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L'UNIVERSITE DE RENNES 1

ECOLE DOCTORALE N° 601
Mathématiques et Sciences et Technologies
de l'Information et de la Communication
Spécialité : *Mathématiques et leurs interactions*

Par
Zhining LIU

Deux résultats sur la classification des variétés singulières à classe canonique semi-négative

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Two results on the classification of singular spaces with semi-negative canonical class

Zhining Liu

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Abstract

Abstract. The subject of this thesis is to study the classification problem for singular spaces under two different assumptions on the positivity of the anti-canonical class of the spaces and their singularities in these two different setups . We will apply quite different methods for these two assumptions.

In the first part, we study the classification problem for polarized varieties. For the positivity of the anti-canonical classes, we assume that the varieties have high nefvalue, or in other words, their anti-canonical classes are quite positive. We give a complete list of isomorphism classes for normal polarized varieties with high nefvalue. This generalizes classical work on the smooth case by Fujita, Beltrametti and Sommese. As a consequence we obtain that polarized varieties with slc singularities and high nefvalue, are birationally equivalent to projective bundles over nodal curves.

In the second part, we consider a specific class of singular spaces, namely the orbifolds. An orbifold has quotient singularities. Hence we have milder singularities in this context compared to those considered in first part. We also assume that these orbifolds are compact Kähler with nef anti-canonical classes in the orbifold sense. We will study the topology of these orbifolds by characterizing their orbifold fundamental groups. In this part, we will fully exploit the orbifold assumption by applying results from differential geometry and metric geometry on orbifolds. We will show that a compact Kähler orbifold with nef anti-canonical class has virtually nilpotent orbifold fundamental group.

Keywords: polarized varieties, nefvalue, slc singularities, birational geometry, orbifolds, Margulis lemma, fundamental groups

Résumé. Le sujet de cette thèse est d'étudier le problème de classification des espaces singuliers sous deux hypothèses différentes sur la positivité de la classe anti-canonique des espaces et de leurs singularités dans ces deux conditions différentes. Nous appliquerons des méthodes assez différentes dans ces deux contextes.

Dans la première partie, nous étudions un problème de classification des variétés polarisées. Pour la positivité des classes anti-canoniques, nous supposons que les variétés ont une *nefvalue* élevée, ou en d'autres termes, leurs classes anti-canoniques sont assez positives. Nous donnons une liste complète des classes d'isomorphisme des variétés polarisées normales avec une nefvalue élevée. Cela généralise le travail classique sur le cas lisse de Fujita, Beltramitti et Sommese. En conséquence, nous obtenons que les variétés polarisées avec des singularités slc et une nefvalue élevée sont birationnellement équivalentes à des fibrés projectifs sur des courbes nodales.

Dans la deuxième partie, nous considérons une classe spécifique d'espaces singuliers, à savoir les orbifoldes. Une orbifolde a des singularités quotients. Par conséquence, nous avons des singularités mieux contrôlées dans ce contexte par rapport à celles considérées dans la première partie. Nous supposons également que ces orbifoldes sont kähleriennes compactes avec des classes anti-canoniques nef au sens des orbifoldes. Nous étudierons la topologie de ces orbifoldes à travers leurs groupes fondamentaux orbifoldes. Dans cette partie, nous exploiterons pleinement l'hypothèse orbifolde en appliquant des résultats de géométrie différentielle et de la géométrie métrique sur orbifolds. Nous montrerons qu'une orbifolde kählerienne compacte dont la classe anti-canonique est nef a un groupe fondamental orbifolde virtuellement nilpotent.

Mots-clés: variétés polarisées, nefvalue, singularités slc, géométrie birationelle, orbifoldes, lemme de Margulis, groupes fundamentaux

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Résumé étendu

Partie 1: Classification des variété à singularités slc de *valeur nef* élevées

Quand on étudie une variété polarisée (X, L) avec X ayant singularité klt, il se trouve que le *morphisme de valeur nef* de (X, L) est utile. Supposons que K_X n'est pas nef et posons $\tau(L) := \inf\{t \in \mathbb{Q} : K_X + tL \text{ est nef}\}$. Par le théorème sur l'absense de point base de Kawamata, on a que $K_X + \tau(L)L$ est sans point base. Pour un entier m suffisammment divisible, on sait que le système linéaire $|m(K_X + \tau(L)L|$ définit un morphisme $\phi_{|m(K_X+\tau(L)L|} : X \to \mathbb{P}_{\mathbb{C}}$ de X vers un espace projectif dont la factorisation de Stein $\phi : X \to Y$ ne dépend pas de m. On a $L = \phi^*(\mathcal{O}_Y(1))$. On peut comprendre (X, L) en etudiant les propriétés des fibres générales de ϕ.

Par exemple, Andreatta a prouvé le résultat suivant dans [And95].

Theorem 0.1 ([And95, Theorem 2.1.]). *Soit X une variété projective à singularités klt et soit L un fibré en droites sur X. Soit $\phi : X \to Z$ un morphisme surjectif à fibres connexes entre variétés normales. Supposons que L est ϕ-ample et $K_X + \tau L \sim_{\mathbb{Q},\phi} 0$ pour certain $\tau \in \mathbb{Q}^+$. Soient $F_1 = \phi^{-1}(z)$ une fibre non-triviale, $F \subset F_1$ une des ces composants irréductibles, et F' la normalisation de F. On note par L' l'image réciproque de L sur F'. Soient $\lfloor \tau \rfloor$ la partie entière de τ et $\tau' := \lceil \tau \rceil = -\lfloor -\tau \rfloor$. Alors, on a alors:*

(I,1) $\dim(F) \geq \tau - 1$;
(I,2) si $\dim(F) < \tau$, *alors* $F \cong \mathbb{P}^{\tau'-1}$ *et* $L|_F = \mathcal{O}_{\mathbb{P}^{\tau'-1}}(1)$;
(I,3) si $\dim(F) < \tau + 1$, *alors* $\Delta(F', L') = 0$.

Si de plus on a $\dim(F) > \dim(X) - \dim(Z)$, *alors:*

(II,1) $\dim(F) \geq \tau$;
(II,2) si $\dim(F) = \tau$, *alors* $F \cong \mathbb{P}^{\tau}$ *et* $L|_F = \mathcal{O}_{\mathbb{P}^{\tau}}(1)$;
(II,3) si $\dim(F) < \tau + 1$, *alors* $\Delta(F', L') = 0$,

Si toutes les composantes de la fibre F_1 satisfont $\dim(F) < \tau(L)$, *dans cas (I.2) ou* $\dim(F) \leq \tau(L)$ *dans cas (II.3), alors F_1 est en fait irréductible.*

On voit de ce théorème que $\tau(L) > \dim(X) - 1$ est une condition très restrictive sur (X, L). En fait, quand X a singularités terminales, la classification pour (X, L) est complète pour L des valeur nef supérieure à $n - 1$.

Proposition 0.2 ([BS11, Proposition 7.2.2.]). *Soit (X, L) une variété polarisée. Supposons X à singularités terminales. Soit $\phi : X \to Z$ le morphisme valeur nef. Alors on est dans l'un des cas suivants:*

(1) $\tau = n + 1$ *et* $(X, L) \cong (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$;
(2) $\tau = n$ *et* $(X, L) \cong (Q, \mathcal{O}_Q(1))$ *où* $Q \subset \mathbb{P}^{n+1}$ *est une hyperquadrique dans* $\mathbb{P}^{n+1}$ *si* $K_X + nL \equiv_{\text{num}} \mathcal{O}_X$;

(3) $\tau = n$ *et* Z *est une courbe lisse et* $(X, L) \cong (\mathbb{P}(E), \mathcal{O}_{\mathbb{P}(E)}(1))$ *où* E *est un fibré vectoriel sur* Z *et* $\phi : X \to Z$ *est le morphisme structurel;*
(4) $\tau < n$ *et* $K_X + nL$ *est nef et big.*

Proposition 0.3 ([BS11, Proposition 7.2.4.])**.** *Soit* (X, L) *une variété polarisée. Supposons* X *à singularités terminales* $\mathbb{Q}$*-factorielles et* $n = \dim(X) \geq 2$*. Supposons que* $K_X + nL$ *est ample et* $\tau > n-1$*. Alors nous avons* $\tau = n - \frac{1}{2}$ *et* $(X, L) = C_n(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$ *est un cône généralisé sur* $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$*.*

Dans la partie 1, nous explorerons d'abord le problème de classification pour les variétés polarisées normales (X, L) avec nefvalue élévée en autorisant des singularités plus "sauvages" que terminales pour X. Nous donnerons la classification suivante.

Theorem 0.4 (=Theorem 3.14). *Soit* (X, L) *une variété polarisée de dimension* n*. Supposons que* K_X *est* $\mathbb{Q}$*-Cartier et* $K_X + (n-1)L \notin \mathrm{Pseff}(X)$*. Alors* (X, L) *est l'une des paires suivantes:*

(1) $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$*;*
(2.i) $(\mathbb{P}(\mathcal{V}), \mathcal{O}_{\mathbb{P}(\mathcal{V})}(1))$*, où* $\mathcal{V}$ *est un fibré vectoriel ample de rang* n *sur une courbe lisse* C*;*
(2.ii) $C_n(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(a))$*, un cône généralisé avec* $a \geq 3$*;*
(3) $(Q, \mathcal{O}_{\mathbb{P}^{n+1}}(1))$*, où* $Q \subset \mathbb{Q}^{n+1}$ *est une hyperquadrique;*
(4) $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$*;*
(5) un cône généralisé $C_n(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$ *sur* $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$*.*

La stratégie de cette classification consiste tout d'abord à établir une classification pour (X', L') quasi-polarisée avec X' à singularités canoniques et $K_{X'} + (\dim(X') - 1)L') \notin \mathrm{Pseff}(X')$. Ceci occupe la Section 3.1. Le point clé est d'utiliser le programme de modèles minimaux (MMP) pour réduire le problème à la classification de (X'', L'') avec $\tau(L'') > \dim(X'') - 1$. Nous pouvons ensuite appliquer une modification canonique $\mu : X^{\mathrm{can}} \to X$ et réduire le problème à la classification de la variété quasi-polarisée $(X^{\mathrm{can}}, \mu^*(L))$.

Avec la méthode similaire, nous étalibrons aussi un résultat de la classification pour une paire log canonique (X, Δ) avec $(K_X + \Delta) + (\dim(X) - 1)L \notin \mathrm{Pseff}(X)$.

Proposition 0.5 (=Corollary 3.19)**.** *Soit* (X, Δ) *une paire log canonique avec* $\Delta \neq 0$ *un diviseur réduit. Supposons que* L *est un fibré en droites ample sur* X *et* $(K_X + \Delta) + (n-1)L \notin \mathrm{Pseff}(X)$*, où* $n = \dim(X)$*. Alors nous avons l'un des cas suivants:*

(1) $(X, L) \cong (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$*,* $\Delta \equiv_{\mathrm{num}} H$ *est un diviseur irréductible où* H *est un hyperplan de* $\mathbb{P}^n$*;*
(2.i) il existe un $(\mathbb{P}^{n-1}, \mathcal{O}_{\mathbb{P}^{n-1}}(1))$*-fibré* $(\mathbb{P}(E), \mathcal{O}_{\mathbb{P}(E)}(1))$ *sur une courbe lisse* C*, et un morphisme birationelle* $\mu : \mathbb{P}(E) \to X$ *tel que* $\mu^*(L) \cong \mathcal{O}_{\mathbb{P}(E)}(1)$ *et* $\Delta = \sum F_i$ *est une somme finie où* $F_i \cong \mathbb{P}^{n-1}$ *sont les images de fibres générales distinctes de* π *par* μ*;*
(2.ii) $(X, L) = (\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(1)), \mathcal{O}_{\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(1))}(1))$ *avec* $1 < a$ *et* $\Delta = D$ *est irréductible, où* D *est l'unique section de* $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(1)) \to \mathbb{P}^1$ *tel que* $D \equiv_{\mathrm{num}} \mathcal{O}_{\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(1))}(1)) - af$*, où* f *est une fibre générale;*

(3.i) $(X, L) \cong (Q, \mathcal{O}_{\mathbb{P}^{n+1}}(1))$, *où* $Q \subset \mathbb{P}^{n+1}$ *est une hyperquadrique de rang* 3, *le diviseur* Δ *est un hyperplan dans* Q *et* $[\Delta] = \frac{1}{2}[H \cap Q]$ *où* H *est un hyperplan dans* $\mathbb{P}^{n+1}$*;*

(3.ii) $(X, L) \cong (Q, \mathcal{O}_{\mathbb{P}^{n+1}}(1))$, *où* $Q \subset \mathbb{P}^{n+1}$ *est une hyperquadrique de rang* 4. *Si nous écrivons* $Q = \mathrm{Proj}\left(\frac{\mathbb{C}[x_0, \ldots, x_{n+1}]}{(x_0x_1 - x_2x_3)}\right)$, *alors* $\Delta = D$ *est irréductible et* D *est le cône de sommet* $\mathbb{P}^{n-3}$ *sur* $\mathbb{P}^1 \times \mathrm{pt}$ *ou* $\mathrm{pt} \times \mathbb{P}^1$. *En particulier,* $D \cong \mathbb{P}^{n-1}$*;*

Finalement, nous affranchissons de la condition de normalité sur les variétés. Pour une variété non-normale, nous pouvons toujours considérer sa normalisation $\nu : \bar{X} \to X$. Quand X a des singularitiés slc, nous avons que $(\bar{X}, \bar{D})$ est log canonique où $\bar{D}$ est le *diviseur conducteur*. Nous donnons la classification suivante.

