha^4 + \| \|U^m\|_\alpha^2 - \|U^{m-1}\|_\alpha^2 \|^2)$, therefore

$$\begin{aligned}
&\frac{1}{4} \left(\|U^m\|_\alpha^4 - \|U^{m-1}\|_\alpha^4 + \frac{1}{2} \left| \|U^m\|_\alpha^2 - \|U^{m-1}\|_\alpha^2 \right|^2 + \|U^m - U^{m-1}\|_\alpha^2 \|U^m\|_\alpha^2 \right. \\
&\quad \left. + k\nu \|U^m\|_{h,\alpha}^2 \|U^m\|_\alpha^2 \right) \leq \frac{2kC_D^4 K_3^4}{\nu^3} + \frac{2kC_D^4 K_4^4}{\nu^3} \|U^{m-1}\|_\alpha^4 \\
&\quad + \left(g(t_{m-1}, U^{m-1}) \Delta_m W, U^{m-1} \right) \|U^{m-1}\|_\alpha^2 + 4 \|g(t_{m-1}, U^{m-1})\|_{\mathcal{L}_2(K, \mathbb{L}^2)}^4 \|\Delta_m W\|_K^4 \\
&\quad + 5 \|g(t_{m-1}, U^{m-1})\|_{\mathcal{L}_2(K, \mathbb{L}^2)}^2 \|\Delta_m W\|_K^2 \|U^{m-1}\|_\alpha^2,
\end{aligned} \tag{4.26}$$

Proceeding as (4.20), the penultimate term can be estimated as follows

$$\mathbb{E} \left[\|g(t_{m-1}, U^{m-1})\|_{\mathcal{L}_2(K, \mathbb{L}^2)}^4 \|\Delta_m W\|_K^4 \right] \lesssim K_1^4 \text{Tr}(Q)^2 k^2 + K_2^4 \text{Tr}(Q)^2 k^2 \mathbb{E} \left[\|U^{m-1}\|_\alpha^4 \right]. \tag{4.27}$$

Next, we bound the last term on the right-hand side of (4.26)

$$\begin{aligned}
&\mathbb{E} \left[\|g(t_{m-1}, U^{m-1})\|_{\mathcal{L}_2(K, \mathbb{L}^2)}^2 \|\Delta_m W\|_K^2 \|U^{m-1}\|_\alpha^2 \right] \\
&\lesssim K_1^2 k \text{Tr}(Q) \mathbb{E} \left[\|U^{m-1}\|_\alpha^2 \right] + K_2^2 \text{Tr}(Q) k \mathbb{E} \left[\|U^{m-1}\|_\alpha^4 \right].
\end{aligned} \tag{4.28}$$

The third term on the right-hand side of (4.26) vanishes after taking its expectation, thanks to the measurability of the iterates U^m , $m \in \{1, \dots, M\}$. We collect and plug the above estimates back in (4.26), and we sum it up over m from $m = 1$ to $m = M$. Then, we apply the mathematical expectation, and employ the discrete Grönwall lemma to get

$$\max_{1 \leq m \leq M} \mathbb{E} \left[\|U^m\|_\alpha^4 \right] \leq C_{T,2}, \tag{4.29}$$

where $C_{T,2} > 0$ does not depend on α , k and h . We also get by Grönwall lemma the following two estimates:

$$\frac{1}{4} \mathbb{E} \left[\sum_{m=1}^M \|U^m - U^{m-1}\|_\alpha^2 \|U^m\|_\alpha^2 \right] \leq C_{T,2} \text{ and } \frac{k\nu}{4} \mathbb{E} \left[\sum_{m=1}^M \|U^m\|_{h,\alpha}^2 \|U^m\|_\alpha^2 \right] \leq C_{T,2}.$$

It remains to show that $\mathbb{E} \left[\max_{1 \leq m \leq M} \|U^m\|_\alpha^4 \right] \leq C_{T,2}$. To do so, we follow the technique which was employed in the previous step (A priori energy estimate) by summing up inequality (4.26) over m from 1 to $\ell \geq 1$. We will only need to control the following stochastic term:

$$\begin{aligned}
&\mathbb{E} \left[\max_{1 \leq \ell \leq M} \sum_{m=1}^{\ell} \left(g(t_{m-1}, U^{m-1}) \Delta_m W, U^{m-1} \right) \|U^{m-1}\|_\alpha^2 \right] \\
&\lesssim \mathbb{E} \left[\left(k \sum_{m=1}^M \|g(t_{m-1}, U^{m-1})\|_{\mathcal{L}_2(K, \mathbb{L}^2)}^2 \|U^{m-1}\|_\alpha^6 \right)^{\frac{1}{2}} \right] \\
&\leq \mathbb{E} \left[\frac{1}{8} \|U^0\|_{\mathbb{H}^1}^4 + \frac{1}{8} \max_{1 \leq m \leq M} \|U^m\|_\alpha^4 + 4K_1^2 k \sum_{m=1}^M \|U^{m-1}\|_\alpha^2 + 4K_2^2 k \sum_{m=1}^M \|U^{m-1}\|_\alpha^4 \right].
\end{aligned}$$

Collecting all estimates together and using (4.29) complete the proof of estimate (iii). \square

As stated in the above proof, uniqueness of $\{(U^m, V^m)\}_{m=1}^M$ might not be afforded in the whole probability set Ω . This is on account of the nonlinear state of the proposed numerical scheme that comes into view in $(U^m \times (\nabla \times V^m), \varphi)$. However, the iterates' uniqueness can be shown to hold in a subspace Ω_ε of Ω such that $\varepsilon = \varepsilon(k, h) > 0$ and $\mathbb{P}(\Omega_\varepsilon) \rightarrow 1$ as $\varepsilon \rightarrow 0$.

Lemma 4.4.2 Let $\{(U^m, V^m)\}_{m=1}^M$ be the iterates of Algorithm 4.1, (S_1) - (S_3) be valid, and ε be a positive real number. Let Ω_ε be a sample probability subset defined by

$$\Omega_\varepsilon := \left\{ \omega \in \Omega \mid \max_{1 \leq m \leq M} \|U^m\|_{\mathbb{L}^2}^4 \leq \frac{1}{\varepsilon} \right\} \subset \Omega.$$

Then, the probability measure of Ω_ε approaches 1 as the value of ε decreases i.e. $\mathbb{P}(\Omega_\varepsilon) \geq 1 - \varepsilon C_{T,2}$ where $C_{T,2}$ is coming out from Lemma 4.4.1, and iterates $\{(U^m, V^m)\}_{m=1}^M$ are \mathbb{P} -a.s. unique in Ω_ε provided that $\frac{k}{h^2} < \frac{27\nu^3\varepsilon}{2048C_D^4 C^2(1 + \alpha_{max}^4 C^4)}$ where the constant C emerges from inequality (4.15) and $C_D > 0$ depends only on the domain D .

Proof: Let $\{(U_1^m, V_1^m)\}_{m=1}^M$ and $\{(U_2^m, V_2^m)\}_{m=1}^M$ be two solutions to Algorithm 4.1 such that $U_1^0 = U_2^0$ and $V_1^0 = V_2^0$. We denote by u^m the difference $U_1^m - U_2^m$, and by v^m the difference $V_1^m - V_2^m$. Subtracting the equations that are satisfied by (U_1^m, V_1^m) and (U_2^m, V_2^m) returns

$$\begin{cases} (v^m - v^{m-1}, \varphi) + k\nu (\nabla v^m, \nabla \varphi) - k(U_1^m \times (\nabla \times V_1^m), \varphi) + k(U_2^m \times (\nabla \times V_2^m), \varphi) \\ = k \langle f(t_{m-1}, U_1^{m-1}) - f(t_{m-1}, U_2^{m-1}), \varphi \rangle \\ + \left([g(t_{m-1}, U_1^{m-1}) - g(t_{m-1}, U_2^{m-1})] \Delta_m W, \varphi \right), \quad \forall \varphi \in \mathbb{V}_h, \\ (v^m, \psi) = (u^m, \psi) + \alpha^2 (\nabla u^m, \nabla \psi), \quad \forall \psi \in \mathbb{V}_h. \end{cases} \quad (4.30)$$

In order to demonstrate the uniqueness, we shall argue by induction on m . For $m = 1$, it holds that

$$(v^1, \varphi) + k\nu (\nabla v^1, \nabla \varphi) = k(U_1^1 \times (\nabla \times V_1^1), \varphi) - k(U_2^1 \times (\nabla \times V_2^1), \varphi), \quad (4.31)$$

where the right-hand side of equation (4.30)₁ vanishes because $U_1^0 = U_2^0$. Moreover,

$$\begin{aligned} & (U_1^1 \times (\nabla \times V_1^1), u^1) - (U_2^1 \times (\nabla \times V_2^1), u^1) \\ & = (u^1 \times (\nabla \times V_1^1), u^1) + (U_2^1 \times (\nabla \times V_1^1), u^1) - (U_2^1 \times (\nabla \times V_2^1), u^1) = (U_2^1 \times (\nabla \times v^1), u^1), \end{aligned}$$

where the first term on the right-hand side vanishes due to Proposition 4.2.1-(i). For clarity's sake, denote $\|\cdot\|_{h,\alpha}^2 := \|\nabla \cdot\|_{\mathbb{L}^2}^2 + \alpha^2 \|\Delta^h \cdot\|_{\mathbb{L}^2}^2$. Therefore, we replace φ with u^1 in equation (4.31) and use Lemma 4.2.1 to obtain:

$$\begin{aligned} \|u^1\|_\alpha^2 + k\nu \|u^1\|_{h,\alpha}^2 & = k(U_2^1 \times (\nabla \times v^1), u^1) \leq C_D k \|U_2^1\|_{\mathbb{L}^4} \|\nabla v^1\|_{\mathbb{L}^2} \|u^1\|_{\mathbb{L}^2}^{1/2} \|\nabla u^1\|_{\mathbb{L}^2}^{1/2} \\ & \leq C_D k \|U_2^1\|_{\mathbb{L}^4} \|\nabla u^1\|_{\mathbb{L}^2}^{3/4} \|u^1\|_{\mathbb{L}^2}^{1/2} + C_D k \alpha^2 \|U_2^1\|_{\mathbb{L}^4} \|\nabla \Delta^h u^1\|_{\mathbb{L}^2} \|u^1\|_{\mathbb{L}^2}^{1/2} \|\nabla u^1\|_{\mathbb{L}^2}^{1/2} =: I + J, \end{aligned} \quad (4.32)$$

where Proposition 4.2.1-(iv) has been employed alongside identity $v^1 = u^1 - \alpha^2 \Delta^h u^1$ of Lemma 4.2.1 and the Young inequality. The first term on the right-hand side of equation (4.32) can be handled through the Young inequality as follows:

$$I \leq \frac{128C_D^4}{27\nu^3} k \|U_2^1\|_{\mathbb{L}^4}^4 \|u^1\|_{\mathbb{L}^2}^2 + \frac{k\nu}{2} \|\nabla u^1\|_{\mathbb{L}^2}^2 \leq \frac{128C_D^4}{27\nu^3} k \|U_2^1\|_{\mathbb{L}^4}^4 \|u^1\|_{\mathbb{L}^2}^2 + \frac{k\nu}{2} \|u^1\|_{h,\alpha}^2.$$

On the other hand, the condition $\alpha \leq \alpha_{max} h$, the inverse inequality (4.15), estimates $\|\nabla \cdot\|_{\mathbb{L}^2} \leq \|\cdot\|_{h,\alpha}$, $\alpha \|\Delta^h \cdot\|_{\mathbb{L}^2} \leq \|\cdot\|_{h,\alpha}$, and the Young inequality imply

$$\begin{aligned} J & \leq C_D k \alpha_{max} C \|U_2^1\|_{\mathbb{L}^4} \|\nabla u^1\|_{\mathbb{L}^2}^{1/2} \alpha \|\Delta^h u^1\|_{\mathbb{L}^2} \|u^1\|_{\mathbb{L}^2}^{1/2} \leq C_D k \alpha_{max} C \|U_2^1\|_{\mathbb{L}^4} \|u\|_{h,\alpha}^{3/2} \|u^1\|_{h,\alpha}^{1/2} \\ & \leq \frac{128C_D^4 \alpha_{max}^4 C^4}{27\nu^3} k \|U_2^1\|_{\mathbb{L}^4}^4 \|u^1\|_{\mathbb{L}^2}^2 + \frac{k\nu}{2} \|u^1\|_{h,\alpha}^2. \end{aligned}$$

Collecting the achieved estimates for I and J , plugging them back in equation (4.32) and using the Ladyzhenskaya inequality (see [83, Lemma I.1]), and the inverse estimate (4.15) yield

$$\begin{aligned} \|u^1\|_{\mathbb{L}^2}^2 &\leq \|u^1\|_{\alpha}^2 \leq \frac{128C_D^4}{27\nu^3}(1 + \alpha_{max}^4 C^4)k \|U_2^1\|_{\mathbb{L}^4}^4 \|u^1\|_{\mathbb{L}^2}^2 \\ &\leq \frac{2048C_D^4 C^2 k}{27\nu^3 h^2}(1 + \alpha_{max}^4 C^4) \|U_2^1\|_{\mathbb{L}^2}^4 \|u^1\|_{\mathbb{L}^2}^2, \end{aligned}$$

which implies eventually

$$\left(1 - \frac{2048C_D^4 C^2 k}{27\nu^3 h^2}(1 + \alpha_{max}^4 C^4) \|U_2^1\|_{\mathbb{L}^2}^4\right) \|u^1\|_{\mathbb{L}^2}^2. \quad (4.33)$$

That being said, we still need $\|U_2^1\|_{\mathbb{L}^2}^4$ to be controlled in a sample probability subset to achieve iterates' uniqueness. To this end, let $\varepsilon > 0$, and $\Omega_\varepsilon := \left\{\omega \in \Omega \mid \max_{1 \leq m \leq M} \|U_2^m\|_{\mathbb{L}^2}^4 \leq \frac{1}{\varepsilon}\right\} \subset \Omega$. Subsequently, by multiplying estimate (4.33) by the indicator function $\mathbb{1}_{\Omega_\varepsilon}$, it follows:

$$\left(1 - \frac{2048C_D^4 C^2 k}{27\nu^3 h^2 \varepsilon}(1 + \alpha_{max}^4 C^4)\right) \mathbb{1}_{\Omega_\varepsilon} \|u^1\|_{\mathbb{L}^2}^2 \leq 0.$$

By assuming that the coefficient of $\|u^1\|_{\mathbb{L}^2}^2$ is positive; that is, $\frac{k}{h^2} < \frac{27\nu^3 \varepsilon}{2048C_D^4 C^2 (1 + \alpha_{max}^4 C^4)}$, we infer that u^1 is null a.e. in D and \mathbb{P} -a.s. in Ω_ε . On the other hand, Lemma 4.2.1-(i) implies that $v^1 = u^1 - \alpha^2 \Delta^h u^1 = 0$, \mathbb{P} -a.s. in Ω_ε . Hence, for $m = 1$, uniqueness has been ensured. We suppose that $u^{m-1} = v^{m-1} = 0$ a.e. in D and \mathbb{P} -a.s. in Ω_ε . By arguing in a similar way to what have been carried out in this proof for $m = 1$, it can be readily demonstrated that $v^m = u^m = 0$ \mathbb{P} -a.s. in Ω_ε and a.e. in D , which terminates the uniqueness demonstration. Furthermore, by virtue of Markov's inequality and Lemma 4.4.1-(iii), it holds that $\mathbb{P}(\Omega_\varepsilon) \geq 1 - \varepsilon \mathbb{E} \left[\max_{1 \leq m \leq M} \|U_1^m\|_{\mathbb{L}^2}^4 \right] \geq 1 - \varepsilon C_{T,2}$. \square

Remark 4.4.1 *The existence of iterates $\{(U^m, V^m)\}_{m=1}^M$ was accomplished after eliminating the pressure $\{\Pi^m\}_{m=1}^M$ from Algorithm 4.1. To recover the existence of $\{\Pi^m\}_{m=1}^M$, we make use of the De Rham theorem, see for instance [84, Theorem 4.1].*

Lemma 4.4.3 *Assume the hypothesis of Lemma 4.4.1. Iterates $\{U^m\}_{m=1}^M$ satisfy the following estimate:*

$$\mathbb{E} \left[\left(k\nu \sum_{m=1}^M \left(\|\nabla U^m\|_{\mathbb{L}^2}^2 + \alpha^2 \|\Delta^h U^m\|_{\mathbb{L}^2}^2 \right) \right)^{2^{p-1}} \right] \leq C_{T,p},$$

where $C_{T,p} > 0$ has same ingredients as that of Lemma 4.4.1.

Proof: The demonstration is straightforward; it can be illustrated as follows: we pick the adequate inequality from equation (4.19):

$$\begin{aligned} \frac{k\nu}{2} \sum_{m=1}^M \|U^m\|_{h,\alpha}^2 &\leq \frac{1}{2} \|U^0\|_{\alpha}^2 + \frac{C_D^2 T K_3^2}{\nu} + \frac{C_D^2 K_4^2 T}{\nu} \max_{1 \leq m \leq M} \|U^{m-1}\|_{\alpha}^2 \\ &+ \sum_{m=1}^M \|g(t_{m-1}, U^{m-1}) \Delta_m W\|_{\mathbb{L}^2}^2 + \max_{1 \leq \ell \leq M} \sum_{m=1}^{\ell} \left(g(t_{m-1}, U^{m-1}) \Delta_m W, U^{m-1} \right) \end{aligned}$$

Next, we raise the above inequality to the power 2^{p-1} and use the estimate $|a + b|^q \leq 2^{q-1}(|a|^q + |b|^q)$, $\forall a, b \in \mathbb{R}, \forall q \geq 1$. It holds that

$$\begin{aligned} & \left(\frac{k\nu}{2} \sum_{m=1}^M \|U^m\|_{h,\alpha}^2 \right)^{2^{p-1}} \lesssim \|U^0\|_{\alpha}^{2^p} + \left(\frac{C_D^2 T K_3^2}{\nu} \right)^{2^{p-1}} + \left(\frac{C_D^2 K_4^2 T}{\nu} \right)^{2^{p-1}} \max_{1 \leq m \leq M} \|U^{m-1}\|_{\alpha}^{2^p} \\ & + \left(\sum_{m=1}^M \|g(t_{m-1}, U^{m-1}) \Delta_m W\|_{\mathbb{L}^2}^2 \right)^{2^{p-1}} + \left(\max_{1 \leq \ell \leq M} \sum_{m=1}^{\ell} \left(g(t_{m-1}, U^{m-1}) \Delta_m W, U^{m-1} \right) \right)^{2^{p-1}}. \end{aligned}$$

We will only focus on bounding the mathematical expectation of the last two terms. By the Burkholder-Davis-Gundy inequality, assumption (S_2) , and Lemma (4.4.1)-(iii), it follows

$$\begin{aligned} & \mathbb{E} \left[\left(\sum_{m=1}^M \|g(t_{m-1}, U^{m-1}) \Delta_m W\|_{\mathbb{L}^2}^2 \right)^{2^{p-1}} \right] \leq M^{2^{p-1}} \mathbb{E} \left[\max_{1 \leq m \leq M} \left\| \int_{t_{m-1}}^{t_m} g(t_{m-1}, U^{m-1}) dW(t) \right\|_{\mathbb{L}^2}^{2^p} \right] \\ & \lesssim M^{2^{p-1}} \mathbb{E} \left[k^{2^{p-1}} \|g(t_{m-1}, U^{m-1})\|_{\mathcal{L}_2(K, \mathbb{L}^2)}^{2^p} \right] \lesssim T^{2^{p-1}} K_1^{2^p} + T^{2^{p-1}} K_2^{2^p} \mathbb{E} \left[\max_{1 \leq m \leq M} \|U^{m-1}\|_{\alpha}^{2^p} \right] \\ & \leq T^{2^{p-1}} K_1^{2^p} + T^{2^{p-1}} K_2^{2^p} C_{T,p} \end{aligned}$$

On the other hand,

$$\begin{aligned} & \mathbb{E} \left[\left(\max_{1 \leq \ell \leq M} \sum_{m=1}^{\ell} \left(g(t_{m-1}, U^{m-1}) \Delta_m W, U^{m-1} \right) \right)^{2^{p-1}} \right] \\ & \lesssim \mathbb{E} \left[\left(k \sum_{m=1}^M \|g(t_{m-1}, U^{m-1})\|_{\mathcal{L}_2(K, \mathbb{L}^2)}^2 \|U^{m-1}\|_{\mathbb{L}^2}^2 \right)^{2^{p-2}} \right] \\ & \leq \mathbb{E} \left[\left(k \sum_{m=1}^M 2K_1^2 + 2(K_1^2 + K_2^2) \|U^{m-1}\|_{\alpha}^4 \right)^{2^{p-2}} \right] \\ & \leq \mathbb{E} \left[\left(2TK_1^2 + 2T(K_1^2 + K_2^2) \max_{1 \leq m \leq M} \|U^{m-1}\|_{\alpha}^4 \right)^{2^{p-2}} \right] \\ & \lesssim (2TK_1^2)^{2^{p-2}} + [2T(K_1^2 + K_2^2)]^{2^{p-2}} C_{T,p}, \end{aligned}$$

thanks to the Burkholder-Davis-Gundy inequality, assumption (S_2) , Lemma 4.4.1-(iii), inequalities $x^2 \leq 1 + x^4, \forall x \in \mathbb{R}$ and $|a + b|^q \leq 2^{q-1}(|a|^q + |b|^q)$. This completes the proof of this lemma. \square

In order to obtain a priori estimates for $\{V^m\}_{m=1}^M$ in Sobolev spaces, uniformly in α , we shall assume that $\alpha \leq \alpha_{max} h$ for some $\alpha_{max} > 0$ independent of h and k . We will present in Lemma 4.4.4 some preliminary estimates.

Lemma 4.4.4 *Let $\{(U^m, V^m)\}_{m=1}^M$ be the iterates of Algorithm 4.1 and $0 < \alpha \leq \alpha_{max} h$, where $\alpha_{max} > 0$ independent of α, h and k . Then, for all $m \in \{1, \dots, M\}$ and \mathbb{P} -a.s.*

- (i) $\|V^m\|_{\mathbb{L}^2} \leq \mathcal{C}_1 \|U^m\|_{\alpha}$,
- (ii) $\|\nabla V^m\|_{\mathbb{L}^2}^2 \leq \mathcal{C}_1 \left(\|\nabla U^m\|_{\mathbb{L}^2}^2 + \alpha^2 \|\Delta^h U^m\|_{\mathbb{L}^2}^2 \right)$,
- (iii) $\|V^{m+\ell} - V^m\|_{\mathbb{L}^2} \leq \mathcal{C}_1 \|U^{m+\ell} - U^m\|_{\alpha}$, for all $\ell \in \{1, \dots, M - m\}$,

where $\mathcal{C}_1 > 0$ depends only on α_{max} and the constant C of the inverse inequality (4.15).

Proof: Let $m \in \{1, \dots, M\}$. From equation (4.17)₂, taking $\psi = V^m$ and applying the Cauchy-Schwarz and Young inequalities yield

$$\|V^m\|_{\mathbb{L}^2}^2 \leq \|U^m\|_{\mathbb{L}^2}^2 + \frac{1}{4}\|V^m\|_{\mathbb{L}^2}^2 + \frac{\alpha^2}{\epsilon}\|\nabla U^m\|_{\mathbb{L}^2}^2 + \frac{\epsilon\alpha^2}{4}\|\nabla V^m\|_{\mathbb{L}^2}^2,$$

where $\epsilon > 0$. Taking $\epsilon = \frac{1}{\alpha_{max}^2 C^2}$ and applying the inverse inequality (4.15) complete the proof of assertion (i). On the other hand, by Lemma 4.2.1-(i), $\nabla V^m = \nabla U^m - \alpha^2 \nabla \Delta^h U^m$, \mathbb{P} -a.s. and a.e. in D . Thus,

$$\|\nabla V^m\|_{\mathbb{L}^2}^2 \leq 2\|\nabla U^m\|_{\mathbb{L}^2}^2 + 2\alpha_{max}^2 C^2 \alpha^2 \|\Delta^h U^m\|_{\mathbb{L}^2}^2,$$

thanks to the inverse inequality (4.15). Estimate (iii) has similar proof to that of assertion (i). \square

Clearly, one must incorporate Lemmas 4.4.1 and 4.4.4 to obtain:

Lemma 4.4.5 *Let $\{V^m\}_{m=1}^M$ be the iterates of Algorithm 4.1. Assume that assumptions (S₁)-(S₃) are fulfilled and that $0 < \alpha \leq \alpha_{max} h$, for some $\alpha_{max} > 0$ independent of k and h . Then,*

$$\mathbb{E} \left[\max_{1 \leq m \leq M} \|V^m\|_{\mathbb{L}^2}^2 + \frac{k\nu}{2} \sum_{m=1}^M \|\nabla V^m\|_{\mathbb{L}^2}^2 + \frac{1}{4} \sum_{m=1}^M \|V^m - V^{m-1}\|_{\mathbb{L}^2}^2 \right] \leq C'_T,$$

where $C'_T > 0$ does not depend on α, k and h .

We terminate this section with a local monotonicity property associated with the trilinear term of the underlying equations, as stated in the following proposition.

Proposition 4.4.1 *Assume that $\alpha \leq \alpha_{max} h$ for some $\alpha_{max} > 0$ independent of k and h , and that*

$L_f \leq \frac{\nu}{\sqrt{2}C_P^2}$ and $L_g \leq \frac{1}{\sqrt{\text{Tr}(Q)}} \left(\frac{\nu}{2C_P^2} - \frac{L_f^2 C_P^2}{\nu} \right)^{1/2}$, where C_P represents here the Poincaré constant.

For $v_h^1, v_h^2 \in \mathbb{V}_h$, let \bar{u}_h^1 and \bar{u}_h^2 be their discrete differential filters, respectively. Denote $w_h = \bar{u}_h^1 - \bar{u}_h^2$. There is a constant $\mathcal{K} > 0$ depending only on D, α_{max} and the inverse inequality's constant C such that

$$\begin{aligned} & \left\langle \nu \Delta(v_h^1 - v_h^2) + \bar{u}_h^1 \times (\nabla \times v_h^1) - \bar{u}_h^2 \times (\nabla \times v_h^2) + f(\cdot, \bar{u}_h^1) - f(\cdot, \bar{u}_h^2) - \frac{\mathcal{K}}{\nu^3} \|\bar{u}_h^2\|_{\mathbb{L}^4}^4 w_h, w_h \right\rangle \\ & + \text{Tr}(Q) \left\| g(\cdot, \bar{u}_h^1) - g(\cdot, \bar{u}_h^2) \right\|_{\mathcal{L}_2(K; \mathbb{L}^2)}^2 \leq 0. \end{aligned} \quad (4.34)$$

Proof: The first target in this proof will be the estimate

$$\|\nabla(v_h^1 - v_h^2)\|_{\mathbb{L}^2} \leq (1 + C^2 \alpha_{max}^2) \|\nabla(\bar{u}_h^1 - \bar{u}_h^2)\|_{\mathbb{L}^2}. \quad (4.35)$$

Indeed, from Lemma 4.2.1-(i), we get $\nabla(v_h^1 - v_h^2) = \nabla(\bar{u}_h^1 - \bar{u}_h^2) - \alpha^2 \nabla \Delta^h(\bar{u}_h^1 - \bar{u}_h^2)$ a.e. in D . Therefore, a simple application of the inverse inequalities (4.15), (4.16) and the hypothesis $\alpha \leq \alpha_{max} h$ justifies (4.35). On the other hand,

$$\begin{aligned} & \left| \left(\bar{u}_h^1 \times (\nabla \times v_h^1) - \bar{u}_h^2 \times (\nabla \times v_h^2), w_h \right) \right| = \left| \left(\bar{u}_h^2 \times (\nabla \times (v_h^1 - v_h^2)), w_h \right) \right| \\ & \leq C_D \|\bar{u}_h^2\|_{\mathbb{L}^4} \|\nabla(v_h^1 - v_h^2)\|_{\mathbb{L}^2} \|w_h\|_{\mathbb{L}^2}^{1/2} \|\nabla w_h\|_{\mathbb{L}^2}^{1/2} \leq C_D (1 + C^2 \alpha_{max}^2) \|\bar{u}_h^2\|_{\mathbb{L}^4} \|\nabla w_h\|_{\mathbb{L}^2}^{3/2} \|w_h\|_{\mathbb{L}^2}^{1/2} \\ & \leq \frac{\nu}{4} \|\nabla w_h\|_{\mathbb{L}^2}^2 + \frac{\mathcal{K}}{\nu^3} \|\bar{u}_h^2\|_{\mathbb{L}^4}^4 \|w_h\|_{\mathbb{L}^2}^2, \end{aligned} \quad (4.36)$$

for some constant $\mathcal{K} > 0$ depending on C, α_{max} and C_D , where Proposition 4.2.1-(i), (iv), estimate (4.35) and Young's inequality were employed. In addition, assumption (S_2) implies

$$\begin{aligned} \left| \langle f(\cdot, \bar{u}_h^1) - f(\cdot, \bar{u}_h^2), w_h \rangle \right| &\leq L_f C_P \|w_h\|_\alpha \|\nabla w_h\|_{\mathbb{L}^2} \leq \frac{L_f^2 C_P^2}{\nu} \|w_h\|_\alpha^2 + \frac{\nu}{4} \|\nabla w_h\|_{\mathbb{L}^2}^2 \\ &\leq \frac{L_f^2 C_P^4}{\nu} (\|\nabla w_h\|_{\mathbb{L}^2}^2 + \alpha^2 \|\Delta^h w_h\|_{\mathbb{L}^2}^2) + \frac{\nu}{4} \|\nabla w_h\|_{\mathbb{L}^2}^2, \end{aligned} \quad (4.37)$$

where the Poincaré inequality and $\|\nabla \varphi_h\|_{\mathbb{L}^2} \leq C_P \|\Delta^h \varphi_h\|_{\mathbb{L}^2}, \forall \varphi_h \in \mathbb{V}_h$, were used in the last inequality. Similarly, by virtue of assumptions (S_2) , the norm equivalence (4.5), and the Poincaré inequality, one gets

$$\|g(\cdot, \bar{u}_h^1) - g(\cdot, \bar{u}_h^2)\|_{\mathcal{L}_2(K; \mathbb{L}^2)}^2 \leq L_g^2 C_P^2 \|\nabla w_h\|_{\mathbb{L}^2}^2.$$

By Lemma 4.2.1-(i) and identity 4.13, it holds that

$$\nu \left(\nabla(v_h^1 - v_h^2), \nabla w_h \right) = \nu \|\nabla w_h\|_{\mathbb{L}^2}^2 + \nu \alpha^2 \|\Delta^h w_h\|_{\mathbb{L}^2}^2.$$

The sum of the former and the latter identities along with inequalities (4.36) and (4.37) yields estimate (4.34). \square

Remark 4.4.2 *The assumed conditions on L_f and L_g in Proposition 4.4.1 are mainly imposed to maintain the monotonicity. They appear in this context due to the dependence of both f and g on the solution. In other words, if $f = f(t)$ and $g = g(t)$, these conditions would no longer make sense. Observe, in addition, that one could have omitted the factor $\frac{1}{\sqrt{\text{Tr}(Q)}}$ by adjusting the Lipschitz-continuity of the diffusion coefficient g to $\|g(\cdot, z_1) - g(\cdot, z_2)\|_{\mathcal{L}_2(Q^{1/2}(K); \mathbb{L}^2)} \leq L_g \|z_1 - z_2\|_\alpha$.*

4.5 . Convergence

All the previous analysis relied on $\{(U^m, V^m)\}_{m=1}^M$, which does not depend explicitly on the time variable. To investigate the convergence in continuous-time spaces, e.g. $L^2(\Omega; L^2(0, T; \mathbb{H}_0^1))$, we need to define the following processes

$$\left(\mathcal{U}_{k,h}^-(t, x), \mathcal{V}_{k,h}^-(t, x) \right) := \left(U^{m-1}(x), V^{m-1}(x) \right), \quad \forall (t, x) \in [t_{m-1}, t_m) \times D, \quad (4.38)$$

$$\left(\mathcal{U}_{k,h}^+(t, x), \mathcal{V}_{k,h}^+(t, x) \right) := \left(U^m(x), V^m(x) \right), \quad \forall (t, x) \in (t_{m-1}, t_m] \times D, \quad (4.39)$$

$$(f^-(t, \cdot), g^-(t, \cdot)) = (f(t_{m-1}, \cdot), g(t_{m-1}, \cdot)), \quad \forall t \in [t_{m-1}, t_m). \quad (4.40)$$

Discrete derivation with respect to time will be required later on. For this purpose, we list a few rules in the proposition below.

Proposition 4.5.1 *Denote by d_t the discrete derivation defined by $d_t z^m = \frac{z^m - z^{m-1}}{k}$, for all $m \in \{1, \dots, M\}$. Let $z^+, z^- : [0, T] \rightarrow \mathbb{R}$ be the piecewise constant functions defined by $z^+(t) := z^m$ for all $t \in (t_{m-1}, t_m]$, and $z^-(t) := z^{m-1}$ for all $t \in [t_{m-1}, t_m)$. The following properties hold true:*

$$(i) \quad d_t(\zeta^+ \xi^+) = \zeta^+ d_t \xi^+ + \xi^- d_t \zeta^+.$$

$$(ii) \quad \int_0^T \zeta^+ d_t \xi^+(t) dt = \zeta^+(T) \xi^+(T) - \zeta^-(0) \xi^-(0) - \int_0^T (d_t \zeta^+(t)) \xi^- dt.$$

$$(iii) \quad d_t e^{\zeta^+} = e^{\zeta^-} d_t \zeta^+ + e^\eta \frac{(\zeta^+ - \zeta^-)^2}{2k}, \quad \text{for some } \eta \in (\zeta^-, \zeta^+).$$

Proof: See [24, Appendix B]. □

The remaining subsection of this section is solely devoted to giving adequate proof for solutions' existence. For further analysis, the reader may refer to Section 4.6.

4.5.1 . Convergence when $\alpha \leq \alpha_{max}h$

The demonstration technique that has been followed here keeps Skorokhod's theorem out of the game. This avoidance is valid by virtue of Proposition 4.4.1. We need to go through a few steps to illustrate the convergence of Algorithm 4.1.

Step 1: Boundedness

The following sequences $\{\mathcal{U}_{k,h}^+\}_{k,h}$, $\{\mathcal{V}_{k,h}^+\}_{k,h}$, $\{f^-(\cdot, \mathcal{U}_{k,h}^-)\}_{k,h}$ and $\{g^-(\cdot, \mathcal{U}_{k,h}^-)\}_{k,h}$ are respectively bounded in the following spaces $L^2(\Omega; L^\infty(0, T; \mathbb{L}^2)) \cap L^2(\Omega; L^2(0, T; \mathbb{H}_0^1))$, $L^2(\Omega; L^2(0, T; \mathbb{H}^{-1}))$ and $L^2(\Omega; L^2(0, T; \mathcal{L}_2(K; \mathbb{L}^2)))$ by virtue of Lemmas 4.4.1, 4.4.5 and assumption (S_2) . Therefore, there are $u, v \in L^2(\Omega; L^\infty(0, T; \mathbb{L}^2)) \cap L^2(0, T; \mathbb{H}_0^1)$, $F_0 \in L^2(\Omega; L^2(0, T; \mathbb{H}^{-1}))$, $G_0 \in L^2(\Omega; L^2(0, T; \mathcal{L}_2(K; \mathbb{L}^2)))$ and two subsequences denoted by $\{\mathcal{U}_{k',h'}^+\}_{k',h'}$, $\{\mathcal{V}_{k',h'}^+\}_{k',h'}$ such that

$$\mathcal{U}_{k',h'}^+ \rightharpoonup u \quad \& \quad \mathcal{V}_{k',h'}^+ \rightharpoonup v \quad \text{in } L^2(\Omega; L^2(0, T; \mathbb{H}_0^1)), \quad (4.41)$$

$$\mathcal{U}_{k',h'}^+ \xrightarrow{*} u \quad \& \quad \mathcal{V}_{k',h'}^+ \xrightarrow{*} v \quad \text{in } L^2(\Omega; L^\infty(0, T; \mathbb{L}^2)), \quad (4.42)$$

$$f^-(\cdot, \mathcal{U}_{k',h'}^-) \rightharpoonup F_0 \quad \text{in } L^2(\Omega; L^2(0, T; \mathbb{H}^{-1})), \quad (4.43)$$

$$g^-(\cdot, \mathcal{U}_{k',h'}^-) \rightharpoonup G_0 \quad \text{in } L^2(\Omega; L^2(0, T; \mathcal{L}_2(K; \mathbb{L}^2))). \quad (4.44)$$

Let $\varphi \in D(A)$. Set $\varphi_h = \Pi_h \varphi$ and $R(\mathcal{U}_{k,h}^+) := \nu \Delta \mathcal{V}_{k,h}^+ + \mathcal{U}_{k,h}^+ \times (\nabla \times \mathcal{V}_{k,h}^+)$. By summing equation (4.17) over m from 1 to M , we achieve:

$$\begin{aligned} \int_0^T \langle R(\mathcal{U}_{k',h'}^+), \varphi_{h'} \rangle dt &= \left(\mathcal{V}_{k',h'}^+(T) - \mathcal{V}_{k',h'}^-(0), \varphi_{h'} \right) - \int_0^T \langle f^-(t, \mathcal{U}_{k',h'}^-), \varphi_{h'} \rangle dt \\ &\quad - \left(\int_0^T g^-(t, \mathcal{U}_{k',h'}^-) dW(t), \varphi_{h'} \right). \end{aligned}$$

Since all terms on the right-hand side converge after applying the mathematical expectation, thanks to (4.41)-(4.44), we define the operator R_0 as

$$\mathbb{E} \left[\int_0^T \langle R_0(t), \varphi \rangle dt \right] = \lim_{k',h' \rightarrow 0} \mathbb{E} \left[\int_0^T \langle R(\mathcal{U}_{k',h'}^+), \Pi_{h'} \varphi \rangle dt \right], \quad \forall \varphi \in D(A).$$

Subsequently, the limiting function v fulfills, for all $\varphi \in D(A)$, for all $t \in [0, T]$ and \mathbb{P} -a.s. the following equation:

$$(v(t), \varphi) - \int_0^t \langle R_0(s), \varphi \rangle ds = (v_0, \varphi) + \int_0^t \langle F_0(s), \varphi \rangle ds + \left(\int_0^t G_0(s) dW(s), \varphi \right). \quad (4.45)$$

Besides the convergence results (4.41)-(4.44), we will also need the following:

$$f^-(\cdot, \mathcal{U}_{k',h'}^+) \rightharpoonup F_0 \quad \text{in } L^2(\Omega; L^2(0, T; \mathbb{H}^{-1})), \quad (4.46)$$

$$g^-(\cdot, \mathcal{U}_{k',h'}^+) \rightharpoonup G_0 \quad \text{in } L^2(\Omega; L^2(0, T; \mathcal{L}_2(K; \mathbb{L}^2))). \quad (4.47)$$

Convergence (4.46) can be illustrated as follows: $\{f^-(\cdot, \mathcal{U}_{k',h'}^+)\}_{k',h'}$ is bounded in $L^2(\Omega; L^2(0, T; \mathbb{H}^{-1}))$, thanks to assumption (S_2) and Lemma 4.4.1. Thus, there are \bar{F}_0 and a subsequence of $\{f^-(\cdot, \mathcal{U}_{k',h'}^+)\}_{k',h'}$ (still denoted $f^-(\cdot, \mathcal{U}_{k',h'}^+)$) such that $f^-(\cdot, \mathcal{U}_{k',h'}^+) \rightharpoonup \bar{F}_0$ in $L^2(\Omega; L^2(0, T; \mathbb{H}^{-1}))$. To unify the limiting functions F_0 and \bar{F}_0 , we let $\varphi \in L^2(\Omega; L^2(0, T; \mathbb{H}_0^1))$. Thus,

$$\begin{aligned} \langle \bar{F}_0(t) - F_0(t), \varphi(t) \rangle &\leq \langle \bar{F}_0(t) - f^-(t, \mathcal{U}_{k',h'}^+), \varphi(t) \rangle + L_f \|\mathcal{U}_{k',h'}^+ - \mathcal{U}_{k',h'}^-\|_\alpha \|\varphi(t)\|_{\mathbb{H}^1} \\ &\quad + \langle f^-(t, \mathcal{U}_{k',h'}^-) - F_0(t), \varphi(t) \rangle. \end{aligned}$$

Integrating with respect to t and applying the mathematical expectation while taking into account the strong convergence toward 0 of $\mathbb{E} \left[\int_0^T \|\mathcal{U}_{k',h'}^+ - \mathcal{U}_{k',h'}^-\|_\alpha^2 dt \right]$ (thanks to Lemma 4.4.1) yield $F_0 = \bar{F}_0$ in $L^2(\Omega; L^2(0, T; \mathbb{H}^{-1}))$. Convergence (4.47) follows similarly.

Step 2: u and v are equal

From equation (4.17), there holds $\mathcal{V}_{k',h'}^+ = \mathcal{U}_{k',h'}^+ - \alpha^2 \Delta^{h'} \mathcal{U}_{k',h'}^+$, \mathbb{P} -a.s. and a.e. in $(0, T) \times D$. Moreover, for all $\varphi \in L^2(\Omega; L^2(0, T; \mathbb{L}^2))$, we have

$$\begin{aligned} \alpha^2 \mathbb{E} \left[\int_0^T \left(\Delta^{h'} \mathcal{U}_{k',h'}^+, \varphi(t) \right) dt \right] &\leq \alpha \|\varphi\|_{L^2(\Omega; L^2(0, T; \mathbb{L}^2))} \mathbb{E} \left[\int_0^T \alpha^2 \|\Delta^{h'} \mathcal{U}_{k',h'}^+\|_{\mathbb{L}^2}^2 dt \right]^{\frac{1}{2}} \\ &\leq \alpha_{max} h' \|\varphi\|_{L^2(\Omega; L^2(0, T; \mathbb{L}^2))} C_T \rightarrow 0 \text{ as } k', h' \rightarrow 0, \end{aligned}$$

where Lemma 4.4.1-(ii) is exploited along with the hypothesis $\alpha \leq \alpha_{max} h$. As a result, $\{\Delta^{h'} \mathcal{U}_{k',h'}^+\}_{k',h'}$ is weakly convergent to 0 in $L^2(\Omega; L^2(0, T; \mathbb{L}^2))$. Consequently, it follows from convergence (4.41) and the relationship which was stated in the beginning of this step that $u = v$ \mathbb{P} -a.s. and a.e. in $(0, T) \times D$.

Step 3: Identification of R_0, F_0 and G_0

We shall denote, from this step onwards, $(k', h') = (k, h)$ for the sake of clarity. For a given $z \in C([0, T]; D(A))$, we define $z_h^+(t) = \Pi_h z(t_m)$ for all $t \in (t_{m-1}, t_m]$ and all $m \in \{0, 1, \dots, M\}$. We also denote by \bar{z}_h^+ its discrete differential filter. For all $m \in \{1, \dots, M\}$, set

$$r_m := \frac{2\mathcal{K}}{\nu^3} k \sum_{i=1}^m \|\bar{z}_h(t_i)\|_{\mathbb{L}^4}^4,$$

and we associate with it the piecewise constant function $r^+(t) = r_m$ when $t \in (t_{m-1}, t_m]$. The constant \mathcal{K} in r_m emerges from Proposition 4.4.1 and will play a relevant role in the upcoming analysis. We finally define a non-increasing function $\rho: [0, T] \rightarrow \mathbb{R}_+$ verifying $\rho(0) = 1$ and such that its discrete version reads $\rho_m := e^{-r_m}$, for all $m \in \{1, \dots, M\}$ and we assign ρ^+ and ρ^- the usual piecewise constant definition. We replace φ with U^m in equation (4.17), employ the Cauchy-Schwarz and Young inequalities, then apply the mathematical expectation to achieve:

$$\begin{aligned} \mathbb{E} \left[\|U^m\|_\alpha^2 - \|U^{m-1}\|_\alpha^2 - 2k \langle R(U^m) + f(t_{m-1}, U^{m-1}), U^m \rangle \right] \\ \leq \mathbb{E} \left[\|g(t_{m-1}, U^{m-1}) \Delta_m W\|_{\mathbb{L}^2}^2 \right]. \end{aligned} \quad (4.48)$$

Afterwards, we multiply (4.48) by ρ_{m-1} and sum it over m from 1 to M . It follows:

$$\begin{aligned} \mathbb{E} \left[\rho^+(T) \|\mathcal{U}_{k,h}^+(T)\|_\alpha^2 - \|\mathcal{U}_{k,h}^-(0)\|_\alpha^2 \right] &\leq \mathbb{E} \left[2 \int_0^T \rho^-(t) \langle R(\mathcal{U}_{k,h}^+) + f^-(t, \mathcal{U}_{k,h}^+), \mathcal{U}_{k,h}^+ \rangle dt \right] \\ &+ \mathbb{E} \left[2 \int_0^T \rho^-(t) \langle f^-(t, \mathcal{U}_{k,h}^-) - f^-(t, \mathcal{U}_{k,h}^+), \mathcal{U}_{k,h}^+ \rangle dt \right] \\ &+ \mathbb{E} \left[\int_0^T \rho^-(t) \|g^-(t, \mathcal{U}_{k,h}^-)\|_{\mathcal{L}_2(Q^{1/2}(K); \mathbb{L}^2)}^2 dt \right] + \mathbb{E} \left[\int_0^T \|\mathcal{U}_{k,h}^+\|_\alpha^2 d_t \rho^+ dt \right], \end{aligned} \quad (4.49)$$

where the first two terms on the left-hand side in inequality (4.48) were handled through Proposition 4.5.1 as follows:

$$\begin{aligned} \sum_{m=1}^M \rho_{m-1} \left(\|U^m\|_\alpha^2 - \|U^{m-1}\|_\alpha^2 \right) &= \int_0^T \rho^-(t) d_t \|\mathcal{U}_{k,h}^+\|_\alpha^2 dt \\ &= \rho^+(T) \|\mathcal{U}_{k,h}^+(T)\|_\alpha^2 - \int_0^T \|\mathcal{U}_{k,h}^+\|_\alpha^2 d_t \rho^+(t) dt, \end{aligned}$$

and the right-hand side of (4.48) is treated with Itô's isometry. Taking into account the discrete derivation (see Proposition 4.5.1) and adjusting a few terms in equation (4.49), we obtain

$$\begin{aligned}
& \mathbb{E} \left[\rho^+(T) \|\mathcal{U}_{k,h}^+(T)\|_{\mathbb{L}^2}^2 - \|\mathcal{U}_{k,h}^-(0)\|_{\mathbb{L}^2}^2 \right] \leq \alpha^2 \mathbb{E} \left[\|\nabla \mathcal{U}_{k,h}^-(0)\|_{\mathbb{L}^2}^2 \right] + \mathbb{E} \left[\int_0^T \|\mathcal{U}_{k,h}^+ - \bar{z}_h^+\|_{\alpha}^2 dt \rho^+ dt \right] \\
& + \mathbb{E} \left[\int_0^T \left\{ 2 \left((\mathcal{U}_{k,h}^+, \bar{z}_h^+ \right)_{\alpha} - \|\bar{z}_h^+\|_{\alpha}^2 \right) dt \rho^+ dt \right\} \right] + \mathbb{E} \left[2 \int_0^T \rho^-(t) \langle f^-(t, \mathcal{U}_{k,h}^-) - f^-(t, \mathcal{U}_{k,h}^+), \mathcal{U}_{k,h}^+ \rangle dt \right] \\
& + \mathbb{E} \left[2 \int_0^T \rho^-(t) \langle R(\mathcal{U}_{k,h}^+) - R(\bar{z}_h^+) + f^-(t, \mathcal{U}_{k,h}^+) - f^-(t, \bar{z}_h^+), \mathcal{U}_{k,h}^+ - \bar{z}_h^+ \rangle dt \right] \\
& + \mathbb{E} \left[2 \int_0^T \rho^-(t) \langle R(\mathcal{U}_{k,h}^+) - R(\bar{z}_h^+) + f^-(t, \mathcal{U}_{k,h}^+) - f^-(t, \bar{z}_h^+), \bar{z}_h^+ \rangle dt \right] \\
& + \mathbb{E} \left[2 \int_0^T \rho^-(t) \langle R(\bar{z}_h^+) + f^-(t, \bar{z}_h^+), \mathcal{U}_{k,h}^+ \rangle dt \right] + \mathbb{E} \left[\int_0^T \rho^-(t) \left\{ \|g^-(t, \mathcal{U}_{k,h}^-) - g^-(t, \mathcal{U}_{k,h}^+)\|_{\mathcal{L}_2^Q}^2 \right. \right. \\
& + \|g^-(t, \mathcal{U}_{k,h}^+) - g^-(t, \bar{z}_h^+)\|_{\mathcal{L}_2^Q}^2 - \|g^-(t, \bar{z}_h^+)\|_{\mathcal{L}_2^Q}^2 + 2 \left(g^-(t, \mathcal{U}_{k,h}^+), g^-(t, \bar{z}_h^+) \right)_{\mathcal{L}_2^Q} \\
& \left. \left. + 2 \left(g^-(t, \mathcal{U}_{k,h}^-) - g^-(t, \mathcal{U}_{k,h}^+), g^-(t, \mathcal{U}_{k,h}^+) \right)_{\mathcal{L}_2^Q} \right\} dt \right] =: I_1 + \dots + I_7 + I_8^Q + \dots + I_{12}^Q.
\end{aligned}$$

where we recall that $((\cdot, \cdot)_{\alpha}) := (\cdot, \cdot) + \alpha^2 (\nabla \cdot, \nabla \cdot)$, $R(\bar{z}_h^+) = \nu \Delta \bar{z}_h^+ + \bar{z}_h^+ \times (\nabla \times \bar{z}_h^+)$, and \mathcal{L}_2^Q is a shorthand for $\mathcal{L}_2(Q^{1/2}(K); \mathbb{L}^2)$. Since $z \in C([0, T]; D(A))$, it follows straightforwardly that z_h^+ and \bar{z}_h^+ converge toward z in $L^p(0, T; \mathbb{H}^1)$ as $k, h \rightarrow 0$, for all $p \geq 1$ (e.g. [81, Theorem 4.0.1]). It suffices now to study the limit of all I_1, \dots, I_{12}^Q . It is easy to see that $I_1 \rightarrow 0$ as $k, h \rightarrow 0$ because $\mathcal{U}_{k,h}^-(0) = U^0$ and $\|U^0\|_{\mathbb{H}^1}$ is uniformly bounded in h . By Proposition 4.5.1,

$$I_2 \leq \mathbb{E} \left[\int_0^T -\frac{2\mathcal{K}}{\nu^3} \rho^- \|\bar{z}_h^+\|_{\mathbb{L}^4}^4 \|\mathcal{U}_{k,h}^+ - \bar{z}_h^+\|_{\mathbb{L}^2}^2 dt \right] + \mathbb{E} \left[\int_0^T \|\mathcal{U}_{k,h}^+ - \bar{z}_h^+\|_{\alpha}^2 e^{\eta} \frac{(r^+ - r^-)^2}{2k} dt \right] =: I_{2,1} + I_{2,2},$$

for some $\eta \in (-r^+, -r^-)$. By a simple application of Proposition 4.4.1, it follows that $I_5 + I_{2,1} + I_9^Q \leq 0$. We also mention that $I_{2,2} \rightarrow 0$ as $k, h \rightarrow 0$ by utilizing Lemma 4.4.1 and the fact that

$$\frac{(r^+ - r^-)^2}{2k} = \frac{2\mathcal{K}^2}{\nu^6} k \|\bar{z}_h^+\|_{\mathbb{L}^4}^8.$$

Furthermore, we know by Lemma 4.4.1-(ii) that $\mathbb{E} \left[\int_0^T \|\mathcal{U}_{k,h}^+ - \mathcal{U}_{k,h}^-\|_{\alpha}^2 dt \right]$ goes to 0 as $k, h \rightarrow 0$,

therefore I_4, I_8^Q and I_{12}^Q converge to 0, thanks to assumption (S_2) . Collecting the recently derived limits and using convergences (4.41)-(4.47), we acquire:

$$\begin{aligned}
& \lim_{k,h \rightarrow 0} \mathbb{E} \left[\rho^+(T) \|\mathcal{U}_{k,h}^+(T)\|_{\mathbb{L}^2}^2 - \|\mathcal{U}_{k,h}^-(0)\|_{\mathbb{L}^2}^2 \right] \leq \mathbb{E} \left[\int_0^T \left\{ 2(v, z) - \|z\|_{\mathbb{L}^2}^2 \right\} \partial_t \rho(t) dt \right] \\
& + \mathbb{E} \left[2 \int_0^T \rho(t) \langle R_0(t) - R(z) + F_0(t) - f(t, z), z \rangle dt \right] + \mathbb{E} \left[2 \int_0^T \rho(t) \langle R(z) + f(t, z), v \rangle dt \right] \quad (4.50) \\
& + \mathbb{E} \left[\int_0^T \rho(t) \left\{ -\|g(t, z)\|_{\mathcal{L}_2^Q}^2 + 2(G_0(t), g(t, z))_{\mathcal{L}_2^Q} \right\} dt \right]
\end{aligned}$$

Next, Itô's formula employed to the stochastic process $(t, v) \mapsto \rho(t) \|v(t)\|_{\mathbb{L}^2}^2$ (where v fulfills equation (4.45)) together with inequality (4.50), condition $\alpha \leq \alpha_{max} h$, convergence $U^0 \rightarrow \bar{u}_0 = v_0$ as $h \rightarrow 0$, and the fact that $\mathbb{E} [\rho(T) \|v(T)\|_{\mathbb{L}^2}^2] \leq \liminf \mathbb{E} [\rho^+(T) \|\mathcal{U}_{k,h}^+(T)\|_{\mathbb{L}^2}^2]$ grant:

$$\begin{aligned}
& \mathbb{E} \left[\int_0^T \partial_t \rho(t) \|v(t) - z(t)\|_{\mathbb{L}^2}^2 dt \right] + \mathbb{E} \left[\int_0^T \rho(t) \|G_0(t) - g(t, z(t))\|_{\mathcal{L}_2(Q^{1/2}(K); \mathbb{L}^2)}^2 dt \right] \\
& \leq \mathbb{E} \left[2 \int_0^T \rho(t) \langle R_0(t) - R(z(t)) + F_0(t) - f(t, z(t)), z(t) - v(t) \rangle dt \right], \quad (4.51)
\end{aligned}$$

for all $z \in C([0, T]; D(A))$. Particularly, inequality (4.51) holds true for all $z \in L^4(\Omega; L^\infty(0, T; \mathbb{H})) \cap L^2(\Omega; L^2(0, T; \mathbb{V}))$ due to the density of $C([0, T]; D(A))$ in $L^4(\Omega; L^\infty(0, T; \mathbb{H})) \cap L^2(\Omega; L^2(0, T; \mathbb{V}))$. Hence, taking $z = v$ yields $G_0 = g(\cdot, v)$ in $L^2(\Omega; L^2(0, T; \mathcal{L}_2(Q^{1/2}(K); \mathbb{L}^2)))$. Therewith, plugging $z = v + \lambda w$, for $w \in L^2(\Omega; L^\infty(0, T; \mathbb{H})) \cap L^2(\Omega; L^2(0, T; \mathbb{V}))$ and $\lambda > 0$, into inequality (4.51) implies the following

$$\begin{aligned} & \lambda \mathbb{E} \left[\int_0^T \partial_t \rho(t) \|w(t)\|_{\mathbb{L}^2}^2 dt \right] \\ & \leq \mathbb{E} \left[2 \int_0^T \rho(t) \langle R_0(t) - R(v + \lambda w) + F_0(t) - f(t, v + \lambda w), w(t) \rangle dt \right]. \end{aligned}$$

Taking into account the hemi-continuity of operator R and the fact the f is Lipschitz-continuous with respect to its space-variable, one may take $\lambda \rightarrow 0$ to obtain

$$R_0 + F_0 = R(v) + f(\cdot, v) \text{ in } L^2(\Omega; L^2(0, T; \mathbb{H}^{-1})).$$

4.6 . Further properties, numerical experiments, and conclusion

Section 4.5 gave an insight into the limiting functions and the existence of solutions. Yet, it has not provided the divergence-free property which must be associated with v . The following proposition treats this issue.

