



HAL
open science

Asymptotic distribution of a time-inhomogeneous kinetic model

Emeline Luirard

► **To cite this version:**

Emeline Luirard. Asymptotic distribution of a time-inhomogeneous kinetic model. Analysis of PDEs [math.AP]. Université de Rennes, 2022. English. NNT : 2022REN1S041 . tel-03903575

HAL Id: tel-03903575

<https://theses.hal.science/tel-03903575v1>

Submitted on 16 Dec 2022

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

THÈSE DE DOCTORAT DE

L'UNIVERSITÉ DE RENNES 1

ÉCOLE DOCTORALE N° 601
*Mathématiques et Sciences et Technologies
de l'Information et de la Communication*
Spécialité : *Mathématiques et leurs Interactions*

Par

Emeline LUIRARD

Loi limite de modèles cinétiques inhomogènes en temps

Thèse présentée et soutenue à Rennes, le 27 Juin 2022
Unité de recherche : IRMAR (UMR CNRS 6625)

Rapporteurs avant soutenance :

Patrick CATTIAUX Professeur, Université Paul Sabatier
Nicolas PRIVAULT Professeur, Nanyang Technological University

Composition du Jury :

Présidente :	Fabienne CASTELL	Professeure, Université Aix-Marseille
Examineurs :	Vlad BALLY	Professeur, Université Gustave Eiffel
	Nils BERGLUND	Professeur, Université d'Orléans
	Fabienne CASTELL	Professeure, Université Aix-Marseille
	Camille TARDIF	Maître de conférence, Sorbonne Université
Dir. de thèse :	Mihai GRADINARU	Professeur, Université de Rennes 1

“Je suis moi et les décisions d’une montagne pour imposante qu’elle soit n’y peuvent rien ! Je suis moi et j’ai l’intention de poursuivre mon chemin où qu’il me mène !”

— Ellana, *Le Pacte des Marchombres* de Pierre Bottero

REMERCIEMENTS

C'est une nouvelle page qui se tourne, et je suis fière d'être arrivée jusqu'au bout de cette aventure ! J'aimerais remercier toutes les personnes qui m'ont permis de réussir, de près ou de loin. Je vais faire de mon mieux pour n'oublier personne.

J'exprime tout d'abord ma reconnaissance à Mihai GRADINARU, mon directeur de thèse, pour son encadrement tout au long de mon doctorat et pour le temps qu'il m'a consacré. Merci pour ta gentillesse et ta grande disponibilité. Merci également de m'avoir fait découvrir le monde de la recherche.

Je remercie Patrick CATTIAUX et Nicolas PRIVAULT de m'avoir fait l'honneur de rapporter mon mémoire de thèse. Merci pour vos relectures précieuses et soignées. Merci également à Vlad BALLY, Nils BERGLUND, Fabienne CASTELL et Camille TARDIF d'avoir accepté d'être dans le jury.

Je tiens également à remercier tout particulièrement Jürgen ANGST pour nos échanges, ses nombreux conseils et son soutien. Merci à Francois BOLLEY et Karine BEAUCHARD d'avoir veillé de loin ces derniers mois.

Bien sûr je ne serai pas là où j'en suis aujourd'hui si je ne prenais pas plaisir à faire des mathématiques ! Alors merci également à tous.tes les (autres) professeur.es de mathématiques qui m'ont inspirée tout au long de ma scolarité. Pour n'en citer que quelques-uns, merci à Mr MOINE et ses carrés magiques, Mme MARTINENT, Mr GOUTTENOIRE, Judicaël COURANT pour m'avoir appris la rigueur mathématique, Bruno ARSAC (pour tes encouragements tout au long de mon parcours), Thibault DEHEUVELS, Jérémy LEBORGNE, Jean-Christophe BRETON, Matthieu ROMAGNY, Félix ULMER, Hélène GUERRIN, Benjamin BOUTIN, Dominique CERVEAU.

J'ai effectué ma thèse dans le cadre agréable de la tour des maths. J'en remercie tous les membres, chercheur.euses, enseignant.es, bibliothécaires, secrétaires. Merci tout particulièrement à Adeline, Bernard, Barbara, Rozenn, Nathalie, Hélène, Marie-Aude, Éric, Olivier, Dominique.

J'ai également eu la chance de pouvoir enseigner les mathématiques en licence de bi-

ologie. Ces moments d'oxygène ont été essentiels pendant ces trois années. J'ai toujours pris plaisir à enseigner et je remercie mes étudiant.es d'avoir entretenu cette bonne ambiance. Merci également à Jean-Christophe de m'avoir fait confiance pour préparer les étudiant.es de la prépa agreg à l'option Proba-Stats.

Bien sûr, je désire remercier les personnes avec qui j'ai partagé le bureau 334 pendant ces trois années, Markus, Thibault, Marie, Louis et Marc. Merci pour les bons moments partagés, les balades, les délicieux gâteaux, les discussions sur la recherche, l'enseignement ou encore la danse, les conseils en pâtisserie, et surtout vos encouragements et votre soutien. La crêperie du bureau 334 aura embaumé le couloir quelques mois ! Je pense aussi à nos décomptes en dessin avant les vacances et aux beaux dessins de géométrie qui fleurissent sur le tableau depuis quelques mois !

Je pense également à Josselin. Tes promenades parmi les étages t'ont souvent mené au bureau 334. Merci pour tous tes bons conseils et tes origamis qui décorent mon bureau. Merci Fabrice pour ta bonne humeur et tes conseils d'ancien. Je ne peux oublier le bureau 232, terre d'accueil. Merci à François, Pierre, Rémi et Thomas pour les discussions à refaire le monde, les bons moments. Merci Thomas pour tes relectures précieuses et toutes nos discussions. Faire des maths à plusieurs, c'est tellement mieux ! Merci également à Laura et Florian pour avoir partagé toutes les bonnes infos et pour vos conseils avisés.

Je tiens à remercier Alice d'être là et de m'avoir épaulée dans les moments difficiles. Je suis arrivée au bout ! Merci également pour ta bonne humeur. On est trop fortes !

Merci à tous les doctorant.es qui ont partagé ma vie au labo. Je remercie le groupe des doctorant.es se retrouvant à la pause du midi. Merci en particulier à Raoul, Titouan et Xabier pour nos sorties culturelles qui nous ont fait reprendre pied dans le monde réel, Antoine et Mégane pour vos bons conseils, Lisa, Lucien, Lucile, Milan, Mériadec, Emeric, Grégoire, Maxime. Certains d'entre vous sont devenus de très bons amis et j'espère que cela continuera après mon départ du labo.

J'aimerais également remercier tous les doctorant.es et chercheur.euses dont j'ai croisé la route, Armand, Baptiste, Barbara, Clément, Laetitia, Loïc, Maxime, Michel, Zoé, ... Une route parfois virtuelle au sein de la famille des doctorant.es du Discord. Merci Mathilde et Eva pour cette belle idée ! Toutes ces petites lanternes qui ont éclairé mon chemin et m'ont permis d'avancer. C'est un endroit où on se sent toujours écouté.es, soutenu.es et encouragé.es, j'y ai fait de belles rencontres. Merci aux *rusheur.euses*, à Erwan, Félix, Floriane, Galaad, Giom, Marie, Marina, Mendeleïev, Nico, Paul et tous les

autres, pour les reprises de chansons, les repas en Salon de Thé, les discussions, les sorties IRL et surtout merci de rendre cet espace bienveillant.

J'aimerais également remercier mes ami.es, beaucoup sont éloigné.es géographiquement mais c'est toujours un plaisir de se retrouver. Adrien, Béranger, Camille, Corinne (ma partenaire de poterie !), Cyril, Florette, Léo (quel temps précieux j'ai gagné grâce à Git et Zotéro !), Lucie, Marie, Maxence, William et Yan.

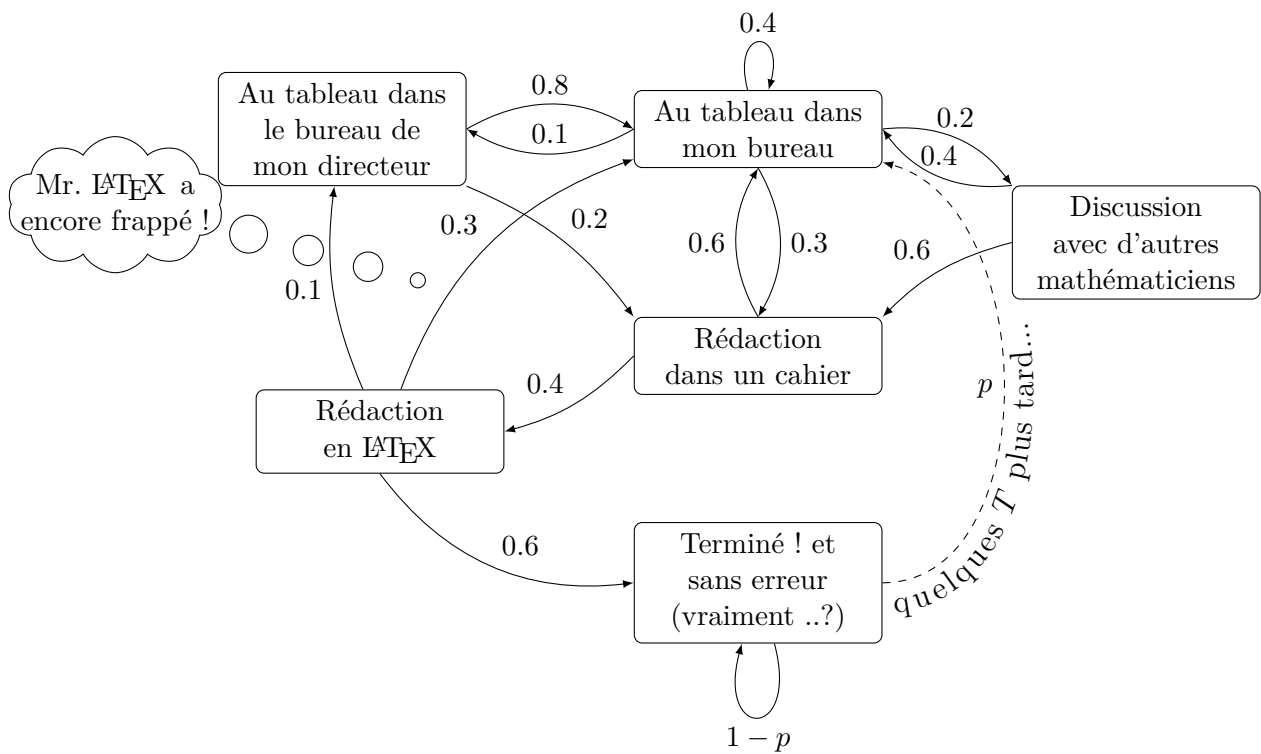
Puisque, comme dit ma grand-mère, pour bien travailler, il faut bien manger, j'aimerais remercier mon maraîcher préféré, Bruno, ainsi que sa femme, pour tous les bons légumes qu'ils font pousser. Mention spéciale pour toutes les variétés de courges que j'ai pu découvrir ! Merci pour toutes les bonnes idées de recettes et pour le moment de causerie du mardi matin.

Je ne peux pas clore ce chapitre sans remercier tous mes proches de croire en moi et de me soutenir, ma mère, ma grand-mère, mon père (ici et là haut), Jean-Paul (si on était partis avec moins d'avance, j'en serai peut être jamais là, qui sait !), Claude et mon neveu Roman.

Je pense tout spécialement à ma mère et à ma grand-mère. Un grand merci pour votre amour, et votre soutien qui m'a permis d'avancer dans la voie que j'ai choisie. Vous m'avez toujours encouragée. Merci maman d'avoir résolu ces carrés magiques et autres casse-têtes avec moi, c'est le début de ma passion pour les maths ! Merci à ma marraine, Nathalie, pour ses bons conseils "Une brique après l'autre".

Enfin je souhaite remercier tout spécialement Joris, mon roc. Merci d'être là et d'avancer dans la vie à mes côtés.

Soient $T \in \{\text{jours, mois, semaines}\}$ et p beaucoup trop proche de 1.



La chaîne de Markov de ma thèse en mathématiques...

TABLE DES MATIÈRES

Plan du manuscrit	1
Introduction	3
1 Processus cinétiques	3
1.1 Système cinétique et physique	3
1.2 Force extérieure aléatoire	4
1.3 Étude de systèmes stochastiques cinétiques	9
2 Dynamique stochastique	11
2.1 Précédents travaux	12
2.2 Quelques cas particuliers	16
2.3 Heuristique	19
2.4 Résultats et discussions	21
2.5 Quelques généralisations du processus directeur	31
3 Dynamique stochastique dans un potentiel confinant	34
3.1 Heuristique	34
3.2 Résultats et discussions	35
4 Perspectives de recherche	41
4.1 Généralisation à une force de frottement non linéaire	41
4.2 Force aléatoire de Feller	41
4.3 Généralisation de la force de frottement et du potentiel	42
4.4 Force de frottement aléatoire	42
I Stochastic dynamics	43
1 Behavior of a time-inhomogeneous Kolmogorov type diffusion	45
1.1 Introduction	46
1.2 Notations and main results	48
1.3 Existence and non-explosion of solution	50
1.4 Moment estimates of the velocity process	57

1.5	Proof of the asymptotic behavior of the solution	59
1.5.1	Asymptotic behavior in the super-critical regime	59
1.5.2	Asymptotic behavior in the critical regime	62
1.5.3	Asymptotic behavior in the sub-critical regime	67
1.6	An extension to multiplicative noise	71
1.6.1	Clock change	71
1.6.2	Diffusion coefficient with scaling property	72
1.A	Some technical results	75
2	Behavior of a time-inhomogeneous kinetic Lévy-driven model	79
2.1	Introduction	80
2.2	Notations and main results	82
2.3	Existence up to explosion	85
2.4	Moment estimates and non-explosion of the velocity process	86
2.5	Proof of the asymptotic behavior of the solution	94
2.5.1	Asymptotic behavior in the super-critical regime	95
2.5.2	Asymptotic behavior in the critical regime	96
2.5.3	Asymptotic behavior in the sub-critical regime	102
2.6	Extended results in the Lévy case	104
2.6.1	Large time behavior of the Lévy driving process	105
2.6.2	Proofs of Theorems 2.6.1 and 2.6.2	110
2.6.3	Lévy driving process with a Brownian component	116
2.A	Some technical results	118
II	Stochastic dynamics in a confining potential	125
	Introduction	127
3	Behavior of the non-autonomous system driven by a Brownian motion	133
3.1	Notations and main results	133
3.2	Existence up to explosion	135
3.3	Moment estimates and non-explosion	136
3.4	Asymptotic behavior of the solution	137
3.4.1	Convergence of the f.d.d. in the super-critical regime	138
3.4.2	Convergence of the f.d.d. in the critical and sub-critical regimes . .	140

3.5	Proof of Corollary 3.1.2	143
4	Behavior of the non-autonomous system driven by a Lévy process	145
4.1	Notations and main results	145
4.1.1	Settings on Lévy processes and functional spaces	145
4.1.2	Main results	146
4.2	Existence up to explosion	148
4.3	Moment estimates and non-explosion	149
4.4	Asymptotic behavior of the solution	153
4.4.1	Convergence of the f.d.d. in the super-critical regime	155
4.4.2	Convergence of the f.d.d. in the critical and sub-critical regime . . .	158
4.5	Proof of Corollary 4.1.2	163
A	Study of the deterministic underlying ODE	165
B	Some technical results	168
	Bibliography	173

PLAN DU MANUSCRIT

Dans cette thèse, nous étudions le comportement asymptotique de solutions de systèmes stochastiques cinétiques inhomogènes en temps de la forme

$$\begin{cases} dV_t = dL_t - \frac{F(V_t)}{t^\beta} dt - \nabla \mathcal{U}(X_t) dt, \\ dX_t = V_t dt. \end{cases}$$

Dans ce modèle, β est un réel positif et L est un processus de Lévy. La fonction F est une force de rappel, vérifiant des propriétés d'invariance d'échelle et \mathcal{U} représente un potentiel confinant. Ainsi, le processus V représente la vitesse d'une particule de position X , évoluant dans un potentiel \mathcal{U} , et soumise à une force de frottement F . Les interactions entre la particule et son environnement sont modélisées par le processus L . Notons qu'en temps long, la force attractive F est modérée par la présence du facteur $t^{-\beta}$.

L'enjeu est alors de comprendre comment interagissent les différentes forces, et de montrer que le processus $(X_{t/\varepsilon}, V_{t/\varepsilon})_{t>0}$, correctement renormalisé, admet une limite en loi explicite lorsque ε tend vers zéro.

Le manuscrit est découpé en deux parties, la première s'intéresse à l'étude du système en l'absence de potentiel confinant, *i.e.* $\mathcal{U} = 0$, la seconde au cas où \mathcal{U} est un potentiel quadratique. Chacune des deux parties est elle-même subdivisée en différents chapitres, suivant le type de processus directeur L considéré, qu'on suppose être soit un mouvement brownien, soit un processus de Lévy α -stable.

Dans l'introduction qui suit, nous décrivons brièvement les motivations du modèle et les résultats existant dans la littérature, en expliquant les différentes approches utilisées. Dans chacun des cadres considérés, nous présentons alors la synthèse des contributions de cette thèse. Dans le cas d'une particule non-confinée, les résultats sont détaillés dans la Section 2.4 (Théorèmes 2.5.1 et 2.5.2), tandis que ceux correspondant au cas d'une particule confinée dans un potentiel quadratique sont détaillés dans la Section 3.2 (Théorèmes 3.1.1 et 3.1.2). Nous précisons systématiquement l'approche et les méthodes employées.

Nous concluons l'introduction par quelques perspectives de recherche.

La suite du manuscrit suit le plan suivant.

Partie 1 : Dynamique stochastique

On se place, ici, dans le cadre d'un système cinétique où le processus vitesse V est défini indépendamment du processus position X , *i.e.* $\mathcal{U} = 0$.

Chapitre 1 : Cas d'un système dirigé par un mouvement brownien

Ce chapitre est adapté de

- M. Gradinaru and E. Luirard, *Asymptotic Behavior for a Time-Inhomogeneous Kolmogorov Type Diffusion*, 2021, arXiv: 2004.11576

Chapitre 2 : Cas d'un système dirigé par un processus de Lévy α -stable

- M. Gradinaru and E. Luirard, *Asymptotic Behavior for a Time-Inhomogeneous Stochastic Differential Equation Driven by an α -Stable Lévy Process*, 2021, arXiv: 2112.07287

Partie 2 : Dynamique stochastique dans un potentiel quadratique

Dans cette partie, les équations vérifiées par les processus vitesse et position sont couplées, via le potentiel quadratique \mathcal{U} .

- T. Cavallazzi and E. Luirard, *Asymptotic Behavior of a Time-Inhomogeneous Stochastic System in the Quadratic Potential*, in preparation.

Chapitre 3 : Cas d'un système dirigé par un mouvement brownien

Chapitre 4 : Cas d'un système dirigé par un processus de Lévy α -stable

INTRODUCTION

1 Processus cinétiques

1.1 Système cinétique et physique

Intéressons-nous à une particule, de masse 1, soumise à des forces extérieures. Supposons que la seule force qui s'exerce sur elle soit une force de frottement f . Alors, le principe fondamental de la dynamique s'écrit

$$\begin{cases} v'_t = -f(t, v_t), \\ x'_t = v_t. \end{cases} \quad (1)$$

Énonçons quelques propriétés sur la force de frottement f . Si la particule n'a pas de vitesse, il n'y a pas de frottement, ainsi en tout temps t , $f(t, 0) = 0$. Par ailleurs, notons sgn la fonction signe, avec la convention $\text{sgn}(0) = 0$. Puisqu'une force de frottement s'oppose au mouvement de la particule, on peut supposer, sans perte de généralité, qu'il existe une fonction $b : \mathbb{R}^+ \times \mathbb{R} \rightarrow \mathbb{R}^+$ telle que, pour tout $v \in \mathbb{R}$ et $t \in \mathbb{R}^+$, on ait $f(t, v) = \text{sgn}(v)b(v, t)$. Il découle de ces propriétés que 0 est un point stable et attractif.

Donnons, dans un premier temps, quelques exemples de forces de frottement classiques ne dépendant pas de la variable temporelle.

- En mécanique classique, à petite vitesse, la force de frottement est supposée linéaire, de la forme $v \mapsto -\rho v$ avec $\rho > 0$.
- Lorsque la vitesse est plus importante, on suppose, d'après la dynamique des fluides, que le frottement est quadratique.

De manière plus générale, on pourra choisir b de la forme $v \mapsto \rho |v|^\gamma$, où $\gamma \in \mathbb{R}^+$.

Par ailleurs, il existe également des forces de frottement qui évoluent avec le temps. Citons quelques exemples.

- Un skieur de fond subit une force de frottement proportionnelle au temps qui passe, au fur et à mesure que le fart de ses skis s'use.

- La pierre de curling est soumise à une force de frottement qui diminue avec les balayages des joueurs.
- La force de frottement d'une fusée qui décolle diminue avec son éloignement à la Terre. En effet, la pression atmosphérique diminue avec le changement d'altitude et, avec elle, la force de traînée.

Ainsi, il est intéressant de rajouter une dépendance en temps à la fonction b . Dans la suite, cette dépendance sera supposée polynomiale, de la forme $t^{-\beta}$, où $\beta \in \mathbb{R}$.

Pour résumer, on supposera dorénavant que la force de frottement est de la forme $(t, v) \mapsto -\rho \operatorname{sgn}(v) |v|^\gamma t^{-\beta}$, où $\rho > 0$, $\gamma \in \mathbb{R}^+$ et $\beta \in \mathbb{R}$ sont des paramètres du système. Par souci de simplicité, on notera F la fonction $v \mapsto \rho \operatorname{sgn}(v) |v|^\gamma$.

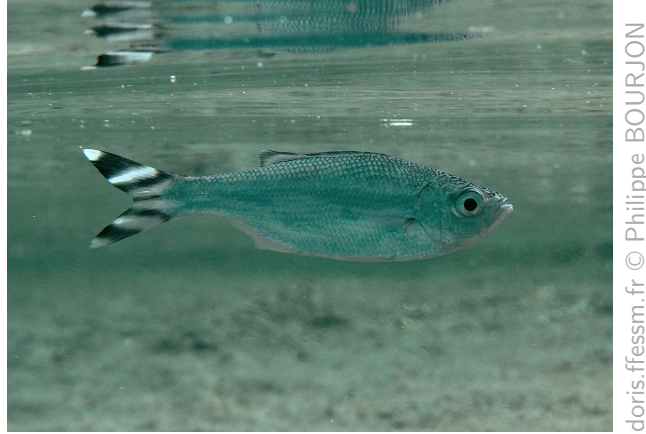
1.2 Force extérieure aléatoire

En 1908, dans [Lan08], le physicien français Paul Langevin s'intéresse à décrire le mouvement d'une particule en suspension dans un liquide. En prenant en compte la mécanique de Newton, il propose une modélisation qui semble être plus réaliste que le modèle brownien. Pendant un intervalle de temps dt , la particule subit un grand nombre de chocs microscopiques, ce qui maintient son agitation. Ces interactions entre la particule et son environnement sont modélisées par une force de perturbation aléatoire. Les chocs étant supposés indépendants, identiquement distribués, centrés et de carré intégrable, le théorème central limite (formalisé par le principe d'invariance de Donsker en 1951) suggère de modéliser cette force par un bruit blanc. Pour être plus précise, en prenant les notations différentielles et en notant B un mouvement brownien, le système (1) devient le système dégénéré suivant

$$\begin{cases} dV_t = dB_t - V_t dt, \\ dX_t = V_t dt. \end{cases} \quad (2)$$

C'est la première apparition d'une Équation Différentielle Stochastique (EDS). Les EDS plus générales sont apparues au milieu du XX^e siècle, par les approches différentes mais complémentaires de Kiyoshi Itô et Wolfgang Döblin. On pourra se référer à [Yor08], pour plus de détails à ce sujet.

Comme pour l'étude des Équations Différentielles Ordinaires (EDO), le caractère linéaire de ce système rend son étude plus simple : on connaît une solution explicite de (2), à

Figure 1 – Photo d'un jeune *Kuhlia mugil*.

savoir,

$$\begin{cases} V_t = e^{-t}v_0 + \int_0^t e^{s-t} dB_s, \\ X_t = x_0 + v_0(1 - e^{-t}) + \int_0^t (1 - e^{s-t}) dB_s = x_0 + v_0 + B_t - V_t. \end{cases} \quad (3)$$

Remarquons que le processus X est gaussien mais n'est pas un processus de Markov. Dans [CCM10], les auteurs introduisent le modèle *Persistent Turning Walker*, inspiré du mouvement d'une espèce de poisson vivant à la Réunion, le *Kuhlia mugil*.

La dynamique du poisson est décrite par le système cinétique suivant, faisant intervenir la position $x \in \mathbb{R}^2$, l'angle vitesse $\theta \in \mathbb{R}$ et la courbure $\kappa \in \mathbb{R}$,

$$\begin{cases} d\kappa_t = \sqrt{2}\alpha dB_t - \kappa_t dt, \\ d\theta_t = \kappa_t dt, \\ dx_t = e^{i\theta_t} dt. \end{cases} \quad (4)$$

Le terme aléatoire représente l'exploration de l'environnement par le poisson, tandis que la force de rappel vers l'origine traduit le fait que le poisson a tendance à reprendre une trajectoire rectiligne. Remarquons que le couple (κ, θ) est l'équivalent du couple (V, X) du système (2).

Le comportement asymptotique du système (4) a précédemment été étudié dans [DM08] à l'aide d'outils propres aux Équations aux Dérivées Partielles (EDP). En effet, ce système cinétique peut être traduit en une EDP : l'équation de *Fokker-Planck*. Plus précisément, la densité de probabilité $p(t, x, \theta, \kappa)$, si elle existe, vérifie l'équation suivante (aussi appelée équation de Kolmogorov)

$$\partial_t p + e^{i\theta} \cdot \nabla_x p + \kappa \partial_\theta p - \partial_\kappa(\kappa p) - \alpha^2 \partial_{\kappa\kappa}^2 p = 0, \quad t \geq 0, \quad x \in \mathbb{R}, \quad \theta \in \mathbb{R}, \quad \kappa \in \mathbb{R}.$$

Cette densité représente la probabilité de trouver le poisson à l’instant t dans un petit voisinage de (x, θ, κ) . Citons [Ber21, Section 1.3.4 p. 26-29] pour plus de détails sur l’équation de Fokker-Planck.

Le système d’EDS est étudié d’un point de vue stochastique dans [CCM10]. Les auteurs montrent par des outils d’analyse stochastique que le poisson atteint un comportement “stationnaire” proche de celui d’un mouvement brownien. Puisque ce dernier est récurrent en dimension 2, on en déduit que le poisson aura tendance à se déplacer dans tout l’espace, et à revenir en tout point une infinité de fois.

Comme nous l’avons vu précédemment, les forces de frottement ne sont pas nécessairement linéaires, ni homogènes en temps. Le système (2) pourrait donc prendre la forme plus générale suivante

$$\begin{cases} dV_t = dB_t - \rho \frac{\text{sgn}(V_t) |V_t|^\gamma}{t^\beta} dt, \\ dX_t = V_t dt. \end{cases} \quad (\text{SKE}_B)$$

L’équation de Fokker-Planck associée s’écrit alors, pour $t > 0$, $x \in \mathbb{R}$, $v \in \mathbb{R}$,

$$\partial_t p(t, x, v) + v \partial_x p(t, x, v) - \partial_v \left[\rho \frac{\text{sgn}(v) |v|^\gamma}{t^\beta} p(t, x, v) \right] - \frac{1}{2} \partial_{vv}^2 p(t, x, v) = 0.$$

Une trajectoire de la solution de (SKE_B) a été simulée et est représentée dans la Figure 2.

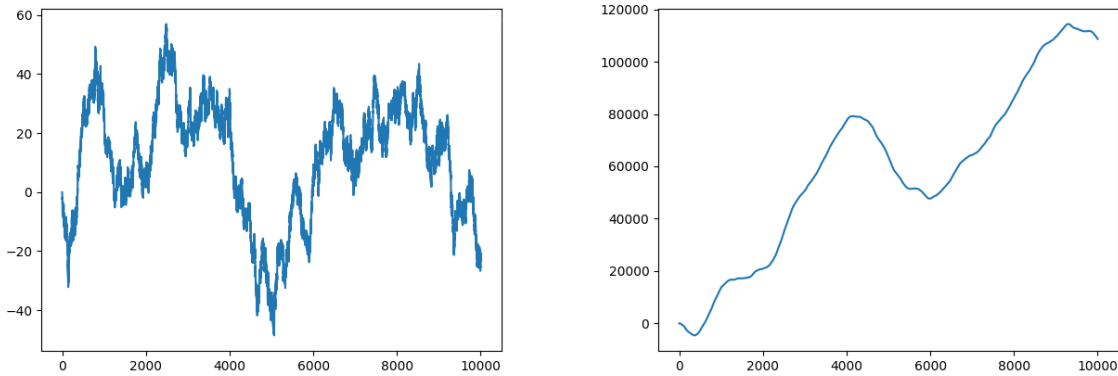


Figure 2 – Exemple de trajectoire des processus vitesse (à gauche) et position (à droite), dirigés par un mouvement brownien, pour $\rho = 1$, $\gamma = 1$ et $\beta = 2$.

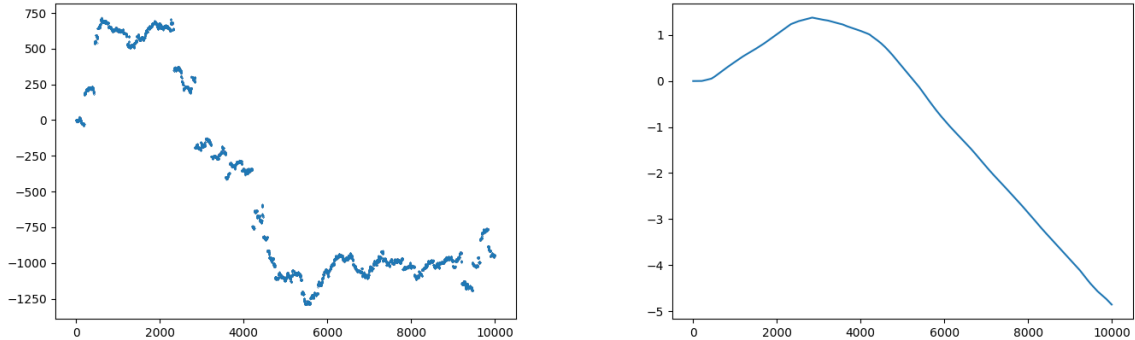


Figure 3 – Exemple de trajectoire des processus vitesse (à gauche) et position (à droite), dirigés par un processus de Lévy (1.3)-stable, pour $\rho = 1$, $\gamma = 1$ et $\beta = 2$.

Force aléatoire discontinue : les processus de Lévy

En outre, certains travaux s'intéressent à des forces de perturbation qui ne sont pas nécessairement supposées continues, mais qui ont des sauts de type Lévy. Citons par exemple [Dit99], où l'évolution du climat, étudiée à partir de la température de calottes glaciaires du Groenland, est modélisée par un processus de Langevin, dirigé notamment par un processus α -stable.

Ainsi, une généralisation possible de (SKE_B) est le système suivant, dirigé par un processus de Lévy L ,

$$\begin{cases} dV_t = dL_t - \rho \frac{\text{sgn}(V_t) |V_t|^\gamma}{t^\beta} dt, \\ dX_t = V_t dt. \end{cases} \quad (SKE_L)$$

Une trajectoire du processus solution est représentée par la Figure 3.

L'équivalent de l'équation de Fokker-Planck pour une force de perturbation de type Lévy est l'équation de Fokker-Planck fractionnaire, faisant intervenir des dérivées fractionnaires. Pour plus d'informations à ce sujet, on pourra se référer à [CG00].

Rappelons brièvement la définition et quelques propriétés des processus de Lévy.

Définition 1.1. Soit $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ un espace probabilisé filtré. On dit qu'un processus $(\mathcal{F}_t)_{t \geq 0}$ -adapté $(L_t)_{t \geq 0}$ est un *processus de Lévy* si

- $L_0 = 0$ p.s,
- $t \mapsto L_t$ est presque sûrement continue à droite avec des limites à gauche (càdlàg),
- pour tout $0 \leq s \leq t$, les accroissements $L_t - L_s$ sont indépendants de \mathcal{F}_s et station-

naires.

De plus, notons $\mathcal{S}(\alpha, \sigma, \beta, \mu)$ la loi stable, où $\alpha \in (0, 2]$ est le paramètre de stabilité. Sa fonction caractéristique est donnée, pour $\xi \in \mathbb{R}$, par

$$\psi(\xi) := \exp \left(i\mu\xi - \sigma|\xi|^\alpha \left(1 - i\beta \tan \left(\frac{\pi\alpha}{2} \right) \text{sgn}(\xi) \right) \right),$$

avec la convention $\beta \tan \left(\frac{\pi\alpha}{2} \right) = 0$, si $\alpha = 1$. Lorsque $L_t - L_s \sim \mathcal{S}(\alpha, (t-s)^{\frac{1}{\alpha}}, 0, 0)$, on dit que L est un *processus de Lévy α -stable symétrique*.

Remarquons que le mouvement brownien est un processus de Lévy 2-stable. C'est le seul processus stable symétrique dont les trajectoires sont continues. Puisque pour tout $c > 0$, on a l'invariance $(c^{\frac{1}{\alpha}} L_t)_{t \geq 0} \stackrel{\mathcal{L}}{=} (L_{ct})_{t \geq 0}$, les processus de Lévy stables sont aux processus càdlàg ce qu'est le mouvement brownien aux processus continus.

Par ailleurs, on connaît une écriture explicite pour un processus α -stable.

Proposition 1.2. *Soit L un processus de Lévy α -stable, avec $\alpha \in (0, 2)$. Sa mesure de Lévy est donnée par*

$$\nu(dz) = \frac{a_+ \mathbb{1}_{\{z > 0\}} + a_- \mathbb{1}_{\{z < 0\}}}{|z|^{1+\alpha}} dz, \quad \text{avec } a_+, a_- \geq 0 \text{ et } a_+ + a_- > 0.$$

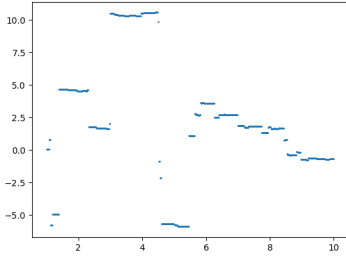
Elle vérifie

$$\int_{\mathbb{R}^*} (1 \wedge z^2) \nu(dz) < +\infty.$$

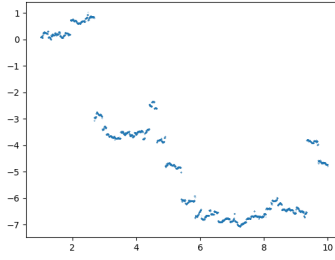
De plus, d'après la décomposition d'Itô-Lévy, il existe une mesure aléatoire de Poisson N et sa mesure compensée \tilde{N} telles que, pour tout $t \geq 0$,

$$L_t = \begin{cases} \int_0^t \int_{\mathbb{R}^*} z N(ds, dz) & \text{si } \alpha \in (0, 1), \\ \int_0^t \int_{\{0 < |z| < 1\}} z \tilde{N}(ds, dz) + \int_0^t \int_{\{|z| \geq 1\}} z N(ds, dz) & \text{si } \alpha = 1, \\ \int_0^t \int_{\mathbb{R}^*} z \tilde{N}(ds, dz) & \text{si } \alpha \in (1, 2). \end{cases}$$

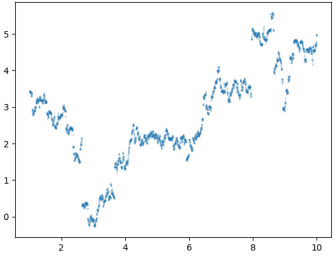
Plus α est petit, plus $|z|^{-1-\alpha}$ décroît pour $0 < |z| < 1$ et croît pour $|z| > 1$. Ainsi, d'après la forme de sa mesure de Lévy, un processus α -stable se déplace essentiellement par de grands sauts lorsque α est proche de 0, et par de petits sauts lorsque α est proche de 2 (voir Figure 4). On pourra se référer à [Bas03] pour plus de détails sur les EDS dirigées par des processus de Lévy.



(a) Processus (1/2)-stable symétrique



(b) Processus (1)-stable symétrique



(c) Processus (3/2)-stable symétrique

Figure 4 – Exemples de trajectoire de processus de Lévy stables.

1.3 Étude de systèmes stochastiques cinétiques

Face au système (SKE_L) , les questions naturelles qui se posent sont semblables à celles pour un système d'EDO.

- Existe-t-il une solution ? Est-elle unique ?
- La solution, si elle existe, explose-t-elle en temps fini ?
- Quel est le comportement en temps long de cette solution ? Peut-on trouver un taux de convergence r_ε tel que $r_\varepsilon X_{t/\varepsilon}$ “converge”, quand $\varepsilon \rightarrow 0$, vers une limite non nulle ?

Dans ce dernier point, la notion de convergence est à définir. Soit E un espace de fonctions à valeurs réelles, muni d'une topologie \mathcal{E} . Soient $(Z_t)_{t>0}$ et $(Z_t^\infty)_{t>0}$ deux processus de E . Définissons $(Z_t^{(\varepsilon)}) := (r_\varepsilon Z_{t/\varepsilon})$ le processus renormalisé avec un taux de convergence approprié. On peut s'intéresser à diverses notions de convergence. Citons entre autres, de la plus faible à la plus forte.

- La convergence unidimensionnelle : si Z_t converge en loi vers Z^∞ dans \mathbb{R} , lorsque $t \rightarrow +\infty$. On la notera

$$Z_t \xrightarrow[t \rightarrow +\infty]{} Z^\infty.$$

- La convergence des marginales de rang fini : si pour tout sous-ensemble fini S de $(0, +\infty)$, le vecteur $(Z_t^{(\varepsilon)})_{t \in S}$ converge en loi vers $(Z_t^\infty)_{t \in S}$ dans \mathbb{R}^S , lorsque $\varepsilon \rightarrow 0$. On la notera

$$(Z_t^{(\varepsilon)})_{t>0} \xrightarrow[\varepsilon \rightarrow 0]{\text{f.d.}} (Z_t^\infty)_{t>0}.$$

- La convergence du processus : si $(Z_t^{(\varepsilon)})_{t>0}$ converge en loi vers $(Z_t^\infty)_{t>0}$ dans (E, \mathcal{E}) ,

lorsque $\varepsilon \rightarrow 0$. On la notera

$$(Z_t^{(\varepsilon)})_{t>0} \xrightarrow{\varepsilon \rightarrow 0} (Z_t^\infty)_{t>0}.$$

Remarque 1.3. *Il est également possible de s'intéresser à des convergences presque sûres de type loi du logarithme itéré.*

Soit (V, X) la solution de (SKE_L) . Dans la suite, on appelle processus renormalisé le processus $(V^{(\varepsilon)}, X^{(\varepsilon)}) := (r_{\varepsilon,V}V_{t/\varepsilon}, r_{\varepsilon,X}X_{t/\varepsilon})_{t \geq \varepsilon t_0}$, pour des taux de convergence $r_{\varepsilon,V}$ et $r_{\varepsilon,X}$ bien choisis.

Comme expliqué dans [Off14],

Il semble vain de chercher à donner des critères généraux sur le comportement en temps long de tels processus, nous pouvons cependant nous intéresser à ceux dont le terme de diffusion et le potentiel possèdent certaines conditions d'invariance d'échelle.

C'est pourquoi on se concentre dans la suite au système cinétique de la forme (SKE_L) .

Remarque 1.4. *En reprenant l'approche par l'équation de Fokker-Planck, la convergence du processus $(r_{\varepsilon,X}X_{t/\varepsilon}, r_{\varepsilon,V}V_{t/\varepsilon})_{t \geq \varepsilon t_0}$ implique, sous condition d'existence d'une densité, la limite faible suivante, sur l'espace des mesures de probabilités,*

$$\lim_{\varepsilon \rightarrow 0} r_{\varepsilon,X}^{-1} r_{\varepsilon,V}^{-1} p_{t/\varepsilon}(r_{\varepsilon,X}^{-1} x, r_{\varepsilon,V}^{-1} v), \quad t > 0, \quad x \in \mathbb{R}, \quad v \in \mathbb{R}.$$

De plus, étant donné que la composante aléatoire n'est présente que sur la coordonnée vitesse, le modèle est dit *dégénéré*. L'étude du comportement asymptotique de la solution permet alors de comprendre comment cette force aléatoire se propage sur la coordonnée position.

Le processus de Kolmogorov. Intéressons-nous tout d'abord à l'exemple fondamental du processus de Kolmogorov

$$\begin{cases} dV_t = dB_t, \\ dX_t = V_t dt. \end{cases} \quad (5)$$

Ce processus de Markov a été introduit par Kolmogorov dans [Kol29]. Son générateur infinitésimal est l'opérateur différentiel d'ordre deux $\mathcal{L} := \frac{1}{2}\Delta + x\nabla$.

Le processus centré X est gaussien et la convergence de ses lois de dimension finie est

caractérisée par la convergence de son noyau de covariance

$$K(s, t) := \mathbb{E}[X_t X_s] = s^2 \left(\frac{t}{2} - \frac{s}{6} \right).$$

On remarque donc que la famille de processus $\left\{ \left(\varepsilon^{\frac{1}{2}} V_{t/\varepsilon}, \varepsilon^{\frac{3}{2}} X_{t/\varepsilon} \right)_{t \geq 0}, \varepsilon > 0 \right\}$ est stationnaire, de même loi que (V, X) .

Remarque 1.5. *Le système (SKE_B) peut être vu comme un processus de Kolmogorov perturbé par la force de frottement f .*

Le processus de Langevin. Regardons maintenant du côté du processus de Langevin, solution de (2). La solution est, elle aussi, explicite, donnée par (3).

Le processus vitesse V est ergodique. On remarque que la limite $\lim_{t \rightarrow +\infty} \mathbb{E}[V_t^2]$ existe, ce qui traduit le fait que l'énergie cinétique moyenne de la particule devient constante. D'autre part, le processus position X s'exprime à partir d'une intégrale de Wiener, c'est à dire une intégrale de la forme $\int_0^t h(s) dB_s$, où l'intégrande h est une fonction déterministe. C'est un processus gaussien et à l'aide d'un TCL pour les martingales locales ([JS03, Théorème VIII-3.11 p. 473]), on peut montrer que $(\sqrt{\varepsilon} X_{t/\varepsilon})_{t \geq 0}$ converge en loi vers un mouvement brownien quand $\varepsilon \rightarrow 0$. On dit alors que le comportement asymptotique est *diffusif*.

Remarquons que la composante position perd en régularité à la limite puisque le mouvement brownien est presque sûrement nulle part dérivable. Notons que cette asymptotique coïncide avec le modèle brownien d'une particule en suspension dans l'eau.

Ces cas particuliers mettent en évidence des comportements nouveaux à la limite : les taux de convergence diffèrent, les propriétés du processus limite également. Dans la suite, nous chercherons à déterminer les bons taux de convergence pour lesquels il y a convergence des processus.

2 Dynamique stochastique

Nous présentons, dans un premier temps, des travaux existant sur le comportement asymptotique du système (SKE_B) .

2.1 Précédents travaux

Convergence des marginales unidimensionnelles du processus vitesse

La convergence des marginales unidimensionnelles de la diffusion V , solution du système (SKE $_B$) dirigé par un mouvement brownien standard, a été étudiée dans [GO13]. Trois régimes asymptotiques sont mis en évidence en fonction de la position des paramètres γ et β par rapport à la droite critique $2\beta = \gamma + 1$.

L'inhomogénéité temporelle de la dérive apporte des difficultés. La stratégie est de se ramener à une diffusion “presque” homogène, à l'aide d'une transformation d'échelles, en tirant avantage de la forme du coefficient de dérive et de l'auto-similarité du mouvement brownien. Cela repose sur l'observation que si $\varphi : [0, t_1) \rightarrow [t_0, +\infty)$ est un \mathcal{C}^2 -difféomorphisme, alors

$$(W_t)_{t \geq 0} := \left(\int_0^t \frac{dB_{\varphi(s)}}{\sqrt{\varphi'(s)}} \right)_{t \geq 0}$$

reste un mouvement brownien standard.

Les auteurs introduisent donc le processus transformé suivant,

$$V_s^{(\varphi)} := \frac{V_{\varphi(s)}}{\sqrt{\varphi'(s)}}, \quad s \in [0, t_1),$$

qui est solution de l'EDS

$$dV_s^{(\varphi)} = dW_s - \rho \frac{\varphi'(s)^{\frac{\gamma+1}{2}}}{\varphi(s)^\beta} \operatorname{sgn}(V_s^{(\varphi)}) |V_s^{(\varphi)}|^\gamma ds - \frac{\varphi''(s)}{\varphi'(s)} \frac{V_s^{(\varphi)}}{2} ds, \quad V_0^{(\varphi)} = \frac{V_{\varphi(0)}}{\sqrt{\varphi'(0)}}.$$

La tactique étant de faire apparaître un terme homogène en temps dans la dérive, deux transformations d'échelles apparaissent naturellement :

- la *transformation exponentielle*, qui vérifie $\varphi'_e = \varphi_e$ et $\varphi_e(0) = t_0$,
- la *transformation puissance*, qui vérifie $\varphi'_\gamma = \varphi_\gamma^{\frac{2\beta}{\gamma+1}}$ et $\varphi_\gamma(0) = t_0$.

La transformation exponentielle a aussi été utilisée dans [Bre68] pour transformer un processus α -stable en un processus de Markov stationnaire. Dans [AW09], elle est utile pour se ramener à un processus satisfaisant les hypothèses du théorème de Motoo.

Lorsque $2\beta = \gamma + 1$, les deux transformations sont identiques et la diffusion $V^{(\varphi)}$ est homogène en temps. L'étude du processus se ramène alors à celle de sa fonction d'échelle

et de sa mesure vitesse. On pourra se reporter par exemple à [KS98, Chap. 5.5] et [Kal02, Chap. 23] pour plus de détails sur ces deux outils.

Afin de conclure de part et d'autre de la droite critique, le lemme asymptotique suivant est fondamental, couplé au théorème de comparaison des EDS [RY05, Théorème 3.7 p. 394].

Lemme 2.1 (Gradinaru, Offret). *Soient σ , σ_∞ , b et b_∞ des fonctions continues et B un mouvement brownien standard. Soient Z et H deux diffusions, uniques solutions faibles de*

$$dZ_s = \sigma(s, Z_s) dB_s + b(s, Z_s) ds \quad \text{et} \quad dH_s = \sigma_\infty(H_s) dB_s + b_\infty(H_s) ds.$$

Supposons que (Z, H) soit asymptotiquement homogène et ν -ergodique, au sens où

$$H_s \xrightarrow{s \rightarrow +\infty} \nu,$$

et, les limites suivantes soient satisfaites, uniformément sur tout compact,

$$\lim_{s \rightarrow +\infty} \sigma(s, z) = \sigma_\infty(z) \quad \text{et} \quad \lim_{s \rightarrow +\infty} b(s, z) = b_\infty(z).$$

Supposons de plus que Z soit borné en probabilité, c'est-à-dire que pour tout $\varepsilon > 0$, il existe $r > 0$ tel que

$$\sup_{s \geq 0} \mathbb{P}(|Z_s| \geq r) < \varepsilon.$$

Alors,

$$Z_s \xrightarrow{s \rightarrow +\infty} \nu.$$

Soient $\Lambda_{\rho, \gamma}$ et $\Pi_{\rho, \gamma}$ les mesures de probabilités définies par

$$\Lambda_{\rho, \gamma}(x) := c^{-1} e^{-\frac{x^2}{2}} e^{-U_{\rho, \gamma, 0}(x)} \quad \text{et} \quad \Pi_{\rho, \gamma}(x) := k^{-1} e^{-U_{\rho, \gamma, 0}(x)},$$

où c et k sont les constantes de normalisation et

$$U_{\rho, \gamma, \beta}(t, x) := \frac{2\rho}{\gamma + 1} |x|^{\gamma+1} t^{-\beta}, \quad t \geq t_0, \quad x \in \mathbb{R}.$$

Observons que $\Lambda_{\rho, \gamma}$ peut être vue comme un “mélange” de la loi gaussienne et de la loi $\Pi_{\rho, \gamma}$. Il en découle alors l'un des résultats principaux de [GO13].

Théorème 2.1.1 (Gradinaru, Offret). *Soit V la solution de (SKE_B) .*

- *Si $2\beta > \gamma + 1$, alors la force de frottement est asymptotiquement négligeable et V se comporte comme une loi normale*

$$\frac{V_t}{\sqrt{t}} \xrightarrow[t \rightarrow +\infty]{} \mathcal{N}(0, 1).$$

- *Si $2\beta = \gamma + 1$, alors V n'est plus asymptotiquement gaussienne*

$$\frac{V_t}{\sqrt{t}} \xrightarrow[t \rightarrow +\infty]{} \Lambda_{\rho, \gamma}.$$

- *Si $2\beta < \gamma + 1$, alors le comportement asymptotique de V dépend fortement de la force de frottement*

$$\frac{V_t}{t^{\frac{\beta}{\gamma+1}}} \xrightarrow[t \rightarrow +\infty]{} \Pi_{\rho, \gamma}.$$

Il est naturel de se demander ce qu'il advient de la convergence en tant que processus de V et de celle du processus position X . Cela fait l'objet du premier chapitre de ce manuscrit. Notons que la convergence du processus V ne découle pas de manière immédiate de celle de ses marginales unidimensionnelles. En revanche, la convergence des marginales de rang fini de V , associée à la tension du processus permet de conclure à la convergence du processus vitesse.

Modèle cinétique homogène en temps

Dans [FT21], les auteurs étudient le modèle cinétique suivant, dirigé par un mouvement brownien et dépendant du paramètre $\rho > 0$,

$$\begin{cases} dV_t = dB_t - \frac{\rho}{2} \frac{V_t}{1 + V_t^2} dt, \\ dX_t = V_t dt. \end{cases} \quad (6)$$

Ce modèle fut déjà étudié avec la théorie des EDP dans [NP15], [CNP19] et [LP19]. Puisque les coefficients de (6) ne dépendent pas du temps, les outils propres aux EDS homogènes en temps sont disponibles. Citons par exemple la fonction d'échelle et la mesure vitesse. Le résultat principal de [FT21] est le suivant.

Théorème 2.1.2 (Fournier, Tardif). *Soit (V_t, X_t) la solution de (6). Soient $(W_t)_{t \geq 0}$ un mouvement brownien, $(S_t^{(\alpha)})_{t \geq 0}$ un processus α -stable symétrique avec $\alpha \in (0, 2)$, tel que*

$\mathbb{E} [\exp(i\xi S_t^{(\alpha)})] = \exp(-t |\xi|^\alpha)$, et $(U_t^{(\delta)})_{t \geq 0}$ un processus de Bessel symétrique de dimension $\delta \in (0, 1)$. Les expressions des constantes $\sigma_\rho > 0$ sont données dans [FT21, p. 5].

- Si $\rho > 5$,

$$(\varepsilon^{\frac{1}{2}} X_{t/\varepsilon})_{t \geq 0} \xrightarrow[\varepsilon \rightarrow 0]{f.d.} (\sigma_\rho W_t)_{t \geq 0}.$$

- Si $\rho = 5$,

$$(|\varepsilon \log(\varepsilon)|^{\frac{1}{2}} X_{t/\varepsilon})_{t \geq 0} \xrightarrow[\varepsilon \rightarrow 0]{f.d.} (\sigma_5 W_t)_{t \geq 0}.$$

- Si $\rho \in (1, 5)$, on pose $\alpha = \frac{\rho+1}{3}$

$$(\varepsilon^{\frac{1}{\alpha}} X_{t/\varepsilon})_{t \geq 0} \xrightarrow[\varepsilon \rightarrow 0]{f.d.} (\sigma_\rho S_t^{(\alpha)})_{t \geq 0}.$$

- Si $\rho = 1$,

$$(|\varepsilon \log(\varepsilon)|^{\frac{3}{2}} X_{t/\varepsilon})_{t \geq 0} \xrightarrow[\varepsilon \rightarrow 0]{f.d.} (\sigma_1 S_t^{(2/3)})_{t \geq 0}.$$

- Si $\rho \in (0, 1)$,

$$(\varepsilon^{\frac{1}{2}} V_{t/\varepsilon}, \varepsilon^{\frac{3}{2}} X_{t/\varepsilon})_{t \geq 0} \xrightarrow[\varepsilon \rightarrow 0]{} \left(U_t^{(1-\rho)}, \int_0^t U_s^{(1-\rho)} ds \right)_{t \geq 0}.$$

On observe une transition entre la normalisation du mouvement brownien $\varepsilon^{\frac{1}{2}}$ et celle de la primitive d'un mouvement brownien $\varepsilon^{\frac{3}{2}}$.

La preuve de la convergence pour $\rho > 5$ est classique et repose sur la méthode martingale. On détaillera celle-ci à la section suivante.

Lorsque $\rho \in (0, 1)$, la stratégie de preuve est d'approcher le processus vitesse V par un processus de Bessel de dimension $1 - \rho$, écrit, suivant la représentation de Feller, comme la fonctionnelle d'un mouvement brownien changé de temps. La limite du couple découle alors du *continuous mapping theorem*.

Pour traiter le cas $\rho \in (1, 5)$, l'idée est d'écrire le processus X comme la fonctionnelle d'un mouvement brownien changé de temps. Un résultat de représentation des processus α -stables (voir [BY87]) permet de conclure.

Cette méthode a été réutilisée et généralisée pour obtenir des théorèmes asymptotiques de fonctionnelles additives de V , dans [Bét21]. Notons que ce n'est plus possible pour un système dont les coefficients dépendent du temps. En revanche, le comportement asymptotique de certains cas particuliers de (SKE_B) peut être étudié de manière directe. Décrivons quelques exemples, avant de présenter l'heuristique ainsi que les résultats dans

le cas général.

2.2 Quelques cas particuliers

Le processus de Kolmogorov

Lorsqu'il n'y a pas de force de frottement, *i.e.* $F = 0$, la solution du système (SKE_B) est le processus de Kolmogorov solution de (5). Puisque la famille de processus

$$\left\{ \left(\varepsilon^{\frac{1}{2}} V_{t/\varepsilon}, \varepsilon^{\frac{3}{2}} X_{t/\varepsilon} \right)_{t \geq 0}, \varepsilon > 0 \right\}$$

est stationnaire, de même loi que (V, X) , alors,

$$\left(\varepsilon^{\frac{1}{2}} V_{t/\varepsilon}, \varepsilon^{\frac{3}{2}} X_{t/\varepsilon} \right)_{t \geq 0} \stackrel{\mathcal{L}}{=} \left(B_t, \int_0^t B_s ds \right)_{t \geq 0}.$$

Le processus de Langevin

Le système (2) est homogène en temps et linéaire et le processus V est ergodique. On note \tilde{V} une variable aléatoire ayant pour loi sa mesure invariante. Par ailleurs, l'expression explicite de X est donnée par (3). En étudiant la convergence de la variation quadratique de l'intégrale de Wiener définissant X , on conclut à l'aide d'un TCL pour les martingales locales ([JS03, Théorème VIII-3.11 p. 473]) et en suivant le schéma de preuve [FT21, Théorème 3], que

$$\left(V_{t/\varepsilon}, \sqrt{\varepsilon} X_{t/\varepsilon} \right)_{t \geq 0} \xrightarrow{\varepsilon \rightarrow 0} \left(\tilde{V}, B_t \right)_{t \geq 0}.$$

Le cas homogène en temps

Lorsque la force de frottement est indépendante du temps, *i.e.* $\beta = 0$, on a alors à notre disposition les outils spécifiques aux diffusions homogènes en temps. L'étude de la mesure vitesse et de la fonction d'échelle de V permet de conclure que le processus vitesse est ergodique. Notons \tilde{V} une variable aléatoire distribuée selon sa mesure invariante. À défaut de connaître l'expression explicite de X , comme dans le cas linéaire, on peut utiliser la *méthode martingale*. Cette méthode a notamment été utilisée dans [CCM10], [EG15] et [FT21]. Notons $\mathcal{L} := \frac{1}{2} \partial_{vv}^2 - F \partial_v$ le générateur infinitésimal de l'EDS vérifiée par le processus V . On commence par résoudre l'équation de Poisson $\mathcal{L}g = -\text{Id}$. En appliquant

la formule d'Itô, on obtient alors, pour tout $t \geq 0$ et $\varepsilon > 0$,

$$r_{\varepsilon, X} X_{t/\varepsilon} = r_{\varepsilon, X} [x_0 + g(v_0)] - r_{\varepsilon, X} g(V_{t/\varepsilon}) + r_{\varepsilon, X} \int_0^{t/\varepsilon} g'(V_s) dB_s. \quad (7)$$

Le choix $r_{\varepsilon, X} = \sqrt{\varepsilon}$ découle, une nouvelle fois, d'un théorème central limite pour les martingales locales ([JS03, Théorème VIII-3.22 p. 476]). En effet, notons $M^{(\varepsilon)}$ l'intégrale stochastique de (7). On déduit du théorème ergodique appliqué à V qu'il existe une constante $\kappa > 0$, dépendant de g et de \mathcal{L} , telle que

$$\langle M^{(\varepsilon)} \rangle_t \xrightarrow{\varepsilon \rightarrow 0} \kappa t.$$

Ainsi, on a la convergence suivante

$$\left(\sqrt{\varepsilon} \int_0^{t/\varepsilon} g'(V_s) dB_s \right)_{t \geq 0} \xrightarrow{\varepsilon \rightarrow 0} (\mathcal{B}_{\kappa t})_{t \geq 0}.$$

En outre, en suivant la stratégie de [EG15, Section 3.2], on peut montrer que

$$\sup_{t \geq 0} \left| \sqrt{\varepsilon} g(V_{t/\varepsilon}) \right| \xrightarrow[\varepsilon \rightarrow 0]{\mathbb{P}} 0.$$

Par conséquent, d'après [Bil99, Théorème 3.1 p. 27], on obtient la convergence de la composante position,

$$\left(\sqrt{\varepsilon} X_{t/\varepsilon} \right)_{t \geq 0} \xrightarrow{\varepsilon \rightarrow 0} (\mathcal{B}_{\kappa t})_{t \geq 0}.$$

La convergence du couple s'en déduit, comme dans le cas linéaire,

$$\left(V_{t/\varepsilon}, \sqrt{\varepsilon} X_{t/\varepsilon} \right)_{t \geq 0} \xrightarrow{\varepsilon \rightarrow 0} (\tilde{V}, \mathcal{B}_{\kappa t})_{t \geq 0}.$$

Le cas linéaire et inhomogène

Intéressons-nous maintenant au processus de Langevin inhomogène en temps. Il est possible de résoudre explicitement le système dont il est solution,

$$\begin{cases} dV_t = dB_t - \rho \frac{V_t}{t^\beta} dt, \\ dX_t = V_t dt. \end{cases}$$

Le processus V est gaussien. Son expression est explicite et donnée, pour tout $t \geq t_0 > 0$, par

$$V_t = v_0 + \begin{cases} e^{-\rho \frac{t^{1-\beta}}{1-\beta}} \int_{t_0}^t e^{\rho \frac{s^{1-\beta}}{1-\beta}} dB_s & \text{si } \beta \neq 1, \\ \frac{1}{t^\rho} \int_{t_0}^t s^\rho dB_s & \text{si } \beta = 1. \end{cases} \quad (8)$$

Son comportement asymptotique découle soit de la théorie des martingales continues, soit de celle des processus gaussiens.

Supposons tout d'abord que $\beta \geq 1$. Alors, l'étude de la variation quadratique de V permet de conclure, grâce au théorème central limite pour les martingales locales ([JS03, Théorème VIII-3.22 p. 476]) que

$$\left(\varepsilon^{\frac{1}{2}} V_{t/\varepsilon} \right)_{t \geq \varepsilon t_0} \xrightarrow{\varepsilon \rightarrow 0} (\mathcal{B}_{\kappa t})_{t > 0},$$

où κ est définie par

$$\kappa := \begin{cases} 1 & \text{si } \beta > 1, \\ \frac{1}{1+2\rho} & \text{si } \beta = 1. \end{cases}$$

Par ailleurs, définissons la fonction $g_\varepsilon : V \mapsto \left(V_t, \int_{\varepsilon t_0}^t V_s ds \right)_{t \geq \varepsilon t_0}$. Celle-ci converge vers $g : V \mapsto \left(V_t, \int_0^t V_s ds \right)_{t > 0}$. En adaptant la preuve du *continuous mapping theorem*, il est possible de montrer que

$$g_\varepsilon(V^{(\varepsilon)}) \xrightarrow{\varepsilon \rightarrow 0} g(\mathcal{B}_{\kappa \bullet}).$$

Puisque $\left(\varepsilon^{\frac{1}{2}} V_{t/\varepsilon}, \varepsilon^{\frac{3}{2}} X_{t/\varepsilon} \right)_{t \geq \varepsilon t_0} = g_\varepsilon(V^{(\varepsilon)})$, on obtient donc la convergence suivante

$$\left(\varepsilon^{\frac{1}{2}} V_{t/\varepsilon}, \varepsilon^{\frac{3}{2}} X_{t/\varepsilon} \right)_{t \geq \varepsilon t_0} \xrightarrow{\varepsilon \rightarrow 0} \left(\mathcal{B}_{\kappa t}, \int_0^t \mathcal{B}_{\kappa s} ds \right)_{t > 0}.$$

Traisons maintenant le cas $\beta < 1$. Rappelons qu'un processus gaussien est caractérisé par son espérance et son noyau de covariance. Or, on peut montrer que, pour tout $(s, t) \in [\varepsilon t_0, +\infty)^2$,

$$\lim_{\varepsilon \rightarrow 0} \text{Cov} \left(\varepsilon^{\frac{\beta}{2}} V_{s/\varepsilon}, \varepsilon^{\frac{\beta}{2}} V_{t/\varepsilon} \right) = \frac{1}{2\rho} (s \wedge t)^\beta \mathbb{1}_{\{s=t\}} := K_V(s, t).$$

Notons $(\mathcal{V}_t)_{t \geq 0}$ le processus gaussien centré de noyau de covariance K_V . On en déduit alors que

$$\left(\varepsilon^{\frac{\beta}{2}} V_{t/\varepsilon}\right)_{t \geq 0} \xrightarrow[\varepsilon \rightarrow 0]{\text{f.d.}} (\mathcal{V}_t)_{t > 0}.$$

Si le processus $V^{(\varepsilon)}$ était tendu, alors on pourrait conclure à la convergence du processus $(\varepsilon^{1+\frac{\beta}{2}} X_{t/\varepsilon})_{t \geq \varepsilon t_0}$. Le cas $\beta = 0$ nous indique que ce n'est pas possible. Puisque la convergence de V n'a lieu que pour les marginales de rang fini, l'argument utilisé lorsque $\beta \geq 1$ pour conclure à la convergence du processus position ne fonctionne plus.

En revanche, en reprenant la stratégie de la méthode martingale, on peut appliquer la formule d'Itô à $t^\beta V_t$ dans le but de décomposer X_t comme la somme d'une intégrale de Wiener et d'un terme qui tend vers 0 dans L^1 . Le noyau de covariance du processus gaussien converge vers $K_X : (s, t) \mapsto \frac{(s \wedge t)^{1+2\beta}}{1+2\beta}$. Ainsi, en définissant $(\mathcal{X}_t)_{t \geq 0}$ le processus gaussien centré de noyau de covariance K_X , on obtient que

$$\left(\varepsilon^{\beta+\frac{1}{2}} X_{t/\varepsilon}\right)_{t \geq 0} \xrightarrow[\varepsilon \rightarrow 0]{\text{f.d.}} (\mathcal{X}_t)_{t > 0}.$$

Les stratégies de preuve pour traiter ces cas particuliers sont propres aux EDS homogènes en temps ou aux processus gaussiens. Elles ne pourront donc pas être utilisées dans le cas général. Cependant, elles mettent en lumière des idées qu'il est possible d'adapter. On tentera notamment de se ramener à des processus homogènes.

2.3 Heuristique

On se place, à partir de maintenant, dans le cadre général d'un processus directeur qui est un Lévy α -stable, pour $\alpha \in (0, 2]$. Afin d'expliquer l'heuristique du comportement asymptotique des solutions de (SKE_L) , nous commençons par donner les équations vérifiées par les processus renormalisés. Pour tout $t \geq \varepsilon t_0$, on a

$$V_t^{(\varepsilon)} = r_{\varepsilon, V} V_{t/\varepsilon} = r_{\varepsilon, V} (v_0 - L_{t_0}) + r_{\varepsilon, V} L_{t/\varepsilon} - r_{\varepsilon, V}^{1-\gamma} \varepsilon^{\beta-1} \int_{\varepsilon t_0}^t r_{\varepsilon}^{\gamma} F(r_{\varepsilon, V}^{-1} V_s^{(\varepsilon)}) s^{-\beta} ds, \quad (9)$$

$$X_t^{(\varepsilon)} = r_{\varepsilon, X} X_{t/\varepsilon} = r_{\varepsilon, X} x_0 + \int_{\varepsilon t_0}^t \underbrace{\frac{r_{\varepsilon, X}}{\varepsilon}}_{=r_{\varepsilon, V}} V_{s/\varepsilon} ds. \quad (10)$$

En reprenant l'idée du cas linéaire inhomogène, on peut déduire de la convergence du processus $V^{(\varepsilon)}$ et d'un *continuous mapping theorem* adapté, la convergence du processus

$X^{(\varepsilon)}$. Alors, le processus limite serait cinétique, *i.e.* de la forme $(\mathcal{V}, \int_0^t \mathcal{V})_{t>0}$.