THEOREM 0.6 (=Proposition 3.20). *Soient* X *une variété non-normale projective de dimension* n *à singularitiés slc et* L *un fibré en droites ample sur* X. *Supposons que* $K_X + (n-1)L \notin \mathrm{Pseff}(X)$. *Soient* $\pi : \bar{X} \to X$ *la normalisation de* X *et* $D \subset X$, $\bar{D} \subset \bar{X}$ *les conducteurs. Alors nous avons:*

Il existe une courbe nodale C', *un fibré vectoriel* E' *de rang* n, *des fibres distinctes* $F_1, F_2, \ldots, F_m$ *de* $\mathbb{P}(E')$ *et un morphisme birationelle* $\mu : \mathbb{P}(E') \to X$ *tel que* $\mu^*(L) = \mathcal{O}_{\mathbb{P}(E')}(1)$ *et* $D = \sum_{1 \le i \le m} \mu(F_i)$.

Partie 2: Groupes fondamentaux des orbifoldes kählériennes à fibr'es anticanoniques nef

Etant donné une variété compacte kählérienne X, le fameux théorème de Calabi-Yau nous dit que nous pouvons trouver une forme de Kähler ω sur X tel que la première classe de Chern $c_1(X) = c_1(-K_X)$ est représentée par $\frac{1}{2\pi}\mathrm{Ricci}_\omega$. Donc la positivité du fibré anti-canonique de X se traduit en certaines positivitiés de la courbure de Ricci de ω. Lors de l'étude dugroupe fondamental de X avec certaines conditions de positivitiés sur $-K_X$, nous pouvons nous appuyer sur des résultats sur de géométrie differentielle pour les variétés riemanniennes avec certaines conditions de positivitiés sur leurs courbures de Ricci.

Nous donnons une version reformulée d'un théorème de Kobayashi.

THEOREM 0.7 ([Kob61, Theorem A]). *Une variété compacte de Fano est simplement connexe.*

Avec l'aide du théorème de Calabi-Yau, le théorème de Kobayashi peut être prouvé en appliquant le théorème de Myers (*cf.* [GHL04, 3.85]).

Quand $-K_X$ est nef, nous pouvons, avec l'aide du théorème de Aubin-Yau (*cf.* Theorem 5.15), montrer que pour chaque $\epsilon > 0$, il existe une forme de Kähler ω_ϵ tel que

$$\mathrm{Ricci}_{\omega_\epsilon} \ge -\epsilon\omega_\epsilon.$$

En multipliant ω_ϵ par un scalaire $0 < \lambda \ll 1$ et posant $\omega'_\epsilon := \lambda\omega_\epsilon$, nous avons $\mathrm{Ricci}_{\omega'_\epsilon} \ge -(\dim(X) - 1)\omega'_\epsilon$. Dans [CC96], Cheeger et Colding ont prouvé un résultat profond

Lemme 0.8 (Lemme de Margulis, version géométrique, [CC96, Theorem 8.7.])**.** *Soit $n > 0$ un entier naturel. Il existe un constante universel $C = C(n)$ dépendant seulement de n tel que:*

Pour chaque variété compacte M de dimension n et $\mathrm{Ric}_g \geq -(n-1)g$, on a que le morphisme induit par l'inclusion

$$\pi_1(B_g(p,r),p) \to \pi_1(M,p)$$

a une image virtuellement nilpotente pour tout $r < C(n)$.

En utilisant le thèorème de Aubin-Yau pour construire une metrique spécifique ω sur X, tel que

(1) $\mathrm{Ricci}_\omega \geq -(2\dim_{\mathbb{C}}(X) - 1)\omega$ et
(2) il existe $r_0 < C(2\dim_{\mathbb{C}}(X))$ tel que $B_\omega(p, r_0) = X$.

et appliquant le lemme au-desus directement, Mihai Păun a prouvé le résultat suivant

Theorem 0.9 ([Pău97, Theorem 1]). *Soit (X, ω) une variété compacte kählérienne dont le fibré anti-canonique $-K_X$ est nef. Alors son groupe fondamental $\pi_1(X)$ est virtuellement nilpotent.*

Dans la partie 2, le but principal est de généraliser le résultat de Păun au cadre orbifoldes. Supposons que nous avons une version pour orbifolde du résultat de Cheeger et Colding. Nous pouvons ensuite adapter la preuve de Păun facilement, comme la géométrie differentielle des orbifoldes est bien établie. Équipons l'espace sous-jacent $X = |\mathcal{X}|$ d'une orbifolde riemannienne $(\mathcal{X}, g)$ avec une distance natuelle d (*cf.* Section 6.2), et appliquons une version algébrique du lemme de Margulis dans [BGT12]. Nous prouvons d'abord un lemme de Margulis orbifolde.

Lemme 0.10 (=Lemma 7.12)**.** *Soit $n \geq 1$ un entier. Il existe $\alpha = \alpha(n) > 0$ tel que ce qui suit est vrai. Si $\mathcal{X}$ est une orbifolde riemannienne complète avec sa courbure de Ricci minoré par $\mathrm{Ric} \geq -(n-1)$ et Γ un sous groupe de $\mathrm{Isom}(|\mathcal{X}|)$ agisant proprement discontinuement sur $|\mathcal{X}|$. Alors pour tout $x \in |\mathcal{X}|$, le "presque-stabiliseur"*

$$\Gamma_\alpha(x) := \langle \{\gamma \in \Gamma \mid \mathrm{d}(\gamma \cdot x, x) < \alpha\} \rangle$$

est virtuellement nilpotent.

Nous pouvons donc imiter la démonstration de Păun pour montrer que:

Theorem 0.11 (=Theorem 4.10=Theorem 8.13). *Soit $(\mathcal{X}, \omega)$ une orbifolde effective kählérienne à l'espace sous-jacent $X = |\mathcal{X}|$ compact. Si le orbi-fibré anti-canonique $K_{\mathcal{X}}^{-1}$ est nef, alors $\pi_1^{\mathrm{orb}}(\mathcal{X})$ est virtuellement nilpotent.*

Pour une variété compacte kählérienne X, on peut considérer son morphisme d'Albanese $\mathrm{Alb}_X : X \to A(X)$ (voir Definition 2.33). Posons $Y := \mathrm{Alb}_X(X) \subset A(X)$ l'image de X dans $A(X)$ et $r : \tilde{Y} \to Y$ un modèle lisse de Y. Frédéric Campana a montré comment on peut décrire la suite centrale de $\pi_1(X)$ par la suite centrale de $\pi_1(\tilde{Y})$ dans [Cam95, Théorème 2.2]. En particulier, nous observons que pour une variété compacte kählerienne X dont le groupe fondamental group $\pi_1(X)$ est virtuellement nilpotent, si Alb_X est surjectif à fibres connexes, nous avons que $\pi_1(X)$ est virtuellement abélien.

Pour une orbifolde compacte $\mathcal{X}$, nous pouvons considerer sa representation en log paire (X, Δ_X), où Δ_X est un $\mathbb{Q}$-diviseur à coefficients standard. Pour toute paire

(X, Δ_X) à singularités klt, nous pouvons définir un groupe fondamental $\pi_1(X, \Delta_X)$, et quand la paire represente une orbifolde $\mathcal{X}$, il existe un isomorphe canonique $\pi_1(X, \Delta_X) \cong \pi_1^{\mathrm{orb}}(\mathcal{X})$ (*cf.* Proposition 5.50). Quand X est lisse, nous avons une surjection $\pi_1(X, \Delta_X) \twoheadrightarrow \pi_1(X)$. Donc si $\pi_1^{\mathrm{orb}}(\mathcal{X})$ est virtuellement nilpotent et X est lisse tel que Alb_X est surjectif à fibres connexes, nous peut montrer que $\pi_1^{\mathrm{orb}}(\mathcal{X})$ est virtuallement abélien. Pour une orbifolde générale (X, Δ_X) avec X non nécessairement lisse, nous considérons une résolution $r : Y \to X$. Suivant cette idée, nous prouvons le résultat suivant.

THEOREM 0.12 (=Theorem 4.11=Theorem 9.1). *Soit* (X, Δ) *une orbifolde projective avec* $-(K_X + \Delta)$ *nef. Le groupe fondamental* $\pi_1(X, \Delta)$ *est virtuellement abélien.*

Summary

Part 1: Classification of slc varieties with high nef value

When studying a polarized variety (X, L) with X having klt singularities, a useful tool is its nefvalue morphism. Suppose that K_X is not nef and set

$$\tau(L) := \inf\{t \in \mathbb{Q} : K_X + tL \text{ is nef }\}.$$

Then by Kawamata basepoint-free theorem, we have that $K_X + \tau(L)L$ has no base-point. For $m > 0$ divisible enough, we know that the linear system $|m(K_X + \tau(L)L|$ defines a morphism $\phi_{|m(K_X+\tau(L)L|} : X \to \mathbb{P}_{\mathbb{C}}$ from X to some projective spaces, whose Stein factorization $\phi : X \to Y$ dose not depend on m. We have that $L = \phi^*(\mathcal{O}_Y(1))$. We can study (X, L) by studying the properties of general fibers of ϕ.

For example, Andreatta proved the following result in [And95].

Theorem 0.13 ([And95, Theorem 2.1.]). *Let X be a projective variety with klt singularities and let L be a line bundle on X. Let $\phi : X \to Z$ be a surjective morphism between normal varieties with connected fiber. Suppose that L is ϕ-ample and $K_X + \tau L \sim_{\mathbb{Q},\phi} 0$ for some $\tau \in \mathbb{Q}^+$. Let $F_1 = \phi^{-1}(z)$ be a non-trivial fiber, $F \subset F_1$ be one of its irreducible components, F' be the normalization of F and let L' be the pullback of L on F'. Let $\lfloor \tau \rfloor$ be the integral part of τ and $\tau' = \lceil \tau \rceil = -\lfloor -\tau \rfloor$. Then we have the following*

(I,1) $\dim(F) \geq \tau - 1$*;*
(I,2) If $\dim(F) < \tau$*, then* $F \cong \mathbb{P}^{\tau'-1}$ *and* $L|_F = \mathcal{O}_{\mathbb{P}^{\tau'-1}}(1)$*;*
(I,3) If $\dim(F) < \tau + 1$*, then* $\Delta(F', L') = 0$*,*

If moreover $\dim(F) > \dim(X) - \dim(Z)$*, then*

(II,1) $\dim(F) \geq \tau$*;*
(II,2) If $\dim(F) = \tau$*, then* $F \cong \mathbb{P}^{\tau}$ *and* $L|_F = \mathcal{O}_{\mathbb{P}^{\tau}}(1)$*;*
(II,3) If $\dim(F) < \tau + 1$*, then* $\Delta(F', L') = 0$*,*

If all components of the fiber F satisfy $\dim(F) < \tau(L)$*, in case (I.2) or* $\dim(F) \leq \tau(L)$ *in case (II.3), then F is actually irreducible.*

We see from the theorem that $\tau(L) > \dim(X) - 1$ will be a very restrictive condition. In fact, when X has terminal singularities, the classification for (X, L) is complete when the nefvalue of L is larger than $n - 1$.

Proposition 0.14 ([BS11, Proposition 7.2.2.])**.** *Let (X, L) be a polarized variety. Suppose that X has terminal singularities. Let $\phi : X \to Z$ be the nefvalue morphism. Then we have one of the following*

(1) $\tau = n + 1$ *and* $(X, L) \cong (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$*;*

(2) $\tau = n$ *and* $(X, L) \cong (Q, \mathcal{O}_Q(1))$ *where* $Q \subset \mathbb{P}^{n+1}$ *is a hyperquadric in* $\mathbb{P}^{n+1}$ *if* $K_X + nL \equiv_{\text{num}} \mathcal{O}_X$*;*

(3) $\tau = n$ *and* Z *is a smooth curve and* $(X, L) \cong (\mathbb{P}(E), \mathcal{O}_{\mathbb{P}(E)}(1))$ *where* E *is a vector bundle over* Z *and* $\phi : X \to Z$ *is the structure morphism;*

(4) $\tau < n$ *and* $K_X + nL$ *is nef and big.*

Proposition 0.15 ([BS11, Proposition 7.2.4.])**.** *Let* (X, L) *be a polarized variety. Suppose that* X *has* $\mathbb{Q}$*-factorial terminal singularities and* $n = \dim(X) \geq 2$*. Suppose that* $K_X + nL$ *is ample and* $\tau > n - 1$*. Then we have* $\tau = n - \frac{1}{2}$ *and* $(X, L) = C_n(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$ *is a generalized cone over* $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$*.*

In Part 1, we will first explore the classification problem for a normal polarized variety (X, L) with high nefvalue by allowing wilder singularities than terminal for X. We will give the following classification.

THEOREM 0.16 (=Theorem 3.14)*. Let* (X, L) *be a polarized variety of dimension* n*. Suppose that* K_X *is* $\mathbb{Q}$*-Cartier and* $K_X + (n-1)L \notin \text{Pseff}(X)$*. Then* (X, L) *is one of the following:*

(1) $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$*;*

(2.i) $(\mathbb{P}(\mathcal{V}), \mathcal{O}_{\mathbb{P}(\mathcal{V})}(1))$*, where* $\mathcal{E}$ *is a rank* n *ample vector bundle over a smooth curve* C*;*

(2.ii) $C_n(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(a))$ *be a generalized cone with* $a \geq 3$*;*

(3) $(Q, \mathcal{O}_{\mathbb{P}^{n+1}}(1))$*, where* $Q \subset \mathbb{Q}^{n+1}$ *is a hyperquadric;*

(4) $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$*;*

(5) a generalized cone $C_n(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$ *over* $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$*.*

The strategy of this classification is first to establish a classification for quasi-polarized (X', L') with X' having canonical singularities and $K_{X'} + (\dim(X') - 1)L') \notin \text{Pseff}(X')$. This is done in Section 3.1. The key point is to run an MMP to reduce the problem to classifying (X'', L'') with $\tau(L'') > \dim(X'') - 1$. We may then apply a canonical modification $\mu : X^{\text{can}} \to X$ and reduce the problem to the classification for quasi-polarized $(X^{\text{can}}, \mu^*(L))$.