Proposition 4.6.1 *The limiting function v which was found in Section 4.5 is divergence-free almost everywhere in $(0, T) \times D$ and \mathbb{P} -almost surely.*

Proof: To demonstrate that v is divergence-free, it suffices to show that $\{div(\mathcal{U}_{k,h}^+)\}_{k,h}$ converges weakly in $L^2(\Omega; L^2(0, T; \mathbb{L}^2))$ toward 0, thanks to convergence (4.41). To this end, we evoke the Lagrange interpolation $\mathcal{I}_h : C^3(D) \rightarrow L_h$ (c.f. [21, Theorem 4.4.4]). For $z \in C^3(D)$, we have

$$\begin{aligned} \mathbb{E} \left[\int_0^T \left(div(\mathcal{U}_{k,h}^+), z \right) dt \right] &= \mathbb{E} \left[\int_0^T \left(div(\mathcal{U}_{k,h}^+), z - \mathcal{I}_h z \right) dt \right] + \mathbb{E} \left[\int_0^T \left(div(\mathcal{U}_{k,h}^+), \mathcal{I}_h z \right) dt \right] \\ &\lesssim \mathbb{E} \left[\int_0^T \|\nabla \mathcal{U}_{k,h}^+\|_{L^2} dt \right] \|z - \mathcal{I}_h z\|_{L^2} \xrightarrow[k,h \rightarrow 0]{} 0, \end{aligned}$$

where the second term in the first equality vanishes because $\{\mathcal{U}_{k,h}^+\}_{k,h}$ is weakly divergence-free. \square

4.6.1 . Convergence of LANS- α to NSE in 2D

Assume $\alpha \leq \alpha_{max} h$ for some $\alpha_{max} > 0$ independent of k and h , and $U^0 \rightarrow \bar{u}_0$ in $L^4(\Omega; \mathbb{H}^1)$ as $h \rightarrow 0$. In Subsection 4.5.1, we proved that the process v satisfies \mathbb{P} -a.s. and for all $(t, \varphi) \in [0, T] \times \mathbb{V}$ the following equation:

$$\begin{aligned} & (v(t), \varphi) + \nu \int_0^t (\nabla v(s), \nabla \varphi) ds - \int_0^t \langle v(s) \times (\nabla \times v(s)), \varphi \rangle ds \\ & = (v_0, \varphi) + \int_0^t \langle f(s, v(s)), \varphi \rangle ds + \left(\int_0^t g(s, v(s)) dW(s), \varphi \right), \end{aligned}$$

where we recall that $u = v$, \mathbb{P} -a.s. and a.e in $(0, T) \times D$. The above equation does not represent yet the Navier-Stokes problem we are looking for because the trilinear term $\langle v(s) \times (\nabla \times v(s)), \varphi \rangle$ looks unfamiliar. However, by Proposition 4.2.1-(iii),

$$-\langle v(s) \times (\nabla \times v(s)), \varphi \rangle = \langle [v(s) \cdot \nabla] v(s), \varphi \rangle,$$

where $\langle [\varphi \cdot \nabla]v(s), v(s) \rangle = 0$ because $\varphi \in \mathbb{V}$ and $v \in \mathbb{H}_0^1$ (see for instance [107, Chapter 2, Lemma 1.3]). Moreover, by a standard technique (e.g. [95]), it is easy to check from equation (4.45), that v is continuous in time i.e. $v \in L^2(\Omega; C([0, T]; \mathbb{H}))$. Hence, v is a solution of equations (4.2) in the sense of Definition 4.2.3. Additionally, owing to [66, Proposition 4.1], the process v is unique and it follows that the whole sequences $\{\mathcal{U}_{k,h}^+\}_{k,h}$ and $\{\mathcal{V}_{k,h}^+\}_{k,h}$ are convergent.

4.6.2 . Numerical experiments

This part is devoted to giving computational experiments in 2D for the stochastic Navier-Stokes equations through Algorithm 4.1 when the spatial scale α fulfills $\alpha \leq \alpha_{max}h$. Since our primary objective is to compare solutions' behavior of LANS- α to that of Navier-Stokes, we provide simulation of solutions to the latter equations as well through a non-linear scheme covered in [24, Algorithm 1]. The implementation hereafter is performed using the open source finite element software FEniCS [87]. We employ the lower order Taylor-Hood (P_2 - P_1) element for the spatial discretization within a mixed finite element framework. The chosen domain is a unit square $D = (0, 1)^2$ along the time interval $[0, T]$ with $T = 1$. The initial condition $\bar{u}_0 = (\partial_y \psi, -\partial_x \psi)$ is divergence-free and vanishes on the boundary ∂D , where $\psi(x, y) := 10 \sin(100xy^2)x^2(1-x)^2y^2(1-y)^2$, and the viscosity ν is set to 1. On the other hand, the source term $f(\omega, t, u) = e^{-\frac{\omega^2}{2}} \sin(t)u$ is considered in a way to satisfy assumption (S_2) along with the inequality $L_f \leq \nu/\sqrt{2}C_P^2$, where in the present case, $L_f = 1$. It is worth mentioning that C_P , which is the Poincaré constant in this case, is less than $diam(D)/\pi = \sqrt{2}/\pi$ (see [33]). Finally, the corresponding noise that drives the equations of interest will be considered additive for the sake of simplicity i.e. the drift coefficient g will play the identity operator role.

Q-Wiener process approximation For computational purposes, we must deal with a truncated form of the series (4.3). We consider two independent $H_0^1(D)$ -valued Wiener processes W_1 and W_2 such that $W = (W_1, W_2)$. For $J \geq 1$, the used increments are expressed by $\Delta_m W_\ell \approx k^{1/2} \sum_{i,j=1}^J (\lambda_{i,j}^\ell)^{1/2} \xi_{i,j}^{\ell,m} e_{i,j}$, $\ell \in \{1, 2\}$, where J is set to 10 for the simulations down below, and for all $i, j \in \mathbb{N}$ and $(x, y) \in D$, the basis elements $e_{i,j} := 2 \sin(i\pi x) \sin(j\pi y)$ represent the Laplace eigenfunctions with Dirichlet boundary conditions on D . For $\ell \in \{1, 2\}$, $\{\{\xi_{i,j}^{\ell,m}\}_{i,j}\}_m$ is a family of independent identically distributed standard normal random variables, and $\lambda_{i,j}^\ell := \frac{1}{(i+j)^2}$ for $\ell \in \{1, 2\}$. Note that the coefficients $\lambda_{i,j}$ could have been set to 1 if the Wiener process is desired to be cylindrical.

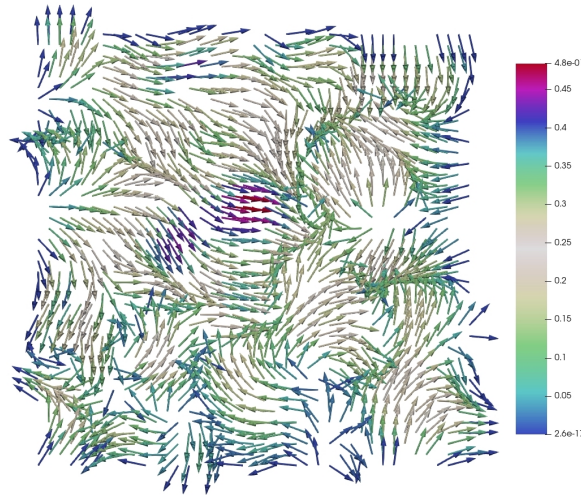


Figure 4.1 – Wiener process $W = (W_1, W_2)$

Consider the spatial scale $\alpha = 0.001h$, the space discretization step size $h \approx 0.03$ and the time discretization step size $k = 0.01$.

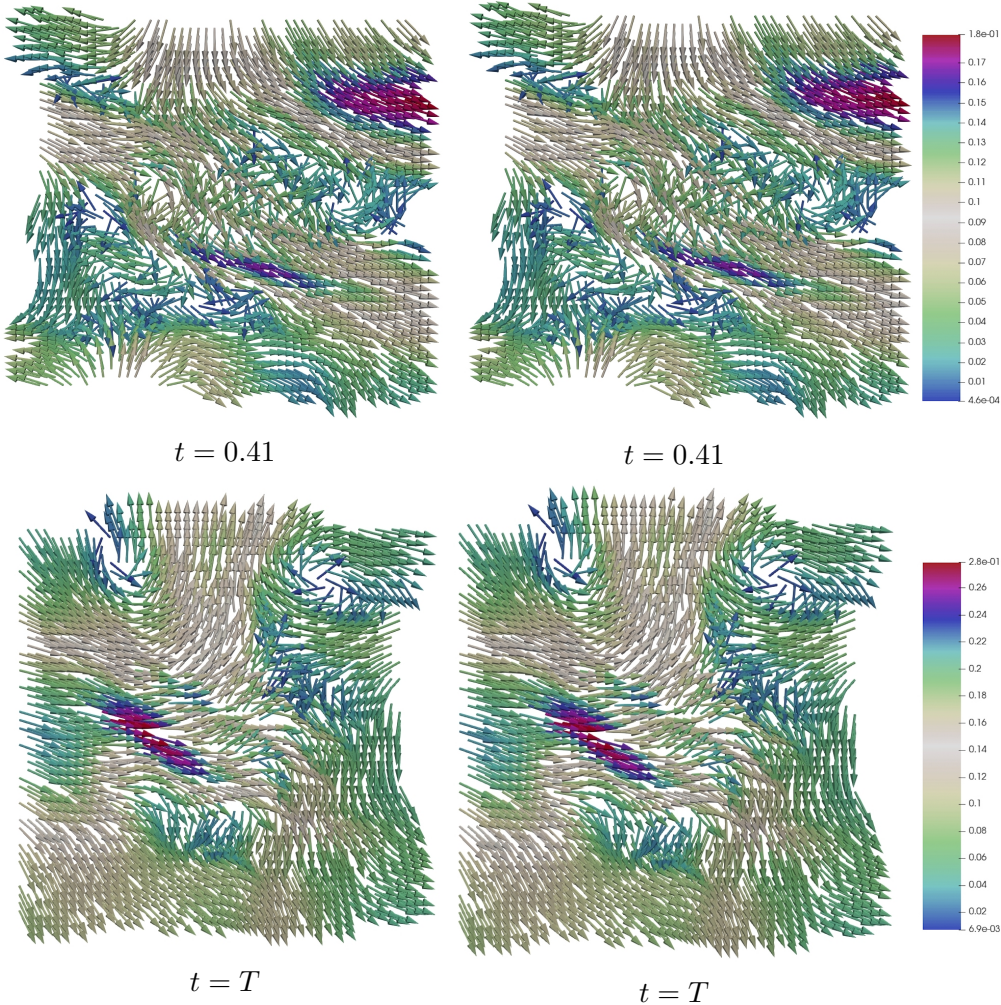


Figure 4.2 – Velocity fields of LANS- α (left) and NS (right) at time $t \in \{0.41, 1\}$

Since both equations (4.1) and (4.2) are related in this chapter, we choose two different time values in $[0, T]$, and plot the associated figures side by side. This allows us to compare the solutions' behavior together with the occurring differences. Observe that both LANS- α and NS solutions behave similarly with a tiny variation in values (observable via high resolution monitors). Such a difference was expected since we are dealing here with approximate computations, not to mention the considered space discretization's step h which is not too close to 0, yet its code execution is costly. We also provide the following pressure figures which are barely distinguishable.

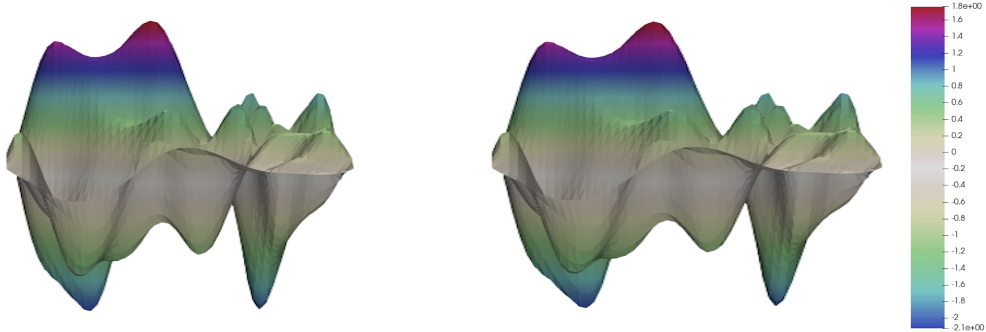


Figure 4.3 – Pressure fields of LANS- α (left) and NS (right) at time $t = 0.41$

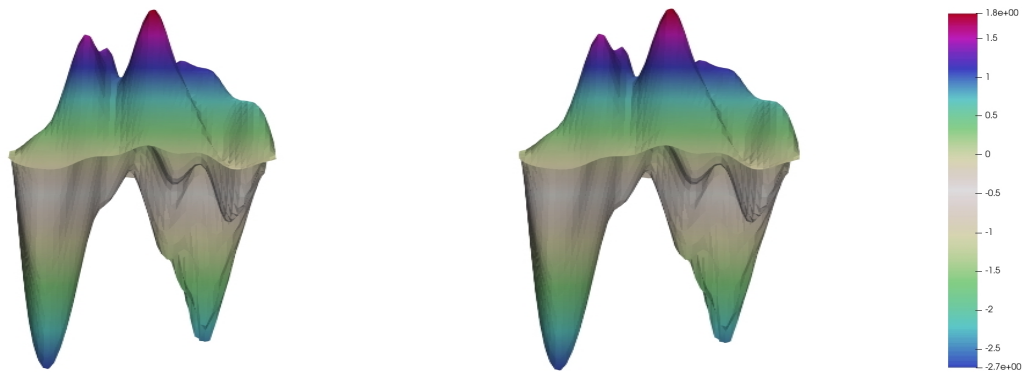


Figure 4.4 – Pressure fields of LANS- α (left) and NS (right) at time $t = T$

In conclusion, as long as the spatial scale α is kept to a minimum, the stochastic Lagrangian averaged Navier-Stokes model is a reliable technique to approximate solutions to the stochastic Navier-Stokes equations in two dimensions. Although Algorithm 4.1 is not investigated for three-dimensional domains in this chapter, it might also establish an authentic numerical scheme for the stochastic Navier-Stokes problem. The followed demonstration technique, however, must be adjusted so that a weak (martingale) solution is obtained at the limit to be coherent with the solutions' sense in three dimensions.

5 - The stochastic LANS- α model: characteristics, numerical and convergence analysis

5.1 . Introduction

The foremost target of this chapter is the numerical analysis of the three-dimensional stochastic Lagrangian averaged Navier-Stokes (LANS- α) equations with a fixed spatial scale α . In other words, the quantity α will be independent of all discretization parameters, so its value is maintained when passing to the limit. These equations are investigated across a finite time interval $[0, T]$, $T > 0$, in a bounded convex polyhedral domain, denoted $D \subset \mathbb{R}^3$, within non-periodic (or periodic) boundary conditions, and are given by the following system:

$$\begin{cases} \partial_t \left(\bar{u} - \alpha^2 \Delta \bar{u} \right) - \nu \Delta \left(\bar{u} - \alpha^2 \Delta \bar{u} \right) + \bar{u} \times \left(\nabla \times \left(\bar{u} - \alpha^2 \Delta \bar{u} \right) \right) - \nabla \bar{p} \\ \quad = f(\cdot, \bar{u}) + g(\cdot, \bar{u}) \frac{\partial W}{\partial t}, & \text{in } [0, T] \times D, \\ \operatorname{div}(\bar{u}) = 0, & \text{in } [0, T] \times D, \\ \bar{u}(0, \cdot) = \bar{u}_0, & \text{in } D. \end{cases} \quad (5.1)$$

The unknowns are the \mathbb{R}^3 -valued fluid velocity vector \bar{u} and the pressure field $\bar{p} \in \mathbb{R}$. The positive constant ν is the kinematic fluid viscosity, f is an outer force depending on time, randomness, and \bar{u} , g is a diffusion coefficient that gives the endowed noise a multiplicative aspect, W is an infinite-dimensional (possibly cylindrical) Wiener process, and \bar{u}_0 is an initial condition of the fluid.

The followed approach in this chapter relies on finite element methods to approximate the space variables. Weakly divergence-free basis functions will be required to ensure the discrete pressure's existence, meaning that a discrete LBB condition will be exploited later in this chapter. Although this constraint could have been tackled through a penalty method, which is treated for the stochastic Navier-Stokes equations in Chapter 6, its properties remain a reliable paradigm to which one can refer for further improvements. Two possible finite element compositions of system (5.1) can be derived: a C^1 piecewise polynomials-based finite element method and a continuous piecewise polynomials-based version. The former technique deals with the genuine form of the underlying equations i.e. as a fourth-order problem and requires more regularities regarding the approximate functions. The latter strategy, which will be privileged here due to its better reputation, transforms identity (5.1)₁ into two equations; that is, a second-order problem pops out, as carried out in Chapter 4. Concerning the finite element method whose basis functions are C^1 piecewise polynomials, a particular section will be dedicated to it for the sake of providing the needed configurations together with an associated algorithm whose implementation might be challenging. Despite the latter drawback, the provided supplementary smoothness ensures finer stability features for numerical schemes to the Navier-Stokes equations, and turns out to be more efficient regarding the convergence rate.

One good feature of equations (5.1) is the existence of a three-dimensional, unique, strong solution in the probabilistic sense [29]. This results in conserving the originally introduced probability space and Wiener process W , thanks to the high spatial regularity which emerges from the biharmonic operator. That being said, any compactness argument arising from the Skorokhod theorem will be obsolete in this chapter and will be replaced by a special technique relying on identifications of limiting functions of an obtained equation through weakly and weakly-* convergent sequences and involving an auxiliary numerical scheme that approximates a Stokes problem driven by an additive noise. This is on account of the considered finite element method whose basis functions are not strongly divergence-free; notably, they

do not allow mixing the proposed numerical scheme with the continuous form of the weak formulation.

The stochastic LANS- α model is meaningless if the spatial scale α is chosen carelessly because, as well-known, it represents a closure model to the stochastic Navier-Stokes equations. When α takes on high values, system (5.1) becomes irrelevant, amplifies the pressure's amplitude considerably, and may no longer model a given fluid. It gets more visible in the a priori estimates of the velocity \bar{u} where α^2 is always multiplied by $\|\Delta\bar{u}\|_{L^2}$. In addition to that, dividing by α^2 and letting α go to 0 reflect a lousy behavior of the equations of interest. As we will find out later on, lowering the values of α keeps the velocity and pressure outcomes tremendously comparable to that of the Navier-Stokes problem in both two and three dimensions.

This chapter is sorted out in the following way. Section 5.2 contains notations, preliminaries, assumptions, and mathematical framework for the LANS- α model. Section 5.3 is devoted to providing theoretical properties of the non-discretized solutions of equations (5.1), including the solution's concept when the underlying system is broken down into a coupled problem. Section 5.4 sets the necessary tools to discretize both space and time domains, and provides associated properties. The primary theorem of this chapter is given in Section 5.5, followed by a preparatory analysis part (Section 5.6) where a priori estimates, solvability and uniqueness of iterates are inspected. The proposed (nonlinear) numerical scheme's convergence is conducted within Section 5.7. On account of some uniqueness shortcomings of the investigated nonlinear numerical scheme, an associated linear version is provided in Section 5.8, and its convergence is examined. A continuously differentiable finite element approximation of equations (5.1) is proposed in Section 5.9 whose implementation is left to the reader. We close this chapter with some numerical experiments corresponding to multiple choices of α together with a conclusion (Section 5.10).

5.2 . Mathematical framework and preliminaries

We recall that this chapter deals with a three-dimensional bounded convex domain $D \subset \mathbb{R}^3$ whose boundary is denoted ∂D , and a finite time interval $[0, T]$ such that $T > 0$. We denote by $\vec{n}: \partial D \rightarrow \mathbb{R}^3$ the unit outward normal vector field to ∂D , by $L^2(D)$ the Lebesgue space, and by $H^m(D)$, $m \in \mathbb{N} \setminus \{0\}$ the Sobolev space of order m . All this chapter long, the following function spaces will be adapted:

$$\begin{aligned} \mathbb{H} &:= \left\{ v \in [L^2(D)]^3 \mid \operatorname{div}(v) = 0 \text{ a.e. in } D, \text{ and } v \cdot \vec{n} = 0 \text{ a.e. on } \partial D \right\}, \\ \mathbb{V} &:= \left\{ v \in [H_0^1(D)]^3 \mid \operatorname{div}(v) = 0 \text{ a.e. in } D \right\}, \\ D(A) &:= [H^2(D)]^3 \cap \mathbb{V}, \end{aligned}$$

where $A := -\mathcal{P}\Delta: D(A) \rightarrow \mathbb{H}$ is the Stokes operator, and $\mathcal{P}: [L^2(D)]^3 \rightarrow \mathbb{H}$ is the Leray projector. Operator A is auto-adjoint, invertible, has inverse compact, and grants a complete orthonormal basis to $D(A)$ (see for instance [96]). From now on, we shall denote by blackboard bold letters all \mathbb{R}^3 -valued function spaces, such as \mathbb{L}^2 which stands for $[L^2(D)]^3$.

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P})$ be a filtered probability space satisfying the "usual conditions" [80]; namely,

- i. all the \mathbb{P} -null sets are contained in \mathcal{F}_0 ,
- ii. $\mathcal{F}_t = \bigcap_{s>t} \mathcal{F}_s$.

Given a separable Hilbert space K , let $\{W(t), t \in [0, T]\}$ be a K -valued cylindrical Wiener process defined by

$$W(t) = \sum_{k \geq 1} \beta^k(t) w_k, \quad \forall t \in [0, T], \quad (5.2)$$

where $\{\beta^k(\cdot), k \in \mathbb{N} \setminus \{0\}\}$ is a sequence of independent identically distributed \mathbb{R} -valued Brownian motions on the probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P})$, and $\{w_k, k \geq 1\}$ is a complete orthonormal basis of the Hilbert space K . The Wiener process W can also be of trace class, meaning that its covariance has finite trace. However, we keep in this chapter the cylindrical settings to indicate a way to generalize the studies in Chapters 4 and 6.

Given a normed vector space X , its norm will be denoted $\|\cdot\|_X$ throughout this chapter. An additional norm that will come into view is

$$\|\cdot\|_\alpha^2 := \|\cdot\|_{\mathbb{L}^2}^2 + \alpha^2 \|\nabla \cdot\|_{\mathbb{L}^2}^2.$$

Depending on the range values of α , three equivalences with the H^1 -norm can be provided:

- i. if $\alpha < 1$, then $\alpha \|\cdot\|_{\mathbb{H}^1} < \|\cdot\|_\alpha < \|\cdot\|_{\mathbb{H}^1}$,
- ii. if $\alpha > 1$, then $\|\cdot\|_{\mathbb{H}^1} < \|\cdot\|_\alpha < \alpha \|\cdot\|_{\mathbb{H}^1}$,
- iii. if $\alpha = 1$, then $\|\cdot\|_\alpha = \|\cdot\|_{\mathbb{H}^1}$.

The cases $\alpha = 1$ and $\alpha > 1$ will not be favored here because, as mentioned before, α should be considered as small as possible.

The nonlinear term $u \times (\nabla \times v)$ that figures in equations (5.1) has the following features.

Proposition 5.2.1

- (i) $(u \times (\nabla \times v), w) = -(w \times (\nabla \times v), u)$ for all $u, v, w \in \mathbb{H}^1$. In particular, $(u \times (\nabla \times v), u) = 0$.
- (ii) $|(u \times (\nabla \times v), w)| \leq C_D \|u\|_{\mathbb{L}^2}^{1/2} \|\nabla u\|_{\mathbb{L}^2}^{1/2} \|\nabla v\|_{\mathbb{L}^2} \|\nabla w\|_{\mathbb{L}^2}$, for all $u, v, w \in \mathbb{H}_0^1$ (Only in 3D).
- (iii) $\|u \times (\nabla \times v)\|_{\mathbb{H}^{-1}} \leq C_D \|u\|_{\mathbb{H}^1}^{1/2} \|u\|_{\mathbb{H}^2}^{1/2} \|v\|_{\mathbb{L}^2}$, for all $v \in \mathbb{L}^2$ and $u \in \mathbb{H}^2 \cap \mathbb{H}_0^1$ (Only in 3D).

Proof: Demonstration of identity (i) is covered in Proposition 4.2.1 of Chapter 4. To justify assertion (ii), we make use of $(u \times (\nabla \times v), w) = ([w \cdot \nabla]v, u) - ([u \cdot \nabla]v, w) =: I - J$ which is illustrated in Proposition 4.2.1. Indeed,

$$|(u \times (\nabla \times v), w)| \leq 2\|u\|_{\mathbb{L}^3} \|\nabla v\|_{\mathbb{L}^2} \|w\|_{\mathbb{L}^6} \leq C_D \|u\|_{\mathbb{L}^2}^{1/2} \|\nabla u\|_{\mathbb{L}^2}^{1/2} \|\nabla v\|_{\mathbb{L}^2} \|\nabla w\|_{\mathbb{L}^2},$$

thanks to the Hölder and Poincaré inequalities, the embedding $\mathbb{H}^1 \hookrightarrow \mathbb{L}^6$, and the three-dimensional Gagliardo-Nirenberg inequality $\|a\|_{\mathbb{L}^3} \leq C_D \|a\|_{\mathbb{L}^2}^{1/2} \|\nabla a\|_{\mathbb{L}^2}^{1/2}$, for all $a \in \mathbb{H}^1$. To prove (iii), we shall start from the same equality that was employed to figure out estimate (ii). Indeed, performing an integration by parts on I yields $I = -([w \cdot \nabla]u, v) - ([div(w)]u, v)$. Therefore, by the Hölder, Agmon, and Gagliardo-Nirenberg inequalities in \mathbb{R}^3 , we get

$$|I| \leq \|w\|_{\mathbb{L}^6} \|\nabla u\|_{\mathbb{L}^3} \|v\|_{\mathbb{L}^2} + \|u\|_{\mathbb{L}^\infty} \|div(w)\|_{\mathbb{L}^2} \|v\|_{\mathbb{L}^2} \leq C_D \|u\|_{\mathbb{H}^1}^{1/2} \|u\|_{\mathbb{H}^2}^{1/2} \|v\|_{\mathbb{L}^2}.$$

Integrating J by parts, it follows $J = -([u \cdot \nabla]w, v) - ([div(u)]v, w)$. Applying once again the same inequalities that were employed to illustrate the bound of I , we obtain

$$|J| \leq \|u\|_{\mathbb{L}^\infty} \|\nabla w\|_{\mathbb{L}^2} \|v\|_{\mathbb{L}^2} + \|v\|_{\mathbb{L}^2} \|div(u)\|_{\mathbb{L}^3} \|w\|_{\mathbb{L}^6} \leq C_D \|u\|_{\mathbb{H}^1}^{1/2} \|u\|_{\mathbb{H}^2}^{1/2} \|v\|_{\mathbb{L}^2}.$$

Consequently, $|(u \times (\nabla \times v), w)| \leq 2C_D \|u\|_{\mathbb{H}^1}^{1/2} \|u\|_{\mathbb{H}^2}^{1/2} \|v\|_{\mathbb{L}^2}$. Throughout this proof, functions u, v , and w were assumed smooth enough. To achieve inequalities (ii) and (iii) within the required regularities, we argue by density. \square

The following properties will be employed frequently in the sequel:

$$(a - b, a) = \frac{1}{2} \left(\|a\|_{\mathbb{L}^2}^2 - \|b\|_{\mathbb{L}^2}^2 + \|a - b\|_{\mathbb{L}^2}^2 \right) \quad \text{for all } a, b \in \mathbb{L}^2, \quad (5.3)$$

$$x^p \leq 1 + x^q, \text{ for all } x \geq 0 \text{ and } 1 \leq p \leq q < +\infty, \quad (5.4)$$

$$|a + b|^p \leq 2^{p-1} (|a|^p + |b|^p), \text{ for all } a, b \in \mathbb{R} \text{ and } p \geq 1. \quad (5.5)$$

Hereafter, for given Banach spaces E and F , the notations $M_{\mathcal{F}_t}^p(0, T; E)$, $\mathcal{L}_2(E, F)$, (\cdot, \cdot) , $\langle \cdot, \cdot \rangle$, \lesssim will denote the space of all \mathcal{F}_t -progressively measurable processes in $L^p(\Omega \times (0, T), d\mathbb{P} \times dt, E)$, the space of all Hilbert-Schmidt operators from E to F , the $L^2(D)$ -scalar product, the duality product between $H_0^1(D)$ and its dual space $H^{-1}(D)$, and a shorthand for \leq up to a non-null universal constant.

Assumptions

(S₁) $\bar{u}_0 \in L^{2p}(\Omega, \mathcal{F}_0, \mathbb{P}; \mathbb{V})$, for some $p \in [1, +\infty]$.

(S₂) $f \in L^2(\Omega; C([0, T]; \mathbb{H}^{-1}))$ and $g \in L^2(\Omega; C([0, T]; \mathcal{L}_2(K, \mathbb{L}^2)))$ are \mathcal{F}_t -progressively measurable and satisfy $d\mathbb{P} \otimes dt$ -a.e. in $\Omega \times (0, T)$:

$$\begin{aligned} \|f(\cdot, u) - f(\cdot, v)\|_{\mathbb{H}^{-1}} &\leq L_f \|u - v\|_\alpha, \quad \forall u, v \in \mathbb{V}, \\ \|g(\cdot, u) - g(\cdot, v)\|_{\mathcal{L}_2(K, \mathbb{L}^2)} &\leq L_g \|u - v\|_\alpha, \quad \forall u, v \in \mathbb{V}, \\ \|g(\cdot, u)\|_{\mathcal{L}_2(K, \mathbb{L}^2)} &\leq K_1 + K_2 \|u\|_\alpha, \quad \forall u \in \mathbb{V}, \\ \|f(\cdot, u)\|_{\mathbb{H}^{-1}} &\leq K_3 + K_4 \|u\|_\alpha, \quad \forall u \in \mathbb{V}, \end{aligned}$$

for some time-independent nonnegative constants $K_1, K_2, K_3, K_4, L_f, L_g$.

f and g could have been considered square-integrable with respect to time instead of continuous by employing specific time-approximation methods, see Remark 4.2.1.

5.3 . Theoretical aspects

For simplicity's sake, the external force f is set to 0 throughout this section.

System 5.1 was only investigated in the literature in its fourth-order form. This section will, therefore, be concerned with its theoretical properties when it is turned into a second-order problem. We begin first by introducing necessary tools such as the continuous differential filter whose selection is fundamental.

Definition 5.3.1 (Continuous differential filter)

Let v be an \mathbb{L}^2 -valued given vector field. Its continuous differential filter \bar{u} is part of the unique solution $(\bar{u}, \bar{p}) \in \mathbb{V} \times L_0^2(D)$ of the following steady Stokes problem:

$$\begin{cases} \bar{u} - \alpha^2 \Delta \bar{u} + \nabla \bar{p} = v, & \text{in } D, \\ \operatorname{div}(\bar{u}) = 0, & \text{in } D, \\ \bar{u} = 0, & \text{on } \partial D. \end{cases} \quad (5.6)$$

System (5.6) is seen as a Stokes problem because one can assemble \bar{u} and v ; namely $v - \bar{u}$ at the right-hand side, to obtain eventually an external force that depends on the solution \bar{u} . The proposed continuous differential filter, which represents a variant of the one introduced in [62], is not the only option that exists in the literature. For instance, the Gaussian filter is one of the unanimous alternatives where its application to the steady Navier-Stokes equations and the Stokes problem was reviewed in [53, 79]. For Large Eddy Simulations (LES), regularly employed differential filters are inspected in [3, 100]. The variational formulation of system (5.6) is a foremost objective because it will participate in building the weak formulation of equations (5.1), and it is given by

$$\begin{cases} \text{Find a solution } \bar{u} \in \mathbb{V} \text{ such that} \\ (\bar{u}, \varphi) + \alpha^2 (\nabla \bar{u}, \nabla \varphi) = (v, \varphi), \text{ for all } \varphi \in \mathbb{V}. \end{cases} \quad (5.7)$$

Since α is fixed, and due to the domain's property, the membership of v in \mathbb{L}^2 implies that $\bar{u} \in D(A)$. That is, $\mathbb{E} \left[\|\bar{u}\|_{D(A)}^{2p} \right] \leq C_D \alpha^{-2} \mathbb{E} \left[\|v\|_{\mathbb{L}^2}^{2p} \right]$, thanks to the Agmon-Douglis-Nirenberg theorem (c.f. [22, Theorem 9.32]). Given that the differential operator $I + \alpha^2 A: D(A) \rightarrow \mathbb{H}$ is bijective, equation (5.7) becomes equivalent to $(I + \alpha^2 A)^{-1} v = \bar{u}$. Furthermore, we define v_0 by $v_0 = (I + \alpha^2 A) \bar{u}_0$, which belongs to \mathbb{H} , thanks to assumption (S_1) and the aforementioned inequality.

5.3.1 . Concept of solutions

In the light of the precedent analysis together with Chapters 3 and 4, a solution to equations (5.1) is defined as follows.

Definition 5.3.2 *Let $T > 0$ be a final time and assume (S_1) - (S_2) . A $\mathbb{V} \times \mathbb{H}$ -valued stochastic process $\{(\bar{u}(t), v(t)), t \in [0, T]\}$ is said to be a variational solution to problem (5.1) if it fulfills the following conditions:*

- (i) $\bar{u} \in L^2(\Omega; L^2(0, T; D(A)) \cap L^2(\Omega; L^\infty(0, T; \mathbb{V}))$,
- (ii) $v \in L^2(\Omega; L^2(0, T; \mathbb{V})) \cap L^2(\Omega; L^\infty(0, T; \mathbb{H}))$,
- (iii) \mathbb{P} -almost surely, \bar{u} is weakly continuous with values in \mathbb{V} , and v is continuous with values in \mathbb{H} ,
- (iv) for all $t \in [0, T]$, \bar{u} satisfies the following equation \mathbb{P} -almost surely

$$\begin{cases} (v(t), \varphi) + \nu \int_0^t (\nabla v(s), \nabla \varphi) ds - \int_0^t (\bar{u}(s) \times (\nabla \times v(s)), \varphi) ds \\ \quad = (v_0, \varphi) + \left(\int_0^t g(s, \bar{u}(s)) dW(s), \varphi \right), \quad \forall \varphi \in \mathbb{V}, \\ (v(t), \psi) = (\bar{u}(t), \psi) + \alpha^2 (\nabla \bar{u}(t), \nabla \psi), \quad \forall \psi \in \mathbb{V}. \end{cases} \quad (5.8)$$

The weak formulation (5.8) is derived from the fourth-order problem that was examined in [29]. This derivation took place in Section 4.2.1 of Chapter 4. The weak continuity of \bar{u} is related to the strong continuity of v and equation (5.8)₂. Indeed, when projecting the above-proposed formulation onto finite-dimensional spaces (for example, through the Faedo-Galerkin method), continuity of v is achieved first when passing to the limit. Then, under identity (5.8)₂, the weak continuity of \bar{u} follows.

The next lemma states relationships between the norms of v and \bar{u} .

Lemma 5.3.1 *Let $\{(\bar{u}(t), v(t)), t \in [0, T]\}$ be a solution to equations (5.1) in the sense of Definition 5.3.2. Then, $\{\bar{u}(t), t \in [0, T]\}$ is a continuous differential filter to $\{v(t), t \in [0, T]\}$ in the sense of Definition 5.3.1, and the following inequalities hold for all $t \in [0, T]$:*

- (i) $\|\bar{u}(t)\|_\alpha \leq \|v(t)\|_{\mathbb{L}^2}$,
- (ii) $\|v(t)\|_{\mathbb{L}^2} \leq \|\bar{u}(t)\|_{\mathbb{L}^2} + \alpha^2 \|A\bar{u}(t)\|_{\mathbb{L}^2}$,
- (iii) $\|\nabla \bar{u}(t)\|_\alpha \leq \frac{1}{\alpha} \|v(t)\|_{\mathbb{L}^2}$,
- (iv) $\|A\bar{u}(t)\|_\alpha \leq \frac{1}{\alpha} \|\nabla v(t)\|_{\mathbb{L}^2}$.

Proof: From equation (5.8)₂, it is straightforward that \bar{u} is the continuous differential filter of v . In addition, setting $\psi = \bar{u}(t)$ in equation (5.8)₂ yields $\|\bar{u}(t)\|_\alpha^2 = (v(t), \bar{u}(t)) \leq \|v(t)\|_{\mathbb{L}^2} \|\bar{u}(t)\|_\alpha$, which implies assertion (i) after simplifying by $\|\bar{u}(t)\|_\alpha$. Next, we replace ψ with $v(t)$ to achieve $\|v(t)\|_{\mathbb{L}^2}^2 \leq \|\bar{u}(t)\|_{\mathbb{L}^2} \|v(t)\|_{\mathbb{L}^2} + \alpha^2 \|A\bar{u}(t)\|_{\mathbb{L}^2} \|v(t)\|_{\mathbb{L}^2}$. Thereby, simplifying by $\|v(t)\|_{\mathbb{L}^2}$ terminates the demonstration of inequality (ii). Similarly, substituting ψ by $A\bar{u}(t)$ returns

$$\|\nabla \bar{u}(t)\|_{\mathbb{L}^2}^2 + \alpha^2 \|A\bar{u}(t)\|_{\mathbb{L}^2}^2 = (v(t), A\bar{u}(t)) \leq \|v(t)\|_{\mathbb{L}^2} \|A\bar{u}(t)\|_{\mathbb{L}^2} \leq \frac{1}{2\alpha^2} \|v(t)\|_{\mathbb{L}^2}^2 + \frac{\alpha^2}{2} \|A\bar{u}(t)\|_{\mathbb{L}^2}^2,$$

thanks to the Cauchy-Schwarz and Young inequalities. Subsequently, $\frac{1}{2}\|\nabla\bar{u}(t)\|_\alpha^2 \leq \frac{1}{2\alpha^2}\|v(t)\|_{\mathbb{L}^2}^2$, and estimate (iii) follows. Finally, letting $\psi = A^2\bar{u}(t)$ gives

$$\|A\bar{u}(t)\|_{\mathbb{L}^2}^2 + \alpha^2\|A^{3/2}\bar{u}(t)\|_{\mathbb{L}^2}^2 = (A^{1/2}v(t), A^{3/2}\bar{u}(t)) \leq \frac{1}{2\alpha^2}\|\nabla v(t)\|_{\mathbb{L}^2}^2 + \frac{\alpha^2}{2}\|A^{3/2}\bar{u}(t)\|_{\mathbb{L}^2}^2.$$

Hence, $\|A\bar{u}(t)\|_\alpha \leq \frac{1}{\alpha}\|\nabla v(t)\|_{\mathbb{L}^2}$. \square

Remark 5.3.1 *The analysis in Lemma 5.3.1 inspects the function that grants better properties to equations (5.1). That is, when the space-regularity becomes higher, a $1/\alpha$ factor appears in the upper bound in terms of v regarding the process \bar{u} . This signifies that one must concentrate on the regularity of v if no $1/\alpha$ factors were desired in the study. In point of fact, considering v as an \mathbb{H}^1 -valued process instead of \mathbb{L}^2 eliminates the $1/\alpha$ factor of estimates (iii) and (iv).*

5.3.2 . A priori estimates

Due to the newly proposed formulation, additional a priori estimates will come into light. We exploit the next lemma to exhibit typical and high space-regularity estimates.

Lemma 5.3.2 *Let $T > 0$, $\alpha < 1$, $p \in [1, +\infty)$, and (S_1) - (S_2) be satisfied. If $\{(\bar{u}(t), v(t)), \in [0, T]\}$ is a solution to equations (5.1) in the sense of Definition 5.3.2, then the following estimates hold true:*

$$\begin{aligned} (i) \quad & \mathbb{E} \left[\sup_{0 \leq t \leq T} \|\bar{u}(t)\|_\alpha^{2p} + \left(\nu \int_0^T \|\nabla\bar{u}(t)\|_\alpha^2 dt \right)^{2p-1} \right] \leq C_1, \\ (ii) \quad & \mathbb{E} \left[\sup_{0 \leq t \leq T} \|v(t)\|_{\mathbb{L}^2}^{2p} + \left(\frac{\nu}{2} \int_0^T \|\nabla v(t)\|_{\mathbb{L}^2}^2 dt \right)^{2p-1} \right] \leq C_2(\alpha, \nu), \\ (iii) \quad & \mathbb{E} \left[\sup_{0 \leq t \leq T} \|\nabla\bar{u}(t)\|_\alpha^{2p} + \left(\frac{\nu}{2} \int_0^T \|A\bar{u}(t)\|_\alpha^2 dt \right)^{2p-1} \right] \leq \frac{C_2(\alpha, \nu)}{\alpha^{2p}}, \end{aligned}$$

where $C_1 > 0$ depends on K_1, K_2, T, p, D , and $\|\bar{u}_0\|_{L^{2p}(\Omega; \mathbb{V})}$, and $C_2(\alpha, \nu) > 0$ depends on $D, C_1, K_1, K_2, p, T, \alpha, \|v_0\|_{L^{2p+2}(\Omega; \mathbb{L}^2)}$, and ν .