Intéressons-nous donc d'abord à la convergence du processus $V^{(\varepsilon)}$. Seuls les deux derniers termes de (9) jouent un rôle dans la convergence du processus. Plusieurs situations peuvent alors se présenter. Elles correspondent aux deux principales méthodes permettant de montrer qu'un processus converge.

- La plus classique consiste à montrer la convergence des marginales de rang fini du processus puis à vérifier un critère de tension.
- La seconde repose sur la décomposition du processus comme somme d'un processus plus connu qui converge et d'un terme qui tend vers 0 en probabilité, uniformément sur tout compact.

Grâce à l'auto-similarité des processus de Lévy α -stables, la seconde semble être la plus simple à mettre en œuvre. En effet, pour tout $\varepsilon > 0$, le processus $(\varepsilon^{\frac{1}{\alpha}} L_{t/\varepsilon})_{t \geq 0}$ est un Lévy α -stable. Il suffit donc de montrer que le dernier terme de (9) tend vers 0 en probabilité, uniformément sur tout compact. Néanmoins, ce n'est pas toujours vrai, ce terme étant parfois compensé par le processus directeur. Dans ce cas, on utilisera la méthode classique.

Remarque 2.2. *En fait, de manière plus générale, il existe des processus de Lévy tels que $(r_\varepsilon L_{t/\varepsilon})_{t \geq 0}$ converge, pour une bonne renormalisation r_ε . On donne des résultats dans la Proposition 2.8.*

On distingue ainsi trois régimes asymptotiques, en fonction de la position des coefficients de frottement γ et β par rapport à α . On se place sous l'hypothèse $\alpha > 1$, afin de pouvoir définir le rapport $q := \frac{\beta}{\gamma + \alpha - 1}$, et ainsi simplifier la disjonction des cas.

- Si $q > \frac{1}{\alpha}$, la force de rappel induite par le frottement est asymptotiquement négligeable. Le processus vitesse-position se comporte comme la diffusion de Kolmogorov.
- En régime critique, *i.e.* $q = \frac{1}{\alpha}$, la force de frottement compense la force aléatoire. La loi limite est la loi cinétique d'un "mélange" entre les lois limites des deux autres régimes. Elle dépend des paramètres de la force de frottement.
- Lorsque $q < \frac{1}{\alpha}$, le processus vitesse tend à diverger vers l'infini via la force aléatoire, mais est très vite rappelé vers 0 par la dérive. Le processus directeur n'est pas asymptotiquement négligeable. De plus, les marginales de rang fini du processus vitesse limite dépendent des paramètres de la force de frottement. Comme nous l'avons vu dans le paragraphe traitant du cas linéaire inhomogène, l'étude du cas homogène en temps $q = 0$ semble indiquer qu'il ne peut y avoir de convergence

du processus $V^{(\varepsilon)}$, mais seulement de ses marginales de rang fini. On perd donc le caractère cinétique du processus limite, et avec celui-ci la régularité de sa composante position.

Lorsque le système est dirigé par un mouvement brownien, on met en évidence le changement de régimes dans la vidéo [Lui22a], ainsi que dans la Figure 5. La Figure 6 représente le système dirigé par un processus α -stable, dans les différents régimes. Une vidéo est également disponible ([Lui22b]).

2.4 Résultats et discussions

Nous généralisons dans la première partie de ce manuscrit les résultats de [GO13] à la convergence du processus (V, X) dirigé par un mouvement brownien standard (Chapitre 1) puis par un processus de Lévy α -stable (Chapitre 2). Notons que cela répond à une question ouverte posée dans [Off14].

Existence, unicité et explosion

L'étude de l'existence et de la non-explosion de solutions dans le cadre brownien est détaillée dans [GO13]. Lorsque le processus directeur n'est plus supposé continu, et que la dérive est localement lipschitzienne, on déduit des résultats classiques ([Sit05, Lemme 115 p. 78 et Théorème 137 p. 104] et [IW81, Théorème 9.1 p. 231]) l'existence et l'unicité d'une solution locale. Lorsque la dérive est une fonction hölderienne, on conclut à l'aide de [CZZ17, Remarque 1.3]. Enfin, pour montrer que le temps d'explosion du processus solution est presque sûrement infini, on démontre le lemme suivant.

Lemme 2.3. *Soit $(Y_t)_{t \geq t_0}$ la solution locale d'une EDS dirigée par un processus de Lévy. On définit, pour $r \geq 0$, le temps d'arrêt $\tau_r := \inf\{t \geq t_0, |Y_t| \geq r\}$ et $\tau_\infty := \lim_{r \rightarrow +\infty} \tau_r$ le temps d'explosion de Y . Supposons qu'il existe deux fonctions mesurables positives ϕ et b telles que*

- (i) ϕ est croissante et $\lim_{r \rightarrow \infty} \phi(r) = +\infty$,
- (ii) b est à valeurs finies,
- (iii) et pour tout $t \geq t_0$,

$$\sup_{r \geq 0} \mathbb{E} [\phi(|Y_{t \wedge \tau_r}|)] \leq b(t).$$

Alors $\tau_\infty = +\infty$ p.s.

On est donc ramené à étudier les moments du processus vitesse V .

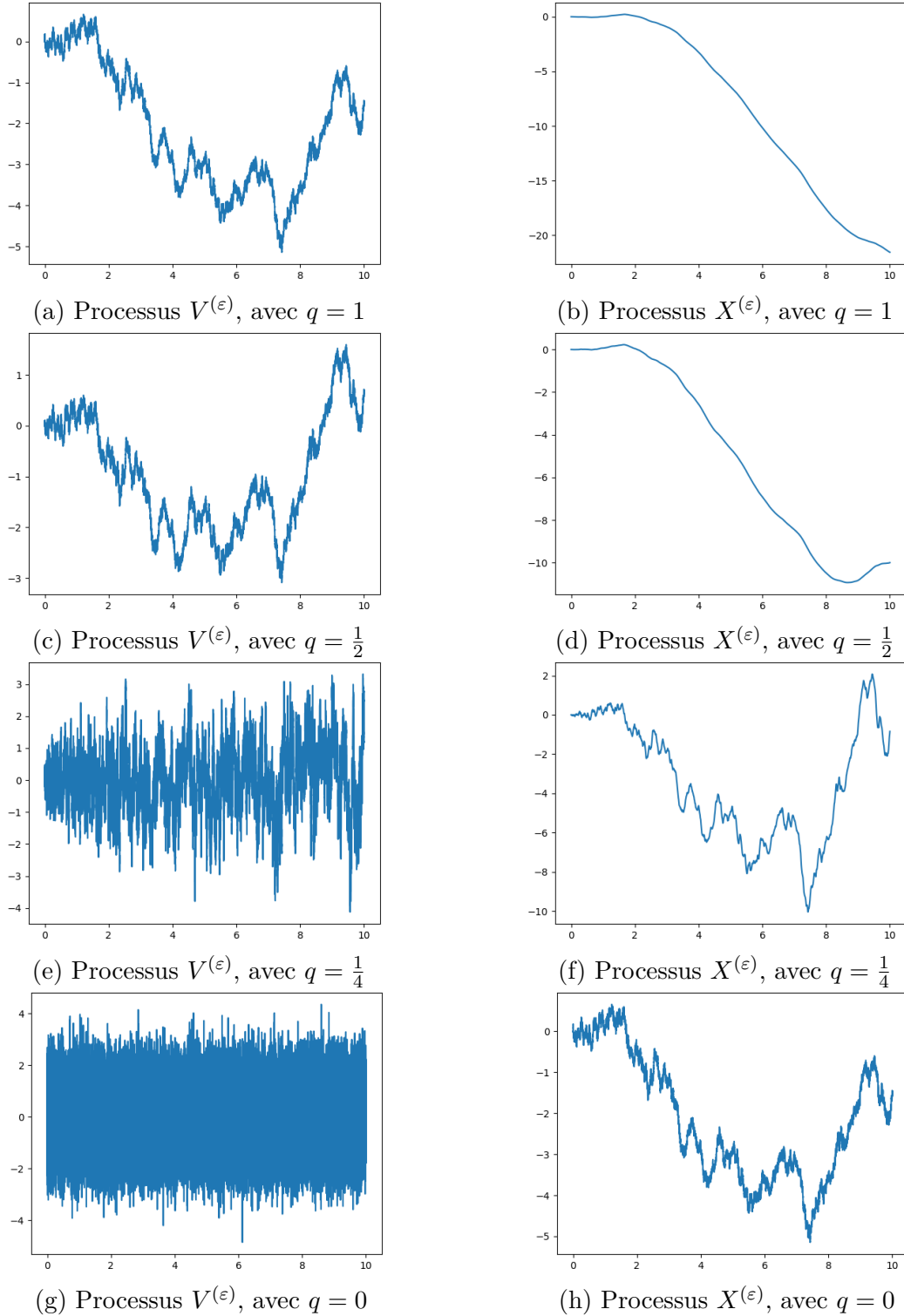


Figure 5 – Exemples de trajectoire des processus vitesse (à gauche) et position (à droite) renormalisés, dans le cas d'un système linéaire dirigé par la même réalisation d'un mouvement brownien. [Lui22a] est une vidéo illustrant l'évolution des processus, en fonction de q .

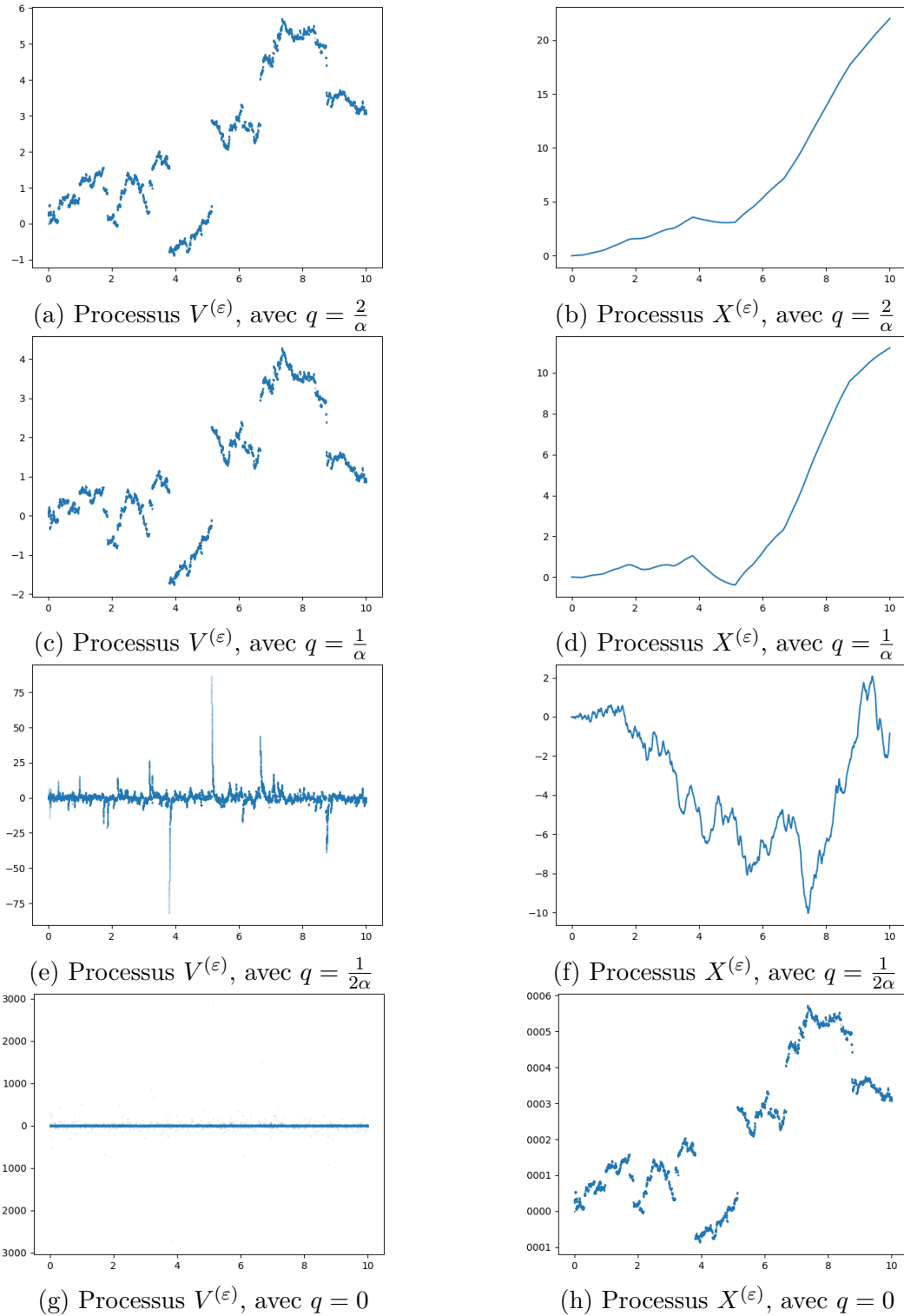


Figure 6 – Exemples de trajectoire des processus vitesse (à gauche) et position (à droite) renormalisés, dans le cas d'un système linéaire dirigé par la même réalisation d'un (1.3)-stable. [Lui22b] est une vidéo illustrant l'évolution des processus, en fonction de q .

Estimée des moments

L'étude des moments occupe un large pan de ce manuscrit. Lorsque V est une diffusion, on montre que V a des moments de tout ordre, comme son processus directeur. En appliquant la formule d'Itô pour la fonction $x \mapsto x^2$ et en utilisant l'attractivité vers l'origine de la dérive, on conclut avec l'inégalité de Jensen pour les moments d'ordre $\kappa \in [0, 2]$. Lorsque $\kappa \geq 2$, on conclut de proche en proche en utilisant le moment d'ordre $\kappa - 2$.

Le cas de processus à sauts est plus complexe. Le calcul des moments d'un processus de Lévy α -stable repose soit sur son auto-similarité, soit sur la connaissance de sa fonction caractéristique. Les estimées des moments utilisées dans [LP06] reposent sur la première méthode, tandis que la deuxième méthode a été utilisée dans [DS15, Théorème 3.1 c)]. Cependant, on ne connaît pas à priori la fonction caractéristique de la solution d'une EDS dirigée par un processus de Lévy α -stable. D'autre part, un processus α -stable n'admet que des moments d'ordre $\kappa \in [0, \alpha)$, or la fonction $f : x \mapsto |x|^\kappa$ n'est pas \mathcal{C}^2 . On ne peut donc pas appliquer directement la formule d'Itô-Lévy. L'idée est donc d'approcher f par une suite de fonctions \mathcal{C}^2 .

Lorsque $\alpha \in (0, 1)$, on fait apparaître le processus de Lévy α -stable $L_t^+ := \sum_{s \leq t} |\Delta L_s|$. Tandis que si $\alpha \in (1, 2)$, l'idée clé pour obtenir une borne optimale des moments est de découper les sauts du processus directeur au niveau $\xi \mapsto \xi^{\frac{1}{\alpha}}$. Cette méthode a notamment été utilisée par [CM20]. En fait, si L est un processus de Lévy α -stable, alors par la décomposition d'Itô-Lévy, pour tout $t \geq t_0$,

$$L_t - L_{t_0} = \int_{t_0}^t \int_{|z| \leq \xi^{\frac{1}{\alpha}}} z \widetilde{N}(ds, dz) + \int_{t_0}^t \int_{|z| > \xi^{\frac{1}{\alpha}}} z N(ds, dz) - \int_{t_0}^t \int_{|z| > \xi^{\frac{1}{\alpha}}} z \nu(dz) ds.$$

Observons que $L_t^{(1)} := \int_{t_0}^t \int_{|z| \leq \xi^{\frac{1}{\alpha}}} z \widetilde{N}(ds, dz)$ et $L_t^{(2)} := \int_{t_0}^t \int_{|z| > \xi^{\frac{1}{\alpha}}} z N(ds, dz)$ vérifient la propriété d'auto-similarité au temps ξ

$$L_\xi^{(1)} \stackrel{\mathcal{L}}{=} \xi^{\frac{1}{\alpha}} L_1^{(1)} \quad \text{et} \quad L_\xi^{(2)} \stackrel{\mathcal{L}}{=} \xi^{\frac{1}{\alpha}} L_1^{(2)}.$$

On applique alors la formule d'Itô avec la fonction $v \mapsto (\eta + v^2)^{\frac{\kappa}{2}}$. Puis, on estime chacun des termes en suivant le schéma de preuve de [DS15, Théorème 3.1]. Celui-ci repose sur l'inégalité de Kunita ([App09, Théorème 4.4.23 p. 265]), une formule de Taylor, et les propriétés hölderiennes des fonctions puissances. En optimisant selon η , on déduit la proposition suivante.

Proposition 2.4. Soit V la solution de (SKE_L) . Pour tous $\alpha \in (0, 2]$, $\gamma \in \mathbb{R}$, $\beta \in \mathbb{R}$ et $\kappa \in [0, \alpha)$, il existe une constante $C_{\gamma, \kappa, \beta, t_0}$ telle que,

$$\forall t \geq t_0, \quad \mathbb{E}[|V_t|^\kappa] \leq C_{\gamma, \kappa, \beta, t_0} t^{\frac{\kappa}{\alpha}}. \quad (11)$$

Lorsque $\alpha = 2$, l'inégalité (11) est satisfaite pour tout $\kappa \geq 0$.

Remarque 2.5. Le résultat reste vrai pour toute solution d'EDS avec une dérive attractive (i.e. telle que pour tout $v \in \mathbb{R}$, on ait $vF(v, \cdot) \geq 0$) de la forme

$$dV_t = dL_t - F(t, V_t) dt.$$

Lorsque la dérive n'est plus supposée attractive vers 0, la preuve repose sur l'inégalité de Jensen et un lemme de Grönwall généralisé (Lemme 1.A.1).

Comportement asymptotique

Énonçons les principaux résultats pour $F : v \mapsto \rho \operatorname{sgn}(v) |v|^\gamma$.

Dans le cadre brownien, on pose $q := \frac{\beta}{\gamma + 1}$ et on munit l'espace des fonctions continues $\mathcal{C}((0, +\infty), \mathbb{R})$ de la topologie uniforme sur tout compact

$$d_u : (f, g) \in \mathcal{C}((0, +\infty), \mathbb{R})^2 \mapsto \sum_{n=1}^{+\infty} \frac{1}{2^n} \min \left(1, \sup_{[\frac{1}{n}, n]} |f - g| \right).$$

Théorème 2.5.1. Soient $\gamma \geq 0$ et $(V_t, X_t)_{t \geq t_0}$ la solution de (SKE_B) .

- **Régime sur-critique.** Supposons $2q > 1$. Soit $(\mathcal{B}_t)_{t \geq 0}$ un mouvement brownien. Alors,

$$\left(\varepsilon^{\frac{1}{2}} V_{t/\varepsilon}, \varepsilon^{\frac{3}{2}} X_{t/\varepsilon} \right)_{t \geq \varepsilon t_0} \xrightarrow{\varepsilon \rightarrow 0} \left(\mathcal{B}_t, \int_0^t \mathcal{B}_s ds \right)_{t > 0}.$$

- **Régime critique.** Supposons $2q = 1$. On note \widetilde{H} le processus stationnaire solution de l'EDS dirigée par un mouvement brownien $(W_t)_{t \geq 0}$

$$d\widetilde{H}_s = dW_s - \frac{\widetilde{H}_s}{2} ds - F(\widetilde{H}_s) ds,$$

telle que $\widetilde{H}_{-\infty}$ suit la loi invariante. On note $(\mathcal{V}_t)_{t \geq 0} := (\sqrt{t} \widetilde{H}_{\log(t)})_{t \geq 0}$.

Alors,

$$\left(\varepsilon^{\frac{1}{2}}V_{t/\varepsilon}, \varepsilon^{\frac{3}{2}}X_{t/\varepsilon}\right)_{t \geq \varepsilon t_0} \xrightarrow[\varepsilon \rightarrow 0]{} \left(\mathcal{V}_t, \int_0^t \mathcal{V}_s ds\right)_{t > 0}.$$

- **Régime sous-critique.** Supposons $2q < 1$. On note \widehat{H} le processus stationnaire, issu de sa loi invariante Π , solution de l'EDS dirigée par un mouvement brownien $(W_t)_{t \geq 0}$

$$d\widehat{H}_s = dW_s - F(\widehat{H}_s) ds.$$

On note $(\mathcal{V}_t)_{t \geq 0}$ le processus ayant pour marginales la mesure image de la mesure produit $\Pi^{\otimes d}$ par l'application $T(u_1, \dots, u_d) := (t_1^q u_1, \dots, t_d^q u_d)$. Alors,

$$\left(\varepsilon^q V_{t/\varepsilon}\right)_{t \geq \varepsilon t_0} \xrightarrow[\varepsilon \rightarrow 0]{f.d.} (\mathcal{V}_t)_{t > 0}.$$

De plus, si $\gamma = 1$ et $\beta \in (-\frac{1}{2}, 1)$, on définit le processus gaussien $(\mathcal{X}_t)_{t \geq 0}$ centré et de noyau de covariance défini par

$$\forall s, t \geq 0, \quad K(s, t) := \frac{(s \wedge t)^{1+2\beta}}{\rho^2(1+2\beta)}.$$

Alors,

$$\left(\varepsilon^{\beta+\frac{1}{2}}X_{t/\varepsilon}\right)_{t \geq \varepsilon t_0} \xrightarrow[\varepsilon \rightarrow 0]{f.d.} (\mathcal{X}_t)_{t > 0}.$$

Définissons

$$\Lambda := \{\lambda : \mathbb{R}^+ \rightarrow \mathbb{R}^+, \text{ continue croissante t.q. } \lambda(0) = 0, \lim_{t \rightarrow +\infty} \lambda(t) = +\infty\}$$

et

$$k_n(t) = \begin{cases} 1 & \text{si } \frac{1}{n} \leq t \leq n, \\ n+1-t & \text{si } n < t < n+1, \\ 0 & \text{si } n+1 \leq t. \end{cases}$$

On munit l'espace des fonctions càdlàg $\mathcal{D}((0, +\infty), \mathbb{R})$ de la topologie de Skorokhod d_s définie, pour tout $(f, g) \in \mathcal{D}((0, +\infty), \mathbb{R})^2$ par

$$\sum_{n=1}^{+\infty} \frac{1}{2^n} \left(1 \wedge \inf \left\{ a, \exists \lambda \in \Lambda, \sup_{s \neq t} \left| \log \frac{\lambda(t) - \lambda(s)}{t - s} \right| \leq a, \sup_{t \geq \frac{1}{n}} |k_n(t) (f \circ \lambda(t) - g(t))| \leq a \right\} \right).$$

Pour $\alpha \in [1, 2)$, on rappelle que $q = \frac{\beta}{\gamma + \alpha - 1}$. Dans ce cas, les régimes dépendent de la

position de q par rapport à $\frac{1}{\alpha}$. On donne les résultats dans le cas général $\alpha \in (0, 2)$.

Théorème 2.5.2. *Soient $\alpha \in (0, 2)$, $\gamma \in (1 - \frac{\alpha}{2}, \alpha)$ et $(V_t, X_t)_{t \geq t_0}$ la solution de (SKE_L), dirigé par un processus de Lévy α -stable.*

- **Régime sur-critique.** *Supposons $\beta > 1 + \frac{\gamma-1}{\alpha}$. Soit $(\mathcal{S}_t)_{t \geq 0}$ de même loi que $(L_t)_{t \geq 0}$.*

Alors,

$$(\varepsilon^{\frac{1}{\alpha}} V_{t/\varepsilon}, \varepsilon^{1+\frac{1}{\alpha}} X_{t/\varepsilon})_{t \geq \varepsilon t_0} \xrightarrow[\varepsilon \rightarrow 0]{} \left(\mathcal{S}_t, \int_0^t \mathcal{S}_s ds \right)_{t > 0}.$$

- **Régime critique.** *Supposons $\beta = 1 + \frac{\gamma-1}{\alpha}$. On note \widetilde{H} le processus stationnaire solution de l'EDS dirigée par un processus α -stable R , de même loi que L ,*

$$d\widetilde{H}_s = dR_s - \frac{\widetilde{H}_s}{\alpha} ds - F(\widetilde{H}_s) ds,$$

telle que $\widetilde{H}_{-\infty}$ suit la loi invariante. On note $(\mathcal{V}_t)_{t \geq 0} := (t^{\frac{1}{\alpha}} \widetilde{H}_{\log(t)})_{t \geq 0}$.

Alors,

$$(\varepsilon^{\frac{1}{\alpha}} V_{t/\varepsilon}, \varepsilon^{1+\frac{1}{\alpha}} X_{t/\varepsilon})_{t \geq \varepsilon t_0} \xrightarrow[\varepsilon \rightarrow 0]{} \left(\mathcal{V}_t, \int_0^t \mathcal{V}_s ds \right)_{t > 0}.$$

- **Régime sous-critique.** *Supposons $\alpha > 1$, $\gamma \geq 1$ et $\beta < 1 + \frac{\gamma-1}{\alpha}$. On note \widehat{H} le processus stationnaire, issu de sa loi invariante Π , solution de l'EDS dirigée par un processus α -stable R , de même loi que L ,*

$$d\widehat{H}_s = dR_s - F(\widehat{H}_s) ds.$$

On note $(\mathcal{V}_t)_{t \geq 0}$ le processus ayant pour marginales la mesure image de la mesure produit $\Pi^{\otimes d}$ par l'application $T(u_1, \dots, u_d) := (t_1^q u_1, \dots, t_d^q u_d)$. Alors,

$$\left(\varepsilon^q V_{t/\varepsilon} \right)_{t \geq \varepsilon t_0} \xrightarrow[\varepsilon \rightarrow 0]{f.d.} (\mathcal{V}_t)_{t > 0}.$$

On représente les résultats sous la forme d'un schéma (Figure 7), qui illustre l'influence des termes de (9). On observe une transition de phase continue des taux de convergence $r_{\varepsilon, V}$ et $r_{\varepsilon, X}$: de la normalisation du processus de Langevin ($q = 0$) à celle du processus de Kolmogorov ($q \geq \frac{1}{\alpha}$). Le changement de régimes est également visible sur la vidéo de simulation [Lui22a].

Détaillons les schémas de preuve.

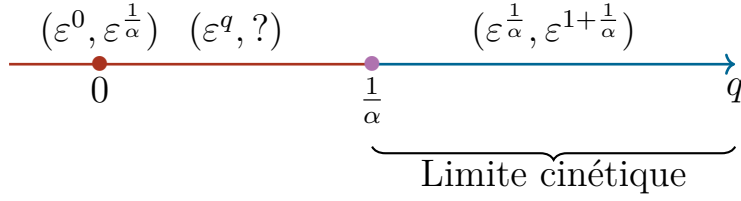


Figure 7 – Transition de phase

On montre, dans un premier temps, à partir d'un *continuous mapping theorem* adapté, que si le processus $V^{(\varepsilon)}$ converge alors $X^{(\varepsilon)}$ converge également. Cela implique que le processus limite est cinétique.

Régime sur-critique. La preuve est la même, suivant si $\alpha = 2$ ou $\alpha \in (0, 2)$. Par auto-similarité, le processus $(L_t^{(\varepsilon)})_{t \geq 0} := (\varepsilon^{\frac{1}{\alpha}} L_{t/\varepsilon})_{t \geq 0}$ a la même loi que L . De plus, on montre, à l'aide de l'estimée des moments, en notant $r := \min(\beta - 1 + \frac{1-\gamma}{\alpha}, \frac{1}{\alpha}) > 0$, que

$$\mathbb{E} \left[\sup_{\varepsilon t_0 \leq t \leq T} |V_t^{(\varepsilon)} - L_t^{(\varepsilon)}| \right] = \begin{cases} O(\varepsilon^r) & \text{si } \beta \neq 1 + \frac{\gamma}{\alpha}, \\ O(\varepsilon^{\frac{1}{\alpha}} |\ln(\varepsilon)|) & \text{sinon.} \end{cases}$$

Régime critique. S'inspirant des travaux de [GO13], on s'intéresse aux transformations d'échelles de V . D'après [ST94, Proposition 3.4.1 p. 124], le processus suivant est un processus α -stable de même loi que L

$$(R_t)_{t \geq 0} := \left(\int_0^t \frac{dL_{\varphi(s)}}{\varphi'(s)^{\frac{1}{\alpha}}} \right)_{t \geq 0}.$$

Ainsi, on obtient un résultat généralisant [GO13, Proposition 2.1].

Proposition 2.6. *Soit $\varphi : [0, t_1) \rightarrow [t_0, +\infty)$ un \mathcal{C}^2 -difféomorphisme. Si V est solution de (SKE_L) , alors $V^{(\varphi)} := \frac{V_{\varphi}}{\varphi'^{\frac{1}{\alpha}}}$ est solution de*

$$dV_s^{(\varphi)} = dR_s - \frac{\varphi'(s)^{1+\frac{\gamma-1}{\alpha}}}{\varphi(s)^\beta} F(V_s^{(\varphi)}) ds - \frac{\varphi''(s)}{\varphi'(s)} \frac{V_s^{(\varphi)}}{\alpha} ds, \quad \text{avec } V_0^{(\varphi)} = \frac{V_{\varphi(0)}}{\varphi'(0)^{\frac{1}{\alpha}}}. \quad (12)$$

Inversement, si $V^{(\varphi)}$ est solution de (12), alors V est solution de (SKE_L) , avec

$$L_t - L_{t_0} := \int_{t_0}^t (\varphi' \circ \varphi^{-1})^{\frac{1}{\alpha}}(s) dR_{\varphi^{-1}(s)}.$$

En régime critique, le processus $V^{(e)}$ est solution de l'EDS homogène en temps

$$dV_s^{(e)} = dR_s - \frac{V_s^{(e)}}{\alpha} ds - \rho \operatorname{sgn} \left(V_s^{(e)} \right) \left| V_s^{(e)} \right|^\gamma ds.$$

On montre dans un premier temps la convergence des marginales de rang fini de $V^{(e)}$. Donnons les idées pour les marginales de dimension 2.

Pour cela, on remarque que le processus $V^{(e)}$ admet une probabilité invariante Λ . Et puisque pour tout $(s, t) \in [\varepsilon t_0, +\infty)^2$, la différence

$$\varphi_\varepsilon^{-1} \left(\frac{t}{\varepsilon} \right) - \varphi_\varepsilon^{-1} \left(\frac{s}{\varepsilon} \right) = \log \left(\frac{t}{s} \right)$$

ne dépend pas de ε , on obtient par stationnarité de $V^{(e)}$ que, pour tout $\varepsilon \rightarrow 0$,

$$\left(V_{\log(s)+\log((t_0\varepsilon)^{-1})}^{(e)}, V_{\log(t)+\log((t_0\varepsilon)^{-1})}^{(e)} \right) \stackrel{\mathcal{L}}{=} \left(V_{\log(s)}^{(e)}, V_{\log(t)}^{(e)} \right).$$

Par ailleurs, en s'inspirant de [CCM10], on montre le lemme général suivant.

Lemme 2.7. *Soit H un processus fortement ergodique de mesure invariante ν , solution d'une EDS dirigée par un processus de Lévy. Soit $\phi : [t_0, +\infty) \rightarrow \mathbb{R}$ une fonction continue telle que $\lim_{s \rightarrow +\infty} \phi(s) = +\infty$.*

Alors, pour tout $d \in \mathbb{N}^$, toute fonction continue bornée $\psi : \mathbb{R}^d \rightarrow \mathbb{R}$, et pour tout $(t_1, \dots, t_d) \in [\varepsilon t_0, +\infty)^d$, on a*

$$\mathbb{E} \left[\psi \left(H_{\phi(\varepsilon^{-1}t_1)}, \dots, H_{\phi(\varepsilon^{-1}t_d)} \right) \middle| H_0 = h_0 \right] - \mathbb{E} \left[\psi \left(H_{\phi(\varepsilon^{-1}t_1)}, \dots, H_{\phi(\varepsilon^{-1}t_d)} \right) \middle| H_0 \sim \nu \right] \xrightarrow{\varepsilon \rightarrow 0} 0.$$

Le processus $V^{(e)}$ satisfait les conditions du lemme, par conséquent, pour tout $(s, t) \in [\varepsilon t_0, +\infty)^2$,

$$\left(\varepsilon^{\frac{1}{\alpha}} V_{s/\varepsilon}, \varepsilon^{\frac{1}{\alpha}} V_{t/\varepsilon} \right) \xrightarrow{\varepsilon \rightarrow 0} T * \mathcal{L} \left(V_{\log(s)}^{(e)}, V_{\log(t)}^{(e)} \right),$$

où $T * \mu$ désigne la mesure image de μ par l'application linéaire $T(u, v) := (s^{\frac{1}{\alpha}} u, t^{\frac{1}{\alpha}} v)$.

Dans un second temps, on vérifie le critère de tension de Kolmogorov ([Kal02, Corollaire 16.9 p. 313]) ou d'Aldous ([Bil99, Théorème 16.10 p. 178]), à l'aide de la propriété de Markov du processus de Lévy directeur et de l'estimée des moments du processus vitesse.

Régime sous-critique. En suivant [GO13], la stratégie est cette fois d'étudier la transformation d'échelles puissance. D'après la Proposition 2.6, le processus $V^{(q)}$ est alors solution de l'EDS dirigée par un processus α -stable R de même loi que L ,

$$dV_s^{(q)} = dR_s - \rho \operatorname{sgn}(V_s^{(q)}) |V_s^{(q)}|^\gamma ds - q\varphi^{\alpha q-1}(s)V_s^{(q)} ds.$$

Contrairement au régime critique, l'EDS n'est pas homogène en temps. Cependant, en mimant l'idée du lemme asymptotique (Lemme 2.1), on montre que $V^{(q)}$ est proche de la solution de

$$dH_s = dR_s - \rho \operatorname{sgn}(H_s) |H_s|^\gamma ds.$$

Plus précisément, on prouve que, pour tout $(s, t) \in [\varepsilon t_0, +\infty)^2$,

$$\left\| \left(H_{\varphi^{-1}(\varepsilon^{-1}s)}, H_{\varphi^{-1}(\varepsilon^{-1}t)} \right) - \left(V_{\varphi^{-1}(\varepsilon^{-1}s)}^{(q)}, V_{\varphi^{-1}(\varepsilon^{-1}t)}^{(q)} \right) \right\| \xrightarrow[\varepsilon \rightarrow 0]{\mathbb{P}} 0.$$

Pour cela, il est important de ne pas majorer trop brutalement les termes dans l'égalité suivante

$$d\left(H - V^{(q)}\right)_t = -\left(F(H_t) - F(V_t^{(q)})\right) dt + q\varphi^{\alpha q-1}(t)V_t^{(q)} dt.$$

En effet, c'est le terme $-\left(F(H_t) - F(V_t^{(q)})\right)$ qui permet de conclure.

Lorsque F est linéaire, on reproduit la méthode de variation des constantes. Néanmoins, lorsque la dérive n'est pas linéaire, il est utile de remarquer que F^{-1} est $\frac{1}{\gamma}$ -hölderienne, ce qui permet de se ramener au cas linéaire.

Dans un second temps, la preuve de la convergence des marginales de rang fini du processus $H^{(\varphi, \varepsilon)} := (H_{\varphi^{-1}(\varepsilon^{-1}t)})_{t \geq \varepsilon t_0}$ est inspirée de [CCM10], en travaillant avec le semi-groupe associé à H . En effet, contrairement au régime critique, on a ici pour tout $(s, t) \in [\varepsilon t_0, +\infty)^2$,

$$\varphi^{-1}\left(\frac{t}{\varepsilon}\right) - \varphi^{-1}\left(\frac{s}{\varepsilon}\right) \xrightarrow[\varepsilon \rightarrow 0]{} +\infty.$$

Puisque H admet une mesure invariante, on s'attend à ce que les marginales de $H^{(\varphi, \varepsilon)}$ se décorrèlent lorsque $\varepsilon \rightarrow 0$. Le potentiel de l'EDS vérifiée par H étant convexe, le processus vérifie une inégalité de Poincaré (voir [Bob99]). On conclut à la convergence des marginales de rang fini de $V^{(\varepsilon)}$ en utilisant un résultat de décroissance exponentielle (voir [BGL14, Théorème 4.2.5 p. 183]).

2.5 Quelques généralisations du processus directeur

Processus de Lévy général

Il est ensuite naturel de se demander ce que devient le comportement asymptotique du système (SKE_L) , s'il est dirigé par un processus de Lévy général. Par exemple, qu'en est-il si on tronque les grands sauts ? Notons que la propriété d'auto-similarité n'est alors plus vérifiée.

Plaçons-nous dans un premier temps sous l'hypothèse que le processus directeur est un processus de Lévy sans partie brownienne. On suppose que sa mesure de Lévy ν vérifie l'une des deux hypothèses suivantes :

$$\nu(z) = \frac{g(z)}{|z|^{1+\alpha}} \mathbb{1}_{\{z \neq 0\}}, \text{ où } g \text{ est une fonction positive mesurable telle que} \quad (H_1^{\nu, \alpha})$$

$$c^+ := \lim_{z \rightarrow +\infty} g(z) \geq 0, \quad c^- := \lim_{z \rightarrow -\infty} g(z) \geq 0,$$

ou

$$\text{pour un certain } \alpha_0 > 1, \quad \int_{|z| \geq 1} |z|^{\alpha_0} \nu(dz) < +\infty. \quad (H_2^{\nu, \alpha_0})$$

Par exemple, la mesure de Lévy d'un processus α -stable tronqué satisfait (H_2^{ν, α_0}) pour tout $\alpha_0 > 1$, tandis que celle d'un processus α -stable tempéré vérifie $(H_1^{\nu, \alpha})$ (avec $g(z) := e^{-\lambda|z|}$, où $\lambda > 0$).

On introduit également l'hypothèse suivante :

$$\nu(z) = \frac{g(z)}{|z|^{1+\alpha}} \mathbb{1}_{\{z \neq 0\}}, \text{ où } g \text{ est une fonction positive mesurable telle que} \quad (H_g)$$

$$\int_1^{+\infty} \frac{|g(z) - g(-z)|}{z^\alpha} dx < +\infty.$$

En s'inspirant de [RT11, Proposition 1], on obtient le comportement asymptotique du processus directeur L .

Proposition 2.8. *Soit $(L_t)_{t \geq 0}$ un processus de Lévy de triplet caractéristique $(0, \nu, b)$, par rapport à la fonction $h : z \mapsto -1 \vee (z \wedge 1)$. Les paramètres ν^* et b^* sont explicités dans la Proposition 2.6.3.*

- (i) *Si ν satisfait $(H_1^{\nu, \alpha})$ avec $\alpha = 1$, $c^+ = c^- = c$ et (H_g) . Alors le processus $(\varepsilon L_{t/\varepsilon})_{t \geq 0}$ converge vers le processus de Lévy 1-stable \mathcal{L} , de triplet caractéristique $(0, \nu^*, b^*)$.*
- (ii) *Si ν satisfait $(H_1^{\nu, \alpha})$ avec $\alpha \in (1, 2)$, et (H_g) . Si $b^* \neq 0$, alors le processus $(\varepsilon L_{t/\varepsilon})_{t \geq 0}$*

converge vers le processus déterministe \mathcal{L} de triplet $(0, 0, b^*)$.

(iii) Si ν satisfait (H_2^{ν, α_0}) et $b^* \neq 0$, alors le processus $(\varepsilon L_{t/\varepsilon})_{t \geq 0}$ converge vers le processus déterministe L^* de triplet $(0, 0, b^*)$.

(iv) Si $\alpha \in (0, 1)$, et ν satisfait $(H_1^{\nu, \alpha})$, alors le processus $(\varepsilon^{\frac{1}{\alpha}} L_{t/\varepsilon})_{t \geq 0}$ converge vers le processus de Lévy α -stable \mathcal{L} de mesure de Lévy ν^* .

(v) Si $\alpha \in (1, 2)$, $b = 0$ et ν vérifie $(H_1^{\nu, \alpha})$, alors le processus $(\varepsilon^{\frac{1}{\alpha}} L_{t/\varepsilon})_{t \geq 0}$ converge vers le processus de Lévy \mathcal{L} de mesure de Lévy ν^* et de centre b^* .

On déduit de ce résultat, le comportement asymptotique de $(V^{(\varepsilon)}, X^{(\varepsilon)})$ en régime sur-critique. Le taux de convergence de $V^{(\varepsilon)}$ est donné par celui de $L^{(\varepsilon)}$ et le processus limite est $(\mathcal{L}_t, \int_0^t \mathcal{L}_s ds)_{t \geq 0}$.

Par ailleurs, on montre que si le bruit directeur est un processus de Lévy général, alors la partie brownienne n'amène pas de régularité. Plus précisément, notons $(V^{(1)}, X^{(1)})$ la solution de

$$\begin{cases} dV_t^{(1)} = dL_t - \rho \frac{\text{sgn}(V_t^{(1)}) |V_t^{(1)}|^\gamma}{t^\beta} dt, & V_{t_0}^{(1)} = v_0 \in \mathbb{R}, \\ dX_t^{(1)} = V_t^{(1)} dt, & X_{t_0}^{(1)} = x_0 \in \mathbb{R}, \end{cases} \quad (13)$$

et $(V^{(2)}, X^{(2)})$ la solution de

$$\begin{cases} dV_t^{(2)} = dL_t + dB_t - \rho \frac{\text{sgn}(V_t^{(2)}) |V_t^{(2)}|^\gamma}{t^\beta} dt, & V_{t_0}^{(2)} = v_0 \in \mathbb{R}, \\ dX_t^{(2)} = V_t^{(2)} dt, & X_{t_0}^{(2)} = x_0 \in \mathbb{R}. \end{cases} \quad (14)$$

Alors le processus $(V^{(2)}, X^{(2)})$ a le même comportement que $(V^{(1)}, X^{(1)})$, ce dernier étant donné par le Théorème 2.5.2. La preuve de ce résultat repose sur le fait que

$$(r_{\varepsilon, V} V_{t/\varepsilon}^{(1)} - r_{\varepsilon, V} V_{t/\varepsilon}^{(2)}, r_{\varepsilon, X} X_{t/\varepsilon}^{(1)} - r_{\varepsilon, X} X_{t/\varepsilon}^{(2)})_{t \geq \varepsilon t_0}$$

converge vers 0 en probabilité, uniformément sur tout compact. Ceci découle de l'inégalité de Burkholder-Davis-Gundy ([App09, Théorème 4.4.22 p. 263]) et du lemme de Grönwall.

Bruit multiplicatif brownien

On peut également se demander ce qu'il advient du comportement asymptotique de (SKE_B) lorsque le terme de diffusion est plus général, par exemple,

$$dV_t = \sigma(t, V_t) dB_t - \frac{F(V_t)}{t^\beta} dt, \quad t \geq t_0. \quad (SDE_\sigma)$$

Cela faisait l'objet d'une ouverture de [Off14]. Deux cas sont étudiés dans la Section 1.6 du Chapitre 1. Dans un premier temps, si le coefficient de diffusion σ ne dépend que du temps t , cela revient à faire un changement d'horloge, et la martingale locale

$$M_t^{(\varepsilon)} := \int_{t_0}^{t/\varepsilon} \sigma(s) dB_s$$

est un processus gaussien centré dont on peut étudier le comportement en temps long. Notre schéma de preuve en régime sur-critique s'adapte donc. En revanche, pour les régimes critique et sous-critique, nous faisons appel à un changement de temps et d'espace. Celui-ci ne permet plus de conclure. En effet, quelque soit le choix de φ , le terme de diffusion $\sigma(\varphi(s), V_s^{(\varphi)} \sqrt{\varphi'(s)}) dW_s$ reste inhomogène en temps.

Cela nous amène donc à étudier un deuxième type de coefficient de diffusion. On suppose qu'il existe une fonction f telle que

$$\text{pour tout } (s, v) \in [0, +\infty) \times \mathbb{R}, \quad \sigma(s, v) = f\left(\frac{v}{\sqrt{s}}\right).$$

Il est alors possible d'utiliser le processus changé de temps et d'espace $V^{(\varphi)}$ pour conclure au comportement asymptotique de V en suivant le même schéma de preuve que [GO13, Théorèmes 4.1, 4.6 et 4.9]. Notons qu'on obtient ainsi le comportement asymptotique unidimensionnel de V . Le comportement du processus X reste un problème ouvert.

3 Dynamique stochastique dans un potentiel confinant

Intéressons-nous maintenant à une particule se déplaçant dans un récipient dont la forme est donnée par une fonction \mathcal{U} . Son énergie potentielle est égale à \mathcal{U} à une constante près. Cette nouvelle force s'exerçant sur la particule est conservative, rendant le système hamiltonien. La conservation de l'énergie assure qu'elle dérive (par rapport à la position) de l'énergie potentielle. Dans la Partie II, nous traitons le cas particulier d'un récipient de la forme $\mathcal{U} : x \mapsto \frac{x^2}{2}$. Plus précisément, nous nous intéressons à l'influence du potentiel quadratique sur le comportement asymptotique du système hamiltonien amorti suivant

$$\begin{cases} dV_t = dL_t - \frac{\text{sgn}(V_t) |V_t|^\gamma}{t^\beta} dt - X_t dt, \\ dX_t = V_t dt. \end{cases} \quad (\text{SKE}_c)$$

Remarque 3.1. *Le système (SKE_c) peut également modéliser une particule accrochée à un ressort. Elle est soumise à une force de rappel de la forme $-kx$. Celle-ci dérive de l'énergie potentielle élastique $\frac{1}{2}kx^2$.*

Contrairement à (SKE_L) , ce système est couplé. Il n'est donc plus possible d'étudier la convergence du processus vitesse V sans étudier celle de X . L'existence et l'unicité d'une solution pour ces systèmes dégénérés, dirigés par un mouvement brownien, ont précédemment été étudiées dans [Fed+17], [WZ15], [Zha16] et [HMC18]. Lorsque le processus directeur est un processus de Lévy, on pourra citer [Zha14] où les coefficients sont supposés homogènes en temps, et [MM21] lorsque la dérive dépend du temps.

Le comportement asymptotique de systèmes hamiltoniens perturbés par une force aléatoire est traité dans [AK94]. Le phénomène d'amortissement résultant de la force de frottement b est étudié dans [Wu01].

3.1 Heuristique

Afin de s'affranchir du couplage des deux équations, l'idée est de voir (SKE_c) comme un système en dimension 2. On note, pour tout $t \geq t_0$ et $v \in \mathbb{R}$,

$$Z_t := \begin{pmatrix} X_t \\ V_t \end{pmatrix}, \quad S_t := \begin{pmatrix} 0 \\ L_t \end{pmatrix}, \quad A := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \Gamma := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{et} \quad F(t, v) := \begin{pmatrix} 0 \\ \text{sgn}(v) \frac{|v|^\gamma}{t^\beta} \end{pmatrix}.$$

Alors, le système (SKE_c) s'écrit

$$dZ_t = \Gamma dS_t + AZ_t dt - F(t, V_t) dt. \quad (15)$$

Remarquons que A est la matrice de rotation d'angle $\frac{\pi}{2}$ et que e^{tA} est la matrice de rotation d'angle $-t$. On voit ainsi apparaître l'oscillateur harmonique sous-jacent au modèle

$$\begin{cases} v'_t = -x_t \\ x'_t = v_t. \end{cases}$$

L'oscillateur empêche le processus $Z^{(\varepsilon)}$ de converger, car celui-ci possède une composante angulaire. Il est en revanche possible d'étudier la convergence de ses marginales unidimensionnelles. Cependant, pour espérer obtenir une convergence en tant que processus, il est nécessaire de retirer la partie oscillante. C'est pourquoi on s'intéresse au processus $Y_t := e^{-tA}(V_t, X_t)^T$, à la place de Z_t .

Comme en l'absence de potentiel, on met en évidence trois régimes asymptotiques, en fonction de la position des coefficients de frottement γ et β par rapport à α . Cependant, le processus limite n'est plus cinétique, en effet le potentiel quadratique sous-jacent au modèle facilite la propagation du bruit sur la composante position. Le taux de convergence et la limite de la composante position sont donc différents.

Le changement de régimes est illustrée par la Figure 8 dans le cas d'un système dirigé par un mouvement brownien, et par la Figure 9 lorsque le processus directeur est un processus de Lévy α -stable.

3.2 Résultats et discussions

Le processus Y est donné par

$$dY_t = e^{-tA}\Gamma dS_t - e^{-tA}F(t, V_t) dt.$$

Par conséquent, son étude se ramène à celle d'une particule libre, mais en dimension 2. La robustesse du schéma de preuve en dimension supérieure nous est utile.

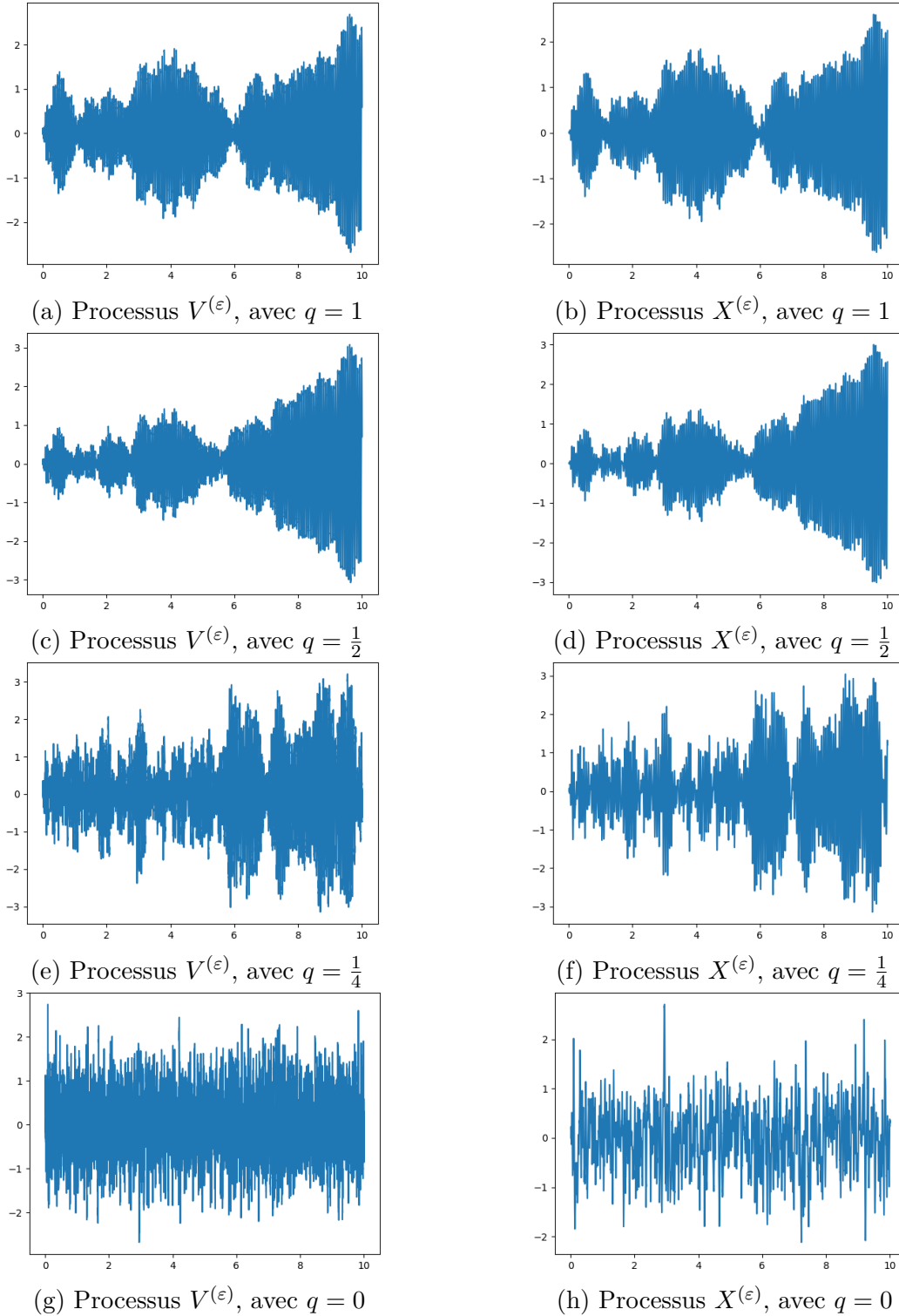


Figure 8 – Exemples de trajectoire des processus vitesse (à gauche) et position (à droite) renormalisés, dans le cas d’un système linéaire dans un potentiel quadratique, et dirigé par la même réalisation d’un mouvement brownien. [Lui22c] est un groupe de vidéos illustrant l’évolution des processus, en fonction de q .

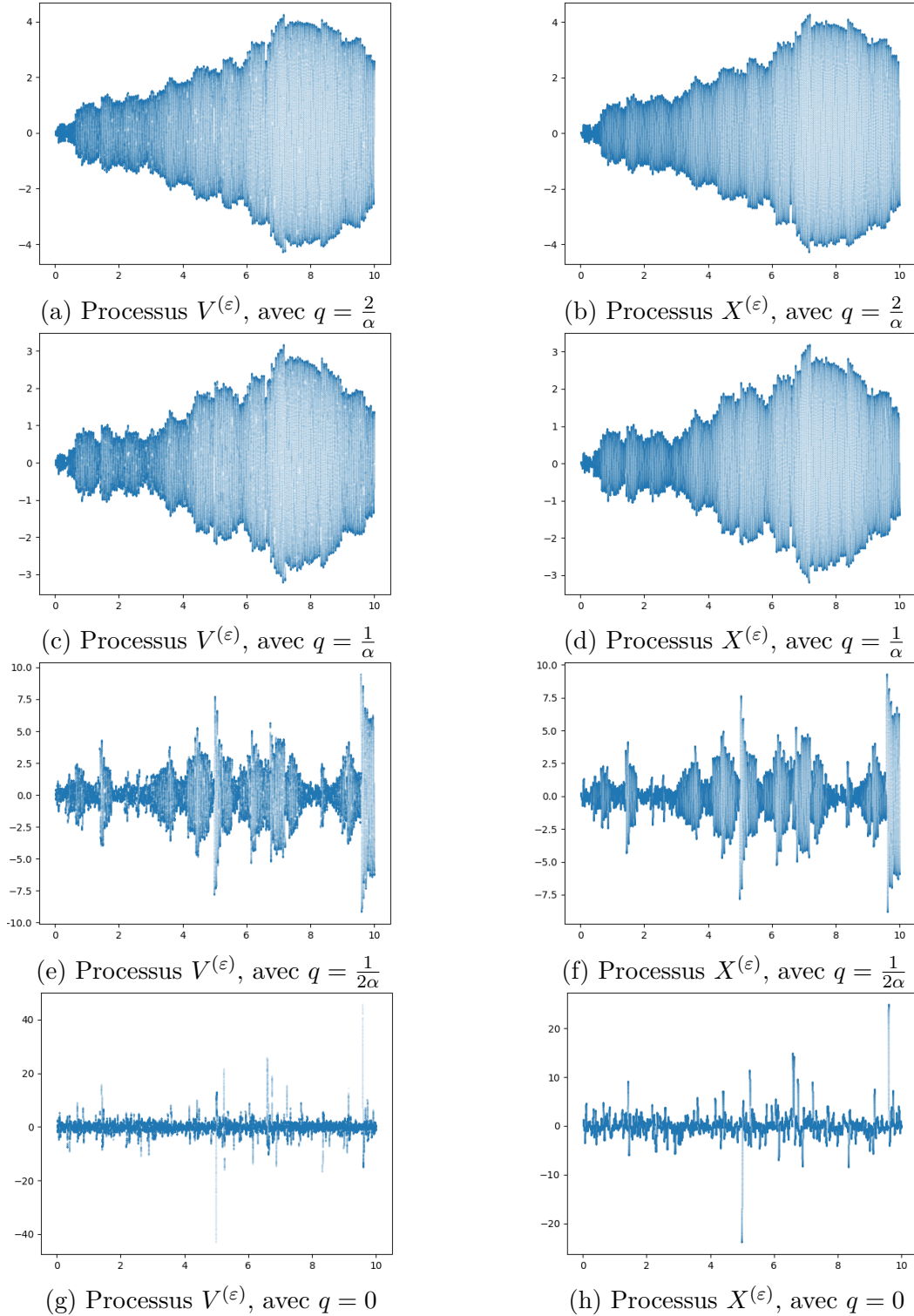


Figure 9 – Exemples de trajectoire des processus vitesse (à gauche) et position (à droite) renormalisés, dans le cas d'un système linéaire dans un potentiel quadratique, et dirigé par la même réalisation d'un (1.3)-stable. [Lui22c] est un groupe de vidéos illustrant l'évolution des processus, en fonction de q .

Existence, unicité, explosion et estimée des moments

Lorsque la dérive est localement lipschitzienne, on conclut par des résultats classiques à l'existence d'une solution locale. Lorsque la dérive est seulement h"olderienne, l'étude est plus complexe. Pour conclure, on utilise [HMC18, Théorème 1] et [MM21, Théorème 1]. La non-explosion découle, comme dans le cas d'une particule libre, de l'estimée des moments, via le Lemme 2.3. Le calcul des estimées s'adapte directement de la preuve des Chapitres 1 et 2 à la dimension supérieure.

Comportement asymptotique

Citons les principaux résultats obtenus.

Théorème 3.1.1. *Soient $\alpha = 2$ et $\gamma \geq 0$. Soit $(V_t, X_t)_{t \geq t_0}$ la solution de (SKE_c) . On pose $q = \frac{\beta}{\gamma + 1}$. Soit \mathcal{B} un mouvement brownien sur \mathbb{R}^2 .*

- **Régime sur-critique.** *Supposons $2q > 1$. Alors le processus $(\sqrt{\varepsilon} Y_{t/\varepsilon})_{t \geq \varepsilon t_0}$ converge vers $(\mathcal{B}_{\frac{t}{2}})_{t > 0}$.*
- **Régime critique.** *Supposons $2q = 1$ et $\gamma = 1$. Alors le processus $(\sqrt{\varepsilon} Y_{t/\varepsilon})_{t \geq \varepsilon t_0}$ converge vers le processus gaussien $(\frac{1}{\sqrt{2t}} \int_0^t \sqrt{s} d\mathcal{B}_s)_{t > 0}$.*
- **Régime sous-critique.** *Supposons $2q < 1$, $\gamma = 1$ et $\beta > \frac{1}{2}$. Alors les lois de dimension finie de $(\varepsilon^{\frac{\beta}{2}} Y_{t/\varepsilon})_{t \geq \varepsilon t_0}$ convergent vers celles du processus gaussien centré de noyau de covariance donné, pour $(s, t) \in (0, +\infty)^2$, par $K(s, t) = \frac{1}{2} s^\beta \mathbb{1}_{\{s=t\}}$.*

Théorème 3.1.2. *Soit $\alpha \in (0, 2)$. Supposons que L est un processus de Lévy α -stable symétrique, d'exposant caractéristique $\xi \mapsto -a \|\xi\|^\alpha$ pour $a > 0$. Soient $\gamma \in (0, \alpha)$ et $(V_t, X_t)_{t \geq t_0}$ la solution de (SKE_c) . On pose $q := \frac{\beta}{\gamma + \alpha - 1}$. Soit \mathcal{L} un processus de Lévy α -stable sur \mathbb{R}^2 , invariant par rotation, d'exposant caractéristique*

$$\xi \in \mathbb{R}^2 \mapsto -\tilde{C} \|\xi\|^\alpha, \quad \text{avec } \tilde{C} := \frac{a}{2\pi} \int_0^{2\pi} |\cos(x)|^\alpha dx.$$

- **Régime sur-critique.** *Supposons $\alpha \in (1, 2)$ et $\alpha q > 1$. Alors le processus $(\varepsilon^{\frac{1}{\alpha}} Y_{t/\varepsilon})_{t \geq \varepsilon t_0}$ converge vers $(\mathcal{L}_t)_{t > 0}$.*
- **Régime critique.** *Supposons $\alpha q = 1$ et $\gamma = 1$. Alors le processus $(\varepsilon^{\frac{1}{\alpha}} Y_{t/\varepsilon})_{t \geq \varepsilon t_0}$ converge vers le processus $(\frac{1}{\sqrt{t}} \int_0^t \sqrt{s} d\mathcal{L}_s)_{t > 0}$.*

- **Régime sous-critique.** Supposons $\alpha q < 1$, $\gamma = 1$ et $\beta > \frac{1}{2}$. Alors, pour tout $(t_1, \dots, t_d) \in (0, +\infty)^d$, les marginales de rang fini $(\varepsilon^{\frac{\beta}{\alpha}} Y_{t_1/\varepsilon}, \dots, \varepsilon^{\frac{\beta}{\alpha}} Y_{t_d/\varepsilon})$ convergent en loi vers la mesure produit $\mu_{t_1} \otimes \dots \otimes \mu_{t_d}$, où μ_t est la loi de fonction caractéristique

$$\xi \mapsto \exp\left(-\frac{2}{\alpha} \tilde{C} \|\xi\|^\alpha t^\beta\right).$$

Le potentiel \mathcal{U} confine la particule. Cela se lit, par exemple dans le régime sur-critique brownien, sur la variance du processus limite : X_t se comporte asymptotiquement comme une loi $\mathcal{N}(0, \frac{t}{2})$ en potentiel confinant, mais comme une loi $\mathcal{N}(0, \frac{t^3}{3})$ en l'absence de potentiel.

Notons également que les composantes vitesse et position de Z (*i.e.* un mélange des processus vitesse et position initiaux) deviennent asymptotiquement indépendantes lorsque le processus directeur est brownien, ce qui n'est pas le cas si ce dernier est un processus de Lévy α -stable.

Détaillons les schémas de preuve.

Régime sur-critique. On retrouve l'idée utilisée dans le régime sur-critique pour une particule libre. On définit, pour $t \geq \varepsilon t_0$,

$$M_t^{(\varepsilon)} := \varepsilon^{\frac{1}{\alpha}} \int_{t_0}^{t/\varepsilon} e^{-sA} \Gamma dS_s = \varepsilon^{\frac{1}{\alpha}} \int_{t_0}^{t/\varepsilon} \begin{pmatrix} -\sin(s) \\ \cos(s) \end{pmatrix} dL_s.$$

On étudie dans un premier temps la convergence des marginales de rang fini du processus $M^{(\varepsilon)}$. La théorie des processus gaussiens permet de conclure lorsque $\alpha = 2$. En effet, rappelons que la convergence d'un processus gaussien est caractérisée par la convergence de son espérance et de son noyau de covariance. Lorsque $\alpha < 2$, la convergence n'a plus de caractérisation aussi simple. Cependant, $M^{(\varepsilon)}$ est une intégrale de Wiener-Lévy, c'est à dire l'intégrale d'une fonction déterministe contre un processus α -stable, ainsi ses accroissements sont indépendants. On peut donc se ramener à l'étude de la convergence des accroissements de l'intégrale, via leur fonction caractéristique.

On vérifie ensuite un critère de tension : le critère de Kolmogorov ([Kal02, Corollaire 16.9 p. 313]) ou d'Aldous ([Bil99, Théorème 16.10 p. 178]).

Enfin, en notant $r := \min(\beta - 1 + \frac{1-\gamma}{\alpha}, \frac{1}{\alpha}) > 0$, on montre à l'aide de l'estimée des moments

que

$$\mathbb{E} \left[\sup_{\varepsilon t_0 \leq t \leq T} |Y_t^{(\varepsilon)} - M_t^{(\varepsilon)}| \right] = \begin{cases} O_{\varepsilon \rightarrow 0}(\varepsilon^r) & \text{si } \beta \neq 1 + \frac{\gamma}{\alpha}, \\ O_{\varepsilon \rightarrow 0}(\varepsilon^{\frac{1}{\alpha}} |\ln(\varepsilon)|) & \text{sinon.} \end{cases}$$

Régimes critique et sous-critique. Ces deux régimes sont plus complexes qu'en l'absence de potentiel confinant. En effet, les composantes vitesse et position étant couplées, une transformation d'échelles ne permet plus de conclure. Pour passer outre cette difficulté, nous traitons le cas linéaire $\gamma = 1$. On voit cette fois-ci apparaître l'oscillateur harmonique amorti

$$x''(t) + \frac{x'(t)}{t^\beta} + x(t) = 0, \quad t \geq t_0. \quad (16)$$

La stratégie de preuve repose alors sur la théorie des équations différentielles ordinaires. En effet, on peut étudier le comportement asymptotique d'un système fondamental de solutions de (16). Notons que si $\beta = 1$, le résultat est déjà connu, puisque les fonctions de Bessel de première et de deuxième espèce d'ordre 0 forment une base de l'espace des solutions.

Notons R la résolvante associée à (16). En notant f l'ordre de grandeur de la résolvante, et en définissant, pour tout $t \geq 0$,

$$\Phi_t := \frac{e^{-tA} R_t}{f(t)} \quad \text{et} \quad M_t^{(\varepsilon)} := v_\varepsilon f(t/\varepsilon) \int_{t_0}^{t/\varepsilon} R_s^{-1} \Gamma \, dW_s,$$

on peut décomposer le processus renormalisé sous la forme

$$Y_t^{(\varepsilon)} = v_\varepsilon f(t/\varepsilon) \Phi_{t/\varepsilon} R_{t_0}^{-1} Z_0 + \Phi_{t/\varepsilon} \widetilde{M}_t^{(\varepsilon)}.$$

La convergence de $Y^{(\varepsilon)}$ découle de la convergence de Φ vers la matrice identité I_2 et de celle de $\widetilde{M}^{(\varepsilon)}$, en utilisant le développement asymptotique des solutions de (16). On conclut à la convergence de $\widetilde{M}^{(\varepsilon)}$ avec les mêmes outils que dans le régime sur-critique.

4 Perspectives de recherche

On présente dans cette section quelques pistes pouvant ouvrir la voie à de futurs travaux.

4.1 Généralisation à une force de frottement non linéaire

Dans les Chapitres 1 et 2, nous avons obtenu la convergence du processus position dans le cas d'une force de frottement linéaire. De plus, la preuve du comportement asymptotique du processus vitesse-position dans le potentiel quadratique, en régimes critique et sous-critique, repose sur la linéarité de la force de frottement F .