With similar method, we will also establish a classification result for a log canonical pair (X, Δ) with $(K_X + \Delta) + (\dim(X) - 1)L \notin \text{Pseff}(X)$.

Proposition 0.17 (=Corollary 3.19)**.** *Let* (X, Δ) *be a log canonical pair, with* $\Delta \neq 0$ *a reduced divisor. Suppose that* L *is an ample line bundle on* X *and* $(K_X + \Delta) + (n-1)L \notin \text{Pseff}(X)$*, where* $n = \dim(X)$*. Then we have one of the following:*

(1) $(X, L) \cong (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$*,* $\Delta \equiv_{\text{num}} H$ *is a prime divisor where* H *is a hyperplane of* $\mathbb{P}^n$*;*

(2.i) There is a $(\mathbb{P}^{n-1}, \mathcal{O}_{\mathbb{P}^{n-1}}(1))$*-bundle* $(\mathbb{P}(E), \mathcal{O}_{\mathbb{P}(E)}(1))$ *over a smooth curve* C*, and a birational morphism* $\mu : \mathbb{P}(E) \to X$ *such that* $\mu^*(L) \cong \mathcal{O}_{\mathbb{P}(E)}(1)$ *and* $\Delta = \sum F_i$ *is a finite sum where* $F_i \cong \mathbb{P}^{n-1}$ *are images of distinct general fibers of* π *by* μ*;*

(2.ii) $(X, L) = (\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(1)), \mathcal{O}_{\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(1))}(1))$ *with* $1 < a$ *and* $\Delta = D$ *is irreducible, where* D *is the unique section of* $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(1)) \to \mathbb{P}^1$ *such that* $D \equiv_{\text{num}} \mathcal{O}_{\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(1))}(1)) - af$*, where* f *is a general fiber;*

(3.i) $(X, L) \cong (Q, \mathcal{O}_{\mathbb{P}^{n+1}}(1))$, *where* $Q \subset \mathbb{P}^{n+1}$ *is a* $\mathrm{rk}(Q) = 3$ *hyperquadric, the boundary divisor* Δ *is a hyperplane in* Q *and* $[\Delta] = \frac{1}{2}[H \cap Q]$ *where* H *is a hyperplane in* $\mathbb{P}^{n+1}$*;*

(3.ii) $(X, L) \cong (Q, \mathcal{O}_{\mathbb{P}^{n+1}}(1))$, *where* $Q \subset \mathbb{P}^{n+1}$ *is a* $\mathrm{rk}(Q) = 4$ *hyperquadric. If we write* $Q = \mathrm{Proj}\left(\frac{\mathbb{C}[x_0, \ldots, x_{n+1}]}{(x_0x_1 - x_2x_3)}\right)$, *then* $\Delta = D$ *is prime and* D *is the cone with vertex* $\mathbb{P}^{n-3}$ *over* $\mathbb{P}^1 \times \mathrm{pt}$ *or* $\mathrm{pt} \times \mathbb{P}^1$. *In particular,* $D \cong \mathbb{P}^{n-1}$*;*

Finally, we may drop the normality condition. For a non-normal variety, we can always consider its normalization $\nu : \bar{X} \to X$. When X has slc singularities, we have that $(\bar{X}, \bar{D})$ is log canonical where $\bar{D}$ is the conductor divisor. We give the following classification.

THEOREM 0.18 (=Proposition 3.20). *Let* X *be a non-normal slc projective variety of dimension* n *and* L *an ample line bundle over* X. *Suppose that* $K_X + (n-1)L \notin \mathrm{Pseff}(X)$. *Let* $\pi : \bar{X} \to X$ *be the normalization of* X *and* $D \subset X$, $\bar{D} \subset \bar{X}$ *the conductors. Then we have:*

There is a nodal curve C'*,a rank* n*-vector bundle* E', *distinct fibers* $F_1, F_2, \ldots, F_m$ *of* $\mathbb{P}(E')$ *and a birational morphism* $\mu : \mathbb{P}(E') \to X$ *such that* $\mu^*(L) = \mathcal{O}_{\mathbb{P}(E')}(1)$ *and* $D = \sum_{1 \le i \le m} \mu(F_i)$.

Part 2: Fundamental group of Kähler orbifolds with nef anti-canonical bundle

Given a compact Kähler manifold X, the celebrated Calabi-Yau theorem tells us that we can find a Kähler form ω on X such that the first Chern class $c_1(X) = c_1(-K_X)$ can be represented by $\frac{1}{2\pi}\mathrm{Ricci}_\omega$. Hence the positivity of the anti-canonical bundle of X can be translated to some positivity of the Ricci curvature of X. When studying the fundamental group of X with certain positivity conditions on $-K_X$, we may tap the results on differential geometry for Riemannian manifolds with certain positivity conditions on its Ricci curvature.

We give a reformulated version of Kobayashi's theorem

THEOREM 0.19 ([Kob61, Theorem A]). *A compact Fano manifold is simply connected.*

With Calabi-Yau theorem, this theorem could be proved by using Myers' theorem (*cf.* [GHL04, 3.85]).

When $-K_X$ is nef, we can, with the help of the Aubin-Yau theorem (*cf.* Theorem 5.15), show that for each $\epsilon > 0$, there exists a Kähler form ω_ϵ such that

$$\mathrm{Ricci}_{\omega_\epsilon} \ge -\epsilon\omega_\epsilon.$$

By multiplying ω_ϵ with a scalar $0 < \lambda \ll 1$ and setting $\omega'_\epsilon := \lambda\omega_\epsilon$, we will have that $\mathrm{Ricci}_{\omega'_\epsilon} \ge -(\dim(X) - 1)\omega'_\epsilon$. In [CC96], Cheeger and Colding proved a deep result:

Lemma 0.20. *(Geometric Margulis lemma, [CC96, Theorem 8.7.]) Let* $n > 0$ *be a natural number. There exists a universal constant* $C = C(n)$ *only depending on* n *such that the following holds:*

For any compact manifold M of dimension n and $\mathrm{Ric}_g \geq -(n-1)g$*, one has that the morphism induced by the inclusion*

$$\pi_1(B_g(p,r),p) \to \pi_1(M,p)$$

has virtually nilpotent image for any $r < C(n)$.

By using Aubin-Yaun theorem to construct a specific metric ω on X, such that

(1) $\mathrm{Ricci}_\omega \geq -(2\dim_{\mathbb{C}}(X)-1)\omega$ and
(2) there exists $r_0 < C(2\dim_{\mathbb{C}}(X))$ such that $B_\omega(p,r_0) = X$.

and applying the above lemma directly, Mihai Păun proved the following result:

THEOREM 0.21 ([Pău97, Theorem 1]). *Let (X,ω) be a compact Kähler manifold whose anti-canonical bundle $-K_X$ nef. Then its fundamental group $\pi_1(X)$ is virtually nilpotent.*

In Part 2, the main goal is to generalize Păun's result to Kähler orbifolds with nef anti-canonical orbi-bundle. Suppose that we have an orbifold version of Cheeger and Colding's result. We may then adapt Păun's proof easily, as the differential geometry of orbifolds is well-established. Equip the underline space $X = |\mathcal{X}|$ of a Riemannian orbifold $(\mathcal{X}, g)$ with a natural metric d (*cf.* Section 6.2), and apply an algebraic Margulis lemma in [BGT12]. We first prove an orbifold Margulis lemma.

Lemma 0.22 (=Lemma 7.12)**.** *Let $n \geq 1$ be an integer. There exists $\alpha = \alpha(n) > 0$ such that the following holds true:*

Suppose that $\mathcal{X}$ is a complete Riemannian orbifold of dimension n with its Ricci curvature bounded by $\mathrm{Ric} \geq -(n-1)$ *and Γ a subgroup of $Isom(|\mathcal{X}|)$ acting properly discontinuously by isometries on $|\mathcal{X}|$. Then for every $x \in |\mathcal{X}|$, the "almost stabliser"*

$$\Gamma_\alpha(x) := \langle \{\gamma \in \Gamma : \mathrm{d}(\gamma \cdot x, x) < \alpha\} \rangle$$

is virtually nilpotent.

We may then mimic Păun's proof to show

THEOREM 0.23 (=Theorem 4.10=Theorem 8.13). *Let $(\mathcal{X},\omega)$ be a compact Kähler orbifold. If the anti-canonical bundle $K_{\mathcal{X}}^{-1}$ is nef, then $\pi_1^{\mathrm{orb}}(\mathcal{X})$ is virtually nilpotent.*

For a Kähler manifold X, we can also consider its Albanese morphism $\mathrm{Alb}_X : X \to A(X)$ (see Definition 2.33). Set $Y := \mathrm{Alb}_X(X) \subset A(X)$ the image of X in $A(X)$ and $r : \tilde{Y} \to Y$ a smooth model of Y. Frédéric Campana shows how one can describe the central series of $\pi_1(X)$ by the central series of $\pi_1(\tilde{Y})$ in [Cam95, Théorème 2.2]. In particular, one observes that for a compact Kähler manifold X whose fundamental group $\pi_1(X)$ is virtually nilpotent, if Alb_X is surjective, we have that $\pi_1(X)$ is virtually Abelian.

For a complex orbifold $\mathcal{X}$, we may consider its log pair representation (X, Δ_X), where Δ_X is a $\mathbb{Q}$-divisor with standard coefficients. For any klt pair (X, Δ_X), we can define a fundamental group $\pi_1(X, \Delta_X)$, and when the pair represents an orbifold $\mathcal{X}$, there is a canonical isomorphism $\pi_1(X, \Delta_X) \cong \pi_1^{\mathrm{orb}}(\mathcal{X})$ (*cf.* Proposition 5.50). When X is smooth, we have a surjection $\pi_1(X, \Delta_X) \twoheadrightarrow \pi_1(X)$. Hence if $\pi_1^{\mathrm{orb}}(\mathcal{X})$ has virtually nilpotent and X is smooth such that Alb_X is surjective, we can show $\pi_1^{\mathrm{orb}}(\mathcal{X})$ is virtually Abelian. For

general (X, Δ_X), we consider its resolution $r : Y \to X$. Following this idea, we prove the following result.

THEOREM 0.24 (=Theorem 4.11=Theorem 9.1). *Let (X, Δ) be a projective orbifold pair with $-(K_X + \Delta)$ nef. The orbifold fundamental group $\pi_1(X, \Delta)$ is virtually Abelian.*

Part 1

Classification of slc varieties with high nef value

CHAPTER 1

Introduction

In [KO73], Kobayashi and Ochiai proved the following classifying result concerning smooth projective varieties.

THEOREM 1.1 ([KO73, Theorem 1.1.]). *Let X be a smooth projective variety of dimension n with an ample line bundle L such that*

(1) $L^n = 1$*;*
(2) $h^0(X, L) \geq n + 1$.

Then (X, L) is isomorphic to $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$.

The result has later been generalized to singular varieties:

THEOREM 1.2 (Generalized Kobayashi-Ochiai Theorem, *cf.*[BS11, Theorem 3.1.6]). *Let X be an n-dimensional connected normal projective scheme and L an ample line bundle on X. Then we are in one of the following situations:*

- $(X, L) \cong (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$ *if and only if* $K_X + (n+1)L \equiv_{\text{num}} \mathcal{O}_X$*;*
- $(X, L) \cong (Q, \mathcal{O}_Q(1))$ *where* $Q \subset \mathbb{P}^{n+1}$ *is a hyperquadric in* $\mathbb{P}^{n+1}$ *if and only if* $K_X + nL \equiv_{\text{num}} \mathcal{O}_X$.

A projective variety X together with an ample line bundle L on X is called a polarized variety and is denoted by (X, L). To study polarized varieties, Fujita introduces the Δ-genus $\Delta(X, L) := n + L^n - h^0(X, L)$ of polarized varieties, which encodes the dimension of the variety X and L^n, and develops classification theories for polarized varieties with small Δ-genus under certain assumptions on the singularities of X and positivity on L. For Fujita's work, we refer to [Fuj90, Chapter 1].

When X has terminal singularities, set $\tau(L) := \inf\{t \in \mathbb{R} : K_X + tL \text{ is nef}\}$ which is called the *nefvalue* of X. If K_X is not nef, Kawamata's rationality theorem shows that $\tau(L)$ is a rational number. Let $a > 0$ be an integer such that $a(K_X + \tau(L)L)$ is a Cartier divisor. Then Kawamata-Shokurov's base point free theorem shows that for $\mathbb{N} \ni b \gg 0$, the divisor $ba(K_X + \tau(L)L$ has no base point. It is then a classical results in birational geometry that the ring

$$R(X, a(K_X + \tau(L)L)) := \bigoplus_{m \geq 0} (\mathrm{H}^0(X, ma(K_X + \tau(L)L))$$

is normal and finitely generated over $\mathbb{C}$. We call the normal variety

$$Z := \mathrm{Proj}(R(X, a(K_X + \tau(L)L))$$

an adjoint model for (X, L). There exists a canonical morphism given by sections of $a(K_X + \tau(L)L)$

$$\phi : X \to Z$$

called the *nefvalue morphism*. It is thus natural to study (X, L) via ϕ.

