



**HAL**  
open science

# Two results on the classification of singular spaces with semi-negative canonical class

Zhining Liu

► **To cite this version:**

Zhining Liu. Two results on the classification of singular spaces with semi-negative canonical class. Algebraic Geometry [math.AG]. Université de Rennes, 2022. English. NNT : 2022REN1S048 . tel-03908080

**HAL Id: tel-03908080**

**<https://theses.hal.science/tel-03908080>**

Submitted on 20 Dec 2022

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# THESE DE DOCTORAT DE

L'UNIVERSITE DE RENNES 1

ECOLE DOCTORALE N° 601

*Mathématiques et Sciences et Technologies  
de l'Information et de la Communication*

Spécialité : *Mathématiques et leurs interactions*

Par

**Zhining LIU**

## **Deux résultats sur la classification des variétés singulières à classe canonique semi-négative**

Thèse présentée et soutenue à Rennes, le 13 juillet 2022

Unité de recherche : IRMAR – UMR CNRS 6625

### **Rapporteurs avant soutenance :**

Enrica Floris    Maître de conférences, Université de Poitiers  
Mihai Păun    Professeur, Universität Bayreuth

### **Composition du Jury :**

Président :	Christophe Mourougane	Professeur, Université de Rennes 1
Examineurs :	Junyan Cao	Professeur, Université Côte d'Azur
	Enrica Floris	Maître de conférences, Université de Poitiers
	Christophe Mourougane	Professeur, Université de Rennes 1
	Mihai Păun	Professeur, Universität Bayreuth
Dir. de thèse :	Benoît Claudon	Professeur, Université de Rennes 1
Dir. de thèse :	Andreas Höring	Professeur, Université Côte d'Azur



**Two results on the classification of singular  
spaces with semi-negative canonical class**

Zhining Liu



## Remerciements

J'ai eu la chance d'avoir deux excellents mathématiciens Benoît Claudon et Andreas H\"oring comme directeurs de thèse, qui m'ont amené à une aventure bizzare dans la géométrie complexe. Je tiens d'abord à leur exprimer ma profonde gratitude. Pendant les trois plus une années, ils ont répondu à mes nombreuses questions et donné de nombreuses suggestions inspirantes. Au cours de leurs discussions avec moi, ils m'ont montré comment les "working mathematicians" font de la recherche, ce qui est inestimable pour moi. Ils m'ont donné la liberté de m'aventurer dans d'autres mathématiques et m'ont ramené à la géométrie complexe lorsque je dérivais trop loin. Je suis reconnaissant à la gentillesse de Benoît d'avoir toléré les ennuis que j'ai causés. Je suis également profondément impressionné par la façon dont Andreas organise sa vie quotidienne de manière ordonnée.

Je voudrais ensuite remercier mes rapporteurs de thèse Enrica Floris et Mihai Păun. Leurs remarques m'ont permis d'améliorer la première version de ma thèse. Je remercie également Junyan Cao et Chirstophe Mourougane d'avoir accepté de faire partie de mon jury. J'ai également eu des discussions très utiles avec eux.

Je remercie également les équipes de l'IRMAR et du LJAD pour les bons environnements de recherche dont j'ai pu profiter .

J'ai rencontré de nombreux confrères doctorants/post-doctorants et j'ai eu des échanges intéressants avec eux, parmi lesquels je nomme: Alice, Antoine, Arame, Cécile, Severio, Titouan, Victor, Virgie, Zhixin . . . . . Je tiens également à remercier mes ami exploreurs dans le la monde de géométrie algébrique depuis les années de Master: Haowen et Jieao.

Ma thèse est financée par l'allocation spécifique de l'Ens. Je remercie l'Ens d'avoir financé mes six années en France, qui sont désormais la meilleure période de ma vie.

Enfin, je remercie mes parents pour leur soutien inconditionnel.



## Abstract

ABSTRACT. The subject of this thesis is to study the classification problem for singular spaces under two different assumptions on the positivity of the anti-canonical class of the spaces and their singularities in these two different setups . We will apply quite different methods for these two assumptions.

In the first part, we study the classification problem for polarized varieties. For the positivity of the anti-canonical classes, we assume that the varieties have high nefvalue, or in other words, their anti-canonical classes are quite positive. We give a complete list of isomorphism classes for normal polarized varieties with high nefvalue. This generalizes classical work on the smooth case by Fujita, Beltrametti and Sommese. As a consequence we obtain that polarized varieties with slc singularities and high nefvalue, are birationally equivalent to projective bundles over nodal curves.

In the second part, we consider a specific class of singular spaces, namely the orbifolds. An orbifold has quotient singularities. Hence we have milder singularities in this context compared to those considered in first part. We also assume that these orbifolds are compact Kähler with nef anti-canonical classes in the orbifold sense. We will study the topology of these orbifolds by characterizing their orbifold fundamental groups. In this part, we will fully exploit the orbifold assumption by applying results from differential geometry and metric geometry on orbifolds. We will show that a compact Kähler orbifold with nef anti-canonical class has virtually nilpotent orbifold fundamental group.

**Keywords:** polarized varieties, nefvalue, slc singularities, birational geometry, orbifolds, Margulis lemma, fundamental groups



RÉSUMÉ. Le sujet de cette thèse est d'étudier le problème de classification des espaces singuliers sous deux hypothèses différentes sur la positivité de la classe anti-canonique des espaces et de leurs singularités dans ces deux conditions différentes. Nous appliquerons des méthodes assez différentes dans ces deux contextes.

Dans la première partie, nous étudions un problème de classification des variétés polarisées. Pour la positivité des classes anti-canoniques, nous supposons que les variétés ont une *nefvalue* élevée, ou en d'autres termes, leurs classes anti-canoniques sont assez positives. Nous donnons une liste complète des classes d'isomorphisme des variétés polarisées normales avec une *nefvalue* élevée. Cela généralise le travail classique sur le cas lisse de Fujita, Beltramitti et Sommese. En conséquence, nous obtenons que les variétés polarisées avec des singularités slc et une *nefvalue* élevée sont birationnellement équivalentes à des fibrés projectifs sur des courbes nodales.

Dans la deuxième partie, nous considérons une classe spécifique d'espaces singuliers, à savoir les orbifolles. Une orbifolde a des singularités quotients. Par conséquence, nous avons des singularités mieux contrôlées dans ce contexte par rapport à celles considérées dans la première partie. Nous supposons également que ces orbifolles sont kähleriennes compactes avec des classes anti-canoniques nef au sens des orbifolles. Nous étudierons la topologie de ces orbifolles à travers leurs groupes fondamentaux orbifolles. Dans cette partie, nous exploiterons pleinement l'hypothèse orbifolde en appliquant des résultats de géométrie différentielle et de la géométrie métrique sur orbifolds. Nous montrerons qu'une orbifolde kählerienne compacte dont la classe anti-canonique est nef a un groupe fondamental orbifolde virtuellement nilpotent.

**Mots-clés:** variétés polarisées, *nefvalue*, singularités slc, géométrie birationnelle, orbifolles, lemme de Margulis, groupes fondamentaux

## Contents

Remerciements	3
Abstract	5
Résumé étendu	9
Partie 1: Classification des variété à singularités slc de <i>valeur nef</i> élevées	9
Partie 2: Groupes fondamentaux des orbifolds kählériennes à fibrés anticanoniques nef	11
Summary	15
Part 1: Classification of slc varieties with high nef value	15
Part 2: Fundamental group of Kähler orbifolds with nef anti-canonical bundle	17
<b>Part 1. Classification of slc varieties with high nef value</b>	<b>21</b>
Chapter 1. Introduction	23
Chapter 2. Preliminaries	29
2.1. Conventions	29
2.2. Singularities of Pairs, MMP	31
2.3. Minimal model program for pairs	32
2.4. Some results on modifications	37
2.5. Slc singularities	38
2.6. Miscellaneous constructions	41
Chapter 3. Polarized varieties with high nef value	43
3.1. Canonical polarized varieties	43
3.2. Normal polarized varieties	52
3.3. Semi-log canonical polarized varieties	64
<b>Part 2. Fundamental group of Kähler orbifolds with nef anti-canonical bundle</b>	<b>67</b>
Chapter 4. Introduction	69
Chapter 5. Preliminaries	73
5.1. Conventions	73
5.2. Metric spaces	74
5.3. Differential geometry	75
5.4. Kähler geometry	76

5.5. Classical Orbifolds	77
5.6. Complex Orbifolds	82
5.7. Orbi-bundles	84
Chapter 6. Riemannian orbifolds	91
6.1. Differential calculus on orbifold	91
6.2. Metric structures on orbifolds	92
6.3. Volume comparisons	96
Chapter 7. Orbifold coverings and generalized Margulis lemma	99
7.1. Metric geometry of orbifold coverings	99
7.2. Dirichlet domains and generalized Margulis lemma	100
Chapter 8. Main theorem	103
Chapter 9. Projective case	111
<b>Appendice</b>	115
Appendix A. Groupoids	117
1.1. Orbifold Groupoids	117
1.2. $\mathcal{G}$ -bundles	120
Bibliography	123

## Résumé étendu

### Partie 1: Classification des variété à singularités slc de valeur nef élevées

Quand on étudie une variété polarisée  $(X, L)$  avec  $X$  ayant singularité klt, il se trouve que le *morphisme de valeur nef* de  $(X, L)$  est utile. Supposons que  $K_X$  n'est pas nef et posons  $\tau(L) := \inf\{t \in \mathbb{Q} : K_X + tL \text{ est nef}\}$ . Par le théorème sur l'absence de point base de Kawamata, on a que  $K_X + \tau(L)L$  est sans point base. Pour un entier  $m$  suffisamment divisible, on sait que le système linéaire  $|m(K_X + \tau(L)L)|$  définit un morphisme  $\phi_{|m(K_X + \tau(L)L)|} : X \rightarrow \mathbb{P}_{\mathbb{C}}$  de  $X$  vers un espace projectif dont la factorisation de Stein  $\phi : X \rightarrow Y$  ne dépend pas de  $m$ . On a  $L = \phi^*(\mathcal{O}_Y(1))$ . On peut comprendre  $(X, L)$  en étudiant les propriétés des fibres générales de  $\phi$ .

Par exemple, Andreatta a prouvé le résultat suivant dans [And95].

**THEOREM 0.1** ([And95, Theorem 2.1.]). *Soit  $X$  une variété projective à singularités klt et soit  $L$  un fibré en droites sur  $X$ . Soit  $\phi : X \rightarrow Z$  un morphisme surjectif à fibres connexes entre variétés normales. Supposons que  $L$  est  $\phi$ -ample et  $K_X + \tau L \sim_{\mathbb{Q}, \phi} 0$  pour certain  $\tau \in \mathbb{Q}^+$ . Soient  $F_1 = \phi^{-1}(z)$  une fibre non-triviale,  $F \subset F_1$  une des composantes irréductibles, et  $F'$  la normalisation de  $F$ . On note par  $L'$  l'image réciproque de  $L$  sur  $F'$ . Soient  $\lfloor \tau \rfloor$  la partie entière de  $\tau$  et  $\tau' := \lceil \tau \rceil = -\lfloor -\tau \rfloor$ . Alors, on a alors:*

- (I,1)  $\dim(F) \geq \tau - 1$ ;
- (I,2) si  $\dim(F) < \tau$ , alors  $F \cong \mathbb{P}^{\tau'-1}$  et  $L|_F = \mathcal{O}_{\mathbb{P}^{\tau'-1}}(1)$ ;
- (I,3) si  $\dim(F) < \tau + 1$ , alors  $\Delta(F', L') = 0$ .

Si de plus on a  $\dim(F) > \dim(X) - \dim(Z)$ , alors:

- (II,1)  $\dim(F) \geq \tau$ ;
- (II,2) si  $\dim(F) = \tau$ , alors  $F \cong \mathbb{P}^{\tau}$  et  $L|_F = \mathcal{O}_{\mathbb{P}^{\tau}}(1)$ ;
- (II,3) si  $\dim(F) < \tau + 1$ , alors  $\Delta(F', L') = 0$ ,

Si toutes les composantes de la fibre  $F_1$  satisfont  $\dim(F) < \tau(L)$ , dans cas (I.2) ou  $\dim(F) \leq \tau(L)$  dans cas (II.3), alors  $F_1$  est en fait irréductible.

On voit de ce théorème que  $\tau(L) > \dim(X) - 1$  est une condition très restrictive sur  $(X, L)$ . En fait, quand  $X$  a singularités terminales, la classification pour  $(X, L)$  est complète pour  $L$  des valeur nef supérieure à  $n - 1$ .

**Proposition 0.2** ([BS11, Proposition 7.2.2.]). *Soit  $(X, L)$  une variété polarisée. Supposons  $X$  à singularités terminales. Soit  $\phi : X \rightarrow Z$  le morphisme valeur nef. Alors on est dans l'un des cas suivants:*

- (1)  $\tau = n + 1$  et  $(X, L) \cong (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$ ;
- (2)  $\tau = n$  et  $(X, L) \cong (Q, \mathcal{O}_Q(1))$  où  $Q \subset \mathbb{P}^{n+1}$  est une hyperquadrique dans  $\mathbb{P}^{n+1}$  si  $K_X + nL \equiv_{\text{num}} \mathcal{O}_X$ ;

- (3)  $\tau = n$  et  $Z$  est une courbe lisse et  $(X, L) \cong (\mathbb{P}(E), \mathcal{O}_{\mathbb{P}(E)}(1))$  où  $E$  est un fibré vectoriel sur  $Z$  et  $\phi : X \rightarrow Z$  est le morphisme structurel;
- (4)  $\tau < n$  et  $K_X + nL$  est nef et big.

**Proposition 0.3** ([BS11, Proposition 7.2.4.]). *Soit  $(X, L)$  une variété polarisée. Supposons  $X$  à singularités terminales  $\mathbb{Q}$ -factorielles et  $n = \dim(X) \geq 2$ . Supposons que  $K_X + nL$  est ample et  $\tau > n - 1$ . Alors nous avons  $\tau = n - \frac{1}{2}$  et  $(X, L) = C_n(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$  est un cône généralisé sur  $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$ .*

Dans la partie 1, nous explorerons d'abord le problème de classification pour les variétés polarisées normales  $(X, L)$  avec nefvalue élevée en autorisant des singularités plus "sauvages" que terminales pour  $X$ . Nous donnerons la classification suivante.

**THEOREM 0.4** (=Theorem 3.14). *Soit  $(X, L)$  une variété polarisée de dimension  $n$ . Supposons que  $K_X$  est  $\mathbb{Q}$ -Cartier et  $K_X + (n - 1)L \notin \text{Pseff}(X)$ . Alors  $(X, L)$  est l'une des paires suivantes:*

- (1)  $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$ ;
- (2.i)  $(\mathbb{P}(\mathcal{V}), \mathcal{O}_{\mathbb{P}(\mathcal{V})}(1))$ , où  $\mathcal{V}$  est un fibré vectoriel ample de rang  $n$  sur une courbe lisse  $C$ ;
- (2.ii)  $C_n(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(a))$ , un cône généralisé avec  $a \geq 3$ ;
- (3)  $(Q, \mathcal{O}_{\mathbb{P}^{n+1}}(1))$ , où  $Q \subset \mathbb{Q}^{n+1}$  est une hyperquadrique;
- (4)  $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$ ;
- (5) un cône généralisé  $C_n(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$  sur  $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$ .

La stratégie de cette classification consiste tout d'abord à établir une classification pour  $(X', L')$  quasi-polarisée avec  $X'$  à singularités canoniques et  $K_{X'} + (\dim(X') - 1)L' \notin \text{Pseff}(X')$ . Ceci occupe la Section 3.1. Le point clé est d'utiliser le programme de modèles minimaux (MMP) pour réduire le problème à la classification de  $(X'', L'')$  avec  $\tau(L'') > \dim(X'') - 1$ . Nous pouvons ensuite appliquer une modification canonique  $\mu : X^{\text{can}} \rightarrow X$  et réduire le problème à la classification de la variété quasi-polarisée  $(X^{\text{can}}, \mu^*(L))$ .

Avec la méthode similaire, nous établissons aussi un résultat de la classification pour une paire log canonique  $(X, \Delta)$  avec  $(K_X + \Delta) + (\dim(X) - 1)L \notin \text{Pseff}(X)$ .

**Proposition 0.5** (=Corollary 3.19). *Soit  $(X, \Delta)$  une paire log canonique avec  $\Delta \neq 0$  un diviseur réduit. Supposons que  $L$  est un fibré en droites ample sur  $X$  et  $(K_X + \Delta) + (n - 1)L \notin \text{Pseff}(X)$ , où  $n = \dim(X)$ . Alors nous avons l'un des cas suivants:*

- (1)  $(X, L) \cong (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$ ,  $\Delta \equiv_{\text{num}} H$  est un diviseur irréductible où  $H$  est un hyperplan de  $\mathbb{P}^n$ ;
- (2.i) il existe un  $(\mathbb{P}^{n-1}, \mathcal{O}_{\mathbb{P}^{n-1}}(1))$ -fibré  $(\mathbb{P}(E), \mathcal{O}_{\mathbb{P}(E)}(1))$  sur une courbe lisse  $C$ , et un morphisme birationnelle  $\mu : \mathbb{P}(E) \rightarrow X$  tel que  $\mu^*(L) \cong \mathcal{O}_{\mathbb{P}(E)}(1)$  et  $\Delta = \sum F_i$  est une somme finie où  $F_i \cong \mathbb{P}^{n-1}$  sont les images de fibres générales distinctes de  $\pi$  par  $\mu$ ;
- (2.ii)  $(X, L) = (\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(1)), \mathcal{O}_{\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(1))}(1))$  avec  $1 < a$  et  $\Delta = D$  est irréductible, où  $D$  est l'unique section de  $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(1)) \rightarrow \mathbb{P}^1$  tel que  $D \equiv_{\text{num}} \mathcal{O}_{\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(1))}(1) - af$ , où  $f$  est une fibre générale;

- (3.i)  $(X, L) \cong (Q, \mathcal{O}_{\mathbb{P}^{n+1}}(1))$ , où  $Q \subset \mathbb{P}^{n+1}$  est une hyperquadrique de rang 3, le diviseur  $\Delta$  est un hyperplan dans  $Q$  et  $[\Delta] = \frac{1}{2}[H \cap Q]$  où  $H$  est un hyperplan dans  $\mathbb{P}^{n+1}$ ;
- (3.ii)  $(X, L) \cong (Q, \mathcal{O}_{\mathbb{P}^{n+1}}(1))$ , où  $Q \subset \mathbb{P}^{n+1}$  est une hyperquadrique de rang 4. Si nous écrivons  $Q = \text{Proj} \left( \frac{\mathbb{C}[x_0, \dots, x_{n+1}]}{(x_0x_1 - x_2x_3)} \right)$ , alors  $\Delta = D$  est irréductible et  $D$  est le cône de sommet  $\mathbb{P}^{n-3}$  sur  $\mathbb{P}^1 \times \text{pt}$  ou  $\text{pt} \times \mathbb{P}^1$ . En particulier,  $D \cong \mathbb{P}^{n-1}$ ;

Enfin, nous affranchissons de la condition de normalité sur les variétés. Pour une variété non-normale, nous pouvons toujours considérer sa normalisation  $\nu : \bar{X} \rightarrow X$ . Quand  $X$  a des singularités slc, nous avons que  $(\bar{X}, \bar{D})$  est log canonique où  $\bar{D}$  est le diviseur conducteur. Nous donnons la classification suivante.

**THEOREM 0.6 (=Proposition 3.20).** *Soient  $X$  une variété non-normale projective de dimension  $n$  à singularités slc et  $L$  un fibré en droites ample sur  $X$ . Supposons que  $K_X + (n-1)L \notin \text{Pseff}(X)$ . Soient  $\pi : \bar{X} \rightarrow X$  la normalisation de  $X$  et  $D \subset X$ ,  $\bar{D} \subset \bar{X}$  les conducteurs. Alors nous avons:*

*Il existe une courbe nodale  $C'$ , un fibré vectoriel  $E'$  de rang  $n$ , des fibres distinctes  $F_1, F_2, \dots, F_m$  de  $\mathbb{P}(E')$  et un morphisme birationnelle  $\mu : \mathbb{P}(E') \rightarrow X$  tel que  $\mu^*(L) = \mathcal{O}_{\mathbb{P}(E')}(1)$  et  $D = \sum_{1 \leq i \leq m} \mu(F_i)$ .*

## Partie 2: Groupes fondamentaux des orbifolds kählériennes à fibrés anticanoniques nef

Etant donné une variété compacte kählérienne  $X$ , le fameux théorème de Calabi-Yau nous dit que nous pouvons trouver une forme de Kähler  $\omega$  sur  $X$  tel que la première classe de Chern  $c_1(X) = c_1(-K_X)$  est représentée par  $\frac{1}{2\pi}\text{Ricci}_\omega$ . Donc la positivité du fibré anti-canonique de  $X$  se traduit en certaines positivités de la courbure de Ricci de  $\omega$ . Lors de l'étude du groupe fondamental de  $X$  avec certaines conditions de positivités sur  $-K_X$ , nous pouvons nous appuyer sur des résultats sur de géométrie différentielle pour les variétés riemanniennes avec certaines conditions de positivités sur leurs courbures de Ricci.

Nous donnons une version reformulée d'un théorème de Kobayashi.

**THEOREM 0.7 ([Kob61, Theorem A]).** *Une variété compacte de Fano est simplement connexe.*

Avec l'aide du théorème de Calabi-Yau, le théorème de Kobayashi peut être prouvé en appliquant le théorème de Myers (cf. [GHL04, 3.85]).

Quand  $-K_X$  est nef, nous pouvons, avec l'aide du théorème de Aubin-Yau (cf. **Theorem 5.15**), montrer que pour chaque  $\epsilon > 0$ , il existe une forme de Kähler  $\omega_\epsilon$  tel que

$$\text{Ricci}_{\omega_\epsilon} \geq -\epsilon\omega_\epsilon.$$

En multipliant  $\omega_\epsilon$  par un scalaire  $0 < \lambda \ll 1$  et posant  $\omega'_\epsilon := \lambda\omega_\epsilon$ , nous avons  $\text{Ricci}_{\omega'_\epsilon} \geq -(\dim(X) - 1)\omega'_\epsilon$ . Dans [CC96], Cheeger et Colding ont prouvé un résultat profond

**Lemme 0.8** (Lemme de Margulis, version géométrique, [CC96, Theorem 8.7.]). *Soit  $n > 0$  un entier naturel. Il existe un constante universel  $C = C(n)$  dépendant seulement de  $n$  tel que:*

*Pour chaque variété compacte  $M$  de dimension  $n$  et  $\text{Ric}_g \geq -(n-1)g$ , on a que le morphisme induit par l'inclusion*

$$\pi_1(B_g(p, r), p) \rightarrow \pi_1(M, p)$$

*a une image virtuellement nilpotente pour tout  $r < C(n)$ .*

En utilisant le théorème de Aubin-Yau pour construire une métrique spécifique  $\omega$  sur  $X$ , tel que

- (1)  $\text{Ricci}_\omega \geq -(2 \dim_{\mathbb{C}}(X) - 1)\omega$  et
- (2) il existe  $r_0 < C(2 \dim_{\mathbb{C}}(X))$  tel que  $B_\omega(p, r_0) = X$ .

et appliquant le lemme au-dessus directement, Mihai Păun a prouvé le résultat suivant

**THEOREM 0.9** ([Pău97, Theorem 1]). *Soit  $(X, \omega)$  une variété compacte kählérienne dont le fibré anti-canonique  $-K_X$  est nef. Alors son groupe fondamental  $\pi_1(X)$  est virtuellement nilpotent.*

Dans la partie 2, le but principal est de généraliser le résultat de Păun au cadre orbifoldes. Supposons que nous avons une version pour orbifoldes du résultat de Cheeger et Colding. Nous pouvons ensuite adapter la preuve de Păun facilement, comme la géométrie différentielle des orbifoldes est bien établie. Équipped l'espace sous-jacent  $X = |\mathcal{X}|$  d'une orbifolde riemannienne  $(\mathcal{X}, g)$  avec une distance naturelle  $d$  (cf. Section 6.2), et appliquons une version algébrique du lemme de Margulis dans [BGT12]. Nous prouvons d'abord un lemme de Margulis orbifoldes.

**Lemme 0.10** (=Lemma 7.12). *Soit  $n \geq 1$  un entier. Il existe  $\alpha = \alpha(n) > 0$  tel que ce qui suit est vrai. Si  $\mathcal{X}$  est une orbifolde riemannienne complète avec sa courbure de Ricci minoré par  $\text{Ric} \geq -(n-1)$  et  $\Gamma$  un sous groupe de  $\text{Isom}(|\mathcal{X}|)$  agissant proprement discontinuement sur  $|\mathcal{X}|$ . Alors pour tout  $x \in |\mathcal{X}|$ , le "presque-stabilisateur"*

$$\Gamma_\alpha(x) := \langle \{\gamma \in \Gamma \mid d(\gamma \cdot x, x) < \alpha\} \rangle$$

*est virtuellement nilpotent.*

Nous pouvons donc imiter la démonstration de Păun pour montrer que:

**THEOREM 0.11** (=Theorem 4.10=Theorem 8.13). *Soit  $(\mathcal{X}, \omega)$  une orbifolde effective kählérienne à l'espace sous-jacent  $X = |\mathcal{X}|$  compact. Si le orbi-fibré anti-canonique  $K_{\mathcal{X}}^{-1}$  est nef, alors  $\pi_1^{\text{orb}}(\mathcal{X})$  est virtuellement nilpotent.*

Pour une variété compacte kählérienne  $X$ , on peut considérer son morphisme d'Albanese  $\text{Alb}_X : X \rightarrow A(X)$  (voir Definition 2.33). Posons  $Y := \text{Alb}_X(X) \subset A(X)$  l'image de  $X$  dans  $A(X)$  et  $r : \tilde{Y} \rightarrow Y$  un modèle lisse de  $Y$ . Frédéric Campana a montré comment on peut décrire la suite centrale de  $\pi_1(X)$  par la suite centrale de  $\pi_1(\tilde{Y})$  dans [Cam95, Théorème 2.2]. En particulier, nous observons que pour une variété compacte kählérienne  $X$  dont le groupe fondamental  $\pi_1(X)$  est virtuellement nilpotent, si  $\text{Alb}_X$  est surjectif à fibres connexes, nous avons que  $\pi_1(X)$  est virtuellement abélien.

Pour une orbifoldes compacte  $\mathcal{X}$ , nous pouvons considérer sa représentation en log paire  $(X, \Delta_X)$ , où  $\Delta_X$  est un  $\mathbb{Q}$ -diviseur à coefficients standard. Pour toute paire

$(X, \Delta_X)$  à singularités klt, nous pouvons définir un groupe fondamental  $\pi_1(X, \Delta_X)$ , et quand la paire représente une orbifold  $\mathcal{X}$ , il existe un isomorphe canonique  $\pi_1(X, \Delta_X) \cong \pi_1^{\text{orb}}(\mathcal{X})$  (cf. [Proposition 5.50](#)). Quand  $X$  est lisse, nous avons une surjection  $\pi_1(X, \Delta_X) \twoheadrightarrow \pi_1(X)$ . Donc si  $\pi_1^{\text{orb}}(\mathcal{X})$  est virtuellement nilpotent et  $X$  est lisse tel que  $\text{Alb}_X$  est surjectif à fibres connexes, nous peut montrer que  $\pi_1^{\text{orb}}(\mathcal{X})$  est virtuellement abélien. Pour une orbifold générale  $(X, \Delta_X)$  avec  $X$  non nécessairement lisse, nous considérons une résolution  $r : Y \rightarrow X$ . Suivant cette idée, nous prouvons le résultat suivant.

**THEOREM 0.12** (= [Theorem 4.11](#) = [Theorem 9.1](#)). *Soit  $(X, \Delta)$  une orbifold projective avec  $-(K_X + \Delta)$  nef. Le groupe fondamental  $\pi_1(X, \Delta)$  est virtuellement abélien.*





## Summary

### Part 1: Classification of slc varieties with high nef value

When studying a polarized variety  $(X, L)$  with  $X$  having klt singularities, a useful tool is its nefvalue morphism. Suppose that  $K_X$  is not nef and set

$$\tau(L) := \inf\{t \in \mathbb{Q} : K_X + tL \text{ is nef}\}.$$

Then by Kawamata basepoint-free theorem, we have that  $K_X + \tau(L)L$  has no base-point. For  $m > 0$  divisible enough, we know that the linear system  $|m(K_X + \tau(L)L)|$  defines a morphism  $\phi_{|m(K_X + \tau(L)L)|} : X \rightarrow \mathbb{P}_{\mathbb{C}}$  from  $X$  to some projective spaces, whose Stein factorization  $\phi : X \rightarrow Y$  dose not depend on  $m$ . We have that  $L = \phi^*(\mathcal{O}_Y(1))$ . We can study  $(X, L)$  by studying the properties of general fibers of  $\phi$ .

For example, Andreatta proved the following result in [And95].

**THEOREM 0.13** ([And95, Theorem 2.1.]). *Let  $X$  be a projective variety with klt singularities and let  $L$  be a line bundle on  $X$ . Let  $\phi : X \rightarrow Z$  be a surjective morphism between normal varieties with connected fiber. Suppose that  $L$  is  $\phi$ -ample and  $K_X + \tau L \sim_{\mathbb{Q}, \phi} 0$  for some  $\tau \in \mathbb{Q}^+$ . Let  $F_1 = \phi^{-1}(z)$  be a non-trivial fiber,  $F \subset F_1$  be one of its irreducible components,  $F'$  be the normalization of  $F$  and let  $L'$  be the pullback of  $L$  on  $F'$ . Let  $\lceil \tau \rceil$  be the integral part of  $\tau$  and  $\tau' = \lceil \tau \rceil = -\lfloor -\tau \rfloor$ . Then we have the following*

- (I,1)  $\dim(F) \geq \tau - 1$ ;
- (I,2) If  $\dim(F) < \tau$ , then  $F \cong \mathbb{P}^{\tau'-1}$  and  $L|_F = \mathcal{O}_{\mathbb{P}^{\tau'-1}}(1)$ ;
- (I,3) If  $\dim(F) < \tau + 1$ , then  $\Delta(F', L') = 0$ ,

If moreover  $\dim(F) > \dim(X) - \dim(Z)$ , then

- (II,1)  $\dim(F) \geq \tau$ ;
- (II,2) If  $\dim(F) = \tau$ , then  $F \cong \mathbb{P}^{\tau}$  and  $L|_F = \mathcal{O}_{\mathbb{P}^{\tau}}(1)$ ;
- (II,3) If  $\dim(F) < \tau + 1$ , then  $\Delta(F', L') = 0$ ,

If all components of the fiber  $F$  satisfy  $\dim(F) < \tau(L)$ , in case (I.2) or  $\dim(F) \leq \tau(L)$  in case (II.3), then  $F$  is actually irreducible.

We see from the theorem that  $\tau(L) > \dim(X) - 1$  will be a very restrictive condition. In fact, when  $X$  has terminal singularities, the classification for  $(X, L)$  is complete when the nefvalue of  $L$  is larger than  $n - 1$ .

**Proposition 0.14** ([BS11, Proposition 7.2.2.]). *Let  $(X, L)$  be a polarized variety. Suppose that  $X$  has terminal singularities. Let  $\phi : X \rightarrow Z$  be the nefvalue morphism. Then we have one of the following*

- (1)  $\tau = n + 1$  and  $(X, L) \cong (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$ ;

- (2)  $\tau = n$  and  $(X, L) \cong (Q, \mathcal{O}_Q(1))$  where  $Q \subset \mathbb{P}^{n+1}$  is a hyperquadric in  $\mathbb{P}^{n+1}$  if  $K_X + nL \equiv_{\text{num}} \mathcal{O}_X$ ;
- (3)  $\tau = n$  and  $Z$  is a smooth curve and  $(X, L) \cong (\mathbb{P}(E), \mathcal{O}_{\mathbb{P}(E)}(1))$  where  $E$  is a vector bundle over  $Z$  and  $\phi : X \rightarrow Z$  is the structure morphism;
- (4)  $\tau < n$  and  $K_X + nL$  is nef and big.

**Proposition 0.15** ([BS11, Proposition 7.2.4.]). *Let  $(X, L)$  be a polarized variety. Suppose that  $X$  has  $\mathbb{Q}$ -factorial terminal singularities and  $n = \dim(X) \geq 2$ . Suppose that  $K_X + nL$  is ample and  $\tau > n - 1$ . Then we have  $\tau = n - \frac{1}{2}$  and  $(X, L) = C_n(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$  is a generalized cone over  $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$ .*

In Part 1, we will first explore the classification problem for a normal polarized variety  $(X, L)$  with high nefvalue by allowing wilder singularities than terminal for  $X$ . We will give the following classification.

**THEOREM 0.16** (=Theorem 3.14). *Let  $(X, L)$  be a polarized variety of dimension  $n$ . Suppose that  $K_X$  is  $\mathbb{Q}$ -Cartier and  $K_X + (n - 1)L \notin \text{Pseff}(X)$ . Then  $(X, L)$  is one of the following:*

- (1)  $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$ ;
- (2.i)  $(\mathbb{P}(\mathcal{V}), \mathcal{O}_{\mathbb{P}(\mathcal{V})}(1))$ , where  $\mathcal{E}$  is a rank  $n$  ample vector bundle over a smooth curve  $C$ ;
- (2.ii)  $C_n(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(a))$  be a generalized cone with  $a \geq 3$ ;
- (3)  $(Q, \mathcal{O}_{\mathbb{P}^{n+1}}(1))$ , where  $Q \subset \mathbb{Q}^{n+1}$  is a hyperquadric;
- (4)  $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$ ;
- (5) a generalized cone  $C_n(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$  over  $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$ .

The strategy of this classification is first to establish a classification for quasi-polarized  $(X', L')$  with  $X'$  having canonical singularities and  $K_{X'} + (\dim(X') - 1)L' \notin \text{Pseff}(X')$ . This is done in Section 3.1. The key point is to run an MMP to reduce the problem to classifying  $(X'', L'')$  with  $\tau(L'') > \dim(X'') - 1$ . We may then apply a canonical modification  $\mu : X^{\text{can}} \rightarrow X$  and reduce the problem to the classification for quasi-polarized  $(X^{\text{can}}, \mu^*(L))$ .

With similar method, we will also establish a classification result for a log canonical pair  $(X, \Delta)$  with  $(K_X + \Delta) + (\dim(X) - 1)L \notin \text{Pseff}(X)$ .

**Proposition 0.17** (=Corollary 3.19). *Let  $(X, \Delta)$  be a log canonical pair, with  $\Delta \neq 0$  a reduced divisor. Suppose that  $L$  is an ample line bundle on  $X$  and  $(K_X + \Delta) + (n - 1)L \notin \text{Pseff}(X)$ , where  $n = \dim(X)$ . Then we have one of the following:*

- (1)  $(X, L) \cong (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$ ,  $\Delta \equiv_{\text{num}} H$  is a prime divisor where  $H$  is a hyperplane of  $\mathbb{P}^n$ ;
- (2.i) There is a  $(\mathbb{P}^{n-1}, \mathcal{O}_{\mathbb{P}^{n-1}}(1))$ -bundle  $(\mathbb{P}(E), \mathcal{O}_{\mathbb{P}(E)}(1))$  over a smooth curve  $C$ , and a birational morphism  $\mu : \mathbb{P}(E) \rightarrow X$  such that  $\mu^*(L) \cong \mathcal{O}_{\mathbb{P}(E)}(1)$  and  $\Delta = \sum F_i$  is a finite sum where  $F_i \cong \mathbb{P}^{n-1}$  are images of distinct general fibers of  $\pi$  by  $\mu$ ;
- (2.ii)  $(X, L) = (\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(1)), \mathcal{O}_{\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(1))}(1))$  with  $1 < a$  and  $\Delta = D$  is irreducible, where  $D$  is the unique section of  $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(1)) \rightarrow \mathbb{P}^1$  such that  $D \equiv_{\text{num}} \mathcal{O}_{\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(1))}(1) - af$ , where  $f$  is a general fiber;

- (3.i)  $(X, L) \cong (Q, \mathcal{O}_{\mathbb{P}^{n+1}}(1))$ , where  $Q \subset \mathbb{P}^{n+1}$  is a  $\text{rk}(Q) = 3$  hyperquadric, the boundary divisor  $\Delta$  is a hyperplane in  $Q$  and  $[\Delta] = \frac{1}{2}[H \cap Q]$  where  $H$  is a hyperplane in  $\mathbb{P}^{n+1}$ ;
- (3.ii)  $(X, L) \cong (Q, \mathcal{O}_{\mathbb{P}^{n+1}}(1))$ , where  $Q \subset \mathbb{P}^{n+1}$  is a  $\text{rk}(Q) = 4$  hyperquadric. If we write  $Q = \text{Proj} \left( \frac{\mathbb{C}[x_0, \dots, x_{n+1}]}{(x_0x_1 - x_2x_3)} \right)$ , then  $\Delta = D$  is prime and  $D$  is the cone with vertex  $\mathbb{P}^{n-3}$  over  $\mathbb{P}^1 \times \text{pt}$  or  $\text{pt} \times \mathbb{P}^1$ . In particular,  $D \cong \mathbb{P}^{n-1}$ ;

Finally, we may drop the normality condition. For a non-normal variety, we can always consider its normalization  $\nu : \bar{X} \rightarrow X$ . When  $X$  has slc singularities, we have that  $(\bar{X}, \bar{D})$  is log canonical where  $\bar{D}$  is the conductor divisor. We give the following classification.

**THEOREM 0.18** (=Proposition 3.20). *Let  $X$  be a non-normal slc projective variety of dimension  $n$  and  $L$  an ample line bundle over  $X$ . Suppose that  $K_X + (n-1)L \notin \text{Pseff}(X)$ . Let  $\pi : \bar{X} \rightarrow X$  be the normalization of  $X$  and  $D \subset X$ ,  $\bar{D} \subset \bar{X}$  the conductors. Then we have:*

*There is a nodal curve  $C'$ , a rank  $n$ -vector bundle  $E'$ , distinct fibers  $F_1, F_2, \dots, F_m$  of  $\mathbb{P}(E')$  and a birational morphism  $\mu : \mathbb{P}(E') \rightarrow X$  such that  $\mu^*(L) = \mathcal{O}_{\mathbb{P}(E')}(1)$  and  $D = \sum_{1 \leq i \leq m} \mu(F_i)$ .*

## Part 2: Fundamental group of Kähler orbifolds with nef anti-canonical bundle

Given a compact Kähler manifold  $X$ , the celebrated Calabi-Yau theorem tells us that we can find a Kähler form  $\omega$  on  $X$  such that the first Chern class  $c_1(X) = c_1(-K_X)$  can be represented by  $\frac{1}{2\pi}\text{Ricci}_\omega$ . Hence the positivity of the anti-canonical bundle of  $X$  can be translated to some positivity of the Ricci curvature of  $X$ . When studying the fundamental group of  $X$  with certain positivity conditions on  $-K_X$ , we may tap the results on differential geometry for Riemannian manifolds with certain positivity conditions on its Ricci curvature.

We give a reformulated version of Kobayashi's theorem

**THEOREM 0.19** ([Kob61, Theorem A]). *A compact Fano manifold is simply connected.*

With Calabi-Yau theorem, this theorem could be proved by using Myers' theorem (cf. [GHL04, 3.85]).

When  $-K_X$  is nef, we can, with the help of the Aubin-Yau theorem (cf. Theorem 5.15), show that for each  $\epsilon > 0$ , there exists a Kähler form  $\omega_\epsilon$  such that

$$\text{Ricci}_{\omega_\epsilon} \geq -\epsilon\omega_\epsilon.$$

By multiplying  $\omega_\epsilon$  with a scalar  $0 < \lambda \ll 1$  and setting  $\omega'_\epsilon := \lambda\omega_\epsilon$ , we will have that  $\text{Ricci}_{\omega'_\epsilon} \geq -(\dim(X) - 1)\omega'_\epsilon$ . In [CC96], Cheeger and Colding proved a deep result:

**Lemma 0.20.** (Geometric Margulis lemma, [CC96, Theorem 8.7.]) *Let  $n > 0$  be a natural number. There exists a universal constant  $C = C(n)$  only depending on  $n$  such that the following holds:*

For any compact manifold  $M$  of dimension  $n$  and  $\text{Ric}_g \geq -(n-1)g$ , one has that the morphism induced by the inclusion

$$\pi_1(B_g(p, r), p) \rightarrow \pi_1(M, p)$$

has virtually nilpotent image for any  $r < C(n)$ .

By using Aubin-Yaun theorem to construct a specific metric  $\omega$  on  $X$ , such that

- (1)  $\text{Ricci}_\omega \geq -(2 \dim_{\mathbb{C}}(X) - 1)\omega$  and
- (2) there exists  $r_0 < C(2 \dim_{\mathbb{C}}(X))$  such that  $B_\omega(p, r_0) = X$ .

and applying the above lemma directly, Mihai Păun proved the following result:

**THEOREM 0.21** ([Pău97, Theorem 1]). *Let  $(X, \omega)$  be a compact Kähler manifold whose anti-canonical bundle  $-K_X$  nef. Then its fundamental group  $\pi_1(X)$  is virtually nilpotent.*

In Part 2, the main goal is to generalize Păun's result to Kähler orbifolds with nef anti-canonical orbi-bundle. Suppose that we have an orbifold version of Cheeger and Colding's result. We may then adapt Păun's proof easily, as the differential geometry of orbifolds is well-established. Equip the underline space  $X = |\mathcal{X}|$  of a Riemannian orbifold  $(\mathcal{X}, g)$  with a natural metric  $d$  (cf. Section 6.2), and apply an algebraic Margulis lemma in [BGT12]. We first prove an orbifold Margulis lemma.

**Lemma 0.22** (=Lemma 7.12). *Let  $n \geq 1$  be an integer. There exists  $\alpha = \alpha(n) > 0$  such that the following holds true:*

*Suppose that  $\mathcal{X}$  is a complete Riemannian orbifold of dimension  $n$  with its Ricci curvature bounded by  $\text{Ric} \geq -(n-1)$  and  $\Gamma$  a subgroup of  $\text{Isom}(|\mathcal{X}|)$  acting properly discontinuously by isometries on  $|\mathcal{X}|$ . Then for every  $x \in |\mathcal{X}|$ , the "almost stabiliser"*

$$\Gamma_\alpha(x) := \langle \{\gamma \in \Gamma : d(\gamma \cdot x, x) < \alpha\} \rangle$$

*is virtually nilpotent.*

We may then mimic Păun's proof to show

**THEOREM 0.23** (=Theorem 4.10=Theorem 8.13). *Let  $(\mathcal{X}, \omega)$  be a compact Kähler orbifold. If the anti-canonical bundle  $K_{\mathcal{X}}^{-1}$  is nef, then  $\pi_1^{\text{orb}}(\mathcal{X})$  is virtually nilpotent.*

For a Kähler manifold  $X$ , we can also consider its Albanese morphism  $\text{Alb}_X : X \rightarrow A(X)$  (see Definition 2.33). Set  $Y := \text{Alb}_X(X) \subset A(X)$  the image of  $X$  in  $A(X)$  and  $r : \tilde{Y} \rightarrow Y$  a smooth model of  $Y$ . Frédéric Campana shows how one can describe the central series of  $\pi_1(X)$  by the central series of  $\pi_1(\tilde{Y})$  in [Cam95, Théorème 2.2]. In particular, one observes that for a compact Kähler manifold  $X$  whose fundamental group  $\pi_1(X)$  is virtually nilpotent, if  $\text{Alb}_X$  is surjective, we have that  $\pi_1(X)$  is virtually Abelian.

For a complex orbifold  $\mathcal{X}$ , we may consider its log pair representation  $(X, \Delta_X)$ , where  $\Delta_X$  is a  $\mathbb{Q}$ -divisor with standard coefficients. For any klt pair  $(X, \Delta_X)$ , we can define a fundamental group  $\pi_1(X, \Delta_X)$ , and when the pair represents an orbifold  $\mathcal{X}$ , there is a canonical isomorphism  $\pi_1(X, \Delta_X) \cong \pi_1^{\text{orb}}(\mathcal{X})$  (cf. Proposition 5.50). When  $X$  is smooth, we have a surjection  $\pi_1(X, \Delta_X) \twoheadrightarrow \pi_1(X)$ . Hence if  $\pi_1^{\text{orb}}(\mathcal{X})$  has virtually nilpotent and  $X$  is smooth such that  $\text{Alb}_X$  is surjective, we can show  $\pi_1^{\text{orb}}(\mathcal{X})$  is virtually Abelian. For

general  $(X, \Delta_X)$ , we consider its resolution  $r : Y \rightarrow X$ . Following this idea, we prove the following result.

**THEOREM 0.24** (=Theorem 4.11=Theorem 9.1). *Let  $(X, \Delta)$  be a projective orbifold pair with  $-(K_X + \Delta)$  nef. The orbifold fundamental group  $\pi_1(X, \Delta)$  is virtually Abelian.*



## Part 1

# Classification of slc varieties with high nef value





## CHAPTER 1

### Introduction

In [KO73], Kobayashi and Ochiai proved the following classifying result concerning smooth projective varieties.

**THEOREM 1.1** ([KO73, Theorem 1.1.]). *Let  $X$  be a smooth projective variety of dimension  $n$  with an ample line bundle  $L$  such that*

- (1)  $L^n = 1$ ;
- (2)  $h^0(X, L) \geq n + 1$ .

*Then  $(X, L)$  is isomorphic to  $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$ .*

The result has later been generalized to singular varieties:

**THEOREM 1.2** (Generalized Kobayashi-Ochiai Theorem, cf.[BS11, Theorem 3.1.6]). *Let  $X$  be an  $n$ -dimensional connected normal projective scheme and  $L$  an ample line bundle on  $X$ . Then we are in one of the following situations:*

- $(X, L) \cong (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$  if and only if  $K_X + (n + 1)L \equiv_{\text{num}} \mathcal{O}_X$ ;
- $(X, L) \cong (Q, \mathcal{O}_Q(1))$  where  $Q \subset \mathbb{P}^{n+1}$  is a hyperquadric in  $\mathbb{P}^{n+1}$  if and only if  $K_X + nL \equiv_{\text{num}} \mathcal{O}_X$ .

A projective variety  $X$  together with an ample line bundle  $L$  on  $X$  is called a polarized variety and is denoted by  $(X, L)$ . To study polarized varieties, Fujita introduces the  $\Delta$ -genus  $\Delta(X, L) := n + L^n - h^0(X, L)$  of polarized varieties, which encodes the dimension of the variety  $X$  and  $L^n$ , and develops classification theories for polarized varieties with small  $\Delta$ -genus under certain assumptions on the singularities of  $X$  and positivity on  $L$ . For Fujita's work, we refer to [Fuj90, Chapter 1].

When  $X$  has terminal singularities, set  $\tau(L) := \inf\{t \in \mathbb{R} : K_X + tL \text{ is nef}\}$  which is called the *nefvalue* of  $X$ . If  $K_X$  is not nef, Kawamata's rationality theorem shows that  $\tau(L)$  is a rational number. Let  $a > 0$  be an integer such that  $a(K_X + \tau(L)L)$  is a Cartier divisor. Then Kawamata-Shokurov's base point free theorem shows that for  $\mathbb{N} \ni b \gg 0$ , the divisor  $ba(K_X + \tau(L)L)$  has no base point. It is then a classical results in birational geometry that the ring

$$R(X, a(K_X + \tau(L)L)) := \bigoplus_{m \geq 0} (H^0(X, ma(K_X + \tau(L)L)))$$

is normal and finitely generated over  $\mathbb{C}$ . We call the normal variety

$$Z := \text{Proj}(R(X, a(K_X + \tau(L)L)))$$

an adjoint model for  $(X, L)$ . There exists a canonical morphism given by sections of  $a(K_X + \tau(L)L)$

$$\phi : X \rightarrow Z$$

called the *nefvalue morphism*. It is thus natural to study  $(X, L)$  via  $\phi$ .

It turns out with the help of nefvalue morphism, the classification for  $(X, L)$  is complete when the nefvalue of  $L$  is larger than  $n - 1$ .

**Proposition 1.3** ([BS11, Proposition 7.2.2.]). *Let  $(X, L)$  be a polarized variety. Suppose that  $X$  has terminal singularities. Let  $\phi : X \rightarrow Z$  be the nefvalue morphism and  $\tau$  the nefvalue of  $L$ . Then we have one of the following*

- (1)  $\tau = n + 1$  and  $(X, L) \cong (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$ ;
- (2)  $\tau = n$  and  $(X, L) \cong (Q, \mathcal{O}_Q(1))$  where  $Q \subset \mathbb{P}^{n+1}$  is a hyperquadric in  $\mathbb{P}^{n+1}$  if  $K_X + nL \equiv_{\text{num}} \mathcal{O}_X$ ;
- (3)  $\tau = n$  and  $Z$  is a smooth curve and  $(X, L) \cong (\mathbb{P}(E), \mathcal{O}_{\mathbb{P}(E)}(1))$  where  $E$  is a vector bundle over  $Z$  and  $\phi : X \rightarrow Z$  is the structure morphism;
- (4)  $\tau < n$  and  $K_X + nL$  is nef and big.

**Proposition 1.4** ([BS11, Proposition 7.2.4.]). *Let  $(X, L)$  be a polarized variety. Suppose that  $X$  has  $\mathbb{Q}$ -factorial terminal singularities and  $n = \dim(X) \geq 2$ . Suppose that  $K_X + nL$  is ample and  $\tau > n - 1$ . Then we have that  $\tau = n - \frac{1}{2}$  and that  $(X, L) = C_n(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$  is a generalized cone over  $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$ .*

The study of polarized varieties has a natural counterpart in the study in the  $\mathbb{Q}$ -Fano foliations. For a foliation  $\mathcal{F} \subset T_X$  of rank  $r$  on a projective variety  $X$  of dimension  $n$ , we define the canonical class  $K_{\mathcal{F}}$  of the foliation  $\mathcal{F}$  to be the divisor class satisfying  $\mathcal{O}_X(-K_{\mathcal{F}}) \cong \det(\mathcal{F})$ , where  $\det(\mathcal{F})$  is defined to be  $(\wedge^r \mathcal{F})^{**}$ . When  $-K_{\mathcal{F}}$  is  $\mathbb{Q}$ -Cartier and ample, we call  $\mathcal{F}$  a  $\mathbb{Q}$ -Fano foliation and define its *index*  $i_{\mathcal{F}}$  to be the largest positive rational number such that  $-K_{\mathcal{F}} \sim_{\mathbb{Q}} i_{\mathcal{F}}H$  for an ample Cartier divisor  $H$  on  $X$ . In [AD14], Araujo and Druel established a Kobayashi-Ochiai type theorem ([AD14, Theorem 1.2]), which gives an upper bound of the index of a  $\mathbb{Q}$ -Fano foliation  $\mathcal{F}$  and gives a description when the upper bound is reached.

When a foliation  $\mathcal{F}$  is algebraically integrable, one can define naturally general log leaves of  $\mathcal{F}$  (cf. [AD14, Definition 3.11]). A *general log leaf*  $(\tilde{F}, \tilde{\Delta})$  comprises a normalization of the closure of a general leaf  $F$  of  $\mathcal{F}$  and an effective Weil  $\mathbb{Q}$ -divisor  $\tilde{\Delta}$ . Let  $e : \tilde{F} \rightarrow F$  be the normalization map. Then  $\tilde{\Delta}$  is given by  $K_{\tilde{F}} + \tilde{\Delta} \equiv_{\text{num}} e^*K_{\mathcal{F}}$ . It turns out that understanding the log general leaves helps to study of algebraically integrable foliations. This motivates us to consider classification problem for  $(X, \Delta)$ , where  $X$  is a variety and  $\Delta$  a Weil  $\mathbb{Q}$ -divisor.

When an algebraically integrable foliation  $\mathcal{F}$  is  $\mathbb{Q}$ -Fano, we have the equality  $-(K_{\tilde{F}} + \tilde{\Delta}) \equiv_{\text{num}} i_{\mathcal{F}}(e^*H)$ . Hence one may very well try to establish a pair version of [Theorem 1.2](#). In fact, Fujino and Miyamoto proved the following

**THEOREM 1.5** ([FM21, Theorem 1], see also [AD14, Theorem 1.1]). *Let  $(X, \Delta)$  be a projective klt pair such that  $-(K_X + \Delta)$  is ample. Assume that  $-(K_X + \Delta) \equiv_{\text{num}} rH$  for some Cartier divisor  $H$  on  $X$  with  $r > n = \dim(X)$ . Then  $X$  is isomorphic to  $\mathbb{P}^n$  with  $\mathcal{O}_X(H) = \mathcal{O}_{\mathbb{P}^n}(1)$ .*

The result of Fujino and Miyamoto assumes mild singularities on the pair  $(X, \Delta)$  and a divisibility condition of the log canonical bundle  $K_X + \Delta$ . However, with a foliation  $\mathcal{F}$ , its log general leaf  $(\tilde{F}, \tilde{\Delta})$  is *a priori* just normal. On the other hand, for the classification theorems [Proposition 1.3](#) and [Proposition 1.4](#), we do not need divisibility. However we

do need  $-K_X$  to be very positive. Thus one may try to weaken the conditions and consider the classification problems

- (1) Classify the triple  $(X, \Delta, L)$  where  $(X, \Delta)$  is log canonical,  $L$  is ample and  $K_X + (\dim(X) - 1)L \notin \text{Pseff}(X)$ ;
- (2) Classify the pair  $(X, L)$  where  $X$  is a projective variety with singularities wilder than normal,  $L$  is ample and  $K_X + (\dim(X) - 1)L \notin \text{Pseff}(X)$ .

We call a projective variety  $X$  together with a nef and big line bundle  $L$  a *quasi-polarized variety*. For a quasi-polarized variety  $(X, L)$  where  $X$  is canonical and  $\mathbb{Q}$ -factorial, we may run a MMP which contracts all  $L$ -trivial extremal rays and get a polarized variety  $(X', L')$  (see [Lemma 3.5](#)). By using Andreatta's result [Theorem 3.7](#) which describes the general fibers of extremal contractions, we can reduce the problem of classifying  $(X', L')$  with high nefvalue to the problem of classifying polarized variety with  $\Delta$ -genus zero. We generalize [Proposition 1.3](#) and [Proposition 1.4](#) as follows.

**THEOREM 1.6 (=Theorem 3.13).** *Let  $X$  be a variety with canonical  $\mathbb{Q}$ -factorial singularities and  $L$  a nef and big line bundle on  $X$ . Suppose that  $K_X + (n-1)L \notin \text{Pseff}(X)$ . Then we have one of the following cases:*

- (1)  $(X, L) \sim_{\text{bir}} (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$ ;
- (2)  $(X, L)$  is birational equivalent to a  $(\mathbb{P}^{n-1}, \mathcal{O}_{\mathbb{P}^{n-1}}(1))$ -bundle over a smooth curve  $C$ ;
- (3)  $(X, L) \sim_{\text{bir}} (Q, \mathcal{O}_{\mathbb{P}^{n+1}}(1))$ , where  $Q \subset \mathbb{Q}^{n+1}$  is a hyperquadric;
- (4)  $(X, L) \sim_{\text{bir}} (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}^2(2))$ ;
- (5)  $(X, L) \sim_{\text{bir}} C_n(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}^2(2))$ , where  $C_n(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}^2(2))$  is a generalised cone over  $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}^2(2))$

The drawback of letting  $L$  be nef and big, is that after running MMP, we don't have isomorphism and even have indeterminacies.

For a normal variety  $X$ , we have modifications  $\mu : X' \rightarrow X$  for  $X$  such that  $X'$  has mild singularities and  $K_{X'}$  is  $\mu$ -ample. A good reference for these modifications is [[Kol13](#), Chapter 1]. For a polarized variety  $(X, L)$ , with  $X$  normal, we may take a canonical modifications  $\mu : X' \rightarrow X$  for  $X$  and consider the quasi-polarized variety  $(X', L')$ . We have the following result.

**THEOREM 1.7 (=Theorem 3.14).** *Let  $(X, L)$  be a polarized normal variety of dimension  $n$ . Suppose that  $K_X$  is  $\mathbb{Q}$ -Cartier and  $K_X + (n-1)L \notin \text{Pseff}(X)$ . Then we have one of the following cases:*

- (1)  $(X, L) \cong (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$ ;
- (2.i)  $(X, L) \cong (\mathbb{P}(\mathcal{V}), \mathcal{O}_{\mathbb{P}(\mathcal{V})}(1))$ , where  $\mathcal{E}$  is a rank  $n$  ample vector bundle over a smooth curve  $C$ ;
- (2.ii)  $(X, L) \cong C_n(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(a))$  be a generalized cone with  $a \geq 3$ ;
- (3)  $(X, L) \cong (Q, \mathcal{O}_{\mathbb{P}^{n+1}}(1))$ , where  $Q \subset \mathbb{Q}^{n+1}$  is a hyperquadric;
- (4)  $(X, L) \cong (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}^2(2))$ ;
- (5)  $(X, L) \cong C_n(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}^2(2))$ , where  $C_n(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}^2(2))$  is a generalised cone over  $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}^2(2))$ .