Le cas d'une force de frottement plus générale fait l'objet de travaux en cours.

4.2 Force aléatoire de Feller

Puisque les accroissements d'un processus de Lévy L sont indépendants et stationnaires, le comportement des sauts de L en t ne dépend ni de t ni de L_{t-} . En ce sens, les processus de Feller sont une généralisation des processus de Lévy. En effet, on associe à un processus de Feller, une famille de triplet $\{(b(x), Q(x), \nu(x, dz)), x \in \mathbb{R}\}$. Son générateur infinitésimal est donné, pour f assez régulière, par

$$\mathcal{A}f(x) = b(x)\nabla f(x) + \frac{1}{2}Q(x)\Delta f(x) + \int_{z \neq 0} \left[f(x+z) - f(x) - \nabla f(x)z\mathbb{1}_{|z|<1} \right] \nu(x, dz).$$

Ainsi, un processus de Feller se comporte localement comme un processus de Lévy. C'est pourquoi, on appelle parfois ces processus, des processus de type Lévy. On pourra se référer, par exemple, à [Küh] pour plus de détails sur ces processus.

On peut donc se demander ce que devient le comportement du système (SKE_L) lorsque le processus directeur est un processus de Feller général. Par exemple, on peut s'intéresser à un processus de Feller qui se comporte localement comme un processus α -stable. Pour une fonction hölderienne $\alpha : \mathbb{R} \rightarrow (0, 2]$, ne s'approchant pas de zéro, le générateur d'un tel processus est donné par

$$\mathcal{A}f(x) = \int_{\{z \neq 0\}} \left[f(x+z) - f(x) - \nabla f(x)z\mathbb{1}_{\{|z|<1\}} \right] \frac{c_{\alpha(x)}}{|z|^{1+\alpha(x)}} dz.$$

Cependant, notons que comme pour les processus de Lévy généraux, nous perdons à priori la propriété d'auto-similarité, qui est essentielle dans nos preuves.

4.3 Généralisation de la force de frottement et du potentiel

Plusieurs modèles peuvent être considérés. Une première idée consiste à généraliser (SKE_c) à une particule se déplaçant dans un paysage plus général. Par exemple, qu'en est-il s'il y a un double puits du type $\mathcal{U} : x \mapsto x^4 - x^2$?

On peut également s'intéresser au système (SKE_L) où la force de frottement dépend de la position, ou ne possède pas un mais deux points fixes stables.

4.4 Force de frottement aléatoire

Un autre axe de recherche consiste à s'intéresser à une force de frottement aléatoire. Soient $(B_t)_{t \geq 0}$ un mouvement brownien standard et $\beta \in \mathbb{R}$. Soit $(W(x))_{x \in \mathbb{R}}$ un mouvement brownien sur \mathbb{R} , indépendant de B . Il est défini, à partir de deux mouvements browniens standards indépendants W^+ et W^- , par

$$W(x) := W_x^+ \mathbb{1}_{\{x > 0\}} + W_{-x}^- \mathbb{1}_{\{x < 0\}}, \quad x \in \mathbb{R}.$$

On se place en environnement aléatoire et on considère la solution formelle du système

$$\begin{cases} dV_t = dB_t - \frac{1}{2} \frac{W'(V_t)}{t^\beta} dt, \\ dX_t = V_t dt. \end{cases} \quad (17)$$

Le processus vitesse a été étudié dans [Off14]. Heuristiquement, à environnement fixé, le processus position X semble se comporter comme le système (SKE_B) avec $\gamma = -\frac{1}{2}$. Le cas $\gamma < 0$ n'a pas été étudié, mais on s'attend à ce que les résultats soient similaires à ceux obtenus dans le Chapitre 1.

Puisque pour tout $t \geq 0$, on a l'invariance $T_t W(x) := e^{-\frac{t}{4}} W(e^{\frac{t}{2}} x) \stackrel{\mathcal{L}}{=} W(x)$, le régime critique ($\beta = \frac{1}{4}$) fait apparaître l'EDS formelle, qui est, en loi, homogène en temps,

$$dH_t = d\tilde{B}_t - \frac{1}{2} \partial_x F(t, H_t) dt, \quad \text{où } F(t, x) := \frac{x^2}{2} + T_t W(x).$$

On devrait donc pouvoir adapter la preuve du régime critique du Chapitre 1 dans ce cas. Le régime sur-critique, pourtant le plus simple dans ce manuscrit, semble plus complexe puisque l'écriture (17) est formelle.

PART I

Time-inhomogeneous stochastic dynamics in a free potential

Contents

1	Behavior of a time-inhomogeneous Kolmogorov type diffusion	45
1.1	Introduction	46
1.2	Notations and main results	48
1.3	Existence and non-explosion of solution	50
1.4	Moment estimates of the velocity process	57
1.5	Proof of the asymptotic behavior of the solution	59
1.5.1	Asymptotic behavior in the super-critical regime	59
1.5.2	Asymptotic behavior in the critical regime	62
1.5.3	Asymptotic behavior in the sub-critical regime	67
1.6	An extension to multiplicative noise	71
1.6.1	Clock change	71
1.6.2	Diffusion coefficient with scaling property	72
1.A	Some technical results	75
2	Behavior of a time-inhomogeneous kinetic Lévy-driven model	79
2.1	Introduction	80
2.2	Notations and main results	82
2.3	Existence up to explosion	85
2.4	Moment estimates and non-explosion of the velocity process	86
2.5	Proof of the asymptotic behavior of the solution	94
2.5.1	Asymptotic behavior in the super-critical regime	95
2.5.2	Asymptotic behavior in the critical regime	96
2.5.3	Asymptotic behavior in the sub-critical regime	102
2.6	Extended results in the Lévy case	104
2.6.1	Large time behavior of the Lévy driving process	105
2.6.2	Proofs of Theorems 2.6.1 and 2.6.2	110
2.6.3	Lévy driving process with a Brownian component	116
2.A	Some technical results	118

BEHAVIOR OF A TIME-INHOMOGENEOUS KOLMOGOROV TYPE DIFFUSION

Abstract: We study a kinetic stochastic model with a non-linear time-inhomogeneous friction force and a Brownian-type random force. More precisely, a Kolmogorov type diffusion (V, X) is considered: here, X is the position of the particle, and V is its velocity. The process V is solution to a stochastic differential equation driven by a one-dimensional Brownian motion, with a drift of the form $t^{-\beta}F(v)$. The function F satisfies some homogeneity condition, and β is a real number. The behavior in large time of the process (V, X) is proved by using stochastic analysis tools.

Keywords: kinetic stochastic equation; time-inhomogeneous diffusion; explosion time; scaling transformation; asymptotic distribution; ergodicity.

MSC2010 Subject Classification: Primary 60J60; Secondary 60H10; 60J65; 60F17.

This chapter is based on the article [GL21a], submitted.

1.1 Introduction

In several domains as fluids dynamics, statistical mechanics, or biology, a number of models are based on the Fokker-Planck and Langevin equations driven by Brownian motion, could be non-linear or driven by other random processes. For example, in [CCM10] the persistent turning walker model was introduced, inspired by the modelling of fish motion. An associated two-component Kolmogorov type diffusion solves a kinetic system based on an Ornstein-Uhlenbeck Gaussian process, and the authors study the large-time behavior of this model by using appropriate tools from stochastic analysis. One of the natural questions is to understand the behavior in large-time of the solution to the corresponding stochastic differential equation (SDE). Although the tools of partial differential equations allow us to ask this kind of question, and since these models are probabilistic, tools based on stochastic processes could be more natural to use.

In the last decade, the asymptotic study of solutions to non-linear Langevin's type was the subject of an important number of papers, see [CNP19], [EG15] and [FT21]. For instance, in [FT21] the following system is studied

$$V_t = v_0 + B_t - \frac{\rho}{2} \int_0^t F(V_s) ds \quad \text{and} \quad X_t = x_0 + \int_0^t V_s ds.$$

In other words, one considers a particle moving such that its velocity is a diffusion with an invariant measure behaving like $(1 + |v|^2)^{-\rho/2}$, as $|v| \rightarrow +\infty$. The authors prove that for large-time, after a suitable rescaling, the position process behaves as a Brownian motion or other stable processes, following the values of ρ . Results have been extended to additive functional of V in [Bét21]. It should be noticed that these cited papers use the standard tools associated with time-homogeneous equations, as invariant measure, scale function, and speed measure. Several of them will not be available when the drag force depends explicitly on time. In [GO13], a non-linear SDE driven by a Brownian motion but having time-inhomogeneous drift coefficient was studied, and its large-time behavior was described. Moreover, sharp rates of convergence are proved for the 1-dimensional marginal of the solution. In the present paper, we consider the velocity process as satisfying the same kind of SDE.

Let us describe our framework: we consider a one-dimensional time-inhomogeneous stochastic kinetic model driven by a Brownian motion. We denote by $(X_t)_{t>0}$ the process describing the position of a particle at time t and having the velocity V_t . The velocity

process $(V_t)_{t>0}$ is supposed to solve a Brownian-driven SDE with a drag force, varying in time:

$$dV_t = dB_t - b(t, V_t) dt \quad \text{and} \quad X_t = X_0 + \int_0^t V_s ds.$$

This system can be viewed as a perturbation of the classical two-component Kolmogorov diffusion

$$dV_t = dB_t \quad \text{and} \quad X_t = X_0 + \int_0^t V_s ds.$$

In the present paper the drift is supposed to grow slowly to infinity, and it will be supposed to be of the form $t^{-\beta}F(v)$, with $\beta \in \mathbb{R}$ and F satisfying some homogeneity condition. It describes a one-dimensional particle subject to a friction force and undergoing many small random shocks. A natural question is to understand the behavior of the process (V, X) in large time. More precisely we look for the limit in distribution of $(r_{\varepsilon, V}V_{t/\varepsilon}, r_{\varepsilon, X}X_{t/\varepsilon})_t$, as $\varepsilon \rightarrow 0$, for some rates of convergence. Our results are proved on the product of path spaces and consequently contain those of [GO13].

If $F = 0$, it is not difficult to see that the rescaled position process $(\varepsilon^{\frac{1}{2}}V_{t/\varepsilon}, \varepsilon^{\frac{3}{2}}X_{t/\varepsilon})_t$ converges in distribution towards the Kolmogorov diffusion $(B_t, \int_0^t B_s ds)_t$. We prove that this kinetic behavior still holds for sufficiently “small at infinity” drag force. The strategy to tackle this problem is based on estimates of moments of the velocity process. The main result can then be extended to the case of a drift being equally weighted in some sense as the random noise. It either offsets the random noise (critical regime) or swings with it (sub-critical regime).

As suggested at the beginning of the introduction, other random noises can be considered. In [GL21b], the case of a Lévy random noise is analyzed. The case of a stochastic system evolving in a quadratic potential is the purpose of another work (see [CL]).

The organization of the chapter is as follows: in the next section, we introduce notations, and we state our main results. Results about existence and non-explosion of solutions are stated in Section 1.3. Estimates of the moments of the velocity process are given in Section 1.4 while the proofs of our main results are presented in Section 1.5. We deal with an extension to multiplicative noise in the last section.

1.2 Notations and main results

Let $(B_t)_{t \geq 0}$ be a standard Brownian motion, β a real number and F a continuous function which is supposed to satisfy

$$\text{for some } \gamma \in \mathbb{R}, \forall v \in \mathbb{R}, \lambda > 0, F(\lambda v) = \lambda^\gamma F(v). \quad (H_\gamma)$$

In the following, sgn is the sign function with convention $\text{sgn}(0) = 0$. As an example of function satisfying (H_γ) one can keep in mind $F : v \mapsto \text{sgn}(v) |v|^\gamma$ (see also [GO13]).

Remark 1.2.1. *If a function π satisfies (H_γ) , then for all $x \in \mathbb{R}$, $\pi(x) = \pi(\text{sgn}(x)) |x|^\gamma$.*

We consider the following one-dimensional stochastic kinetic model, for $t \geq t_0 > 0$,

$$dV_t = dB_t - t^{-\beta} F(V_t) dt, \quad V_{t_0} = v_0 > 0, \quad \text{and} \quad dX_t = V_t dt, \quad X_{t_0} = x_0 \in \mathbb{R}. \quad (\text{SKE}_B)$$

Most of the convergences take place in the space of continuous functions $\mathcal{C}((0, +\infty), \mathbb{R})$ endowed by the uniform topology

$$d_u : (f, g) \in \mathcal{C}((0, +\infty), \mathbb{R})^2 \mapsto \sum_{n=1}^{+\infty} \frac{1}{2^n} \min \left(1, \sup_{[\frac{1}{n}, n]} |f - g| \right).$$

For a family $((Z_t^{(\varepsilon)})_{t>0})_{\varepsilon>0}$ of continuous processes, we write

$$(Z_t^{(\varepsilon)})_{t>0} \xrightarrow[\varepsilon \rightarrow 0]{} (Z_t)_{t>0},$$

if $(Z_t^{(\varepsilon)})_{t>0}$ converges in distribution to $(Z_t)_{t>0}$ in $\mathcal{C}((0, +\infty), \mathbb{R})$, as $\varepsilon \rightarrow 0$.

We write

$$(Z_t^{(\varepsilon)})_{t>0} \xrightarrow[\varepsilon \rightarrow 0]{\text{f.d.d.}} (Z_t)_{t>0},$$

if for all finite subsets $S \subset (0, +\infty)$, the vector $(Z_t^{(\varepsilon)})_{t \in S}$ converges in distribution to $(Z_t)_{t \in S}$ in \mathbb{R}^S , as $\varepsilon \rightarrow 0$.

Let us state our main results. Set $q := \frac{\beta}{\gamma + 1}$.

Theorem 1.2.2. *Consider $\gamma \geq 0$, and $q > \frac{1}{2}$. Assume that (H_γ) is satisfied. Let $(V_t, X_t)_{t \geq t_0}$ be the solution to (SKE_B) and $(\mathcal{B}_t)_{t \geq 0}$ be a standard Brownian motion. Furthermore, if $\gamma \geq 1$, we suppose that for all $v \in \mathbb{R}$, $vF(v) \geq 0$.*

Then,

$$\left(\varepsilon^{\frac{1}{2}}V_{t/\varepsilon}, \varepsilon^{\frac{3}{2}}X_{t/\varepsilon}\right)_{t \geq \varepsilon t_0} \xrightarrow{\varepsilon \rightarrow 0} \left(\mathcal{B}_t, \int_0^t \mathcal{B}_s \, ds\right)_{t \geq 0}.$$

Theorem 1.2.3. Consider $\gamma \geq 0$ and $q = \frac{1}{2}$. Assume that (H_γ) is satisfied. Let $(V_t, X_t)_{t \geq t_0}$ be the solution to (SKE_B) . If $\gamma \geq 1$, we suppose furthermore that for all $v \in \mathbb{R}$, $vF(v) \geq 0$. Call \widetilde{H} the eternal ergodic process, solution to the homogeneous SDE

$$dH_s = dW_s - \frac{H_s}{2} \, ds - F(H_s) \, ds,$$

such that the distribution of $\widetilde{H}_{-\infty}$ is the invariant measure, where $(W_t)_{t \geq 0}$ is again a standard Brownian motion. Setting $\Lambda_{F, t_1, \dots, t_d}$ for the finite dimensional distributions (f.d.d.) of \widetilde{H} , we call $(\mathcal{V}_t)_{t \geq 0}$ the process whose f.d.d. are $T * \Lambda_{F, \log(t_1), \dots, \log(t_d)}$, the pushforward measure of $\Lambda_{F, \log(t_1), \dots, \log(t_d)}$ by the linear map $T(u_1, \dots, u_d) := (\sqrt{t_1}u_1, \dots, \sqrt{t_d}u_d)$. Indeed, we have $(\mathcal{V}_t)_{t \geq 0} = (\sqrt{t}\widetilde{H}_{\log(t)})_{t \geq 0}$.

Then,

$$\left(\varepsilon^{\frac{1}{2}}V_{t/\varepsilon}, \varepsilon^{\frac{3}{2}}X_{t/\varepsilon}\right)_{t \geq \varepsilon t_0} \xrightarrow{\varepsilon \rightarrow 0} \left(\mathcal{V}_t, \int_0^t \mathcal{V}_s \, ds\right)_{t \geq 0}.$$

Remark 1.2.4. The one-dimensional distribution of $(\mathcal{V}_t)_{t \geq 0}$ has already been explicitly computed (see Theorem 4.1 in [GO13]).

Theorem 1.2.5. Consider $\gamma \geq 1$ and $q < \frac{1}{2}$. Assume that $F : v \mapsto \rho \operatorname{sgn}(v) |v|^\gamma$ with $\rho > 0$. Let $(V_t, X_t)_{t \geq t_0}$ be the solution to (SKE_B) . Call \widehat{H} the ergodic process, solution to the homogeneous SDE

$$dH_s = dW_s - F(H_s) \, ds,$$

where $(W_t)_{t \geq 0}$ is a standard Brownian motion. Call Π_F its invariant measure. We call $(\mathcal{Y}_t)_{t \geq 0}$ the process whose f.d.d. are $T * (\Pi_F^{\otimes d})$, the pushforward measure of $\Pi_F^{\otimes d}$ by the linear map $T(u_1, \dots, u_d) := (t_1^q u_1, \dots, t_d^q u_d)$.

Then,

$$\left(\varepsilon^q V_{t/\varepsilon}\right)_{t \geq \varepsilon t_0} \xrightarrow[\varepsilon \rightarrow 0]{\text{f.d.d.}} (\mathcal{Y}_t)_{t \geq 0}.$$

Moreover, in the linear case (i.e. $\gamma = 1$) and if $\beta > -\frac{1}{2}$, we define $(\mathcal{X}_t)_{t \geq 0}$ the centered Gaussian process with covariance function $K(s, t) := (\rho^2(1 + 2\beta))^{-1}(s \wedge t)^{1+2\beta}$.

Then, as $\varepsilon \rightarrow 0$,

$$\left(\varepsilon^{\beta + \frac{1}{2}} X_{t/\varepsilon}\right)_{t \geq \varepsilon t_0} \xrightarrow[\varepsilon \rightarrow 0]{\text{f.d.d.}} (\mathcal{X}_t)_{t \geq 0}. \quad (1.1)$$

Remark 1.2.6. If $\beta = 0$, one can prove using the martingale method, that $(\sqrt{\varepsilon}X_{t/\varepsilon})_{t \geq 0}$ converges towards a Brownian motion. Assume, by way of contradiction, that the process

$(\varepsilon^q V_{t/\varepsilon})_{t \geq \varepsilon t_0}$ would converge (i.e. were tight), then by the continuous mapping theorem, the process $(\varepsilon X_{t/\varepsilon})_{t \geq 0}$ should converge. This is a contradiction with (1.1). Here is why we deal only with the finite-dimensional convergence of the velocity process.

Remark 1.2.7. Let us give a comment on a generalization to greater dimension. Fix $d \geq 1$. We denote by a dot the usual scalar product and $\|\cdot\|$ its associated norm. Consider the following one-dimensional stochastic kinetic model driven by a d -dimensional standard Brownian motion, defined, for $t \geq t_0 > 0$, by

$$dV_t = dB_t - \frac{V_t \|V_t\|^{\gamma-1}}{t^\beta} dt, \quad V_{t_0} = v_0 \in \mathbb{R}^d, \quad \text{and} \quad dX_t = V_t dt, \quad X_{t_0} = x_0 \in \mathbb{R}^d. \quad (\text{SKE}^d)$$

Note that the drift function $F : v \mapsto v \|v\|^{\gamma-1}$ satisfies

$$\forall v \in \mathbb{R}, \quad v \cdot F(v) \geq 0.$$

Replacing $|\cdot|$ by $\|\cdot\|$, one could prove a similar result as in dimension one.

1.3 Existence and non-explosion of solution

In this section, we will prove the existence of solution to (SKE_B) up to explosion time and the non explosion of such solution with additional assumption.

In the following, we suppose that $\gamma > -1$ and set $\Omega = \overline{\mathcal{C}}([t_0, +\infty))$ the set of continuous functions, that equal $+\infty$ after their (possibly infinite) explosion time. Following the idea used in [GO13], we first perform a change of time in (SKE_B) in order to produce at least one time-homogeneous coefficient in the transformed equation. For every \mathcal{C}^2 -diffeomorphism $\varphi : [0, t_1) \rightarrow [t_0, +\infty)$, let introduce the scaling transformation Φ_φ defined, for $\omega \in \Omega$, by

$$\Phi_\varphi(\omega)(s) := \frac{\omega(\varphi(s))}{\sqrt{\varphi'(s)}}, \quad \text{with } s \in [0, t_1).$$

The result containing the change of time transformation can be found in Proposition 2.1 p. 187 in [GO13].

Let V be solution to the equation (SKE_B) . Thanks to Lévy's characterization theorem of the Brownian motion, $(W_t)_{t \geq 0} := \left(\int_0^t \frac{dB_{\varphi(s)}}{\sqrt{\varphi'(s)}} \right)_{t \geq 0}$ is a standard Brownian motion. Then,

by a change of variable $t = \varphi(s)$, one gets

$$V_{\varphi(t)} - V_{\varphi(0)} = \int_0^t \sqrt{\varphi'(s)} dW_s - \int_0^t \frac{F(V_{\varphi(s)})}{\varphi(s)^\beta} \varphi'(s) ds.$$

The integration by parts formula yields

$$d \left(\frac{V_{\varphi(s)}}{\sqrt{\varphi'(s)}} \right) = dW_s - \frac{\sqrt{\varphi'(s)}}{\varphi(s)^\beta} F(V_{\varphi(s)}) ds - \frac{\varphi''(s)}{2\varphi'(s)} \frac{V_{\varphi(s)}}{\sqrt{\varphi'(s)}} ds.$$

As a consequence, we can state the following result in our context.

Proposition 1.3.1. *If V is a solution to the equation (SKE_B) , then $V^{(\varphi)} := \Phi_\varphi(V)$ is a solution to*

$$dV_s^{(\varphi)} = dW_s - \frac{\sqrt{\varphi'(s)}}{\varphi(s)^\beta} F(\sqrt{\varphi'(s)} V_s^{(\varphi)}) ds - \frac{\varphi''(s)}{\varphi'(s)} \frac{V_s^{(\varphi)}}{2} ds, \quad V_0^{(\varphi)} = \frac{V_{\varphi(0)}}{\sqrt{\varphi'(0)}}, \quad (1.2)$$

where $W_t = \int_0^t \frac{dB_{\varphi(s)}}{\sqrt{\varphi'(s)}}$.

If $V^{(\varphi)}$ is a solution to (1.2), then $\Phi_\varphi^{-1}(V^{(\varphi)})$ is a solution to the equation (SKE_B) , where

$$B_t - B_{t_0} := \int_{t_0}^t \sqrt{(\varphi' \circ \varphi^{-1})(s)} dW_{\varphi^{-1}(s)}.$$

Furthermore, uniqueness in law, pathwise uniqueness or strong existence hold for the equation (SKE_B) if and only if they hold for the equation (1.2).

In the following, we will use two particular changes of time, depending on which term of (1.2) should become time-homogeneous.

- *The exponential change of time:* Setting $\varphi_e : t \mapsto t_0 e^t$, the exponential scaling transformation is defined by $\Phi_e(\omega) : s \in \mathbb{R}^+ \mapsto \frac{\omega_{t_0 e^s}}{\sqrt{t_0 e^{\frac{s}{2}}}}$, for $\omega \in \Omega$. Thanks to Proposition 1.3.1, the process $V^{(e)} := \Phi_e(V)$ satisfies the equation

$$dV_s^{(e)} = dW_s - \frac{V_s^{(e)}}{2} ds - t_0^{\frac{1}{2}-\beta} e^{(\frac{1}{2}-\beta)s} F(\sqrt{t_0 e^{\frac{s}{2}}} V_s^{(e)}) ds,$$

where $(W_t)_{t \geq 0}$ is a standard Brownian motion.

- *The power change of time:* for $q = \frac{\beta}{\gamma+1}$, consider $\varphi_q \in \mathcal{C}^2([0, t_1])$ the solution to the Cauchy problem

$$\varphi_q' = \varphi_q^{2q}, \quad \varphi_q(0) = t_0.$$

Clearly, $\varphi_q(t) = (t_0^{1-2q} + (1-2q)t)^{1/(1-2q)}$, when $2q \neq 1$, and $\varphi_q = \varphi_e$, when $2q = 1$. The time t_1 satisfies $t_1 = +\infty$, when $2q \leq 1$, and $t_1 = t_0^{1-2q}(2q-1)^{-1}$, when $2q > 1$. The power scaling transformation is defined by $\Phi_q(\omega) : s \in \mathbb{R}^+ \mapsto \frac{\omega(\varphi_q(s))}{\varphi_q(s)^q}$. The process $V^{(q)} := V^{(\varphi_q)}$ satisfies the equation

$$dV_s^{(q)} = dW_s - \varphi_q^{-\gamma q}(s)F\left(\sqrt{\varphi_q'(s)}V_s^{(q)}\right) ds - q\varphi_q^{2q-1}(s)V_s^{(q)} ds, \quad (1.3)$$

where $(W_t)_{t \geq 0}$ is a standard Brownian motion.

We now study the existence and the explosion of the solution to (SKE_B) under the homogeneity assumption (H_γ) . As a consequence, the power scaling process $V^{(q)}$ satisfies the equation

$$dV_s^{(q)} = dW_s - F(V_s^{(q)}) ds - q\varphi_q^{2q-1}(s)V_s^{(q)} ds, \quad s \in [0, t_1), \quad (1.4)$$

which can be written, when $2q > 1$, as

$$dV_s^{(q)} = dW_s - F(V_s^{(q)}) ds - \delta \frac{V_s^{(q)}}{t_1 - s} ds, \quad s \in [0, t_1), \quad (1.5)$$

where $\delta := \frac{q}{2q-1}$. Proposition 3.2, p. 188 in [GO13] can be stated in the present situation.

Proposition 1.3.2. *For $\gamma > -1$, there exists a pathwise unique strong solution to (SKE_B) , defined up to the explosion time.*

Remark 1.3.3. *In the linear case ($\gamma = 1$), the drift and the diffusion terms are Lipschitz and satisfy locally linear growth. The existence and non-explosion of V follow from Theorem 2.9, p. 289, in [KS98].*

Proof. We sketch the proof in our context. Remark first that, since $\gamma > -1$, $x \mapsto |x|^\gamma$ is locally integrable. Leaving out the third term on the right-hand side of (1.4), one gets a time-homogeneous equation:

$$dH_s = dW_s - F(H_s) ds, \quad s \in [0, t_1). \quad (1.6)$$

By using Proposition 2.2, p. 28, in [CE05], there exists a unique weak solution H to this time-homogeneous equation (1.6) defined up to the explosion time. Moreover, the Girsanov transformation induces a linear bijection between weak solutions defined up to

the explosion time to equations (1.4) and (1.6). It follows that there exists a unique weak solution $V^{(q)}$ to equation (1.4). Therefore, by using the bijection induced by the change of time (Proposition 1.3.1), there exists a unique weak solution V to equation (SKE_B). Besides, by using Corollary 3.4 and Proposition 3.2, pp. 389-390, in [RY05], pathwise uniqueness holds for the equation (SKE_B). The conclusion follows (Theorem 1.7, p. 368, in [RY05]). \square

Proposition 1.3.4.

- When $\gamma \leq 1$ or for all $v \in \mathbb{R}$, $vF(v) \geq 0$, then the explosion time of V is a.s. infinite.
- If $2q > 1$, then $\mathbb{P}(\tau_\infty = +\infty) > 0$.
- If $\gamma > 1$ and $(F(-1), F(1)) \in ((0, +\infty) \times [0, +\infty)) \cup (\mathbb{R} \times (-\infty, 0))$, then we have $\mathbb{P}(\tau_\infty = +\infty) < 1$, where τ_∞ denotes the explosion time of V .

Remark 1.3.5. If $2q < 1$ and $F(1) = -F(-1) < 0$, it follows from Proposition 3.6 p. 9 in [GO13] that the explosion time of V is finite a.s.

Remark 1.3.6. We sketch the proof by using the same argument as in [GO13]. Actually, one can conclude of the non explosion of the solution by using the moment estimates computed in Proposition 1.4.1, as in Chapter 2 (see Lemma 2.4.1).

Proof. This proof is inspired by those of Propositions 3.6 and 3.7 in [GO13]. We split the proof into several steps.

STEP 1. Assume first that $\gamma \leq 1$ or $vF(v) \geq 0$. We will use a criterion of non explosion stated in [SV06].

Call \mathcal{L}_t the time-inhomogeneous infinitesimal generator of V , then

$$\mathcal{L}_t := \frac{1}{2} \frac{\partial^2}{\partial x^2} - \frac{F(x)}{t^\beta} \frac{\partial}{\partial x}, \quad t \geq t_0. \tag{1.7}$$

Let ψ be a twice continuous differentiable positive function such that

$$\text{for all } |x| \geq 1, \psi(x) = 1 + x^2, \text{ for all } |x| \leq \frac{1}{2}, \psi(x) = 1, \text{ and } \psi \geq 1, \text{ on } \mathbb{R}.$$

Note that ψ does not depend on time. Hence, $(\partial_t + \mathcal{L}_t)\psi = \mathcal{L}_t\psi$.

Fix $T \geq t_0$ and call c_T the supremum of the continuous function $\mathcal{L}_t\psi$ on $[t_0, T] \times [-1, 1]$.

Then, for all $|x| \leq 1$ and $t \in [t_0, T]$,

$$\mathcal{L}_t \psi(x) \leq c_T \leq c_T \psi(x).$$

Moreover, for all $|x| > 1$ and $t \in [t_0, T]$, for a positive constant C ,

$$\mathcal{L}_t \psi(x) = -2x \frac{F(x)}{t^\beta} + 1 \leq \begin{cases} 1 \leq \psi(x) & \text{if for all } v \in \mathbb{R}, vF(v) \geq 0, \\ 2 \max(|F(1)|, |F(-1)|)x^2 + 1 \leq C\psi(x) & \text{if } \gamma \leq 1. \end{cases}$$

So, by using Theorem 10.2.1, p. 254, in [SV06], we deduce that τ_∞ is infinite a.s.

STEP 2. In this step, we suppose that $2q > 1$. We follow the ideas of the proof of Proposition 3.7, pp. 191-192, in [GO13]. We first show that $\mathbb{P}(\tau_\infty = +\infty) > 0$.

Let $V^{(q)}$ be the pathwise unique strong solution to equation (1.5). And denote by b , the δ -Brownian bridge, the pathwise unique strong solution to equation

$$db_s = dW_s - \delta \frac{b_s}{t_1 - s} ds, \quad b_0 = x_0, \quad s \in [0, t_1]. \quad (1.8)$$

Note that the equation (1.8) is obtained from (1.5) by omitting the second term on the right-hand side. Setting $\tau_\infty^{(q)}$ for the explosion time of $V^{(q)}$, then $\tau_\infty^{(q)} \in [0, t_1] \cup \{+\infty\}$ a.s. and $\{\tau_\infty^{(q)} \geq t_1\} = \{\tau_\infty = +\infty\}$. Note that b becomes continuous on $[0, t_1]$ by setting $b_{t_1} = 0$ a.s.

Fix $n \geq 1$, and define for all $s \in [0, t_1]$,

$$T_n := \inf \left\{ s \in [0, t_1], |V_s^{(q)}| \geq n \right\}, \quad \sigma_n := \inf \left\{ s \in [0, t_1], |b_s| \geq n \right\},$$

and

$$\mathcal{E}(s) := \exp \left(\int_0^s -F(b_u) dW_u - \frac{1}{2} \int_0^s F(b_u)^2 du \right).$$

Since $\gamma > 1 \geq 0$, we have

$$\begin{aligned} \mathbb{E} \left[\exp \left(\frac{1}{2} \int_0^{s \wedge \sigma_n} F(b_u)^2 du \right) \right] &\leq \mathbb{E} \left[\exp \left(\frac{1}{2} \int_0^{s \wedge \sigma_n} n^{2\gamma} \max(F(1)^2, F(-1)^2) du \right) \right] \\ &\leq \exp \left(\frac{t_1}{2} n^{2\gamma} \max(F(1)^2, F(-1)^2) \right). \end{aligned}$$

We observe that Novikov's condition applies to $(\mathcal{E}_{s \wedge \sigma_n})_{s \geq 0}$. Therefore, by using the Girsanov transformation between b and $V^{(q)}$, we can write for every $n \geq 1$, $s \in [0, t_1]$ and

$A \in \mathcal{F}_s$,

$$\mathbb{E} \left[\mathbb{1}_A \left(V_{\bullet \wedge T_n}^{(q)} \right) \mathbb{1}_{T_n > s} \right] = \mathbb{E} \left[\mathbb{1}_A (b_{\bullet \wedge \sigma_n}) \mathcal{E}(s \wedge \sigma_n) \mathbb{1}_{\sigma_n > s} \right].$$

Letting $n \rightarrow +\infty$, by dominated convergence theorem and Fatou's lemma, we obtain

$$\mathbb{E} \left[\mathbb{1}_A \left(V^{(q)} \right) \mathbb{1}_{\tau_\infty^{(q)} > s} \right] \geq \mathbb{E} \left[\mathbb{1}_A (b) \mathcal{E}(s) \right].$$

Hence, $\mathbb{P}(\tau_\infty = +\infty) = \mathbb{P}(\tau_\infty^{(q)} \geq t_1) \geq \mathbb{E}[\mathcal{E}(t_1)] > 0$.

STEP 3. Assume now that $\gamma > 1$ and $(F(-1), F(1)) \in ((0, +\infty)) \times [0, +\infty)) \cup (\mathbb{R} \times (-\infty, 0))$. We will show that $\mathbb{P}(\tau_\infty = +\infty) < 1$ when $F(1) > 0$ and $F(-1) > 0$. Our strategy is to apply the criterion for explosion stated at Theorem 10.2.1, p. 254 in [SV06]. Pick $T > t_0$ and choose $a \in (1, \gamma)$. One can choose $k \geq 1$ such that $a(a-1)^{-1} < k(T-t_0)$. Introduce the continuous differentiable negative function $f_1 : x \mapsto \frac{-1/2}{1+|x|^a}$, and, for $\mu > 0$, the bounded twice continuous differentiable function

$$G_{1,\mu} : x \mapsto \exp \left(\mu \int_{-\infty}^x f_1(y) dy \right).$$

For all $t \in [t_0, T]$ and all $x \in \mathbb{R}$,

$$\begin{aligned} (\partial_t + \mathcal{L}_t)G_{1,\mu}(x) &= \mathcal{L}_t G_{1,\mu}(x) = \mu G_{1,\mu}(x) \left[F(x)t^{-\beta} |f_1(x)| + \frac{1}{2}f_1'(x) + \frac{\mu}{2}f_1^2(x) \right] \\ &\geq \mu G_{1,\mu}(x) \left[F(x)T^{-\beta} |f_1(x)| + \frac{1}{2}f_1'(x) + \frac{\mu}{2}f_1^2(x) \right]. \end{aligned}$$

Since $|f_1(x)| \underset{|x| \rightarrow +\infty}{\sim} \frac{1}{2}|x|^{-a}$, then we have $\lim_{|x| \rightarrow +\infty} F(x)|f_1(x)| = +\infty$, and using that $\lim_{|x| \rightarrow +\infty} f_1'(x) = 0$, there exists $r \geq 1$ such that, for all $\mu > 0$,

$$(\partial_t + \mathcal{L}_t)G_{1,\mu}(x) \geq \mu G_{1,\mu}(x) \left[F(x)T^{-\beta} |f_1(x)| + \frac{1}{2}f_1'(x) \right] \geq k\mu G_{1,\mu}(x) \text{ on } [t_0, T] \times [-r, r]^c.$$

Moreover, since f_1^2 is bounded away from zero, while $|f_1'|$ is bounded on $[-r, r]$ and since F is non-negative, there exists μ_0 , such that,

$$(\partial_t + \mathcal{L}_t)G_{1,\mu_0}(x) \geq \mu_0 G_{1,\mu_0}(x) \left[\frac{1}{2}f_1'(x) + \frac{\mu_0}{2}f_1^2(x) \right] \geq k\mu_0 G_{1,\mu_0}(x) \text{ on } [t_0, T] \times [-r, r].$$

Hence, for all $t \in [t_0, T]$ and all $x \in \mathbb{R}$, $(\partial_t + \mathcal{L}_t)G_{1,\mu_0}(x) \geq k\mu_0 G_{1,\mu_0}(x)$. Besides, since

$|f_1(x)| \leq 1 \wedge |x|^{-a}$, we get

$$\int_{-\infty}^{x_0} (-f_1(x)) dx \leq \int_{\mathbb{R}} (1 \wedge |x|^{-a}) dx = a(a-1)^{-1} < k(T-t_0).$$

Thus, we get a lower bound

$$G_{1,\mu_0}(x_0) > e^{-k\mu_0(T-t_0)} \geq e^{-k\mu_0(T-t_0)} \sup_{x \in \mathbb{R}} G_{1,\mu_0}(x).$$

Therefore, Theorem 10.2.1, p. 254, in [SV06] applies and V explodes in finite time with positive probability.

When $F(-1) < 0$ and $F(1) < 0$, one can proceed in the same way, using the function $x \mapsto \exp\left(\mu \int_x^{+\infty} f_1(y) dy\right)$ instead of $G_{1,\mu}$, in order to get that $\mathbb{P}(\tau_\infty = +\infty) < 1$.

STEP 4. It remains to show that $\mathbb{P}(\tau_\infty = +\infty) < 1$ when $F(1) < 0$ and $F(-1) > 0$.

As in the previous step, we choose $a \in (1, \gamma)$ and for every $T > t_0$, we choose again $k \geq 1$ such that $a(a-1)^{-1} < k(T-t_0)$. Moreover, it can be noted that there exists a continuous differentiable odd function f_2 , defined on \mathbb{R} , vanishing only at $x = 0$, such that $|f_2(x)| \leq 1 \wedge |x|^{-a}$, and satisfying

$$f_2(x) := kx, \quad x \in \left[-\frac{1}{2k}, \frac{1}{2k}\right], \quad \lim_{|x| \rightarrow +\infty} |x|^\gamma |f_2(x)| = +\infty \quad \text{and} \quad \lim_{|x| \rightarrow +\infty} f_2'(x) = 0.$$

For $\mu > 0$, we introduce the bounded twice continuous differentiable function

$$G_{2,\mu} : x \mapsto \exp\left(\mu \int_0^x f_2(y) dy\right).$$

Then for all $t \in [t_0, T]$ and all $x \in \mathbb{R}$,

$$\begin{aligned} (\partial_t + \mathcal{L}_t)G_{2,\mu}(x) &= \mathcal{L}_t G_{2,\mu}(x) = \mu G_{2,\mu}(x) \left[\frac{|F(x)f_2(x)|}{t^\beta} + \frac{1}{2}f_2'(x) + \frac{\mu}{2}f_2^2(x) \right] \\ &\geq \mu G_{2,\mu}(x) \left[\rho \frac{|x|^\gamma |f_2(x)|}{t^\beta} + \frac{1}{2}f_2'(x) + \frac{\mu}{2}f_2^2(x) \right], \end{aligned}$$

where $\rho = \min\{|F(1)|, |F(-1)|\} > 0$. One can conclude, using the same argument as in the proof of Proposition 3.7, p. 13, in [GO13].

□

1.4 Moment estimates of the velocity process

In this section, we give estimates for the moment of the velocity process. It will be useful to control some stochastic terms appearing later.

Proposition 1.4.1. *Assume that $\gamma \geq 0$ and $\beta \in \mathbb{R}$. The inequality*

$$\forall t \geq t_0, \quad \mathbb{E}[|V_t|^\kappa] \leq C_{\gamma, \kappa, \beta, t_0} t^{\frac{\kappa}{2}}$$

holds for

- $\kappa \in [0, 1]$, when $\gamma < 1$ and $\beta \geq \frac{\gamma+1}{2}$,
- $\kappa \geq 0$, when for all $v \in \mathbb{R}$, $vF(v) \geq 0$.

If $\kappa \in [0, 1]$, $\gamma < 1$ and $\beta < \frac{\gamma+1}{2}$, then

$$\forall t \geq t_0, \quad \mathbb{E}[|V_t|^\kappa] \leq C_{\gamma, \kappa, \beta, t_0} t^{\kappa \frac{1-\beta}{1-\gamma}}.$$

Remark 1.4.2. *When $-1 < \gamma < 0$, it can be proved that for all $t \geq t_0$, $\mathbb{E}[|V_t|] \leq C_{\gamma, \beta, t_0} \sqrt{t}$, without hypothesis of the positivity of the function $v \mapsto vF(v)$.*

Proof. STEP 1. Assume that $\gamma \geq 1$ and that for all $v \in \mathbb{R}$, $vF(v) \geq 0$.

Define, for all $n \geq 0$, the stopping time $T_n := \inf\{t \geq t_0, |V_t| \geq n\}$. By Itô's formula, for all $t \geq t_0$, we have

$$\begin{aligned} V_{t \wedge T_n}^2 &= v_0^2 + \int_{t_0}^{t \wedge T_n} 2V_s dB_s - \int_{t_0}^{t \wedge T_n} 2s^{-\beta} V_s F(V_s) ds + (t \wedge T_n - t_0) \\ &= v_0^2 + \int_{t_0}^t \mathbb{1}_{\{s \leq T_n\}} 2V_s dB_s - \int_{t_0}^{t \wedge T_n} 2s^{-\beta} V_s F(V_s) ds + (t \wedge T_n - t_0) \\ &\leq v_0^2 + \int_{t_0}^t \mathbb{1}_{\{s \leq T_n\}} 2V_s dB_s + (t - t_0). \end{aligned}$$

Since $\int_{t_0}^t 4\mathbb{1}_{\{s \leq T_n\}} V_s^2 ds \leq 4n^2(t - t_0) < +\infty$, taking expectation yields

$$\mathbb{E}[V_{t \wedge T_n}^2] \leq v_0^2 + (t - t_0) \leq C_{t_0} t.$$

Set $\kappa \in [0, 2]$, we obtain by Jensen's inequality that

$$\mathbb{E}[|V_t|^\kappa] \leq \mathbb{E}[|V_t|^2]^{\frac{\kappa}{2}} \leq \left(\liminf_{n \rightarrow +\infty} \mathbb{E}[V_{t \wedge T_n}^2] \right)^{\frac{\kappa}{2}} \leq C_{\kappa, t_0} t^{\frac{\kappa}{2}}. \quad (1.9)$$

When $\kappa > 2$, the function $v \mapsto |v|^\kappa$ is \mathcal{C}^2 , so by Itô's formula, we can write for all $t \geq t_0$,

$$\begin{aligned} |V_{t \wedge T_n}|^\kappa &= |v_0|^\kappa + \int_{t_0}^{t \wedge T_n} \kappa \operatorname{sgn}(V_s) |V_s|^{\kappa-1} dB_s - \int_{t_0}^{t \wedge T_n} \kappa s^{-\beta} |V_s|^{\kappa-1} \operatorname{sgn}(V_s) F(V_s) ds \\ &\quad + \int_{t_0}^{t \wedge T_n} \frac{\kappa(\kappa-1)}{2} |V_s|^{\kappa-2} ds. \end{aligned}$$

In addition, using the hypothesis on the sign of F , we have

$$|V_{t \wedge T_n}|^\kappa \leq |v_0|^\kappa + \int_{t_0}^t \mathbb{1}_{\{s \leq T_n\}} \kappa \operatorname{sgn}(V_s) |V_s|^{\kappa-1} dB_s + \int_{t_0}^{t \wedge T_n} \frac{\kappa(\kappa-1)}{2} |V_s|^{\kappa-2} ds. \quad (1.10)$$

We observe that $\int_{t_0}^t \kappa^2 V_s^{2\kappa-2} \mathbb{1}_{\{s \leq T_n\}} ds \leq \kappa^2 n^{2\kappa-2} (t - t_0) < +\infty$. Taking expectation in (1.10), we obtain

$$\mathbb{E}[|V_t|^\kappa] \leq \liminf_{n \rightarrow +\infty} \mathbb{E}[|V_{t \wedge T_n}|^\kappa] \leq |v_0|^\kappa + \int_{t_0}^t \frac{\kappa(\kappa-1)}{2} \mathbb{E}[|V_s|^{\kappa-2}] ds.$$

When $0 \leq \kappa - 2 \leq 2$, we can upper bound $\mathbb{E}[|V_s|^{\kappa-2}]$ by injecting (1.9) and get

$$\mathbb{E}[|V_t|^\kappa] \leq |v_0|^\kappa + \int_{t_0}^t \frac{\kappa(\kappa-1)}{2} C_{\kappa, t_0} s^{\frac{\kappa-2}{2}} ds \leq C_{\kappa, t_0} s^{\frac{\kappa}{2}}.$$

The same method is then applied inductively to prove the inequality for all $\kappa > 2$.

STEP 2. Assume now that $\gamma \in [0, 1[$.

Fix $\kappa \in [0, 1]$. Then Jensen's inequality yields, for all $t \geq t_0$, $\mathbb{E}[|V_t|^\kappa] \leq \mathbb{E}[|V_t|]^\kappa$, hence it suffices to verify the inequality only for $\kappa = 1$.

Define, for all $n \geq 0$, the stopping time $T_n := \inf\{t \geq t_0, |V_t| \geq n\}$ and let us recall that under the hypothesis (H_γ) , there exists a positive constant K , such that $|F(v)| \leq K|v|^\gamma$.

We can write, for $t \geq t_0$ and $n \geq 0$,

$$\begin{aligned} |V_{t \wedge T_n}| &\leq |v_0 - B_{t_0}| + |B_{t \wedge T_n}| + \int_{t_0}^{t \wedge T_n} s^{-\beta} |F(V_{s \wedge T_n})| ds \\ &\leq |v_0 - B_{t_0}| + |B_{t \wedge T_n}| + \int_{t_0}^{t \wedge T_n} K s^{-\beta} |V_{s \wedge T_n}|^\gamma ds. \end{aligned}$$

By noting that $\gamma \in [0, 1[$ and that $(B_t^2 - t)_{t \geq 0}$ is a martingale, taking expectation we get

$$\begin{aligned} \mathbb{E} [|V_{t \wedge T_n}|] &\leq \mathbb{E} [|v_0 - B_{t_0}|] + \mathbb{E} [|B_{t \wedge T_n}|] + \int_{t_0}^t K s^{-\beta} \mathbb{E} [|V_{s \wedge T_n}|^\gamma] ds \\ &\leq \mathbb{E} [|v_0 - B_{t_0}|] + \sqrt{\mathbb{E} [B_{t \wedge T_n}^2]} + \int_{t_0}^t K s^{-\beta} \mathbb{E} [|V_{s \wedge T_n}|^\gamma] ds \\ &\leq \mathbb{E} [|v_0 - B_{t_0}|] + \sqrt{\mathbb{E} [t \wedge T_n]} + \int_{t_0}^t K s^{-\beta} \mathbb{E} [|V_{s \wedge T_n}|^\gamma] ds \\ &\leq C_{t_0} \sqrt{t} + \int_{t_0}^t K s^{-\beta} \mathbb{E} [|V_{s \wedge T_n}|^\gamma] ds. \end{aligned}$$

The function $g_n : t \mapsto \mathbb{E} [|V_{t \wedge T_n}|]$ is bounded by n . Applying a Grönwall-type lemma (Lemma 1.A.1) and Fatou's lemma, for $\beta \neq 1$ and for all $t \geq t_0$, we end up with

$$\begin{aligned} \mathbb{E} [|V_t|] &\leq \liminf_{n \rightarrow +\infty} \mathbb{E} [|V_{t \wedge T_n}|] \leq C_\gamma \left[C_{t_0} \sqrt{t} + \left(\frac{1-\gamma}{1-\beta} K (t^{1-\beta} - t_0^{1-\beta}) \right)^{\frac{1}{1-\gamma}} \right] \\ &\leq C_{\gamma, \beta, t_0} \begin{cases} \sqrt{t} & \text{if } \beta \geq \frac{\gamma+1}{2}, \\ t^{\frac{1-\beta}{1-\gamma}} & \text{else.} \end{cases} \end{aligned}$$

The case $\beta = 1$ can be treated similarly. □

1.5 Proof of the asymptotic behavior of the solution

This section is devoted to the proof of our main results.

1.5.1 Asymptotic behavior in the super-critical regime

In this section, we assume that $\gamma \geq 0$ and $q > \frac{1}{2}$.

Proof of Theorem 1.2.2. We split the proof into three steps.

STEP 1. We note that it is enough to prove that the process

$$(V_t^{(\varepsilon)})_{t \geq 0} := (\sqrt{\varepsilon} V_{t/\varepsilon})_{t \geq 0}$$

converges in distribution to a Brownian motion in the space of continuous functions $\mathcal{C}([0, +\infty))$ endowed by the uniform topology. To see $V^{(\varepsilon)}$ as a process of $\mathcal{C}([0, +\infty))$,

let us state for all $s \in [0, \varepsilon t_0]$, $V_s^{(\varepsilon)} := V_{\varepsilon t_0}^{(\varepsilon)} = \sqrt{\varepsilon} v_0$.

For every $\varepsilon \in (0, 1]$ and $t \geq \varepsilon t_0$, we can write

$$\varepsilon^{\frac{3}{2}} X_{t/\varepsilon} = \varepsilon^{\frac{3}{2}} x_0 + \int_{\varepsilon t_0}^t V_s^{(\varepsilon)} ds.$$

Clearly, the theorem will be proved once we show that $g_\varepsilon(V_{\bullet}^{(\varepsilon)}) := (V_{\bullet}^{(\varepsilon)}, \int_{\varepsilon t_0}^{\bullet} V_s^{(\varepsilon)} ds)$ converges weakly in $\mathcal{C}([0, +\infty))$ endowed by the uniform topology. Here the mapping $g_\varepsilon : v \mapsto (v_t, \int_{\varepsilon t_0}^t v_s ds)_{t \geq 0}$ is defined and valued on $\mathcal{C}((0, +\infty))$. This mapping is converging, as $\varepsilon \rightarrow 0$, to the continuous mapping $g : v \mapsto (v_t, \int_0^t v_s ds)_{t \geq 0}$.

We have, for every $\varepsilon \in (0, 1]$ and $t \geq \varepsilon t_0$,

$$\begin{aligned} V_t^{(\varepsilon)} &= \sqrt{\varepsilon} V_{t/\varepsilon} = \sqrt{\varepsilon}(v_0 - B_{t_0}) + \sqrt{\varepsilon} B_{t/\varepsilon} - \sqrt{\varepsilon} \int_{t_0}^{t/\varepsilon} F(V_s) s^{-\beta} ds \\ &= \sqrt{\varepsilon}(v_0 - B_{t_0}) + B_t^{(\varepsilon)} - \varepsilon^{\beta - \frac{1}{2}} \int_{\varepsilon t_0}^t F(V_{u/\varepsilon}) u^{-\beta} du. \end{aligned}$$

By self-similarity, $B^{(\varepsilon)} := (\sqrt{\varepsilon} B_{t/\varepsilon})_{t \geq 0}$ has the same distribution as a standard Brownian motion.

Assume that the convergence of the rescaled velocity process is proved in the strong way, that is

$$\forall T > 0, \sup_{\varepsilon t_0 \leq t \leq T} |V_t^{(\varepsilon)} - B_t^{(\varepsilon)}| \xrightarrow[\varepsilon \rightarrow 0]{\mathbb{P}} 0. \quad (1.11)$$

Then it suffices to prove that $g_\varepsilon(B^{(\varepsilon)}) \implies g(\mathcal{B})$ and $d_u(g_\varepsilon(V^{(\varepsilon)}), g_\varepsilon(B^{(\varepsilon)})) \xrightarrow{\mathbb{P}} 0$, as $\varepsilon \rightarrow 0$ (see Theorem 3.1 p. 27 in [Bil99]).

On the one hand, the process $B^{(\varepsilon)}$ being a Brownian motion and $\|\cdot\|$ denoting a norm on \mathbb{R}^2 , the first convergence follows from

$$\forall T > 0, \sup_{\varepsilon t_0 \leq t \leq T} \|g_\varepsilon(\mathcal{B}_t) - g(\mathcal{B}_t)\| \xrightarrow[\varepsilon \rightarrow 0]{\mathbb{P}} 0. \quad (1.12)$$

Let us prove (1.12). For every $\varepsilon t_0 \leq t \leq T$, we get

$$\|g_\varepsilon(\mathcal{B}_t) - g(\mathcal{B}_t)\| = \left| \int_0^{\varepsilon t_0} \mathcal{B}_s ds \right| \leq \int_0^{\varepsilon t_0} |\mathcal{B}_s| ds.$$

Hence,

$$\mathbb{E} \left[\sup_{\varepsilon t_0 \leq t \leq T} \|g_\varepsilon(\mathcal{B}_t) - g(\mathcal{B}_t)\| \right] \leq \int_0^{\varepsilon t_0} \mathbb{E} |\mathcal{B}_s| ds \leq C \int_0^{\varepsilon t_0} \sqrt{s} ds \xrightarrow[\varepsilon \rightarrow 0]{} 0.$$

On the other hand, we prove that

$$\forall T > 0, \sup_{\varepsilon t_0 \leq t \leq T} \|g_\varepsilon(V_t^{(\varepsilon)}) - g_\varepsilon(B_t^{(\varepsilon)})\| \xrightarrow[\varepsilon \rightarrow 0]{\mathbb{P}} 0. \quad (1.13)$$

For every $\varepsilon t_0 \leq t \leq T$, using (1.11)

$$\begin{aligned} \|g_\varepsilon(V_t^{(\varepsilon)}) - g_\varepsilon(B_t^{(\varepsilon)})\| &= |V_t^{(\varepsilon)} - B_t^{(\varepsilon)}| + \left| \int_{\varepsilon t_0}^t V_s^{(\varepsilon)} - B_s^{(\varepsilon)} ds \right| \\ &\leq (1 + T - \varepsilon t_0) \sup_{\varepsilon t_0 \leq t \leq T} |V_t^{(\varepsilon)} - B_t^{(\varepsilon)}| \xrightarrow[\varepsilon \rightarrow 0]{\mathbb{P}} 0. \end{aligned}$$

STEP 2. Let us prove now (1.11).

Recall that under the hypothesis (H_γ) , there exists a positive constant K , such that $(\sqrt{\varepsilon})^\gamma \left| F\left(\frac{V_u^{(\varepsilon)}}{\sqrt{\varepsilon}}\right) \right| \leq K |V_u^{(\varepsilon)}|^\gamma$. Modifying the factor in front of the integral part, we get

$$V_t^{(\varepsilon)} = \sqrt{\varepsilon}(v_0 - B_{t_0}) + \sqrt{\varepsilon}B_{t/\varepsilon} - \varepsilon^{\beta - \frac{(\gamma+1)}{2}} \int_{\varepsilon t_0}^t (\sqrt{\varepsilon})^\gamma F\left(\frac{V_u^{(\varepsilon)}}{\sqrt{\varepsilon}}\right) u^{-\beta} du.$$

It follows that, for all $T > 0$,

$$\begin{aligned} \sup_{\varepsilon t_0 \leq t \leq T} |V_t^{(\varepsilon)} - B_t^{(\varepsilon)}| &\leq \sqrt{\varepsilon} |v_0 - B_{t_0}| + \varepsilon^{\beta - \frac{(\gamma+1)}{2}} \sup_{\varepsilon t_0 \leq t \leq T} \left| \int_{\varepsilon t_0}^t (\sqrt{\varepsilon})^\gamma F\left(\frac{V_u^{(\varepsilon)}}{\sqrt{\varepsilon}}\right) u^{-\beta} du \right| \\ &\leq \sqrt{\varepsilon} |v_0 - B_{t_0}| + \varepsilon^{\beta - \frac{(\gamma+1)}{2}} \int_{\varepsilon t_0}^T K |V_u^{(\varepsilon)}|^\gamma u^{-\beta} du. \end{aligned}$$

Taking the expectation and using moment estimates (Proposition 1.4.1), we obtain, when $\beta \neq \frac{\gamma}{2} + 1$ and since $\beta > \frac{\gamma+1}{2}$,

$$\begin{aligned} \varepsilon^{\beta - \frac{(\gamma+1)}{2}} \mathbb{E} \left[\int_{\varepsilon t_0}^T K |V_u^{(\varepsilon)}|^\gamma u^{-\beta} du \right] &= \varepsilon^{\beta - \frac{(\gamma+1)}{2}} \int_{\varepsilon t_0}^T K \mathbb{E} [|V_u^{(\varepsilon)}|^\gamma] u^{-\beta} du \\ &\leq \varepsilon^{\beta - \frac{(\gamma+1)}{2}} \int_{\varepsilon t_0}^T C_{\gamma, \beta, t_0} u^{\frac{\gamma}{2} - \beta} du \\ &\leq C \left(\varepsilon^{\beta - \frac{(\gamma+1)}{2}} T^{\frac{\gamma}{2} - \beta + 1} - t_0^{\frac{\gamma}{2} - \beta + 1} \sqrt{\varepsilon} \right) \xrightarrow[\varepsilon \rightarrow 0]{} 0. \end{aligned}$$

Hence, setting $r = \min(\frac{1}{2}, \beta - \frac{(\gamma+1)}{2}) > 0$,

$$\mathbb{E} \left[\sup_{\varepsilon t_0 \leq t \leq T} |V_t^{(\varepsilon)} - B_t^{(\varepsilon)}| \right] = O_{\varepsilon \rightarrow 0}(\varepsilon^r).$$

The case $\beta = \frac{\gamma}{2} + 1$ can be treated similarly to get

$$\mathbb{E} \left[\sup_{\varepsilon t_0 \leq t \leq T} |V_t^{(\varepsilon)} - B_t^{(\varepsilon)}| \right] = O_{\varepsilon \rightarrow 0}(\sqrt{\varepsilon} \ln(\varepsilon)).$$

This concludes the proof. □

Remark 1.5.1. *One can observe that the only moment in this proof when we need the condition “ $\gamma < 1$ or for all $v \in \mathbb{R}$, $vF(v) \geq 0$ ” is when we are proving the moment estimates.*

Remark 1.5.2. *One can notice that in this proof, we only need an upper bound of the function F . As a consequence, one can assume the function F to satisfy the following hypothesis:*

$$|F| \leq G \text{ where } G \text{ is a positive function satisfying } (H_\gamma). \quad (H_2^\gamma)$$

This implies that there exists a positive constant K such that, for all $v \in \mathbb{R}$, $|F(v)| \leq K|v|^\gamma$. Obviously (H_2^γ) is a generalization of (H_γ) . As an example of a function satisfying (H_2^γ) (with $\gamma = 0$), one can keep in mind $F : v \mapsto v/(1+v^2)$ (see also [FT21]).

The same method cannot be used to conclude to the existence up to the explosion, since the equation satisfied by the power scaling process (1.3) does not have any time-homogeneous term. Instead, one gets it by using the exponential change of time process and by considering G instead of $|F|$. While, the non-explosion of the solution is ensured by the condition “ $\gamma < 1$ or for all $v \in \mathbb{R}$, $vF(v) \geq 0$ ”.

1.5.2 Asymptotic behavior in the critical regime

Assume in this section that $\beta = \frac{\gamma+1}{2}$.

Proof of Theorem 1.2.3. **STEP 1.** As in the first step of the previous section, it suffices to prove the convergence of the rescaled velocity process $(\sqrt{\varepsilon}V_{t/\varepsilon})_t$.

Keeping same notations, we prove that $g_\varepsilon(V^{(\varepsilon)})$ converges in distribution in $\mathcal{C}([0, +\infty))$

to $g(\mathcal{V})$. In order to see $V^{(\varepsilon)}$ as a process of $\mathcal{C}([0, +\infty))$, let us set for all $s \in [0, \varepsilon t_0]$, $V_s^{(\varepsilon)} := V_{\varepsilon t_0}^{(\varepsilon)} = \sqrt{\varepsilon} v_0$. Call P_ε, P the distribution of $V^{(\varepsilon)}$ and \mathcal{V} respectively. Then, using Portmanteau theorem (see Theorem 2.1 p. 16 in [Bil99]), it suffices to prove that for all bounded and uniformly continuous function $h : \mathcal{C}([0, +\infty)) \times \mathcal{C}([0, +\infty)) \rightarrow \mathbb{R}$,

$$\int_{\mathcal{C}([0, +\infty))^2} h(g_\varepsilon(\omega)) dP_\varepsilon(d\omega) \xrightarrow{\varepsilon \rightarrow 0} \int_{\mathcal{C}([0, +\infty))^2} h(g(\omega)) dP(d\omega).$$

Take a bounded and uniformly continuous function h . By assumption, one knows that $P_\varepsilon \xrightarrow{\varepsilon \rightarrow 0} P$, hence, by Problem 4.12 p. 64 in [KS98], it suffices to prove that the uniformly bounded sequence $(h \circ g_\varepsilon)$ of continuous functions on $\mathcal{C}([0, +\infty))$ converges uniformly on compact subsets of $\mathcal{C}([0, +\infty))$ to the continuous function $h \circ g$. Let K be a compact set of $\mathcal{C}([0, +\infty))$. Then, for all $\omega \in K$, $\max_{[0, \varepsilon t_0]} |\omega|$ is uniformly bounded by a constant, called M .

Fix $\eta > 0$. By the uniform continuity of h , there exists $\delta > 0$ such that for all $\omega \in K$,

$$d_u(g_\varepsilon(\omega), g(\omega)) \leq \delta \implies |h \circ g_\varepsilon(\omega) - h \circ g(\omega)| \leq \eta.$$

However, there exists $\varepsilon_1 > 0$ small enough, such that for all $\varepsilon \leq \varepsilon_1$ and for all $\omega \in K$,

$$d_u(g_\varepsilon(\omega), g(\omega)) \leq C \left| \int_0^{\varepsilon t_0} \omega(s) ds \right| \leq C \varepsilon t_0 M \leq \delta.$$

STEP 2. We first prove the f.d.d. convergence.

The exponential scaling process $V^{(e)}$ satisfies the time-homogeneous equation

$$dV_s^{(e)} = dW_s - \frac{V_s^{(e)}}{2} ds - F(V_s^{(e)}) ds, \quad (1.14)$$

where $(W_t)_{t \geq 0}$ is a standard Brownian motion.

By using Proposition 2.2, p. 28, in [CE05], there exists a unique weak solution H to the time-homogeneous equation (1.14) defined up to the explosion time. Using the bijection induced by the exponential change of time (Proposition 1.3.1), we get

$$\left(\frac{V_{t_0 e^t}}{\sqrt{t_0 e^{t/2}}} \right)_{t \geq 0} = (H_t)_{t \geq 0},$$

as solutions to the same SDE, starting at the same point. This can also be written as

$$\left(\frac{V_t}{\sqrt{t}} \right)_{t \geq t_0} = (H_{\log(t/t_0)})_{t \geq t_0}.$$

So, we have, for all $\varepsilon > 0$, and $(t_1, \dots, t_d) \in [\varepsilon t_0, +\infty)^d$,

$$\left(\frac{V_{\varepsilon^{-1}t_1}}{\sqrt{\varepsilon^{-1}t_1}}, \dots, \frac{V_{\varepsilon^{-1}t_d}}{\sqrt{\varepsilon^{-1}t_d}} \right) = \left(H_{\log(t_1)+\log((\varepsilon t_0)^{-1})}, \dots, H_{\log(t_d)+\log((\varepsilon t_0)^{-1})} \right). \quad (1.15)$$

As in [GO13], the scale function and the speed measure of H are respectively

$$\mathbf{p}(x) := \int_0^x \exp \left(\frac{y^2}{2} + \frac{2}{\gamma+1} \operatorname{sgn}(y) F(\operatorname{sgn}(y)) |y|^{\gamma+1} \right) dy$$

and

$$\nu_F(dx) := \exp \left(-\frac{x^2}{2} - \frac{2}{\gamma+1} \operatorname{sgn}(x) F(\operatorname{sgn}(x)) |x|^{\gamma+1} \right) dx.$$

By the ergodic theorem (Theorem 23.15 p. 465 in [Kal02]), H is Λ_F -ergodic, where Λ_F is the probability measure associated to ν_F . Call \widetilde{H} the solution to the time homogeneous equation (1.14) such that the initial condition $\widetilde{H}_{-\infty}$ has the distribution Λ_F .

For $(t_1, \dots, t_d) \in [\varepsilon t_0, +\infty)^d$, let $\Lambda_{F,t_1, \dots, t_d} := \mathcal{L}(\widetilde{H}_{t_1}, \dots, \widetilde{H}_{t_d})$ be the distribution of the vector $(\widetilde{H}_{t_1}, \dots, \widetilde{H}_{t_d})$. Then, for all $s \in \mathbb{R}$, $\Lambda_{F,t_1, \dots, t_d} = \Lambda_{F,t_1+s, \dots, t_d+s}$. Indeed, thanks to the invariance property of Λ_F , $(\widetilde{H}_t)_{t \in \mathbb{R}}$ and $(\widetilde{H}_{t+s})_{t \in \mathbb{R}}$ satisfy the same SDE, starting at the same distribution. As a consequence, for all $\varepsilon > 0$,

$$\mathcal{L} \left(\widetilde{H}_{\log(t_1)+\log((\varepsilon t_0)^{-1})}, \dots, \widetilde{H}_{\log(t_d)+\log((\varepsilon t_0)^{-1})} \right) = \Lambda_{F, \log(t_1), \dots, \log(t_d)}. \quad (1.16)$$

Moreover, by exponential ergodicity, we can prove that for every continuous and bounded function $\psi : \mathbb{R}^d \rightarrow \mathbb{R}$,

$$\mathbb{E} \left[\psi \left(H_{\log(t_1/(t_0\varepsilon))}, \dots, H_{\log(t_d/(t_0\varepsilon))} \right) \right] - \mathbb{E} \left[\psi \left(\widetilde{H}_{\log(t_1/(t_0\varepsilon))}, \dots, \widetilde{H}_{\log(t_d/(t_0\varepsilon))} \right) \right] \xrightarrow{\varepsilon \rightarrow 0} 0. \quad (1.17)$$

We postpone the proof of this convergence in Step 3.