It turns out with the help of nefvalue morphism, the classification for (X, L) is complete when the nefvalue of L is larger than $n-1$.

Proposition 1.3 ([BS11, Proposition 7.2.2.]). *Let (X, L) be a polarized variety. Suppose that X has terminal singularities. Let $\phi : X \to Z$ be the nefvalue morphism and τ the nefvalue of L. Then we have one of the following*

(1) $\tau = n+1$ and $(X, L) \cong (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$;
(2) $\tau = n$ and $(X, L) \cong (Q, \mathcal{O}_Q(1))$ where $Q \subset \mathbb{P}^{n+1}$ is a hyperquadric in $\mathbb{P}^{n+1}$ if $K_X + nL \equiv_{\text{num}} \mathcal{O}_X$;
(3) $\tau = n$ and Z is a smooth curve and $(X, L) \cong (\mathbb{P}(E), \mathcal{O}_{\mathbb{P}(E)}(1))$ where E is a vector bundle over Z and $\phi : X \to Z$ is the structure morphism;
(4) $\tau < n$ and $K_X + nL$ is nef and big.

Proposition 1.4 ([BS11, Proposition 7.2.4.]). *Let (X, L) be a polarized variety. Suppose that X has $\mathbb{Q}$-factorial terminal singularties and $n = \dim(X) \geq 2$. Suppose that $K_X + nL$ is ample and $\tau > n-1$. Then we have that $\tau = n - \frac{1}{2}$ and that $(X, L) = C_n(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$ is a generalized cone over $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$.*

The study of polarized varieties has a natural counterpart in the study in the $\mathbb{Q}$-Fano foliations. For a foliation $\mathcal{F} \subset T_X$ of rank r on a projective variety X of dimension n, we define the canonical class $K_{\mathcal{F}}$ of the foliation $\mathcal{F}$ to be the divisor class satisfying $\mathcal{O}_X(-K_{\mathcal{F}}) \cong \det(\mathcal{F})$, where $\det(\mathcal{F})$ is defined to be $(\wedge^r \mathcal{F})^{\star\star}$. When $-K_{\mathcal{F}}$ is $\mathbb{Q}$-Cartier and ample, we call $\mathcal{F}$ a $\mathbb{Q}$-Fano foliation and define its *index* $i_{\mathcal{F}}$ to be the largest positive rational number such that $-K_{\mathcal{F}} \sim_{\mathbb{Q}} i_{\mathcal{F}} H$ for an ample Cartier divisor H on X. In [AD14], Araujo and Druel established a Kobayashi-Ochiai type theorem ([AD14, Theorem 1.2]), which gives an upper bound of the index of a $\mathbb{Q}$-Fano foliation $\mathcal{F}$ and gives a description when the upper bound is reached.

When a foliation $\mathcal{F}$ is algebraically integrable, one can define naturally general log leaves of $\mathcal{F}$ (*cf.* [AD14, Definition 3.11]). A *general log leaf* $(\tilde{F}, \tilde{\Delta})$ comprises a normalization of the closure of a general leaf F of $\mathcal{F}$ and an effective Weil $\mathbb{Q}$-divisor $\tilde{\Delta}$. Let $e : \tilde{F} \to F$ be the normalization map. Then $\tilde{\Delta}$ is given by $K_{\tilde{F}} + \Delta \equiv_{\text{num}} e^* K_{\mathcal{F}}$. It turns out that understanding the log general leaves helps to study of algebraically integrable foliations. This motivates us to consider classification problem for (X, Δ), where X is a variety and Δ a Weil $\mathbb{Q}$-divisor.

When an algebraically integrable foliation $\mathcal{F}$ is $\mathbb{Q}$-Fano, we have the equality $-(K_{\tilde{F}} + \Delta) \equiv_{\text{num}} i_{\mathcal{F}}(e^* H)$. Hence one may very well try to establish a pair version of Theorem 1.2. In fact, Fujino and Miyamoto proved the following

THEOREM 1.5 ([FM21, Theorem 1], see also [AD14, Theorem 1.1]). *Let (X, Δ) be a projective klt pair such that $-(K_X + \Delta)$ is ample. Assume that $-(K_X + \Delta) \equiv_{\text{num}} rH$ for some Cartier divisor H on X with $r > n = \dim(X)$. Then X is isomorphic to $\mathbb{P}^n$ with $\mathcal{O}_X(H) = \mathcal{O}_{\mathbb{P}^n}(1)$.*

The result of Fujino and Miyamoto assumes mild singularites on the pair (X, Δ) and a divisibility condition of the log canonical bundle $K_X + \Delta$. However, with a foliation $\mathcal{F}$, its log general leaf $(\tilde{F}, \tilde{\Delta})$ is *a priori* just normal. On the other hand, for the classification theorems Proposition 1.3 and Proposition 1.4, we do not need divisibility. However we

do need $-K_X$ to be very positive. Thus one may try to weaken the conditions and consider the classification problems

(1) Classify the triple (X, Δ, L) where (X, Δ) is log canonical, L is ample and $K_X + (\dim(X) - 1)L \notin \mathrm{Pseff}(X)$;
(2) Classify the pair (X, L) where X is a projective variety with singularities wilder than normal, L is ample and $K_X + (\dim(X) - 1)L \notin \mathrm{Pseff}(X)$.

We call a projective variety X together with a nef and big line bundle L a *quasi-polarized variety*. For a quasi-polarized variety (X, L) where X is canonical and $\mathbb{Q}$-factorial, we may run a MMP which contracts all L-trivial extremal rays and get a polarized variety (X', L') (see Lemma 3.5). By using Andreatta's result Theorem 3.7 which describes the general fibers of extremal contractions, we can reduce the problem of classifying (X', L') with high nefvalue to the problem of classifying polarized variety with Δ-genus zero. We generalize Proposition 1.3 and Proposition 1.4 as follows.

THEOREM 1.6 (=Theorem 3.13). *Let X be a variety with canonical $\mathbb{Q}$-factorial singularities and L a nef and big line bundle on X. Suppose that $K_X + (n-1)L \notin \mathrm{Pseff}(X)$. Then we have one of the following cases:*

(1) $(X, L) \sim_{\mathrm{bir}} (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$;
(2) (X, L) *is birational equivalent to a* $(\mathbb{P}^{n-1}, \mathcal{O}_{\mathbb{P}^{n-1}}(1))$*-bundle over a smooth curve* C;
(3) $(X, L) \sim_{\mathrm{bir}} (Q, \mathcal{O}_{\mathbb{P}^{n+1}}(1))$, *where* $Q \subset \mathbb{Q}^{n+1}$ *is a hyperquadric;*
(4) $(X, L) \sim_{\mathrm{bir}} (\mathbb{P}^2, \mathcal{O}^2_{\mathbb{P}}(2))$;
(5) $(X, L) \sim_{\mathrm{bir}} C_n(\mathbb{P}^2, \mathcal{O}^2_{\mathbb{P}}(2))$, *where* $C_n(\mathbb{P}^2, \mathcal{O}^2_{\mathbb{P}}(2))$ *is a generalised cone over* $(\mathbb{P}^2, \mathcal{O}^2_{\mathbb{P}}(2))$

The drawback of letting L be nef and big, is that after running MMP, we don't have isomorphism and even have indeterminacies.

For a normal variety X, we have modifications $\mu : X' \to X$ for X such that X' has mild singularities and $K_{X'}$ is μ-ample. A good reference for these modifications is [Kol13, Chapter 1]. For a polarized variety (X, L), with X normal, we may take a canonical modifications $\mu : X' \to X$ for X and consider the quasi-polarized variety (X', L'). We have the following result.

THEOREM 1.7 (=Theorem 3.14). *Let (X, L) be a polarized normal variety of dimension n. Suppose that K_X is $\mathbb{Q}$-Cartier and $K_X + (n-1)L \notin \mathrm{Pseff}(X)$. Then we have one of the following cases:*

(1) $(X, L) \cong (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$;
(2.i) $(X, L) \cong (\mathbb{P}(\mathcal{V}), \mathcal{O}_{\mathbb{P}(\mathcal{V})}(1))$, *where $\mathcal{E}$ is a rank n ample vector bundle over a smooth curve C;*
(2.ii) $(X, L) \cong C_n(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(a))$ *be a generalized cone with* $a \geq 3$;
(3) $(X, L) \cong (Q, \mathcal{O}_{\mathbb{P}^{n+1}}(1))$, *where* $Q \subset \mathbb{Q}^{n+1}$ *is a hyperquadric;*
(4) $(X, L) \cong (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$;
(5) $(X, L) \cong C_n(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$, *where* $C_n(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$ *is a generalised cone over* $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$.

In Theorem 1.7, we note that even if in the proof we have taken a modification, in the resulting list we have isomorphism. The reason is that L is ample and birational equivalences between normal polarized varieties are always isomorphisms.

For a log canonical pair (X, Δ) with $(K_X + \Delta) + (\dim(X) - 1)L \notin \mathrm{Pseff}(X)$, a first observation is that as Δ is effective, we will have $K_X + (\dim(X) - 1)L \notin \mathrm{Pseff}(X)$. Hence we will have a list for (X, L) similar to Theorem 1.7. However in this list the Picard number $\rho(X)$ of X is at most 2. Hence for Δ to be an irreducible divisor or more generally reduced divisor, we don't have to many choice. We may thus give a list for (X, Δ, L).

Proposition 1.8 (=Corollary 3.19)**.** *Let (X, Δ) be a log canonical pair, with $\Delta \neq 0$ a reduced divisor. Suppose that L is an ample line bundle on X and $(K_X+\Delta)+(n-1)L \notin \mathrm{Pseff}(X)$, where $n = dim(X)$. Then (X, Δ, L) is one of the following:*

(1) $(X, L) \cong (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$, $\Delta \equiv_{\mathrm{num}} H$ is a prime divisor where H is a hyperplane of $\mathbb{P}^n$;

(2.i) There is a $(\mathbb{P}^{n-1}, \mathcal{O}_{\mathbb{P}^{n-1}}(1))$-bundle $(\mathbb{P}(E), \mathcal{O}_{\mathbb{P}(E)}(1))$ over a smooth curve C, and a birational morphism $\mu : \mathbb{P}(E) \to X$ such that $\mu^*(L) \cong \mathcal{O}_{\mathbb{P}(E)}(1)$ and $\Delta = \sum F_i$ is a finite sum where $F_i \cong \mu(\mathbb{P}^{n-1})$ are images of distinct general fibers of π by μ;

(2.ii) $(X, L) = (\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(1)), \mathcal{O}_{\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(a)\oplus\mathcal{O}_{\mathbb{P}^1}(1))}(1))$ with $a > 1$ and $\Delta = D$ is irreducible, where D is the unique section of $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(a)\oplus\mathcal{O}_{\mathbb{P}^1}(1)) \to \mathbb{P}^1$ such that $D \equiv_{\mathrm{num}} \mathcal{O}_{\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(a)\oplus\mathcal{O}_{\mathbb{P}^1}(1))}(1)) - af$, where f is a general fiber;

(3.i) $(X, L) \cong (Q, \mathcal{O}_{\mathbb{P}^{n+1}}(1))$, where $Q \subset \mathbb{P}^{n+1}$ is a $\mathrm{rk}(Q) = 3$ hyperquadric, the boundary divisor Δ is a hyperplane in Q and $[\Delta] = \frac{1}{2}[H \cap Q]$ where H is a hyperplane in $\mathbb{P}^{n+1}$;

(3.ii) $(X, L) \cong (Q, \mathcal{O}_{\mathbb{P}^{n+1}}(1))$, where $Q \subset \mathbb{P}^{n+1}$ is a $\mathrm{rk}(Q) = 4$ hyperquadirc. If we write $Q = \mathrm{Proj}\left(\frac{\mathbb{C}[x_0, \ldots, x_{n+1}]}{(x_0x_1 - x_2x_3)}\right)$, then $\Delta = D$ is prime and D is the cone with vertex $\mathbb{P}^{n-3}$ over $\mathbb{P}^1 \times \mathrm{pt}$ or $\mathrm{pt} \times \mathbb{P}^1$. In particular, $D \cong \mathbb{P}^{n-1}$;

When the polarized variety (X, L) is not normal, a natural way to study it is consider the normalization $\nu : \bar{X} \to X$ of X and study the polarized variety $(\bar{X}, \nu^*(L))$. However, when considering normalization, there is a natural ideal of $\mathcal{O}_X$ measuring how far X is from being normal, the conductor $\mathfrak{cond}_X := \mathscr{H}om_{\mathcal{O}_X}(\nu_*(\mathcal{O}_{\bar{X}}), \mathcal{O}_X)$. When X has deminormal singularities, the subschemes $D \subset X$ defined by $\mathfrak{cond}_X$ and $\bar{D} \subset \bar{X}$ defined by $\nu_*(\mathcal{I}_{\bar{D}}) = \mathfrak{cond}_X$ are both generically reduced purely codimension 1 subschemes. In particular, the algebraic cycles $[D]$ and $[D']$ are reduced Weil divisors. When X is slc, we also have that $(\bar{X}, \bar{D})$ is log canonical. Hence for polarized variety (X, L) with X slc, we may use Proposition 1.8 to study $(\bar{X}, \nu^*(L))$. We give the following classification.