Proof: The demonstration technique that will be followed here may not be thorough. A more flawless way can be through the Faedo-Galerkin method, which provides the projected quantities with smoothness properties permitting the equation of interest to be well-posed. To illustrate estimate (i), we define the process $F_1(v(t)) := \|(I + \alpha^2 A)^{-1/2}v(t)\|_{\mathbb{L}^2}^2 = \|\bar{u}(t)\|_\alpha^2$. The Fréchet derivatives of F_1 are given by

$$\begin{aligned} DF_1(v) &= 2(I + \alpha^2 A)^{-1}v = 2\bar{u}, \\ D^2F_1(v) &= 2(I + \alpha^2 A)^{-1}. \end{aligned}$$

Next, we apply Itô's formula to F_1 to obtain for all $t \in [0, T]$:

$$\begin{aligned} \|\bar{u}(t)\|_\alpha^2 &= \|\bar{u}_0\|_\alpha^2 + 2 \int_0^t (\bar{u}(s), g(s, \bar{u}(s))dW(s)) \\ &+ 2 \int_0^t \langle -\nu Av(s) + \bar{u}(s) \times (\nabla \times v(s)), \bar{u}(s) \rangle ds + \int_0^t \|(I + \alpha^2 A)^{-\frac{1}{2}}g(s, \bar{u}(s))\|_{\mathcal{L}_2(K, \mathbb{L}^2)}^2 ds. \end{aligned} \quad (5.9)$$

We recall that v can be replaced with $(I + \alpha^2 A)\bar{u}$, and Proposition 5.2.1-(i) cancels out the nonlinear term; namely $\langle \bar{u}(s) \times (\nabla \times v(s)), \bar{u}(s) \rangle = 0$. Therefore,

$$2 \int_0^t \langle -\nu Av(s) + \bar{u}(s) \times (\nabla \times v(s)), \bar{u}(s) \rangle ds = -2\nu \int_0^t \|\nabla\bar{u}(s)\|_\alpha^2 ds.$$

In addition, it is straightforward that $\|(I + \alpha^2 A)^{-1/2} g(s, \bar{u}(s))\|_{\mathcal{L}_2(K, \mathbb{L}^2)} \leq \|g(s, \bar{u}(s))\|_{\mathcal{L}_2(K, \mathbb{L}^2)}$. Thus, raising equation (5.9) to the power 2^{p-1} , using inequality $(|a| + |b|)^p \leq 2^{p-1}(|a|^p + |b|^p)$, and taking the supremum over $t \in [0, T]$ yield:

$$\begin{aligned} & \sup_{0 \leq t \leq T} \|\bar{u}(t)\|_{\alpha}^{2p} + \left(\nu \int_0^T \|\nabla \bar{u}(t)\|_{\alpha}^2 dt \right)^{2^{p-1}} \lesssim \|\bar{u}_0\|_{\alpha}^{2p} \\ & + \sup_{0 \leq t \leq T} \left(\int_0^t (\bar{u}(s), g(s, \bar{u}(s))) dW(s) \right)^{2^{p-1}} + \left(\int_0^T \|g(t, \bar{u}(t))\|_{\mathcal{L}_2(K, \mathbb{L}^2)}^2 dt \right)^{2^{p-1}}. \end{aligned} \quad (5.10)$$

It remains to take the mathematical expectation of the above equation and to estimate the last two terms on the right-hand side. To this end, we have:

$$\begin{aligned} & \mathbb{E} \left[\sup_{0 \leq t \leq T} \left(\int_0^t (\bar{u}(s), g(s, \bar{u}(s))) dW(s) \right)^{2^{p-1}} \right] \lesssim \mathbb{E} \left[\left(\int_0^T \|\bar{u}(t)\|_{\mathbb{L}^2}^2 \|g(t, \bar{u}(t))\|_{\mathcal{L}_2(K, \mathbb{L}^2)}^2 dt \right)^{2^{p-2}} \right] \\ & \lesssim \mathbb{E} \left[\int_0^T \|\bar{u}(t)\|_{\mathbb{L}^2}^{2^{p-1}} (K_1 + K_2 \|\bar{u}(t)\|_{\alpha})^{2^{p-1}} dt \right] \lesssim \mathbb{E} \left[\int_0^T \|\bar{u}(t)\|_{\alpha}^{2^{p-1}} (K_1^{2^{p-1}} + K_2^{2^{p-1}} \|\bar{u}(t)\|_{\alpha}^{2^{p-1}}) dt \right] \\ & \lesssim K_1^{2^{p-1}} T + (K_1^{2^{p-1}} + K_2^{2^{p-1}}) \int_0^T \mathbb{E} \left[\sup_{0 \leq s \leq t} \|\bar{u}(s)\|_{\alpha}^{2p} \right] dt. \end{aligned}$$

thanks to the Burkholder-Davis-Gundy inequality, the Jensen inequality, assumption (S_2) , estimates (5.4) and (5.5). On the other hand,

$$\begin{aligned} & \mathbb{E} \left[\left(\int_0^T \|g(t, \bar{u}(t))\|_{\mathcal{L}_2(K, \mathbb{L}^2)}^2 dt \right)^{2^{p-1}} \right] \lesssim \mathbb{E} \left[\int_0^T (K_1^{2p} + K_2^{2p} \|\bar{u}(t)\|_{\alpha}^{2p}) dt \right] \\ & \leq K_1^{2p} T + K_2^{2p} \int_0^T \mathbb{E} \left[\sup_{0 \leq s \leq t} \|\bar{u}(s)\|_{\alpha}^{2p} \right] dt \end{aligned}$$

where Jensen's inequality, estimate (5.5), and assumption (S_2) were utilized. Consequently, putting it all together and using the Grönwall inequality imply

$$\begin{aligned} & \mathbb{E} \left[\sup_{0 \leq t \leq T} \|\bar{u}(t)\|_{\alpha}^{2p} + \left(\nu \int_0^T \|\nabla \bar{u}(t)\|_{\alpha}^2 dt \right)^{2^{p-1}} \right] \\ & \lesssim \left\{ \mathbb{E} \left[\|\bar{u}_0\|_{\alpha}^{2p} \right] + (K_1^{2^{p-1}} + K_1^{2p}) T \right\} \exp \left(T (K_1^{2^{p-1}} + K_2^{2^{p-1}} + K_2^{2p}) \right). \end{aligned}$$

Next, we prove assertion (ii). To this purpose, let $F_2(v(t)) := \|v(t)\|_{\mathbb{L}^2}^2$. The Fréchet derivatives of F_2 are the following

$$DF_2(v(t)) = 2v(t) \quad \text{and} \quad D^2 F_2(v(t)) = 2Id,$$

where Id is the identity operator. Thus, we apply Itô's formula to the process $F_2(v(t))$ to accomplish for all $t \in [0, T]$:

$$\begin{aligned} & \|v(t)\|_{\mathbb{L}^2}^2 + 2\nu \int_0^t \|\nabla v(s)\|_{\mathbb{L}^2}^2 ds = \|v_0\|_{\mathbb{L}^2}^2 + 2 \int_0^t (v(s), g(s, \bar{u}(s))) dW(s) \\ & + 2 \int_0^t (\bar{u}(s) \times (\nabla \times v(s)), v(s)) ds + \int_0^t \|g(s, \bar{u}(s))\|_{\mathcal{L}_2(K, \mathbb{L}^2)}^2 ds. \end{aligned} \quad (5.11)$$

Taking into account that $v(t) = \bar{u}(t) + \alpha^2 A \bar{u}(t)$ \mathbb{P} -a.s. together with Proposition 5.2.1-(i), the nonlinear term can be formulated and bounded as follows:

$$\begin{aligned} & |(\bar{u}(s) \times (\nabla \times v(s)), v(s))| = \left| -\alpha^2 (A \bar{u}(s) \times (\nabla \times v(s)), \bar{u}(s)) \right| \\ & \leq \alpha^2 C_D \|A \bar{u}(t)\|_{\mathbb{L}^2}^{\frac{1}{2}} \|A^{\frac{3}{2}} \bar{u}(t)\|_{\mathbb{L}^2}^{\frac{1}{2}} \|\nabla v(t)\|_{\mathbb{L}^2} \|\nabla \bar{u}(t)\|_{\mathbb{L}^2} \leq \alpha C_D \|A \bar{u}(t)\|_{\mathbb{L}^2}^{\frac{1}{2}} \|\nabla v(t)\|_{\mathbb{L}^2}^{\frac{3}{2}} \|\nabla \bar{u}(t)\|_{\mathbb{L}^2} \\ & \leq \frac{\alpha^4 C_D^4}{4\nu^3} \|A \bar{u}(t)\|_{\mathbb{L}^2}^2 \|\nabla \bar{u}(t)\|_{\mathbb{L}^2}^4 + \frac{3\nu}{4} \|\nabla v(t)\|_{\mathbb{L}^2}^2 \leq \frac{C_D^4}{4\alpha^2 \nu^3} \|\nabla \bar{u}(t)\|_{\alpha}^2 \|\bar{u}(t)\|_{\alpha}^4 + \frac{3\nu}{4} \|\nabla v(t)\|_{\mathbb{L}^2}^2 \end{aligned}$$

thanks to Proposition 5.2.1-(ii), Lemma 5.3.1-(iv), and Young's inequality. Subsequently, equation (5.11) becomes

$$\begin{aligned} & \|v(t)\|_{\mathbb{L}^2}^2 + \frac{\nu}{2} \int_0^t \|\nabla v(s)\|_{\mathbb{L}^2}^2 ds \leq \|v_0\|_{\mathbb{L}^2}^2 + 2 \int_0^t (v(s), g(s, \bar{u}(s))) dW(s) \\ & + \frac{C_D^4}{2\alpha^2\nu^3} \sup_{0 \leq t \leq T} \|\bar{u}(t)\|_\alpha^4 \int_0^t \|\nabla \bar{u}(t)\|_\alpha^2 dt + \int_0^t \|g(t, \bar{u}(t))\|_{\mathcal{L}_2(K, \mathbb{L}^2)}^2 dt. \end{aligned}$$

Next, we raise the above equation to the power 2^{p-1} and take the supremum over $t \in [0, T]$ to achieve

$$\begin{aligned} & \sup_{0 \leq t \leq T} \|v(t)\|_{\mathbb{L}^2}^{2p} + \left(\frac{\nu}{2} \int_0^T \|\nabla v(t)\|_{\mathbb{L}^2}^2 dt \right)^{2^{p-1}} \lesssim \sup_{0 \leq t \leq T} \left(2 \int_0^t (v(s), g(s, \bar{u}(s))) dW(s) \right)^{2^{p-1}} \\ & + \left(\frac{C_D^4}{2\alpha^2\nu^3} \right)^{2^{p-1}} \sup_{0 \leq t \leq T} \|\bar{u}(t)\|_\alpha^{2^{p+1}} \left(\int_0^T \|\nabla \bar{u}(t)\|_\alpha^2 dt \right)^{2^{p-1}} \\ & + T^{2^{p-1}-1} \int_0^T \|g(t, \bar{u}(t))\|_{\mathcal{L}_2(K, \mathbb{L}^2)}^{2p} dt + \|v_0\|_{\mathbb{L}^2}^{2p} =: I_1 + I_2 + I_3 + \|v_0\|_{\mathbb{L}^2}^{2p}. \end{aligned} \quad (5.12)$$

thanks to estimate (5.5) and the Jensen inequality. Now, the Burkholder-Davis-Gundy and Jensen inequalities, assumption (S_2) , estimates (5.5) and (5.4), together with Lemma 5.3.1-(i) are employed gradually to obtain:

$$\begin{aligned} \mathbb{E}[I_1] & \lesssim \mathbb{E} \left[\left(\int_0^T \|v(t)\|_{\mathbb{L}^2}^2 \|g(t, \bar{u}(t))\|_{\mathcal{L}_2(K, \mathbb{L}^2)}^2 dt \right)^{2^{p-2}} \right] \lesssim \mathbb{E} \left[\int_0^T \|v(t)\|_{\mathbb{L}^2}^{2^{p-1}} \|g(t, \bar{u}(t))\|_{\mathcal{L}_2(K, \mathbb{L}^2)}^{2^{p-1}} dt \right] \\ & \lesssim K_1^{2^{p-1}} \mathbb{E} \left[\int_0^T \|v(t)\|_{\mathbb{L}^2}^{2^{p-1}} dt \right] + K_2^{2^{p-1}} \mathbb{E} \left[\int_0^T \|v(t)\|_{\mathbb{L}^2}^{2^{p-1}} \|\bar{u}(t)\|_\alpha^{2^{p-1}} dt \right] \\ & \lesssim K_1^{2^{p-1}} T + (K_1^{2^{p-1}} + K_2^{2^{p-1}}) \mathbb{E} \left[\int_0^T \sup_{0 \leq s \leq t} \|v(s)\|_{\mathbb{L}^2}^{2p} dt \right]. \end{aligned}$$

Additionally, using the Hölder inequality along with assertion (i) of the current lemma imply

$$\mathbb{E}[I_2] \leq \left(\frac{C_D^4}{2\alpha^2\nu^3} \right)^{2^{p-1}} \mathbb{E} \left[\sup_{0 \leq t \leq T} \|\bar{u}(t)\|_\alpha^{2^{p+2}} \right]^{\frac{1}{2}} \mathbb{E} \left[\left(\int_0^T \|\nabla \bar{u}(t)\|_\alpha^2 dt \right)^{2p} \right] \leq \left(\frac{C_D^4}{2\alpha^2\nu^3} \right)^{2^{p-1}} C_1.$$

The remaining term I_3 is controlled as follows:

$$\mathbb{E}[I_3] \lesssim \mathbb{E} \left[\int_0^T (K_1^{2p} + K_2^{2p} \|\bar{u}(t)\|_\alpha^{2p}) dt \right] \leq T^{2^{p-1}} K_1^{2p} + T^{2^{p-1}} K_2^{2p} C_1$$

thanks to assumption (S_2) , estimate (5.5), and assertion (i) of the current lemma. Consequently, collecting all estimates on I_i , $i \in \{1, 2, 3\}$, plugging them back in equation (5.12), and employing the Grönwall lemma yield

$$\begin{aligned} & \mathbb{E} \left[\sup_{0 \leq t \leq T} \|v(t)\|_{\mathbb{L}^2}^{2p} + \left(\frac{\nu}{2} \int_0^T \|\nabla v(t)\|_{\mathbb{L}^2}^2 dt \right)^{2^{p-1}} \right] \\ & \lesssim \left\{ \mathbb{E} [\|v_0\|_{\mathbb{L}^2}^{2p}] + \left(\frac{C_D^4}{\alpha^2\nu^3} \right)^{2^{p-1}} C_1 + T^{2^{p-1}} (K_1^{2p} + K_2^{2p} C_1) + K_1^{2^{p-1}} T \right\} \exp \left(T (K_1^{2^{p-1}} + K_2^{2^{p-1}}) \right). \end{aligned}$$

To justify inequality (iii), it suffices to employ estimates (iii) and (iv) of Lemma 5.3.1 together with assertion (ii) of the current lemma. \square

Remark 5.3.2 *The upper bound of estimate (iii) in Lemma 5.3.2 can be improved regarding the denominator α^{2p} . In fact, applying Itô's formula to the process $F_3(v(t)) := \left\| A^{\frac{1}{2}} (I + \alpha^2 A)^{-\frac{1}{2}} v(t) \right\|_{\mathbb{L}^2}^2$ permits the deletion of the $\frac{1}{\alpha^{2p}}$ factor.*

5.4 . Numerical approach

Given an $M \in \mathbb{N} \setminus \{0\}$, we define the uniform partition $I_k = \{t_\ell\}_{\ell=1}^M$ of the time interval $[0, T]$, with $k := \frac{T}{M}$ being the associated step size, and $t_\ell = k\ell$ symbolizes a node at index $\ell \in \{0, 1, \dots, M\}$.

The domain $D \subset \mathbb{R}^3$ is meshed through a quasi-uniform triangulation, denoted \mathcal{T}_h , such that the index h represents the diameters' maximum of all simplexes of \mathcal{T}_h , and $\bar{D} = \bigcup_{S \in \mathcal{T}_h} \bar{S}$. For $n_1, n_2 \in \mathbb{N} \setminus \{0\}$, we propose the following finite element spaces:

$$\begin{aligned} \mathbb{H}_h &:= \left\{ v_h \in \mathbb{H}_0^1 \cap [C(\bar{D})]^3 \mid v_h|_S \in \mathbb{P}_{n_1}(S), \forall S \in \mathcal{T}_h \right\}, \\ L_h &:= \left\{ q_h \in L_0^2(D) \mid q_h|_S \in \mathbb{P}_{n_2}(S), \forall S \in \mathcal{T}_h \right\}, \\ \mathbb{V}_h &:= \left\{ v_h \in \mathbb{H}_h \mid (\operatorname{div}(v_h), q_h) = 0, \forall q_h \in L_h \right\}. \end{aligned}$$

where $\mathbb{P}_n(S)$ denotes the space of polynomial vector fields with degree less than or equal to $n \in \mathbb{N} \setminus \{0\}$, on a simplex $S \in \mathcal{T}_h$. The finite element pair (\mathbb{H}_h, L_h) may suffer from instability arising from an incompatibility of the velocity and the pressure. We therefore assume that the pair (\mathbb{H}_h, L_h) satisfies the following discrete LBB condition

$$\sup_{v_h \in \mathbb{H}_h \setminus \{0\}} \frac{(\operatorname{div}(v_h), q_h)}{\|\nabla v_h\|_{\mathbb{L}^2}} \geq \beta \|q_h\|_{L^2}, \quad \forall q_h \in L_h, \quad (5.13)$$

for some constant $\beta > 0$ independent of h . The above condition plays a crucial role in determining whether a given problem is well-posed or not and governs the choice of finite element discretizations. We point out that an equal-order velocity-pressure approximation might not be allowed because no pressure stabilizing techniques will be covered in this chapter. The reader may refer to [38, 43, 61, 99, 110] for more information about this approach.

For all $v \in \mathbb{L}^2$ and $u \in \mathbb{H}_0^1$, we define the projection $\Pi_h: \mathbb{L}^2 \rightarrow \mathbb{H}_h$ onto \mathbb{H}_h by

$$(v - \Pi_h v, \varphi_h) = 0, \quad \forall \varphi_h \in \mathbb{V}_h, \quad (5.14)$$

and the discrete Laplace operator $\Delta^h: \mathbb{H}_0^1 \rightarrow \mathbb{V}_h$ by

$$\left(\Delta^h u, \varphi_h \right) = -(\nabla u, \nabla \varphi_h), \quad \forall \varphi_h \in \mathbb{V}_h. \quad (5.15)$$

Since \mathcal{T}_h is a quasi-uniform triangulation of the domain D , the following two estimates are met (c.f. [21]):

for all $v \in \mathbb{H}^2 \cap \mathbb{H}_0^1$, there exists a constant $C > 0$ independent of the mesh size h such that

$$\sum_{j=0}^1 h^j \left\| D^j (v - \Pi_h v) \right\|_{\mathbb{L}^2} \leq Ch^2 \|\Delta v\|_{\mathbb{L}^2}, \quad (5.16)$$

and for all $\ell \in \mathbb{N}$, $1 \leq p, q \leq +\infty$, and $0 \leq m \leq \ell$, there is a constant $C > 0$ independent of h such that

$$\|\varphi_h\|_{\mathbb{W}^{\ell, p}} \leq Ch^{m-\ell+3\min(\frac{1}{p}-\frac{1}{q}, 0)} \|\varphi_h\|_{\mathbb{W}^{m, q}}, \quad \forall \varphi_h \in \mathbb{H}_h. \quad (5.17)$$

Combining identity (5.15) with the inverse inequality (5.17) grants

$$\left\| \Delta^h \varphi_h \right\|_{\mathbb{L}^2} \leq Ch^{-1} \|\nabla \varphi_h\|_{\mathbb{L}^2}, \quad \forall \varphi_h \in \mathbb{V}_h. \quad (5.18)$$

Following the continuous differential filter that was introduced in Definition 5.3.1, we define its discrete version through the variational formulation (5.7) as follows.

Definition 5.4.1 (Discrete differential filter)

Let v be the vector field of Definition 5.3.1. Its discrete differential filter, denoted $\bar{u}_h \in \mathbb{V}_h$, is given by the unique solution of

$$\alpha^2 (\nabla \bar{u}_h, \nabla \varphi_h) + (\bar{u}_h, \varphi_h) = (v, \varphi_h), \quad \forall \varphi_h \in \mathbb{V}_h.$$

The affiliation of the test function φ_h to \mathbb{V}_h , in Definition 5.4.1, eliminates all pressure fields. In point of fact, this discrete differential filter could have been formulated as follows if $\varphi_h, \bar{u}_h \in \mathbb{H}_h$:

$$\alpha^2 (\nabla \bar{u}_h, \nabla \varphi_h) + (\bar{u}_h, \varphi_h) - (\bar{p}_h, \varphi_h) = (v, \varphi_h),$$

for some discrete pressure field $\bar{p}_h \in L_h$ whose existence is ensured through the discrete LBB condition (5.13). The next lemma states a few properties of the discrete differential filter (c.f. [44, Lemma 2.1]).

Lemma 5.4.1 *Let $v = v_h \in \mathbb{V}^h$ and $\bar{u}_h \in \mathbb{V}_h$ be its discrete differential filter. Then,*

$$(i) \quad v_h = \bar{u}_h - \alpha^2 \Delta^h \bar{u}_h \text{ and } \nabla v_h = \nabla \bar{u}_h - \alpha^2 \nabla \Delta^h \bar{u}_h \text{ a.e. in } D.$$

$$(ii) \quad (\nabla v_h, \nabla \bar{u}_h) = \|\nabla \bar{u}_h\|_{\mathbb{L}^2}^2 + \alpha^2 \|\Delta^h \bar{u}_h\|_{\mathbb{L}^2}^2.$$

We deal herein with the same algorithm that was presented and studied in Chapter 4. However, it will be handled from now on in three dimensions, although its application in two dimensions is trivially correct when the spatial scale α does not vanish as k, h go to 0. For the reader's convenience, we state once again the algorithm of Chapter 4.

Algorithm 5.1

For a given $U^0 \in \mathbb{H}_h$, compute V^0 via $(V^0, \psi) = (U^0, \psi) + \alpha^2 (\nabla U^0, \nabla \psi)$, for all $\psi \in \mathbb{H}_h$.

If $U^{m-1}, V^{m-1}, \Pi^{m-1}, \tilde{\Pi}^{m-1}$ are obtained for a certain $m \in \{1, \dots, M\}$, then find a 4-tuple stochastic process $(U^m, V^m, \Pi^m, \tilde{\Pi}^m) \in \mathbb{H}_h \times \mathbb{H}_h \times L_h \times L_h$ such that for all $(\varphi, \psi, \Lambda_1, \Lambda_2) \in \mathbb{H}_h \times \mathbb{H}_h \times L_h \times L_h$, it holds \mathbb{P} -a.s.

$$\begin{cases} (V^m - V^{m-1}, \varphi) + k\nu (\nabla V^m, \nabla \varphi) - k(U^m \times (\nabla \times V^m), \varphi) - k(\Pi^m, \text{div}(\varphi)) \\ \quad = k \langle f(t_{m-1}, U^{m-1}), \varphi \rangle + (g(t_{m-1}, U^{m-1}) \Delta_m W, \varphi), \\ (V^m, \psi) = (U^m, \psi) + \alpha^2 (\nabla U^m, \nabla \psi) - (\tilde{\Pi}^m, \text{div}(\psi)), \\ (\text{div}(U^m), \Lambda_1) = (\text{div}(V^m), \Lambda_2) = 0, \end{cases}$$

where $\Delta_m W = W(t_m) - W(t_{m-1})$ for all $m \in \{1, \dots, M\}$.

The above numerical scheme is obviously nonlinear. An associated linear version is provided in Section 5.8. The initial condition V^0 can also be found through the equation

$$(V^0, \psi) = (U^0, \psi) + \alpha^2 (\nabla U^0, \nabla \psi) - (\tilde{\Pi}^0, \text{div}(\psi)), \quad \forall \psi \in \mathbb{H}_h,$$

where the formula of $\tilde{\Pi}^0$ needs to be defined.

It is worth highlighting that when ψ is considered in \mathbb{V}_h in Algorithm 5.1, the second equation becomes

$$(V^m, \psi) = (U^m, \psi) + \alpha^2 (\nabla U^m, \nabla \psi),$$

which signifies that $\{U^m\}_{m=1}^M$ is a discrete differential filter to $\{V^m\}_{m=1}^M$, according to Definition 5.4.1.

5.5 . Main result

Theorem 5.5.1 *Let $T > 0$, $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, T]}, \mathbb{P})$ be a filtered probability space, and $D \subset \mathbb{R}^3$ be a bounded convex polyhedral domain. Suppose that assumptions (S_1) - (S_2) are fulfilled. Given an $M \in \mathbb{N} \setminus \{0\}$, define the discretization step size $k := \frac{T}{M}$ such that $k \in (0, 1)$ and I_k makes up a uniform partition of the time interval $[0, T]$. Let $h \in (0, 1)$ be the space discretization step size such that the triangulation \mathcal{T}_h is quasi-uniform, and let $\alpha > 0$ be a fixed non-vanishing parameter that does not depend on k and h . Define the finite element triple $(\mathbb{H}_h, L_h, U_h^0)$ such that the pair (\mathbb{H}_h, L_h) satisfies the discrete LBB condition (5.13) and the initial datum U_h^0 ($:= U^0$) belongs to \mathbb{H}_h . The following results are valid:*

- *For a given $(k, h, \alpha) \in (0, 1) \times (0, 1) \times (0, +\infty)$, there is a solution $\left\{ (U^m, V^m, \Pi^m, \tilde{\Pi}^m) \right\}_{m=1}^M$ to Algorithm 5.1 that fulfills Lemmas 5.6.1 and 5.6.3.*
- *For a family $\{k, h\}$ of discretization parameters satisfying $k, h \rightarrow 0$ instantly such that $U^0 \rightarrow \bar{u}_0$ as $h \rightarrow 0$ in the space $L^4(\Omega; \mathbb{H}^1)$ (particularly $\|U^0\|_{L^4(\Omega; \mathbb{H}^1)}$ is uniformly bounded in h), the solution $\left\{ (U^m, V^m, \Pi^m, \tilde{\Pi}^m) \right\}_{m=1}^M$ of Algorithm 5.1 converges toward the unique solution of the Lagrangian averaged Navier-Stokes equations (5.1) in the sense of Definition 5.3.2.*

The uniform boundedness of $\|U^0\|_{\mathbb{H}^1}$ might be achieved independently of the condition $U^0 \rightarrow \bar{u}_0$ as $h \rightarrow 0$, if U^0 is defined by $U^0 := \mathcal{R}_h \bar{u}_0$, where \mathcal{R}_h is the Ritz operator. For additional information, the reader may refer to Section 4.3 of Chapter 4.

5.6 . Stability and solvability

This section is devoted to giving stability properties to Algorithm 5.1. We also inspect its solvability by means of Brouwer's fixed point theorem.

The proposed system in Algorithm 5.1 can be readjusted in the following manner:

$$\begin{cases} (V^m - V^{m-1}, \varphi) + k\nu(\nabla V^m, \nabla \varphi) - k(U^m \times (\nabla \times V^m), \varphi) \\ = k\langle f(t_{m-1}, U^{m-1}), \varphi \rangle + (g(t_{m-1}, U^{m-1})\Delta_m W, \varphi), \quad \forall \varphi \in \mathbb{V}_h. \\ (V^m, \psi) = (U^m, \psi) + \alpha^2(\nabla U^m, \nabla \psi), \quad \forall \psi \in \mathbb{V}_h. \end{cases} \quad (5.19)$$

We exploit the next lemma to illustrate stability features of the iterates $\{(U^m, V^m)\}_{m=1}^M$, and solvability of Algorithm 5.1.

Lemma 5.6.1 *Assume that assumptions (S_1) - (S_2) are valid and let $p \in [2, \infty) \cap \mathbb{N}$. Then, there exists a $\mathbb{V}_h \times \mathbb{V}_h \times L_h \times L_h$ -valued sequence of random variables $\{(U^m, V^m, \Pi^m, \tilde{\Pi}^m)\}_{m=1}^M$ that solves \mathbb{P} -a.s. Algorithm 5.1, and fulfills the following assertions:*

(i) *for any $m \in \{1, \dots, M\}$, the maps $U^m, V^m: \Omega \rightarrow \mathbb{H}_h$ are \mathcal{F}_{t_m} -measurable.*

$$(ii) \quad \mathbb{E} \left[\max_{1 \leq m \leq M} \|U^m\|_\alpha^2 + \frac{k\nu}{2} \sum_{m=1}^M \left(\|\nabla U^m\|_{\mathbb{L}^2}^2 + \alpha^2 \|\Delta^h U^m\|_{\mathbb{L}^2}^2 \right) + \frac{1}{4} \sum_{m=1}^M \|U^m - U^{m-1}\|_\alpha^2 \right] \leq C_T,$$

$$(iii) \quad \mathbb{E} \left[\max_{1 \leq m \leq M} \|U^m\|_\alpha^{2p} + \sum_{m=1}^M \|U^m\|_\alpha^{2p-1} \|U^m - U^{m-1}\|_\alpha^2 + k\nu \sum_{m=1}^M \|U^m\|_\alpha^{2p-1} \left(\|\nabla U^m\|_{\mathbb{L}^2}^2 + \alpha^2 \|\Delta^h U^m\|_{\mathbb{L}^2}^2 \right) \right] \leq C_{T,p},$$

$$(iv) \quad \mathbb{E} \left[\left(k\nu \sum_{m=1}^M \left(\|\nabla U^m\|_{\mathbb{L}^2}^2 + \alpha^2 \|\Delta^h U^m\|_{\mathbb{L}^2}^2 \right) \right)^{2p-1} \right] \leq C_{T,p},$$

where $C_{T,p} > 0$ depends only on $\|U^0\|_{L^{2p}(\Omega; \mathbb{H}^1)}$, T , $\{K_i\}_{i=1}^4$, ν and D , and it is independent of α , k and h . Note that $C_T := C_{T,1}$.

Proof: We only prove assertion (ii) for the sake of showing how to get ride of the assumption that was imposed on the Wiener process in Chapter 4; namely a Q -Wiener process with Q being of trace-class. For the remaining assertions, the reader may check the demonstration of Lemmas 4.4.1 and 4.4.3 in Chapter 4. Using Lemma 5.4.1-(i), identity (5.15), and equality (5.3), we infer that

$$\begin{aligned} (V^m - V^{m-1}, U^m) &= (U^m - U^{m-1}, U^m) + \alpha^2 (\nabla(U^m - U^{m-1}), \nabla U^m) \\ &= \frac{1}{2} \|U^m\|_\alpha^2 - \frac{1}{2} \|U^{m-1}\|_\alpha^2 + \frac{1}{2} \|U^m - U^{m-1}\|_\alpha^2, \end{aligned}$$

and by Lemma 5.4.1-(ii),

$$(\nabla V^m, \nabla U^m) = \|\nabla U^m\|_{\mathbb{L}^2}^2 + \alpha^2 \|\Delta^h U^m\|_{\mathbb{L}^2}^2.$$

Therefore, setting $\varphi = U^m$ in equation (5.19), and employing Proposition 5.2.1-(i), assumption (S_2) , the Cauchy-Schwarz, Poincaré and Young inequalities yield

$$\begin{aligned} &\frac{1}{2} \|U^m\|_\alpha^2 - \frac{1}{2} \|U^{m-1}\|_\alpha^2 + \frac{1}{4} \|U^m - U^{m-1}\|_\alpha^2 + \frac{k\nu}{2} \|\nabla U^m\|_{\mathbb{L}^2}^2 + k\nu\alpha^2 \|\Delta^h U^m\|_{\mathbb{L}^2}^2 \\ &\leq \frac{C_D^2 k}{\nu} \left(K_3^2 + K_4^2 \|U^{m-1}\|_\alpha^2 \right) + \|g(t_{m-1}, U^{m-1}) \Delta_m W\|_{\mathbb{L}^2}^2 + \left(g(t_{m-1}, U^{m-1}) \Delta_m W, U^{m-1} \right). \end{aligned}$$

Next, we sum the above equation over m from 1 to $\ell \in \{2, \dots, M\}$ to achieve:

$$\begin{aligned} &\frac{1}{2} \|U^\ell\|_\alpha^2 + \frac{1}{4} \sum_{m=1}^{\ell} \|U^m - U^{m-1}\|_\alpha^2 + \frac{k\nu}{2} \sum_{m=1}^{\ell} \|\nabla U^m\|_{\mathbb{L}^2}^2 + k\nu\alpha^2 \sum_{m=1}^{\ell} \|\Delta^h U^m\|_{\mathbb{L}^2}^2 \leq \frac{1}{2} \|U^0\|_\alpha^2 \\ &+ \frac{C_D^2 K_3^2 T}{\nu} + \frac{C_D^2 K_4^2}{\nu} k \sum_{m=1}^{\ell} \|U^{m-1}\|_\alpha^2 + \sum_{m=1}^{\ell} \|g(t_{m-1}, U^{m-1}) \Delta_m W\|_{\mathbb{L}^2}^2 \\ &+ \sum_{m=1}^{\ell} \left(g(t_{m-1}, U^{m-1}) \Delta_m W, U^{m-1} \right). \end{aligned} \quad (5.20)$$

Due to the $\mathcal{F}_{t_{m-1}}$ -measurability of U^{m-1} and assumption (S_2) , it holds that

$$\mathbb{E} \left[\left(g(t_{m-1}, U^{m-1}) \Delta_m W, U^{m-1} \right) \right] = 0.$$

Moreover, by Itô's isometry and assumption (S_2) , it follows:

$$\begin{aligned} &\sum_{m=1}^{\ell} \mathbb{E} \left[\left\| g(t_{m-1}, U^{m-1}) \Delta_m W \right\|_{\mathbb{L}^2}^2 \right] = \sum_{m=1}^{\ell} \mathbb{E} \left[\left\| \int_{t_{m-1}}^{t_m} g(t_{m-1}, U^{m-1}) dW(t) \right\|_{\mathbb{L}^2}^2 \right] \\ &= \sum_{m=1}^{\ell} \int_{t_{m-1}}^{t_m} \mathbb{E} \left[\left\| g(t_{m-1}, U^{m-1}) \right\|_{\mathcal{L}_2(K, \mathbb{L}^2)}^2 \right] dt \leq 2TK_1^2 + 2K_2^2 k \sum_{m=1}^{\ell} \mathbb{E} \left[\|U^{m-1}\|_\alpha^2 \right]. \end{aligned} \quad (5.21)$$

Subsequently, a simple application of the discrete Grönwall lemma returns

$$\begin{aligned} &\max_{1 \leq m \leq M} \mathbb{E} \left[\|U^m\|_\alpha^2 \right] + \frac{1}{2} \mathbb{E} \left[\sum_{m=1}^M \|U^m - U^{m-1}\|_\alpha^2 \right] + k\nu \mathbb{E} \left[\sum_{m=1}^M \|\nabla U^m\|_{\mathbb{L}^2}^2 \right] \\ &+ 2k\nu\alpha^2 \mathbb{E} \left[\sum_{m=1}^M \|\Delta^h U^m\|_{\mathbb{L}^2}^2 \right] \\ &\leq \left\{ \mathbb{E} \left[\|U^0\|_\alpha^2 \right] + \frac{2C_D^2 K_3^2 T}{\nu} + 4K_1^2 T \right\} \exp \left(\frac{C_D^2 K_4^2 T}{\nu^3} + 2K_2^2 T \right) =: \mathcal{C}_T. \end{aligned} \quad (5.22)$$

It remains to show that $\mathbb{E} [\max_{1 \leq m \leq M} \|U^m\|_\alpha^2]$ is bounded. To this end, we take the maximum over $\ell \in \{1, \dots, M\}$ in equation (5.20), then apply the mathematical expectation:

$$\begin{aligned} & \frac{1}{2} \max_{1 \leq m \leq M} \|U^m\|_\alpha^2 + \frac{1}{4} \sum_{m=1}^M \|U^m - U^{m-1}\|_\alpha^2 + \frac{k\nu}{2} \sum_{m=1}^M \|\nabla U^m\|_{\mathbb{L}^2}^2 + k\nu\alpha^2 \sum_{m=1}^M \|\Delta^h U^m\|_{\mathbb{L}^2}^2 \\ & \leq \frac{1}{2} \|U^0\|_\alpha^2 + \frac{C_D^2 K_3^2 T}{\nu} + \frac{C_D^2 K_4^2}{\nu} k \sum_{m=1}^M \|U^{m-1}\|_\alpha^2 + \sum_{m=1}^M \|g(t_{m-1}, U^{m-1}) \Delta W\|_{\mathbb{L}^2}^2 \\ & \quad + \max_{1 \leq \ell \leq M} \sum_{m=1}^{\ell} \left(g(t_{m-1}, U^{m-1}) \Delta_m W, U^{m-1} \right) =: \frac{1}{2} \|U^0\|_\alpha^2 + \frac{C_D^2 K_3^2 T}{\nu} + J_1 + J_2 + J_3. \end{aligned}$$

Result (5.22) implies that

$$\mathbb{E} [J_1] \leq \frac{C_D^2 K_4^2 T}{\nu} \mathbb{E} [\|U^0\|_\alpha^2] + \frac{C_D^2 K_4^2 T}{\nu} \mathcal{C}_T.$$

Additionally, thanks to equation (5.21), we get

$$\mathbb{E} [J_2] \leq 2TK_1^2 + 2K_2^2 T \mathbb{E} [\|U^0\|_\alpha^2] + 2K_2^2 T \mathcal{C}_T.$$

On the other hand, by the Burkholder-Davis-Gundy and Young inequalities, assumption (S_2) , along with result (5.22)

$$\begin{aligned} \mathbb{E} [J_3] & \lesssim \mathbb{E} \left[\left(k \sum_{m=1}^M \|g(t_{m-1}, U^{m-1})\|_{\mathcal{L}_2(K, \mathbb{L}^2)}^2 \|U^{m-1}\|_{\mathbb{L}^2}^2 \right)^{1/2} \right] \\ & \leq \frac{1}{4} \mathbb{E} \left[\max_{1 \leq m \leq M} \|U^m\|_\alpha^2 \right] + \mathbb{E} \left[\frac{1}{4} \|U^0\|_\alpha^2 + k \sum_{m=1}^M \|g(t_{m-1}, U^{m-1})\|_{\mathcal{L}_2(K, \mathbb{L}^2)}^2 \right] \\ & \leq \frac{1}{4} \mathbb{E} \left[\max_{1 \leq m \leq M} \|U^m\|_\alpha^2 \right] + \left(\frac{1}{4} + 2K_2^2 T \right) \mathbb{E} [\|U^0\|_\alpha^2] + 2K_1^2 T + 2K_2^2 T \mathcal{C}_T. \end{aligned}$$

Putting it all together, we infer that

$$\begin{aligned} & \mathbb{E} \left[\frac{1}{4} \max_{1 \leq m \leq M} \|U^m\|_\alpha^2 + \frac{1}{4} \sum_{m=1}^M \|U^m - U^{m-1}\|_\alpha^2 + \frac{k\nu}{2} \sum_{m=1}^M \|\nabla U^m\|_{\mathbb{L}^2}^2 + k\nu\alpha^2 \sum_{m=1}^M \|\Delta^h U^m\|_{\mathbb{L}^2}^2 \right] \\ & \leq \left(\frac{3}{4} + \frac{C_D^2 K_4^2 T}{\nu} + 4K_2^2 T \right) \mathbb{E} [\|U^0\|_\alpha^2] + \frac{C_D^2 (K_3^2 + K_4^2 \mathcal{C}_T) T}{\nu} + 4K_1^2 T + 4K_2^2 T \mathcal{C}_T, \end{aligned}$$

which completes the proof. \square

As mentioned in Chapter 4 regarding the uniqueness of iterates $\{(U^m, V^m)\}_{m=1}^M$, which were extracted by means of Brouwer's theorem, it holds true in a sample subset Ω_ε of Ω whose probability measure converges to 1 as ε tends to 0.

Lemma 5.6.2 *Let $\{(U^m, V^m)\}_{m=1}^M$ be a solution to Algorithm 5.1, assumptions (S_1) - (S_2) be satisfied, and ε be a real positive number. We define $\Omega_\varepsilon := \left\{ \omega \in \Omega \mid \max_{1 \leq m \leq M} \|\nabla U^m\|_{\mathbb{L}^2}^4 \leq \frac{1}{\varepsilon} \right\}$ a sample probability subset of Ω . Then, $\mathbb{P}(\Omega_\varepsilon) \geq 1 - \alpha^{-4} C_{T,2} \varepsilon$, with $C_{T,2} > 0$ is provided in Lemma 5.6.1, and iterates $\{(U^m, V^m)\}_{m=1}^M$ are \mathbb{P} -a.s. unique in Ω_ε under the condition $k \left(1 + \frac{C^4 \alpha^4}{h^4} \right) < \frac{27\nu^3 \varepsilon}{128C_D^4}$, where the constant C_D depends only on the domain D , and C appears in the inverse inequality (5.17).*

Proof: Let $\{(U_1^m, V_1^m)\}_{m=1}^M$ and $\{(U_2^m, V_2^m)\}_{m=1}^M$ be two solutions to Algorithm 5.1 such that $U_1^0 = U_2^0$ and $V_1^0 = V_2^0$, and for all $m \in \{0, 1, \dots, M\}$, let $u^m := U_1^m - U_2^m$ and $v^m := V_1^m - V_2^m$. Then, $\{(u^m, v^m)\}_{m=1}^M$ satisfies the following equations:

$$\begin{cases} (v^m - v^{m-1}, \varphi) + k\nu (\nabla v^m, \nabla \varphi) - k (U_1^m \times (\nabla \times V_1^m), \varphi) + k (U_2^m \times (\nabla \times V_2^m), \varphi) \\ = k \langle f(t_{m-1}, U_1^{m-1}) - f(t_{m-1}, U_2^{m-1}), \varphi \rangle \\ + \left([g(t_{m-1}, U_1^{m-1}) - g(t_{m-1}, U_2^{m-1})] \Delta_m W, \varphi \right), \quad \forall \varphi \in \mathbb{V}_h, \\ (v^m, \psi) = (u^m, \psi) + \alpha^2 (\nabla u^m, \nabla \psi), \quad \forall \psi \in \mathbb{V}_h. \end{cases} \quad (5.23)$$

Arguing by induction on m : for $m = 1$, we have

$$(v^1, \varphi) + k\nu (\nabla v^1, \nabla \varphi) = k (U_1^1 \times (\nabla \times V_1^1), \varphi) - k (U_2^1 \times (\nabla \times V_2^1), \varphi), \quad (5.24)$$

where the right-hand side of equation (5.23)₁ vanishes because $U_1^0 = U_2^0$. Furthermore,

$$\begin{aligned} & (U_1^1 \times (\nabla \times V_1^1), u^1) - (U_2^1 \times (\nabla \times V_2^1), u^1) \\ &= (u^1 \times (\nabla \times V_1^1), u^1) + (U_2^1 \times (\nabla \times V_1^1), u^1) - (U_2^1 \times (\nabla \times V_2^1), u^1) = - (u^1 \times (\nabla \times v^1), U_2^1), \end{aligned}$$

thanks to Proposition 5.2.1-(i). For clarity's sake, denote $\|\cdot\|_{h,\alpha}^2 := \|\nabla \cdot\|_{\mathbb{L}^2}^2 + \alpha^2 \|\Delta^h \cdot\|_{\mathbb{L}^2}^2$. That said, we substitute φ by u^1 in equation (5.24) and make use of Lemma 5.4.1, Proposition 5.2.1-(ii) and the Young inequality to achieve:

$$\begin{aligned} & \|u^1\|_{\alpha}^2 + k\nu \|u^1\|_{h,\alpha}^2 = -k (u^1 \times (\nabla \times v^1), U_2^1) \\ &= -k (u^1 \times (\nabla \times u^1), U_2^1) + \alpha^2 k (u^1 \times (\nabla \Delta^h u^1), U_2^1) \\ &\leq C_D k \|u^1\|_{\mathbb{L}^2}^{1/2} \|\nabla u^1\|_{\mathbb{L}^2}^{3/2} \|\nabla U_2^1\|_{\mathbb{L}^2} + C_D k \alpha^2 \|u^1\|_{\mathbb{L}^2}^{1/2} \|\nabla u^1\|_{\mathbb{L}^2}^{1/2} \|\nabla \Delta^h u^1\|_{\mathbb{L}^2} \|\nabla U_2^1\|_{\mathbb{L}^2} \\ &=: I + J. \end{aligned} \quad (5.25)$$

Next, by Young's inequality, it holds that:

$$I \leq \frac{128C_D^4}{27\nu^3} k \|u^1\|_{\mathbb{L}^2}^2 \|\nabla U_2^1\|_{\mathbb{L}^2}^4 + \frac{k\nu}{2} \|\nabla u^1\|_{\mathbb{L}^2}^2 \leq \frac{128C_D^4}{27\nu^3} k \|u^1\|_{\mathbb{L}^2}^2 \|\nabla U_2^1\|_{\mathbb{L}^2}^4 + \frac{k\nu}{2} \|u^1\|_{h,\alpha}^2.$$

Moreover, the inverse inequality (5.17), estimates $\|\nabla \cdot\|_{\mathbb{L}^2} \leq \|\cdot\|_{h,\alpha}$, $\|\Delta^h \cdot\|_{\mathbb{L}^2} \leq \|\cdot\|_{h,\cdot}$, and the Young inequality imply

$$\begin{aligned} J &\leq \frac{C_D k \alpha C}{h} \|u^1\|_{\mathbb{L}^2}^{1/2} \|\nabla u^1\|_{\mathbb{L}^2}^{1/2} \alpha \|\Delta^h u^1\|_{\mathbb{L}^2} \|\nabla U_2^1\|_{\mathbb{L}^2} \\ &\leq \frac{C_D k \alpha C}{h} \|u^1\|_{\mathbb{L}^2}^{1/2} \|u^1\|_{h,\alpha}^{3/2} \|\nabla U_2^1\|_{\mathbb{L}^2} \leq \frac{128C_D^4 \alpha^4 C^4 k}{27\nu^3 h^4} \|u^1\|_{\mathbb{L}^2}^2 \|\nabla U_2^1\|_{\mathbb{L}^2}^4 + \frac{k\nu}{2} \|u^1\|_{h,\alpha}^2. \end{aligned}$$

Therewith, equation (5.25) becomes

$$\|u^1\|_{\alpha}^2 \leq \frac{128C_D^4 k}{27\nu^3} \|\nabla U_2^1\|_{\mathbb{L}^2}^4 \left(1 + \frac{C^4 \alpha^4}{h^4} \right) \|u^1\|_{\alpha}^2. \quad (5.26)$$

Now, for a fixed $\varepsilon > 0$, we introduce the probability subset $\Omega_\varepsilon := \left\{ \omega \in \Omega \mid \max_{1 \leq m \leq M} \|\nabla U_2^m\|_{\mathbb{L}^2}^4 \leq \frac{1}{\varepsilon} \right\}$ whose probability measure is controlled as follows:

$$\mathbb{P}(\Omega_\varepsilon) \geq 1 - \varepsilon \mathbb{E} \left[\max_{1 \leq m \leq M} \|\nabla U_2^m\|_{\mathbb{L}^2}^4 \right] \geq 1 - \frac{\varepsilon}{\alpha^4} \mathbb{E} \left[\max_{1 \leq m \leq M} \|U_1^m\|_{\alpha}^4 \right] \geq 1 - \frac{\varepsilon C_{T,2}}{\alpha^4},$$

thanks to Markov's inequality and Lemma 5.6.1. Therefore, we multiply equation (5.26) by the indicator function $\mathbb{1}_{\Omega_\varepsilon}$ to accomplish:

$$\left[1 - \frac{128C_D^4 k}{27\nu^3 \varepsilon} \left(1 + \frac{C^4 \alpha^4}{h^4} \right) \right] \mathbb{1}_{\Omega_\varepsilon} \|u^1\|_\alpha^2 \leq 0.$$

The above equation implies that $u^1 = 0$ \mathbb{P} -a.s. and a.e. in D under the condition $k \left(1 + \frac{C^4 \alpha^4}{h^4} \right) < \frac{27\nu^3 \varepsilon}{128C_D^4}$. Finally, Lemma 5.4.1 implies that $\mathbb{1}_{\Omega_\varepsilon} v^1 = (I - \alpha^2 \Delta^h) \mathbb{1}_{\Omega_\varepsilon} u^1 = 0$ a.e. in D . The rest of the demonstration can be straightforwardly carried out by induction. \square

The provided a priori estimates in Lemma 5.6.1 are not sufficient to derive the existence of a (continuous) solution when α is fixed. Whence the need of the following lemma which consists of further stability properties for Algorithm 5.1.

Lemma 5.6.3 *Assume that α is fixed away from h and k and that (S_1) - (S_2) are valid. Let $\{V^m\}_{m=1}^M$ be the iterates of Algorithm 5.1. For $p \in [2, +\infty) \cap \mathbb{N}$, it holds*

$$(i) \ \mathbb{E} \left[\max_{1 \leq m \leq M} \|V^m\|_{\mathbb{L}^2}^2 + k\nu \sum_{m=1}^M \|\nabla V^m\|_{\mathbb{L}^2}^2 + \sum_{m=1}^M \|V^m - V^{m-1}\|_{\mathbb{L}^2}^2 \right] \leq C(\alpha),$$

$$(ii) \ \mathbb{E} \left[\max_{1 \leq m \leq M} \|V^m\|_{\mathbb{L}^2}^{2p} + \left(k\nu \sum_{m=1}^M \|\nabla V^m\|_{\mathbb{L}^2}^2 \right)^{2^{p-1}} + \left(\sum_{m=1}^M \|V^m - V^{m-1}\|_{\mathbb{L}^2}^2 \right)^{2^{p-1}} \right] \leq C_p(\alpha),$$

for some constant $C_p(\alpha) > 0$ depending on α , $\|V^0\|_{L^{2p}(\Omega; \mathbb{L}^2)}$, $\|U^0\|_{L^{2p+2}(\Omega; \mathbb{H}_0^1)}$, but not on k and h . Note that $C(\alpha) := C_1(\alpha)$.