To conclude this step, gather (1.15), (1.16) and (1.17) to get

$$\left(\frac{V_{\varepsilon^{-1}t_1}}{\sqrt{\varepsilon^{-1}t_1}}, \dots, \frac{V_{\varepsilon^{-1}t_d}}{\sqrt{\varepsilon^{-1}t_d}} \right) \xrightarrow{\varepsilon \rightarrow 0} \Lambda_{F, \log(t_1), \dots, \log(t_d)}.$$

This can be written as

$$\left(\sqrt{\varepsilon}V_{t_1/\varepsilon}, \dots, \sqrt{\varepsilon}V_{t_d/\varepsilon}\right) \xrightarrow[\varepsilon \rightarrow 0]{} T * \Lambda_{F, \log(t_1), \dots, \log(t_d)},$$

where $T * \Lambda_{F, \log(t_1), \dots, \log(t_d)}$ is the pushforward of the measure $\Lambda_{F, \log(t_1), \dots, \log(t_d)}$ by the linear map $T(u_1, \dots, u_d) := (\sqrt{t_1}u_1, \dots, \sqrt{t_d}u_d)$.

STEP 3. Let us now prove (1.17).

Pick $\varepsilon t_0 \leq s \leq t$. Set $h_0 = v_0 t_0^{-\frac{1}{2}}$. Actually, we prove a more general result, which will also be useful in the last regime. The convergence (1.17) will be a direct consequence of this lemma.

Lemma 1.5.3. *Let H be an exponential ergodic process with invariant measure ν , solution to a SDE driven by a Brownian motion. Pick a continuous function $\phi : [t_0, +\infty) \rightarrow \mathbb{R}$ satisfying $\lim_{s \rightarrow +\infty} \phi(s) = +\infty$.*

Then, for all integer $d \geq 1$, every continuous and bounded function $\psi : \mathbb{R}^d \rightarrow \mathbb{R}$, all $h_0 \in \mathbb{R}$ and all $(t_1, \dots, t_d) \in [\varepsilon t_0, +\infty)^d$,

$$\mathbb{E} \left[\psi \left(H_{\phi(\varepsilon^{-1}t_1)}, \dots, H_{\phi(\varepsilon^{-1}t_d)} \right) \middle| H_0 = h_0 \right] - \mathbb{E} \left[\psi \left(H_{\phi(\varepsilon^{-1}t_1)}, \dots, H_{\phi(\varepsilon^{-1}t_d)} \right) \middle| H_0 \sim \nu \right] \xrightarrow[\varepsilon \rightarrow 0]{} 0.$$

Proof. For the sake of clarity, let us give a proof for $d = 2$. The general case $d \geq 2$ is similar.

Let $\psi : \mathbb{R}^2 \rightarrow \mathbb{R}$ be a continuous and bounded function.

We set $\mu_\varepsilon := \mathcal{L} \left(H_{\phi(\varepsilon^{-1}s)} \middle| H_0 = h_0 \right)$ and use the generalized Markov property of solution to SDEs driven by a Brownian motion (see Corollary 16.9 p. 313 in [Kal02]). This leads to

$$\mathbb{E} \left[\psi \left(H_{\phi(\varepsilon^{-1}s)}, H_{\phi(\varepsilon^{-1}t)} \right) \middle| H_0 = h_0 \right] = \mathbb{E} \left[\psi \left(H_0, H_{\phi(\varepsilon^{-1}t) - \phi(\varepsilon^{-1}s)} \right) \middle| H_0 \sim \mu_\varepsilon \right]$$

and, since Λ_F is invariant,

$$\mathbb{E} \left[\psi \left(H_{\phi(\varepsilon^{-1}s)}, H_{\phi(\varepsilon^{-1}t)} \right) \middle| H_0 \sim \nu \right] = \mathbb{E} \left[\psi \left(H_0, H_{\phi(\varepsilon^{-1}t) - \phi(\varepsilon^{-1}s)} \right) \middle| H_0 \sim \nu \right].$$

Then, we are reduced to prove

$$\mathbb{E} \left[\psi \left(H_0, H_{\phi(\varepsilon^{-1}t) - \phi(\varepsilon^{-1}s)} \right) \middle| H_0 \sim \mu_\varepsilon \right] - \mathbb{E} \left[\psi \left(H_0, H_{\phi(\varepsilon^{-1}t) - \phi(\varepsilon^{-1}s)} \right) \middle| H_0 \sim \nu \right] \xrightarrow[\varepsilon \rightarrow 0]{} 0.$$

Hence, setting $p(t, x, dy) := \mathbb{P}_x(H_t \in dy)$ and $\|\cdot\|_{TV}$ for the total variation norm, we get

$$\begin{aligned} & \left| \mathbb{E} \left[\psi \left(H_0, H_{\phi(\varepsilon^{-1}t) - \phi(\varepsilon^{-1}s)} \right) \middle| H_0 \sim \mu_\varepsilon \right] - \mathbb{E} \left[\psi \left(H_0, H_{\phi(\varepsilon^{-1}t) - \phi(\varepsilon^{-1}s)} \right) \middle| H_0 \sim \nu \right] \right| \\ & \leq \left| \int_{\mathbb{R}} \mathbb{E} \left[\psi \left(H_0, H_{\phi(\varepsilon^{-1}t) - \phi(\varepsilon^{-1}s)} \right) \middle| H_0 = y \right] (\mu_\varepsilon(dy) - \nu(dy)) \right| \\ & \leq \|\psi\|_\infty \int_{\mathbb{R}} |p(\phi(\varepsilon^{-1}s), h_0, dy) - \nu(dy)| \\ & \leq \|\psi\|_\infty \|p(\phi(\varepsilon^{-1}s), h_0, \cdot) - \nu\|_{TV}. \end{aligned}$$

We let $\varepsilon \rightarrow 0$, using the exponential ergodicity of H . \square

STEP 4. Let us prove now the tightness of the family of distributions $V^{(\varepsilon)} = (\sqrt{\varepsilon}V_{t/\varepsilon})_{t \geq \varepsilon t_0}$ on every compact interval $[m, M]$, $0 < m \leq M$. We check the Kolmogorov criterion stated in Problem 4.11 p. 64 in [KS98].

Take ε_0 small enough such that for all $\varepsilon \leq \varepsilon_0$, $\varepsilon t_0 \leq m$. Fix $m \leq s \leq t \leq M$ and $\alpha > 2$. Recalling that $B^{(\varepsilon)}$ is a Brownian motion, using Jensen's inequality, moment estimates (Proposition 1.4.1) and the relation $\beta = \frac{\gamma+1}{2}$, we can write

$$\begin{aligned} \mathbb{E} \left[|V_t^{(\varepsilon)} - V_s^{(\varepsilon)}|^\alpha \right] & \leq C_\alpha \mathbb{E} \left[|B_t^{(\varepsilon)} - B_s^\varepsilon|^\alpha \right] + C_\alpha \mathbb{E} \left[\left| \sqrt{\varepsilon} \int_{s/\varepsilon}^{t/\varepsilon} F(V_u) u^{-\beta} du \right|^\alpha \right] \\ & \leq C_\alpha \mathbb{E} [|B_t - B_s|^\alpha] + C_\alpha \varepsilon^{1-\frac{\alpha}{2}} (t-s)^{\alpha-1} \mathbb{E} \left[\int_{s/\varepsilon}^{t/\varepsilon} |F(V_u)|^\alpha u^{-\beta\alpha} du \right] \\ & \leq C_\alpha \mathbb{E} [|B_{t-s}|^\alpha] + C_\alpha \varepsilon^{1-\frac{\alpha}{2}} (t-s)^{\alpha-1} \int_{s/\varepsilon}^{t/\varepsilon} u^{\frac{\gamma\alpha}{2} - \beta\alpha} du \\ & \leq C_\alpha (t-s)^{\frac{\alpha}{2}} + C_\alpha \varepsilon^{1-\frac{\alpha}{2}} (t-s)^{\alpha-1} \int_{s/\varepsilon}^{t/\varepsilon} u^{-\frac{\alpha}{2}} du \\ & \leq C_\alpha (t-s)^{\frac{\alpha}{2}} + C_\alpha (t-s)^{\alpha-1} (t^{1-\frac{\alpha}{2}} - s^{1-\frac{\alpha}{2}}) \\ & \leq C_\alpha (t-s)^{\frac{\alpha}{2}} + C_{\alpha, m, M} (t-s)^{\alpha-1} \\ & \leq C_{\alpha, m, M} (t-s)^{\frac{\alpha}{2}}. \end{aligned}$$

Since $\alpha > 2$, then $\frac{\alpha}{2} > 1$ and the upper bound does not depend on ε . Furthermore, by moment estimates (Proposition 1.4.1),

$$\sup_{\varepsilon \leq \varepsilon_0} \mathbb{E} \left[|V_m^{(\varepsilon)}| \right] \leq \sqrt{m} < +\infty.$$

CONCLUSION. The previous steps yield weak convergence on every compact set (Theorem

13.1 p. 139 in [Bil99]). The conclusion follows from Theorem 16.7 p. 174 in [Bil99], since all processes considered are continuous. □

Example 1.5.4. We will see that the limiting process \mathcal{V} is more explicit in the linear case ($\gamma = 1$). Choose $F(1) = 1$, $F(-1) = -1$, the process \widetilde{H} solution to (1.14) is in fact an Ornstein-Uhlenbeck process with invariant measure $\Lambda_F(dx) := e^{-\frac{3x^2}{2}} dx$. It is a centered Gaussian process, hence for all s_1, \dots, s_d , its f.d.d. $\Lambda_{F, s_1, \dots, s_d}$ are Gaussian. As a consequence, knowing the covariance function K is enough to provide the distribution of the process. Since \widetilde{H} is a stationary Ornstein-Uhlenbeck process, $K(s, t) = \frac{1}{3}e^{-\frac{3}{2}|t-s|}$. Hence, the limiting process \mathcal{V} having f.d.d $T * \Lambda_{F, \log(t_1), \dots, \log(t_d)}$ is a centered Gaussian process with covariance function $(s, t) \mapsto \frac{1}{3} \frac{(s \wedge t)^2}{s \vee t}$.

1.5.3 Asymptotic behavior in the sub-critical regime

Assume in this section that $\beta < \frac{\gamma+1}{2}$ and $F : v \mapsto \rho \operatorname{sgn}(v) |v|^\gamma$ with $\gamma \geq 1$. For simplicity, we shall write φ instead of φ_q .

Proof of Theorem 1.2.5. STEP 1. We first prove the f.d.d. convergence of the velocity process $(V_t^{(\varepsilon)})_{t \geq \varepsilon t_0} := (\varepsilon^q V_{t/\varepsilon})_{t \geq \varepsilon t_0}$. Again we give a proof only for $d = 2$, since the general case $d \geq 2$ is similar.

The power scaling process $V^{(q)}$, solution to (1.3) satisfies

$$dV_s^{(q)} = dW_s - F\left(V_s^{(q)}\right) ds - q\varphi^{2q-1}(s)V_s^{(q)} ds.$$

We call H the ergodic process solution to the SDE

$$dH_s = dW_s - F\left(H_s\right) ds, \quad \text{with } H_0 = h_0 := v_0 t_0^{-q}. \quad (1.18)$$

We denote by $\Pi_F(dx) := e^{-\frac{2\rho}{\gamma+1}|x|^{\gamma+1}} dx$ its invariant measure. Using the bijection induced by the power change of time (Proposition 1.3.1), as solutions to the same SDE starting at the same point, we have, for all $\varepsilon > 0$, and $(s, t) \in [\varepsilon t_0, +\infty)^2$,

$$\left(\varepsilon^q \frac{V_{\varepsilon^{-1}s}}{s^q}, \varepsilon^q \frac{V_{\varepsilon^{-1}t}}{t^q} \right) = \left(V_{\varphi^{-1}(\varepsilon^{-1}s)}^{(q)}, V_{\varphi^{-1}(\varepsilon^{-1}t)}^{(q)} \right).$$

Using Theorem 3.1 p. 27 in [Bil99], it suffices to prove that for all $(s, t) \in [\varepsilon t_0, +\infty)^2$,

- $\left\| \left(H_{\varphi^{-1}(\varepsilon^{-1}s)}, H_{\varphi^{-1}(\varepsilon^{-1}t)} \right) - \left(V_{\varphi^{-1}(\varepsilon^{-1}s)}^{(q)}, V_{\varphi^{-1}(\varepsilon^{-1}t)}^{(q)} \right) \right\| \xrightarrow{\varepsilon \rightarrow 0} 0.$
- $\left(H_{\varphi^{-1}(\varepsilon^{-1}s)}, H_{\varphi^{-1}(\varepsilon^{-1}t)} \right) \xrightarrow{\varepsilon \rightarrow 0} \Pi_F \otimes \Pi_F.$

STEP 2. We prove that $\mathbb{E} \left[\left(H_t - V_t^{(q)} \right)^2 \right] \xrightarrow{t \rightarrow +\infty} 0.$

We have

$$d \left(H - V^{(q)} \right)_t = - \left(F(H_t) - F(V_t^{(q)}) \right) dt + q\varphi^{2q-1}(t)V_t^{(q)} dt.$$

By straightforward differentiation, we can write

$$d \left(H - V^{(q)} \right)_t^2 = -2 \left(F(H_t) - F(V_t^{(q)}) \right) \left(H_t - V_t^{(q)} \right) dt + 2q\varphi^{2q-1}(t)V_t^{(q)} \left(H_t - V_t^{(q)} \right) dt. \quad (1.19)$$

We set

$$g(t) := \mathbb{E} \left[\left(H_t - V_t^{(q)} \right)^2 \right], \quad t \geq 0.$$

Taking expectation in (1.19), we get

$$g'(t) = -2\mathbb{E} \left[\left(F(H_t) - F(V_t^{(q)}) \right) \left(H_t - V_t^{(q)} \right) \right] + 2q\varphi^{2q-1}(t)\mathbb{E} \left[V_t^{(q)} \left(H_t - V_t^{(q)} \right) \right].$$

Since $\gamma \geq 1$, the function F^{-1} is $\frac{1}{\gamma}$ -Hölder, therefore there exists $C_\gamma > 0$ such that,

$$g'(t) \leq -C_\gamma \mathbb{E} \left[\left(H_t - V_t^{(q)} \right)^{1+\gamma} \right] + 2q\varphi^{2q-1}(t)\mathbb{E} \left[V_t^{(q)} \left(H_t - V_t^{(q)} \right) \right].$$

Then, by Jensen's inequality, since $\gamma \geq 1$,

$$g'(t) \leq -C_\gamma g(t)^{\frac{\gamma+1}{2}} + 2q\varphi^{2q-1}(t)\mathbb{E} \left[V_t^{(q)} \left(H_t - V_t^{(q)} \right) \right].$$

Using Cauchy-Schwarz inequality and moment estimates (Proposition 1.4.1), we have

$$g'(t) \leq -C_\gamma g(t)^{\frac{\gamma+1}{2}} + C |q| \varphi^{q-\frac{1}{2}}(t) \sqrt{g(t)}, \quad g(0) = 0.$$

Note that since $2q < 1$, then $\varphi^{q-\frac{1}{2}}(t) \xrightarrow{t \rightarrow +\infty} 0$, therefore the conclusion follows from

Lemma 1.A.3. Besides, for all $t \geq \varepsilon t_0$, $\mathbb{E} \left[\left(H_{\varphi^{-1}(\varepsilon^{-1}t)} - V_{\varphi^{-1}(\varepsilon^{-1}t)}^{(q)} \right)^2 \right] = g(\varphi^{-1}(\varepsilon^{-1}t)) \xrightarrow{\varepsilon \rightarrow 0} 0.$

STEP 3. Pick $(s, t) \in [\varepsilon t_0, +\infty)^2$. We prove that the solution H to (1.18) satisfies

$$\left(H_{\varphi^{-1}(\varepsilon^{-1}s)}, H_{\varphi^{-1}(\varepsilon^{-1}t)} \right) \xrightarrow{\varepsilon \rightarrow 0} \Pi_F \otimes \Pi_F. \quad (1.20)$$

Observe that

$$\varphi^{-1}(\varepsilon^{-1}t) - \varphi^{-1}(\varepsilon^{-1}s) = \frac{t^{1-2q} - s^{1-2q}}{\varepsilon^{1-2q}} \xrightarrow{\varepsilon \rightarrow 0} +\infty. \quad (1.21)$$

By Lemma 1.5.3, for every continuous and bounded function ψ , we can write

$$\mathbb{E} \left[\psi \left(H_{\varphi^{-1}(\varepsilon^{-1}s)}, H_{\varphi^{-1}(\varepsilon^{-1}t)} \right) \middle| H_0 = h_0 \right] - \mathbb{E} \left[\psi \left(H_{\varphi^{-1}(\varepsilon^{-1}s)}, H_{\varphi^{-1}(\varepsilon^{-1}t)} \right) \middle| H_0 \sim \Pi_F \right] \xrightarrow{\varepsilon \rightarrow 0} 0.$$

Hence, it suffices to prove that for every bounded continuous functions $f, g : \mathbb{R} \rightarrow \mathbb{R}$, the following convergence holds

$$\lim_{\varepsilon \rightarrow 0} \mathbb{E} \left[f \left(H_{\varphi^{-1}(\varepsilon^{-1}s)} \right) g \left(H_{\varphi^{-1}(\varepsilon^{-1}t)} \right) \middle| H_0 \sim \Pi_F \right] = \Pi_F(f) \Pi_F(g).$$

The following reasoning is inspired from the proof of Lemma 3.2 p. 7-8 in [CCM10]. Since H_0 is starting from the invariant measure, up to considering $f - \Pi_F(f)$ and $g - \Pi_F(g)$, we can assume that f and g have zero Π_F -mean. We call $(P_t)_{t \geq 0}$ the semigroup of H , then we get, by invariance property of Π_F ,

$$\begin{aligned} \mathbb{E} \left[f \left(H_{\varphi^{-1}(\varepsilon^{-1}s)} \right) g \left(H_{\varphi^{-1}(\varepsilon^{-1}t)} \right) \middle| H_0 \sim \Pi_F \right] &= \int P_{\varphi^{-1}(\varepsilon^{-1}s)} \left(f P_{\varphi^{-1}(\varepsilon^{-1}t) - \varphi^{-1}(\varepsilon^{-1}s)} g \right) d\Pi_F \\ &= \int f P_{\varphi^{-1}(\varepsilon^{-1}t) - \varphi^{-1}(\varepsilon^{-1}s)} g d\Pi_F. \end{aligned}$$

Note that $U : v \mapsto \frac{|v|^{1+\gamma}}{1+\gamma}$ is a convex function, thus a λ -Poincaré inequality holds for the process H (see [Bob99] p. 1904). This implies the exponential decay of the variance (see Theorem 4.2.5 p. 183 in [BGL14]), i.e. there exists a constant $C > 0$ such that, since Π_F is a probability measure,

$$\begin{aligned} \left| \int f P_{\varphi^{-1}(\varepsilon^{-1}t) - \varphi^{-1}(\varepsilon^{-1}s)} g d\Pi_F \right| &\leq \left\| f P_{\varphi^{-1}(\varepsilon^{-1}t) - \varphi^{-1}(\varepsilon^{-1}s)} g \right\|_2 \\ &\leq \|f\|_\infty \left\| P_{\varphi^{-1}(\varepsilon^{-1}t) - \varphi^{-1}(\varepsilon^{-1}s)} g \right\|_2 \\ &\leq C \|f\|_\infty \|g\|_\infty e^{-\lambda(\varphi^{-1}(\varepsilon^{-1}t) - \varphi^{-1}(\varepsilon^{-1}s))}. \end{aligned}$$

We deduce (1.20) from (1.21).

STEP 4. We prove the convergence of the f.d.d. of the position process.

We set $(X_t^{(\varepsilon)})_{t \geq \varepsilon t_0} := (\varepsilon^{\beta + \frac{1}{2}} X_{t/\varepsilon})_{t \geq \varepsilon t_0}$. Take $\gamma = 1$ and $\beta \in (-\frac{1}{2}, 1)$. Pick $t \geq \varepsilon t_0$. By Itô's

formula applied to $t^\beta V_t$, we get

$$\rho X_t^{(\varepsilon)} = \varepsilon^{\beta+\frac{1}{2}}(t_0^\beta v_0 + x_0) - \varepsilon^{\frac{1-\beta}{2}} t^\beta V_t^{(\varepsilon)} + \varepsilon^{\beta+\frac{1}{2}} \int_{t_0}^{t/\varepsilon} s^\beta dB_s + \varepsilon^{\beta+\frac{1}{2}} \int_{t_0}^{t/\varepsilon} \beta s^{\beta-1} V_s ds.$$

Since $\beta > -\frac{1}{2}$, the first term converges to 0 in probability as $\varepsilon \rightarrow 0$. Moreover, by Itô's formula, for all $t \geq t_0$,

$$\frac{d}{dt} \mathbb{E} [V_t^2] = -2\rho s^{-\beta} \mathbb{E} [V_s^2] + 1.$$

Hence, by comparison theorem for ordinary differential equation,

$$\mathbb{E} [V_t^2] \leq \exp\left(-2\rho \frac{t^{1-\beta}}{1-\beta}\right) \left(v_0^2 + \int_{t_0}^t \exp\left(2\rho \frac{s^{1-\beta}}{1-\beta}\right) ds \right).$$

Using an integration by parts, we deduce that there exists a positive constant C such that, for all $t \geq t_0$,

$$\mathbb{E} [V_t^2] \leq Ct^\beta.$$

As a consequence, we obtain

$$\begin{aligned} \mathbb{E} \left[\left| -\varepsilon^{\frac{1-\beta}{2}} t^\beta V_t^{(\varepsilon)} + \varepsilon^{\beta+\frac{1}{2}} \int_{t_0}^{t/\varepsilon} \beta s^{\beta-1} V_s ds \right| \right] &\leq \varepsilon^{\frac{1-\beta}{2}} t^\beta \mathbb{E} [|V_t^{(\varepsilon)}|] + \varepsilon^{\beta+\frac{1}{2}} \int_{t_0}^{t/\varepsilon} \beta s^{\beta-1} \mathbb{E} [|V_s|] ds \\ &\leq C\varepsilon^{\frac{1}{2}} t^{\frac{3\beta}{2}} + C\varepsilon^{\frac{1-\beta}{2}} t^{\frac{3\beta}{2}} - C\varepsilon^{\beta+\frac{1}{2}} t_0^{\frac{3\beta}{2}} \xrightarrow{\varepsilon \rightarrow 0} 0. \end{aligned}$$

It remains to study the centered Gaussian process $M_t^{(\varepsilon)} := \varepsilon^{\beta+\frac{1}{2}} \int_{t_0}^{t/\varepsilon} s^\beta dB_s$. By Itô's isometry and since $\beta > -\frac{1}{2}$, for all $\varepsilon t_0 \leq s \leq t$, we can write

$$\text{Cov}(M_s^{(\varepsilon)}, M_t^{(\varepsilon)}) = \varepsilon^{2\beta+1} \int_{t_0}^{s/\varepsilon} u^{2\beta} ds \underset{\varepsilon \rightarrow 0}{\sim} \frac{s^{1+2\beta}}{1+2\beta}.$$

Since the convergence of centered Gaussian processes is characterized by the convergence of their covariance function, the conclusion follows from Theorem 3.1 p. 27 in [Bil99]. □

1.6 An extension to multiplicative noise

Let σ be a continuous function on $[0, +\infty) \times \mathbb{R}$ and define for $\rho > 0$, the function $F : v \mapsto \rho \operatorname{sgn}(v) |v|^\gamma$. Consider the one-dimensional SDE defined, for $t \geq t_0$, by

$$dV_t = \sigma(t, V_t) dB_t - F(V_t) dt. \quad (SDE_\sigma)$$

We assume in the following that there exists a unique solution to (SDE_σ) . Notice that the (SDE_σ) is an extension to the one studied in [GO13].

1.6.1 Clock change

If the diffusion function σ depends only on the time parameter, i.e. $\sigma(t, v) = \sigma(t)$, then the diffusion term is a Brownian motion with a “clock change”. Furthermore, in the super-critical regime, the asymptotic behavior depends on the one of the centered Gaussian process

$$M_t^{(\varepsilon)} := \int_{t_0}^{t/\varepsilon} \sigma(s) dB_s.$$

More precisely, if there exists $f \notin L^1(+\infty)$ such that $\sigma^2 \sim_{+\infty} f$, then

$$K^{(\varepsilon)}(s, t) := \operatorname{Cov}(M_s^{(\varepsilon)}, M_t^{(\varepsilon)}) \underset{\varepsilon \rightarrow 0}{\sim} \int_{t_0}^{(s \wedge t)/\varepsilon} f(s) ds.$$

Example 1.6.1. Assume that f is a bounded T -periodic continuous function. One can keep in mind the square of cosinus or sinus functions. Setting $[x]$ for the integral part of a real x , we have

$$\varepsilon K^{(\varepsilon)}(s, t) \underset{\varepsilon \rightarrow 0}{\sim} \left\lfloor \frac{s \wedge t}{T} \right\rfloor \int_{t_0}^{t_0+T} f(u) du.$$

Consequently, $(\sqrt{\varepsilon} M_t^\varepsilon)_{t \geq \varepsilon t_0}$ converges in f.d.d. to the centered Gaussian process with covariance function $K(s, t) := \left\lfloor \frac{s \wedge t}{T} \right\rfloor \int_{t_0}^{t_0+T} f(u) du$.

For example, if $f \in \{\cos^2, \sin^2\}$, then $K(s, t) = \left\lfloor \frac{s \wedge t}{2\pi} \right\rfloor \pi$. Moment estimates can be computed as in Section 1.4, since the diffusion coefficient is bounded. The tightness criterion checking follows the same line as in Section 1.5.2. As a conclusion, the process $(\sqrt{\varepsilon} V_{t/\varepsilon})_{t \geq \varepsilon t_0}$ converges to the centered Gaussian process with covariance function K .

In both critical and sub-critical regimes, the changed-of-time process is inefficient to conclude. Indeed, the general diffusion coefficient does not satisfy a scaling transformation and thus the changed-of-time process get a time-inhomogeneous diffusion term.

1.6.2 Diffusion coefficient with scaling property

Let us switch now to another appropriate diffusion coefficient. The scaling transformation Φ_φ , applied to an SDE with multiplicative noise, leads to the following result.

Proposition 1.6.2. *Let $\varphi : [0, t_1) \rightarrow [t_0, \infty)$ be a C^2 -diffeomorphism. If V is a solution to the equation (SDE_σ) , then $V^{(\varphi)} := \Phi_\varphi(V)$ is a solution to*

$$dV_s^{(\varphi)} = \sigma\left(\varphi(s), V_s^{(\varphi)} \sqrt{\varphi'(s)}\right) dW_s - \frac{\varphi'(s)^{\frac{\gamma+1}{2}}}{\varphi(s)^\beta} F(V_s^{(\varphi)}) ds - \frac{\varphi''(s)}{\varphi'(s)} \frac{V_s^{(\varphi)}}{2} ds, \quad V_0^{(\varphi)} = \frac{V_{\varphi(0)}}{\sqrt{\varphi'(0)}}, \quad (1.22)$$

where $W_t := \int_0^t \frac{dB_{\varphi(s)}}{\sqrt{\varphi'(s)}}$.

If $V^{(\varphi)}$ is a solution to the equation (1.22), then $\Phi_\varphi^{-1}(V^{(\varphi)})$ is a solution to the equation (SDE_σ) , where $B_t - B_{t_0} := \int_{t_0}^t \sqrt{(\varphi' \circ \varphi^{-1})(s)} dW_{\varphi^{-1}(s)}$.

Furthermore, uniqueness in law, pathwise uniqueness or strong existence hold for the equation (SDE_σ) if and only if they hold for the equation (1.2).

Pick a continuous function f and suppose that

$$\text{for all } (s, v) \in [0, +\infty) \times \mathbb{R}, \quad \sigma(s, v) = f\left(\frac{v}{\sqrt{s}}\right).$$

We suppose that the following usual condition holds: there exists a positive increasing function h such that

$$\begin{aligned} \forall (x, y) \in \mathbb{R}^2, \quad |f(x) - f(y)| &\leq h(|x - y|), \\ \text{and } \int_{0+} h^{-2}(u) du &= +\infty. \end{aligned} \quad (1.23)$$

Furthermore, assume that the function f satisfies the conditions

$$\begin{aligned} \int_{-\infty}^0 \exp\left(\int_0^x \frac{s}{f^2(s)} ds\right) dx &= \int_0^{+\infty} \exp\left(\int_0^x \frac{s}{f^2(s)} ds\right) dx = +\infty, \\ \text{and } x \mapsto \frac{1}{f^2(x)} \exp\left(-\int_0^x \frac{s}{f^2(s)} ds\right) &\in L^1(\mathbb{R}). \end{aligned} \quad (H_1^f)$$

We introduce another assumption on f , which will sometimes be imposed in the sequel

$$\int_{-\infty}^0 \exp\left(\int_0^x \frac{2F(s)}{f^2(s)} ds\right) dx = \int_0^{+\infty} \exp\left(\int_0^x \frac{2F(s)}{f^2(s)} ds\right) dx = +\infty, \quad (H_2^f)$$

and $x \mapsto \frac{1}{f^2(x)} \exp\left(-\int_0^x \frac{2F(s)}{f^2(s)} ds\right) \in L^1(\mathbb{R})$.

As an example of a function satisfying (H_1^f) one can keep in mind $f : x \mapsto |x|^a$ with $a \in [0, \frac{1}{2})$. Moreover, the function $f : x \mapsto |x|^a$ with $a \in [0, \frac{1+\gamma}{2} \wedge \frac{1}{2})$ satisfies (H_1^f) and (H_2^f) .

- On the one hand, the transformation associated to the exponential change of time $\varphi_e : t \mapsto t_0 e^t$ satisfies

$$dV_s^{(e)} = f(V_s^{(e)}) dW_s - t_0^{\frac{\gamma+1}{2}-\beta} e^{(\frac{\gamma+1}{2}-\beta)s} F(V_s^{(e)}) ds - \frac{V_s^{(e)}}{2} ds. \quad (1.24)$$

- On the other hand, the transformation associated to the power change of time

$$\varphi_q : t \mapsto (t_0^{1-2q} + (1-2q)t)^{\frac{1}{1-2q}}$$

satisfies

$$dV_s^{(q)} = f(V_s^{(q)}) dW_s - F(V_s^{(q)}) ds - q\varphi_q^{2q-1} V_s^{(q)} ds. \quad (1.25)$$

Remark 1.6.3. *Following the same lines as in the proof of Proposition 1.3.2, one can prove that there exists a pathwise unique strong solution to (SDE_σ) , defined up to the explosion time. Hence, by Proposition 1.6.2, there exists a pathwise unique strong solution to (1.24) and (1.25).*

We introduce the following probability distributions, where k and k^\pm denotes normalization constants,

$$\mu(dx) := \frac{k^+}{f^2(x)} \exp\left(\int_0^x \frac{-2s}{f^2(s)} ds\right) dx,$$

$$\Lambda(dx) := \frac{k}{f^2(x)} \exp\left(\int_0^x \frac{-2F(s) - 2s}{f^2(s)} ds\right) dx,$$

and

$$\Pi(dx) := \frac{k^-}{f^2(x)} \exp\left(\int_0^x \frac{-2F(s)}{f^2(s)} ds\right) dx.$$

Theorem 1.6.4. Consider $\gamma \geq 0$. Let $(V_t)_{t \geq t_0}$ be the solution to (SDE_σ) such that for all $(s, v) \in [0, +\infty) \times \mathbb{R}$, $\sigma(s, v) = f\left(\frac{v}{\sqrt{s}}\right)$. Assume that (H_1^f) is satisfied.

(i) Assume that $2q > 1$. Then,

$$\frac{V_t}{\sqrt{t}} \xrightarrow[t \rightarrow +\infty]{} \mu.$$

(ii) Assume that $2q = 1$. Then,

$$\frac{V_t}{\sqrt{t}} \xrightarrow[t \rightarrow +\infty]{} \Lambda.$$

(iii) Assume that $2q > 1$ and that (H_2^f) is satisfied. Then,

$$\frac{V_t}{t^q} \xrightarrow[t \rightarrow +\infty]{} \Pi.$$

Proof. In the sequel, the ergodic property of time-homogeneous processes follows from Theorem 23.15 p. 465, in [Kal02] by studying their scale function and speed measure.

(i) Assume that $2q > 1$. Following the proof of Theorem 4.6 in [GO13], we prove that the asymptotic behavior of V is related to the one of the process solution to

$$dH_s = f(H_s) dW_s - \frac{H_s}{2} ds.$$

It is equivalent to prove

$$V_t^{(e)} \implies \mu.$$

Thanks to (H_1^f) , the process H is ergodic. Moreover, define for $v \in \mathbb{R}$,

$$b^-(v) := -\rho |v|^\gamma \mathbb{1}_{v \geq 0} - \frac{v}{2},$$

$$b^+(v) := \rho |v|^\gamma \mathbb{1}_{v \leq 0} - \frac{v}{2},$$

and denote by V^\pm the pathwise unique strong solution to equations

$$dV_s^\pm = f(V_s^\pm) dB_s + b^\pm(V_s^\pm) ds, \quad V_0^\pm = V_0^{(e)}.$$

By using (H_1^f) , V^+ and V^- are ergodic processes, and are thus bounded in probability. Using a comparison theorem (see Theorem 1.3 in [Yam73]), we prove that, for all $t \geq 0$, $V_t^- \leq V_t^{(e)} \leq V_t^+$ a.s. This implies that $V^{(e)}$ is bounded in probability. The conclusion follows from the asymptotic lemma (Lemma 4.5 in [GO13]).

- (ii) Assume that $2q = 1$. Let $V^{(e)}$ be the solution to the time-homogeneous SDE (1.24). Thanks to (H_1^f) , the process $V^{(e)}$ is ergodic. The conclusion is a direct consequence of the ergodic theorem.
- (iii) Assume that $2q < 1$. As in the proof of Theorem 4.9 in [GO13], it is equivalent to prove that

$$V_t^{(q)} \xrightarrow[t \rightarrow +\infty]{} \Pi. \quad (1.26)$$

Denote by Z the ergodic process solution to

$$dZ_s = f(Z_s) dW_s - F(Z_s) ds. \quad (1.27)$$

Thanks to (H_2^f) , Z is ergodic and satisfies (1.26). Let $u > 0$ be such that

$$\sup_{t \geq u} |q| \phi^{2q-1}(t) \leq 1.$$

Introduce Z^\pm the pathwise unique strong solution to

$$dZ_s^\pm = f(Z_s^\pm) dW_s - F(Z_s^\pm) ds \pm |Z_s^\pm| \mathbb{1}_{\pm Z_s^\pm \geq 0} ds, \quad Z_u^\pm = V_u^{(q)}.$$

Using a comparison theorem (see Theorem 1.3 in [Yam73]), we obtain that, for all $t \geq u$, $Z_t^- \leq V_t^{(q)} \leq Z_t^+$ a.s. By using (H_2^f) , Z^+ and Z^- are ergodic processes and thus $V^{(q)}$ is bounded in probability. The conclusion follows again from the asymptotic lemma (Lemma 4.5 in [GO13]).

□

1.A Some technical results

Let us state and prove a Grönwall-type lemma which has been used to get moment estimates.

Lemma 1.A.1 (Grönwall-type lemma). *Fix $r \in [0, 1)$ and $t_0 \in \mathbb{R}$. Assume that g is a non-negative real-valued function, b is a positive function and a is a differentiable real-valued function. Moreover, suppose that the function bg^r is continuous. If*

$$\forall t \geq t_0, g(t) \leq a(t) + \int_{t_0}^t b(s)g(s)^r ds, \quad (1.28)$$

then,

$$\forall t \geq t_0, \quad g(t) \leq 2^{\frac{1}{1-r}} \left[a(t) + \left((1-r) \int_{t_0}^t b(s) \, ds \right)^{\frac{1}{1-r}} \right].$$

Proof. For $t \geq t_0$, since $r \geq 0$,

$$g(t)^r \leq \left(a(t) + \int_{t_0}^t b(s)g(s)^r \, ds \right)^r,$$

then, multiplying by $b(t) > 0$,

$$b(t)g(t)^r \leq b(t) \left(a(t) + \int_{t_0}^t b(s)g(s)^r \, ds \right)^r.$$

Now, let us make appear the derivative of H

$$a'(t) + b(t)g(t)^r \leq a'(t) + b(t) \left(a(t) + \int_{t_0}^t b(s)g(s)^r \, ds \right)^r,$$

that is

$$\frac{a'(t) + b(t)g(t)^r}{\left(a(t) + \int_{t_0}^t b(s)g(s)^r \, ds \right)^r} \leq b(t) + \frac{a'(t)}{\left(a(t) + \int_{t_0}^t b(s)g(s)^r \, ds \right)^r} \leq b(t) + \frac{a'(t)}{a(t)^r}.$$

Integrating, since $r \neq 1$, we obtain

$$\frac{1}{1-r} \left[\left(a(t) + \int_{t_0}^t b(s)g(s)^r \, ds \right)^{1-r} - a(t_0)^{1-r} \right] \leq \frac{1}{1-r} \left[a(t)^{1-r} - a(t_0)^{1-r} \right] + \int_{t_0}^t b(s) \, ds$$

or equivalently, setting H for the right-hand side of (1.28) and using that $r < 1$, we get

$$H(t)^{1-r} \leq a(t)^{1-r} + (1-r) \int_{t_0}^t b(s) \, ds.$$

Since $\frac{1}{1-r} > 0$ and using (1.28)

$$g(t) \leq \left(a(t)^{1-r} + (1-r) \int_{t_0}^t b(s) \, ds \right)^{\frac{1}{1-r}} \leq C_r \left[a(t) + \left((1-r) \int_{t_0}^t b(s) \, ds \right)^{\frac{1}{1-r}} \right].$$

This concludes the proof of the lemma. □

Remark 1.A.2. Call H the right-hand side of (1.28). If g is not continuous, note that the function H is continuous and satisfies (1.28) (since b is positive and $g \leq H$). Therefore,

one can apply the lemma to H and then use the inequality $g \leq H$.

Let us now state and prove the following result.

Lemma 1.A.3. *Let b be a function such that $\lim_{t \rightarrow +\infty} b(t) = 0$. Pick $a > 0$ and $\gamma \geq 1$. Let g be a continuously differentiable positive function satisfying*

$$g'(t) \leq -ag(t)^{\frac{\gamma+1}{2}} + b(t)\sqrt{g(t)}, \quad t \geq 0. \quad (1.29)$$

Then, $g(t) \xrightarrow[t \rightarrow +\infty]{} 0$.

Proof. Pick $\varepsilon > 0$. Let t_1 be a positive real such that for all $t \geq t_1$, $|b(t)| \leq \frac{a}{2}\varepsilon^{\frac{\gamma}{2}}$.

STEP 1. We first show that there exists $t^* \geq t_1$, such that $g(t^*) \leq \varepsilon$.

Assume, by way of contradiction, that it is not the case. Thus, one can consider the function $y = \sqrt{g}$, which satisfies

$$2y'(t) \leq -ay(t)^\gamma + b(t), \quad t \geq t_1. \quad (1.30)$$

For all $t \geq t_1$, we have

$$2y'(t) \leq -a\varepsilon^{\frac{\gamma}{2}} + \frac{a}{2}\varepsilon^{\frac{\gamma}{2}} \leq -\frac{a}{2}\varepsilon^{\frac{\gamma}{2}}.$$

As a consequence, for all $t \geq t_1$,

$$2\sqrt{\varepsilon} < 2y(t) \leq 2y(t_1) - (t - t_1)\frac{a}{2}\varepsilon^{\frac{\gamma}{2}} \xrightarrow[t \rightarrow +\infty]{} -\infty.$$

This is a contradiction.

STEP 2. We show that for all $t \geq t^*$, $g(t) \leq \varepsilon$.

Define $T = \inf\{t \geq t^*, g(t) > \varepsilon\}$. By continuity of the function g , we have $g(T) = \varepsilon$.

Hence,

$$g'(T) \leq -a\varepsilon^{\frac{\gamma+1}{2}} + \frac{a}{2}\varepsilon^{\frac{\gamma}{2}}\sqrt{\varepsilon} < -\frac{a}{2}\varepsilon^{\frac{\gamma+1}{2}} < 0.$$

Therefore, there exists $\delta_0 > 0$, such that for all $0 < \delta \leq \delta_0$,

$$g(T + \delta) < g(T) = \varepsilon.$$

This is a contradiction with the definition of T .

□

Acknowledgements

The authors would like to thank Jürgen Angst for valuable exchanges and suggestions about this work and Thomas Cavallazzi for his careful reading of the manuscript. We would also like to thank the anonymous Referees for her/his careful reading of the manuscript and useful advice.

BEHAVIOR OF A TIME-INHOMOGENEOUS KINETIC LÉVY-DRIVEN MODEL

Abstract: We study a one-dimensional kinetic stochastic model driven by a Lévy process with a non-linear time-inhomogeneous drift. More precisely, the process (V, X) is considered, where X is the position of the particle and its velocity V is the solution to a stochastic differential equation with a drift of the form $t^{-\beta}F(v)$. The driving process can be a stable Lévy process of index α or a general Lévy process under appropriate assumptions. The function F satisfies a homogeneity condition and β is a real number. The behavior in large time of the process (V, X) is proved and the precise rate of convergence is pointed out by using stochastic analysis tools. To this end, we compute the moment estimates of the velocity process.

Keywords: kinetic stochastic equation; time-inhomogeneous stochastic differential equation; Lévy process; explosion time; scaling transformation; asymptotic distribution; ergodicity; tightness.

MSC2020 Subject Classification: Primary 60H10; Secondary 60F17; 60G52; 60G18; 60J65.

This chapter [GL21b] has been submitted.

2.1 Introduction

In this chapter, we consider a one-dimensional stochastic kinetic model driven by a Lévy process.

$$dV_t = dL_t - F(V_t)t^{-\beta} dt \quad \text{and} \quad X_t = X_0 + \int_0^t V_s ds. \quad (2.1)$$

The process $(V_t, X_t)_{t>0}$ may be thought of as the velocity and position processes of a particle subject to a friction force $F(v)t^{-\beta}$ and interacting with its environment.

Our purpose is to study the long-time behavior of solutions to (2.1) where L is an α -stable (non-symmetric) Lévy process. More precisely, we look for the convergence in distribution of the process $(V_{t/\varepsilon}, X_{t/\varepsilon})_{t>0}$, as $\varepsilon \rightarrow 0$, with an appropriate rate.

It is a simple observation when $F = 0$ to see that the rescaled process $(\varepsilon^{\frac{1}{\alpha}}V_{t/\varepsilon}, \varepsilon^{1+\frac{1}{\alpha}}X_{t/\varepsilon})_{t>0}$ converges in distribution towards the Kolmogorov process $(\mathcal{S}_t, \int_0^t \mathcal{S}_s ds)_{t>0}$, where \mathcal{S} has the same distribution as the driving process.

The goal of the present paper is to extend the results obtained in [GL21a], where the driving process is a Brownian motion ($\alpha = 2$).

The study of stochastic differential equations (SDEs) driven by a Lévy process is a topic of great interest (see [Bas03] for a survey). The α -stable perturbation is a generalization of the Gaussian case, and it is also motivated by some Langevin-type models in stochastic climate dynamics (see [Dit99]). So far, most of the papers present results about existence and uniqueness of solution, see for instance [AS09], [Don18], [Kur07], [Pil13], [CSZ18] and [CZZ17]. The coefficients of the studied SDE are often supposed to be time-homogeneous (see for instance [AS09] and [Don18]). Accordingly, the case of time-dependent coefficients is scarcely studied (see [CZZ17], [Kur07] and [Zha13]). In this situation, the usual tools associated with time-homogeneous equation may no longer be invoked.

Furthermore, few papers (see [AS09], [Pri+12], [Rek20]) present results about the asymptotic behavior of the solution of such SDEs. For instance, in [AS09] the authors give conditions for asymptotic stability of the solutions to a SDE driven by a Brownian motion and a compensated Poisson process, with coefficients that are supposed to satisfy usual global Lipschitz and growth assumptions. In [Pri+12], the authors establish the exponential ergodicity of the solutions to a SDE driven by an α -stable process, where the drift coefficient is supposed to be the sum of two components, one linear and the other

bounded. In these papers, coefficients are time-homogeneous. In a number of articles, the small noise influence of the solutions is analyzed. To our knowledge, the only works considering the long-time behavior are [FT21], in a time-homogeneous setting, and the present one.

Let us explain heuristically what the intuition of our analysis is. In long-time regime, we observe three schemes, depending on the balance between the space and time coefficients of the drift function with respect to α , the parameter of stability of the driving process. When the drag force is sufficiently “small at infinity”, the convergence towards the Kolmogorov process $(\mathcal{S}, \int_0^\cdot \mathcal{S})$ still holds. When the two terms in the stochastic equation of the velocity process offset, we still get a kinetic process of the form $(\mathcal{V}, \int_0^\cdot \mathcal{V})$, as limiting process. Though the process \mathcal{V} no longer has the same distribution as the driving Lévy process. Alternatively, when the drift swings with the random noise, the limiting process is no longer kinetic.

Proofs are mainly based on moment estimates and on the self-similarity of the driving process. By their scaling property, Lévy stable processes are natural extensions of the Brownian motion. However, the jump component of the Lévy noise brings difficulties. Indeed, by contrast with a Brownian motion, an α -stable Lévy process can only have moments of order $\kappa \in [0, \alpha)$. Thus, moment estimation of the velocity process stands as a significant part of our study (see Section 2.4). Moment estimates of Lévy and Lévy-type processes were studied in [LP06], [Küh17] and [DS15]. Nevertheless, the methods used can not be easily adapted to the solutions to a SDE. In fact, the key idea will be to make a non-homogeneous cutting of the jumps size of the driving process. As explained in [CM20] and references therein, the cutting threshold $\xi \mapsto \xi^{\frac{1}{\alpha}}$ makes appear integral terms satisfying the scaling property (see the proof of Proposition 2.4.4).

The proof of the critical and sub-critical cases (see Theorem 2.2.4) significantly relies on a change in both space and time, taking advantage of the scaling property of the driving process, to be close to a stationary time-homogeneous SDE, as performed in [AW09] and [GO13].

In addition, extensions for the solution to the SDE driven by a general Lévy process are stated in Theorems 2.6.1 and 2.6.2. Let us point out that the case analyzed in Section 2.6.3 agrees with the equation (1) p. 1442 in [Dit99], where the described Langevin equation has two noise terms, a white noise and a pure-jump noise. To this end, we first study the asymptotic behavior of the rescaled Lévy driving process $(r_\varepsilon L_{t/\varepsilon})_{t \geq 0}$, where r_ε is some

rate of convergence. Under appropriate assumptions on its Lévy measure, it converges in distribution, as ε goes to zero (see Proposition 2.6.3 below).

Here is the structure of the paper. In Section 2.2, we introduce some notations and state our main results. Theorems 2.2.2 and 2.2.4 are extension for the α -stable setting of the main results stated in [GL21a] in case of a Brownian driving process. We study the existence of the solution to (2.1) in Section 2.3. In Section 2.4, we give estimates of the moment, which also ensure the non-explosion of the velocity process. The proofs of our main results are presented in Section 2.5. Finally, Section 2.6 is devoted to the study of the convergence of a rescaled Lévy process and to the extension of Theorem 2.2.2.

2.2 Notations and main results

Let $(L_t)_{t \geq 0}$ be a Lévy process. Throughout the paper, we deal with L as an α -stable Lévy process with $\alpha \in (0, 2)$. We call ν its Lévy measure, given by

$$\nu(dz) = \frac{a_+ \mathbb{1}_{\{z > 0\}} + a_- \mathbb{1}_{\{z < 0\}}}{|z|^{1+\alpha}} dz, \quad \text{with } a_+, a_- \geq 0 \text{ and } a_+ + a_- > 0.$$

As a Lévy measure, it satisfies $\int_{\mathbb{R}^*} (1 \wedge z^2) \nu(dz) < +\infty$. By Lévy-Itô's decomposition, L is a pure-jump Lévy process and there exists a Poisson point measure N and its compensated Poisson measure \widetilde{N} such that, for all $t \geq 0$,

$$L_t = \begin{cases} \int_0^t \int_{\mathbb{R}^*} z N(ds, dz) & \text{if } \alpha \in (0, 1), \\ \int_0^t \int_{\{0 < |z| < 1\}} z \widetilde{N}(ds, dz) + \int_0^t \int_{\{|z| \geq 1\}} z N(ds, dz) & \text{if } \alpha = 1, \\ \int_0^t \int_{\mathbb{R}^*} z \widetilde{N}(ds, dz) & \text{if } \alpha \in (1, 2). \end{cases} \quad (2.2)$$

In Section 2.6, the case of a generalized Lévy driving process will be discussed.

The space of continuous functions $\mathcal{C}((0, +\infty), \mathbb{R})$ is endowed with the uniform topology

$$d_u : (f, g) \in \mathcal{C}((0, +\infty), \mathbb{R})^2 \mapsto \sum_{n=1}^{+\infty} \frac{1}{2^n} \min \left(1, \sup_{[\frac{1}{n}, n]} |f - g| \right).$$

Set

$$\Lambda := \{ \lambda : \mathbb{R}^+ \rightarrow \mathbb{R}^+, \text{ continuous and increasing function s.t. } \lambda(0) = 0, \lim_{t \rightarrow +\infty} \lambda(t) = +\infty \}$$

and

$$k_n(t) = \begin{cases} 1 & \text{if } \frac{1}{n} \leq t \leq n, \\ n+1-t & \text{if } n < t < n+1, \\ 0 & \text{if } n+1 \leq t. \end{cases}$$

The space of right-continuous with left limits (càdlàg) functions $\mathcal{D}((0, +\infty), \mathbb{R})$ is endowed with the Skorokhod topology d_s defined for $(f, g) \in \mathcal{D}((0, +\infty), \mathbb{R})^2$ by

$$\sum_{n=1}^{+\infty} \frac{1}{2^n} \left(1 \wedge \inf \left\{ a, \exists \lambda \in \Lambda, \sup_{s \neq t} \left| \log \frac{\lambda(t) - \lambda(s)}{t - s} \right| \leq a, \sup_{t \geq \frac{1}{n}} |k_n(t) (f \circ \lambda(t) - g(t))| \leq a \right\} \right).$$

For simplicity, we shall write \mathcal{C} and \mathcal{D} for $\mathcal{C}((0, +\infty), \mathbb{R})$ and $\mathcal{D}((0, +\infty), \mathbb{R})$, respectively. For a family $((Z_t^{(\varepsilon)})_{t>0})_{\varepsilon>0}$ of càdlàg processes, we write

$$(Z_t^{(\varepsilon)})_{t>0} \xrightarrow[\varepsilon \rightarrow 0]{} (Z_t)_{t>0},$$

if $(Z_t^{(\varepsilon)})_{t>0}$ converges in distribution to $(Z_t)_{t>0}$ in \mathcal{D} , as $\varepsilon \rightarrow 0$.

We write

$$(Z_t^{(\varepsilon)})_{t>0} \xrightarrow[\varepsilon \rightarrow 0]{\text{f.d.d.}} (Z_t)_{t>0},$$

if for all finite subsets $S \subset (0, +\infty)$, the vector $(Z_t^{(\varepsilon)})_{t \in S}$ converges in distribution to $(Z_t)_{t \in S}$ in \mathbb{R}^S , as $\varepsilon \rightarrow 0$.

Let β a real number and F a continuous function satisfying

$$\text{for some } \gamma \in \mathbb{R}, \forall v \in \mathbb{R}, \lambda > 0, F(\lambda v) = \lambda^\gamma F(v). \quad (H_\gamma)$$

We introduce another assumption on F , which will sometimes be imposed in the sequel.

$$\begin{aligned} & \text{When (i) } \alpha \in (0, 1] \text{ or (ii) } \alpha \in (1, 2) \text{ and } \gamma \geq 1, \\ & \text{we suppose furthermore that for all } v \in \mathbb{R}, vF(v) \geq 0. \end{aligned} \quad (H_{\text{sgn}})$$

We take an interest into the following one-dimensional stochastic kinetic model defined, for $t \geq t_0 > 0$, by

$$dV_t = dL_t - t^{-\beta} F(V_t) dt, \text{ with } V_{t_0} = v_0 > 0, \quad \text{and} \quad dX_t = V_t dt, \text{ with } X_{t_0} = x_0 \in \mathbb{R}. \quad (\text{SKE}_L)$$

In the following, sgn is the sign function with the convention that $\text{sgn}(0) = 0$. The abbreviation a.s. stands for almost surely. We denote by C some positive constants, which may change from line to line. We use the subscripts to indicate the parameters on which it depends. For instance, $C_{t_0, \alpha}$ denotes a constant depending on the parameters t_0 and α .

Remark 2.2.1. *If a function π satisfies (H_γ) , then for all $x \in \mathbb{R}$, $\pi(x) = \pi(\text{sgn}(x)) |x|^\gamma$.*

As an example of a function satisfying (H_γ) one can keep in mind $F : v \mapsto \text{sgn}(v) |v|^\gamma$ (see also [GO13]).

Let us state our main results.

Theorem 2.2.2. *Consider $\gamma \in (1 - \frac{\alpha}{2}, \alpha)$. Assume that (H_γ) and (H_{sgn}) are satisfied, and $\beta > 1 + \frac{\gamma-1}{\alpha}$. Let $(V_t, X_t)_{t \geq t_0}$ be the solution to (SKE_L) and $(\mathcal{S}_t)_{t \geq 0}$ be an α -stable process, having same distribution as $(L_t)_{t \geq 0}$.*

Then, in the space \mathcal{D} ,

$$(\varepsilon^{\frac{1}{\alpha}} V_{t/\varepsilon}, \varepsilon^{1+\frac{1}{\alpha}} X_{t/\varepsilon})_{t \geq \varepsilon t_0} \xrightarrow[\varepsilon \rightarrow 0]{} \left(\mathcal{S}_t, \int_0^t \mathcal{S}_s \, ds \right)_{t > 0}.$$

Remark 2.2.3. *Theorem 2.2.2 is also true when the following hypothesis holds instead of (H_γ) .*

$$\begin{aligned} &F \text{ is such that } (\text{SKE}_L) \text{ has a unique solution up to explosion and} \\ &|F| \leq G \text{ where } G \text{ is a positive function satisfying } (H_\gamma). \end{aligned} \tag{H'_\gamma}$$

For instance, the function $F : v \mapsto \frac{v}{(1+v^2)}$ (see also [FT21]) satisfies (H'_γ) (with $\gamma = 0$).

Theorem 2.2.4. *Consider $\gamma \in (1 - \frac{\alpha}{2}, \alpha)$ and $\beta = 1 + \frac{\gamma-1}{\alpha}$. Assume that (H_γ) and (H_{sgn}) are satisfied. Let $(V_t, X_t)_{t \geq t_0}$ be the solution to (SKE_L) .*

Call \widetilde{H} the eternal ergodic process solving the following SDE driven by an α -stable process $(R_t)_{t \geq 0}$ with same distribution as L and such that the distribution of $\widetilde{H}_{-\infty}$ is the invariant measure,

$$dH_s = dR_s - \frac{H_s}{\alpha} ds - F(H_s) ds.$$

We denote by $\Lambda_{F, t_1, \dots, t_d}$ the finite-dimensional distributions of \widetilde{H} . We call $(\mathcal{V}_t)_{t \geq 0}$ the process having as finite-dimensional distributions the pushforward measure of $\Lambda_{F, \log(t_1), \dots, \log(t_d)}$ by the linear map $T(u_1, \dots, u_d) := (t_1^{1/\alpha} u_1, \dots, t_d^{1/\alpha} u_d)$. Indeed, we have $\mathcal{V} = \left(t^{\frac{1}{\alpha}} \widetilde{H}_{\log(t)} \right)_{t \geq 0}$.

Then, under (H_γ) , the following convergence, in the space \mathcal{D} , holds

$$\left(\varepsilon^{\frac{1}{\alpha}}V_{t/\varepsilon}, \varepsilon^{1+\frac{1}{\alpha}}X_{t/\varepsilon}\right)_{t \geq \varepsilon t_0} \xrightarrow{\varepsilon \rightarrow 0} \left(\mathcal{V}_t, \int_0^t \mathcal{V}_s ds\right)_{t > 0}.$$

Theorem 2.2.5. Consider $\alpha > 1$, $\gamma \in [1, \alpha)$ and $\beta < 1 + \frac{\gamma-1}{\alpha}$. Assume that (H_γ) and (H_{sgn}) are satisfied. Let $(V_t, X_t)_{t \geq t_0}$ be the solution to (SKE_L) . Define $q := \frac{\beta}{\alpha + \gamma - 1} < \frac{1}{\alpha}$. Call \widehat{H} the ergodic process solving the following SDE driven by an α -stable process $(R_t)_{t \geq 0}$ with same distribution as L and starting at its invariant measure,

$$dH_s = dR_s - F(H_s) ds.$$

Call Π_F its invariant measure. We call $(\mathcal{V}_t)_{t \geq 0}$ the process whose f.d.d. are $T * (\Pi_F^{\otimes d})$: the pushforward measure of $\Pi_F^{\otimes d}$ by the linear map $T(u_1, \dots, u_d) := (t_1^q u_1, \dots, t_d^q u_d)$.

Then,

$$\left(\varepsilon^q V_{t/\varepsilon}\right)_{t \geq \varepsilon t_0} \xrightarrow[\varepsilon \rightarrow 0]{\text{f.d.d.}} (\mathcal{V}_t)_{t \geq 0}.$$

Remark 2.2.6. As we will see in Section 2.3, the assumption $\gamma > 1 - \frac{\alpha}{2}$ is needed in order to obtain the existence up to explosion of the solution under the hypothesis (H_γ) .

Remark 2.2.7. Let us point out that, during the proof of Theorem 2.2.2 and Theorem 2.2.4, we employ some moment estimates for the solution V . We state the estimates below.

Assume that (H_{sgn}) is satisfied. We suppose also that the hypothesis on the sign of F holds for $(\alpha, \gamma, \kappa) \in (1, 2) \times [0, 1] \times (1, \alpha)$. Then, for any $\alpha \in (0, 2)$, $\gamma \in \mathbb{R}$, $\beta \in \mathbb{R}$ and $\kappa \in [0, \alpha)$, there exists a constant $C_{\gamma, \kappa, \beta, t_0}$ such that,

$$\forall t \geq t_0, \quad \mathbb{E}[|V_t|^\kappa] \leq C_{\gamma, \kappa, \beta, t_0} t^{\frac{\kappa}{\alpha}}.$$

Note that the above bounds are the best possible, taking $F = 0$.

2.3 Existence up to explosion

In this section, we study the existence of the solution to (SKE_L) up to explosion time.

Remark 2.3.1. Assume that (H_γ) holds. If $0 < \gamma < 1$, then the function F is γ -Hölder and if $\gamma \geq 1$, it is locally Lipschitz.

Proposition 2.3.2. *Assume that (H_γ) is satisfied. There exists a pathwise unique strong solution to (SKE_L) , defined up to the explosion time, provided that*

(i) $1 - \frac{\alpha}{2} < \gamma < 1$ and $\beta \geq 0$ when $\alpha \in (0, 2)$.

(ii) $\gamma \geq 1$ when $\alpha > 1$.

Proof. If $\gamma \in (0, 1)$, the drift coefficient is γ -Hölder (see Remark 2.3.1) and locally bounded, thereby the conclusion of the first point follows from Remark 1.3 in [CZZ17].

Assume now that $\alpha > 1$ and $\gamma \geq 1$. The drift coefficient is locally Lipschitz (see Remark 2.3.1) and locally bounded, so we can apply Lemma 115 p. 78 in [Sit05] to get the pathwise uniqueness. Thanks to Theorem 137 p. 104 in [Sit05], it suffices to prove that there exists a weak solution.

The drift coefficient is continuous with respect to its two variables, so it is a locally bounded and measurable function. By a standard localization argument, using Theorem 9.1 p. 231 in [IW81], since the drift coefficient is locally Lipschitz, there is a unique solution defined up to explosion. \square

2.4 Moment estimates and non-explosion of the velocity process

In this section, we present estimates on moments of the velocity process V solution to (SKE_L) . This will be useful to conclude of the non-explosion of solution to (SKE_L) with Lemma 2.4.1, and to control some terms appearing along the proofs of Theorem 2.2.2 and 2.2.4 in Section 2.5.

Let V be the unique solution up to explosion time to (SKE_L) . For all $r \geq 0$, define the stopping time

$$\tau_r := \inf\{t \geq t_0, |V_t| \geq r\}. \quad (2.3)$$

Set $\tau_\infty := \lim_{r \rightarrow +\infty} \tau_r$ the explosion time of V .

We give first a sufficient condition for the non-explosion of a general process.

Lemma 2.4.1. *Let $(Y_t)_{t \geq t_0}$ be a càdlàg process and τ_∞ its explosion time. Assume that there exist two measurable and non-negative functions ϕ and b such that*

(i) ϕ is non-decreasing and $\lim_{r \rightarrow \infty} \phi(r) = +\infty$,

(ii) b is finite-valued,

(iii) and for all $t \geq t_0$,

$$\sup_{r \geq 0} \mathbb{E} [\phi(|Y_{t \wedge \tau_r}|)] \leq b(t). \quad (2.4)$$

Then $\tau_\infty = +\infty$ a.s.

Proof. Pick $t \geq t_0$. Using the definition of τ_r , the monotony of ϕ and (iii), we get, for all $r \geq 0$,

$$\phi(r) \mathbb{P}(\tau_r \leq t) \leq \mathbb{E} [\phi(Y_{\tau_r}) \mathbb{1}_{\{\tau_r \leq t\}}] \leq \mathbb{E} [\phi(Y_{\tau_r})] \leq b(t).$$

Thus, by Fatou's lemma,

$$0 \leq \mathbb{P}(\tau_\infty \leq t) \leq \liminf_{r \rightarrow \infty} \mathbb{P}(\tau_r \leq t) \leq b(t) \lim_{r \rightarrow \infty} \frac{1}{\phi(r)} = 0.$$

As a consequence,

$$0 \leq \mathbb{P}(\tau_\infty < +\infty) \leq \sum_{t \in \mathbb{Q}} \mathbb{P}(\tau_\infty \leq t) = 0.$$

This concludes the proof. \square

We will show that there exists a constant $C_{\gamma, \kappa, \beta, t_0}$ such that

$$\forall t \geq t_0, \quad \mathbb{E}[|V_t|^\kappa] \leq C_{\gamma, \kappa, \beta, t_0} t^{\frac{\kappa}{\alpha}}. \quad (2.5)$$

Proposition 2.4.2. *Pick $\alpha \in (0, 1)$. Assume that (H_{sgn}) holds. Recall that $(V_t)_{t \geq t_0}$ is the solution to (SKE_L) . For any γ, β , the explosion time τ_∞ is a.s. infinite and for all $\kappa \in [0, \alpha)$, there exists a constant C_{κ, t_0} such that, we have*

$$\forall t \geq t_0, \quad \mathbb{E}[|V_t|^\kappa] \leq C_{\kappa, t_0} t^{\frac{\kappa}{\alpha}}. \quad (2.6)$$

Proof. Fix $t \geq t_0$. Since $\alpha < 1$, the stable process can be written as

$$L_t = \int_0^t \int_{\mathbb{R}^*} z N(ds, dz) = \sum_{s \leq t} \Delta L_s.$$

Fix $\kappa \in [0, \alpha)$. Pick the sequence of \mathcal{C}^2 -functions $f_n : x \mapsto \sqrt{x^2 + \frac{1}{n}}$, which converges uniformly to the function $x \mapsto |x|$ on \mathbb{R} . Then, for all $n \geq 1$, we apply Itô's formula (see

Theorem 32 p. 78 in [Pro05]) to get

$$\begin{aligned} f_n(V_{t \wedge \tau_r}) &= f_n(v_0) - \int_{t_0}^{t \wedge \tau_r} \frac{f'_n(V_s)F(V_s)}{s^\beta} ds + \int_{t_0}^{t \wedge \tau_r} \int_{\mathbb{R}^*} (f_n(V_{s-} + z) - f_n(V_{s-})) N(ds, dz) \\ &\leq f_n(v_0) + \sum_{s \leq t \wedge \tau_r} (f_n(V_{s-} + \Delta L_s) - f_n(V_{s-})). \end{aligned}$$

The term $\int_{t_0}^{t \wedge \tau_r} f'_n(V_s)F(V_s)s^{-\beta} ds$ is non-negative, since (H_{sgn}) holds.