Theorem 1.9 (=Proposition 3.20). *Let X be a non-normal slc projective variety of dimension n and L an ample line bundle over X. Suppose that $K_X+(n-1)L \notin \mathrm{Pseff}(X)$. Let $\pi : \bar{X} \to X$ be the normalization of X and $D \subset X$, $\bar{D} \subset \bar{X}$ the conductors. Then we have:*

There is a nodal curve C',a rank n-vector bundle E', distinct fibers $F_1, F_2, \ldots, F_m$ of $\mathbb{P}(E')$ and a birational morphism $\mu : \mathbb{P}(E') \to X$ such that $\mu^*(L) = \mathcal{O}_{\mathbb{P}(E')}(1)$ and $D = \sum_{1\leq i\leq m} \mu(F_i)$

We see that Theorem 1.9 shortens the list in Proposition 1.8 rather than increasing it. In fact, there is a degree 2 morphism $\bar{D}^\nu \to D^\nu$, where $\bar{D}^\nu$ and D^ν are the normalizations of $\bar{D}$ and D respectively. Hence we need $(L'|_{\bar{D}^\nu})^{n-1}$ divisible by 2, which gives more restriction on $(\bar{X}, \bar{D})$ than the assumption in Proposition 1.8.

Remark 1.10. After running MMP to reduce Theorem 1.6 to the problem of classifying (X', L') with X' canonical and L' ample (see Lemma 3.5), the results is already known for even when X' is klt (*cf.* [And13, Proposition 3.5.]). My personal contribution in the classification is to use modifications to get Theorem 1.7 and Theorem 1.9.

Plan of Part 1. We organize Part 1 as following. In Chapter 2, we give the necessary materials for the results. In particular we give a review for the singularities involved and recall the relevant results in the minimal model program. In Chapter 3, we will give several classifications for polarized varieties (X, L). In Section 3.1, we prove Theorem 1.7 by running MMP (Lemma 3.5) to reduce the proof to proving Lemma 3.8 and Lemma 3.10. In Section 3.2, we prove Theorem 1.7 thanks to canonical modifications Theorem 2.25 and use similar methods to prove Proposition 1.8. In Section 3.3, for a polarized slc variety (X, L), we use Proposition 1.8 on the triple $(\bar{X}, \bar{D}, L')$, where $(\bar{X}, \bar{D})$ is the normalization of X and the conductor divisor on $\bar{X}$ and L' is the pullback of L, to get Theorem 1.9.

CHAPTER 2

Preliminaries

2.1. Conventions

We work over $\mathbb{C}$. The definitions and conventions that we adapt follow [Kol13][KM98].

2.1.1. Divisors.

- A scheme is supposed to be separated and of finite type over $\mathbb{C}$. A variety is a reduced and irreducible scheme over $\mathbb{C}$. A point x in a scheme X is an element x in the underlying topological space of X. A scheme is said to be normal if $\mathcal{O}_{X,x}$ is a normal local domain for all point $x \in X$.
- We denote by $\mathrm{Weil}(X)$ the group of Weil-divisors of a scheme X and $\mathrm{Cl}(X)$ its quotient modulo principal divisors [Sta22, Tag 0BE2]. We define
$$\mathrm{Weil}(X)_{\mathbb{Q}} := \mathrm{Weil}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$$
to be the group of $\mathbb{Q}$-Weil-divisors.
- Let X be a scheme and $\mathcal{F}$ a coherent $\mathcal{O}_X$-module. Let $\mathcal{M}_X$ be the sheaf of germes of meromorphic functions over X. We say that $\mathcal{F}$ is *torsion-free* if the natural map
$$\mathcal{F} \to \mathcal{F} \otimes \mathcal{M}_X$$
is injective.

  The *reflexive hull* of $\mathcal{F}$ is its double dual $\mathscr{H}om_{\mathcal{O}_X}(\mathscr{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X), \mathcal{O}_X)$ and we have a natural morphism
$$j_{\mathcal{F}} : \mathcal{F} \to \mathscr{H}om_{\mathcal{O}_X}(\mathscr{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X), \mathcal{O}_X)$$
by sending local section a of $\mathcal{F}$ to the local section $(\phi \mapsto \phi(a))$. We say that $\mathcal{F}$ is *torsionless* if $j_{\mathcal{F}}$ is an injection. Clearly torsionlessness implies torsion free and we have the inverse when $\mathcal{O}_{X,x}$ is integral for all $x \in X$.

  We call $\mathcal{F}$ a *reflexive module* or a *reflexive sheaf* if $j_{\mathcal{F}}$ is an isomorphism. When $\mathcal{F}$ is reflexive, if there exists an open subset $U \subset X$ such that $\mathcal{F}|_U$ is locally free of rank r, we define the *rank* of $\mathcal{F}$ to be $\mathrm{rk}(\mathcal{F}) = r$. For any reflexive sheaf $\mathcal{F}$, we set
$$\mathcal{F}^{[m]} := \mathscr{H}om_{\mathcal{O}_X}(\mathscr{H}om_{\mathcal{O}_X}(\mathcal{F}^{\otimes m}, \mathcal{O}_X), \mathcal{O}_X).$$
- For a scheme X, its rank one reflexive sheaves form an group under the group operation
$$\mathcal{F}_1 \hat{\otimes} \mathcal{F}_2 := \mathscr{H}om_{\mathcal{O}_X}(\mathscr{H}om_{\mathcal{O}_X}(\mathcal{F}_1 \otimes \mathcal{F}_2, \mathcal{O}_X), \mathcal{O}_X).$$
When X is normal, we may associate for $D \in \mathrm{Weil}(X)$ a rank one reflexive sheaf $\mathcal{O}_X(D)$, which induces a group isomorphism between $\mathrm{Cl}(X)$ and the group of rank one reflexive sheaves (*cf.* [Sta22, Tag 0EBM]).

- We denote by $\mathrm{Div}(X) = \Gamma(X, \mathcal{M}_X^*/\mathcal{O}_X^*)$ the group of Cartier divisors of X and define $\mathrm{Div}(X)_{\mathbb{Q}} := \mathrm{Div}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$.
- Let X be a scheme. We have a natural morphism $\mathrm{cyc} : \mathrm{Div}(X) \to \mathrm{Weil}(X)$ (*cf.* [Gro67, IV.21.6.7.]). The morphism extends naturally to $\mathrm{cyc}_{\mathbb{Q}} : \mathrm{Div}(X)_{\mathbb{Q}} \to \mathrm{Weil}(X)_{\mathbb{Q}}$. We say that a $\mathbb{Q}$-Weil-divisor D of X is $\mathbb{Q}$-*Cartier* if it is in the image of $\mathrm{cyc}_{\mathbb{Q}}$.

  When X is normal, the morphism cyc is injective. Let D be a $\mathbb{Q}$-Weil-divisor of X. We have that the followings are equivalent
  (1) D is $\mathbb{Q}$-Cartier;
  (2) There exists an integer m such that $mD \in \mathrm{Weil}(X)$ and the rank 1 reflexive sheaf $\mathcal{O}_X(mD)$ is invertible.
- Let X be a scheme. For two $\mathbb{Q}$-Cartier divisors D_1 and D_2, we say that D_1 and D_2 are $\mathbb{Q}$-*linearly equivalent*, if $D_1 - D_2$ is a $\mathbb{Q}$-combination of principal divisors and we denote it by $D_1 \sim_{\mathbb{Q}} D_2$; we say that D_1 and D_2 are *numerically equivalent* if for any irreducible curve $C \subset X$, we have that $D_1 \cdot C = D_2 \cdot C$ and we denote this by $D_1 \equiv_{\mathrm{num}} D_2$.
- Let $f : X \dashrightarrow Y$ be a rational map between schemes. Let $Z \subset X$ be a subscheme of X. If f is defined on an open dense subset $Z^0 \subset Z$, we define the *strict transform* of Z by f to be the closure of $f(Z^0)$ in Y. If $g : Y \to X$ is a birational morphism, and $D \in \mathrm{Weil}(X)$ be a prime divisor of X, we denote by $g_*^{-1}(D)$ the strict transform of D by g^{-1}, which is a prime divisor on Y. We may thus define a $\mathbb{Q}$-linear map $g_*^{-1} : \mathrm{Weil}(X)_{\mathbb{Q}} \to \mathrm{Weil}(Y)_{\mathbb{Q}}$ by mappinng $\Delta = \sum d_i D_i$ to $g_*^{-1}(\Delta) := \sum d_i g_*^{-1}(D_i)$, here $g_*^{-1}(D_i) = 0$ if D_i is contained in the closure of $g(\mathrm{exc}(g))$.

2.1.2. Projectivisation. Let X be a scheme and $\mathcal{S} = \oplus_{n \geq 0} \mathcal{S}_n$ a quasi-coherent $\mathbb{N}$-graded $\mathcal{O}_X$-algebra. We recall that the *relative Proj* of $\mathcal{S}$ (*cf.* [Gro60, II.3.1]), the X-scheme $Y = \mathrm{Proj}(\mathcal{S}) \to X$ is defined by gluing over each affine open $\mathrm{Spec}(A) = U \subset X$ the U-scheme $Y_U := \mathrm{Proj}(\Gamma(U, \mathcal{S})) \to U = \mathrm{Spec}(A)$).

For a quasi-coherent $\mathcal{O}_X$-module $\mathcal{E}$, we denote by

$$\mathbb{P}(\mathcal{E}) := \mathrm{Proj}(\oplus_{n \geq 0} \mathrm{Sym}^n(\mathcal{E}))$$

its *projectivisation*.

The projectivisation $\mathbb{P}(\mathcal{E})$ has the following characterization:

Proposition 2.1 ([Gro60, Propostion II.4.2.3])**.** *Let X be a scheme and $\mathcal{E}$ a quasi-coherent $\mathcal{O}_X$-module. For any X-scheme $f : Y \to X$, there is a bijective between X-morphisms $\mathrm{Hom}_X(Y, \mathbb{P}(\mathcal{E}))$ and the rank 1 quotient of $f^*(\mathcal{E})$, that is the set $\{f^*(\mathcal{E}) \twoheadrightarrow \mathcal{L} :$ where $\mathcal{L}$ is a line bundle on $Y\}/\sim$, where the equivalence is given by $\alpha : f^*(\mathcal{E}) \twoheadrightarrow \mathcal{L}_1$ being equivalent to $\beta : f^*(\mathcal{E}) \twoheadrightarrow \mathcal{L}_2$ if there exists a isomorphism $\gamma : \mathcal{L}_1 \to \mathcal{L}_2$ such that $\gamma \circ \alpha = \beta$.*

2.1.3. Positivity notions. Let X be a proper scheme over $\mathbb{C}$ and L a line bundle on X. We follow the notions introduced in [Laz04a].

- We say that L is *ample* if there exists positive integers m, n and a closed embedding $X \hookrightarrow \mathbb{P}^n$ such that $L \cong \mathcal{O}_{\mathbb{P}^n}(1)|_X$.
- We say that L is *nef* if for any curve $C \subset X$, we have that $L \cdot C \geq 0$.

- We say that a Cartier divisor $D \in \mathrm{Div}(X)$ is ample (*resp.* nef) if $\mathcal{O}_X(D)$ is ample (*resp.* nef).
- For a $\mathbb{Q}$-divisor $D \in \mathrm{Div}(X)_{\mathbb{Q}}$, we say that D is ample if D can be written as $D = \sum_i r_i D_i$ with $r_i > 0$ and $D_i \in \mathrm{Div}(X)$ ample divisors.
- For a $\mathbb{Q}$-divisor $D \in \mathrm{Div}(X)_{\mathbb{Q}}$, we say that D is nef if D can be written as $D = \sum_i r_i D_i$ with $r_i \geq 0$ and $D_i \in \mathrm{Div}(X)$ nef divisors.
- Being ample (*resp.* nef) is an numerical property, *i.e.*, if D_1, $D_2 \in \mathrm{Div}(X)_{\mathbb{Q}}$ and $D_1 \equiv_{\mathrm{num}} D_2$ then D_1 is ample (*resp.* nef) iff D_2 is ample (*resp.* nef)
- When X is an irreducible projective variety and $D \in \mathrm{Div}(X)_{\mathbb{Q}}$ is a $\mathbb{Q}$-divisor, we say that D is *big* if $D \equiv_{\mathrm{num}} A + E$ where A is ample and E is effective (*cf.* [Laz04a, Corollary 2.2.7.]). A line bundle L on X is big if there exists a big divisor D such that $L \cong \mathcal{O}_X(D)$.

2.2. Singularities of Pairs, MMP

The general reference for this section is [KM98, Chapter 2, Chapter 3][Kol13, Chapter 2].

Definition 2.2 (Pairs)**.**

(0) Let $D = \sum_i m_i D_i$ be a $\mathbb{Q}$-divisor on a scheme X, where D_i are prime divisors. We call D a *subboundary* if $m_i \leq 1$ for all i; we call D a *boundary* if $0 \leq m_i \leq 1$. For a real number r, we denote by $\lfloor r \rfloor$ its integral part and set $\lceil r \rceil := -\lfloor -r \rfloor$. We set $\lfloor D \rfloor := \sum_i \lfloor m_i \rfloor D_i$ and $\lceil D \rceil := \sum_i \lceil m_i \rceil D_i$. We say that D is *reduced* if $m_i \in \{0, 1\}$. We say that D has *standard coefficients* if $m_i = 1 - \frac{1}{n_i}$ for some $n_i \in \mathbb{N}_{>0}$ for all i.

(1) A *log pair* is a scheme X together with a boundary divisor Δ. We denote the log pair by (X, Δ).

(2) A *pair* is a variety X together with a subboundary divisor Δ such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier. We denote the pair by (X, Δ).

Definition 2.3 (Simple normal crossing)**.** Let X be a scheme and $D = \sum_i m_i D_i$, $m_i \neq 0$, a $\mathbb{Q}$-divisor. We say that (X, D) has *simple normal crossing* (snc), if X and D_i's are smooth and D_i intersects transversely. For a normal scheme Y with a $\mathbb{Q}$-divisor Δ, there is a maximal open $U \subset Y$ such that $(U, \Delta|_U)$ is snc. We set non-snc$(Y, \Delta) := Y \setminus U$.