In [Theorem 1.7](#), we note that even if in the proof we have taken a modification, in the resulting list we have isomorphism. The reason is that  $L$  is ample and birational equivalences between normal polarized varieties are always isomorphisms.

For a log canonical pair  $(X, \Delta)$  with  $(K_X + \Delta) + (\dim(X) - 1)L \notin \text{Pseff}(X)$ , a first observation is that as  $\Delta$  is effective, we will have  $K_X + (\dim(X) - 1)L \notin \text{Pseff}(X)$ . Hence we will have a list for  $(X, L)$  similar to [Theorem 1.7](#). However in this list the Picard number  $\rho(X)$  of  $X$  is at most 2. Hence for  $\Delta$  to be an irreducible divisor or more generally reduced divisor, we don't have to many choice. We may thus give a list for  $(X, \Delta, L)$ .

**Proposition 1.8** (= [Corollary 3.19](#)). *Let  $(X, \Delta)$  be a log canonical pair, with  $\Delta \neq 0$  a reduced divisor. Suppose that  $L$  is an ample line bundle on  $X$  and  $(K_X + \Delta) + (n - 1)L \notin \text{Pseff}(X)$ , where  $n = \dim(X)$ . Then  $(X, \Delta, L)$  is one of the following:*

- (1)  $(X, L) \cong (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$ ,  $\Delta \equiv_{\text{num}} H$  is a prime divisor where  $H$  is a hyperplane of  $\mathbb{P}^n$ ;
- (2.i) There is a  $(\mathbb{P}^{n-1}, \mathcal{O}_{\mathbb{P}^{n-1}}(1))$ -bundle  $(\mathbb{P}(E), \mathcal{O}_{\mathbb{P}(E)}(1))$  over a smooth curve  $C$ , and a birational morphism  $\mu : \mathbb{P}(E) \rightarrow X$  such that  $\mu^*(L) \cong \mathcal{O}_{\mathbb{P}(E)}(1)$  and  $\Delta = \sum F_i$  is a finite sum where  $F_i \cong \mu(\mathbb{P}^{n-1})$  are images of distinct general fibers of  $\pi$  by  $\mu$ ;
- (2.ii)  $(X, L) = (\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(1)), \mathcal{O}_{\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(1))}(1))$  with  $a > 1$  and  $\Delta = D$  is irreducible, where  $D$  is the unique section of  $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(1)) \rightarrow \mathbb{P}^1$  such that  $D \equiv_{\text{num}} \mathcal{O}_{\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(1))}(1) - af$ , where  $f$  is a general fiber;
- (3.i)  $(X, L) \cong (Q, \mathcal{O}_{\mathbb{P}^{n+1}}(1))$ , where  $Q \subset \mathbb{P}^{n+1}$  is a  $\text{rk}(Q) = 3$  hyperquadric, the boundary divisor  $\Delta$  is a hyperplane in  $Q$  and  $[\Delta] = \frac{1}{2}[H \cap Q]$  where  $H$  is a hyperplane in  $\mathbb{P}^{n+1}$ ;
- (3.ii)  $(X, L) \cong (Q, \mathcal{O}_{\mathbb{P}^{n+1}}(1))$ , where  $Q \subset \mathbb{P}^{n+1}$  is a  $\text{rk}(Q) = 4$  hyperquadric. If we write  $Q = \text{Proj} \left( \frac{\mathbb{C}[x_0, \dots, x_{n+1}]}{(x_0x_1 - x_2x_3)} \right)$ , then  $\Delta = D$  is prime and  $D$  is the cone with vertex  $\mathbb{P}^{n-3}$  over  $\mathbb{P}^1 \times \text{pt}$  or  $\text{pt} \times \mathbb{P}^1$ . In particular,  $D \cong \mathbb{P}^{n-1}$ ;

When the polarized variety  $(X, L)$  is not normal, a natural way to study it is consider the normalization  $\nu : \bar{X} \rightarrow X$  of  $X$  and study the polarized variety  $(\bar{X}, \nu^*(L))$ . However, when considering normalization, there is a natural ideal of  $\mathcal{O}_X$  measuring how far  $X$  is from being normal, the conductor  $\text{cond}_X := \mathcal{H}om_{\mathcal{O}_X}(\nu_*(\mathcal{O}_{\bar{X}}), \mathcal{O}_X)$ . When  $X$  has deminormal singularities, the subschemes  $D \subset X$  defined by  $\text{cond}_X$  and  $\bar{D} \subset \bar{X}$  defined by  $\nu_*(\mathcal{I}_{\bar{D}}) = \text{cond}_X$  are both generically reduced purely codimension 1 subschemes. In particular, the algebraic cycles  $[D]$  and  $[\bar{D}]$  are reduced Weil divisors. When  $X$  is slc, we also have that  $(\bar{X}, \bar{D})$  is log canonical. Hence for polarized variety  $(X, L)$  with  $X$  slc, we may use [Proposition 1.8](#) to study  $(\bar{X}, \nu^*(L))$ . We give the following classification.

**THEOREM 1.9** (= [Proposition 3.20](#)). *Let  $X$  be a non-normal slc projective variety of dimension  $n$  and  $L$  an ample line bundle over  $X$ . Suppose that  $K_X + (n - 1)L \notin \text{Pseff}(X)$ . Let  $\pi : \bar{X} \rightarrow X$  be the normalization of  $X$  and  $D \subset X$ ,  $\bar{D} \subset \bar{X}$  the conductors. Then we have:*

*There is a nodal curve  $C'$ , a rank  $n$ -vector bundle  $E'$ , distinct fibers  $F_1, F_2, \dots, F_m$  of  $\mathbb{P}(E')$  and a birational morphism  $\mu : \mathbb{P}(E') \rightarrow X$  such that  $\mu^*(L) = \mathcal{O}_{\mathbb{P}(E')}(1)$  and  $D = \sum_{1 \leq i \leq m} \mu(F_i)$*

We see that [Theorem 1.9](#) shortens the list in [Proposition 1.8](#) rather than increasing it. In fact, there is a degree 2 morphism  $\bar{D}^\nu \rightarrow D^\nu$ , where  $\bar{D}^\nu$  and  $D^\nu$  are the normalizations of  $\bar{D}$  and  $D$  respectively. Hence we need  $(L'|_{\bar{D}^\nu})^{n-1}$  divisible by 2, which gives more restriction on  $(\bar{X}, \bar{D})$  than the assumption in [Proposition 1.8](#).

**Remark 1.10.** After running MMP to reduce [Theorem 1.6](#) to the problem of classifying  $(X', L')$  with  $X'$  canonical and  $L'$  ample (see [Lemma 3.5](#)), the results is already known for even when  $X'$  is klt (*cf.* [[And13](#), Proposition 3.5.]). My personal contribution in the classification is to use modifications to get [Theorem 1.7](#) and [Theorem 1.9](#).

**Plan of Part 1.** We organize Part 1 as following. In [Chapter 2](#), we give the necessary materials for the results. In particular we give a review for the singularities involved and recall the relevant results in the minimal model program. In [Chapter 3](#), we will give several classifications for polarized varieties  $(X, L)$ . In [Section 3.1](#), we prove [Theorem 1.7](#) by running MMP ([Lemma 3.5](#)) to reduce the proof to proving [Lemma 3.8](#) and [Lemma 3.10](#). In [Section 3.2](#), we prove [Theorem 1.7](#) thanks to canonical modifications [Theorem 2.25](#) and use similar methods to prove [Proposition 1.8](#). In [Section 3.3](#), for a polarized slc variety  $(X, L)$ , we use [Proposition 1.8](#) on the triple  $(\bar{X}, \bar{D}, L')$ , where  $(\bar{X}, \bar{D})$  is the normalization of  $X$  and the conductor divisor on  $\bar{X}$  and  $L'$  is the pullback of  $L$ , to get [Theorem 1.9](#).



## CHAPTER 2

### Preliminaries

#### 2.1. Conventions

We work over  $\mathbb{C}$ . The definitions and conventions that we adapt follow [Kol13][KM98].

##### 2.1.1. Divisors.

- A scheme is supposed to be separated and of finite type over  $\mathbb{C}$ . A variety is a reduced and irreducible scheme over  $\mathbb{C}$ . A point  $x$  in a scheme  $X$  is an element  $x$  in the underlying topological space of  $X$ . A scheme is said to be normal if  $\mathcal{O}_{X,x}$  is a normal local domain for all point  $x \in X$ .
- We denote by  $\text{Weil}(X)$  the group of Weil-divisors of a scheme  $X$  and  $\text{Cl}(X)$  its quotient modulo principal divisors [Sta22, Tag 0BE2]. We define

$$\text{Weil}(X)_{\mathbb{Q}} := \text{Weil}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$$

to be the group of  $\mathbb{Q}$ -Weil-divisors.

- Let  $X$  be a scheme and  $\mathcal{F}$  a coherent  $\mathcal{O}_X$ -module. Let  $\mathcal{M}_X$  be the sheaf of germs of meromorphic functions over  $X$ . We say that  $\mathcal{F}$  is *torsion-free* if the natural map

$$\mathcal{F} \rightarrow \mathcal{F} \otimes \mathcal{M}_X$$

is injective.

The *reflexive hull* of  $\mathcal{F}$  is its double dual  $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X), \mathcal{O}_X)$  and we have a natural morphism

$$j_{\mathcal{F}} : \mathcal{F} \rightarrow \mathcal{H}om_{\mathcal{O}_X}(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X), \mathcal{O}_X)$$

by sending local section  $a$  of  $\mathcal{F}$  to the local section  $(\phi \mapsto \phi(a))$ . We say that  $\mathcal{F}$  is *torsionless* if  $j_{\mathcal{F}}$  is an injection. Clearly torsionlessness implies torsion free and we have the inverse when  $\mathcal{O}_{X,x}$  is integral for all  $x \in X$ .

We call  $\mathcal{F}$  a *reflexive module* or a *reflexive sheaf* if  $j_{\mathcal{F}}$  is an isomorphism. When  $\mathcal{F}$  is reflexive, if there exists an open subset  $U \subset X$  such that  $\mathcal{F}|_U$  is locally free of rank  $r$ , we define the *rank* of  $\mathcal{F}$  to be  $\text{rk}(\mathcal{F}) = r$ . For any reflexive sheaf  $\mathcal{F}$ , we set

$$\mathcal{F}^{[m]} := \mathcal{H}om_{\mathcal{O}_X}(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}^{\otimes m}, \mathcal{O}_X), \mathcal{O}_X).$$

- For a scheme  $X$ , its rank one reflexive sheaves form an group under the group operation

$$\mathcal{F}_1 \hat{\otimes} \mathcal{F}_2 := \mathcal{H}om_{\mathcal{O}_X}(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}_1 \otimes \mathcal{F}_2, \mathcal{O}_X), \mathcal{O}_X).$$

When  $X$  is normal, we may associate for  $D \in \text{Weil}(X)$  a rank one reflexive sheaf  $\mathcal{O}_X(D)$ , which induces a group isomorphism between  $\text{Cl}(X)$  and the group of rank one reflexive sheaves (cf. [Sta22, Tag 0EBM]).



- We denote by  $\text{Div}(X) = \Gamma(X, \mathcal{M}_X^*/\mathcal{O}_X^*)$  the group of Cartier divisors of  $X$  and define  $\text{Div}(X)_{\mathbb{Q}} := \text{Div}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ .
- Let  $X$  be a scheme. We have a natural morphism  $\text{cyc} : \text{Div}(X) \rightarrow \text{Weil}(X)$  (cf. [Gro67, IV.21.6.7.]). The morphism extends naturally to  $\text{cyc}_{\mathbb{Q}} : \text{Div}(X)_{\mathbb{Q}} \rightarrow \text{Weil}(X)_{\mathbb{Q}}$ . We say that a  $\mathbb{Q}$ -Weil-divisor  $D$  of  $X$  is  $\mathbb{Q}$ -Cartier if it is in the image of  $\text{cyc}_{\mathbb{Q}}$ .

When  $X$  is normal, the morphism  $\text{cyc}$  is injective. Let  $D$  be a  $\mathbb{Q}$ -Weil-divisor of  $X$ . We have that the followings are equivalent

- (1)  $D$  is  $\mathbb{Q}$ -Cartier;
  - (2) There exists an integer  $m$  such that  $mD \in \text{Weil}(X)$  and the rank 1 reflexive sheaf  $\mathcal{O}_X(mD)$  is invertible.
- Let  $X$  be a scheme. For two  $\mathbb{Q}$ -Cartier divisors  $D_1$  and  $D_2$ , we say that  $D_1$  and  $D_2$  are  $\mathbb{Q}$ -linearly equivalent, if  $D_1 - D_2$  is a  $\mathbb{Q}$ -combination of principal divisors and we denote it by  $D_1 \sim_{\mathbb{Q}} D_2$ ; we say that  $D_1$  and  $D_2$  are numerically equivalent if for any irreducible curve  $C \subset X$ , we have that  $D_1 \cdot C = D_2 \cdot C$  and we denote this by  $D_1 \equiv_{\text{num}} D_2$ .
  - Let  $f : X \dashrightarrow Y$  be a rational map between schemes. Let  $Z \subset X$  be a subscheme of  $X$ . If  $f$  is defined on an open dense subset  $Z^0 \subset Z$ , we define the strict transform of  $Z$  by  $f$  to be the closure of  $f(Z^0)$  in  $Y$ . If  $g : Y \rightarrow X$  is a birational morphism, and  $D \in \text{Weil}(X)$  be a prime divisor of  $X$ , we denote by  $g_*^{-1}(D)$  the strict transform of  $D$  by  $g^{-1}$ , which is a prime divisor on  $Y$ . We may thus define a  $\mathbb{Q}$ -linear map  $g_*^{-1} : \text{Weil}(X)_{\mathbb{Q}} \rightarrow \text{Weil}(Y)_{\mathbb{Q}}$  by mapping  $\Delta = \sum d_i D_i$  to  $g_*^{-1}(\Delta) := \sum d_i g_*^{-1}(D_i)$ , here  $g_*^{-1}(D_i) = 0$  if  $D_i$  is contained in the closure of  $g(\text{exc}(g))$ .

**2.1.2. Projectivisation.** Let  $X$  be a scheme and  $\mathcal{S} = \bigoplus_{n \geq 0} \mathcal{S}_n$  a quasi-coherent  $\mathbb{N}$ -graded  $\mathcal{O}_X$ -algebra. We recall that the relative Proj of  $\mathcal{S}$  (cf. [Gro60, II.3.1]), the  $X$ -scheme  $Y = \text{Proj}(\mathcal{S}) \rightarrow X$  is defined by gluing over each affine open  $\text{Spec}(A) = U \subset X$  the  $U$ -scheme  $Y_U := \text{Proj}(\Gamma(U, \mathcal{S})) \rightarrow U = \text{Spec}(A)$ .

For a quasi-coherent  $\mathcal{O}_X$ -module  $\mathcal{E}$ , we denote by

$$\mathbb{P}(\mathcal{E}) := \text{Proj}(\bigoplus_{n \geq 0} \text{Sym}^n(\mathcal{E}))$$

its projectivisation.

The projectivisation  $\mathbb{P}(\mathcal{E})$  has the following characterization:

**Proposition 2.1** ([Gro60, Propostion II.4.2.3]). *Let  $X$  be a scheme and  $\mathcal{E}$  a quasi-coherent  $\mathcal{O}_X$ -module. For any  $X$ -scheme  $f : Y \rightarrow X$ , there is a bijective between  $X$ -morphisms  $\text{Hom}_X(Y, \mathbb{P}(\mathcal{E}))$  and the rank 1 quotient of  $f^*(\mathcal{E})$ , that is the set  $\{f^*(\mathcal{E}) \twoheadrightarrow \mathcal{L} : \text{where } \mathcal{L} \text{ is a line bundle on } Y\} / \sim$ , where the equivalence is given by  $\alpha : f^*(\mathcal{E}) \twoheadrightarrow \mathcal{L}_1$  being equivalent to  $\beta : f^*(\mathcal{E}) \twoheadrightarrow \mathcal{L}_2$  if there exists a isomorphism  $\gamma : \mathcal{L}_1 \rightarrow \mathcal{L}_2$  such that  $\gamma \circ \alpha = \beta$ .*

**2.1.3. Positivity notions.** Let  $X$  be a proper scheme over  $\mathbb{C}$  and  $L$  a line bundle on  $X$ . We follow the notions introduced in [Laz04a].

- We say that  $L$  is ample if there exists positive integers  $m, n$  and a closed embedding  $X \hookrightarrow \mathbb{P}^n$  such that  $L \cong \mathcal{O}_{\mathbb{P}^n}(1)|_X$ .
- We say that  $L$  is nef if for any curve  $C \subset X$ , we have that  $L \cdot C \geq 0$ .

- We say that a Cartier divisor  $D \in \text{Div}(X)$  is ample (*resp.* nef) if  $\mathcal{O}_X(D)$  is ample (*resp.* nef).
- For a  $\mathbb{Q}$ -divisor  $D \in \text{Div}(X)_{\mathbb{Q}}$ , we say that  $D$  is ample if  $D$  can be written as  $D = \sum_i r_i D_i$  with  $r_i > 0$  and  $D_i \in \text{Div}(X)$  ample divisors.
- For a  $\mathbb{Q}$ -divisor  $D \in \text{Div}(X)_{\mathbb{Q}}$ , we say that  $D$  is nef if  $D$  can be written as  $D = \sum_i r_i D_i$  with  $r_i \geq 0$  and  $D_i \in \text{Div}(X)$  nef divisors.
- Being ample (*resp.* nef) is an numerical property, *i.e.*, if  $D_1, D_2 \in \text{Div}(X)_{\mathbb{Q}}$  and  $D_1 \equiv_{\text{num}} D_2$  then  $D_1$  is ample (*resp.* nef) iff  $D_2$  is ample (*resp.* nef).
- When  $X$  is an irreducible projective variety and  $D \in \text{Div}(X)_{\mathbb{Q}}$  is a  $\mathbb{Q}$ -divisor, we say that  $D$  is *big* if  $D \equiv_{\text{num}} A + E$  where  $A$  is ample and  $E$  is effective (*cf.* [Laz04a, Corollary 2.2.7.]). A line bundle  $L$  on  $X$  is big if there exists a big divisor  $D$  such that  $L \cong \mathcal{O}_X(D)$ .

## 2.2. Singularities of Pairs, MMP

The general reference for this section is [KM98, Chapter 2, Chapter 3][Kol13, Chapter 2].

**Definition 2.2** (Pairs).

- (0) Let  $D = \sum_i m_i D_i$  be a  $\mathbb{Q}$ -divisor on a scheme  $X$ , where  $D_i$  are prime divisors. We call  $D$  a *subboundary* if  $m_i \leq 1$  for all  $i$ ; we call  $D$  a *boundary* if  $0 \leq m_i \leq 1$ . For a real number  $r$ , we denote by  $[r]$  its integral part and set  $\lceil r \rceil := -\lfloor -r \rfloor$ . We set  $\lfloor D \rfloor := \sum_i \lfloor m_i \rfloor D_i$  and  $\lceil D \rceil := \sum_i \lceil m_i \rceil D_i$ . We say that  $D$  is *reduced* if  $m_i \in \{0, 1\}$ . We say that  $D$  has *standard coefficients* if  $m_i = 1 - \frac{1}{n_i}$  for some  $n_i \in \mathbb{N}_{>0}$  for all  $i$ .
- (1) A *log pair* is a scheme  $X$  together with a boundary divisor  $\Delta$ . We denote the log pair by  $(X, \Delta)$ .
- (2) A *pair* is a variety  $X$  together with a subboundary divisor  $\Delta$  such that  $K_X + \Delta$  is  $\mathbb{Q}$ -Cartier. We denote the pair by  $(X, \Delta)$ .

**Definition 2.3** (Simple normal crossing). Let  $X$  be a scheme and  $D = \sum_i m_i D_i$ ,  $m_i \neq 0$ , a  $\mathbb{Q}$ -divisor. We say that  $(X, D)$  has *simple normal crossing* (snc), if  $X$  and  $D_i$ 's are smooth and  $D_i$  intersects transversely. For a normal scheme  $Y$  with a  $\mathbb{Q}$ -divisor  $\Delta$ , there is a maximal open  $U \subset Y$  such that  $(U, \Delta|_U)$  is snc. We set  $\text{non-snc}(Y, \Delta) := Y \setminus U$ .

**Definition 2.4** (Log resolution). Let  $X$  be a variety and  $D$  be a Weil-divisor on  $X$ . A log resolution of  $(X, D)$  is a proper birational morphism  $r : X' \rightarrow X$  such that

- $X'$  is smooth;
- The exceptional locus  $\text{exc}(r)$  has pure codimension 1;
- $(X', r_*^{-1}(D) + \text{exc}(r))$  has simple normal crossing.

The existence of log resolutions is first proved by Hironaka in [Hir64a] [Hir64b]. In [Sza94], Szabó showed a strengthened version that we can even take  $r$  to be an isomorphism over the snc locus of  $(X, D)$ .

**Definition 2.5** (Canonical classes). Let  $X$  be a normal variety. We denote by  $i : X_{\text{reg}} \rightarrow X$  the inclusion morphism. The push-forward  $i_*(\det(\omega_{X_{\text{reg}}}))$  is a rank one reflexive sheaf which corresponds to a divisor class  $K_X$ . We call  $K_X$  the *canonical class* and  $\omega_X = \mathcal{O}_X(K_X)$  the *canonical sheaf*.

When  $X$  is not normal, under some assumptions it is still possible to define the canonical class  $K_X$  of  $X$ . We refer the readers to [Kol13, Definition 1.6.]

**Definition 2.6.** Let  $(X, \Delta)$  be a pair with  $X$  normal. Write  $\Delta = \sum_{j \in J} d_j D_j$  with  $D_j$  prime divisors and  $d_j \in \mathbb{Q}$ . For a birational morphism  $f : Y \rightarrow X$  from a normal variety  $Y$  to  $X$ , we have that

$$K_Y + f_*^{-1}(\Delta) \sim_{\mathbb{Q}} f^*(K_X + \Delta) + \sum_{i \in I} a(E_i, X, \Delta) E_i,$$

where the sum is taken over exceptional divisors of  $f$ . The expression above is unique. We set  $a(F_j, X, \Delta) := -d_j$ , where  $F_j = f_*^{-1}(D_j)$  is the strict transforms of  $D_j$ .

For  $k \in I \cup J$ , we call  $a(G_k, X, \Delta)$  the *discrepancy* of  $G_k$  with respect to  $(X, \Delta)$ , where  $G_k = F_k$  if  $k \in J$  and  $G_k = E_k$  if  $k \in I$ . The discrepancy  $a(G_k, X, \Delta)$  only depends on  $G_k$  but not on  $f$ .

**Definition 2.7.** Let  $X$  be a normal variety. A *divisor over  $X$*  is a triple  $(E, Y, f)$  where  $E \in \text{Weil}(Y)$ ,  $Y$  is a normal variety and  $f : Y \rightarrow X$  is a birational morphism. It turns out that the discrepancy  $a(E, X, \Delta)$  defined in Definition 2.6 is determined by its local ring  $\mathcal{O}_{Y,E}$  in  $K(Y) = K(X)$  but not by the choice of  $f$  and  $Y$  (cf. [KM98, Remark 2.23]). We often omit  $Y$  and  $f$ . We call the closure of  $f(E)$  the *center* of  $E$  and denote it by  $\text{cent}_X(E)$ .

**Definition 2.8.** Let  $(X, \Delta)$  be a pair. We define singularities that we will use in the thesis.

- We say that  $(X, \Delta)$  has *canonical singularities*, if  $X$  is normal and for every exceptional divisor  $E$  over  $X$ , we have that  $a(E, X, \Delta) \geq 0$ ;
- We say that  $(X, \Delta)$  has *Kawamata log terminal singularities* (*klt singularities* for short), if  $X$  is normal and for all  $E$  over  $X$ , we have that  $a(E, X, \Delta) > -1$ ;
- We say that  $(X, \Delta)$  has *log canonical singularities* (*lc singularities* for short), if  $X$  is normal and for all  $E$  over  $X$ , we have that  $a(E, X, \Delta) \geq -1$ ;
- We say that  $(X, \Delta)$  has *divisorial log terminal singularities* (*dlt singularities* for short), if  $X$  is normal and for all  $E$  with  $\text{cent}_X(E) \subset \text{non-snc}(X, \Delta)$ , we have that  $a(E, X, \Delta) > -1$ .

### 2.3. Minimal model program for pairs

The classical references are [KM98] and [Kol13]. We also refer the readers to [Fuj11] for results in the log canonical setup.

**2.3.1. Cones.** We first recall relevant cones that will appear in the thesis.

**Definition 2.9** (Cones 1). Let  $X$  be a projective variety.

- The *Neron-Severi group*  $N^1(X)$  of  $X$  is defined to be the quotient  $N^1(X) := \text{Div}(X) / \equiv_{\text{num}}$ . We denote by  $N^1(X)_{\mathbb{Q}} := N^1(X) \otimes_{\mathbb{Z}} \mathbb{Q}$  and  $N^1(X)_{\mathbb{R}} := N^1(X) \otimes_{\mathbb{Z}} \mathbb{R}$  the group of  $\mathbb{Q}$ -divisors and the group of  $\mathbb{R}$ -divisors respectively. Note that we have the inclusion  $N^1(X)_{\mathbb{Q}} \subset N^1(X)_{\mathbb{R}}$ .
- The Neron-Severi groups  $N^1(X)$  is a free Abelian group of finite rank (cf. [Laz04a, Proposition 1.1.16.]). We define the *Picard number*  $\rho(X)$  of  $X$  to be  $\rho(X) := \text{Rank}(N^1(X))$ . The finite dimensional vector space  $N^1(X)_{\mathbb{R}}$  thus has a well-defined Euclidean topology.

- The *ample cone*  $\text{Amp}(X) \subset \mathbb{N}^1(X)_{\mathbb{R}}$  is defined to be the convex cone generated by ample  $\mathbb{Q}$ -divisors.
- The *nef cone*  $\text{Nef}(X) \subset \mathbb{N}^1(X)_{\mathbb{R}}$  is defined to be the convex cone generated by nef  $\mathbb{Q}$ -divisors. We have that  $\overline{\text{Amp}(X)} = \text{Nef}(X)$  and  $\text{Int}(\text{Nef}(X)) = \text{Amp}(X)$  (cf. [Laz04a, Theorem 1.4.23]).
- The *big cone*  $\text{Big}(X) \subset \mathbb{N}^1(X)_{\mathbb{R}}$  is defined to be the convex cone generated by big  $\mathbb{Q}$ -divisors.
- The *pseudo-effective cone*  $\text{Pseff}(X) \subset \mathbb{N}^1(X)_{\mathbb{R}}$  is defined to be the convex cone generated by the divisor classes that can be represented by effective  $\mathbb{Q}$ -divisors. We have that  $\overline{\text{Big}(X)} = \text{Pseff}(X)$  and  $\text{Int}(\text{Pseff}(X)) = \text{big}(X)$  (cf. [Laz04a, Theorem 2.2.26]).

Dually, we may define cones in the 1-cycles.

**Definition 2.10** (Cones 2). Let  $X$  be quasi-projective variety.

- (1) We define the *numerical equivalence classes of 1-cycles* to be

$$\mathbb{N}_1(X) := \mathbb{Z}_1(X) \otimes_{\mathbb{Z}} \mathbb{R} / \equiv$$

where  $\mathbb{Z}_1(X)$  is group of 1-cycles of  $X$  and the equivalent relation is given by  $C_1 \equiv C_2$  iff for any line bundle  $L$  over  $X$  we have that  $L \cdot C_1 = L \cdot C_2$ . It comes from the definition that we have a perfect pairing

$$\mathbb{N}^1(X)_{\mathbb{R}} \times \mathbb{N}_1(X) \rightarrow \mathbb{R}, (\delta, \gamma) \mapsto \delta \cdot \gamma$$

We define the *Mori cone*  $\overline{\text{NE}}(X)$  of  $X$  to be the closure of the cone in  $\mathbb{N}_1(X)$  generated by cycles represented by positive combinations of irreducible curves.

- (2) Let  $C$  be a cone in a finite dimensional real vector space  $V$ . A *face*  $F$  of  $C$  is a sub-cone of  $C$ . When  $F$  has dimension 1, we call it a *ray*. A face  $F \subset C$  is called *extremal* if  $F$  satisfies the following property:

For any  $x, y \in C$ , we have that  $x + y \in F$  implies  $x \in F$  and  $y \in F$ .

For a linear function  $l \in V^{\vee}$ , we denote  $l < 0$  the sub-cone  $\{x \in C : l(x) < 0\}$ . We say that  $F \subset C$  is *l-negative* if  $F \subset (l < 0)$ .

- (3) Let  $f : X \rightarrow S$  be a projective morphism between two varieties. We define the *relative cone* to be  $\overline{\text{NE}}(X/S) := \ker(f_* : \overline{\text{NE}}(X) \rightarrow \overline{\text{NE}}(S))$ . The relative cone  $\overline{\text{NE}}(X/S) \subset \overline{\text{NE}}(X)$  is an closed extremal face (cf. [Deb01, Lemma 6.7.]). We define the *relative Picard number*  $\rho(X/S)$  to be the dimension of  $\overline{\text{NE}}(X/S)$ . We also denote  $\overline{\text{NE}}(X/S)$  by  $\overline{\text{NE}}(f)$

With the language of cones, we can give a characterization for nef and ample divisors.

**Proposition 2.11** (cf. [Laz04a, Proposition 1.4.28., Theorem 1.4.29.]). *Let  $X$  be a proper variety.*

- (1) *The Mori cone  $\overline{\text{NE}}(X)$  is dual to the nef cone  $\text{Nef}(X)$ , i.e., we have that*

$$\overline{\text{NE}}(X) = \{\gamma \in \mathbb{N}_1(X) : \delta \cdot \gamma \geq 0 \text{ for all } \delta \in \text{Nef}(X)\}$$

- (2) *If  $X$  is projective, then for  $D \in \mathbb{N}^1(X)_{\mathbb{R}}$ , we have that  $D \in \text{Amp}(X)$  iff for any  $\gamma \in \overline{\text{NE}}(X) \setminus 0$  the product  $D \cdot \gamma$  is positive.*

It is also possible to describe the dual cone in  $\mathbb{N}_1(X)$  of the pseudo-effective cone  $\text{Pseff}(X)$ .

**Definition 2.12** ([Laz04b, Definition 11.4.6.]). Let  $X$  be a projective variety of dimension  $n$ . A class  $\gamma \in N_1(X)$  is *movable* if there exists a birational morphism  $\mu : X' \rightarrow X$  from a projective variety  $X'$  to  $X$ , together with  $n - 1$  ample classes  $a_1, a_2, \dots, a_{n-1} \in \text{Amp}(X')$  such that

$$\gamma = \mu_*(a_1 \cdot a_2 \cdot \dots \cdot a_{n-1}).$$

The *movable cone*  $\overline{\text{Mov}}(X) \subset N_1(X)$  is the closed cone generated by all the movable classes.

As  $X$  is projective, the movable cone is also the closed cone generated by the classes of a flat family of curves  $C_t$  in  $X$  such that  $\bigcup_t C_t = X$  ([BDPP13, Theorem 1.5.(ii)]).

We have the following theorem.

**THEOREM 2.13** (BDPP theorem, [BDPP13, Theorem 2.2., Theorem 2.4.]). *Let  $X$  be a projective variety. Then the cones*

$$\overline{\text{Mov}}(X) \text{ and } \text{Pseff}(X)$$

*are dual.*

We finally give some remarks on pseudo-effectiveness to end this subsection. When  $X$  is a normal projective variety, it is possible to define pseudo-effectiveness for reflexive sheaves on  $X$ :

**Definition 2.14** ([HP19, Definition 2.1.]). Let  $X$  be a normal projective variety and  $\mathcal{E}$  a reflexive sheaf on  $X$ . We say that  $\mathcal{E}$  is *pseudo-effective* if there exists an ample divisor  $H$  on  $X$  satisfying the following:

For any  $c > 0$  there exists integers  $j > 0$  and  $i > jc$  such that

$$H^0(X, S^{[i]}(\mathcal{E}) \otimes \mathcal{O}_X(jH)) \neq 0$$

where  $S^{[i]}(\mathcal{E})$  is the double dual of  $\text{Sym}^i(\mathcal{E})$ .

The following lemma implies that the above definition for line bundles coincides with the usual definition by pseudo-effective cones.

**Lemma 2.15** ([HP19, Lemma 2.3.]). *Let  $X$  be a normal projective variety and  $\mathcal{E}$  a reflexive sheaf on  $X$ . Let  $\tau \in N^1(\mathbb{P}(\mathcal{E}))_{\mathbb{R}}$  be the class of the tautological bundle  $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ . We have that  $\tau \in \text{Pseff}(\mathbb{P}(\mathcal{E}))$  iff  $\mathcal{E}$  is pseudo-effective in the sense of [Definition 2.14](#).*

Let  $X$  be a normal projective variety and  $L$  be a line bundle. The structure morphism  $\pi : \mathbb{P}(L) \rightarrow X$  is an isomorphism and  $\mathcal{O}_{\mathbb{P}(L)}(1) \cong \pi^*(L)$ . Hence by above lemma  $L \in \text{Pseff}(X)$  iff  $L$  is pseudo-effective in sense of [Definition 2.14](#). If  $D \in \text{Weil}(X)$  is  $\mathbb{Q}$ -Cartier, suppose that  $mD$  is a Cartier divisor for  $m \in \mathbb{N}$ . Then  $D \in \text{Pseff}(X)$  is equivalent to  $mD \in \text{Pseff}(X)$ . On the other hand we have that  $\mathcal{O}_X(mD) = \mathcal{O}_X(D)^{[m]}$ . It's easy to see that  $\mathcal{O}_X(D)^{[m]}$  is pseudo-effective in sense of [Definition 2.14](#) iff  $\mathcal{O}_X(D)$  is pseudo-effective in sense of [Definition 2.14](#). Hence we get the following:

Let  $D \in \text{Weil}(X)$  be  $\mathbb{Q}$ -Cartier on a normal projective variety. We have that  $D \in \text{Pseff}(X)$  if and only if  $\mathcal{O}_X(D)$  is pseudo-effective.

**2.3.2. Minimal model program for pairs.** We begin by a lemma which says that morphism between projective varieties is determined by its relative cone.

**Lemma 2.16** ([Deb01, Proposition 1.14]). *Let  $X, Y, Y'$  be three projective varieties, and morphisms*

$$\pi : X \rightarrow Y \text{ and } \pi' : X \rightarrow Y'.$$

*Suppose that  $\mathcal{O}_Y \simeq \pi_*(\mathcal{O}_X)$  and  $\overline{\text{NE}}(X/Y) \subset \overline{\text{NE}}(X/Y')$ . Then there exists a unique morphism  $f : Y \rightarrow Y'$  such that  $f \circ \pi = \pi'$ .*

**Lemma 2.16** also shows that it is helpful to understand the structure of the relative cone when studying morphisms between projective varieties. We now state the cone theorem, which gives a description of the structure of the relative cone.

**THEOREM 2.17** (Cone Theorem, [Fuj11, Theorem 1.1.]). *Let  $(X, \Delta)$  be a pair as in Definition 2.2. Suppose that  $(X, \Delta)$  has log canonical singularities. Let  $\pi : X \rightarrow S$  be a projective morphism onto a variety  $S$ . Then*

$$\overline{\text{NE}}(X/S) = \overline{\text{NE}}(X/S)_{K_X + \Delta \geq 0} + \sum R_j$$

with the following properties:

- (1)  $R_j$  is a  $(K_X + \Delta)$ -negative extremal ray;
- (2) For any  $\pi$ -ample  $\mathbb{Q}$ -divisor  $A$ , there are only finite many  $R_j$ 's included in  $(K_X + \Delta + A) \cdot R_j < 0$ . In particular, the  $R_j$ 's are discrete in the half space  $(K_X + \Delta) \cdot R_j < 0$  and the sum of  $R_j$  is indexed over a countable set;
- (3) Let  $F \subset \overline{\text{NE}}(X/S)$  be an extremal face such that  $F \cap \overline{\text{NE}}(X/S)_{K_X + \Delta \geq 0} = 0$ . There exists a contraction morphism  $\text{cont}_F : X \rightarrow Y$  over  $S$ .
  - (i) Let  $C \subset X$  be a curve such that  $\pi(C)$  is a point. We have that  $[C] \in F$  if and only if  $\text{cont}_F(C)$  is a point;
  - (ii)  $\mathcal{O}_Y \cong (\text{cont}_F)_*(\mathcal{O}_X)$ ;
  - (iii) Let  $L$  be a line bundle on  $X$  such that  $L \cdot C = 0$  for any curve  $C$  with  $[C] \in F$ . Then there exists a line bundle  $L_Y$  on  $Y$  such that  $L = \text{cont}_F^*(L_Y)$ .
- (4) Every  $(K_X + \Delta)$ -negative extremal ray  $R_j$  is spanned by a rational curve  $C_j$  with  $0 < -(K_X + \Delta) \cdot C_j \leq 2 \dim(X)$ .

We remark that (3) in **Theorem 2.17** is also known as the *contraction theorem*.

We will also use the Kawamata's rationality theorem.

**THEOREM 2.18** (Rationality theorem, [KM98, Theorem 3.5., Complement 3.6.]). *Let  $(X, \Delta)$  be a proper klt pair with  $\Delta$  effective. Suppose that  $K_X + \Delta$  is not nef. Let  $H$  be a nef and big line Cartier divisor on  $X$ . Then the following number*

$$r(H) := \sup\{t \in \mathbb{R} : H + t(K_X + \Delta) \text{ is nef}\}$$

*is a non-negative rational number. Further more, there exists a  $(K_X + \Delta)$ -negative extremal ray  $R$  such that  $(H + r(H)(K_X + \Delta)) \cdot R = 0$ .*

We give a little more details about the contraction map.

**Lemma 2.19** ([KM98, Proposition 2.5]). *Let  $\phi : X \rightarrow Y$  be a morphism between projective varieties. Suppose that  $X$  is normal and  $\mathbb{Q}$ -factorial. Suppose that  $\overline{\text{NE}}(X/Y)$  is a 1-dimensional cone. Then one of the following holds:*



- (1) (fiber type contraction)  $\dim(X) > \dim(Y)$  and  $\phi$  is said to be a fiber type contraction;
- (2) (divisorial contraction)  $\phi$  is birational and the exceptional locus  $\text{exc}(\phi)$  is an irreducible divisor  $E$  and  $\phi$  is said to be a divisorial contraction;
- (3) (small contraction)  $\phi$  is birational and  $\text{codim}_X(\text{exc}(\phi)) \geq 2$  and  $\phi$  is said to be a small contraction.

Let  $(X, \Delta)$  be a pair and let  $\phi : X \rightarrow X'$  be a birational morphism. Set  $\Delta' = \phi_*(\Delta)$ . We now consider  $(X', \Delta')$ . If  $\phi$  is divisorial and it contracts a  $K_X + \Delta$ -negative extremal ray, then by [KM98, Corollary 3.43, Corollary 3.44], we have that  $(X', \Delta')$  is canonical (*resp.* klt *resp.* dlt *resp.* lc) if  $(X, \Delta)$  is canonical (*resp.* klt *resp.* dlt *resp.* lc).

When  $\text{codim}_X(\text{exc}(\phi)) \geq 2$  and  $K_X + \Delta$  is not relatively numerically trivial over  $X'$ , we know that  $K'_{X'} + \Delta'$  is not  $\mathbb{Q}$ -Cartier.<sup>1</sup> The study of  $(X', \Delta')$  becomes complicated as we can no longer use [Theorem 2.17](#). One way to circumvent this inconvenience is to introduce a new birational operation.

**Definition 2.20** ([KM98, Definition 3.33]). Let  $X$  be a normal variety and  $D$  be a  $\mathbb{Q}$ -divisor such that  $K_X + D$  is  $\mathbb{Q}$ -Cartier. A *flipping contraction* is a proper birational morphism  $f : X \rightarrow Y$  to a normal scheme such that  $-(K_X + D)$  is  $f$ -ample and  $\text{codim}_X(\text{exc}(f)) \geq 2$ .

A  $(K_X + D)$ -flip is a normal variety  $X^+$  together with a proper birational morphism  $f^+ : X^+ \rightarrow Y$  such that:

- (1) Set  $\phi := (f^+)^{-1} \circ f$ , and set  $D^+ = \phi_*(D)$ . Then we have that  $K_{X^+} + D^+$  is  $\mathbb{Q}$ -Cartier;
- (2)  $K_{X^+} + D^+$  is  $f^+$ -ample;
- (3) The exceptional locus  $\text{exc}(f^+)$  has codimension at least 2.

By abuse of language, we also call the birational map  $\phi : X \dashrightarrow X^+$  the  $(K_X + D)$ -flip. We know ([KM98, Corollary 3.43, Corollary 3.44]) that  $(X^+, D^+)$  is canonical (*resp.* klt *resp.* dlt *resp.* lc) if  $(X, D)$  is canonical (*resp.* klt *resp.* dlt *resp.* lc).

We now give a sketch of the inductive procedure known as "running the minimal model program". Let  $(X, \Delta)$  be a klt pair such that  $X$  is  $\mathbb{Q}$ -factorial. Suppose that  $K_X + \Delta$  is not nef. By [Theorem 2.17](#), we may write

$$\overline{\text{NE}}(X) = \overline{\text{NE}}(X)_{(K_X + \Delta) \geq 0} + \sum \mathbb{R}_{\geq 0}[C_i].$$

By (4) of [Theorem 2.17](#), we may consider the contraction morphism for an extremal ray  $R = \mathbb{R}_{\geq 0}[C]$ . Hence we get  $\text{cont}_R : X \rightarrow Y$ .

- (1) If  $\phi$  is fiber type, the MMP terminates with a Mori fiber space;
- (2) If  $\phi$  is divisorial, set  $(X_1, \Delta_1) = (Y, \phi_*(\Delta))$ ;
- (3) If  $\phi$  is small and the  $(K_X + \Delta)$ -flip  $X \dashrightarrow X^+$  exists, set  $(X_1, \Delta_1) = (X^+, (X \dashrightarrow X^+)_*(\Delta))$

If we are in the case (2) or case (3), we may then consider the klt pair  $(X_1, \Delta_1)$  to produce  $(X_2, \Delta_2)$  and so on. Thus we get a sequence of birational maps:

<sup>1</sup>Suppose contrarily that  $K'_{X'} + \Delta'$  is  $\mathbb{Q}$ -Cartier. Take an integer  $m \gg 0$  such that  $m(K_X + \Delta) = D$  and  $m(K'_{X'} + \Delta') = D'$  where  $D, D'$  are Cartier divisors. We know that  $K_X + \Delta$  and  $K'_{X'} + \Delta'$  are identified by  $\phi$  over  $X \setminus \text{exc}(\phi)$ . Hence we have that  $\phi^*(D')|_{X \setminus \text{exc}(\phi)} = D|_{X \setminus \text{exc}(\phi)}$ . As  $X$  is normal and  $\text{codim}_X(\text{exc}(\phi)) \geq 2$ , we have that  $D = \phi^*(D')$  which is relatively numerically trivial, a contradiction.

$$(X, \Delta) = (X_0, \Delta_0) \dashrightarrow (X_1, \Delta_1) \dashrightarrow \cdots \dashrightarrow (X_n, \Delta_n).$$

[KMM87, Figure 2] gives a clear description of this process. When  $X_i \dashrightarrow X_{i+1}$  is a divisorial contraction, the relative Picard number  $\rho(X_i/X_{i+1})$  equals 1. Hence in the above program, we have at most  $\rho(X)$  divisorial contraction. The existence of flips is established in [BCHM10, Corollary 1.4.1]. Hence the problem of termination of MMP is thus the problem of termination of flips. The termination of flips in dimension 3 is proved by Kawamata in [Kaw92]. The termination of flips in dimension  $\geq 4$  is still open.

We end this subsection with two theorems from [BCHM10] that give sufficient condition for a minimal program to end.

**THEOREM 2.21** ([BCHM10, Theorem 1.2.]). *Let  $(X, \Delta)$  be a klt pair. Let  $\pi : X \rightarrow U$  be a projective morphism between quasi-projective varieties.*

*Assume that either  $\Delta$  is  $\pi$ -big and  $K_X + \Delta$  is  $\pi$ -pseudo-effective or  $K_X + \Delta$  is  $\pi$ -big. Then*

- (1)  $K_X + \Delta$  has a log terminal model over  $U$ ,
- (2) if  $K_X + \Delta$  is  $\pi$ -big, then  $K_X + \Delta$  has a log canonical model over  $U$ , and
- (3) the  $\mathcal{O}_U$ -algebra

$$\mathfrak{R}(\pi, K_X + \Delta) = \bigoplus_{m \in \mathbb{N}} \pi_* \mathcal{O}_X(\lfloor m(K_X + \Delta) \rfloor)$$

*is finitely generated.*

**THEOREM 2.22** ([BCHM10, Corollary 1.3.3.]). *Let  $(X, \Delta)$  be a klt pair and suppose  $X$  is  $\mathbb{Q}$ -Cartier. Let  $\pi : X \rightarrow U$  be a projective morphism between normal quasi-projective varieties. Suppose that  $K_X + \Delta$  is not  $\pi$ -pseudo-effective.*

*Then we may run  $f : X \dashrightarrow Y$  a  $(K_X + \Delta)$ -MMP over  $U$  and end with a Mori fiber space  $g : Y \rightarrow W$  over  $U$ .*

## 2.4. Some results on modifications

Let  $(X, \Delta)$  be a pair with  $X$  normal. We can find a birational morphism  $g : X' \rightarrow X$  such that  $(X', g_*^{-1}(\Delta))$  has mild singularities and  $K_{X'} + g_*^{-1}(\Delta)$  has some positivities. We list here the results that we need in the thesis. The reference for this section is [Koll13, Chapter 1.4.].

**Lemma 2.23** (Small  $\mathbb{Q}$ -factorial modification, cf. [Koll13, Corollary 1.37]). *Let  $(X, \Delta)$  be dlt and  $\Delta$  a boundary. Then there is a proper birational morphism  $g : X^{\text{qf}} \rightarrow X$  such that:*

- (1)  $X^{\text{qf}}$  is  $\mathbb{Q}$ -factorial,
- (2)  $g$  is small.

**THEOREM 2.24** (Terminal modification, cf. [Koll13, Theorem 1.33]). *Let  $X$  be a normal, quasi-projective variety and  $\Delta$  a boundary on  $X$  such that  $[\Delta] = 0$ . Then there is a non-unique, projective, birational morphism  $g^{\text{min}} : X^{\text{min}} \rightarrow X$  such that*

- (1)  $(X^{\text{min}}, (g^{\text{min}})_*^{-1}(\Delta))$  is terminal and
- (2)  $K_{X^{\text{min}}} + (g^{\text{min}})_*^{-1}(\Delta)$  is  $g^{\text{min}}$ -nef.

**THEOREM 2.25** ([Koll13] Theorem 1.31). *Let  $X$  be a normal variety and  $\Delta$  a boundary on  $X$ . There exists a unique, projective, birational morphism  $f : X' \rightarrow X$  such that*



- (1)  $(X', f_*^{-1}(\Delta))$  is canonical and
- (2)  $K_{X'} + f_*^{-1}(\Delta)$  is  $f$ -ample

### 2.5. Slc singularities

In this section we consider non-normal schemes. We strictly follow [Kol13, Chapter 5].

**Definition 2.26** (Serre's conditions, cf. [Gro65, Définition IV.5.7.2]). Let  $X$  be a scheme. We say that  $X$  has property  $(S_2)$ , if for all  $x \in X$ , a not necessarily closed point, one of the following holds

- (1) We have that  $\dim(\mathcal{O}_{X,x}) = 0$ ;
- (2) We have that  $\dim(\mathcal{O}_{X,x}) = 1$  and that there exists  $a \in \mathfrak{m}_x \setminus 0$  such that  $a$  is not a zero divisor;
- (3) We have that  $\dim(\mathcal{O}_{X,x}) \geq 2$  and that there exists  $a_1, a_2 \in \mathfrak{m}_x \setminus 0$  such that  $a_1$  is not a zero divisor in  $\mathcal{O}_{X,x}$  and  $a_2$  is not a zero divisor in  $\mathcal{O}_{X,x}/(a_1)$ .

In general, let  $k \geq 0$  be an integer. We say that a coherent  $\mathcal{O}_X$ -module  $\mathcal{F}$  has property  $(S_k)$  if for all  $x \in X$ , we have that  $\text{depth}_{\mathcal{O}_{X,x}}(\mathcal{F}_x) \geq \min\{k, \dim \text{supp}(\mathcal{F}_x)\}$ .

By Serre's criterion for normality (cf. [Gro65, IV.5.8.6]), all normal schemes are  $(S_2)$ .

**Definition 2.27.** Let  $X$  be a scheme.

- (1) We say that  $X$  has a *node* at a not necessarily closed point  $x \in X$ , if  $\mathcal{O}_{X,x} \cong R/(f)$ , where  $(R, \mathfrak{m})$  is a regular local ring of dimension 2,  $f \in \mathfrak{m}^2$  and  $f$  is not a square in  $\mathfrak{m}^2/\mathfrak{m}^3$ ;
- (2) A scheme  $X$  is called *demi-normal* if  $X$  has the property  $(S_2)$  and all codimension 1 points  $x$  of  $X$  are regular points or nodes.

The simplest examples for demi-normal schemes are the nodal curves.

**Definition 2.28** (conductor). Let  $X$  be a reduced scheme and  $\pi : \bar{X} \rightarrow X$  its normalization. The *conductor ideal*

$$\mathbf{cond}_X := \mathcal{H}om_{\mathcal{O}_X}(\pi_*\mathcal{O}_{\bar{X}}, \mathcal{O}_X)$$

is the largest ideal sheaf of  $\mathcal{O}_X$  such that it is also an ideal sheaf of  $\pi_*\mathcal{O}_{\bar{X}}$ . As  $\pi$  is finite, we have a unique ideal sheaf  $\mathbf{cond}_{\bar{X}}$  of  $\bar{X}$  that corresponds to  $\mathbf{cond}_X$ .

We define the conductor schemes to be

$$D := \text{Spec}(\mathcal{O}_X/\mathbf{cond}_X) \text{ and } \bar{D} := \text{Spec}(\mathcal{O}_{\bar{X}}/\mathbf{cond}_{\bar{X}})$$

They fit into the Cartesian square

$$\begin{array}{ccc} \bar{D} & \longrightarrow & \bar{X} \\ \downarrow & & \downarrow \\ D & \longrightarrow & X \end{array}$$

For any  $x \in X$ , we have that the stalk  $\pi_*(\mathcal{O}_{\bar{X}})_x$  is canonically isomorphic to the integral closure of  $\mathcal{O}_{X,x}$  in its total rings of fractional (cf. [Sta22, Tag 0C3B]). When  $\mathcal{O}_{X,x}$  is a normal ring, we have that  $\mathcal{O}_{X,x} = \pi_*(\mathcal{O}_{\bar{X}})_x$  and  $x \notin D$ . Hence  $D$  is the locus of  $X$  where  $\mathcal{O}_{X,x}$  is not integrally closed in  $\mathcal{M}_{X,x}$  and  $\bar{X} \setminus \bar{D} \cong X \setminus D$ . When  $X$  is

(S<sub>2</sub>), the conductor ideal  $\text{cond}_X$  is (S<sub>2</sub>) by [Sta22, Tag 0EBC]. Let  $\xi$  be a generic point of  $D$ . We have that  $\mathcal{O}_{X,\xi}$  is not a normal ring. Hence by Serre's criterion for normality, there exists a  $\eta \in X$  which is a generalization of  $\xi$  such that  $\dim(\mathcal{O}_{X,\eta}) = 1$  and  $\mathcal{O}_{X,\eta}$  is not regular. Thus  $\eta \in D$  and  $\eta = \xi$ . We have that  $D$  is of pure codimension 1. As  $\pi$  is finite, the subscheme  $\bar{D}$  is also of pure codimension 1. By [Kol13, Corollary 2.61.], the subschemes  $D$  and  $\bar{D}$  have (S<sub>1</sub>) property or equivalently they have no embedded points. When  $X$  is demi-normal, both  $D$  and  $\bar{D}$  are reduced.

For a demi-normal scheme  $X$ , its Weil-divisors that do not contain any components of the conductor divisor  $D$  behave like the Weil-divisors on a normal scheme. We give some elaborations.

**Definition 2.29.** Let  $X$  be a demi-normal scheme and  $\pi : \bar{X} \rightarrow X$  its normalization. We denote by  $D$  and  $\bar{D}$  the conductor divisors on  $X$  and  $\bar{X}$  respectively.

- We denote by  $\text{Weil}^*(X) \subset \text{Weil}(X)$  the subgroup of Weil-divisors whose support does not contain any irreducible component of the conductor  $D$ . It is easy to see that  $\text{Weil}^*(X)$  is the image of the canonical map  $i_* : \text{Weil}(X \setminus D) \hookrightarrow \text{Weil}(X)$ , where  $i : X \setminus D \rightarrow X$  is the open inclusion.
- We denote by  $\text{Weil}^*(\bar{X}) \subset \text{Weil}(\bar{X})$  the subgroup of Weil-divisors whose support does not contain any irreducible component of the conductor  $\bar{D}$ . Similarly we have that  $\text{Weil}^*(\bar{X})$  is the image of the canonical map  $j_* : \text{Weil}(\bar{X} \setminus \bar{D}) \hookrightarrow \text{Weil}(\bar{X})$ , where  $j : \bar{X} \setminus \bar{D} \rightarrow \bar{X}$  is the open inclusion.
- A *divisorial sheaf* on  $X$  is a rank one reflexive sheaf that is locally free at the generic points of  $D$ . The isomorphism classes of divisorial sheaves form a group with multiplication given by  $\hat{\otimes}$ .

**Lemma 2.30.** Let  $X$  be a demi-normal scheme and  $\mathcal{F}$  a reflexive module on  $X$ . We denote by  $D$  the conductor of  $\mathcal{F}$ . Suppose that  $\mathcal{F}$  is locally free at the generic points of  $D$ . Then there exists an open subset  $X^0 \subset X$  such that

- (1) The restriction  $\mathcal{F}|_{X^0}$  is locally free;
- (2) Every irreducible component of  $X \setminus X^0$  has codimension at least 2;
- (3) Denote by  $j : X^0 \rightarrow X$  the inclusion. The canonical map  $\mathcal{F} \rightarrow j_*j^*(\mathcal{F})$  is an isomorphism.

**PROOF.** Let  $X^0 \subset X$  be the maximal open subset satisfying Lemma 2.30-(1). For a point  $x \in X$  such that  $\dim \mathcal{O}_{X,x} \leq 1$ , we have that either  $\mathcal{O}_{X,x}$  is a Dedekind domain, hence  $\mathcal{F}_x$  is a free  $\mathcal{O}_{X,x}$ -module or  $x$  is a node and  $\mathcal{F}_x$  is free at  $x$  by hypothesis. Hence for all  $x \in X \setminus X^0$ , we have that  $\dim \mathcal{O}_{X,x} \geq 2$ . Hence we have Lemma 2.30-(2). As  $X$  has the property (S<sub>2</sub>), we have that  $\text{depth}(\mathcal{O}_{X,x}) \geq 2$  for all  $x \in X \setminus X^0$ . This implies that the stalk  $\mathcal{F}_x = \mathcal{H}om_{\mathcal{O}_{X,x}}(\mathcal{H}om_{\mathcal{O}_{X,x}}(\mathcal{F}_x, \mathcal{O}_{X,x}), \mathcal{O}_X)$  has depth at least 2 for all  $x \in X \setminus X^0$ . [Sta22, Tag 0E9I] implies Lemma 2.30-(3).  $\square$

Let  $B$  be a Weil-divisor on a demi-normal scheme  $X$ . Suppose that  $B$  contains no components of the conductor divisor  $D$  of  $X$ . Then there exists an open subset  $X^0 \subset X$  such that the codimension of  $X \setminus X^0$  is at least 2 and  $B|_{X^0}$  is Cartier. Thus we may associate to  $B$  a reflexive  $\mathcal{O}_X(B) := i_*(\mathcal{O}_{X^0}(B|_{X^0}))$ , where  $i : X^0 \rightarrow X$  is the inclusion. The map  $B \mapsto \mathcal{O}_X(B)$  induces an isomorphism from  $\text{Weil}^*(X)$  modulo linear equivalences to the group of isomorphism classes of divisorial sheaves on  $X$ .