Hence, the previous equation can be written as

$$f_n(V_{t \wedge \tau_r}) \leq f_n(v_0) + \sum_{s \leq t \wedge \tau_r} (f_n(V_s) - f_n(V_{s-})).$$

Since $\|f'_n\|_\infty \leq 1$, we deduce that $(f_n(V_s) - f_n(V_{s-})) \leq |\Delta V_s| = |\Delta L_s|$, hence,

$$|V_{t \wedge \tau_r}| \leq f_n(V_{t \wedge \tau_r}) \leq f_n(v_0) + \sum_{s \leq t \wedge \tau_r} |\Delta L_s|.$$

Furthermore, since $\kappa < \alpha < 1$, we have

$$|V_{t \wedge \tau_r}|^\kappa \leq f_n(v_0)^\kappa + \left(\sum_{s \leq t \wedge \tau_r} |\Delta L_s| \right)^\kappa.$$

Taking the expectation, we get

$$\mathbb{E}[|V_{t \wedge \tau_r}|^\kappa] \leq \mathbb{E}[f_n(v_0)^\kappa] + \mathbb{E}\left[\left(\sum_{s \leq t} |\Delta L_s|\right)^\kappa\right].$$

Notice that the process $L_t^+ := \sum_{s \leq t} |\Delta L_s|$ is an α -stable process. Then, since $\kappa < \alpha$, letting $n \rightarrow +\infty$, we obtain

$$\mathbb{E}[|V_{t \wedge \tau_r}|^\kappa] \leq |v_0|^\kappa + \mathbb{E}\left[|L_t^+|^\kappa\right] \leq C_{t_0, \kappa} t^{\frac{\kappa}{\alpha}}.$$

Thanks to Lemma 2.4.1, we can conclude that the explosion time of V is a.s. infinite, and (2.6) follows, letting $r \rightarrow \infty$. \square

Proposition 2.4.3. *Pick $\alpha \in (1, 2)$. Recall that $(V_t)_{t \geq t_0}$ is the solution to (SKE_L) . For any $\gamma \in [0, 1)$ and any $\beta \in \mathbb{R}$, the explosion time τ_∞ is a.s. infinite and for all $\kappa \in [0, 1]$,*

there exists $C_{\gamma,\kappa,\beta,t_0}$ such that we have

$$\forall t \geq t_0, \quad \mathbb{E}[|V_t|^\kappa] \leq C_{\gamma,\kappa,\beta,t_0} \begin{cases} t^{\frac{\kappa}{\alpha}} & \text{if } \frac{\gamma-1}{\alpha} + 1 \leq \beta, \\ t^{\kappa \frac{1-\beta}{1-\gamma}} & \text{else.} \end{cases} \quad (2.7)$$

Proof of Proposition 2.4.3. Assume that $\gamma \in [0, 1)$ and fix $\kappa \in [0, 1]$. Then Jensen's inequality yields, for all $t \geq t_0$, $\mathbb{E}[|V_t|^\kappa] \leq \mathbb{E}[|V_t|]^\kappa$, hence it suffices to verify (2.7) only for $\kappa = 1$.

Recall that under (H_γ) , there exists a positive constant K , such that for all $v \in \mathbb{R}$, $|F(v)| \leq K|v|^\gamma$. Hence, we can write, for any $t \geq t_0$ and $r \geq 0$,

$$\begin{aligned} |V_{(t \wedge \tau_r)-}| &\leq |v_0 - L_{t_0}| + |L_{(t \wedge \tau_r)-}| + \int_{t_0}^{t \wedge \tau_r} s^{-\beta} |F(V_{s \wedge \tau_r})| ds \\ &\leq |v_0 - L_{t_0}| + |L_{(t \wedge \tau_r)-}| + K \int_{t_0}^{t \wedge \tau_r} s^{-\beta} |V_{s \wedge \tau_r}|^\gamma ds. \end{aligned}$$

Since L is an α -stable process, it has a finite first moment, which can be computed. Taking the expectation in the above inequality, we get, by choosing C_{t_0} big enough,

$$\begin{aligned} \mathbb{E}[|V_{(t \wedge \tau_r)-}|] &\leq \mathbb{E}[|v_0 - L_{t_0}|] + \mathbb{E}[|L_{(t \wedge \tau_r)-}|] + K \int_{t_0}^t s^{-\beta} \mathbb{E}[|V_{s \wedge \tau_r}|^\gamma] ds \\ &\leq C_{t_0} t^{\frac{1}{\alpha}} + K \int_{t_0}^t s^{-\beta} \mathbb{E}[|V_{s \wedge \tau_r}|^\gamma] ds. \end{aligned}$$

Recalling that τ_r is given by (2.3), the function $g_r : t \mapsto \mathbb{E}[|V_{(t \wedge \tau_r)-}|]$ is bounded by r . Applying a Grönwall-type lemma (see Lemma 2.A.1), we end up, for $\beta \neq 1$, with

$$\forall t \geq t_0, \quad \mathbb{E}[|V_{(t \wedge \tau_r)-}|] \leq C_\gamma \left[C_{t_0} t^{\frac{1}{\alpha}} + \left(\frac{1-\gamma}{1-\beta} K (t^{1-\beta} - t_0^{1-\beta}) \right)^{\frac{1}{1-\gamma}} \right].$$

The case $\beta = 1$ can be treated similarly. Thanks to Lemma 2.4.1, we conclude that the explosion time of V is a.s. infinite, and (2.7) follows from Fatou's lemma. \square

Proposition 2.4.4. *Pick $\alpha \in [1, 2)$. Assume here that for all $v \in \mathbb{R}$, $vF(v) \geq 0$. Recall that $(V_t)_{t \geq t_0}$ is the solution to (SKE_L) . For any $\gamma \in \mathbb{R}$ and any $\beta \in \mathbb{R}$, the explosion time τ_∞ is a.s. infinite and there exists C_{κ,t_0} such that*

$$\text{for } \kappa \in (0, \alpha), \quad \forall t \geq t_0, \quad \mathbb{E}[|V_t|^\kappa] \leq C_{\kappa,t_0} t^{\frac{\kappa}{\alpha}}. \quad (2.8)$$

Proof. The key idea is to slice the small and big jumps in a non-homogeneous way with respect to the function $\xi \mapsto \xi^{\frac{1}{\alpha}}$. We write the proof in the general setting of $\alpha \in (1, 2)$. When $\alpha = 1$, the proof is similar since ν is symmetric.

Pick $\xi \geq t_0$. As explained in [CM20] and references therein, by using this cutting threshold, the α -stable Lévy driving process can be written as

$$L_t - L_{t_0} = \int_{t_0}^t \int_{|z| \leq \xi^{\frac{1}{\alpha}}} z \widetilde{N}(ds, dz) + \int_{t_0}^t \int_{|z| > \xi^{\frac{1}{\alpha}}} z N(ds, dz) - \int_{t_0}^t \int_{|z| > \xi^{\frac{1}{\alpha}}} z \nu(dz) ds.$$

The two first integrals then satisfy the same scaling property as the α -stable Lévy driving process.

We can compute

$$\int_{|z| > \xi^{\frac{1}{\alpha}}} z \nu(dz) = \frac{a_+ - a_-}{\alpha - 1} \xi^{\frac{1}{\alpha} - 1}.$$

STEP 1. We first apply Itô's formula and estimate the expectation of each term for $\kappa \leq 1$, in order to get (2.8).

Fix $\eta > 0$ and define the \mathcal{C}^2 -function $f : v \mapsto (\eta + v^2)^{\kappa/2}$. For all $t \geq t_0$, by Itô's formula, using that for all $v \in \mathbb{R}$, $vF(v) \geq 0$, we have

$$f(V_{t \wedge \tau_r}) \leq f(V_0) - \frac{a_+ - a_-}{\alpha - 1} \xi^{\frac{1}{\alpha} - 1} \int_{t_0}^t \mathbb{1}_{\{s \leq \tau_r\}} f'(V_s) ds + M_t + R_t + S_t, \quad (2.9)$$

where

$$M_t := \int_{t_0}^t \int_{0 < |z| < \xi^{\frac{1}{\alpha}}} \mathbb{1}_{\{s \leq \tau_r\}} [f(V_{s-} + z) - f(V_{s-})] \widetilde{N}(ds, dz), \quad (2.10)$$

$$R_t := \int_{t_0}^t \int_{|z| \geq \xi^{\frac{1}{\alpha}}} \mathbb{1}_{\{s \leq \tau_r\}} [f(V_{s-} + z) - f(V_{s-})] N(ds, dz), \quad (2.11)$$

$$S_t := \int_{t_0}^t \int_{0 < |z| < \xi^{\frac{1}{\alpha}}} \mathbb{1}_{\{s \leq \tau_r\}} [f(V_s + z) - f(V_s) - z f'(V_s)] \nu(dz) ds. \quad (2.12)$$

Note that, since $\kappa < 1$, for all $v \in \mathbb{R}$,

$$|f'(v)| \leq \kappa \eta^{\frac{\kappa-1}{2}}. \quad (2.13)$$

Moreover, remark that for all $k > \alpha$,

$$\int_{0 < |z| < \xi^{\frac{1}{\alpha}}} |z|^k \nu(dz) = \frac{a_+ + a_-}{k - \alpha} \xi^{\frac{k}{\alpha} - 1}, \quad (2.14)$$

and for all $k < \alpha$,

$$\int_{|z| \geq \xi^{\frac{1}{\alpha}}} |z|^k \nu(dz) = \frac{a_+ + a_-}{\alpha - k} \xi^{\frac{k}{\alpha} - 1}. \quad (2.15)$$

We estimate expectations of M , R and S .

To that end, we first show that the local martingale $(M_t)_{t \geq t_0}$ is a martingale.

Fix $q \geq 2$ and $r \geq 0$. Set

$$I_t(q) := \int_{t_0}^t \int_{0 < |z| < \xi^{\frac{1}{\alpha}}} \mathbb{1}_{\{s \leq \tau_r\}} |f(V_{s-} + z) - f(V_{s-})|^q \nu(dz) ds.$$

Notice that, since for all $|v| \leq r$ and $|z| \leq \xi^{\frac{1}{\alpha}}$, $|f(v+z) - f(v)| \leq \left\| f' \mathbb{1}_{[-(r+\xi^{\frac{1}{\alpha}}), r+\xi^{\frac{1}{\alpha}}]} \right\|_{\infty} |z|$, so we have

$$I_t(q) \leq \left\| f' \mathbb{1}_{[-(r+\xi^{\frac{1}{\alpha}}), r+\xi^{\frac{1}{\alpha}}]} \right\|_{\infty}^q \int_{t_0}^t \int_{0 < |z| < \xi^{\frac{1}{\alpha}}} \mathbb{1}_{\{s \leq \tau_r\}} |z|^q \nu(dz) ds.$$

Hence, it is a finite quantity, since $q \geq 2$ and (2.14) holds. Therefore, for $q \geq 2$, by Kunita's inequality (see Theorem 4.4.23 p. 265 in [App09]), there exists $D_q > 0$ such that

$$\mathbb{E} \left[\sup_{t_0 \leq s \leq t} |M_s|^q \right] \leq D_q \left(\mathbb{E} [I_t(2)^{\frac{q}{2}}] + \mathbb{E} [I_t(q)] \right) < +\infty.$$

Hence, by Theorem 51 p. 38 in [Pro05], M is a martingale.

We estimate now the finite variation part S defined in (2.12). We use a similar idea as in the proof of Theorem 3.1 p. 3863 in [DS15]. Note that for all $v \in \mathbb{R}$,

$$\begin{aligned} |f''(v)| &= \kappa(2 - \kappa)v^2(v^2 + \eta)^{\frac{\kappa}{2} - 2} + \kappa(v^2 + \eta)^{\frac{\kappa}{2} - 1} \\ &= \kappa(2 - \kappa)v^2(v^2 + \eta)^{-1}(v^2 + \eta)^{\frac{\kappa}{2} - 1} + \kappa(v^2 + \eta)^{\frac{\kappa}{2} - 1} \\ &\leq \kappa(3 - \kappa)(v^2 + \eta)^{\frac{\kappa}{2} - 1} \\ &\leq \kappa(3 - \kappa)\eta^{\frac{\kappa}{2} - 1}, \text{ since } \frac{\kappa}{2} - 1 < 0. \end{aligned}$$

Assume that $|z| < \xi^{\frac{1}{\alpha}}$. Using Taylor's formula, we get a.s.

$$|f(V_s + z) - f(V_s) - z f'(V_s)| \leq \frac{1}{2} \kappa(3 - \kappa) \eta^{\frac{\kappa}{2} - 1} z^2.$$

Hence, we get the almost sure following bound

$$\left| \int_{0 < |z| < \xi^{\frac{1}{\alpha}}} (f(V_s + z) - f(V_s) - z f'(V_s)) \nu(dz) \right| \leq \frac{1}{2} \kappa (3 - \kappa) \eta^{\frac{\kappa}{2} - 1} \int_{0 < |z| < \xi^{\frac{1}{\alpha}}} z^2 \nu(dz).$$

Injecting (2.14), we get

$$\left| \int_{0 < |z| < \xi^{\frac{1}{\alpha}}} (f(V_s + z) - f(V_s) - z f'(V_s)) \nu(dz) \right| \leq \frac{1}{2} \kappa (3 - \kappa) \eta^{\frac{\kappa}{2} - 1} \frac{a_+ + a_-}{2 - \alpha} \xi^{\frac{2}{\alpha} - 1}. \quad (2.16)$$

It remains to study the Poisson integral R defined in (2.11), using Theorem 2.3.7 p. 106 in [App09]. Pick $\kappa \leq 1$, by Hölder property of power functions, we can write,

$$\begin{aligned} |f(v + z) - f(v)| &= \left| (\eta + (v + z)^2)^{\frac{\kappa}{2}} - ((v + z)^2)^{\frac{\kappa}{2}} + (v + z)^\kappa - v^\kappa + (v^2)^{\frac{\kappa}{2}} - (\eta + v^2)^{\frac{\kappa}{2}} \right| \\ &\leq 2\eta^{\frac{\kappa}{2}} + |z|^\kappa. \end{aligned}$$

We deduce that

$$\int_{|z| \geq \xi^{\frac{1}{\alpha}}} |f(V_s + z) - f(V_s)| \nu(dz) \leq \eta^{\frac{\kappa}{2}} \nu(|z| \geq \xi^{\frac{1}{\alpha}}) + \int_{|z| \geq \xi^{\frac{1}{\alpha}}} |z|^\kappa \nu(dz).$$

Injecting (2.15), this leads to

$$\int_{|z| \geq \xi^{\frac{1}{\alpha}}} |f(V_s + z) - f(V_s)| \nu(dz) \leq \eta^{\frac{\kappa}{2}} \frac{a_+ + a_-}{\alpha} \xi^{-1} + \frac{a_+ + a_-}{\alpha - \kappa} \xi^{\frac{\kappa}{\alpha} - 1}. \quad (2.17)$$

Gathering (2.13), (2.17) and (2.16), we get

$$\begin{aligned} \mathbb{E} [|V_{t \wedge \tau_r}|^\kappa] &\leq \mathbb{E} [f(V_{t \wedge \tau_r})] \leq \mathbb{E} [f(V_{t_0})] + t \xi^{-1} \times \\ &\left(\kappa \eta^{\frac{\kappa-1}{2}} \frac{a_+ - a_-}{\alpha - 1} \xi^{\frac{1}{\alpha}} + \eta^{\kappa/2} \frac{a_+ + a_-}{\alpha} + \frac{a_+ + a_-}{\alpha - \kappa} \xi^{\frac{\kappa}{\alpha}} + \frac{1}{2} \kappa (3 - \kappa) \eta^{\frac{\kappa}{2} - 1} \frac{a_+ + a_-}{2 - \alpha} \xi^{\frac{2}{\alpha}} \right). \end{aligned} \quad (2.18)$$

Choosing $\eta = t^{\frac{2}{\alpha}}$ and $\xi = t$, we get

$$\begin{aligned} \mathbb{E} [|V_{t \wedge \tau_r}|^\kappa] &\leq \mathbb{E} [f(V_{t_0})] + t^{\frac{\kappa}{\alpha}} \times \left(\kappa \frac{a_+ - a_-}{\alpha - 1} + \frac{a_+ + a_-}{\alpha} + \frac{a_+ + a_-}{\alpha - \kappa} + \frac{1}{2} \kappa (3 - \kappa) \frac{a_+ + a_-}{2 - \alpha} \right) \\ &\leq C_{\kappa, t_0} t^{\frac{\kappa}{\alpha}}. \end{aligned} \quad (2.19)$$

Thanks to Lemma 2.4.1, we can conclude that the explosion time of V is a.s. infinite and

letting $r \rightarrow +\infty$, for all $\kappa \in [0, 1]$,

$$\mathbb{E} [|V_t|^\kappa] \leq C_{\kappa, t_0} t^{\frac{\kappa}{\alpha}}. \quad (2.20)$$

STEP 2. Pick $\kappa \in (1, \alpha)$. We estimate R in another way, using again Theorem 2.3.7 p. 106 in [App09].

By Hölder property of power function and (2.15), we get

$$\begin{aligned} \int_{|z| \geq \xi^{\frac{1}{\alpha}}} |f(V_s + z) - f(V_s)| \nu(dz) &\leq \int_{|z| \geq \xi^{\frac{1}{\alpha}}} |2zV_s + z^2|^{\frac{\kappa}{2}} \nu(dz) \\ &\leq C_\kappa \left(\frac{a_+ + a_-}{\alpha - \kappa} \xi^{\frac{\kappa}{\alpha} - 1} + |V_s|^{\frac{\kappa}{2}} \frac{a_+ + a_-}{\alpha - \frac{\kappa}{2}} \xi^{\frac{\kappa}{2\alpha} - 1} \right). \end{aligned} \quad (2.21)$$

Gathering (2.16), (2.21) and then using that for all $v \in \mathbb{R}$, $|f'(v)| \leq \kappa |v|^{\kappa-1}$,

$$\begin{aligned} \mathbb{E} [|V_{t \wedge \tau_r}|^\kappa] &\leq \mathbb{E} [f(V_{t_0})] + \left(C_\kappa \frac{a_+ + a_-}{\alpha - \kappa} \xi^{\frac{\kappa}{\alpha} - 1} + \frac{1}{2} \kappa (3 - \kappa) \eta^{\frac{\kappa}{2} - 1} \frac{a_+ + a_-}{2 - \alpha} \xi^{\frac{2}{\alpha} - 1} \right) t \\ &\quad + \kappa \frac{a_+ - a_-}{\alpha - 1} \xi^{\frac{1}{\alpha} - 1} \int_{t_0}^t \mathbb{E} [|V_s|^{\kappa-1}] ds + C_\kappa \frac{a_+ + a_-}{\alpha - \frac{\kappa}{2}} \xi^{\frac{\kappa}{2\alpha} - 1} \int_{t_0}^t \mathbb{E} [|V_s|^{\frac{\kappa}{2}}] ds. \end{aligned} \quad (2.22)$$

Injecting (2.20), and choosing $\eta = t^{\frac{2}{\alpha}}$ and $\xi = t$, we get

$$\mathbb{E} [|V_{t \wedge \tau_r}|^\kappa] \leq C_{\kappa, t_0, \alpha} t^{\frac{\kappa}{\alpha}}.$$

Taking $r \rightarrow +\infty$, (2.8) follows. □

Example 2.4.5. Remark that the velocity process V is more explicit in the linear case ($\gamma = 1$), and that the moment estimate is as best as possible. Choose $F(1) = \rho > 0$, $F(-1) = -\rho$. Pick $\beta \neq 1$, so

$$V_t = v_0 + \exp\left(-\rho \frac{t^{1-\beta}}{1-\beta}\right) \int_{t_0}^t \exp\left(\rho \frac{s^{1-\beta}}{1-\beta}\right) dL_s$$

is solution to (SKE $_L$).

Hence, by an integration by parts,

$$V_t = v_0 + L_t - e^{\rho \frac{1}{1-\beta} (t_0^{1-\beta} - t^{1-\beta})} L_{t_0} - e^{-\rho \frac{t^{1-\beta}}{1-\beta}} \int_{t_0}^t \rho s^{-\beta} e^{\rho \frac{s^{1-\beta}}{1-\beta}} L_s ds.$$

Thus,

$$\mathbb{E}[|V_t|] \leq C_{t_0} \left(t^{\frac{1}{\alpha}} + t^{1-\beta+\frac{1}{\alpha}} \right) \leq C_{t_0} t^{\frac{1}{\alpha}}.$$

The case $\beta = 1$ can be treated similarly.

2.5 Proof of the asymptotic behavior of the solution

This section is devoted to the proofs of our main results, Theorems 2.2.2, 2.2.4 and 2.2.5.

Notice that, in the super-critical and critical regimes, it suffices to prove the convergence of the rescaled velocity process $(\varepsilon^{\frac{1}{\alpha}} V_{t/\varepsilon})_{t \geq \varepsilon t_0}$ in the space \mathcal{D} endowed with the Skorokhod topology. Assume for a moment that this convergence is proved.

For $\varepsilon \in (0, 1]$ and $t \geq \varepsilon t_0$ we can write

$$\varepsilon^{1+\frac{1}{\alpha}} X_{t/\varepsilon} = \varepsilon^{1+\frac{1}{\alpha}} x_0 + \int_{\varepsilon t_0}^t V_s^{(\varepsilon)} ds.$$

Let us introduce the mapping $g_\varepsilon : V \mapsto \left(V_t, \int_{\varepsilon t_0}^t V_s ds \right)_{t > 0}$ defined and valued on \mathcal{D} . Clearly, the theorem will be proved once we show that $g_\varepsilon(V_\bullet^{(\varepsilon)})$ converges weakly in \mathcal{D} endowed with the Skorokhod topology. This mapping is converging, as $\varepsilon \rightarrow 0$, to the continuous mapping $g : V \mapsto \left(V_t, \int_0^t V_s ds \right)_{t > 0}$.

To see $V^{(\varepsilon)}$ as a process of $\mathcal{D}([0, +\infty))$, we state, for all $s \in [0, \varepsilon t_0]$, $V_s^{(\varepsilon)} := V_{\varepsilon t_0}^{(\varepsilon)} = \varepsilon^{\frac{1}{\alpha}} v_0$. Call P_ε, P the distribution of $V^{(\varepsilon)}, \mathcal{S}$, respectively. Invoking the Portmanteau theorem (see Theorem 2.1 p. 16 in [Bil99]), it suffices to prove that for all bounded and uniformly continuous function $h : \mathcal{D}([0, +\infty)) \times \mathcal{D}([0, +\infty)) \rightarrow \mathbb{R}$,

$$\int_{\mathcal{D}([0, +\infty))^2} h(g_\varepsilon(\omega)) dP_\varepsilon(d\omega) \xrightarrow{\varepsilon \rightarrow 0} \int_{\mathcal{D}([0, +\infty))^2} h(g(\omega)) dP(d\omega).$$

Pick such a function h . By assumption, the convergence $P_\varepsilon \xrightarrow{\varepsilon \rightarrow 0} P$ holds, hence, using Lemma 2.A.4, it suffices to prove that the uniformly bounded sequence $(h \circ g_\varepsilon)$ of continuous functions on $\mathcal{D}([0, +\infty))$ converges to the continuous function $h \circ g$ uniformly on compact subsets of $\mathcal{D}([0, +\infty))$. Let K be a compact subset of $\mathcal{D}([0, +\infty))$. Then, for all $\omega \in K$, $\max_{[0, \varepsilon t_0]} |\omega|$ is uniformly bounded by a constant, say M . Fix $\eta > 0$. By the uniform continuity of h , there exists $\delta > 0$ such that for all $\omega \in K$,

$$d_u(g_\varepsilon(\omega), g(\omega)) \leq \delta \implies |h \circ g_\varepsilon(\omega) - h \circ g(\omega)| \leq \eta.$$

There exists $\varepsilon_1 > 0$ small enough, such that for all $\varepsilon \leq \varepsilon_1$, for all $\omega \in K$,

$$d_u(g_\varepsilon(\omega), g(\omega)) \leq C \left| \int_0^{\varepsilon t_0} \omega(s) ds \right| \leq C\varepsilon t_0 M \leq \delta.$$

Therefore, we proved that it suffices to prove the convergence of the rescaled velocity process $(\varepsilon^{\frac{1}{\alpha}} V_{t/\varepsilon})_{t \geq \varepsilon t_0}$ in order to prove Theorems 2.2.2 and 2.2.4.

In Sections 2.5.1 and 2.5.2, the aim is to prove the convergence of the velocity process.

2.5.1 Asymptotic behavior in the super-critical regime

In the remainder of this section, we assume that $\gamma \geq 0$ and $\beta > 1 + \frac{\gamma-1}{\alpha}$.

Proof of Theorem 2.2.2. Thanks to a change of variables, we have, for all $\varepsilon \in (0, 1]$ and $t \geq \varepsilon t_0$,

$$\begin{aligned} \varepsilon^{\frac{1}{\alpha}} V_{t/\varepsilon} &= \varepsilon^{\frac{1}{\alpha}} (v_0 - L_{t_0}) + \varepsilon^{\frac{1}{\alpha}} L_{t/\varepsilon} - \varepsilon^{\frac{1}{\alpha}} \int_{t_0}^{t/\varepsilon} F(V_s) s^{-\beta} ds \\ &= \varepsilon^{\frac{1}{\alpha}} (v_0 - L_{t_0}) + \varepsilon^{\frac{1}{\alpha}} L_{t/\varepsilon} - \varepsilon^{\beta-1+\frac{1}{\alpha}} \int_{\varepsilon t_0}^t F(V_{u/\varepsilon}) u^{-\beta} du. \end{aligned}$$

By self-similarity, $L^{(\varepsilon)} := (\varepsilon^{\frac{1}{\alpha}} L_{t/\varepsilon})_{t \geq 0}$ has the same distribution as an α -stable process. As a consequence, thanks to Theorem 3.1 p. 27 in [Bil99] and Lemma 2.A.3, it suffices to prove

$$\forall T > 0 \quad \sup_{\varepsilon t_0 \leq t \leq T} \left| V_t^{(\varepsilon)} - L_t^{(\varepsilon)} \right| \xrightarrow{\mathbb{P}} 0, \quad \text{as } \varepsilon \rightarrow 0. \quad (2.23)$$

Recall that under (H_γ) , there exists a positive constant K , such that

$$\varepsilon^{\frac{\gamma}{\alpha}} \left| F \left(\frac{V_\bullet^{(\varepsilon)}}{\varepsilon^{\frac{1}{\alpha}}} \right) \right| \leq K |V_\bullet^{(\varepsilon)}|^\gamma. \quad (2.24)$$

Modifying the factor in front of the integral, we get

$$V_t^{(\varepsilon)} = \varepsilon^{\frac{1}{\alpha}} (v_0 - L_{t_0}) + L_t^{(\varepsilon)} - \varepsilon^{\beta-1+\frac{1-\gamma}{\alpha}} \int_{\varepsilon t_0}^t \varepsilon^{\frac{\gamma}{\alpha}} F \left(\frac{V_u^{(\varepsilon)}}{\varepsilon^{\frac{1}{\alpha}}} \right) u^{-\beta} du. \quad (2.25)$$

Gathering (2.25) and (2.24), for all $T > 0$, we have,

$$\begin{aligned} \sup_{\varepsilon t_0 \leq t \leq T} |V_t^{(\varepsilon)} - L_t^{(\varepsilon)}| &\leq \varepsilon^{\frac{1}{\alpha}}(v_0 - L_{t_0}) + \varepsilon^{\beta-1+\frac{1-\gamma}{\alpha}} \sup_{\varepsilon t_0 \leq t \leq T} \left| \int_{\varepsilon t_0}^t \varepsilon^{\frac{\gamma}{\alpha}} F\left(\frac{V_u^{(\varepsilon)}}{\varepsilon^{\frac{1}{\alpha}}}\right) u^{-\beta} du \right| \\ &\leq \varepsilon^{\frac{1}{\alpha}}(v_0 - L_{t_0}) + \varepsilon^{\beta-1+\frac{1-\gamma}{\alpha}} \int_{\varepsilon t_0}^T K |V_u^{(\varepsilon)}|^{\gamma} u^{-\beta} du. \end{aligned}$$

Taking the expectation and using the moment estimates on V (see Remark 2.2.7), we obtain, when $\beta \neq \frac{\gamma}{\alpha} + 1$,

$$\begin{aligned} \varepsilon^{\beta-1+\frac{1-\gamma}{\alpha}} \mathbb{E} \left[\int_{\varepsilon t_0}^T K |V_u^{(\varepsilon)}|^{\gamma} u^{-\beta} du \right] &= \varepsilon^{\beta-1+\frac{1-\gamma}{\alpha}} \int_{\varepsilon t_0}^T K \mathbb{E} [|V_u^{(\varepsilon)}|^{\gamma}] u^{-\beta} du \\ &= \varepsilon^{\beta-1+\frac{1}{\alpha}} \int_{\varepsilon t_0}^T K \mathbb{E} [|V_{u/\varepsilon}|^{\gamma}] u^{-\beta} du \\ &\leq \varepsilon^{\beta-1+\frac{1-\gamma}{\alpha}} \int_{\varepsilon t_0}^T C_{\alpha,\beta,t_0} u^{\frac{\gamma}{\alpha}-\beta} du \\ &= C_{\alpha,\beta,t_0} \left(\varepsilon^{\beta-1+\frac{1-\gamma}{\alpha}} T^{\frac{\gamma}{\alpha}-\beta+1} - t_0^{\frac{\gamma}{\alpha}-\beta+1} \varepsilon^{\frac{1}{\alpha}} \right). \end{aligned}$$

Hence, setting $r := \min(\beta - 1 + \frac{1-\gamma}{\alpha}, \frac{1}{\alpha})$ which is positive, since $\beta > 1 + \frac{\gamma-1}{\alpha}$, we get

$$\mathbb{E} \left[\sup_{\varepsilon t_0 \leq t \leq T} |V_t^{(\varepsilon)} - L_t^{(\varepsilon)}| \right] = O_{\varepsilon \rightarrow 0}(\varepsilon^r).$$

The case $\beta = 1 + \frac{\gamma}{\alpha}$ can be treated similarly to get

$$\mathbb{E} \left[\sup_{\varepsilon t_0 \leq t \leq T} |V_t^{(\varepsilon)} - L_t^{(\varepsilon)}| \right] = O_{\varepsilon \rightarrow 0}(\varepsilon^{\frac{1}{\alpha}} \ln(\varepsilon)).$$

This concludes the proof. □

Remark 2.5.1. Observe that, in this proof, we did not use the condition “ $\gamma < 1$ or for all $v \in \mathbb{R}$, $vF(v) \geq 0$ ”, except to get moment estimates.

2.5.2 Asymptotic behavior in the critical regime

We adapt the Proposition 2.1, p. 187 of [GO13] to the α -stable Lévy case.

Pick a \mathcal{C}^2 -diffeomorphism $\varphi : [0, t_1) \rightarrow [t_0, +\infty)$. Let V be the solution to the equation (SKE $_L$). Thanks to Proposition 3.4.1 p. 124 in [ST94], the following process is also an

α -stable process

$$(R_t)_{t \geq 0} := \left(\int_0^t \frac{dL_{\varphi(s)}}{\varphi'(s)^{\frac{1}{\alpha}}} \right)_{t \geq 0}. \quad (2.26)$$

Then, by the change of variables $t = \varphi(s)$, we get

$$V_{\varphi(t)} - V_{\varphi(0)} = \int_0^t \varphi'(s)^{\frac{1}{\alpha}} dR_s - \int_0^t \frac{F(V_{\varphi(s)})}{\varphi(s)^\beta} \varphi'(s) ds.$$

Thanks to an integration by parts, we get

$$d \left(\frac{V_{\varphi(s)}}{\varphi'(s)^{\frac{1}{\alpha}}} \right) = dR_s - \frac{\varphi'(s)^{1-\frac{1}{\alpha}}}{\varphi(s)^\beta} F(V_{\varphi(s)}) ds - \frac{\varphi''(s)}{\alpha \varphi'(s)} \frac{V_{\varphi(s)}}{\varphi'(s)^{\frac{1}{\alpha}}} ds.$$

Set $\Omega = \overline{\mathcal{D}}([t_0, \infty))$ the set of càdlàg functions, that equal ∞ after their (possibly infinite) explosion time. Introduce the scaling transformation Φ_φ defined, for $\omega \in \Omega$, by

$$\Phi_\varphi(\omega)(s) := \frac{\omega(\varphi(s))}{\varphi'(s)^{\frac{1}{\alpha}}}, \text{ with } s \in [0, t_1).$$

As a consequence, we obtain the following result.

Proposition 2.5.2. *If V is a solution to the equation (SKE_L), then $V^{(\varphi)} := \Phi_\varphi(V)$ is a solution to*

$$dV_s^{(\varphi)} = dR_s - \frac{\varphi'(s)^{1-\frac{1}{\alpha}}}{\varphi(s)^\beta} F(\varphi'(s)^{\frac{1}{\alpha}} V_s^{(\varphi)}) ds - \frac{\varphi''(s)}{\varphi'(s)} \frac{V_s^{(\varphi)}}{\alpha} ds, \quad \text{with } V_0^{(\varphi)} = \frac{V_{\varphi(0)}}{\varphi'(0)^{\frac{1}{\alpha}}}, \quad (2.27)$$

where R is given by (2.26).

Conversely, if $V^{(\varphi)}$ is a solution to (2.27), then $\Phi_\varphi^{-1}(V^{(\varphi)})$ is a solution to the equation (SKE_L), where

$$L_t - L_{t_0} := \int_{t_0}^t (\varphi' \circ \varphi^{-1})^{\frac{1}{\alpha}}(s) dR_{\varphi^{-1}(s)}.$$

Furthermore, uniqueness in law, pathwise uniqueness, strong existence hold for the equation (SKE_L) if and only if they hold for the equation (2.27).

In the following, we will focus on the exponential change of time $\varphi_e : t \mapsto t_0 e^t$. This scaling is convenient since it allows to produce a time-homogeneous term in (2.27). Thanks to Proposition 2.5.2, the process $V^{(e)} := \Phi_e(V)$ satisfies the SDE driven by an α -stable

process $(R_t)_{t \geq 0}$,

$$dV_s^{(e)} = dR_s - \frac{V_s^{(e)}}{\alpha} ds - t_0^{1-\frac{1}{\alpha}-\beta} e^{(1-\frac{1}{\alpha}-\beta)s} F\left(t_0^{\frac{1}{\alpha}} e^{\frac{s}{\alpha}} V_s^{(e)}\right) ds. \quad (2.28)$$

Proof of Theorem 2.2.4. Assume in the sequel that $\beta = 1 + \frac{\gamma-1}{\alpha}$.

STEP 1. Firstly we prove the finite-dimensional convergence of the rescaled velocity process. To that end, we reduce the problem to the convergence of a time-homogenous process. Since (H_γ) holds, (2.28) becomes

$$dV_s^{(e)} = dR_s - \frac{V_s^{(e)}}{\alpha} ds - F(V_s^{(e)}) ds. \quad (2.29)$$

Using the bijection Φ_e induced by the exponential change of time (see Proposition 2.5.2), and the unique strong existence of the velocity process V (see Proposition 2.3.2 and Remark 2.2.7), there exists a pathwise unique strong solution H to the time-homogeneous equation (2.29). Hence, we have the equality

$$\left(\frac{V_{t_0 e^t}}{(t_0 e^t)^{1/\alpha}} \right)_{t \geq 0} = (H_t)_{t \geq 0},$$

as two solutions to the same SDE, starting from the same point. We can write the above equality as

$$\left(\frac{V_t}{t^{1/\alpha}} \right)_{t \geq t_0} = (H_{\log(t/t_0)})_{t \geq t_0}.$$

So, we have, for all $\varepsilon > 0$, $d \in \mathbb{N}^*$, and $(t_1, \dots, t_d) \in [\varepsilon t_0, +\infty)^d$,

$$\left(\frac{V_{\varepsilon^{-1}t_1}}{(\varepsilon^{-1}t_1)^{1/\alpha}}, \dots, \frac{V_{\varepsilon^{-1}t_d}}{(\varepsilon^{-1}t_d)^{1/\alpha}} \right) = \left(H_{\log(t_1)+\log((\varepsilon t_0)^{-1})}, \dots, H_{\log(t_d)+\log((\varepsilon t_0)^{-1})} \right). \quad (2.30)$$

Since $\limsup_{|x| \rightarrow +\infty} \frac{-F(x)-x/\alpha}{x} < 0$, it follows from Proposition 0.1 in [Kul09] that the process $(H_t)_{t \geq 0}$ is exponentially ergodic. We denote its invariant measure by Λ_F . Call \widetilde{H} the solution to the time homogeneous equation (2.29), such that the initial condition $\widetilde{H}_{-\infty}$ has the distribution Λ_F . For $t_1, \dots, t_d \in \mathbb{R}^d$, let $\Lambda_{F,t_1, \dots, t_d} := \mathcal{L}(\widetilde{H}_{t_1}, \dots, \widetilde{H}_{t_d})$ be the distribution of $(\widetilde{H}_{t_1}, \dots, \widetilde{H}_{t_d})$. Then, for all $s \geq 0$, $\Lambda_{F,t_1, \dots, t_d} = \Lambda_{F,t_1+s, \dots, t_d+s}$. Indeed, thanks to the invariance property of Λ_F , (\widetilde{H}_\bullet) and $(\widetilde{H}_{\bullet+s})$ satisfy the same SDE, starting

from the same point. As a consequence, we get the stationary limit

$$\lim_{\varepsilon \rightarrow 0} \mathcal{L} \left(\widetilde{H}_{\log(t_1) + \log((\varepsilon t_0)^{-1})}, \dots, \widetilde{H}_{\log(t_d) + \log((\varepsilon t_0)^{-1})} \right) = \Lambda_{F, \log(t_1), \dots, \log(t_d)}. \quad (2.31)$$

Moreover, by exponential ergodicity, we have for every continuous and bounded function $\psi : \mathbb{R}^d \rightarrow \mathbb{R}$,

$$\mathbb{E} \left[\psi \left(H_{\log(t_1/(t_0\varepsilon))}, \dots, H_{\log(t_d/(t_0\varepsilon))} \right) \right] - \mathbb{E} \left[\psi \left(\widetilde{H}_{\log(t_1/(t_0\varepsilon))}, \dots, \widetilde{H}_{\log(t_d/(t_0\varepsilon))} \right) \right] \xrightarrow{\varepsilon \rightarrow 0} 0. \quad (2.32)$$

We postpone the proof of this limit in Step 2.

To conclude this step, gather (2.30), (2.31) and (2.32) to get

$$\left(\frac{V_{\varepsilon^{-1}t_1}}{(\varepsilon^{-1}t_1)^{1/\alpha}}, \dots, \frac{V_{\varepsilon^{-1}t_d}}{(\varepsilon^{-1}t_d)^{1/\alpha}} \right) \xrightarrow{\varepsilon \rightarrow 0} \Lambda_{F, \log(t_1), \dots, \log(t_d)}.$$

This can also be written as

$$\left(\varepsilon^{\frac{1}{\alpha}} V_{t_1/\varepsilon}, \dots, \varepsilon^{\frac{1}{\alpha}} V_{t_d/\varepsilon} \right) \xrightarrow{\varepsilon \rightarrow 0} T * \Lambda_{F, \log(t_1), \dots, \log(t_d)},$$

where $T * \Lambda_{F, \log(t_1), \dots, \log(t_d)}$ is the pushforward of the measure $\Lambda_{F, \log(t_1), \dots, \log(t_d)}$ by the linear map $T(u_1, \dots, u_d) := (t_1^{1/\alpha} u_1, \dots, t_d^{1/\alpha} u_d)$.

STEP 2. Let us now prove (2.32).

For the sake of clarity, let us give a proof for $d = 2$, the general case $d \geq 2$ is similar.

Let $\psi : \mathbb{R}^2 \rightarrow \mathbb{R}$ be a continuous and bounded function. Pick $\varepsilon t_0 \leq s \leq t$. Set $h_0 = v_0 t_0^{-\frac{1}{\alpha}}$, (2.32) is now equivalent to

$$\mathbb{E} \left[\psi \left(H_{\log(s/(t_0\varepsilon))}, H_{\log(t/(t_0\varepsilon))} \right) \middle| H_0 = h_0 \right] - \mathbb{E} \left[\psi \left(H_{\log(s/(t_0\varepsilon))}, H_{\log(t/(t_0\varepsilon))} \right) \middle| H_0 \sim \Lambda_F \right] \xrightarrow{\varepsilon \rightarrow 0} 0.$$

We set $\mu_\varepsilon := \mathcal{L} \left(H_{\log(s/(t_0\varepsilon))} \middle| H_0 = h_0 \right)$. We now use the generalized Markov property of solution to SDE driven by Lévy process. For the sake of completeness, we state and prove it in our context in Appendix (see Lemma 2.A.6). This leads to

$$\mathbb{E} \left[\psi \left(H_{\log(s/(t_0\varepsilon))}, H_{\log(t/(t_0\varepsilon))} \right) \middle| H_0 = h_0 \right] = \mathbb{E} \left[\psi \left(H_0, H_{\log(t/s)} \right) \middle| H_0 \sim \mu_\varepsilon \right]$$

and, since Λ_F is invariant,

$$\mathbb{E} \left[\psi \left(H_{\log(s/(t_0\varepsilon))}, H_{\log(t/(t_0\varepsilon))} \right) \middle| H_0 \sim \Lambda_F \right] = \mathbb{E} \left[\psi \left(H_0, H_{\log(t/s)} \right) \middle| H_0 \sim \Lambda_F \right].$$

Then, we are reduced to prove

$$\mathbb{E} \left[\psi \left(H_0, H_{\log(t/s)} \right) \middle| H_0 \sim \mu_\varepsilon \right] - \mathbb{E} \left[\psi \left(H_0, H_{\log(t/s)} \right) \middle| H_0 \sim \Lambda_F \right] \xrightarrow{\varepsilon \rightarrow 0} 0.$$

The left-hand side can be written as,

$$\int_{\mathbb{R}} \mathbb{E} \left[\psi \left(H_0, H_{\log(t/s)} \right) \middle| H_0 = y \right] (\mu_\varepsilon(dy) - \Lambda_F(dy)).$$

Hence, setting $p(t, x, dy) := \mathbb{P}_x(H_t \in dy)$ and $\|\cdot\|_{TV}$ for the total variation norm, we get

$$\begin{aligned} & \left| \mathbb{E} \left[\psi \left(H_0, H_{\log(t/s)} \right) \middle| H_0 \sim \mu_\varepsilon \right] - \mathbb{E} \left[\psi \left(H_0, H_{\log(t/s)} \right) \middle| H_0 \sim \Lambda_F \right] \right| \\ & \leq \|\psi\|_\infty \int_{\mathbb{R}} |p(\log(s/(t_0\varepsilon)), h_0, dy) - \Lambda_F(dy)| \\ & \leq \|\psi\|_\infty \|p(\log(s/(t_0\varepsilon)), h_0, \cdot) - \Lambda_F\|_{TV}. \end{aligned}$$

This converges to 0, as $\varepsilon \rightarrow 0$, by the exponential ergodicity of H .

STEP 3. Let us prove now the tightness of the family of distributions of the càdlàg process $(V^{(\varepsilon)})_{t \geq \varepsilon t_0} = (\varepsilon^{\frac{1}{\alpha}} V_{t/\varepsilon})_{t \geq \varepsilon t_0}$ on every compact interval $[m, M]$, $0 < m \leq M$.

We check the Aldous's tightness criterion stated at Theorem 16.10 p.178 in [Bil99]. Let a, η, T be positive reals. Let τ be a discrete stopping time with finite range \mathcal{T} , bounded by T . Choose $\delta > 0$ and $\varepsilon > 0$ small enough.

We have, by Jensen's inequality, for $r = \frac{\alpha}{2}$,

$$\mathbb{E} \left[|V_{\tau+\delta}^{(\varepsilon)} - V_\tau^{(\varepsilon)}|^r \right] \leq \mathbb{E} \left[|L_{\tau+\delta}^{(\varepsilon)} - L_\tau^{(\varepsilon)}|^r \right] + \mathbb{E} \left[\int_\tau^{\tau+\delta} K |V_u^{(\varepsilon)}|^\gamma u^{-\beta} du \right]^r.$$

Since $L^{(\varepsilon)}$ is an α -stable process, by the strong Markov property,

$$\mathbb{E} \left[|L_{\tau+\delta}^{(\varepsilon)} - L_\tau^{(\varepsilon)}|^r \right] = \mathbb{E} [\mathbb{E}_{L_\tau} [|L_\delta - L_0|^r]] \leq C \delta^{\frac{r}{\alpha}}.$$

The stopping time has a finite range \mathcal{T} . Hence, we can write

$$\begin{aligned} \mathbb{E} \left[\int_{\tau}^{\tau+\delta} K |V_u^{(\varepsilon)}|^{\gamma} u^{-\beta} du \right] &= \mathbb{E} \left[\mathbb{E} \left[\int_{\tau}^{\tau+\delta} K |V_u^{(\varepsilon)}|^{\gamma} u^{-\beta} du \middle| \tau \right] \right] \\ &= \mathbb{E} \left[\sum_{\tau_i \in \mathcal{T}} \frac{1}{\mathbb{P}(\tau = \tau_i)} \mathbb{E} \left[\mathbb{1}_{\{\tau = \tau_i\}} \int_{\tau_i}^{\tau_i+\delta} K |V_u^{(\varepsilon)}|^{\gamma} u^{-\beta} du \right] \mathbb{1}_{\{\tau = \tau_i\}} \right] \\ &\leq \mathbb{E} \left[\sum_{\tau_i \in \mathcal{T}} \frac{1}{\mathbb{P}(\tau = \tau_i)} \mathbb{E} \left[\int_{\tau_i}^{\tau_i+\delta} K |V_u^{(\varepsilon)}|^{\gamma} u^{-\beta} du \right] \mathbb{1}_{\{\tau = \tau_i\}} \right]. \end{aligned}$$

For each $\tau_i \in \mathcal{T}$, using the relation $\beta = 1 + \frac{(\gamma-1)}{\alpha}$ and the moment estimates on V (see Remark 2.2.7), we obtain

$$\begin{aligned} \mathbb{E} \left[\int_{\tau_i}^{\tau_i+\delta} K |V_u^{(\varepsilon)}|^{\gamma} u^{-\beta} du \right] &= \int_{\tau_i}^{\tau_i+\delta} K \mathbb{E} \left[|V_u^{(\varepsilon)}|^{\gamma} \right] u^{-\beta} du \\ &\leq K \int_{\tau_i}^{\tau_i+\delta} u^{\frac{\gamma}{\alpha} - \beta} du = K \left[(\tau_i + \delta)^{\frac{1}{\alpha}} - \tau_i^{\frac{1}{\alpha}} \right] \\ &\leq K \delta^{1, \frac{1}{\alpha}}. \end{aligned}$$

The term $\delta^{1, \frac{1}{\alpha}}$ has to be read as δ or $\delta^{\frac{1}{\alpha}}$ depending on the fact that $x \mapsto x^{\frac{1}{\alpha}}$ is a Lipschitz continuous function on $[0, T + \delta]$, if $\alpha < 1$, or a $\frac{1}{\alpha}$ -Hölder function, if $\alpha > 1$.

By Markov's inequality, for δ small enough, we have

$$\mathbb{P} \left(|V_{\tau+\delta}^{(\varepsilon)} - V_{\tau}^{(\varepsilon)}| \geq a \right) \leq \frac{K \delta^{r, \frac{r}{\alpha}}}{a^r} \leq \eta.$$

Furthermore, by moment estimates (see Propositions 2.4.2, 2.4.3 and 2.4.4), for all $t \geq \varepsilon t_0$,

$$\sup_{\varepsilon} \left[|V_t^{(\varepsilon)}|^r \right] \leq C t^{\frac{r}{\alpha}}.$$

Hence, using again Markov's inequality, by Corollary and Theorem 16.8 p. 175 in [Bil99], this concludes the proof of the tightness of the velocity process and therefore the proof of Theorem 2.2.4.

□

2.5.3 Asymptotic behavior in the sub-critical regime

Assume in this section that $\beta < 1 + \frac{\gamma-1}{\alpha}$ and $\alpha > 1$. As a consequence, $\alpha q < 1$. We take an interest into the power change of time $\varphi_q : t \mapsto \left(t_0^{1-\alpha q} + (1-\alpha q)t\right)^{\frac{1}{1-\alpha q}}$. Thanks to Proposition 2.5.2, the process $V^{(q)} := \Phi_q(V)$ satisfies the SDE driven by an α -stable process R distributed as L ,

$$dV_s^{(q)} = dR_s - F\left(V_s^{(q)}\right) ds - q\varphi_q^{\alpha q-1} V_s^{(q)} ds. \quad (2.33)$$

For simplicity, we shall write ϕ instead of ϕ_q .

Proof of Theorem 2.2.5. STEP 1. We first prove the finite dimensional convergence of the velocity process $(V_t^{(\varepsilon)})_{t \geq \varepsilon t_0} := (\varepsilon^q V_{t/\varepsilon})_{t \geq \varepsilon t_0}$. We give a proof for $d = 2$, the general case $d \geq 2$ is similar.

We call H the ergodic process solution to

$$dH_s = dR_s - F\left(H_s\right) ds, \quad \text{with } H_0 = h_0 := v_0 t_0^{-q}. \quad (2.34)$$

We denote by Π_F its invariant measure. Using the bijection induced by the power change of time (Proposition 2.5.2), as solutions to the same SDE starting at the same point, we have, for all $\varepsilon > 0$, and $(s, t) \in [\varepsilon t_0, +\infty)^2$,

$$\left(\varepsilon^q \frac{V_{\varepsilon^{-1}s}}{s^q}, \varepsilon^q \frac{V_{\varepsilon^{-1}t}}{t^q}\right) = \left(V_{\varphi^{-1}(\varepsilon^{-1}s)}^{(q)}, V_{\varphi^{-1}(\varepsilon^{-1}t)}^{(q)}\right)$$

Using Theorem 3.1 p. 27 in [Bil99], it suffices to prove that for all $(s, t) \in [\varepsilon t_0, +\infty)^2$

- $\left\| \left(\left(H_{\varphi^{-1}(\varepsilon^{-1}s)}, H_{\varphi^{-1}(\varepsilon^{-1}t)} \right) - \left(V_{\varphi^{-1}(\varepsilon^{-1}s)}^{(q)}, V_{\varphi^{-1}(\varepsilon^{-1}t)}^{(q)} \right) \right) \right\| \xrightarrow{\varepsilon \rightarrow 0} 0$, where $\|\cdot\|$ is a metric on \mathbb{R}^2 .
- $\left(H_{\varphi^{-1}(\varepsilon^{-1}s)}, H_{\varphi^{-1}(\varepsilon^{-1}t)} \right) \xrightarrow{\varepsilon \rightarrow 0} \Pi_F \otimes \Pi_F$.

STEP 2. Pick $\kappa \in (1, \alpha)$. We prove that $\mathbb{E} \left[\left| H_t - V_t^{(q)} \right|^\kappa \right] \xrightarrow{t \rightarrow +\infty} 0$.

We have

$$d\left(H - V^{(q)}\right)_t = -\left(F(H_t) - F(V_t^{(q)})\right) dt + q\varphi^{\alpha q-1}(t)V_t^{(q)} dt.$$

By straightforward differentiation, we can write

$$\begin{aligned} d \left| H - V^{(q)} \right|_t^\kappa &= -\kappa \left| F(H_t) - F(V_t^{(q)}) \right| \left| H_t - V_t^{(q)} \right|^{\kappa-1} dt \\ &\quad + \kappa q \varphi^{\alpha q-1}(t) V_t^{(q)} \operatorname{sgn} \left(H_t - V_t^{(q)} \right) \left| H_t - V_t^{(q)} \right|^{\kappa-1} dt. \end{aligned} \quad (2.35)$$

We set

$$g(t) := \mathbb{E} \left[\left| H_t - V_t^{(q)} \right|^\kappa \right], \quad t \geq 0.$$

Taking expectation in (2.35), we get

$$\begin{aligned} g'(t) &= -\kappa \mathbb{E} \left[\left| F(H_t) - F(V_t^{(q)}) \right| \left| H_t - V_t^{(q)} \right|^{\kappa-1} \right] \\ &\quad + \kappa q \varphi^{\alpha q-1}(t) \mathbb{E} \left[V_t^{(q)} \operatorname{sgn} \left(H_t - V_t^{(q)} \right) \left| H_t - V_t^{(q)} \right|^{\kappa-1} \right]. \end{aligned}$$

Since $\gamma \geq 1$, the function F^{-1} is $\frac{1}{\gamma}$ -Hölder, therefore there exists $C_\gamma > 0$ such that,

$$g'(t) \leq -C_\gamma \mathbb{E} \left[\left| H_t - V_t^{(q)} \right|^{\kappa-1+\gamma} \right] + \kappa |q| \varphi^{\alpha q-1}(t) \mathbb{E} \left[\left| V_t^{(q)} \right| \left| H_t - V_t^{(q)} \right|^{\kappa-1} \right].$$

Then, by Jensen's inequality, since $\gamma \geq 1$,

$$g'(t) \leq -C_\gamma g(t)^{\frac{\kappa-1+\gamma}{\kappa}} + \kappa |q| \varphi^{\alpha q-1}(t) \mathbb{E} \left[\left| V_t^{(q)} \right| \left| H_t - V_t^{(q)} \right|^{\kappa-1} \right].$$

Using Hölder's inequality and moment estimates (Proposition 2.4.2), we have

$$g'(t) \leq -C_\gamma g(t)^{\frac{\kappa-1+\gamma}{\kappa}} + C |q| \varphi^{(\alpha q-1)(1-\frac{1}{\alpha})}(t) g(t)^{\frac{\kappa-1}{\kappa}}, \quad g(0) = 0.$$

Note that since $\alpha > 1$ and $\alpha q < 1$, then $\varphi^{(\alpha q-1)(1-\frac{1}{\alpha})}(t) \xrightarrow[t \rightarrow +\infty]{} 0$, therefore the conclusion follows from Lemma 2.A.7.

Besides, for all $t \geq \varepsilon t_0$, $\mathbb{E} \left[\left| H_{\varphi^{-1}(\varepsilon^{-1}t)} - V_{\varphi^{-1}(\varepsilon^{-1}t)}^{(q)} \right|^\kappa \right] = g(\varphi^{-1}(\varepsilon^{-1}t)) \xrightarrow[\varepsilon \rightarrow 0]{} 0$.

STEP 3. Pick $(s, t) \in [\varepsilon t_0, +\infty)^2$. Similarly, as in [GL21a], one can prove that the solution H to (2.34) satisfies

$$\left(H_{\varphi^{-1}(\varepsilon^{-1}s)}, H_{\varphi^{-1}(\varepsilon^{-1}t)} \right) \xrightarrow[\varepsilon \rightarrow 0]{} \Pi_F \otimes \Pi_F. \quad (2.36)$$

□

2.6 Extended results in the Lévy case

In this section, the driving process of (SKE_L) is supposed to be a general Lévy process L . We denote by (A, ν, b) its generating triplet, with respect to the truncation function $h : z \mapsto -1 \vee (z \wedge 1)$. Here ν is the Lévy measure, $A > 0$ and $b \in \mathbb{R}$. This means that, for $t \geq 0$, in virtue of the Lévy-Khintchine formula (see Theorem 8.1 p. 37 in [Sat99]), the characteristic function of L_t is given by

$$\mathbb{E} \left[e^{iuL_t} \right] = e^{t\psi(u)}, \quad \text{where } \psi(u) := -\frac{Au^2}{2} + ibu + \int_{\mathbb{R}} (e^{iux} - 1 - iuh(x))\nu(dx).$$

Pick $\alpha \in (0, 2)$. The Lévy measure ν of the driving process is supposed to satisfy either

$$\nu(z) = \frac{g(z)}{|z|^{1+\alpha}} \mathbb{1}_{\{z \neq 0\}}, \quad \text{where } g \text{ is a non-negative measurable function such that} \quad (H_1^{\nu, \alpha})$$

$$c^+ := \lim_{z \rightarrow +\infty} g(z) \geq 0, \quad c^- := \lim_{z \rightarrow -\infty} g(z) \geq 0,$$

or

$$\text{for some } \alpha_0 > 1, \quad \int_{|z| \geq 1} |z|^{\alpha_0} \nu(dz) < +\infty. \quad (H_2^{\nu, \alpha_0})$$

Remark that if ν satisfies $(H_1^{\nu, \alpha})$ with $\alpha > 1$, then it satisfies (H_2^{ν, α_0}) . Note that any tempered stable process satisfies $(H_1^{\nu, \alpha})$, and any truncated α -stable process satisfies (H_2^{ν, α_0}) . For clarity, we recall the stochastic kinetic model: for $t \geq t_0 > 0$,

$$dV_t = dL_t - t^{-\beta} F(V_t) dt, \quad \text{with } V_{t_0} = v_0 > 0, \quad \text{and} \quad dX_t = V_t dt, \quad \text{with } X_{t_0} = x_0 \in \mathbb{R}. \quad (\text{SKE}_L)$$

We work under the assumption that there exists a unique solution to (SKE_L) . We will show that Theorem 2.2.2 can be extended with this general Lévy driving process. We suppose first the Lévy process to be without a Brownian component. This case will be discussed in Section 2.6.3. We obtain the two following theorems, depending on which hypothesis is satisfied by the Lévy measure ν .

Theorem 2.6.1. *Consider $\gamma \in [0, \alpha)$ and $\beta \geq 0$. Assume that $(H_1^{\nu, \alpha})$ and (H_{sgn}) are*

satisfied and define

$$p_\alpha(\gamma) := \begin{cases} \frac{\gamma}{\alpha} & \text{if } g \text{ is bounded,} \\ \frac{\gamma}{\alpha} & \text{if } \alpha \in (0, 1), \\ \frac{\gamma}{\alpha} & \text{if } \alpha \in (1, 2), \gamma \in [0, 1) \text{ and } b = 0, \\ \gamma & \text{if } \alpha \in [1, 2), \gamma \in [0, 1) \text{ and } b \neq 0, \\ \gamma & \text{if } \alpha \in (1, 2), \gamma = 1, \\ \frac{\gamma}{\alpha} + \frac{\gamma}{2} & \text{if } \alpha \in (1, 2), \gamma \in (1, \alpha). \end{cases}$$

Let $(V_t, X_t)_{t \geq t_0}$ be the solution to (SKE_L).

Then there exist a rate of convergence ε^θ and a Lévy process \mathcal{L} , given in Proposition 2.6.3 (iii)-(v), such that, as $\varepsilon \rightarrow 0$, if $\beta > 1 + p_\alpha(\gamma) - \theta$, in the space \mathcal{D} ,

$$(\varepsilon^\theta V_{t/\varepsilon}, \varepsilon^{1+\theta} X_{t/\varepsilon})_{t \geq \varepsilon t_0} \xrightarrow[\varepsilon \rightarrow 0]{} \left(\mathcal{L}_t, \int_0^t \mathcal{L}_s ds \right)_{t > 0}.$$

Theorem 2.6.2. Consider $\gamma \geq 0$ and $\beta \geq 0$. Assume that (H_2^{ν, α_0}) is satisfied and define

$$p_{\alpha_0}(\gamma) := \begin{cases} \gamma & \text{if } \gamma \in [0, 1), \\ \min(\gamma, \frac{\gamma}{\alpha_0} + \frac{\gamma}{2}) & \text{if } \gamma \in [0, 1) \text{ and } vF(v) \geq 0, \\ \frac{\gamma}{\alpha_0} + \frac{\gamma}{2} & \text{if } \gamma \in [1, \alpha_0] \text{ and } vF(v) \geq 0. \end{cases}$$

Suppose that $\beta > 1 + p_{\alpha_0}(\gamma) - 1$. Let $(V_t, X_t)_{t \geq t_0}$ be the solution to (SKE_L).

Then there exists a Lévy process \mathcal{L} , given in Proposition 2.6.3 (iii), such that, as $\varepsilon \rightarrow 0$, in the space \mathcal{D} ,

$$(\varepsilon V_{t/\varepsilon}, \varepsilon^2 X_{t/\varepsilon})_{t \geq \varepsilon t_0} \xrightarrow[\varepsilon \rightarrow 0]{} \left(\mathcal{L}_t, \int_0^t \mathcal{L}_s ds \right)_{t > 0}.$$

2.6.1 Large time behavior of the Lévy driving process

Since the Lévy noise is no longer self-similar, we need to study its large-time behavior. We dedicate this subsection to the study of the convergence in distribution of the rescaled noise $(L_t^{(\varepsilon)})_{t \geq 0} := (r_\varepsilon L_{t/\varepsilon})_{t \geq 0}$, for a suitable rate r_ε , tending to 0.

We introduce another assumption on ν , which will sometimes be imposed in the sequel:

$$\nu(z) = \frac{g(z)}{|z|^{1+\alpha}} \mathbb{1}_{\{z \neq 0\}}, \text{ where } g \text{ is a non negative measurable function with} \quad (H_g)$$

$$\int_1^{+\infty} \frac{|g(z) - g(-z)|}{z^\alpha} dx < +\infty.$$

Inspired from [RT11], we get the following result.

Proposition 2.6.3. *Let $(L_t)_{t \geq 0}$ be a Lévy process having generating triplet (A, ν, b) , with respect to the truncation function $h : z \mapsto -1 \vee (z \wedge 1)$.*

- (i) *Assume that the Lévy measure satisfies the condition $(H_1^{\nu, \alpha})$ with $\alpha = 1$, $c^+ = c^- = c$ and (H_g) . Then the process $(\varepsilon L_{t/\varepsilon})_{t \geq 0}$ converges in distribution to the 1-stable Lévy process \mathcal{L} generated by $(0, \nu^*, b^*)$, where*

$$b^* := b + \int_1^{+\infty} \frac{g(z) - g(-z)}{z^{1+\alpha}} (z - h(z)) dz \quad \text{and} \quad \nu^*(dz) := \frac{c \mathbb{1}_{\{z \neq 0\}}}{|z|^2} dz.$$

- (ii) *Suppose that the Lévy measure satisfies the condition $(H_1^{\nu, \alpha})$ with $1 < \alpha < 2$ and (H_g) . If*

$$b^* := b + \int_1^{+\infty} \frac{g(z) - g(-z)}{z^{1+\alpha}} (z - h(z)) dz \neq 0,$$

then the process $(\varepsilon L_{t/\varepsilon})_{t \geq 0}$ converges in distribution to the deterministic Lévy process \mathcal{L} generated by $(0, 0, b^)$.*

- (iii) *If the Lévy measure satisfies (H_2^{ν, α_0}) and $b^* := b + \int_{|z| \geq 1} (z - h(z)) \nu(dz) \neq 0$, then the process $(\varepsilon L_{t/\varepsilon})_{t \geq 0}$ converges in distribution to the deterministic Lévy process \mathcal{L}^* generated by $(0, 0, b^*)$.*

- (iv) *Assuming that $0 < \alpha < 1$, if the Lévy measure satisfies the condition $(H_1^{\nu, \alpha})$, then the process $(\varepsilon^{\frac{1}{\alpha}} L_{t/\varepsilon})_{t \geq 0}$ converges in distribution to the α -stable Lévy process \mathcal{L} with*

$$\nu^*(dz) := \frac{c^+ \mathbb{1}_{\{z > 0\}} + c^- \mathbb{1}_{\{z < 0\}}}{|z|^{1+\alpha}} dz.$$

- (v) *Assuming that $1 < \alpha < 2$ and $b = 0$, if the Lévy measure satisfies the condition $(H_1^{\nu, \alpha})$, then the process $(\varepsilon^{\frac{1}{\alpha}} L_{t/\varepsilon})_{t \geq 0}$ converges in distribution to the Lévy process \mathcal{L}*

with measure and center

$$\nu^*(dz) := \frac{c^+ \mathbb{1}_{\{z>0\}} + c^- \mathbb{1}_{\{z<0\}}}{|z|^{1+\alpha}} dz, \quad b^* := \int_{\mathbb{R}^*} z \nu^*(dz).$$

Proof. By the Lévy-Khintchine's formula, the generating triplet of $(L^{(\varepsilon)})_{t \geq 0} := (r_\varepsilon L_{t/\varepsilon})_{t \geq 0}$ is given by

$$A^\varepsilon = \frac{r_\varepsilon^2}{\varepsilon} A, \quad (2.37)$$

$$\text{for all } B \in \mathcal{B}(\mathbb{R}), \quad \nu^\varepsilon(B) = \varepsilon^{-1} \nu(\{z, zr_\varepsilon \in B\}), \quad (2.38)$$

$$b^\varepsilon = \frac{r_\varepsilon}{\varepsilon} \left[b + \int_{\mathbb{R}^*} \left(r_\varepsilon^{-1} h(r_\varepsilon z) - h(z) \right) \nu(dz) \right]. \quad (2.39)$$

Call (A^*, ν^*, b^*) the generating triplet of the limiting process \mathcal{L} . By Corollary 3.6 p. 415 in [JS03] and Theorem 14.7 p. 81 in [Sat99], we have to check that

$$b^\varepsilon \xrightarrow{\varepsilon \rightarrow 0} b^*, \quad (2.40)$$

$$A^\varepsilon + \int_{\mathbb{R}^*} h^2(z) \nu^\varepsilon(dz) \xrightarrow{\varepsilon \rightarrow 0} A^* + \int_{\mathbb{R}^*} h^2(z) \nu^*(dz), \quad (2.41)$$

and that for any continuous and bounded function f which is zero in a neighborhood of zero,

$$\int_{\mathbb{R}^*} f(z) \nu^\varepsilon(dz) \xrightarrow{\varepsilon \rightarrow 0} \int_{\mathbb{R}^*} f(z) \nu^*(dz). \quad (2.42)$$

- (i) Recall that $\alpha = 1$ and assume that the Lévy measure ν satisfies the conditions $(H_1^{\nu, \alpha})$ and (H_g) .

To prove (2.40), we write b^ε as

$$b^\varepsilon = b + \int_0^{+\infty} \frac{g(z) - g(-z)}{z^{1+\alpha}} (\varepsilon^{-1} h(\varepsilon z) - h(z)) dz.$$

The dominated convergence theorem can be applied, and we show that it converges to

$$b^* = b + \int_1^{+\infty} \frac{g(x) - g(-x)}{x^2} (x - h(x)) dx.$$

Observe that the condition (H_g) was only required for this step.

Afterwards, note that, using a change of variables,

$$\int_{\mathbb{R}^*} h^2(z) \nu^\varepsilon(dz) = r_\varepsilon^\alpha \varepsilon^{-1} \int_{\mathbb{R}^*} h^2(z) \frac{g(zr_\varepsilon^{-1})}{|z|^{1+\alpha}} dz,$$

and thus, we can apply the dominated convergence theorem to prove that the last integral converges to

$$\int_{\mathbb{R}^*} h^2(z) \nu^*(dz).$$

Let f be a continuous and bounded function which is zero in a neighborhood of zero, then, using again a change of variables and applying the dominated convergence theorem,

$$\int_{\mathbb{R}^*} f(z) \nu^\varepsilon(dz) \xrightarrow{\varepsilon \rightarrow 0} \int_{\mathbb{R}^*} f(z) \nu^*(dz).$$

(ii) The proof is similar to the previous one and thus is left to the reader.