Definition 2.4 (Log resolution)**.** Let X be a variety and D be a Weil-divisor on X. A log resolution of (X, D) is a proper birational morphism $r : X' \to X$ such that

- X' is smooth;
- The exceptional locus $\mathrm{exc}(r)$ has pure codimension 1;
- $(X', r_*^{-1}(D) + \mathrm{exc}(r))$ has simple normal crossing.

The existence of log resolutions is first proved by Hironaka in [Hir64a] [Hir64b]. In [Sza94], Szabó showed a strengthened version that we can even take r to be an isomorphism over the snc locus of (X, D).

Definition 2.5 (Canonical classes)**.** Let X be a normal variety. We denote by $i : X_{\mathrm{reg}} \to X$ the inclusion morphism. The push-foward $i_*(\det(\omega_{X_{\mathrm{reg}}}))$ is a rank one reflexive sheaf which corresponds to a divisor class K_X. We call K_X the *canonical class* and $\omega_X = \mathcal{O}_X(K_X)$ the *canonical sheaf*.

When X is not normal, under some assumptions it is still possible to define the canonical class K_X of X. We refer the readers to [Kol13, Definition 1.6.]

Definition 2.6. Let (X, Δ) be a *pair* with X normal. Write $\Delta = \sum_{j\in J} d_j D_j$ with D_j prime divisors and $d_j \in \mathbb{Q}$. For a birational morphism $f : Y \to X$ from a normal variety Y to X, we have that

$$K_Y + f_*^{-1}(\Delta) \sim_{\mathbb{Q}} f^*(K_X + \Delta) + \textstyle\sum_{i\in I} a(E_i, X, \Delta)E_i,$$

where the sum is taken over exceptional divisors of f. The expression above is unique. We set $a(F_j, X, \Delta) := -d_j$, where $F_j = f_*^{-1}(D_j)$ is the strict transforms of D_j.

For $k \in I \cup J$, we call $a(G_k, X, \Delta)$ the *discrepancy* of G_k with respect to (X, Δ), where $G_k = F_k$ if $k \in J$ and $G_k = E_k$ if $k \in I$. The discrepancy $a(G_k, X, \Delta)$ only depends on G_k but not on f.

Definition 2.7. Let X be a normal variety. A *divisor over* X is a triple (E, Y, f) where $E \in \mathrm{Weil}(Y)$, Y is a normal variety and $f : Y \to X$ is a birational morphism. It turns out that the discrepancy $a(E, X, \Delta)$ defined in Definition 2.6 is determined by its local ring $\mathcal{O}_{Y,E}$ in $K(Y) = K(X)$ but not by the choice of f and Y(*cf.* [KM98, Remark 2.23]). We often omit Y and f. We call the closure of $f(E)$ the *center* of E and denote it by $\mathrm{cent}_X(E)$.

Definition 2.8. Let (X, Δ) be a *pair*. We define singularities that we will use in the thesis.

- We say that (X, Δ) has *canonical singularities*, if X is normal and for every exceptional divisor E over X, we have that $a(E, X, \Delta) \geq 0$;
- We say that (X, Δ) has *Kawamata log terminal singularities* (*klt singularities* for short), if X is normal and for all E over X, we have that $a(E, X, \Delta) > -1$;
- We say that (X, Δ) has *log canonical singularities* (*lc singularities* for short), if X is normal and for all E over X, we have that $a(E, X, \Delta) \geq -1$;
- We say that (X, Δ) has *divisorial log terminal* singularities (*dlt singularties* for short), if X is normal and for all E with $\mathrm{cent}_X(E) \subset \text{non-snc}(X, \Delta)$, we have that $a(E, X, \Delta) > -1$

2.3. Minimal model program for pairs

The classical references are [KM98] and [Kol13]. We also refer the readers to [Fuj11] for results in the log canonical setup.

2.3.1. Cones. We first recall relevant cones that will appear in the thesis.

Definition 2.9 (Cones 1)**.** Let X be a projective variety.

- The *Neron-Severi group* $\mathrm{N}^1(X)$ of X is defined to be the quotient $\mathrm{N}^1(X) := \mathrm{Div}(X)/\equiv_{\mathrm{num}}$. We denote by $\mathrm{N}^1(X)_{\mathbb{Q}} := \mathrm{N}^1(X)\otimes_{\mathbb{Z}}\mathbb{Q}$ and $\mathrm{N}^1(X)_{\mathbb{R}} := \mathrm{N}^1(X)\otimes_{\mathbb{Z}}\mathbb{R}$ the group of $\mathbb{Q}$-divisors and the group of $\mathbb{R}$-divisors respectively. Note that we have the inclusion $\mathrm{N}^1(X)_{\mathbb{Q}} \subset \mathrm{N}^1(X)_{\mathbb{R}}$.
- The Neron-Severi groups $\mathrm{N}^1(X)$ is a free Abelian group of finite rank (*cf.* [Laz04a, Proposition 1.1.16.]). We define the *Picard number* $\rho(X)$ of X to be $\rho(X) := \mathrm{Rank}(\mathrm{N}^1(X))$. The finite dimensional vector space $\mathrm{N}^1(X)_{\mathbb{R}}$ thus has a well-defined Euclidean topology.

- The *ample cone* $\mathrm{Amp}(X) \subset \mathrm{N}^1(X)_{\mathbb{R}}$ is defined to be the convex cone generated by ample $\mathbb{Q}$-divisors.
- The *nef cone* $\mathrm{Nef}(X) \subset \mathrm{N}^1(X)_{\mathbb{R}}$ is defined to be the convex cone generated by nef $\mathbb{Q}$-divisors. We have that $\overline{\mathrm{Amp}(X)} = \mathrm{Nef}(X)$ and $\mathrm{Int}(\mathrm{Nef}(X)) = \mathrm{Amp}(X)$ (*cf.* [Laz04a, Theorem 1.4.23]).
- The *big cone* $\mathrm{Big}(X) \subset \mathrm{N}^1(X)_{\mathbb{R}}$ is defined to be the convex cone generated by big $\mathbb{Q}$-divisors.
- The *pseudo-effective cone* $\mathrm{Pseff}(X) \subset \mathrm{N}^1(X)_{\mathbb{R}}$ is defined to be the convex cone generated by the divisor classes that can be represented by effective $\mathbb{Q}$-divisors. We have that $\overline{\mathrm{Big}(X)} = \mathrm{Pseff}(X)$ and $\mathrm{Int}(\mathrm{Pseff}(X)) = \mathrm{big}(X)$ (*cf.* [Laz04a, Theorem 2.2.26])

Dually, we may define cones in the 1-cycles.

Definition 2.10 (Cones 2)**.** Let X be quasi-projective variety.

(1) We define the *numerical equivalence classes of 1-cycles* to be
$$\mathrm{N}_1(X) := \mathrm{Z}_1(X) \otimes_{\mathbb{Z}} \mathbb{R}/\equiv$$
where $Z_1(X)$ is group of 1-cycles of X and the equivalent relation is given by $C_1 \equiv C_2$ iff for any line bundle L over X we have that $L \cdot C_1 = L \cdot C_2$. It comes from the definition that we have a perfect pairing
$$\mathrm{N}^1(X)_{\mathbb{R}} \times \mathrm{N}_1(X) \to \mathbb{R},\ (\delta, \gamma) \mapsto \delta \cdot \gamma$$
We define the *Mori cone* $\overline{\mathrm{NE}}(X)$ of X to be the closure of the cone in $\mathrm{N}_1(X)$ generated by cycles represented by positive combinations of irreducible curves.

(2) Let C be a cone in a finite dimensional real vector space V. A *face* F of C is a sub-cone of C. When F has dimension 1, we call it a *ray*. A face $F \subset C$ is called *extremal* if F satisfies the following property:
For any $x, y \in C$, we have that $x + y \in F$ implies $x \in F$ and $y \in F$.
For a linear function $l \in V^\vee$, we denote $l < 0$ the sub-cone $\{x \in C : l(x) < 0\}$. We say that $F \subset C$ is l-negatve if $F \subset (l < 0)$.

(3) Let $f : X \to S$ be a projective morphism between two varieties. We define the *relative cone* to be $\overline{\mathrm{NE}}(X/S) := \ker(f_* : \overline{\mathrm{NE}}(X) \to \overline{\mathrm{NE}}(S))$. The relative cone $\mathrm{NE}(X/S) \subset \mathrm{NE}(X)$ is an closed extremal face (*cf.* [Deb01, Lemma 6.7.]). We define the *relative Picard number* $\rho(X/S)$ to be the dimension of $\overline{\mathrm{NE}}(X/S)$. We also denote $\overline{\mathrm{NE}}(X/S)$ by $\overline{\mathrm{NE}}(f)$

With the language of cones, we can give a characterization for nef and ample divisors.

Proposition 2.11 (*cf.* [Laz04a, Proposition 1.4.28., Theorem 1.4.29.])**.** *Let X be a proper variety.*

(1) The Mori cone $\overline{\mathrm{NE}}(X)$ is dual to the nef cone $\mathrm{Nef}(X)$, i.e., we have that
$$\overline{\mathrm{NE}}(X) = \{\gamma \in \mathrm{N}_1(X) : \delta \cdot \gamma \geq 0 \text{ for all } \delta \in \mathrm{Nef}(X)\}$$
(2) If X is projective, then for $D \in \mathrm{N}^1(X)_{\mathbb{R}}$, we have that $D \in \mathrm{Amp}(X)$ iff for any $\gamma \in \overline{\mathrm{NE}}(X) \setminus 0$ the product $D \cdot \gamma$ is positive.

It is also possible to describe the dual cone in $\mathrm{N}_1(X)$ of the pseudo-effective cone $\mathrm{Pseff}(X)$.

Definition 2.12 ([Laz04b, Definition 11.4.6.])**.** Let X be a projective variety of dimension n. A class $\gamma \in \mathrm{N}_1(X)$ is *movable* if there exists a birational morphism $\mu : X' \to X$ from a projective variety X' to X, together with $n-1$ ample classes $a_1, a_2, \ldots, a_{n-1} \in \mathrm{Amp}(X')$ such that

$$\gamma = \mu_*(a_1 \cdot a_2 \cdot \ldots \cdot a_{n-1}).$$

The *movable cone* $\overline{\mathrm{Mov}}(X) \subset \mathrm{N}_1(X)$ is the closed cone generated by all the movable classes.

As X is projective, the movable cone is also the closed cone generated by the classes of a flat family of curves C_t in X such that $\bigcup_t C_t = X$ ([BDPP13, Theorem 1.5.(ii)]).

We have the following theorem.

THEOREM 2.13 (BDPP theorem, [BDPP13, Theorem 2.2., Theorem 2.4.]). *Let X be a projective variety. Then the cones*

$$\overline{\mathrm{Mov}}(X) \text{ and } \mathrm{Pseff}(X)$$

are dual.

We finally give some remarks on pseudo-effectiveness to end this subsection. When X a is normal projective variety, it is possible to define pseudo-effectiveness for reflexive sheaves on X:

Definition 2.14 ([HP19, Definition 2.1.])**.** Let X be a normal projective variety and $\mathcal{E}$ a reflexive sheaf on X. We say that $\mathcal{E}$ is *pseudo-effective* if there exists an ample divisor H on X satisfying the following:

For any $c > 0$ there exists integers $j > 0$ and $i > jc$ such that

$$H^0(X, S^{[i]}(\mathcal{E}) \otimes \mathcal{O}_X(jH)) \neq 0$$

where $S^{[i]}(\mathcal{E})$ is the double dual of $\mathrm{Sym}^i(\mathcal{E})$.

The following lemma implies that the above definition for line bundles coincides with the usual definition by pseudo-effective cones.

Lemma 2.15 ([HP19, Lemma 2.3.])**.** *Let X be a normal projective variety and $\mathcal{E}$ a reflexive sheaf on X. Let $\tau \in \mathrm{N}^1(\mathbb{P}(\mathcal{E}))_{\mathbb{R}}$ be the class of the tautological bundle $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$. We have that $\tau \in \mathrm{Pseff}(\mathbb{P}(\mathcal{E}))$ iff $\mathcal{E}$ is pseudo-effective in the sense of Definition 2.14.*

Let X be a normal projective variety and L be a line bundle. The structure morphism $\pi : \mathbb{P}(L) \to X$ is an isomorphism and $\mathcal{O}_{\mathbb{P}(L)}(1) \cong \pi^*(L)$. Hence by above lemma $L \in \mathrm{Pseff}(X)$ iff L is pseudo-effective in sense of Definition 2.14. If $D \in \mathrm{Weil}(X)$ is $\mathbb{Q}$-Cartier, suppose that mD is a Cartier divisor for $m \in \mathbb{N}$. Then $D \in \mathrm{Pseff}(X)$ is equivalent to $mD \in \mathrm{Pseff}(X)$. On the other hand we have that $\mathcal{O}_X(mD) = \mathcal{O}_X(D)^{[m]}$. It's easy to see that $\mathcal{O}_X(D)^{[m]}$ is pseudo-effective in sense of Definition 2.14 iff $\mathcal{O}_X(D)$ is pseudo-effective in sense of Definition 2.14. Hence we get the following:

Let $D \in \mathrm{Weil}(X)$ be $\mathbb{Q}$-Cartier on a normal projective variety. We have that $D \in \mathrm{Pseff}(X)$ if and only if $\mathcal{O}_X(D)$ is pseudo-effective.