Let  $X$  be demi-normal, and  $\pi : \bar{X} \rightarrow X$  be its normalization. Let  $B$  be a Weil-divisor on  $X$ . Write  $B = B_1 + B_2$  where  $B_1$  is the divisorial part. We have that set-theoretically  $\pi^{-1}(B) = \pi^{-1}(B_1) \cup \pi^{-1}(B_2)$ . As  $\text{supp}(B_2) \subset D$ , we have that  $\overline{\pi^{-1}(B_2)} \subset \bar{D}$ . On the other hand, as  $\bar{X} \setminus \bar{D} \cong X \setminus D$ , we have that  $\pi^{-1}(B_1) = \overline{(\pi|_{\bar{X} \setminus \bar{D}})^*(B_1|_{\bar{X} \setminus \bar{D}})}$ . Hence we define the divisor  $j_*(\pi|_{\bar{X} \setminus \bar{D}})^*(B_1|_{\bar{X} \setminus \bar{D}})$  to be the *divisorial part* of the subscheme  $\pi^{-1}(B)$ , where  $j : \bar{X} \setminus \bar{D} \rightarrow \bar{X}$  is the inclusion. We denote the divisorial part of  $\pi^{-1}(B)$  by  $\bar{B}$ . As  $\pi : \bar{X} \setminus \bar{D} \rightarrow X \setminus D$  is an isomorphism. It's easy from the construction to see that  $B \mapsto \bar{B}$  is a bijection from  $\text{Weil}^*(X)$  to  $\text{Weil}^*(\bar{X})$ .

We now combine the above discussion together. We have the following proposition concerning the dualizing sheaves of  $X$  and  $\bar{X}$ . For a divisor  $B \in \text{Weil}^*$  and the dualizing sheaf  $\omega_X$ , we will use the following notation

$$\omega_X^{[m]}(B) := \omega_X^{[m]} \hat{\otimes} \mathcal{O}_X(B),$$

where  $m$  is an integer. If  $m = 1$ , we simplify the notation  $\omega_X^{[1]}(B)$  to  $\omega_X(B)$ . We use similar notations for *all* divisors  $B'$  on  $\bar{X}$  and  $\omega_{\bar{X}}$ .

**Proposition 2.31** ([Rei94, Proposition 2.3]). *Let  $X$  be a reduced scheme, and  $\pi : \bar{X} \rightarrow X$  its normalization. Let  $\omega_X$  and  $\omega_{\bar{X}}$  be the dualizing sheaves of  $X$  and  $\bar{X}$  respectively. We have that*

$$\pi_*(\omega_{\bar{X}}) = \mathcal{H}om_{\mathcal{O}_X}(\pi_*(\mathcal{O}_{\bar{X}}), \omega_X)$$

If  $X$  is  $(S_2)$  and  $\omega_X$  is invertible then

$$\pi_*(\omega_{\bar{X}}) = \mathbf{cond}_X \cdot \omega_X \text{ and } \pi^*(\omega_X) = \mathcal{H}om_{\mathcal{O}_{\bar{X}}}(\mathbf{cond}_{\bar{X}}, \omega_{\bar{X}}) = \omega_{\bar{X}}(\bar{D}).$$

We note that the second equality is equivalent to the first. In fact, by definition we have that  $\pi_*(\mathbf{cond}_{\bar{X}}) = \mathbf{cond}_X$ . Then by the projection formula, we have that  $\pi_*(\mathbf{cond}_{\bar{X}} \cdot \pi^*\omega_X) = \mathbf{cond}_X \cdot \omega_X = \pi_*\omega_{\bar{X}}$ . This implies  $\mathbf{cond}_{\bar{X}} \cdot (\pi^*\omega_X) = \omega_{\bar{X}}$ . Tensoring both sides with  $\mathbf{cond}_{\bar{X}}^{-1}$  and taking reflexive hulls, we get the second equality.

The dualizing sheaf  $\omega_C$  is locally free for a nodal curve  $C$ . For a general demi-normal scheme  $X$ , each nodal point  $x \in X$  is also a double nc point. Hence by the adjunction formula, the dualizing sheaf  $\omega_X$  is locally free at  $x$ . Thus there exists an open  $X^0 \subset X$  with  $\text{codim}_X(X \setminus X^0) \geq 2$  and  $\omega_X|_{X^0}$  is locally free. Apply the above proposition to  $X^0$  and  $\pi^{-1}(X^0)$  then push forward. We get that

$$\pi_*(\omega_{\bar{X}}) = \omega_X(-D) \text{ and } (\pi^*\omega_X)^{**} = \omega_{\bar{X}}(\bar{D}).$$

For any  $B \in \text{Weil}^*(X)$ , we have that

$$(1) \quad (\pi^*\omega_X^{[m]}(B))^{**} \simeq \omega_{\bar{X}}^{[m]}(m\bar{D} + \bar{B})$$

If  $\Delta$  is a  $\mathbb{Q}$ -Cartier  $\mathbb{Q}$ -divisor such that  $m\Delta \in \text{Weil}^*$ , we have that

$$(2) \quad (\pi^*\omega_X^{[m]}(m\Delta))^{**} \simeq \omega_{\bar{X}}^{[m]}(m\bar{D} + m\bar{\Delta})$$

which will also be written as

$$(3) \quad \pi^*(K_X + \Delta) \sim_{\mathbb{Q}} K_{\bar{X}} + \bar{D} + \bar{\Delta}$$

We are now ready to define slc singularities.

**Definition 2.32** ([Kol13, Definition-Lemma 5.10]). Let  $(X, \Delta)$  be a pair as in [Definition 2.2](#). Let  $\pi : \bar{X} \rightarrow X$  be its normalization, the conductors  $\bar{D}$  and  $D$  as in [Definition 2.28](#). The pair  $(X, \Delta)$  is called *semi-log canonical* or *slc* if  $(\bar{X}, \bar{D} + \bar{\Delta})$  is log canonical.

## 2.6. Miscellaneous constructions

**2.6.1. Generalized cones.** We follow the construction in [BS11, 1.1.8.] Let  $V$  be a projective scheme of dimension  $n$  and  $L$  a very ample line bundle over  $V$ . Fix  $N \geq n$  an integer. Set  $E := \oplus^{N-n} \mathcal{O}_V$  and  $p : \mathbb{P}(E \oplus L) \rightarrow V$ . We denote the projectivisation  $\mathbb{P}(E \oplus L)$  by  $X$ . Note that  $E \oplus L$  is globally generated and we have for the tautological bundle  $\xi := \mathcal{O}_{\mathbb{P}(E \oplus L)}(1)$  of  $\mathbb{P}(E \oplus L)$  a surjective morphism  $p^*(E \oplus L) \rightarrow \xi$ . Hence we have a surjective morphism

$$H^0(V, E \oplus L) \otimes_{\mathbb{C}} \mathcal{O}_X \rightarrow \xi.$$

By [Proposition 2.1](#), the above morphism corresponds to a unique morphism

$$\phi_{|\xi|} : X \rightarrow \mathbb{P}(H^0(V, E \oplus L)).$$

We take the Stein factorization of  $\phi$ :

$$\begin{array}{ccc} X & \longrightarrow & \mathbb{P}(H^0(V, E \oplus L)) \\ \psi_{|\xi|} \downarrow & \nearrow & \\ C_N(V, L) & & \end{array}$$

and call  $C_N(V, L)$  the *generalized cone of dimension  $N$  on  $(V, L)$* . As  $\xi$  is big, the scheme  $C_N(V, L)$  has dimension  $N$ . Set  $\xi_L := \mathcal{O}_{\mathbb{P}(H^0(V, E \oplus L))}(1)|_{C_N(V, L)}$ , then  $\xi_L$  is ample.

**2.6.2. Albanese morphism.** We give a sketch on how to construct Albanese morphism for a singular complex variety  $X$ . We first recall the Albanese morphism for smooth compact varieties.

**Definition 2.33** ([Uen75, Definition 9.6.]). Let  $V$  be a compact complex manifold. The *Albanese torus* of  $V$  is a complex torus  $A(V)$  together with a morphism  $\text{Alb}_V : V \rightarrow A(V)$  such that the universal property is satisfied:

For a morphism  $g : V \rightarrow T$  from  $V$  to a complex torus  $T$ , there exists a unique Lie group morphism  $h : A(V) \rightarrow T$  and a unique element  $a \in T$  such that for all  $x \in V$  we have

$$g(x) = h(\text{Alb}_V(x)) + a.$$

The universal property characterizes  $(A(V), \text{Alb}_V)$  up to isomorphisms.

**THEOREM 2.34** ([Uen75, Theorem 9.7.]). *For any smooth compact complex variety  $V$ , its Albanese torus  $(A(V), \text{Alb}_V)$  exists.*

When  $X$  is smooth and projective or more generally compact Kähler, we may construct  $(A(X), \text{Alb}_X)$  by

$$\begin{aligned} A(X) &= H^0(X, \Omega_X)^* / H_1(X, \mathbb{Z}) \\ \text{Alb}_X : x &\longmapsto (\alpha \in H^0(X, \Omega_X) \mapsto \int_{x_0}^x \alpha) \end{aligned}$$

where  $x_0 \in X$  is a point that we fix and the integral is defined over any path that connects  $x_0$  to  $x$ .

Now, suppose that  $X$  is a normal projective variety, not necessarily smooth. Let  $r : Y \rightarrow X$  be a projective resolution of singularities of  $X$ . We consider the following diagram

$$\begin{array}{ccc} Y & \xrightarrow{\text{Alb}_Y} & A(Y) \\ r \downarrow & \nearrow & \\ X & & \end{array}$$

If the rational map  $\text{Alb}_Y \circ r^{-1}$  has no points of indeterminacy, we can show  $(\text{Alb}_Y \circ r^{-1}, A(Y))$  is independent of the choice of the resolution  $r$ . We have the following

**THEOREM 2.35** ([Rei83, Proposition 2.3.]). *Let  $X$  be a normal projective variety and  $r : Y \rightarrow X$  be a projective resolution of singularities of  $X$ . If  $R^1 f_*(\mathcal{O}_Y) = 0$ , then the rational map  $\text{Alb}_Y \circ r^{-1}$  has no points of indeterminacy.*

## Polarized varieties with high nef value

### 3.1. Canonical polarized varieties

In this section, we assume that  $X$  is a projective normal variety with canonical singularities.

First we recall the definition of polarized and quasi-polarized varieties.

**Definition 3.1.** Let  $(X, L)$  be a pair consisting of a projective variety  $X$  and a line bundle  $L$  over  $X$ . we call it

- (1) A *quasi-polarized variety* if  $L$  is nef and big;
- (2) A *polarized variety* if  $L$  is ample.

**Definition 3.2.** Let  $X$  be a normal projective variety such that  $K_X$  is  $\mathbb{Q}$ -cartier and  $K_X$  is not nef. For a big and nef line bundle  $L$  on  $X$ , we define

$$r(L) =: \sup\{t \in \mathbb{R} : tK_X + L \text{ is nef}\}.$$

And set  $\tau(L) = \frac{1}{r(L)}$  with the convention  $\tau(L) = +\infty$  when  $r(L) = 0$ . We call this number  $\tau(L)$  the nefvalue of  $L$ .

By [Theorem 2.18](#), we know that  $r$  is a rational number and that there exists a  $K_X$ -negative extremal ray  $R$  such that  $(rK_X + L) \cdot R = 0$ .

**Lemma 3.3.** *Let  $X$  be a normal projective variety with canonical singularities and  $L$  a big and nef line bundle on  $X$ . Suppose that  $\tau(L)$  is finite. If  $K_X + (n-1)L \notin \text{Pseff}(X)$ , we have that  $\tau(L) > n-1$ .*

PROOF. We know that  $\text{Pseff}(X) = \overline{\text{Big}(X)}$  is a closed cone. Hence there exists an ample  $\mathbb{Q}$ -divisor  $A$ , such that  $K_X + (n-1)L + A$  is not pseudo-effective. If  $\tau(L) \leq n-1$ , we have that

$$K_X + (n-1)L + A = (K_X + \tau(L)L) + (n-1-\tau(L))L + A.$$

That is,  $K_X + (n-1)L + A$  is a sum of a nef and an ample divisor, which is ample, a contradiction.  $\square$

**Definition 3.4.** Let  $(X_1, L_1), (X_2, L_2)$  be two pairs consisting of a normal variety  $X_i$  and a line bundle  $L_i$  on  $X_i$ .

- (1) We say that  $(X_1, L_1)$  is *isomorphic* to  $(X_2, L_2)$ , if there exists an isomorphism  $\phi : X_1 \rightarrow X_2$  such that  $\phi^*(L_2)$  is isomorphic to  $L_1$ . We denote this by  $(X_1, L_1) \cong (X_2, L_2)$ .
- (2) We say that  $(X_1, L_1)$  and  $(X_2, L_2)$  are *birationally equivalent*, if there exists a variety  $X$  and two birational morphism  $\phi_i : X \rightarrow X_i$  such that  $\phi_1^*(L_1)$  is isomorphic to  $\phi_2^*(L_2)$ . We denote this by  $(X_1, L_1) \sim_{\text{bir}} (X_2, L_2)$ .

**Lemma 3.5.** *Let  $X$  be a variety with canonical  $\mathbb{Q}$ -factorial singularities and  $L$  a big and nef line bundle on  $X$ . Suppose that  $K_X + (n-1)L \notin \text{Pseff}(X)$ . Then  $(X, L)$  is birationally equivalent to  $(X', L')$ , where  $X'$  is a normal projective variety with canonical  $\mathbb{Q}$ -factorial singularities,  $K_{X'} + (n-1)L' \notin \text{Pseff}(X')$  and*

- (1) *Either  $\tau(L')$  is finite;*
- (2) *or there is a Mori fiber space structure  $\phi : X' \rightarrow W$  and a rational number  $\tau > (n-1)$  such that  $L'$  is  $\phi$ -ample and  $K_{X'} + \tau L' \sim_{\mathbb{Q}, \phi} 0$ .*

PROOF. By [Theorem 2.17](#), we know that

$$\overline{\text{NE}}(X) = \overline{\text{NE}}(X)_{K_X \geq 0} + \sum \mathbb{R}_{\geq 0}[C_j]$$

where  $C_j$  are  $K_X$ -negative rational curves and the sum is over countably many  $j$ .

- (1) For every  $K_X$ -negative extremal ray  $R = \mathbb{R}_{\geq 0}[C]$ , we have that  $L \cdot C > 0$ . By [Theorem 2.18](#), there exists a  $K_X$ -negative extremal  $C_0$  such that  $r(L) = -\frac{L \cdot C_0}{K_X \cdot C_0} > 0$ . Hence  $r = 0$  only if there exists an  $L$ -trivial  $K_X$ -negative extremal ray.
- (2) There exists a  $K_X$ -negative extremal ray  $R$  such that  $L \cdot R = 0$ . By [Theorem 2.17](#)-(3), we consider the contraction with respect to  $R$ ,  $\text{cont}_R : X \rightarrow Z$ . Note that there exists a line bundle  $L_Z$  on  $Z$  such that  $L \cong \text{cont}_R^*(L_Z)$ .

Hence a basic idea is to run a MMP to contract every  $L$ -trivial extremal rays to get a  $(X', L')$  satisfying case (1). However we may encounter the problem with termination. We now show how to circumvent this problem.

By [Theorem 2.24](#) and [Lemma 2.23](#), we can find a modification  $f : Y \rightarrow X$  such that  $Y$  has  $\mathbb{Q}$ -factorial terminal singularities. Set  $L_Y = f^*L$ . We have that  $L_Y$  is nef and big and  $K_Y + (n-1)L_Y \notin \text{Pseff}(Y)$ . By [[And13](#), Lemma 4.1.], we can find an effective  $\mathbb{Q}$ -divisor  $\Delta$  on  $Y$  such that

$$\Delta \sim_{\mathbb{Q}} (n-1)L_Y \text{ and } (Y, \Delta) \text{ is klt.}$$

Now consider the pair  $(Y, \Delta)$ . We have that  $K_Y + \Delta \notin \text{Pseff}(Y)$ . By [Theorem 2.22](#), we can run a  $(K_Y + \Delta)$ -MMP to get

$$(Y, \Delta) = (Y_0, \Delta_0) \dashrightarrow (Y_1, \Delta_1) \dashrightarrow \cdots \dashrightarrow (Y_s, \Delta_s),$$

with  $Y_s$  a Mori fiber space. Suppose that the map  $\phi_i : Y_i \dashrightarrow Y_{i+1}$  is associated with a  $(K_{Y_i} + \Delta_i)$ -negative extremal ray  $R_i$ . By [[And13](#), Proposition 4.2.], for every  $i = 0, 1, \dots, s$ , we have that

- (1)  $Y_i$  is  $\mathbb{Q}$ -factorial terminal;
- (2)  $\Delta_i \cdot R_i = 0$ ;
- (3) There exists nef and big line bundles  $L_i$  on  $Y_i$  and  $\Delta_i \sim_{\mathbb{Q}} (n-1)L_i$ .

It is then obvious  $K_{Y_i} + (n-1)L_i \notin \text{Pseff}(Y_i)$ .

We then set  $(X', L') := (Y_s, L_s)$ .

- (1) If  $(Y_s, \Delta_s)$  has no  $K_{Y_s}$ -negative extremal ray  $R$  such that  $L_s \cdot R = 0$ , take  $C_0$  a  $K_{X'}$ -negative extremal curve such that  $(r(L')K_{X'} + L') \cdot C_0 = 0$ . Then  $r(L') = -\frac{L' \cdot C_0}{K_{X'} \cdot C_0} > 0$ . Hence the nefvalue of  $L'$ ,  $\tau(L') = \frac{1}{r(L')}$  is finite.

- (2) Otherwise, we consider the Mori fiber space  $\phi_s : Y_s \rightarrow W$  obtained in the above  $(K_Y + \Delta)$ -MMP. Let  $R_s := \text{NE}(\phi_s)$  be the extremal ray of  $\phi_s$ . We claim that  $L_s \cdot R_s > 0$ . Suppose in contrary that  $L_s \cdot R_s = 0$ . Then by [Theorem 2.17](#)-(3), there exists  $L_W$  such that  $\phi_s^*(L_W) = L_s$ . As  $\dim(W) < \dim(Y_s)$ , we have that  $L_s^n = \phi_s^*(L_W^n) = 0$  contradicting  $L_s$  to be nef and big. As  $R_s$  is a  $(K_{Y_s} + \Delta_s)$ -negative extremal ray, we have that  $(K_{Y_s} + (n-1)L_s) \cdot R_s < 0$ . Hence the  $\tau > 0$  such that  $K_{Y_s} + \tau L_s \sim_{\mathbb{Q}, \phi} 0$  satisfies that  $\tau > (n-1)$ .

□

Before we move on to further classification, we first introduce the definition of  $\Delta$ -genus.

**Definition 3.6.** Let  $(X, L)$  be a quasi-polarised variety of dimension  $n$ . We define its  $\Delta$ -genus to be

$$\Delta(X, L) := n + L^n - h^0(X, L)$$

We will use the relative Kobayashi-Ochiai criterion by Andreatta.

**THEOREM 3.7** ([\[And95, Theorem 2.1.\]](#)). *Let  $X$  be a projective variety with klt singularities and let  $L$  be a line bundle on  $X$ . Let  $\phi : X \rightarrow Z$  be a surjective morphism with connected fibers between normal varieties. Suppose that  $L$  is  $\phi$ -ample and  $K_X + \tau L \sim_{\mathbb{Q}, \phi} 0$  for some  $\tau \in \mathbb{Q}^+$ . Let  $F_1 = \phi^{-1}(z)$  be a non-trivial fiber,  $F \subset F_1$  be one of its irreducible components,  $F'$  be the normalization of  $F$  and let  $L'$  be the pullback of  $L$  on  $F$ . Let  $\lfloor \tau \rfloor$  be the integral part of  $\tau$  and  $\tau' = \lceil \tau \rceil = -\lfloor -\tau \rfloor$ .*

$$(I,1) \dim(F) \geq \tau - 1;$$

$$(I,2) \text{ If } \dim(F) < \tau, \text{ then } F \cong \mathbb{P}^{\tau'-1} \text{ and } L|_F = \mathcal{O}_{\mathbb{P}^{\tau'-1}}(1);$$

$$(I,3) \text{ If } \dim(F) < \tau + 1, \text{ then } \Delta(F', L') = 0,$$

If moreover  $\dim(F) > \dim(X) - \dim(Z)$ , then

$$(II,1) \dim(F) \geq \tau;$$

$$(II,2) \text{ If } \dim(F) = \tau, \text{ then } F \cong \mathbb{P}^\tau \text{ and } L|_F = \mathcal{O}_{\mathbb{P}^\tau}(1);$$

$$(II,3) \text{ If } \dim(F) < \tau + 1, \text{ then } \Delta(F', L') = 0,$$

If all components of the fiber satisfy  $\dim(F) < \tau$ , in case (I.2) or  $\dim(F) \leq \tau$  in case (II.3), then the fiber is actually irreducible.

**Lemma 3.8.** *Let  $(X, L)$  be a quasi-polarised variety of dimension  $n$ . Suppose that  $X$  has canonical  $\mathbb{Q}$ -factorial singularities and  $K_X + (n-1)L \notin \text{Pseff}(X)$ . Suppose that there exists a  $K_X$ -negative extremal ray  $R = \mathbb{R}_{\geq 0}[C_0]$  such that  $L \cdot C_0 > 0$ . Then  $(X, L)$  is the one of the following*

$$(1) (X, L) \cong (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)), \text{ and } \tau = n + 1;$$

$$(2) (X, L) \text{ is isomorphic to a } (\mathbb{P}^{n-1}, \mathcal{O}_{\mathbb{P}^{n-1}}(1))\text{-bundle over a smooth curve } C \text{ and } \tau = n;$$

$$(3) \Delta(X, L) = 0, K_X + \tau L \sim_{\mathbb{Q}} \mathcal{O}_X \text{ and } n - 1 < \tau \leq n.$$

**PROOF.** Let  $\phi : X \rightarrow Z$  be the Mori contraction of the extremal ray  $R$ . Set  $t > 0$  to be the rational number such that  $(K_X + tL) \cdot C_0 = 0$ . Let  $F$  be a general fiber of  $\phi$ , then  $(K_X + (n-1)L)|_F \notin \text{Pseff}(F)$ . As  $\overline{\text{NE}}(F) = \mathbb{R}_{\geq 0}[C_0]$ , we have that  $(K_X + (n-1)L) \cdot C_0 < 0$ . Thus  $t > (n-1)$ .



Let  $m$  be a divisible enough integer such that  $mK_X$  is a Cartier divisor and  $mt$  is an integer. The line bundle  $mK_X + mtL$  is  $\phi$ -numerically trivial. By [Theorem 2.17](#)-(3), we know that  $K_X + tL \sim_{\mathbb{Q},\phi} 0$ . As  $\text{NE}(X/Z) = R$ , we have that  $L$  is  $\phi$ -ample. Thus we are in the situation of [Theorem 3.7](#).

We first show that  $\phi$  is not birational. Suppose by contradiction that  $\phi : X \rightarrow Z$  is birational. Let  $F$  be a component of a non trivial fiber  $F_1 = \phi^{-1}(z)$ . By [Theorem 3.7](#) (II,1), we have that  $\dim(F) \geq t > n - 1$ . Thus  $\phi(X)$  is a singleton, a contradiction.

By [Theorem 3.7](#), we know that  $\dim(F) \geq t - 1 > n - 2$ . Thus either  $\dim(F) = n$  or  $\dim(F) = n - 1$ .

- (1) If  $\dim(F) = n$ , we have that  $F = X$  and  $Z = \{z\}$ . Then  $K_X + tL \sim_{\mathbb{Q}} \mathcal{O}_X$  and  $\tau = t$ . If  $t > n$ , [Theorem 3.7](#) (I,2) implies that  $(X, L) = (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$  and  $\tau = n + 1$ . If  $n - 1 < t \leq n$ , we have that  $\dim(F) = n < t + 1$ . By [Theorem 3.7](#) (I,3), we know that  $\Delta(X, L) = 0$ .
- (2)  $\dim(F) = n - 1$ . Let  $F' \subset F_1$  be another component of  $F_1$ . Then [Theorem 3.7](#) implies  $\dim(F') \geq n - 1$ . On the other hand we could not have that  $\dim(F') = n$ , for this would imply that  $F = F' = X$  which has dimension  $n$ , a contradiction. Hence by [Theorem 3.7](#) again, we know that  $F_1$  is irreducible and  $F = F_1$ . As  $\phi$  is not birational, by semi-continuity of dimensions of fibers (*cf.* for example [[Sta22](#), [Tag 02FZ](#)]), for any point  $z'$ ,  $\phi^{-1}(z')$  has positive dimension. By [Theorem 3.7](#) and repeating the argument for  $F$  and  $F_1$ , we know that  $\phi^{-1}(z')$  is irreducible with dimension  $n - 1$ . Then [Theorem 3.7](#) (I,2) implies that for every fiber  $\phi^{-1}(z')$ , we have that  $(\phi^{-1}(z'), L_{\phi^{-1}(z')}) \cong (\mathbb{P}^{n-1}, \mathcal{O}_{\mathbb{P}^{n-1}}(1))$ . Thus we know that  $(X, L)$  is isomorphic to a  $(\mathbb{P}^{n-1}, \mathcal{O}_{\mathbb{P}^{n-1}}(1))$ -bundle over a smooth curve  $C$  and  $\tau = n$ .

□

Thus we are left in the case (3) of [Lemma 3.8](#). In this case, we have:

**Lemma 3.9.** *Let  $(X, L)$  be a quasi-polarized variety of dimension  $n$  with  $\Delta(X, L) = 0$ . Suppose that  $X$  has canonical  $\mathbb{Q}$ -factorial singularities, and that the nefvalue  $\tau = \tau(L)$  of  $L$  satisfies  $n - 1 < \tau(L) \leq n$ . If  $K_X + \tau L \sim_{\mathbb{Q}} \mathcal{O}_X$ , then there exists a birational morphism  $\mu : X \rightarrow Y$  such that*

- (1)  $Y$  has canonical singularities,  $\mu^*(K_Y) = K_X$  ;
- (2) There exists an ample line bundle  $A$  on  $Y$  such that  $\mu^*(A) = L$ ;
- (3)  $\Delta(Y, A) = 0$  and  $K_Y + \tau A \equiv_{\text{num}} \mathcal{O}_Y$ .

PROOF. The divisor  $L - K_X \sim_{\mathbb{Q}} 2\tau L$  is nef and big. Hence we may apply the basepoint-free theorem for  $L$  ([[KM98](#), Theorem 3.3.]), to get that for a sufficient large integer  $b$ ,  $|bL|$  has no basepoints. Now consider the graded algebra

$$R(X, L) =: \bigoplus_{n \geq 0} H^0(X, nL).$$

We have a canonical rational map  $\mu : X \rightarrow \text{Proj}(R(X, L)) =: Y$ . As  $\text{Bs}(|bL|) = \emptyset$ , we know that  $\mu$  has no indeterminacy and  $R(X, L)$  is finite generated (*cf.* [[Deb01](#), Proposition 7.6.]). Hence the ring  $R(X, bL)$  is integral and normal. As  $L$  is big, the morphism  $\mu$  is birational and  $L := \mu^*(\mathcal{O}_Y(1))$  (*cf.* [[Deb01](#), Lemma 7.10.]). By setting  $A := \mathcal{O}_Y(1)$ , we get (2).

We now take a divisible enough  $m$  such that  $mK_X$  is Cartier, the number  $m\tau$  is an integer and  $mK_X + m\tau L \sim_{\mathbb{Z}} 0$ . Denote the exceptional locus of  $\mu$  by  $E$  and  $\nu : Y \setminus \mu(E) \rightarrow X \setminus E$  the inverse of  $\mu$ . We have that

$$\mathcal{O}_Y(mK_Y)|_{Y \setminus \mu(E)} \sim \nu^*(\mathcal{O}_X|_{X \setminus E}) \sim \nu^*(-m\tau L|_{X \setminus E}) \sim -m\tau A|_{Y \setminus \mu(E)}$$

The rank one reflexive sheaf  $\mathcal{O}_Y(mK_Y)$  and the line bundle  $-m\tau A$  agree outside a subset whose codimension is at least 2. Hence  $\mathcal{O}_Y(mK_Y)$  is a line bundle and  $K_Y$  is  $\mathbb{Q}$ -Cartier. We thus have the equalities  $K_Y = -\tau A$  and  $\mu^*(K_Y) = K_X$ . Hence  $\mu$  is crepant and  $Y$  has canonical singularities. We get (1). By projection formula, we have that  $K_Y + \tau A = \mu_*(K_X + \tau L) = \mathcal{O}_Y$  and  $\Delta(Y, A) = n + A^n - h^0(Y, A) = n + L^n - h^0(X, L) = 0$ .  $\square$

Hence it rest for us to classify the polarized variety  $(X, L)$  with  $L$  ample,  $n - 1 < \tau(L) \leq n$ ,  $\Delta(X, L) = 0$  and  $K_X + \tau(L)L \cong \mathcal{O}_X$ . We have the following:

**Lemma 3.10.** *Let  $(X, L)$  be a polarized variety with  $L$  ample,  $n - 1 < \tau(L) \leq n$ ,  $\Delta(X, L) = 0$  and  $K_X + \tau(L)L \cong_{\text{num}} \mathcal{O}_X$ . Suppose that  $X$  has canonical singularities. Then one of the following occurs:*

- (1)  $(X, L) \cong (Q, \mathcal{O}_{\mathbb{P}^{n+1}}(1))$ , where  $Q \subset \mathbb{Q}^{n+1}$  is a hyperquadric;
- (2)  $(X, L)$  is a  $\mathbb{P}^{n-1}$ -bundle over  $\mathbb{P}^1$  and the restriction of  $L$  to each fiber is  $\mathcal{O}_{\mathbb{P}^{n-1}}(1)$ ;
- (3)  $(X, L) \cong (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$ ;
- (4)  $(X, L) \cong C_n(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$  is a generalised cone over  $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$

PROOF. If  $\tau(L) = n$ , the divisor  $K_X + nL$  is numerically trivial. Then [Theorem 1.2](#) implies that  $(X, L) \cong (Q, \mathcal{O}_{\mathbb{P}^{n+1}}(1))$ , where  $Q \subset \mathbb{P}^{n+1}$  is a hyperquadric. Hence we are in case (1).

From now on we may assume that  $\tau(L) < n$ . As  $L$  is ample, we have that  $K_X + nL \cong_{\text{num}} (n - \tau(L))L$  is ample.

By Fujita's classification theorem for polarized varieties with  $\Delta$ -genus zero (cf. [\[Fuj90, Theorem 5.10 and Theorem 5.15\]](#) [\[BS11, Proposition 3.1.2.\]](#)), besides the four cases given above in [Lemma 3.10](#), there are two more possibilities for  $(X, L)$ :

- (i) Either  $(X, L) \cong (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$ ,
- (ii) or  $(X, L)$  is a generalized cone over  $(V, L_V)$ , where  $V \subset X$  is a smooth submanifold,  $L|_V = L_V$  is very ample and  $\Delta(V, L_V) = 0$ .

Case (i) is impossible, since  $\tau(\mathcal{O}_{\mathbb{P}^n}(1)) = n + 1$ . Hence we need to investigate case (ii). Set  $r := n - \dim(V)$ , we have by definition of the generalized cone the following diagram

$$\begin{array}{ccccc} \mathbb{P}(\mathcal{O}_V^{\oplus r}) = V \times \mathbb{P}^{r-1} & \xrightarrow{\text{pr}_2} & \mathbb{P}^{r-1} & & \\ \downarrow i & & \downarrow & & \\ \mathbb{P}(L_V) & \longrightarrow & \mathbb{P}(\mathcal{O}_V^{\oplus r} \oplus L_V) & \xrightarrow{\psi|_{\xi}} & C_n(V, L_V) = X \\ & \searrow \cong & \downarrow \pi & & \\ & & V & & \end{array}$$

where  $\xi = \mathcal{O}_{\mathbb{P}(\mathcal{O}_V^{\oplus r} \oplus L_V)}(1)$  is the tautological bundle. The identification of  $V \cong \mathbb{P}(L_V)$  is given by the quotient morphism  $\mathcal{O}_V^{\oplus r} \oplus L_V \twoheadrightarrow L_V$ .

We claim that outside  $\mathbb{P}(\mathcal{O}_V^{\oplus r})$  the morphism  $\psi_{|\xi|}$  induces an isomorphism onto its image. Take  $z \in C_n(V, L)$  such that  $\psi_{|\xi|}^{-1}(z)$  has positive dimension. In particular, there exists a curve  $C_1$  such that  $\psi_{|\xi|}(C_1) = \{z\}$ . Since  $\mathcal{O}_V^{\oplus r} \oplus L_V$  is globally generated, the morphism  $\psi_{|\xi|}$  restricted to each fiber of  $\pi$  is an embedding. Hence  $\pi$  maps  $C_1$  bijectively to its image  $C$ . By generic smoothness, there is an open subset  $U \subset C$  such that  $\pi : C_0 := \pi^{-1}(U) \rightarrow U$  is an isomorphism. We may regard  $C_0$  as a section of  $\pi$  defined over  $U$ . That is

$$\begin{array}{ccc} \mathbb{P}((\mathcal{O}_V^{\oplus r} \oplus L_V)|_U) & \longrightarrow & \mathbb{P}(\mathcal{O}_V^{\oplus r} \oplus L_V) \\ \sigma \uparrow \downarrow & & \downarrow \\ U & \longrightarrow & V \end{array}$$

The section  $\sigma$  is defined by a quotient  $\rho = (\rho_1, \rho_2) : (\mathcal{O}_V^{\oplus r} \oplus L_V)|_U \rightarrow M$ , with  $M$  a line bundle on  $U$ . The morphism  $\rho$  has a decomposition into  $\rho_1 : \mathcal{O}_V^{\oplus r} \rightarrow M$  and  $\rho_2 : L_V \rightarrow M$ . As  $\psi_{|\xi|} \circ \sigma(U) = \{z\}$ , we know that  $M \cong \sigma^*(\xi|_{\mathbb{P}((\mathcal{O}_V^{\oplus r} \oplus L_V)|_U)})$  is trivial. As  $h^0(\text{Hom}_{\mathcal{O}_U}(L_V|_U, \mathcal{O}_U)) = h^0(U, L_V^\vee|_U) = 0$ , we have that  $\rho_2 = 0$ . Hence the quotient is given by  $\rho_1 : \mathcal{O}_U^{\oplus n-1} \rightarrow \mathcal{O}_U$ . We know that  $C_0 = U \subset \mathbb{P}(\mathcal{O}_V^{\oplus r})$  and hence  $C = \overline{C_0} \subset \mathbb{P}(\mathcal{O}_V^{\oplus r})$ .

As  $V = \mathbb{P}(L_V)$  is smooth, we have the short exact sequence

$$0 \rightarrow T_{\mathbb{P}(L_V)} \rightarrow T_{\mathbb{P}(\mathcal{O}_V^{\oplus r} \oplus L_V)}|_{\mathbb{P}(L_V)} \rightarrow N_{\mathbb{P}(L_V)/\mathbb{P}(\mathcal{O}_V^{\oplus r} \oplus L_V)} \rightarrow 0.$$

We have thus

$$(4) \quad \omega_{\mathbb{P}(\mathcal{O}_V^{\oplus r} \oplus L_V)}|_{\mathbb{P}(L_V)} = \omega_{\mathbb{P}(L_V)}^\vee \otimes \wedge^r N_{\mathbb{P}(L_V)/\mathbb{P}(\mathcal{O}_V^{\oplus r} \oplus L_V)}.$$

The canonical bundle formula gives us

$$\omega_{\mathbb{P}(\mathcal{O}_V^{\oplus r} \oplus L_V)} = \pi^*(\omega_V \otimes L_V) \otimes \xi^{\otimes -(r+1)}.$$

With  $\xi|_V = L_V$ , we know that  $\omega_{\mathbb{P}(\mathcal{O}_V^{\oplus r} \oplus L_V)}|_V = \omega_V \otimes L_V^{\otimes -r}$ . Thus [Equation \(4\)](#) gives

$$\wedge^r N_{\mathbb{P}(L_V)/\mathbb{P}(\mathcal{O}_V^{\oplus r} \oplus L_V)} = L_V^{\otimes r}.$$

As  $\mathbb{P}(L_V)$  is disjoint from the singular locus  $\mathbb{P}^{r-1} \subset X$ , we also have the exact sequence

$$0 \rightarrow T_{\mathbb{P}(L_V)} \rightarrow T_X|_{\mathbb{P}(L_V)} \rightarrow N_{\mathbb{P}(L_V)/X} \rightarrow 0.$$

Hence

$$\omega_X^\vee|_{\mathbb{P}(L_V)} = \omega_{\mathbb{P}(L_V)}^\vee \otimes \wedge^r N_{\mathbb{P}(L_V)/X}.$$

Note that  $N_{\mathbb{P}(L_V)/X} = N_{\mathbb{P}(L_V)/\mathbb{P}(\mathcal{O}_V^{\oplus r} \oplus L_V)}$ . Hence  $\omega_X|_V = \omega_V \otimes L_V^{\otimes -r}$ . Then we have that

$$\omega_X \otimes L_V^{\otimes n}|_V = \omega_V \otimes L_V^{\otimes (n-r)}.$$

Hence the divisor  $K_V + \dim(V)L_V$  is ample.

If  $\dim(V) \geq 2$ , apply [\[Fuj90, Theorem 5.10\]](#) again for  $(V, L_V)$ . We know that  $(V, L_V)$  is one of the following:

- $(\mathbb{P}^{\dim(V)}, \mathcal{O}_{\mathbb{P}^{\dim(V)}}(1))$ ; or

- $(Q, \mathcal{O}_Q(1))$ , where  $Q \subset \mathbb{P}^{\dim(V)+1}$  is a hyperquadric; or
- $(\mathbb{P}(\mathcal{E}), \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1))$  where  $\mathcal{E}$  is an ample vector bundle of rank  $\dim(V)$  over  $\mathbb{P}^1$ ; or
- $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$

Suppose first that  $\dim(V) = 2$ . If  $(V, L)$  is  $(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))$  or  $(Q, \mathcal{O}_{\mathbb{P}^3}(1))$ , the divisor  $K_V + 2L_V$  will not be ample. If  $(V, L)$  is a  $\mathbb{P}^1$ -bundle over  $\mathbb{P}^1$ , then  $K_V + 2L_V$  is trivial on each fiber, contradicting to the fact that  $K_V + 2L_V$  is ample. Hence  $(V, L) \cong (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$ .

If  $\dim(V) = 1$ , we have that  $(V, L_V) \cong (\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(a))$  with  $a \geq 3$ . By the following [Lemma 3.11](#) we know that for  $n \geq 2$ , a generalized cone  $C_n(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(a))$  has singularities worse than canonical.

Hence when  $(X, L)$  is a generalized cone, we have that  $(X, L) \cong C_n(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$ .  $\square$

**Lemma 3.11.** *Let  $(X, L) = C_n(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(a))$  be a generalized cone with  $a \geq 3$  and  $n \geq 2$ . Then*

- (1)  $X$  has klt singularities and  $X$  is not canonical;
- (2) the nefvalue of  $L$  is  $n - \frac{a-2}{a}$ ;
- (3)  $K_X + (n-1)L$  is not pseudo-effective.

PROOF. Consider the following commutative diagram

$$\begin{array}{ccc}
 D = \mathbb{P}^1 \times \mathbb{P}^{n-2} & \xrightarrow{\text{pr}_2} & \mathbb{P}^{n-2} \\
 \downarrow i & & \downarrow \\
 T = \mathbb{P}(\mathcal{O}^{\oplus n-1} \oplus \mathcal{O}(a)) & \xrightarrow{\psi|_{\xi}} & C_n(\mathbb{P}^1, \mathcal{O}(a)) \\
 \downarrow \pi & & \\
 \mathbb{P}^1 & & 
 \end{array}
 ,$$

where  $\xi$  is the tautological bundle  $\mathcal{O}_T(1)$ . The inclusion  $i : D \rightarrow T$  is given by the quotient  $\mathcal{O}^{\oplus n-1} \oplus \mathcal{O}(a) \rightarrow \mathcal{O}^{\oplus n-1}$ . The map  $\psi|_{\xi}|_D : D \rightarrow \psi|_{\xi}(D)$  onto its image  $\mathbb{P}^{n-2}$  is identified to  $\text{pr}_2$ . We have that  $\xi|_D \cong \text{pr}_2^*(\mathcal{O}_{\mathbb{P}^{n-2}}(1))$  and  $\pi \circ i = \text{pr}_1$ .

By the canonical bundle formula, we have that

$$(5) \quad K_T = \pi^*(\mathcal{O}_{\mathbb{P}^1}(a-2)) - n\xi,$$

$$(6) \quad K_D = \text{pr}_1^*(\mathcal{O}_{\mathbb{P}^1}(-2)) - (n-1)\xi|_D.$$

We can thus compute

$$(7) \quad \mathcal{O}_T(D)|_D = K_D - K_T|_D$$

$$(8) \quad = \text{pr}_1^*(\mathcal{O}_{\mathbb{P}^1}(-a)) + \text{pr}_2^*(\mathcal{O}_{\mathbb{P}^{n-2}}(1)).$$

Consider  $\mathbb{P}^1 \times \{\text{pt}\} \subset D$ , which is mapped isomorphically to  $\mathbb{P}^1$  by  $\pi$ . We have that

$$\mathcal{O}_T(D) \cdot (\mathbb{P}^1 \times \{\text{pt}\}) = -a < 0,$$

$$K_T \cdot (\mathbb{P}^1 \times \{\text{pt}\}) = a - 2 > 0.$$

We have that

$$(9) \quad K_T = \psi|_{\xi}^*(K_{C_n(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(a))}) + bD.$$

Intersect both sides with  $\mathbb{P}^1 \times \{\text{pt}\}$ . We find that  $0 > b = -\frac{a-2}{a} > -1$ , hence  $C_n(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(a))$  has klt singularities and does not have canonical singularities. As  $\mathbb{P}^{n-2} \subset C_n(\mathbb{P}^1, \mathcal{O}(a))$  has inverse image  $D$  by  $\psi|_{\xi|}$ , we see that  $\psi|_{\xi|} : (T, D) \rightarrow (C_n(\mathbb{P}^1, \mathcal{O}(a)), \mathbb{P}^{n-2})$  is a log resolution of  $(C_n(\mathbb{P}^1, \mathcal{O}(a)), \mathbb{P}^{n-2})$ .

We now compute the nefvalue  $\tau = \tau(L)$  of  $L$ . As  $L$  is very ample, we know that  $K_X + \tau L$  is nef but not ample. By Equation (9), we have that

$$K_T - bD + \tau\xi = \psi|_{\xi|}^*(K_X + \tau L).$$

Thus the restriction  $(K_T - bD + \tau\xi)|_D$  is nef on  $D$ . By Equation (5), Equation (6) and Equation (8), we have that

$$(K_T - bD + \tau\xi)|_D = (\tau - (n + b))\text{pr}_2^*(\mathcal{O}_{\mathbb{P}^{n-2}}(1)).$$

Hence  $\tau \geq n + b$ . Let  $C \subset T$  be a curve. We will consider its intersection with  $K_T - bD + \tau\xi = \pi^*(\mathcal{O}_{\mathbb{P}^1}(a-2)) + (\tau - n)\xi - bD$ . We have 3 situations:

- (1) The curve  $C$  is contained in  $D$ . Then  $(K_T - bD + \tau\xi) \cdot C = (\tau - (n + b))\text{pr}_2^*(\mathcal{O}_{\mathbb{P}^{n-2}}(1)) \cdot C \geq 0$ .
- (2) The generic point of  $C$  is not in  $D$  and  $C$  is contracted by  $\pi$ . Then  $C$  is a curve in a fiber  $F$  of  $\pi$ . We have that  $F = \mathbb{P}^{n-1}$  and  $\xi|_F = \mathcal{O}_{\mathbb{P}^{n-1}}(1)$ . If  $C \cap D = \emptyset$ , then  $D \cdot C = 0$  and  $(K_T - bD + \tau\xi) \cdot C = (\tau - n)(\xi \cdot C)$ ; if  $C \cap D \neq \emptyset$ , we have that  $D|_F = \mathbb{P}^{n-2} = \xi|_F$  and  $(K_T - bD + \tau\xi) \cdot C = (\tau - n - b)(\xi \cdot C)$ .
- (3) The generic point of  $C$  is not in  $D$  and  $\pi : C \rightarrow \mathbb{P}^1$  is a finite morphism. We identify  $\mathbb{P}^1$  with  $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(a)) \subset T$ . The finite morphism is thus given by

$$\begin{array}{ccc} C & \longrightarrow & \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(a)) \\ & & \swarrow \\ & & T \end{array}$$

As  $\xi|_{\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(a))} = \mathcal{O}_{\mathbb{P}^1}(a)$ , we have that

$$(\pi^*(\mathcal{O}_{\mathbb{P}^1}(a-2)) + (\tau - n)\xi - bD) \cdot C = \deg(C/\mathbb{P}^1)(a-2 + (\tau - n)a) = 0.$$

We claim that  $\tau = n + b$ . By case (2), we know that  $\tau \geq n + b$  and  $K_T - bD + (n + b)\xi = \phi|_{\xi|}^*(K_X + (n + b)L)$  is nef. On the other hand, any curve  $C \subset X$  is dominated by a curve  $C' \subset T$ . Hence  $\tau(L) = n + b$ .

Finally we show that  $K_X + (n - 1)L \notin \text{Pseff}(X)$ . In fact, suppose by contradiction that  $K_X + (n - 1)L$  is pseudo-effective. Then so is  $K_T - bD + (n - 1)\xi = \phi|_{\xi|}^*(K_X + (n - 1)L)$ . We restrict the line bundle to  $\{\text{pt}\} \times \mathbb{P}^{n-2} \subset D$ . The divisor  $-(1 + b)\text{pr}_2^*(\mathcal{O}_{\mathbb{P}^{n-2}}(1))$  is pseudo-effective. As  $-1 < b < 0$ , we know that  $(1 + b)\text{pr}_2^*(\mathcal{O}_{\mathbb{P}^{n-2}}(1)) \in \text{Pseff}(\mathbb{P}^{n-2}) \setminus \{0\}$ , a contradiction.  $\square$

**Lemma 3.12.** *Let  $(X, L) = C_n(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(a))$  be a generalized cone with  $a \leq 2$  and  $n \geq 2$ . Then  $K_X + (n - 1)L \in \text{Pseff}(X)$ .*

**PROOF.** We use the same notation as in Lemma 3.11. Using the same computation as in Lemma 3.11, we have that

$$(10) \quad K_T = \psi|_{\xi|}^*(K_{C_n(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(a))}) + bD.$$

Intersect with  $\mathbb{P}^1 \times \{\text{pt}\}$ . We find that  $b = -\frac{a-2}{a} \geq 0$ . Hence  $(X, L)$  has canonical singularity.

Suppose by contradiction that  $K_X + (n-1)L$  is not pseudo-effective. As  $\rho(X) = 1$ , **Lemma 3.8** implies that  $\Delta(X, L) = 0$  and  $K_X + \tau L \sim_{\mathbb{Q}} \mathcal{O}_X$  and  $n-1 < \tau(L) \leq n$ . The computation in **Lemma 3.10** shows that  $K_{\mathbb{P}^1} + \mathcal{O}_{\mathbb{P}^1}(a) = \mathcal{O}_{\mathbb{P}^1}(a-2)$  is ample. This is a contradiction.  $\square$

Combining all the precedent results, we have the following:

**THEOREM 3.13 (=Theorem 1.6).** *Let  $X$  be a variety with canonical  $\mathbb{Q}$ -factorial singularities and  $L$  a big and nef line bundle on  $X$ . Suppose that  $K_X + (n-1)L \notin \text{Pseff}(X)$ . Then we have one of the following cases:*

- (1)  $(X, L) \sim_{\text{bir}} (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$ ;
- (2)  $(X, L)$  is birational equivalent to a  $(\mathbb{P}^{n-1}, \mathcal{O}_{\mathbb{P}^{n-1}}(1))$ -bundle over a smooth curve  $C$ ;
- (3)  $(X, L) \sim_{\text{bir}} (Q, \mathcal{O}_{\mathbb{P}^{n+1}}(1))$ , where  $Q \subset \mathbb{Q}^{n+1}$  is a hyperquadric;
- (4)  $(X, L) \sim_{\text{bir}} (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$ ;
- (5)  $(X, L) \sim_{\text{bir}} C_n(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$ , where  $C_n(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$  is a generalised cone over  $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$

**PROOF.** By **Lemma 3.5**, there exists a birational equivalence  $(X, L) \sim_{\text{bir}} (X', L')$ , where  $X'$  is a normal projective variety with canonical  $\mathbb{Q}$ -factorial singularity,  $K_{X'} + (n-1)L' \notin \text{Pseff}(X')$  and

- (0-i) Either  $\tau(L')$  is finite;
- (0-ii) or there is a Mori fiber space structure  $\phi : X' \rightarrow W$  and a rational number  $\tau > (n-1)$  such that  $L'$  is  $\phi$ -ample and  $K_{X'} + \tau L' \sim_{\mathbb{Q}, \phi} 0$ .

In the first case, we have that  $r(L') = \frac{1}{\tau(L')} > 0$ . Hence by Kawamata rationality theorem there exists an  $K'_{X'}$ -negative extremal ray  $R_0 = \mathbb{R}_{\geq 0}[C_0]$  such that  $(r(L')K_{X'} + L') \cdot C_0 = 0$ . Hence  $L' \cdot C_0 > 0$ . In the second case, take  $R_0 = \mathbb{R}_{\geq 0}[C_0]$  to be the extremal ray associated to  $\phi$ . Then  $L' \cdot C_0 > 0$ .

Applying **Lemma 3.8** on  $(X', L')$ , we get that  $(X', L')$  is the one of the following

- (1)  $(X', L') \cong (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$ , and  $\tau = n+1$ ;
- (2-i)  $(X', L')$  is isomorphic to a  $(\mathbb{P}^{n-1}, \mathcal{O}_{\mathbb{P}^{n-1}}(1))$ -bundle over a smooth curve  $C$  and  $\tau = n$ ;
- (\*)  $\Delta(X', L') = 0$ ,  $K_{X'} + \tau L' \equiv_{\text{num}} \mathcal{O}_X$  and  $n-1 < \tau \leq n$ .

If we are in case (\*), apply **Lemma 3.9**. We have a birational morphism  $\mu : X' \rightarrow X''$  such that

- (a)  $X''$  has canonical singularities,  $\mu^*(K_{X''}) = K_{X'}$ ;
- (b) There exists an ample line bundle  $L''$  on  $X''$  such that  $\mu^*(L'') = L'$ ;
- (c)  $\Delta(X'', L'') = 0$  and  $K_{X''} + \tau L'' \equiv_{\text{num}} \mathcal{O}_{X''}$ .

In particular  $(X', L') \sim_{\text{bir}} (X'', L'')$ . Now apply **Lemma 3.10** to  $(X'', L'')$ . We have that  $(X'', L'')$  is isomorphic to the following pair:

- (3)  $(X'', L'') \cong (Q, \mathcal{O}_{\mathbb{P}^{n+1}}(1))$ , where  $Q \subset \mathbb{Q}^{n+1}$  is a hyperquadric;
- (2-ii)  $(X'', L'')$  is a  $\mathbb{P}^{n-1}$ -bundle over  $\mathbb{P}^1$  and  $L$  restricted to each fiber is  $\mathcal{O}_{\mathbb{P}^{n-1}}(1)$ ;
- (4)  $(X'', L'') \cong (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$ ;

(5)  $(X'', L'') \cong C_n(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$  is a generalised cone over  $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$

Thus we get the list stated in [Theorem 3.13](#) □

### 3.2. Normal polarized varieties

With the help of canonical modification [Theorem 2.25](#), we can give a classification theorem for normal polarized varieties with  $\mathbb{Q}$ -Gorenstein singularities.

**THEOREM 3.14** (= [Theorem 1.7](#)). *Let  $(X, L)$  be a polarized normal variety of dimension  $n$ . Suppose that  $K_X$  is  $\mathbb{Q}$ -Cartier and  $K_X + (n - 1)L \notin \text{Pseff}(X)$ . Then we have one of the following cases :*

- (1)  $(X, L) \cong (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$ ;
- (2.i)  $(X, L) \cong (\mathbb{P}(\mathcal{V}), \mathcal{O}_{\mathbb{P}(\mathcal{V})}(1))$ , where  $\mathcal{V}$  is a rank  $n$  ample vector bundle over a smooth curve  $C$ ;
- (2.ii)  $(X, L) \cong C_n(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(a))$  with  $a \geq 3$ , where  $C_n(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(a))$  is a generalised cone;
- (3)  $(X, L) \cong (Q, \mathcal{O}_{\mathbb{P}^{n+1}}(1))$ , where  $Q \subset \mathbb{P}^{n+1}$  is a hyperquadric;
- (4)  $(X, L) \cong (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$ ;
- (5)  $(X, L) \cong C_n(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$ , a generalised cone over  $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$ .

**PROOF.** Apply [Theorem 2.25](#) to the pair  $(X, 0)$ . We get the canonical modification  $f : X' \rightarrow X$  with  $K_{X'}$  being  $f$ -ample. We take a further step, applying [Lemma 2.23](#) to get a small  $\mathbb{Q}$ -factorial modification  $g : Y \rightarrow X'$  of  $X'$ . We denote the composition  $g \circ f$  by  $\mu$ . As  $g$  is small, we have that  $K_Y = g^*(K_{X'})$  is  $\mu$ -nef. Note that  $\mu|_{\mu^{-1}(X_{\text{reg}})} : \mu^{-1}(X_{\text{reg}}) \rightarrow X_{\text{reg}}$  is an isomorphism. For the canonical sheaves  $\omega_Y = \mathcal{O}_Y(K_Y)$  and  $\omega_X = \mathcal{O}_X(K_X)$ , we know that  $\mu_*(\omega_Y)|_{X_{\text{reg}}} \cong \omega_X|_{X_{\text{reg}}}$ . Note that  $\mu_*(\omega_Y)$  is torsion-free, so we have an injection  $\mu_*(K_Y) \hookrightarrow K_X$ . By the projection formula we have an injection

$$\mathcal{O}_X(\mu_*(K_Y + (n - 1)\mu^*L)) \hookrightarrow \mathcal{O}_X(K_X + (n - 1)L).$$

As  $K_X + (n - 1)L$  is not pseudo-effective, we know that neither is  $K_Y + (n - 1)\mu^*(L)$ . We set  $\mu^*(L) := M$ . As  $M$  is nef and big, we know that  $M \in \text{Pseff}(Y)$ . Note that  $K_Y$  is not pseudo-effective, hence it is not nef.

Let  $R = \mathbb{R}_{\geq 0}[C]$  be a  $K_Y$ -negative extremal ray, with  $C \subset Y$  a rational curve. Then  $K_Y \cdot C < 0$ . As  $K_Y$  is  $\mu$ -nef, we know that  $C$  is not contracted by  $\mu$ . Hence  $\mu(C) \subset X$  has dimension 1. Since  $L$  is ample, we have that  $M \cdot C = \deg(C/\mu(C))L \cdot \mu(C) > 0$ . Thus for any  $K_Y$ -negative extremal ray  $R$ , we have that  $M \cdot R > 0$ . [Lemma 3.5](#) implies that  $r(M) > 0$  and  $\tau(M) > n - 1$ . By [Lemma 3.8](#) applied to  $(Y, M)$ , we have one of the following cases:

- (i)  $(Y, M) \cong (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$ , and  $\tau = n + 1$ , or
- (ii)  $(Y, M)$  is isomorphic to a  $(\mathbb{P}^{n-1}, \mathcal{O}_{\mathbb{P}^{n-1}}(1))$ -bundle over a smooth curve  $C$  and  $\tau = n$ , or
- (iii)  $\Delta(Y, M) = 0$ ,  $K_Y + \tau M \sim_{\mathbb{Q}} \mathcal{O}_Y$  and  $n - 1 < \tau \leq n$ .

In case (i), we have a birational morphism  $\mu : \mathbb{P}^n \rightarrow X$  with  $\mu^*(L) = \mathcal{O}_{\mathbb{P}^n}(1)$ . We have that  $\overline{\text{NE}}(\mathbb{P}^n/X) = 0$  since both  $L$  and  $\mathcal{O}_{\mathbb{P}^n}(1)$  are ample. By [Lemma 2.16](#), the morphism  $\mu$  is an isomorphism. We have case (1) in [Theorem 3.14](#).