(iii) In this point, we assume that (H_2^{ν, α_0}) holds.

The convergence (2.40) follows from the dominated convergence theorem.

Using the explicit form of the truncation function, we get, for $r_\varepsilon < 1$,

$$\begin{aligned} \int_{\mathbb{R}^*} h^2(z) \nu^\varepsilon(dz) &= \int_{\mathbb{R}^*} \varepsilon^{-1} h^2(r_\varepsilon z) \nu(dz) \\ &= \varepsilon \int_{0 < |z| < 1} z^2 \nu(dz) + \int_{1 \leq |z| \leq \varepsilon^{-1}} \varepsilon z^2 \nu(dz) + \int_{|z| > \varepsilon^{-1}} \varepsilon^{-1} \nu(dz). \end{aligned} \tag{2.43}$$

Using the property of a Lévy measure, the first term in (2.43) converges to zero, as $\varepsilon \rightarrow 0$, by assumption. Then, the last two terms in (2.43) are lower than

$$\varepsilon^{\alpha_0-1} \int_{1 < |z| < \varepsilon^{-1}} |z|^{\alpha_0} \nu(dz) + \varepsilon^{\alpha_0-1} \int_{|z| \geq \varepsilon^{-1}} |z|^{\alpha_0} \nu(dz),$$

which converges to zero, when ε goes to zero, since $\alpha_0 > 1$.

Let f be a continuous and bounded function and assume that there exists $\delta > 0$ such that $f(z) = 0$ for all $|z| \leq \delta$, thus,

$$\int_{\mathbb{R}^*} f(z) \nu^\varepsilon(dz) = \int_{|z| > \frac{\delta}{\varepsilon}} \varepsilon^{-1} f(r_\varepsilon z) \nu(dz) \leq C \varepsilon^{\alpha_0-1} \int_{|z| > \frac{\delta}{\varepsilon}} |z|^{\alpha_0} \nu(dz).$$

This vanishes as $\varepsilon \rightarrow 0$.

(iv) Take $0 < \alpha < 1$.

Using the explicit form of h , giving in Proposition 2.6.3, we have,

$$\begin{aligned} b^\varepsilon &= \varepsilon^{\frac{1}{\alpha}-1}b + \int_{-\infty}^{-\varepsilon^{-\frac{1}{\alpha}}} [\varepsilon^{\frac{1}{\alpha}-1} - \varepsilon^{-1}] \nu(dz) + \int_{\varepsilon^{-\frac{1}{\alpha}}}^{+\infty} [\varepsilon^{-1} - \varepsilon^{\frac{1}{\alpha}-1}] \nu(dz) \\ &\quad + \varepsilon^{\frac{1}{\alpha}-1} \int_{-\varepsilon^{-\frac{1}{\alpha}}}^{-1} (z+1) \nu(dz) + \varepsilon^{\frac{1}{\alpha}-1} \int_1^{\varepsilon^{-\frac{1}{\alpha}}} (z-1) \nu(dz). \end{aligned}$$

Since g has finite limits at infinity, for any $\delta > 0$, we can choose $\eta > 1$ big enough so that $|g(z) - c^+| < \delta$ for $z \geq \eta$, and $|g(z) - c^-| < \delta$ for $z \leq -\eta$.

Hence,

$$\begin{aligned} \limsup_{\varepsilon \rightarrow 0} \varepsilon^{\frac{1}{\alpha}-1} \int_1^{\varepsilon^{-\frac{1}{\alpha}}} (z-1) \nu(dz) &= \limsup_{\varepsilon \rightarrow 0} \varepsilon^{\frac{1}{\alpha}-1} \int_\eta^{\varepsilon^{-\frac{1}{\alpha}}} (z-1) \nu(dz) \\ &\leq \limsup_{\varepsilon \rightarrow 0} \varepsilon^{\frac{1}{\alpha}-1} (c^+ + \delta) \int_\eta^{\varepsilon^{-\frac{1}{\alpha}}} \frac{x-1}{x^{1+\alpha}} dx = \frac{c^+ + \delta}{1-\alpha}. \end{aligned}$$

Similarly,

$$\begin{aligned} \liminf_{\varepsilon \rightarrow 0} \varepsilon^{\frac{1}{\alpha}-1} \int_1^{\varepsilon^{-\frac{1}{\alpha}}} (z-1) \nu(dz) &= \liminf_{\varepsilon \rightarrow 0} \varepsilon^{\frac{1}{\alpha}-1} \int_\eta^{\varepsilon^{-\frac{1}{\alpha}}} (z-1) \nu(dz) \\ &\geq \liminf_{\varepsilon \rightarrow 0} \varepsilon^{\frac{1}{\alpha}-1} (c^+ - \delta) \int_\eta^{\varepsilon^{-\frac{1}{\alpha}}} \frac{z-1}{z^{1+\alpha}} dz = \frac{c^+ - \delta}{1-\alpha}. \end{aligned}$$

The choice of δ being arbitrary, we get

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^{\frac{1}{\alpha}-1} \int_1^{\varepsilon^{-\frac{1}{\alpha}}} (z-1) \nu(dz) = \frac{c^+}{1-\alpha}.$$

If ε is small enough, then we can upper bound

$$\begin{aligned} \limsup_{\varepsilon \rightarrow 0} \int_{\varepsilon^{-\frac{1}{\alpha}}}^{+\infty} [\varepsilon^{-1} - \varepsilon^{\frac{1}{\alpha}-1}] \nu(dz) &\leq \limsup_{\varepsilon \rightarrow 0} (\varepsilon^{-1} - \varepsilon^{\frac{1}{\alpha}-1}) (c^+ + \delta) \nu([\varepsilon^{-\frac{1}{\alpha}}, +\infty)) \\ &= \limsup_{\varepsilon \rightarrow 0} (1 - \varepsilon^{\frac{1}{\alpha}}) \frac{c^+ + \delta}{\alpha} = \frac{c^+ + \delta}{\alpha}. \end{aligned}$$

Moreover,

$$\begin{aligned} \liminf_{\varepsilon \rightarrow 0} \int_{\varepsilon^{-\frac{1}{\alpha}}}^{+\infty} [\varepsilon^{-1} - \varepsilon^{\frac{1}{\alpha}-1}] \nu(dz) &\geq \liminf_{\varepsilon \rightarrow 0} (\varepsilon^{-1} - \varepsilon^{\frac{1}{\alpha}-1}) (c^+ - \delta) \nu([\varepsilon^{-\frac{1}{\alpha}}, +\infty)) \\ &= \liminf_{\varepsilon \rightarrow 0} (1 - \varepsilon^{\frac{1}{\alpha}}) \frac{c^+ - \delta}{\alpha} = \frac{c^+ - \delta}{\alpha}. \end{aligned}$$

Similarly, this leads to

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^{\frac{1}{\alpha}-1} \int_{-\varepsilon^{-\frac{1}{\alpha}}}^{-1} (z-1) \nu(dz) = -\frac{c^-}{1-\alpha},$$

and,

$$\lim_{\varepsilon \rightarrow 0} \int_{-\infty}^{-\varepsilon^{-\frac{1}{\alpha}}} [\varepsilon^{\frac{1}{\alpha}-1} - \varepsilon^{-1}] \nu(dz) = -\frac{c^-}{\alpha}.$$

Hence, we obtain

$$\lim_{\varepsilon \rightarrow 0} b^\varepsilon = \frac{c^+ - c^-}{\alpha(1-\alpha)}.$$

Since, $\int_{\mathbb{R}^*} h(z) \nu^*(dz) = \frac{c^+ - c^-}{\alpha(1-\alpha)}$, the drift coefficient of the limiting process equals zero.

The proof of (2.41) and (2.42) are identical to the one done in (i).

(v) Take $1 < \alpha < 2$ and assume that $b = 0$.

After a change of variables, we can apply the dominated convergence theorem to

$$b^\varepsilon = \int_{\mathbb{R}^*} \left(h(y) - \varepsilon^{\frac{1}{\alpha}} h(y\varepsilon^{-\frac{1}{\alpha}}) \right) \frac{g(y\varepsilon^{-\frac{1}{\alpha}})}{|y|^{1+\alpha}} dy.$$

The proof of (2.41) and (2.42) are identical to the one done in (i).

□

2.6.2 Proofs of Theorems 2.6.1 and 2.6.2

In this section, we suppose that $\beta > 1 + p_\alpha(\gamma) - \theta$, where θ is the exponent of the rate of convergence given in Proposition 2.6.3 and $p_\alpha(\gamma)$ is given in the statement of Theorems 2.6.1 and 2.6.2. Recall that $(V_t)_{t \geq t_0}$ is the solution to (SKE_L).

Moment estimates of the velocity process

As in Section 2.4, we will show that there exists a constant $C_{\gamma, \kappa, \beta, t_0}$ such that

$$\forall t \geq t_0, \quad \mathbb{E}[|V_t|^\kappa] \leq C_{\gamma, \kappa, \beta, t_0, b} t^{p_\alpha(\gamma, \kappa)}, \quad (2.44)$$

where $p_\alpha(\gamma, \kappa)$ has to be detailed.

Proposition 2.6.4. *Pick $\alpha \in (0, 1)$. Assume $(H_1^{\nu, \alpha})$ and (H_{sgn}) . For any γ, β , the explosion time τ_∞ is a.s. infinite and for all $\kappa \in [0, \alpha)$, there exists $C_{\kappa, t_0, b}$ such that, we have*

$$\forall t \geq t_0, \quad \mathbb{E}[|V_t|^\kappa] \leq C_{\kappa, t_0, b} t^{\frac{\kappa}{\alpha}}, \quad \text{or equivalently } p_\alpha(\gamma, \kappa) = \frac{\kappa}{\alpha}.$$

Proof. The proof is analogous to the proof of Proposition 2.4.2. The term $\int_{t_0}^t b f'_n(V_s) ds$ is bounded by $|b|t$ since $\|f'_n\|_\infty \leq 1$. Moreover, since ν satisfies $(H_1^{\nu, \alpha})$ with $\alpha < 1$, the process $L_t^+ := \sum_{s \leq t} |\Delta L_s|$ satisfies the conditions of Theorem 3.1 c) in [DS15]. Thus, for all $\kappa \in [0, \alpha)$, for all $t \geq t_0$,

$$\mathbb{E}\left[|L_t^+|^\kappa\right] \leq C_{t_0, \kappa} t^{\frac{\kappa}{\alpha}}.$$

The estimates for V follows. □

Proposition 2.6.5. *Assume either (H_2^{ν, α_0}) or $(H_1^{\nu, \alpha})$ with $\alpha \in (1, 2)$. For any $\gamma \in [0, 1)$ and any $\beta \in \mathbb{R}$ the explosion time τ_∞ is a.s. infinite and for all $\kappa \in [0, 1]$, there exists $C_{\gamma, \kappa, \beta, t_0}$ and C_{γ, t_0} such that under $(H_1^{\nu, \alpha})$, we have*

$$\forall t \geq t_0, \quad \mathbb{E}[|V_t|^\kappa] \leq C_{\gamma, \kappa, \beta, t_0} t^{\frac{\kappa}{\alpha}} + C_{\gamma, t_0} |b|^\kappa t^\kappa. \quad (2.45)$$

Or equivalently $p_\alpha(\gamma, \kappa) = \frac{\kappa}{\alpha}$ if $b = 0$, and $p_\alpha(\gamma, \kappa) = \kappa$, else. And under (H_2^{ν, α_0}) , there exists $C_{\gamma, \kappa, \beta, t_0, b}$ such that we have

$$\forall t \geq t_0, \quad \mathbb{E}[|V_t|^\kappa] \leq C_{\gamma, \kappa, \beta, t_0, b} t^\kappa.$$

Proof. We explain the differences with respect to the proof of Proposition 2.4.3. Under each hypothesis on the Lévy measure ν , the Lévy process has a finite first moment. We write L_t as the sum $bt + \widehat{L}_t$, where \widehat{L} is the Lévy process without the drift part.

We get,

$$\mathbb{E}\left[|V_{(t \wedge \tau_r)-}|^\kappa\right] \leq C_{t_0} |b|^\kappa t + \mathbb{E}\left[|\widehat{L}_{(t \wedge \tau_r)-}|^\kappa\right] + K \int_{t_0}^t s^{-\beta} \mathbb{E}\left[|V_{s \wedge \tau_r}|^\kappa\right] ds.$$

The proof of Theorem 3.1 (a) p. 3861 in [DS15] can be adapted to estimate the moment of the Lévy process stopped at the stopping time τ_r , given by (2.3).

- (i) If the Lévy measure ν satisfies $(H_1^{\nu, \alpha})$ with $1 < \alpha < 2$, then it satisfies the conditions of Theorem 3.1 (a) and (c) in [DS15]. Thus, for all $\kappa \in [0, 1]$, there exists $C_{t_0, \kappa}$ such

that, for all $t \geq t_0$,

$$\mathbb{E} \left[\left| \widehat{L}_{(t \wedge \tau_r)-} \right|^\kappa \right] \leq C_{t_0, \kappa} t,$$

and,

$$\mathbb{E} \left[\left| \widehat{L}_t \right|^\kappa \right] \leq C_{t_0, \kappa} t^{\frac{\kappa}{\alpha}}.$$

(ii) If the Lévy measure ν satisfies (H_2^{ν, α_0}) , then it satisfies the conditions of Theorem 3.1 (a) in [DS15]. Thus, for all $\kappa \in [0, 1]$, there exists $C_{t_0, \kappa}$ such that, for all $t \geq t_0$,

$$\mathbb{E} \left[\left| \widehat{L}_{(t \wedge \tau_r)-} \right|^\kappa \right] \leq C_{t_0, \kappa} t.$$

Applying the Grönwall-type lemma again (see Lemma 2.A.1) and Fatou's lemma, for $\beta \neq 1$, we end up with

$$\forall t \geq t_0, \mathbb{E} \left[\left| V_{(t \wedge \tau_r)-} \right| \right] \leq C_\gamma \left[C_{t_0, \beta} t + \left(\frac{1-\gamma}{1-\beta} K (t^{1-\beta} - t_0^{1-\beta}) \right)^{\frac{1}{1-\gamma}} \right].$$

The case $\beta = 1$ can be done in a similar manner.

We conclude that the explosion time of V is a.s. infinite. To refine the estimates under the hypothesis $(H_1^{\nu, \alpha})$, we apply again the Grönwall-type lemma to

$$\mathbb{E} [|V_t|] \leq C_{t_0} |b| t + \mathbb{E} [|\widehat{L}_t|] + K \int_{t_0}^t s^{-\beta} \mathbb{E} [|V_s|]^\gamma ds.$$

This proves (2.45) since $\frac{\gamma-1}{\alpha} + 1 - \beta \leq 0$.

□

Proposition 2.6.6. *Assume that for all $v \in \mathbb{R}$, $vF(v) \geq 0$. Pick $\gamma \in \mathbb{R}$ and $\beta \in \mathbb{R}$. For each of the following cases, the explosion time τ_∞ is a.s. infinite.*

(i) *Assume that $(H_1^{\nu, \alpha})$ holds with a bounded function g and $\alpha \in [1, 2)$, there exists C_{κ, t_0} such that*

$$\text{for } \kappa \in (0, \alpha), \forall t \geq t_0, \mathbb{E} [|V_t|^\kappa] \leq C_{\kappa, t_0} t^{\frac{\kappa}{\alpha}}, \text{ or equivalently } p_\alpha(\gamma, \kappa) = \frac{\kappa}{\alpha}. \quad (2.46)$$

(ii) *Assume that $(H_1^{\nu, \alpha})$ holds with $\alpha \in [1, 2)$. Then,*

for $\kappa \in [0, 1]$ (resp. $\kappa \in [0, 1)$, if $\alpha = 1$), there exists C_{κ, t_0} such that

$$\forall t \geq t_0, \mathbb{E} [|V_t|^\kappa] \leq C_{\kappa, t_0} t^\kappa, \text{ or equivalently } p_\alpha(\gamma, \kappa) = \kappa; \quad (2.47)$$

for $\kappa \in (1, \alpha)$, $\gamma \in [0, 1)$ and $b = 0$, there exists C_{κ, t_0} such that

$$\forall t \geq t_0, \mathbb{E}[|V_t|^\kappa] \leq C_{\kappa, t_0} t^{\frac{3\kappa}{2\alpha}}, \text{ or equivalently } p_\alpha(\gamma, \kappa) = \frac{3\kappa}{2\alpha}; \quad (2.48)$$

for $\kappa \in (1, \alpha)$, there exists C_{κ, t_0} such that

$$\forall t \geq t_0, \mathbb{E}[|V_t|^\kappa] \leq C_{\kappa, t_0} t^{\frac{\kappa}{\alpha} + \frac{\kappa}{2}}, \text{ or equivalently } p_\alpha(\gamma, \kappa) = \frac{\kappa}{\alpha} + \frac{\kappa}{2}. \quad (2.49)$$

(iii) Assume (H_2^{ν, α_0}) . Then,

for $\kappa \in [0, 1]$, there exists C_{κ, t_0} such that,

$$\forall t \geq t_0, \mathbb{E}[|V_t|^\kappa] \leq C_{\kappa, t_0} t^\kappa, \text{ or equivalently } p(\gamma, \kappa) = \kappa; \quad (2.50)$$

for $\kappa \in [0, \alpha_0]$, there exists C_{κ, t_0} such that

$$\forall t \geq t_0, \mathbb{E}[|V_t|^\kappa] \leq C_{\kappa, t_0} t^{\frac{\kappa}{\alpha_0} + \frac{\kappa}{2}}, \text{ or equivalently } p(\gamma, \kappa) = \frac{\kappa}{\alpha_0} + \frac{\kappa}{2}. \quad (2.51)$$

Proof. We highlight only the differences with respect to the proof of Proposition 2.4.4. In the following, we assume that $\alpha \neq 1$, the proof is similar for $\alpha = 1$.

STEP A. Assume that $(H_1^{\nu, \alpha})$ holds with a bounded function g .

STEP A1. Pick $\kappa \in [0, 1]$. We adapt the estimates of the Itô's formula's terms.

There is an additional term in (2.9), given, for $t \geq t_0$, by

$$\int_{t_0}^t \mathbb{1}_{\{s \leq \tau_r\}} f'(V_s) b \, ds.$$

If g is a bounded function, the other terms of (2.9) can be estimated in the same way, and (2.19) becomes

$$\begin{aligned} \mathbb{E}[|V_{t \wedge \tau_r}|^\kappa] &\leq \mathbb{E}[f(V_{t_0})] + t^{\frac{\kappa}{\alpha} + 1 - \frac{1}{\alpha}} \kappa |b| + t^{\frac{\kappa}{\alpha}} \sup_{\mathbb{R}} |g| \times \\ &\quad \left(\kappa \frac{a_+ - a_-}{\alpha - 1} + \frac{a_+ + a_-}{\alpha} + \frac{a_+ + a_-}{\alpha - \kappa} + \frac{1}{2} \kappa (3 - \kappa) \frac{a_+ + a_-}{2 - \alpha} \right) \leq C_{\kappa, t_0, b} t^{\frac{\kappa}{\alpha}}. \end{aligned} \quad (2.52)$$

This gives the proof of (2.46) for $\kappa \in [0, 1]$.

STEP A2. Pick $\kappa \in (1, \alpha)$. We estimate R , given by (2.11) in another way.

The inequality (2.22) becomes

$$\begin{aligned} \mathbb{E}[|V_{t \wedge \tau_r}|^\kappa] &\leq \mathbb{E}[f(V_{t_0})] + t \left(C_\kappa \frac{a_+ + a_-}{\alpha - \kappa} \xi^{\frac{\kappa}{\alpha} - 1} + \frac{1}{2} \kappa (3 - \kappa) \eta^{\frac{\kappa}{2} - 1} \frac{a_+ + a_-}{2 - \alpha} \xi^{\frac{2}{\alpha} - 1} \right) \\ &\quad + \kappa \left(\frac{a_+ - a_-}{\alpha - 1} \xi^{\frac{1}{\alpha} - 1} + |b| \right) \int_{t_0}^t \mathbb{E}[|V_s|^{\kappa-1}] ds + C_\kappa \frac{a_+ + a_-}{\alpha - \frac{\kappa}{2}} \xi^{\frac{\kappa}{2\alpha} - 1} \int_{t_0}^t \mathbb{E}[|V_s|^{\frac{\kappa}{2}}] ds. \end{aligned} \quad (2.53)$$

The inequality (2.46) follows as in the proof of Proposition 2.4.4.

STEP B. When working under (H_2^{ν, α_0}) , we will pick $\kappa \leq \alpha_0$ during the proof. Assume that either $(H_1^{\nu, \alpha})$ holds with an unbounded function g or (H_2^{ν, α_0}) holds, then we are not able to estimate the following terms

$$\int_{|z| > \xi^{\frac{1}{\alpha}}} z \nu(dz), \quad \int_{0 < |z| < \xi^{\frac{1}{\alpha}}} |z|^k \nu(dz), \quad \text{and} \quad \int_{|z| \geq \xi^{\frac{1}{\alpha}}} |z|^k \nu(dz).$$

Hence, we apply the same proof scheme with small and big jumps sliced at 1.

STEP B1. This leads to a similar bound as in (2.19):

$$\begin{aligned} \mathbb{E}[|V_{t \wedge \tau_r}|^\kappa] &\leq \mathbb{E}[f(V_{t \wedge \tau_r})] \leq \mathbb{E}[f(V_{t_0})] + \int_{t_0}^t |b| \mathbb{E}[|f'(V_{s \wedge \tau_r})|] ds + \mathbb{E}[t \wedge \tau_r] \times \\ &\quad \left(\eta^{\kappa/2} \nu(|z| \geq 1) + \int_{|z| \geq 1} |z|^\kappa \nu(dz) + \frac{1}{2} \kappa (3 - \kappa) \eta^{\frac{\kappa}{2} - 1} \int_{0 < |z| < 1} z^2 \nu(dz) \right) \leq C_{\kappa, t_0} t. \end{aligned}$$

By Jensen's inequality, we can deduce (2.47) and (2.50).

STEP B2. Pick $\kappa \in (1, \alpha)$. We estimate R , given by (2.11) in another way.

By classical Hölder inequality,

$$\int_{|z| \geq 1} |f(V_s + z) - f(V_s)| \nu(dz) \leq \int_{|z| \geq 1} |2zV_s + z^2|^{\frac{\kappa}{2}} \nu(dz) \leq C(1 + |V_s|^{\frac{\kappa}{2}}) \int_{|z| \geq 1} |z|^\kappa \nu(dz) \quad (2.54)$$

The last integral is finite. Gathering (2.16), (2.54) and then using (2.47), (2.50) or Proposition 2.6.5,

$$\mathbb{E}[|V_{t \wedge \tau_r}|^\kappa] \leq \mathbb{E}[f(V_{t \wedge \tau_r})] \leq C_{\kappa, t_0} t + C \int_{t_0}^t \mathbb{E}[|V_s|^{\frac{\kappa}{2}}] ds + \int_{t_0}^t |b| \mathbb{E}[|V_s|^{\kappa-1}] ds.$$

Taking $r \rightarrow +\infty$, we can conclude that $p_\alpha(\gamma, \kappa) = 1 + p_\alpha(\gamma, \kappa/2)$.

STEP B3. We refine the estimates.

Fix $\kappa \in [0, \alpha)$. There exists ε_1 such that, $\kappa \leq \alpha - \varepsilon$ and $\alpha - \varepsilon > 1$. Hence, we can write

$$\mathbb{E} \left[|V_t|^{\alpha-\varepsilon} \right] \leq C_{\kappa, t_0} t^{1+p_\alpha(\gamma, (\alpha-\varepsilon)/2)}.$$

Using Jensen's inequality, we get

$$\mathbb{E} [|V_t|^\kappa] \leq C_{\kappa, t_0} t^{\frac{\kappa}{\alpha-\varepsilon}(1+p_\alpha(\gamma, (\alpha-\varepsilon)/2))},$$

and it suffices to let $\varepsilon \rightarrow 0$ to conclude. This concludes the proof of (2.48) and (2.49). The last step is identical under $(H_1^{\nu, \alpha})$ and can be done with α_0 instead of $\alpha - \varepsilon$, under (H_2^{ν, α_0}) . This concludes the proof of (2.51). □

Proofs of Theorems 2.6.1 and 2.6.2

We are now in position to give the proofs of Theorems 2.6.1 and 2.6.2. Assume that either $(H_1^{\nu, \alpha})$ or (H_2^{ν, α_0}) is satisfied.

As in the α -stable case (see Section 2.5), it suffices to prove the convergence of the rescaled velocity process $(V_t^{(\varepsilon)})_{t>0} := (r_\varepsilon V_{t/\varepsilon})_{t>0}$.

Thanks to Proposition 2.6.3, there exists $\theta \in \{1, \frac{1}{\alpha}\}$ such that $L^{(\varepsilon)} := (\varepsilon^\theta L_{t/\varepsilon})_{t \geq 0}$ converges in distribution.

For the sake of simplicity, we omit the dependencies of p with respect to α and α_0 . Assume that $\gamma \geq 0$ and $\beta > 1 + p(\gamma) - \theta$.

We can write, for $T > 0$,

$$\sup_{\varepsilon t_0 \leq t \leq T} |V_t^{(\varepsilon)} - L_t^{(\varepsilon)}| \leq r_\varepsilon (v_0 - L_{t_0}) + r_\varepsilon^{1-\gamma} \varepsilon^{\beta-1} \int_{\varepsilon t_0}^T K |V_u^{(\varepsilon)}|^\gamma u^{-\beta} du.$$

Using the moment estimates on V (see Section 2.6.2), this leads, with $\beta \neq p(\gamma) + 1$, to

$$r_\varepsilon^{1-\gamma} \varepsilon^{\beta-1} \mathbb{E} \left[\int_{\varepsilon t_0}^T K |V_u^{(\varepsilon)}|^\gamma u^{-\beta} du \right] \leq C \left(r_\varepsilon \varepsilon^{\beta-1-p(\gamma)} T^{p(\gamma)-\beta+1} - t_0^{p(\gamma)-\beta+1} r_\varepsilon \right).$$

Hence, setting $q := \min(\beta - 1 + \theta - p(\gamma), \theta)$, which is a positive number since $\beta > 1 + p(\gamma) - \theta$, we obtain

$$\mathbb{E} \left[\sup_{\varepsilon t_0 \leq t \leq T} |V_t^{(\varepsilon)} - L_t^{(\varepsilon)}| \right] \underset{\varepsilon \rightarrow 0}{=} O(\varepsilon^q).$$

The case $\beta = 1 + p(\gamma)$ can be treated similarly. This concludes both of the proofs.

2.6.3 Lévy driving process with a Brownian component

We extend the results given in Theorems 2.2.2, 2.2.4, 2.6.1 and 2.6.2 to a driving process having a Brownian component. To this end, we decompose the Lévy noise as the sum of a Brownian part $(B_t)_{t \geq 0}$ and a Lévy part without a Brownian component $(L_t)_{t \geq 0}$.

Remark 2.6.7. *The reasoning is the same with a Lévy process \widehat{L} instead of the Brownian driving process. Indeed, let \widehat{r}_ε be the rate such that $(\widehat{r}_\varepsilon \widehat{L}_{t/\varepsilon})_{t \geq 0}$ converges. The conclusion follows, provided that $\widehat{r}_\varepsilon^{-1} r_\varepsilon$ tends to 0.*

Pick $\rho \geq 0$. We consider the following one-dimensional stochastic kinetic model, for $t \geq t_0 > 0$,

$$\begin{aligned} dV_t^{(2)} &= dL_t + dB_t - t^{-\beta} \rho \operatorname{sgn}(V_t^{(2)}) |V_t^{(2)}|^\gamma dt, & V_{t_0}^{(2)} &= v_0 \in \mathbb{R}, \\ \text{and } dX_t^{(2)} &= V_t^{(2)} dt, & X_{t_0}^{(2)} &= x_0 \in \mathbb{R}. \end{aligned} \tag{SKE2}$$

We compare this solution with the solution to the equation driven by a Lévy process L without a Brownian component. From now on, for the sake of convenience, the solutions to (SKE $_L$) with $F(v) = \rho \operatorname{sgn}(v) |v|^\gamma$, will be denoted by $(V_t^{(1)}, X_t^{(1)})_{t \geq t_0}$ instead of $(V_t, X_t)_{t \geq t_0}$. The asymptotic behavior of the latter SDE is given by Theorems 2.2.2, 2.2.4, 2.6.1 and 2.6.2. We will show that the Brownian part has no contribution.

Let us first point out some results about existence up to explosion of (SKE2).

Proposition 2.6.8.

- (i) *When $\gamma \geq 1$, (SKE2) admits a unique solution up to explosion.*
- (ii) *Pick $\alpha \in (1, 2)$. If (SKE2) is driven by the sum of a Brownian motion and an α -stable process, then, if $0 \leq \gamma < 1$, there exists a solution to (SKE2).*

Proof.

- (i) Since the drift coefficient is locally Lipschitz, by a standard localization argument, using Theorem 3.1 and Remark 2 p. 338-339 in [Kun04], there is a unique solution defined up to explosion.

(ii) Since the drift coefficient is a continuous function, using a standard localization argument, we can apply Theorem 3.1 p. 866 in [Kur07] to conclude. \square

Assume in the following that (SKE2) admits a unique solution up to explosion.

Theorem 2.6.9. *Consider $\gamma \in [0, \alpha)$ and $\beta \geq 0$. Let $(V_t^{(2)}, X_t^{(2)})_{t \geq t_0}$ be a solution to (SKE2). The Lévy noise is supposed to be the sum of a Brownian noise $(B_t)_{t \geq 0}$ and a Lévy process $(L_t)_{t \geq 0}$ without Brownian component. Suppose also that (SKE_L) satisfies the conditions of either Theorems 2.2.2, 2.2.4, 2.6.1 or 2.6.2. Then $(V^{(2)}, X^{(2)})$ has the same asymptotic behavior as $(V^{(1)}, X^{(1)})$, which is given in the cited theorems.*

Proof. For $i \in \{1, 2\}$, let us introduce the following rescaled processes:

$$(V_t^{(i,\varepsilon)})_{t \geq \varepsilon t_0} := (r_\varepsilon V_{t/\varepsilon}^{(i)})_{t \geq \varepsilon t_0}, \quad (X_t^{(i,\varepsilon)})_{t \geq \varepsilon t_0} := (\varepsilon r_\varepsilon X_{t/\varepsilon}^{(i)})_{t \geq \varepsilon t_0}, \quad \text{and} \quad (B_t^{(\varepsilon)})_{t \geq 0} := (\sqrt{\varepsilon} B_{t/\varepsilon})_{t \geq 0}.$$

We write, for all $t \geq \varepsilon t_0$,

$$V_t^{(1,\varepsilon)} - V_t^{(2,\varepsilon)} = r_\varepsilon B_{t_0} - r_\varepsilon \varepsilon^{-\frac{1}{2}} B_t^{(\varepsilon)} - r_\varepsilon^{1-\gamma} \varepsilon^{\beta-1} \int_{\varepsilon t_0}^t r_\varepsilon^\gamma \left(F \left(\frac{V_u^{(1,\varepsilon)}}{r_\varepsilon^\gamma} \right) - F \left(\frac{V_u^{(2,\varepsilon)}}{r_\varepsilon^\gamma} \right) \right) u^{-\beta} du.$$

Hence, by Itô's formula, we get

$$\begin{aligned} (V_t^{(1,\varepsilon)} - V_t^{(2,\varepsilon)})^2 &= r_\varepsilon^2 B_{t_0}^2 - 2 \int_{\varepsilon t_0}^t (V_s^{(1,\varepsilon)} - V_s^{(2,\varepsilon)}) r_\varepsilon \varepsilon^{-\frac{1}{2}} dB_s^{(\varepsilon)} + r_\varepsilon^2 \varepsilon^{-1} (t - \varepsilon t_0) \\ &\quad - 2 r_\varepsilon^{1-\gamma} \varepsilon^{\beta-1} \int_{\varepsilon t_0}^t (V_s^{(1,\varepsilon)} - V_s^{(2,\varepsilon)}) r_\varepsilon^\gamma \left(F \left(\frac{V_s^{(1,\varepsilon)}}{r_\varepsilon^\gamma} \right) - F \left(\frac{V_s^{(2,\varepsilon)}}{r_\varepsilon^\gamma} \right) \right) u^{-\beta} du. \end{aligned}$$

The last term on the right-hand side of the upper equality is positive since F is an increasing function.

Moreover, we can apply BDG inequalities (see Theorem 73 p. 222 in [Pro05]) to the local martingale, defined by

$$M_t := -2 \int_{\varepsilon t_0}^t (V_s^{(1,\varepsilon)} - V_s^{(2,\varepsilon)}) r_\varepsilon \varepsilon^{-\frac{1}{2}} dB_s^{(\varepsilon)}, \quad t \geq \varepsilon t_0.$$

This yields, for each $T > 0$,

$$\mathbb{E} \left[\sup_{\varepsilon t_0 \leq t \leq T} M_t^2 \right] \leq \mathbb{E} \left[4 r_\varepsilon^2 \varepsilon^{-1} \int_{\varepsilon t_0}^T (V_s^{(1,\varepsilon)} - V_s^{(2,\varepsilon)})^2 ds \right].$$

For the sake of clarity, given x and y , we introduce $x \lesssim y$ to mean that there exists a positive constant C such that $x \leq Cy$. We get

$$\begin{aligned} \mathbb{E} \left[\sup_{\varepsilon t_0 \leq t \leq T} \left(V_t^{(1,\varepsilon)} - V_t^{(2,\varepsilon)} \right)^4 \right] &\lesssim r_\varepsilon^4 \mathbb{E} \left[B_{t_0}^4 \right] + r_\varepsilon^4 \varepsilon^{-2} T^2 + \mathbb{E} \left[\sup_{\varepsilon t_0 \leq t \leq T} M_t^2 \right] \\ &\lesssim r_\varepsilon^4 \mathbb{E} \left[B_{t_0}^4 \right] + r_\varepsilon^4 \varepsilon^{-2} T^2 + 4r_\varepsilon^2 \varepsilon^{-1} \int_{\varepsilon t_0}^T \mathbb{E} \left[\left(V_s^{(1,\varepsilon)} - V_s^{(2,\varepsilon)} \right)^2 \right] ds. \end{aligned}$$

Hence, using that for all real a , $a^2 \leq a^4 + 1$, we deduce that

$$\begin{aligned} \mathbb{E} \left[\sup_{\varepsilon t_0 \leq t \leq T} \left(V_t^{(1,\varepsilon)} - V_t^{(2,\varepsilon)} \right)^4 \right] \\ \lesssim r_\varepsilon^4 \mathbb{E} \left[B_{t_0}^4 \right] + r_\varepsilon^4 \varepsilon^{-2} T^2 + 4r_\varepsilon^2 \varepsilon^{-1} T + 4r_\varepsilon^2 \varepsilon^{-1} \int_{\varepsilon t_0}^T \mathbb{E} \left[\sup_{\varepsilon t_0 \leq t \leq s} \left(V_t^{(1,\varepsilon)} - V_t^{(2,\varepsilon)} \right)^4 \right] ds. \end{aligned}$$

Applying Grönwall's lemma, up to stopping times, we get

$$\mathbb{E} \left[\sup_{\varepsilon t_0 \leq t \leq T} \left(V_t^{(1,\varepsilon)} - V_t^{(2,\varepsilon)} \right)^4 \right] \lesssim \left(r_\varepsilon^4 \mathbb{E} \left[B_{t_0}^4 \right] + r_\varepsilon^4 \varepsilon^{-2} T^2 + 4r_\varepsilon^2 \varepsilon^{-1} T \right) \exp \left(4r_\varepsilon^2 \varepsilon^{-1} T \right).$$

We deduce also that

$$\begin{aligned} \mathbb{E} \left[\sup_{\varepsilon t_0 \leq t \leq T} \left(X_t^{(1,\varepsilon)} - X_t^{(2,\varepsilon)} \right)^4 \right] &\leq T^4 \mathbb{E} \left[\sup_{\varepsilon t_0 \leq t \leq T} \left(V_t^{(1,\varepsilon)} - V_t^{(2,\varepsilon)} \right)^4 \right] \\ &\lesssim T^4 \left(r_\varepsilon^4 \mathbb{E} \left[B_{t_0}^4 \right] + r_\varepsilon^4 \varepsilon^{-2} T^2 + 4r_\varepsilon^2 \varepsilon^{-1} T \right) \exp \left(4r_\varepsilon^2 \varepsilon^{-1} T \right). \end{aligned}$$

Hence, $\left(V_t^{(1,\varepsilon)} - V_t^{(2,\varepsilon)}, X_t^{(1,\varepsilon)} - X_t^{(2,\varepsilon)} \right)_{t \geq \varepsilon t_0}$ converges uniformly towards 0 in probability on compacts, as ε goes to zero.

The conclusion follows from Theorems 2.2.2, 2.2.4, 2.6.1 and 2.6.2, Theorem 3.1 p. 27 in [Bil99] and Lemma 2.A.3. \square

2.A Some technical results

Let us state first a Grönwall-type lemma which has been used to get moment estimates. The proof can be found in [GL21a].

Lemma 2.A.1 (Grönwall-type lemma). *Fix $r \in [0, 1)$ and $t_0 \in \mathbb{R}$. Assume that g is a non-negative real-valued function, b is a positive function and a is a differentiable real-valued function. Moreover, suppose that the function bg^r is a continuous function.*

Assume that

$$\forall t \geq t_0, g(t) \leq a(t) + \int_{t_0}^t b(s)g(s)^r ds. \quad (2.55)$$

Then, setting $C_r := 2^{\frac{1}{1-r}}$,

$$\forall t \geq t_0, g(t) \leq C_r \left[a(t) + \left((1-r) \int_{t_0}^t b(s) ds \right)^{\frac{1}{1-r}} \right].$$

Remark 2.A.2. Call H the right-hand side of (2.55). If g is not continuous, note that the function H is still continuous and satisfies (2.55) (since b is positive and $g \leq H$). So, one can apply the lemma to H and thereafter use the inequality $g \leq H$.

We state now a technical lemma about the convergence in the spaces \mathcal{C} and \mathcal{D} . We recall that the space of continuous functions \mathcal{C} is endowed with the uniform topology

$$d_u : (f, g) \in \mathcal{C}((0, +\infty), \mathbb{R})^2 \mapsto \sum_{n=1}^{+\infty} \frac{1}{2^n} \min \left(1, \sup_{[\frac{1}{n}, n]} |f - g| \right).$$

Set

$$\Lambda := \{ \lambda : \mathbb{R}^+ \rightarrow \mathbb{R}^+, \text{ continuous and increasing function s.t. } \lambda(0) = 0, \lim_{t \rightarrow +\infty} \lambda(t) = +\infty \}$$

and

$$k_n(t) = \begin{cases} 1 & \text{if } \frac{1}{n} \leq t \leq n, \\ n+1-t & \text{if } n < t < n+1, \\ 0 & \text{if } n+1 \leq t. \end{cases}$$

The space of càdlàg functions \mathcal{D} is endowed with the Skorokhod topology d_s defined for $(f, g) \in \mathcal{D}((0, +\infty))^2$ by

$$\sum_{n=1}^{+\infty} \frac{1}{2^n} \left(1 \wedge \inf \left\{ a, \exists \lambda \in \Lambda, \sup_{s \neq t} \left| \log \frac{\lambda(t) - \lambda(s)}{t - s} \right| \leq a, \sup_{t \geq \frac{1}{n}} |k_n(t) (f \circ \lambda(t) - g(t))| \leq a \right\} \right).$$

Lemma 2.A.3.

(i) The uniform distance is finer than the Skorokhod one i.e. $d_s \leq d_u$.

(ii) Let $(f_\varepsilon)_{\varepsilon \geq 0}, (h_\varepsilon)_{\varepsilon \geq 0}$ be two sequences of functions of \mathcal{D} . If for all $n \geq 1$,

$$\sup_{t \in [\frac{1}{n}, n]} |f_\varepsilon(t) - h_\varepsilon(t)| \xrightarrow[\varepsilon \rightarrow 0]{\mathbb{P}} 0,$$

then $d(f_\varepsilon, h_\varepsilon) \xrightarrow[\varepsilon \rightarrow 0]{\mathbb{P}} 0$, where $d \in \{d_u, d_s\}$.

Proof. Let f, g be two càdlàg functions. The first point is true using the definition of the metrics d_s and d_u and noting that

$$\inf \left\{ a, \exists \lambda \in \Lambda, \sup_{s \neq t} \left| \log \frac{\lambda(t) - \lambda(s)}{t - s} \right| \leq a, \sup_{t \geq \frac{1}{n}} |k_n(t) (f(\lambda(t)) - g(t))| \leq a \right\} \leq \sup_{[\frac{1}{n+1}, n+1]} |f - g|.$$

Let us now prove the second part. Assume that for all $n \geq 1$, $\sup_{[\frac{1}{n}, n]} |f_\varepsilon - h_\varepsilon| \xrightarrow[\varepsilon \rightarrow 0]{\mathbb{P}} 0$, as $\varepsilon \rightarrow 0$. Fix $\eta > 0$ and choose $N > 0$ such that $\sum_{n=N+1}^{+\infty} \frac{1}{2^n} \leq \frac{\eta}{2}$. Then,

$$d_s(f_\varepsilon, h_\varepsilon) \leq d_u(f_\varepsilon, h_\varepsilon) \leq \frac{\eta}{2} + \sum_{n=1}^N \frac{1}{2^n} \sup_{[\frac{1}{n}, n]} |f_\varepsilon - h_\varepsilon|.$$

It follows, by setting $\eta' := \frac{\eta}{2} \left(\sum_{n=1}^{+\infty} \frac{1}{2^n} \right)^{-1}$, that

$$\mathbb{P}(d(f_\varepsilon, h_\varepsilon) > \eta) \leq \sum_{n=1}^N \mathbb{P} \left(\sup_{[\frac{1}{n}, n]} |f_\varepsilon - h_\varepsilon| > \eta' \right) \xrightarrow[\varepsilon \rightarrow 0]{} 0.$$

□

For the sake of completeness, we state and improve the result of Problem 4.12 p. 64 in [KS98], on a general metric space.

Lemma 2.A.4. *Let S be a Polish metric space endowed with a Borel σ -field \mathcal{S} . Suppose that $(P_n)_{n \geq 1}$ is a sequence of probability measures on (S, \mathcal{S}) which converges weakly to a probability measure P . Suppose, in addition, that the sequence $(f_n)_{n \geq 1}$ of real-valued continuous functions on S is uniformly bounded and converges to a continuous function f , the convergence being uniform on compact subsets of S . Then, we have*

$$\lim_{n \rightarrow +\infty} \int_S f_n(\omega) dP_n(\omega) = \int_S f(\omega) dP(\omega).$$

Proof. Notice that, since $(P_n)_{n \geq 1}$ converges weakly thus, it is tight. So, for each $\varepsilon > 0$, there exists a compact subset K of S such that for any $n \geq 1$, $P_n(K) \geq 1 - \varepsilon$.

Let us decompose

$$\int_S f_n dP_n - \int_S f dP = A + B + C + D,$$

where

$$A := \int_{S \setminus K} f_n dP_n, \quad B := \int_K (f_n - f) dP_n, \quad C := \int_{S \setminus K} f dP_n, \quad \text{and} \quad D := \int_S f dP_n - \int_S f dP.$$

Let M be a bound for the sequence (f_n) . Thus, by the choice of K ,

$$|A| \leq MP_n(S \setminus K) \leq M\varepsilon.$$

The third integral can be treated analogously. Besides, since the sequence (f_n) converges uniformly on K to f , there exists n_ε such that for all $n \geq n_\varepsilon$, $\sup_K |f_n - f| \leq \varepsilon$. Thereby, we get

$$|B| \leq \varepsilon P_n(K).$$

The last integral is smaller than ε for n large enough, since P_n converges weakly to P , and this concludes the proof. \square

Remark 2.A.5. *Lemma 2.A.4 could be applied with $S = \mathcal{C}([0, +\infty))$ or $\mathcal{D}([0, +\infty))$. However, the result for $S = \mathcal{C}([0, +\infty))$ is already contained in Problem 4.12 p. 64 in [KS98].*

Lemma 2.A.6. *Let $(Y_t^y)_{t \geq 0}$ be the solution to a time-homogeneous SDE driven by an α -stable process,*

$$\begin{cases} dY_t = dL_t + b(Y_t) dt \\ Y_0 = y \end{cases} \quad (2.56)$$

The measurable function b is supposed to be such that (2.56) has a pathwise unique strong solution. Then $(Y_t)_{t \geq 0}$ is a Markov process.

Namely, for any $d \geq 1$, $0 \leq t_1 \leq \dots \leq t_d$, $u \geq 0$ and any bounded measurable function $\phi : \mathbb{R}^d \rightarrow \mathbb{R}$,

$$\mathbb{E} \left[\phi(Y_{t_1+u}^y, \dots, Y_{t_d+u}^y) \middle| \mathcal{F}_u \right] = \mathbb{E} \left[\phi(Y_{t_1}^z, \dots, Y_{t_d}^z) \right]_{z=Y_u^y}. \quad (2.57)$$

Proof. We give a proof for $d = 2$, the general case being similar. Call $(Y_t^{s,y})$ the solution to $dY_t = dL_t + b(Y_t) dt$, satisfying $Y_s = y$. Let $\phi : \mathbb{R}^2 \rightarrow \mathbb{R}$ be a bounded measurable

function. Pick $u \geq 0$. Consider, for $y \in \mathbb{R}$ and $u \leq s \leq t$ the function

$$G(y, s, t, u) := (Y_s^{u,y}, Y_t^{u,y}) = \left(y + L_s - L_u + \int_u^s b(Y_h) dh, y + L_t - L_u + \int_u^t b(Y_h) dh \right).$$

Pick $0 \leq s \leq t$. Using pathwise uniqueness, $(Y_{s+u}^y, Y_{t+u}^y) = G(Y_u^y, s+u, t+u, u)$. Moreover, by time-homogeneity of the SDE, $(Y_{s+u}^{u,y})_{s \geq 0}$ and $(Y_s^y)_{s \geq 0}$ have the same distribution. As a consequence, $G(y, s+u, t+u, u) = G(y, s, t, 0)$. Besides, by Markov property of Lévy processes, the function G is independent of \mathcal{F}_u . Hence,

$$\begin{aligned} \mathbb{E} \left[\phi(Y_{s+u}^y, Y_{t+u}^y) \middle| \mathcal{F}_u \right] &= \mathbb{E} \left[\phi \circ G(Y_u^y, s+u, t+u, u) \middle| \mathcal{F}_u \right] = \mathbb{E} \left[\phi \circ G(z, s, t, 0) \middle| \mathcal{F}_u \right]_{z=Y_u^y} \\ &= \mathbb{E} [\phi \circ G(z, s, t, 0)]_{z=Y_u^y} = \mathbb{E} [\phi(Y_s^z, Y_t^z)]_{z=Y_u^y} \end{aligned}$$

This concludes the proof. \square

Let us now state and prove the following result.

Lemma 2.A.7. *Let b be a function such that $\lim_{t \rightarrow +\infty} b(t) = 0$. Pick $a > 0$, $\gamma \geq 1$ and $\kappa > 1$. Let g be a continuously differentiable positive function satisfying*

$$g'(t) \leq -ag(t)^{\frac{\kappa+\gamma-1}{\kappa}} + b(t)g(t)^{\frac{\kappa-1}{\kappa}}, \quad t \geq 0. \quad (2.58)$$

Then, $g(t) \xrightarrow[t \rightarrow +\infty]{} 0$.

Proof. Pick $\varepsilon > 0$. Let t_1 be a positive real such that for all $t \geq t_1$, $|b(t)| \leq \frac{a}{2}\varepsilon^{\frac{\gamma}{\kappa}}$.

STEP 1. We first show that there exists $t^* \geq t_1$, such that $g(t^*) \leq \varepsilon$.

Assume, by way of contradiction, that it is not the case. Thus, one can consider the function $y = g^{\frac{1}{\kappa}}$, which satisfies

$$\kappa y'(t) \leq -ay(t)^\gamma + b(t), \quad t \geq t_1. \quad (2.59)$$

For all $t \geq t_1$, we have

$$\kappa y'(t) \leq -a\varepsilon^{\frac{\gamma}{\kappa}} + \frac{a}{2}\varepsilon^{\frac{\gamma}{\kappa}} \leq -\frac{a}{2}\varepsilon^{\frac{\gamma}{\kappa}}.$$

As a consequence, for all $t \geq t_1$,

$$\kappa\varepsilon^{\frac{1}{\kappa}} < \kappa y(t) \leq \kappa y(t_1) - (t - t_1)\frac{a}{2}\varepsilon^{\frac{\gamma}{\kappa}} \xrightarrow[t \rightarrow +\infty]{} -\infty.$$

This is a contradiction.

STEP 2. We show that for all $t \geq t^*$, $g(t) \leq \varepsilon$.

Define $T = \inf\{t \geq t^*, g(t) > \varepsilon\}$. By continuity of the function g , we have $g(T) = \varepsilon$.

Hence,

$$g'(T) \leq -a\varepsilon^{\frac{\kappa+\gamma-1}{\kappa}} + \frac{a}{2}\varepsilon^{\frac{\gamma}{\kappa}}\varepsilon^{\frac{\kappa-1}{\kappa}} < -\frac{a}{2}\varepsilon^{\frac{\kappa+\gamma-1}{\kappa}} < 0.$$

Therefore, there exists $\delta_0 > 0$, such that for all $0 < \delta \leq \delta_0$,

$$g(T + \delta) < g(T) = \varepsilon.$$

This is a contradiction with the definition of T .

□

Acknowledgements

The authors would like to thank Nicolas Fournier and Thomas Cavallazzi for helpful discussions about moment estimates. We would also like to thank Vlad Bally for his suggestion which gives rise to the Section 2.6. We are grateful to Thomas Cavallazzi for his careful reading of the manuscript.

PART II

Time-inhomogeneous stochastic dynamics in the quadratic potential

Contents

Introduction	127
3 Behavior of the non-autonomous system driven by a Brownian motion	133
3.1 Notations and main results	133
3.2 Existence up to explosion	135
3.3 Moment estimates and non-explosion	136
3.4 Asymptotic behavior of the solution	137
3.4.1 Convergence of the f.d.d. in the super-critical regime	138
3.4.2 Convergence of the f.d.d. in the critical and sub-critical regimes . .	140
3.5 Proof of Corollary 3.1.2	143
4 Behavior of the non-autonomous system driven by a Lévy process	145
4.1 Notations and main results	145
4.1.1 Settings on Lévy processes and functional spaces	145
4.1.2 Main results	146
4.2 Existence up to explosion	148
4.3 Moment estimates and non-explosion	149
4.4 Asymptotic behavior of the solution	153
4.4.1 Convergence of the f.d.d. in the super-critical regime	155
4.4.2 Convergence of the f.d.d. in the critical and sub-critical regime . . .	158
4.5 Proof of Corollary 4.1.2	163
A Study of the deterministic underlying ODE	165
B Some technical results	168

Abstract: We consider a particle evolving in the quadratic potential, and subject to a time-inhomogeneous frictional force and to a random force. The couple of its velocity and position is solution to a stochastic differential equation driven by an α -stable Lévy process, and the frictional force is of the form $t^{-\beta} \operatorname{sgn}(v)|v|^\gamma$. We identify three regimes, depending on the balance between β , γ and α , for the behavior in long-time of the couple velocity-position, with a suitable rescaling.

Keywords: kinetic stochastic equation; time-inhomogeneous stochastic differential equation; Lévy process; scaling transformation; asymptotic distribution; asymptotic expansion of solutions to ordinary differential equations.

MSC2020 Subject Classification: Primary 60H10; Secondary 60F17; 60G52; 60G18; 60J65; 34E05.

This part is a work in collaboration with Thomas Cavallazzi.

INTRODUCTION

In this part, we consider a particle, with velocity $V \in \mathbb{R}$ and position $X \in \mathbb{R}$, evolving in the quadratic potential $\mathcal{U} : x \mapsto \frac{x^2}{2}$, subject to a time-inhomogeneous frictional force b and to a random force. This random force models the particle's interaction with its environment. Using Newton's second law, the dynamics of the particle can be described by the following stochastic damping Hamiltonian system, driven by an α -stable process L with $\alpha \in (0, 2]$,

$$\begin{cases} dV_t = dL_t - b(t, V_t) dt - \nabla \mathcal{U}(X_t) dt, \\ dX_t = V_t dt, \\ (V_{t_0}, X_{t_0}) = (v_0, x_0). \end{cases} \quad (\text{SKE}_c)$$

Stochastic Hamiltonian systems have been widely studied in the time-homogeneous setting. An interesting problem is to understand their asymptotic behavior (see [AK94]). The case of damping systems appears in [Wu01] (see also references therein). The long-time behavior of a particle evolving in a free potential, i.e. when $\mathcal{U} = 0$, has already been studied, see e.g. [GO13], [FT21], [GL21a] and [GL21b]. When the random force is supposed to be Brownian, i.e. $\alpha = 2$, a non-linear Langevin's type SDE with time-homogeneous coefficients was studied in [FT21]. The system is of the form

$$V_t = v_0 + B_t - \frac{\rho}{2} \int_0^t b(V_s) ds \quad \text{and} \quad X_t = x_0 + \int_0^t V_s ds,$$

with, for example, $b(v) = \frac{v}{1+v^2}$. Using a suitable rescaling, the authors show that the position process behaves asymptotically as a Brownian motion, a stable process, or an integrated Bessel process, depending on the value of ρ . For example, in the normal diffusive regime, the rescaled velocity-position process behaves asymptotically as an ergodic process and a Brownian motion. In the integrated Bessel regime, the Bessel process and its time integral are the asymptotic distributions. However, the tools used therein, such as invariant measure, scale function and speed measure, are limited to time-homogeneous coefficients.

In [GO13] and [GL21a], the drift coefficient b was allowed to depend on time under a homogeneity condition. More precisely, in [GO13], the authors study the convergence in

distribution, when t tends to $+\infty$, of $r_t V_t$, for a certain rate of convergence r_t . In [GL21a], the authors extend to convergence of processes the results obtained in [GO13] for the couple velocity-position. Namely, the authors study the limit in distribution of the rescaled process $(r_{\varepsilon,V} V_{t/\varepsilon}, r_{\varepsilon,X} X_{t/\varepsilon})_t$ for two appropriate rates of convergence $r_{\varepsilon,V}$ and $r_{\varepsilon,X}$. Results were further generalized in [GL21b] to a Lévy driving process, supposed to be an α -stable process. To be more precise, if b is of the form $t^{-\beta} \operatorname{sgn}(v) |v|^\gamma$, the authors highlight three schemes, depending on the balance between β , γ and α , the index of stability of L . When the frictional force is sufficiently “small at infinity”, the rescaled process behaves as if there was no frictional force and thus converges in distribution towards the Kolmogorov process $(\mathcal{L}, \int_0^\cdot \mathcal{L})$, where \mathcal{L} has the same distribution as the driving process. When the two forces offset, the limiting process is still of kinetic form $(\mathcal{V}, \int_0^\cdot \mathcal{V})$, but the process \mathcal{V} is henceforth ergodic. Whereas, when the drag force swings with the random noise, the limiting process is no longer kinetic. The rescaled velocity process converges in finite dimensional distributions towards a product measure. The proofs are essentially based on the self-similarity of the driving process and on moment estimates of the velocity process.

Degenerate systems has been intensively studied for several years. In particular, the existence and uniqueness of solutions to degenerate SDEs have been discussed in many papers. These models are called degenerate because the noise is only present in one component of the system but can be transferred into others by drift terms. The well-posedness of these systems, when their deterministic version is ill-posed, can be proved by taking advantage of the regularizing effect of the noise and of its propagation through the whole system. The case of Brownian degenerate SDEs has been of course widely explored, see e.g. [Fed+17], [WZ15], [Zha16], [HMC18]. The time-dependence is treated in the last three cited papers. The case of a Lévy driving process is more recent, see e.g. [Zha14] for a time-homogeneous setting, and [MM21] for drifts depending on time.

In this part, we are interested in the long-time behavior of the solution to (SKE_c) , driven either by a Brownian motion, i.e. $\alpha = 2$, or by an α -stable Lévy process, with $\alpha \in (0, 2)$. We consider a time-inhomogeneous frictional force given by $t^{-\beta} \operatorname{sgn}(v) |v|^\gamma$, where β and γ are non-negative parameters. More precisely, our goal is to study the asymptotic behavior, as $\varepsilon \rightarrow 0$, of the rescaled velocity-position process $(Z_t^{(\varepsilon)})_t := \left(r_\varepsilon (X_{t/\varepsilon}, V_{t/\varepsilon})^T \right)_t$, for an appropriate rate of convergence r_ε . One of our motivations is to study how the presence of the quadratic potential influences the results obtained in [GL21a; GL21b]

through a confining effect on the position X . Our system without noise and frictional force is nothing else than the classical harmonic oscillator

$$\begin{cases} v'_t = -x_t, \\ x'_t = v_t. \end{cases}$$

The intrinsic oscillatory behavior induced by the quadratic potential prevents the rescaled process $Z^{(\varepsilon)}$ from converging as a process. However, we will prove that each of its one-dimensional marginal distributions, for example its distribution at time $t = 1$, converges towards either a Gaussian distribution with independent coordinates in the Brownian case or towards a symmetric α -stable distribution when $\alpha \in (0, 2)$. In order to obtain the convergence of the whole process, the key idea is to remove the oscillations present in the system. Namely, we set $Y_t := \Theta_{-t}(X_t, V_t)^T$, where Θ_{-t} is the rotation on \mathbb{R}^2 of angle $-t$ defined by

$$\Theta_{-t} := \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix}.$$

As in [GL21a; GL21b], we highlight three regimes depending on the position of $q := \frac{\beta}{\gamma + \alpha - 1}$ with respect to $\frac{1}{\alpha}$. The rate of convergence r_ε is given by $r_\varepsilon := \varepsilon^{q \wedge \frac{1}{\alpha}}$, where \wedge denotes the minimum. Actually, the shift of regimes depends on the position of β with respect to $\frac{\gamma + \alpha - 1}{\alpha}$. Consequently, the condition $q > \frac{1}{\alpha}$ should be read as $\beta > \frac{\gamma + \alpha - 1}{\alpha}$, but we keep the notation q for the sake of simplicity, since we deal with parameters satisfying $\gamma + \alpha - 1 > 0$. In the super-critical regime, i.e. $q > \frac{1}{\alpha}$, we prove in Theorems 3.1.1 and 4.1.1 that the rescaled process $(Y_t^{(\varepsilon)})_t := (r_\varepsilon Y_{t/\varepsilon})_t$ converges in distribution towards either a standard Brownian motion on \mathbb{R}^2 when $\alpha = 2$, or a rotationally invariant stable process when $\alpha < 2$. In the critical regime, i.e. $q = \frac{1}{\alpha}$, and in the sub-critical regime, i.e. $q < \frac{1}{\alpha}$, we treat only the linear case $\gamma = 1$, and we assume that $\beta \in (\frac{1}{2}, 1]$. In the critical regime, we prove that the rescaled process converges in distribution, while in the sub-critical regime, we obtain only the convergence in finite dimensional distributions. In every case, the limit is explicit.

We deduce in Corollaries 3.1.2 and 4.1.2 the convergence in distribution of $t^{-q \wedge \frac{1}{\alpha}} Z_t$, as $t \rightarrow +\infty$. Let us mention that the rate of convergence of the position X is different from that found in [GL21a; GL21b], when $\mathcal{U} = 0$. Indeed, contrary to the free potential system, the position process is somehow more diffusive. This is due to the structure of our model. Namely, the presence of the quadratic potential allows the noise to propagate more effi-

ciently from the velocity component to the position one (see [Fed+17] for more details). This explains why both the limiting processes and the rate of convergence are different between our work and [GL21a; GL21b]. Let us also note that the position process grows more slowly in our case than when $\mathcal{U} = 0$. For example, in the Brownian super-critical regime, X_t behaves asymptotically as $\mathcal{N}(0, \frac{t}{2})$ in our framework, but as $\mathcal{N}(0, \frac{t^3}{3})$ in the free potential one. This difference can also be seen in moment estimates established for the position process X (see Remarks 3.3.2 and 4.3.2). This can be explained by the fact that the quadratic potential confines the particle through a spring force. Let us also notice that in the Brownian case, the velocity and position processes become asymptotically independent contrary to the α -stable case (see Remark 4.1.3).

In our model, the particle is no longer free. Consequently, both equations are intrinsically linked to each other. Therefore, we can no longer separate by components the study of the velocity-position process. We thus write the system (SKE_c) in a vector viewpoint, as done in [Fed+17], and use a variation of constants method to bring back to the study of a two-dimensional system in a free potential. We then adapt the methods used in [GL21a; GL21b]. The behavior depends again on the balance between the drag force and the random force. In the super-critical regime, the proof essentially relies of the self-similarity of the driving process and on moment estimates of V and X . In the critical and sub-critical regimes, we need to restrict ourselves to a linear drag force, i.e. $\gamma = 1$, in order to rely on the study of the asymptotic behavior of the underlying non-autonomous ordinary differential equation (ODE).

When the driving process is Brownian, we take advantage of the theory of Gaussian processes. The convergence is thus characterized by the study of the mean and covariance functions. In the case of a stable driving process, we need to study the convergence, in distribution and as a process, of a Wiener-Lévy integral (i.e. the integral of a deterministic function integrated against a stable process). The key point here is to use the fact that a Wiener-Lévy integral is a process with independent increments.

We consider the case of a Brownian driving process in Chapter 3 and we follow the same structure for a pure-jump α -stable driving process in Chapter 4. For the sake of clarity, we opt for separating the two cases since the tools used are different. In each chapter, we introduce some notations and state our main results. We start studying the existence of the solution to (SKE_c) and then give estimates of the moment. The proofs of

the main results are then presented. Finally, we state and prove some technical results in appendix.

Throughout this part, for $x, y \in \mathbb{R}^2$, $\|x\|$ represent the Euclidean norm of x , and $x \cdot y$ the inner product of x and y . If $x \in \mathbb{R}^2$, for each $i \in \{1, 2\}$, $x^{(i)}$ denotes its i -th component. We call I_2 the identity matrix of dimension 2 and A^T is the transpose matrix of a matrix A . We denote by C some positive constant, which may change from line to line, and we use subscripts to indicate the parameters on which it depends when it is necessary.

BEHAVIOR OF THE NON-AUTONOMOUS DEGENERATE SYSTEM IN THE QUADRATIC POTENTIAL, DRIVEN BY A BROWNIAN MOTION

3.1 Notations and main results

Throughout the chapter, to keep standard notations, we denote by B a standard Brownian motion. The space of continuous functions $\mathcal{C}((0, +\infty), \mathbb{R})$ is endowed with the uniform metric on compact subsets

$$d_u : (f, g) \in \mathcal{C}((0, +\infty), \mathbb{R})^2 \mapsto \sum_{n=1}^{+\infty} \frac{1}{2^n} \left(1 \wedge \sup_{[\frac{1}{n}, n]} |f - g| \right),$$

where \wedge denotes the minimum. For simplicity, we shall write \mathcal{C} for $\mathcal{C}((0, +\infty), \mathbb{R})$.

Let us fix $t_0 > 0$, $\beta \geq 0$, $\gamma > 0$ and $(v_0, x_0) \in \mathbb{R}^2$. We consider the following system of SDEs, defined on the time interval $[t_0, +\infty)$,

$$\begin{cases} dV_t = dB_t - \operatorname{sgn}(V_t) \frac{|V_t|^\gamma}{t^\beta} dt - X_t dt, \\ dX_t = V_t dt, \\ (V_{t_0}, X_{t_0}) = (v_0, x_0). \end{cases} \quad (3.1)$$

The previous system can be written in a vector viewpoint. Indeed, we set, for all $t \geq t_0$

and $v \in \mathbb{R}$,

$$Z_t := \begin{pmatrix} X_t \\ V_t \end{pmatrix}, W_t := \begin{pmatrix} 0 \\ B_t \end{pmatrix}, A := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \Gamma := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } F(t, v) := \begin{pmatrix} 0 \\ \text{sgn}(v) \frac{|v|^\gamma}{t^\beta} \end{pmatrix}.$$

Thereby, the system (3.1) can be rewritten as

$$\begin{cases} dZ_t = \Gamma dW_t + AZ_t dt - F(t, V_t) dt, \\ Z_{t_0} = z_0 := (x_0, v_0). \end{cases} \quad (3.2)$$

Notice that the matrix A is the rotation matrix of angle $\frac{\pi}{2}$ and that, for all $t \in \mathbb{R}$,

$$\Theta_{-t} := e^{tA} = \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix},$$

which is the rotation matrix of angle $-t$.

To state the main results of our paper, we define, for any $t \geq t_0$, $Y_t := e^{-tA}Z_t$. We easily check, with Itô's formula, that Y is given by

$$dY_t = e^{-tA}\Gamma dS_t - e^{-tA}F(t, V_t) dt. \quad (SDE_Y)$$

We are now able to state our results on the long-time behavior of these systems.