2.3.2. Minimal model program for pairs. We begin by a lemma which says that morphism between projective varieties is determined by its relative cone.

Lemma 2.16 ([Deb01, Proposition 1.14])**.** *Let X, Y, Y' be three projective varieties, and morphisms*

$$\pi : X \to Y \text{ and } \pi' : X \to Y'.$$

Suppose that $\mathcal{O}_Y \simeq \pi_*(\mathcal{O}_X)$ and $\overline{\mathrm{NE}}(X/Y) \subset \overline{\mathrm{NE}}(X/Y')$. Then there exists a unique morphism $f : Y \to Y'$ such that $f \circ \pi = \pi'$.

Lemma 2.16 also shows that it is helpful to understand the structure of the relative cone when studying morphisms between projective varieties. We now state the cone theorem, which gives a description of the structure of the relative cone.

THEOREM 2.17 (Cone Theorem, [Fuj11, Theorem 1.1.]). *Let (X, Δ) be a pair as in Definition 2.2. Suppose that (X, Δ) has log canonical singularites. Let $\pi : X \to S$ be a projective morphism onto a variety S. Then*

$$\overline{\mathrm{NE}}(X/S) = \overline{\mathrm{NE}}(X/S)_{K_X+\Delta\geq 0} + \sum R_j$$

with the following properties:

(1) R_j is a $(K_X + \Delta)$-negative extremal ray;
(2) For any π-ample $\mathbb{Q}$-divisor A, there are only finite many R_j's included in $(K_X + \Delta + A) < 0$. In particular, the R_j's are discrete in the half space $(K_X + \Delta) < 0$ and the sum of R_j is indexed over a countable set;
(3) Let $F \subset \overline{\mathrm{NE}}(X/S)$ be an extremal face such that $F \cap \overline{\mathrm{NE}}(X/S)_{K_X+\Delta\geq 0} = 0$. There exists a contraction morphism $\mathrm{cont}_F : X \to Y$ over S.
  (i) Let $C \subset X$ be a curve such that $\pi(C)$ is a point. We have that $[C] \in F$ if and only if $\mathrm{cont}_F(C)$ is a point;
  (ii) $\mathcal{O}_Y \cong (\mathrm{cont}_F)_*(\mathcal{O}_X)$;
  (iii) Let L be a line bundle on X such that $L \cdot C = 0$ for any curve C with $[C] \in F$. Then there exists a line bundle L_Y on Y such that $L = \mathrm{cont}_F^*(L_Y)$.
(4) Every $(K_X + \Delta)$-negative extremal ray R_j is spanned by a rational curve C_j with $0 < -(K_X + \Delta) \cdot C_j \leq 2\dim(X)$.

We remark that (3) in Theorem 2.17 is also known as the *contraction theorem.*
We will also use the Kawamata's rationality theorem.

THEOREM 2.18 (Rationality theorem, [KM98, Theorem 3.5., Complement 3.6.]). *Let (X, Δ) be a propre klt pair with Δ effective. Suppose that $K_X + \Delta$ is not nef. Let H be a nef and big line Cartier divisor on X. Then the following number*

$$r(H) := \sup\{t \in R : H + t(K_X + \Delta) \text{ is nef}\}$$

is a non-negative rational number. Further more, there exists a $(K_X + \Delta)$-negative extremal ray R such that $(H + r(H)(K_X + \Delta)) \cdot R = 0$.

We give a little more details about the contraction map.

Lemma 2.19 ([KM98, Propostion 2.5])**.** *Let $\phi : X \to Y$ be a morphism between projective varieties. Suppose that X is normal and $\mathbb{Q}$-factorial. Suppose that $\overline{\mathrm{NE}}(X/Y)$ is a 1-dimensional cone. Then one of the following holds:*

(1) *(fiber type contraction)* $\dim(X) > \dim(Y)$ *and* ϕ *is said to be a fiber type contraction;*
(2) *(divisorial contraction)* ϕ *is birational and the exceptional locus* $\mathrm{exc}(\phi)$ *is is an irreducible divisor* E *and* ϕ *is said to be a divisorial contraction;*
(3) *(small contraction)* ϕ *is birational and* $\mathrm{codim}_X(\mathrm{exc}(\phi)) \geq 2$ *and* ϕ *is said to be a small contraction.*

Let (X, Δ) be a pair and let $\phi : X \to X'$ be a birational morphism. Set $\Delta' = \phi_*(\Delta)$. We now consider (X', Δ'). If ϕ is divisorial and it contracts a $K_X + \Delta$-negative extremal ray, then by [KM98, Corollary 3.43, Corallary 3.44], we have that (X', Δ') is canonical (*resp.* klt *resp.* dlt *resp.* lc) if (X, Δ) is canonical (*resp.* klt *resp.* dlt *resp.* lc).

When $\mathrm{codim}_X(\mathrm{exc}(\phi)) \geq 2$ and $K_X + \Delta$ is not relatively numerically trivial over X', we know that $K'_X + \Delta'$ is not $\mathbb{Q}$-Cartier. [1] The study of (X', Δ') becomes complicated as we can no longer use Theorem 2.17. One way to circumvent this inconvenience is to introduce a new birational operation.

Definition 2.20 ([KM98, Definition 3.33])**.** Let X be a normal variety and D be a $\mathbb{Q}$-divisor such that $K_X + D$ is $\mathbb{Q}$-Cartier. A *flipping contraction* is a proper birational morphism $f : X \to Y$ to a normal scheme such that $-(K_X + D)$ is f-ample and $\mathrm{codim}_X(\mathrm{exc}(f)) \geq 2$.

A $(K_X + D)$-*flip* is a normal variety X^+ together with a proper birational morphism $f^+ : X^+ \to Y$ such that:

(1) Set $\phi := (f^+)^{-1} \circ f$, and set $D^+ = \phi_*(D)$. Then we have that $K_{X^+} + D^+$ is $\mathbb{Q}$-Cartier;
(2) $K_{X^+} + D^+$ is f^+-ample;
(3) The exceptional locus $\mathrm{exc}(f^+)$ has codimension at least 2.

By abuse of language, we also call the birational map $\phi : X \dashrightarrow X^+$ the $(K_X + D)$-flip. We know ([KM98, Corollary 3.43, Corallary 3.44]) that (X^+, D^+) is canonical (*resp.* klt *resp.* dlt *resp.* lc) if (X, D) is canonical (*resp.* klt *resp.* dlt *resp.* lc).

We now give a sketch of the inductive procedure known as "running the minimal model program". Let (X, Δ) be a klt pair such that X is $\mathbb{Q}$-factorial. Suppose that $K_X + \Delta$ is not nef. By Theorem 2.17, we may write

$$\overline{\mathrm{NE}}(X) = \overline{\mathrm{NE}}(X)_{(K_X+\Delta)\geq 0} + \sum \mathbb{R}_{\geq 0}[C_i].$$

By (4) of Theorem 2.17, we may consider the contraction morphism for an extremal ray $R = \mathbb{R}_{\geq 0}[C]$. Hence we get $\mathrm{cont}_R : X \to Y$.

(1) If ϕ is fiber type, the MMP terminates with a Mori fiber space;
(2) If ϕ is divisorial, set $(X_1, \Delta_1) = (Y, \phi_*(\Delta))$;
(3) If ϕ is small and the $(K_X+\Delta)$-flip $X \dashrightarrow X^+$ exists, set $(X_1, \Delta_1) = (X^+, (X \dashrightarrow X^+)_*(\Delta))$

If we are in the case (2) or case (3), we may then consider the klt pair (X_1, Δ_1) to produce (X_2, Δ_2) and so on. Thus we get a sequence of birational maps:

[1] Suppose contrarily that $K'_X + \Delta'$ is $\mathbb{Q}$-Cartier. Take an integer $m \gg 0$ such that $m(K_X + \Delta) = D$ and $m(K'_X + \Delta') = D'$ where D,D' are Cartier divisors. We know that $K_X + \Delta$ and $K'_X + \Delta$ are identified by ϕ over $X \setminus \mathrm{exc}(\phi)$. Hence we have that $\phi^*(D')|_{X\setminus\mathrm{exc}(\phi)} = D|_{X\setminus\mathrm{exc}(\phi)}$. As X is normal and $\mathrm{codim}_X(\mathrm{exc}(\phi)) \geq 2$, we have that $D = \phi^*(D')$ which is relatively numerically trivial, a contradiction.

$$(X, \Delta) = (X_0, \Delta_0) \dashrightarrow (X_1, \Delta_1) \dashrightarrow \cdots \dashrightarrow (X_n, \Delta_n).$$

[KMM87, Figure 2] gives a clear description of this process. When $X_i \dashrightarrow X_{i+1}$ is a divisorial contraction, the relative Picard number $\rho(X_i/X_{i+1})$ equals 1. Hence in the above program, we have at most $\rho(X)$ divisorial contraction. The existence of flips is established in [BCHM10, Corollary 1.4.1]. Hence the problem of termination of MMP is thus the problem of termination of flips. The termination of flips in dimension 3 is proved by Kawamata in [Kaw92]. The termination of flips in dimension ≥ 4 is still open.

We end this subsection with two theorems from [BCHM10] that give sufficient condition for a minimal program to end.

THEOREM 2.21 ([BCHM10, Theorem 1.2.]). *Let (X, Δ) be a klt pair. Let $\pi : X \to U$ be a projective morphism between quasi-projective varieties.*

Assume that either Δ is π-big and $K_X + \Delta$ is π-pseudo-effective or $K_X + \Delta$ is π-big. Then

(1) $K_X + \Delta$ has a log terminal model over U,
(2) if $K_X + \Delta$ is π-big, then $K_X + \Delta$ has a log canonical model over U, and
(3) the $\mathcal{O}_U$-algebra

$$\mathfrak{R}(\pi, K_X + \Delta) = \oplus_{m \in \mathbb{N}} \pi_* \mathcal{O}_X(\lfloor m(K_X + \Delta) \rfloor)$$

is finitely generated.

THEOREM 2.22 ([BCHM10, Corollary 1.3.3.]). *Let (X, Δ) be a klt pair and suppose X is $\mathbb{Q}$-Cartier. Let $\pi : X \to U$ be a projective morphism between normal quasi-projective varieties. Suppose that $K_X + \Delta$ is not π-pseudo-effective.*

Then we may run $f : X \dashrightarrow Y$ a $(K_X + \Delta)$-MMP over U and end with a Mori fiber space $g : Y \to W$ over U.

2.4. Some results on modifications

Let (X, Δ) be a *pair* with X normal. We can find a birational morphism $g : X' \to X$ such that $(X', g_*^{-1}(\Delta))$ has mild singularities and $K_{X'} + g_*^{-1}(\Delta)$ has some positivities. We list here the results that we need in the thesis. The reference for this section is [Kol13, Chapter 1.4.].

Lemma 2.23 (Small $\mathbb{Q}$-factorial modification, *cf.* [Kol13, Corollary 1.37])**.** *Let (X, Δ) be dlt and Δ a boundary. Then there is a proper birational morphism $g : X^{\mathrm{qf}} \to X$ such that:*

(1) X^{qf} is $\mathbb{Q}$-factorial,
(2) g is small.

THEOREM 2.24 (Terminal modification, *cf.* [Kol13, Theorem 1.33]). *Let X be a normal, quasi-projective variety and Δ a boundary on X such that $\lfloor \Delta \rfloor = 0$. Then there is a non-unique, projective, birational morphism $g^{\min} : X^{\min} \to X$ such that*

(1) $(X^{\min}, (g^{\min})_*^{-1}\Delta)$ is terminal and
(2) $K_{X^{\min}} + (g^{\min})_*^{-1}\Delta)$ is $g^{\min}$-nef.

THEOREM 2.25 ([Kol13] Theorem 1.31). *Let X be a normal variety and Δ a boundary on X. There exists a unique, projective, birational morphism $f : X' \to X$ such that*

(1) $(X', f_*^{-1}(\Delta))$ *is canonical and*
(2) $K_{X'} + f_*^{-1}(\Delta)$ *is* f*-ample*

2.5. Slc singularities

In this section we consider non-normal schemes. We strictly follow [Kol13, Chapter 5].

Definition 2.26 (Serre's conditions, *cf.* [Gro65, Définition IV.5.7.2])**.** Let X be a scheme. We say that X has property (S_2), if for all $x \in X$, a not necessarily closed point, one of the following holds

(1) We have that $\dim(\mathcal{O}_{X,x}) = 0$;
(2) We have that $\dim(\mathcal{O}_{X,x}) = 1$ and that there exists $a \in \mathfrak{m}_x \setminus 0$ such that a is not a zero divisor;
(3) We have that $\dim(\mathcal{O}_{X,x}) \geq 2$ and that there exists a_1, $a_2 \in \mathfrak{m}_x \setminus 0$ such that a_1 is not a zero divisor in $\mathcal{O}_{X,x}$ and a_2 is not a zero divisor in $\mathcal{O}_{X,x}/(a_1)$.

In general, let $k \geq 0$ be an integer. We say that a coherent $\mathcal{O}_X$-module $\mathcal{F}$ has propery $(\mathrm{S_k})$ if for all $x \in X$, we have that $\operatorname{depth}_{\mathcal{O}_{X,x}}(\mathcal{F}_x) \geq \min\{k, \dim \operatorname{supp}(\mathcal{F}_x)\}$.

By Serre's criterion for normality (*cf.* [Gro65, IV.5.8.6]), all normal schemes are (S_2).

Definition 2.27. Let X be a scheme.