Proof: We replace φ by V^m in equation (5.19), then apply identity (5.3) to get

$$\begin{aligned} & \frac{1}{2} \|V^m\|_{\mathbb{L}^2}^2 - \frac{1}{2} \|V^{m-1}\|_{\mathbb{L}^2}^2 + \frac{1}{2} \|V^m - V^{m-1}\|_{\mathbb{L}^2}^2 + k\nu \|\nabla V^m\|_{\mathbb{L}^2}^2 \\ & = k (U^m \times (\nabla \times V^m), V^m) + k \langle f(t_{m-1}, U^{m-1}), V^m \rangle + (g(t_{m-1}, U^{m-1}) \Delta_m W, V^m). \end{aligned}$$

Taking the sum over m from 1 to $\ell \in \{1, \dots, M\}$, then the maximum over ℓ yields

$$\begin{aligned} & \frac{1}{2} \max_{1 \leq \ell \leq M} \|V^\ell\|_{\mathbb{L}^2}^2 + k\nu \sum_{m=1}^M \|\nabla V^m\|_{\mathbb{L}^2}^2 + \frac{1}{4} \sum_{m=1}^M \|V^m - V^{m-1}\|_{\mathbb{L}^2}^2 \leq \frac{1}{2} \|V^0\|_{\mathbb{L}^2}^2 \\ & + k \sum_{m=1}^M |(U^m \times (\nabla \times V^m), V^m)| + k \sum_{m=1}^M \|f(t_{m-1}, U^{m-1})\|_{\mathbb{H}^{-1}} \|V^m\|_{\mathbb{H}^1} \\ & + \sum_{m=1}^M \|g(t_{m-1}, U^{m-1}) \Delta_m W\|_{\mathbb{L}^2}^2 + \max_{1 \leq \ell \leq M} \sum_{m=1}^{\ell} (g(t_{m-1}, U^{m-1}) \Delta_m W, V^{m-1}). \end{aligned} \tag{5.27}$$

We exploit the identity $V^m = U^m - \alpha^2 \Delta^h U^m$ and Proposition 5.2.1-(i) to write

$$(U^m \times (\nabla \times V^m), V^m) = \alpha^2 (\Delta^h U^m \times (\nabla \times V^m), U^m). \tag{5.28}$$

Moreover, plugging $\psi = U^m$ in equation (5.19)₂ and using the Cauchy-Schwarz, Young and Poincaré inequalities return $\|\nabla U^m\|_{\mathbb{L}^2} \leq C_D \alpha^{-1} \|\nabla V^m\|_{\mathbb{L}^2}$. Further, by Lemma 5.4.1-(i), it follows

$$\alpha^2 \|\nabla \Delta^h U^m\|_{\mathbb{L}^2} \leq \|\nabla U^m\|_{\mathbb{L}^2} + \|\nabla V^m\|_{\mathbb{L}^2} \leq (C_D \alpha^{-1} + 1) \|\nabla V^m\|_{\mathbb{L}^2}.$$

Therefore, identity (5.28), Proposition 5.2.1-(ii), the norm equivalence $\alpha \|\cdot\|_{\mathbb{H}^1} \leq \|\cdot\|_\alpha \leq \|\cdot\|_{\mathbb{H}^1}$, the Hölder and Young inequalities imply

$$\begin{aligned} k \sum_{m=1}^M |(U^m \times (\nabla \times V^m), V^m)| &\leq C_D \alpha k \sum_{m=1}^M \|U^m\|_\alpha \|\nabla V^m\|_{\mathbb{L}^2} \|\Delta^h U^m\|_{\mathbb{L}^2}^{\frac{1}{2}} \alpha^{-1} \sqrt{C_D \alpha^{-1} + 1} \|\nabla V^m\|_{\mathbb{L}^2}^{\frac{1}{2}} \\ &\leq C_D \alpha^{-1/2} \sqrt{C_D \alpha^{-1} + 1} \left(k \sum_{m=1}^M \|U^m\|_\alpha^4 \alpha^2 \|\Delta^h U^m\|_{\mathbb{L}^2}^2 \right)^{1/4} \left(k \sum_{m=1}^M \|\nabla V^m\|_{\mathbb{L}^2}^2 \right)^{3/4} \\ &\leq \frac{27C_D^4 (C_D + \alpha)^2}{4\nu^3 \alpha^4} k \sum_{m=1}^M \|U^m\|_\alpha^4 \alpha^2 \|\Delta^h U^m\|_{\mathbb{L}^2}^2 + \frac{k\nu}{4} \sum_{m=1}^M \|\nabla V^m\|_{\mathbb{L}^2}^2. \end{aligned}$$

Furthermore, employing assumption (S_2) , the Poincaré and Young inequalities gives

$$k \sum_{m=1}^M \|f(t_{m-1}, U^{m-1})\|_{\mathbb{H}^{-1}} \|V^m\|_{\mathbb{H}^1} \leq \frac{2C_D^2}{\nu} k \sum_{m=1}^M (K_3^2 + K_4^2 \|U^{m-1}\|_\alpha^2) + \frac{k\nu}{4} \sum_{m=1}^M \|\nabla V^m\|_{\mathbb{L}^2}^2.$$

Collecting all inequalities together, equation (5.27) becomes

$$\begin{aligned} \frac{1}{2} \max_{1 \leq m \leq M} \|V^m\|_{\mathbb{L}^2}^2 + \frac{k\nu}{2} \sum_{m=1}^M \|\nabla V^m\|_{\mathbb{L}^2}^2 + \frac{1}{4} \sum_{m=1}^M \|V^m - V^{m-1}\|_{\mathbb{L}^2}^2 &\leq \frac{1}{2} \|V^0\|_{\mathbb{L}^2}^2 \\ + \frac{27C_D^4 (C_D + \alpha)^2}{4\nu^3 \alpha^4} k \sum_{m=1}^M \|U^m\|_\alpha^4 \alpha^2 \|\Delta^h U^m\|_{\mathbb{L}^2}^2 + \frac{2C_D^2 T K_3^2}{\nu} + \frac{2C_D^2 K_4^2 T}{\nu} \max_{1 \leq m \leq M} \|U^{m-1}\|_\alpha^2 &\quad (5.29) \\ + \sum_{m=1}^M \|g(t_{m-1}, U^{m-1}) \Delta_m W\|_{\mathbb{L}^2}^2 + \max_{1 \leq \ell \leq M} \sum_{m=1}^{\ell} \left(g(t_{m-1}, U^{m-1}) \Delta_m W, V^{m-1} \right). & \end{aligned}$$

Next, we take the mathematical expectation of equation (5.29) to bound each term on its right-hand side. We have

$$\mathbb{E} \left[k \sum_{m=1}^M \|U^m\|_\alpha^4 \alpha^2 \|\Delta^h U^m\|_{\mathbb{L}^2}^2 \right] \leq \mathbb{E} \left[k \sum_{m=1}^M \|U^m\|_{\mathbb{L}^2}^4 \left(\|\nabla U^m\|_{\mathbb{L}^2}^2 + \alpha^2 \|\Delta^h U^m\|_{\mathbb{L}^2}^2 \right) \right] \leq C_{T,3},$$

thanks to Lemma 5.6.1-(iii). Moreover, making use of estimate (5.21) and Lemma 5.6.1-(ii), we infer that

$$\mathbb{E} \left[\sum_{m=1}^M \|g(t_{m-1}, U^{m-1}) \Delta_m W\|_{\mathbb{L}^2}^2 \right] \leq 2TK_1^2 + 2TK_2^2 \mathbb{E} [\|U^0\|_\alpha^2] + 2K_2^2 TC_T.$$

To control the last term on the right-hand side of equation (5.29), we shall employ the Burkholder-Davis-Gundy and Young inequalities, assumption (S_2) , and Lemma 5.6.1-(ii)

$$\begin{aligned} \mathbb{E} \left[\max_{1 \leq \ell \leq M} \sum_{m=1}^{\ell} \left(g(t_{m-1}, U^{m-1}) \Delta_m W, V^{m-1} \right) \right] &\lesssim \mathbb{E} \left[\left(k \sum_{m=1}^M \|g(t_{m-1}, U^{m-1})\|_{\mathcal{L}_2(K, \mathbb{L}^2)}^2 \|V^{m-1}\|_{\mathbb{L}^2}^2 \right)^{\frac{1}{2}} \right] \\ &\leq \frac{1}{4} \mathbb{E} \left[\max_{1 \leq m \leq M} \|V^m\|_{\mathbb{L}^2}^2 \right] + \mathbb{E} \left[\frac{1}{4} \|V^0\|_{\mathbb{L}^2}^2 + k \sum_{m=1}^M \|g(t_{m-1}, U^{m-1})\|_{\mathcal{L}_2(K, \mathbb{L}^2)}^2 \right] \\ &\leq \frac{1}{4} \mathbb{E} \left[\max_{1 \leq m \leq M} \|V^m\|_{\mathbb{L}^2}^2 \right] + \frac{1}{4} \mathbb{E} [\|V^0\|_{\mathbb{L}^2}^2] + 2K_2^2 T \mathbb{E} [\|U^0\|_\alpha^2] + 2K_1^2 T + 2K_2^2 TC_T. \end{aligned}$$

The first term on the right-hand side of the above equation must be absorbed in the left-hand side of estimate (5.29) after taking its mathematical expectation. This finishes up the demonstration of assertion (i). To illustrate estimate (ii), we shall raise inequality (5.29) to the power 2^{p-1} , apply estimate (5.5),

then take the mathematical expectation:

$$\begin{aligned}
& \mathbb{E} \left[\max_{1 \leq m \leq M} \|V^m\|_{\mathbb{L}^2}^{2p} + \left(k\nu \sum_{m=1}^M \|\nabla V^m\|_{\mathbb{L}^2}^2 \right)^{2p-1} + \left(\sum_{m=1}^M \|V^m - V^{m-1}\|_{\mathbb{L}^2}^2 \right)^{2p-1} \right] \lesssim \mathbb{E} \left[\|V^0\|_{\mathbb{L}^2}^{2p} \right] \\
& + \frac{C_D^{2p+1} (C_D + \alpha)^{2p}}{\nu^{3 \times 2^{p-1}} \alpha^{2p+1}} \mathbb{E} \left[\left(k \sum_{m=1}^M \|U^m\|_{\alpha}^4 \alpha^2 \|\Delta^h U^m\|_{\mathbb{L}^2}^2 \right)^{2p-1} \right] + \frac{C_D^{2p} T^{2p-1} K_3^{2p}}{\nu^{2p-1}} \\
& + \frac{C_D^{2p} T^{2p-1} K_4^{2p}}{\nu^{2p-1}} \mathbb{E} \left[\max_{1 \leq m \leq M} \|U^{m-1}\|_{\alpha}^{2p} \right] + \mathbb{E} \left[\left(\sum_{m=1}^M \|g(t_{m-1}, U^{m-1}) \Delta_m W\|_{\mathbb{L}^2}^2 \right)^{2p-1} \right] \\
& + \mathbb{E} \left[\left(\max_{1 \leq \ell \leq M} \sum_{m=1}^{\ell} \left(g(t_{m-1}, U^{m-1}) \Delta_m W, V^{m-1} \right) \right)^{2p-1} \right] \\
& =: \mathbb{E} \left[\|V^0\|_{\mathbb{L}^2}^{2p} \right] + J_1 + \frac{C_D^{2p} T^{2p-1} K_3^{2p}}{\nu^{2p-1}} + J_2 + J_3 + J_4.
\end{aligned}$$

Using the Cauchy-Schwarz inequality and Lemma 5.6.1, it holds that

$$\begin{aligned}
J_1 & \leq \frac{C_D^{2p+1} (C_D + \alpha)^{2p}}{\nu^{3 \times 2^{p-1}} \alpha^{2p+1}} \mathbb{E} \left[\max_{1 \leq m \leq M} \|U^m\|_{\alpha}^{2p+2} \right]^{\frac{1}{2}} \mathbb{E} \left[\left(k \sum_{m=1}^M \left(\|\nabla U^m\|_{\mathbb{L}^2}^2 + \alpha^2 \|\Delta^h U^m\|_{\mathbb{L}^2}^2 \right) \right)^{2p} \right]^{\frac{1}{2}} \\
& \leq \frac{C_D^{2p+1} (C_D + \alpha)^{2p}}{\nu^{3 \times 2^{p-1}} \alpha^{2p+1}} \sqrt{C_{T,p+2}} \sqrt{C_{T,p+1}}.
\end{aligned}$$

J_2 can be readily bounded through Lemma 5.6.1-(ii). On the other hand, we have

$$\begin{aligned}
J_3 & \leq M^{2p-1} \mathbb{E} \left[\max_{1 \leq m \leq M} \left\| \int_{t_{m-1}}^{t_m} g(t_{m-1}, U^{m-1}) dW(t) \right\|_{\mathbb{L}^2}^{2p} \right] \\
& \lesssim M^{2p-1} \mathbb{E} \left[\left(\int_{t_{m-1}}^{t_m} \|g(t_{m-1}, U^{m-1})\|_{\mathcal{L}_2(K, \mathbb{L}^2)}^2 dt \right)^{2p-1} \right] \leq T^{2p-1} \mathbb{E} \left[\max_{1 \leq m \leq M} \left(K_1 + K_2 \|U^{m-1}\|_{\alpha} \right)^{2p} \right] \\
& \lesssim T^{2p-1} K_1^{2p} + T^{2p-1} K_2^{2p} \mathbb{E} \left[\|U^0\|_{\alpha}^{2p} \right] + T^{2p-1} K_2^{2p} C_{T,p},
\end{aligned}$$

thanks to the Burkholder-Davis-Gundy inequality, assumption (S_2) , estimate $|a+b|^q \leq 2^{q-1}(|a|^q + |b|^q)$, and Lemma 5.6.1-(iii). Similarly, J_4 can be controlled by the use of Burkholder-Davis-Gundy's inequality, assumption (S_2) , Young's inequality, estimate $|a+b|^q \leq 2^{q-1}(|a|^q + |b|^q)$, and Lemma 5.6.1-(iii):

$$\begin{aligned}
J_4 & \lesssim \mathbb{E} \left[\left(k \sum_{m=1}^M \|g(t_{m-1}, U^{m-1})\|_{\mathcal{L}_2(K, \mathbb{L}^2)} \|V^{m-1}\|_{\mathbb{L}^2}^2 \right)^{2p-2} \right] \\
& \leq \frac{1}{4} \mathbb{E} \left[\max_{1 \leq m \leq M} \|V^m\|_{\mathbb{L}^2}^{2p} \right] + \mathbb{E} \left[\frac{1}{4} \|V^0\|_{\mathbb{L}^2}^{2p} + \left(T \max_{1 \leq m \leq M} \left(K_1 + K_2 \|U^{m-1}\|_{\alpha} \right)^2 \right)^{2p-1} \right] \\
& \leq \frac{1}{4} \mathbb{E} \left[\max_{1 \leq m \leq M} \|V^m\|_{\mathbb{L}^2}^{2p} \right] + \frac{1}{4} \mathbb{E} \left[\|V^0\|_{\mathbb{L}^2}^{2p} \right] + T^{2p-1} K_2^{2p} \mathbb{E} \left[\|U^0\|_{\alpha}^{2p} \right] + T^{2p-1} (K_1^{2p} + K_2^{2p} C_{T,p}).
\end{aligned}$$

Putting it all together and absorbing the first term on the right-hand side of the above equation terminate the proof. \square

Remark 5.6.1 According to the proof of Lemma 5.6.3, one can assume the relation $\nu^3 \alpha^4 \geq 1$ to tackle the non-uniformness in α of the obtained estimate.

5.7 . Convergence

The first target in this section is to convert the time-independent iterates $\{(U^m, V^m)\}_{m=1}^M$ to time-dependent processes. To this purpose, we define the following functions:

$$\left(\mathcal{U}_{k,h}^-(t, x), \mathcal{V}_{k,h}^-(t, x)\right) := \left(U^{m-1}(x), V^{m-1}(x)\right), \quad \forall (t, x) \in [t_{m-1}, t_m) \times D, \quad (5.30)$$

$$\left(\mathcal{U}_{k,h}^+(t, x), \mathcal{V}_{k,h}^+(t, x)\right) := \left(U^m(x), V^m(x)\right), \quad \forall (t, x) \in (t_{m-1}, t_m] \times D, \quad (5.31)$$

$$\left(f^-(t, \cdot), g^-(t, \cdot)\right) = \left(f(t_{m-1}, \cdot), g(t_{m-1}, \cdot)\right), \quad \forall t \in [t_{m-1}, t_m). \quad (5.32)$$

The following proposition presents a few discrete derivation formulae on which the convergence analysis will rely (c.f. [24, Appendix B]).

Proposition 5.7.1 *We define the discrete derivation with respect to time by $d_t v^m = \frac{v^m - v^{m-1}}{k}$, for all $m \in \{1, \dots, M\}$. Let $v^+, v^- : [0, T] \rightarrow \mathbb{R}$ be the piecewise constant functions defined by $v^+(t) := v^m$ for all $t \in (t_{m-1}, t_m]$, and $v^-(t) := v^{m-1}$ for all $t \in [t_{m-1}, t_m)$. Then, it holds that*

$$(i) \quad d_t(v^+ u^+) = v^+ d_t u^+ + u^- d_t v^+.$$

$$(ii) \quad \int_0^T v^+(t) d_t u^+(t) dt = v^+(T) u^+(T) - v^-(0) u^-(0) - \int_0^T \left(d_t v^+(t)\right) u^-(t) dt.$$

$$(iii) \quad d_t e^{v^+} = e^{v^-} d_t v^+ + e^\eta \frac{(v^+ - v^-)^2}{2k}, \quad \text{for some } \eta \in (v^-, v^+).$$

We exploit the following subsection to demonstrate the convergence of Algorithm 5.1 toward the unique solution of equations (5.1).

5.7.1 . Convergence when α is fixed

Step 1: Boundedness

We aim here at bounding each term of equation (5.19) in a reflexive Banach space. By virtue of Lemmas 5.6.1 and 5.6.3, the sequences $\{\mathcal{U}_{k,h}^+\}_{k,h}$ and $\{\mathcal{V}_{k,h}^+\}_{k,h}$ are bounded in $L^{2p}(\Omega; L^\infty(0, T; \mathbb{L}^2) \cap L^2(0, T; \mathbb{H}_0^1))$ and $L^{2p}(\Omega; L^\infty(0, T; \mathbb{H}_0^1) \cap L^2(0, T; \mathbb{H}^1))$, respectively. In addition, one may bound the sequence $\{\mathcal{U}_{k,h}^+ \times (\nabla \times \mathcal{V}_{k,h}^+)\}_{k,h}$ in $L^{2p}(\Omega; L^2(0, T; \mathbb{H}^{-1}))$ as follows:

$$\begin{aligned} & \mathbb{E} \left[\left(\int_0^T \|\mathcal{U}_{k,h}^+ \times (\nabla \times \mathcal{V}_{k,h}^+)\|_{\mathbb{H}^{-1}}^2 dt \right)^{2p-1} \right] \leq C_D \mathbb{E} \left[\left(\int_0^T \|\nabla \mathcal{U}_{k,h}^+\|_{\mathbb{L}^2}^2 \|\nabla \mathcal{V}_{k,h}^+\|_{\mathbb{L}^2}^2 dt \right)^{2p-1} \right] \\ & \leq C_D \mathbb{E} \left[\sup_{0 \leq t \leq T} \|\nabla \mathcal{U}_{k,h}^+\|_{\mathbb{L}^2}^{2p+1} \right]^{1/2} \mathbb{E} \left[\left(\int_0^T \|\nabla \mathcal{V}_{k,h}^+\|_{\mathbb{L}^2}^2 dt \right)^{2p-1} \right]^{1/2}, \end{aligned}$$

thanks to Proposition 5.2.1-(ii). The right-hand side is bounded by a constant due to Lemmas 5.6.1-(iii) and 5.6.3-(ii). Moreover, for $p \geq 1$, $\{f^-(\cdot, \mathcal{U}_{k,h}^-)\}_{k,h}$ is bounded in $L^{2p}(\Omega; L^2(0, T; \mathbb{H}^{-1}))$; indeed, making use of assumption (S_2) , inequality (5.5), and Lemma 5.6.1-(iii), it follows

$$\begin{aligned} & \mathbb{E} \left[\left(\int_0^T \|f^-(t, \mathcal{U}_{k,h}^-)\|_{\mathbb{H}^{-1}}^2 dt \right)^{2p-1} \right] \leq \mathbb{E} \left[\left(2K_3^2 T + 2K_4^2 \int_0^T \|\mathcal{U}_{k,h}^-\|_\alpha^2 dt \right)^{2p-1} \right] \\ & \leq 2^{2p-1-1} \mathbb{E} \left[2^{2p-1} K_3^{2p} T^{2p-1} + 2^{2p-1} K_4^{2p} T^{2p-1} \max_{1 \leq m \leq M} \|U^{m-1}\|_\alpha^{2p} \right] \leq 2^{2p-1} T^{2p-1} (K_3^{2p} + K_4^{2p} C_{T,p}). \end{aligned}$$

By applying the same technique, one obtained the boundedness of $\{g^-(\cdot, \mathcal{U}_{k,h}^-)\}_{k,h}$ in the reflexive Banach space $L^{2p}(\Omega; L^2(0, T; \mathcal{L}_2(K, \mathbb{L}^2)))$. Therewith, by virtue of Lemmas 5.6.1-(iii)-(iv), 5.6.3-(ii),

there are two subsequences $\{\mathcal{V}_{k',h'}^+\}_{k',h'}$ and $\{\mathcal{U}_{k',h'}^+\}_{k',h'}$ of $\{\mathcal{V}_{k,h}^+\}_{k,h}$ and $\{\mathcal{U}_{k,h}^+\}_{k,h}$, respectively, permitting the following convergences

$$\mathcal{V}_{k',h'}^+ \rightharpoonup v_\alpha \quad \& \quad \mathcal{U}_{k',h'}^+ \rightharpoonup u_\alpha \quad \text{in } L^{2p}(\Omega; L^2(0, T; \mathbb{H}_0^1)), \quad (5.33)$$

$$\mathcal{V}_{k',h'}^+ \overset{*}{\rightharpoonup} v_\alpha \quad \text{in } L^{2p}(\Omega; L^\infty(0, T; \mathbb{L}^2)), \quad (5.34)$$

$$\mathcal{U}_{k',h'}^+ \overset{*}{\rightharpoonup} u_\alpha \quad \text{in } L^{2p}(\Omega; L^\infty(0, T; \mathbb{H}_0^1)), \quad (5.35)$$

$$\mathcal{U}_{k',h'}^+ \times (\nabla \times \mathcal{V}_{k',h'}^+) \rightharpoonup B_0 \quad \text{in } L^{2p}(\Omega; L^2(0, T; \mathbb{H}^{-1})), \quad (5.36)$$

$$f^-(\cdot, \mathcal{U}_{k',h'}^-) \rightharpoonup F_0 \quad \text{in } L^{2p}(\Omega; L^2(0, T; \mathbb{H}^{-1})), \quad (5.37)$$

$$g^-(\cdot, \mathcal{U}_{k',h'}^-) \rightharpoonup G_0 \quad \text{in } L^{2p}(\Omega; L^2(0, T; \mathcal{L}_2(K; \mathbb{L}^2))), \quad (5.38)$$

for some functions $v_\alpha \in L^{2p}(\Omega; L^\infty(0, T; \mathbb{L}^2) \cap L^2(0, T; \mathbb{H}_0^1))$, $u_\alpha \in L^{2p}(\Omega; L^\infty(0, T; \mathbb{H}_0^1) \cap L^2(0, T; \mathbb{H}_0^1))$, $B_0 \in L^{2p}(\Omega; L^2(0, T; \mathbb{H}^{-1}))$, $F_0 \in L^{2p}(\Omega; L^2(0, T; \mathbb{H}^{-1}))$ and $G_0 \in L^{2p}(\Omega; L^2(0, T; \mathcal{L}_2(K; \mathbb{L}^2)))$.

Remark 5.7.1 For $p \in [1, +\infty)$, since $L^{2p}(\Omega) \hookrightarrow L^2(\Omega)$ continuously, all the above obtained convergence results hold particularly in $L^2(\Omega; \mathcal{X})$, where \mathcal{X} corresponds to the space of each convergence.

Step 2: $v_\alpha = u_\alpha + \alpha^2 Au_\alpha$ and properties of u_α

By convergence (5.33) and Lemmas-5.6.1-(iv), 5.6.3-(ii), it holds

$$\mathbb{E} \left[\left(\int_0^T \|\nabla v_\alpha(t)\|_{\mathbb{L}^2}^2 dt \right)^{2p-1} \right] \leq C_p(\alpha) \quad \text{and} \quad \mathbb{E} \left[\left(\int_0^T \|\nabla u_\alpha(t)\|_{\mathbb{L}^2}^2 dt \right)^{2p-1} \right] \leq C_{T,p}.$$

Moreover, by convergences (5.34)-(5.35) and Lemmas 5.6.1-(iii), 5.6.3-(ii), we get

$$\mathbb{E} \left[\sup_{0 \leq t \leq T} \|v_\alpha(t)\|_{\mathbb{L}^2}^{2p} \right] \leq C_p(\alpha) \quad \text{and} \quad \mathbb{E} \left[\sup_{0 \leq t \leq T} \|u_\alpha(t)\|_{\mathbb{H}^1}^{2p} \right] \leq C_{T,p}.$$

Let $w \in L^2(\Omega; L^2(0, T; D(A)))$ be arbitrary. From equation (5.19), it follows that

$$\mathbb{E} \left[\int_0^T (\mathcal{V}_{k',h'}^+, \Pi_{h'} w(t)) dt \right] = \mathbb{E} \left[\int_0^T (\mathcal{U}_{k',h'}^+, \Pi_{h'} w(t)) dt \right] + \alpha^2 \mathbb{E} \left[\int_0^T (\nabla \mathcal{U}_{k',h'}^+, \nabla \Pi_{h'} w(t)) dt \right].$$

Taking into account that $\Pi_{h'} w \rightarrow w$ as $h' \rightarrow 0$, strongly in $L^2(\Omega; L^2(0, T; \mathbb{H}^1))$ together with convergence (5.33) and the embedding $L^{2p}(\Omega) \hookrightarrow L^2(\Omega)$, one gets

$$\mathbb{E} \left[\int_0^T (v_\alpha(t), w(t)) dt \right] = \mathbb{E} \left[\int_0^T (u_\alpha(t), w(t)) dt \right] + \alpha^2 \mathbb{E} \left[\int_0^T (\nabla u_\alpha(t), \nabla w(t)) dt \right],$$

for all $w \in L^2(\Omega; L^2(0, T; D(A)))$, which implies $v_\alpha = u_\alpha + \alpha^2 Au_\alpha$ in $L^2(\Omega; L^2(0, T; \mathbb{H}^{-1}))$. Owing to convergences (5.34) and (5.35), $\alpha^2 Au_\alpha = v_\alpha - u_\alpha \in L^2(\Omega; L^\infty(0, T; \mathbb{L}^2))$. Subsequently, $v_\alpha = u_\alpha + \alpha^2 Au_\alpha$, \mathbb{P} -a.s. and a.e. in $(0, T) \times D$. This implies

$$\begin{aligned} \mathbb{E} \left[\left(\alpha^4 \int_0^T \|Au_\alpha(t)\|_{\mathbb{L}^2}^2 dt \right)^{2p-1} \right] &\leq 2^{2p-1} T^{2p-1} \mathbb{E} \left[\sup_{0 \leq t \leq T} \|v_\alpha(t)\|_{\mathbb{L}^2}^{2p} + \sup_{0 \leq t \leq T} \|u_\alpha(t)\|_{\mathbb{L}^2}^{2p} \right] \\ &\leq 2^{2p-1} T^{2p-1} (C_p(\alpha) + C_{T,p}), \end{aligned}$$

thanks to inequality (5.5).

The weak time-continuity of the process u_α (i.e. $u_\alpha \in L^2(\Omega; C_w([0, T]; \mathbb{H}^1))$) can be illustrated through the time-continuity of v_α (see step 3).

Step 3: Auxiliary scheme

Owing to equation (5.19) and the convergence results (5.33)-(5.38), the stochastic process $v_\alpha(t)$, $t \in [0, T]$ belongs to $L^2(\Omega; C([0, T]; \mathbb{L}^2))$ (e.g. [95]), and fulfills \mathbb{P} -a.s., for all $t \in [0, T]$ and $\varphi \in \mathbb{V}$ the equation

$$\begin{aligned} & (v_\alpha(t), \varphi) + \nu \int_0^t (\nabla v_\alpha(s), \nabla \varphi) ds - \int_0^t \langle B_0(s), \varphi \rangle ds \\ &= (v_0, \varphi) + \int_0^t \langle F_0(s), \varphi \rangle ds + \int_0^t (G_0(s) dW(s), \varphi). \end{aligned} \quad (5.39)$$

In order to identify the obtained limiting functions in Step 1 with their counterparts, we need to do a subtraction in one way or another allowing us to appear the difference between the abstract and the solution-dependent functions, for instance $\|F_0 - f(\cdot, u_\alpha)\|$ for some norm $\|\cdot\|$ to be determined later. Since neither the employed finite element method involves strong divergence-free vector fields as test functions nor the space of strongly divergence-free vector fields is included in the space of weakly divergence-free vectors fields, the subtraction of equation (5.39) from the scheme (5.19) does not seem to make any sense. Instead, we propose a fully discrete auxiliary scheme arising from equation (5.39) and permitting the subtraction we mentioned shortly before. We will make use of the time and space discretizations that were introduced in Section 5.4. We define a starting point of the auxiliary scheme $(V_\alpha^0, U_\alpha^0) = (V^0, U^0)$ and the discrete versions of B_0, F_0 and G_0 as follows: for all $m \in \{1, \dots, M\}$,

$$\begin{aligned} B_0^m &= \frac{1}{k} \int_{t_{m-1}}^{t_m} B_0(t) dt, \\ (F_0^0, G_0^0) &= (F_0(0), G_0(0)), \quad \text{and} \quad (F_0^m, G_0^m) = \left(\frac{1}{k} \int_{t_{m-1}}^{t_m} F_0(t) dt, \frac{1}{k} \int_{t_{m-1}}^{t_m} G_0(t) dt \right). \end{aligned} \quad (5.40)$$

Such approximations will be required within the last step for the sake of obtaining strong convergence in time toward their non-discretized counterparts. For all $(\varphi, \psi) \in \mathbb{V}_h \times \mathbb{V}_h$ and for every $m \in \{1, \dots, M\}$, the auxiliary scheme reads

$$\begin{cases} (V_\alpha^m - V_\alpha^{m-1}, \varphi) + k\nu(\nabla V_\alpha^m, \nabla \varphi) = k \langle B_0^m, \varphi \rangle + k \langle F_0^{m-1}, \varphi \rangle + (G_0^{m-1} \Delta_m W, \varphi), \\ (V_\alpha^m, \psi) = (U_\alpha^m, \psi) + \alpha^2 (\nabla U_\alpha^m, \nabla \psi). \end{cases} \quad (5.41)$$

Equation (5.39) can be considered as a stochastic Stokes problem driven by an additive noise with diffusion coefficient $G_0 \in L^2(\Omega; L^\infty(0, T; \mathcal{L}_2(K; \mathbb{L}^2)))$, initial datum $v_0 \in L^4(\Omega; \mathbb{L}^2)$ and a source term $B_0 + F_0 \in L^2(\Omega; L^2(0, T; \mathbb{H}^{-1}))$. For all $m \in \{1, \dots, M\}$, define the following piecewise constant processes:

$$\begin{aligned} (\mathcal{V}_\alpha^+(t), \mathcal{U}_\alpha^+(t), B_0^+(t)) &= (V_\alpha^m, U_\alpha^m, B_0^m), \quad \forall t \in (t_{m-1}, t_m], \\ (\mathcal{V}_\alpha^-(t), \mathcal{U}_\alpha^-(t), F_0^-(t), G_0^-(t)) &= (V_\alpha^{m-1}, U_\alpha^{m-1}, F_0^{m-1}, G_0^{m-1}), \quad \forall t \in [t_{m-1}, t_m). \end{aligned}$$

For a complete investigation of scheme (5.41), the reader may refer to [56], from which we pick the following convergence results as $k, h \rightarrow 0$ (see Theorem 6 therein):

$$\mathcal{V}_\alpha^+ \rightarrow v_\alpha \text{ in } L^2(\Omega; L^\infty(0, T; \mathbb{L}^2)) \quad \text{and} \quad \mathcal{V}_\alpha^- \rightarrow v_\alpha \text{ in } L^2(\Omega; L^2(0, T; \mathbb{H}_0^1)). \quad (5.42)$$

The above convergences are in the strong sense. Taking now the second equation of (5.41) and plugging it back into its first one, we get

$$\left((U_\alpha^m - U_\alpha^{m-1}, \varphi) \right)_\alpha + k\nu((U_\alpha^m, \varphi))_{\alpha, h} = k \langle B_0^m, \varphi \rangle + k \langle F_0^{m-1}, \varphi \rangle + (G_0^{m-1} \Delta_m W, \varphi),$$

for all $\varphi \in \mathbb{V}_h$, where $((\cdot, \cdot))_\alpha := (\cdot, \cdot) + \alpha^2 (\nabla \cdot, \nabla \cdot)$ and $((\cdot, \cdot))_{\alpha, h} := (\nabla \cdot, \nabla \cdot) + \alpha^2 (\Delta^h \cdot, \Delta^h \cdot)$ can be treated in the same way as the \mathbb{L}^2 and the \mathbb{H}_0^1 -inner products, respectively. Thereby, applying once again [56, Theorem 6] yields the following strong convergence as $k, h \rightarrow 0$:

$$\mathcal{U}_\alpha^+ \rightarrow u_\alpha \text{ in } L^2(\Omega; L^\infty(0, T; \mathbb{H}^1)). \quad (5.43)$$

Notice that the limiting function in (5.43) is u_α which turns out to be true after adjusting the \mathbb{L}^2 and the \mathbb{H}_0^1 -inner products in equation (5.39) to fit the framework of that of the scheme, through the identity $v_\alpha = u_\alpha + \alpha^2 Au_\alpha$ that was illustrated in Step 2. We point out that $((\cdot, \cdot))_{\alpha, h}$ corresponds in this case to $(\nabla \cdot, \nabla \cdot) + \alpha^2 (\Delta \cdot, \Delta \cdot)$, which makes sense because $u_\alpha \in L^2(\Omega; L^2(0, T; D(A)))$. We still need to exhibit the a priori estimates satisfied by $\{V_\alpha^m\}_m$ and $\{U_\alpha^m\}_m$ before moving on to the next step. To this end, we replace φ by V_α^m in equation (5.41), then we follow the same demonstration technique of Lemma 5.6.3 to obtain eventually for all $p \in [1, +\infty)$,

$$\mathbb{E} \left[\max_{1 \leq m \leq M} \|V_\alpha^m\|_{\mathbb{L}^2}^{2p} + \left(k\nu \sum_{m=1}^M \|\nabla V_\alpha^m\|_{\mathbb{L}^2}^2 \right)^{2p-1} + \left(\sum_{m=1}^M \|V_\alpha^m - V_\alpha^{m-1}\|_{\mathbb{L}^2}^2 \right)^{2p-1} \right] \leq C_{\alpha, v}, \quad (5.44)$$

where $C_{\alpha, v} > 0$ does not depend on k and h . On the other hand, replacing φ in scheme (5.41) with U_α^m , and following the proof steps of Lemma 5.6.1, we acquire for all $p \in [1, +\infty)$:

$$\mathbb{E} \left[\max_{1 \leq m \leq M} \|U_\alpha^m\|_\alpha^{2p} + \sum_{m=1}^M \|U_\alpha^m - U_\alpha^{m-1}\|_\alpha^2 + \left(k\nu \sum_{m=1}^M \|U_\alpha^m\|_{\alpha, h}^2 \right)^{2p-1} \right] \leq C_{\alpha, u}, \quad (5.45)$$

for some $C_{\alpha, u} > 0$ independent of k and h , where $\|\cdot\|_{\alpha, h}^2 := \|\nabla \cdot\|_{\mathbb{L}^2}^2 + \alpha^2 \|\Delta \cdot\|_{\mathbb{L}^2}^2$.

Step 4: Identification of B_0, F_0 and G_0

From now on, the indices k' and h' that were derived in Step 1, will be denoted k and h for the sake of clarity. For each $n \in \mathbb{N} \setminus \{0\}$ and for all $m \in \{0, 1, \dots, M\}$, we define the discrete stopping time

$$\tau_m^n := \min \left(t_m, \min_{0 \leq \ell \leq m} \left\{ t_\ell \mid \|u_\alpha(t_\ell)\|_\alpha^2 + \int_0^{t_\ell} \|u_\alpha(s)\|_{D(A)}^2 ds > n \right\} \right).$$

Obviously, τ_m^n belongs to the time discretization grid points $\{t_0, \dots, t_M\}$. It is worth mentioning that when $k \rightarrow 0$ (i.e. $M \rightarrow +\infty$), τ_m^n converges \mathbb{P} -a.s. toward the following \mathcal{F}_s -stopping time

$$\tau^n := \min \left(t, \inf \left\{ s \in [0, t] \mid \|u_\alpha(s)\|_\alpha^2 + \int_0^s \|u_\alpha(\lambda)\|_{D(A)}^2 d\lambda > n \right\} \right).$$

For all $m \in \{1, \dots, M\}$, we define the discrete weight

$$\rho_m := \exp \left(-\eta_1 t_m - \eta_2 \int_0^{t_m} \|u_\alpha(s)\|_{\mathbb{H}^1} \|u_\alpha(s)\|_{\mathbb{H}^2} ds \right) =: \rho(t_m),$$

where $\eta_1, \eta_2 > 0$ are to be fixed later. Clearly, ρ_m is \mathcal{F}_{t_m} -measurable and is non-increasing. In addition to that, we need to deal with a piecewise-constant version of ρ_m , which is why we introduce first the notations

$$i^+(t) = t_m, \quad \forall t \in (t_{m-1}, t_m] \quad \text{and} \quad i^-(t) = t_{m-1}, \quad \forall t \in [t_{m-1}, t_m).$$

We therefore set

$$\rho^+(t) = \rho(i^+(t)), \quad \forall t \in (t_{m-1}, t_m] \quad \text{and} \quad \rho^-(t) = \rho(i^-(t)), \quad \forall t \in [t_{m-1}, t_m).$$

We subtract both equations (5.19) and (5.41), then apply identity (5.3):

$$\begin{aligned} & \frac{1}{2} \|U_\alpha^m - U^m\|_\alpha^2 - \frac{1}{2} \|U_\alpha^{m-1} - U^{m-1}\|_\alpha^2 + k\nu \|U_\alpha^m - U^m\|_{\alpha, h}^2 \\ & \leq k \langle B_0^m - U^m \times (\nabla \times V^m), U_\alpha^m - U^m \rangle + k \langle F_0^{m-1} - f(t_{m-1}, U^{m-1}), U_\alpha^m - U^m \rangle \\ & + \left([G_0^{m-1} - g(t_{m-1}, U^{m-1})] \Delta_m W, U_\alpha^{m-1} - U^{m-1} \right) \\ & + \frac{1}{2} \| [G_0^{m-1} - g(t_{m-1}, U^{m-1})] \Delta_m W \|_{\mathbb{L}^2}^2. \end{aligned} \quad (5.46)$$

The next step would be to multiply equation (5.46) by ρ_{m-1} and sum it over m . However, we must clarify a few identities before. Let $\ell \in \{1, \dots, M\}$ be arbitrary. By Proposition 5.7.1, it holds that

$$\begin{aligned} \sum_{m=1}^{\ell} \rho_{m-1} \left(\|U_{\alpha}^m - U^m\|_{\alpha}^2 - \|U_{\alpha}^{m-1} - U^{m-1}\|_{\alpha}^2 \right) &= \int_0^{t_{\ell}} \rho^{-}(t) d_t \|U_{\alpha}^{+} - U_{k,h}^{+}\|_{\alpha}^2 dt \\ &= \rho^{+}(t_{\ell}) \|U_{\alpha}^{+}(t_{\ell}) - U_{k,h}^{+}(t_{\ell})\|_{\alpha}^2 - \int_0^{t_{\ell}} \|U_{\alpha}^{+} - U_{k,h}^{+}\|_{\alpha}^2 d_t \rho^{+}(t) dt. \end{aligned} \quad (5.47)$$

Moreover,

$$\begin{aligned} \sum_{m=1}^{\ell} \rho_{m-1} \left([G_0^{m-1} - g(t_{m-1}, U^{m-1})] \Delta_m W, U_{\alpha}^{m-1} - U^{m-1} \right) \\ = \int_D \int_0^{t_{\ell}} \rho^{-}(t) \left(U_{\alpha}^{-} - U_{k,h}^{-} \right) \cdot [G_0^{-}(t) - g^{-}(t, U_{k,h}^{-})] dW(t) dx =: M_1(t_{\ell}). \end{aligned} \quad (5.48)$$

And,

$$\begin{aligned} \sum_{m=1}^{\ell} \rho_{m-1} \| [G_0^{m-1} - g(t_{m-1}, U^{m-1})] \Delta_m W \|_{\mathbb{L}^2}^2 \\ = \sum_{m=1}^{\ell} \left\| \int_{t_{m-1}}^{t_m} \sqrt{\rho^{-}(t)} [G_0^{-}(t) - g^{-}(t, U_{k,h}^{-})] dW(t) \right\|_{\mathbb{L}^2}^2 =: M_2(t_{\ell}). \end{aligned} \quad (5.49)$$

Subsequently, we multiply equation (5.46) by ρ_{m-1} , sum it over m from 1 to $\ell \in \{1, \dots, M\}$, make use of (5.47)-(5.49), replace afterwards the node t_{ℓ} by the discrete stopping time τ_{m-1}^n , and then apply the mathematical expectation to get

$$\begin{aligned} \frac{1}{2} \mathbb{E} \left[\rho^{+}(\tau_{m-1}^n) \|U_{\alpha}^{+}(\tau_{m-1}^n) - U_{k,h}^{+}(\tau_{m-1}^n)\|_{\alpha}^2 \right] + \nu \mathbb{E} \left[\int_0^{\tau_{m-1}^n} \rho^{-}(t) \|U_{\alpha}^{+} - U_{k,h}^{+}\|_{\alpha,h}^2 dt \right] \\ \leq \mathbb{E} \left[\int_0^{\tau_{m-1}^n} \rho^{-}(t) \left\langle B_0^{+}(t) - U_{k,h}^{+} \times (\nabla \times \mathcal{V}_{k,h}^{+}) + F_0^{-}(t) - f^{-}(t, U_{k,h}^{-}), U_{\alpha}^{+} - U_{k,h}^{+} \right\rangle dt \right] \\ + \frac{1}{2} M_2(\tau_{m-1}^n) + \frac{1}{2} \int_0^{\tau_{m-1}^n} \|U_{\alpha}^{+} - U_{k,h}^{+}\|_{\alpha}^2 d_t \rho^{+}(t) dt, \end{aligned} \quad (5.50)$$

where $\mathbb{E} [M_1(\tau_{m-1}^n)] = 0$ due to assumption (S_2) which implies that the integrand of M_1 is a martingale, together with the fact that τ_{m-1}^n is a stopping time. Next, we handle $M_2(\tau_{m-1}^n)$ as follows:

$$\begin{aligned} \mathbb{E} [M_2(\tau_{m-1}^n)] &= \sum_{i=1}^{m-1} \mathbb{E} \left[\left\| \int_{\tau_{i-1}^n}^{\tau_i^n} \sqrt{\rho^{-}(t)} [G_0^{-}(t) - g^{-}(t, U_{k,h}^{-})] dW(t) \right\|_{\mathbb{L}^2}^2 \right] \\ &= \sum_{i=1}^{m-1} \mathbb{E} \left[\int_{\tau_{i-1}^n}^{\tau_i^n} \rho^{-}(t) \|G_0^{-}(t) - g^{-}(t, U_{k,h}^{-})\|_{\mathcal{L}_2(K, \mathbb{L}^2)}^2 dt \right] \\ &= \mathbb{E} \left[\int_0^{\tau_{m-1}^n} \rho^{-}(t) \|G_0^{-}(t) - g^{-}(t, U_{k,h}^{-})\|_{\mathcal{L}_2(K, \mathbb{L}^2)}^2 dt \right], \end{aligned} \quad (5.51)$$

thanks to the Itô isometry. On the other hand,

$$\begin{aligned} \|G_0^{-}(t) - g^{-}(t, U_{k,h}^{-})\|_{\mathcal{L}_2(K, \mathbb{L}^2)}^2 &= \|g^{-}(t, u_{\alpha}(t)) - g^{-}(t, U_{k,h}^{-})\|_{\mathcal{L}_2(K, \mathbb{L}^2)}^2 \\ &+ 2 \left(G_0^{-}(t) - g^{-}(t, U_{k,h}^{-}), G_0^{-}(t) - g^{-}(t, u_{\alpha}(t)) \right)_{\mathcal{L}_2(K, \mathbb{L}^2)} - \|g^{-}(t, u_{\alpha}(t)) - G_0^{-}(t)\|_{\mathcal{L}_2(K, \mathbb{L}^2)}^2 \\ &\leq 2L_g^2 \|u_{\alpha}(t) - U_{\alpha}^{-}\|_{\alpha}^2 + 2L_g^2 \|U_{\alpha}^{-} - U_{k,h}^{-}\|_{\alpha}^2 \\ &+ 2 \left(G_0^{-}(t) - g^{-}(t, U_{k,h}^{-}), G_0^{-}(t) - g^{-}(t, u_{\alpha}(t)) \right)_{\mathcal{L}_2(K, \mathbb{L}^2)} - \|g^{-}(t, u_{\alpha}(t)) - G_0^{-}(t)\|_{\mathcal{L}_2(K, \mathbb{L}^2)}^2, \end{aligned} \quad (5.52)$$

where the Lipschitz-continuity of g has been utilized. Before heading toward the calculation of $d_t \rho^{+}(t)$, we define

$$\gamma^{+}(t) := -\eta_1 t_m - \eta_2 \int_0^{t_m} \|u_{\alpha}(s)\|_{\mathbb{H}^1} \|u_{\alpha}(s)\|_{\mathbb{H}^2} ds, \quad \text{for all } t \in (t_{m-1}, t_m], \quad \text{for all } m \in \{1, \dots, M\}.$$

The function γ^- can be defined accordingly. We point out that $\rho^+ = e^{\gamma^+}$. By Proposition 5.7.1, it holds for $t \in (t_{m-1}, t_m)$,

$$d_t \rho^+(t) = -\eta_1 \rho^-(t) - \eta_2 \rho^-(t) \frac{1}{k} \int_{i^-(t)}^{i^+(t)} \|u_\alpha(s)\|_{\mathbb{H}^1} \|u_\alpha(s)\|_{\mathbb{H}^2} ds + e^{\xi(t)} \frac{(\gamma^+ - \gamma^-)^2}{2k},$$

for all $m \in \{1, \dots, M\}$, for some $\xi \in (\gamma^-, \gamma^+)$. Now, for each $m \in \{1, \dots, M\}$, there is a $\zeta_m \in (t_{m-1}, t_m)$ such that

$$\int_{t_{m-1}}^{t_m} \|u_\alpha(s)\|_{\mathbb{H}^1} \|u_\alpha(s)\|_{\mathbb{H}^2} ds = k \|u_\alpha(\zeta_m)\|_{\mathbb{H}^1} \|u_\alpha(\zeta_m)\|_{\mathbb{H}^2},$$

thanks to the mean value theorem. Thus, for all $m \in \{1, \dots, M\}$ and $t \in (t_{m-1}, t_m)$,

$$d_t \rho^+(t) = -\eta_1 \rho^-(t) - \eta_2 \rho^-(t) \|u_\alpha(\zeta_m)\|_{\mathbb{H}^1} \|u_\alpha(\zeta_m)\|_{\mathbb{H}^2} + e^{\xi(t)} \frac{(\gamma^+ - \gamma^-)^2}{2k}. \quad (5.53)$$

Furthermore, we take advantage of all (i), (ii) and (iii) of Proposition 5.2.1 to write for all $t \in (t_{m-1}, t_m)$:

$$\begin{aligned} \langle B_0^+(t) - \mathcal{U}_{k,h}^+ \times (\nabla \times \mathcal{V}_{k,h}^+), \mathcal{U}_\alpha^+ - \mathcal{U}_{k,h}^+ \rangle &\leq \langle B_0^+(t) - \mathcal{U}_\alpha^+ \times (\nabla \times \mathcal{V}_\alpha^+), \mathcal{U}_\alpha^+ - \mathcal{U}_{k,h}^+ \rangle \\ &+ C_D \|\nabla(\mathcal{U}_\alpha^+ - u_\alpha(t))\|_{\mathbb{L}^2} \|\nabla(\mathcal{V}_\alpha^+ - \mathcal{V}_{k,h}^+)\|_{\mathbb{L}^2} \|\nabla(\mathcal{U}_\alpha^+ - \mathcal{U}_{k,h}^+)\|_{\mathbb{L}^2} \\ &+ C_D \|\nabla(u_\alpha(t) - u_\alpha(\zeta_m))\|_{\mathbb{L}^2} \|\nabla(\mathcal{V}_\alpha^+ - \mathcal{V}_{k,h}^+)\|_{\mathbb{L}^2} \|\nabla(\mathcal{U}_\alpha^+ - \mathcal{U}_{k,h}^+)\|_{\mathbb{L}^2} \\ &+ C_D \|u_\alpha(\zeta_m)\|_{\mathbb{H}^1}^{\frac{1}{2}} \|u_\alpha(\zeta_m)\|_{\mathbb{H}^2}^{\frac{1}{2}} \|\mathcal{V}_\alpha^+ - \mathcal{V}_{k,h}^+\|_{\mathbb{L}^2} \|\nabla(\mathcal{U}_\alpha^+ - \mathcal{U}_{k,h}^+)\|_{\mathbb{L}^2}. \end{aligned} \quad (5.54)$$

The last term can be bounded through Young's inequality by

$$\frac{2C_D^2 \max(C_D, \alpha^2)^2}{\nu \alpha^4} \|u_\alpha(\zeta_m)\|_{\mathbb{H}^1} \|u_\alpha(\zeta_m)\|_{\mathbb{H}^2} \|\mathcal{U}_\alpha^+ - \mathcal{U}_{k,h}^+\|_\alpha^2 + \frac{\nu}{4} \|\mathcal{U}_\alpha^+ - \mathcal{U}_{k,h}^+\|_{\alpha,h}^2,$$

where the estimate $\|\mathcal{V}_\alpha^+ - \mathcal{V}_{k,h}^+\|_{\mathbb{L}^2} \leq \max(C_D, \alpha^2) \|\mathcal{U}_\alpha^+ - \mathcal{U}_{k,h}^+\|_{\alpha,h}$ was employed. Moreover, by assumption (S_2) , Young's inequality and $\|\cdot\|_{\mathbb{H}^1} \leq C_D \|\cdot\|_{\alpha,h}$, it follows

$$\begin{aligned} \langle F_0^-(t) - f^-(t, \mathcal{U}_{k,h}^-), \mathcal{U}_\alpha^+ - \mathcal{U}_{k,h}^+ \rangle &\leq \langle F_0^-(t) - f^-(t, \mathcal{U}_\alpha^-), \mathcal{U}_\alpha^+ - \mathcal{U}_{k,h}^+ \rangle \\ &+ \frac{L_f^2 C_D^2}{\nu} \|\mathcal{U}_\alpha^- - \mathcal{U}_{k,h}^-\|_\alpha^2 + \frac{\nu}{4} \|\mathcal{U}_\alpha^+ - \mathcal{U}_{k,h}^+\|_{\alpha,h}^2. \end{aligned} \quad (5.55)$$

On the other hand, since $U_\alpha^0 = U^0$ and $(\rho_m)_m$ is non-increasing, the following holds:

$$\int_0^{\tau_{m-1}^n} \rho^- \|\mathcal{U}_\alpha^- - \mathcal{U}_{k,h}^-\|_\alpha^2 dt \leq \int_0^{\tau_{m-1}^n} \rho^+ \|\mathcal{U}_\alpha^+ - \mathcal{U}_{k,h}^+\|_\alpha^2 dt \leq \int_0^{\tau_{m-1}^n} \rho^- \|\mathcal{U}_\alpha^+ - \mathcal{U}_{k,h}^+\|_\alpha^2 dt. \quad (5.56)$$

By setting $\eta_1 = 2 \frac{L_f^2 C_D^2}{\nu} + 2L_g^2$ and $\eta_2 = \frac{4C_D^2 \max(C_D, \alpha^2)^2}{\nu \alpha^4}$ and after assembling the obtained estimates

(5.51)-(5.56) together, equation (5.50) becomes

$$\begin{aligned}
& \frac{1}{2} \mathbb{E} \left[\rho^+(\tau_{m-1}^n) \|\mathcal{U}_\alpha^+(\tau_{m-1}^n) - \mathcal{U}_{k,h}^+(\tau_{m-1}^n)\|_\alpha^2 + \nu \int_0^{\tau_{m-1}^n} \rho^- \|\mathcal{U}_\alpha^+ - \mathcal{U}_{k,h}^+\|_{\alpha,h}^2 dt \right. \\
& \left. + \int_0^{\tau_{m-1}^n} \rho^- \|g^-(t, u_\alpha(t)) - G_0^-(t)\|_{\mathcal{L}_2(K; \mathbb{L}^2)}^2 dt \right] \\
& \leq \mathbb{E} \left[\int_0^{\tau_{m-1}^n} e^{\xi} \frac{(\gamma^+ - \gamma^-)^2}{4k} \|\mathcal{U}_\alpha^+ - \mathcal{U}_{k,h}^+\|_\alpha^2 dt \right. \\
& \left. + \int_0^{\tau_{m-1}^n} \rho^- \langle B_0^+ - \mathcal{U}_\alpha^+ \times (\nabla \times \mathcal{V}_\alpha^+), \mathcal{U}_\alpha^+ - \mathcal{U}_{k,h}^+ \rangle dt \right. \\
& \left. + C_D \int_0^{\tau_{m-1}^n} \rho^- \|\nabla(\mathcal{U}_\alpha^+ - \mathcal{U}_{k,h}^+)\|_{\mathbb{L}^2} \|\nabla(\mathcal{V}_\alpha^+ - \mathcal{V}_{k,h}^+)\|_{\mathbb{L}^2} \|\nabla(\mathcal{U}_\alpha^+ - u_\alpha(t))\|_{\mathbb{L}^2} dt \right. \\
& \left. + C_D \int_0^{\tau_{m-1}^n} \rho^- \|\nabla(u_\alpha(t) - u_\alpha(\zeta_m))\|_{\mathbb{L}^2} \|\nabla(\mathcal{V}_\alpha^+ - \mathcal{V}_{k,h}^+)\|_{\mathbb{L}^2} \|\nabla(\mathcal{U}_\alpha^+ - \mathcal{U}_{k,h}^+)\|_{\mathbb{L}^2} dt \right. \\
& \left. + \int_0^{\tau_{m-1}^n} \rho^- \langle F_0^-(t) - f^-(t, \mathcal{U}_\alpha^-), \mathcal{U}_\alpha^+ - \mathcal{U}_{k,h}^+ \rangle dt + L_g^2 \int_0^{\tau_{m-1}^n} \rho^- \|u_\alpha(t) - \mathcal{U}_\alpha^-\|_\alpha^2 dt \right. \\
& \left. + \int_0^{\tau_{m-1}^n} \rho^- \left(G_0^-(t) - g^-(t, \mathcal{U}_{k,h}^-), G_0^-(t) - g^-(t, u_\alpha(t)) \right)_{\mathcal{L}_2(K; \mathbb{L}^2)} dt \right] =: I + \dots + VII. \tag{5.57}
\end{aligned}$$

Step 4.1: $(\rho_m)_m$ is strongly convergent in $L^4(\Omega; L^2(0, T))$

We recall the notation $\rho_m = \rho^+(t) = \rho(i^+(t))$ for $t \in (t_{m-1}, t_m]$. We have

$$\begin{aligned}
& \mathbb{E} \left[\left(\int_0^T |\rho(t) - \rho(i^+(t))|^2 dt \right)^2 \right] \leq T \mathbb{E} \left[\int_0^T \left| e^{-\eta_1(t-i^+(t)) - \eta_2 \int_{i^+(t)}^t \|u_\alpha(s)\|_{\mathbb{H}^1} \|u_\alpha(s)\|_{\mathbb{H}^2} ds} - 1 \right|^4 dt \right] \\
& \leq T \eta_1 \int_0^T |t - i^+(t)| dt + T \eta_2 \mathbb{E} \left[\int_0^T \int_t^{i^+(t)} \|u_\alpha(s)\|_{\mathbb{H}^1} \|u_\alpha(s)\|_{\mathbb{H}^2} ds dt \right],
\end{aligned}$$

where Jensen's inequality and $|e^{-|x|} - 1|^4 \leq |x|^4$ were employed in the first and second inequalities, respectively. Since for each $t \in (t_{m-1}, t_m)$, we have $|t - i^+(t)| \leq k$, the first term goes to 0 as $k \rightarrow 0$. Similarly, the second term converges to 0 by a simple application of the dominated convergence theorem.