In case (ii), we have a birational morphism  $\mu : \mathbb{P}(\mathcal{V}) \rightarrow X$ , such that  $K_{\mathbb{P}(\mathcal{V})}$  is  $\mu$ -nef. We denote by  $\xi$  the pull-back  $\mathcal{O}_{\mathbb{P}(\mathcal{V})}(1) = \mu^*(L)$ . We know that  $\xi$  is nef and big.

We first note that  $\xi$  is ample if and only if  $\mu$  is an isomorphism. In fact, if  $\mu$  is an isomorphism, then  $\xi$  is ample. Conversely, if  $\xi$  is ample, we have that  $\overline{\text{NE}}(\mathbb{P}(\mathcal{V})/X) = 0$  and by [Lemma 2.16](#)  $\mu$  is an isomorphism. In this case, we have that

$$K_{\mathbb{P}(\mathcal{V})} + (n-1)\xi = \pi^*(K_C + \det \mathcal{V}) - \xi$$

is not pseudo-effective. In fact, the general fiber  $f$  is from a covering family and we have that  $K_{\mathbb{P}(\mathcal{V})} + (n-1)\xi|_f = \mathcal{O}_f(-1)$ . [Theorem 2.13](#) implies that  $K_{\mathbb{P}(\mathcal{V})} + (n-1)\xi$  is not pseudo-effective. Thus we get (2,i).

Now suppose that  $\xi$  is not ample. Then  $\mu$  is not an isomorphism. We have the following diagram:

$$\begin{array}{ccc} (\mathbb{P}(\mathcal{V}), \xi) & \xrightarrow{\mu} & (X, L) \\ \pi \downarrow & & \\ C & & \end{array}$$

We know that  $\rho(\mathbb{P}(\mathcal{V})) = 2$ . As  $\mu \neq \pi$ , we have that

$$\overline{\text{NE}}(\mathbb{P}(\mathcal{V})) = \text{NE}(\pi) + \text{NE}(\mu).$$

We denote a general fiber of  $\pi$  by  $f$ . By [[Ful11](#), Page 450], we know that  $\overline{\text{NE}}(\mathbb{P}(\mathcal{V}))$  has as extremal rays  $\mathbb{R}_{\geq 0}\xi^{n-2}f$  and  $\mathbb{R}_{\geq 0}(\xi^{n-1} + \nu^{(n-1)}\xi^{n-2}f)$  for some  $\nu^{(n-1)} \in \mathbb{Q}$ . Note that  $\mathbb{P}^1 = \xi^{n-2}f$  is contracted by  $\pi$ . Hence  $\text{NE}(\pi) = \mathbb{R}_{\geq 0}\xi^{n-2}f$ . We have that  $K_{\mathbb{P}(\mathcal{V})} = \pi^*(K_C + \det(\mathcal{V})) - n\xi$ . Hence  $K_{\mathbb{P}(\mathcal{V})} \cdot \xi^{n-2}f = -n$ . Thus  $\pi$  is the Mori contraction associated to the extremal ray  $\mathbb{R}_{\geq 0}\xi^{n-2}f$ . As  $\overline{\text{NE}}(\mu)$  is an extremal ray, we know that  $\mu$  is an extremal contraction and [Lemma 2.19](#) implies that  $\mu$  is either small or divisorial.

If  $\mu$  is small, we have that  $K_{\mathbb{P}(\mathcal{V})} = \mu^*(K_X)$ . As  $\rho(X) = 1$ , we have that  $K_X \equiv_{\text{num}} mL$  for some  $m \in \mathbb{Q}$ . Hence  $K_{\mathbb{P}(\mathcal{V})} \equiv_{\text{num}} m\xi$ . We have that

$$m = m\xi \cdot \xi^{n-2}f = K_{\mathbb{P}(\mathcal{V})} \cdot \xi^{n-2}f = -n.$$

Thus we get that  $K_X + nL \equiv_{\text{num}} \mathcal{O}_X$ . By [Theorem 1.2](#), we have that  $(X, L) \cong (Q, \mathcal{O}_Q(1))$  where  $Q \subset \mathbb{P}^{n+1}$  is a hyperquadric. Hence we are in case (3) of [Theorem 3.14](#).

If  $\mu$  is divisorial, we denote the exceptional divisor by  $E = \text{exc}(\mu)$ . Note that  $\mu^*(L) = \xi = \mathcal{O}_{\mathcal{V}}(1)$  is nef, hence  $\mathcal{V}$  is nef. We have a unique exact sequence of locally free sheaves:

$$0 \rightarrow \mathcal{A} \rightarrow \mathcal{V} \rightarrow \mathcal{Q} \rightarrow 0$$

with  $\mathcal{A}$  is an ample vector bundle and  $\mathcal{Q}$  is numerically flat. If  $l \subset \mathbb{P}(\mathcal{V})$  is a curve such that  $\xi \cdot l = 0$ , then  $l$  is contained in  $\mathbb{P}(\mathcal{Q})$ . Thus we have that  $E \subset \mathbb{P}(\mathcal{Q})$ . In particular,  $\text{rk}(\mathcal{Q}) = n-1$  and  $E = \mathbb{P}(\mathcal{Q})$ . We denote the bundle morphism by  $\pi' : \mathbb{P}(\mathcal{Q}) \rightarrow C$ . Now



we compute  $E|_E$ .

$$\begin{aligned} E|_E &= (K_{\mathbb{P}(\mathcal{Q})} - K_{\mathbb{P}(\mathcal{V})})|_E \\ &= \pi'^*(K_C + \det \mathcal{Q}) - (n-1)\xi|_E - (\pi'^*(K_C + \det \mathcal{V}) - n\xi)|_E \\ &= \pi'^*(\det \mathcal{Q} - \det \mathcal{V}) + \xi|_E \\ &= \pi'^*(-\mathcal{A}) + \xi|_E. \end{aligned}$$

Take a rational curve  $l$  that is in the fiber of  $\pi'$ . We have that  $E \cdot l = E|_E \cdot l = 1$ . Now write  $K_{\mathbb{P}(\mathcal{V})} = \mu^*(K_X) + \lambda E$ . As  $\rho(X) = 1$ , there exists some  $m \in \mathbb{Q}$  such that  $K_X \equiv_{\text{num}} mL$ . As  $K_X + (n-1)L \equiv_{\text{num}} (m+n-1)L \notin \text{Pseff}(X)$ , we have that  $m+n < 1$ . Intersect with  $l$  and get

$$-n = K_{\mathbb{P}(\mathcal{V})} \cdot l = (\mu^*(K_X) + \lambda E) \cdot l = (m\xi + \lambda E) \cdot l = m + \lambda.$$

Hence  $\lambda = -m - n > -1$  and  $X$  has klt singularities. A  $\pi'$ -fiber is isomorphic to  $\mathbb{P}^{n-2}$  and is mapped isomorphically onto its image by  $\mu$ . Hence each non-trivial  $\mu$ -fiber has dimension 1. As  $X$  has klt singularities, *a fortiori*  $(X, 0)$  is dlt. Applying [HM07, Corollary 1.5-(1)] to the birational morphism  $\mu$ , each  $\mu$ -fiber is rationally chain connected. Hence a non trivial fiber has  $\mathbb{P}^1$  as its normalization. We have thus a finite map  $\pi'|_{\mathbb{P}^1} : \mathbb{P}^1 \rightarrow C$ . Thus  $C = \mathbb{P}^1$  and  $\mathcal{V} = \mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}^{\oplus(n-1)}$ .

Consider the morphism  $\psi : \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}^{\oplus(n-1)}) \rightarrow C_n(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(a))$ . We know that  $\psi$  does not contract the extremal ray  $\overline{\text{NE}}(\pi)$ . Hence  $\overline{\text{NE}}(\psi) = \overline{\text{NE}}(\mu)$  and by [Lemma 2.16](#) we have  $X = C_n(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(a))$ . As  $L$  and the restriction of  $\mathcal{O}_{\mathbb{P}(H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(a)))}$  to  $C_n(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(a))$  agree outside a subscheme of codimension at least 2, we have that  $(X, L) = C_n(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(a))$ . As  $K_X + (n-1)L \notin \text{Pseff}(X)$ , [Lemma 3.12](#) implies  $a \geq 3$ . Now [Lemma 3.11](#) shows that for all  $a \geq 3$ , we have that  $K_X + (n-1)L \notin \text{Pseff}(X)$  and  $X$  is klt. Thus we get (2,ii)

If we are in case (iii), apply [Lemma 3.9](#) to  $(Y, M)$ . We have a crepant resolution  $\nu : Y \rightarrow Y_{\text{can}}$  with an ample divisor  $A$  on  $Y_{\text{can}}$  such that  $\nu^*(A) = M$ , the  $\Delta$ -genus satisfies  $\Delta(Y_{\text{can}}, A) = 0$  and  $K_{Y_{\text{can}}} + \tau A \equiv_{\text{num}} \mathcal{O}_{Y_{\text{can}}}$ . By [Lemma 3.10](#), the polarized variety  $(Y_{\text{can}}, A)$  is isomorphic to one of the following:

- (a)  $(Q, \mathcal{O}_{\mathbb{P}^{n+1}}(1))$ , where  $Q \subset \mathbb{Q}^{n+1}$  is a hyperquadric;
- (b) a  $\mathbb{P}^{n-1}$ -bundle over  $\mathbb{P}^1$  and  $L$  restricted to each fiber is  $\mathcal{O}_{\mathbb{P}^{n-1}}(1)$ ;
- (c)  $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$ ;
- (d) a generalised cone  $C_n(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$  over  $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$ .

Case (b) is a special case of (ii) treated above. In case (a),(c) and (d), we have the following diagram

$$\begin{array}{ccc} (Y, M) & \xrightarrow{\mu} & (X, L) \\ \downarrow \nu & \nearrow h & \\ (Y_{\text{can}}, A) & & \end{array}$$

where  $h$  is a birational map *a priori* not necessarily defined on all  $Y_{\text{can}}$ . We now show that  $h$  is indeed an isomorphism and  $h^*(L) = A$ . Let  $C \subset Y$  be a curve. We have that

$$\nu^*(A) \cdot C = M \cdot C = \mu^*(L) \cdot C.$$

As  $A$  and  $L$  are both ample, we have that  $\text{NE}(\mu) = \text{NE}(\nu)$ . [Lemma 2.16](#) implies that  $h$  is an isomorphism. As  $h^*(L)$  agrees with  $A$  outside a subscheme of codimension at least 2, we have that  $h^*(L) = A$ . Hence we get case (3), (4), (5) in [Theorem 3.14](#).  $\square$

Using similar methods, we can classify log pairs  $(X, D)$  with  $D$  a prime Weil divisor. First let us recall:

**Definition 3.15.** Let  $n > 0$  be a integer. A *hyperquadric*  $Q$  in the projective space  $\mathbb{P}^n = \text{Proj}(\mathbb{C}[x_0, x_1, \dots, x_n])$  is a subscheme whose ideal sheaf has the form  $\mathcal{O}_{\mathbb{P}^n}(-D)$  where  $D = \text{div}(s)$  for some  $s \in H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(2))$ . As  $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(2))$  is canonically isomorphic to  $(\mathbb{C}[x_0, x_1, \dots, x_n])_2$ , we can write  $s = \sum_{0 \leq i \leq n, 0 \leq j \leq n} a_{ij} x_i x_j$  with  $a_{ij} \in \mathbb{C}$ . We define  $\text{rk}(Q)$  the *rank* of the hyperquadric  $Q$  to be the rank of the matrix  $A + A^t$ , where  $A_{ij} = a_{ij}$ . If  $s'$  is another section of  $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(2))$  such that  $\mathcal{O}_{\mathbb{P}^n}(-\text{div}(s')) = \mathcal{O}_{\mathbb{P}^n}(-D)$ , then there exists a unique  $r \in \mathbb{C} \setminus 0$  such that  $rs = s'$ . Hence the rank of  $Q$  is well defined.

We have the following classification.

**THEOREM 3.16.** *Let  $(X, D)$  be a log canonical pair,  $D \subset X$  a prime divisor,  $\dim(X) = n$ . Suppose that  $L$  is an ample line bundle on  $X$  and  $(K_X + D) + (n-1)L \notin \text{Pseff}(X)$ . Then  $(X, D, L)$  is one of the following:*

- (1)  $(X, L) \cong (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$  and  $D$  is a hyperplane in  $\mathbb{P}^n$ ;
- (2.i) There is a  $(\mathbb{P}^{n-1}, \mathcal{O}_{\mathbb{P}^{n-1}}(1))$ -bundle  $(\mathbb{P}(E), \mathcal{O}_{\mathbb{P}(E)}(1))$  over a smooth curve  $C$ , and a birational morphism  $\mu : \mathbb{P}(E) \rightarrow X$  such that  $\mu^*(L) \cong \mathcal{O}_{\mathbb{P}(E)}(1)$  and  $D \cong \mu(\mathbb{P}^{n-1})$  is the image of a general fiber of  $\pi$  by  $\mu$  and  $\mathbb{P}^{n-1} \rightarrow D$  has degree 1;
- (2.ii)  $(X, L) = (\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(1)), \mathcal{O}_{\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(1))}(1))$  with  $a > 1$  and  $D$  is the unique section of  $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(1)) \rightarrow \mathbb{P}^1$  such that

$$D \equiv_{\text{num}} \mathcal{O}_{\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(1))}(1) - af,$$

where  $f$  is a general fiber;

- (3.i)  $(X, L) \cong (Q, \mathcal{O}_{\mathbb{P}^{n+1}}(1))$ , where  $Q \subset \mathbb{P}^{n+1}$  is a  $\text{rk}(Q) = 3$  hyperquadric and  $D \cong \mathbb{P}^{n-1}$  is a hyperplane in  $Q$  with  $D \equiv_{\text{num}} \frac{1}{2} \mathcal{O}_Q(1)$ ;
- (3.ii)  $(X, L) \cong (Q, \mathcal{O}_{\mathbb{P}^{n+1}}(1))$ , where  $Q \subset \mathbb{P}^{n+1}$  is a  $\text{rk}(Q) = 4$  hyperquadric. If we write  $Q = \text{Proj} \left( \frac{\mathbb{C}[x_0, \dots, x_{n+1}]}{(x_0 x_1 - x_2 x_3)} \right)$ , then  $D$  is the cone with vertex  $\mathbb{P}^{n-3}$  over  $\mathbb{P}^1 \times \text{pt}$  or  $\text{pt} \times \mathbb{P}^1$ . In particular,  $D \cong \mathbb{P}^{n-1}$ .

**PROOF.** As in the proof of [Theorem 3.14](#), we apply [Theorem 2.25](#) then [Lemma 2.23](#) to the normal variety  $X$ . We get a birational morphism  $\mu : Y \rightarrow X$  such that  $Y$  has  $\mathbb{Q}$ -factorial canonical singularities, the canonical class  $K_Y$  is  $\mu$ -nef and  $\mu$  is isomorphic over the regular locus of  $X$ . We set  $D' = \mu_*^{-1}(D)$ . If we denote by  $\omega_Y = \mathcal{O}_Y(K_Y)$  and by  $\omega_X = \mathcal{O}_X(K_X)$  the canonical sheaves, we know that

$$\mu_*(\omega_Y \otimes \mathcal{O}_Y(D'))|_{X_{\text{reg}}} \cong (\omega_X \otimes \mathcal{O}_X(D))|_{X_{\text{reg}}}$$

Note  $\mu_*(\omega_Y \otimes \mathcal{O}_Y(D'))$  is torsion-free, so we have an injection  $\mu_*(\omega_Y \otimes \mathcal{O}_Y(D')) \hookrightarrow \omega_X \otimes \mathcal{O}_X(D)$ . Tensoring with  $\mu^*(L^{\otimes n-1})$ , we have an injection

$$\mu_*(\omega_Y \otimes \mathcal{O}_Y(D') \otimes \mu^*(L^{\otimes n-1})) \mapsto \omega_X \otimes \mathcal{O}_X(D) \otimes L^{\otimes n-1}.$$

As  $(K_X + D) + (n-1)L$  is not pseudo-effective, neither is  $(K_Y + D') + (n-1)\mu^*(L)$ . We set  $\mu^*(L) =: M$ . As  $D'$  is effective, the divisor  $K_Y + (n-1)M$  is not pseudo-effective.

We note that as  $K_Y$  is  $\mu$ -nef and  $M = \mu^*(L)$ , for any  $K_Y$ -negative extremal ray  $R$ , we have that  $M \cdot R > 0$ . Hence we may argue as in [Theorem 3.14](#) by applying [Lemma 3.8](#). We thus get:

- (a) Either  $(Y, M) \cong (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$  and  $\tau = n+1$ , or
- (b)  $(Y, M)$  is isomorphic to a  $(\mathbb{P}^{n-1}, \mathcal{O}_{\mathbb{P}^{n-1}}(1))$ -bundle over a smooth curve  $C$  and  $\tau = n$ , or
- (c)  $\Delta(Y, M) = 0$ ,  $K_Y + \tau M \sim_{\mathbb{Q}} \mathcal{O}_Y$  and  $n-1 < \tau \leq n$ .

In case (a),  $\mu$  is an isomorphism. We know that  $D \sim \mathcal{O}_{\mathbb{P}^n}(a)$  for some  $a \geq 1$ . We have that  $K_X + (n-1)L + D = \mathcal{O}_{\mathbb{P}^n}(a-2)$ . Hence the only possible choice is  $a = 1$  and  $D$  is a hyperplane and we are in case (1) of [Theorem 3.16](#).

In case (b), we have a diagram

$$\begin{array}{ccc} (\mathbb{P}(\mathcal{V}), \xi) & \xrightarrow{\mu} & (X, L) \\ \pi \downarrow & & \\ C & & \end{array}$$

where  $\xi = \mathcal{O}_{\mathbb{P}(\mathcal{V})}(1)$  and  $K_{\mathbb{P}(\mathcal{V})}$  is  $\mu$ -nef. As  $\tau(\xi) = n$ , we know that  $K_{\mathbb{P}(\mathcal{V})} + n\xi = \pi^*(K_C + \det(\mathcal{V}))$  is nef.

First we assume that  $\mu$  is an isomorphism. In this case  $\mathcal{V}$  is ample. Let  $F = \mathbb{P}^{n-1}$  be a general fiber of  $\pi$ . Suppose that  $D|_F = \mathcal{O}_F(d)$  with  $d \geq 0$ . We have the following equality

$$(K_{\mathbb{P}(\mathcal{V})} + D + (n-1)\xi)|_F = (\pi^*(K_C + \det(\mathcal{V})) + D - \xi)|_F = \mathcal{O}_F(d-1).$$

If  $d = 0$ , then  $D$  is one of the general fiber. In fact, if for a general fiber  $F$ , we have that  $D \cap F \neq \emptyset$  and  $D \not\subseteq F$ , there exists a curve  $l \subset F \setminus D$  such that  $l \cap D \neq \emptyset$ . We have that  $D \cdot l > 0$ , a contradiction. Let  $l$  be a rational curve in  $F$ . We have that  $(K_{\mathbb{P}(\mathcal{V})} + D + (n-1)\xi) \cdot l = -1$ . Since  $F$  is a member of a covering family, [Theorem 2.13](#) implies that  $K_{\mathbb{P}(\mathcal{V})} + (n-1)\xi + F$  is not pseudo-effective. We thus are in case (2) of [Theorem 3.16](#).

If  $d > 0$ , by [Lemma 3.17](#), we have that  $n = \dim(\mathbb{P}(\mathcal{V})) = 2$ . We first show that  $C = \mathbb{P}^1$ . The non pseudo-effective divisor in question  $K_X + D + (n-1)L$  thus becomes  $K_{\mathbb{P}(\mathcal{V})} + D + \xi$ . We have that  $(K_{\mathbb{P}(\mathcal{V})} + D + \xi)|_F = \mathcal{O}_F(d-1)$  which is nef. As  $K_{\mathbb{P}(\mathcal{V})} + D + \xi$  is not nef, there exists an extremal ray  $R'$  which is not generated by the fiber of  $\pi$ , such that  $(K_{\mathbb{P}(\mathcal{V})} + D + \xi) \cdot R' < 0$ . In particular, we have that  $R'$  is  $(K_{\mathbb{P}(\mathcal{V})} + D)$ -negative. By [Theorem 2.17](#)-(4) we know that  $R' = \mathbb{R}_{\geq 0}[l]$  for a rational curve  $l$ . Note that  $l$  maps finitely onto  $C$ . Hence  $C \cong \mathbb{P}^1$ .

[Lemma 3.18](#) implies that  $(X, D, L)$  is either in cases (2.i), (2.ii) of [Theorem 3.16](#) or  $(X, L)$  is a hyperquadric of rank 4, which will be dealt in the following case (c1).

Assume, from now on, that  $\mu$  is not an isomorphism. We know that

$$\overline{\text{NE}}(\mathbb{P}(\mathcal{V})) = \overline{\text{NE}}(\mu) + \overline{\text{NE}}(\pi),$$

and  $\pi$  contracts the extremal ray  $\mathbb{R}_{\geq 0}\xi^{n-2}f$ . The birational morphism  $\mu$  is either small or divisorial.

If  $\mu$  is small, by construction, we have that  $K_{\mathbb{P}(\mathcal{V})} + D' = \mu^*(K_X + D)$ . Let  $F$  be a general fiber of  $\pi$ . We have that  $D'|_F = \mathcal{O}_F(d)$  for some  $d \geq 0$ . As  $K_X + D$  is  $\mathbb{Q}$ -Cartier and  $\rho(X) = 1$ , we have that  $K_X + D \equiv_{\text{num}} mL$  for some  $m \in \mathbb{Q}$ . Hence  $K_{\mathbb{P}(\mathcal{V})} + D' \equiv_{\text{num}} \mu^*mL$ . Intersect with  $\xi^{n-2}f$ . We get that  $-n + d = m$ . Hence  $K_X + D + (n-1)L \equiv_{\text{num}} (d-1)L$ . Thus we have that  $d = 0$  and  $D' \sim F$ . As  $D' = \mu_*^{-1}(D)$  by definition, we get that  $D' \rightarrow D$  has degree 1. We are thus in case (2.i) of [Theorem 3.16](#).

If  $\mu$  is divisorial, we denote the exceptional divisor by  $E = \text{exc}(\mu)$ . [[KM98](#), Proposition 3.36.] implies that  $X$  is  $\mathbb{Q}$ -factorial. In particular  $K_X$  is  $\mathbb{Q}$ -Cartier. We have a unique exact sequence of locally free sheaves:

$$0 \rightarrow \mathcal{A} \rightarrow \mathcal{V} \rightarrow \mathcal{Q} \rightarrow 0$$

with  $\mathcal{A}$  is an ample vector bundle and  $\mathcal{Q}$  is numerically flat. And we know that  $E = \mathbb{P}(\mathcal{Q})$  and  $E \cdot \xi^{n-2}f = 1$ . Let  $F$  be a general fiber of  $\pi$ . There exists a  $d \geq 0$  such that  $D'|_F = \mathcal{O}_F(d)$ . As  $K_X + D$  is  $\mathbb{Q}$ -Cartier and  $\rho(X) = 1$ , there exists an  $m \in \mathbb{Q}$  such that  $K_X + D \equiv_{\text{num}} mL$ . Then  $K_X + D + (n-1)L \equiv_{\text{num}} (m+n-1)L \notin \text{Pseff}(X)$ . Hence  $m+n < 1$ . We now have that

$$K_{\mathbb{P}(\mathcal{V})} + D' = \mu^*(K_X + D) + \lambda E.$$

Intersect both sides with  $\xi^{n-2}f$ . We get that  $-n + d = m + \lambda$ . Since  $-(m+n) > -1$ , we have that  $\lambda \geq -1 + d$ . Now we claim that  $d = 0$ . Suppose that by contrary  $d \geq 1$ . Recall that  $(K_{\mathbb{P}(\mathcal{V})} + D' + (n-1)\xi)|_F = \mathcal{O}_F(d-1)$ . As  $K_{\mathbb{P}(\mathcal{V})} + D' + (n-1)\xi$  is not nef, we know that  $\text{NE}(\mu)$  is an  $(K_{\mathbb{P}(\mathcal{V})} + D')$ -negative extremal ray. By [Theorem 2.17-\(iii\)](#), there is a rational curve  $l$  whose class  $[l]$  is in  $\overline{\text{NE}}(\mu)$ . As  $l$  maps finitely onto  $C$ , we know that  $C \cong \mathbb{P}^1$ . Hence  $(X, L) = C_n(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(a))$ . We have by assumption that  $K_X + (n-1)L \notin \text{Pseff}(X)$ . Hence [Lemma 3.12](#) implies that  $a \geq 3$ . [Lemma 3.11](#) implies that  $K_X \equiv_{\text{num}} (-n + \frac{a-2}{a})L$ . Suppose that  $D \equiv_{\text{num}} m_2L$  for some  $m_2 \in \mathbb{Q}^+$ . For  $\mathbb{P}^1 \subset F$  mapped isomorphically to its image, we have that  $m_2 = m_2\xi \cdot \mathbb{P}^1 = \mu^*(D) \cdot \mathbb{P}^1 = D \cdot \mu(\mathbb{P}^1) \in \mathbb{N}$ . Hence  $m_2 \geq 1$  and  $K_X + D + (n-1)L = (\frac{a-2}{a} + (m_2-1))L \in \text{Pseff}(X)$ , a contradiction. This proves the claim. As  $D' = \mu_*^{-1}(D)$  by definition, we get that  $D' \rightarrow D$  has degree 1. Thus  $D \cong \mu(\mathbb{P}^{n-1})$  is the image of a general fiber of  $\pi$  and we are in case (2.i) of [Theorem 3.16](#).

If we are in case (c), apply [Lemma 3.9](#) to  $(Y, M)$ . We have a crepant resolution  $\nu : Y \rightarrow Y_{\text{can}}$  with an ample divisor  $A$  on  $Y_{\text{can}}$  such that  $\nu^*(A) = M$ ,  $\Delta(Y_{\text{can}}, A) = 0$  and  $K_{Y_{\text{can}}} + \tau A \equiv_{\text{num}} 0$ . By [Lemma 3.10](#), we have one of the following:

- (c1)  $(Y_{\text{can}}, A) \cong (Q, \mathcal{O}_{\mathbb{P}^{n+1}}(1))$ , where  $Q \subset \mathbb{Q}^{n+1}$  is a hyperquadric;
- (c2)  $(Y_{\text{can}}, A)$  is a  $\mathbb{P}^{n-1}$ -bundle over  $\mathbb{P}^1$  and the restriction of  $A$  to each fiber is  $\mathcal{O}_{\mathbb{P}^{n-1}}(1)$ ;
- (c3)  $(Y_{\text{can}}, A) \cong (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$ ;
- (c4)  $(Y_{\text{can}}, A) \cong C_n(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$  is a generalised cone over  $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$ .

We have the following diagram

$$\begin{array}{ccc}
(Y, M) & \xrightarrow{\mu} & (X, L) \\
\downarrow \nu & \nearrow h & \\
(Y_{\text{can}}, A) & & 
\end{array}$$

where  $h$  is an isomorphism and  $\mu^*(L) = M = \nu^*(A)$ , with  $(Y_{\text{can}}, A)$  being one of the above four pairs.

In case (c1), after an automorphism of  $\mathbb{P}^{n+1} = \text{Proj}(\mathbb{C}[x_0, \dots, x_{n+1}])$ , the hyperquadric  $Q$  is given by the homogeneous ideal  $I_r = (\sum_{0 \leq i \leq r} x_i^2) \subset \mathbb{C}[x_0, \dots, x_{n+1}]$  for some  $r \geq 2$ . By [Har77, Exercise II.6.5], the class group  $\text{Cl}(Q)$  of  $Q$  is the following:

- When  $r = 2$ , the divisor  $\frac{1}{2}[\mathcal{O}_Q(1)]$  is an integral divisor and  $\text{Cl}(Q) = \mathbb{Z} \cdot \frac{1}{2}[\mathcal{O}_Q(1)]$ . Hence  $D$  is numerically equivalent to a hyperplane  $\mathbb{P}^{n-1}$  in  $Q$  and  $K_X + (n-1)L + D = -\frac{1}{2}\mathcal{O}_Q(1)$  is not pseudo-effective. We are thus in case (3.i) of [Theorem 3.16](#).
- When  $r = 3$ , there is an isomorphism  $\text{Cl}(Q) \cong \mathbb{Z} \oplus \mathbb{Z}$ . Note that here we can write

$$Q = \text{Proj} \left( \frac{\mathbb{C}[x_0, \dots, x_{n+1}]}{(x_0x_1 - x_2x_3)} \right),$$

which is a cone of vertex  $\mathbb{P}^{n-3} = \{x_1 = x_2 = x_3 = 0\} \subset \mathbb{P}^{n+1}$  with base  $\mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3 = \{x_4 = \dots = x_{n+1} = 0\} \subset \mathbb{P}^{n+1}$  (cf. [Har77, Exercise I.5.12.(d)]). If we consider the inclusions  $\mathbb{P}^3 \subset \mathbb{P}^4 \subset \dots \subset \mathbb{P}^n \subset \mathbb{P}^{n+1}$ , then  $Q$  is also obtained by taking projective cone in the sense of [Har77, Exercise I.2.10] of  $\mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3$  successively. By [Har77, Exercise II.6.3.(a)], we know that  $\text{Cl}(\mathbb{P}^1 \times \mathbb{P}^1) \cong \text{Cl}(Q)$ . For a hyperplan  $H \subset \mathbb{P}^{n+1}$ ,  $H \cap Q$  has type  $(1, 1)$ . The cone over  $\mathbb{P}^1 \times \text{pt}$  has type  $(1, 0)$  and the cone over  $\text{pt} \times \mathbb{P}^1$  has type  $(0, 1)$ . Thus  $D$  has type  $(1, 0)$  or type  $(0, 1)$ . We are thus in case (3.ii) of [Theorem 3.16](#).

- When  $r \geq 4$ ,  $\text{Cl}(Q) = \mathbb{Z} \cdot [\mathcal{O}_Q(1)]$ . Hence  $D = d[\mathcal{O}_Q(1)]$ , and  $K_X + (n-1)L + D \equiv_{\text{num}} \mathcal{O}_Q(d-1)$  is pseudo-effective. Thus this situation is excluded.

Case (c2) is already treated in case (b) and we are in case (2.i) of [Theorem 3.16](#).

In case (c3), the divisor  $D$  is linearly equivalent to  $\mathcal{O}_{\mathbb{P}^2}(1)$ . The divisor  $K_X + (n-1)L + D$  is numerically trivial and hence is pseudo-effective, a contradiction. Hence case (c3) does not happen.

In case (c4), we proceed as in [Lemma 3.11](#). We consider the following diagram

$$\begin{array}{ccc}
E = \mathbb{P}^2 \times \mathbb{P}^{n-3} & \xrightarrow{\text{pr}_2} & \mathbb{P}^{n-3} \\
\downarrow i & & \downarrow \\
T = \mathbb{P}(\mathcal{O}^{\oplus n-2} \oplus \mathcal{O}(2)) & \xrightarrow{\psi_{|\xi|}} & C_n(\mathbb{P}^2, \mathcal{O}(2)) = X \\
\downarrow \pi & & \\
\mathbb{P}^2 & & 
\end{array}$$

Since  $T$  is a projective bundle, by [Ful98, Theorem 3.3.(b)] we have that

$$\text{Cl}(T) = \mathbb{Z}[\pi^*(\mathcal{O}_{\mathbb{P}^2}(1))] \oplus \mathbb{Z}[\xi].$$

On the other hand, since  $E = \text{exc}(\psi|_{\xi})$  is contracted, the morphism  $(\psi|_{\xi})_* : \text{Cl}(T) \rightarrow \text{Cl}(X)$  is surjective and  $\text{rk}(\text{Cl}(X)) = 1$ . We have that  $\psi|_{\xi}^*(L) = \xi$ . Thus  $(\psi|_{\xi})_*([\xi]) = [L] \neq 0$ . To determine  $\text{Cl}(X)$ , one just need to know the image  $(\psi|_{\xi})_*\pi^*([\mathcal{O}_{\mathbb{P}^2}(1)])$ . Let  $H$  be a Weil divisor on  $T$  such that  $\mathcal{O}_T(H) = \pi^*(\mathcal{O}_{\mathbb{P}^2}(1))$ . For example, we can take  $H$  to be  $\pi^{-1}(l)$  where  $l \subset \mathbb{P}^2$  is a linear subspace. Then it's easy to see that  $H \neq E$ . Set  $G := (\psi|_{\xi})_*H$ . As  $L$  is ample, we know that  $[L] \neq 0$  in  $\text{Cl}(X) \otimes \mathbb{Q}$ . Take  $m \in \mathbb{Q}$  such that  $[G] = m[L]$  in  $\text{Cl}(X) \otimes \mathbb{Q}$ . We have that

$$(11) \quad \psi|_{\xi}^*(G) \sim_{\mathbb{Q}} (\psi|_{\xi})_*^{-1}(G) + aE,$$

with  $(\psi|_{\xi})_*^{-1}(G) = H$ . By the canonical bundle formula, we have

$$\begin{aligned} K_T &= \pi^*(\mathcal{O}_{\mathbb{P}^2}(-1)) - (n-1)\xi \quad \text{and} \\ K_E &= \text{pr}_1^*(\mathcal{O}_{\mathbb{P}^2}(-2)) - (n-2)\xi|_E. \end{aligned}$$

Hence we have

$$\mathcal{O}_E(E) = \text{pr}_1^*(\mathcal{O}_{\mathbb{P}^2}(-2)) \otimes \text{pr}_2^*(\mathcal{O}_{\mathbb{P}^{n-3}}(1)).$$

Let  $C_1 = \mathbb{P}^1 \times \{\text{pt}\} \subset E$ . Then  $E \cdot C_1 = -2$ . We intersect both sides of [Equation \(11\)](#) with  $C_1$ . As  $(\psi|_{\xi})_*(C_1) = 0$ , by the projection formula  $(\psi|_{\xi})^*(G) \cdot C_1 = 0$ . By applying the projection formula to the morphism  $\pi|_H : H \rightarrow \mathbb{P}^2$ , we get that  $H \cdot C_1 = 1$ . Hence  $a = \frac{1}{2}$ . Thus we have

$$(12) \quad m[\xi] = \pi^*[\mathcal{O}_{\mathbb{P}^2}(1)] + \frac{1}{2}E.$$

Let  $F = \mathbb{P}^{n-2}$  be a fibre of  $\pi$  such that  $F \cap E \neq \emptyset$ . Then  $E \cap F = \mathbb{P}^{n-3} \subset \mathbb{P}^{n-2} = F$ . Take  $C_2 = \mathbb{P}^1 \subset F$  and intersect both side of [Equation \(12\)](#) with  $C_2$ . We have that  $\xi \cdot C_2 = 1$  and  $\pi^*[\mathcal{O}_{\mathbb{P}^2}(1)] \cdot C_2 = 0$  and  $E \cdot C_2 = 1$ . Thus we get that  $m = \frac{1}{2}$ . Hence we know the  $\text{Cl}(X) \otimes \mathbb{Q} = \mathbb{Q} \cdot \frac{1}{2}[L]$ . Suppose  $[(\psi|_{\xi})_*^{-1}(D)] = m_1\pi^*[(\mathcal{O}_{\mathbb{P}^2}(1))] + m_2[\xi]$  with  $m_1, m_2$  natural numbers. We have  $D = (\psi|_{\xi})_*(\psi|_{\xi})_*^{-1}D \sim_{\mathbb{Q}} (\frac{m_1}{2} + m_2)L$ . Being a generalized cone,  $X$  is  $\mathbb{Q}$ -factorial. For  $\mathbb{Q}$ -Cartier divisor  $K_X$  we have

$$K_X = (\psi|_{\xi})_*(K_T) = (\psi|_{\xi})_*(\pi^*(\mathcal{O}(-1)) - (n-1)\xi) = -(n - \frac{1}{2})[L].$$

Now  $K_X + D + (n-1)L \equiv_{\text{num}} (\frac{m_1-1}{2} + m_2)L$  is pseudo-effective. We thus exclude case (c4). □

**Lemma 3.17.** *Let  $(X, D) = (\mathbb{P}(\mathcal{V}), D)$  be a log canonical pair, where  $\pi : \mathbb{P}(\mathcal{V}) \rightarrow C$  is a projective bundle over a smooth curve  $C$  and  $\mathcal{V}$  is an ample vector bundle of rank  $n$ . If for a general fiber  $F$ , we have that  $D|_F = \mathcal{O}_F(d)$  for some  $d > 0$ . Then  $\dim(X) = \dim(\mathbb{P}(\mathcal{V})) = 2$ .*

PROOF. We have that

$$(K_{\mathbb{P}(\mathcal{V})} + D + (n-1)\xi)|_F = (\pi^*(K_C + \det(\mathcal{V})) + D - \xi)|_F = \mathcal{O}_F(d-1).$$

We take a *thrifty dlt modification* for  $(\mathbb{P}(\mathcal{V}), D)$  as in [[Kol13](#), Corollary 1.36.], i.e., a proper birational morphism  $f : \mathbb{P}(\mathcal{V})^{\text{dlt}} \rightarrow \mathbb{P}(\mathcal{V})$  with a boundary divisor  $\Delta^{\text{dlt}}$  such that:

- (1)  $(\mathbb{P}(\mathcal{V})^{\text{dlt}}, \Delta^{\text{dlt}})$  has dlt singularities;
- (2)  $f^*(K_{\mathbb{P}(\mathcal{V})} + D) \sim_{\mathbb{Q}} K_{\mathbb{P}(\mathcal{V})^{\text{dlt}}} + \Delta^{\text{dlt}}$ ;
- (3)  $K_{\mathbb{P}(\mathcal{V})^{\text{dlt}}} + \Delta^{\text{dlt}}$  is  $f$ -nef;
- (4)  $\mathbb{P}(\mathcal{V})^{\text{dlt}}$  is  $\mathbb{Q}$ -factorial.

Thus we have that

$$\mathbb{P}(\mathcal{V})^{\text{dlt}} \xrightarrow{f} \mathbb{P}(\mathcal{V}) \xrightarrow{\pi} C.$$

We set  $g = \pi \circ f$  and  $\xi' = f^*\xi$ . Then we have that

$$f^*(K_{\mathbb{P}(\mathcal{V})} + D + (n-1)\xi) \equiv_{\text{num}} K_{\mathbb{P}(\mathcal{V})^{\text{dlt}}} + \Delta^{\text{dlt}} + (n-1)\xi'.$$

As  $f$  is surjective, we have that  $f_*$  preserves numerical equivalence. By the projection formula we have that

$$f_*(K_{\mathbb{P}(\mathcal{V})^{\text{dlt}}} + \Delta^{\text{dlt}} + (n-1)\xi') \equiv_{\text{num}} f_*f^*(K_{\mathbb{P}(\mathcal{V})} + D + (n-1)\xi) = K_{\mathbb{P}(\mathcal{V})} + D + (n-1)\xi.$$

Hence  $K_{\mathbb{P}(\mathcal{V})^{\text{dlt}}} + \Delta^{\text{dlt}} + (n-1)\xi'$  cannot be pseudo-effective. So there exists an extremal ray  $R$  of  $\overline{\text{NE}}(\mathbb{P}(\mathcal{V})^{\text{dlt}})$  such that

$$(K_{\mathbb{P}(\mathcal{V})^{\text{dlt}}} + \Delta^{\text{dlt}} + (n-1)\xi') \cdot R < 0.$$

For  $0 < \epsilon \ll 1$ , we have that

$$(13) \quad (K_{\mathbb{P}(\mathcal{V})^{\text{dlt}}} + (1-\epsilon)\Delta^{\text{dlt}} + (n-1)\xi') \cdot R < 0.$$

We note that  $\xi' \cdot R = f^*(L) \cdot R > 0$ , for otherwise any curve  $l$  such that  $[l] \in R$  is contracted by  $f$ , which means  $(K_{\mathbb{P}(\mathcal{V})^{\text{dlt}}} + \Delta^{\text{dlt}}) \cdot R \geq 0$ , a contradiction. Hence  $R$  is in fact a  $(K_{\mathbb{P}(\mathcal{V})^{\text{dlt}}} + \Delta^{\text{dlt}})$ -negative extremal ray.

By [Theorem 2.17](#)-(3) we get the contraction morphism  $\text{cont}_R : \mathbb{P}(\mathcal{V})^{\text{dlt}} \rightarrow Y$  which contracts the ray  $R$ . Let  $S \subset \mathbb{P}(\mathcal{V})^{\text{dlt}}$  be a fiber of  $\text{cont}_R$ . If  $\dim(S) \geq 2$ , there exists a curve  $l \subset S$  that is contracted to a point by  $g$ . As  $K_{\mathbb{P}(\mathcal{V})^{\text{dlt}}} + \Delta^{\text{dlt}}$  is  $f$ -nef, the  $K_{\mathbb{P}(\mathcal{V})^{\text{dlt}}} + \Delta^{\text{dlt}}$ -negative curve  $l$  can not be contracted to a point by  $f$ . Hence  $l$  maps finitely onto a curve  $l' \subset F$ . Now we have that

$$\begin{aligned} (K_{\mathbb{P}(\mathcal{V})^{\text{dlt}}} + \Delta^{\text{dlt}} + (n-1)\xi') \cdot l &= (f^*(K_{\mathbb{P}(\mathcal{V})} + D + (n-1)\xi)) \cdot l \\ &= \deg(l/l')(K_{\mathbb{P}(\mathcal{V})} + D + (n-1)\xi) \cdot l \\ &= \deg(l/l')\mathcal{O}_F(d-1) \cdot l' \\ &\geq 0, \end{aligned}$$

a contradiction. Hence any fiber of  $\text{cont}_R$  has dimension at most 1. Let  $E \subset \text{exc}(\text{cont}_R)$  be an irreducible component of the exceptional locus of  $\text{cont}_R$ . We thus have that

$$\dim(E) - \dim(\text{cont}_R(E)) \leq 1.$$

We know that for  $0 < \epsilon \ll 1$ , the pair  $(\mathbb{P}(\mathcal{V})^{\text{dlt}}, (1-\epsilon)\Delta^{\text{dlt}})$  has klt singularities (*cf.* [\[KM98, Proposition 2.41.\]](#)). For sufficiently small  $\epsilon$ , the divisor  $-(K_{\mathbb{P}(\mathcal{V})^{\text{dlt}}} + (1-\epsilon)\Delta^{\text{dlt}})$  is still  $\text{cont}_R$ -ample. The estimate of the length of extremal ray [\[Deb01, Theorem 7.46.\]](#) for klt pairs shows that the rational curves  $l \in R$  cover  $E$  and there exists a rational curve  $l_\epsilon \in R$  such that



$$0 < -(K_{\mathbb{P}(\mathcal{V})}^{\text{dlt}} + (1 - \epsilon)\Delta^{\text{dlt}}) \cdot l_\epsilon \leq 2.$$

For any curve  $l$  whose class  $[l]$  is in  $R$ , we have that  $\xi' \cdot l \geq 1$ . Combining these two inequalities with [Equation \(13\)](#), we have that

$$0 > (K_{\mathbb{P}(\mathcal{V})}^{\text{dlt}} + (1 - \epsilon)\Delta^{\text{dlt}} + (n - 1)\xi') \cdot l_\epsilon \geq -2 + (n - 1).$$

Hence  $n = 2$ . □

**Lemma 3.18.** *Let  $\mathcal{V}$  be a rank 2 ample vector bundle over  $\mathbb{P}^1$ . Set  $(X, L) := (\mathbb{P}(\mathcal{V}), \mathcal{O}_{\mathbb{P}(\mathcal{V})}(1))$ . Suppose that  $D$  is a prime Weil divisor on  $X$  and  $K_X + D + L$  is not pseudo-effective. Then we have one of the following:*

- (1) *Either  $D \cong \mathbb{P}^1$  is a fiber of the structure map  $\pi : \mathbb{P}(\mathcal{V}) \rightarrow \mathbb{P}^1$ ; or*
- (2)  *$(X, L) = (\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(1)), \mathcal{O}_{\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(1))}(1))$  with  $a > 1$  and  $D$  is the unique section of  $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(1)) \rightarrow \mathbb{P}^1$  such that*

$$D \equiv_{\text{num}} \mathcal{O}_{\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(1))}(1) - af,$$

*where  $f$  is a general fiber; or*

- (3)  $\mathcal{V} = \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1)$ .

PROOF. As  $\mathcal{V}$  is ample, we know that  $\mathcal{V} \cong \mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(b)$  with  $a, b > 0$ . We may suppose that  $a \geq b > 0$ . Set  $e = a - b \geq 0$ . Set  $\mathcal{W} = \mathcal{V} \otimes \mathcal{O}_{\mathbb{P}^1}(-a)$ . We have that  $X_e := \mathbb{P}(\mathcal{W}) \cong \mathbb{P}(\mathcal{V})$ . We denote by  $p : \mathbb{P}(\mathcal{W}) \rightarrow \mathbb{P}^1$  the projection. By [\[Har77, Lemma II.7.9\]](#), we know that  $(X, L) \cong (X_e, \mathcal{O}_{X_e}(1) \otimes p^*(\mathcal{O}_{\mathbb{P}^1}(a)))$ . From now on, we follow the convention in [\[Har77, Notation V.2.8.1\]](#). We denote the general fiber of  $p$  by  $f'$ . Note that  $\mathcal{W}$  satisfies the assumption in [\[Har77, Propostion 2.8.\]](#). Hence by [\[Har77, Propostion 2.8.\]](#) there exists a section  $C_0$  of  $p$  such that  $\mathcal{O}_{X_e}(1) \cong \mathcal{O}_{X_e}(C_0)$ . [\[Har77, Propostion 2.9.\]](#) implies  $C_0^2 = -e$ . Hence if  $a \neq b$ , we have that  $C_0$  is unique. We know that  $L \equiv_{\text{num}} C_0 + af'$  and [\[Har77, Lemma 2.10.\]](#) implies that  $K_{X_e} \sim -2C_0 + (-2 - e)f'$ . Assume that  $D \sim xC_0 + d'f'$ , with  $x, d'$  being integers. We have that

$$K_X + L + D \equiv_{\text{num}} (x - 1)C_0 + (d' + b - 2)f'.$$

As  $D$  is a prime divisor, [\[Har77, Corollary V.2.18-\(b\)\]](#) implies one of the following:

- i  $x = 0, d' = 1$ , and  $K_X + L + D \equiv_{\text{num}} -C_0 + (b - 1)f' \notin \text{Pseff}(X)$ ;
- ii  $x = 1, d' = 0$  and  $K_X + L + D \equiv_{\text{num}} (b - 2)f'$ , which is not pseudo-effective iff  $b = 1$ ;
- iii  $x > 0, d' > xe$ . Note  $d' + b - 2 \geq 0$ . so we have that  $K_X + L + D$ , being a positive combination of effective divisors, is pseudo-effective;
- iv  $e > 0, x > 0$ , and  $d' = xe$ . Again  $d' + b - 2 \geq 0$ . So we have that  $K_X + L + D$ , being a positive combination of effective divisors, is pseudo-effective.

In case (i), the divisor  $D$  is a fiber of  $p$ , which maps isomorphically to a fiber of  $\pi$  under the canonical isomorphism  $\mathbb{P}(\mathcal{W}) \cong \mathbb{P}(\mathcal{V})$ . Hence we are again in case (1) of [Lemma 3.18](#).

In case (ii), if  $a > b = 1$ , as  $D$  is irreducible, [\[Har77, Proposition V.2.20-\(a\)\]](#) implies that  $D = C_0$ . Hence  $D$  is the unique section of  $\pi : \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(1)) \rightarrow \mathbb{P}^1$  such that  $D \equiv_{\text{num}} \xi - \pi^*(\mathcal{O}_{\mathbb{P}^1}(a))$ . Hence we are in case (2) of [Lemma 3.18](#). If  $a = b = 1$ , then  $(X, L) \cong (\mathbb{P}^1 \times \mathbb{P}^1, (1, 1))$  which is a rank 4 hyperquadric in  $\mathbb{P}^3$ . Thus we are in case (3) of [Lemma 3.18](#). □



The proof of [Theorem 3.16](#) can be adapted to the case where the boundary  $\Delta$  is not irreducible.

**Corollary 3.19** (= [Proposition 1.8](#)). *Let  $(X, \Delta)$  be a log canonical pair, with  $\Delta \neq 0$  a reduced divisor. Suppose that  $L$  is an ample line bundle on  $X$  and  $(K_X + \Delta) + (n-1)L \notin \text{Pseff}(X)$ , where  $n = \dim(X)$ . Then we have one of the following:*

- (1)  $(X, L) \cong (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$ ,  $\Delta \equiv_{\text{num}} H$  is a prime divisor where  $H$  is a hyperplane of  $\mathbb{P}^n$ ;
- (2.i) There is a  $(\mathbb{P}^{n-1}, \mathcal{O}_{\mathbb{P}^{n-1}}(1))$ -bundle  $(\mathbb{P}(E), \mathcal{O}_{\mathbb{P}(E)}(1))$  over a smooth curve  $C$ , and a birational morphism  $\mu : \mathbb{P}(E) \rightarrow X$  such that  $\mu^*(L) \cong \mathcal{O}_{\mathbb{P}(E)}(1)$  and  $\Delta = \sum F_i$  is a finite sum where  $F_i \cong \mu(\mathbb{P}^{n-1})$  are images of distinct general fibers of  $\pi$  by  $\mu$  and for each  $i$ , the morphism  $\mathbb{P}^{n-1} \rightarrow F_i$  has degree 1;
- (2.ii)  $(X, L) = (\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(1)), \mathcal{O}_{\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(1))}(1))$  with  $a > 1$  and  $\Delta = D$  is irreducible, where  $D$  is the unique section of  $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(1)) \rightarrow \mathbb{P}^1$  such that  $D \equiv_{\text{num}} \mathcal{O}_{\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(1))}(1) - af$ , where  $f$  is a general fiber;
- (3.i)  $(X, L) \cong (Q, \mathcal{O}_{\mathbb{P}^{n+1}}(1))$ , where  $Q \subset \mathbb{P}^{n+1}$  is a  $\text{rk}(Q) = 3$  hyperquadric, the boundary divisor  $\Delta$  is a hyperplane in  $Q$  and  $[\Delta] = \frac{1}{2}[H \cap Q]$  where  $H$  is a hyperplane in  $\mathbb{P}^{n+1}$ ;
- (3.ii)  $(X, L) \cong (Q, \mathcal{O}_{\mathbb{P}^{n+1}}(1))$ , where  $Q \subset \mathbb{P}^{n+1}$  is a  $\text{rk}(Q) = 4$  hyperquadric. If we write  $Q = \text{Proj} \left( \frac{\mathbb{C}[x_0, \dots, x_{n+1}]}{(x_0x_1 - x_2x_3)} \right)$ , then  $\Delta = D$  is prime and  $D$  is the cone with vertex  $\mathbb{P}^{n-3}$  over  $\mathbb{P}^1 \times \text{pt}$  or  $\text{pt} \times \mathbb{P}^1$ . In particular,  $D \cong \mathbb{P}^{n-1}$ ;

PROOF. Let  $D$  be a component of  $\Delta$ .

First we suppose that  $X$  is  $\mathbb{Q}$ -factorial. The pair  $(X, D)$  satisfies the assumptions of [Theorem 3.16](#). Hence we have a classification for  $(X, D, L)$ . If we are in case (2.i) of [Theorem 3.16](#) for  $(X, D)$ , for a general fiber  $F$ , we cannot have another  $D'$  such that  $D'|_F = \mathcal{O}_F(d)$  with  $d \geq 1$ . Hence all the components will be fibers. We are thus in case (2.i) of [Corollary 3.19](#). If we are in case (2.ii) of [Theorem 3.16](#) for  $(X, D)$ , the divisor  $D$  is the unique section such that  $D \equiv_{\text{num}} \mathcal{O}_{\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(1))}(1) - af$ . Thus we have that  $\Delta = D$  is irreducible. And we are in case (2.ii) of [Corollary 3.19](#). If we are in case (1) or (3) of [Theorem 3.16](#) for  $(X, D)$ , we cannot add another prime divisor. Hence  $\Delta = D$  and we are in case (1) or (3) of [Corollary 3.19](#).

In general, the proof is identical to [Theorem 3.16](#) by adjusting our argument for  $(X, D)$  to  $(X, \Delta)$ . We apply [Theorem 2.25](#) and [Lemma 2.23](#) to the normal variety  $X$ . We get a birational morphism  $\mu : Y \rightarrow X$  such that  $Y$  has  $\mathbb{Q}$ -factorial canonical singularities,  $K_Y$  is  $\mu$ -nef and  $\mu$  is isomorphic over regular points of  $X$ . We set  $\Delta' = \mu_*^{-1}(\Delta)$ . Then  $\Delta'$  is reduced. Let  $\omega_Y = \mathcal{O}_Y(K_Y)$  and  $\omega_X = \mathcal{O}_X(K_X)$  be the canonical sheaves. We have an injection

$$\mu_*(\omega_Y \otimes \mathcal{O}_Y(\Delta') \otimes \mu^*(L^{\otimes n-1})) \hookrightarrow \omega_X \otimes \mathcal{O}_X(\Delta) \otimes L^{\otimes n-1}.$$

As  $(K_X + \Delta) + (n-1)L$  is not pseudo-effective, neither is  $(K_Y + \Delta') + (n-1)\mu^*(L)$ . We set  $\mu^*(L) =: M$ . As  $\Delta'$  is effective, the divisor  $K_Y + (n-1)M$  is not pseudo-effective.

By applying [Lemma 3.8](#), we get:

- (a)  $(Y, M) \cong (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$ , and  $\tau = n + 1$ ;

(b)  $(Y, M)$  is isomorphic to a  $(\mathbb{P}^{n-1}, \mathcal{O}_{\mathbb{P}^{n-1}}(1))$ -bundle over a smooth curve  $C$  and  $\tau = n$ ;

(c)  $\Delta(Y, M) = 0$ ,  $K_Y + \tau M \sim_{\mathbb{Q}} \mathcal{O}_Y$  and  $n - 1 < \tau(M) \leq n$ .

If we are in case (a), we know that  $\mu$  is an isomorphism. The divisor  $\Delta$  is given by  $\mathcal{O}_{\mathbb{P}^n}(a)$  for some  $a \geq 1$ . We have that  $K_X + (n - 1)L + D = \mathcal{O}_{\mathbb{P}^n}(a - 2)$ . Hence the only possible choice is  $a = 1$  and  $\Delta = D$  is a hyperplane. We are thus in case (1) of [Corollary 3.19](#).

If we are in case (b), we have a diagram

$$\begin{array}{ccc} (\mathbb{P}(\mathcal{V}), \xi) & \xrightarrow{\mu} & (X, L) \\ \pi \downarrow & & \\ C & & \end{array}$$

where  $\xi = \mathcal{O}_{\mathbb{P}(\mathcal{V})}(1)$  and  $K_{\mathbb{P}(\mathcal{V})}$  is  $\mu$ -nef.

- If  $\mu$  is an isomorphism, then  $X$  is  $\mathbb{Q}$ -factorial, and we are in case (2.i) or case (2.ii) of [Corollary 3.19](#);
- If  $\mu$  is small, the variety  $X$  cannot be  $\mathbb{Q}$ -factorial. We have that  $K_{\mathbb{P}(\mathcal{V})} + \Delta' = \mu^*(K_X + \Delta)$ . Let  $F$  be a general fiber of  $\pi$ . Then we have that  $\Delta'|_F = \mathcal{O}_F(d)$  for some integer  $d \geq 0$ . As  $K_X + \Delta$  is  $\mathbb{Q}$ -Cartier and  $\rho(X) = 1$ , there exists a rational number  $m$  such that  $K_X + \Delta \equiv_{\text{num}} mL$ . Hence  $K_{\mathbb{P}(\mathcal{V})} + \Delta' \equiv_{\text{num}} m\mu^*L$ . Intersect with  $\xi^{n-2}f$  and we get that  $-n + d = m$ . The divisor  $K_X + \Delta + (n - 1)L \equiv_{\text{num}} (d - 1)L$  is not pseudo-effective. Hence  $d = 0$ . If we write  $\Delta' = \sum D'_i$  with  $D'_i$  distinct prime divisors. We have that  $D'_i|_F = \mathcal{O}_F(0)$ . Thus the  $D'_i$ 's are distinct general fibers. We get that  $D_i = \mu(D'_i)$  and we are in case (2.i) of [Corollary 3.19](#).
- If  $\mu$  is divisorial, then  $X$  is again  $\mathbb{Q}$ -factorial (*cf.* [\[KM98, Corollary 3.18\]](#)) and we are in case (2.i) of [Corollary 3.19](#).