Theorem 3.1.1. *Define $q := \frac{\beta}{\gamma+1}$, $r_\varepsilon := \varepsilon^{q \wedge \frac{1}{2}}$ and set $(Y_t^{(\varepsilon)})_{t \geq \varepsilon t_0} := (r_\varepsilon Y_{t/\varepsilon})_{t \geq \varepsilon t_0}$. Let \mathcal{B} be a standard Brownian motion on \mathbb{R}^2 .*

- (i) *(Super-critical regime i.e. $2q > 1$). The rescaled process $Y^{(\varepsilon)}$ converges in distribution in \mathcal{C} towards $(\mathcal{B}_{\frac{t}{2}})_{t>0}$.*
- (ii) *(Critical regime i.e. $2q = 1$). Assume that $\gamma = 1$. The rescaled process $Y^{(\varepsilon)}$ converges in distribution in \mathcal{C} towards the Gaussian process $(\frac{1}{\sqrt{2t}} \int_0^t \sqrt{s} d\mathcal{B}_s)_{t>0}$.*
- (iii) *(Sub-critical regime i.e. $2q < 1$). Assume that $\gamma = 1$ and $\beta > \frac{1}{2}$. The rescaled process $Y^{(\varepsilon)}$ converges in finite dimensional distributions towards the centered Gaussian process with covariance kernel $K(s, t) = \frac{1}{2}s^\beta \mathbb{1}_{\{s=t\}} I_2$.*

The question of the convergence of the rescaled process associated with Z is treated in the following corollary.

Corollary 3.1.2. *Let us define $(Z_t^{(\varepsilon)})_{t \geq \varepsilon t_0} := (r_\varepsilon Z_{t/\varepsilon})_{t \geq \varepsilon t_0}$, where $r_\varepsilon := \varepsilon^{q \wedge \frac{1}{2}}$. The rescaled process $Z^{(\varepsilon)}$ does not converge in distribution. However, under the assumptions of the preceding theorem, we deduce from Theorem 3.1.1 the convergence in distribution of $r_{1/t} Z_t$ towards explicit limits, as $t \rightarrow +\infty$.*

The limit is either $\mathcal{N}(0, \frac{1}{2}I_2)$ in the super-critical and sub-critical regimes, or $\mathcal{N}(0, \frac{1}{4}I_2)$ in the critical regime.

Remark 3.1.3. *The rate of convergence of the position process is the same as for the velocity process. This is not true in the free potential case, i.e. when $\mathcal{U} = 0$ (see [GL21a]). Indeed, our rate of convergence for X is smaller. For example in the Brownian super-critical regime, the variance of X_t heuristically behaves as $\frac{t}{2}$ in large time, while in [GL21a], it behaves as $\frac{t^3}{3}$. This is quite natural because the quadratic potential tends to confine the particle through a spring force, so the particle spreads out more slowly than without potential.*

3.2 Existence up to explosion

Theorem 3.2.1. *The system of SDEs (3.1) admits a unique (global) strong solution if $\gamma \in (0, 1]$. And if $\gamma > 1$, there exists a unique strong solution defined up to its explosion time τ_∞ .*

Proof. In the case $\gamma > 1$, the coefficients of the SDE (3.2) are locally Lipschitz continuous with respect to the space variable, locally uniformly in time. Thus, by Theorem 21.3 p. 415 in [Kal02], there exists a unique solution up to explosion. The argument is standard, see [SG03] for details.

Assume now that $\gamma \leq 1$. We will use Theorem 1 in [HMC18]. Keeping the same notations, we have in our case, for any $(x, v) \in \mathbb{R}^2$ and $t \geq t_0$, $F_1(t, v, x) := -\text{sgn}(v)|v|^\gamma t^{-\beta} - x$, $F_2(t, v, x) := v$ and $\sigma(t, v, x) = 1$. Assumptions **(ML)** and **(UE)** in [HMC18] are obviously satisfied. Let us now remark that F_1 is γ -Hölder with respect to $v \in \mathbb{R}$ uniformly with respect to $t \geq t_0$ and $x \in \mathbb{R}$, and is Lipschitz continuous with respect to x , uniformly with respect to t and v . With the notations used in [HMC18], we have $\beta_1 = \gamma$ and $\beta_2 = 1$. Thus, Assumption **(T β)** is satisfied. Finally, we check that Assumption **(H η)** is satisfied. Since $\partial_v F_2 = 1$, we can conclude, taking η small enough and $\mathcal{E}_1 = \{1\}$, that there exists a unique strong solution to (3.1).

□

3.3 Moment estimates and non-explosion

In this section, we state and prove moment estimates of Z . It will be useful to control some stochastic terms appearing later. For all $n \geq 0$, define the stopping time

$$\tau_n := \inf\{t \geq t_0, \|Z_t\| \geq n\}.$$

Set $\tau_\infty := \lim_{n \rightarrow +\infty} \tau_n$ the explosion time of Z .

Proposition 3.3.1. *The explosion time of Z is a.s. infinite and, for all $\kappa \geq 0$,*

$$\mathbb{E} [\|Z_t\|^\kappa] \leq C_{\kappa, t_0} t^{\frac{\kappa}{2}}. \quad (3.3)$$

Remark 3.3.2. *Let us mention that the moment estimate obtained for the position process X is a priori smaller in our case than in the free potential case [GL21a]. It is explained by the confining effect of the quadratic potential.*

Proof. The proof is adapted from [GL21a] to two-dimensional processes. For the sake of completeness, we sketch the proof in our context.

Using Itô's formula applied to the function $f : (x, v) \mapsto x^2 + v^2$ and the fact that for all $z \in \mathbb{R}^2$, $z \cdot Az = 0$, we deduce that, for all $t \geq t_0$,

$$\|Z_{t \wedge \tau_n}\|^2 \leq \|z_0\|^2 + \int_{t_0}^t 2\mathbb{1}_{\{s \leq \tau_n\}} Z_s \cdot (\Gamma dW_s) - \int_{t_0}^{t \wedge \tau_n} 2Z_s \cdot F(s, V_s) ds + (t - t_0).$$

Remark that for any $s \geq t_0$, $Z_s \cdot F(s, V_s) = V_s \operatorname{sgn}(V_s) |V_s|^\gamma s^{-\beta} \geq 0$. Taking expectation yields

$$\mathbb{E} [\|Z_{t \wedge \tau_n}\|^2] \leq \|z_0\|^2 + (t - t_0) \leq C_{t_0} t.$$

Thanks to Lemma B.0.1, we can conclude that the explosion time of Z is a.s. infinite. Set $\kappa \in [0, 2]$, so, by Jensen's inequality and Fatou's lemma

$$\mathbb{E} [\|Z_{t \wedge \tau_\infty}\|^\kappa] \leq \mathbb{E} [\|Z_{t \wedge \tau_\infty}\|^2]^{\frac{\kappa}{2}} \leq \left(\liminf_{n \rightarrow \infty} \mathbb{E} [\|Z_{t \wedge \tau_n}\|^2] \right)^{\frac{\kappa}{2}} \leq C_{\kappa, t_0} t^{\frac{\kappa}{2}}. \quad (3.4)$$

This leads to (3.3).

When $\kappa > 2$, $v \mapsto \|v\|^\kappa$ is a \mathcal{C}^2 -function, so by Itô's formula, for all $t \geq t_0$,

$$\begin{aligned} \|Z_{t \wedge \tau_n}\|^\kappa &\leq \|z_0\|^\kappa + \int_{t_0}^{t \wedge \tau_n} \kappa \|Z_s\|^{\kappa-2} Z_s \cdot (\Gamma dW_s) - \int_{t_0}^{t \wedge \tau_n} \kappa \|Z_s\|^{\kappa-2} Z_s \cdot F(s, V_s) ds \\ &\quad + \int_{t_0}^{t \wedge \tau_n} C_\kappa \|Z_s\|^{\kappa-2} ds. \end{aligned}$$

In addition, it follows from the hypothesis on the sign of the drift function that

$$\|Z_{t \wedge \tau_n}\|^\kappa \leq \|z_0\|^\kappa + \int_{t_0}^t \kappa \mathbb{1}_{\{s \leq \tau_n\}} \|Z_s\|^{\kappa-2} Z_s \cdot (\Gamma dW_s) + \int_{t_0}^{t \wedge \tau_n} C_\kappa \|Z_s\|^{\kappa-2} ds. \quad (3.5)$$

Taking expectation in (3.5), we have

$$\mathbb{E} [\|Z_{t \wedge \tau_\infty}\|^\kappa] \leq \liminf_{n \rightarrow \infty} \mathbb{E} [\|Z_{t \wedge \tau_n}\|^\kappa] \leq \|v_0\|^\kappa + \int_{t_0}^t C_\kappa \mathbb{E} [\|Z_s\|^{\kappa-2}] ds.$$

When $0 \leq \kappa - 2 \leq 2$, we can upper bound $\mathbb{E} [\|Z_s\|^{\kappa-2}]$ by injecting (3.4) and get

$$\mathbb{E} [\|Z_{t \wedge \tau_\infty}\|^\kappa] \leq \|v_0\|^\kappa + \int_{t_0}^t C_{\kappa, t_0} s^{\frac{\kappa-2}{2}} ds \leq C_{\kappa, t_0} s^{\frac{\kappa}{2}}.$$

The method is then applied inductively to prove the inequality for all $\kappa > 2$. \square

3.4 Asymptotic behavior of the solution

We gather in this section the proof of Theorem 3.1.1. The strategy is to prove the convergence of the finite dimensional distributions (f.d.d.) of the process $Y^{(\varepsilon)}$, as $\varepsilon \rightarrow 0$, and its tightness in the critical and super-critical regimes. We first focus on the tightness.

Lemma 3.4.1. *If $2q \geq 1$, then the family $\{(\sqrt{\varepsilon} Y_{t/\varepsilon})_{t \geq \varepsilon t_0}, \varepsilon > 0\}$ is tight on every compact interval $[m, M]$, with $0 < m \leq M$.*

Proof. We use the Kolmogorov criterion stated in Problem 4.11 p. 64 in [KS98].

Take ε_0 small enough such that for all $\varepsilon \leq \varepsilon_0$, we have $\varepsilon t_0 \leq m$. Fix $m \leq s \leq t \leq M$ and $a > 4$. Define, for $t \geq \varepsilon t_0$, the local martingale term appearing in (SDE_Y)

$$M_t^{(\varepsilon)} := \sqrt{\varepsilon} \int_{t_0}^{t/\varepsilon} e^{-sA} \Gamma dW_s = \sqrt{\varepsilon} \int_{t_0}^{t/\varepsilon} \begin{pmatrix} -\sin(s) \\ \cos(s) \end{pmatrix} dB_s. \quad (3.6)$$

Using Jensen's inequality, moment estimates (see Proposition 3.3.1) and Burkholder-Davis-Gundy's inequality (see Theorem 4.4.22 p. 263 in [App09]), we have

$$\begin{aligned}
 \mathbb{E} \left[\left\| Y_t^{(\varepsilon)} - Y_s^{(\varepsilon)} \right\|^a \right] &\leq C_a \mathbb{E} \left[\left\| M_t^{(\varepsilon)} - M_s^{(\varepsilon)} \right\|^a \right] + C_a \mathbb{E} \left[\left\| \sqrt{\varepsilon} \int_{s/\varepsilon}^{t/\varepsilon} e^{-uA} F(u, V_u) \, du \right\|^a \right] \\
 &\leq C_a \mathbb{E} \left[\left\| M_t^{(\varepsilon)} - M_s^{(\varepsilon)} \right\|^a \right] + C_a \varepsilon^{1-\frac{a}{2}} (t-s)^{a-1} \mathbb{E} \left[\int_{s/\varepsilon}^{t/\varepsilon} \|F(u, V_u)\|^a \, du \right] \\
 &\leq C_a \mathbb{E} \left[\left(\text{Tr} \left(\left\langle M^{(\varepsilon)} - M_s^{(\varepsilon)} \right\rangle_t \right) \right)^{a/2} \right] + C_a \varepsilon^{1-\frac{a}{2}} (t-s)^{a-1} \int_{s/\varepsilon}^{t/\varepsilon} u^{\frac{\gamma a}{2} - \beta a} \, du \\
 &\leq C_a (t-s)^{\frac{a}{2}} + C_{a,m,M} \varepsilon^{\alpha(\beta - \frac{\gamma+1}{2})} (t-s)^{a-1} \\
 &\leq C_{a,m,M} (t-s)^{\frac{a}{2}}.
 \end{aligned}$$

Since $\frac{a}{2} > 2$ and $\beta \geq \frac{\gamma+1}{2}$ the upper bound is independent of $\varepsilon \leq 1$. Furthermore, by moment estimates (Proposition 3.3.1),

$$\sup_{\varepsilon \leq \varepsilon_0} \mathbb{E} \left[\left\| Y_m^{(\varepsilon)} \right\| \right] \leq \sqrt{m} < \infty.$$

Thus, Kolmogorov's criterion can be applied, proving the tightness result. \square

We will now prove the convergence of the finite-dimensional distributions of $Y^{(\varepsilon)}$. Thanks to the previous lemma, this will yield the weak convergence on every compact set (see Theorem 13.1 p. 139 in [Bil99]). The convergence in distribution on the whole space \mathcal{C} will follow, for $2q \geq 1$, from Theorem 16.7 p. 174 in [Bil99], since all processes considered are continuous.

3.4.1 Convergence of the f.d.d. in the super-critical regime

Assume here that $2q > 1$. Recall that $(Y_t^{(\varepsilon)})_{t \geq \varepsilon t_0} := (\sqrt{\varepsilon} Y_{t/\varepsilon})_{t \geq \varepsilon t_0}$.

Proof of Theorem 3.1.1 (i).

STEP 1. We first prove the convergence of the f.d.d. of the local martingale term $M^{(\varepsilon)}$ appearing in (SDE_Y) .

Recall that the stochastic integral $M^{(\varepsilon)}$ was defined in (3.6). It is a centered Gaussian process with covariance kernel defined, for any $(s, t) \in [\varepsilon t_0, +\infty)^2$, by

$$K^{(\varepsilon)}(s, t) := \begin{pmatrix} \text{Cov}(M_s^{(\varepsilon)}) & \text{Cov}(M_s^{(\varepsilon)}, M_t^{(\varepsilon)}) \\ \text{Cov}(M_t^{(\varepsilon)}, M_s^{(\varepsilon)}) & \text{Cov}(M_t^{(\varepsilon)}) \end{pmatrix},$$

where

$$\text{Cov}(M_s^{(\varepsilon)}, M_t^{(\varepsilon)}) = \begin{pmatrix} \text{Cov}(M_s^{(\varepsilon,1)}, M_t^{(\varepsilon,1)}) & \text{Cov}(M_s^{(\varepsilon,1)}, M_t^{(\varepsilon,2)}) \\ \text{Cov}(M_s^{(\varepsilon,2)}, M_t^{(\varepsilon,1)}) & \text{Cov}(M_s^{(\varepsilon,2)}, M_t^{(\varepsilon,2)}) \end{pmatrix},$$

and $\text{Cov}(M_s^{(\varepsilon)}) = \text{Cov}(M_s^{(\varepsilon)}, M_s^{(\varepsilon)})$. Thus, the convergence of the f.d.d. of $M^{(\varepsilon)}$ reduces on the study of the limit of $K^{(\varepsilon)}$, when ε converges to 0. Let us fix $\varepsilon t_0 \leq s \leq t$. Using that $M^{(\varepsilon)}$ has independent increments and by Itô's isometry, we find that

$$\text{Cov}(M_s^{(\varepsilon)}, M_t^{(\varepsilon)}) = \begin{pmatrix} \varepsilon \int_{t_0}^{s/\varepsilon} \sin(u)^2 du & -\varepsilon \int_{t_0}^{s/\varepsilon} \sin(u) \cos(u) du \\ -\varepsilon \int_{t_0}^{s/\varepsilon} \sin(u) \cos(u) du & \varepsilon \int_{t_0}^{s/\varepsilon} \cos(u)^2 du \end{pmatrix}.$$

We get that, for all $0 < s \leq t$,

$$\text{Cov}(M_s^{(\varepsilon)}, M_t^{(\varepsilon)}) \xrightarrow{\varepsilon \rightarrow 0} \frac{1}{2} \begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix}.$$

We recognize the covariance kernel of the process $(\mathcal{B}_{\frac{t}{2}})_{t>0}$, where \mathcal{B} denotes a standard Brownian motion on \mathbb{R}^2 . Since mean and covariance functions characterize Gaussian process (see Lemma 13.1 (i) p. 250 in [Kal02]), we have thus proved that $(M_t^{(\varepsilon)})_{t \geq \varepsilon t_0}$ converges in f.d.d. towards $(\mathcal{B}_{\frac{t}{2}})_{t>0}$.

STEP 2. Pick $T > 0$. We prove that

$$\mathbb{E} \left[\sup_{\varepsilon t_0 \leq t \leq T} \|Y_t^{(\varepsilon)} - M_t^{(\varepsilon)}\| \right] \xrightarrow{\varepsilon \rightarrow 0} 0.$$

We have

$$\sup_{\varepsilon t_0 \leq t \leq T} \|Y_t^{(\varepsilon)} - M_t^{(\varepsilon)}\| \leq \sqrt{\varepsilon} \|z_0\| + \sqrt{\varepsilon} \int_{t_0}^{T/\varepsilon} \|e^{-sA} F(s, V_s)\| ds.$$

We use moment estimates (Proposition 3.3.1) to get

$$\begin{aligned} \mathbb{E} \left[\sqrt{\varepsilon} \int_{t_0}^{T/\varepsilon} \|e^{-sA} F(s, V_s)\| ds \right] &= \mathbb{E} \left[\sqrt{\varepsilon} \int_{t_0}^{T/\varepsilon} \|F(s, V_s)\| ds \right] \\ &\leq \mathbb{E} \left[\sqrt{\varepsilon} \int_{t_0}^{T/\varepsilon} |V_s|^\gamma s^{-\beta} ds \right] \\ &\leq \sqrt{\varepsilon} C_{\kappa, t_0} \int_{t_0}^{T/\varepsilon} s^{\frac{\gamma}{2} - \beta} ds \\ &\leq C_{\kappa, t_0} (\varepsilon^{\beta - \frac{\gamma+1}{2}} T^{\frac{\gamma}{2} - \beta + 1} - \sqrt{\varepsilon} t_0^{\frac{\gamma}{2} - \beta + 1}). \end{aligned}$$

Hence, setting $r := \min(\beta - \frac{\gamma+1}{2}, \frac{1}{2})$, which is positive by assumption, we get

$$\mathbb{E} \left[\sup_{\varepsilon t_0 \leq t \leq T} \|Y_t^{(\varepsilon)} - M_t^{(\varepsilon)}\| \right] = O_{\varepsilon \rightarrow 0}(\varepsilon^r).$$

We conclude the proof by using Theorem 3.1 p. 27 in [Bil99].

□

3.4.2 Convergence of the f.d.d. in the critical and sub-critical regimes

In this section, we consider the linear case, i.e. $\gamma = 1$. Pick $\beta \in (\frac{1}{2}, 1]$. Recall that

$$(Y_t^{(\varepsilon)})_{t \geq \varepsilon t_0} := (\varepsilon^q Y_{t/\varepsilon})_{t \geq \varepsilon t_0},$$

where $q = \frac{\beta}{\gamma+1}$.

Proof of Theorem 3.1.1 (ii) and (iii). Leaving out the Brownian term, the underlying ODE of our system is the following

$$x''(t) + \frac{x'(t)}{t^\beta} + x(t) = 0, \quad t \geq t_0. \quad (3.7)$$

Pick the basis of solutions given in Lemma A.0.3 and denote by R its resolvent matrix. By Itô's formula, we get, for all $t \geq t_0$,

$$\tilde{Y}_t := R_t^{-1} Z_t = R_{t_0}^{-1} Z_0 + \int_{t_0}^t R_s^{-1} \Gamma \, dW_s.$$

Let us define f the rate of decrease of R (see Lemma A.0.3) by

$$\forall t > 0, f(t) := \begin{cases} \frac{1}{\sqrt{t}} & \text{if } \beta = 1, \\ \exp\left(-\frac{t^{1-\beta}}{2(1-\beta)}\right) & \text{else.} \end{cases}$$

Set, for $t \geq \varepsilon t_0$,

$$\Phi_t := \frac{e^{-tA} R_t}{f(t)} \quad \text{and} \quad \tilde{M}_t^{(\varepsilon)} := \varepsilon^q f\left(\frac{t}{\varepsilon}\right) \int_{t_0}^{t/\varepsilon} R_s^{-1} \Gamma \, dW_s. \quad (3.8)$$

Pick $t \geq \varepsilon t_0$. To study the convergence of $Y^{(\varepsilon)}$ we decompose it into

$$Y_t^{(\varepsilon)} = \varepsilon^q f\left(\frac{t}{\varepsilon}\right) \Phi_{t/\varepsilon} R_{t_0}^{-1} Z_0 + \Phi_{t/\varepsilon} \widetilde{M}_t^{(\varepsilon)}. \quad (3.9)$$

Let us notice that, for any $t > 0$, $\varepsilon^q f\left(\frac{t}{\varepsilon}\right)$ converges to 0, as $\varepsilon \rightarrow 0$.

STEP 1. We first note the convergence of Φ .

Using the asymptotic expansion of the resolvent matrix (Lemma A.0.3), we can write, for $t \geq \varepsilon t_0$,

$$\Phi_t = I_2 + O_{t \rightarrow \infty}(t^{1-2\beta}).$$

As a consequence, since $1 - 2\beta < 0$, $\Phi_{t/\varepsilon}$ converges to the identity matrix I_2 , as $\varepsilon \rightarrow 0$.

Moreover, we obtain that

$$\varepsilon^q f\left(\frac{t}{\varepsilon}\right) \Phi_{t/\varepsilon} R_{t_0}^{-1} Z_0 \xrightarrow{\varepsilon \rightarrow 0} 0.$$

As a consequence, we can forget the first term appearing in the decomposition (3.9) of $Y^{(\varepsilon)}$ (see Theorem 3.1 p. 27 in [Bil99]).

STEP 2. We compute now the covariance kernel of $Y^{(\varepsilon)}$ defined in (3.9).

It is defined, for $(s, t) \in [\varepsilon t_0, +\infty)^2$, by

$$K^{(\varepsilon)}(s, t) := \begin{pmatrix} \text{Cov}(Y_s^{(\varepsilon)}) & \text{Cov}(Y_s^{(\varepsilon)}, Y_t^{(\varepsilon)}) \\ \text{Cov}(Y_t^{(\varepsilon)}, Y_s^{(\varepsilon)}) & \text{Cov}(Y_t^{(\varepsilon)}) \end{pmatrix},$$

where

$$\text{Cov}(Y_s^{(\varepsilon)}, Y_t^{(\varepsilon)}) = \begin{pmatrix} \text{Cov}(Y_s^{(\varepsilon,1)}, Y_t^{(\varepsilon,1)}) & \text{Cov}(Y_s^{(\varepsilon,1)}, Y_t^{(\varepsilon,2)}) \\ \text{Cov}(Y_s^{(\varepsilon,2)}, Y_t^{(\varepsilon,1)}) & \text{Cov}(Y_s^{(\varepsilon,2)}, Y_t^{(\varepsilon,2)}) \end{pmatrix},$$

and $\text{Cov}(Y_s^{(\varepsilon)}) = \text{Cov}(Y_s^{(\varepsilon)}, Y_s^{(\varepsilon)})$.

Moreover, we get, for all $(s, t) \in [\varepsilon t_0, +\infty)^2$,

$$\text{Cov}\left(\Phi_{s/\varepsilon} \widetilde{M}_s^{(\varepsilon)}, \Phi_{t/\varepsilon} \widetilde{M}_t^{(\varepsilon)}\right) = \Phi_{s/\varepsilon} \text{Cov}(\widetilde{M}_s^{(\varepsilon)}, \widetilde{M}_t^{(\varepsilon)}) \Phi_{t/\varepsilon}^T.$$

Using the expression of the Wronskian obtained in Lemma A.0.3, we obtain, for all $t \geq \varepsilon t_0$,

$$\widetilde{M}_t^{(\varepsilon)} = \varepsilon^q f(t/\varepsilon) \int_{t_0}^{t/\varepsilon} f(u)^{-2} \begin{pmatrix} -y_2(u) \\ y_1(u) \end{pmatrix} dB_u.$$

It is a centered Gaussian process and for any $\varepsilon t_0 \leq s \leq t$, we have

$$\text{Cov}(\widetilde{M}_s^{(\varepsilon)}, \widetilde{M}_t^{(\varepsilon)}) = \varepsilon^\beta f(t/\varepsilon)f(s/\varepsilon) \int_{t_0}^{s/\varepsilon} f(u)^{-4} \begin{pmatrix} y_2^2(u) & -y_2(u)y_1(u) \\ -y_2(u)y_1(u) & y_1^2(u) \end{pmatrix} du.$$

Using the asymptotic expansion of the solutions and Lemma B.0.3, we can write, for all $\varepsilon t_0 < s \leq t$,

$$\begin{aligned} \text{Cov}(\widetilde{M}_s^{(\varepsilon)}, \widetilde{M}_t^{(\varepsilon)}) &= \varepsilon^\beta f(t/\varepsilon)f(s/\varepsilon) \int_{t_0}^{s/\varepsilon} f(u)^{-2} \begin{pmatrix} \sin^2(u) & -\sin(u)\cos(u) \\ -\sin(u)\cos(u) & \cos^2(u) \end{pmatrix} du \\ &\quad + \underset{\varepsilon \rightarrow 0}{O} \left(\varepsilon^{2\beta-1} f(t/\varepsilon)f(s/\varepsilon)^{-1} \right). \end{aligned}$$

Moreover, using asymptotic expansions of these integrals (see Lemmas B.0.2 and B.0.3),

$$\begin{aligned} \varepsilon^\beta f(t/\varepsilon)f(s/\varepsilon) \int_{t_0}^{s/\varepsilon} f(u)^{-2} \cos^2(u) du &= \varepsilon^\beta f(t/\varepsilon)f(s/\varepsilon) \frac{1}{2} \int_{t_0}^{s/\varepsilon} f(u)^{-2} du \\ &\quad + \underset{\varepsilon \rightarrow 0}{o} \left(f(t/\varepsilon)f(s/\varepsilon)^{-1} \right). \end{aligned}$$

The same equality holds for

$$\varepsilon^\beta f(t/\varepsilon)f(s/\varepsilon) \int_{t_0}^{s/\varepsilon} f(u)^{-2} \sin^2(u) du,$$

and we have

$$\varepsilon^\beta f(t/\varepsilon)f(s/\varepsilon) \int_{t_0}^{s/\varepsilon} f(u)^{-2} \cos(u) \sin(u) du = \underset{\varepsilon \rightarrow 0}{o} \left(f(t/\varepsilon)f(s/\varepsilon)^{-1} \right).$$

Thanks to Lemma B.0.3, this leads to

$$\begin{aligned} \text{Cov}(\widetilde{M}_s^{(\varepsilon)}, \widetilde{M}_t^{(\varepsilon)}) &= \left[\frac{1}{2} \varepsilon^\beta f(t/\varepsilon)f(s/\varepsilon) \int_{t_0}^{s/\varepsilon} f(u)^{-2} du \right] I_2 + \underset{\varepsilon \rightarrow 0}{o} \left(f(t/\varepsilon)f(s/\varepsilon)^{-1} \right) \\ &= k_\beta \frac{f(t/\varepsilon)}{f(s/\varepsilon)} s^\beta I_2 + \underset{\varepsilon \rightarrow 0}{o} \left(f(t/\varepsilon)f(s/\varepsilon)^{-1} \right), \end{aligned}$$

where

$$k_\beta := \begin{cases} \frac{1}{4} & \text{if } \beta = 1, \\ \frac{1}{2} & \text{else.} \end{cases}$$

Thus, we have proved the convergence of the f.d.d. of $Y^{(\varepsilon)}$. Note that when $2q = 1$,

we recognize the covariance kernel of the process $\left(\frac{1}{\sqrt{2t}} \int_0^t \sqrt{s} \, d\mathcal{B}_s\right)_{t>0}$, where \mathcal{B} denotes a standard Brownian motion on \mathbb{R}^2 .

□

Remark 3.4.2. *The proof relies on the asymptotic expansion of the resolvent matrix of (3.7). We were able to prove it only for $\beta \in \left(\frac{1}{2}, 1\right]$. However, if $\beta = 0$, the resolvent matrix is explicit and following the same lines, we can prove that $\left(Z_{t/\varepsilon}\right)_{t \geq \varepsilon t_0}$ converges in f.d.d. towards a centered Gaussian process with covariance kernel $(s, t) \mapsto \frac{1}{2} I_2 \mathbb{1}_{\{s=t\}}$. This behavior can be explained by the fact that the frictional force does not decrease along time. This cancels somehow the rotation bearing, which prevents $Z^{(\varepsilon)}$ from converging as a process when $\beta > 0$.*

3.5 Proof of Corollary 3.1.2

Proof of Corollary 3.1.2. We start by proving the convergence in distribution of $v_{1/T} Z_T$, as $T \rightarrow +\infty$. We claim that it follows from Theorem 3.1.1. Indeed, it is enough to remark that the convergence results stated in Theorem 3.1.1 imply the convergence in distribution of the marginal distribution at time $t = 1$ of $Y^{(\varepsilon)}$. Let us also recall that $Z_T = e^{TA} Y_T$. Setting $T = \frac{1}{\varepsilon}$, the convergence of $v_{1/T} Z_T$ is therefore a direct consequence of Lemma B.0.4.

We now show that the rescaled process $Z^{(\varepsilon)}$ does not converge in distribution. We do the proof only in the super-critical regime. Assume by contradiction that it is the case. Hence, each of its coordinates shall converge too. We thus have the convergence of the rescaled process $X^{(\varepsilon)}$. Using (SDE_Y) , we can write

$$\sqrt{\varepsilon} X_{t/\varepsilon} = \sqrt{\varepsilon} x_0 + \sqrt{\varepsilon} \int_{t_0}^{t/\varepsilon} \sin\left(\frac{t}{\varepsilon} - s\right) \, dB_s - \sqrt{\varepsilon} \int_{t_0}^{t/\varepsilon} \sin\left(\frac{t}{\varepsilon} - s\right) F(V_s) s^{-\beta} \, ds.$$

As in the proof of Theorem 3.1.1, the last term converges in probability uniformly on compact intervals towards zero. Hence, the following term shall converge in distribution

$$I_t^{(\varepsilon)} := \sqrt{\varepsilon} \int_{t_0}^{t/\varepsilon} \sin\left(\frac{t}{\varepsilon} - s\right) \, dB_s.$$

The process $(I_t^{(\varepsilon)})_{t \geq \varepsilon t_0}$ is Gaussian, thereby its limit shall be Gaussian too and its covariance function shall converge (see Lemma 13.1 (i) p. 250 in [Kal02]). However, using Itô's

isometry, one can compute, for $\varepsilon t_0 \leq s \leq t$,

$$\begin{aligned} \mathbb{E} \left[I_t^{(\varepsilon)} I_s^{(\varepsilon)} \right] &= \varepsilon \int_{t_0}^{s/\varepsilon} \sin \left(\frac{t}{\varepsilon} - u \right) \sin \left(\frac{s}{\varepsilon} - u \right) du \\ &= \varepsilon \frac{1}{2} \left[\cos \left(\frac{t-s}{\varepsilon} \right) \left(\frac{s}{\varepsilon} - t_0 \right) + \frac{1}{2} \left(\sin \left(\frac{t-s}{\varepsilon} \right) - \sin \left(\frac{t-s}{\varepsilon} - 2t_0 \right) \right) \right] \\ &= \frac{1}{2} s \cos \left(\frac{t-s}{\varepsilon} \right) + o_{\varepsilon \rightarrow 0}(1). \end{aligned}$$

This term does not converge if $s \neq t$, and that concludes the proof. □

BEHAVIOR OF THE NON-AUTONOMOUS DEGENERATE SYSTEM IN THE QUADRATIC POTENTIAL AND DRIVEN BY A LÉVY PROCESS

4.1 Notations and main results

4.1.1 Settings on Lévy processes and functional spaces

Throughout the paper, we denote by L a symmetric α -stable Lévy process on \mathbb{R} with $\alpha \in (0, 2]$. We call ν its Lévy measure, which can be written as $\nu(dz) = a |z|^{-1-\alpha} \mathbb{1}_{\{z \neq 0\}} dz$ with $a > 0$. As a Lévy measure, it satisfies $\int_{\mathbb{R}^*} (1 \wedge z^2) \nu(dz) < +\infty$. Let us also denote by ψ the characteristic exponent of the stable process L . It follows from Theorem 14.15 p. 86 in [Sat99] that, for all $\xi \in \mathbb{R}$,

$$\psi(\xi) = -a|\xi|^\alpha. \quad (4.1)$$

We denote by N the Poisson random measure associated with L and by \widetilde{N} its compensated Poisson measure. Using Lévy-Itô's decomposition, we have, for all $t \geq 0$,

$$L_t = \begin{cases} \int_0^t \int_{\mathbb{R}^*} z N(ds, dz) & \text{if } \alpha \in (0, 1), \\ \int_0^t \int_{\{0 < |z| < 1\}} z \widetilde{N}(ds, dz) + \int_0^t \int_{\{|z| \geq 1\}} z N(ds, dz) & \text{if } \alpha = 1, \\ \int_0^t \int_{\mathbb{R}^*} z \widetilde{N}(ds, dz) & \text{if } \alpha \in (1, 2). \end{cases}$$

Set $\Lambda := \{\lambda : \mathbb{R}^+ \rightarrow \mathbb{R}^+, \text{continuous and increasing s.t. } \lambda(0) = 0, \lim_{t \rightarrow +\infty} \lambda(t) = +\infty\}$ and

$$k_n(t) = \begin{cases} 1 & \text{if } \frac{1}{n} \leq t \leq n, \\ n + 1 - t & \text{if } n < t < n + 1, \\ 0 & \text{if } n + 1 \leq t. \end{cases}$$

The space of right-continuous with left limits (càdlàg) functions $\mathcal{D}((0, +\infty), \mathbb{R})$, i.e. the functions defined on $(0, +\infty)$ which are càdlàg on every compact subinterval of $(0, +\infty)$, is endowed with the Skorokhod metric d_s defined for $(f, g) \in \mathcal{D}((0, +\infty), \mathbb{R})^2$ by

$$\sum_{n=1}^{+\infty} \frac{1}{2^n} \left(1 \wedge \inf \left\{ a, \exists \lambda \in \Lambda, \sup_{s \neq t} \left| \log \frac{\lambda(t) - \lambda(s)}{t - s} \right| \leq a, \sup_{t \geq \frac{1}{n}} |k_n(t) (f \circ \lambda(t) - g(t))| \leq a \right\} \right).$$

For simplicity, we shall write \mathcal{D} for $\mathcal{D}((0, +\infty), \mathbb{R})$.

4.1.2 Main results

Let us fix $t_0 > 0$, $\beta \geq 0$, $\gamma > 0$ and $(v_0, x_0) \in \mathbb{R}^2$. We consider the following system of SDEs, defined on the time interval $[t_0, +\infty)$,

$$\begin{cases} dV_t = dL_t - \text{sgn}(V_t) \frac{|V_t|^\gamma}{t^\beta} dt - X_t dt, \\ dX_t = V_t dt, \\ (V_{t_0}, X_{t_0}) = (v_0, x_0). \end{cases} \quad (\text{SKE})$$

Following the same lines as in the previous chapter, we rewrite (SKE) as

$$\begin{cases} dZ_t = \Gamma dS_t + AZ_t dt - F(t, V_t) dt, \\ Z_{t_0} = z_0 := (x_0, v_0)^T, \end{cases} \quad (4.2)$$

where for all $t \geq t_0$ and $v \in \mathbb{R}$,

$$Z_t := \begin{pmatrix} X_t \\ V_t \end{pmatrix}, \quad S_t := \begin{pmatrix} 0 \\ L_t \end{pmatrix}, \quad A := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \Gamma := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad F(t, v) := \begin{pmatrix} 0 \\ \text{sgn}(v) \frac{|v|^\gamma}{t^\beta} \end{pmatrix}.$$

To state the main results of our paper, we define, for any $t \geq t_0$, $Y_t := e^{-tA} Z_t$, which

is given by

$$dY_t = e^{-tA} \Gamma dS_t - e^{-tA} F(t, V_t) dt. \quad (SDE_Y)$$

We are now able to state our results on the long-time behavior of these systems.

Theorem 4.1.1. *Assume that $\gamma \in (0, \alpha)$. Define $q := \frac{\beta}{\gamma + \alpha - 1}$, $r_\varepsilon := \varepsilon^{q \wedge \frac{1}{\alpha}}$ and set $(Y_t^{(\varepsilon)})_{t \geq \varepsilon t_0} := (r_\varepsilon Y_{t/\varepsilon})_{t \geq \varepsilon t_0}$. Let \mathcal{L} be a rotationally invariant stable process on \mathbb{R}^2 , whose characteristic exponent is given by*

$$\xi \in \mathbb{R}^2 \mapsto -\tilde{C} \|\xi\|^\alpha, \quad \text{with } \tilde{C} := \frac{a}{2\pi} \int_0^{2\pi} |\cos(x)|^\alpha dx.$$

- (i) *(Super-critical regime i.e. $\alpha q > 1$). Assume that $\alpha \in (1, 2)$. The rescaled process $Y^{(\varepsilon)}$ converges in distribution in \mathcal{D} towards $(\mathcal{L}_t)_{t>0}$.*
- (ii) *(Critical regime i.e. $\alpha q = 1$). Assume that $\gamma = 1$. The rescaled process $Y^{(\varepsilon)}$ converges in distribution in \mathcal{D} towards the Lévy-type process $(\frac{1}{\sqrt{t}} \int_0^t \sqrt{s} d\mathcal{L}_s)_{t>0}$.*
- (iii) *(Sub-critical regime i.e. $\alpha q < 1$). Assume that $\gamma = 1$ and $\beta > \frac{1}{2}$. Then, for all $(t_1, \dots, t_d) \in (0, +\infty)^d$, $(Y_{t_1}^{(\varepsilon)}, \dots, Y_{t_d}^{(\varepsilon)})$ converges in distribution towards the product measure $\mu_{t_1} \otimes \dots \otimes \mu_{t_d}$, where μ_t is the distribution with characteristic function*

$$\xi \in \mathbb{R}^2 \mapsto \exp\left(-\frac{2}{\alpha} \tilde{C} \|\xi\|^\alpha t^\beta\right).$$

The question of the convergence of the rescaled process associated with Z is treated in the following corollary.

Corollary 4.1.2. *Let us define $(Z_t^{(\varepsilon)})_{t \geq \varepsilon t_0} := (r_\varepsilon Z_{t/\varepsilon})_{t \geq \varepsilon t_0}$, where $r_\varepsilon := \varepsilon^{q \wedge \frac{1}{\alpha}}$. The rescaled process $Z^{(\varepsilon)}$ does not converge in distribution. However, under the assumptions of the preceding theorem, we deduce from Theorem 4.1.1 the convergence in distribution of $r_{1/t} Z_t$ towards explicit limits, as $t \rightarrow +\infty$.*

Keeping the same notations as in the preceding theorem, the characteristic function of the limit is given, for all $\xi \in \mathbb{R}^2$, by

- (i) $\exp(-\tilde{C} \|\xi\|^\alpha)$ *in the super-critical regime,*
- (ii) $\exp\left(-\left(1 + \frac{\alpha}{2}\right)^{-1} \tilde{C} \|\xi\|^\alpha\right)$ *in the critical regime,*
- (iii) $\exp\left(-\frac{2}{\alpha} \tilde{C} \|\xi\|^\alpha\right)$ *in the sub-critical regime.*

Remark 4.1.3. *Let us notice that in the Brownian setting, the position X and the velocity V become independent in large time. However, this is false in the stable case. Indeed,*

the limit is a rotationally invariant stable process on \mathbb{R}^2 , which cannot have independent coordinates.

Remark 4.1.4. *The rate of convergence of the position process is the same as for the velocity process. This is not true in the free potential case, i.e. when $\mathcal{U} = 0$ (see [GL21b]). Indeed, our rate of convergence for X is smaller. For example in the Brownian super-critical regime, the variance of X_t heuristically behaves as $\frac{t}{2}$ in large time, while in [GL21a], it behaves as $\frac{t^3}{3}$. This is quite natural because the quadratic potential tends to confine the particle through a spring force, so the particle spreads out more slowly than without potential.*

Remark 4.1.5. *The condition $\alpha \in (1, 2)$ and the symmetry of ν are only required to ensure the well-posedness of (SKE) when $\gamma < 1$. Nevertheless, proofs of the moment estimates and of the asymptotic behavior can be written for a non-symmetric α -stable Lévy driving process with $\alpha \in (0, 2)$.*

4.2 Existence up to explosion

Theorem 4.2.1. *Assume that $\alpha > 1$. The system (SKE) admits a unique weak solution if $\gamma \in (0, 1]$. If $\gamma > 1$, there exists a unique strong solution defined up to its explosion time τ_∞ .*

Proof. In the case $\gamma > 1$, the coefficients of the SDE (4.2) satisfied by $Z = (X, V)$ are locally Lipschitz continuous with respect to the space variable, locally uniformly in time. So we can apply Lemma 115 p. 78 in [Sit05] to get the pathwise uniqueness. The drift coefficient is continuous with respect to its two variables, so it is a locally bounded and measurable function. By a standard localization argument, using Theorem 9.1 p. 231 in [IW81], since the drift coefficient is locally Lipschitz, there is a unique solution defined up to explosion.

Assume now that $\gamma \leq 1$. We check that we can use Theorem 1 in [MM21]. Using the same notations, we have

$$\tilde{A} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

and for any $(t, x_1, x_2) \in [t_0, +\infty) \times \mathbb{R}^2$, $F_1(t, x_1, x_2) = \text{sgn}(x_1)|x_1|^\gamma t^{-\beta}$, $F_2(t, x_1, x_2) = 0$ and $\sigma(t, x_1, x_2) = 1$. Assumptions **(UE)** and **(ND)** are clearly satisfied. Since F_2 does not depend on x_1 and since $[\tilde{A}]_{2,1} = 1$ is different from 0, we deduce that Assumption **(H)**

is satisfied. We easily check that Theorem 1 in [MM21] can be applied with $\beta_1 = \gamma$, and $\beta_2 = 1$. □

Remark 4.2.2. For $\alpha \in (0, 2)$, employing the technique of Picard iteration and the interlacing procedure, one can deduce that (4.2) has a unique solution in the linear setting $\gamma = 1$ (see [App09, p. 375]).

4.3 Moment estimates and non-explosion

Let Z be the unique solution up to explosion time to (4.2). As in the continuous setting, define, for all $r \geq 0$, the stopping time

$$\tau_r := \inf\{t \geq t_0, \|Z_t\| \geq r\}.$$

Set $\tau_\infty := \lim_{r \rightarrow +\infty} \tau_r$ the explosion time of Z . For the sake of simplicity, since there is no jump on the position component, for $z \in \mathbb{R}$, we shall write $Z_{s-} + z$ for $(X_s, V_{s-} + z)$ in the following.

We adapt the proof of [GL21b] to two-dimensional processes.

Proposition 4.3.1. Pick $\alpha \in (0, 1)$. For any γ, β , the explosion time τ_∞ is a.s. infinite and for all $\kappa \in [0, \alpha)$, there exists a constant C_{κ, t_0} such that, we have

$$\forall t \geq t_0, \mathbb{E}[\|Z_t\|^\kappa] \leq C_{\kappa, t_0} t^{\frac{\kappa}{\alpha}}. \tag{4.3}$$

Remark 4.3.2. Note that, as in the Brownian case, the moment estimates obtained for the position process X is a priori smaller in our case than in the free potential case [GL21b]. It is explained by the confining effect of the quadratic potential.

Proof. Fix $t \geq t_0$ and $\kappa \in [0, \alpha)$. Since $\alpha < 1$, the symmetric stable process can be written as

$$L_t = \int_0^t \int_{\mathbb{R}^*} z N(ds, dz) = \sum_{s \leq t} \Delta L_s.$$

Pick the sequence of \mathcal{C}^2 -functions $f_n : (x, v) \mapsto \sqrt{x^2 + v^2 + \frac{1}{n}}$, which converges uniformly to $(x, v) \mapsto \|(x, v)\|$ on \mathbb{R}^2 . Then, for all $n \geq 1$ and $r \geq 0$, we apply Itô's formula (see Theorem 33 p. 81 in [Pro05]). We use the fact that for all $y \in \mathbb{R}^2$, $y \cdot Ay = 0$, and observe

that for any $s \geq t_0$ and $(x, v) \in \mathbb{R}^2$, $(x, v) \cdot F(s, v) = |v|^{\gamma+1} s^{-\beta} \geq 0$. Namely, we get

$$f_n(Z_{t \wedge \tau_r}) \leq f_n(z_0) + \sum_{s \leq t \wedge \tau_r} (f_n(Z_s) - f_n(Z_{s-})).$$

Since $\sup\{\|\nabla f_n(y)\|, y \in \mathbb{R}^2\} \leq 1$, we deduce, from Taylor-Lagrange's inequality that for all $s \leq t \wedge \tau_r$, $|f_n(Z_s) - f_n(Z_{s-})| \leq \|\Delta Z_s\| = |\Delta L_s|$. It follows that

$$\|Z_{t \wedge \tau_r}\| \leq f_n(Z_{t \wedge \tau_r}) \leq f_n(z_0) + \sum_{s \leq t \wedge \tau_r} |\Delta L_s|.$$

Furthermore, since $\kappa < \alpha < 1$, we have

$$\mathbb{E} [\|Z_{t \wedge \tau_r}\|^\kappa] \leq \mathbb{E} [f_n(z_0)^\kappa] + \mathbb{E} \left[\left(\sum_{s \leq t} |\Delta L_s| \right)^\kappa \right].$$

Notice that the process $L_t^+ := \sum_{s \leq t} |\Delta L_s|$ is an α -stable process. Then, since $\kappa < \alpha$, letting $n \rightarrow +\infty$, we obtain

$$\mathbb{E} [\|Z_{t \wedge \tau_r}\|^\kappa] \leq \|z_0\|^\kappa + \mathbb{E} [L_t^{+\kappa}] \leq C_{t_0, \kappa} t^{\frac{\kappa}{\alpha}}.$$

Thanks to Lemma B.0.1, we can conclude that the explosion time of Z is a.s. infinite and Fatou's lemma yields (4.3). \square

Proposition 4.3.3. *Pick $\alpha \in [1, 2)$. For any $\gamma \geq 0$ and $\beta \geq 0$, the explosion time τ_∞ is a.s. infinite and for $\kappa \in (0, \alpha)$, there exists C_{κ, t_0} such that*

$$\forall t \geq t_0, \quad \mathbb{E} [\|Z_t\|^\kappa] \leq C_{\kappa, t_0} t^{\frac{\kappa}{\alpha}}. \quad (4.4)$$

Proof. The key idea is to slice the small and big jumps in a non-homogeneous way with respect to the characteristic scale of an α -stable process $\xi \mapsto \xi^{\frac{1}{\alpha}}$.

Pick $\xi \geq t_0$. The α -stable symmetric Lévy driving process can be written as

$$L_t - L_{t_0} = \int_{t_0}^t \int_{|z| \leq \xi^{\frac{1}{\alpha}}} z \widetilde{N}(ds, dz) + \int_{t_0}^t \int_{|z| > \xi^{\frac{1}{\alpha}}} z N(ds, dz).$$

STEP 1. We first apply Itô's formula (see Theorem 4.47 p. 251 in [App09]) and estimate the expectation of each term for $\kappa \leq 1$, in order to get (4.4).

Fix $\eta > 0$ to be chosen latter and define the \mathcal{C}^2 -function $f : (x, v) \mapsto (\eta + x^2 + v^2)^{\kappa/2}$. We

use the fact that for all $y \in \mathbb{R}^2$, $y \cdot Ay = 0$, and observe that for any $s \geq t_0$ and $(x, v) \in \mathbb{R}^2$, $(x, v)^T \cdot F(s, v) = |v|^{\gamma+1} s^{-\beta} \geq 0$. For all $t \geq t_0$, by Itô's formula, we have

$$f(Z_{t \wedge \tau_r}) \leq f(V_0) + M_t + R_t + S_t,$$

where

$$M_t := \int_{t_0}^t \int_{0 < |z| < \xi^{\frac{1}{\alpha}}} \mathbb{1}_{\{s \leq \tau_r\}} (f(Z_{s-} + z) - f(Z_{s-})) \widetilde{N}(ds, dz),$$

$$R_t := \int_{t_0}^t \int_{|z| \geq \xi^{\frac{1}{\alpha}}} \mathbb{1}_{\{s \leq \tau_r\}} (f(Z_{s-} + z) - f(Z_{s-})) N(ds, dz), \quad (4.5)$$

$$S_t := \int_{t_0}^t \int_{0 < |z| < \xi^{\frac{1}{\alpha}}} \mathbb{1}_{\{s \leq \tau_r\}} [f(Z_{s-} + z) - f(Z_{s-}) - df(Z_{s-}) \cdot z] \nu(dz) ds. \quad (4.6)$$

Moreover, remark that for all $k > \alpha$,

$$\int_{0 < |z| < \xi^{\frac{1}{\alpha}}} |z|^k \nu(dz) = \frac{2a}{k - \alpha} \xi^{\frac{k}{\alpha} - 1}, \quad (4.7)$$

and for all $k < \alpha$,

$$\int_{|z| \geq \xi^{\frac{1}{\alpha}}} |z|^k \nu(dz) = \frac{2a}{\alpha - k} \xi^{\frac{k}{\alpha} - 1}. \quad (4.8)$$

We estimate expectations of M , R and S .

To that end, we first show that the local martingale $(M_t)_{t \geq t_0}$ is a martingale. Fix $q \geq 2$ and $r \geq 0$. Moreover, we set

$$I_t(q) := \int_{t_0}^t \int_{0 < |z| < \xi^{\frac{1}{\alpha}}} \mathbb{1}_{\{s \leq \tau_r\}} |f(Z_{s-} + z) - f(Z_{s-})|^q \nu(dz) ds.$$

Thanks to Taylor-Lagrange inequality, for all $\|(x, v)\| \leq r$ and $|z| \leq \xi^{\frac{1}{\alpha}}$,

$$|f(x, v + z) - f(x, v)| \leq \sup\{\|\nabla f(y)\|, \|y\| \in [-r - \xi^{1/\alpha}, r + \xi^{1/\alpha}]\} |z| \leq C_{r, \xi, \kappa} |z|,$$

so we have

$$I_t(q) \leq C_{r, \xi, \kappa} \int_{t_0}^t \int_{0 < |z| < \xi^{\frac{1}{\alpha}}} \mathbb{1}_{\{s \leq \tau_r\}} |z|^q \nu(dz) ds.$$

Hence, it is a finite quantity, since $q \geq 2$ and (4.7) holds. Therefore, for $q \geq 2$, by Kunita's

inequality (see Theorem 4.4.23 p. 265 in [App09]), there exists $D_q > 0$ such that

$$\mathbb{E} \left[\sup_{t_0 \leq s \leq t} |M_s|^q \right] \leq D_q \left(\mathbb{E} [I_t(2)^{\frac{q}{2}}] + \mathbb{E} [I_t(q)] \right) < +\infty.$$

Hence, by Theorem 51 p. 38 in [Pro05], M is a martingale.

We estimate now the finite variation part S defined in (4.6). Note that for all $(x, v) \in \mathbb{R}^2$, the Hessian matrix of f is given by

$$\text{Hess}(f)(x, v) = \kappa(x^2 + v^2 + \eta)^{\frac{\kappa}{2}-1} \begin{pmatrix} 1 + (\kappa - 2) \frac{x^2}{x^2 + v^2 + \eta} & (\kappa - 2) \frac{xv}{x^2 + v^2 + \eta} \\ (\kappa - 2) \frac{xv}{x^2 + v^2 + \eta} & 1 + (\kappa - 2) \frac{v^2}{x^2 + v^2 + \eta} \end{pmatrix}.$$

Its matrix norm is bounded by $C_\kappa \eta^{\frac{\kappa}{2}-1}$.

Assume that $|z| < \xi^{\frac{1}{\alpha}}$. Using Taylor- Lagrange's inequality and injecting (4.7) we get the almost sure following bound, for all $s \geq t_0$,

$$\left| \int_{0 < |z| < \xi^{\frac{1}{\alpha}}} (f(Z_{s-} + z) - f(Z_{s-}) - \nabla f(Z_{s-}) \cdot z) \nu(dz) \right| \leq C_\kappa \eta^{\frac{\kappa}{2}-1} \frac{2a}{2 - \alpha} \xi^{\frac{2}{\alpha}-1}. \quad (4.9)$$

It remains to study the Poisson integral R defined in (4.5). Pick $\kappa \leq 1$, by Hölder property of power functions and (4.8), we deduce that

$$\int_{|z| \geq \xi^{\frac{1}{\alpha}}} |f(Z_{s-} + z) - f(Z_{s-})| \nu(dz) \leq \eta^{\frac{\kappa}{2}} \frac{2a}{\alpha} \xi^{-1} + \frac{2a}{\alpha - \kappa} \xi^{\frac{\kappa}{\alpha}-1}. \quad (4.10)$$

Moment estimate of the Poisson integral follows from Theorem 2.3.7 p. 106 in [App09]. Gathering (4.10) and (4.9), we obtain

$$\mathbb{E} [\|Z_{t \wedge \tau_r}\|^\kappa] \leq \mathbb{E} [f(Z_{t \wedge \tau_r})] \leq \mathbb{E} [f(Z_{t_0})] + t \xi^{-1} \left(\eta^{\kappa/2} \frac{2a}{\alpha} + \frac{2a}{\alpha - \kappa} \xi^{\frac{\kappa}{\alpha}} + C_\kappa \eta^{\frac{\kappa}{2}-1} \frac{2a}{2 - \alpha} \xi^{\frac{2}{\alpha}} \right).$$

Choosing $\eta = t^{\frac{2}{\alpha}}$ and $\xi = t$, we get

$$\mathbb{E} [\|Z_{t \wedge \tau_r}\|^\kappa] \leq \mathbb{E} [f(Z_{t_0})] + t^{\frac{\kappa}{\alpha}} \left(\frac{2a}{\alpha} + \frac{2a}{\alpha - \kappa} + C_\kappa \frac{2a}{2 - \alpha} \right) \leq C_{\kappa, t_0} t^{\frac{\kappa}{\alpha}}.$$

Thanks to Lemma B.0.1, we can conclude that the explosion time of Z is a.s. infinite, and

letting $r \rightarrow +\infty$ with Fatou's lemma, for all $\kappa \in [0, 1]$,

$$\mathbb{E} [\|Z_t\|^\kappa] \leq C_{\kappa, t_0} t^{\frac{\kappa}{\alpha}}. \quad (4.11)$$

STEP 2. Pick $\kappa \in (1, \alpha)$. We estimate R in another way, using again Theorem 2.3.7 p. 106 in [App09].

By the Hölder property of power function and (4.8), we get

$$\begin{aligned} \int_{|z| \geq \xi^{\frac{1}{\alpha}}} |f(Z_{s-} + z) - f(Z_{s-})| \nu(dz) &\leq \int_{|z| \geq \xi^{\frac{1}{\alpha}}} |2zV_{s-} + z^2|^{\frac{\kappa}{2}} \nu(dz) \\ &\leq C_\kappa \left(\frac{2a}{\alpha - \kappa} \xi^{\frac{\kappa}{\alpha} - 1} + |V_{s-}|^{\frac{\kappa}{2}} \frac{2a}{\alpha - \frac{\kappa}{2}} \xi^{\frac{\kappa}{2\alpha} - 1} \right). \end{aligned} \quad (4.12)$$

Gathering (4.9) and (4.12), one has

$$\begin{aligned} \mathbb{E} [\|Z_{t \wedge \tau_r}\|^\kappa] &\leq \mathbb{E} [f(Z_{t_0})] + t \left(C_\kappa \frac{2a}{\alpha - \kappa} \xi^{\frac{\kappa}{\alpha} - 1} + C_\kappa \eta^{\frac{\kappa}{2} - 1} \frac{2a}{2 - \alpha} \xi^{\frac{2}{\alpha} - 1} \right) \\ &\quad + C_\kappa \frac{2a}{\alpha - \frac{\kappa}{2}} \xi^{\frac{\kappa}{2\alpha} - 1} \int_{t_0}^t \mathbb{E} [|V_s|^{\frac{\kappa}{2}}] ds. \end{aligned}$$

Injecting (4.11) applied with $\frac{\kappa}{2}$, choosing $\eta = t^{\frac{2}{\alpha}}$ and $\xi = t$, we get

$$\mathbb{E} [\|Z_{t \wedge \tau_r}\|^\kappa] \leq C_{\kappa, t_0, \alpha} t^{\frac{\kappa}{\alpha}}.$$

The conclusion of the proof follows, letting $r \rightarrow +\infty$. □

4.4 Asymptotic behavior of the solution

We gather in this section the proof of Theorem 4.1.1. The strategy is to prove the convergence of the f.d.d. of the process $Y^{(\varepsilon)}$, and then its tightness both in the super-critical and critical regimes. We first prove the tightness when $\alpha q \geq 1$.

Lemma 4.4.1. *Assume that $\alpha q \geq 1$, then the family $\left\{ (\varepsilon^{\frac{1}{\alpha}} Y_{t/\varepsilon})_{t \geq \varepsilon t_0}, \varepsilon > 0 \right\}$ is tight on every compact interval $[m, M]$, for $0 < m \leq M$.*

Proof. We check the Aldous's tightness criterion stated in Theorem 16.10 p. 178 in [Bil99]. Let a, η, T be positive reals. Let τ be a discrete stopping time with finite range \mathcal{T} ,

bounded by T and fix $\delta > 0$ and $\varepsilon > 0$ small enough to be chosen latter. Define, for $t \geq \varepsilon t_0$, the Wiener-Lévy integral appearing in (SDE_Y)

$$M_t^{(\varepsilon)} := \varepsilon^{\frac{1}{\alpha}} \int_{t_0}^{t/\varepsilon} e^{-sA} \Gamma \, dS_s = \varepsilon^{\frac{1}{\alpha}} \int_{t_0}^{t/\varepsilon} \begin{pmatrix} -\sin(s) \\ \cos(s) \end{pmatrix} dL_s. \quad (4.13)$$

We use Jensen's inequality to get, for $r = \frac{\alpha}{2}$,

$$\mathbb{E} \left[\left\| Y_{\tau+\delta}^{(\varepsilon)} - Y_{\tau}^{(\varepsilon)} \right\|^r \right] \leq \mathbb{E} \left[\left\| \int_{\tau/\varepsilon}^{(\tau+\delta)/\varepsilon} \varepsilon^{\frac{1}{\alpha}} e^{-uA} \Gamma \, dS_u \right\|^r \right] + \mathbb{E} \left[\varepsilon^{\frac{1}{\alpha}} \int_{\tau/\varepsilon}^{(\tau+\delta)/\varepsilon} |V_u|^\gamma u^{-\beta} \, du \right]^r.$$

The stopping time has a finite range \mathcal{T} , hence, we can write

$$\begin{aligned} \mathbb{E} \left[\left\| \int_{\tau/\varepsilon}^{(\tau+\delta)/\varepsilon} \varepsilon^{\frac{1}{\alpha}} e^{-uA} \Gamma \, dS_u \right\|^r \right] &= \mathbb{E} \left[\mathbb{E} \left[\left\| \int_{\tau/\varepsilon}^{(\tau+\delta)/\varepsilon} \varepsilon^{\frac{1}{\alpha}} e^{-uA} \Gamma \, dS_u \right\|^r \middle| \mathcal{T} \right] \right] \\ &= \mathbb{E} \left[\sum_{\tau_i \in \mathcal{T}} \frac{\mathbb{1}_{\{\tau=\tau_i\}}}{\mathbb{P}(\tau = \tau_i)} \mathbb{E} \left[\mathbb{1}_{\{\tau=\tau_i\}} \left\| \int_{\tau_i/\varepsilon}^{(\tau_i+\delta)/\varepsilon} \varepsilon^{\frac{1}{\alpha}} e^{-uA} \Gamma \, dS_u \right\|^r \right] \right] \\ &\leq \mathbb{E} \left[\sum_{\tau_i \in \mathcal{T}} \frac{\mathbb{1}_{\{\tau=\tau_i\}}}{\mathbb{P}(\tau = \tau_i)} \mathbb{E} \left[\left\| \int_{\tau_i/\varepsilon}^{(\tau_i+\delta)/\varepsilon} \varepsilon^{\frac{1}{\alpha}} e^{-uA} \Gamma \, dS_u \right\|^r \right] \right]. \end{aligned}$$

Besides, we can use the self-similarity of S , Lemma 5.2 in [DF13], and the fact that e^{tA} is a rotation matrix for any $t \in \mathbb{R}$, to compute for each $\tau_i \in \mathcal{T}$ and δ small enough

$$\mathbb{E} \left[\left\| \int_{\tau_i/\varepsilon}^{(\tau_i+\delta)/\varepsilon} \varepsilon^{\frac{1}{\alpha}} e^{-uA} \Gamma \, dS_u \right\|^r \right] = \mathbb{E} \left[\left\| \int_{\tau_i}^{\tau_i+\delta} e^{-\frac{u}{\varepsilon}A} \Gamma \, dS_u \right\|^r \right] \leq C_r \delta^{\frac{r}{\alpha}}.$$

Since $\tau \in [m, M]$ a.s., the last term can be handled as in Section 5.2 in [GL21b] using moment estimates of V (see Propositions 4.3.1 and 4.3.3) to have

$$\mathbb{E} \left[\varepsilon^{\frac{1}{\alpha}} \int_{\tau/\varepsilon}^{(\tau+\delta)/\varepsilon} K |V_u|^\gamma u^{-\beta} \, du \right]^r \leq C_{m,M} \varepsilon^{r\beta-r\frac{\gamma+\alpha-1}{\alpha}}.$$

Since $\eta > 0$ and by Markov's inequality, we obtain for δ and ε small enough

$$\mathbb{P} \left(\left\| Y_{\tau+\delta}^{(\varepsilon)} - Y_{\tau}^{(\varepsilon)} \right\| \geq a \right) \leq \frac{C_r \delta^{\frac{r}{\alpha}} + C_{m,M} \varepsilon^{r\beta-r\frac{\gamma+\alpha-1}{\alpha}}}{a^r} \leq \eta.$$

Moreover, by Markov's inequality and the moment estimates again, we deduce that for

all $t \in [m, M]$,

$$\lim_{a \rightarrow +\infty} \limsup_{\varepsilon \rightarrow 0} \mathbb{P} \left(\|Y_t^{(\varepsilon)}\| \geq a \right) \leq \lim_{a \rightarrow +\infty} \limsup_{\varepsilon \rightarrow 0} \frac{\mathbb{E} \left[\left\| Y_t^{(\varepsilon)} \right\|^r \right]}{a^r} \leq \lim_{a \rightarrow +\infty} \frac{Ct^{\frac{r}{\alpha}}}{a^r} = 0.$$

By Corollary and Theorem 16.8 p. 175 in [Bil99], this concludes the proof of the tightness on every compact interval of $(0, +\infty)$. \square

We will now prove the convergence of the f.d.d. of $Y^{(\varepsilon)}$. Thanks to the previous lemma, this will yield the weak convergence on every compact set (see Theorem 13.1 p. 139 in [Bil99]) in the super-critical and critical regimes. The convergence in distribution on the whole space \mathcal{D} will follow from Theorem 16.7 p. 174 in [Bil99], since all processes considered are càdlàg.

4.4.1 Convergence of the f.d.d. in the super-critical regime

Assume that $\alpha q > 1$. Recall that $(Y_t^{(\varepsilon)})_{t \geq \varepsilon t_0} := (\varepsilon^{\frac{1}{\alpha}} Y_{t/\varepsilon})_{t \geq \varepsilon t_0}$.

Proof of Theorem 4.1.1 (i).

STEP 1. We first prove the convergence of the f.d.d. of the Wiener-Lévy integral appearing in (SDE_Y) .

Recall that the local martingale $M^{(\varepsilon)}$ was defined in (4.13).

STEP 1A. We begin with the convergence in distribution of $M_{s,t}^{(\varepsilon)} := M_t^{(\varepsilon)} - M_s^{(\varepsilon)}$, for $\varepsilon t_0 \leq s \leq t$.