Step 4.2: Convergence of I, \dots, VII to 0

We have

$$\gamma^+ - \gamma^- = -\eta_1 k - \eta_2 \int_{i^-(t)}^{i^+(t)} \|u_\alpha(s)\|_{\mathbb{H}^1} \|u_\alpha(s)\|_{\mathbb{H}^2} ds, \quad \text{for all } t \in (t_{m-1}, t_m).$$

By Jensen's inequality, one gets

$$\frac{(\gamma^+ - \gamma^-)^2}{4k} \leq \frac{\eta_1^2}{2} k + \frac{\eta_2^2}{2} \int_{i^-(t)}^{i^+(t)} \|u_\alpha(s)\|_{\mathbb{H}^1}^2 \|u_\alpha(s)\|_{\mathbb{H}^2}^2 ds.$$

This implies

$$\begin{aligned}
I & \leq T \eta_1^2 k (C_{\alpha, u} + C_T) + \frac{\eta_2^2}{2} \mathbb{E} \left[\sup_{0 \leq t \leq T} \|\mathcal{U}_\alpha^+ - \mathcal{U}_{k,h}^+\|_\alpha^2 \sum_{m=1}^M \int_{t_{m-1}}^{t_m} \int_{t_{m-1}}^{t_m} \|u_\alpha(s)\|_{\mathbb{H}^1}^2 \|u_\alpha(s)\|_{\mathbb{H}^2}^2 ds dt \right] \\
& \leq T \eta_1^2 k (C_{\alpha, u} + C_T) + \frac{\eta_2^2 k}{2} \mathbb{E} \left[\sup_{0 \leq t \leq T} \|\mathcal{U}_\alpha^+ - \mathcal{U}_{k,h}^+\|_\alpha^2 \sup_{0 \leq t \leq T} \|u_\alpha(t)\|_{\mathbb{H}^1}^2 \int_0^T \|u_\alpha(s)\|_{\mathbb{H}^2}^2 ds \right] \rightarrow 0,
\end{aligned}$$

thanks to Lemma 5.6.1, estimate (5.45), and Step 2. Moving on to II , we have

$$\begin{aligned}
II & = \mathbb{E} \left[\int_0^{\tau^n} \rho^- \langle B_0^+ - \mathcal{U}_\alpha^+ \times (\nabla \times \mathcal{V}_\alpha^+), \mathcal{U}_\alpha^+ - \mathcal{U}_{k,h}^+ \rangle dt \right] \\
& \quad + \mathbb{E} \left[\int_{\tau^n}^{\tau_{m-1}^n} \rho^- \langle B_0^+ - \mathcal{U}_\alpha^+ \times (\nabla \times \mathcal{V}_\alpha^+), \mathcal{U}_\alpha^+ - \mathcal{U}_{k,h}^+ \rangle dt \right] = II_1 + II_2.
\end{aligned}$$

We set $II_1 = II_{1,1} + II_{1,2}$ where

$$\begin{aligned} II_{1,1} &:= \mathbb{E} \left[\int_0^{\tau^n} (\rho^- - \rho) \langle B_0^+ - \mathcal{U}_\alpha^+ \times (\nabla \times \mathcal{V}_\alpha^+), \mathcal{U}_\alpha^+ - \mathcal{U}_{k,h}^+ \rangle dt \right] \\ &\leq \|\rho^- - \rho\|_{L^4(\Omega; L^2(0, T))} \mathbb{E} \left[\sup_{0 \leq t \leq T} \|\mathcal{U}_\alpha^+ - \mathcal{U}_{k,h}^+\|_{\mathbb{H}^1}^4 \right]^{\frac{1}{4}} \mathbb{E} \left[\int_0^T \|B_0^+ - \mathcal{U}_\alpha^+ \times (\nabla \times \mathcal{V}_\alpha^+)\|_{\mathbb{H}^{-1}}^2 dt \right]^{\frac{1}{2}} \rightarrow 0 \end{aligned}$$

due to the strong convergence of $(\rho_m)_m$ (see Step 4.1) along with the boundedness of the remaining terms. More precisely, $\mathbb{E} \left[\sup_{0 \leq t \leq T} \|\mathcal{U}_\alpha^+ - \mathcal{U}_{k,h}^+\|_{\mathbb{H}^1}^4 \right]$ is controlled through Lemma 5.6.1 and estimate (5.45), $\mathcal{U}_\alpha^+ \times (\nabla \times \mathcal{V}_\alpha^+)$ obeys the same bounding technique as $\mathcal{U}_{k,h}^+ \times (\nabla \times \mathcal{V}_{k,h}^+)$ in Step 1 where we associate with it the estimates (5.44) and (5.45). By the definition of B_0^m together with [107, Lemma 4.5], we get $\mathbb{E} \left[\int_0^T \|B_0^+\|_{\mathbb{H}^{-1}}^2 dt \right] \leq \mathbb{E} \left[\int_0^T \|B_0(t)\|_{\mathbb{H}^{-1}}^2 dt \right]$ which is bounded through the convergence (5.36). On the other hand,

$$\begin{aligned} II_{1,2} &:= \mathbb{E} \left[\int_0^{\tau^n} \rho(t) \langle B_0^+ - \mathcal{U}_\alpha^+ \times (\nabla \times \mathcal{V}_\alpha^+), \mathcal{U}_\alpha^+ - \mathcal{U}_{k,h}^+ \rangle dt \right] \\ &= \mathbb{E} \left[\int_0^{\tau^n} \rho(t) \langle B_0^+ - B_0(t), \mathcal{U}_\alpha^+ - \mathcal{U}_{k,h}^+ \rangle dt + \int_0^{\tau^n} \rho(t) \langle B_0(t) - u_\alpha(t) \times (\nabla \times v_\alpha(t)), \mathcal{U}_\alpha^+ - \mathcal{U}_{k,h}^+ \rangle dt \right. \\ &\quad \left. + \int_0^{\tau^n} \rho(t) \langle u_\alpha(t) \times (\nabla \times v_\alpha(t)) - \mathcal{U}_\alpha^+ \times (\nabla \times \mathcal{V}_\alpha^+), \mathcal{U}_\alpha^+ - \mathcal{U}_{k,h}^+ \rangle dt \right] =: J_1 + J_2 + J_3. \end{aligned}$$

Due to the specific construction of B_0^m together with [107, Lemma 4.9], it holds that $B_0^+ \rightarrow B_0$ in $L^2(\Omega; L^2(0, T; \mathbb{H}^{-1}))$ as $k, h \rightarrow 0$. Moreover, $\mathcal{U}_\alpha^+ - \mathcal{U}_{k,h}^+ \rightarrow 0$ in $L^2(\Omega; L^2(0, T; \mathbb{H}^1))$ thanks to convergences (5.33) and (5.43). Therefore, $J_1 \rightarrow 0$ as $k, h \rightarrow 0$. Similarly, $J_2 \rightarrow 0$ as $k, h \rightarrow 0$ because $\mathbb{1}_{[0, \tau^n]} \rho (B_0 - u_\alpha \times (\nabla \times v_\alpha)) \in L^2(\Omega; L^2(0, T; \mathbb{H}^{-1}))$ along with the weak convergence toward 0 of $\mathcal{U}_\alpha^+ - \mathcal{U}_{k,h}^+$ in $L^2(\Omega; L^2(0, T; \mathbb{H}^1))$. Making use of Proposition 5.2.1-(ii)-(iii), one gets

$$\begin{aligned} &\mathbb{E} \left[\int_0^T \left\| \mathbb{1}_{[0, \tau^n]}(t) \rho(t) \left(u_\alpha(t) \times (\nabla \times v_\alpha(t)) - \mathcal{U}_\alpha^+ \times (\nabla \times \mathcal{V}_\alpha^+) \right) \right\|_{\mathbb{H}^{-1}}^2 dt \right] \\ &\leq 2\mathbb{E} \left[\int_0^T \|u_\alpha(t) \times (\nabla \times (v_\alpha(t) - \mathcal{V}_\alpha^+))\|_{\mathbb{H}^{-1}}^2 dt \right] + 2\mathbb{E} \left[\int_0^T \|(u_\alpha(t) - \mathcal{U}_\alpha^+) \times (\nabla \times \mathcal{V}_\alpha^+)\|_{\mathbb{H}^{-1}}^2 dt \right] \\ &\lesssim \mathbb{E} \left[\sup_{0 \leq t \leq T} \|v_\alpha(t) - \mathcal{V}_\alpha^+\|_{\mathbb{L}^2}^2 \int_0^T \|u_\alpha(t)\|_{\mathbb{H}^2}^2 dt \right] + \mathbb{E} \left[\sup_{0 \leq t \leq T} \|\nabla(u_\alpha(t) - \mathcal{U}_\alpha^+)\|_{\mathbb{L}^2}^2 \int_0^T \|\nabla \mathcal{V}_\alpha^+\|_{\mathbb{L}^2}^2 dt \right]. \end{aligned}$$

Using the generalized Hölder inequality, the first term on the right-hand side can be controlled by

$$\mathbb{E} \left[\sup_{0 \leq t \leq T} \|v_\alpha(t) - \mathcal{V}_\alpha^+\|_{\mathbb{L}^2}^4 \right]^{\frac{1}{4}} \|v_\alpha - \mathcal{V}_\alpha^+\|_{L^2(\Omega; L^\infty(0, T; \mathbb{L}^2))} \mathbb{E} \left[\left(\int_0^T \|u_\alpha(t)\|_{\mathbb{H}^2}^2 dt \right)^4 \right]^{\frac{1}{4}},$$

which tends to 0 as $k, h \rightarrow 0$. Indeed, its first term can be bounded through estimate (5.44), convergence (5.34), and Step 2, its second term goes to 0 by (5.42), and its third term is also bounded by Step 2. Similarly, using the same techniques together with convergence (5.43) imply the convergence toward 0 of the second term on the right-hand side. Therefore, one infers that $J_3 \rightarrow 0$ as $k, h \rightarrow 0$. For II_2 , since all terms of the integrand own estimates with high-moments, one can easily show that $II_2 \rightarrow 0$ after a simple application of the Cauchy-Schwarz inequality as follows:

$$\begin{aligned} |II_2| &\leq \mathbb{E} \left[\int_0^T \mathbb{1}_{[\tau^n, \tau_{m-1}^n]}(t) dt \right]^{\frac{1}{2}} \mathbb{E} \left[\sup_{0 \leq t \leq T} \|\mathcal{U}_\alpha^+ - \mathcal{U}_{k,h}^+\|_{\mathbb{H}^1}^2 \int_0^T \|B_0^+(t) - \mathcal{U}_\alpha^+ \times (\nabla \times \mathcal{V}_\alpha^+)\|_{\mathbb{H}^{-1}}^2 dt \right]^{\frac{1}{2}} \\ &\leq \mathbb{E} [\tau_{m-1}^n - \tau^n]^{\frac{1}{2}} \mathbb{E} \left[\sup_{0 \leq t \leq T} \|\mathcal{U}_\alpha^+ - \mathcal{U}_{k,h}^+\|_{\mathbb{H}^1}^4 \right]^{\frac{1}{4}} \mathbb{E} \left[\left(\int_0^T \|B_0^+(t) - \mathcal{U}_\alpha^+ \times (\nabla \times \mathcal{V}_\alpha^+)\|_{\mathbb{H}^{-1}}^2 dt \right)^2 \right]^{\frac{1}{4}}. \end{aligned}$$

The first term $\mathbb{E}[|\tau_{m-1}^n - \tau^n|]$ converges toward 0 because $\tau_{m-1}^n \rightarrow \tau^n$ \mathbb{P} -a.s. as $k \rightarrow 0$. The two remaining terms can be easily controlled through Lemma 5.6.1-(iii), equation (5.45), and the convergence result (5.36). In addition to that, $III \rightarrow 0$, thanks to convergence (5.43), Lemma 5.6.1, estimates (5.45) and (5.44). Similarly, $IV \rightarrow 0$ due to the time-continuity of u_α in \mathbb{H}^1 , as mentioned in Step 2. The term V can be handled in a similar way to II by taking into consideration the construction of F_0^m along with assumption (S_2) . Further,

$$VI \leq 2L_g^2 \mathbb{E} \left[\int_0^T \|u_\alpha(t) - \mathcal{U}_\alpha^+\|_\alpha^2 dt \right] + 2L_g^2 \mathbb{E} \left[\int_0^T \|\mathcal{U}_\alpha^+ - \mathcal{U}_\alpha^-\|_\alpha^2 dt \right].$$

The first term converges to 0, thanks to convergence (5.43). The second term can be rewritten as follows:

$$2L_g^2 \mathbb{E} \left[k \sum_{m=1}^M \|U_\alpha^m - U_\alpha^{m-1}\|_\alpha^2 \right] \leq 2L_g^2 k C_{\alpha,u} \rightarrow 0,$$

by virtue of estimate (5.45). It remains to treat VII . To this end, it will be split into two terms as follows

$$\begin{aligned} VII &= \mathbb{E} \left[\int_0^{\tau^n} \rho^-(t) \left(G_0^- - g^-(t, \mathcal{U}_{k,h}^-), G_0^- - g^-(t, u_\alpha(t)) \right)_{\mathcal{L}_2(K; \mathbb{L}^2)} dt \right. \\ &\quad \left. + \int_{\tau^n}^{\tau_{m-1}^n} \rho^-(t) \left(G_0^- - g^-(t, \mathcal{U}_{k,h}^-), G_0^- - g^-(t, u_\alpha(t)) \right)_{\mathcal{L}_2(K; \mathbb{L}^2)} dt \right] =: VII_1 + VII_2. \end{aligned}$$

We have

$$\begin{aligned} VII_1 &= \mathbb{E} \left[\int_0^{\tau^n} (\rho^- - \rho(t)) \left(G_0^- - g^-(t, \mathcal{U}_{k,h}^-), G_0^- - g^-(t, u_\alpha(t)) \right)_{\mathcal{L}_2(K; \mathbb{L}^2)} dt \right. \\ &\quad \left. + \int_0^{\tau^n} \rho(t) \left(G_0^- - g^-(t, \mathcal{U}_{k,h}^-), G_0^- - g^-(t, u_\alpha(t)) \right)_{\mathcal{L}_2(K; \mathbb{L}^2)} dt \right] =: VII_{1,1} + VII_{1,2}. \end{aligned}$$

$VII_{1,1}$ can be handled in the same way as that of $II_{1,1}$ to achieve convergence to 0. For $VII_{1,2}$,

$$\begin{aligned} VII_{1,2} &= \mathbb{E} \left[\int_0^{\tau^n} \rho(t) \left(G_0^- - G_0(t), G_0^- - g^-(t, u_\alpha(t)) \right)_{\mathcal{L}_2(K; \mathbb{L}^2)} dt \right. \\ &\quad + \int_0^{\tau^n} \rho(t) \left(G_0(t) - g^-(t, \mathcal{U}_{k,h}^-), G_0^- - G_0(t) \right)_{\mathcal{L}_2(K; \mathbb{L}^2)} dt \\ &\quad + \int_0^{\tau^n} \rho(t) \left(G_0(t) - g^-(t, \mathcal{U}_{k,h}^-), G_0(t) - g(t, u_\alpha(t)) \right)_{\mathcal{L}_2(K; \mathbb{L}^2)} dt \\ &\quad \left. + \int_0^{\tau^n} \rho(t) \left(G_0(t) - g^-(t, \mathcal{U}_{k,h}^-), g(t, u_\alpha(t)) - g^-(t, u_\alpha(t)) \right)_{\mathcal{L}_2(K; \mathbb{L}^2)} dt \right]. \end{aligned}$$

Owing to the construction of G_0^m together with [107, Lemma 4.9], it holds that $G_0^- \rightarrow G_0$ in the space $L^2(\Omega; L^2(0, T; \mathcal{L}_2(K; \mathbb{L}^2)))$. Adding on top of that the boundedness properties of the terms G_0^- , $g^-(\cdot, u_\alpha)$, G_0 and $g^-(\cdot, \mathcal{U}_{k,h}^-)$ (such as Lemma 5.6.1, Step 2, and assumption (S_2)), we infer that the first and second terms on the right-hand side go to 0 as $k, h \rightarrow 0$. Third term also goes to 0 due to convergence (5.38) and the fact that $\mathbb{1}_{[0, \tau^n]} \rho(G_0 - g(\cdot, u_\alpha)) \in L^2(\Omega; L^2(0, T; \mathcal{L}_2(K; \mathbb{L}^2)))$. Similarly, the fourth term vanishes when $k, h \rightarrow 0$ by virtue of the weak convergence (5.38) and the strong convergence $g^-(\cdot, u_\alpha) \rightarrow g(\cdot, u_\alpha)$ in $L^2(\Omega; L^2(0, T; \mathcal{L}_2(K; \mathbb{L}^2)))$ which emerges from the continuity of g with respect to t (see assumption (S_2)). Finally, $VII_2 \rightarrow 0$ as $k, h \rightarrow 0$ because its integrand is uniformly bounded in $L^2(\Omega; L^\infty(0, T; \mathcal{L}_2(K; \mathbb{L}^2)))$ and $\tau_{m-1}^n \rightarrow \tau^n$ \mathbb{P} -a.s. as $k \rightarrow 0$. Putting it all together to conclude from equation (5.57) the following:

$$\begin{aligned} \lim_{k, h \rightarrow 0} \mathbb{E} \left[\int_0^{\tau_{m-1}^n} \rho^-(t) \|\mathcal{U}_\alpha^+ - \mathcal{U}_{k,h}^+\|_{\alpha, h}^2 dt \right] &= \lim_{k, h \rightarrow 0} \mathbb{E} \left[\int_0^{\tau_{m-1}^n} \rho^-(t) \|g^-(t, u_\alpha(t)) - G_0^-(t)\|_{\mathcal{L}_2(K; \mathbb{L}^2)}^2 dt \right] \\ &= 0. \end{aligned}$$

For $t \in (0, \tau_{m-1}^n)$, we know through the discrete stopping time τ_{m-1}^n that $\rho^-(t) > e^{-\eta_1 T - \eta_2 n}$. Subsequently, by making use of $\|\mathcal{V}_\alpha^+ - \mathcal{V}_{k,h}^+\|_{\mathbb{L}^2} \leq \max(C_D, \alpha^2) \|\mathcal{U}_\alpha^+ - \mathcal{U}_{k,h}^+\|_{\alpha,h}$, it follows

$$\lim_{k,h \rightarrow 0} \mathbb{E} \left[\int_0^{\tau_{m-1}^n} \|\mathcal{V}_\alpha^+ - \mathcal{V}_{k,h}^+\|_{\mathbb{L}^2}^2 dt \right] = \lim_{k,h \rightarrow 0} \mathbb{E} \left[\int_0^{\tau_{m-1}^n} \|g^-(t, u_\alpha(t)) - G_0^-(t)\|_{\mathcal{L}_2(K; \mathbb{L}^2)}^2 dt \right] = 0,$$

which implies

$$\lim_{k,h \rightarrow 0} \mathbb{E} \left[\int_0^{\tau^n} \|\mathcal{V}_\alpha^+ - \mathcal{V}_{k,h}^+\|_{\mathbb{L}^2}^2 dt \right] = \lim_{k,h \rightarrow 0} \mathbb{E} \left[\int_0^{\tau^n} \|g^-(t, u_\alpha(t)) - G_0^-(t)\|_{\mathcal{L}_2(K; \mathbb{L}^2)}^2 dt \right] = 0. \quad (5.58)$$

Indeed, it suffices to write $\int_0^{\tau^n} = \int_0^{\tau_{m-1}^n} + \int_{\tau_{m-1}^n}^{\tau^n}$. The first integral on the right converges to 0 as already shown and the second one goes to 0 as well because $\tau_{m-1}^n \rightarrow \tau^n$ \mathbb{P} -a.s. as $k \rightarrow 0$ along with the associated estimates of each integrand, such as Lemma 5.6.3, inequality (5.44), assumption (S_2) , convergence (5.38), and Step 2. Consequently, it follows that

$$\begin{aligned} \mathbb{E} \left[\int_0^{\tau^n} \|G_0(t) - g(t, u_\alpha(t))\|_{\mathcal{L}_2(K; \mathbb{L}^2)}^2 dt \right] &\leq 2\mathbb{E} \left[\int_0^{\tau^n} \|G_0(t) - G_0^-(t)\|_{\mathcal{L}_2(K; \mathbb{L}^2)}^2 dt \right] \\ &+ \int_0^{\tau^n} \|G_0^-(t) - g^-(t, u_\alpha(t))\|_{\mathcal{L}_2(K; \mathbb{L}^2)}^2 dt + \int_0^{\tau^n} \|g^-(t, u_\alpha(t)) - g(t, u_\alpha(t))\|_{\mathcal{L}_2(K; \mathbb{L}^2)}^2 dt \rightarrow 0, \end{aligned}$$

thanks to the strong convergence in $L^2(\Omega; L^2(0, T; \mathcal{L}_2(K; \mathbb{L}^2)))$ of G_0^m to G_0 together with (5.58) and the time-continuity of g by assumption (S_2) . Taking into account that $\{\tau^n\}_n$ is increasing to $t \in [0, T]$ leads to $G_0 = g(\cdot, u_\alpha)$ in $L^2(\Omega; L^2(0, T; \mathcal{L}_2(K; \mathbb{L}^2)))$. On the other hand,

$$\mathbb{E} \left[\int_0^{\tau^n} \|\mathcal{V}_{k,h}^+ - v_\alpha(t)\|_{\mathbb{L}^2}^2 dt \right] \leq 2\mathbb{E} \left[\int_0^{\tau^n} \|\mathcal{V}_{k,h}^+ - \mathcal{V}_\alpha^+\|_{\mathbb{L}^2}^2 dt + \int_0^{\tau^n} \|\mathcal{V}_\alpha^+ - v_\alpha(t)\|_{\mathbb{L}^2}^2 dt \right] \rightarrow 0, \quad (5.59)$$

by convergences (5.58) and (5.42). Similarly, by $\|\mathcal{U}_{k,h}^+ - \mathcal{U}_\alpha^+\|_\alpha \lesssim \|\mathcal{V}_{k,h}^+ - \mathcal{V}_\alpha^+\|_{\mathbb{L}^2}$, (5.58) and (5.43),

$$\mathbb{E} \left[\int_0^{\tau^n} \|\mathcal{U}_{k,h}^+ - u_\alpha(t)\|_\alpha^2 dt \right] \leq 2\mathbb{E} \left[\int_0^{\tau^n} \|\mathcal{U}_{k,h}^+ - \mathcal{U}_\alpha^+\|_\alpha^2 dt + \int_0^{\tau^n} \|\mathcal{U}_\alpha^+ - u_\alpha(t)\|_\alpha^2 dt \right] \rightarrow 0. \quad (5.60)$$

For $z \in M_{\mathcal{F}_t}^\infty(0, T; \mathbb{H}_0^1)$, we have,

$$\begin{aligned} &\left| \mathbb{E} \left[\int_0^{\tau^n} \langle u_\alpha(t) \times (\nabla \times v_\alpha(t)) - \mathcal{U}_{k,h}^+ \times (\nabla \times \mathcal{V}_{k,h}^+), z(t) \rangle dt \right] \right| \\ &\leq \|z\|_{M_{\mathcal{F}_t}^\infty(0, T; \mathbb{H}_0^1)} \mathbb{E} \left[\int_0^{\tau^n} \|u_\alpha(t) \times (\nabla \times (v_\alpha(t) - \mathcal{V}_{k,h}^+))\|_{\mathbb{H}^{-1}} dt \right] \\ &\quad + \|z\|_{M_{\mathcal{F}_t}^\infty(0, T; \mathbb{H}_0^1)} \mathbb{E} \left[\int_0^{\tau^n} \|(u_\alpha(t) - \mathcal{U}_{k,h}^+) \times (\nabla \times \mathcal{V}_{k,h}^+)\|_{\mathbb{H}^{-1}} dt \right] \\ &\leq C_D \|z\|_{M_{\mathcal{F}_t}^\infty(0, T; \mathbb{H}_0^1)} \mathbb{E} \left[\int_0^T \|u_\alpha(t)\|_{\mathbb{H}^1}^2 dt \right]^{\frac{1}{4}} \mathbb{E} \left[\int_0^T \|u_\alpha(t)\|_{\mathbb{H}^2}^2 dt \right]^{\frac{1}{4}} \mathbb{E} \left[\int_0^{\tau^n} \|\mathcal{V}_{k,h}^+ - v_\alpha(t)\|_{\mathbb{L}^2}^2 dt \right]^{\frac{1}{2}} \\ &\quad + C_D \|z\|_{M_{\mathcal{F}_t}^\infty(0, T; \mathbb{H}_0^1)} \mathbb{E} \left[\int_0^{\tau^n} \|\nabla(\mathcal{U}_{k,h}^+ - u_\alpha(t))\|_{\mathbb{L}^2}^2 dt \right]^{\frac{1}{2}} \mathbb{E} \left[\int_0^T \|\nabla \mathcal{V}_{k,h}^+\|_{\mathbb{L}^2}^2 dt \right]^{\frac{1}{2}} \rightarrow 0, \text{ as } k, h \rightarrow 0, \end{aligned}$$

where Proposition 5.2.1-(ii)-(iii), the Cauchy-Schwarz and the generalized Hölder inequalities, Step 2, Lemma 5.6.1, convergence (5.59) and (5.60) were applied. As a result, the above convergence together with (5.36) yield

$$\begin{aligned} \mathbb{E} \left[\int_0^{\tau^n} \langle B_0(t) - u_\alpha(t) \times (\nabla \times v_\alpha(t)), z(t) \rangle dt \right] &= \mathbb{E} \left[\int_0^{\tau^n} \langle B_0(t) - \mathcal{U}_{k,h}^+ \times (\nabla \times \mathcal{V}_{k,h}^+), z(t) \rangle dt \right] \\ &+ \int_0^{\tau^n} \langle \mathcal{U}_{k,h}^+ \times (\nabla \times \mathcal{V}_{k,h}^+) - u_\alpha(t) \times (\nabla \times v_\alpha(t)), z(t) \rangle dt \rightarrow 0, \quad \forall z \in M_{\mathcal{F}_t}^\infty(0, T; \mathbb{H}_0^1). \end{aligned}$$

Now, since the space $M_{\mathcal{F}_t}^\infty(0, T; \mathbb{H}_0^1)$ is dense in $L^2(\Omega; L^2(0, T; \mathbb{H}_0^1))$ and $\{\tau^n\}$ is increasing to $t \in [0, T]$, we infer the identity $B_0 = u_\alpha \times (\nabla \times v_\alpha)$ in $L^2(\Omega; L^2(0, T; \mathbb{H}^{-1}))$. It remains to identify F_0 with its counterpart. To this purpose, let $z \in L^2(\Omega; L^2(0, T; \mathbb{H}_0^1))$. We have

$$\begin{aligned} \langle f(t, u_\alpha(t)) - f^-(t, \mathcal{U}_{k,h}^-), z(t) \rangle &\leq \langle f(t, u_\alpha(t)) - f^-(t, u_\alpha(t)), z(t) \rangle + L_f \|z(t)\|_{\mathbb{H}^1} \|u_\alpha(t) - \mathcal{U}_{k,h}^+\|_\alpha \\ &\quad + L_f \|z(t)\|_{\mathbb{H}^1} \|\mathcal{U}_{k,h}^+ - \mathcal{U}_{k,h}^-\|_\alpha. \end{aligned}$$

Therefore, the time-continuity of f , convergence (5.60) and the fact that

$$\mathbb{E} \left[\int_0^T \|\mathcal{U}_{k,h}^+ - \mathcal{U}_{k,h}^-\|_\alpha^2 dt \right] = k \mathbb{E} \left[\sum_{m=1}^M \|U^m - U^{m-1}\|_\alpha^2 \right] \leq k C_T \rightarrow 0, \quad (\text{by Lemma 5.6.1})$$

ensure the weak convergence $f^-(\cdot, \mathcal{U}_{k,h}^-) \rightharpoonup f(\cdot, u_\alpha)$ in $L^2(\Omega; L^2(0, T; \mathbb{H}^{-1}))$. The latter together with convergence (5.37) permit the following

$$\begin{aligned} \mathbb{E} \left[\int_0^{\tau^n} \langle f(t, u_\alpha(t)) - F_0(t), z(t) \rangle dt \right] &= \mathbb{E} \left[\int_0^{\tau^n} \langle f(t, u_\alpha(t)) - f^-(t, \mathcal{U}_{k,h}^-), z(t) \rangle dt \right] \\ &\quad + \mathbb{E} \left[\int_0^{\tau^n} \langle f^-(t, \mathcal{U}_{k,h}^-) - F_0(t), z(t) \rangle dt \right] \rightarrow 0 \quad \text{as } k, h \rightarrow 0, \quad \forall z \in L^2(\Omega; L^2(0, T; \mathbb{H}_0^1)). \end{aligned}$$

Consequently, $F_0 = f(\cdot, u_\alpha)$ in $L^2(\Omega; L^2(0, T; \mathbb{H}^{-1}))$.

Step 5: Conclusion of all above steps

Assume $\alpha > 0$ a non-vanishing parameter, and $U^0 \rightarrow \bar{u}_0$ in $L^4(\Omega; \mathbb{H}^1)$ as $h \rightarrow 0$. According to what we have carried out so far, the stochastic process (u_α, v_α) satisfies, \mathbb{P} -a.s. and for all $(t, \varphi, \psi) \in (0, T) \times \mathbb{V} \times \mathbb{V}$, equation (5.8) together with $(v_\alpha(t), \psi) = (u_\alpha(t), \psi) + \alpha^2 (\nabla u_\alpha(t), \nabla \psi)$. We also had $v_\alpha \in L^2(\Omega; C([0, T]; \mathbb{H}))$ according to Step 3, which implies that u_α is weakly continuous with values in \mathbb{V} , \mathbb{P} -almost surely. Therewith, u_α makes up a solution of equation (5.1) in the sense of Definition 5.3.2. Taking advantage of [29, Theorem 4.4], we infer that u_α is unique and that the whole sequence $\{(\mathcal{U}_{k,h}^+, \mathcal{V}_{k,h}^+)\}_{k,h}$ is convergent.

The below proposition examines the divergence of the obtained limiting function u_α .

Proposition 5.7.2 *The limiting function u_α that was provided by convergence (5.33), and which turned out to be a solution to equation (5.1), is divergence-free almost everywhere in $(0, T) \times D$ and \mathbb{P} -almost surely.*

Proof: To show that u_α is divergence-free, it is sufficient to demonstrate that $\{\text{div}(\mathcal{U}_{k,h}^+)\}_{k,h}$ converges weakly in $L^2(\Omega; L^2(0, T; \mathbb{L}^2))$ toward 0, by virtue of (5.33). For this purpose, we summon the Lagrange interpolation $\mathcal{I}_h: C^3(D) \rightarrow L_h$ (c.f. [21, Theorem 4.4.4]). For $v \in C^3(D)$, we have

$$\begin{aligned} \mathbb{E} \left[\int_0^T (\text{div}(\mathcal{U}_{k,h}^+), v) dt \right] &= \mathbb{E} \left[\int_0^T (\text{div}(\mathcal{U}_{k,h}^+), v - \mathcal{I}_h v) dt \right] + \mathbb{E} \left[\int_0^T (\text{div}(\mathcal{U}_{k,h}^+), \mathcal{I}_h v) dt \right] \\ &\lesssim \mathbb{E} \left[\int_0^T \|\nabla \mathcal{U}_{k,h}^+\|_{L^2} dt \right] \|v - \mathcal{I}v\|_{L^2} \xrightarrow[k, h \rightarrow 0]{} 0, \end{aligned}$$

where the second term in the first equality vanishes because $\{\mathcal{U}_{k,h}^+\}_{k,h}$ is weakly divergence-free. \square

5.8 . Linear version of Algorithm 5.1

On account of the lousy uniqueness features of Algorithm 5.1, an associated linear version will be provided in this section. Its convergence analysis will rely on the proofing technique of Algorithm 5.1, and its iterates will be unique \mathbb{P} -almost surely.

Algorithm 5.2

For a given $\mathcal{U}^0 \in \mathbb{H}_h$,

project the initial condition v^0 through $(\mathcal{V}^0, \psi) = (\mathcal{U}^0, \psi) + \alpha^2 (\nabla \mathcal{U}^0, \nabla \psi)$, for all $\psi \in \mathbb{H}_h$.

If $\mathcal{U}^{m-1}, \mathcal{V}^{m-1}, \pi^{m-1}, \tilde{\pi}^{m-1}$ are known for a certain $m \in \{1, \dots, M\}$, then find a 4-tuple stochastic process $(\mathcal{U}^m, \mathcal{V}^m, \pi^m, \tilde{\pi}^m) \in \mathbb{H}_h \times \mathbb{H}_h \times L_h \times L_h$ such that for all $(\varphi, \psi, \Lambda_1, \Lambda_2) \in \mathbb{H}_h \times \mathbb{H}_h \times L_h \times L_h$, it holds \mathbb{P} -a.s.

$$\begin{cases} (\mathcal{V}^m - \mathcal{V}^{m-1}, \varphi) + k\nu(\nabla \mathcal{V}^m, \nabla \varphi) - k(\mathcal{U}^m \times (\nabla \times \mathcal{V}^{m-1}), \varphi) - k(\pi^m, \text{div}(\varphi)) \\ \quad = k\langle f(t_{m-1}, \mathcal{U}^{m-1}), \varphi \rangle + (g(t_{m-1}, \mathcal{U}^{m-1})\Delta_m W, \varphi), \\ (\mathcal{V}^m, \psi) = (\mathcal{U}^m, \psi) + \alpha^2 (\nabla \mathcal{U}^m, \nabla \psi) - (\tilde{\pi}^m, \text{div}(\psi)), \\ (\text{div}(\mathcal{U}^m), \Lambda_1) = (\text{div}(\mathcal{V}^m), \Lambda_2) = 0, \end{cases}$$

where $\Delta_m W = W(t_m) - W(t_{m-1})$ for all $m \in \{1, \dots, M\}$.

Observe that the only difference between Algorithms 5.1 and 5.2 is the corresponding nonlinear term. For Algorithm 5.2, we choose to explicit in time its second variable because owing to Proposition 5.2.1-(i), if both first and third arguments are equal, the whole nonlinear term vanishes. This fact guarantees that iterates $\{(\mathcal{U}^m, \mathcal{V}^m)\}_{m=1}^M$ of Algorithm 5.2 satisfy all a priori estimates that were illustrated in Section 5.6. In addition, taking φ and ψ in \mathbb{V}_h allows the following reformulation of the proposed linear numerical scheme:

$$\begin{cases} (\mathcal{V}^m - \mathcal{V}^{m-1}, \varphi) + k\nu(\nabla \mathcal{V}^m, \nabla \varphi) - k(\mathcal{U}^m \times (\nabla \times \mathcal{V}^{m-1}), \varphi) \\ \quad = k\langle f(t_{m-1}, \mathcal{U}^{m-1}), \varphi \rangle + (g(t_{m-1}, \mathcal{U}^{m-1})\Delta_m W, \varphi), \\ (\mathcal{V}^m, \psi) = (\mathcal{U}^m, \psi) + \alpha^2 (\nabla \mathcal{U}^m, \nabla \psi). \end{cases} \quad (5.61)$$

To illustrate the convergence of iterates $\{(\mathcal{U}^m, \mathcal{V}^m)\}_{m=1}^M$, one shall follow the same demonstration technique that was utilized for Algorithm 5.1 in Section 5.7. However, there will be an additional assumption on the starting point \mathcal{V}^0 to guarantee a uniform bound in h . This pops up when we try to control each term of system (5.56) in a suitable Banach space. Indeed, making use of Proposition 5.2.1-(ii), the Cauchy-Schwarz inequality, Lemmas 5.6.1-(iii) and 5.6.3-(ii) leads to:

$$\begin{aligned} \mathbb{E} \left[k \sum_{m=1}^M \left\| \mathcal{U}^m \times (\nabla \times \mathcal{V}^{m-1}) \right\|_{\mathbb{H}^{-1}}^2 \right] &\leq C_D \mathbb{E} \left[k \sum_{m=1}^M \|\mathcal{U}^m\|_{\mathbb{L}^2} \|\nabla \mathcal{U}^m\|_{\mathbb{L}^2} \|\nabla \mathcal{V}^{m-1}\|_{\mathbb{L}^2}^2 \right] \\ &\leq \frac{C_D}{\alpha} \mathbb{E} \left[\max_{1 \leq m \leq M} \|\mathcal{U}^m\|_{\alpha}^2 k \sum_{m=1}^M \|\nabla \mathcal{V}^{m-1}\|_{\mathbb{L}^2}^2 \right] \\ &\leq \frac{C_D}{\alpha} \mathbb{E} \left[\max_{1 \leq m \leq M} \|\mathcal{U}^m\|_{\alpha}^4 \right]^{1/2} \mathbb{E} \left[\left(k \sum_{m=1}^M \|\nabla \mathcal{V}^{m-1}\|_{\mathbb{L}^2}^2 \right)^2 \right]^{1/2} \\ &\leq \frac{\sqrt{2}C_D C_{T,2}}{\alpha} \mathbb{E} \left[T^2 \|\nabla \mathcal{V}^0\|_{\mathbb{L}^2}^4 + \left(k \sum_{m=1}^M \|\nabla \mathcal{V}^m\|_{\mathbb{L}^2}^2 \right)^2 \right]^{1/2} \\ &\leq \frac{\sqrt{2}C_D C_{T,2}}{\alpha} \left(\mathbb{E} \left[T^2 \|\nabla \mathcal{V}^0\|_{\mathbb{L}^2}^4 \right] + C_2(\alpha) \right)^{1/2}. \end{aligned}$$

According to the above calculation, in order for $\{\mathcal{U}^m \times (\nabla \times \mathcal{V}^{m-1})\}_{m=1}^M$ to be bounded in the space $L^2(\Omega; L^2(0, T; \mathbb{H}^{-1}))$, the initial datum's \mathbb{H}_0^1 -norm $\|\mathcal{V}^0\|_{\mathbb{H}_0^1}$ needs to be uniformly bounded in h . From now onwards, all the steps that were conducted in Section 5.7.1 can be applied to prove the convergence of iterates $\{(\mathcal{U}^m, \mathcal{V}^m)\}_{m=1}^M$ toward the solution of equations (5.1) in the sense of Definition 5.3.2.

The main interest of Algorithm 5.2 is the uniqueness of its iterates that holds \mathbb{P} -a.s. in all Ω .

Lemma 5.8.1 *Let $T > 0$, and assumptions (S_1) - (S_2) be fulfilled. If $\{(\mathcal{U}^m, \mathcal{V}^m)\}_{m=1}^M$ is a solution to Algorithm 5.2, then it is unique \mathbb{P} -almost surely.*

Proof: We only indicate here the modifications that occur in the proof of Lemma 5.6.2 when we deal with Algorithm 5.2. For this purpose, let $\{(\mathcal{U}_1^m, \mathcal{V}_1^m)\}_{m=1}^M$ and $\{(\mathcal{U}_2^m, \mathcal{V}_2^m)\}_{m=1}^M$ be two solutions to Algorithm 5.2 such that $\mathcal{U}_1^0 = \mathcal{U}_2^0$ and $\mathcal{V}_1^0 = \mathcal{V}_2^0$. For all $m \in \{0, 1, \dots, M\}$, set $u^m := \mathcal{U}_1^m - \mathcal{U}_2^m$ and $v^m := \mathcal{V}_1^m - \mathcal{V}_2^m$. Therefore, equation (5.24) becomes

$$(v^1, \varphi) + k\nu (\nabla v^1, \nabla \varphi) = k (u^1 \times (\nabla \times \mathcal{V}_1^0), \varphi), \quad \forall \varphi \in \mathbb{V}_h,$$

because $\mathcal{V}_1^0 = \mathcal{V}_2^0$. Next, we substitute φ by u^1 and apply Proposition 5.2.1-(i) and Lemma 5.4.1 to infer

$$\|u^1\|_\alpha^2 + k\nu \|\nabla u^1\|_{\mathbb{L}^2}^2 + k\nu \alpha^2 \|\Delta^h u^1\|_{\mathbb{L}^2}^2 = 0.$$

Consequently, $u^1 = 0$ \mathbb{P} -a.s and a.e. in D . This implies that $v^1 = 0$ \mathbb{P} -a.s. and a.e. in D , thanks to Lemma 5.4.1-(i). Arguing by induction completes the proof. \square

5.9 . Numerical approach through C^1 piecewise polynomials-based finite element methods

We devote this section to finite element methods whose basis functions are continuously differentiable across the domain D . The primary objective is to define the mathematical framework and to provide this context with an associated algorithm, which may be challenging to implement but has good stability properties. C^1 finite element methods establish a more advantageous sparsity pattern that is less time-consuming regarding the non-conforming methods based on polynomials of the same order. It generates stable condition numbers and improves solvers' speed (c.f. [82]). If possible, implementations of a continuously differentiable finite element method are done through some well-known libraries such as Firedrake [8], oomph-lib [73] that has the Bell elements, and scikit-fem [72] that offers the Argyris elements. The time-discretization that was introduced in Section 5.4 will be maintained, only the space-discretization will be highlighted.

We begin by proposing a quasi-uniform triangulation \mathcal{T}_h to the domain D that turns it into a partition of simplexes of maximal diameter $h > 0$. The finite element spaces are given by:

$$\begin{aligned} \mathcal{H}_h &:= \left\{ v_h \in [C^1(\bar{D})]^3 \cap \mathbb{H}_0^2 \mid u_h|_S \in \mathbb{P}_{n_1}(S), \forall S \in \mathcal{T}_h \right\}, \\ \mathcal{L}_h &:= \left\{ q_h \in L_0^2(D) \mid q_h|_S \in \mathbb{P}_{n_2}(S), \forall S \in \mathcal{T}_h \right\}, \end{aligned}$$

where $\mathbb{H}_0^2 := \left\{ u \in \mathbb{H}^2 \mid u|_{\partial D} = \frac{\partial u}{\partial n}|_{\partial D} = 0 \right\}$, and n_1, n_2 are two positive integers. The finite element pair $(\mathcal{H}_h, \mathcal{L}_h)$ must satisfy the discrete LBB condition:

$$\sup_{\varphi_h \in \mathcal{H}_h \setminus \{0\}} \frac{(\operatorname{div}(\varphi_h), q_h)}{\|\nabla \varphi_h\|_{\mathbb{L}^2}} \geq \beta \|q_h\|_{L^2}, \quad \forall q_h \in \mathcal{L}_h,$$

for some constant $\beta > 0$ independent of h . For instance, one can opt for Bogner-Fox-Schmit (BFS) rectangle to approximate the velocity, and $P_2^{disc} \cdot Q_3 / P_2^{disc}$ to approximate the pressure field (c.f. [39]). In addition to that, we define the projection $P_h: \mathbb{L}^2 \rightarrow \mathcal{H}_h$ onto \mathcal{H}_h by

$$(v - P_h v, \varphi_h) = 0, \quad \forall \varphi_h \in \mathcal{H}_h, \text{ and } v \in \mathbb{L}^2.$$

By virtue of the quasi-uniform feature of \mathcal{T}_h , the projector P_h is stable in \mathbb{H}^1 . That is, there is a constant $\zeta > 0$ independent of the mesh size h such that $\|P_h v\|_{\mathbb{H}^1} \leq \zeta \|v\|_{\mathbb{H}^1}$ for all $v \in \mathbb{H}^1$.