If we are in case (c), apply [Lemma 3.9](#) to  $(Y, M)$ . We have a crepant resolution  $\nu : Y \rightarrow Y_{\text{can}}$  with an ample divisor  $A$  on  $Y_{\text{can}}$  such that  $\nu^*(A) = M$ , the  $\Delta$ -genus  $\Delta(Y_{\text{can}}, A) = 0$  and  $K_{Y_{\text{can}}} + \tau A \equiv_{\text{num}} \mathcal{O}_{Y_{\text{can}}}$ . By [Lemma 3.10](#), we have one of the following cases:

- (c1)  $(Y_{\text{can}}, A) \cong (Q, \mathcal{O}_{\mathbb{P}^{n+1}}(1))$ , where  $Q \subset \mathbb{Q}^{n+1}$  is a hyperquadric;
- (c2)  $(Y_{\text{can}}, A)$  is a  $\mathbb{P}^{n-1}$ -bundle over  $\mathbb{P}^1$  and the restriction of  $L$  to each fiber is  $\mathcal{O}_{\mathbb{P}^{n-1}}(1)$ ;
- (c3)  $(Y_{\text{can}}, A) \cong (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}^2(2))$ ;
- (c4)  $(Y_{\text{can}}, A) \cong C_n(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}^2(2))$  is a generalised cone over  $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}^2(2))$ .

We have the following diagram

$$\begin{array}{ccc} (Y, M) & \xrightarrow{\mu} & (X, L) \\ \nu \downarrow & \nearrow h & \\ (Y_{\text{can}}, A) & & \end{array}$$

such that  $h$  is an isomorphism and  $\mu^*(L) = M = \nu^*(A)$  with  $(Y_{\text{can}}, A)$  being one of the above four pairs. The case (c2) is treated in case (b). If we are in (c1), (c3), (c4), we have that  $X$  is  $\mathbb{Q}$ -factorial and  $(X, \Delta, L)$  is classified in the beginning of the proof. Hence we get the list as in [Corollary 3.19](#).  $\square$

### 3.3. Semi-log canonical polarized varieties

Let  $X$  be a demi-normal variety. Let  $\pi : \bar{X} \rightarrow X$  be its normalization. We have that the conductor divisors  $\bar{D} \subset \bar{X}$  and  $D \subset X$  are reduced. The pair  $(\bar{X}, \bar{D})$  is log canonical when  $X$  is semi-log canonical ([Definition 2.32](#)). Suppose now that we have an ample line bundle  $L$  on  $X$ . We may consider using the classification results [Corollary 3.19](#) for  $(\bar{X}, \bar{D}, \pi^*(L))$  to get a classification for  $(X, L)$

We now state the following:

**Proposition 3.20** (=Theorem 1.9). *Let  $X$  be a non-normal slc projective variety of dimension  $n$  and  $L$  an ample line bundle over  $X$ . Suppose that  $K_X + (n-1)L \notin \text{Pseff}(X)$ . Let  $\pi : \bar{X} \rightarrow X$  be the normalization of  $X$  and  $D \subset X$ ,  $\bar{D} \subset \bar{X}$  the conductors. Then we have:*

*There is a nodal curve  $C'$ , a rank  $n$ -vector bundle  $E'$ , distinct fibers  $F_1, F_2, \dots, F_m$  of  $\mathbb{P}(E')$  and a birational morphism  $\mu : \mathbb{P}(E') \rightarrow X$  such that*

- $\mu^*(L) = \mathcal{O}_{\mathbb{P}(E')}(1)$  and
- $D = \sum_{1 \leq i \leq m} \mu(F_i)$

PROOF. We know by definition that  $(\bar{X}, \bar{D})$  is log canonical. Note that the absolute normalization  $\pi : \bar{X} \rightarrow X$  is finite (cf. [Sta22, Tag 0BXR]). Hence  $\pi^*(L)$  is ample. We have by [Equation \(3\)](#) that

$$\pi^*(K_X + (n-1)L) = K_{\bar{X}} + \bar{D} + (n-1)\pi^*(L).$$

Let  $C \subset X$  be a movable curve in  $X$  such that  $(K_X + (n-1)L) \cdot C < 0$ . Let  $C' \subset \bar{X}$  be a movable curve that dominates  $C$ . Then by the projection formula  $(K_{\bar{X}} + \bar{D} + (n-1)\pi^*(L)) \cdot C' = \deg(C'/C)(K_X + (n-1)L) \cdot C < 0$ . Hence by [Theorem 2.13](#), the divisor  $K_{\bar{X}} + \bar{D} + (n-1)\pi^*(L)$  is not pseudo-effective.

Note that  $D$  and  $\bar{D}$  are reduced, and if we denote by  $\bar{D}^\nu$ ,  $D^\nu$  respectively their normalizations, then  $\pi$  induces a degree 2 map  $\nu : \bar{D}^\nu \rightarrow D^\nu$  and there is a Galois involution  $\tau : \bar{D}^\nu \rightarrow \bar{D}^\nu$  which is generically fixed point free (cf. [Kol13, 5.2]). Thus we have the following diagram

$$\begin{array}{ccccc} \tau \curvearrowright \bar{D}^\nu & \longrightarrow & \bar{D} & \longrightarrow & \bar{X} \\ \downarrow \nu & & \downarrow & & \downarrow \pi \\ D^\nu & \longrightarrow & D & \longrightarrow & X \end{array} .$$

Since  $\nu : \bar{D}^\nu \rightarrow D^\nu$  has degree 2, we have by the projection formula that

$$\pi^*(L)|_{\bar{D}^\nu}^{n-1} = \deg(\nu) \cdot (L|_{D^\nu}^{n-1}) \in 2\mathbb{Z}.$$

We now apply [Corollary 3.19](#) to  $(\bar{X}, \bar{D}, \pi^*(L))$ . We have one of the following:

- (1)  $(\bar{X}, \pi^*(L)) \cong (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$ ,  $\bar{D} = H$  is a prime divisor where  $H$  is a hyperplane of  $\mathbb{P}^n$ ;

- (2.i) There is a  $(\mathbb{P}^{n-1}, \mathcal{O}_{\mathbb{P}^{n-1}}(1))$ -bundle  $(\mathbb{P}(E), \mathcal{O}_{\mathbb{P}(E)}(1))$  over a smooth curve  $C$ , and a birational morphism  $\mu : \mathbb{P}(E) \rightarrow \bar{X}$  such that  $\mu^*(\pi^*L) \cong \mathcal{O}_{\mathbb{P}(E)}(1)$  and  $\bar{D} = \sum F_i$  is a finite sum where  $F_i \cong \mu(\mathbb{P}^{n-1})$  are images of distinct general fibers by  $\mu$  and  $\deg(\mathbb{P}^{n-1}/F_i) = 1$  ;
- (2.ii)  $(\bar{X}, \pi^*L) = (\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(1)), \mathcal{O}_{\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(1))}(1))$  with  $a > 1$  and  $\bar{D} = C$ , where  $C$  is the unique section of  $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(1)) \rightarrow \mathbb{P}^1$  such that  $C \equiv_{\text{num}} \mathcal{O}_{\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(1))}(1) - af$ , where  $f$  is a general fiber;
- (3.i)  $(\bar{X}, \pi^*L) \cong (Q, \mathcal{O}_{\mathbb{P}^{n+1}}(1))$ , where  $Q \subset \mathbb{P}^{n+1}$  is a  $\text{rk}(Q) = 3$  hyperquadric, the divisor  $\bar{D}$  is a hyperplane in  $Q$  and  $[\bar{D}] = \frac{1}{2}[H \cap Q]$  where  $H$  is a hyperplane in  $\mathbb{P}^{n+1}$ ;
- (3.ii)  $(\bar{X}, \pi^*L) \cong (Q, \mathcal{O}_{\mathbb{P}^{n+1}}(1))$ , where  $Q \subset \mathbb{P}^{n+1}$  is a  $\text{rk}(Q) = 4$  hyperquadric. If we write  $Q = \text{Proj} \left( \frac{\mathbb{C}[x_0, \dots, x_{n+1}]}{(x_0x_1 - x_2x_3)} \right)$ , then  $\bar{D}$  is prime and  $\bar{D}$  is the cone with vertex  $\mathbb{P}^{n-3}$  over  $\mathbb{P}^1 \times \text{pt}$  or  $\text{pt} \times \mathbb{P}^1$ . In particular,  $\bar{D} \cong \mathbb{P}^{n-1}$ ;

In case (1), we have that  $\bar{D}^\nu = \bar{D} \cong \mathbb{P}^{n-1}$  is smooth and  $\pi^*(L)|_{\bar{D}^\nu} = \mathcal{O}_{\mathbb{P}^{n-1}}(1)$ . As  $\pi^*(L)|_{\bar{D}^\nu}^{n-1} = 1$  is odd. We exclude case (1).

In case (2.i), we have that  $\bar{D} = \sum_{1 \leq i \leq k} F_i$  for  $k$  a natural number and the morphism  $\mu : \prod_{1 \leq i \leq k} \mathbb{P}^{n-1} \rightarrow \bar{D}$  factors through  $\bar{D}^\nu \rightarrow \bar{D}$  [Sta22, Tag 035Q]-(4). Hence  $\pi^*(L)|_{\bar{D}^\nu}^{n-1} = k$  and  $k$  is even. As  $\deg(\mathbb{P}^{n-1}/F_i) = 1$ , we have that

$$\pi^*(L)|_{F_i}^{n-1} = \deg(\mathbb{P}^{n-1}/F_i)(\mathcal{O}_{\mathbb{P}(E)}(1)|_{\mathbb{P}^{n-1}})^{n-1} = 1.$$

Thus each irreducible component of  $D$  has pre-image consisting of two of the  $F_i$ 's. We have thus the diagram

$$\begin{array}{ccc} \mathbb{P}(E) & \xrightarrow{\mu} & \bar{X} \xrightarrow{\pi} X \\ p \downarrow & & \\ C & & \end{array}$$

Set  $k = 2m$ . We write  $D = \sum_{1 \leq i \leq m} D_i$ . We denote the two components of  $\bar{D}$  that are mapped onto  $D_i$  by  $F_{i,1}$  and  $F_{i,2}$ . Let  $x_{i,1}$  (resp.  $x_{i,2}$ ) be the point of  $C$  such that  $\mu(p^{-1}(x_{i,1})) = F_{i,1}$  (resp.  $\mu(p^{-1}(x_{i,2})) = F_{i,2}$ ). As  $C$  is smooth, we may glue  $x_{i,1}$  and  $x_{i,2}$ . We thus get a nodal curve  $C'$  together with a quotient morphism  $q : C \rightarrow C'$  such that there exists a rank  $n$  vector bundle  $E'$  on  $C'$  satisfying  $q^*(E') = E$ . The morphism  $\pi \circ \mu$  thus factors through  $\mathbb{P}(E)$ , i.e. we have the following commutative diagram:

$$\begin{array}{ccc} \mathbb{P}(E) & \xrightarrow{\mu} & \bar{X} \\ \downarrow & & \downarrow \pi \\ \mathbb{P}(E') & \xrightarrow{\mu'} & X \end{array}$$

The morphism  $\mu'$  is birational. If we denote  $x_i = p(x_{i,1})$  and  $F_i$  the fiber of  $x_i$  in  $\mathbb{P}(E')$ , we have that  $D_i = \mu'(F_i)$ . Thus we have the result of [Proposition 3.20](#).

In case (2.ii), we have that  $\pi^*(L) \cdot C = a - e = 1$ . Hence we exclude this case.

In case (3.i),  $\bar{D}$  is irreducible and  $\pi^*(L)|_{\bar{D}}^{n-1} = 1$ . Hence we also exclude this case.

In case (3.ii),  $\bar{D} \cong \mathbb{P}^{n-1}$  and  $\pi^*(L)|_{\bar{D}} = \mathcal{O}_{\mathbb{P}^{n-1}}(1)$  which is not divisible by 2. Hence we exclude this case, too.  $\square$

## Part 2

# Fundamental group of Kähler orbifolds with nef anti-canonical bundle



## CHAPTER 4

### Introduction

When studying compact Kähler manifolds, we want to understand their topology. In particular, we expect to know what is the fundamental group of a compact Kähler manifold under some geometric assumptions. For a compact Kähler manifold  $X$  whose anti-canonical bundle has certain positivity, it turns out that the fundamental group  $\pi_1(X)$  of  $X$  is quite small.

In [Kob61], Kobayashi proved the following

**THEOREM 4.1** ([Kob61, Theorem A]). *A compact Kähler manifold  $X$  with positive definitive Ricci tensor is simply connected.*

Kobayashi's proof is straightforward. First apply Myer's theorem (cf. [GHL04, 3.85]) to get that the universal covering  $\pi : \tilde{X} \rightarrow X$  is a  $k$ -fold covering with  $k \in \mathbb{N}$ . Then note, by Kodaira's vanishing theorem, that the Euler characteristics  $\chi(\mathcal{O}_X) = \chi(\mathcal{O}_{\tilde{X}}) = 1$ . As  $\pi$  is a  $k$ -fold covering, we have that  $\chi(\tilde{X}) = k\chi(X)$ . Hence  $k = 1$  and  $X$  is simply connected.

By Aubin-Yau's theorem (Theorem 5.15), we know that for a compact Kähler manifold  $X$ , its first Chern class  $c_1(X) = c_1(-K_X)$  can be represented by  $\frac{1}{2\pi}\text{Ricci}_\omega$  for some Kähler form  $\omega$  on  $X$ . Hence we may reformulate Theorem 4.1 as

**THEOREM 4.2.** *A compact Fano manifold is simply connected.*

When a compact Kähler manifold  $X$  has first Chern class  $c_1(X) = 0$ , we have the following Beauville-Bogomolov decomposition theorem.

**THEOREM 4.3** ([Bea83, Théorème 1]). *Let  $X$  be a compact Kähler manifold with  $c_1(X) = 0$ . We have the following*

- *Let  $\tilde{X} \rightarrow X$  be the universal cover of  $X$ . Then we can write  $\tilde{X}$  as a product*

$$\tilde{X} \simeq \mathbb{C}^k \times \prod_i Y_i \times \prod_j S_j$$

*where  $Y_i$ 's are irreducible Calabi-Yau manifolds and  $S_j$ 's are irreducible hyperkähler manifolds. The product is unique up to re-ordering.*

- *There exists a finite cover  $X' \rightarrow X$  such that  $X'$  can be written as a product*

$$X' \simeq T \times \prod_i Y_i \times \prod_j S_j$$

*where  $T$  is a complex torus.*

We write separately the direct result of the previous theorem.

**Corollary 4.4.** *Let  $X$  be a compact Kähler manifold with zero first Chern class. Then the fundamental group  $\pi_1(X)$  of  $X$  is virtually Abelian.*



From the point of view of cones of line bundles, we know that the closure of ample cone  $\text{Amp}(X)$  is the nef cone  $\text{Nef}(X)$ . Hence we expect some restrictions on  $\pi_1(X)$  when  $-K_X$  is nef. In [Pău97], Mihai Păun proved the following result

**THEOREM 4.5** ([Pău97, Theorem 1]). *Let  $(X, \omega)$  be a compact Kähler manifold whose anti-canonical bundle  $-K_X$  is nef. Then its fundamental group  $\pi_1(X)$  is virtually nilpotent.*

A key material in the proof of **Theorem 4.5** is the following geometric Margulis lemma by Cheeger-Colding.

**Lemma 4.6** ([CC96, Theorem 8.7.]). *Let  $n > 0$  be a natural number. There exists a universal constant  $C = C(n)$  depending only on  $n$  such that the following holds:*

*For any compact manifold  $M$  of dimension  $n$  and  $\text{Ric}_g \geq -(n-1)g$ , the morphism induced by the inclusion*

$$\pi_1(B_g(p, r), p) \rightarrow \pi_1(M, p)$$

*has virtually nilpotent image for any  $r < C(n)$ .*

Păun proved **Theorem 4.5** by using Aubin-Yau theorem to construct a specific metric  $\omega$  on  $X$ , such that

- (1)  $\text{Ricci}_\omega \geq -(2 \dim_{\mathbb{C}}(X) - 1)\omega$  and
- (2) there exists a positive  $r_0 < C(2 \dim_{\mathbb{C}}(X))$  such that  $B_\omega(p, r_0) = X$ .

Then he applied **Lemma 4.6** directly.

The proof of **Theorem 4.2**, **Theorem 4.3** and **Theorem 4.5** use tools from differential geometry. On the other hand, when given a Kähler manifold  $X$ , we naturally consider its Albanese morphism  $\text{Alb}_X : X \rightarrow A(X)$  (see **Definition 2.33**). Set  $Y := \text{Alb}_X(X) \subset A(X)$  the image of  $X$  in  $A(X)$  and  $r : \tilde{Y} \rightarrow Y$  a smooth model of  $Y$ . Frédéric Campana shows how one could describe the central series of  $\pi_1(X)$  by the central series of  $\pi_1(\tilde{Y})$  in [Cam95, Théorème 2.2]. In particular, one observes that for a compact Kähler manifold  $X$  whose fundamental group  $\pi_1(X)$  has polynomial growth, if  $\text{Alb}_X$  is surjective, we have that  $\pi_1(X)$  is virtually Abelian.

When a Kähler manifold  $X$  is projective, it had been shown that  $\text{Alb}_X$  is surjective (cf. [Zha96, Theorem 1]) at the time when Păun proved **Theorem 4.5**. Later, it was proven by Junyan Cao that  $\text{Alb}_X : X \rightarrow A(X)$  is a fibration provided  $X$  projective and  $-K_X$  nef ([Cao19, Theorem 1.2]). The Kähler case is due to Păun:

**Proposition 4.7** ([Pău17, Theorem 1.7]). *Let  $X$  be a compact Kähler manifold whose anti-canonical bundle  $-K_X$  is nef. Then the Albanese morphism*

$$\text{Alb}_X : X \rightarrow A(X)$$

*is surjective.*

Combined with **Theorem 4.5**, we have:

**THEOREM 4.8** (cf. [Pău97, Theorem 2]). *Let  $X$  be a compact Kähler manifold whose anti-canonical bundle  $-K_X$  is nef. Then the fundamental group  $\pi_1(X)$  is virtually Abelian.*

The heavy use of tools in differential geometry in the proofs of the various results considering fundamental groups indicates that we may consider the problems of fundamental groups of Kähler spaces with singularities where the tools in differential geometry

can still be applied. A very good candidate is the similar problem for Kähler orbifolds, *i.e.*, Kähler spaces with quotient singularities.

An orbifold is a geometric object whose local model is  $\mathbb{C}^n/G$  where  $G$  is a finite subgroup of  $\text{Aut}(\mathbb{C}^n)$ . As a natural generalization of manifolds, since its introduction by Satake in 1956 ([Sat56]), many results in differential geometry of manifolds have been generalized to orbifold case by adapting existing proofs for manifold case.

**Theorem 4.2** and **Corollary 4.4** has been generalized in the following form.

**THEOREM 4.9** ([CC14, Theorem 4.2.]). *Let  $\mathcal{X} = (X, \Delta)$  be a compact Kähler orbifold (see **Definition 5.37**) with non-negative first Chern class  $c_1(\mathcal{X})$ . Then  $\pi_1^{\text{orb}}(\mathcal{X})$  is virtually Abelian. More precisely,*

- *If  $c_1(\mathcal{X}) > 0$ , then  $\pi_1^{\text{orb}}(\mathcal{X})$  is finite;*
- *If  $c_1(\mathcal{X}) = 0$ , then  $\pi_1^{\text{orb}}(\mathcal{X})$  is virtually Abelian of even rank bounded by  $2 \dim(X)$ .*

We mention that there are even more general results for wilder singularities. A recent work by S.Matsumura and J.Wang gives a decomposition theorem ([MW21, Corollary 1.2]) for projective klt pair  $(X, \Delta)$  with  $-(K_X + \Delta)$  nef which can be viewed as a generalization of **Theorem 4.3**. L.Braun have a deep results showing that for a klt pair  $(X, \Delta)$  with  $-(K_X + \Delta)$  nef and big, if  $\Delta$  has standard coefficients, then its fundamental group  $\pi_1(X, \Delta)$  is finite ([Bra21, Theorem 2]), which is the best result of the generalizations of **Theorem 4.2** by far.

In this part of the thesis, we will adapt Păun's arguments in [Pău97] to orbifold case and generalize his results **Theorem 4.5** and **Theorem 4.8** to Kähler orbifolds. We prove the following

**THEOREM 4.10** (= **Theorem 8.13**, Main Theorem). *Let  $(\mathcal{X}, \omega)$  be a compact Kähler orbifold. If the anti-canonical bundle  $K_{\mathcal{X}}^{-1}$  is nef, then  $\pi_1^{\text{orb}}(\mathcal{X})$  is virtually nilpotent.*

**THEOREM 4.11** (= **Theorem 9.1**). *Let  $(X, \Delta)$  be a projective orbifold pair with  $-(K_X + \Delta)$  nef. The orbifold fundamental group  $\pi_1(X, \Delta)$  is virtually Abelian.*

**Plan of Part 2.** This part is organized as the following. In **Chapter 5**, we give the necessary materials. In **Chapter 6**, we examine the metric space structure and Bishop-Gromov theorem (**Theorem 6.22**) on Riemannian orbifold. In **Chapter 7**, we use a result in [BGT12] to show a version of orbifold Margulis lemma (**Lemma 7.12**). In **Chapter 8**, we adapt Păun's argument by using **Lemma 7.12** to show **Theorem 4.10**. In **Chapter 9**, we use the pair model for orbifolds to apply the Albanese morphism argument as in the smooth case to obtain **Theorem 4.11**.



## CHAPTER 5

### Preliminaries

#### 5.1. Conventions

In Part 2, we deal with complex spaces instead of schemes over  $\mathbb{C}$ . We use the standard notions as in [GR84]. The conventions that we summarize here are parallel to Section 2.1.

- A *complex model space* is a ringed space  $(V(\mathcal{I}), \mathcal{O}_D/\mathcal{I})$ , where  $D \subset \mathbb{C}^n$  is an open connected subset and  $\mathcal{I} \subset \mathcal{O}_D$  is an ideal sheaf of finite type.
- A *complex space* is a locally ringed space  $(X, \mathcal{O}_X)$  over  $\mathbb{C}$  such that for each point  $x \in X$ , there exists an open subset  $U_x \ni x$  and  $(U_x, \mathcal{O}_{U_x})$  is isomorphic as locally ringed space over  $\mathbb{C}$  to a complex model space. A *complex manifold* of dimension  $n \in \mathbb{N}$  is a complex space which is locally isomorphic to the complex model space  $\mathbb{C}^n$  at every point.
- A *morphism* between two complex space  $X$  and  $Y$  is a morphism between  $X$  and  $Y$  considered as locally ringed space over  $\mathbb{C}$ .
- We say  $X$  is *reduced* (resp. *normal*) if for all  $x \in X$  the local ring  $\mathcal{O}_{X,x}$  is *reduced* (resp. *normal*). For a reduced complex space  $X$ , there is a canonical monomorphism  $\mathcal{O}_X \hookrightarrow \mathcal{C}_X$ , where  $\mathcal{C}_X$  is the sheaf of germs of continuous functions valued in  $\mathbb{C}$  over  $X$ .
- Let  $X$  and  $Y$  be two reduced complex spaces. Then  $f$  induces a morphism  $f^\# : \mathcal{C}_Y \rightarrow f_*\mathcal{C}_X$  by  $\mathcal{C}_Y \ni s \mapsto s \circ f \in \mathcal{C}_X$ , which gives a morphism  $(f, f^\#) : (X, \mathcal{C}_X) \rightarrow (Y, \mathcal{C}_Y)$  between locally ringed space over  $\mathbb{C}$ . We  $f$  is *holomorphic* if  $f^\#(\mathcal{O}_Y) \subset \text{im}(f_*\mathcal{O}_X \rightarrow f_*\mathcal{C}_X)$ . In this case  $(f, f^\#) : (X, \mathcal{O}_X) \rightarrow (Y, \mathcal{O}_Y)$  is a morphism between complex spaces.
- Let  $X$  be a reduced complex space. A *prime divisor*  $D$  of  $X$  is an irreducible analytic subspace of codimension 1. A *Weil-divisor* is a formal linear combination

$$F = \sum a_i D_i$$

where  $a_i \in \mathbb{Z}$  and  $D_i$  is a prime divisor and the sum is required to be locally finite. We denote by  $\text{Weil}(X)$  the group of Weil-divisors of a reduced complex space  $X$ .

- Let  $X$  be a reduced complex space. We denote by  $\text{Div}(X) = \Gamma(X, \mathcal{M}_X^*/\mathcal{O}_X^*)$  the group of *Cartier divisors* of  $X$  and set  $\text{Div}(X)_{\mathbb{Q}} := \text{Div}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ .
- Let  $X$  be a normal complex space. We say a  $\mathbb{Q}$ -divisor  $D$  of  $X$  is  $\mathbb{Q}$ -Cartier, if there exists an integer  $m$  such that  $mD \in \text{Weil}(X)$  and  $\mathcal{O}_X(mD)$  is invertible. For two  $\mathbb{Q}$ -Cartier divisors  $D_1$  and  $D_2$ , we say that  $D_1$  and  $D_2$  are  $\mathbb{Q}$ -linearly equivalent, if  $D_1 - D_2$  is a  $\mathbb{Q}$ -combination of principal divisors and we denote

it by  $D_1 \sim_{\mathbb{Q}} D_2$ ; we say that  $D_1$  and  $D_2$  are numerically equivalent if for any irreducible curve  $C \subset X$ , we have that  $D_1 \cdot C = D_2 \cdot C$  and we denote this by  $D_1 \equiv_{\text{num}} D_2$ .

- We will use Serre's "GAGA principle" [Ser56] and its refinements [Gro03, Exposé XII] in the thesis.

## 5.2. Metric spaces

In this subsection, we recall basic notions in metric geometry. We follow [BH99, Chapter I.1, Chapter I.3]

**Definition 5.1.** Let  $(X, d)$  be a metric space.

- A *path* is a continuous map from a compact interval  $[a, b] \subset \mathbb{R}$  to  $X$ ;
- Let  $c : [a, b] \rightarrow X$  be a path, the *length*  $l(c)$  of  $c$  is defined as

$$l(c) := \sup \left\{ \sum_{1 \leq i \leq n} d(c(t_{i-1}), c(t_i)) \right\},$$

where the supremum is taken over all partitions  $a = t_0 \leq t_1 \leq \dots \leq t_{n-1} \leq t_n = b$  for  $[a, b]$ . When  $l(c) < \infty$ , we say that  $c$  is *rectifiable*;

- A *geodesic* is a map  $c : I \rightarrow X$  where  $I \subset \mathbb{R}$  is an interval, such that for any  $t, t' \in I$ , we have that  $d(c(t), c(t')) = |t - t'|$ ;
- We say  $(X, d)$  is a *geodesic space* or *geodesically convex*, if for any  $x, y \in X$ , we could find a geodesic  $c$  joining  $x$  and  $y$ ;
- A *local geodesic* is a map  $c : I \rightarrow X$  where  $I \subset \mathbb{R}$  is an interval, such that for any  $t$ , there exists  $\epsilon > 0$  such that for all  $|t' - t| < \epsilon$ , we have that  $d(c(t), c(t')) = |t - t'|$

For any metric space  $(X, d)$ , we may associate a new metric  $d_i$  called the *inner metric*.

**Definition 5.2.** Let  $(X, d)$  be a metric space. For any  $x, y \in X$ , we define  $d_i(x, y)$  by

$$d_i(x, y) = \inf l(c)$$

where the infimum is taken over all rectifiable path  $c : [a, b] \rightarrow X$  such that  $c(a) = x$  and  $c(b) = y$ . We set  $d_i(x, y) = \infty$  if no such path exists. We call  $d_i : X \times X \rightarrow [0, \infty]$  the *inner metric* of  $d$ .

We have the basic results:

**Proposition 5.3** ([BH99, Proposition 3.2.]). *Let  $(X, d)$  be a metric space. Let  $d_i$  be the inner metric of  $d$ . Then*

- (1)  $(X, d_i)$  is a metric space, and  $d_i \geq d$ ;
- (2) For any rectifiable path  $c$  in  $(X, d)$ , we have that  $c$  is also a path in  $(X, d_i)$  and its length with respect to  $d_i$  equals to its length with respect to  $d$ ;
- (3) We may consider the inner metric  $(d_i)_i$  of  $d_i$ . Then  $(d_i)_i = d_i$ .

**Definition 5.4.** Let  $(X, d)$  be a metric space. We call  $(X, d)$  a *length space*, if we have  $d = d_i$ .

We end this section by recall the Hopf-Rinow theorem.

**THEOREM 5.5** ([BH99, Proposition 3.7.]). *Let  $(X, d)$  be a length space. If  $X$  is complete and locally compact, then*

- (1) *every closed bounded subset of  $X$  is compact, and*
- (2)  *$X$  is a geodesic space.*

In practice, to show that any two points of a compact metric space  $(X, d)$  can be joined by a geodesic, we can try to prove that  $(X, d)$  is a length space.

### 5.3. Differential geometry

In this subsection, we also consider real manifold. For basic notions on Riemannian geometry, the general references are [GHL04][Pet16][Sak96].

**Definition 5.6.** Let  $(M, g)$  be a (real) Riemannian manifold and  $\nabla$  its Levi-Civita connection.

- (1) The *curvature tensor*  $R$  of  $(M, g)$  is a  $(3, 1)$ -tensor given by

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z,$$

where  $X, Y, Z$  are local vector fields. We may also use  $R$  to denote the associated  $(4, 0)$ -tensor *i.e.*

$$R(X, Y, Z, W) = g(R(X, Y)Z, W)$$

- (2) We define the *Ricci curvature* by contracting  $R$ :

$$\text{Ric}_g(v, w) = \text{tr}(x \mapsto R(x, v)w)$$

For any piece-wise  $C^1$  curve  $\gamma : I = [a, b] \rightarrow M$ , there is a length associated to  $g$ . We denote it by

$$l_g(c) := \int_a^b \sqrt{g(c'(t), c'(t))} dt.$$

With this length function  $l_g$ , for any connected Riemannian manifold  $(M, g)$ , we define a metric  $d_g$  by

$$d_g(x, y) := \inf l_g(\gamma),$$

where the infimum is taken over all piece-wise  $C^1$  curves that join  $x$  and  $y$ . For a piece-wise  $C^1$  curve  $\gamma : I = [a, b] \rightarrow M$ , regarding it as a path as in [Definition 5.1](#), we may define its length  $l_{d_g}(c)$  as in [Definition 5.1](#). We see easily that  $l_{d_g}(c) \leq l_g(c)$ .

**Definition 5.7.** Let  $(M, g)$  be a Riemannian manifold. A *geodesic with respect to  $g$*  is a smooth curve  $c : [a, b] \rightarrow M$  such that

$$c^*(\nabla) \frac{d}{dt} \dot{c}(t) = 0,$$

for all  $t \in (a, b)$ .

The definition is a little confusing with [Definition 5.1](#). However we have:

**Proposition 5.8** ([GHL04, 2.91, 2.92]). *Let  $(M, g)$  be a connected Riemannian manifold and  $d_g$  the associated metric. Then*

- (1) *The topology induced by  $d_g$  coincides with the topology of  $M$ ;*

- (2) Let  $m_0 \in M$ . There exists a neighborhood  $U$  of  $m_0$  and  $\epsilon > 0$  such that for any  $p, q \in U$ , there exists a unique geodesic with respect to  $g$ , whose image is contained in  $U$ , joining  $p$  and  $q$  and  $L_g(c) = d_g(p, q) < \epsilon$ .

The first consequence is the following:

**Lemma 5.9.** *Let  $(M, g)$  be a Riemannian manifold and  $d_g$  the associated metric. The metric space  $(M, d_g)$  is a length space.*

PROOF. Let  $d_i$  be the inner metric of  $d_g$ . Then  $d_i(x, y) = \inf l_{d_g}(c)$  where  $c$  is a rectifiable path that joins  $x$  and  $y$ . By [Proposition 5.8](#), we know that these paths are exactly continuous curves in  $M$ . Then  $d_i(x, y) \leq l_{d_g}(c) \leq l_g(c)$  for any piece-wise  $C^1$ -curve that joins  $x$  and  $y$ . Hence  $d_i(x, y) \leq \inf l_g(c) = d_g(x, y)$ .  $\square$

Suppose that  $c$  is a geodesic with respect to  $g$ . Then

$$\frac{d}{dt}g(\dot{c}(t), \dot{c}(t)) = 2g(c^*(\nabla)_{\frac{d}{dt}}\dot{c}(t), \dot{c}(t)) = 0$$

shows that we can always reparametrize  $c$  such that  $l_g(c|_{[t_1, t_2]}) = |t_1 - t_2|$ . Then a compactness argument with [Proposition 5.8](#) shows that  $c$  is a local geodesic in  $(M, d_g)$ . Hence we can always say "Let  $c$  be a local geodesic in the Riemannian manifold  $(M, g)$ " without ambiguity.

**Definition 5.10** ([\[GHL04, 2.84-2.88\]](#)). Let  $(M, g)$  be a Riemannian manifold. Let  $p \in M$  and  $v \in T_pM$ . There exists a unique local geodesic  $c_v : I \rightarrow M$  such that  $c(0) = p$  and  $\dot{c}(0) = v$ . There exists an open  $0 \in U \subset T_pM$  such that for any  $w \in U$  there exists  $\epsilon > 0$  such that  $c_w$  is defined on  $(0 - \epsilon, 1 + \epsilon)$ . We define the *exponential map*  $\exp_p : U \rightarrow M$  by  $w \mapsto c_w(1)$ . The exponential map is smooth and is a diffeomorphism around  $0 \in T_pM$ .

#### 5.4. Kähler geometry

We recall some elements of Kähler geometry. The reference for this section is [\[Dem12\]\[Huy05\]\[Voi02\]](#).

**Definition 5.11.** Let  $X$  be a complex manifold. A *Kähler form* is a closed real  $(1, 1)$ -form  $\omega$ . A *Kähler manifold*  $(X, \omega)$  is a complex manifold together with a Kähler form.

**Remark 5.12.** We will sometimes call a complex manifold  $X$  Kähler if it admits a Kähler form. This abuse of language is a common practice among complex geometers.

For a Kähler manifold  $(X, \omega)$ , we can get the associated Riemannian metric  $g$  by  $g(x, y) = \omega(x, Jy)$ , where  $J : TX \rightarrow TX$  is the almost complex structure of  $X$  and  $x, y \in TX$  are real tangent vectors. Let  $\nabla$  be the Levi-Civita connection of  $g$ . We have that  $\nabla J = 0$ . Conversely, given a Hermitian manifold  $(X, g)$ , the associated  $(1, 1)$ -form  $\omega(x, y) := g(Jx, y)$  is Kähler if  $\nabla J = 0$  (*cf.* [\[Huy05, Proposition 4.8.A\]](#)).

**Definition 5.13.** Let  $(X, \omega)$  be a Kähler manifold and  $g$  its associated Riemannian metric. We define the *Ricci form* to be

$$\text{Ricci}_\omega(x, y) := \text{Ric}_g(Jx, y).$$

Locally, taking a holomorphic coordinate  $(z^1, z^2, \dots, z^n)$ , we can write

$$\begin{aligned}\omega &= g_{i\bar{j}} dz^i \wedge d\bar{z}^j, \\ g &= g_{i\bar{j}} dz^i \otimes d\bar{z}^j.\end{aligned}$$

Here we use the Einstein convention.

**Lemma 5.14** (cf. [Szé14, Lemma 1.22]). *Let  $(X, \omega)$  be a Kähler manifold. Let  $(z^1, z^2, \dots, z^n)$  be a holomorphic coordinate and write  $\text{Ric}_g = R_{i\bar{j}} dz^i \otimes d\bar{z}^j$ . Then we have that*

$$R_{i\bar{j}} = -\partial_i \partial_{\bar{j}} \log \det(g_{p\bar{q}}).$$

In particular, we get that

$$\text{Ricci}_\omega = -\sqrt{-1} \partial \bar{\partial} \log \det(g_{p\bar{q}}).$$

We finally state the celebrated Aubin-Yau theorem. We will use its orbifold version in the thesis. The statement here is for cultural reasons.

**THEOREM 5.15** ([Aub78, Théorème 3][Yau78, Theorem 1]). *Let  $(X, \omega)$  be a compact Kähler manifold. For any smooth function  $f$  on  $X$  and real number  $\lambda \geq 0$ . The equation*

$$(MA) \quad \log M(\phi) = \lambda \phi + f$$

where  $M(\phi) = \frac{(\omega + \sqrt{-1} \partial \bar{\partial} \phi)^n}{\omega^n}$  is the Monge-Ampère operator, has a unique admissible solution if  $\lambda > 0$  and has a unique admissible solution up to a constant if  $\lambda = 0$ .

We remark that the case  $\lambda > 0$  is proven by T. Aubin and  $\lambda = 0$  is proven by S.-T. Yau. The latter has a much harder  $C^0$ -estimates.

## 5.5. Classical Orbifolds

In this section, we only deal with *effective orbifolds*. The general reference are the original papers and lecture notes by Satake [Sat56][Sat57] and Thurston [Thu79, Chapter 13].

**Definition 5.16** (cf. [Sat56, 2.Definition]). Let  $X$  be a topological space. Fix  $n \geq 0$ .

- (1) An  $n$ -dimensional real *orbifold chart* on  $X$  is a triple consisting of an open subset  $\tilde{U} \subset \mathbb{R}^n$ , a finite subgroup  $G$  of  $\text{Aut}(\tilde{U})$  and a homeomorphism  $\phi : \tilde{U}/G \rightarrow U$ , where  $U$  is an open subset of  $X$ .
- (2) Suppose that  $U \subset V$  are two open subsets of  $X$ . A *chart embedding*  $\lambda : (\tilde{U}, G, \phi) \rightarrow (\tilde{V}, H, \psi)$  is a smooth embedding  $\lambda : \tilde{U} \rightarrow \tilde{V}$  such that  $\psi \circ \lambda = \phi$ .
- (3) An *orbifold atlas* on  $X$  is a family  $\mathcal{U} = \{(\tilde{U}, G, \phi)\}$  of orbifold charts such that
  - (a)  $\{U = \phi(\tilde{U})\}$  covers  $X$ ; and
  - (b) for any  $x \in X$  covered by  $U$  and  $V$ , there exists a third orbifold chart  $(\tilde{W}, K, \mu)$  with  $x \in W$  and two chart embedding  $(\tilde{W}, K, \mu) \rightarrow (\tilde{U}, G, \phi)$  and  $(\tilde{W}, K, \mu) \rightarrow (\tilde{V}, H, \psi)$ .
- (4) An atlas  $\mathcal{V}$  is said to *refine*  $\mathcal{U}$  if every chart of  $\mathcal{V}$  embeds into some chart of  $\mathcal{U}$ . Two atlas are said to be *equivalent* if they have a common refinement.
- (5) One could proceed with open subsets in  $\mathbb{C}^n$  to form analytic charts and atlases.



Note that we don't need that the fixed points of the group  $G$  has codimension at least 2, as imposed by Satake. We refer to [MP97, Appendix] for details.

**Definition 5.17** ([ALR07, Definition 1.2.]). A real (*resp.* complex) *effective orbifold*  $\mathcal{X}$  of dimension  $n$  is a collection of the following data:

- (i) A topological space  $X$  which is Hausdorff and second countable;
- (ii) An equivalence class  $[\mathcal{U}]$  of real (*resp.* complex)  $n$ -dimensional orbifold atlas.

We often use  $|\mathcal{X}|$  to denote the underlying topological space.

**Definition 5.18.** Let  $\mathcal{X}$  be an orbifold and  $x \in |\mathcal{X}|$  a point. Let  $(\tilde{U}, G, \phi)$  be a chart around  $x$  and  $\tilde{x}$  a pre-image of  $x$  by  $\phi$ . We define the *local group* at  $x$  by

$$G_x := \{g \in G : g \cdot \tilde{x} = \tilde{x}\}.$$

Note that  $G_x$  is only defined up to conjugacy.

**Definition 5.19.** For an effective orbifold  $\mathcal{X} = (X, \mathcal{U})$ , we define its *singular locus* by

$$X_{\text{sing}} := \{x \in X : G_x \neq 1\}.$$

We set  $X_{\text{reg}} := X - X_{\text{sing}}$ . It is an open subset of  $X$  and the orbifold structure restricted to  $X_{\text{reg}}$  makes  $X_{\text{reg}}$  a manifold.

**Remark 5.20.** It's possible to define *ineffective* orbifolds via atlas, however we have to require more compatibility conditions (*i.e.* *descent*). See [PST16, Definition 4.10].

**Definition 5.21.** Let  $\mathcal{X} = (X, [\mathcal{U}])$  be an orbifold. We say that  $\mathcal{X}$  is *compact* or  $\mathcal{X}$  is a *compact orbifold*, if  $X$  is compact.

The following notion is due to T. Satake.

**Definition 5.22** ([Sat56, 4][ALR07, Definition 1.3.]). A *smooth* (*resp.* *holomorphic*) map  $f$  between to orbifolds  $\mathcal{X} = (X, \mathcal{U})$  and  $\mathcal{Y} = (Y, \mathcal{V})$  is a continuous map  $f : X \rightarrow Y$  on the underlying spaces, such that the following *local lifting property* is satisfied:

For any  $x \in X$  if we denote  $y = f(x)$ , there exists a chart  $(\tilde{U}, G, \phi)$  for  $x$  and a chart  $(\tilde{V}, H, \psi)$  for  $y$  and a smooth (*resp.* *holomorphic*) map  $\tilde{f} : \tilde{U} \rightarrow \tilde{V}$  such that the following diagram commutes

$$\begin{array}{ccc} \tilde{U} & \xrightarrow{\tilde{f}} & \tilde{V} \\ \downarrow & & \downarrow \\ U & \xrightarrow{f} & V \end{array}.$$

We call  $\tilde{f}$  a *local lifting* of  $f$ .

We will see later that for a complex orbifold  $\mathcal{X}$  we can associate an (analytic) klt pair  $(X, \Delta)$ . However the corresponding morphism are in general not smooth in the sense of smooth morphism between complex spaces. Regretfully, the name and definition are widely used.

**Example 5.23** (Frame Bundle). Let  $\mathcal{X} = (X, \mathcal{U})$  be a real orbifold of dimension  $n$ . By a partition of unity argument, we can construct for each atlas  $(\tilde{U}_i, G_i)$  a Riemmanian metric  $g_i$  such that if  $\lambda : (\tilde{U}_i, G_i) \rightarrow (\tilde{U}_j, G_j)$  is an chart embedding. then  $\lambda^*(g_j) = g_i$ .

We fix such a family  $\{g_i\}$  (cf. [Definition 6.1](#)). In particular, on each chart  $(\tilde{U}_i, G_i, U_i)$ , we have a metric  $g_{i,\tilde{x}}$  on  $T_{\tilde{x}}\tilde{U}_i$ .

Fix a chart  $(\tilde{U}_{i_0}, G_{i_0}, U_{i_0})$ . In this paragraph we only deal with this chart. Hence we drop the index for the simplicity of notation. For each  $\tilde{x} \in \tilde{U}$ , we define the frame  $F_{\tilde{x}}$  to be

$$F_{\tilde{x}} := \{p \in \text{Hom}_{\mathbb{R}}(\mathbb{R}^n, T_{\tilde{x}}\tilde{U}) : p \text{ is an isometry between } \mathbb{R}^n \text{ and } (T_{\tilde{x}}\tilde{U}, g_{\tilde{x}})\}.$$

Here the metric on  $\mathbb{R}^n$  is the standard one. The frame bundle over  $\tilde{U}$  is defined to be

$$\text{Fr}(\tilde{U}) := \coprod_{\tilde{x} \in \tilde{U}} F_{\tilde{x}},$$

together with the natural projection  $p : \text{Fr}(\tilde{U}) \rightarrow \tilde{U}$ . It's not hard to see that  $p$  can be locally trivialized as  $V \times O(n, \mathbb{R})$ . Hence  $p : \text{Fr}(\tilde{U}) \rightarrow \tilde{U}$  is a fiber bundle. The compact Lie group  $O(n, \mathbb{R})$  acts from right on  $\text{Fr}(\tilde{U})$  by  $(p.A)(v) := p(Av)$ , where  $p \in F_{\tilde{x}}$  and  $v \in \mathbb{R}^n$ . Hence  $p : \text{Fr}(\tilde{U}) \rightarrow \tilde{U}$  is a right  $O(n, \mathbb{R})$ -bundle. The group  $G$  acts from left on  $\text{Fr}(\tilde{U})$  by  $\alpha.(\tilde{x}, p) := (\alpha \cdot \tilde{x}, T_{\tilde{x}}\alpha \circ p)$ . Note that the actions of  $G$  and  $O(n, \mathbb{R}^n)$  commutes:

$$\alpha.((\tilde{x}, p).A) = (\alpha.(\tilde{x}, p)).A.$$

As  $G$  acts faithfully on  $\tilde{U}$ , it acts freely on  $\text{Fr}(\tilde{U})$ . The quotient  $G \backslash \text{Fr}(\tilde{U})$  is a manifold sitting in the following diagram

$$\begin{array}{ccc} \text{Fr}(\tilde{U}) & \xrightarrow{p} & \tilde{U} \\ \downarrow & & \downarrow \\ G \backslash \text{Fr}(\tilde{U}) & \longrightarrow & G \backslash \tilde{U} = U \end{array}.$$

The  $O(n, \mathbb{R})$ -action on  $\text{Fr}(\tilde{U})$  induces an  $O(n, \mathbb{R})$ -action on  $G \backslash \text{Fr}(\tilde{U})$ . Let  $[\tilde{x}, p]$  be a class in  $G \backslash \text{Fr}(\tilde{U})$ . We have that  $[\tilde{x}, p]A = [\tilde{x}, p.A]$ . Let  $x$  be the image of  $\tilde{x}$  under the map  $\tilde{U} \rightarrow U$ . Then the isotropy group of  $O(n, \mathbb{R})$  at  $[\tilde{x}, p]$  is  $G_x$ . We have that  $G \backslash \text{Fr}(\tilde{U}) / O(n, \mathbb{R})$  is isomorphic to  $U$ .

Let  $(\tilde{U}_{i_1}, G_{i_1}, U_{i_1})$  be a second chart. We denote it by  $(\tilde{V}, H, V)$ . Suppose that we have an chart embedding  $\lambda : \tilde{V} \rightarrow \tilde{U}$ . The embedding  $\lambda$  induces a unique group monomorphism  $\lambda_* : H \rightarrow G$  such that  $\lambda$  is  $\lambda_*$ -equivariant. From the above construction, we see that  $\lambda$  induces a  $O(n, \mathbb{R})$ -bundle morphism  $\text{Fr}(\lambda) : \text{Fr}(\tilde{V}) \rightarrow \text{Fr}(\tilde{U})$  which is  $\lambda_*$ -equivariant. Thus we get a commutative diagram:

$$\begin{array}{ccccc} & & \text{Fr}(\tilde{U}) & \xrightarrow{p} & \tilde{U} \\ & \nearrow \text{Fr}(\lambda) & \downarrow p & \nearrow \lambda & \downarrow \\ \text{Fr}(\tilde{V}) & & \tilde{V} & & U \\ \downarrow & & \downarrow & & \downarrow \\ G \backslash \text{Fr}(\tilde{U}) & \longrightarrow & G \backslash \text{Fr}(\tilde{V}) & \longrightarrow & V \end{array}.$$

By gluing up all the  $G_i \backslash \text{Fr}\tilde{U}_i \rightarrow U_i$ , we get a manifold together with a smooth map  $p : \text{Fr}(\mathcal{X}) \rightarrow \mathcal{X}$ , called the *frame bundle* of  $\mathcal{X}$ .

**5.5.1. Covering maps of orbifolds.** In this section we recall Thurston's definition covering map of and its basic properties. The references here are [[Thu79](#), Chapter 13] [[Cho04](#)] and [[Cho12](#), Chapter 4].

**Definition 5.24** ([Thu79, Definition 13.2.2.]). A smooth map  $p : \mathcal{X} \rightarrow \mathcal{Y}$  between two orbifolds  $\mathcal{X}$  and  $\mathcal{Y}$  is called an *orbifold covering map*, if for any  $y \in Y$ , there exists an orbifold chart  $(\tilde{V}, G, \psi)$  such that each component  $U_i$  of  $p^{-1}(V)$  has  $(\tilde{V}, G_i, \phi_i)$  as a chart for some subgroup  $G_i$  of  $G$  and  $p$  is lifted with respect to this chart as identity, *i.e.*, we have the following commutative diagram

$$\begin{array}{ccc} (\tilde{V}, G_i) & \xrightarrow{\text{id}} & (\tilde{V}, G) \\ \downarrow & & \downarrow \\ U_i & \xrightarrow{p} & V \end{array}$$

We call such a neighborhood  $V$  an *elementary neighborhood* with respect to  $p$  at  $y$ .

**Proposition 5.25** ([Cho04, Proposition 6]). *Let  $p : \mathcal{X} \rightarrow \mathcal{Y}$  be a covering map and  $p \in Y$ . Let  $(\tilde{V}, G)$  be a chart at  $y$ . If  $\tilde{V}$  is simply connected then  $V = \tilde{V}/G$  is an elementary neighborhood with respect to  $y$ .*

The above proposition gives the following corollary:

**Corollary 5.26.** *Let  $p : \mathcal{X} \rightarrow \mathcal{Y}$  be a covering map and  $p \in Y$ . The elementary neighborhoods at  $y$  with respect to  $p$  form a basis of the topology of  $Y$  at  $y$ .*

**Definition 5.27.** Let  $\mathcal{X}$  be a connected orbifold,  $x \in X_{\text{reg}}$ . We call a covering map  $p : (\tilde{\mathcal{X}}, \tilde{x}) \rightarrow (\mathcal{X}, x)$  a *universal covering* if  $|\tilde{\mathcal{X}}|$  is connected and for any covering  $q : (\mathcal{Y}, y) \rightarrow (\mathcal{X}, x)$  there exists a unique smooth map  $f : (\tilde{\mathcal{X}}, \tilde{x}) \rightarrow (\mathcal{Y}, y)$  such that  $p = q \circ f$ .

The following theorem is due to Thurston.

**THEOREM 5.28** ([Thu79, Proposition 13.2.4.], *cf.* also [Cho04, Proposition 8]). *For any orbifold  $\mathcal{X}$ , its universal cover exists and is unique up to an isomorphism.*

**Definition 5.29** ([Thu79, Proposition 13.2.5.]). Let  $p : \tilde{\mathcal{X}} \rightarrow \mathcal{X}$  be the universal covering of  $\mathcal{X}$ . The orbifold *fundamental group*  $\pi_1^{\text{orb}}(\mathcal{X})$  is defined to be  $\text{Aut}(p)$ .

The following proposition is an analogue for the Galois correspondence in the theory of covering spaces for topological spaces.

**Proposition 5.30** ([Cho04, Corollary 2]). *Let  $\mathcal{X}$  be an orbifold and  $p : \tilde{\mathcal{X}} \rightarrow \mathcal{X}$  be the universal covering of  $\mathcal{X}$ .*

- (1) *The orbifold fundamental group  $\pi_1^{\text{orb}}(\mathcal{X})$  acts by automorphisms on  $\tilde{\mathcal{X}}$ . For any  $x \in X_{\text{reg}}$ , set  $F_x := p^{-1}(x) \subset \tilde{X}_{\text{reg}}$ . Then  $\pi_1^{\text{orb}}(\mathcal{X})$  acts freely and transitively on  $F_x$ .*
- (2) *There is a one-one correspondence between the isomorphism classes of orbifold coverings of  $\mathcal{X}$  and the conjugacy classes of subgroups of  $\pi_1^{\text{orb}}(\mathcal{X})$ .*

We conclude this section by the following lemma.

**Lemma 5.31.** *Let  $p : \mathcal{Y} \rightarrow \mathcal{X}$  be an orbifold covering map between two  $n$ -dimensional real orbifolds. Then there exists an  $O(n, \mathbb{R})$ -equivariant covering map  $\text{Fr}(p) : \text{Fr}(\mathcal{Y}) \rightarrow \text{Fr}(\mathcal{X})$ , such that the following diagram commutes:*

$$\begin{array}{ccc} \mathrm{Fr}(\mathcal{Y}) & \xrightarrow{\mathrm{Fr}(p)} & \mathrm{Fr}(\mathcal{X}) . \\ \downarrow & & \downarrow \\ \mathcal{Y} & \xrightarrow{p} & \mathcal{X} \end{array}$$

PROOF. Let  $(\tilde{U}, G, U)$  be an elementary neighborhood with respect to  $p$ . Let  $V$  be a component of  $p^{-1}(U)$ . Then we have a subgroup  $H \subset G$  and a chart  $(\tilde{U}, H, V)$  of  $\mathcal{Y}$  such that

$$\begin{array}{ccc} (\tilde{U}, H) & \xrightarrow{\mathrm{id}} & (\tilde{U}, G) . \\ \downarrow & & \downarrow \\ V & \xrightarrow{p} & U \end{array}$$

It is then immediate that the  $O(n, \mathbb{R})$ -equivariant map  $H \backslash \mathrm{Fr}(\tilde{U}) \rightarrow G \backslash \mathrm{Fr}(\tilde{U})$  is a covering map between manifolds and it fits in the commutative diagram

$$\begin{array}{ccc} H \backslash \mathrm{Fr}(\tilde{U}) & \xrightarrow{\mathrm{Fr}(p)} & G \backslash \mathrm{Fr}(\tilde{U}) . \\ \downarrow & & \downarrow \\ V & \xrightarrow{p} & U \end{array}$$

Thus we only need to show that the local constructions glue together. Let  $U' \subset U$  be an elementary neighborhood with respect to  $p$  contained in  $U$  and  $V' \subset V$  a component of  $p^{-1}(U')$  contained in  $V$ . Suppose that the charts for  $V'$  and  $U'$  are  $(\tilde{U}', H', V')$  and  $(\tilde{U}', H', U')$ . If we have charts embeddings  $\rho : (\tilde{U}', H', V') \rightarrow (\tilde{U}, H, V)$  and  $\lambda : (\tilde{U}', H', U') \rightarrow (\tilde{U}, G, U)$ . Then we have the following commutative diagram

$$\begin{array}{ccccc} & & G' \backslash \mathrm{Fr}(\tilde{U}') & \xrightarrow{\quad} & G \backslash \mathrm{Fr}(\tilde{U}) . \\ & \nearrow & \downarrow & & \downarrow \\ H' \backslash \mathrm{Fr}(\tilde{U}') & \xrightarrow{\quad} & H \backslash \mathrm{Fr}(\tilde{U}) & \xrightarrow{\quad} & G \backslash \mathrm{Fr}(\tilde{U}) \\ \downarrow & & \downarrow & & \downarrow \\ & \nearrow & U' & \xrightarrow{\quad} & U \\ \downarrow & & \downarrow & & \downarrow \\ V' & \xrightarrow{\quad} & V & \xrightarrow{\quad} & U \end{array}$$

The maps in the upper square do not depend on the choices of embeddings. Now we cover  $X$  by elementary neighborhoods. Then  $Y$  is covered by components of the inverse images of these elementary neighborhoods by  $p$ . Let  $V_1, V_2$  be components of  $p^{-1}(U_1), p^{-1}(U_2)$  respectively. As elementary neighborhoods form a base of topology, there exists an elementary neighborhood  $U_3$  and a component  $V_3$  of  $U_3$  such that  $V_3$  embeds into  $V_1$  and  $V_2$ , and  $U_3$  embeds into  $U_1$  and  $U_2$ . We may thus glue  $\mathrm{Fr}(V_1) \rightarrow \mathrm{Fr}(U_1)$  and  $\mathrm{Fr}(V_2) \rightarrow \mathrm{Fr}(U_2)$  over  $\mathrm{Fr}(V_3) \rightarrow \mathrm{Fr}(U_3)$  which ends the proof.  $\square$

### 5.6. Complex Orbifolds

A good reference on this section is [GK07] and [DM93, Chapter 14].

We begin by recalling Cartan's quotient theorem.

**Definition 5.32** (Holomorphic quotient, *cf.* [BM19, Definition 3.8.11]). Let  $M$  be a reduced complex space. A holomorphic map  $\pi : M \rightarrow N$  from  $M$  to a reduced complex space  $N$  is a *holomorphic quotient map* if

- (1) The morphism  $\pi$  regarded as a map between topological spaces is a topological quotient;
- (2) For any holomorphic map  $f : M \rightarrow N_1$  which is constant on the fiber of  $\pi$ , the natural map  $N \rightarrow N_1$  is holomorphic.