To this end, we study the characteristic function $\phi_{s,t}^{(\varepsilon)}$ of $M_{s,t}^{(\varepsilon)}$. Let us recall that ψ denotes the characteristic exponent of L , and is given, for all $\xi \in \mathbb{R}$, by

$$\psi(\xi) = -a|\xi|^\alpha.$$

The characteristic function of the Wiener-Lévy integral can be computed as p. 105 in

[Sat99], hence one has, for all $\xi := (u, v) \in \mathbb{R}^2$,

$$\begin{aligned}
 \phi_{s,t}^{(\varepsilon)}(\xi) &= \mathbb{E} \left(\exp \left[-iu\varepsilon^{\frac{1}{\alpha}} \int_{s/\varepsilon}^{t/\varepsilon} \sin(y) \, dL_y + iv\varepsilon^{\frac{1}{\alpha}} \int_{s/\varepsilon}^{t/\varepsilon} \cos(y) \, dL_y \right] \right) \\
 &= \mathbb{E} \left(\exp \left[i\varepsilon^{\frac{1}{\alpha}} \int_{s/\varepsilon}^{t/\varepsilon} (-u \sin(y) + v \cos(y)) \, dL_y \right] \right) \\
 &= \exp \left(\int_{s/\varepsilon}^{t/\varepsilon} \psi \left(\varepsilon^{\frac{1}{\alpha}} [-u \sin(y) + v \cos(y)] \right) \, dy \right) \\
 &= \exp \left(-a\varepsilon \int_{s/\varepsilon}^{t/\varepsilon} |-u \sin(y) + v \cos(y)|^\alpha \, dy \right).
 \end{aligned}$$

Using Lemma B.0.2, we deduce that $\phi_{s,t}^{(\varepsilon)}(\xi)$ converges, as $\varepsilon \rightarrow 0$, to

$$\exp \left(-a(t-s) \frac{1}{2\pi} \int_0^{2\pi} |-u \sin(y) + v \cos(y)|^\alpha \, dy \right).$$

STEP 1B. We now compute explicitly the scale parameter of the stable limiting process. We denote by λ the uniform probability distribution on the circle \mathbb{S}^1 . Thanks to a change of variable and the symmetry of λ , setting $\omega := \frac{\xi}{\|\xi\|}$ for $\xi = (u, v) \in \mathbb{R}^2 \setminus \{0\}$, we have

$$\begin{aligned}
 \frac{1}{2\pi} \int_0^{2\pi} |-u \sin(y) + v \cos(y)|^\alpha \, dy &= \int_{\mathbb{S}^1} |\xi \cdot \lambda|^\alpha \, d\lambda \\
 &= \|\xi\|^\alpha \int_{\mathbb{S}^1} |\omega \cdot \lambda|^\alpha \, d\lambda.
 \end{aligned}$$

Since λ is rotationally invariant, we deduce that $\int_{\mathbb{S}^1} |\omega \cdot \lambda|^\alpha \, d\lambda$ does not depend on $\omega \in \mathbb{S}^1$. Taking $\omega = (1, 0)^T$, we set

$$\tilde{C} := \frac{a}{2\pi} \int_0^{2\pi} |\cos(x)|^\alpha \, dx. \tag{4.14}$$

We have thus proved that, for any $\xi \in \mathbb{R}^2$,

$$\phi_{s,t}^{(\varepsilon)}(\xi) \xrightarrow{\varepsilon \rightarrow 0} \exp \left(-(t-s) \tilde{C} \|\xi\|^\alpha \right).$$

Thus, the following convergence in distribution holds

$$M_{s,t}^{(\varepsilon)} = M_t^{(\varepsilon)} - M_s^{(\varepsilon)} \xrightarrow{\varepsilon \rightarrow 0} \mathcal{L}_{t-s}. \tag{4.15}$$

Following the same lines, we show that, for any $t > 0$,

$$M_t^{(\varepsilon)} \xrightarrow[\varepsilon \rightarrow 0]{} \mathcal{L}_t. \quad (4.16)$$

STEP 1C. We now prove the convergence in f.d.d. of $M^{(\varepsilon)}$ to \mathcal{L} , as ε tends to 0.

Let us fix $0 < t_1 \leq t_2 \leq \dots \leq t_d$. Note that $(M_t^{(\varepsilon)})_{t \geq \varepsilon t_0}$ is a càdlàg process with independent increments, since the integrands in its definition are deterministic and because L is a Lévy process. Thus, the random variables $(M_{t_1}^{(\varepsilon)}, M_{t_1, t_2}^{(\varepsilon)}, \dots, M_{t_{d-1}, t_d}^{(\varepsilon)})$ are mutually independent. We deduce from the convergence results established in (4.15) and (4.16), and the fact that \mathcal{L} has stationary and independent increments that

$$(M_{t_1}^{(\varepsilon)}, M_{t_1, t_2}^{(\varepsilon)}, \dots, M_{t_{d-1}, t_d}^{(\varepsilon)}) \xrightarrow[\varepsilon \rightarrow 0]{} (\mathcal{L}_{t_1}, \mathcal{L}_{t_2} - \mathcal{L}_{t_1}, \dots, \mathcal{L}_{t_d} - \mathcal{L}_{t_{d-1}}).$$

The continuous mapping theorem yields the convergence in f.d.d. of $M^{(\varepsilon)}$ to \mathcal{L} .

STEP 2. Pick $T > 0$. We prove that

$$\mathbb{E} \left[\sup_{\varepsilon t_0 \leq t \leq T} \|Y_t^{(\varepsilon)} - M_t^{(\varepsilon)}\| \right] \xrightarrow[\varepsilon \rightarrow 0]{} 0.$$

We have

$$\sup_{\varepsilon t_0 \leq t \leq T} \|Y_t^{(\varepsilon)} - M_t^{(\varepsilon)}\| \leq \varepsilon^{\frac{1}{\alpha}} \|z_0\| + \varepsilon^{\frac{1}{\alpha}} \int_{t_0}^{T/\varepsilon} \|e^{-sA} F(s, V_s)\| ds.$$

We use moment estimates (Propositions 4.3.1 and 4.3.3) to get

$$\begin{aligned} \mathbb{E} \left[\varepsilon^{\frac{1}{\alpha}} \int_{t_0}^{T/\varepsilon} \|e^{-sA} F(s, V_s)\| ds \right] &= \mathbb{E} \left[\varepsilon^{\frac{1}{\alpha}} \int_{t_0}^{T/\varepsilon} \|F(s, V_s)\| ds \right] \\ &\leq \mathbb{E} \left[\varepsilon^{\frac{1}{\alpha}} \int_{t_0}^{T/\varepsilon} |V_s|^\gamma s^{-\beta} ds \right] \\ &\leq \varepsilon^{\frac{1}{\alpha}} C_{\kappa, t_0} \int_{t_0}^{T/\varepsilon} s^{\frac{\gamma}{\alpha} - \beta} ds \\ &\leq C_{\kappa, t_0} (\varepsilon^{\beta - \frac{\gamma + \alpha - 1}{\alpha}} T^{\frac{\gamma}{\alpha} - \beta + 1} - \varepsilon^{\frac{1}{\alpha}} t_0^{\frac{\gamma}{\alpha} - \beta + 1}). \end{aligned}$$

Hence, setting $r := \min(\beta - \frac{\gamma + \alpha - 1}{\alpha}, \frac{1}{\alpha})$, which is positive by assumption, we get

$$\mathbb{E} \left[\sup_{\varepsilon t_0 \leq t \leq T} \|Y_t^{(\varepsilon)} - M_t^{(\varepsilon)}\| \right] = O(\varepsilon^r). \quad (4.17)$$

The conclusion follows from Theorem 3.1 p. 27 in [Bil99]. \square

4.4.2 Convergence of the f.d.d. in the critical and sub-critical regime

In this section, we consider the linear case, i.e. $\gamma = 1$ and we assume that $\beta \in (\frac{1}{2}, 1]$. Recall that

$$(Y_t^{(\varepsilon)})_{t \geq \varepsilon t_0} = (\varepsilon^q Y_{t/\varepsilon})_{t \geq \varepsilon t_0}.$$

Proof of Theorem 4.1.1 (ii) and (iii). The proof follows the same lines as in the Brownian setting. Recall that leaving out the Brownian term, the underlying ODE is the following

$$x''(t) + \frac{x'(t)}{t^\beta} + x(t) = 0, \quad t \geq t_0. \quad (4.18)$$

We pick again the basis of solutions given by Lemma A.0.3, and we still denote by R its resolvent matrix and by f its rate of decrease. Recall that it is given by

$$\forall t > 0, f(t) := \begin{cases} \frac{1}{\sqrt{t}} & \text{if } \beta = 1, \\ \exp\left(-\frac{t^{1-\beta}}{2(1-\beta)}\right) & \text{else.} \end{cases} \quad (4.19)$$

We set, for all $t \geq \varepsilon t_0$,

$$\widetilde{M}_t^{(\varepsilon)} := \varepsilon^q f(t/\varepsilon) \int_{t_0}^{t/\varepsilon} R_s^{-1} \Gamma \, dS_s. \quad (4.20)$$

Keeping the same notations as in the Brownian case, we decompose $(Y_t)_{t \geq t_0} = (e^{-tA} Z_t)_{t \geq t_0}$ into

$$\varepsilon^q Y_{t/\varepsilon} = \varepsilon^q f(t/\varepsilon) \Phi_{t/\varepsilon} R_{t_0}^{-1} Z_0 + \Phi_{t/\varepsilon} \widetilde{M}_t^{(\varepsilon)}.$$

Reasoning as in the Brownian case, it remains to study the convergence of $\widetilde{M}^{(\varepsilon)}$ since the first term converges towards 0. Using the expression of the Wronskian obtained in Lemma A.0.3, we obtain, for all $t \geq \varepsilon t_0$,

$$\widetilde{M}_t^{(\varepsilon)} = \varepsilon^q f(t/\varepsilon) \int_{t_0}^{t/\varepsilon} f(u)^{-2} \begin{pmatrix} -y_2(u) \\ y_1(u) \end{pmatrix} dL_u.$$

Let us fix $0 < s < t$. We study the convergence in distribution of the couple $(\widetilde{M}_s^{(\varepsilon)}, \widetilde{M}_t^{(\varepsilon)})$ when ε tends to 0. The convergence in distribution of a general d -dimensional distribution $(\widetilde{M}_{t_1}^{(\varepsilon)}, \dots, \widetilde{M}_{t_d}^{(\varepsilon)})$ relies on the same computations.

Let us fix $(\xi_1, \xi_2) \in \mathbb{R}^2 \times \mathbb{R}^2$. Using that L has independent increments, the characteristic

function $\tilde{\phi}_{s,t}^{(\varepsilon)}$ of $(\tilde{M}_s^{(\varepsilon)}, \tilde{M}_t^{(\varepsilon)})$ is given by

$$\begin{aligned} \tilde{\phi}_{s,t}^{(\varepsilon)}(\xi_1, \xi_2) &= \mathbb{E} \left[\exp \left(i\varepsilon^q \left[f(s/\varepsilon) \xi_1 \cdot \int_{t_0}^{s/\varepsilon} f(u)^{-2} \begin{pmatrix} -y_2(u) \\ y_1(u) \end{pmatrix} dL_u \right. \right. \right. \\ &\quad \left. \left. \left. + f(t/\varepsilon) \xi_2 \cdot \int_{t_0}^{t/\varepsilon} f(u)^{-2} \begin{pmatrix} -y_2(u) \\ y_1(u) \end{pmatrix} dL_u \right] \right) \right] \\ &= \mathbb{E} \left[\exp \left(i\varepsilon^q (f(s/\varepsilon) \xi_1 + f(t/\varepsilon) \xi_2) \cdot \int_{t_0}^{s/\varepsilon} f(u)^{-2} \begin{pmatrix} -y_2(u) \\ y_1(u) \end{pmatrix} dL_u \right) \right] \\ &\quad \times \mathbb{E} \left[\exp \left(i\varepsilon^q f(t/\varepsilon) \xi_2 \cdot \int_{s/\varepsilon}^{t/\varepsilon} f(u)^{-2} \begin{pmatrix} -y_2(u) \\ y_1(u) \end{pmatrix} dL_u \right) \right]. \end{aligned}$$

Let us recall that the characteristic exponent of L is given, for all $\xi \in \mathbb{R}$, by

$$\psi(\xi) = -a|\xi|^\alpha.$$

The characteristic function of the Wiener-Lévy integral can be computed as p. 105 in [Sat99], hence one has

$$\begin{aligned} \tilde{\phi}_{s,t}^{(\varepsilon)}(\xi_1, \xi_2) &= \exp \left(-a\varepsilon^\beta \int_{t_0}^{s/\varepsilon} f(u)^{-2\alpha} \left| (f(s/\varepsilon) \xi_1 + f(t/\varepsilon) \xi_2) \cdot \begin{pmatrix} -y_2(u) \\ y_1(u) \end{pmatrix} \right|^\alpha du \right) \\ &\quad \times \exp \left(-a\varepsilon^\beta \int_{s/\varepsilon}^{t/\varepsilon} f(u)^{-2\alpha} \left| f(t/\varepsilon) \xi_2 \cdot \begin{pmatrix} -y_2(u) \\ y_1(u) \end{pmatrix} \right|^\alpha du \right). \end{aligned}$$

Using the asymptotic expansion of the resolvent matrix (Lemma A.0.3), we can write, for any $u \geq t_0$,

$$\begin{pmatrix} -y_2(u) \\ y_1(u) \end{pmatrix} = f(u) \left[\begin{pmatrix} -\sin(u) \\ \cos(u) \end{pmatrix} + g(u) \right],$$

where $g : [t_0, +\infty) \rightarrow \mathbb{R}^2$ is a function satisfying for all $u \geq t_0$,

$$|g(u)| \leq Cu^{1-2\beta}.$$

We set

$$K_1^{(\varepsilon)} := \exp \left(-a\varepsilon^\beta \int_{t_0}^{s/\varepsilon} f(u)^{-\alpha} \left| (f(s/\varepsilon) \xi_1 + f(t/\varepsilon) \xi_2) \cdot \left[\begin{pmatrix} -\sin(u) \\ \cos(u) \end{pmatrix} + g(u) \right] \right|^\alpha du \right) \quad (4.21)$$

and

$$K_2^{(\varepsilon)} := \exp \left(-a\varepsilon^\beta \int_{s/\varepsilon}^{t/\varepsilon} f(u)^{-\alpha} \left| f\left(\frac{t}{\varepsilon}\right) \xi_2 \cdot \left[\begin{pmatrix} -\sin(u) \\ \cos(u) \end{pmatrix} + g(u) \right] \right|^\alpha du \right).$$

We thus obtain

$$\tilde{\phi}_{s,t}^{(\varepsilon)}(\xi_1, \xi_2) = K_1^{(\varepsilon)} \times K_2^{(\varepsilon)}, \quad (4.22)$$

STEP 1. We start by justifying that we can omit g to study the limit when $\varepsilon \rightarrow 0$. More precisely, we prove that, for all function $\zeta : \mathbb{R} \rightarrow \mathbb{R}^2$ such that $\|\zeta(\varepsilon)\| f(s/\varepsilon)^{-1} = O_{\varepsilon \rightarrow 0}(1)$,

$$R^{(\varepsilon)} := \varepsilon^\beta \int_{t_0}^{s/\varepsilon} f(u)^{-\alpha} \left| \left| \zeta(\varepsilon) \cdot \left[\begin{pmatrix} -\sin(u) \\ \cos(u) \end{pmatrix} + g(u) \right] \right|^\alpha - \left| \zeta(\varepsilon) \cdot \left[\begin{pmatrix} -\sin(u) \\ \cos(u) \end{pmatrix} \right] \right|^\alpha \right| du \xrightarrow{\varepsilon \rightarrow 0} 0. \quad (4.23)$$

Thanks to the mean value theorem applied to $|\cdot|^\alpha$ (for $\alpha \geq 1$) or by Hölder property (for $\alpha < 1$), and the domination of g , we obtain that, for some constant $C > 0$,

$$R^{(\varepsilon)} \leq C\varepsilon^\beta \|\zeta(\varepsilon)\|^\alpha \int_{t_0}^{s/\varepsilon} f(u)^{-\alpha} u^{1-2\beta} du = O_{\varepsilon \rightarrow 0}(\|\zeta(\varepsilon)\|^\alpha f(s/\varepsilon)^{-\alpha} \varepsilon^{2\beta-1}),$$

where the last equality follows from Lemma B.0.3. This proves (4.23) since $\beta > \frac{1}{2}$.

STEP 2. We focus on the first term $K_1^{(\varepsilon)}$ defined in (4.21). Since f is decreasing, notice that

$$\zeta(\varepsilon) := f(s/\varepsilon) \xi_1 + f(t/\varepsilon) \xi_2 = O_{\varepsilon \rightarrow 0}(f(s/\varepsilon)).$$

Then we have to study the convergence of $I^{(\varepsilon)}$ defined by

$$I^{(\varepsilon)} := a\varepsilon^\beta \int_{t_0}^{s/\varepsilon} f(u)^{-\alpha} \left| (f(s/\varepsilon) \xi_1 + f(t/\varepsilon) \xi_2) \cdot \begin{pmatrix} -\sin(u) \\ \cos(u) \end{pmatrix} \right|^\alpha du.$$

Its limit differs according to the value of β .

STEP 2A. Assume first that $\beta = 1$. Then, using the expression of f (see (4.19)),

$$I^{(\varepsilon)} = a\varepsilon^{1+\frac{\alpha}{2}} \int_{t_0}^{s/\varepsilon} f(u)^{-\alpha} \left| \left(\frac{\xi_1}{\sqrt{s}} + \frac{\xi_2}{\sqrt{t}} \right) \cdot \begin{pmatrix} -\sin(u) \\ \cos(u) \end{pmatrix} \right|^\alpha du.$$

We proved in Step 2b of the super-critical regime that there exists a constant $\tilde{C} > 0$ given in (4.14) such that, for all $\zeta \in \mathbb{R}^2$,

$$\frac{a}{2\pi} \int_0^{2\pi} \left| \zeta \cdot \begin{pmatrix} -\sin(u) \\ \cos(u) \end{pmatrix} \right|^\alpha du = \tilde{C} \|\zeta\|^\alpha. \quad (4.24)$$

Using Lemma B.0.2, we can compute the following asymptotic expansion

$$I^{(\varepsilon)} = \varepsilon^{1+\frac{\alpha}{2}} \tilde{C} \left\| \frac{\xi_1}{\sqrt{s}} + \frac{\xi_2}{\sqrt{t}} \right\|^\alpha \int_{t_0}^{s/\varepsilon} f(u)^{-\alpha} du + \underset{\varepsilon \rightarrow 0}{o} \left(\varepsilon^{1+\frac{\alpha}{2}} \int_{t_0}^{s/\varepsilon} f(u)^{-\alpha} du \right).$$

Therefore, it follows from Lemma B.0.3 that

$$K_1^{(\varepsilon)} \xrightarrow{\varepsilon \rightarrow 0} \exp \left(-\tilde{C} \left(1 + \frac{\alpha}{2} \right)^{-1} \left\| \frac{\xi_1}{\sqrt{s}} + \frac{\xi_2}{\sqrt{t}} \right\|^\alpha s^{1+\frac{\alpha}{2}} \right).$$

STEP 2B. Let us consider now $\beta \in \left(\frac{1}{2}, 1\right)$. Let us notice that $I^{(\varepsilon)}$ can be decomposed into the sum

$$I^{(\varepsilon)} = I_1^{(\varepsilon)} + I_2^{(\varepsilon)} \quad (4.25)$$

of the two following terms

$$I_1^{(\varepsilon)} := a\varepsilon^\beta \int_{t_0}^{s/\varepsilon} f(u)^{-\alpha} \left| f\left(\frac{s}{\varepsilon}\right) \xi_1 \cdot \begin{pmatrix} -\sin(u) \\ \cos(u) \end{pmatrix} \right|^\alpha du$$

and

$$I_2^{(\varepsilon)} := a\varepsilon^\beta \int_{t_0}^{s/\varepsilon} f(u)^{-\alpha} \left[\left| \left(f\left(\frac{s}{\varepsilon}\right) \xi_1 + f\left(\frac{t}{\varepsilon}\right) \xi_2 \right) \cdot \begin{pmatrix} -\sin(u) \\ \cos(u) \end{pmatrix} \right|^\alpha - \left| f\left(\frac{s}{\varepsilon}\right) \xi_1 \cdot \begin{pmatrix} -\sin(u) \\ \cos(u) \end{pmatrix} \right|^\alpha \right] du.$$

Using again either the mean value theorem or the Hölder property, and Lemma B.0.3, we get since $\beta < 1$ that for some constant $C > 0$,

$$|I_2^{(\varepsilon)}| \leq C\varepsilon^\beta f(s/\varepsilon)^{\alpha-1} f(t/\varepsilon) \int_{t_0}^{s/\varepsilon} f(u)^{-\alpha} du = \underset{\varepsilon \rightarrow 0}{O} \left(f(t/\varepsilon) f(s/\varepsilon)^{-1} \right) = \underset{\varepsilon \rightarrow 0}{o} (1). \quad (4.26)$$

Using Lemma B.0.2, we can compute the following asymptotic expansion of $I_1^{(\varepsilon)}$

$$\begin{aligned} I_1^{(\varepsilon)} &= a\varepsilon^\beta f\left(\frac{s}{\varepsilon}\right)^\alpha \int_{t_0}^{s/\varepsilon} f(u)^{-\alpha} \left| \xi_1 \cdot \begin{pmatrix} -\sin(u) \\ \cos(u) \end{pmatrix} \right|^\alpha du \\ &= a\varepsilon^\beta f\left(\frac{s}{\varepsilon}\right)^\alpha \left[\frac{1}{2\pi} \int_0^{2\pi} \left| \xi_1 \cdot \begin{pmatrix} -\sin(u) \\ \cos(u) \end{pmatrix} \right|^\alpha du \int_{t_0}^{s/\varepsilon} f(u)^{-\alpha} du + o_{\varepsilon \rightarrow 0} \left(\int_{t_0}^{s/\varepsilon} f(u)^{-\alpha} du \right) \right]. \end{aligned}$$

Thanks to (4.24) and the asymptotic expansion's results given in Lemma B.0.3, there exists an explicit constant $k_{\beta,\alpha}$ given in Lemma B.0.3, such that

$$I_1^{(\varepsilon)} \xrightarrow{\varepsilon \rightarrow 0} k_{\beta,\alpha} \tilde{C} s^\beta \|\xi_1\|^\alpha. \quad (4.27)$$

Combining (4.23), (4.25), (4.26) and (4.27), we have proved that $K_1^{(\varepsilon)}$, defined in (4.21), converges as $\varepsilon \rightarrow 0$ towards

$$\exp\left(-k_{\beta,\alpha} \tilde{C} s^\beta \|\xi_1\|^\alpha\right).$$

STEP 3. It remains to deal with the limit of $K_2^{(\varepsilon)}$. Notice that

$$\zeta(\varepsilon) := f(t/\varepsilon) \xi_2 = O_{\varepsilon \rightarrow 0}(f(t/\varepsilon)) = O_{\varepsilon \rightarrow 0}(f(s/\varepsilon)).$$

Hence, thanks to Step 1, we are reduced to study, for $r \in \{s, t\}$,

$$J_r^{(\varepsilon)} := a\varepsilon^\beta f\left(\frac{t}{\varepsilon}\right)^\alpha \int_{t_0}^{r/\varepsilon} f(u)^{-\alpha} \left| \xi_2 \cdot \begin{pmatrix} -\sin(u) \\ \cos(u) \end{pmatrix} \right|^\alpha du. \quad (4.28)$$

Asymptotic expansion's results (Lemmas B.0.2 and B.0.3) and (4.24) yield

$$\begin{aligned} J_r^{(\varepsilon)} &= \tilde{C} \|\xi_2\|^\alpha \varepsilon^\beta f(t/\varepsilon)^\alpha \int_{t_0}^{r/\varepsilon} f(u)^{-\alpha} du + o_{\varepsilon \rightarrow 0}(f(t/\varepsilon)^\alpha f(r/\varepsilon)^{-\alpha}) \\ &= \tilde{C} \|\xi_2\|^\alpha k_{\beta,\alpha} r^\beta f(t/\varepsilon)^\alpha f(r/\varepsilon)^{-\alpha} + o_{\varepsilon \rightarrow 0}(f(t/\varepsilon)^\alpha f(r/\varepsilon)^{-\alpha}). \end{aligned}$$

Hence,

$$J_t^{(\varepsilon)} \xrightarrow{\varepsilon \rightarrow 0} \tilde{C} \|\xi_2\|^\alpha k_{\beta,\alpha} t^\beta \quad (4.29)$$

and

$$J_s^{(\varepsilon)} \xrightarrow{\varepsilon \rightarrow 0} \tilde{C} \|\xi_2\|^\alpha k_{\beta,\alpha} s^\beta \left(\frac{s}{t}\right)^{\frac{\alpha}{2}} \mathbb{1}_{\{\beta=1\}}. \quad (4.30)$$

Since

$$K_2^{(\varepsilon)} = \exp\left(-J_t^{(\varepsilon)} + J_s^{(\varepsilon)}\right),$$

we thus obtain that, for all $0 < s \leq t$,

$$\tilde{\phi}_{s,t}^{(\varepsilon)}(\xi_1, \xi_2) \xrightarrow{\varepsilon \rightarrow 0} \begin{cases} \exp\left(-k_{\beta,\alpha} \tilde{C} s^\beta \|\xi_1\|^\alpha\right) \exp\left(-k_{\beta,\alpha} \tilde{C} t^\beta \|\xi_2\|^\alpha\right) & \text{if } \beta < 1, \\ \exp\left(-k_{\beta,\alpha} \tilde{C} \left[\left\|\frac{\xi_1}{\sqrt{s}} + \frac{\xi_2}{\sqrt{t}}\right\|^\alpha s^{1+\frac{\alpha}{2}} + \|\xi_2\|^\alpha t - \|\xi_2\|^\alpha \left(\frac{s}{t}\right)^{\frac{\alpha}{2}} s\right]\right) & \text{if } \beta = 1. \end{cases}$$

STEP 4. We can compute the characteristic function of the process $\left(\frac{1}{\sqrt{t}} \int_0^t \sqrt{s} d\mathcal{L}_s\right)_{t>0}$ in the same manner, and thus recognize the limiting process in the critical regime. \square

Remark 4.4.2. *As in the Brownian setting, if $\beta = 0$, the resolvent matrix is explicit and following the same lines, we can prove that $\left(Z_{t/\varepsilon}\right)_{t \geq \varepsilon t_0}$ converges in f.d.d. towards the product of the measure μ , whose characteristic function is given by*

$$\xi \mapsto \exp\left(-a \int_0^{+\infty} e^{-\alpha u} \|\xi \cdot h(u)\|^\alpha du\right),$$

h being an explicit periodic function depending on the resolvent matrix.

4.5 Proof of Corollary 4.1.2

Proof of Corollary 4.1.2. We start by proving the convergence in distribution of $v_{1/T} Z_T$. Reasoning as in the Brownian setting, it follows from Theorem 4.1.1 that $v_{1/T} Y_T$ converges. The conclusion is a consequence of Lemma B.0.4, noting that the limiting distribution is invariant under rotations thanks to the expression of its characteristic function.

Let us now prove that the rescaled process $Z^{(\varepsilon)}$ does not converge in distribution. We state the proof in the super-critical regime. Assume by contradiction that it is the case. Reasoning as in the Brownian case, we prove that this implies the convergence in distribution of the process $I^{(\varepsilon)}$ defined, for $t \geq \varepsilon t_0$, by

$$I_t^{(\varepsilon)} := \varepsilon^{\frac{1}{\alpha}} \int_{t_0}^{t/\varepsilon} \sin\left(\frac{t}{\varepsilon} - u\right) dL_u.$$

In particular, for $s < t$, the random variable $I_t^{(\varepsilon)} - I_s^{(\varepsilon)}$ shall converge in distribution.

Let us denote by $\phi^{(\varepsilon)}$ the characteristic function of $I_t^{(\varepsilon)} - I_s^{(\varepsilon)}$, which is supposed to converge

on \mathbb{R} . Using that L has independent increments, and setting

$$\phi^{(\varepsilon),1} := \mathbb{E} \left[\exp \left(\varepsilon^{\frac{1}{\alpha}} \int_{s/\varepsilon}^{t/\varepsilon} \sin \left(\frac{t}{\varepsilon} - u \right) dL_u \right) \right],$$

$$\phi^{(\varepsilon),2} := \mathbb{E} \left[\exp \left(\varepsilon^{\frac{1}{\alpha}} \int_{t_0}^{s/\varepsilon} \sin \left(\frac{t}{\varepsilon} - u \right) - \sin \left(\frac{s}{\varepsilon} - u \right) dL_u \right) \right],$$

we have $\phi^{(\varepsilon)}(1) = \phi^{(\varepsilon),1} \phi^{(\varepsilon),2}$.

Recall that ψ defined in (4.1), denotes the characteristic exponent of L . Using a change of variables, we have in particular

$$\begin{aligned} \phi^{(\varepsilon),1} &= \exp \left(\int_{s/\varepsilon}^{t/\varepsilon} \psi \left(\varepsilon^{\frac{1}{\alpha}} \sin \left(\frac{t}{\varepsilon} - u \right) \right) du \right) = \exp \left(-a\varepsilon \int_{s/\varepsilon}^{t/\varepsilon} \left| \sin \left(\frac{t}{\varepsilon} - u \right) \right|^\alpha du \right) \\ &= \exp \left(-a\varepsilon \int_0^{(t-s)/\varepsilon} |\sin(u)|^\alpha du \right). \end{aligned}$$

Lemma B.0.2 ensures that $\phi^{(\varepsilon),1}$ has a limit when ε converges to 0. Similarly, we obtain

$$\begin{aligned} \phi^{(\varepsilon),2} &= \exp \left(\int_{t_0}^{s/\varepsilon} \psi \left(\varepsilon^{\frac{1}{\alpha}} \sin \left(\frac{t}{\varepsilon} - u \right) - \sin \left(\frac{s}{\varepsilon} - u \right) \right) du \right) \\ &= \exp \left(-a\varepsilon \int_{t_0}^{s/\varepsilon} \left| \sin \left(\frac{t}{\varepsilon} - u \right) - \sin \left(\frac{s}{\varepsilon} - u \right) \right|^\alpha du \right) \\ &= \exp \left(-a2^\alpha \left| \sin \left(\frac{t-s}{2\varepsilon} \right) \right|^\alpha \varepsilon \int_{t_0}^{s/\varepsilon} \left| \cos \left(\frac{t+s}{2\varepsilon} - u \right) \right|^\alpha du \right) \\ &= \exp \left(-a2^\alpha \left| \sin \left(\frac{t-s}{2\varepsilon} \right) \right|^\alpha \varepsilon \int_{\frac{t-s}{2\varepsilon}}^{\frac{t+s}{2\varepsilon}-t_0} |\cos(u)|^\alpha du \right). \end{aligned} \quad (4.31)$$

The change of variables $u = v + \pi$ yields, for all $\varepsilon > 0$,

$$\int_0^{2\pi} |\cos(u)|^\alpha \operatorname{sgn} \left(\sin \left(\frac{t-s}{2\varepsilon} \right) \cos(u) \right) du = 0.$$

Thus, Lemma B.0.2 ensures that

$$\varepsilon \int_{\frac{t-s}{2\varepsilon}}^{\frac{t+s}{2\varepsilon}-t_0} |\cos(u)|^\alpha du \quad \xrightarrow{\varepsilon \rightarrow 0} \quad \frac{s}{2\pi} \int_0^{2\pi} |\cos(u)|^\alpha du.$$

Coming back to (4.31), we see that $\phi^{(\varepsilon),2}$ does not converge when ε tends to 0. This is a contradiction. \square

STUDY OF THE DETERMINISTIC UNDERLYING ODE

The deterministic ODE behind the system is the following

$$x''(t) + \frac{x'(t)}{t^\beta} + x(t) = 0, \quad t \geq t_0. \quad (\text{A.1})$$

The solutions form a vector space of dimension 2. Let us take two solutions y_1 and y_2 which are linearly independent. Then, we introduce the fundamental system of solutions (resolvent matrix) R to (A.1) defined, for $t \geq t_0$, by

$$R_t = \begin{pmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{pmatrix}.$$

It satisfies, for all $t \geq t_0$,

$$R_t' = \begin{pmatrix} 0 & 1 \\ -1 & -\frac{1}{t^\beta} \end{pmatrix} R_t.$$

We recall that the Wronskian w is defined, for all $t \geq t_0$, by

$$w(t) = y_1(t)y_2'(t) - y_1'(t)y_2(t).$$

Let us finally set, for $t > 0$,

$$f(t) := \begin{cases} \frac{1}{\sqrt{t}} & \text{if } \beta = 1, \\ \exp\left(-\frac{t^{1-\beta}}{2(1-\beta)}\right) & \text{else.} \end{cases} \quad (\text{A.2})$$

Lemma A.0.1. *Pick $\beta \in \left(\frac{1}{2}, 1\right]$ and consider a solution y to (A.1). Then, there exist*

$a \in \mathbb{R}$ and $\phi \in [0, 2\pi)$ such that

$$y(t) = af(t) \cos(t + \phi) + \underset{t \rightarrow \infty}{O} \left(f(t)t^{1-2\beta} \right)$$

and

$$y'(t) = -af(t) \sin(t + \phi) + \underset{t \rightarrow \infty}{O} \left(f(t)t^{1-2\beta} \right).$$

Proof. Let us set, for $t \geq t_0$, $u(t) = f(t)^{-1}y(t)$. We easily check that u satisfies

$$u''(t) + u(t) [1 + h(t)] = 0,$$

where $h(t) := \frac{f''(t)}{f(t)} + \frac{f'(t)}{f(t)t^\beta} = \underset{t \rightarrow +\infty}{O} (t^{-2\beta})$. Following the proof of the method of variation of parameters, there exists $a_0, b_0 \in \mathbb{R}$ such that, for any $t \geq t_0$,

$$u(t) = a_0 \cos(t) + b_0 \sin(t) - \int_{t_0}^t u(s)h(s) \sin(t-s) ds.$$

Using that $h \in L^1((t_0, +\infty))$ since $\beta > \frac{1}{2}$, we obtain by Grönwall's lemma that the function u is bounded on $[t_0, +\infty)$. We deduce that both the functions $s \mapsto u(s)h(s) \cos(s)$ and $s \mapsto u(s)h(s) \sin(s)$ belong to $L^1((t_0, +\infty))$. Thus, up to changing the constants a_0 and b_0 , one has, for all $t \geq t_0$,

$$u(t) = a_0 \cos(t) + b_0 \sin(t) + \sin(t) \int_t^\infty u(s)h(s) \cos(s) ds - \cos(t) \int_t^\infty u(s)h(s) \sin(s) ds. \quad (\text{A.3})$$

It follows from the fact that u is bounded that

$$u(t) = a_0 \cos(t) + b_0 \sin(t) + \underset{t \rightarrow +\infty}{O} \left(\int_t^\infty \frac{ds}{s^{2\beta}} \right).$$

Thus, there exist $a \in \mathbb{R}$ and $\phi \in [0, 2\pi)$ such that

$$u(t) = a \cos(t + \phi) + \underset{t \rightarrow +\infty}{O} (t^{1-2\beta}).$$

This proves the asymptotic expansion of y .

Differentiating (A.3) and using that $h(t) = \underset{t \rightarrow +\infty}{O} (t^{-2\beta})$, we prove that

$$u'(t) = -a \sin(t + \phi) + \underset{t \rightarrow +\infty}{O} (t^{1-2\beta}).$$

Since u is bounded and $f'(t) = \underset{t \rightarrow +\infty}{O}(f(t)t^{-\beta})$, we finally obtain that

$$y'(t) = f'(t)u(t) + f(t)u'(t) = f(t)u'(t) + \underset{t \rightarrow +\infty}{O}(f(t)t^{-\beta}).$$

This concludes the proof of the asymptotic expansion of y' since $\beta \geq 2\beta - 1$. \square

Remark A.0.2. Note that if $\beta = 1$, the Bessel functions of the first kind J_0 and of the second kind Y_0 form a basis of solutions. Their asymptotic expansions can be found in [Wat44, Chap VII].

Lemma A.0.3. There exists a basis of solutions y_1 and y_2 to (A.1) such that the resolvent matrix R satisfies

$$R_t = f(t) \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix} + \underset{t \rightarrow \infty}{O}(f(t)t^{1-2\beta}) = f(t)e^{tA} + \underset{t \rightarrow \infty}{O}(f(t)t^{1-2\beta}).$$

Moreover, its Wronskian w is given, for any $t \geq t_0$, by $w(t) = f(t)^2$.

Proof. It is well-known that the Wronskian satisfies, for all $t \geq t_0$,

$$w'(t) = -\frac{1}{t^\beta}w(t).$$

Thus, there exists $w_0 \in \mathbb{R} \setminus \{0\}$ such that, for all $t \geq t_0$, $w(t) = w_0 f(t)^2$. Moreover, thanks to Lemma A.0.1, for $i \in \{1, 2\}$, there exist $a_i \in \mathbb{R}$ and $\phi_i \in [0, 2\pi)$ such that

$$y_i(t) = a_i f(t) \cos(t + \phi_i) + \underset{t \rightarrow \infty}{O}(f(t)t^{1-2\beta})$$

and

$$y_i'(t) = -a_i f(t) \sin(t + \phi_i) + \underset{t \rightarrow \infty}{O}(f(t)t^{1-2\beta}).$$

As a consequence,

$$w(t) = -a_1 a_2 f(t)^2 \sin(\phi_2 - \phi_1) + \underset{t \rightarrow \infty}{O}(f(t)^2 t^{2-4\beta}).$$

But since $w(t) = w_0 f(t)^2$, it implies that $a_i \neq 0$ and $\phi_2 \not\equiv \phi_1[\pi]$.

Up to dividing by a_i , we can assume that $a_i = 1$, and up to considering a linear combination of y_1 and y_2 , we can assume that $\phi_1 = 0$ and $\phi_2 = -\frac{\pi}{2}$. Thus, we have $w_0 = 1$. This concludes the proof. \square

SOME TECHNICAL RESULTS

We collect here some technical results used in our proofs. Recall first a sufficient condition for the non-explosion of the solution to a SDE. The proof can be found in [GL21a].

Lemma B.0.1. *Let $(Y_t)_{t \geq t_0}$ be a càdlàg process, solution to a SDE. For all $n \geq 0$, define the stopping time*

$$\tau_n := \inf\{t \geq t_0, \|Y_t\| \geq n\}. \quad (\text{B.1})$$

Set $\tau_\infty := \lim_{n \rightarrow +\infty} \tau_n$ the explosion time of Y . Assume that there exist two measurable and non-negative functions ϕ and b such that

- (i) ϕ is non-decreasing and $\lim_{n \rightarrow \infty} \phi(n) = +\infty$,
- (ii) b is finite-valued,
- (iii) and for all $t \geq t_0$,

$$\sup_{n \geq 0} \mathbb{E} [\phi(|Y_{t \wedge \tau_n}|)] \leq b(t).$$

Then $\tau_\infty = +\infty$ a.s.

We now state and prove a result on the periodic-averaging phenomenon.

Lemma B.0.2. *Let us fix $t_0 > 0$ and $h : [t_0, +\infty) \rightarrow \mathbb{R}$ a continuous m -periodic function, with $m > 0$. Let $g : [t_0, +\infty) \rightarrow \mathbb{R}^+$ be a continuously differentiable function which is not integrable on $[t_0, +\infty)$. We assume moreover that*

- (i) $g'(t) = \underset{t \rightarrow +\infty}{o}(g(t))$,
- (ii) $g(t) = \underset{t \rightarrow +\infty}{o}\left(\int_{t_0}^t g(u) \, du\right)$.

Then,

$$\int_{t_0}^t g(u)h(u) \, du = \left[\frac{1}{m} \int_{t_0}^{t_0+m} h(u) \, du \right] \int_{t_0}^t g(u) \, du + \underset{t \rightarrow +\infty}{o}\left(\int_{t_0}^t g(u) \, du\right).$$

Let us remark that the functions g_1 and g_2 defined for $t \in \mathbb{R}$, $r \geq 0$ and $\beta \in [0, 1)$ by $g_1(t) := t^r$ and $g_2(t) := \exp(rt^{1-\beta})$, satisfy the preceding assumptions made on g .

Proof. Let us define $\tilde{h} := h - \frac{1}{m} \int_{t_0}^{t_0+m} h(u) du$, and \tilde{H} a primitive of \tilde{h} . The function \tilde{H} is bounded on $[t_0, +\infty)$ since the average of \tilde{h} on its period is equal to 0. To prove the lemma, we only need to justify that

$$\int_{t_0}^t g(u) \tilde{h}(u) du = o_{t \rightarrow +\infty} \left(\int_{t_0}^t g(u) du \right).$$

By integration by parts, we obtain that, for all $t \geq t_0$,

$$\int_{t_0}^t g(u) \tilde{h}(u) du = g(t) \tilde{H}(t) - g(t_0) \tilde{H}(t_0) - \int_{t_0}^t g'(u) \tilde{H}(u) du.$$

Using the fact that \tilde{H} is bounded, that $g'(t) = o_{t \rightarrow +\infty}(g(t))$ and that $\int_{t_0}^{\infty} g(u) du = +\infty$, we deduce that

$$\int_{t_0}^t g'(u) \tilde{H}(u) du = o_{t \rightarrow +\infty} \left(\int_{t_0}^t g(u) du \right).$$

The conclusion follows from the fact that $g(t) \tilde{H}(t) - g(t_0) \tilde{H}(t_0) = o_{t \rightarrow +\infty} \left(\int_{t_0}^t g(u) du \right)$, since we have assumed that $g(t) = o_{t \rightarrow +\infty} \left(\int_{t_0}^t g(u) du \right)$ and that $\int_{t_0}^{\infty} g(u) du = +\infty$. \square

Lemma B.0.3. *Let f be given by (A.2) for $\beta \in (\frac{1}{2}, 1]$, and pick $\alpha \in (0, 2]$. Define*

$$k_{\beta, \alpha} := \begin{cases} (1 + \alpha/2)^{-1} & \text{if } \beta = 1, \\ \frac{2}{\alpha} & \text{else.} \end{cases}$$

Then for any $t > 0$, we have

$$\int_{t_0}^{t/\varepsilon} f(u)^{-\alpha} u^{1-2\beta} du = O_{\varepsilon \rightarrow 0}(f(t/\varepsilon)^{-\alpha} \varepsilon^{\beta-1}),$$

and

$$\int_{t_0}^{t/\varepsilon} f(u)^{-\alpha} du = k_{\beta, \alpha} f\left(\frac{t}{\varepsilon}\right)^{-\alpha} \left(\frac{t}{\varepsilon}\right)^{\beta} + o_{\varepsilon \rightarrow 0}(f(t/\varepsilon)^{-\alpha} \varepsilon^{-\beta}).$$

Proof. When $\beta = 1$, the results follow from direct computations because of the expression of f . Assume now that $\beta \in (\frac{1}{2}, 1)$. For the first point, the integration by parts formula

ensures that

$$\begin{aligned}
 \int_{t_0}^{t/\varepsilon} f(u)^{-\alpha} u^{1-2\beta} du &= \frac{2}{\alpha} \left[f(u)^{-\alpha} u^{1-\beta} \right]_{t_0}^{t/\varepsilon} - \frac{2}{\alpha} (1-\beta) \int_{t_0}^{t/\varepsilon} f(u)^{-\alpha} u^{-\beta} du \\
 &= O_{\varepsilon \rightarrow 0}(f(t/\varepsilon)^{-\alpha} \varepsilon^{\beta-1}) + O_{\varepsilon \rightarrow 0}(f(t/\varepsilon)^{-\alpha}) \\
 &= O_{\varepsilon \rightarrow 0}(f(t/\varepsilon)^{-\alpha} \varepsilon^{\beta-1}).
 \end{aligned}$$

For the second asymptotic expansion, it follows again from an integration by parts that

$$\begin{aligned}
 \int_{t_0}^{t/\varepsilon} f(u)^{-\alpha} du &= \int_{t_0}^{t/\varepsilon} f(u)^{-\alpha} u^{-\beta} u^{\beta} du \\
 &= \frac{2}{\alpha} \left[f(u)^{-\alpha} u^{\beta} \right]_{t_0}^{t/\varepsilon} - \frac{2}{\alpha} \beta \int_{t_0}^{t/\varepsilon} f(u)^{-\alpha} u^{\beta-1} du.
 \end{aligned}$$

Remarking that $f(u)^{-\alpha} u^{\beta-1} = o_{u \rightarrow +\infty}(f(u)^{-\alpha})$, since $\beta < 1$, we deduce that

$$\int_{t_0}^{t/\varepsilon} f(u)^{-\alpha} u^{\beta-1} du = o_{\varepsilon \rightarrow 0} \left(\int_{t_0}^{t/\varepsilon} f(u)^{-\alpha} du \right).$$

We obtain that

$$\int_{t_0}^{t/\varepsilon} f(u)^{-\alpha} du \underset{\varepsilon \rightarrow 0}{\sim} \frac{2}{\alpha} f\left(\frac{t}{\varepsilon}\right)^{-\alpha} \left(\frac{t}{\varepsilon}\right)^{\beta}$$

This ends the proof. □

Lemma B.0.4. *Let $(X_n)_n$ be a sequence of random variables with values in \mathbb{R}^2 , and which converges in distribution to a random variable X . We assume that the distribution of X is invariant under rotations, i.e. for any orthogonal matrix $R \in \mathcal{M}_2(\mathbb{R})$, the random variables X and RX have the same distribution. Then for all sequence $(R_n)_n$ of orthogonal matrices in $\mathcal{M}_2(\mathbb{R})$, we have*

$$R_n X_n \xrightarrow[n \rightarrow +\infty]{\Longrightarrow} X.$$

Proof. Let us denote by ϕ_Z the characteristic function of a random variable Z . Using Theorem 5.3 p. 86 in [Kal02], we know that $(\phi_{X_n})_n$ converges to ϕ_X uniformly on every compact subset of \mathbb{R}^2 . The characteristic function of the random variable $Y_n := R_n X_n$ is given by

$$\xi \mapsto \phi_{Y_n}(\xi) = \phi_{X_n}(R_n \xi).$$

Thus, by assumption, we have, for all $\xi \in \mathbb{R}^2$,

$$\phi_X(R_n \xi) = \phi_X(\xi).$$

It follows that, for any $\xi \in \mathbb{R}^2$ and $n \geq 0$,

$$|\phi_{Y_n}(\xi) - \phi(\xi)| = |\phi_{X_n}(R_n\xi) - \phi_X(R_n\xi)| \leq \sup_{z \in \mathbb{R}^2, \|z\| = \|\xi\|} |\phi_{X_n}(z) - \phi_X(z)|,$$

which converges to 0, as $n \rightarrow +\infty$. This ends the proof of the lemma. \square

Acknowledgements

The authors would like to thank Paul-Eric Chaudru de Raynal and Mihai Gradinaru for their supervision and advice. The authors would also like to thank Jürgen Angst for having suggested this subject and useful discussions.

BIBLIOGRAPHY

- [AK94] S. Albeverio and A. Klar, « Long Time Behavior of Nonlinear Stochastic Oscillators: The One-dimensional Hamiltonian Case », *in: J. Math. Phys.* 35.8 (1994), pp. 4005–4027, DOI: 10.1063/1.530839 (cit. on pp. 34, 128).
- [App09] D. Applebaum, *Levy Processes and Stochastic Calculus*, Second, Cambridge Studies in Advanced Mathematics, Cambridge University Press, 2009 (cit. on pp. 24, 32, 91–93, 138, 149, 150, 152, 153).
- [AS09] D. Applebaum and M. Siakalli, « Asymptotic Stability of Stochastic Differential Equations Driven by Lévy Noise », *in: Journal of Applied Probability* 46.4 (2009), pp. 1116–1129, DOI: 10.1239/jap/1261670692 (cit. on p. 80).
- [AW09] J. Appleby and H. Wu, « Solutions of Stochastic Differential Equations Obeying the Law of the Iterated Logarithm, with Applications to Financial Markets », *in: Electron. J. Probab.* 14 (2009), pp. 912–959, DOI: 10.1214/EJP.v14-642 (cit. on pp. 12, 81).
- [BGL14] D. Bakry, I. Gentil, and M. Ledoux, *Analysis and Geometry of Markov Diffusion Operators*, vol. 348, Grundlehren Der Mathematischen Wissenschaften, Cham: Springer International Publishing, 2014, DOI: 10.1007/978-3-319-00227-9 (cit. on pp. 30, 69).
- [Bas03] R. F. Bass, « Stochastic Differential Equations Driven by Symmetric Stable Processes », *in: Séminaire de Probabilités XXXVI*, Lecture Notes in Mathematics, Springer, 2003, pp. 302–313, DOI: 10.1007/978-3-540-36107-7_11 (cit. on pp. 8, 80).
- [Ber21] N. Berglund, *Long-Time Dynamics of Stochastic Differential Equations*, 2021, arXiv: 2106.12998 (cit. on p. 6).
- [Bét21] L. Béthencourt, *Stable Limit Theorems for Additive Functionals of One Dimensional Diffusion Processes*, 2021, arXiv: 2104.06027 (cit. on pp. 15, 46).
- [BY87] P. Biane and M. Yor, « Valeurs Principales Associées Aux Temps Locaux Browniens », *in: Bull. Sci. Math. (2)* 111.1 (1987), pp. 23–101 (cit. on p. 15).

-
- [Bil99] P. Billingsley, *Convergence of Probability Measures*, 2nd ed, Wiley Series in Probability and Statistics. Probability and Statistics Section, Wiley, 1999 (cit. on pp. 17, 29, 39, 60, 63, 67, 70, 94, 95, 100–102, 118, 138, 140, 141, 153, 155, 157).
- [Bob99] S. G. Bobkov, « Isoperimetric and Analytic Inequalities for Log-Concave Probability Measures », *in: The Annals of Probability* 27.4 (1999), pp. 1903–1921 (cit. on pp. 30, 69).
- [Bre68] L. Breiman, « A Delicate Law of the Iterated Logarithm for Non-Decreasing Stable Processes », *in: Ann. Math. Statist.* 39.6 (1968), pp. 1818–1824, DOI: 10.1214/aoms/1177698014 (cit. on p. 12).
- [CCM10] P. Cattiaux, D. Chafai, and S. Motsch, « Asymptotic Analysis and Diffusion Limit of the Persistent Turning Walker Model », *in: Asymptotic Analysis* 67.1–2 (2010), pp. 17–31, DOI: 10.3233/ASY-2009-0969, arXiv: 0811.0600 (cit. on pp. 5, 6, 16, 29, 30, 46, 69).
- [CNP19] P. Cattiaux, E. Nasreddine, and M. Puel, « Diffusion Limit for Kinetic Fokker-Planck Equation with Heavy Tails Equilibria: The Critical Case », *in: Kinetic & Related Models* 12.4 (2019), p. 727, DOI: 10.3934/krm.2019028 (cit. on pp. 14, 46).
- [CL] T. Cavallazzi and E. Luirard, *Asymptotic Behavior of a Time-Inhomogeneous Stochastic System in the Quadratic Potential* (cit. on pp. 2, 47).
- [CM20] P.-E. Chaudru De Raynal and S. Menozzi, *On Multidimensional Stable-Driven Stochastic Differential Equations with Besov Drift*, 2020, arXiv: 1907.12263 (cit. on pp. 24, 81, 90).
- [CG00] A. V. Chechkin and V. Y. Gonchar, « Linear Relaxation Processes Governed by Fractional Symmetric Kinetic Equations », *in: J. Exp. Theor. Phys.* 91.3 (2000), pp. 635–651, DOI: 10.1134/1.1320100, arXiv: cond-mat/9910091 (cit. on p. 7).
- [CSZ18] Z.-Q. Chen, R. Song, and X. Zhang, « Stochastic Flows for Lévy Processes with Hölder Drifts », *in: Revista Matemática Iberoamericana* 34 (2018), DOI: 10.4171/rmi/1042, arXiv: 1501.04758 (cit. on p. 80).

-
- [CZZ17] Z.-Q. Chen, X. Zhang, and G. Zhao, *Well-Posedness of Supercritical SDE Driven by Lévy Processes with Irregular Drifts*, 2017, arXiv: 1709.04632 (cit. on pp. 21, 80, 86).
- [CE05] A. S. Cherny and H. J. Engelbert, *Singular Stochastic Differential Equations*, Lecture Notes in Mathematics 1858, Springer, 2005 (cit. on pp. 52, 63).
- [DF13] A. Debussche and N. Fournier, « Existence of Densities for Stable-like Driven SDE's with Hölder Continuous Coefficients. », *in: Journal of Functional Analysis* 264 (2013), pp. 1757–1778, DOI: 10.1016/j.jfa.2013.01.009 (cit. on p. 154).
- [DM08] P. Degond and S. Motsch, « Large Scale Dynamics of the Persistent Turning Walker Model of Fish Behavior », *in: J Stat Phys* 131.6 (2008), pp. 989–1021, DOI: 10.1007/s10955-008-9529-8 (cit. on p. 5).
- [DS15] C.-S. Deng and R. L. Schilling, « On Shift Harnack Inequalities for Subordinate Semigroups and Moment Estimates for Lévy Processes », *in: Stochastic Processes and their Applications* 125.10 (2015), pp. 3851–3878, DOI: 10.1016/j.spa.2015.05.013 (cit. on pp. 24, 81, 91, 111, 112).
- [Dit99] P. D. Ditlevsen, « Observation of α -Stable Noise Induced Millennial Climate Changes from an Ice-Core Record », *in: Geophysical Research Letters* 26 (1999), pp. 1441–1444, DOI: 10.1029/1999GL900252 (cit. on pp. 7, 80, 81).
- [Don18] Y. Dong, « Jump Stochastic Differential Equations with Non-Lipschitz and Superlinearly Growing Coefficients », *in: Stochastics* 90.5 (2018), pp. 782–806, DOI: 10.1080/17442508.2017.1421195 (cit. on p. 80).
- [EG15] R. Eon and M. Gradinaru, « Gaussian Asymptotics for a Non-Linear Langevin Type Equation Driven by an α -Stable Lévy Noise », *in: Electron. J. Probab.* 20 (2015), DOI: 10.1214/EJP.v20-4068 (cit. on pp. 16, 17, 46).
- [Fed+17] E. Fedrizzi et al., « Regularity of Stochastic Kinetic Equations », *in: Electron. J. Probab.* 22 (2017), DOI: 10.1214/17-EJP65 (cit. on pp. 34, 129, 131).
- [FT21] N. Fournier and C. Tardif, « One Dimensional Critical Kinetic Fokker-Planck Equations, Bessel and Stable Processes », *in: Communications in Mathematical Physics* 381 (2021), DOI: 10.1007/s00220-020-03903-0, arXiv: 1805.09728 (cit. on pp. 14–16, 46, 62, 81, 84, 128).

-
- [GL21a] M. Gradinaru and E. Luirard, *Asymptotic Behavior for a Time-Inhomogeneous Kolmogorov Type Diffusion*, 2021, arXiv: 2004.11576 (cit. on pp. 2, 45, 80, 82, 103, 118, 128–131, 135, 136, 148, 168).
- [GL21b] M. Gradinaru and E. Luirard, *Asymptotic Behavior for a Time-Inhomogeneous Stochastic Differential Equation Driven by an α -Stable Lévy Process*, 2021, arXiv: 2112.07287 (cit. on pp. 2, 47, 79, 128–131, 148, 149, 154).
- [GO13] M. Gradinaru and Y. Offret, « Existence and Asymptotic Behaviour of Some Time-Inhomogeneous Diffusions », in: *Ann. Inst. H. Poincaré Probab. Statist.* 49.1 (2013), pp. 182–207, DOI: 10.1214/11-AIHP469 (cit. on pp. 12, 13, 21, 28, 30, 33, 46–50, 52–54, 56, 64, 71, 74, 75, 81, 84, 96, 128, 129).
- [HMC18] I. Honore, S. Menozzi, and P.-E. Chaudru de Raynal, *Strong Regularization by Brownian Noise Propagating through a Weak Hörmander Structure*, 2018, arXiv: 1810.12225 [math] (cit. on pp. 34, 38, 129, 135).
- [IW81] N. Ikeda and S. Watanabe, *Stochastic Differential Equations and Diffusion Processes*, North-Holland Mathematical Library 24, Elsevier, Academic Press, 1981 (cit. on pp. 21, 86, 148).
- [JS03] J. Jacod and A. N. Shiryaev, *The General Theory of Stochastic Processes, Semimartingales and Stochastic Integrals*, Grundlehren Der Mathematischen Wissenschaften, Berlin, Heidelberg: Springer, 2003, DOI: 10.1007/978-3-662-05265-5_1 (cit. on pp. 11, 16–18, 107).
- [Kal02] O. Kallenberg, *Foundations of Modern Probability*, Probability and Its Applications, New York, NY: Springer New York, 2002, DOI: 10.1007/978-1-4757-4015-8 (cit. on pp. 13, 29, 39, 64, 65, 74, 135, 139, 143, 170).
- [KS98] I. Karatzas and S. Shreve, *Brownian Motion and Stochastic Calculus*, Second, Graduate Texts in Mathematics, New York: Springer-Verlag, 1998, DOI: 10.1007/978-1-4612-0949-2 (cit. on pp. 13, 52, 63, 66, 120, 121, 137).
- [Kol29] A. Kolmogoroff, « Über das Gesetz des iterierten Logarithmus », in: *Mathematische Annalen* 101 (1929), pp. 126–135 (cit. on p. 10).
- [Küh] F. Kühn, *Lévy Matters VI - Lévy-Type Processes: Moments, Construction and Heat Kernel Estimates*, Springer (cit. on p. 41).

-
- [Küh17] F. Kühn, « Existence and Estimates of Moments for Lévy-type Processes », *in: Stochastic Processes and their Applications* 127.3 (2017), pp. 1018–1041, DOI: 10.1016/j.spa.2016.07.008 (cit. on p. 81).
- [Kul09] A. M. Kulik, « Exponential Ergodicity of the Solutions to SDE's with a Jump Noise », *in: Stochastic Processes and their Applications* 119.2 (2009), pp. 602–632, DOI: 10.1016/j.spa.2008.02.006 (cit. on p. 98).
- [Kun04] H. Kunita, « Stochastic Differential Equations Based on Lévy Processes and Stochastic Flows of Diffeomorphisms », *in: Real and Stochastic Analysis: New Perspectives*, Trends in Mathematics, Boston, MA: Birkhäuser, 2004, pp. 305–373, DOI: 10.1007/978-1-4612-2054-1_6 (cit. on p. 116).
- [Kur07] V. P. Kurenok, « Stochastic Equations with Time-Dependent Drift Driven by Levy Processes », *in: J Theor Probab* 20.4 (2007), pp. 859–869, DOI: 10.1007/s10959-007-0086-x (cit. on pp. 80, 117).
- [Lan08] P. Langevin, « Sur la théorie du mouvement brownien », *in: Comptes-Rendus de l'Académie des Sciences* 146 (1908), pp. 530–532 (cit. on p. 4).
- [LP19] G. Lebeau and M. Puel, « Diffusion Approximation for Fokker Planck with Heavy Tail Equilibria: A Spectral Method in Dimension 1 », *in: Commun. Math. Phys.* 366.2 (2019), pp. 709–735, DOI: 10.1007/s00220-019-03315-9 (cit. on p. 14).
- [Lui22a] E. Luirard, *Simulation for "Asymptotic Behavior for a Time-Inhomogeneous Kolmogorov Type Diffusion"*, 2022, DOI: 10.5281/zenodo.6472854 (cit. on pp. 21, 22, 27).
- [Lui22b] E. Luirard, *Simulation for "Asymptotic Behavior for a Time-Inhomogeneous Stochastic Differential Equation Driven by an α -Stable Lévy Process"*, 2022, DOI: 10.5281/zenodo.6472878 (cit. on pp. 21, 23).
- [Lui22c] E. Luirard, *Simulation for "Asymptotic Behavior of a Time-Inhomogeneous Stochastic System in the Quadratic Potential "*, 2022, DOI: 10.5281/zenodo.6472888 (cit. on pp. 36, 37).
- [LP06] H. Luschgy and G. Pagès, « Moment Estimates for Lévy Processes », *in: Electronic Communications in Probability* 13 (2006), DOI: 10.1214/ECP.v13-1397, arXiv: math/0607282 (cit. on pp. 24, 81).

-
- [MM21] L. Marino and S. Menozzi, *Weak Well-Posedness for Degenerate SDEs Driven by Lévy Processes*, 2021, arXiv: 2107.04325 (cit. on pp. 34, 38, 129, 148, 149).
- [NP15] E. Nasreddine and M. Puel, « Diffusion Limit of Fokker-Planck Equation with Heavy Tail Equilibria », *in: ESAIM: M2AN* 49.1 (2015), pp. 1–17, DOI: 10.1051/m2an/2014020 (cit. on p. 14).
- [Off14] Y. Offret, « Invariant Distributions and Scaling Limits for Some Diffusions in Time-Varying Random Environments », *in: Probab. Theory Relat. Fields* 158.1 (2014), pp. 1–38, DOI: 10.1007/s00440-012-0475-7 (cit. on pp. 10, 21, 33, 42).
- [Pil13] A. Pilipenko, « On Existence and Properties of Strong Solutions of One-Dimensional Stochastic Equations with an Additive Noise », *in: Theory of Stochastic Processes* 18.2 (2013), arXiv: 1306.0212 (cit. on p. 80).
- [Pri+12] E. Priola et al., « Exponential Ergodicity and Regularity for Equations with Lévy Noise », *in: Stochastic Processes and their Applications* 122.1 (2012), pp. 106–133, DOI: 10.1016/j.spa.2011.10.003 (cit. on p. 80).
- [Pro05] P. E. Protter, *Stochastic Integration and Differential Equations*, Second, Stochastic Modelling and Applied Probability, Berlin Heidelberg: Springer-Verlag, 2005, DOI: 10.1007/978-3-662-10061-5 (cit. on pp. 88, 91, 117, 149, 152).
- [Rek20] J. Reker, *Short-Time Behavior of Solutions to Lévy-driven SDEs*, 2020, arXiv: 2008.00526 (cit. on p. 80).
- [RY05] D. Revuz and M. Yor, *Continuous Martingales and Brownian Motion*, Berlin; Heidelberg: Springer, 2005 (cit. on pp. 13, 53).
- [RT11] M. Rosenbaum and P. Tankov, « Asymptotic Results for Time-Changed Lévy Processes Sampled at Hitting Times », *in: Stochastic Processes and their Applications* 121.7 (2011), pp. 1607–1632, DOI: 10.1016/j.spa.2011.03.013 (cit. on pp. 31, 106).
- [SG03] G. Samorodnitsky and M. Grigoriu, « Tails of Solutions of Certain Nonlinear Stochastic Differential Equations Driven by Heavy Tailed Lévy Motions », *in: Stochastic Processes and their Applications* 105.1 (2003), pp. 69–97, DOI: 10.1016/S0304-4149(03)00002-4 (cit. on p. 135).
- [ST94] G. Samorodnitsky and M. S. Taqqu, *Stable Non-Gaussian Random Processes: Stochastic Models with Infinite Variance*, CRC Press, 1994 (cit. on pp. 28, 96).

-
- [Sat99] K.-i. Sato, *Lévy Processes and Infinitely Divisible Distributions*, Cambridge Studies in Advanced Mathematics 68, Cambridge, U.K. ; New York: Cambridge University Press, 1999 (cit. on pp. 104, 107, 145, 156, 159).
- [Sit05] R. Situ, *Theory of Stochastic Differential Equations with Jumps and Applications: Mathematical and Analytical Techniques with Applications to Engineering*, Mathematical and Analytical Techniques with Applications to Engineering, Springer US, 2005, DOI: 10.1007/b106901 (cit. on pp. 21, 86, 148).
- [SV06] D. W. Stroock and S. R. S. Varadhan, *Multidimensional Diffusion Processes*, Classics in Mathematics, Berlin Heidelberg: Springer-Verlag, 2006, DOI: 10.1007/3-540-28999-2 (cit. on pp. 53–56).
- [WZ15] F.-Y. Wang and X. Zhang, *Degenerate SDEs in Hilbert Spaces with Rough Drifts*, 2015, arXiv: 1501.04150 (cit. on pp. 34, 129).
- [Wat44] G. N. Watson, *A Treatise on the Theory of Bessel Functions*, Second, Cambridge University Press, 1944 (cit. on p. 167).
- [Wu01] L. Wu, « Large and Moderate Deviations and Exponential Convergence for Stochastic Damping Hamiltonian Systems », *in: Stochastic Processes and their Applications* 91.2 (2001), pp. 205–238, DOI: 10.1016/S0304-4149(00)00061-2 (cit. on pp. 34, 128).
- [Yam73] T. Yamada, « On a Comparison Theorem for Solutions of Stochastic Differential Equations and Its Applications », *in: Journal of Mathematics of Kyoto University* 13.3 (1973), pp. 497–512, DOI: 10.1215/kjm/1250523321 (cit. on pp. 74, 75).
- [Yor08] M. Yor, *Du pli cacheté de Döblin aux équations différentielles stochastiques : conférence du mercredi 23 janvier 2008*, 2008 (cit. on p. 4).
- [Zha13] X. Zhang, « Stochastic Differential Equations with Sobolev Drifts and Driven by α -Stable Processes », *in: Ann. Inst. H. Poincaré Probab. Statist.* 49.4 (2013), pp. 1057–1079, DOI: 10.1214/12-AIHP476 (cit. on p. 80).
- [Zha14] X. Zhang, « Densities for SDEs Driven by Degenerate α -Stable Processes », *in: The Annals of Probability* 42.5 (2014), pp. 1885–1910, DOI: 10.1214/13-AOP900 (cit. on pp. 34, 129).
- [Zha16] X. Zhang, *Stochastic Hamiltonian Flows with Singular Coefficients*, 2016, arXiv: 1606.04360 (cit. on pp. 34, 129).

Titre : Loi limite de modèles cinétiques inhomogènes en temps

Mot clés : modèle stochastique cinétique ; équation différentielle stochastique inhomogène en temps ; processus de Lévy ; loi limite ; transformation d'échelles

Résumé : Nous étudions, dans cette thèse, le comportement asymptotique de solutions de systèmes cinétiques inhomogènes, dirigés par un processus de Lévy L . Plus précisément, on s'intéresse à la dynamique d'une particule, évoluant dans un potentiel \mathcal{U} , et soumise à la fois à une force de frottement F et à une force extérieure aléatoire L . La force F est attractive et vérifie des propriétés d'invariance d'échelle. Elle est altérée par la présence d'un facteur $t^{-\beta}$.

La première partie de ce manuscrit correspond à l'étude du système en l'absence de potentiel confinant, tandis que la seconde s'intéresse à la présence d'un potentiel quadratique.

L'enjeu est de comprendre comment interagissent les différentes forces afin de montrer que le processus vitesse-position, correctement renormalisé, admet une limite en loi explicite.

Title: Asymptotic distribution of a time-inhomogeneous kinetic model

Keywords: kinetic stochastic equation; time-inhomogeneous stochastic differential equation; Lévy process; asymptotic distribution; scaling transformation

Abstract: We study the asymptotic behavior of some stochastic kinetic inhomogeneous models driven by a Lévy process L . It describes the dynamics of a particle, evolving in a potential \mathcal{U} , subject to a random force L and a frictional force F . The drag force is supposed to satisfy some scaling properties and to be weakened along the time. In the first part, we study the free potential sys-

tem. Alternatively, in the second part, the particle is supposed to evolve in a quadratic potential.

The issue of this thesis is to understand the balance between the drag force and the random force, with the aim of highlighting an explicit distributional limit of the couple velocity-position.