In the light of the above details, the numerical scheme that approximates equations (5.1) is given by the following algorithm.

Algorithm 5.3 Set $\mathbb{U}^0 = P_h \bar{u}_0$.

For each $m \in \{1, \dots, M\}$ such that $(\mathbb{U}^{m-1}, \mathbb{p}^{m-1})$ is known, find an $\mathcal{H}_h \times \mathcal{L}_h$ -valued pair $(\mathbb{U}^m, \mathbb{p}^m)$ that satisfies \mathbb{P} -a.s. the following equations:

$$\begin{cases} \left(\mathbb{U}^m - \mathbb{U}^{m-1}, \varphi \right) + \alpha^2 \left(\nabla(\mathbb{U}^m - \mathbb{U}^{m-1}), \nabla \varphi \right) + k\nu \left(\nabla \mathbb{U}^m, \nabla \varphi \right) + k\nu \alpha^2 \left(\Delta \mathbb{U}^m, \Delta \varphi \right) \\ \quad - k \left(\mathbb{U}^m \times \left(\nabla \times (\mathbb{U}^m - \alpha^2 \Delta \mathbb{U}^m) \right), \varphi \right) - k \left(\mathbb{p}^m, \operatorname{div}(\varphi) \right) \\ \quad = k \left\langle f(t_{m-1}, \mathbb{U}^{m-1}), \varphi \right\rangle + \left\langle g(t_{m-1}, \mathbb{U}^{m-1}) \Delta_m W, \varphi \right\rangle, \quad \forall \varphi \in \mathcal{H}_h, \\ \left(\operatorname{div}(\mathbb{U}^m), q_h \right) = 0, \quad \forall q_h \in \mathcal{L}_h. \end{cases}$$

where $\Delta_m W := W(t_m) - W(t_{m-1})$ for all $m \in \{1, \dots, M\}$.

Note that Algorithm 5.3 can be proven to converge toward the unique strong solution of equations (5.1) in the sense of Definition 5.3.2, through the exact same technique of Section 5.7. In point of fact, its demonstration will turn out to be easier than that of Section 5.7 because the discrete Laplace operator is replaced with its continuous version here. We only show the stability properties of $\{\mathbb{U}^m\}_{m=1}^M$ in the below lemma, whose demonstration is left to the reader.

Lemma 5.9.1 Let $T > 0$ be a fixed final time, $p \in [2, \infty) \cap \mathbb{N}$, and assumptions (S_1) - (S_2) be valid. If $\{(\mathbb{U}^m, \mathbb{p}^m)\}_{m=1}^M$ is a solution to Algorithm 5.3, the following estimates hold true:

$$\begin{aligned} (i) \quad & \mathbb{E} \left[\max_{1 \leq m \leq M} \|\mathbb{U}^m\|_\alpha^2 + k\nu \sum_{m=1}^M \|\nabla \mathbb{U}^m\|_\alpha^2 + \sum_{m=1}^M \|\mathbb{U}^m - \mathbb{U}^{m-1}\|_\alpha^2 \right] \leq \tilde{C}_T, \\ (ii) \quad & \mathbb{E} \left[\max_{1 \leq m \leq M} \|\mathbb{U}^m\|_\alpha^{2p} + k\nu \sum_{m=1}^M \|\mathbb{U}^m\|_\alpha^{2p-1} \|\nabla \mathbb{U}^m\|_\alpha^2 + \sum_{m=1}^M \|\mathbb{U}^m\|_\alpha^{2p-1} \|\mathbb{U}^m - \mathbb{U}^{m-1}\|_\alpha^2 \right] \leq \tilde{C}_{T,p}, \\ (iii) \quad & \mathbb{E} \left[\left(k\nu \sum_{m=1}^M \|\nabla \mathbb{U}^m\|_\alpha^2 \right)^{2p-1} \right] \leq \tilde{C}_{T,p}, \end{aligned}$$

where \tilde{C}_T and $\tilde{C}_{T,p}$ do not depend on k , h , and α . Note that $\|\nabla \cdot\|_\alpha^2 = \|\nabla \cdot\|_{\mathbb{L}^2}^2 + \alpha^2 \|\Delta \cdot\|_{\mathbb{L}^2}^2$.

Remark 5.9.1 To facilitate the demonstration, it is recommended to eliminate the pressure \mathbb{p}^m from Algorithm 5.3 by considering the test function φ in $\mathcal{V}_h := \left\{ u_h \in \mathcal{H}_h \mid (\operatorname{div}(u_h), q_h) = 0, \forall q_h \in \mathcal{L}_h \right\}$.

5.10 . Numerical experiments and conclusion

We provide in this section numerical simulations of Algorithm 5.1 with different values of the spatial scale α to check the acceptable ranges through comparing the outcomes to those of Chapter 4. Since the discrete LBB condition (5.13) is imposed, we choose the lower order Taylor-Hood elements (P_2-P_1) to discretize the velocity-pressure pair. Down below are used the following configurations: the domain $D = (0, 1)^2$, the final time $T = 1$, the divergence-free initial condition $\bar{u}_0 = (\partial_y \psi, -\partial_x \psi)$ with

$$\psi(x, y) := 10 \sin(100xy^2) x^2 (1-x)^2 y^2 (1-y)^2,$$

the kinematic viscosity $\nu = 1$, the outer force $f(\omega, t, u) = e^{-\frac{\omega^2}{2}} \sin(t) u$ fulfills assumption (S_2) , and the drift coefficient $g = Id$.

The Wiener process W that is utilized in the simulation down below is a Q -Wiener process with Q being of trace class. This is for the sake of comparing the obtained results to those of Chapter 4. Nevertheless, we display in the following figures the form of cylindrical and Q -Wiener processes.

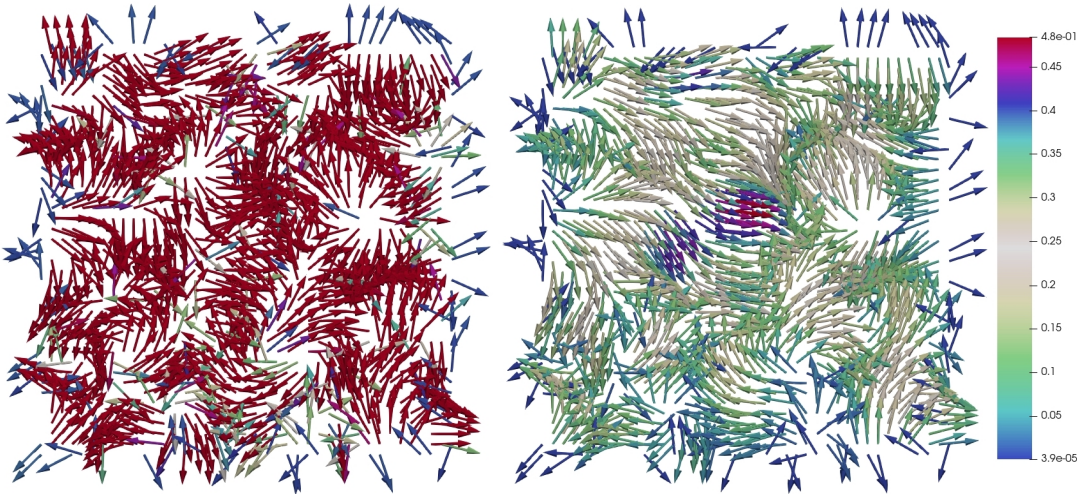


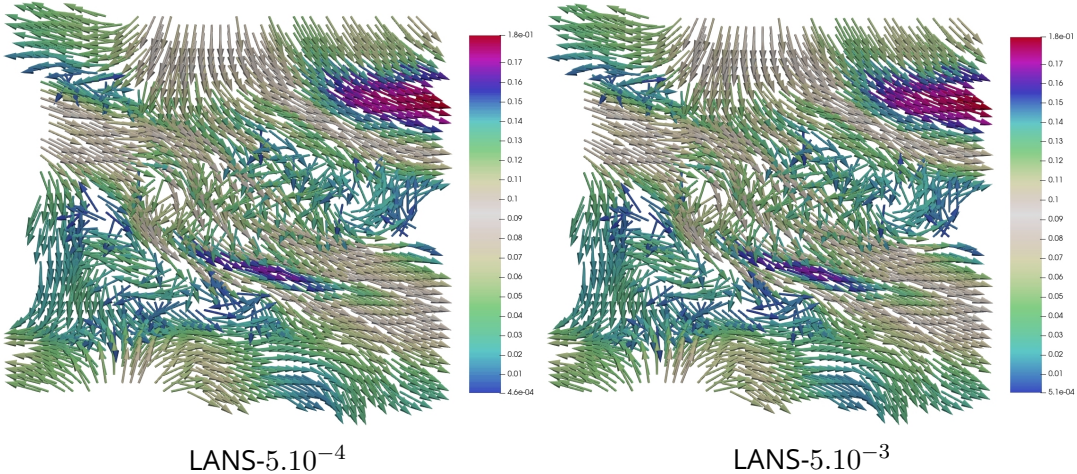
Figure 5.1 – Cylindrical Wiener process (left) and Q -Wiener process (right)

Figure 5.1 shows that the behavior of a cylindrical Wiener process is more chaotic than a Q -Wiener process. From this moment onwards, we consider two independent $H_0^1(D)$ -valued Wiener processes W_1 and W_2 such that $W = (W_1, W_2)$. The Wiener increments $\Delta_m W$ are approximated by the following formula

$$\Delta_m W_\ell \approx \sqrt{k} \sum_{i,j=1}^{10} \sqrt{\lambda_{i,j}^\ell} \xi_{i,j}^{\ell,m} e_{i,j}, \quad \ell \in \{1, 2\},$$

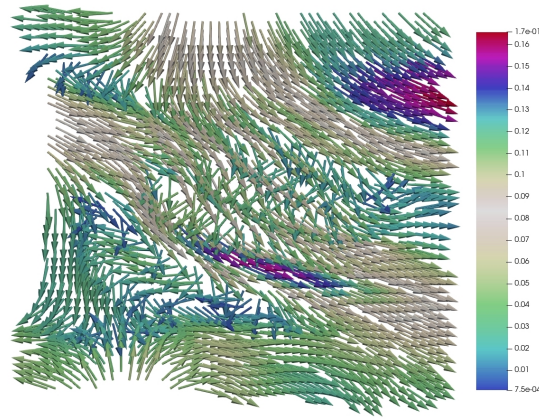
where the frequency 10 is chosen arbitrarily. Moreover, for all $(x, y) \in D$ and $i, j \in \mathbb{N}$, $e_{i,j}(x, y) = 2\sin(i\pi x)\sin(j\pi y)$ are the Laplace eigenfunctions arising from Dirichlet boundary conditions. For $\ell \in \{1, 2\}$, the family $\left\{ \left\{ \xi_{i,j}^{\ell,m} \right\}_{i,j} \right\}^{\ell,m}$ consists of independent identically distributed standard normal variables, and $\lambda_{i,j} := \frac{1}{(i+j)^2}$.

We set $h \approx 0.03$, $k = 0.01$, and we consider three different values of α : $5 \cdot 10^{-4}$, $5 \cdot 10^{-3}$, and 0.05 . We show down below three figures, each corresponds to a value of α at time $t = 0.41$, and each equipped with a color bar in order to compare their values to those of Chapter 4.



LANS- $5 \cdot 10^{-4}$

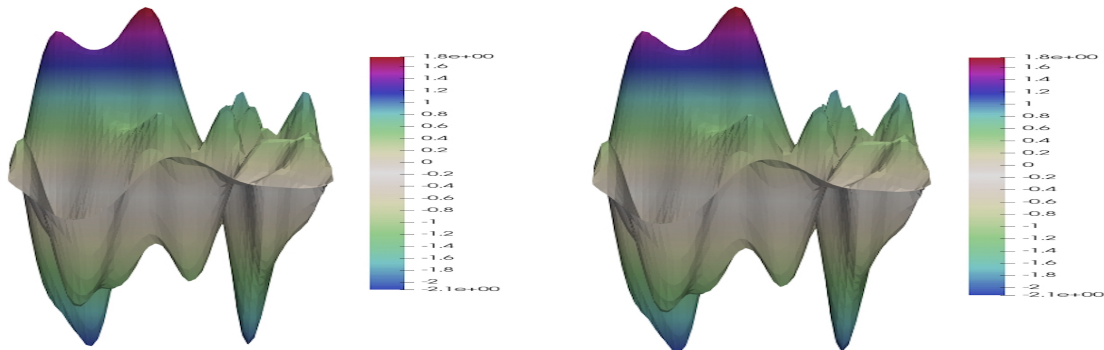
LANS- $5 \cdot 10^{-3}$



LANS- 5.10^{-2}

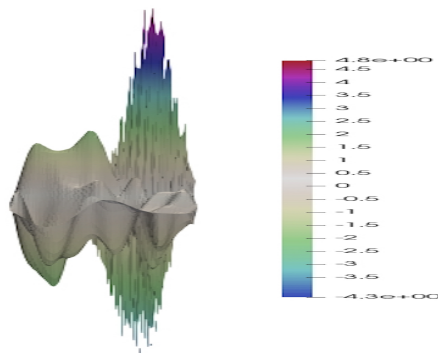
Velocity fields at time $t = 0.41$ of LANS- α for $\alpha \in \{5.10^{-4}, 5.10^{-3}, 5.10^{-2}\}$

Observe that the velocity fields' behavior when $\alpha = 5.10^{-3}$ is tremendously comparable with that of Chapter 4, although the chosen value of α is not as small. There is however, a remarkable velocity flow variation each time α increases in value. Therewith, the LANS- α equations might not be an alternative model for the NSEs when the spatial scale α is somewhat large. Beside the mentioned variation of the velocity, the pressure field is also heavily impacted by the modification of α as it appears in the upcoming figures. We point out that as α increases, the pressure gains an enormous amplitude which prevent the corresponding simulation outcome to be visible, especially when α exceeds 0.5.



LANS- 5.10^{-4}

LANS- 5.10^{-3}



LANS- 5.10^{-2}

Pressure fields at time $t = 0.41$ of LANS- α for $\alpha \in \{5.10^{-4}, 5.10^{-3}, 5.10^{-2}\}$

6 - Numerical approximation of the stochastic Navier-Stokes equations through artificial compressibility

6.1 . Introduction

The first thought that springs to mind when it comes to the numerical simulation of the Navier-Stokes equations (NSEs) is the complexity of the occurring situation, which can be represented by turbulent behaviors and physical processes by which energy becomes not only unavailable but irrecoverable in any form. The notorious NSEs are widely-known for their essential role in modeling phenomena that emerge from aeronautical science, thermo-hydraulics, ocean dynamics, and so on. They read in this chapter's context:

$$\begin{cases} \partial_t v - \nu \Delta v + [v \cdot \nabla]v + \nabla p = f + g(v)\dot{W}, \\ \operatorname{div}(v) = 0, \\ v(0, \cdot) = v_0, \end{cases} \quad (6.1)$$

with $v = v(\omega, t, x)$ being the fluid velocity, $p = p(\omega, t, x)$ is the pressure, $f = f(\omega, t, x)$ embodies an external force, g represents the diffusion coefficient, and $\nu > 0$ is the fluid kinematic viscosity. The term W is regarded as a Wiener process admitting a trace-class covariance operator, with the notation $\dot{W} = \partial_t W(\omega, t, x)$.

The present chapter deals with numerical approximations of the two-dimensional incompressible NSEs driven by multiplicative noise, equipped with homogeneous Dirichlet boundary conditions, within a bounded polygonal domain of \mathbb{R}^2 . Since the construction of divergence-free subspaces is not an effortless task (see for instance [17, 54, 94]), the attention will be turned toward a variant of the underlying equations involving a pseudo-compressibility method, avoiding divergence-free fields, and owning the unique, strong solution of the NSEs when passing to the limit, under a few assumptions. To be more accurate, the model which will undergo the discretization later on satisfies:

$$\begin{cases} \partial_t v^\varepsilon - \nu \Delta v^\varepsilon + [v^\varepsilon \cdot \nabla]v^\varepsilon + \frac{1}{2}[\operatorname{div}(v^\varepsilon)]v^\varepsilon + \nabla p^\varepsilon = f + g(v^\varepsilon)\dot{W}, \\ \varepsilon \partial_t p^\varepsilon + \operatorname{div}(v^\varepsilon) = 0, \\ (v^\varepsilon(0, \cdot), p^\varepsilon(0, \cdot)) = (v_0, p_0), \end{cases} \quad (6.2)$$

where v^ε and p^ε are the associated fluid velocity and pressure, respectively. The parameter $\varepsilon > 0$ represents a small scale that will eventually tend to zero with the other discretization parameters to recover a solution to equations (6.1), and (v_0, p_0) is the initial condition. The supplementary term $\frac{1}{2}[\operatorname{div}(v^\varepsilon)]v^\varepsilon$ ensures the well-posedness of the model (6.2), which is why it cannot be taken out. Notice that alternative configurations (also known as penalty methods) might have been possible, especially for the mass conservation equation of problem (6.2). For instance,

$$\begin{aligned} \varepsilon p^\varepsilon + \operatorname{div}(v^\varepsilon) &= 0, \\ \varepsilon \Delta p^\varepsilon - \operatorname{div}(v^\varepsilon) &= 0 \quad \text{with} \quad \frac{\partial p^\varepsilon}{\partial n} = 0, \\ \varepsilon \Delta \partial_t p^\varepsilon - \operatorname{div}(v^\varepsilon) &= 0 \quad \text{with} \quad \frac{\partial}{\partial n}(\partial_t p^\varepsilon) = 0, \text{ and } p^\varepsilon(0, \cdot) = p_0. \end{aligned}$$

The reader may refer to [101], [102], [105], and [108] for thorough deterministic studies of the above mentioned techniques, including the one considered here. The convergence rate of the Stokes problem driven by a multiplicative noise and subject to an artificial compressibility was conducted in [55] where optimal rates are obtained.

The mass conservation equation in problem (6.2) returns, in terms of regularity, good a priori estimates for the pressure p^ε (see [88, Proposition 3.1]), which may be taken advantage of during the convergence rate analysis. In point of fact, the pressure's lack of time-regularity in equations (6.1) (see for instance [84, Theorem 4.1]) has a negative effect on the convergence rate of those equations, which appears through the time-rate $O(\Delta t^{-1})$, as it was illustrated in [31, Corollary 4.2].

Problem (6.2) was theoretically investigated in [88] where the authors conducted the existence and uniqueness properties of the associated solution. The proof technique therein consists of the local monotonicity property of the sum of the Stokes operator and the nonlinear term. A discrete version of this method will be considered in the present chapter in order to demonstrate the convergence of the proposed numerical scheme and to avoid the Skorokhod theorem as well.

Finite element analysis of system (6.2) will be carried out hereafter, allowing the space variables to be discretized across the domain D . The proposed approximate finite element spaces for the velocity vector v^ε and the pressure field p^ε consist of continuous piecewise polynomials whose degrees can be opted arbitrarily without any constraint, unlike the case of a saddle point problem where a discrete inf-sup condition must be imposed, leading to restrictive choices. Time discretization relies on the Euler method and is offered in two options: linear and nonlinear (Algorithms 6.1 and 6.2). As broadly known, an implicit numerical scheme gathers more stability properties than an explicit version. This appears in both Algorithms 6.1 and 6.2, especially regarding the initial datum's regularity. In contrast, when it comes to iterates' uniqueness, explicit numerical schemes for stochastic partial differential equations perform better than implicit ones. Finally, in order for the proposed numerical scheme to convergence toward the unique strong solution of equations (6.1), the spatial and temporal discretization parameters along with the scale ε should vanish at the same time.

Unlike article [105] where a numerical scheme for the deterministic version of equations (6.2) is investigated, there will be no need for $\Delta t/\varepsilon$ to converge toward 0 when $\Delta t, \varepsilon \rightarrow 0$, with Δt being the time discretization step size, thanks to the finite element method and the used demonstration technique herein. According to the artificial compressibility method that has been chosen here, the supplementary term $\varepsilon \partial_t p^\varepsilon$ allows the pressure to gain time-regularity that is not traditional for the incompressible Navier-Stokes equations (6.1). This extra regularity is usually linked to the penalty parameter when studying the convergence rate. For instance, penalizing by $\varepsilon \Delta p^\varepsilon$ instead of $\varepsilon \partial_t p^\varepsilon$ imposes that $h^2/\sqrt{\varepsilon}$ should go to 0 as $\varepsilon, h \rightarrow 0$, with h being the space discretization step size, as illustrated in [55, Theorem 5.9]. for the stochastic time-dependent Stokes problem.

This chapter is split into five sections and is organized as follows. Section 6.2 provides the adequate preliminaries and configurations, including the required assumptions, solutions' definitions to problems (6.1), (6.2), and the numerical scheme. Section 6.3 is devoted to giving the main theorem of this chapter. Solvability, stability, and convergence of the numerical approximation are given in Section 6.4 along with a linear version of the proposed numerical scheme. This same section grants a small analysis scope concerned with the best choice of the scale ε in terms of the discretization parameters when comparing the proposed numerical scheme with a saddle point-based approach. Section 6.5 supplies the reader with pieces of evidence through numerical experiments and comparisons with other schemes. The last section concludes all the work in this chapter.

6.2 . Notations, materials and algorithm

Let $T > 0$ be a finishing time. Given a bounded polygonal domain $D \subset \mathbb{R}^2$ (for simplicity's sake), denote by ∂D its boundary, and by $\vec{n}: \partial D \rightarrow \mathbb{R}^2$ its corresponding unit outward normal vector field.

Function spaces in the Navier-Stokes framework are commonly denoted by \mathbb{H} and \mathbb{V} and are defined by

$$\begin{aligned}\mathcal{V} &:= \left\{ z \in [C_c^\infty(D)]^2 \mid \operatorname{div}(z) = 0 \text{ in } D \right\}, \\ \mathbb{H} &:= \left\{ z \in \left(L^2(D) \right)^2 \mid \operatorname{div}(z) = 0 \text{ a.e. in } D, z \cdot \vec{n} = 0 \text{ a.e. on } \partial D \right\}, \\ \mathbb{V} &:= \left\{ z \in \left(H_0^1(D) \right)^2 \mid \operatorname{div}(z) = 0 \text{ a.e. in } D \right\},\end{aligned}$$

where $C_c^\infty(D)$ denotes the space of $C^\infty(D)$ functions with compact support. The vector spaces will be henceforth indicated by blackboard bold letters for clarity's sake (e.g. $\mathbb{H}^1 = (H^1(D))^2$). The inner product of the Lebesgue space \mathbb{L}^2 and the duality product between \mathbb{H}_0^1 and \mathbb{H}^{-1} are denoted by (\cdot, \cdot) and $\langle \cdot, \cdot \rangle$, respectively. The parameter ε of equation (6.2) satisfies all this chapter long the condition $\varepsilon \leq 1$, the Gelfand triple $(\mathbb{H}_0^1, \mathbb{L}^2, \mathbb{H}^{-1})$ will solely be employed, and the trilinear form

$$\hat{b}(u, v, w) := ([u \cdot \nabla]v, w) + \frac{1}{2} ([\operatorname{div}(u)]v, w)$$

will be linked to equation (6.2). Two operators can be associated with \hat{b} ; the trilinear form $b(u, v, w) := ([u \cdot \nabla]v, w)$ that arises from the NSEs and the bilinear operator $\hat{B}: \mathbb{H}_0^1 \times \mathbb{H}_0^1 \rightarrow \mathbb{H}^{-1}$ which reads: $\langle \hat{B}(u, v), w \rangle = \hat{b}(u, v, w)$, for all $u, v, w \in \mathbb{H}_0^1$. The upcoming proposition lists a few properties of the trilinear form \hat{b} (cf. [101]).

Proposition 6.2.1 (i) $\hat{b}: \mathbb{H}_0^1 \times \mathbb{H}_0^1 \times \mathbb{H}_0^1 \rightarrow \mathbb{R}$ is continuous.

(ii) $\hat{b}(u, v, v) = 0$ for all $u, v \in \mathbb{H}_0^1$.

(iii) $|\hat{b}(u, v, w)| \leq C_D \|u\|_{\mathbb{L}^2}^{\frac{1}{2}} \|\nabla u\|_{\mathbb{L}^2}^{\frac{1}{2}} \|\nabla v\|_{\mathbb{L}^2} \|\nabla w\|_{\mathbb{L}^2}$, for all $u, v, w \in \mathbb{H}_0^1$.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space endowed with a filtration $(\mathcal{F}_t)_{0 \leq t \leq T}$ such that \mathcal{F}_0 contains all the null sets and $\mathcal{F}_t = \bigcap_{s>t} \mathcal{F}_s$. Let K be a separable Hilbert space equipped with a complete orthonormal basis $\{w_k, k \geq 1\}$, and Q be a nuclear operator on K such that w_k is an eigenvector of Q for all $k \geq 1$. The noise term W will be considered hereafter as a Q -Wiener process and it is defined by

$$W(t, x) = \sum_{k \geq 1} \sqrt{q_k} \beta^k(t) w_k(x),$$

where $\{\beta^k(\cdot), k \geq 1\}$ is a sequence of independent and identically distributed real-valued Brownian motions, and $\{q_k, k \geq 1\}$ is the eigenvalue of Q that is associated with w_k . With that said, the required assumptions are listed below.

Assumptions

(S₁) $Q: K \rightarrow K$ is a symmetric positive definite nuclear operator.

(S₂) For $p \in [2, +\infty)$, $v_0 \in L^{2p}(\Omega; \mathbb{L}^2)$ and $p_0 \in L^{2p}(\Omega; L^2(D))$ are \mathcal{F}_0 -measurable.

(S₃) For $p \in [1, +\infty)$, $f \in L^{2p}(\Omega; L^2(0, T; \mathbb{H}^{-1}))$ and $g: \mathbb{L}^2 \rightarrow \mathcal{L}_2(K, \mathbb{L}^2)$ satisfies

$$\begin{aligned}\|g(u) - g(v)\|_{\mathcal{L}_2(\sqrt{Q}(K), \mathbb{L}^2)} &\leq L_g \|u - v\|_{\mathbb{L}^2}, \quad \forall u, v \in \mathbb{L}^2, \\ \|g(u)\|_{\mathcal{L}_2(K, \mathbb{L}^2)} &\leq K_1 + K_2 \|u\|_{\mathbb{L}^2}, \quad \forall u \in \mathbb{L}^2,\end{aligned}$$

for some positive time-independent constants K_1, K_2, L_g such that $L_g \leq \sqrt{\frac{\nu}{2C_P^2}}$, where C_P is the Poincaré constant.

Remark 6.2.1 Assumption (S_1) can be straightforwardly eliminated and replaced with a cylindrical Wiener process, which is obviously a more general framework. This generalization is carried out in Chapter 5 and can be performed here as well.

Throughout this chapter, the writing $x \lesssim y$ designates $x \leq cy$ for a universal constant $c \geq 0$, the constant C_D may vary from one calculation to another; however, it will depend only on the domain D , and finally the symbol $\mathcal{L}_2(X, Y)$ refers to the space of Hilbert-Schmidt operators from X to Y , where X and Y are two Hilbert spaces.

6.2.1 . Concept of solutions

According to [88], a solution to equations (6.2) satisfies the following definition.

Definition 6.2.1 Given a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P})$, a stochastic process v^ε is said to be a strong solution to equations (6.2) under assumptions (S_1) - (S_3) if it belongs to the space $L^2(\Omega; C([0, T]; \mathbb{L}^2)) \cap L^2(\Omega; L^2(0, T; \mathbb{H}_0^1))$, and it satisfies for all $t \in [0, T]$, \mathbb{P} -a.s.

$$\begin{cases} (v^\varepsilon(t), \varphi) + \nu \int_0^t (\nabla v^\varepsilon(s), \nabla \varphi) ds + \int_0^t \hat{b}(v^\varepsilon(s), v^\varepsilon(s), \varphi) ds - \int_0^t (p^\varepsilon(s), \operatorname{div}(\varphi)) ds \\ = (v_0, \varphi) + \int_0^t \langle f(s), \varphi \rangle ds + \left(\int_0^t g(v^\varepsilon(s)) dW(s), \varphi \right), \quad \forall \varphi \in \mathbb{H}_0^1, \\ (\varepsilon \partial_t p^\varepsilon(t) + \operatorname{div}(v^\varepsilon(t)), q) = 0, \quad \forall q \in L^2(D), \end{cases}$$

along with the energy inequality

$$\mathbb{E} \left[\sup_{0 \leq t \leq T} \left(\|v^\varepsilon(t)\|_{\mathbb{L}^2}^p + \varepsilon \|p^\varepsilon(t)\|_{\mathbb{L}^2}^p \right) e^{-\delta t} + p\nu \int_0^T \|\nabla v^\varepsilon(t)\|_{\mathbb{L}^2}^2 \|v^\varepsilon(t)\|_{\mathbb{L}^2}^{p-2} e^{-\delta t} dt \right] \leq C,$$

for all $p \in [2, +\infty)$, $\varepsilon > 0$, $\delta > 0$, and for some constant $C > 0$ depending only on $\delta, p, T, v_0, p_0, f, K_1, K_2$ and ε .

On the other hand, a solution to problem (6.1) in 2D can be defined as follows.

Definition 6.2.2 Assume (S_1) - (S_3) and let $T > 0$. A stochastic process v on a given filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P})$ is a strong solution to equations (6.1) if it belongs to $L^2(\Omega; C([0, T]; \mathbb{H})) \cap L^2(\Omega; L^2(0, T; \mathbb{V}))$, and it fulfills for all $0 \leq t \leq T$, \mathbb{P} -a.s.

$$\begin{aligned} & (v(t), \varphi) + \nu \int_0^t (\nabla v(s), \nabla \varphi) ds + \int_0^t ([v(s) \cdot \nabla]v(s), \varphi) ds \\ & = (v_0, \varphi) + \int_0^t \langle f(s), \varphi \rangle ds + \left(\int_0^t g(v(s)) dW(s), \varphi \right), \quad \forall \varphi \in \mathbb{V}. \end{aligned}$$

6.2.2 . Discretization

The time interval $[0, T]$ will be decomposed into $M \in \mathbb{N} \setminus \{0\}$ subintervals with equidistant nodes $\{t_\ell\}_{\ell=0}^M =: I_k$ for simplicity's sake. The corresponding step is denoted by $k := \frac{T}{M}$.

The spatial domain D , which is assumed to be convex, bounded and polygonal, will be covered by a quasi-uniform triangulation \mathcal{T}_h , with h being the diameters' maximum of all triangles. Let \mathbb{H}_h be a subspace of \mathbb{H}_0^1 consisting of $[C(\bar{D})]^2$ -valued piecewise polynomials over \mathcal{T}_h , and fulfilling for all $m \geq 2$:

$$\inf_{v_h \in \mathbb{H}_h} \{ \|v - v_h\|_{\mathbb{L}^2} + h \|\nabla(v - v_h)\|_{\mathbb{L}^2} \} \leq Ch^m \|v\|_{\mathbb{H}^m}, \quad \forall v \in \mathbb{H}_0^1 \cap \mathbb{H}^m. \quad (6.3)$$

The quasi-uniformity of \mathcal{T}_h permits the inverse inequality (cf. [21, Lemma 4.5.3]):

$$\|v_h\|_{\mathbb{H}^\ell} \leq \mathcal{C} h^{m-\ell} \|v_h\|_{\mathbb{H}^m}, \quad \forall v_h \in \mathbb{H}_h, \quad \forall 0 \leq m \leq \ell, \quad (6.4)$$

for some $\mathcal{C} > 0$ independent of h . Let L_h be a subspace of $L_0^2(D)$ consisting of $C(\bar{D})$ piecewise polynomial functions over \mathcal{T}_h , and satisfying for all $m \geq 1$:

$$\inf_{p_h \in L_h} \|p - p_h\|_{\mathbb{L}^2} \leq Ch^m \|p\|_{\mathbb{H}^m}, \quad \forall p \in L_0^2(D) \cap H^m(D). \quad (6.5)$$

For $(v, p) \in \mathbb{L}^2 \times L^2(D)$, the associated orthogonal projections are denoted $\Pi_h: \mathbb{L}^2 \rightarrow \mathbb{H}_h$ and $\rho_h: L^2(D) \rightarrow L_h$ and are defined by the following identities, respectively:

$$(v - \Pi_h v, \varphi_h) = 0, \quad \forall \varphi_h \in \mathbb{H}_h \quad \text{and} \quad (p - \rho_h p, q_h) = 0, \quad \forall q_h \in L_h. \quad (6.6)$$

Thanks to the pseudo-compressibility method which is provided by equations (6.2), the finite element pair (\mathbb{H}_h, L_h) is not forced to satisfy the discrete LBB condition.

For the sake of clarity, the notations φ^+ and φ^- will designate throughout this chapter piecewise constant functions with respect to time. For instance,

$$\begin{aligned} \varphi^+(t) &:= \varphi^m, \quad \forall t \in (t_{m-1}, t_m], \quad \text{and} \\ \varphi^-(t) &:= \varphi^{m-1}, \quad \forall t \in [t_{m-1}, t_m), \end{aligned}$$

for the a given sequence $\{\varphi^m\}_m$. The discrete derivation with respect to time will also intervene later on. For this purpose, the below proposition (cf. [24, Appendix B]) lists a few associated properties.

Proposition 6.2.2 *Given a sequence $\{\varphi^m\}_m$, the discrete derivative is defined by $d_t \varphi^m = \frac{\varphi^m - \varphi^{m-1}}{k}$, for all $m \in \{1, \dots, M\}$, and it fulfills the following assertions:*

- (i) $d_t(\varphi^+ \psi^+) = \varphi^+ d_t \psi^+ + \psi^- d_t \varphi^+$,
- (ii) $\int_0^T \varphi^+ d_t \psi^+ dt = \varphi^+(T) \psi^+(T) - \varphi^-(0) \psi^-(0) - \int_0^T (d_t \varphi^+) \psi^- dt$,
- (iii) $d_t e^{\varphi^+} = e^{\varphi^-} d_t \varphi^+ + e^{\delta} \frac{(\varphi^+ - \varphi^-)^2}{2k}$, for some $\delta \in (\varphi^-, \varphi^+)$.

Relying on Definition 6.2.1 and the space-time discretization, the numerical scheme which will be studied throughout the rest of this chapter is given by:

Algorithm 6.1 *Let $m \in \{1, \dots, M\}$ and $(v_h^0, p_h^0) \in \mathbb{H}_h \times L_h$ be a starting point. For a given $(V_\varepsilon^{m-1}, \Pi_\varepsilon^{m-1}) \in \mathbb{H}_h \times L_h$ such that $(V_\varepsilon^0, \Pi_\varepsilon^0) := (v_h^0, p_h^0)$, find $(V_\varepsilon^m, \Pi_\varepsilon^m) \in \mathbb{H}_h \times L_h$ that satisfies*

$$\begin{cases} (V_\varepsilon^m - V_\varepsilon^{m-1}, \varphi_h) + k\nu(\nabla V_\varepsilon^m, \nabla \varphi_h) + k\hat{b}(V_\varepsilon^m, V_\varepsilon^m, \varphi_h) - k(\Pi_\varepsilon^m, \text{div}(\varphi_h)) \\ = k\langle f^m, \varphi_h \rangle + (g(V_\varepsilon^{m-1})\Delta_m W, \varphi_h), \quad \forall \varphi_h \in \mathbb{H}_h, \\ \frac{\varepsilon}{k}(\Pi_\varepsilon^m - \Pi_\varepsilon^{m-1}, q_h) + (\text{div}(V_\varepsilon^m), q_h) = 0, \quad \forall q_h \in L_h, \end{cases}$$

where for all $m \in \{1, \dots, M\}$, $f^m := \frac{1}{k} \int_{t_{m-1}}^{t_m} f(t) dt$ and $\Delta_m W := W(t_m) - W(t_{m-1})$.

The initial datum (v_h^0, p_h^0) of Algorithm 6.1 is required to be uniformly bounded in $\mathbb{L}^2 \times L^2(D)$ with respect to h . To this end, it suffices to consider $v_h^0 = \Pi_h v_0$ and $p_h^0 = \rho_h p_0$ because both projectors Π_h and ρ_h are stable in L^2 (cf. [52]):

$$\|\Pi_h u\|_{\mathbb{L}^2} \leq \|u\|_{\mathbb{L}^2}, \quad \forall u \in \mathbb{L}^2 \quad \text{and} \quad \|\rho_h q\|_{L^2} \leq \|q\|_{L^2}, \quad \forall q \in L^2. \quad (6.7)$$

Owing to [107, Lemma III.4.5], there holds

$$k \sum_{m=1}^M \|f^m\|_{\mathbb{H}^{-1}}^2 \leq \int_0^T \|f(t)\|_{\mathbb{H}^{-1}}^2 dt. \quad (6.8)$$

6.3 . Main result

Theorem 6.3.1 For $T > 0$, let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P})$ be a filtered probability space, $D \subset \mathbb{R}^2$ be a polygonal domain, and assumptions (S_1) - (S_3) be satisfied. Given a positive integer M , define the discretization step size $k := \frac{T}{M}$ such that $k \in (0, 1)$ and I_k forms a uniform partition of the time interval $[0, T]$. Let $\varepsilon \in (0, 1)$ be a given penalty parameter, and $h \in (0, 1)$ be the space discretization step size such that the triangulation \mathcal{T}_h is quasi-uniform. Define the finite element triple $(\mathbb{H}_h, L_h, (v_h^0, p_h^0))$ such that the initial datum (v_h^0, p_h^0) belongs to (\mathbb{H}_h, L_h) . Then, the following results are true:

- For a given triple $(k, h, \varepsilon) \in (0, 1)^3$, there exists a solution $\{(V_\varepsilon^m, \Pi_\varepsilon^m)\}_{m=1}^M$ to Algorithm 6.1 satisfying Lemmata 6.4.1, 6.4.2, and 6.4.3.
- For a family $\{k, h, \varepsilon\}$ of parameters fulfilling $k, h, \varepsilon \rightarrow 0$ instantly, such that the initial datum $v_h^0 \rightarrow v_0$ as $h \rightarrow 0$ in $L^2(\Omega; \mathbb{L}^2)$, the solution $\{(V_\varepsilon^m, \Pi_\varepsilon^m)\}_{m=1}^M$ of Algorithm 6.1 converges toward the unique strong solution of the stochastic Navier-Stokes equations (6.1) in the sense of Definition 6.2.2.

All this chapter long, the penalty parameter ε is meant to be a vanishing scale, just as the discretization parameters k and h . The passage to the limit in ε , k and h will be instant, meaning that none of the mentioned parameters should vanish on its own.

The convergence of Algorithm 6.1 can also be investigated with a fixed non-vanishing ε to obtain a solution to equations (6.2) in the sense of Definition 6.2.1. Then, one can take advantage of [88, Proposition 4.1] to gain a solution to equations (6.1) in the sense of Definition 6.2.2. However, this idea is beyond the scope of this chapter.

6.4 . Discussion

6.4.1 . Existence and uniqueness of solutions

This section is devoted to giving existence and uniqueness properties to the iterates $\{(V_\varepsilon^m, \Pi_\varepsilon^m)\}_{m=1}^M$. The solvability of Algorithm 6.1 and the measurability of its iterates are handled first in the following lemma.

Lemma 6.4.1 Let $T > 0$ be fixed. Under assumptions (S_1) - (S_3) , Algorithm 6.1 has at least one discrete solution. Moreover, for all $m \in \{1, \dots, M\}$, the processes $V_\varepsilon^m: \Omega \rightarrow \mathbb{H}_h$ and $\Pi_\varepsilon^m: \Omega \rightarrow L_h$ are \mathcal{F}_{t_m} -measurable.

Proof: The solvability of Algorithm 6.1 can be proven by induction. Indeed, assume that iterates V_ε^ℓ and Π_ε^ℓ exist for all $\ell \in \{1, \dots, m-1\}$. The existence of $(V_\varepsilon^m, \Pi_\varepsilon^m)$ is therefore the target. To this end, let $E := \mathbb{H}_0^1 \times L_0^2$, and $B_\varepsilon: E \rightarrow E$ be defined by

$$\begin{aligned} (B_\varepsilon(u, p), (v, q))_{\mathbb{L}^2 \times L^2} &= \left(u - V_\varepsilon^{m-1}(\omega), v \right) + k\nu \left(\nabla u, \nabla v \right) + k\hat{b}(u, u, v) - k(p, \operatorname{div}(v)) - k(f^m, v) \\ &\quad - \left(g(V_\varepsilon^{m-1}(\omega)) \Delta_m W(\omega), v \right) + \varepsilon \left(p - \Pi_\varepsilon^{m-1}(\omega), q \right) + k(\operatorname{div}(u), q), \end{aligned}$$

for all $(u, p), (v, q) \in E$, and for almost all $\omega \in \Omega$. The symbol $(\cdot, \cdot)_{\mathbb{L}^2 \times L^2}$ denotes the velocity-pressure $\mathbb{L}^2 \times L^2(D)$ -inner product. Thanks to Proposition 6.2.1-(i), the continuity of B_ε can be tackled easily. Through the application of Proposition 6.2.1-(ii), the Poincaré and Young inequalities, estimate (6.8), and assumption (S_3) , one obtains

$$\begin{aligned} (B_\varepsilon(u, p), (u, p))_{\mathbb{L}^2 \times L^2} &\geq \frac{1}{2} \|u\|_{\mathbb{L}^2}^2 - \frac{1}{2} \|V_\varepsilon^{m-1}\|_{\mathbb{L}^2}^2 + k\nu \|\nabla u\|_{\mathbb{L}^2}^2 - k \|f^m\|_{\mathbb{H}^{-1}} \|u\|_{\mathbb{H}^1} \\ &\quad - \|g(V_\varepsilon^{m-1})\|_{\mathcal{L}_2(K, \mathbb{L}^2)} \|\Delta_m W\|_K \|u\|_{\mathbb{L}^2} + \frac{\varepsilon}{2} \|p\|_{L^2}^2 - \frac{\varepsilon}{2} \|\Pi_\varepsilon^{m-1}\|_{L^2}^2 \geq \frac{1}{4} \|u\|_{\mathbb{L}^2}^2 + \frac{\varepsilon}{2} \|p\|_{L^2}^2 \\ &\quad - \frac{1}{2} \|V_\varepsilon^{m-1}\|_{\mathbb{L}^2}^2 - \frac{\varepsilon}{2} \|\Pi_\varepsilon^{m-1}\|_{L^2}^2 - \frac{C_D^2}{4\nu} \|f\|_{L^2(0, T; \mathbb{H}^{-1})}^2 - (K_1 + K_2 \|V_\varepsilon^{m-1}\|_{\mathbb{L}^2})^2 \|\Delta_m W\|_K^2 \geq 0, \end{aligned}$$

for all $(u, p) \in E_h(\omega) := \left\{ (v, q) \in \mathbb{H}_h \times L_h \mid \|v\|_{\mathbb{L}^2} \geq \sqrt{S(\omega)}, \|q\|_{L^2} \geq \|\Pi_\varepsilon^{m-1}(\omega)\|_{L^2} \right\}$, where

$$S(\omega) := 2\|V_\varepsilon^{m-1}(\omega)\|_{\mathbb{L}^2}^2 + \frac{C_D^2}{\nu} \|f\|_{L^2(0,T;\mathbb{H}^{-1})}^2 + 4(K_1 + K_2\|V_\varepsilon^{m-1}(\omega)\|_{\mathbb{L}^2})^2 \|\Delta_m W\|_K^2.$$

Both $S(\omega)$ and $\|\Pi_\varepsilon^{m-1}(\omega)\|_{L^2}$ are \mathbb{P} -a.s. finite, thanks to the induction supposition. With that said, the Brouwer fixed point theorem [65, Corollary IV.1.1] implies the existence of at least one $(u_\omega, p_\omega) \in \mathbb{H}_h \times L_h$ such that $B_\varepsilon(u_\omega, p_\omega) = (0, 0)$, $\|u_\omega\|_{\mathbb{L}^2} \leq \sqrt{S(\omega)}$ and $\|p_\omega\|_{L^2} \leq \|\Pi_\varepsilon^{m-1}\|_{L^2}$. Therewith, it suffices to set $(V_\varepsilon^m, \Pi_\varepsilon^m) = (u_\omega, p_\omega)$. On the other hand, the measurability of $\{(V_\varepsilon^m, \Pi_\varepsilon^m)\}_{m=1}^M$ can be also demonstrated by induction. The idea consists in expressing the newly obtained iterates (u_ω, p_ω) in terms of the existing ones. This can be done through a universally Borel-measurable selector function $\sigma: \mathbb{H}_h \times L_h \times K \rightarrow \mathbb{H}_h \times L_h$. For instance, $(u_\omega, p_\omega) = \sigma(V_\varepsilon^{m-1}, \Pi_\varepsilon^{m-1}, \Delta_m W)$, and the \mathcal{F}_{t_m} -measurability arises from the Brownian increment $\Delta_m W$. The reader may refer to [47, Page 744] for a detailed approach. \square

Lemma 6.4.1 dealt with the existence of a discrete solution which might not be unique. In point of fact, uniqueness in the whole probability set Ω does not seem to hold due to the nonlinearity interaction. Also, a contraction argument does not perform well in the discrete settings because the discrete time-derivative of an exponential function leads to a supplementary term which blocks the demonstration (see Proposition 6.2.2-(iii)). However, it can be proven that iterates' uniqueness holds true in a sample subset of Ω as demonstrated in the following lemma.

Lemma 6.4.2 *Assume (S_1) - (S_3) and let $\delta > 0$ be a small constant. Solutions $\{(V_\varepsilon^m, \Pi_\varepsilon^m)\}_{m=1}^M$ to Algorithm 6.1 are \mathbb{P} -almost surely unique within either of the following probability subsets:*

$$(i) \ \Omega_\delta^1 := \left\{ \omega \in \Omega \mid k \sum_{m=1}^M \|V_\varepsilon^m\|_{\mathbb{L}^4}^4 \leq \frac{1}{\delta} \right\} \text{ provided that } \frac{1}{\nu^3 \delta} \leq 4c_0^3,$$

$$(ii) \ \Omega_\delta^2 := \left\{ \omega \in \Omega \mid \max_{1 \leq m \leq M} \|V_\varepsilon^m\|_{\mathbb{L}^2}^4 \leq \frac{1}{\delta} \right\} \text{ provided that } \frac{k}{\nu^3 \delta h^2} \leq \frac{2c_0^3}{\mathcal{C}^2},$$

for some universal constant $c_0 \in (0, 3^{-1}2^{\frac{2}{3}}]$. Furthermore, $\mathbb{P}(\Omega_\delta^1) \geq 1 - \delta \mathbb{E} \left[k \sum_{m=1}^M \|V_\varepsilon^m\|_{\mathbb{L}^4}^4 \right]$ and $\mathbb{P}(\Omega_\delta^2) \geq 1 - \delta \mathbb{E} \left[\max_{1 \leq m \leq M} \|V_\varepsilon^m\|_{\mathbb{L}^2}^4 \right]$.