**THEOREM 5.33** (Cartan's quotient theorem, *cf.* [Car54, Théorème 1]). *Let  $X$  be a complex manifold. Let  $G \subset \text{Aut}(X)$  be a finite automorphism group of  $X$ . Then there exists a normal complex space structure on the topological quotient  $X/G$  such that  $\pi : X \rightarrow X/G$  is a holomorphic quotient.*

By the universal property of holomorphic quotient, we know that the complex space  $X/G$  is defined up to a unique isomorphism.

Let  $\mathcal{X} = (X, \mathcal{U})$  be a complex orbifold of dimension  $n$  and  $(\tilde{U}, G, \phi)$  an orbifold chart of  $\mathcal{X}$ . By **Theorem 5.33**, we have a unique normal complex space structure on  $U$  such that  $\phi : \tilde{U} \rightarrow U$  is a holomorphic quotient. Note that here  $U$  and  $\phi$  is fixed, if we choose the complex space structure on  $U$  to be that  $\mathcal{O}_U \subset \mathcal{C}_U$ , then it is unique rather than unique up to a unique isomorphism. Let  $\lambda : (\tilde{V}, H, \psi) \rightarrow (\tilde{U}, G, \phi)$  be a chart embedding. By **Theorem 5.33** again, we know that the inclusion  $V \hookrightarrow U$  makes  $V$  an open sub complex space of  $U$ . We can thus equip a unique normal complex structure on  $X$ . The complex dimension of  $X$  is  $n$ .

**Definition 5.34** (*cf.* [GK07, Section 2]). Let  $\pi : X \rightarrow Y$  be a surjective finite morphism between normal complex spaces. The *ramification divisor*  $R(\pi)$  of  $\pi$  is defined as following:

- (1) If  $X$  and  $Y$  are both smooth, let  $(z_1, z_2, \dots, z_n)$  and  $(w_1, w_2, \dots, w_n)$  be local coordinates of  $X$  and  $Y$  respectively. Then  $R(\pi)$  is locally given by the equation

$$\det\left(\frac{\partial \pi_i}{\partial w_j}\right) = 0,$$

where  $\pi_i = z_i \circ \pi$ .

- (2) For general  $X$  and  $Y$ , set  $X' := \pi^{-1}(Y_{\text{reg}}) \cap X_{\text{reg}}$  and  $Y' := \pi(X')$ . We have that  $X'$  is open and  $\text{codim}_X(X') \geq 2$ . Set  $\pi' := \pi|_{X'} : X' \rightarrow Y'$ . We define  $R(\pi)$  to be the closure of  $R(\pi')$  in  $X$ .

**Definition 5.35** (*cf.* [GK07, Section 2]). Let  $\pi : X \rightarrow Y$  be a surjective finite morphism between normal complex spaces. We may write its ramification divisor as

$$R = R(\pi) = \sum r_j R_j,$$

where  $r_j \in \mathbb{N}$  and  $R_j$ 's are prime divisors. Set  $R_{\text{red}} := \sum R_i$ .

Set  $Y'' := Y' \setminus (\pi_*(R)_{\text{sing}} \cup \pi(R_{\text{sing}}))$  and  $X'' := \pi^{-1}(Y'')$ . We have that the complements of  $X''$  and  $Y''$  both have codimension  $\geq 2$ . For any  $x \in X''$ , either  $x$  is in

exactly one component  $R_j$  of  $R$  and we define the *ramification order of  $\pi$  at  $x$*  to be  $\text{ord}_\pi(x) := r_j + 1$ ; either  $x \notin R_{\text{red}}$  and we define the *ramification order of  $\pi$  at  $x$*  to be  $\text{ord}_\pi(x) = 1$ .

For a prime divisor  $D$  on  $X$ , the ramification orders of  $\pi$  are defined for general points of  $D$  and they are equal, we call this number *the ramification order of  $\pi$  along  $D$*  and denote it by  $\text{ord}_\pi(D)$ .

**Definition 5.36** (Branching divisor for a Galois analytic covering). We call a surjective finite morphism  $\pi : X \rightarrow Y$  a *Galois analytic covering* if  $\text{Gal}(\pi) := \{f \in \text{Aut}(X) \mid f \circ \pi = \pi\}$  acts transitively on any fiber  $\pi^{-1}(y)$ . Let  $R(\pi) = \sum r_t R_t$  be the ramification divisor. Suppose that  $B_i$  is an irreducible component of  $\pi_* R(\pi)$ . Suppose that  $R_j$  and  $R_k$  are two irreducible component of  $R(\pi)$  such that  $\pi_*(R_j) = \pi_*(R_k) = B_i$ . Then there exists an element  $f \in \text{Gal}(\pi)$  such that  $f_*(R_j) = R_k$ . One sees easily  $r_j = r_k$ . We may thus assign a *multiplicity*

$$\text{mult}_\pi(B_i) := 1 - \frac{1}{r_j + 1} = 1 - \frac{1}{\text{ord}_\pi(x)}$$

for  $B_i$ , where  $x$  is a general point in the fiber of  $\pi$ . We define the *branching divisor* to be

$$B := \sum \text{mult}_\pi(B_l) B_l,$$

which is an effective  $\mathbb{Q}$ -divisor.

For a Galois analytical covering  $\pi : Y \rightarrow X$ , we have the equation of  $\mathbb{Q}$ -Weil divisors *classes*

$$(14) \quad K_Y = \pi^*(K_X + B).$$

Let  $\mathcal{X}$  be a complex orbifold with  $X$  its underlying normal complex space. Let  $\lambda : (\tilde{U}, G, \phi) \rightarrow (\tilde{V}, H, \psi)$  be a chart embedding. We first note that  $\phi$  and  $\psi$  are Galois analytic coverings. Hence we may define  $B(\phi)$  on  $U$  and  $B(\psi)$  on  $V$ . It's also easy to see that  $\lambda^*(B(\psi)) = B(\phi)$ . Hence by the rule of assigning multiplicities in [Definition 5.36](#), we have that

$$B(\phi)|_U = B(\psi).$$

Gluing all the  $(U, B(\phi))$  together, we get a log pair  $(X, \Delta_X)$ . Conversely, we may use the log pair to encode its orbifold structure.

**Definition 5.37** (cf. [\[CC14, Definition 3.1\]](#)). A log pair  $(X, \Delta)$  is an *orbifold pair* if  $\Delta$  is a  $\mathbb{Q}$ -Weil divisor of the form

$$\Delta = \sum \left(1 - \frac{1}{m_i}\right) D_i,$$

where  $m_i \geq 2$  are integers and  $(X, \Delta)$  satisfies the locally uniformizable condition: there exists finite morphisms  $\phi_j : U_j \rightarrow X$  such that

- (1)  $\phi_j(U_j) \subset X$  is open and  $\bigcup \phi_j(U_j) = X$ ;
- (2)  $\phi_j : U_j \rightarrow \phi_j(U_j)$  is a Galois analytical cover and it's branching divisor  $B(\phi_j) = \Delta|_{\phi_j(U_j)}$

The orbifold pair  $(X, \Delta)$  uniquely determines a complex orbifold structure on the topological space  $X$ . From now on, when dealing with complex orbifold, we will use [Definition 5.17](#) and [Definition 5.37](#) interchangeably.

Let  $(X, \Delta)$  be a complex orbifold. We know that  $X$  has *quotient singularities* (that is  $X$  is locally given by  $U/G$  for a complex manifold  $U$  and a finite group  $G$  acting on  $U$ ). We have the following

**THEOREM 5.38** ([\[KM98, Proposition 5.15\]](#)). *Let  $X$  be a reduced complex space with quotient singularities only. Then  $X$  has rational singularities and  $X$  is  $\mathbb{Q}$ -factorial.*

We say that  $X$  has *rational singularities* if for any resolution  $r : Y \rightarrow X$ , one has  $\mathcal{O}_X = r_*(\mathcal{O}_Y)$  and  $R^i r_*(\mathcal{O}_Y) = 0$  for any  $i > 0$ . Hence by [Theorem 2.35](#), we have an Albanese morphism  $\text{Alb}_X : X \rightarrow A(X)$  defined on all  $X$ .

For a log pair  $(X, \Delta)$  with standard coefficients, there is a natural definition of its fundamental group.

**Definition 5.39.** Let  $(X, \Delta)$  be a klt pair with  $\Delta = \sum(1 - \frac{1}{m_i})D_i$  where  $m_i \geq 2$  are integers. We define its *fundamental group*  $\pi_1(X, \Delta)$  to be

$$\pi_1(X_{\text{reg}} \setminus |\Delta|)/N,$$

where  $N$  is the normal group generated by  $\gamma_j^{m_j}$ , with  $\gamma_j$  a small loop around the component  $D_j$  with multiplicity  $1 - \frac{1}{m_j}$  of  $\Delta$ .

For a complex orbifold  $\mathcal{X} = (X, \Delta)$ , we thus associate, *a priori*, two fundamental groups to it. We will show in the next section these two groups are canonically isomorphic.

## 5.7. Orbi-bundles

**5.7.1. Orbi-bundles.** We will use groupoid theory for orbifolds introduced by Moerdijk and Pronk [\[MP97\]](#) to study the orbi-bundles. We put the basic definitions and the properties of groupoids in [Appendix A](#). The reader can find in [\[MP97\]](#) [\[Moe02\]](#) and [\[ALR07, Chapters 1 and 2\]](#) all the technical results.

Recall we defined smooth morphism in [Definition 5.22](#). Though with the name *smooth*, we note that smooth map does not behaves well. For example, we don't know whether a smooth morphism  $f : \mathcal{X} \rightarrow \mathcal{Y}$  induces pullback morphism of differential forms. And the author does not know if for  $x \in |X|$ , the smooth map  $f$  induces a morphism of local groups  $G_x \rightarrow G_{f(x)}$ . To overcome this problem, we use the notion of *strong morphism* introduced by Moerdijk and Pronk [\[MP97\]](#).

The other motivation to consider orbifolds as groupoids is to describe the universal covering orbifolds more concretely. Recall that we have defined the orbifold fundamental groups as Galois groups of deck transformations. It is not immediate that we have an orbifold Van-Kampen theorem. On the other hand, the quite mild singularities on orbifolds makes it tempting to realize orbifold fundamental groups by certain "homotopy class of loops" as in the case for fundamental groups of topological space. By considering an orbifold as a groupoid  $\mathcal{G}$ , we can use the  $\mathcal{G}$ -paths and  $\mathcal{G}$ -homotopies to describe its orbifold fundamental group. We refer the reader to [\[BH99, Chapter  \$\mathcal{G}\$ \]](#) [\[Cho12, Chapter 4.7.\]](#).

[MP97] deals with orbifolds by identifying them with certain groupoids and define the maps between orbifolds to be the ones induced by morphisms between groupoids.

Here is Moerdijk and Pronk's definition (*cf.* [MP97, Theorem 4.1.] [ALR07, Definition 1.48.]) of orbifolds.

**Definition 5.40.**

- (1) An orbifold structure on a paracompact Hausdorff space  $X$  consists of an orbifold groupoid  $\mathcal{G}$  and a homeomorphism  $f : |\mathcal{G}| \rightarrow X$ . We say that two orbifold structures  $f : |\mathcal{G}| \rightarrow X$  and  $g : |\mathcal{H}| \rightarrow X$  are equivalent, if there exists an equivalence of groupoids  $\phi : \mathcal{G} \rightarrow \mathcal{H}$  such that  $f = g \circ |\phi|$ .
- (2) An orbifold  $\mathcal{X}$  is a space  $X$  together with a class of equivalent orbifold structure. An element  $f : |\mathcal{G}| \rightarrow X$  is called a presentation of the orbifold  $\mathcal{X}$ .

Let  $(X, \mathcal{G}, f : |\mathcal{G}| \rightarrow X)$  be an orbifold in the sense of Definition 5.40. Take  $x \in X$  and  $\tilde{x} \in G_0$  one of its pre-image. By Proposition A.8, we have an orbifold chart  $(U_{\tilde{x}}, G_{\tilde{x}}) \rightarrow f(U_{\tilde{x}}/G_{\tilde{x}})$  around  $x$ . It's easy to see that we can get an atlas consisting of all these charts. Hence we get an orbifold  $(X, \mathcal{U})$  in the sense of Definition 5.17. We call  $\mathcal{U}$  the orbifold atlas associated to  $\mathcal{G}$ . If  $\mathcal{G}$  is Morita equivalent to  $\mathcal{H}$ , then their associated atlases are equivalent. In [MP97], Moerdijk and Pronk proved that for any orbifold  $(X, \tilde{\mathcal{U}})$  in the sense of Definition 5.17, we can get a unique (up to equivalence) groupoid representation of  $X$ .

**THEOREM 5.41** ([MP97, Theorem 4.1.]). *Let  $\mathcal{X} = (X, \mathcal{U})$  be an orbifolds in the sense of Definition 5.17. There exists up to a Morita equivalence, a unique effective orbifold groupoid  $\mathcal{G}$  and a homeomorphism  $|\mathcal{G}| \rightarrow X$ , such that the associated atlas  $\mathcal{V}$  of  $\mathcal{G}$  is equivalent to  $\mathcal{U}$ .*

Thus we may interchange freely both definitions of orbifolds: in terms of atlas or in terms of groupoid. We now give the defintion of strong maps.

**Definition 5.42.** Let  $\mathcal{X}, \mathcal{Y}$  be two orbifolds. A *strong map*  $(f, [[F]])$  from  $\mathcal{X}$  to  $\mathcal{Y}$  consists of a continuous map  $f : |\mathcal{X}| \rightarrow |\mathcal{Y}|$  and a  $\mathcal{R}$ -class  $[F]$  of arrows in the category  $\mathcal{HS}$  (Definition A.11), satisfying the following conditions:

- (1) There are representations  $\mathcal{G}, \mathcal{H}$  of  $\mathcal{X}$  and  $\mathcal{Y}$  respectively;
- (2)  $F : \mathcal{X} \rightarrow \mathcal{Y}$  is a groupoid morphism;
- (3) There is a commutative diagram

$$\begin{array}{ccc} |\mathcal{G}| & \xrightarrow{|F|} & |\mathcal{H}| \\ \downarrow & & \downarrow \\ |\mathcal{X}| & \xrightarrow{f} & |\mathcal{Y}| \end{array} .$$

Let  $\mathcal{X}, \mathcal{Y}$  be two orbifolds. From the definition above, we see that the set of strong maps from  $\mathcal{X}$  to  $\mathcal{Y}$  is canonically bijective to  $\text{Hom}_{\text{HS}}(\mathcal{G}_{\mathcal{X}}, \mathcal{G}_{\mathcal{Y}})$ , for any representations  $|\mathcal{G}_{\mathcal{X}}| \rightarrow \mathcal{X}$  of  $\mathcal{X}$  and  $|\mathcal{G}_{\mathcal{Y}}| \rightarrow \mathcal{Y}$  of  $\mathcal{Y}$ .

The first component of a strong map is clearly smooth. From the above definition, we see that  $(f, [[F]])$  induces morphisms between local groups. For simplicity, we will drop the brackets and denote strong maps by  $(f, F)$ . We note that an orbifold covering maps always comes from a strong map.



**Lemma 5.43.** *Let  $p : \mathcal{Y} \rightarrow \mathcal{X}$  be an orbifold covering. There exists a morphism between groupoids  $F : \mathcal{G}_{\mathcal{Y}} \rightarrow \mathcal{G}_{\mathcal{X}}$  representing  $p$ , where  $\mathcal{G}_{\mathcal{Y}}$  and  $\mathcal{G}_{\mathcal{X}}$  are orbifold groupoids representing  $\mathcal{Y}$  and  $\mathcal{X}$ .*

PROOF. [MP97, Theorem 4.1.] implies that we have representations  $\mathcal{G}_{\mathcal{Y}}$  of  $\mathcal{Y}$  and  $\mathcal{G}_{\mathcal{X}}$  of  $\mathcal{X}$  such that  $\mathcal{G}_{\mathcal{Y}}$  is Morita equivalent to  $O(n, \mathbb{R}) \ltimes \text{Fr}(\mathcal{Y})$  and  $\mathcal{G}_{\mathcal{X}}$  is Morita equivalent to  $O(n, \mathbb{R}) \ltimes \text{Fr}(\mathcal{X})$ . Lemma 5.31 implies that we have a Lie groupoid morphism  $\text{Fr}(p)$  which induces  $p$  on the orbit space. As  $\text{Hom}_{\mathcal{H}\mathcal{S}}(\mathcal{G}_{\mathcal{Y}}, \mathcal{G}_{\mathcal{X}})$  is isomorphic to  $\text{Hom}_{\mathcal{H}\mathcal{S}}(O(n, \mathbb{R}) \ltimes \text{Fr}(\mathcal{Y}), O(n, \mathbb{R}) \ltimes \text{Fr}(\mathcal{X}))$ , the class  $[[\text{Fr}(p)]]$  induces  $p$ .  $\square$

We finish our comment on strong maps by citing the pathology from [Ler10].

**Example 5.44** ([Ler10, Lemma 3.41.]). Let  $S^1 \subset \mathbb{C}^2$  be the unit circle considered as an orbifold groupoid  $\{S^1 \rightrightarrows S^1\}$ . Set  $U_1 := S^1 \setminus +1$  and  $U_2 := S^1 \setminus -1$ . Consider the quotient groupoid  $\mathbb{Z}/2\mathbb{Z} \ltimes *$ , where  $*$  is a singleton. There are exactly two distinct strong maps  $(f_j, F_j)$ ,  $j = 1, 2$  from  $\{S^1 \rightrightarrows S^1\}$  to  $\mathbb{Z}/2\mathbb{Z} \ltimes *$ . However  $(f_1, F_1)|_{U_i} = (f_2, F_2)|_{U_i}$  for  $i = 1, 2$ .

We now give our definition of orbi-vector bundles on orbifolds.

**Definition 5.45.** Let  $\mathcal{X}$  be an orbifold. A real (resp. complex) vector bundle of rank  $r$  is a strong map  $p : \mathcal{V} \rightarrow \mathcal{X}$  together with the following

- (1) A representation  $f : \mathcal{G} \rightarrow |\mathcal{X}|$  of  $\mathcal{X}$ ;
- (2) A real (resp. complex) left- $\mathcal{G}$ -vector bundle  $E$  (Definition A.12) on  $G_0$ ;
- (3) A homeomorphism  $g : \mathcal{G} \ltimes E \rightarrow |\mathcal{V}|$ , such that  $g$  gives a representation of the orbifold  $\mathcal{V}$  and  $\mathcal{G} \ltimes E \rightarrow \mathcal{G}$  represents  $p$

As in the manifold case, we need another definition for holomorphic orbi-vector bundles. Our definition of *holomorphic bundle* is given in Definition 8.4. There is also other issues on how to define sections, see Remark 8.5. A clean definition is to consider a complex orbifold  $\mathcal{X}$  as a stack fibered in the category of complex manifolds  $\text{CompMan}$ , then define a holomorphic vector bundle as a representable map  $V \rightarrow X$  covered by  $V \times_{U_i} \mathcal{X} \cong \mathbb{C}^n \times U_i$  for some cover  $\{U_i \rightarrow A\}$  of the atlas  $A$  of  $\mathcal{X}$ . We refer the reader to [Par20].

Let  $\mathcal{V} \rightarrow \mathcal{X}$  be an orbi-vector bundle represented by  $\mathcal{E} = \mathcal{G} \ltimes E \rightarrow \mathcal{G}$ . For a  $x \in G_0$ , we take a neighborhood  $U_x$  of  $x$  such that  $G_x \ltimes U_x \cong \mathcal{G}|_{U_x}$  as in Proposition A.8. We may take  $U_x$  sufficiently small such that there is a trivialization  $E|_{U_x} \cong U_x \times \mathbb{F}^r$ . Then there is an isomorphism  $\mathcal{E}|_{U_x} \cong G_x \ltimes (U_x \times \mathbb{F}^r)$ . The actions of  $G_x$  fits into a commutative diagram

$$\begin{array}{ccc} G_x \times (U_x \times \mathbb{F}^r) & \longrightarrow & U_x \times \mathbb{F}^r \\ \downarrow & & \downarrow \\ G_x \times U_x & \longrightarrow & U_x \end{array}$$

Hence  $(U_x \times \mathbb{F}^r)/G_x$  and  $U_x/G_x$  are orbifold charts of  $\mathcal{V}$  and  $\mathcal{X}$  respectively. Note that for any  $y \in U_x/G_x$ , its fiber  $|\mathcal{V}|_y$  is isomorphic to  $\mathbb{F}^r/G'$ , where  $G'$  is a subgroup of  $G_x$ .

**Definition 5.46.** Let  $\pi : E \rightarrow G_0$  be a left- $\mathcal{G}$ -vector bundle. A  $\mathcal{G}$ -section of  $E$  over  $U \subset G_0$  is a section  $s : U \rightarrow E$  of  $\pi$  such that for any  $g \in G_1$ , we have that  $g \cdot s(x) = s(g \cdot x)$ . Note that  $s$  induces a morphism  $\mathcal{G}|_U \rightarrow (\mathcal{G} \ltimes E)|_U$ .

If the orbi-vector bundle  $p : \mathcal{V} \rightarrow \mathcal{X}$  is represented by  $\mathcal{G} \times E \rightarrow \mathcal{G}$ , we say that  $\sigma : U/\mathcal{G} \rightarrow \mathcal{V}$  is a section of  $p$  if  $\sigma$  corresponds to a  $\mathcal{G}$ -section  $s$  of  $E$ .

**Example 5.47.** If  $E$  and  $F$  are left- $\mathcal{G}$ -bundles for the topological groupoid  $\mathcal{G}$ ,  $E \otimes F$ ,  $\wedge^p E$ ,  $\text{Sym}^n(E)$ ,  $\text{Hom}(E, F)$  and  $E^\vee$  have natural left- $\mathcal{G}$ -bundle structures, and we denote their associated groupoids by  $\mathcal{E} \otimes \mathcal{F}$ ,  $\wedge^p \mathcal{E}$ ,  $\text{Sym}^n(\mathcal{E})$ ,  $\text{Hom}(\mathcal{E}, \mathcal{F})$  and  $\mathcal{E}^\vee$  respectively.

**Example 5.48.** Let  $\mathcal{X}$  be a real orbifold of dimension  $n$  and  $\mathcal{U}$  be its atlas. For each chart  $(\tilde{U}_i, G_i, \phi_i)$ , we associate a  $G_i$ -space  $(T\tilde{U}_i, G_i)$ , where  $g_i$  acts on  $T\tilde{U}_i$  by its tangent associated to its action on  $\tilde{U}_i$ . We identify  $[v_i] \in T\tilde{U}_i/G_i$  and  $[v_j] \in T\tilde{U}_j/G_j$  if there are orbifold chart embeddings  $\rho_i : \tilde{V} \rightarrow \tilde{U}_i$  and  $\rho_j : \tilde{V} \rightarrow \tilde{U}_j$  such that  $T\rho_i(v_i) = T\rho_j(v_j)$ . The topological space  $\{\sqcup_i \tilde{U}_i/G_i\}/([v_i] \sim [v_j])$  has an orbifold atlas, whose charts are  $(T\tilde{U}_i, G_i)$ . We denote this orbifold by  $T\mathcal{X}$ . By construction there is a natural projection  $T\mathcal{X} \rightarrow \mathcal{X}$ . If  $|\mathcal{G}| \rightarrow |\mathcal{X}|$  is a representation of  $\mathcal{X}$ , then  $T\mathcal{G} := \mathcal{G} \times TG_0$  is a representation of  $T\mathcal{X}$  and the projection  $T\mathcal{X} \rightarrow \mathcal{X}$  is represented by  $T\mathcal{G} \rightarrow \mathcal{G}$ .

Hence when considering tangent bundles and cotangent bundles of  $\mathcal{X}$ , we don't distinguish the representation groupoids  $\mathcal{G}$  used to construct the representation  $T\mathcal{G}$  of  $T\mathcal{X}$ .

We thus define *tensors* and *forms* to be the orbi-sections of their corresponding bundles (cf. Definition 6.9). From the construction of tangent bundle  $T\mathcal{X}$  of  $\mathcal{X}$ , we see that giving a  $p$ -form  $\omega$  over  $O \subset X$  an open subset of  $X = |\mathcal{X}|$  is equivalent to find a cover of  $O$  by orbifold charts  $(U_i, G_i)$  and  $G_i$ -invariant  $p$ -forms  $\omega_i$  on  $U_i$  such that for any chart embedding  $\lambda : U_i \rightarrow U_j$ , we have  $\lambda^*(\omega_j) = \omega_i$ . We can say the same for  $p$ -tensors.

With this description of forms, we now give the following

**Lemma 5.49.** *Let  $p : \mathcal{X} \rightarrow \mathcal{Y}$  be an orbifold covering map. Then  $p$  induces a morphism between local groups and pullbacks for tensors and forms.*

PROOF. Let  $\omega$  be a differential form on  $\mathcal{Y}$ . As elementary neighborhoods with respect to  $p$  form a base of topology, we can consider  $\omega$  as a family of invariant forms over a covering by elementary neighborhoods.

Let  $(\tilde{V}, H) \rightarrow V$  be an elementary neighborhood of  $\mathcal{Y}$ . For a component  $U \subset p^{-1}(V)$ , we have a chart  $(\tilde{V}, G) \rightarrow U$  where  $G$  is a subgroup of  $H$  and  $f$  is lifted by  $\text{id}_{\tilde{V}}$ . If  $\omega$  is represented by the  $H_y$ -invariant form  $\omega_{\tilde{V}}$  on  $\tilde{V}$ , we set the invariant form over  $(\tilde{V}, G) \rightarrow U$  to be  $\omega_{\tilde{V}, U}$ . If  $[\tilde{V}'/H'] = V' \subset V$  is another elementary neighborhood, after shrinking  $V'$ , we may assume that there exists an chart embedding  $(\tilde{V}', H') \rightarrow (\tilde{V}, H)$ . Let  $U' \subset p^{-1}(V')$  be the component that is contained in  $U$ , and  $(\tilde{V}', G') \rightarrow U'$  be a chart. For any  $x \in U'$ , there exists a chart  $(\tilde{W}, G_x) \rightarrow W$  such that  $(\tilde{W}, G_x)$  embeds into both  $(\tilde{V}', G')$  and  $(\tilde{V}, G)$ . We may suppose that  $\tilde{W} \subset \tilde{V}'$  and  $\tilde{x} \in \tilde{W}$  maps to  $x$  such that  $G_x = \{g \in G' : g\tilde{x} = \tilde{x}\}$ . Set  $H_x := \{h \in G' : h\tilde{x} = \tilde{x}\}$ . Then  $G_x = H_x \cap G'$ . Shrink  $\tilde{W}$  if necessary. We may assume that  $\tilde{W}$  is  $H_x$  invariant and simply connected. Let  $O$  be the image of  $\tilde{W}$  in  $V'$ . Then  $O$  is open and  $(\tilde{W}, H_x) \rightarrow O$  is an elementary chart of  $\mathcal{Y}$  with respect to  $p$ . Now  $(\tilde{W}, H_x)$  embeds into  $(\tilde{V}', H')$  by construction. From the construction, we have that

$$\omega_{\tilde{V}', U'}|_{\tilde{W}} = \omega_{\tilde{V}'}|_{\tilde{W}} = \omega_{\tilde{V}}|_{\tilde{W}} = \omega_{\tilde{V}, U}|_{\tilde{W}}.$$

This means that if we associate  $\omega$  to the family  $\{\omega_{\tilde{V},U}\}$ , the family will satisfy the compatibility condition hence define a form on  $\mathcal{Y}$ . We denote this form by  $p^*(\omega)$ .

From the construction above, we also see that if  $\omega$  is represented by  $\tilde{\omega}$  on an elementary chart  $(\tilde{V}, H)$ , then on each component  $U_i = [\tilde{V}/G_i]$ , the pullback  $p^*(\omega)$  is also represented by  $\tilde{\omega}$ .  $\square$

**5.7.2. Comparison of fundamental groups.** For any category  $C$ , we can associate functorially a topological space  $BC$ , its *classifying space*. When  $F : C \rightarrow C'$  is an equivalence, the continuous map  $BF : BC \rightarrow BC'$  is an homotopy equivalence [Seg68, Proposition 2.1.]. If  $\mathcal{G}$  and  $\mathcal{H}$  are two Morita equivalent groupoids, then  $B\mathcal{G}$  and  $B\mathcal{H}$  have the same homotopy type. Thus for an orbifold  $\mathcal{X}$ , we may define its orbifold homotopy group  $\pi_n^{\text{orb}}(\mathcal{X})$  to be  $\pi_n(B\mathcal{G})$ , where  $\mathcal{G}$  is any groupoid representation of  $\mathcal{X}$ . We refer the reader to [ALR07, Section 1.4] for more details. Combining [ALR07, Proposition 2.17] and [BH99, Corollary III.3.19], we know that  $\pi_1(B\mathcal{G})$  is isomorphic to the group  $\pi_1(X, \mathcal{G})$  of  $\mathcal{G}$ -homotopy classes of  $\mathcal{G}$ -loops defined in [BH99, Definition III.3.6.]. [Cho12, Theorem 4.7.4.] implies that  $\pi_1(X, \mathcal{G})$  is isomorphic to the fundamental group defined as deck transformation in Definition 5.29.

If in addition  $\mathcal{X} = (X, \Delta)$  is a complex orbifold, we have another fundamental group  $\pi_1(X, \Delta)$  as in Definition 5.39. It is well-known among experts that  $\pi_1(X, \Delta)$  is isomorphic to the above three groups. As we can not find a reference, we give a proof communicated to us by P.Eyssidieux [Eys].

**Proposition 5.50.** *Let  $\mathcal{X} = (X, \Delta)$  be a complex orbifold. There exists a canonical isomorphism*

$$\pi_1^{\text{orb}}(\mathcal{X}) \cong \pi_1(X, \Delta)$$

PROOF. We consider the frame bundle  $\text{Fr}(\mathcal{X}) \rightarrow \mathcal{X}$ . Suppose that  $\mathcal{G}$  is an orbifold groupoid representing  $\mathcal{X}$ . Then [MP97, Theorem 4.1.] implies that  $U(n, \mathbb{C}) \times \text{Fr}(\mathcal{X})$  is Morita equivalent to  $\mathcal{G}$ . We have thus  $\pi_1^{\text{orb}}(\mathcal{X}) = \pi_1([\text{Fr}(\mathcal{X})/U(n, \mathbb{C})])$ , where  $[\text{Fr}(\mathcal{X})/U(n, \mathbb{C})]$  is considered as a topological stack. By [Noo14, Example 5.6.], we have the following exact sequences

$$(15) \quad \pi_1(U(n, \mathbb{C})) \rightarrow \pi_1(\text{Fr}(\mathcal{X})) \rightarrow \pi_1^{\text{orb}}(\mathcal{X}) \rightarrow 1$$

Let  $U \subset X$  be the snc locus of  $(X, \Delta)$ . We have that  $X \setminus U$  has codimension at least 2. As  $p : \text{Fr}(\mathcal{X}) \rightarrow X$  is equidimensional, we have that  $\text{Fr}(\mathcal{X}) \setminus p^{-1}(U)$  has codimension at least 2, too. Note that as a complex variety  $\text{Fr}(\mathcal{X})$  is smooth. Hence we have

$$\pi_1(\text{Fr}(\mathcal{X})) = \pi_1(p^{-1}(U)).$$

Set  $\mathcal{O}$  to be the open sub-orbifold  $(U, \Delta|_U)$ . We have the following commutative diagram

$$\begin{array}{ccccccc} \pi_1(U(n, \mathbb{C})) & \longrightarrow & \pi_1(p^{-1}(U)) & \longrightarrow & \pi_1^{\text{orb}}(\mathcal{O}) & \longrightarrow & 1 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \pi_1(U(n, \mathbb{C})) & \longrightarrow & \pi_1(\text{Fr}(\mathcal{X})) & \longrightarrow & \pi_1^{\text{orb}}(\mathcal{X}) & \longrightarrow & 1 \end{array}$$

By four lemma, we see that  $\pi_1^{\text{orb}}(\mathcal{O}) = \pi_1^{\text{orb}}(\mathcal{X})$ . As  $U \setminus |\Delta|_U = X_{\text{reg}} \setminus |\Delta|$ , we see from definition that

$$\pi_1(U, \Delta|_U) = \pi_1(X, \Delta).$$

Hence we just need to show that  $\pi_1(U, \Delta|_U) = \pi_1^{\text{orb}}(\mathcal{O})$ . As  $(U, \Delta|_U)$  has as chart

$$(z_1, \dots, z_i, z_{i+1}, \dots, z_n) \mapsto (z_1^{m_1}, \dots, z_i^{m_i}, z_{i+1}, \dots, z_n)$$

when  $\mathcal{O}$  is covered by such a single chart, we have that  $\pi_1(U, \Delta|_U) = \pi_1^{\text{orb}}(\mathcal{O})$ . The general case is by applying Van-Kampen theorem.  $\square$

**5.7.3. Integration and the de Rham cohomology.** We recall the integration of orbi-forms and the subsequent de Rham cohomology of orbifolds.

**Definition 5.51.** Let  $\mathcal{X}$  be a real orbifold of dimension  $n$ . we say that  $\mathcal{X}$  is *orientable* if there exists a non-vanishing  $n$ -form  $\alpha$  on  $\mathcal{X}$ . We say that a chart  $(\tilde{U}, G, \phi)$  is compatible with this orientation if  $\phi^*(\alpha) = \lambda \cdot \text{vol}_{\mathbb{R}^n}$ , where  $\lambda$  is a positive function. For  $\omega$  an  $n$ -form supported in  $U$ , we define its *integration* by

$$\int_{\mathcal{X}} \omega := \frac{1}{|G|} \int_{\tilde{U}} \phi^* \omega.$$

For general case, we can cover  $\mathcal{X}$  by charts  $(\tilde{U}_i, G_i, \phi_i)$ , take a partition of unity  $\rho_i$  with respects to  $\{U_i\}$ , and define

$$\int_{\mathcal{X}} \omega := \sum_i \int_{\mathcal{X}} \rho_i \omega.$$

We give a quick argument that the integration is well-defined. Suppose first that  $\omega$  is supported in  $U$ , and  $(\tilde{U}_i, G_i, \phi_i)$  are charts that are embedded via  $\lambda_i$  to the chart  $(\tilde{U}, G, \phi)$  (cf. [MP97, page 5 Remark (6)]). Then  $G_i$  is a subgroup of  $G$  and all the distinct embedding of  $(\tilde{U}_i, G_i, \phi_i)$  into  $(\tilde{U}, G, \phi)$  will be  $g \cdot \lambda_i$ , where  $g \cdot G_i$  forms the cosets. Thus

$$\begin{aligned} \int_{\mathcal{X}} \omega &= \frac{1}{|G|} \int_{\tilde{U}} \phi^* \omega \\ &= \frac{1}{|G|} \sum_i \int_{\tilde{U}} (\rho_i \circ \phi) \phi^* \omega \\ &= \sum_i \sum_g \frac{1}{|G|} \int_{g\lambda(\tilde{U}_i)} (\rho_i \circ \phi) \phi^* \omega \\ &= \sum_i \frac{1}{|G_i|} \int_{\tilde{U}_i} (\rho_i \circ \phi_i) \phi_i^* \omega \end{aligned}$$

where in the forth equation  $g$  runs through the representatives of cosets  $G/G_i$ .

For the general case, suppose that  $\{\tilde{U}_i\}, \{\tilde{V}_j\}$  are two coverings of  $\text{supp}(\omega)$  by orbifold charts. We take a third covering  $\{\tilde{W}_k\}$  which refines both  $\{\tilde{U}_i\}$  and  $\{\tilde{V}_j\}$ , *i.e.* each  $\tilde{W}_k$  embeds into some  $\tilde{U}_{i(k)}$  and  $\tilde{V}_{j(k)}$ . Then it reduces the argument to a single chart and its refinement. Integration is thus well-defined.

Let  $\mathcal{A}^p(\mathcal{X})$  be the global orbi- $p$ -forms on  $\mathcal{X}$ . We see that the exterior differential  $d$  maps  $\mathcal{A}^p$  to  $\mathcal{A}^{p-1}$ . Hence, it makes sense to consider the de Rham cohomology of  $\mathcal{X}$ . We recall some basic results.

**Proposition 5.52.** *Let  $\mathcal{X}$  be a real  $n$ -orbifold and  $X = |\mathcal{X}|$  be its underlying topological space. We have a canonical isomorphism  $H^p(X, \mathbb{R}) \cong H_{\text{dR}}^p(\mathcal{X})$ .*

PROOF. We give a direct proof. Consider the sheaf  $C_{\mathcal{X}}^{\infty}$  on  $X$ , given by  $V \mapsto \text{Mor}(V, \mathbb{R})$ . Note that  $\mathcal{X}$  has a partition of unity by smooth function with respect to any open cover. Thus the sheaf  $\mathcal{A}^p$  is fine and acyclic (cf. [Voi02, Definition 4.35 and Proposition 4.36]). On the other hand, the complex  $\mathcal{A}^{\bullet}$  is a resolution of  $\mathbb{R}_X$ . Hence we have the canonical isomorphism  $H^p(X, \mathbb{R}) \cong H_{\text{dR}}^p(\mathcal{X})$ .  $\square$

In [Sat56, section 7], Satake showed that there is a canonical morphism

$$H_p^{\text{sing}}(\mathcal{X}, \mathbb{R}) \rightarrow \check{H}_p(\mathcal{U}, \mathbb{R}),$$

where the latter is the Čech homology group. If we define  $\check{H}_p(X, \mathbb{R}) = \varprojlim \check{H}_p(\mathcal{U}, \mathbb{R})$ , then we have

**Proposition 5.53** (cf. [Sat56, Theorem 2]). *Let  $\mathcal{X}$  be a real  $n$ -orbifold with  $X = |\mathcal{X}|$  its underlying space. We have a canonical isomorphism  $H_p^{\text{sing}}(\mathcal{X}, \mathbb{R}) \rightarrow \check{H}_p(X, \mathbb{R})$*

As  $\check{H}^p(\mathcal{U}, \mathbb{R})$  is dual to  $\check{H}_p(\mathcal{U}, \mathbb{R})$ , there is an isomorphism  $H_{\text{dR}}^{n-p}(\mathcal{X}) \cong H_p^{\text{sing}}(X, \mathbb{R})$ . Moreover, we have the Poincaré duality for orbifolds:

**Proposition 5.54.** *Let  $\mathcal{X}$  be a compact real  $n$ -orbifold, the natural map*

$$\begin{cases} H_{\text{dR}}^p(\mathcal{X}) \times H_{\text{dR}}^{n-p}(\mathcal{X}) & \longrightarrow \mathbb{R} \\ (\omega, \theta) & \longmapsto \int_{\mathcal{X}} \omega \wedge \theta \end{cases}$$

*is a perfect pairing.*

## Riemannian orbifolds

From this chapter onward, we only deal with effective orbifolds..

### 6.1. Differential calculus on orbifold

**Definition 6.1.** A *Riemmanian orbifold* is a pair  $(\mathcal{X}, g)$  where  $\mathcal{X}$  is an orbifold and  $g$  is an orbi-section of  $(T^2\mathcal{X})^\vee$  satisfying the following equivalent conditions:

- (i) If  $\mathcal{X}$  is represented by  $\mathcal{G}$ , and  $g$  corresponds to  $\sigma : G_0 \rightarrow (T^2G_0)^\vee$  then  $\sigma$  is a Riemannian metric on  $G_0$ ;
- (ii) There exists a family of charts  $\{(\tilde{U}_i, G_i)\}$  covering  $X$ , with  $G_i$ -invariant metrics  $\tilde{g}_i$  representing  $g$  over  $[\tilde{U}_i/G]$ .

Most operators on Riemannian manifolds can be generalized to Riemannian orbifolds. We begin to treat some basic results on covariant derivatives on orbifolds.

Let  $\mathcal{X}$  be an orbifold and  $\{\mathcal{G}, f : |\mathcal{G}| \rightarrow |\mathcal{X}|\}$  being a groupoid representation of  $\mathcal{X}$ . We know from [Example 5.48](#) that  $T^p\mathcal{X} \otimes T^q(T\mathcal{X}^\vee)$  is represented by  $T^p\mathcal{G} \otimes T^q(T\mathcal{G}^\vee)$ . A  $(p, q)$ -tensor over an open subset  $U$  of  $|\mathcal{X}|$  is thus a collection of  $G_i$ -invariant  $(p, q)$ -tensor over  $\tilde{U}_i$  such that  $\{U_i = \tilde{U}_i/G_i\}$  cover  $U$ . For  $T = X_1 \otimes X_2 \otimes \cdots \otimes X_p \otimes S$ , we have

$$g \cdot T = g_*(X_1) \otimes g_*(X_2) \otimes \cdots \otimes g_*(X_p) \otimes (g^{-1})^*(S).$$

As all the calculation can be performed locally, in the following we consider a local model  $(U, H, g)$  where  $U \subset \mathbb{R}^n$  is an open subset, with  $H$  a finite subgroup of  $\text{Aut}(U)$ , and  $g$  a Riemannian metric on  $U$  which is  $H$ -invariant. For any  $h \in H$ , we note that the action of  $h$  on  $x \in U$  by  $L_h(x)$  or  $h \cdot x$ . Also for any smooth function  $f$ , we define the  $H$ -action on  $f$  by  $h \cdot f = f \circ L_{h^{-1}}$ . A easy consequence for this adaption is that for any  $(p, q)$ -tensor  $T$ , we have that  $h \cdot (fT) = (h \cdot f)(h \cdot T)$ .

Let  $T$  be a  $(0, p)$ -tensor over  $U$ , and  $X_i$  vector field over  $U$ , where  $1 \leq i \leq p$ . We have that

$$\begin{aligned} h^*(T)(X_1, X_2, \dots, X_p)(x) &= T(hx)(T_x L_h X_1(x), \dots, T_x L_h X_p(x)) \\ &= T(hx)(h_* X(xh), \dots, h_* X(xh)) \\ &= T(h_* X_1, \dots, h_* X_p)(hx). \end{aligned}$$

Thus  $T$  being  $H$ -invariant is characterized by

$$T(X_1, X_2, \dots, X_p) = T(h_*(X_1), \dots, h_*(X_p)) \circ L_h$$

for any  $h \in H$  and any vector fields  $X_1, \dots, X_p$ .

The metric  $g$  being  $H$ -invariant, we infer that  $L_h$  is an isometry for any  $h \in H$  and we have  $h_*(\nabla_X Y) = \nabla_{(h_* X)}(h_* Y)$  where  $\nabla$  is the Levi-Civita connection of the metric  $g$ .

The following easy lemma is an example that differential operators on Riemannian manifolds have natural generalizations to Riemannian orbifolds.

**Lemma 6.2.** *Let  $T$  be a  $(0, p)$ -tensor which is  $H$ -invariant. Then the  $(0, p + 1)$ -tensor  $\nabla T$  is  $H$ -invariant.*

PROOF. Let  $X_0, \dots, X_p$  be  $p + 1$  vector fields. The lemma follows from the following computation:

$$\begin{aligned}
& ((\nabla T)(h_*(X_0), h_*(X_1), \dots, h_*(X_p))) \circ L_h \\
&= ((\nabla_{h_*(X_0)} T)(h_*(X_1), \dots, h_*(X_p))) \circ L_h \\
&= \left( (h_*(X_0) \cdot T(h_*(X_1), \dots, h_*(X_p)) \right. \\
&\quad \left. - \sum_i T(h_*(X_1), \dots, \nabla_{h_*(X_0)} h_*(X_i), \dots, h_*(X_p)) \right) \circ L_h \\
&= \left( (X_0 \cdot (T(h_*(X_1), \dots, h_*(X_p)) \circ L_h) \right) \circ L_{h^{-1}} \\
&\quad \left. - \sum_i T(h_*(X_1), \dots, h_*(\nabla_{X_0} X_i), \dots, h_*(X_p)) \right) \circ L_h \\
&= X_0 \cdot T(X_1, \dots, X_p) - \sum_i T(X_1, \dots, \nabla_{X_0} X_i, \dots, X_p) \\
&= (\nabla T)(X_0, X_1, \dots, X_p).
\end{aligned}$$

□

As differential commutes with pull-back, we see that if  $\omega$  is an invariant  $p$ -form, so is  $d\omega$ . In particular, if  $f$  is  $H$ -invariant smooth function, the  $df$  is an invariant 1-form and  $\nabla f = (df)^\sharp$  is an invariant vector field.

Let  $W, V$  be two orbi-vector fields over  $\mathcal{X}$ . Take  $(\tilde{U}, G)$  a chart for  $\mathcal{X}$  such that  $W, V$  are represented by  $G$ -invariant fields  $\tilde{W}, \tilde{V}$  respectively. Then  $h_*(\nabla_{\tilde{W}} \tilde{V}) = \nabla_{\tilde{W}} \tilde{V}$  for any  $h \in G$ . If  $\lambda : (\tilde{U}', G') \rightarrow (\tilde{U}, G)$  is a chart embedding, and  $\tilde{W}', \tilde{V}'$  are the representations of  $W, V$  on  $\tilde{U}'$ , then  $\nabla_{\tilde{W}'} \tilde{V}' = \lambda^*(\nabla_{\tilde{W}} \tilde{V})$  as  $\lambda$  is an isometry. Thus all the local representations glue back to an orbi-vector field. We may thus define:

**Definition 6.3.** Let  $W, V$  be two orbi-vector fields over  $(\mathcal{X}, g)$ , represented by  $\tilde{W}_i, \tilde{V}_i$  on a covering  $\{(\tilde{U}_i, G_i)\}$  respectively. Let  $\nabla_i$  be the Levi-Civita connection on  $\tilde{U}_i$ , then there is a unique vector field  $\nabla_W V$  on  $\mathcal{X}$  corresponding to the family  $\nabla_{i\tilde{W}_i} \tilde{V}_i$ . We define the association  $\nabla : W, V \mapsto \nabla_W V$  as the *Levi-Civita connection* on  $(\mathcal{X}, g)$ .

If  $\tilde{R}_i$  is the curvature tensor of  $(\tilde{U}_i, \tilde{g}_i)$ , we may glue them to an orbi-tensor  $R$ . We call this tensor the curvature of  $\mathcal{X}$ . Similarly, we can glue all the  $\text{Ric}_i$  to get an orbi-tensor  $\text{Ric}_g$  on  $\mathcal{X}$ .

## 6.2. Metric structures on orbifolds

Let  $\phi : (\tilde{U}, H) \rightarrow U$  be a chart on  $\mathcal{X}$ , with  $\tilde{g}$  representing  $g$  locally. Let  $\tilde{p} \in \tilde{U}$  be a pre-image of the point  $p \in U$ . If  $\tilde{c} : [0, \epsilon) \rightarrow \tilde{U}$  is a local geodesic emanating from  $\tilde{p}$ , as  $H$  acts by isometry on  $\tilde{U}$ , we know that  $g \cdot \tilde{c} = L_g \circ \tilde{c}$  is a local geodesic emanating

from  $g \cdot \tilde{p}$ . If  $V \in T_{\tilde{p}}\tilde{U} = \tilde{c}'(0)$ , then  $TL_g(V) = g \cdot V = (L_g \circ \tilde{c})'(0) \in T_{g\tilde{p}}\tilde{U}$ . In the orbi-fibre  $T_p\mathcal{X} = \mathbb{R}^n/G_p$ , the vectors  $V$  and  $g \cdot V$  represent same orbi-vector. Hence we set  $c_{[V]} := \phi \circ \tilde{c}$ , and it the *geodesic* emanating from  $p$  determined by the orbi-vector  $[V]$ . It is obvious that the definition does not depend on the choice of orbi-chart.

From the construction, we also note that for  $v \in T_p\mathcal{X}$ , the geodesic  $c_v : I \rightarrow \mathcal{X}$  is smooth.

**Definition 6.4.** Let  $(\mathcal{X}, g)$  be a Riemannian orbifold and  $p \in X = |\mathcal{X}|$  be a point. Set  $O := \{v \in T_p\mathcal{X} : c_v \text{ is defined on } [0, 1]\}$ . We define the *exponential map*  $\exp_p : O \rightarrow X$  to be  $\exp(v) = c_v(1)$ .

As a topological map, the exponential map  $\exp_p$  is continuous. Note that  $[0] \in O$  and  $\exp_p[0] = p$ . If  $(\tilde{U}_p, G_p)$  is a fundamental chart at  $p$ , then  $\exp_p$  has a local lifting  $\exp_{\tilde{p}} : \tilde{\Omega} \rightarrow \tilde{U}_p$ , where  $\tilde{\Omega} \subset T_{\tilde{p}}\tilde{U}$  is an  $G_p$ -invariant open subset containing 0, and  $\exp_{\tilde{p}}$  is the classical Riemannian exponential map. We know that  $\exp_p$  restricts to some  $W = [\tilde{W}/G_p]$  gives an open embedding.

We also note that if  $p \in X_{\text{reg}}$  is a regular point, then the geodesics are identical to the Riemannian ones around  $p$ . Hence so is the exponential map  $\exp_p$ .

**Remark 6.5.** With the Levi-Civita connection defined on  $(\mathcal{X}, g)$ , one may consider define a covariant connection along a smooth curve  $c : I \rightarrow \mathcal{X}$ . However, even when  $c$  is lifted as  $\tilde{c} : I \rightarrow \tilde{U}$ , we don't know if all the lifts are of the form  $g \cdot \tilde{c}$ . Another hurdle for mere smooth curves is that the definition of orbi-vector fields along them. One of the possible definition is to restrict the curves to be strong curve, i.e.  $c : I \rightarrow \mathcal{X}$  is strong. In this situation, we could pull the orbi-vector bundle  $T\mathcal{X}$  together with  $\nabla$  back on  $I$  via  $c$ . If the strong curve  $c$  has image in  $X_{\text{reg}}$ , the definition coincides with the classical one.

We follow the treatment of [Bor93] for the metric aspects of Riemannian orbifolds.

Let  $(\mathcal{X}, g)$  be a Riemannian orbifold and  $(\tilde{U}, G)$  a chart for  $\mathcal{X}$ . Suppose that  $g$  is represented over  $\tilde{U}$  by  $\tilde{g}$ . Then  $(\tilde{U}, d_{\tilde{g}})$  is a well-defined metric space. If  $\lambda : \tilde{V} \rightarrow \tilde{U}$  is an orbifold embedding, then  $\lambda$  is an isometry (of metric spaces). If for a continuous curve  $c : I \rightarrow X$ , we have local lifts on charts that cover  $c(I)$ , we can then define the length of  $c$  by adding the lengths of its local liftings. We now precise the definition.

First, we have

**THEOREM 6.6** ([Bre72, Chapter 2, Lemma 6.1]). *Let  $X$  be a left  $G$ -space, with  $G$  a compact Lie group. Let  $f : I \rightarrow X/G$  be any path. Then there exists a lifting  $f' : I \rightarrow X$  covering  $f$ , i.e., we have  $p \circ f' = f$ .*

Now go back to the Riemannian orbifold  $(\mathcal{X}, g)$ . By a compactness argument, for a path  $c : [0, 1] \rightarrow X = |\mathcal{X}|$ , there exists a partition  $0 = t_0 < t_1 < \dots < t_k = 1$ , and orbifold charts  $(\tilde{U}_i, G_i)$ ,  $1 \leq i \leq k$  such that  $c|_{[t_{i-1}, t_i]}$  has image in  $U_i$  and lifting  $\tilde{c}_i$  in  $\tilde{U}_i$ .

As the liftings are not unique, we give the following definition

**Definition 6.7.** Let  $(\mathcal{X}, g)$  be a Riemannian orbifold with underlying space  $X = |\mathcal{X}|$ . Let  $c : [0, 1] \rightarrow X$  be a path. Let  $\mathcal{P}$  be the set of the local liftings that glue back to  $c$ , i.e. an element of  $\mathcal{P}$  is a triple  $(A, B, C)$  where  $A$  is a partition  $0 = t_0 < t_1 < \dots < t_k = 1$ ,



$B$  is a family of chart  $(\tilde{U}_i, G_i)$ ,  $1 \leq i \leq k$  and  $C$  is a family of curves  $\tilde{c}_i : [t_{i-1}, t_i] \rightarrow \tilde{U}_i$  such that  $\tilde{c}_i$  cover  $c|_{[t_{i-1}, t_i]}$ . We define the *length* of  $c$  to be

$$L_g(c) = \inf_{\mathcal{P}} \sum L_i(\tilde{c}_i).$$

If  $c_1 : [0, 1] \rightarrow X$ ,  $c_2 : [0, 1] \rightarrow X$  are two curves such that  $c_1(1) = c_2(0)$ , then we may consider the curve  $c_1 * c_2 : [0, 1] \rightarrow X$  defined by  $t \in [0, \frac{1}{2}] \mapsto c_1(2t)$  and  $t \in [\frac{1}{2}, 1] \mapsto c_2(2t - 1)$ . It's obvious that  $L_g(c_1 * c_2) = L_g(c_1) + L_g(c_2)$ . Hence the definition is coherent with the intuition of the length of a curve.

**Lemma 6.8.** *Let  $(\mathcal{X}, g)$  be a Riemannian orbifold and  $(\tilde{U}, G)$  a chart for  $\mathcal{X}$ . Let  $c : [0, 1] \rightarrow \tilde{U}/G$  and  $\tilde{x}$  be a pre-image of  $x = c(0)$ . If there exists a partition  $0 = t_0 < t_1 < \dots < t_k = 1$ , orbifold charts  $(\tilde{U}_i, G_i)$ ,  $1 \leq i \leq k$  such that  $c|_{[t_{i-1}, t_i]}$  has image in  $U_i$  and lifting  $\tilde{c}_i$  in  $\tilde{U}_i$ , then there is a lifting  $\tilde{c} : I \rightarrow \tilde{U}$  such that  $L(\tilde{c}) = \sum L_i(\tilde{c}_i)$ .*

PROOF. Suppose that we have constructed  $\tilde{c} : [0, t_i] \rightarrow \tilde{U}$  that lift  $c|_{[0, t_i]}$  whose length equals to  $\sum_{j \leq i} L_i(\tilde{c}_j)$ . We now extend  $\tilde{c}$  on  $[t_i, t_{i+1}]$ . The projections of  $\tilde{c}(t_i)$  and  $\tilde{c}_i(t_i)$  on  $X$  are both  $c(t_i)$ . Hence by the definition of orbifolds, there exists a chart  $(\tilde{V}, \tilde{y})$  at  $c(t_i)$  and two chart embeddings  $\lambda : \tilde{V} \rightarrow \tilde{U}$  and  $\rho : \tilde{V} \rightarrow \tilde{U}_i$  such that  $\lambda(\tilde{y}) = \tilde{c}(t_i)$  and  $\rho(\tilde{y}) = \tilde{c}_i(t_i)$ . If  $\rho(\tilde{V})$  contains  $\tilde{c}_i([t_i, t_i + \epsilon])$ , we may then extend  $\tilde{c}$  on  $[t_i, t_i + \epsilon]$  via  $\lambda \circ (\rho)^{-1} \circ \tilde{c}_i$ . Note that  $\rho$  and  $\lambda$  are isometries. Hence  $L(\tilde{c}|_{[t_i, t_i + \epsilon]}) = L_i(\tilde{c}_i|_{[t_i, t_i + \epsilon]})$ . A compactness argument shows that we can construct  $\tilde{c}$  on  $[t_i, t_{i+1}]$  with  $L(\tilde{c}|_{[t_i, t_{i+1}]}) = L_i(\tilde{c}_i)$ .  $\square$

Hence for a curve  $c : I \rightarrow \tilde{U}/G$ , we may define its length by only considering its liftings on  $\tilde{U}$ .

**Definition 6.9.** Let  $(\mathcal{X}, g)$  be a Riemannian orbifold and  $L_g$  the length function on paths. Let  $x, y$  be two points of  $X$ . We define the *distance*  $d_g(x, y)$  of  $x$  and  $y$  by

$$d_g(x, y) = \inf L_g(\gamma)$$

where the infimum is taken over all the curves that join  $x$  and  $y$ , with the convention  $\inf_{\emptyset} = \infty$ .

We see easily that  $d_g : X \times X \rightarrow \mathbb{R}_{\geq 0}$ ,  $(x, y) \mapsto d_g(x, y)$  is a distance on  $X$  and  $d(x, y) = \infty$  iff  $x$  and  $y$  are in different components of  $X$ .