Proof: Assume that $\{(V_\varepsilon^m, \Pi_\varepsilon^m)\}_{m=1}^M$ and $\{(U_\varepsilon^m, P_\varepsilon^m)\}_{m=1}^M$ are solutions to Algorithm 6.1 starting from the same initial condition (v_h^0, p_h^0) . For all $m \in \{0, 1, \dots, M\}$, let $Z_\varepsilon^m := V_\varepsilon^m - U_\varepsilon^m$ and $Q_\varepsilon^m := \Pi_\varepsilon^m - P_\varepsilon^m$. Then, iterates $\{(Z_\varepsilon^m, Q_\varepsilon^m)\}_{m=1}^M$ satisfy for all $m \in \{1, \dots, M\}$ and \mathbb{P} -a.s. the following equations

$$\begin{cases} \left(Z_\varepsilon^m - Z_\varepsilon^{m-1}, \varphi_h \right) + k\nu (\nabla Z_\varepsilon^m, \nabla \varphi_h) + k \langle \hat{B}(V_\varepsilon^m, V_\varepsilon^m) - \hat{B}(U_\varepsilon^m, U_\varepsilon^m), Z_\varepsilon^m \rangle \\ \quad - k (Q_\varepsilon^m, \operatorname{div}(\varphi_h)) = ([g(V_\varepsilon^{m-1}) - g(U_\varepsilon^{m-1})] \Delta_m W, \varphi_h), \quad \forall \varphi_h \in \mathbb{H}_h, \\ \frac{\varepsilon}{k} (Q_\varepsilon^m - Q_\varepsilon^{m-1}, q_h) + (\operatorname{div}(Z_\varepsilon^m), q_h) = 0, \quad \forall q_h \in L_h. \end{cases} \quad (6.9)$$

Observe that the stochastic term in equation (6.9) can be eliminated if one had $V_\varepsilon^{m-1} = U_\varepsilon^{m-1}$. Since $U_\varepsilon^0 = V_\varepsilon^0 = v_h^0$, an induction argument seems to be legitimate. Indeed, for $m = 1$ and $(\varphi_h, q_h) = (Z_\varepsilon^1, Q_\varepsilon^1)$, equation (6.9) turns into

$$\begin{aligned} & \|Z_\varepsilon^1\|_{\mathbb{L}^2}^2 + \varepsilon \|Q_\varepsilon^1\|_{L^2}^2 + k\nu \|\nabla Z_\varepsilon^1\|_{\mathbb{L}^2}^2 = k \langle \hat{B}(U_\varepsilon^1, U_\varepsilon^1) - \hat{B}(V_\varepsilon^1, V_\varepsilon^1), Z_\varepsilon^1 \rangle \\ & \leq 2k \|\nabla Z_\varepsilon^1\|_{\mathbb{L}^2}^{\frac{3}{2}} \|Z_\varepsilon^1\|_{\mathbb{L}^2}^{\frac{1}{2}} \|V_\varepsilon^1\|_{\mathbb{L}^4} \leq 3.2^{-\frac{2}{3}} c_0 k\nu \|\nabla Z_\varepsilon^1\|_{\mathbb{L}^2}^2 + \frac{k}{4\nu^3 c_0^3} \|Z_\varepsilon^1\|_{\mathbb{L}^2}^2 \|V_\varepsilon^1\|_{\mathbb{L}^4}^4, \end{aligned} \quad (6.10)$$

where the first inequality employs the estimate

$$\left| \langle \hat{B}(u, u) - \hat{B}(v, v), z \rangle \right| \leq 2 \|\nabla(u - v)\|_{\mathbb{L}^2}^{3/2} \|u - v\|_{\mathbb{L}^2}^{1/2} \|z\|_{\mathbb{L}^4},$$

for all $u, v, z \in \mathbb{H}_0^1$ (see the proof in [88, Lemma 2.3]), and the second inequality uses Young's inequality for some constant $c_0 \in (0, 3^{-1}2^{\frac{2}{3}}]$. Subsequently, equation (6.10) becomes

$$\left(1 - k4^{-1}\nu^{-3}c_0^{-3}\|V_\varepsilon^1\|_{\mathbb{L}^4}^4\right) \|Z_\varepsilon^1\|_{\mathbb{L}^2}^2 + \varepsilon \|Q_\varepsilon^1\|_{L^2}^2 \leq 0. \quad (6.11)$$

One way to obtain uniqueness is by multiplying equation (6.11) by the indicator function $\mathbb{1}_{\Omega_\delta^1}$ which grants

$$(1 - 4^{-1}\nu^{-3}c_0^{-3}\delta^{-1}) \mathbb{1}_{\Omega_\delta^1} \|Z_\varepsilon^1\|_{\mathbb{L}^2}^2 + \varepsilon \mathbb{1}_{\Omega_\delta^1} \|Q_\varepsilon^1\|_{L^2}^2 \leq 0.$$

It follows that $Z_\varepsilon^1 = Q_\varepsilon^1 = 0$ a.e. in D and \mathbb{P} -a.s. in Ω_δ^1 provided that the coefficient of $\|Z_\varepsilon^1\|_{\mathbb{L}^2}^2$ is positive. The second way for uniqueness consists in multiplying equation (6.11) by $\mathbb{1}_{\Omega_\delta^2}$ after employing the inverse estimate (6.4). That is, $\|V_\varepsilon^1\|_{\mathbb{L}^4}^4 \leq 2\|V_\varepsilon^1\|_{\mathbb{L}^2}^2 \|\nabla V_\varepsilon^1\|_{\mathbb{L}^2}^2 \leq 2\mathcal{C}^2 h^{-2} \|V_\varepsilon^1\|_{\mathbb{L}^2}^4$, where the first inequality is due to Ladyzhenskaya (see [83, Lemma I.1]). Therefore, equation (6.11) turns into

$$(1 - 2^{-1}\nu^{-3}c_0^{-3}\mathcal{C}^2\delta^{-1}h^{-2}k) \mathbb{1}_{\Omega_\delta^2} \|Z_\varepsilon^1\|_{\mathbb{L}^2}^2 + \varepsilon \mathbb{1}_{\Omega_\delta^2} \|Q_\varepsilon^1\|_{L^2}^2 \leq 0$$

which implies $Z_\varepsilon^1 = Q_\varepsilon^1 = 0$ a.e. in D and \mathbb{P} -a.s. in Ω_δ^2 provided the coefficient of $\|Z_\varepsilon^1\|_{\mathbb{L}^2}^2$ is positive. With that being said, it suffices to assume that $Z_\varepsilon^{m-1} = Q_\varepsilon^{m-1} = 0$ a.e. in D , \mathbb{P} -a.s. in either Ω_δ^1 or Ω_δ^2 , and re-apply the same technique to obtain a similar result for the rank m . Finally, estimates of $\mathbb{P}(\Omega_\delta^1)$ and $\mathbb{P}(\Omega_\delta^2)$ derive from the Markov inequality. \square

Remark 6.4.1 Picking between Ω_δ^1 and Ω_δ^2 in Lemma 6.4.2 depends on the choice of the viscosity ν . Observe that the condition $\frac{1}{\nu^3\delta} \leq 4c_0^3$ does not allow δ to be small when ν is tiny. Therewith, choosing ν large (resp. small) corresponds to Ω_δ^1 (resp. Ω_δ^2). Moreover, lower bounds associated with $\mathbb{P}(\Omega_\delta^1)$ and $\mathbb{P}(\Omega_\delta^2)$ in Lemma 6.4.2 are finite as illustrated in Lemma 6.4.3. It is worth mentioning that

$$\mathbb{E} \left[k \sum_{m=1}^M \|V_\varepsilon^m\|_{\mathbb{L}^4}^4 \right] \lesssim \mathbb{E} \left[\max_{1 \leq m \leq M} \|V_\varepsilon^m\|_{\mathbb{L}^2}^4 \right]^{\frac{1}{2}} \mathbb{E} \left[\left(k \sum_{m=1}^M \|\nabla V_\varepsilon^m\|_{\mathbb{L}^2}^2 \right)^2 \right]^{\frac{1}{2}},$$

thanks to the Ladyzhenskaya and Cauchy-Schwarz inequalities.

6.4.2 . A priori bounds and convergence

The first part of this section is dedicated to achieving stability of Algorithm 6.1, whose convergence toward the unique solution of equations (6.1) is handled in the second part.

A priori bounds

Lemma 6.4.3 Let $p \in [2, +\infty) \cap \mathbb{N}$ be fixed and assumptions (S_1) - (S_3) be satisfied. Then, iterates $\{(V_\varepsilon^m, \Pi_\varepsilon^m)\}_{m=1}^M$ of Algorithm 6.1 fulfill the following estimates:

$$(i) \mathbb{E} \left[\max_{1 \leq m \leq M} \|V_\varepsilon^m\|_{\mathbb{L}^2}^2 + k\nu \sum_{m=1}^M \|\nabla V_\varepsilon^m\|_{\mathbb{L}^2}^2 + \sum_{m=1}^M \|V_\varepsilon^m - V_\varepsilon^{m-1}\|_{\mathbb{L}^2}^2 \right] \leq C_1,$$

$$(ii) \mathbb{E} \left[\max_{1 \leq m \leq M} \|\Pi_\varepsilon^m\|_{L^2}^2 + \sum_{m=1}^M \|\Pi_\varepsilon^m - \Pi_\varepsilon^{m-1}\|_{L^2}^2 \right] \leq \frac{C_1}{\varepsilon},$$

$$(iii) \mathbb{E} \left[\max_{1 \leq m \leq M} \|V_\varepsilon^m\|_{\mathbb{L}^2}^{2p} + \left(k\nu \sum_{m=1}^M \|\nabla V_\varepsilon^m\|_{\mathbb{L}^2}^2 \right)^{2p-1} + \left(\sum_{m=1}^M \|V_\varepsilon^m - V_\varepsilon^{m-1}\|_{\mathbb{L}^2}^2 \right)^{2p-1} \right] \leq C_p,$$

$$(iv) \mathbb{E} \left[\max_{1 \leq m \leq M} \|\Pi_\varepsilon^m\|_{L^2}^{2p} + \left(\sum_{m=1}^M \left\| \Pi_\varepsilon^m - \Pi_\varepsilon^{m-1} \right\|_{L^2}^2 \right)^{2p-1} \right] \leq \varepsilon^{-2p-1} C_p,$$

for some constant $C_p \geq 0$ depending only on $\|v_0\|_{L^{2p}(\Omega; \mathbb{L}^2)}$, $\|p_0\|_{L^{2p}(\Omega; L^2)}$, D , ν , $\|f\|_{L^{2p}(\Omega; L^2(0, T; \mathbb{H}^{-1}))}$, T , $Tr(Q)$, K_1 , p and K_2 , with $C_1 = C_{p=1}$.

Proof: Replace (φ_h, q_h) by $(V_\varepsilon^m, \Pi_\varepsilon^m)$ in Algorithm 6.1 and employ the identity $(a-b, a) = \frac{1}{2}(\|a\|_{\mathbb{L}^2}^2 - \|b\|_{\mathbb{L}^2}^2 - \|a-b\|_{\mathbb{L}^2}^2)$ together with Proposition 6.2.1-(ii), Cauchy-Schwarz, Poincaré and Young's inequalities:

$$\begin{aligned} & \frac{1}{2} \|V_\varepsilon^m\|_{\mathbb{L}^2}^2 - \frac{1}{2} \|V_\varepsilon^{m-1}\|_{\mathbb{L}^2}^2 + \frac{1}{4} \|V_\varepsilon^m - V_\varepsilon^{m-1}\|_{\mathbb{L}^2}^2 + \frac{\varepsilon}{2} \|\Pi_\varepsilon^m\|_{L^2}^2 - \frac{\varepsilon}{2} \|\Pi_\varepsilon^{m-1}\|_{L^2}^2 \\ & + \frac{\varepsilon}{2} \|\Pi_\varepsilon^m - \Pi_\varepsilon^{m-1}\|_{L^2}^2 + \frac{k\nu}{2} \|\nabla V_\varepsilon^m\|_{\mathbb{L}^2}^2 \leq \frac{C_D^2 k}{2\nu} \|f^m\|_{\mathbb{H}^{-1}}^2 + \|g(V_\varepsilon^{m-1})\Delta_m W\|_{\mathbb{L}^2}^2 \\ & + \left(g(V_\varepsilon^{m-1})\Delta_m W, V_\varepsilon^{m-1} \right). \end{aligned} \quad (6.12)$$

Summing equations (6.12) over m from 1 to $\ell \in \{1, \dots, M\}$, then applying the mathematical expectation, condition $\varepsilon \leq 1$, estimates (6.7) and (6.8) yield

$$\begin{aligned} & \mathbb{E} \left[\|V_\varepsilon^\ell\|_{\mathbb{L}^2}^2 + \varepsilon \|\Pi_\varepsilon^\ell\|_{L^2}^2 + k\nu \sum_{m=1}^{\ell} \|\nabla V_\varepsilon^m\|_{\mathbb{L}^2}^2 + \frac{1}{2} \sum_{m=1}^{\ell} \|V_\varepsilon^m - V_\varepsilon^{m-1}\|_{\mathbb{L}^2}^2 \right. \\ & \left. + \sum_{m=1}^{\ell} \varepsilon \|\Pi_\varepsilon^m - \Pi_\varepsilon^{m-1}\|_{L^2}^2 \right] \\ & \leq \mathbb{E} \left[\|v_0\|_{\mathbb{L}^2}^2 + \|p_0\|_{L^2}^2 + \frac{C_D^2}{\nu} \|f\|_{L^2(0, T; \mathbb{H}^{-1})}^2 + 2 \sum_{m=1}^{\ell} \|g(V_\varepsilon^{m-1})\Delta_m W\|_{\mathbb{L}^2}^2 \right], \end{aligned} \quad (6.13)$$

where the mathematical expectation of last term in equation (6.12) vanishes due to the $\mathcal{F}_{t_{m-1}}$ -measurability of V_ε^{m-1} together with assumption (S_3) . On the other hand, the last term of inequality (6.13) can be handled through Itô's isometry and assumption (S_3) as follows:

$$\begin{aligned} & \mathbb{E} \left[\|g(V_\varepsilon^{m-1})\Delta_m W\|_{\mathbb{L}^2}^2 \right] = \mathbb{E} \left[\left\| \int_{t_{m-1}}^{t_m} g(V_\varepsilon^{m-1}) dW(t) \right\|_{\mathbb{L}^2}^2 \right] = k \mathbb{E} \left[\left\| g(V_\varepsilon^{m-1}) \right\|_{\mathcal{L}_2(\sqrt{Q}(K), \mathbb{L}^2)}^2 \right] \\ & \leq 2k Tr(Q) K_1^2 + 2k Tr(Q) K_2^2 \mathbb{E} \left[\|V_\varepsilon^{m-1}\|_{\mathbb{L}^2}^2 \right]. \end{aligned} \quad (6.14)$$

Thus, the discrete Grönwall inequality implies

$$\begin{aligned} & \max_{1 \leq m \leq M} \mathbb{E} \left[\|V_\varepsilon^m\|_{\mathbb{L}^2}^2 + \varepsilon \|\Pi_\varepsilon^m\|_{L^2}^2 \right] + \sum_{m=1}^M \mathbb{E} \left[k\nu \|\nabla V_\varepsilon^m\|_{\mathbb{L}^2}^2 + \frac{1}{2} \|V_\varepsilon^m - V_\varepsilon^{m-1}\|_{\mathbb{L}^2}^2 \right] \\ & + \mathbb{E} \left[\sum_{m=1}^M \varepsilon \|\Pi_\varepsilon^m - \Pi_\varepsilon^{m-1}\|_{L^2}^2 \right] \leq C_1, \end{aligned} \quad (6.15)$$

where $C_1 > 0$ depends only on $\|v_0\|_{L^2(\Omega; \mathbb{L}^2)}$, $\|p_0\|_{L^2(\Omega; L^2)}$, D , ν , $\|f\|_{L^2(\Omega; L^2(0, T; \mathbb{H}^{-1}))}$, T , $Tr(Q)$, K_1 and K_2 . To terminate the proof of estimates (i) and (ii), it suffices to reconsider equation (6.12), sum it over m from 1 to $\ell \in \{1, \dots, M\}$, take the maximum over ℓ , then apply the mathematical expectation to get

$$\begin{aligned} & \mathbb{E} \left[\max_{1 \leq \ell \leq M} (\|V_\varepsilon^\ell\|_{\mathbb{L}^2}^2 + \varepsilon \|\Pi_\varepsilon^\ell\|_{L^2}^2) \right] \leq \mathbb{E} \left[\|v_0\|_{\mathbb{L}^2}^2 + \|p_0\|_{L^2}^2 + C_D^2 \nu^{-1} \|f\|_{L^2(0, T; \mathbb{H}^{-1})}^2 \right. \\ & \left. + 2 \sum_{m=1}^M \|g(V_\varepsilon^{m-1})\Delta_m W\|_{\mathbb{L}^2}^2 + 2 \max_{1 \leq \ell \leq M} \sum_{m=1}^{\ell} \left(g(V_\varepsilon^{m-1})\Delta_m W, V_\varepsilon^{m-1} \right) \right]. \end{aligned} \quad (6.16)$$

The penultimate term is estimated in inequality (6.14). The last term is controlled by

$$\begin{aligned} &\lesssim 6\mathbb{E} \left[\left(k \sum_{m=1}^M \|g(V_\varepsilon^{m-1})\|_{\mathcal{L}_2(K, \mathbb{L}^2)}^2 \|V_\varepsilon^{m-1}\|_{\mathbb{L}^2}^2 \right)^{\frac{1}{2}} \right] \\ &\leq \frac{3}{4}\mathbb{E} \left[\max_{1 \leq \ell \leq M} \|V_\varepsilon^\ell\|_{\mathbb{L}^2}^2 \right] + \mathbb{E} \left[\frac{3}{4}\|v_h^0\|_{\mathbb{L}^2}^2 + 3k \sum_{m=1}^M (K_1^2 + K_2^2 \|V_\varepsilon^{m-1}\|_{\mathbb{L}^2}^2) \right], \end{aligned}$$

where Young's inequality and assumption (S_3) are used together with the Davis inequality which is applicable since the integrand can be considered as a simple function with respect to time. Obviously, the first term on the right-hand side must be absorbed into the left side of equation (6.16) and the remaining terms can be readily controlled through estimates (6.7) and (6.15). This completes the proof of assertions (i) and (ii). Estimates (iii) and (iv) can be demonstrated as follows: let $p \geq 2$ be an integer. Summing equation (6.12) over m from 1 to $\ell \in \{1, \dots, M\}$, making use of estimates (6.7), (6.8), then raising both sides to the power 2^{p-1} yield

$$\begin{aligned} &\max_{1 \leq \ell \leq M} (\|V_\varepsilon^\ell\|_{\mathbb{L}^2}^{2p} + \varepsilon^{2p-1} \|\Pi_\varepsilon^\ell\|_{\mathbb{L}^2}^{2p}) + \left(\sum_{m=1}^M (\|V_\varepsilon^m - V_\varepsilon^{m-1}\|_{\mathbb{L}^2}^2 + \varepsilon \|\Pi_\varepsilon^m - \Pi_\varepsilon^{m-1}\|_{\mathbb{L}^2}^2) \right)^{2p-1} \\ &+ \left(k\nu \sum_{m=1}^M \|\nabla V_\varepsilon^m\|_{\mathbb{L}^2}^2 \right)^{2p-1} \lesssim \|v_0\|_{\mathbb{L}^2}^{2p} + \|p_0\|_{\mathbb{L}^2}^{2p} + C_D \nu^{-2p-1} \|f\|_{L^2(0, T; \mathbb{H}^{-1})}^{2p} \\ &+ \left(\sum_{m=1}^M \|g(V_\varepsilon^{m-1}) \Delta_m W\|_{\mathbb{L}^2}^2 \right)^{2p-1} + \left(\max_{1 \leq \ell \leq M} \sum_{m=1}^{\ell} (g(V_\varepsilon^{m-1}) \Delta_m W, V_\varepsilon^{m-1}) \right)^{2p-1}. \end{aligned} \quad (6.17)$$

The mathematical expectation of the penultimate term is estimated through assumption (S_3) , inequality $|a+b|^q \leq 2^{q-1}(|a|^q + |b|^q)$, the Burkholder-Davis-Gundy (see [23, Theorem 2.4]) and Young inequalities as follows:

$$\begin{aligned} &\mathbb{E} \left[\left(\sum_{m=1}^M \|g(V_\varepsilon^{m-1}) \Delta_m W\|_{\mathbb{L}^2}^2 \right)^{2p-1} \right] \leq M^{2p-1-1} \mathbb{E} \left[\sum_{m=1}^M \left\| \int_{t_{m-1}}^{t_m} g(V_\varepsilon^{m-1}) dW(t) \right\|_{\mathbb{L}^2}^{2p} \right] \\ &\lesssim M^{2p-1-1} \sum_{m=1}^M \mathbb{E} \left[\left(\int_{t_{m-1}}^{t_m} \|g(V_\varepsilon^{m-1})\|_{\mathcal{L}_2(K, \mathbb{L}^2)}^2 dt \right)^{2p-1} \right] \\ &\leq T^{2p-1-1} \mathbb{E} \left[k \sum_{m=1}^M \|g(V_\varepsilon^{m-1})\|_{\mathcal{L}_2(K, \mathbb{L}^2)}^{2p} \right] \lesssim T^{2p-1} K_1^{2p} + T^{2p-1} K_2^{2p} \mathbb{E} \left[k \sum_{m=1}^M \|V_\varepsilon^{m-1}\|_{\mathbb{L}^2}^{2p} \right]. \end{aligned} \quad (6.18)$$

The last term of equation (6.17) can be controlled through the Burkholder-Davis-Gundy inequality, assumption (S_3) , the Young and Hölder inequalities as follows:

$$\begin{aligned} &\mathbb{E} \left[\left(\max_{1 \leq \ell \leq M} \sum_{m=1}^{\ell} (g(V_\varepsilon^{m-1}) \Delta_m W, V_\varepsilon^{m-1}) \right)^{2p-1} \right] \lesssim \mathbb{E} \left[\left(k \sum_{m=1}^M \|g(V_\varepsilon^{m-1})\|_{\mathcal{L}_2(K, \mathbb{L}^2)}^2 \|V_\varepsilon^{m-1}\|_{\mathbb{L}^2}^2 \right)^{2p-2} \right] \\ &\leq \mathbb{E} \left[\max_{1 \leq m \leq M} \|V_\varepsilon^{m-1}\|_{\mathbb{L}^2}^{2p-1} k^{2p-1} M^{2p-1-1} \sum_{m=1}^M \|g(V_\varepsilon^{m-1})\|_{\mathcal{L}_2(K, \mathbb{L}^2)}^{2p} \right] \\ &\leq \frac{1}{4} \mathbb{E} \left[\max_{1 \leq m \leq M} \|V_\varepsilon^m\|_{\mathbb{L}^2}^{2p} \right] + \frac{1}{4} \mathbb{E} \left[\|v_0\|_{\mathbb{L}^2}^{2p} \right] + T^{2p-2} \mathbb{E} \left[\left(k \sum_{m=1}^M \|g(V_\varepsilon^{m-1})\|_{\mathcal{L}_2(K, \mathbb{L}^2)}^2 \right)^2 \right] \\ &\leq \frac{1}{4} \mathbb{E} \left[\max_{1 \leq m \leq M} \|V_\varepsilon^m\|_{\mathbb{L}^2}^{2p} \right] + \frac{1}{4} \mathbb{E} \left[\|v_0\|_{\mathbb{L}^2}^{2p} \right] + T^{2p-1} \mathbb{E} \left[k \sum_{m=1}^M (K_1 + K_2 \|V_\varepsilon^{m-1}\|_{\mathbb{L}^2})^{2p-1} \right] \\ &\lesssim \frac{1}{4} \mathbb{E} \left[\max_{1 \leq m \leq M} \|V_\varepsilon^m\|_{\mathbb{L}^2}^{2p} \right] + \frac{1}{4} \mathbb{E} \left[\|v_0\|_{\mathbb{L}^2}^{2p} \right] + T^{2p} K_1^{2p-1} + T^{2p-1} K_2^{2p-1} \mathbb{E} \left[k \sum_{m=1}^M \|V_\varepsilon^{m-1}\|_{\mathbb{L}^2}^{2p} \right]. \end{aligned}$$

Putting it all together and applying the discrete Grönwall inequality to equation (6.17) complete the proof. \square

Convergence

Stability properties that were derived in Lemma 6.4.3 will play a crucial role in this part, especially to offer convergence results to $\{(V_\varepsilon^m, \Pi_\varepsilon^m)\}_{m=1}^M$ as $\varepsilon, k, h \rightarrow 0$. For this purpose, a few new notations must be summoned along with one important lemma consisting of a monotonicity property that allows the convergence of Algorithm 6.1 toward equations (6.1) to occur. For all $m \in \{1, \dots, M\}$, the new notations read:

$$\begin{aligned} (\mathcal{V}_{\varepsilon,k,h}^+(t), \Pi_{\varepsilon,k,h}^+(t)) &:= (V_\varepsilon^m, \Pi_\varepsilon^m), \quad \forall t \in (t_{m-1}, t_m], \\ (\mathcal{V}_{\varepsilon,k,h}^-(t), \Pi_{\varepsilon,k,h}^-(t)) &:= (V_\varepsilon^{m-1}, \Pi_\varepsilon^{m-1}), \quad \forall t \in [t_{m-1}, t_m). \end{aligned}$$

There will also be similar notations in the upcoming part such as f^+ and r^- ; the reader may refer to section 6.2.2 for an adequate definition. Note that it is not mandatory for ε to be dependent on the discretization parameters k and h . If so, it suffices that $\varepsilon = \varepsilon(k, h) \rightarrow 0$ as $k, h \rightarrow 0$.

For instance, the penalty parameter ε may be linked to the time discretization step size k in a way that k/ε tends to 0 when both k, ε vanish. This idea is exposed in the below proposition, but will not be utilized for the convergence analysis of Algorithm 6.1.

Proposition 6.4.1 *Let $\{(V_\varepsilon^m, \Pi_\varepsilon^m)\}_{m=1}^M$ be the iterates of Algorithm 6.1. Then,*

$$\mathbb{E} \left[\sum_{m=1}^M \varepsilon \left\| \Pi_\varepsilon^m - \Pi_\varepsilon^{m-1} \right\|_{L^2}^2 \right] \leq 2C_1 \frac{k}{\varepsilon},$$

where C_1 appears in Lemma 6.4.3.

Proof: Let $q \in L^2(D) \setminus \{0\}$. By identity (6.6), it holds that $(\Pi_\varepsilon^m - \Pi_\varepsilon^{m-1}, q) = (\Pi_\varepsilon^m - \Pi_\varepsilon^{m-1}, \rho_h q)$. Therefore, using Algorithm 6.1, one obtains

$$\varepsilon(\Pi_\varepsilon^m - \Pi_\varepsilon^{m-1}, q) = -k(\operatorname{div}(V_\varepsilon^m), \rho_h q) \leq \sqrt{2}k \|\nabla V_\varepsilon^m\|_{\mathbb{L}^2} \|q\|_{L^2}, \quad (6.19)$$

thanks to the Cauchy-Schwarz inequality, the stability of ρ_h in $L^2(D)$ and the estimate $\|\operatorname{div}(\cdot)\|_{L^2} \leq \sqrt{d} \|\nabla \cdot\|_{\mathbb{L}^2}$, with $d = 2$ is the dimension. Therefore,

$$\sup_{q \in L^2(D) \setminus \{0\}} \frac{\varepsilon(\Pi_\varepsilon^m - \Pi_\varepsilon^{m-1}, q)}{\|q\|_{L^2}} \leq k\sqrt{2} \|\nabla V_\varepsilon^m\|_{\mathbb{L}^2}.$$

Since $L^2(D)$ is the pivot space, the supremum in the above equation turns into $\varepsilon \|\Pi_\varepsilon^m - \Pi_\varepsilon^{m-1}\|_{L^2}$. Therefore, squaring both sides of the above equation, taking the sum over m from 1 to M , then applying the mathematical expectation return the following

$$\mathbb{E} \left[\sum_{m=1}^M \varepsilon \|\Pi_\varepsilon^m - \Pi_\varepsilon^{m-1}\|_{L^2}^2 \right] \leq \frac{2k}{\varepsilon} \mathbb{E} \left[k \sum_{m=1}^M \|\nabla V_\varepsilon^m\|_{\mathbb{L}^2}^2 \right].$$

Finally, a simple application of Lemma 6.4.3-(i) completes the proof. \square

Going back to the convergence demonstration of Algorithm 6.1, the following lemma states a monotonicity property of the operator $u \mapsto -\nu \Delta u + \hat{B}(u, u)$. This feature together with the Lipschitz-continuity of the diffusion coefficient g allow the avoidance of the Skorokhod theorem which forces the filtered probability space that was defined in Section 6.2 to be exchanged with a new one.

Lemma 6.4.4 Assume that $L_g \leq \sqrt{\frac{\nu}{2C_P^2}}$ where $C_P > 0$ is the Poincaré constant, and let $u, w \in \mathbb{H}_0^1$. For $z := u - w$, the following inequality holds true:

$$\left\langle -\nu\Delta z + \hat{B}(u, u) - \hat{B}(w, w) + \frac{27}{2\nu^3} \|w\|_{\mathbb{L}^4}^4 z, z \right\rangle - \|g(u) - g(w)\|_{\mathcal{L}_2(\sqrt{Q}(K), \mathbb{L}^2)}^2 \geq 0.$$

Proof: From [88, Lemma 2.4], it holds that

$$\langle -\nu\Delta z + \hat{B}(u, u) - \hat{B}(w, w) + \frac{27}{2\nu^3} \|w\|_{\mathbb{L}^4}^4 z, z \rangle \geq \frac{\nu}{2} \|\nabla z\|_{\mathbb{L}^2}^2.$$

It suffices now to subtract from both sides the term $\|g(u) - g(w)\|_{\mathcal{L}_2(\sqrt{Q}(K), \mathbb{L}^2)}^2$, use assumption (S_3) , then employ the Poincaré inequality. \square

Besides Lemma 6.4.4, it is worth highlighting the strong convergence of $\{g(\mathcal{V}_{\varepsilon, k, h}^+) - g(\mathcal{V}_{\varepsilon, k, h}^-)\}_{k, h}$ in $L^2(\Omega; L^2(0, T; \mathcal{L}_2(K, \mathbb{L}^2)))$, which can be illustrated through assumption (S_3) and Lemma 6.4.3-(i) as follows

$$\mathbb{E} \left[\int_0^T \|g(\mathcal{V}_{\varepsilon, k, h}^+) - g(\mathcal{V}_{\varepsilon, k, h}^-)\|_{\mathcal{L}_2(K, \mathbb{L}^2)}^2 dt \right] \leq L_g^2 k \mathbb{E} \left[\sum_{m=1}^M \|V_\varepsilon^m - V_\varepsilon^{m-1}\|_{\mathbb{L}^2}^2 \right] \leq L_g^2 C_1 k \rightarrow 0. \quad (6.20)$$

The convergence demonstration down below is broken down into steps for clarity's sake.

Step1: Weak convergence and divergence-free

By virtue of Lemma 6.4.3, the sublinearity of g (see assumption (S_3)) and inequality (6.7), the sequences $\{\mathcal{V}_{\varepsilon, k, h}^+\}_{\varepsilon, k, h}$, $\{\sqrt{\varepsilon}\Pi_{\varepsilon, k, h}^+\}_{\varepsilon, k, h}$, $\{g(\mathcal{V}_{\varepsilon, k, h}^-)\}_{\varepsilon, k, h}$ are bounded in the Banach spaces $L^2(\Omega; L^\infty(0, T; \mathbb{L}^2) \cap L^2(0, T; \mathbb{H}_0^1))$, $L^2(\Omega; L^\infty(0, T; L^2(D)))$ and $L^2(\Omega; L^2(0, T; \mathcal{L}_2(K, \mathbb{L}^2)))$, respectively. Therefore, the Banach-Alaoglu theorem ensures the existence of the limiting functions $v \in L^2(\Omega; L^\infty(0, T; \mathbb{L}^2) \cap L^2(0, T; \mathbb{H}_0^1))$, $\chi \in L^2(\Omega; L^\infty(0, T; L^2(D)))$, $G_0 \in L^2(\Omega; L^2(0, T; \mathcal{L}_2(K, \mathbb{L}^2)))$ and two subsequences (still denoted as their original sequences) $\{\mathcal{V}_{\varepsilon, k, h}^+\}_{\varepsilon, k, h}$, $\{\sqrt{\varepsilon}\Pi_{\varepsilon, k, h}^+\}_{\varepsilon, k, h}$ such that

$$\mathcal{V}_{\varepsilon, k, h}^+ \overset{*}{\rightharpoonup} v \quad \text{in } L^2(\Omega; L^\infty(0, T; \mathbb{L}^2)), \quad (6.21)$$

$$\mathcal{V}_{\varepsilon, k, h}^+ \rightharpoonup v \quad \text{in } L^2(\Omega; L^2(0, T; \mathbb{H}_0^1)), \quad (6.22)$$

$$\sqrt{\varepsilon}\Pi_{\varepsilon, k, h}^+ \overset{*}{\rightharpoonup} \chi \quad \text{in } L^2(\Omega; L^\infty(0, T; L^2(D))), \quad (6.23)$$

$$g(\mathcal{V}_{\varepsilon, k, h}^-) \rightharpoonup G_0 \quad \text{in } L^2(\Omega; L^2(0, T; \mathcal{L}_2(K, \mathbb{L}^2))). \quad (6.24)$$

Besides convergence (6.24), it is also possible to acquire $g(\mathcal{V}_{\varepsilon, k, h}^+) \rightharpoonup G_0$ in $L^2(\Omega; L^2(0, T; \mathcal{L}_2(K, \mathbb{L}^2)))$ as follows: for all $\phi \in L^2(\Omega; L^2(0, T; \mathcal{L}_2(K, \mathbb{L}^2)))$,

$$\begin{aligned} \left(g(\mathcal{V}_{\varepsilon, k, h}^+) - G_0(t), \phi(t) \right)_{\mathcal{L}_2(K, \mathbb{L}^2)} &= \left(g(\mathcal{V}_{\varepsilon, k, h}^+) - g(\mathcal{V}_{\varepsilon, k, h}^-), \phi(t) \right)_{\mathcal{L}_2(K, \mathbb{L}^2)} \\ &\quad + \left(g(\mathcal{V}_{\varepsilon, k, h}^-) - G_0(t), \phi(t) \right)_{\mathcal{L}_2(K, \mathbb{L}^2)}. \end{aligned} \quad (6.25)$$

Now, integrate with respect to t , take the mathematical expectation, use results (6.20) and (6.24) to complete the proof.

The obtained function v is divergence-free. Indeed, let $q \in C_c^\infty(D)$ be a scalar function. From Algorithm 6.1, one has $\varepsilon (\Pi_\varepsilon^m - \Pi_\varepsilon^{m-1}, \rho_h q) = -k (\operatorname{div}(V_\varepsilon^m), \rho_h q)$. Summing both sides over m from 1 to M leads to $\int_0^T (\operatorname{div}(\mathcal{V}_{\varepsilon, k, h}^+), \rho_h q) dt = \varepsilon (\Pi_\varepsilon^0, \rho_h q) - \sqrt{\varepsilon} (\sqrt{\varepsilon}\Pi_{\varepsilon, k, h}^+(T), \rho_h q)$. The mathematical expectation of the right-hand side goes to 0 as $\varepsilon, k, h \rightarrow 0$ due to convergence (6.23) and estimate (6.7). Hence,

$$\mathbb{E} \left[\int_0^T (\operatorname{div}(\mathcal{V}_{\varepsilon, k, h}^+), q) dt \right] = \mathbb{E} \left[\int_0^T (\operatorname{div}(\mathcal{V}_{\varepsilon, k, h}^+), q - \rho_h q) dt \right] + \mathbb{E} \left[\int_0^T (\operatorname{div}(\mathcal{V}_{\varepsilon, k, h}^+), \rho_h q) dt \right]$$

converges toward 0 as $\varepsilon, k, h \rightarrow 0$, thanks to estimate (6.5) and convergence $\operatorname{div}(\mathcal{V}_{\varepsilon, k, h}^+) \rightarrow \operatorname{div}(v)$ in $L^2(\Omega; L^2(0, T; L^2(D)))$ which follows straightforwardly from result (6.22). Subsequently, $\operatorname{div}(\mathcal{V}_{\varepsilon, k, h}^+) \rightharpoonup 0$ in $L^2(\Omega; L^2(0, T; L^2(D)))$ which implies $\operatorname{div}(v) = 0$ \mathbb{P} -a.s. and a.e. in $(0, T) \times D$.

Let $\mathcal{R}: \mathbb{H}_0^1 \rightarrow \mathbb{H}^{-1}$ be defined by $\mathcal{R}(u) := -\nu \Delta u + \hat{B}(u, u)$, for all $u \in \mathbb{H}_0^1$. From Algorithm 6.1, and for all $\varphi \in \mathcal{V}$ such that $\varphi_h := \Pi_h \varphi$, it follows

$$\begin{aligned} \int_0^T \langle \mathcal{R}(\mathcal{V}_{\varepsilon, k, h}^+) + \nabla \Pi_{\varepsilon, k, h}^+, \varphi_h \rangle dt &= - \left(\mathcal{V}_{\varepsilon, k, h}^+(T) - \mathcal{V}_{\varepsilon, k, h}^-(0), \varphi_h \right) + \int_0^T \langle f^+, \varphi_h \rangle dt \\ &\quad + \left(\int_0^T g(\mathcal{V}_{\varepsilon, k, h}^-) dW(t), \varphi_h \right). \end{aligned} \quad (6.26)$$

Owing to results (6.21) and (6.24) along with the strong convergence of f^+ in $L^2(\Omega; L^2(0, T; \mathbb{H}^{-1}))$ (see [107, Lemma III.4.9]), the mathematical expectation of the right-hand side of equation (6.26) is convergent. Therewith, define \mathcal{R}_0 by

$$\mathbb{E} \left[\int_0^T \langle \mathcal{R}_0(t), \varphi \rangle dt \right] = \lim_{\varepsilon, k, h \rightarrow 0} \mathbb{E} \left[\int_0^T \langle \mathcal{R}(\mathcal{V}_{\varepsilon, k, h}^+) + \nabla \Pi_{\varepsilon, k, h}^+, \Pi_h \varphi \rangle dt \right], \quad \forall \varphi \in \mathcal{V}.$$

Subsequently, the limiting function v satisfies \mathbb{P} -a.s. and for all $(t, \varphi) \in [0, T] \times \mathcal{V}$ the following:

$$(v(t) - v_0, \varphi) + \int_0^t \langle \mathcal{R}_0(s), \varphi \rangle ds = \int_0^t \langle f(s), \varphi \rangle ds + \left(\int_0^t G_0(s) dW(s), \varphi \right). \quad (6.27)$$

Step 2: Identification of \mathcal{R}_0 and G_0

For $\sigma \in C([0, T], \mathcal{V})$, define the finite element space of weakly divergence-free functions:

$$\mathbb{V}_h := \{u_h \in \mathbb{H}_h \mid (\operatorname{div}(u_h), q_h) = 0, \forall q_h \in L_h\}.$$

Let $P_h: \mathbb{L}^2 \rightarrow \mathbb{V}_h$ be the projection operator from \mathbb{L}^2 onto \mathbb{V}_h such that for all $u \in \mathbb{L}^2$,

$$(P_h u, \varphi_h) = (u, \varphi_h), \quad \forall \varphi_h \in \mathbb{V}_h.$$

The space \mathbb{V}_h is required because the function σ that was introduced shortly before is divergence-free and shall be projected onto a finite element space that owns a null divergence constraint. This will become clearer when the discrete pressure $\{\Pi_\varepsilon^m\}_{m=1}^M$ is dealt with in the sequel, especially because the limit of iterates $\{(V_\varepsilon^m, \Pi_\varepsilon^m)\}_{m=1}^M$ takes place in divergence-free spaces, as shown earlier for v . It is worth pointing out that the finite element space \mathbb{V}_h does not interact with $\{V_\varepsilon^m\}_{m=1}^M$ and $\{\Pi_\varepsilon^m\}_{m=1}^M$. In other words, the sequence $\{V_\varepsilon^m\}_{m=1}^M$ will remain non divergence-free and will never belong to \mathbb{V}_h . Now, for all $m \in \{1, \dots, M\}$, denote $\sigma_h^+(t) := \sigma_h^m := P_h \sigma(t_m)$ and define $r^+(t) := r^m := \frac{27}{\nu^3} k \sum_{n=1}^m \|\sigma_h^n\|_{\mathbb{L}^4}^4$ for all $t \in (t_{m-1}, t_m]$, together with an exponential non-increasing function $\eta: [0, T] \rightarrow \mathbb{R}$ verifying $\eta(0) = 0$, and having the discrete forms $\eta^+(t) := \eta^m := e^{-r^+(t)}$ for all $t \in (t_{m-1}, t_m]$ and $\eta^-(t) := \eta^{m-1} := e^{-r^-(t)}$ for all $t \in [t_{m-1}, t_m)$. Setting $(\varphi_h, q_h) = (V_\varepsilon^m, \Pi_\varepsilon^m)$ in Algorithm 6.1, using Cauchy-Schwarz and Young's inequalities, identity $(a - b, a) = \frac{1}{2} \|a\|_{\mathbb{L}^2}^2 - \frac{1}{2} \|b\|_{\mathbb{L}^2}^2 + \frac{1}{2} \|a - b\|_{\mathbb{L}^2}^2$, and finally multiplying by η^{m-1} yield

$$\begin{aligned} \eta^{m-1} (\|V_\varepsilon^m\|_{\mathbb{L}^2}^2 - \|V_\varepsilon^{m-1}\|_{\mathbb{L}^2}^2) + 2\eta^{m-1} k \langle \mathcal{R}(V_\varepsilon^m) + \nabla \Pi_\varepsilon^m, V_\varepsilon^m \rangle &\leq 2\eta^{m-1} k \langle f^m, V_\varepsilon^m \rangle \\ + \eta^{m-1} \|g(V_\varepsilon^{m-1}) \Delta_m W\|_{\mathbb{L}^2}^2 + 2\eta^{m-1} \left(g(V_\varepsilon^{m-1}) \Delta_m W, V_\varepsilon^{m-1} \right). \end{aligned} \quad (6.28)$$

Note that $\sum_{m=1}^M \eta^{m-1} (\|V_\varepsilon^m\|_{\mathbb{L}^2}^2 - \|V_\varepsilon^{m-1}\|_{\mathbb{L}^2}^2) = \int_0^T \eta^-(t) dt \| \mathcal{V}_{\varepsilon, k, h}^+ \|_{\mathbb{L}^2}^2 dt$, and through equation (6.14), it holds that $\mathbb{E} [\|g(V_\varepsilon^{m-1}) \Delta_m W\|_{\mathbb{L}^2}^2] = k \mathbb{E} [\|g(V_\varepsilon^{m-1})\|_{\mathcal{L}_2(\sqrt{Q}(K), \mathbb{L}^2)}^2]$. Therefore, taking the sum

over m from 1 to M , employing Proposition 6.2.2-(ii), then applying the mathematical expectation to equation (6.28) give

$$\begin{aligned} & \mathbb{E} \left[\eta^+(T) \|\mathcal{V}_{\varepsilon,k,h}^+(T)\|_{\mathbb{L}^2}^2 - \|\mathcal{V}_{\varepsilon,k,h}^-(0)\|_{\mathbb{L}^2}^2 \right] \leq \mathbb{E} \left[\int_0^T \|\mathcal{V}_{\varepsilon,k,h}^+\|_{\mathbb{L}^2}^2 dt \eta^+ dt \right] \\ & - \mathbb{E} \left[\int_0^T \eta^-(t) \langle 2\mathcal{R}(\mathcal{V}_{\varepsilon,k,h}^+) + 2\nabla \Pi_{\varepsilon,k,h}^+, \mathcal{V}_{\varepsilon,k,h}^+ \rangle dt \right] + \mathbb{E} \left[2 \int_0^T \eta^-(t) \langle f^+, \mathcal{V}_{\varepsilon,k,h}^+ \rangle dt \right] \\ & + \mathbb{E} \left[\int_0^T \eta^-(t) \|g(\mathcal{V}_{\varepsilon,k,h}^-)\|_{\mathcal{L}_2(\sqrt{Q}(K), \mathbb{L}^2)}^2 dt \right] =: I + II + III + IV, \end{aligned} \quad (6.29)$$

where the last term on the right-hand side of equation (6.28) vanishes after taking its expectation due to assumption (S_3) and the measurability of $\{V_\varepsilon^m\}_m$ (see Lemma 6.4.1). By virtue of Proposition 6.2.2-(iii), it follows that $d_t \eta^+ = -\frac{27}{\nu^3} \eta^- \|\sigma_h^+\|_{\mathbb{L}^4}^4 + \frac{27^2 k}{2\nu^6} e^{\delta(t)} \|\sigma_h^+\|_{\mathbb{L}^4}^8$, for some $\delta \in (-r^+, -r^-)$. Therefore,

$$I = -\mathbb{E} \left[\int_0^T \eta^-(t) \frac{27}{\nu^3} \|\sigma_h^+\|_{\mathbb{L}^4}^4 \|\mathcal{V}_{\varepsilon,k,h}^+\|_{\mathbb{L}^2}^2 dt \right] + \frac{27^2 k}{2\nu^6} \mathbb{E} \left[\int_0^T \|\mathcal{V}_{\varepsilon,k,h}^+\|_{\mathbb{L}^2}^2 e^{\delta(t)} \|\sigma_h^+\|_{\mathbb{L}^4}^8 dt \right] =: I_1 + I_2.$$

Obviously, I_2 goes to 0 as $k, h, \varepsilon \rightarrow 0$ thanks to Lemma 6.4.3. I_1 can be rewritten as follows

$$\begin{aligned} I_1 &= -\frac{27}{\nu^3} \mathbb{E} \left[\int_0^T \eta^- \|\sigma_h^+\|_{\mathbb{L}^4}^4 \|\mathcal{V}_{\varepsilon,k,h}^+ - \sigma_h^+\|_{\mathbb{L}^2}^2 dt \right] \\ & - \frac{27}{\nu^3} \mathbb{E} \left[\int_0^T \eta^- \|\sigma_h^+\|_{\mathbb{L}^4}^4 \left\{ 2 \left(\mathcal{V}_{\varepsilon,k,h}^+, \sigma_h^+ \right) - \|\sigma_h^+\|_{\mathbb{L}^2}^2 \right\} dt \right] =: I_{1,1} + I_{1,2}. \end{aligned}$$

Making use of result (6.22) along with the strong convergence of $\{\sigma_h^m\}_m$ to σ in $C([0, T]; \mathbb{V})$, it can be easily shown that $I_{1,2} \rightarrow -\frac{27}{\nu^3} \mathbb{E} \left[\int_0^T \eta(t) \|\sigma(t)\|_{\mathbb{L}^4}^4 \left\{ 2 \left(v(t), \sigma(t) \right) - \|\sigma(t)\|_{\mathbb{L}^2}^2 \right\} dt \right]$. On the other hand,

$$\begin{aligned} II &= -\mathbb{E} \left[\int_0^T \eta^- \langle 2\mathcal{R}(\mathcal{V}_{\varepsilon,k,h}^+) - 2\mathcal{R}(\sigma_h^+), \mathcal{V}_{\varepsilon,k,h}^+ - \sigma_h^+ \rangle dt \right] - \mathbb{E} \left[\int_0^T \eta^- \langle 2\nabla \Pi_{\varepsilon,k,h}^+, \mathcal{V}_{\varepsilon,k,h}^+ - \sigma_h^+ \rangle dt \right] \\ & - \mathbb{E} \left[\int_0^T \eta^- \langle 2\mathcal{R}(\mathcal{V}_{\varepsilon,k,h}^+) + 2\nabla \Pi_{\varepsilon,k,h}^+ - 2\mathcal{R}(\sigma_h^+), \sigma_h^+ \rangle dt \right] - \mathbb{E} \left[\int_0^T \eta^- \langle 2\mathcal{R}(\sigma_h^+), \mathcal{V}_{\varepsilon,k,h}^+ \rangle dt \right] \\ & =: II_1 + II_2 + II_3 + II_4. \end{aligned}$$

By an integration by parts, II_2 can be rewritten as follows:

$$II_2 = 2\mathbb{E} \left[\int_0^T \eta^-(t) \left(\Pi_{\varepsilon,k,h}^+, \operatorname{div}(\mathcal{V}_{\varepsilon,k,h}^+) \right) dt \right] = 2\mathbb{E} \left[k \sum_{m=1}^M \eta^{m-1} \left(\Pi_\varepsilon^m, \operatorname{div}(V_\varepsilon^m) \right) \right],$$

because $\sigma_h^+ \in \mathbb{V}_h$ i.e. $(\Pi_{\varepsilon,k,h}^+, \operatorname{div}(\sigma_h^+)) = 0$. Therefore, making use of Algorithm 6.1 yields

$$\begin{aligned} II_2 &= -2\varepsilon \mathbb{E} \left[\sum_{m=1}^M \eta^{m-1} \left(\Pi_\varepsilon^m - \Pi_\varepsilon^{m-1}, \Pi_\varepsilon^m \right) \right] \\ &= -\varepsilon \mathbb{E} \left[\sum_{m=1}^M \eta^{m-1} \left(\|\Pi_\varepsilon^m\|_{L^2}^2 - \|\Pi_\varepsilon^{m-1}\|_{L^2}^2 + \|\Pi_\varepsilon^m - \Pi_\varepsilon^{m-1}\|_{L^2}^2 \right) \right] \\ &= -\varepsilon \mathbb{E} \left[\int_0^T \eta^- dt \|\Pi_{\varepsilon,k,h}^+\|_{L^2}^2 \right] - \varepsilon \mathbb{E} \left[\sum_{m=1}^M \eta^{m-1} \|\Pi_\varepsilon^m - \Pi_\varepsilon^{m-1}\|_{L^2}^2 \right] =: II_{2,1} + II_{2,2}, \end{aligned}$$

thanks to the identity $(a - b, a) = \frac{1}{2} (\|a\|_{L^2}^2 - \|b\|_{L^2}^2 + \|a - b\|_{L^2}^2)$. Observe first that $II_{2,2} \leq 0$, which is true because $\{\eta^m\}_m$ is a nonnegative sequence. Moreover, by Proposition 6.2.2-(ii), it holds that

$$\begin{aligned} II_{2,1} &= \varepsilon \mathbb{E} \left[\int_0^T \|\Pi_{\varepsilon,k,h}^+\|_{L^2}^2 dt \eta^+(t) dt \right] - \varepsilon \mathbb{E} \left[\eta^+(T) \|\Pi_{\varepsilon,k,h}^+(T)\|_{L^2}^2 - \|\Pi_{\varepsilon,k,h}^-(0)\|_{L^2}^2 \right] \\ &\leq \varepsilon \mathbb{E} \left[\int_0^T \|\Pi_{\varepsilon,k,h}^+\|_{L^2}^2 dt \eta^+(t) dt \right] + \varepsilon \mathbb{E} \left[\|\Pi_{\varepsilon,k,h}^-(0)\|_{L^2}^2 \right] =: II_{2,1,1} + II_{2,1,2}. \end{aligned}$$