Let  $(\tilde{U}_x, G_x)$  be a fundamental chart at  $x \in X$ . As  $\tilde{U}_x$  is a Riemannian manifold, **Proposition 5.8-(2)** implies that there is a  $\delta > 0$  such that for any  $\tilde{y}, \tilde{z} \in B_\delta(\tilde{x})$  there is a unique geodesic  $\tilde{c}$  with endpoints  $\tilde{y}$  and  $\tilde{z}$ , such that  $L(\tilde{c}) = \tilde{d}(\tilde{y}, \tilde{z})$ . By taking the projection  $\tilde{U} \rightarrow U$  and combine **Lemma 6.8**, we have

**Lemma 6.10.** *Let  $(\mathcal{X}, g)$  be a Riemannian orbifold with underlying space  $X = |\mathcal{X}|$  and  $d_g$  the distance on  $X$ . The metric topology induced by  $d_g$  is the same as the original topology on  $X$ . For any  $x \in X$ , there exists  $\delta > 0$  such that for any  $y, z \in B_\delta(x) \subset X$ , there exists a unique geodesic  $c$  with endpoints  $y$  and  $z$  and  $L_g(c) = d_g(y, z)$ .*

A direct consequence of **Lemma 6.10** is:

**Corollary 6.11.** *Let  $(\mathcal{X}, g)$  be a Riemannian orbifold and  $c : I \rightarrow X$  be a geodesic. Then with respect to the metric  $d_g$ , the path  $c$  is locally minimizing, i.e. for any  $t \in I$*

there exists  $\epsilon > 0$  such that  $\forall t_1, t_2 \in (t - \epsilon, t + \epsilon)$  we have that

$$L_d(c|_{[t_1, t_2]}) = d_g(c(t_1), c(t_2)).$$

**Proposition 6.12.** *Let  $(\mathcal{X}, g)$  be a Riemannian orbifold. For a local geodesic  $c : [0, 1] \rightarrow X$ , we have that  $L_g(c) = L_d(c)$ .*

PROOF. Suppose that  $c : [0, 1] \rightarrow X$  be a local geodesic. By compactness, using [Corollary 6.11](#) and [Lemma 6.10](#), we have an  $\epsilon > 0$  such that for  $s, t \in [0, 1]$ ,  $|s - t| < \epsilon$ ,

$$d_g(c(s), c(t)) = L_d(c|_{[s, t]}) \leq L_g(c|_{[s, t]}) = d_g(c(s), c(t)).$$

Hence

$$L_g(c) = \sum L_g(c|_{[t_i, t_{i+1}]}) = \sum L_d(c|_{[t_i, t_{i+1}]}) = L_d(c). \quad \square$$

**Proposition 6.13** (cf. [[Bor93](#), Page 6]). *Let  $(\mathcal{X}, g)$  be a Riemannian orbifold with underline space  $X = |\mathcal{X}|$ . With the metric  $d_g$  in [Definition 6.9](#), the metric space  $(X, d_g)$  is a length space.*

PROOF. Let  $d_i$  denote the inner metric associated with  $d_g$ . We only need to show  $d_g \geq d_i$ . Suppose that  $d_g(x, y) < \infty$ . Let  $\epsilon > 0$  be a positive number. By definition, there exists a curve  $\gamma : [0, 1] \rightarrow X$  such that  $\gamma(0) = x, \gamma(1) = y$  and  $L_g(\gamma) \leq d_g(x, y) + \epsilon$ . By [Lemma 6.10](#), there exists a  $\delta > 0$  such that:

for any  $t, t' \in [0, 1]$ , if  $|t - t'| < \delta$ , there exists a minimizing geodesic  $c$  with endpoints  $\gamma(t), \gamma(t')$  whose length  $L_g(c)$  is  $d_g(\gamma(t), \gamma(t'))$ .

Let  $0 = t_0 < t_1 < \dots < t_n = 1$  be a partition of  $[0, 1]$  with  $t_{i+1} - t_i < \delta$  and  $c_i$  the corresponding geodesic. Set  $c := *_i c_i$ . We have that

$$\begin{aligned} d_g(x, y) + \epsilon &\geq L_g(\gamma) = \sum L_g(\gamma|_{[t_i, t_{i+1}]}) \geq \sum d_g([t_i, t_{i+1}]) = \sum L_g(c_i) \\ &\geq \sum L_d(c_i) = L_d(c) \geq d_i(x, y). \end{aligned} \quad \square$$

As any geodesic is locally minimizing, combining this fact with [Corollary 6.11](#), we have:

**Lemma 6.14.** *Let  $(\mathcal{X}, g)$  be a Riemannian orbifold. If  $(X, d_g)$  is complete, then any two point can be joined by a minimizing geodesic.*

**Remark 6.15.** In the case of a Riemannian manifold  $(M, g)$ , one associates a metric  $d_g$  on  $M$  by defining  $d_g(x, y) = \inf L_g(c)$ , where  $c$  is a piece-wise smooth curve and  $L_g(c) = \int |c'(t)| dt$ . For the metric  $d_g$ , we can associate another length  $L_d$  as in [Definition 5.1](#). One sees easily  $L_g \geq L_d$  and it is a classical result that  $L_g = L_d$ . The author does not know if this still holds in the orbifold setting.

For a complete orbifold  $(\mathcal{X}, g)$ , its geodesics have a good property:

**THEOREM 6.16** ([[Bor93](#), Proposition 15]). *Let  $(\mathcal{X}, g)$  be a Riemannian orbifold with underline space  $X = |\mathcal{X}|$ . Let  $\gamma : I = [0, 1] \rightarrow \mathcal{X}$  be a minimizing geodesic. Set  $p := \gamma(0)$  and  $q := \gamma(1)$ . Then we have one of the following mutually exclusive conditions:*

- (1)  $\gamma(I) \subset X_{\text{sing}}$
- (2)  $\gamma(I) \cap X_{\text{sing}} \subset \{p, q\}$ .

Hence for  $p, q \in X_{\text{reg}}$ , the minimizing geodesic  $\gamma$  joining  $p$  and  $q$  lies completely in  $X_{\text{reg}}$ . In this situation  $\gamma$  is also the minimizing geodesic in the Riemannian manifold  $X_{\text{reg}}$ .

**Corollary 6.17.** *Let  $(\mathcal{X}, g)$  be a Riemannian orbifold. If  $(X, d_g)$  is complete, then  $(X_{\text{reg}}, g)$  is a convex Riemannian manifold.*

Let  $(M, g)$  be a convex Riemannian manifold,  $p \in M$ . Let  $u \in U_p M$  be a unit tangent vector and  $\gamma_u$  be the geodesic emanating from  $p$  with  $\gamma'_u(0) = u$ . We define

$$t(u) := \sup\{t > 0 : \gamma_u \text{ is defined on } [0, t] \text{ and } \gamma_u|_{[0, t]} \text{ is minimal and } t \in [0, \infty]\}.$$

We may define the *cut locus* in the convex situation by

**Definition 6.18.** Let  $M$  be a convex Riemannian manifold and  $p \in M$ . We define

$$\tilde{C}_p := \{t(u)u : u \in U_p M \text{ such that } t(u) < \infty\} \quad \text{and} \quad C_p := \exp_p(\tilde{C}_p)$$

to be the *tangent cut locus* and *cut locus* of  $p$  respectively. We also set

$$\tilde{I}_p := \{tu : u \in U_p M, 0 < t < t(u)\} \quad \text{and} \quad I_p := \exp_p(\tilde{I}_p).$$

Then we have similar results as in the complete case:

**Lemma 6.19.** *Let  $M$  be a convex Riemannian manifold and  $p \in M$ . We use the notations  $\tilde{C}_p, C_p, \tilde{I}_p$  and  $I_p$  as in [Definition 6.18](#). Then we have the following*

- (1)  $I_p$  is a connected open neighborhood of  $0 \in T_p M$ .
- (2)  $\tilde{I}_p \cap C_p = \emptyset$ ,  $M = I_p \cup C_p$ , and  $\tilde{I}_p = M$ .
- (3)  $\exp_p : \tilde{I}_p \rightarrow I_p$  is a diffeomorphism.
- (4)  $C_p$  has measure 0.

PROOF. See [[Sak96](#), Proposition III.4.1 and Lemma III.4.4]. Note that though all the statements in *loc. cit.* are for complete manifold  $M$ , the proofs only use the fact that  $M$  is convex.  $\square$

### 6.3. Volume comparisons

We first recall the classical Bishop-Gromov comparison theorem:

**THEOREM 6.20.** *Let  $k$  be a real number. Let  $(M^n, g)$  be a convex Riemannian manifold with  $\text{Ric}_g \geq (n-1)k$ . Let  $v(n, k, r)$  be the volume of a ball of radius  $r$  in the model space with constant curvature  $k$ . The volume ratio*

$$r \mapsto \frac{\text{vol}_g(B(p, r))}{v(n, k, r)}$$

*is a non-increasing function whose limit is 1 as  $r \rightarrow 0$ .*

PROOF. See [[Pet16](#), Lemma 7.1.4]. Note that though all the statement in *loc. cit.* is for complete manifold  $M$ , the proof only uses the fact  $\exp_p : \tilde{I}_p \rightarrow I_p$  is a diffeomorphism and that  $C_p$  has measure zero.  $\square$

Before we state the orbifold version Bishop-Gromov theorem, we first need to define the measure on  $X$  for a Riemannian orbifold  $(\mathcal{X}, g)$ .

Let  $(\mathcal{X}, g)$  be a Riemannian orbifold. Let  $(\tilde{U}, \tilde{g})$  be a chart. After taking an orientation of  $\tilde{U}$ , we have a unique volume form  $\text{vol}_{\tilde{g}}$ . Hence  $\text{vol}_{\tilde{g}}$  is defined up to a sign. If  $O \subset X$  is an open subset covered by  $\{U_i = \tilde{U}_i/G_i\}$ , we can take a partition of unity  $\rho_i$  subordinate to  $\{U_i\}$  and define

$$\text{vol}_g(O) := \sum_i \left| \int_{\mathcal{X}} \rho_i \text{vol}_{\tilde{g}_i} \right|.$$

The integral on orbifold is defined in [Definition 5.51](#). It's easy to see that  $\text{vol}_g(O)$  is well-defined.

**Lemma 6.21** (cf. [\[Bor93, Lemma 18\]](#)). *Let  $(\mathcal{X}, g)$  be a Riemannian orbifold and  $\text{vol}_g$  be its canonical measure on  $X$ . Then  $X_{\text{sing}}$  has measure zero.*

PROOF. Note that  $X_{\text{sing}}$  is a closed subset of  $X$ . Hence it is measurable. As  $X$  is second countable, we may cover  $X_{\text{sing}}$  by countable many local charts. Hence it suffices to show for  $(\tilde{U}, G)$  the non-free point set has measure zero. As  $G$  is finite, it is trivial.  $\square$

Now we state the orbifold Bishop-Gromov theorem.

**THEOREM 6.22** (cf. [\[Bor93, Proposition 20\]](#)). *Let  $k$  be a real number. Let  $(\mathcal{X}, g)$  be a Riemannian orbifold with  $\text{Ric}_g \geq (n-1)k$ . Let  $v(n, k, r)$  be the volume of a ball of radius  $r$  in the model space with constant curvature  $k$ . The volume ratio*

$$r \mapsto \frac{\text{vol}_g(B(p, r))}{v(n, k, r)}$$

*is a non-increasing function whose limit is  $|G_p|$  as  $r \rightarrow 0$ .*

PROOF. For the reader's convenience, we reproduce the proof here.

First suppose that  $p \in X_{\text{reg}}$  is a regular point. We have that  $\text{vol}_g(B(p, r)) = \text{vol}_g(B(p, r) \cap X_{\text{reg}})$ . Set  $B'(p, r) = B(p, r) \cap X_{\text{reg}}$  and denote by  $\text{vol}'_g$  the volume on  $X_{\text{reg}}$ . We have that

$$\frac{\text{vol}_g(B(p, r))}{v(n, k, r)} = \frac{\text{vol}'_g(B'(p, r))}{v(n, k, r)}.$$

By [Theorem 6.20](#), we get the sought result.

If  $p \in X_{\text{sing}}$  is a singular point, let us pick  $(p_i)_{i \geq 1} \in X_{\text{reg}}$  a sequence that converges to  $p$ . Then  $\text{vol}_g(B(p, r)) = \lim_{i \rightarrow \infty} \text{vol}_g(B(p_i, r))$ . For any  $r' > r$ , by the results for regular points, we have that

$$\frac{\text{vol}_g(B(p_i, r))}{\text{vol}_g(B(p_i, r'))} \geq \frac{v(n, k, r)}{v(n, k, r')}.$$

After taking the limit as  $i \rightarrow \infty$ , we have that

$$\frac{\text{vol}_g(B(p, r))}{\text{vol}_g(B(p, r'))} \geq \frac{v(n, k, r)}{v(n, k, r')}.$$

For  $r$  small enough, we have the inclusion  $B(p, r) \subset \tilde{U}_p/G_p$ , with  $(\tilde{U}_p, G_p)$  a fundamental chart at  $p$ . If we denote by  $\tilde{B}(\tilde{p}, r)$  the ball of radius  $r$  in  $\tilde{U}_p$  centered at  $\tilde{p}$ , we then have:

$$\text{vol}_{\tilde{g}} \tilde{B}(\tilde{p}, r) = |G_p| \cdot \text{vol}_g(B(p, r)).$$

The limit  $|G_p|$  as  $r \rightarrow 0$  follows from [Theorem 6.20](#). □

## Orbifold coverings and generalized Magulis lemma

### 7.1. Metric geometry of orbifold coverings

Let  $p : \mathcal{Y} \rightarrow \mathcal{X}$  be an orbifold covering map. Suppose that we have a Riemannian metric  $g$  on  $\mathcal{X}$ . By [Lemma 5.49](#), we have a pull-back Riemannian metric  $p^*(g)$  on  $\mathcal{Y}$ . We denote by  $d_Y$  the metric induced by  $p^*(g)$  on  $Y = |\mathcal{Y}|$ . Then the map  $p$  induces a morphism of metric spaces  $(Y, d_Y) \rightarrow (X, d_X)$  such that  $d_Y(a, b) \geq d_X(p(a), p(b))$ .

For a Galois covering  $p : \mathcal{Y} \rightarrow \mathcal{X}$ , the distances  $d_X$  and  $d_Y$  are nicely related:

**Proposition 7.1** ([\[Lan20, Lemma 2.8\]](#)). *Let  $p : \mathcal{Y} \rightarrow \mathcal{X}$  be a Galois covering map and  $\text{Gal}(p)$  be its Galois group. Then  $X = Y/\text{Gal}(p)$  and  $d_X$  is the quotient metric of  $d_Y$  by  $\text{Gal}(p)$ .*

For our purpose, we state the following lemma.

**Lemma 7.2.** *Let  $(\mathcal{X}, g)$  be a Riemannian orbifold and  $p : \mathcal{Y} \rightarrow \mathcal{X}$  an orbifold covering of  $\mathcal{X}$ . We equip  $\mathcal{Y}$  with the Riemannian metric  $p^*(g)$ . If  $(X, d_X)$  is complete, then  $(Y, d_Y)$  is complete.*

PROOF. Let  $\{y_n\}$  be a Cauchy sequence in  $(Y, d_Y)$ . We may assume that  $\{y_n\}$  lies in a compact subset  $K$  of a component  $V$  of the pre-image  $p^{-1}(U)$  of an elementary neighborhood  $U = [\tilde{U}/G]$ . Hence there exists  $H \leq G$  such that  $V = [\tilde{U}/H]$ . Let  $U \ni x_n = p(y_n)$  be the image of  $y_n$  under  $p$ . Then  $x_n \in p(K)$ . As  $p$  does not increase distances, the sequence  $\{x_n\}$  is a Cauchy sequence and has a limit  $\lim_{n \rightarrow \infty} x_n = x \in p(K) \subset U$ . Take  $\tilde{x} \in \tilde{U}$  be a lifting of  $x$ . Then there exist  $\tilde{x}_n$  lifting  $x_n$  such that  $\tilde{x}_n \rightarrow \tilde{x}$ . Note that for each  $n$  there exists  $g_n \in G$  such that  $g_n \tilde{x}_n$  lifts  $y_n$ . As  $G$  is finite, there exists a  $g \in G$  that appears infinitely many times in  $\{g_n\}$ . Hence there is a sub-sequence  $g \cdot \tilde{x}_{n_i} \rightarrow g \cdot \tilde{x}$ . If  $y$  is the image of  $g \cdot \tilde{x}$  in  $V$ , we will have  $y_n \rightarrow y$ .  $\square$

By [Theorem 6.6](#), we see that for any covering  $p : \mathcal{X}' \rightarrow \mathcal{X}$  and path  $c : [0, 1] \rightarrow \mathcal{X}$  with  $c(0) = x \in X$  and  $x' \in p^{-1}(x)$ , there exists a lifting  $c' : [0, 1] \rightarrow \mathcal{X}'$  starts from  $x'$ . However, this will not be unique. Consider  $(\mathbb{R}^2, \mathbb{Z}/2\mathbb{Z})$  with the action  $(x, y) \rightarrow (-x, y)$ . The path  $t \mapsto (t, 0)$  has liftings  $t \mapsto (t, 0)$  and  $t \mapsto (-t, 0)$  in  $\mathbb{R}^2$ . If  $\text{id}_{\tilde{U}} : (\tilde{U}, G_{x'}) \rightarrow (\tilde{U}, G_x)$  is a local lifting of  $p$  and  $c$  is a geodesic, then it has at least  $[G_x : G_{x'}]$  lifting.

We finish this section by an easy observation.

**Lemma 7.3.** *Let  $(\mathcal{X}, g)$  be a Riemannian orbifold. Let  $\pi : \mathcal{X}' \rightarrow \mathcal{X}$  be an orbifold covering with Galois group  $\text{Gal}(p)$ . If we endow  $\mathcal{X}'$  with the pullback metric  $\pi^*(g)$ , then  $\text{Gal}(p)$  acts on  $|\mathcal{X}'|$  by isometries.*

PROOF. Let  $x'_1, x'_2$  be two points of  $|\mathcal{X}'|$ . Suppose that  $c : [0, 1] \rightarrow |\mathcal{X}'|$  is a path such that  $c(0) = x'_1$  and  $c(1) = x'_2$ . Let  $\gamma$  be an element of  $\text{Gal}(p)$ . We have that

$L_{\pi^*(g)}(\gamma c) = L_{\pi^*(g)}(c)$ . Hence

$$\begin{aligned} d(x'_1, x'_2) &= \inf_{c:[0,1] \rightarrow |\mathcal{X}'|, c \text{ connects } x'_1 \text{ and } x'_2} L_{\pi^*(g)}(c) \\ &= \inf_{c:[0,1] \rightarrow |\mathcal{X}'|, c \text{ connects } x'_1 \text{ and } x'_2} L_{\pi^*(g)}(\gamma c) \\ &= d(\gamma x'_1, \gamma x'_2). \end{aligned}$$

□

## 7.2. Dirichlet domains and generalized Margulis lemma

Let  $(\mathcal{X}, g)$  be an Riemannian orbifold. Let  $\pi : \mathcal{X}' \rightarrow \mathcal{X}$  be the universal covering of  $(\mathcal{X}, g)$  and  $\Gamma$  be its Galois group. We endow  $\mathcal{X}'$  with the pullback metric  $p^*(g)$ . Then  $(X', d')$  is a complete length space. **Lemma 7.3** shows that  $\Gamma$  acts by isometries on  $X'$ .

Suppose now that  $\text{diam}(X) \leq 1$ . Let  $x_0 \in \pi^{-1}(X_{\text{reg}})$  be a regular point. We now show some basic properties of the *Dirichlet domain* of  $\Gamma$  based at  $x_0$ :

$$F := \{p \in X' : d'(x_0, p) \leq d'(\gamma \cdot x_0, p) \text{ for all } \gamma \in \Gamma\}.$$

**Lemma 7.4.**  $\text{diam}(F) \leq 2$ .

PROOF. Let  $p$  be a point in  $F$ . As  $X'$  is complete, there is a minimizing geodesic  $c' : [0, 1] \rightarrow X'$  joining  $x_0$  and  $p$  such that  $L_d(c') = d'(x_0, p)$ . In particular,  $c'$  is a local geodesic. We consider  $c := \pi \circ c'$ , which joins  $\pi(x_0)$  and  $\pi(p)$ . If  $c$  is not a minimizing geodesic, then there is a  $c_1$  which is a minimizing geodesic that joins  $\pi(x_0)$  and  $\pi(p)$ . We have that  $d_g(\pi(x_0), \pi(p)) < L_d(c) = L_g(c)$ . As  $c_1$  is a geodesic, it has a lifting  $c'_1$  starting at  $x_0$ . As  $\pi \circ c'_1 = c_1 \neq c = \pi \circ c'$ , we have that  $c'_1 \neq c'$ . Take  $\gamma \in \Gamma$  such that  $\gamma \cdot p = c'_1(1)$ . Then

$$d'(x_0, \gamma p) \leq L_d(c'_1) = L_d(c_1) < L_d(c) = d'(x_0, p).$$

Thus  $c$  is also a segment and  $L(c) = L(c') \leq 1$ . Thus  $\text{diam}(F) \leq 2$ . □

We consider the subset of  $\Gamma$  defined by

$$S := \{\gamma \in \Gamma : d(\gamma \cdot x_0, x_0) \leq 4\}.$$

One sees easily that  $S$  is symmetric and contains 1. We have

**Lemma 7.5.**  $S$  generates  $\Gamma$ .

PROOF. We have that  $\cup \gamma \cdot F = X'$ . We take a segment  $c : [0, 1] \rightarrow X'$  that joins  $x_0$  and  $\gamma_0 \cdot x_0$ . As  $c(I)$  is compact, the set  $c(I)$  is contained in a ball  $B_r(x_0)$  that meets only finitely many translates  $\gamma \cdot F$  of  $F$ . Hence  $c$  passes through finitely many  $\gamma \cdot F$ . We list these elements by  $1 = \gamma_1, \dots, \gamma_k = \gamma_0$ , ordered by the time when  $c$  enters  $\gamma_i \cdot F$ . Note that they are not necessarily different. Then  $\gamma_i \cdot F \cap \gamma_{i+1} \cdot F \neq \emptyset$  and we thus have that  $d'(\gamma_i x_0, \gamma_{i+1} x_0) \leq 4$ . Finally we remark that

$$\gamma_0 = \gamma_k = \prod_{i=1}^{k-1} (\gamma_{i+1} \cdot \gamma_i^{-1}) \in S^{k-1}.$$

□

**Lemma 7.6.** Let  $r > 0$  be an integer.  $B(x_0, r) \subset S^r \cdot F \subset B(x_0, 3r + 2)$ .

PROOF. If  $p \in S^r \cdot F$ , one can write  $p = \gamma_1 \dots \gamma_r \cdot q$  with  $q \in F$  and  $\gamma_i \in S$ . It yields

$$d'(x_0, p) \leq d'(\gamma_1 \dots \gamma_r \cdot x_0, x_0) + d'(\gamma_1 \dots \gamma_r \cdot x_0, \gamma_1 \dots \gamma_r \cdot q) \leq 2 + 4r.$$

If  $p \in B(x_0, r)$ , as  $\bigcup \gamma \cdot F = X'$ , there exists  $\gamma_0 \in \Gamma$  such that  $\gamma_0^{-1} \cdot p \in F$ . Then  $d'(x_0, \gamma \cdot x_0) \leq d'(x_0, p) + d'(\gamma_0 \cdot x_0, p) \leq r + 1$ . Hence  $\gamma_0 \in S^r$ .  $\square$

For any  $\gamma \neq 1$ ,  $\gamma \cdot F \cap F \subset \partial F$  and we have  $\partial F = \bigcup_{\gamma \neq 1} (\gamma \cdot F \cap F)$ .

**Lemma 7.7.** *Let  $\mu$  be the canonical measure associated with  $p^*(g)$ . Then  $\mu(\partial F) = 0$ .*

PROOF. Let  $p \in \partial F$  be a point. If  $\pi(p) \in X_{\text{reg}}$ , then  $p \in X'_{\text{reg}}$ . Suppose that  $p \in F \cap \gamma \cdot F$  for some  $\gamma \neq 1$ . Then  $d'(x_0, p) = d(\gamma \cdot x_0, p)$ . The proof of [Lemma 7.5](#) shows that we can then find two distinct segments on  $X$  joining  $\pi(x_0)$  and  $p$ . By [Theorem 6.16](#), these two segments are minimizing geodesics in the convex manifold  $X_{\text{reg}}$ . Hence  $\pi(p)$  lies in the cut locus  $C_{\pi(x_0)}$  of  $x_0$  in  $X_{\text{reg}}$ . Thus we have that  $\pi(\partial F) \subset C_{\pi(x_0)} \cup X_{\text{sing}}$ . Now cover  $\pi(\partial F)$  by elementary neighborhoods with respect to  $\pi$ . By second countability, we may find countably many neighborhoods  $(\tilde{U}_i, G_i, \phi_i)$ . Note that  $\phi_i^{-1}(C_{\pi(x_0)} \cup X_{\text{sing}})$  has measure 0 in  $\tilde{U}_i$  and  $\partial F$  is covered by countably many  $\tilde{U}_i/H_{ij}$ . Hence  $\mu(\partial F) = 0$ .  $\square$

With [Theorem 6.22](#) and [Proposition 7.1](#), the Margulis lemma for fundamental groups of compact manifolds with Ricci curvature bounded below by [\[BGT12\]](#) holds for the orbifold case:

**Proposition 7.8** (cf. [\[BGT12, Corollary 11.13\]](#)). *Given  $n \in \mathbb{N}$ , there is  $\epsilon = \epsilon(n) > 0$  such that:*

*for any  $n$ -dimensional compact Riemannian orbifold  $(\mathcal{X}, g)$  with underlying space  $|X|$ , if  $\text{Ric}_g \geq -\epsilon$  and  $\text{diam}(X) \leq 1$ , then  $\pi_1^{\text{orb}}(\mathcal{X})$  is virtually nilpotent.*

PROOF. Let  $\pi, x_0, F$ , be defined as above. With [Lemma 7.6](#) and [Lemma 7.7](#), we have that

$$\frac{|S^r|}{|S|} \leq \frac{\mu(B(x_0, 4r + 2))}{\mu(B(x_0, 1))}.$$

By [Theorem 6.22](#), we have

$$\frac{\mu(B(x_0, r))}{\mu(B(x_0, 1))} \leq \frac{v(n, -\epsilon, r)}{v(n, -\epsilon, 1)}.$$

Let  $\omega_n$  be the volume of  $(n - 1)$ -dimensional unit sphere in Euclidean space  $\mathbb{R}^n$ . Then  $v(n, -\epsilon, r) = \omega_n \int_0^r \left(\frac{\sinh(\sqrt{\epsilon}t)}{\sqrt{\epsilon}}\right)^{n-1} dt$ . The latter tends to  $\omega_n r^n/n$  when  $\epsilon$  tends to 0. Thus for any  $R_0 \geq 1$ , there exists  $\epsilon_0 = \epsilon_0(d, R_0)$  such that

$$\frac{|S^r|}{|S|} \leq 2(4r + 2)^n$$

for all  $r \leq R_0$ , provided that  $0 < \epsilon < \epsilon_0$ . The existence of  $\epsilon = \epsilon(d)$  follows from [\[BGT12, Corollary 11.5\]](#).  $\square$

For the main theorem, we introduce the following notion.



**Definition 7.9.** Let  $(X, d)$  be a metric space. We say that  $X$  to have *bounded packing* with *packing constant*  $K$  if there exists  $K > 0$  such that every ball of radius 4 in  $X$  can be covered by at most  $K$  balls of radius 1.

**Lemma 7.10.** *Let  $(\mathcal{X}, g)$  be a complete Riemannian orbifold with  $\text{Ric}_g > -(n-1)$ . Then  $(X, d_g)$  has bounded packing with packing constant  $K = K(n)$ .*

PROOF. Let  $p$  be a point in  $X$ . For the ball  $B(p, 5)$  and a ball  $B(q, \frac{1}{2}) \subset B(p, 5)$ , by [Theorem 6.22](#) we have that

$$\frac{\text{vol}_g(B(p, 5))}{\text{vol}_g(B(q, \frac{1}{2}))} \leq \frac{\text{vol}_g(B(q, 10))}{\text{vol}_g(B(q, \frac{1}{2}))} \leq \frac{v(n, -1, 10)}{v(n, -1, \frac{1}{2})} = K(n).$$

Let  $\{B(q_i, \frac{1}{2})\}$  be a family of disjoint balls that is contained  $B(p, 5)$  such that for any  $q \neq q_i$ , if  $B(q, \frac{1}{2}) \subset B(p, 5)$ , then  $B(q, \frac{1}{2})$  intersects with one of the  $B(q_i, \frac{1}{2})$ . We know that the family has most  $K(n)$  balls. Note that the balls  $B(q_i, 1)$  cover  $B(p, 4)$ . Hence we have the packing constant  $K = K(n)$ .  $\square$

Let  $(\mathcal{X}, g)$  be a complete Riemannian orbifold and  $p : \tilde{\mathcal{X}} \rightarrow \mathcal{X}$  its universal covering. For any point  $x \in X = |\mathcal{X}|$ , its (topological) fiber  $p^{-1}(x) \subset \tilde{\mathcal{X}} = |\tilde{\mathcal{X}}|$  is a discrete space. Let us pick  $\tilde{x} \in p^{-1}(x)$ . We have that

$$\min\{\tilde{d}(\tilde{x}, \tilde{x}')\} > 0,$$

where the minimum is taken for all  $\tilde{x}' \in p^{-1}(x) \setminus \{\tilde{x}\}$ . We thus see that  $\pi_1^{\text{orb}}(\mathcal{X})$  acts on  $\tilde{X}$  discretely, *i.e.*, for any  $\tilde{x} \in \tilde{X}$ , for any bounded set  $\Sigma \subset \tilde{X}$ , the set

$$\{\gamma \in \pi_1^{\text{orb}}(\mathcal{X}) : \gamma \cdot \tilde{x} \in \Sigma\}$$

is finite.

Finally we recall the generalized Margulis lemma established in [\[BGT12\]](#)

**THEOREM 7.11** ([\[BGT12, Corollary 11.17\]](#)). *Let  $K \geq 1$  be a parameter. There exists  $\epsilon(K) > 0$ , such that the following is true:*

*Suppose that  $X$  is a metric space with packing constant  $K$  and  $\Gamma$  is a subgroup of isometries of  $X$  that acts discretely. Then for every  $x \in X$  the "almost stabiliser"*

$$\Gamma_\epsilon(x) := \langle \{\gamma \in \Gamma : d(\gamma \cdot x, x) < \epsilon\} \rangle$$

*is virtually nilpotent.*

With [Lemma 7.10](#), applying [Theorem 7.11](#) to complete Riemannian orbifolds, we get the following lemma.

**Lemma 7.12** (*cf.* [\[BGT12, Corollary 11.19\]](#)). *Let  $n \geq 1$  be an integer. There exists  $\alpha = \alpha(n) > 0$  such that the following holds true:*

*Let  $\mathcal{X}$  be a complete Riemannian orbifold with its Ricci curvature bounded by  $\text{Ric} \geq -(n-1)$  and  $\Gamma$  be a subgroup of  $\text{Isom}(|\mathcal{X}|)$  acting properly discontinuously by isometries on  $|\mathcal{X}|$ . Then for every  $x \in |\mathcal{X}|$ , the "almost stabiliser"*

$$\Gamma_\alpha(x) := \langle \{\gamma \in \Gamma : d(\gamma \cdot x, x) < \alpha\} \rangle$$

*is virtually nilpotent.*

## CHAPTER 8

### Main theorem

In this section and onwards, we only deal with complex orbifolds.

**Definition 8.1.** Let  $\mathcal{X}$  be an orbifold and  $p : \mathcal{E} \rightarrow \mathcal{X}$  be a complex orbi-vector bundle over  $\mathcal{X}$ , represented by the left  $\mathcal{G}$ -space  $E$  such that  $p$  is represented by  $\mathcal{G} \times E \rightarrow \mathcal{G}$  (cf. [Definition 5.45](#)). An *Hermitian metric* on  $\mathcal{E}$  is a map  $h : |\mathcal{E}| \times_{|\mathcal{X}|} |\mathcal{E}| \rightarrow \mathbb{C}$  such that  $h$  lifts to a map  $\tilde{h} : E \times_{\mathcal{G}_0} E \rightarrow \mathbb{C}$  and  $\tilde{h}$  is Hermitian and  $\mathcal{G}$ -invariant.

One can always get a Hermitian metric on an orbi-vector bundle by partition of unity:

**Proposition 8.2** (cf. [\[Par20, Lemma 5.1\]](#)). *Let  $\mathcal{X}$  be an orbifold and  $\mathcal{E}$  be a complex orbi-vector bundle on  $\mathcal{X}$ . Then there exists an Hermitian metric on  $\mathcal{E}$ .*

For a complex orbifold groupoid  $\mathcal{G} = \{G_1 \rightrightarrows G_0\}$ , we know that the structure maps between  $G_i$  are holomorphic (actually they are étale). In particular, for any arrow  $g : x \rightarrow y$  in  $G_1$ , the induced local diffeomorphism  $U_x \rightarrow U_y$  is biholomorphic. Its tangent groupoid is  $T\mathcal{G} = G_1 \times TG_0$ . We see that the almost complex structure  $J$ , differential operators  $d$ ,  $\partial$  and  $\bar{\partial}$  are  $G_1$ -invariant. Thus for a complex orbifold  $\mathcal{X}$ , we have the decomposition  $T_{\mathbb{C}}\mathcal{X} = T^{1,0}\mathcal{X} \oplus T^{0,1}\mathcal{X}$ . We define the *anti-canonical bundle* of  $\mathcal{X}$  to be  $K_{\mathcal{X}}^{-1} = \det(T^{1,0}\mathcal{X})$ .

**Definition 8.3.** Let  $\mathcal{X}$  be a complex orbifold. A *Kähler form* on  $\mathcal{X}$  is a closed real  $(1,1)$ -form  $\omega \in \Gamma(X, (T^2\mathcal{X})^{\vee} \cap (T^{(1,1)}\mathcal{X})^{\vee})$  such that  $\omega(-, J-)$  defines a Riemannian metric on  $\mathcal{X}$ .

We give a definition of holomorphic orbi-vector bundles that suits our later discussion.

**Definition 8.4.** Let  $\mathcal{X}$  be a complex orbifold,  $\mathcal{E} \rightarrow \mathcal{X}$  a complex orbi-vector bundle over  $\mathcal{X}$ . A holomorphic orbi-vector bundle structure over  $\mathcal{E}$  is a representation  $\mathcal{G} \times E \rightarrow \mathcal{G}$  of  $\mathcal{E} \rightarrow \mathcal{X}$  together with a holomorphic vector bundle structure on  $E \rightarrow \mathcal{G}_0$ . We call  $\mathcal{E}$  together with the holomorphic structure a *holomorphic orbi-vector bundle*.

With the above definition, we can see that  $(T\mathcal{X}, J)$  and  $K_{\mathcal{X}}$  carry natural holomorphic structure. It is thus considered as holomorphic orbi-vector bundle in the rest of the article.

**Remark 8.5.** The definition is not optimal. Suppose that  $\phi : \mathcal{H} \rightarrow \mathcal{G}$  is an equivalence, then one will have a nature holomorphic vector bundle  $\phi_0^*(E) \rightarrow \mathcal{H}_0$ . One should consider  $\phi^*(\mathcal{G} \times E) = \mathcal{H} \times \phi_0^*(E)$  gives same holomorphic structure on  $\mathcal{E} \rightarrow \mathcal{X}$ . Hence the right definition will be the

{representation + holomorphic structure on the representation} *mod* "equivalence".

Due to the inability of the author, we can not give a satisfying equivalence relation. However, the given definition suffices for our purpose.

Let  $\mathcal{E} \rightarrow \mathcal{X}$  be a holomorphic orbi-vector bundle represented by  $\mathcal{G} \times E \rightarrow \mathcal{G}$ , where  $\mathcal{G} = [G_1 \rightrightarrows G_0]$  is a groupoid representation of  $\mathcal{X}$ . The holomorphic structure on  $E$  induces a natural complex structure on  $E$ , Hence  $\mathcal{G} \times E$  is a complex orbifold groupoid. For  $x \in G_0$ , by [Proposition A.8](#) we can take a neighbourhood  $U_x \subset G_0$  of  $x$  such that  $\mathcal{G}|_{U_x} \cong G_x \times U_x$ . Shrink  $U_x$  if necessary. We may assume  $E$  is trivialized by a holomorphic frame over  $U_x$ , i.e.,  $E|_{U_x} \cong \mathbb{C}^r \times U_x$ . Then  $(\mathcal{G} \times E)|_{U_x} \cong G_x \times (\mathbb{C}^r \times U_x)$  as complex orbifold groupoid. In particular,  $G_x \times (\mathbb{C}^r \times U_x) \rightarrow \mathbb{C}^r \times U_x$  is holomorphic.

For an element  $g \in G_x$ , the action of  $g$  is  $(v, y) \mapsto (g(y) \cdot v, y)$ , where  $g(y) \in GL(r, \mathbb{C})$ . Thus  $y \mapsto g(y)$  is holomorphic and  $g \in G_x$  transfers holomorphic section of  $E$  on  $U_x$  to a holomorphic section. With the same argument, one could show  $g : x \rightarrow y$  transfers a local holomorphic section around  $x$  to a local holomorphic section around  $y$ . We then define the *holomorphic section* of  $\mathcal{E}$  to be a  $\mathcal{G}$ -invariant holomorphic section in  $E$ .

Let  $s$  be a  $\mathcal{G}$ -invariant holomorphic section of  $E$ , and  $g : x \rightarrow y$  be an arrow in  $G_1$ . Suppose that  $s$  is defined around  $x$  and  $y$ , such that  $s = \sum \phi_i e_i$  around  $x$  and  $s = \sum \psi_j f_j$  around  $y$ , where  $\{e_i\}$  and  $\{f_j\}$  are holomorphic frames. We have that  $g \cdot e_i = \sum_j A_{ji} f_j$  for some holomorphic functions  $A_{ji}$  around  $y$  and  $A_{ji}$  is invertible. Now around  $x$ , we have that  $\bar{\partial}_E(s) = \sum_i \bar{\partial} \phi_i \otimes e_i$ . Around  $y$ , we have that

$$\begin{aligned} g \cdot \bar{\partial}_E(s) &= \sum_i (g^{-1})^* \bar{\partial} \phi_i \otimes g \cdot e_i = \sum_{i,j} (g^{-1})^* \bar{\partial} \phi_i A_{ji} \otimes f_j \\ &= \sum_j \bar{\partial} \psi_j \otimes f_j = \bar{\partial}_E(s). \end{aligned}$$

Thus the Dolbeault operator  $\bar{\partial}_E$  passes to an *orbifold Dolbeault operator*  $\bar{\partial}_{\mathcal{E}}$  on  $\mathcal{E}$ .

Let  $\mathcal{L} \rightarrow \mathcal{X}$  be a holomorphic orbi-line bundle and  $h$  an Hermitian metric on  $\mathcal{L}$ . Let  $\mathcal{G} \times L \rightarrow \mathcal{G}$  be a representation of  $\mathcal{L}$  and  $\tilde{h}$  be the  $\mathcal{G}$ -invariant metric on  $L$ . For an arrow  $g : x \rightarrow y$ , we consider two trivializations by holomorphic sections  $e$  and  $f$  around  $x$  and  $y$  respectively. Suppose that  $g \cdot e(w) = \phi(g^{-1}w)f(w)$ . We have the local matrices for  $\tilde{h}$  to be  $h_1 = \tilde{h}(e, e)$  and  $h_2 = \tilde{h}(f, f)$ . From the equality  $h(g \cdot e, g \cdot e) = h(e, e)$ , we see that

$$h_2(w) \phi(g^{-1}w) \overline{\phi(g^{-1}w)} = h_1(g^{-1}w).$$

Hence

$$\partial \bar{\partial}(\log \circ h_2) = \partial \bar{\partial}(\log \circ h_1 \circ L_{g^{-1}}) = (g^{-1})^* \partial \bar{\partial}(\log \circ h_1) = g \cdot \partial \bar{\partial}(\log \circ h_1),$$

which means that the Chern curvature  $\Theta_{\tilde{h}} = -\frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \tilde{h}$  is  $\mathcal{G}$ -invariant, hence corresponds to an orbifold section of the complex vector bundle  $\mathcal{L}$ .

We are now ready to give the definition of nefness.

**Definition 8.6.** Let  $\mathcal{X}$  be a compact Kähler orbifold and  $\mathcal{L} \rightarrow \mathcal{X}$  a holomorphic line bundle on  $\mathcal{X}$ . We fix a Kähler form  $\omega$  on  $\mathcal{X}$ . We say that  $\mathcal{L}$  is *nef*, if for any  $\epsilon > 0$ , there exists a Hermitian metric  $h_{\epsilon}$  on  $\mathcal{L}$  such that its Chern curvature  $\Theta_{h_{\epsilon}}$  satisfies

$$\Theta_{h_{\epsilon}} \geq -\epsilon \omega.$$

**Remark 8.7.** Suppose that  $\mathcal{L} \rightarrow \mathcal{X}$  is represented by  $\mathcal{G} \times L \rightarrow \mathcal{G}$ . If  $G_0$  is compact, it is obvious that  $L$  is then a nef line bundle on  $G_0$ . However, in general  $G_0$  is not compact, and it makes no sense to say  $L$  is nef or not.

Let  $\mathcal{X}$  be a compact Kähler orbifold whose anti-canonical bundle  $K_{\mathcal{X}}^{-1}$  is nef. We fix a Kähler metric  $\omega$  on  $\mathcal{X}$ . We repeat the technique used by in [DPS93] to construct a sequence of Kähler metrics  $\{\omega_\epsilon\}$  in the same cohomology class of  $\omega$  such that the Ricci form  $\text{Ricci}_{\omega_\epsilon} \geq -\epsilon\omega_\epsilon$ .

For any  $\epsilon > 0$ , since  $K_{\mathcal{X}}^{-1}$  is nef, we have a Hermitian metric  $h_\epsilon$  on  $K_{\mathcal{X}^{-1}}$ , such that  $u_\epsilon = \Theta_{h_\epsilon} \geq -\epsilon\omega$ . It is thus sufficient to search  $\omega_\epsilon$  such that

$$(16) \quad \text{Ricci}_{\omega_\epsilon} = -\epsilon\omega_\epsilon + \epsilon\omega + u_\epsilon.$$

The  $\partial\bar{\partial}$ -lemma still holds in the orbifold setting:

**Lemma 8.8** ([Bai56, Theorem H, Theorem K]). *Let  $(\mathcal{X}, \omega)$  be a Kähler orbifold such that  $X = |\mathcal{X}|$  is compact. If  $\alpha$  is a  $d$ -exact  $(p, q)$ -form, then  $\alpha$  is  $\partial\bar{\partial}$ -exact.*

Hence we may write  $u_\epsilon = \text{Ricci}_\omega + \sqrt{-1}\partial\bar{\partial}f_\epsilon$ . And to search  $\omega_\epsilon$  is the same as search a potential  $\phi_\epsilon$  such that  $\omega_\epsilon = \omega + \sqrt{-1}\partial\bar{\partial}\phi_\epsilon$ . Equation (16) on  $\omega_\epsilon$  is thus equivalent to

$$(17) \quad \frac{(\omega + \sqrt{-1}\partial\bar{\partial}\phi_\epsilon)^n}{\omega^n} = \exp(\epsilon\phi_\epsilon - f_\epsilon)$$

By the following theorem, Equation (17) has a unique solution.

**THEOREM 8.9** (Aubin-Yau Theorem, cf. [Fau19, Theorem 1.1 and Section 6]). *Let  $(\mathcal{X}, \omega)$  be a compact Kähler orbifold. For any smooth function  $f$  on  $\mathcal{X}$  and  $\lambda > 0$ . The equation*

$$(MA) \quad \log M(\phi) = \lambda\phi + f,$$

where  $M(\phi) := \frac{(\omega + \sqrt{-1}\partial\bar{\partial}\phi)^n}{\omega^n}$  is the Monge-Ampère operator, has a unique admissible solution.

Thus we have

**Lemma 8.10.** *Let  $\mathcal{X}$  be a compact Kähler orbifold with  $-K_{\mathcal{X}}$  nef. Fix a Kähler metric  $\omega$  on  $\mathcal{X}$ . For  $\epsilon > 0$ , there exists a Kähler metric  $\omega_\epsilon$  cohomologous to  $\omega$ , and the Ricci form of  $\omega_\epsilon$  satisfying*

$$\text{Ricci}_{\omega_\epsilon} \geq -\epsilon\omega_\epsilon.$$

To prove our main results, we first note that  $\pi_1^{\text{orb}}(\mathcal{X})$  is finitely generated. In fact we have

**Lemma 8.11** ([MP99, Corollary 1.2.5.]). *Let  $\mathcal{X}$  be an orbifold and  $\mathcal{U}$  be an atlas of  $\mathcal{X}$ . There exists an atlas  $\mathcal{V}$  for  $\mathcal{X}$  such that*

- (1)  $\mathcal{V}$  refines  $\mathcal{U}$ ;
- (2) For every chart  $(\tilde{V}, H, \psi)$  in  $\mathcal{V}$ , both  $\tilde{V}$  and  $V = \psi(\tilde{V}) \subset |\mathcal{X}|$  are contractible;
- (3) The intersection of finitely many chart is empty or again a chart in  $\mathcal{V}$ .

Let  $\mathcal{X}$  be a compact orbifold. We may take a finite atlas  $\mathcal{V}$  by [Lemma 8.11](#). Note that each open sub-orbifold  $[\tilde{V}/H]$  has fundamental group

$$\pi_1^{\text{orb}}([\tilde{V}/H]) \cong H.$$

By orbifold Van-Kampen theorem (*cf.* [\[BH99, Exercise III.ℳ.3.10\]](#)), we know that  $\pi_1^{\text{orb}}(\mathcal{X})$  is finitely generated.

We note that the following lemma, which is proved in manifold case in [\[DPS93\]](#), holds true in the orbifold case with exactly the same proof.

**Lemma 8.12** (*cf.* [\[DPS93, Lemma 1.3.\]](#)). *Let  $\pi : \tilde{\mathcal{X}} \rightarrow \mathcal{X}$  be the universal covering of  $\mathcal{X}$ . Let  $U$  be a connected compact subset of  $\tilde{\mathcal{X}} = |\tilde{\mathcal{X}}|$ . Then for any  $\delta > 0$ , there exists a closed subset  $U_{\epsilon, \delta} \subset U$  such that  $\text{vol}_{\omega}(U \setminus U_{\epsilon, \delta}) < \delta$  and  $\text{diam}_{\omega_{\epsilon}}(U_{\epsilon, \delta}) < C_1 \delta^{-\frac{1}{2}}$ , where  $C_1$  is a constant independent of  $\epsilon$  and  $\delta$ .*

PROOF. Suppose first that there exists a chart  $(\tilde{V} \subset \mathbb{C}^n, G, V)$  of  $\tilde{\mathcal{X}}$ , a  $G$ -invariant compact subset  $K$  such that  $K/G = U$  and a  $G$ -invariant open subset  $W \subset \tilde{V}$  such that its closure  $\bar{W} \subset \tilde{V}$  is compact and  $K \subset W$ . We further suppose that  $V$  is a component of  $\pi^{-1}(\tilde{V}/G_0)$ , where  $(\tilde{V}, G_0)$  is an elementary neighborhood of  $\mathcal{X}$  with respect to  $\pi$  and  $G$  is a subset of  $G_0$ .

Let  $\tilde{\omega}_{\epsilon}$  be the  $G$ -invariant form on  $\tilde{V}$ . On  $\tilde{V}$ , we have two metrics: the Euclidean one  $\omega_{\text{euc}}$  of  $\mathbb{C}^n$  and the Riemannian one  $\tilde{g}_{\epsilon}$  corresponding to  $\tilde{\omega}_{\epsilon}$ . As  $W$  is compact, we have a constant  $C_0 = C_0(W)$  such that

$$\frac{1}{C_0} \tilde{\omega} \leq \omega_{\text{euc}} \leq C_0 \tilde{\omega}$$

Here, we consider these real  $(1, 1)$ -forms as positive currents. If  $\beta$  is a positive  $(1, 1)$ -form, we have that

$$\frac{1}{C_0} \tilde{\omega} \wedge \tilde{\omega}_{\text{euc}}^{n-2} \wedge \beta \leq \omega_{\text{euc}}^{n-1} \wedge \beta \leq C_0 \tilde{\omega} \wedge \tilde{\omega}_{\text{euc}}^{n-2} \wedge \beta$$

Iterating  $(n - 1)$  times, we have that

$$\frac{1}{C_0^{n-1}} \tilde{\omega}^{n-1} \leq \omega_{\text{euc}}^{n-1} \leq C_0^{n-1} \tilde{\omega}^{n-1}$$

As  $\omega$  and  $\omega_{\epsilon}$  are in the same cohomology class, we have that

$$\int_{\mathcal{X}} \omega_{\epsilon} \wedge \omega^{n-1} = \int_{\mathcal{X}} \omega \wedge \omega^{n-1}$$

Hence we have that

$$\int_W \tilde{\omega}_{\epsilon} \wedge \omega_{\text{euc}}^{n-1} \leq C_0^{n-1} |G_0| \int_{\mathcal{X}} \omega_{\epsilon} \wedge \omega^{n-1} = C_0^{n-1} |G_0| \int_{\mathcal{X}} \omega \wedge \omega^{n-1} =: C_1$$

For any  $x_1, x_2 \in K \times K$ , we consider the path  $c_{x_1, x_2} : [0, 1] \rightarrow \tilde{V}$ ,  $t \mapsto (1 - t)x_1 + tx_2$ . As  $W$  is convex and contains  $K$ , the path  $c_{x_1, x_2}$  is contained in  $W$ . We denote by  $\langle -, - \rangle$  and  $\langle -, - \rangle_{\epsilon}$  the Riemannian metric on  $TW = W \times \mathbb{C}^n$  induced by  $\omega_{\text{euc}}$  and  $\tilde{\omega}_{\epsilon}$  respectively. We also use  $|\bullet|$  to denote the norm induced by  $\langle -, - \rangle$ .

By Cauchy-Schwartz inequality, we have that

$$\begin{aligned}
\int_{K \times K} l_{\tilde{g}_\epsilon}^2(c_{x_1, x_2}) dx_1 dx_2 &= \int_{K \times K} \left( \int_0^1 (\tilde{g}_\epsilon(\dot{c}_{x_1, x_2}(t), \dot{c}_{x_1, x_2}(t)))^{\frac{1}{2}} dt \right)^2 dx_1 dx_2 \\
(*) \quad &\leq |x_2 - x_1| \int_0^1 dt \int_{K \times K} \langle v(t, x_1, x_2), v(t, x_1, x_2) \rangle_\epsilon dx_1 dx_2
\end{aligned}$$

where  $v(t, x_1, x_2) := ((1-t)x_1 + tx_2, \frac{x_2 - x_1}{|x_2 - x_1|}) \in W \times \mathbb{C}^n = TW$  and  $dx_1, dx_2$  are the volumes with respect to  $\omega_{\text{euc}}$ .

Now

$$\begin{aligned}
&\int_0^{\frac{1}{2}} dt \int_{K \times K} \langle v(t, x_1, x_2), v(t, x_1, x_2) \rangle_\epsilon \left(\frac{1}{2}\right)^{2n} dx_1 dx_2 \\
&\leq \int_0^{\frac{1}{2}} dt \int_{K \times K} \langle v(t, x_1, x_2), v(t, x_1, x_2) \rangle_\epsilon (1-t)^{2n} dx_1 dx_2 \\
&\leq \int_0^{\frac{1}{2}} dt \int_{W \times K} \langle u_{x_2}(y), u_{x_2}(y) \rangle_\epsilon dy dx_2
\end{aligned}$$

where  $u_{x_2} : y \mapsto (y, \frac{x_2 - y}{|x_2 - y|})$  is a vector field on  $W$ . Note that  $|u_{x_2}(y)| = 1$  for all  $y \in W$ .

Let  $z_i : \mathbb{C}^n \rightarrow \mathbb{C}$  be the  $i$ -th projection. We have that

$$\begin{aligned}
\omega_{\text{euc}} &= \sum_{1 \leq i \leq n} \frac{\sqrt{-1}}{2} dz_i \wedge d\bar{z}_i \\
\tilde{\omega}_\epsilon &= \sum_{1 \leq i, j \leq n} \frac{\sqrt{-1}}{2} h_{ij} dz_i \wedge d\bar{z}_j \\
\tilde{\omega}_\epsilon \wedge \omega_{\text{euc}}^{n-1} &= (n-1)! \text{Tr}(H) \left(\frac{\sqrt{-1}}{2}\right)^n \bigwedge_{1 \leq i \leq n} dz_i \wedge d\bar{z}_i
\end{aligned}$$

where  $H = (h_{ij})$  is a strictly positive Hermitian matrix. If we consider the  $\mathbb{C}$ -linear extension of the tensor  $\tilde{g}_\epsilon$  in  $(T_{\mathbb{C}}^*W)^{\otimes 2}$ , we have that

$$\tilde{g}_\epsilon = \sum_{1 \leq i, j \leq n} \frac{h_{ij}}{2} (dz_i \otimes d\bar{z}_j + d\bar{z}_j \otimes dz_i)$$

Thus the matrix of  $\tilde{g}_\epsilon$  with respect to the basis  $(\partial/\partial z_1, \dots, \partial/\partial z_n, \partial/\partial \bar{z}_1, \dots, \partial/\partial \bar{z}_n)$  is

$$\begin{pmatrix} 0 & H \\ H^t & 0 \end{pmatrix}.$$

Hence the maximal eigenvalue of  $\tilde{g}_\epsilon$  is at most the maximal eigenvalue of  $H$ , which is less than  $\text{Tr}(H)$ . Thus for any vector  $u$  such that  $|u| = 1$ , we have that  $\langle u, u \rangle_\epsilon \leq \text{Tr}(H)$ .

We have that

$$\begin{aligned}
& \int_0^{\frac{1}{2}} dt \int_{K \times K} \langle v(t, x_1, x_2), v(t, x_1, x_2) \rangle_\epsilon dx_1 dx_2 \\
& \leq 2^{2n} \int_0^{\frac{1}{2}} dt \int_{W \times K} \langle u_{x_2}(y), u_{x_2}(y) \rangle_\epsilon dy dx_2 \\
& \leq 2^2 \int_0^{\frac{1}{2}} dt \int_{W \times K} \frac{1}{(n-1)!} \tilde{\omega}_\epsilon \wedge \omega_{\text{euc}}^{n-1} dx_2 \\
& \leq 2^{2n-1} \text{vol}(K) C_1
\end{aligned}$$

Similarly, we have that

$$\int_{\frac{1}{2}}^1 dt \int_{K \times K} \langle v(t, x_1, x_2), v(t, x_1, x_2) \rangle_\epsilon dx_1 dx_2 \leq 2^{2n-1} \text{vol}(K) C_1$$

Combined with [Equation \(\\*\)](#), we have that

$$\begin{aligned}
\int_{K \times K} l_{\tilde{g}_\epsilon}^2(c_{x_1, x_2}) dx_1 dx_2 & \leq 2^{2n} \text{diam}(K) \text{vol}(K) C_1 \\
& \leq C_2
\end{aligned}$$

where  $\text{diam}(-)$  is the diameter with respect to  $\omega_{\text{euc}}$  and  $C_2$  is a constant independent of  $\epsilon$ .

We set

$$S := \{(x_1, x_2) \in K \times K \mid l_{\tilde{g}_\epsilon}(c_{x_1, x_2}) > (C_2/\delta)^{\frac{1}{2}}\}$$

The above estimate implies that  $\text{vol}(S) < \delta$ . Set  $S(x_1) := \{x_2 \in K : (x_1, x_2) \in S\}$ . Now we consider the set

$$Q := \{x_1 \in K \mid \text{vol}(S(x_1)) \geq \frac{1}{2} \text{vol}(K)\}$$

Then by Fubini, we have that  $\text{vol}(Q) \leq 2\delta \text{vol}(K)$ . For  $x_1, x_2 \in K \setminus Q$ , we have that  $\text{vol}(S(x_i)) < \frac{1}{2} \text{vol}(K)$ . Thus we have that

$$(K \setminus S(x_1)) \cap (K \setminus S(x_2)) \neq \emptyset.$$

If  $y$  is an element of the above set, then  $(x_1, y) \notin S$  and  $(x_2, y) \notin S$ . Hence we have that

$$l_{\tilde{g}_\epsilon}(c_{x_1, x_2}) \leq l_{\tilde{g}_\epsilon}(c_{x_1, y}) + l_{\tilde{g}_\epsilon}(c_{y, x_2}) \leq 2(C_2/\delta)^{\frac{1}{2}}$$

By continuity, for any  $x_1, x_2 \in \overline{K \setminus Q}$ , we have that  $l_{\tilde{g}_\epsilon}(c_{x_1, x_2}) \leq 2(C_2/\delta)^{\frac{1}{2}}$ .

Let  $U_{\epsilon, \delta}$  be the image of  $\overline{K \setminus Q}$  in  $V$ . Then as  $K$  is  $G$ -variant, we have that

$$\text{vol}_\omega(K \setminus (\bigcup_{\alpha \in G} \alpha \cdot \overline{K \setminus Q})) \leq \text{vol}_\omega(K \setminus \overline{K \setminus Q}) \leq C_3 \text{vol}(Q) < \frac{2C_3\delta}{\text{vol}(K)}$$

where  $C_3 = C_0^n$ .

Hence we have that

$$\text{vol}_\omega(U \setminus U_{\epsilon, \delta}) < \frac{2C_3\delta}{|G| \text{vol}(K)}.$$

If  $a_1, a_2$  are two points in  $U_{\epsilon, \delta}$ , we may take  $x_1 \in \overline{K \setminus Q}$  and  $x_2 \in \overline{K \setminus Q}$  to be one of the inverse images in  $\tilde{V}$  of  $a_1$  and  $a_2$  respectively. We have thus that

$$d_{\omega_\epsilon}(a_1, a_2) \leq l_{\tilde{g}_\epsilon}(c_{x_1, x_2}) \leq 2(C_1/\delta)^{\frac{1}{2}}.$$

This implies that  $\text{diam}_{\omega_\epsilon}(U_{\epsilon, \delta}) \leq 4(C_1/\delta)^{\frac{1}{2}}$ . By scaling  $\delta$ , we have the result for  $U = K/G$ .

For a general  $U \subset \tilde{X}$ , there exists a finite open covering  $U \subset \bigcup_{1 \leq i \leq N} U_i$  such that

- for each  $i$ , the open set  $U_i$  is connected;
- for each  $i$ , there exists a chart  $(\tilde{V}_i, H_i, V_i)$  of  $\tilde{X}$  and an  $H_i$ -invariant convex open  $W_i \subset \tilde{V}_i$  satisfying that  $W_i/H_i$  contains  $U_i$  and  $\overline{W}_i$  is compact.

As  $U$  is connected, after removing all the  $U_j$  that do not intersect  $U$ , we can arrange  $U_i$  such that  $U_i \cap U_{i+1} \neq \emptyset$ . Now the hypothesis in the first paragraph applies to  $\overline{U}_i$ . Hence for  $\delta > 0$ , we get subsets  $U_{i, \epsilon, \delta}$  such that

(1) for each  $i$ , we have that

$$\text{vol}_\omega(\overline{U}_i \setminus U_{i, \epsilon, \delta}) < \min \left\{ \frac{\delta}{N}, \min_{1 \leq j \leq N-1} \left\{ \frac{1}{2} \text{vol}_\omega(U_j \cap U_{j+1}) \right\} \right\};$$

(2) for each  $i$ , we have that  $\text{diam}_{\omega_\epsilon}(U_{i, \epsilon, \delta}) < C'_i \delta^{-\frac{1}{2}}$ , where  $C'_i$  is independent of  $\epsilon$  and  $\delta$ .

Condition (1) implies that  $U_{i, \epsilon, \delta} \cap U_{i+1, \epsilon, \delta} \neq \emptyset$ . Set

$$C_1 := N^2 \cdot \max_{1 \leq i \leq N} \{C'_i\}.$$

Thus  $\text{diam}_{\omega_\epsilon}(\bigcup_{1 \leq i \leq N} U_{i, \epsilon, \delta}) \leq C_1 \delta^{-\frac{1}{2}}$  and

$$\text{vol}_\omega \left( \bigcup_{1 \leq i \leq N} U_i \setminus \bigcup_{1 \leq i \leq N} U_{i, \epsilon, \delta} \right) \leq \sum_{1 \leq i \leq N} \text{vol}_\omega(U_i \setminus U_{i, \epsilon, \delta}) < \delta.$$

Set  $U_{\epsilon, \delta} := U \cap (\bigcup_{1 \leq i \leq N} U_{i, \epsilon, \delta})$ . We have that

$$\text{vol}_\omega(U \setminus U_{\epsilon, \delta}) \leq \text{vol}_\omega \left( \bigcup_{1 \leq i \leq N} U_i \setminus \bigcup_{1 \leq i \leq N} U_{i, \epsilon, \delta} \right) < \delta.$$

This finishes the proof.  $\square$

**THEOREM 8.13** (=Theorem 4.10). *Let  $(\mathcal{X}, \omega)$  be a compact Kähler orbifold. If the anti-canonical bundle  $K_{\mathcal{X}}^{-1}$  is nef, then  $\pi_1^{\text{orb}}(\mathcal{X})$  is virtually nilpotent.*

**PROOF.** We can reproduce the argument by [DPS93] and [Pău97] in the manifold case. Let  $\omega_\epsilon$  be the sequence of Kähler metrics as in Lemma 8.10, and  $\tilde{\mathcal{X}} \rightarrow \mathcal{X}$  be the universal covering of  $\mathcal{X}$ . We fix a finite system of generators  $\{\gamma_i\}$  of  $\pi_1^{\text{orb}}(\mathcal{X})$ . Let  $\alpha = \alpha(2n)$  be the constant in Lemma 7.12. It suffices to show that there exists  $\omega'$  such that all the generators  $\gamma_i$  are contained in the almost stabiliser  $\Gamma_\alpha$  with respect to the distance  $d_{\omega'}$  in Lemma 7.12.

Recall that the fundamental domain of  $\text{Gal}(\tilde{\mathcal{X}} \rightarrow \mathcal{X})$  is defined to be

$$F := \{p \in X' \mid d_{\tilde{\omega}}(x_0, p) \leq d_{\tilde{\omega}}(\gamma \cdot x_0, p) \text{ for all } \gamma \in \pi_1^{\text{orb}}(\mathcal{X})\}.$$



We see easily that  $F$  is path connected. Thus we can take a connected compact subset  $U \subset \tilde{X}$  which contains  $F$ . As  $\{\gamma_i\}$  is finite, we may take  $U$  large enough, such that  $U \cap \gamma_j U \neq \emptyset$  for all  $j$ . We choose a sufficiently small  $\delta > 0$  such that  $\delta < \frac{1}{4} \text{vol}_\omega(U \cap \gamma_j U)$  and  $\delta < \frac{1}{2} \text{vol}_\omega F$ . By [Lemma 8.12](#), there exists a subset  $U_{\epsilon, \delta} \subset U$ , such that  $\text{diam}_{\omega_\epsilon}(U_{\epsilon, \delta}) < C_1 \delta^{\frac{1}{2}} := C$ . By the choice of  $\delta$ , we know that  $U_{\epsilon, \delta} \cap \gamma_j U_{\epsilon, \delta} \neq \emptyset$ . Fix a  $\tilde{x}_0 \in U_{\epsilon, \delta}$ . We know that  $d_{\omega_\epsilon}(\tilde{x}_0, \gamma_j \tilde{x}_0) < C$ . We set  $\tilde{\omega}_\epsilon := \frac{\epsilon}{2n-1} \omega_\epsilon$ . Then  $\text{Ricci}_{\tilde{\omega}_\epsilon} \geq -(2n-1)\tilde{\omega}_\epsilon$  and  $d_{\tilde{\omega}_\epsilon}(\tilde{x}_0, \gamma_j \tilde{x}_0) < \frac{\epsilon}{2n-1} C$ . For  $\epsilon$  sufficient small, we see that  $\frac{\epsilon}{2n-1} C < \alpha$ .  $\square$

## CHAPTER 9

### Projective case

We will consider complex orbifolds as klt pairs described in [Section 5.6](#).

Let  $\mathcal{X} = (X, [G_1 \rightrightarrows G_0])$  be a complex orbifold of dimension  $n$  and  $(X = |\mathcal{X}|, \Delta)$  its associated orbifold pair. For any  $x \in G_0$  by [Proposition A.8](#), there exists open neighborhood  $x \ni U_x \subset G_x$ , such that  $\pi_x : U_x \rightarrow U_x/G_x \subset |\mathcal{X}|$  gives an orbifold chart. [Theorem 5.33](#) implies that the topological quotient  $U_x/G_x$  has a unique normal complex space structure such that  $\pi_x : U_x \rightarrow U_x/G_x$  is a Galois analyticial covering. Hence  $U_x/G_x \subset X$  is an open sub-variety and  $\pi_x : U_x \rightarrow U_x/G_x$  is a local uniformization as in [Definition 5.37](#). The map  $\pi : G_0 \rightarrow X$  is holomorphic. As  $G_0$  is Kähler, [[Var89](#), Proposition 3.3.1] implies  $X$  is a complex Kähler space. We consider the canonical bundle  $K_{\mathcal{X}}$ . From [Definition 8.4](#), we know that it is represented by  $\det(\Omega_{G_0})$ . We denote by  $K_{G_0}$  its canonical class. Then by [Equation \(14\)](#), we have that

$$(18) \quad K_{G_0}|_{U_x} = \pi_x^*(K_X + \Delta).$$

If  $U_x/G_x \subset U_y/G_y$ , we have an element  $g \in G_1$  which induces an embedding  $\rho_g : U_x \rightarrow U_y$  and we have the following commutative diagram:

$$\begin{array}{ccc} U_x & \xrightarrow{\rho_g} & U_y \\ \downarrow \pi_x & & \downarrow \pi_y \\ U_x/G_x & \longrightarrow & U_y/G_y \end{array} .$$

Thus the local equations [Equation \(18\)](#) glue together to

$$K_{G_0} = \pi^*(K_X + \Delta)$$

We may thus regard  $K_X + \Delta$  as the canonical class of  $(X, \Delta)$ . As  $(X, \Delta)$  has klt singularities, let  $a(X)$  be the index of  $(X, \Delta)$ , *i.e.*, the minimal natural number such that  $a(X)(K_X + \Delta)$  is a Cartier divisor. Let  $\mathcal{O}_X(a(X)(K_X + \Delta))$  be the associated line bundle on  $X$ . Then we have that

$$\mathcal{O}_{G_0}(K_{G_0})^{\otimes a(X)} = \pi^*\mathcal{O}_X(a(X)(K_X + \Delta)).$$

Let  $h$  be an Hermitian metric on  $K_{\mathcal{X}}$ . The Hermitian metric  $h^{\otimes a(X)}$  on  $K_{\mathcal{X}}^{\otimes a(X)}$  induces an Hermitian metric on  $\mathcal{O}_X(a(X)(K_X + \Delta))$ , since  $h$  is  $G_1$ -invariant. On the other hand, the pullback of an Hermitian metric on  $\mathcal{O}_X(a(X)(K_X + \Delta))$  by  $\pi$  will induce an Hermitian metric on  $K_{\mathcal{X}}^{\otimes a(X)}$ . In particular, if  $X$  is compact, the orbi-bundle  $K_{\mathcal{X}}^{-1}$  being nef is equivalent to  $-(K_X + \Delta)$  being nef.

We have the following result.

**THEOREM 9.1** (=Theorem 4.11). *Let  $\mathcal{X}$  be a compact Kähler orbifold with  $-K_{\mathcal{X}}$  nef. Let  $(X, \Delta)$  be the associated orbifold pair of  $\mathcal{X}$ . If  $X$  is projective, then  $\pi_1^{\text{orb}}(\mathcal{X})$  is virtually Abelian.*

**PROOF.** By Theorem 8.13, there exists a nilpotent subgroup  $\Gamma < \pi^{\text{orb}}(\mathcal{X})$  such that its index  $[\pi^{\text{orb}}(\mathcal{X}) : \Gamma]$  is finite. Let  $\tilde{\mathcal{X}}$  be the finite cover of  $\mathcal{X}$  such that  $\pi_1^{\text{orb}}(\tilde{\mathcal{X}}) = \Gamma$ , as indicated in Proposition 5.30. For any  $\epsilon > 0$  and the Hermitian metric  $h_\epsilon$  such that  $\Theta_{h_\epsilon} \geq -\epsilon\omega$ , we may take the induced metric on  $-K_{\tilde{\mathcal{X}}}$ . In particular, the orbibundle  $-K_{\tilde{\mathcal{X}}}$  is nef.

To simplify the notation, we may assume that  $\pi_1^{\text{orb}}(\mathcal{X})$  is nilpotent. Let  $(X, \Delta)$  be the associated orbifold pair. Then its anti-canonical bundle  $-(K_X + \Delta)$  is nef. [Zha05, Corollary 2] implies that the Albanese map

$$\alpha_X : X \dashrightarrow \text{Alb}(X)$$

is dominant. Recall that the Albanese map of  $X$  is defined by the Albanese morphism of its smooth model. Let  $r : Y \rightarrow (X, \Delta)$  be a log resolution, we have the following commutative

$$\begin{array}{ccc} Y & \xrightarrow{r} & X \\ \alpha_Y \downarrow & & \swarrow \alpha_X \\ \text{Alb}(Y) =: \text{Alb}(X) & & \end{array}$$

Recall that  $X$  has rational singularities (Theorem 5.38). Theorem 2.35 implies that  $\alpha_X$  is defined on all  $X$ . Hence  $\alpha_X$  is surjective. We have that

$$K_Y + r_*^{-1}(\Delta) = r^*(K_X + \Delta) + \sum a_j E_j,$$

where  $E_j \subset \text{exc}(r)$  are the irreducible exceptional divisors. We have also that

$$(19) \quad Y \setminus (r_*^{-1}(\Delta) \cup \text{exc}(r)) = X_{\text{reg}} \setminus |\Delta|.$$

If  $l_j$  is a loop in  $Y$  around  $E_j$ , the composition  $r \circ l_j$  will be a loop in  $X$ . If  $l_j$  is small enough, the loop  $r \circ l_j$  will be contained in an open subset  $U$  of  $X$  such that  $(U, \Delta|_U) = [\tilde{U}/G]$  with  $\tilde{U}$  simply connected. Then  $\pi_1(U, \Delta|_U) = G$  and we can find a divisible enough  $n_j$  such that  $(r \circ l_j)^{n_j}$  is the unit in  $\pi_1(X, \Delta)$ . We set

$$\Delta_Y := r_*^{-1}(\Delta) + \sum (1 - \frac{1}{n_j}) E_j.$$

By Equation (19), the choice of  $n_j$  and Definition 5.39, we have that

$$\pi_1(Y, \Delta_Y) = \pi_1(X, \Delta).$$

As  $r : Y \rightarrow X$  is a log resolution, the support  $|\Delta_Y|$  of  $\Delta_Y$  is snc. Assume that  $r_*^{-1}(D_i)$ ,  $1 \leq i \leq i_0$  and  $E_j$ ,  $1 \leq j \leq j_0$  are all the components of  $\Delta_Y$  passing through a point  $y \in Y$ . There is a holomorphic chart  $(U, \phi)$  centered at  $y$ , such that  $\phi_*(\Delta_Y|_U)$  is the branching divisor of the following map

$$(z_1, \dots, z_{i_0}, z_{i_0+1}, \dots, z_{i_0+j_0}, \dots, z_n) \mapsto (z_1^{m_1}, \dots, z_{i_0}^{m_{i_0}}, z_{i_0+1}^{n_1}, \dots, z_{i_0+j_0}^{n_j}, z_{i_0+j_0+1}, \dots, z_n)$$

Hence  $(Y, \Delta_Y)$  is an orbifold pair.

[Cam01, lemme 1.9.9] implies that for the orbifold pair  $(Y, \Delta_Y)$ , there exists a short exact sequence

$$1 \rightarrow K \rightarrow \pi_1(Y, \Delta_Y) \rightarrow \pi_1(Y) \rightarrow 1$$

with the group  $K$  generated by torsion elements.

As  $\pi_1(Y, \Delta_Y) = \pi_1(X, \Delta)$  is nilpotent, the quotient  $\pi_1(Y)$  is also nilpotent. By [Hir38], for any nilpotent group  $N$  of finite type, the torsion element of  $N$  forms a finite normal subgroup  $N_{\text{tor}} \trianglelefteq N$  and the nilpotent limit of  $N$  is  $N/N_{\text{tor}}$ . By the above exact sequence, we have that

$$\pi_1(Y, \Delta_Y)/\pi_1(Y, \Delta_Y)_{\text{tor}} = \pi_1(Y)/\pi_1(Y)_{\text{tor}}.$$

As  $\alpha_Y$  is surjective, [Cam95, Théorème 2.2] implies that

$$\pi_1(Y)/\pi_1(Y)_{\text{tor}} = \pi_1(\text{Alb}(X)).$$

By [Cla07, Lemme A.0.1], we know that  $\pi_1(Y, \Delta_Y)$  is then virtually Abelian. As  $\pi_1(Y, \Delta_Y)$  has finite index in  $\pi_1^{\text{orb}}(\mathcal{X})$ , we know that  $\pi_1^{\text{orb}}(\mathcal{X})$  is also virtually Abelian.  $\square$

**Remark 9.2.** As one can see from the proof of [Theorem 9.1](#), the hypothesis  $X$  is projective is used only to show that the Albanese morphism is surjective. One may reformulate [Theorem 9.1](#) as following:

If the covering  $(X', \Delta')$  corresponds to the nilpotent subgroup of  $\pi_1(X, \Delta)$  has surjective Albanese morphism, then  $\pi_1(X, \Delta)$  is virtually Abelian.



# Appendice



## APPENDIX A

# Groupoids

We recall here some basic facts in Lie groupoids. For the general results on categories, we refer the reader to [ML98]. For the Lie groupoids, we refer the reader to [Mac05][MM03]. In this chapter, all groupoids are small.

### 1.1. Orbifold Groupoids

**Definition A.1.** A *topological groupoid*  $\mathcal{G}$  consists of a topological space  $G_0$  of objects and a topological space  $G_1$  of arrows, together with five continuous structure maps listed below:

1. The source map  $s : G_1 \rightarrow G_0$ , which assigns to each arrow  $g \in G_1$  its source  $s(g)$ .
2. The target map  $t : G_1 \rightarrow G_0$ , which assigns to each arrow  $g \in G_1$  its target  $t(g)$ . For any two objects  $x, y \in G_0$ , one writes  $g : x \rightarrow y$  to indicate that  $g \in G_1$  is an arrow with  $s(g) = x$  and  $t(g) = y$ .
3. The composition map  $m : G_1 \times_{s \times t} G_1 \rightarrow G_1$ . If  $h : y \rightarrow z$ ,  $g : x \rightarrow y$ , then  $hg = m(h, g) : x \rightarrow z$ . We have  $m$  to be associative, that is  $m(m(h, g), f) = m(h, m(g, f))$  for any three composable  $h, g, f$ .
4. The identity map  $u : G_0 \rightarrow G_1$  which is a two-sided unit for the composition.
5. The inverse map  $i : G_1 \rightarrow G_1$ . If  $g : x \rightarrow y \in G_1$ , then  $g^{-1} = i(g) : y \rightarrow x$  is the two sided inverse to  $g$ , *i.e.*, we have  $g \circ i(g) = u(y)$  and  $i(g) \circ g = u(x)$ .

One can also consider the topological groupoid as a groupoid (*i.e.* a category whose morphisms are all isomorphisms) equipped with two topological structure on the sets of objects and morphisms such that the structural maps are continuous. We then define the

**Definition A.2.** A *Lie groupoid* is a topological groupoid  $\mathcal{G}$  where  $G_0$  and  $G_1$  are smooth manifolds and all the structure maps  $s, t, m, u$  and  $i$  are smooth. Furthermore,  $s$  and  $t$  are required to be submersions (hence  $G_1 \times_{s \times t} G_1$  is a manifold).

With the topological/smooth structures on  $G_0$  and  $G_1$ , one can define the morphisms between groupoids to be functors with required continuity/smoothness on the sets of objects and morphisms. Similarly, a groupoid natural transformation will require the continuity on the functions assign each objects in source groupoid to the morphisms in the target groupoid.

**Example A.3.** Let  $M$  be a topological space  $K$  a topological group acting on  $M$ . One defines a topological groupoid  $K \ltimes M$ , by setting  $(K \ltimes M)_0 = M$  and  $(K \ltimes M)_1 = K \times M$ , with  $(g, x) : x \rightarrow gx$ . This groupoid is called the action groupoid or translation groupoid



associated to the group action. Note that if  $M$  is a manifold and  $K$  a Lie group, then  $K \ltimes M$  becomes a Lie groupoid.

**Definition A.4.** Let  $\mathcal{G}$  be a Lie groupoid. For a point  $x \in G_0$ , we define the *isotropy group*  $G_x$  of  $\mathcal{G}$  to be  $s^{-1}(x) \cap t^{-1}(x)$ . And we define the orbit space  $|\mathcal{G}|$  of  $\mathcal{G}$  to be the quotient of  $G_0$  by the equivalence relation  $x \sim y$  iff  $\exists g : x \rightarrow y$ .

**Lemma A.5** ([Mac05, Corollary 1.4.11]). *Let  $\mathcal{G}$  be a Lie groupoid. Set  $G_x^y := s^{-1}(x) \cap t^{-1}(y)$ . Then  $G_x^y$  is a smooth manifold, and the morphism  $m : G_x^z \times G_x^y \rightarrow G_x^z$  is smooth. In particular,  $G_x$  is a Lie group.*

Now we define types of groupoids.

**Definition A.6.** Let  $\mathcal{G}$  be a Lie groupoid.

- (a)  $\mathcal{G}$  is proper, if  $(s, t) : G_1 \rightarrow G_0 \times G_0$  is proper.
- (b)  $\mathcal{G}$  is called a foliation groupoid if each isotropy group  $G_x$  is discrete.
- (c)  $\mathcal{G}$  is étale, if  $s$  and  $t$  are local diffeomorphisms. In this case, one defines the dimension of  $\mathcal{G}$  to be  $\dim(\mathcal{G}) := \dim(G_0) = \dim(G_1)$ .

For a  $\mathcal{G}$  is proper étale, we have that  $G_x^y$  is finite. Let  $G_1 \ni g : x \rightarrow y$  be an arrow of  $\mathcal{G}_1$ . As  $s$  and  $t$  are diffeomorphisms around  $g$ ,  $x$  and  $y$ , we get (via  $t \circ s^{-1}$ ) a local diffeomorphism  $\phi_g : U_x \rightarrow U_y$ . After shrinking  $U_x$  and  $U_y$ , we get a morphism  $\phi_x^y : G_x^y \rightarrow \text{Diff}(U_x, U_y)$ . One can prove that  $\phi(hg) = \phi(h) \circ \phi(g)$ . In particular,  $\phi : G_x \rightarrow \text{Diff}(U_x)$  is a group morphism.

**Definition A.7.** We define an *orbifold groupoid* to be a proper étale Lie groupoid. An orbifold groupoid  $\mathcal{G}$  is *effective* if  $\forall x \in G_0$ ,  $\phi : G_x \rightarrow \text{Diff}(U_x)$  is injective.

It turns out that the local structure of an orbifold groupoid  $\mathcal{G}$  around  $x \in G_0$  is completely determined by the local group  $G_x$ . More precisely, we have the following

**Proposition A.8** (cf. [ALR07, Proposition 1.44]). *Let  $\mathcal{G}$  be an orbifold groupoid and  $x$  an element in  $G_0$ . For any neighborhood  $G_0 \supset U$  of  $x$ , there exists an open neighborhood  $N_x \subset U$ , such that the restriction of  $\mathcal{G}$  over  $N_x$  is isomorphic, as Lie groupoid, to the translation groupoid  $G_x \ltimes N_x$  and the quotient space  $N_x/G_x$  is an embedded open subsets of  $|\mathcal{G}|$  via the natural morphism  $\mathcal{G}|_{N_x} \hookrightarrow \mathcal{G}$ .*

**Definition A.9.** A morphism  $\phi : \mathcal{H} \rightarrow \mathcal{G}$  between Lie groupoids is called an *equivalence* between Lie groupoids if both conditions below are satisfied:

- (i) the map

$$t\pi_1 : G_1 \times_{\phi} H_0 \rightarrow G_0$$

defined on the fibered product of manifolds is a surjective submersion;

- (ii) The commutative diagram

$$\begin{array}{ccc} H_1 & \xrightarrow{\phi_1} & G_1 \\ (s,t) \downarrow & & \downarrow (s,t) \\ H_0 \times H_0 & \xrightarrow{\phi \times \phi} & G_0 \times G_0 \end{array}$$

is Cartesian.

Note that a homomorphism  $\phi : \mathcal{H} \rightarrow \mathcal{G}$  induces a continuous map  $|\mathcal{H}| \rightarrow |\mathcal{G}|$ . When  $\phi$  is an equivalence, the induced map on orbit spaces is an homeomorphism.

There this a more subtle equivalent relation between groupoids called *Morita equivalence*.

**Definition A.10.** Two Lie groupoids  $\mathcal{G}$  and  $\mathcal{G}'$  are said to be *Morita equivalent*, if there exists a third groupoid  $\mathcal{H}$  and two equivalences

$$\mathcal{G} \leftarrow \mathcal{H} \rightarrow \mathcal{G}'.$$

We denote the Morita equivalence by  $G \sim_{\text{Morita}} G'$ .

It's not hard to see that Morita equivalence is an equivalence relation. In fact suppose that  $\mathcal{G} \leftarrow \mathcal{H} \rightarrow \mathcal{G}'$  and  $\mathcal{G}' \leftarrow \mathcal{H}' \rightarrow \mathcal{G}''$  are two Morita equivalences. We get a Morita equivalence by considering the following diagram

$$\begin{array}{ccccc} (\mathcal{G} \times_{\mathcal{H}} \mathcal{G}') \times_{\mathcal{G}'} (\mathcal{G}' \times_{\mathcal{H}'} \mathcal{G}'') & \longrightarrow & \mathcal{G}' \times_{\mathcal{H}'} \mathcal{G}'' & \longrightarrow & \mathcal{G}'' \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{G} \times_{\mathcal{H}} \mathcal{G}' & \longrightarrow & \mathcal{G}' & \longrightarrow & \mathcal{H}' \\ \downarrow & & \downarrow & & \\ \mathcal{G} & \longrightarrow & \mathcal{H} & & \end{array}$$

where the product  $\mathcal{G} \times_{\mathcal{H}} \mathcal{G}'$  etc. are 2-fiber products of groupoids. As Lie groupoid equivalences are stable under 2-fiber products. We have a Morita equivalence between  $\mathcal{G}$  and  $\mathcal{G}''$ .

**Definition A.11.** We recall several categories related to groupoids.

- (1) We denote by  $\text{LieGrpd}$  the (strict) 2-category of small Lie groupoids.
- (2) We denote by  $\text{Gp}$  the 1-category defined as follows:
  - (a)  $\text{Gp}$  has objects  $\text{Ob}(\text{Gp}) = \text{Ob}(\text{LieGrpd})$
  - (b) For  $G_1, G_2$  two groupoids, their morphism is  $\text{Hom}_{\text{Gp}}(G_1, G_2) = \{1\text{-arrow of } \text{Hom}_{\text{LieGrpd}}(G_1, G_2)\} / \sim$ , where  $f, g \in \text{Hom}_{\text{LieGrpd}}(G_1, G_2)$  are equivalent if there exists a 2-morphism  $\alpha : f \Rightarrow g$ .
- (3) Let  $W := \{[e] \text{ is an arrow in } \text{Gp}: e \text{ is an equivalence of Lie groupoid (Definition A.9)}\}$  be the subset of arrows of  $\text{Gp}$ . The Hilsun-Skandalis category  $\mathcal{HS}$  is the localization  $\text{Gp}(W^{-1})$  of  $\text{Gp}$  by  $W$ .

We note that  $\mathcal{HS}$  is in fact defined as a category with Lie groupoids as objects and isomorphic bi-bundles as morphisms. However [Ler10, Proposition 3.39.] shows that  $\text{Gp}(W^{-1})$  and  $\mathcal{HS}$  are equivalent. Let  $G, H$  be two groupoids. Let  $G \leftarrow K \rightarrow H$  be a Morita equivalence. Then its obvious that  $\text{Hom}_{\mathcal{HS}}(G, L)$  is bijective to  $\text{Hom}_{\mathcal{HS}}(H, L)$ . We may thus define a pre-relation  $\mathcal{R}_0$  on the arrows of  $\mathcal{HS}$ . Let  $G$  and  $H$  be Morita equivalent. We say that two 1-arrows  $f \in \text{Hom}_{\mathcal{HS}}(G, L)$  and  $g \in \text{Hom}_{\mathcal{HS}}(H, L)$  are  $f\mathcal{R}_0g$  if  $f$  is mapped to  $g$  in the bijection  $\text{Hom}_{\mathcal{HS}}(G, L) \cong \text{Hom}_{\mathcal{HS}}(H, L)$  induced by the Morita equivalence. We denote by  $\mathcal{R}$  the relation generated by  $\mathcal{R}_0$ .

## 1.2. $\mathcal{G}$ -bundles

Let  $\mathcal{G} = [G_1 \rightrightarrows G_0]$  be a groupoid. The concept of  $\mathcal{G}$ -space is a natural generalization of  $G$ -bundles. We refer the reader to [ALR07, Chapter 2] for more details. We only define vector bundles.

**Definition A.12.** Let  $\mathcal{G}$  be a topological groupoid,  $G_1$  its arrows and  $G_0$  its objects. A left- $\mathcal{G}$ -vector bundle is a triple  $(E, \pi, \mu)$ , where  $\pi : E \rightarrow G_0$  is a vector bundle over  $G_0$  and  $\mu : G_1 \times_{s, \pi} E \rightarrow E$  is a continuous map, satisfying the following

- (1)  $\mu(g'g, e) = \mu(g', \mu(g, e))$ ;
- (2)  $\mu(1, e) = e$ ;
- (3)  $\mu(g, -) : E_{s(g)} \rightarrow E_{t(g)}$  is a linear isomorphism.

Let  $E$  be a left- $\mathcal{G}$ -vector bundle over  $\mathcal{G}$ . We associate with  $E$  a groupoid  $\mathcal{G} \times E$  in the following way:

Set  $E$  to be the objects and  $G_1 \times_{s, \pi} E$  to be the arrows. The source map is  $(g, e) \mapsto e$  and target map is  $(g, e) \mapsto \mu(g, e)$ . If  $e \in E_x$ , the identity arrow of  $e$  is  $(1_x, e)$ . The inverse arrow of  $(g, x)$  is  $(g^{-1}, gx)$ . The map  $\pi : E \rightarrow G_0$  extends to a morphism  $\mathcal{G} \times E \rightarrow \mathcal{G}$  by taking the objects map  $\pi$  and the arrows map  $(g, x) \mapsto g$ . We also denote  $\mathcal{G} \times E$  by  $\mathcal{E}$

**Example A.13** (Pullback of vector bundles). Let  $\mathcal{H}, \mathcal{G}$  be two groupoids and  $\phi : \mathcal{H} \rightarrow \mathcal{G}$  a morphism between groupoid. Let  $E$  be a left- $\mathcal{G}$ -vector bundle over  $\mathcal{G}$ . Consider the pullback vector bundle  $\phi_0^*(E) \rightarrow H_0$ . It has a natural left- $\mathcal{H}$ -vector bundle structure  $\nu : H_1 \times_{s, \phi_0^*} \phi_0^*(E) \rightarrow \phi_0^*(E)$ , by defining  $\nu(h, (x, e)) = \mu(\phi_1(h), e)$ . Its associated groupoid  $\mathcal{H} \times \phi_0^*(E)$  fits in the following commutative diagram

$$\begin{array}{ccc} \mathcal{H} \times \phi_0^*(E) & \longrightarrow & \mathcal{E} \\ \downarrow & & \downarrow \\ \mathcal{H} & \longrightarrow & \mathcal{G} \end{array}$$

Thus it makes sense to call  $\mathcal{H} \times \phi_0^*(E)$  the pullback of  $\mathcal{E}$  and denote it by  $\phi^*(\mathcal{E})$ .

**Example A.14** (Pushforward by equivalence). Suppose that  $\phi : \mathcal{H} \rightarrow \mathcal{G}$  is an equivalence between two orbifold groupoids. Then  $\phi$  as a functor is an equivalence. Applying Proposition A.8 to both  $\mathcal{H}$  and  $\mathcal{G}$ , we have that  $\phi_0$  is a local diffeomorphism. Let  $E$  be a left- $\mathcal{H}$ -vector bundle. We will now construct a left- $\mathcal{G}$ -bundle called the *pushforward* of  $E$ .

Let  $x \in G_0$  be a point in  $G_0$ . The morphism  $\phi_0$  is an equivalence implies that there exists a  $y_1 \in H_0$  and an arrow  $G_1 \ni g_1 : x \rightarrow x_1$  such that  $x_1 = \phi_0(y_1)$ . Let  $\psi_1$  be the local inverse of  $\phi_0$  at  $y_1$  and  $x_1$ . We thus get a vector bundle  $g_1^* \psi_1^*(E)$  around  $x$ . We denote the vector bundle by  $F_{x, g_1, y_1}$ . If there is another  $y_2 \in H_0$  and another arrow  $G_1 \ni g_2 : x \rightarrow x_2$  such that  $x_2 = \phi_0(y_2)$ , denoting by  $\psi_2$  the local inverse of  $\phi_0$  at  $y_2$  and  $x_2$ , we will get another vector bundle  $F_{x, g_2, y_2} := g_2^* \psi_2^*(E)$ . As  $\phi$  is an equivalence, we have a unique  $h \in H_1$  such that  $\phi_1(h) = g_1 g_2^{-1}$ . There is a canonical isomorphism  $F_{x, g_1, y_1} \rightarrow F_{x, g_2, y_2}$  induced by  $h$ .

In fact, let  $H_{y_1}, H_{y_2} \subset H_0$  be two neighbourhoods of  $y_1$  and  $y_2$  respectively such that  $h : y_2 \rightarrow y_1$  induces a diffeomorphism  $H_{y_2} \cong H_{y_1}$ . As  $E$  is a left- $\mathcal{H}$ -bundle, we have the following commutative diagram

$$\begin{array}{ccc}
E|_{H_{y_2}} & \xrightarrow{h} & E|_{H_{y_1}} \\
\downarrow & & \downarrow \\
H_{y_2} & \xrightarrow{h} & H_{y_1}
\end{array}$$

And hence a canonical isomorphism  $h : E|_{H_{y_2}} \cong h^*(E|_{H_{y_1}})$  induced by  $h$ . Now we have canonical isomorphisms:

$$g_2^* \psi_2^*(E|_{H_{y_2}}) \cong g_2^* \psi_2^* h^*(E|_{H_{y_1}}) \cong g_2^*(g_1 g_2^{-1})^* \psi_1^*(E|_{H_{y_1}}) \cong g_1^* \psi_1^*(E|_{H_{y_1}}).$$

Thus we get an open covering  $\{U_{x,g,y}\}$  of  $G_0$  and a family of vector bundles  $F_{x,g,y}$  over  $U_{x,g,y}$ . We can verify that they satisfy the cocycle condition and hence glue up to a vector bundle on  $G_0$ . From the construction, this vector bundle is actually a left- $\mathcal{G}$ -vector bundle. We call it the pushforward of  $\mathcal{E}$  and denote it by  $\phi_*(\mathcal{E})$ . We have the relation  $\phi^* \phi_*(\mathcal{E}) = \mathcal{E}$ .

If either  $\mathcal{H}$  or  $\mathcal{G}$  fails to be orbifold groupoid, the  $\phi_0$  fails in general to be local diffeomorphism. This happens, for example when we consider the Morita equivalence  $e : \mathcal{G} \leftarrow \mathcal{K} \rightarrow \mathcal{H}$ . Even  $\mathcal{G}$  and  $\mathcal{H}$  are orbifold groupoids,  $\mathcal{K}$  needs not to be an orbifold groupoid. Hence the above explicit construction can not be used on Morita equivalence. However, we have

**Proposition A.15** (cf. [MP97, page 11 Remark (4)]). *Let  $e : \mathcal{G} \leftarrow \mathcal{K} \rightarrow \mathcal{H}$  be a Morita equivalence between two topological groupoids. Then  $e$  induces an equivalence  $e^* : Sh(\mathcal{H}) \rightarrow Sh(\mathcal{G})$  between two topoi.*



## Bibliography

- [AD14] Carolina Araujo and Stéphane Druel. On codimension 1 del Pezzo foliations on varieties with mild singularities. *Math. Ann.*, 360(3-4):769–798, 2014. [24](#)
- [ALR07] Alejandro Adem, Johann Leida, and Yongbin Ruan. *Orbifolds and Stringy Topology*. Cambridge Tracts in Mathematics. Cambridge University Press, 2007. [78](#), [84](#), [85](#), [88](#), [118](#), [120](#)
- [And95] Marco Andreatta. Some remarks on the study of good contractions. *manuscripta mathematica*, 87, 1995. [9](#), [15](#), [45](#)
- [And13] Marco Andreatta. Minimal model program with scaling and adjunction theory. *Int. J. Math.*, 24(2):13, 2013. Id/No 1350007. [27](#), [44](#)
- [Aub78] Thierry Aubin. équation du type monge-ampère sur les variétés kählériennes compactes. *Bulletin des Sciences Mathématiques*, 102:63–95, 1978. [77](#)
- [Bai56] Walter L. Baily. The decomposition theorem for v-manifolds. *American Journal of Mathematics*, 78(4):862–888, 1956. [105](#)
- [BCHM10] Caucher Birkar, Paolo Cascini, Christopher D. Hacon, and James McKernan. Existence of minimal models for varieties of log general type. *J. Am. Math. Soc.*, 23(2):405–468, 2010. [37](#)
- [BDPP13] Sébastien Boucksom, Jean-Pierre Demailly, Mihai Păun, and Thomas Peternell. The pseudo-effective cone of a compact Kähler manifold and varieties of negative Kodaira dimension. *J. Algebr. Geom.*, 22(2):201–248, 2013. [34](#)
- [Bea83] Arnaud Beauville. Variétés kählériennes dont la première classe de Chern est nulle. *J. Differ. Geom.*, 18:755–782, 1983. [69](#)
- [BGT12] Emmanuel Breuillard, Ben Green, and Terence Tao. The structure of approximate groups. *Publications Mathématiques de l’IHÉS*, 116:115–221, 2012. [12](#), [18](#), [71](#), [101](#), [102](#)
- [BH99] Martin R. Bridson and André Haefliger. *Metric Spaces of Non-Positive Curvature*, volume 319 of *Grundlehren der mathematischen Wissenschaften*. Springer-Verlag Berlin Heidelberg, 1999. [74](#), [75](#), [84](#), [88](#), [106](#)
- [BM19] Daniel Barlet and Jón Magnússon. *Complex Analytic Cycles I*, volume 356 of *Grundlehren der mathematischen Wissenschaften*. Springer, Cham, 2019. [82](#)
- [Bor93] Joseph E. Borzellino. Orbifolds of maximal diameter. *Indiana Univ. Math. J.*, 42(1):37–53, 1993. [93](#), [95](#), [97](#)
- [Bra21] Lukas Braun. The local fundamental group of a Kawamata log terminal singularity is finite. *Invent. Math.*, 226(3):845–896, 2021. [71](#)
- [Bre72] Glen Bredon. *Introduction to Compact Transformation Groups*, volume 46 of *Pure and Applied Mathematics*. Academic Press, 1 edition, 1972. [93](#)

- [BS11] Mauro C. Beltrametti and Andrew J. Sommese. *The Adjunction Theory of Complex Projective Varieties*. De Gruyter, 2011. [9](#), [10](#), [15](#), [16](#), [23](#), [24](#), [41](#), [47](#)
- [Cam95] Frédéric Campana. Remarques sur les groupes de kähler nilpotents. *Annales scientifiques de l'École Normale Supérieure*, 4e série, 28(3):307–316, 1995. [12](#), [18](#), [70](#), [113](#)
- [Cam01] Frédéric Campana. Ensembles de Green-Lazarsfeld et quotients résolubles des groupes de Kähler. *J. Algebr. Geom.*, 10(4):599–622, 2001. [113](#)
- [Cao19] Junyan Cao. Albanese maps of projective manifolds with nef anticanonical bundles. *Ann. Sci. Éc. Norm. Supér. (4)*, 52(5):1137–1154, 2019. [70](#)
- [Car54] Henri Cartan. Quotient d'une variété analytique par un groupe discret d'automorphismes. *Séminaire Henri Cartan*, 6, 1953-1954. talk:12. [82](#)
- [CC96] Jeff Cheeger and Tobias H. Colding. Lower bounds on ricci curvature and the almost rigidity of warped products. *Annals of Mathematics*, 144(1):189–237, 1996. [11](#), [12](#), [17](#), [70](#)
- [CC14] Frédéric Campana and Benoît Claudon. Abelianity conjecture for special compact kähler 3-folds. *Proceedings of the Edinburgh Mathematical Society*, 57(1):55–78, 2014. [71](#), [83](#)
- [Cho04] Suhyoung Choi. Geometric structures on orbifolds and holonomy representations. *Geometriae Dedicata*, 104(1):161–199, 2004. [79](#), [80](#)
- [Cho12] Suhyoung Choi. *Geometric structures on 2-orbifolds. Exploration of discrete symmetry*, volume 27 of *MSJ Mem.* Tokyo: Mathematical Society of Japan, 2012. [79](#), [84](#), [88](#)
- [Cla07] Benoît Claudon. Déformation de variétés kählériennes compactes : invariance de la  $\Gamma$ -dimension et extension de sections pluricanoniques. Phd Thesis, 2007. [113](#)
- [Deb01] Olivier Debarre. *Higher-Dimensional Algebraic Geometry*. Universitext. Springer, New York, NY, 2001. [33](#), [35](#), [46](#), [60](#)
- [Dem12] Jean-Pierre Demailly. Complex analytic and differential geometry. <https://www-fourier.ujf-grenoble.fr/~demailly/manuscripts/agbook.pdf>, 2012. [76](#)
- [DM93] Pierre Deligne and George Daniel Mostow. *Commensurabilities among lattices in  $PU(1, n)$* , volume 132 of *Ann. Math. Stud.* Princeton, NJ: Princeton University Press, 1993. [82](#)
- [DPS93] Jean-Pierre Demailly, Thomas Peternell, and Michael Schneider. Kähler manifolds with numerically effective ricci class. *Compositio Mathematica*, 89(2):217–240, 1993. [105](#), [106](#), [109](#)
- [Eys] Philippe Eyssidieux. Private communication. [88](#)
- [Fau19] Mitchell Faulk. On yau's theorem for effective orbifolds. *Expositiones Mathematicae*, 37(4):382–409, 2019. [105](#)
- [FM21] Osamu Fujino and Keisuke Miyamoto. A characterization of projective spaces from the Mori theoretic viewpoint. *Osaka J. Math.*, 58(4):827–837, 2021. [24](#)
- [Fuj90] Takao Fujita. *Classification Theory of Polarized Varieties*. London Mathematical Society Lecture Note Series. Cambridge University Press, 1990. [23](#), [47](#), [48](#)

- [Fuj11] Osamu Fujino. Fundamental theorems for the log minimal model program. *Publ. Res. Inst. Math. Sci.*, 47(3):727–789, 2011. [32](#), [35](#)
- [Ful98] William Fulton. *Intersection theory*, volume 2. Berlin: Springer, 1998. [58](#)
- [Ful11] Mihai Fulger. The cones of effective cycles on projective bundles over curves. *Math. Z.*, 269(1-2):449–459, 2011. [53](#)
- [GHL04] Sylvestre Gallot, Dominique Hulin, and Jacques Lafontaine. *Riemannian geometry*. Berlin: Springer, 2004. [11](#), [17](#), [69](#), [75](#), [76](#)
- [GK07] Alessandro Ghigi and János Kollár. Kähler-Einstein metrics on orbifolds and Einstein metrics on spheres. *Comment. Math. Helv.*, 82(4):877–802, 2007. [82](#)
- [GR84] Hans Grauert and Reinhold Remmert. *Coherent Analytic Sheaves*, volume 265 of *Grundlehren der mathematischen Wissenschaften*. Springer-Verlag Berlin Heidelberg, 1984. [73](#)
- [Gro60] Alexander Grothendieck. Éléments de géométrie algébrique. I: Le langage des schémas. II: Étude globale élémentaire de quelques classe de morphismes. III: Étude cohomologique des faisceaux cohérents (première partie). *Publ. Math., Inst. Hautes Étud. Sci.*, 4:1–228, 1960. [30](#)
- [Gro65] A. Grothendieck. Éléments de géométrie algébrique. IV: Étude locale des schémas et des morphismes de schémas. (Seconde partie). *Publ. Math., Inst. Hautes Étud. Sci.*, 24:1–231, 1965. [38](#)
- [Gro67] A. Grothendieck. Éléments de géométrie algébrique. IV: Étude locale des schémas et des morphismes de schémas (Quatrième partie). Rédigé avec la collaboration de Jean Dieudonné. *Publ. Math., Inst. Hautes Étud. Sci.*, 32:1–361, 1967. [30](#)
- [Gro81] Mikhael Gromov. Groups of polynomial growth and expanding maps (with an appendix by jacques tits). *Publications Mathématiques de l’IHÉS*, 53:53–78, 1981.
- [Gro03] A. Grothendieck, editor. *Séminaire de géométrie algébrique du Bois Marie 1960-61. Revêtements étales et groupe fondamental (SGA 1). Un séminaire dirigé par Alexander Grothendieck. Augmenté de deux exposés de M. Raynaud.*, volume 3 of *Doc. Math. (SMF)*. Paris: Société Mathématique de France, 2003. [74](#)
- [Har77] Robin Hartshorne. *Algebraic geometry*, volume 52. Springer, Cham, 1977. [58](#), [61](#)
- [Hir38] K. A. Hirsch. On infinite soluble groups (ii). *Proceedings of the London Mathematical Society*, s2-44:336–344, 1938. [113](#)
- [Hir64a] H. Hironaka. Resolution of singularities of an algebraic variety over a field of characteristic zero. I. *Ann. Math. (2)*, 79:109–203, 1964. [31](#)
- [Hir64b] Heisuke Hironaka. Resolution of singularities of an algebraic variety over a field of characteristic zero. II. *Ann. Math. (2)*, 79:205–326, 1964. [31](#)
- [HM07] Christopher D. Hacon and James McKernan. On Shokurov’s rational connectedness conjecture. *Duke Math. J.*, 138(1):119–136, 2007. [54](#)
- [HP19] Andreas Höring and Thomas Peternell. Algebraic integrability of foliations with numerically trivial canonical bundle. *Invent. Math.*, 216(2):395–419, 2019. [34](#)



- [Huy05] Daniel Huybrechts. *Complex geometry. An introduction*. Berlin: Springer, 2005. 76
- [Kaw92] Yujiro Kawamata. Termination of log flips for algebraic 3-folds. *Int. J. Math.*, 3(5):653–659, 1992. 37
- [KM98] Janos Kollár and Shigefumi Mori. *Birational Geometry of Algebraic Varieties*. Cambridge Tracts in Mathematics. Cambridge University Press, 1998. 29, 31, 32, 35, 36, 46, 57, 60, 63, 84
- [KMM87] Yujiro Kawamata, Katsumi Matsuda, and Kenji Matsuki. Introduction to the minimal model problem. Algebraic geometry, Proc. Symp., Sendai/Jap. 1985, Adv. Stud. Pure Math. 10, 283-360 (1987)., 1987. 37
- [KO73] Shoshichi Kobayashi and Takushiro Ochiai. Characterizations of complex projective spaces and hyperquadrics. *J. Math. Kyoto Univ.*, 13:31–47, 1973. 23
- [Kob61] Shoshichi Kobayashi. On compact Kähler manifolds with positive definite Ricci tensor. *Ann. Math. (2)*, 74:570–574, 1961. 11, 17, 69
- [Kol13] János Kollár. *Singularities of the Minimal Model Program*. Cambridge Tracts in Mathematics. Cambridge University Press, 2013. 25, 29, 31, 32, 37, 38, 39, 41, 59, 64
- [Lan20] Christian Lange. Orbifolds from a metric viewpoint. *Geometriae Dedicata*, 209(1):43–57, 2020. 99
- [Laz04a] Robert Lazarsfeld. *Positivity in algebraic geometry. I. Classical setting: line bundles and linear series*, volume 48 of *Ergeb. Math. Grenzgeb., 3. Folge*. Berlin: Springer, 2004. 30, 31, 32, 33
- [Laz04b] Robert Lazarsfeld. *Positivity in Algebraic Geometry II*. *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge / A Series of Modern Surveys in Mathematics*. Springer-Verlag Berlin Heidelberg, 2004. 34
- [Ler10] Eugene Lerman. Orbifolds as stacks? *Enseign. Math. (2)*, 56(3-4):315–363, 2010. 86, 119
- [Mac05] Kirill C. H. Mackenzie. *General Theory of Lie Groupoids and Lie Algebroids*. London Mathematical Society Lecture Note Series. Cambridge University Press, 2005. 117, 118
- [ML98] Saunders Mac Lane. *Categories for the working mathematician.*, volume 5 of *Grad. Texts Math.* New York, NY: Springer, 2nd ed edition, 1998. 117
- [MM03] Ieke Moerdijk and J. Mrčun. *Introduction to foliations and Lie groupoids*, volume 91 of *Camb. Stud. Adv. Math.* Cambridge: Cambridge University Press, 2003. 117
- [Moe02] Ieke Moerdijk. Orbifolds as groupoids: an introduction, 2002. 84
- [MP97] Ieke Moerdijk and Dorret A. Pronk. Orbifolds, sheaves and groupoids. *K-Theory*, 12(1):3–21, 1997. 78, 84, 85, 86, 88, 89, 121
- [MP99] Ieke Moerdijk and Dorret A. Pronk. Simplicial cohomology of orbifolds. *Indagationes Mathematicae*, 10(2):269–293, 1999. 105
- [MW21] Shin-ichi Matsumura and Juanyong Wang. Structure theorem for projective klt pairs with nef anti-canonical divisor, 2021. 71
- [Noo14] Behrang Noohi. Fibrations of topological stacks. *Adv. Math.*, 252:612–640, 2014. 88

- [Par20] John Pardon. Enough vector bundles on orbispaces, 2020. [86](#), [103](#)
- [Pău97] Mihai Păun. Sur le groupe fondamental des variétés kählériennes compactes à classe de ricci numériquement effective. *Comptes Rendus de l'Académie des Sciences - Series I - Mathematics*, 324(11):1249–1254, 1997. [12](#), [18](#), [70](#), [71](#), [109](#)
- [Pău17] Mihai Păun. Relative adjoint transcendental classes and albanese map of compact kähler manifolds with nef ricci curvature. *Advanced Studies in Pure Mathematics*, 74:335–356, 2017. [70](#)
- [Pet16] Peter Petersen. *Riemannian Geometry*, volume 171 of *Graduate Texts in Mathematics*. Springer International Publishing, 3 edition, 2016. [75](#), [96](#)
- [PST16] Dorette Pronk, Laura Scull, and Matteo Tommasini. Atlases for ineffective orbifolds, 2016. [78](#)
- [Rei83] Miles Reid. Projective morphisms according to kawamata. 1983. [42](#)
- [Rei94] Miles Reid. Nonnormal del Pezzo surfaces. *Publ. Res. Inst. Math. Sci.*, 30(5):695–727, 1994. [40](#)
- [Sak96] Takashi Sakai. *Riemannian Geometry*, volume 149 of *Translations of Mathematical Monographs*. American Mathematical Society, 1996. [75](#), [96](#)
- [Sat56] Ichiro Satake. On a generalization of the notion of manifold. *Proceedings of the National Academy of Sciences*, 42(6):359–363, 1956. [71](#), [77](#), [78](#), [90](#)
- [Sat57] Ichiro Satake. The Gauss-Bonnet Theorem for V-manifolds. *Journal of the Mathematical Society of Japan*, 9(4):464 – 492, 1957. [77](#)
- [Seg68] Graeme Segal. Classifying spaces and spectral sequences. *Publ. Math., Inst. Hautes Étud. Sci.*, 34:105–112, 1968. [88](#)
- [Ser56] Jean-Pierre Serre. Géométrie algébrique et géométrie analytique. *Ann. Inst. Fourier*, 6:1–42, 1956. [74](#)
- [Sta22] The Stacks project authors. The stacks project. <https://stacks.math.columbia.edu>, 2022. [29](#), [38](#), [39](#), [46](#), [64](#), [65](#)
- [Sza94] Endre Szabó. Divisorial log terminal singularities. *J. Math. Sci., Tokyo*, 1(3):631–639, 1994. [31](#)
- [Szé14] Gábor Székelyhidi. *An Introduction to Extremal Kähler Metrics*, volume 152 of *Graduate Studies in Mathematics*. American Mathematical Society, 2014. [77](#)
- [Thu79] William Thurston. *Geometry and topology of three-manifolds*. Princeton University, 1979. [77](#), [79](#), [80](#)
- [Uen75] Kenji Ueno. *Classification theory of algebraic varieties and compact complex spaces. Notes written in collaboration with P. Cherenack*, volume 439. Springer, Cham, 1975. [41](#)
- [Var89] Jean Varouchas. Kähler spaces and proper open morphisms. *Math. Ann.*, 283(1):13–52, 1989. [111](#)
- [Voi02] Claire Voisin. *Hodge Theory and Complex Algebraic Geometry I*, volume 1 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, 2002. [76](#), [90](#)
- [Yau78] Shing-Tung Yau. On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation. I. *Commun. Pure Appl. Math.*, 31:339–411, 1978. [77](#)

- [Zha96] Qi Zhang. On projective manifolds with nef anticanonical bundles. *J. Reine Angew. Math.*, 478:57–60, 1996. [70](#)
- [Zha05] Qi Zhang. On projective varieties with nef anticanonical divisors. *Mathematische Annalen*, 332(3):697–703, 2005. [112](#)



**Titre :** Deux résultats sur la classification des variétés singulières à classe canonique semi-négative

**Mots clés :** variétés polarisées, singularités slc, géométrie birationnelle, orbifolles, lemme de Margulis, groupes fondamentaux

**Résumé :** Le sujet de cette thèse est d'étudier le problème de classification des espaces singuliers sous deux hypothèses différentes sur la positivité de la classe anti-canonique des espaces et de leurs singularités dans ces deux conditions différentes. Nous appliquerons des méthodes assez différentes dans ces deux contextes.

Dans la première partie, nous étudions un problème de classification des variétés polarisées. Pour la positivité des classes anti-canoniques, nous supposons que les variétés ont une nefvalue élevée, ou en d'autres termes, leurs classes anti-canoniques sont assez positives. Nous donnons une liste complète des classes d'isomorphisme des variétés polarisées normales avec une nefvalue élevée. Cela généralise le travail classique sur le cas lisse de Fujita, Beltramitti et Sommese. En conséquence, nous obtenons que les variétés polarisées avec des singularités slc et une nefvalue élevée sont birationnellement équivalentes à des fibrés projectifs sur des courbes nodales.

Dans la deuxième partie, nous considérons une classe spécifique d'espaces singuliers, à savoir les orbifolles. Une orbifolle a des singularités quotients. Par conséquent, nous avons des singularités mieux contrôlées dans ce contexte par rapport à celles considérées dans la première partie. Nous supposons également que ces orbifolles sont kähleriennes compactes avec des classes anti-canoniques nef au sens des orbifolles. Nous étudions la topologie de ces orbifolles à travers leurs groupes fondamentaux orbifolles. Dans cette partie, nous exploiterons pleinement l'hypothèse orbifolle en appliquant des résultats de géométrie différentielle et de la géométrie métrique sur orbifolds. Nous montrerons qu'une orbifolle kählerienne compacte dont la classe anti-canonique est nef a un groupe fondamental orbifolle virtuellement nilpotent.

**Title :** Two results on the classification of singular spaces with semi-negative canonical class

**Keywords :** polarized varieties, slc singularities, birational geometry, orbifolds, Margulis lemma, fundamental groups

**Abstract :** The subject of this thesis is to study the classification problem for singular spaces under two different assumptions on the positivity of the anti-canonical class of the spaces and their singularities in these two different setups. We will apply quite different methods for these two assumptions.

In the first part, we study the classification problem for polarized varieties. For the positivity of the anti-canonical classes, we assume that the varieties have high nefvalue, or in other words, their anti-canonical classes are quite positive. We give a complete list of isomorphism classes for normal polarized varieties with high nefvalue. This generalizes classical work on the smooth case by Fujita, Beltramitti and Sommese. As a consequence we obtain that polarized varieties with slc singularities and high nefvalue, are birationally equivalent to projective bundles over nodal curves.

In the second part, we consider a specific class of singular spaces, namely the orbifolds. An orbifold has quotient singularities. Hence we have milder singularities in this context compared to those considered in first part. We also assume that these orbifolds are compact Kähler with nef anti-canonical classes in the orbifold sense. We will study the topology of these orbifolds by characterizing their orbifold fundamental groups. In this part, we will fully exploit the orbifold assumption by applying results from differential geometry and metric geometry on orbifolds. We will show that a compact Kähler orbifold with nef anti-canonical class has virtually nilpotent orbifold fundamental group.