Proposition 6.2.2-(iii) and Lemma 6.4.3-(ii) imply

$$\begin{aligned} II_{2,1,1} &= \varepsilon \mathbb{E} \left[\int_0^T \|\Pi_{\varepsilon,k,h}^+\|_{L^2}^2 \left(-\frac{27}{\nu^3} \eta^- \|\sigma_h^+\|_{\mathbb{L}^4}^4 + \frac{27^2 k}{2\nu^6} e^{\delta(t)} \|\sigma_h^+\|_{\mathbb{L}^4}^8 \right) dt \right] \\ &\leq \frac{27^2}{2\nu^6} \|\sigma\|_{L^\infty(0,T;\mathbb{V})}^8 \mathbb{E} \left[\int_0^T \varepsilon \|\Pi_{\varepsilon,k,h}^+\|_{L^2}^2 \right] k \leq \frac{27^2 T}{2\nu^6} \|\sigma\|_{L^\infty(0,T;\mathbb{V})}^8 \mathbb{E} \left[\varepsilon \max_{1 \leq m \leq M} \|\Pi_\varepsilon^m\|_{L^2}^2 \right] k \\ &\leq \frac{27^2 T C_1}{2\nu^6} \|\sigma\|_{L^\infty(0,T;\mathbb{V})}^8 k \rightarrow 0 \text{ as } k \rightarrow 0, \end{aligned}$$

for some $\delta(t) \in (-r^+(t), -r^-(t))$. Furthermore, since the projector ρ_h is stable in $L^2(D)$, it follows that

$$II_{2,1,2} = \varepsilon \mathbb{E} \left[\|p_h^0\|_{L^2}^2 \right] \leq \varepsilon \mathbb{E} \left[\|p_0\|_{L^2}^2 \right] \rightarrow 0 \text{ as } \varepsilon \rightarrow 0.$$

Moreover, since $\{\sigma_h^m\}_m$ is strongly convergent toward σ in $C([0, T]; \mathbb{V})$, and by the definition of operator \mathcal{R}_0 , one obtains $II_3 \rightarrow -\mathbb{E} \left[\int_0^T \eta(t) \langle 2\mathcal{R}_0(t) - 2\mathcal{R}(\sigma(t)), \sigma(t) \rangle dt \right]$ as $k, h, \varepsilon \rightarrow 0$. Similarly,

$II_4 \rightarrow -\mathbb{E} \left[\int_0^T \eta(t) \langle 2\mathcal{R}(\sigma(t)), v(t) \rangle dt \right]$, thanks to convergence (6.22). As mentioned in Step 1, $\{f^m\}_m$ converges strongly toward f in $L^2(\Omega; L^2(0, T; \mathbb{H}^{-1}))$. The latter together with convergence (6.22) imply that $III \rightarrow \mathbb{E} \left[2 \int_0^T \eta(t) \langle f(t), v(t) \rangle dt \right]$. Moving on to term IV , it can be reformulated as follows:

$$\begin{aligned} IV &= \mathbb{E} \left[\int_0^T \eta^- \left\{ \|g(\mathcal{V}_{\varepsilon,k,h}^-) - g(\mathcal{V}_{\varepsilon,k,h}^+)\|_{\mathcal{L}_2^Q}^2 + \|g(\mathcal{V}_{\varepsilon,k,h}^+) - g(\sigma_h^+)\|_{\mathcal{L}_2^Q}^2 - \|g(\sigma_h^+)\|_{\mathcal{L}_2^Q}^2 \right. \right. \\ &\quad \left. \left. + 2 \left(g(\mathcal{V}_{\varepsilon,k,h}^+); g(\sigma_h^+) \right)_{\mathcal{L}_2^Q} + 2 \left(g(\mathcal{V}_{\varepsilon,k,h}^-) - g(\mathcal{V}_{\varepsilon,k,h}^+); g(\mathcal{V}_{\varepsilon,k,h}^+) \right)_{\mathcal{L}_2^Q} \right\} dt \right] := IV_1 + \dots + IV_5, \end{aligned}$$

where $\mathcal{L}_2^Q := \mathcal{L}_2(\sqrt{Q}(K), \mathbb{L}^2)$. From equation (6.20), it holds that $IV_1 \rightarrow 0$. Furthermore, Lemma 6.4.4 yields $I_{1,1} + II_1 + IV_2 \leq 0$, the strong convergence of $\{\sigma_h^m\}_m$ together with result (6.25) grant both $IV_3 \rightarrow -\mathbb{E} \left[\int_0^T \eta(t) \|g(\sigma(t))\|_{\mathcal{L}_2^Q}^2 dt \right]$ and $IV_4 \rightarrow \mathbb{E} \left[2 \int_0^T \eta(t) (G_0(t), g(\sigma(t)))_{\mathcal{L}_2^Q} dt \right]$. Finally, $IV_5 \rightarrow 0$ by virtue of convergences (6.20) and (6.25). Putting it all together, equation (6.29) becomes

$$\begin{aligned} \lim_{\varepsilon, k, h \rightarrow 0} \mathbb{E} \left[\eta^+(T) \|\mathcal{V}_{\varepsilon,k,h}^+(T)\|_{\mathbb{L}^2}^2 - \|\mathcal{V}_{\varepsilon,k,h}^-(0)\|_{\mathbb{L}^2}^2 \right] &\leq \mathbb{E} \left[\int_0^T \eta'(t) \left\{ 2(v, \sigma) - \|\sigma\|_{\mathbb{L}^2}^2 \right\} dt \right] \\ &- 2\mathbb{E} \left[\int_0^T \eta(t) \left\{ \langle \mathcal{R}_0 - \mathcal{R}(\sigma), \sigma \rangle + \langle \mathcal{R}(\sigma) - f(t), v \rangle + \frac{1}{2} \|g(\sigma)\|_{\mathcal{L}_2^Q}^2 - (G_0, g(\sigma))_{\mathcal{L}_2^Q} \right\} dt \right], \end{aligned} \quad (6.30)$$

where $\eta(t) = \exp\left(-\frac{27}{\nu^3} \int_0^t \|\sigma(s)\|_{\mathbb{L}^4}^4 ds\right)$. Taking into account that $\mathbb{E} [\eta(T) \|v(T)\|_{\mathbb{L}^2}^2 - \|v_0\|_{\mathbb{L}^2}^2]$ is smaller than the left-hand side of equation (6.30) (thanks to result (6.21)) and applying Itô's formula to the process $(t, v) \mapsto \eta(t) \|v\|_{\mathbb{L}^2}^2$ (recall that v satisfies equation (6.27)) lead to

$$\begin{aligned} &\mathbb{E} \left[\int_0^T \eta'(t) \|v(t) - \sigma(t)\|_{\mathbb{L}^2}^2 dt \right] + \mathbb{E} \left[\int_0^T \eta(t) \|G_0(t) - g(\sigma(t))\|_{\mathcal{L}_2(\sqrt{Q}(K), \mathbb{L}^2)}^2 dt \right] \\ &\leq 2\mathbb{E} \left[\int_0^T \eta(t) \langle \mathcal{R}(\sigma(t)) - \mathcal{R}_0(t), \sigma(t) - v(t) \rangle dt \right], \quad \forall \sigma \in C([0, T]; \mathbb{V}). \end{aligned} \quad (6.31)$$

Arguing by density, it can be shown that inequality (6.31) holds for all $\sigma \in L^4(\Omega; L^\infty(0, T; \mathbb{H})) \cap L^2(\Omega; L^2(0, T; \mathbb{V}))$. Therefore, setting $\sigma = v$ yields $G_0 = g(v)$ \mathbb{P} -a.s. and a.e. in $[0, T] \times D$. With that said, the second term on the left-hand side of equation (6.31) cancels out. To identify \mathcal{R}_0 , it suffices to consider $\sigma = v + \mu u$ for $\mu > 0$ and $u \in L^4(\Omega; L^\infty(0, T; \mathbb{H})) \cap L^2(\Omega; L^2(0, T; \mathbb{V}))$. Subsequently,

$$\mu \mathbb{E} \left[\int_0^T \eta'(t) \|u(t)\|_{\mathbb{L}^2}^2 dt \right] \leq 2 \mathbb{E} \left[\int_0^T \eta(t) \langle \mathcal{R}(v(t) + \mu u(t)) - \mathcal{R}_0(t), u(t) \rangle dt \right].$$

Letting $\mu \rightarrow 0$ and taking into consideration the hemicontinuity of the operator \mathcal{R} , one infers that $\mathbb{E} \left[\int_0^T \eta(t) \langle \mathcal{R}(v(t)) - \mathcal{R}_0(t), u(t) \rangle dt \right] \geq 0$, for all $u \in L^4(\Omega; L^\infty(0, T; \mathbb{H})) \cap L^2(\Omega; L^2(0, T; \mathbb{V}))$. Consequently, $\mathcal{R}_0 = \mathcal{R}(v)$ in $L^2(\Omega; L^2(0, T; \mathbb{H}^{-1}))$.

Step 3: Verification of v as NSE solution

The obtained function v is henceforth a solution to equations (6.1) in the sense of Definition 6.2.2. Indeed, the identifications in Step 2 turn equation (6.27) into

$$\begin{aligned} & (v(t), \varphi) + \nu \int_0^t (\nabla v(s), \nabla \varphi) ds + \int_0^t \langle \hat{B}(v(s), v(s)), \varphi \rangle ds \\ &= (v_0, \varphi) + \int_0^t \langle f(s), \varphi \rangle ds + \left(\int_0^t g(v(s)) dW(s), \varphi \right), \quad \forall \varphi \in \mathbb{V}. \end{aligned}$$

By definition, $\hat{B}(v, v) = ([v \cdot \nabla] + \frac{1}{2} \operatorname{div}(v))v = [v \cdot \nabla]v$, thanks to Step 2, where the null divergence of v was illustrated. Finally, $v \in L^2(\Omega; C([0, T]; \mathbb{L}^2))$ can be easily proven via equation (6.27) by using the standard approach in [95].

Step 4: Convergence of the whole sequence

Convergence results that were discovered within Step 1 are all up to a subsequence. However, due to the uniqueness of v (see [91, Proposition 3.2]), it follows that the whole sequence $\{\mathcal{V}_{\varepsilon, k, h}^+\}_{\varepsilon, k, h}$ is convergent toward v .

6.4.3 . A linear version of Algorithm 6.1

In terms of simulations, a less time-consuming numerical scheme can be embodied through a linear Algorithm. This can be made up using a linearization of the trilinear term in Algorithm 6.1 as follows:

Algorithm 6.2 Starting from an initial datum $(v_h^0, p_h^0) \in \mathbb{H}_h \times L_h$, if $(V_\varepsilon^{m-1}, \Pi_\varepsilon^{m-1}) \in \mathbb{H}_h \times L_h$ is known for some $m \in \{1, \dots, M\}$, find $(V_\varepsilon^m, \Pi_\varepsilon^m) \in \mathbb{H}_h \times L_h$ that satisfies \mathbb{P} -a.s. the following:

$$\begin{cases} \left(V_\varepsilon^m - V_\varepsilon^{m-1}, \varphi_h \right) + k\nu (\nabla V_\varepsilon^m, \nabla \varphi) + k\hat{b}(V_\varepsilon^{m-1}, V_\varepsilon^m, \varphi_h) - (\Pi_\varepsilon^m, \operatorname{div}(\varphi_h)) \\ \quad = k \langle f^m, \varphi_h \rangle + \left(g(V_\varepsilon^{m-1}) \Delta_m W, \varphi_h \right), \quad \forall \varphi \in \mathbb{H}_h, \\ \frac{\varepsilon}{k} (\Pi_\varepsilon^m - \Pi_\varepsilon^{m-1}, q_h) + (\operatorname{div}(V_\varepsilon^m), q_h) = 0, \quad \forall q_h \in L_h, \end{cases}$$

where $f^m, \Delta_m W$ are defined in Algorithm 6.1 and $(V_\varepsilon^0, \Pi_\varepsilon^0) := (v_h^0, p_h^0)$.

Observe that $\hat{b}(V_\varepsilon^{m-1}, V_\varepsilon^m, V_\varepsilon^m) = 0$, thanks to Proposition 6.2.1-(ii). Thus, iterates $\{(V_\varepsilon^m, \Pi_\varepsilon^m)\}_{m=1}^M$ of Algorithm 6.2 satisfy Lemmas 6.4.1, 6.4.3 and they fulfill better uniqueness properties than those of Algorithm 6.1, as demonstrated in Lemma 6.4.5. However, due to the infamous properties of \hat{b} , the initial datum v_h^0 should undergo a new assumption which consists of a uniform bound in h of $\|\nabla v_h^0\|_{\mathbb{L}^2}$, as explained beneath the proof of Lemma 6.4.5.

Lemma 6.4.5 Iterates $\{(V_\varepsilon^m, \Pi_\varepsilon^m)\}_{m=1}^M$ of Algorithm 6.2 are unique \mathbb{P} -a.s. in Ω and a.e. in $[0, T] \times D$.

Proof: Let $\{(V_\varepsilon^m, \Pi_\varepsilon^m)\}_{m=1}^M$ and $\{(U_\varepsilon^m, P_\varepsilon^m)\}_{m=1}^M$ be two solutions to Algorithm 6.2 such that the initial data $(V_\varepsilon^0, \Pi_\varepsilon^0) = (U_\varepsilon^0, P_\varepsilon^0) = (v_h^0, p_h^0)$. Denote $Z_\varepsilon^m := V_\varepsilon^m - U_\varepsilon^m$ and $Q_\varepsilon^m := \Pi_\varepsilon^m - P_\varepsilon^m$, for all $m \in \{0, 1, \dots, M\}$. The following equation is \mathbb{P} -a.s. satisfied by $\{(Z_\varepsilon^m, Q_\varepsilon^m)\}_{m=1}^M$:

$$\begin{cases} \left(Z_\varepsilon^m - Z_\varepsilon^{m-1}, \varphi_h \right) + k\nu \left(\nabla Z_\varepsilon^m, \nabla \varphi_h \right) + k \left\langle \hat{B}(V_\varepsilon^{m-1}, V_\varepsilon^m) - \hat{B}(U_\varepsilon^{m-1}, U_\varepsilon^m), \varphi_h \right\rangle \\ - k \left(Q_\varepsilon^m, \operatorname{div}(\varphi_h) \right) = \left([g(V_\varepsilon^{m-1}) - g(U_\varepsilon^{m-1})] \Delta_m W, \varphi_h \right), \quad \forall \varphi \in \mathbb{H}_h, \\ \frac{\varepsilon}{k} \left(Q_\varepsilon^m - Q_\varepsilon^{m-1}, q_h \right) + \left(\operatorname{div}(Z_\varepsilon^m), q_h \right) = 0, \quad \forall q_h \in L_h. \end{cases} \quad (6.32)$$

For $m = 1$, it follows that $g(V_\varepsilon^0) - g(U_\varepsilon^0) = 0$ and $\hat{B}(V_\varepsilon^0, V_\varepsilon^1) - \hat{B}(U_\varepsilon^0, U_\varepsilon^1) = \hat{B}(V_\varepsilon^0, Z_\varepsilon^1)$. Hence, setting $(\varphi_h, q_h) = (Z_\varepsilon^1, Q_\varepsilon^1)$ in equations (6.32) and using Proposition 6.2.1-(ii) yield $\|Z_\varepsilon^1\|_{\mathbb{L}^2}^2 + \varepsilon \|Q_\varepsilon^1\|_{L^2}^2 + k\nu \|\nabla Z_\varepsilon^1\|_{\mathbb{L}^2}^2 = 0$ which implies $Z_\varepsilon^1 = Q_\varepsilon^1 = 0$ \mathbb{P} -a.s. and a.e. in $[0, T] \times D$. Arguing by induction completes the proof. \square

All steps that were conducted in Section 6.4.2 are applicable to Algorithm 6.2, except for Lemma 6.4.4 which does not suit the associated bilinear operator \hat{B} since its variables are not identical. Therefore, a slight adjustment should take place, and it consists of the following:

In Step 1 of Section 6.4.2, $\mathcal{R}(\mathcal{V}_{\varepsilon,k,h}^+)$ shall be substituted by a new operator

$$\mathcal{S}(\mathcal{V}_{\varepsilon,k,h}^-, \mathcal{V}_{\varepsilon,k,h}^+) := -\nu \Delta \mathcal{V}_{\varepsilon,k,h}^+ + \hat{B}(\mathcal{V}_{\varepsilon,k,h}^-, \mathcal{V}_{\varepsilon,k,h}^+)$$

and \mathcal{R}_0 by \mathcal{S}_0 which is defined by

$$\left[\int_0^T \langle \mathcal{S}_0(t), \varphi \rangle dt \right] = \lim_{\varepsilon, k, h \rightarrow 0} \mathbb{E} \left[\int_0^T \langle \mathcal{S}(\mathcal{V}_{\varepsilon,k,h}^-, \mathcal{V}_{\varepsilon,k,h}^+) + \nabla \Pi_{\varepsilon,k,h}^+, \Pi_h \varphi dt \rangle \right], \quad \forall \varphi \in \mathcal{V}.$$

Equation (6.29) remains unchanged because $\langle \mathcal{S}(\mathcal{V}_{\varepsilon,k,h}^-, \mathcal{V}_{\varepsilon,k,h}^+), \mathcal{V}_{\varepsilon,k,h}^+ \rangle = \langle \mathcal{R}(\mathcal{V}_{\varepsilon,k,h}^+), \mathcal{V}_{\varepsilon,k,h}^+ \rangle$, thanks to Proposition 6.2.1-(ii). However, when passing to the limit, term II_3 in Step 2 is not suitable for \mathcal{S}_0 , which is why it can be modified by employing Proposition 6.2.1-(ii) as follows:

$$\begin{aligned} II_3' &= -2\mathbb{E} \left[\int_0^T \eta^- \left\langle \mathcal{S}(\mathcal{V}_{\varepsilon,k,h}^-, \mathcal{V}_{\varepsilon,k,h}^+) + \nabla \Pi_{\varepsilon,k,h}^+ - \mathcal{S}(\sigma_h^-, \sigma_h^+), \sigma_h^+ \right\rangle dt \right] \\ &\quad - 2\mathbb{E} \left[\int_0^T \eta^- \left\langle \hat{B}(\mathcal{V}_{\varepsilon,k,h}^+ - \mathcal{V}_{\varepsilon,k,h}^-, \mathcal{V}_{\varepsilon,k,h}^+), \sigma_h^+ \right\rangle dt \right] := II_{3,1}' + II_{3,2}'. \end{aligned}$$

$II_{3,1}'$ goes to $-2\mathbb{E} \left[\int_0^T \eta(t) \langle \mathcal{S}_0(t) - \mathcal{S}(\sigma(t), \sigma(t)), \sigma(t) \rangle dt \right]$ as $\varepsilon, k, h \rightarrow 0$. Consequently, the whole proof of Section 6.4.2 becomes applicable to Algorithm 6.2, provided that $II_{3,2}'$ goes to 0. To this end, denote $\mathcal{Z}_{\varepsilon,k,h} = \mathcal{V}_{\varepsilon,k,h}^+ - \mathcal{V}_{\varepsilon,k,h}^-$ and utilize Proposition 6.2.1-(iii) to ensure:

$$\begin{aligned} \int_0^T \eta(t) \left\langle \hat{B}(\mathcal{Z}_{\varepsilon,k,h}, \mathcal{V}_{\varepsilon,k,h}^+), \sigma_h^+ \right\rangle dt &\lesssim \int_0^T \|\mathcal{Z}_{\varepsilon,k,h}\|_{\mathbb{L}^2}^{\frac{1}{2}} \|\nabla \mathcal{Z}_{\varepsilon,k,h}\|_{\mathbb{L}^2}^{\frac{1}{2}} \|\nabla \mathcal{V}_{\varepsilon,k,h}^+\|_{\mathbb{L}^2} \|\nabla \sigma_h^+\|_{\mathbb{L}^2} dt \\ &\lesssim k^{\frac{1}{4}} \left(\sum_{m=1}^M \|V_\varepsilon^m - V_\varepsilon^{m-1}\|_{\mathbb{L}^2}^2 \right)^{\frac{1}{4}} \left(k \sum_{m=1}^M \|\nabla(V_\varepsilon^m - V_\varepsilon^{m-1})\|_{\mathbb{L}^2}^2 \right)^{\frac{1}{4}} \left(k \sum_{m=1}^M \|\nabla V_\varepsilon^m\|_{\mathbb{L}^2}^2 \right)^{\frac{1}{2}}, \end{aligned}$$

thanks to the Hölder inequality and the high regularity of σ . Therewith,

$$II_{3,2}' \lesssim k^{\frac{1}{4}} \mathbb{E} \left[\sum_{m=1}^M \|V_\varepsilon^m - V_\varepsilon^{m-1}\|_{\mathbb{L}^2}^2 \right]^{\frac{1}{4}} \mathbb{E} \left[k \sum_{m=1}^M \|\nabla(V_\varepsilon^m - V_\varepsilon^{m-1})\|_{\mathbb{L}^2}^2 \right]^{\frac{1}{4}} \mathbb{E} \left[k \sum_{m=1}^M \|\nabla V_\varepsilon^m\|_{\mathbb{L}^2}^2 \right]^{\frac{1}{2}}.$$

The first and third expectations are bounded by virtue of Lemma 6.4.3-(i). Additionally, the second expectation, after undergoing a triangle inequality, can be controlled in a similar way provided that $\|\nabla v_h^0\|_{\mathbb{L}^2}$ is uniformly bounded in h . Consequently, $II_{3,2}' \lesssim k^{\frac{1}{4}} \rightarrow 0$.

One way of ensuring uniform boundedness in h of $\|\nabla v_h^0\|_{\mathbb{L}^2}$ is through the Ritz (also known as elliptic) operator $\mathcal{R}_h: \mathbb{H}_0^1 \rightarrow \mathbb{H}_h$, which is stable in \mathbb{H}_0^1 (see for instance [109]). In other words, setting $v_h^0 = \mathcal{R}_h v_0$ gets the job done, as long as $v_0 \in \mathbb{H}_0^1$. Another way is to use the already defined projection Π_h which can be an alternative for \mathcal{R}_h . This is true since the triangulation \mathcal{T}_h is quasi-uniform (see [45, Theorem 4]). With being said, an additional theorem can be given.

Theorem 6.4.1 *For $T > 0$, let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P})$ be a filtered probability space, $D \subset \mathbb{R}^2$ be a polygonal domain, assumptions (S_1) - (S_3) be satisfied, and $v_0 \in L^2(\Omega; \mathbb{H}_0^1)$. Given a positive integer M , define the discretization step size $k := \frac{T}{M}$ such that $k \in (0, 1)$ and I_k forms a uniform partition of the time interval $[0, T]$. Let $\varepsilon \in (0, 1)$ be a given penalty parameter, and $h \in (0, 1)$ be the space discretization step size such that the triangulation \mathcal{T}_h is quasi-uniform. Define the finite element triple $(\mathbb{H}_h, L_h, (v_h^0, p_h^0))$ such that the initial datum (v_h^0, p_h^0) belongs to (\mathbb{H}_h, L_h) , and $v_h^0 \in \{\mathcal{R}_h v_0, \Pi_h v_0\}$. Then, the following results hold:*

- For a given $(k, h, \varepsilon) \in (0, 1)^3$, there is a discrete solution $\{(V_\varepsilon^m, \Pi_\varepsilon^m)\}_{m=1}^M$ to Algorithm 6.2 satisfying Lemmas 6.4.1, 6.4.3, and 6.4.5.
- For a family $\{k, h, \varepsilon\}$ of parameters fulfilling $k, h, \varepsilon \rightarrow 0$ instantly, such that $v_h^0 \rightarrow 0$ as $h \rightarrow 0$ in the space $L^2(\Omega; \mathbb{L}^2)$, the solution $\{(V_\varepsilon^m, \Pi_\varepsilon^m)\}_{m=1}^M$ of Algorithm 6.2 converges toward the unique strong solution of stochastic Navier-Stokes equations (6.1) in the sense of Definition 6.2.2.

6.4.4 . How to properly choose ε regarding saddle point-based schemes?

For simplicity's sake and knowing that the Stokes problem establishes an insight into the Navier-Stokes equations, the primary aim of this section will be to evaluate a Stokes version of Algorithm 6.1 against a non-penalty-based numerical scheme of the following stochastic Stokes problem:

$$\begin{cases} \partial_t u - \nu \Delta u + \nabla p = f + g(u) \dot{W}, & \text{in } (0, T) \times D, \\ \operatorname{div}(u) = 0, & \text{in } (0, T) \times D, \\ u(0, \cdot) = v_0, & \text{in } D. \end{cases} \quad (6.33)$$

in order to choose the parameter ε efficiently in terms of the time discretization step size. The finite element spaces \mathbb{H}_h and L_h will be maintained throughout this section, and the discrete LBB (also known as inf-sup) condition

$$\sup_{\varphi_h \in \mathbb{H}_h} \frac{(\operatorname{div}(\varphi_h), q_h)}{\|\nabla \varphi_h\|_{\mathbb{L}^2}} \geq \beta \|q_h\|_{L^2}, \quad \forall q_h \in L_h, \quad (6.34)$$

will be required since numerical schemes of Stokes and Navier-Stokes problems which deal with saddle point approximations cannot converge without it. The constant $\beta > 0$ does not depend on the mesh size h . With that said, it is now meaningful to state the convective-free version of Algorithm 6.1:

$$\begin{cases} (U_\varepsilon^m - U_\varepsilon^{m-1}, \varphi_h) + k\nu (\nabla U_\varepsilon^m, \nabla \varphi_h) - k(p_\varepsilon^m, \operatorname{div}(\varphi_h)) \\ = k \langle f^m, \varphi_h \rangle + (g(U_\varepsilon^{m-1}) \Delta_m W, \varphi_h), \quad \forall \varphi_h \in \mathbb{H}_h, \\ \frac{\varepsilon}{k} (p_\varepsilon^m - p_\varepsilon^{m-1}, q_h) + (\operatorname{div}(U_\varepsilon^m), q_h) = 0, \quad \forall q_h \in L_h, \end{cases} \quad (6.35)$$

together with the following saddle point-based numerical scheme of the Stokes problem:

$$\begin{cases} (U^m - U^{m-1}, \varphi_h) + k\nu (\nabla U^m, \nabla \varphi_h) - k(p^m, \operatorname{div}(\varphi_h)) \\ = k \langle f^m, \varphi_h \rangle + (g(U^{m-1}) \Delta_m W, \varphi_h), \quad \forall \varphi_h \in \mathbb{H}_h, \\ (\operatorname{div}(U^m), q_h) = 0, \quad \forall q_h \in L_h. \end{cases} \quad (6.36)$$

Here, $\Delta_m W$ and f^m are identical to those of Algorithm 6.1, and the starting points $U_\varepsilon^0 = U^0 = \Pi_h v_0$. The convergence analysis of scheme (6.36) along with its convergence rate are provided in [56]. To

come up with effective and adequate conditions upon the parameter ε , it suffices to investigate the quantity $\|U_\varepsilon^m - U^m\|$. This is logical because if u denotes the solution of Stokes equations (6.33), then $\|U_\varepsilon^m - u(t_m)\| \leq \|U_\varepsilon^m - U^m\| + \|U^m - u(t_m)\|$ grants the rate at which scheme (6.35) might converge. To this purpose, subtracting equations (6.35) and (6.36) yields

$$\begin{aligned} & \left(U_\varepsilon^m - U^m - (U_\varepsilon^{m-1} - U^{m-1}) - [g(U_\varepsilon^{m-1}) - g(U^{m-1})] \Delta_m W, \varphi_h \right) + k(p_\varepsilon^m - p_\varepsilon^m, \operatorname{div}(\varphi_h)) \\ & = k\nu (\nabla(U^m - U_\varepsilon^m), \nabla\varphi_h) \leq k\nu \|\nabla(U^m - U_\varepsilon^m)\|_{\mathbb{L}^2} \|\nabla\varphi_h\|_{\mathbb{L}^2}, \quad \forall \varphi_h \in \mathbb{H}_h \setminus \{0\}. \end{aligned} \quad (6.37)$$

Dividing by $\|\nabla\varphi_h\|_{\mathbb{L}^2}$, taking the supremum over $\varphi_h \in \mathbb{H}_h \setminus \{0\}$, and employing the LBB condition (6.34) imply

$$\|p_\varepsilon^m - p_\varepsilon^m\|_{L^2} \leq \frac{\nu}{\beta} \|\nabla(U^m - U_\varepsilon^m)\|_{\mathbb{L}^2}, \quad \forall m \in \{1, \dots, M\}. \quad (6.38)$$

Estimate (6.38) is true because $\omega \mapsto \sup_{\varphi_h \in \mathbb{H}_h} \frac{(U_\varepsilon^m - U^m - (U_\varepsilon^{m-1} - U^{m-1}) - [g(U_\varepsilon^{m-1}) - g(U^{m-1})] \Delta_m W, \varphi_h)}{\|\nabla\varphi_h\|_{\mathbb{L}^2}}$ is non-negative which results from the fact that \mathbb{H}_h is a vector space. In other words, this supremum can be roughly seen as the \mathbb{H}^{-1} -norm of $U_\varepsilon^m - U^m - (U_\varepsilon^{m-1} - U^{m-1}) - [g(U_\varepsilon^{m-1}) - g(U^{m-1})] \Delta_m W$. On the other hand, setting $\varphi_h = U_\varepsilon^m - U^m$ in equation (6.37), using identity $2(a - b, a) = \|a\|_{\mathbb{L}^2}^2 - \|b\|_{\mathbb{L}^2}^2 + \|a - b\|_{\mathbb{L}^2}^2$, the Cauchy-Schwarz and Young inequalities return

$$\begin{aligned} & \frac{1}{2} \|U_\varepsilon^m - U^m\|_{\mathbb{L}^2}^2 - \frac{1}{2} \|U_\varepsilon^{m-1} - U^{m-1}\|_{\mathbb{L}^2}^2 + k\nu \|\nabla(U_\varepsilon^m - U^m)\|_{\mathbb{L}^2}^2 \leq k(p_\varepsilon^m - p_\varepsilon^m, \operatorname{div}(U_\varepsilon^m)) \\ & + \left([g(U_\varepsilon^{m-1}) - g(U^{m-1})] \Delta_m W, U_\varepsilon^{m-1} - U^{m-1} \right) + \frac{1}{2} \left\| [g(U_\varepsilon^{m-1}) - g(U^{m-1})] \Delta_m W \right\|_{\mathbb{L}^2}^2, \end{aligned} \quad (6.39)$$

where $(p_\varepsilon^m - p_\varepsilon^m, \operatorname{div}(U^m)) = 0$, thanks to scheme (6.36). Summing the above equation over m from 1 to an arbitrary $\ell \in \{1, \dots, M\}$, taking its mathematical expectation, employing Itô's isometry to the last term on its right-hand side together with assumption (S_3) and making use of the identity $U_\varepsilon^0 = U^0$ yield

$$\begin{aligned} & \mathbb{E} \left[\frac{1}{2} \|U_\varepsilon^\ell - U^\ell\|_{\mathbb{L}^2}^2 + k\nu \sum_{m=1}^{\ell} \|\nabla(U_\varepsilon^m - U^m)\|_{\mathbb{L}^2}^2 \right] \leq \mathbb{E} \left[k \sum_{m=1}^{\ell} (p_\varepsilon^m - p_\varepsilon^m, \operatorname{div}(U_\varepsilon^m)) \right] \\ & + \frac{L_g^2}{2} \mathbb{E} \left[k \sum_{m=1}^{\ell} \|U_\varepsilon^{m-1} - U^{m-1}\|_{\mathbb{L}^2}^2 \right], \end{aligned} \quad (6.40)$$

where the mathematical expectation of the penultimate term in equation (6.39) vanishes due to assumption (S_3) and the measurability of $\{U_\varepsilon^m\}_{m=1}^M$ and $\{U^m\}_{m=1}^M$. Attention will now turn toward the first term on the right-hand side of equation (6.40) which will eventually hand the upper-bound in terms of ε . Using equations (6.35), one obtains

$$\begin{aligned} J & := \mathbb{E} \left[k \sum_{m=1}^{\ell} (p_\varepsilon^m - p_\varepsilon^m, \operatorname{div}(U_\varepsilon^m)) \right] = -\mathbb{E} \left[\varepsilon \sum_{m=1}^{\ell} (p_\varepsilon^m - p_\varepsilon^{m-1}, p_\varepsilon^m - p_\varepsilon^m) \right] \\ & \leq \frac{\varepsilon\nu}{\beta} \mathbb{E} \left[\sum_{m=1}^{\ell} \|p_\varepsilon^m - p_\varepsilon^{m-1}\|_{L^2} \|\nabla(U_\varepsilon^m - U^m)\|_{\mathbb{L}^2} \right] \\ & \leq \frac{\sqrt{\varepsilon\nu}}{\beta} \mathbb{E} \left[\sum_{m=1}^M \varepsilon \|p_\varepsilon^m - p_\varepsilon^{m-1}\|_{L^2}^2 \right]^{1/2} \mathbb{E} \left[\sum_{m=1}^{\ell} \|\nabla(U_\varepsilon^m - U^m)\|_{\mathbb{L}^2}^2 \right]^{1/2} \\ & \leq \frac{\sqrt{\varepsilon\nu}}{4\beta^2} \mathbb{E} \left[\varepsilon \sum_{m=1}^M \|p_\varepsilon^m - p_\varepsilon^{m-1}\|_{L^2}^2 \right] + \sqrt{\varepsilon\nu} \mathbb{E} \left[\sum_{m=1}^{\ell} \|\nabla(U_\varepsilon^m - U^m)\|_{\mathbb{L}^2}^2 \right] \\ & \leq \frac{\sqrt{\varepsilon\nu}}{4\beta^2} C_1 + \sqrt{\varepsilon\nu} \mathbb{E} \left[\sum_{m=1}^{\ell} \|\nabla(U_\varepsilon^m - U^m)\|_{\mathbb{L}^2}^2 \right], \end{aligned} \quad (6.41)$$

thanks to the Cauchy-Schwarz and Young inequalities, estimate (6.38), and Lemma 6.4.3-(ii). In order to handle the last term on the right-hand side of equation (6.41), the penalty parameter ε shall undergo an assumption; that is, $\sqrt{\varepsilon} \leq k$. This way, it becomes absorbable in the left-hand side of equation (6.40). Finally, plug the result of estimate (6.41) in inequality (6.40) and make use of the discrete Grönwall inequality to achieve

$$\frac{1}{2} \max_{1 \leq m \leq M} \mathbb{E} \left[\|U_\varepsilon^m - U^m\|_{\mathbb{L}^2}^2 \right] + \mathbb{E} \left[(k - \sqrt{\varepsilon}) \nu \sum_{m=1}^M \|\nabla(U_\varepsilon^m - U^m)\|_{\mathbb{L}^2}^2 \right] \leq \tilde{C} \sqrt{\varepsilon}, \quad (6.42)$$

for some constant $\tilde{C} > 0$ depending only on β, C_1, ν, L_g and T .

Estimate (6.42) seems to have the best upper-bound amongst the other possible ways of estimation. Besides, some calculation techniques may be inconsistent with the assumption $\sqrt{\varepsilon} \leq k$. For instance, Proposition 6.4.1 could have been employed for the estimation of the term J , especially for the penultimate inequality in equation (6.41):

$$J \leq \frac{C_1 \nu}{2\beta^2} \frac{k}{\sqrt{\varepsilon}} + \sqrt{\varepsilon} \nu \mathbb{E} \left[\sum_{m=1}^M \|\nabla(U_\varepsilon^m - U^m)\|_{\mathbb{L}^2}^2 \right],$$

however, the second term on the right-hand side of the above inequality becomes non absorbable in the left-hand side of equation (6.40). Indeed, if the assumption $\sqrt{\varepsilon} \leq k$ is imposed, the obtained rate $\frac{k}{\sqrt{\varepsilon}}$ will no longer go to 0, which is senseless.

6.5 . Numerical experiments

The implementation within this section will be carried out through Algorithm 6.2 and a saddle point based-numerical scheme [24, Algorithm 3]:

Algorithm 6.3 *Let $M \in \mathbb{N}$ and $V^0 = v_h^0 \in \mathbb{H}_h$ be given. For every $m \in \{1, \dots, M\}$, find an $\mathbb{H}_h \times L_h$ -valued (V^m, Π^m) such that*

$$\begin{cases} (V^m - V^{m-1}, \varphi_h) + k\nu (\nabla V^m, \nabla \varphi_h) + k\hat{b}(V^{m-1}, V^m, \varphi_h) - k(\Pi^m, \text{div}(\varphi_h)) \\ \quad = k\langle f^m, \varphi_h \rangle + (g(V^{m-1})\Delta_m W, \varphi_h), \quad \forall \varphi_h \in \mathbb{H}_h, \\ (\text{div}(V^m), q_h) = 0, \quad \forall q_h \in L_h, \end{cases}$$

which will play the reference role with respect to the values of the parameter ε . The domain's meshing is carried out through the open source finite element mesh generator Gmsh [63], the implementation of the aforementioned algorithms is executed by the open source finite element software FEniCS [87], and the visualization is ensured via Paraview [2]. The simulation's configuration down below is set as follows: $T = 1$, $\nu = 1$, $h = 0.16$, $\varepsilon = 0.02$, $k = 0.01$. For the sake of comparison, the space discretization will be conducted by the lower order Taylor-Hood (P_2/P_1) finite element for both algorithms 6.2 and 6.3. The initial data u_0 and p_0 are set to 0 which means that $v_h^0 = (0, 0)$ and $p_h^0 = 0$. The domain D is an L -shaped geometry whose figure and mesh are the following:

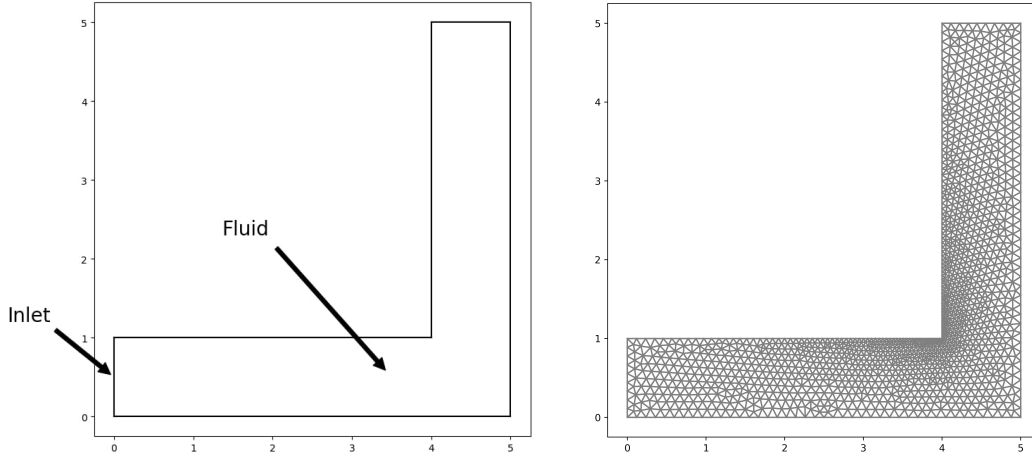


Figure 6.1 – The domain D and its mesh

The boundary condition

$$v(x, y) = \begin{cases} (1, 0) & \text{if } (x, y) \in \{0\} \times [0, 1], \\ (0, 0) & \text{elsewhere,} \end{cases}$$

is non-homogeneous, which is possible since a simple lifting technique can take the problem's boundary condition back to a homogeneous setting. The source term f takes on the value $(0, 0)$ and the diffusion coefficient plays the role of the identity operator $g = Id$ i.e. the considered noise is additive. The Wiener increment $\Delta_m W$ is approximated as follows:

let $J \in \mathbb{N}$ be non-zero, and W_1, W_2 be two independent $H_0^1(D)$ -valued Wiener processes such that $W = (W_1, W_2)$. Then,

$$\Delta_m W_\ell \approx \sqrt{k} \sum_{i,j=1}^J \sqrt{\lambda_{i,j}^\ell} \xi_{i,j}^{\ell,m} e_{i,j}, \quad \ell \in \{1, 2\}.$$

The parameter J takes on the value 5, the eigenvalues of the nuclear operator Q are expressed by $\lambda_{i,j}^\ell = \frac{1}{(i+j)^2}$ for all $i, j \in \mathbb{N}$, $\{(\xi_{i,j}^{1,m}, \xi_{i,j}^{2,m})\}_{i,j}^m$ is a family of independent identically distributed normal random variables, and

$$e_{i,j}(x, y) = \frac{2}{5} \sin(i\pi x/5) \sin(j\pi y/5) \quad \text{for all } i, j \in \mathbb{N} \text{ and } (x, y) \in D.$$

Although $\{e_{i,j}\}_{i,j}$ may not be the best choice for an L -shaped domain (because they represent the Laplace eigenfunctions on the square $(0, 5)^2$ with a Dirichlet boundary condition), they can be thought of herein as a restriction to D . The explicit formula of the Laplace eigenfunctions on an L -shaped domain is unknown as it is explained in [98]. With all that being said, it is now possible to exhibit the simulation results:

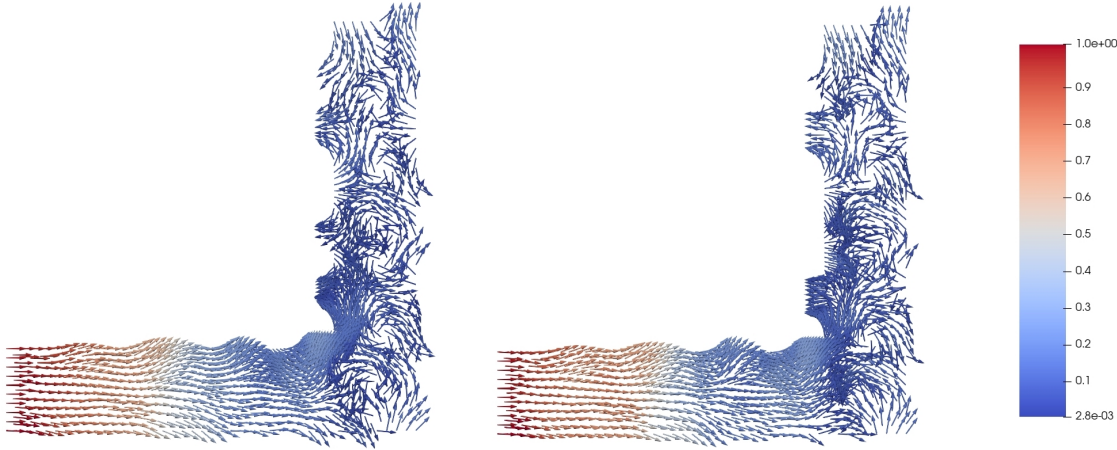


Figure 6.2 – One realization of V^M (left) and V_ϵ^M (right) at time $T = 1$ for $\epsilon = 0.021$

The main difference between the above two representations appears on the domain’s boundary ∂D , particularly on the vertical segment $\{4\} \times [1, 5]$. This is because the penalty parameter ϵ was not chosen small enough. In fact, as ϵ gets smaller, the difference between V^m and V_ϵ^m becomes indistinguishable. This fact is illustrated in an accurate way down below where the relationship between ϵ and the error $\mathbb{E} \left[\left\| V^M - V_\epsilon^M \right\|_{\mathbb{L}^2}^2 \right]$ is exposed:

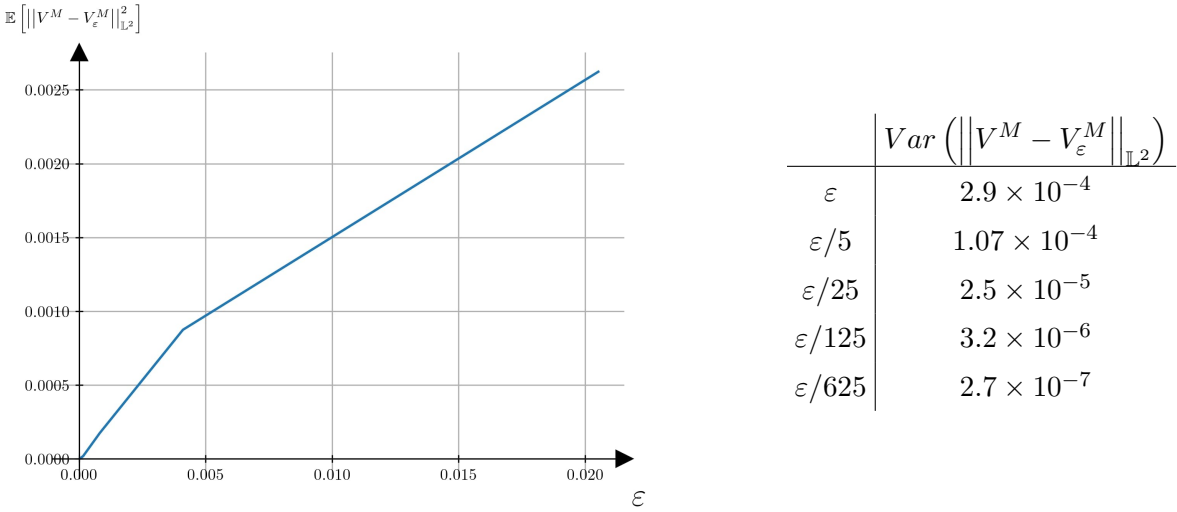


Figure 6.3 – Error and error-variance in terms of ϵ

The computed error in figure 6.3 uses a Monte-Carlo method with 1000 realizations. The obtained curve was expected; it emphasizes the fact that ϵ should be taken as small as possible in order to guarantee accurate outcomes.

The velocity simulation in Figure 6.2 exhibits a remarkable difference between the outcomes of Algorithms 6.2 and 6.3, especially on the boundary ∂D . This is because the penalty parameter ϵ is not chosen small enough. For instance, setting $\epsilon = 10^{-5}$ renders the following almost indistinguishable figures:

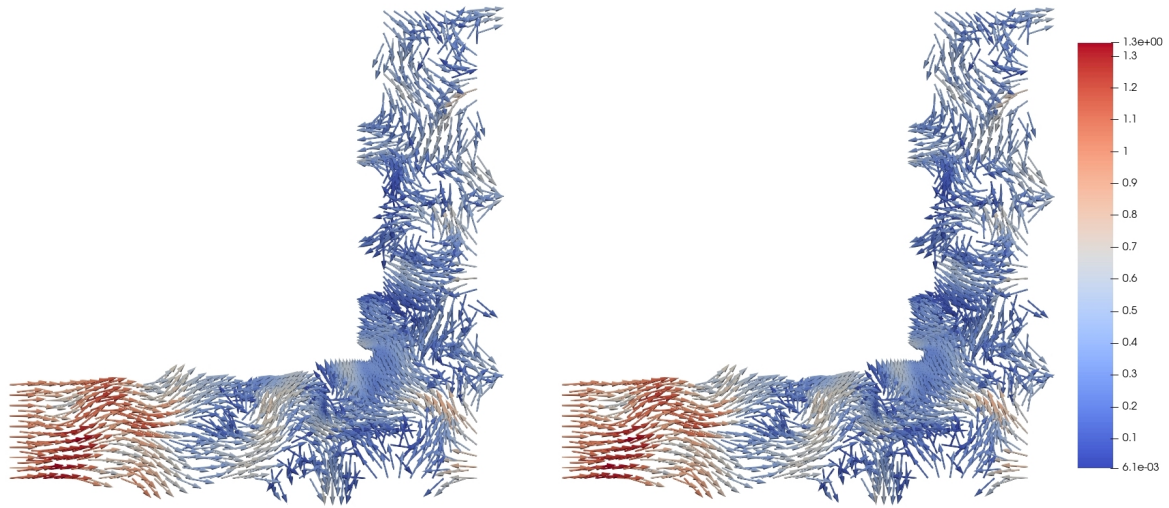


Figure 6.4 – One realization of V^M (left) and V_ε^M (right) at time $T = 1$ for $\varepsilon = 10^{-5}$

The above illustration maintains the same configurations of Figure 6.2, except for the Wiener process W , which is considered cylindrical (see Remark 6.2.1). Its increments are approximated through the formula

$$\Delta_m W_\ell \approx \sqrt{k} \sum_{i,j=1}^5 \xi_{i,j}^{\ell,m} e_{i,j}, \quad \ell \in \{1, 2\}.$$

6.6 . Conclusion

This chapter provides a new approach to simulating the two-dimensional stochastic incompressible Navier-Stokes equations. The introduced technique can be thought of as a compromise between (strongly) divergence-free finite element methods and saddle point problems where a discrete LBB condition is required to prove the pressure's existence. No relationships were assumed between ε and the discretization parameters k and h , although it could have been possible. If so, ε must be solely linked to k on account of the penalizing term $\varepsilon \partial_t p^\varepsilon$ which offers a supplementary time regularity, meaning that h should not intervene. Whereas, in the case where the additional term was, for example, $\varepsilon \Delta \partial_t p^\varepsilon$; that is, the mass conservation equation of system (6.2) had the following form:

$$\operatorname{div}(v^\varepsilon) + \varepsilon \Delta \partial_t p^\varepsilon = 0,$$

then, the parameter ε may be expressed in terms of both discretization step sizes k and h because the Laplace operator offers a supplementary space-regularity to the pressure field p^ε , which needs to disappear when passing to the limit in order to recover a solution to the stochastic incompressible Navier-Stokes equations. In Section 6.4.4, ε was also linked to the time discretization step size k under a particular numerical scheme that involves a discrete LBB condition. This given relationship can dramatically deviate if another numerical method is selected to be compared with the proposed algorithm herein. After all, the most accurate assumption involving both k and ε can come to light during the convergence rate study of Algorithm 6.1 (or 6.2).

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