(1) We say that X has a *node* at a not necessarily closed point $x \in X$, if $\mathcal{O}_{X,x} \cong R/(f)$, where $(R, \mathfrak{m})$ is a regular local ring of dimension 2, $f \in \mathfrak{m}^2$ and f is not a square in $\mathfrak{m}^2/\mathfrak{m}^3$;
(2) A scheme X is called *demi-normal* if X has the property (S_2) and all codimension 1 points x of X are regular points or nodes.

The simplest examples for demi-normal schemes are the nodal curves.

Definition 2.28 (conductor)**.** Let X be a reduced scheme and $\pi : \bar{X} \to X$ its normalization. The *conductor ideal*

$$\mathfrak{cond}_X := \mathscr{H}om_{\mathcal{O}_X}(\pi_*\mathcal{O}_{\bar{X}}, \mathcal{O}_X)$$

is the largest ideal sheaf of $\mathcal{O}_X$ such that it is also an ideal sheaf of $\pi_*\mathcal{O}_{\bar{X}}$. As π is finite, we have a unique ideal sheaf $\mathfrak{cond}_{\bar{X}}$ of $\bar{X}$ that corresponds to $\mathfrak{cond}_X$.

We define the conductor schemes to be

$$D := \operatorname{Spec}(\mathcal{O}_X / \mathfrak{cond}_X) \text{ and } \bar{D} := \operatorname{Spec}(\mathcal{O}_{\bar{X}} / \mathfrak{cond}_{\bar{X}})$$

They fit into the Cartesian square

$$\begin{tikzcd} \bar{D} \arrow[r] \arrow[d] & \bar{X} \arrow[d] \\ D \arrow[r] & X \end{tikzcd}$$

For any $x \in X$, we have that the stalk $\pi_*(\mathcal{O}_{\bar{X}})_x$ is canonically isomorphic to the integral closure of $\mathcal{O}_{X,x}$ in its total rings of fractional (*cf.* [Sta22, Tag 0C3B]). When $\mathcal{O}_{X,x}$ is a normal ring, we have that $\mathcal{O}_{X,x} = \pi_*(\mathcal{O}_{\bar{X}})_x$ and $x \notin D$. Hence D is the locus of X where $\mathcal{O}_{X,x}$ is not integrally closed in $\mathcal{M}_{X,x}$ and $\bar{X} \setminus \bar{D} \cong X \setminus D$. When X is

(S_2),the condutor ideal $\mathfrak{cond}_X$ is (S_2) by [Sta22, Tag 0EBC]. Let ξ be a generic point of D. We have that $\mathcal{O}_{X,\xi}$ is not a normal ring. Hence by Serre's criterion for normality, there exits a $\eta \in X$ which is a generalization of ξ such that $\dim(\mathcal{O}_{X,\eta}) = 1$ and $\mathcal{O}_{X,\eta}$ is not regular. Thus $\eta \in D$ and $\eta = \xi$. We have that D is of pure codimension 1. As π is finite, the subscheme $\bar{D}$ is also of pure codimension 1. By [Kol13, Corollary 2.61.], the subschemes D and $\bar{D}$ have (S_1) property or equivalently they have no embedded points. When X is demi-normal, both D and $\bar{D}$ are reduced.

For a demi-normal scheme X, its Weil-divisors that do not contain any components of the conductor divisor D behave like the Weil-divisors on a normal scheme. We give some elaborations.

Definition 2.29. Let X be a demi-normal scheme and $\pi : \bar{X} \to X$ its normalization. We denote by D and $\bar{D}$ the conductor divisors on X and $\bar{X}$ respectively.

- We denote by $\mathrm{Weil}^*(X) \subset \mathrm{Weil}(X)$ the subgroup of Weil-divisors whose support does not contain any irreducible component of the conductor D. Its easy to see that $\mathrm{Weil}^*(X)$ is the image of the canonical map $i_* : \mathrm{Weil}(X \setminus D) \hookrightarrow \mathrm{Weil}(X)$, where $i : X \setminus D \to X$ is the open inclusion.
- We denote by $\mathrm{Weil}^*(\bar{X}) \subset \mathrm{Weil}(\bar{X})$ the subgroup of Weil-divisors whose support does not contain any irreducible component of the conductor $\bar{D}$. Similarly we have that $\mathrm{Weil}^*(\bar{X})$ is the image of the canonical map $j_* : \mathrm{Weil}(\bar{X} \setminus \bar{D}) \hookrightarrow \mathrm{Weil}(\bar{X})$, where $j : \bar{X} \setminus \bar{D} \to \bar{X}$ is the open inclusion.
- A *divisorial sheaf* on X is a rank one reflexive sheaf that is locally free at the generic points of D. The isomorphic classes of divisorial sheaves form a group with multiplication given by $\hat{\otimes}$.

Lemma 2.30. *Let X be a demi-normal scheme and $\mathcal{F}$ a reflexive module on X. We denote by D the conductor of $\mathcal{F}$. Suppose that $\mathcal{F}$ is locally free at the generic points of D. Then there exists an open subset $X^0 \subset X$ such that*

(1) The restriction $\mathcal{F}|_{X^0}$ is locally free;
(2) Every irreducible components of $X \setminus X^0$ has codimension at least 2;
(3) Denote by $j : X^0 \to X$ the inclusion. The canonical map $\mathcal{F} \to j_* j^*(\mathcal{F})$ is an isomorphism.

Proof. Let $X^0 \subset X$ be the maximal open subset satisfying Lemma 2.30-(1). For a point $x \in X$ such that $\dim \mathcal{O}_{X,x} \leq 1$, we have that either $\mathcal{O}_{X,x}$ is a Dedekind domain, hence $\mathcal{F}_x$ is a free $\mathcal{O}_{X,x}$-module or x is a node and $\mathcal{F}_x$ is free at x by hypothesis. Hence for all $x \in X \setminus X^0$, we have that $\dim \mathcal{O}_{X,x} \geq 2$. Hence we have Lemma 2.30-(2). As X has the property (S_2), we have that $\mathrm{depth}(\mathcal{O}_{X,x}) \geq 2$ for all $x \in X \setminus X^0$. This implies that the stalk $\mathcal{F}_x = \mathscr{H}om_{\mathcal{O}_{X,x}}(\mathscr{H}om_{\mathcal{O}_{X,x}}(\mathcal{F}_x, \mathcal{O}_{X,x}), \mathcal{O}_X)$ has depth at least 2 for all $x \in X \setminus X^0$. [Sta22, Tag 0E9I] implies Lemma 2.30-(3). □

Let B be a Weil-divisor on a demi-normal scheme X. Suppose that B contains no components of the conductor divisor D of X. Then there exists an open subset $X^0 \subset X$ such that the codimension of $X \setminus X^0$ is at least 2 and $B|_{X^0}$ is Cartier. Thus we may associate B a reflexive $\mathcal{O}_X(B) := i_*(\mathcal{O}_{X^0}(B|_{X^0}))$, where $i : X^0 \to X$ is the inclusion. The map $B \mapsto \mathcal{O}_X(B)$ induces an isomorphism from $\mathrm{Weil}^*(X)$ modulo linear equivalences to the group of isomorphic classes of divisorial sheaves on X.

Let X be demi-normal, and $\pi : \bar{X} \to X$ be its normalization. Let B be a Weil-divisor on X. Write $B = B_1 + B_2$ where B_1 is the divisorial part. We have that set-theoretically $\pi^{-1}(B) = \pi^{-1}(B_1) \cup \pi^{-1}(B_2)$. As $\mathrm{supp}(B_2) \subset D$, we have that $\pi^{-1}(B_2) \subset \bar{D}$. On the other hand, as $\bar{X} \setminus \bar{D} \cong X \setminus D$, we have that $\pi^{-1}(B_1) = \overline{(\pi|_{\bar{X}\setminus\bar{D}})^*(B_1|_{\bar{X}\setminus\bar{D}})}$. Hence we define the divisor $j_*(\pi|_{\bar{X}\setminus\bar{D}})^*(B_1|_{\bar{X}\setminus\bar{D}})$ to be the *divisorial part* of the subscheme $\pi^{-1}(B)$, where $j : \bar{X} \setminus \bar{D} \to \bar{X}$ is the inclusion. We denote the divisorial part of $\pi^{-1}(B)$ by $\bar{B}$. As $\pi : \bar{X} \setminus \bar{D} \to X \setminus D$ is an isomorphism. It's easy from the construction to see that $B \mapsto \bar{B}$ is a bijection from $\mathrm{Weil}^*(X)$ to $\mathrm{Weil}^*(\bar{X})$.

We now combine the above discussion together. We have the following proposition concerning the dualizing sheaves of X and $\bar{X}$. For a divisor $B \in \mathrm{Weil}^*$ and the dualizing sheaf ω_X, we will use the following notation

$$\omega_X^{[m]}(B) := \omega_X^{[m]} \hat{\otimes} \mathcal{O}_X(B),$$

where m is an integer. If $m = 1$, we simplify the notation $\omega_X^{[1]}(B)$ to $\omega_X(B)$. We use similar notations for *all* divisors B' on $\bar{X}$ and $\omega_{\bar{X}}$.

Proposition 2.31 ([Rei94, Proposition 2.3])**.** *Let X be a reduced scheme, and $\pi : \bar{X} \to X$ its normalization. Let ω_X and $\omega_{\bar{X}}$ be the dualizing sheaves of X and $\bar{X}$ respectively. We have that*

$$\pi_*(\omega_{\bar{X}}) = \mathscr{H}om_{\mathcal{O}_X}(\pi_*(\mathcal{O}_{\bar{X}}), \omega_X)$$

If X is (S_2) *and ω_X is invertible then*

$$\pi_*(\omega_{\bar{X}}) = \mathfrak{cond}_X \cdot \omega_X \text{ and } \pi^*(\omega_X) = \mathscr{H}om_{\mathcal{O}_{\bar{X}}}(\mathfrak{cond}_{\bar{X}}, \omega_{\bar{X}}) = \omega_{\bar{X}}(\bar{D}).$$

We note that the second equality is equivalent to the first. In fact, by definition we have that $\pi_*(\mathfrak{cond}_{\bar{X}}) = \mathfrak{cond}_X$. Then by the projection formula, we have that $\pi_*(\mathfrak{cond}_{\bar{X}} \cdot \pi^*\omega_X) = \mathfrak{cond}_X \cdot \omega_X = \pi_*\omega_{\bar{X}}$. This implies $\mathfrak{cond}_{\bar{X}} \cdot (\pi^*\omega_X) = \omega_{\bar{X}}$. Tensoring both sides with $\mathfrak{cond}_{\bar{X}}^{-1}$ and taking reflexive hulls, we get the second equality.

The dualizing sheaf ω_C is locally free for a nodal curve C. For a general demi-normal scheme X, each nodal point $x \in X$ is also a double nc point. Hence by the adjunction formula, the dualizing sheaf ω_X is locally free at x. Thus there exists an open $X^0 \subset X$ with $\mathrm{codim}_X(X \setminus X^0) \geq 2$ and $\omega_X|_{X^0}$ is locally free. Apply the above proposition to X^0 and $\pi^{-1}(X^0)$ then push forward. We get that

$$\pi_*(\omega_{\bar{X}}) = \omega_X(-D) \text{ and } (\pi^*\omega_X)^{\star\star} = \omega_{\bar{X}}(\bar{D}).$$

For any $B \in \mathrm{Weil}^*(X)$, we have that

$$(\pi^*\omega_X^{[m]}(B))^{\star\star} \simeq \omega_{\bar{X}}^{[m]}(m\bar{D} + \bar{B}) \tag{1}$$

If Δ is a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor such that $m\Delta \in \mathrm{Weil}^*$, we have that

$$(\pi^*\omega_X^{[m]}(m\Delta))^{\star\star} \simeq \omega_{\bar{X}}^{[m]}(m\bar{D} + m\bar{\Delta}) \tag{2}$$

which will also be written as

$$\pi^*(K_X + \Delta) \sim_{\mathbb{Q}} K_{\bar{X}} + \bar{D} + \bar{\Delta} \tag{3}$$

We are now ready to define slc singularities.

Definition 2.32 ([Kol13, Definition-Lemma 5.10])**.** Let (X, Δ) be a pair as in Definition 2.2. Let $\pi : \bar{X} \to X$ be its normalization, the conductors $\bar{D}$ and D as in Definition 2.28. The pair (X, Δ) is called *semi-log canonical* or *slc* if $(\bar{X}, \bar{D} + \bar{\Delta})$ is log canonical.

2.6. Miscellaneous constructions

2.6.1. Generalized cones. We follow the construction in [BS11, 1.1.8.] Let V be a projective scheme of dimension n and L a very ample line bundle over V. Fix $N \geq n$ an integer. Set $E := \oplus^{N-n}\mathcal{O}_V$ and $p : \mathbb{P}(E \oplus L) \to V$. We denote the projectivisation $\mathbb{P}(E \oplus L)$ by X. Note that $E \oplus L$ is globally generated and we have for the tautological bundle $\xi := \mathcal{O}_{\mathbb{P}(E\oplus L)}(1)$ of $\mathbb{P}(E \oplus L)$ a surjective morphism $p^*(E \oplus L) \to \xi$. Hence we have a surjective morphism

$$H^0(V, E \oplus L) \otimes_{\mathbb{C}} \mathcal{O}_X \twoheadrightarrow \xi.$$

By Proposition 2.1, the above morphism corresponds to a unique morphism

$$\phi_{|\xi|} : X \to \mathbb{P}(H^0(V, E \oplus L)).$$

We take the Stein factorization of ϕ:

$$\begin{tikzcd} X \arrow[r] \arrow[d, "\psi_{|\xi|}"'] & \mathbb{P}(H^0(V, E \oplus L)) \\ C_N(V, L) \arrow[ur] & \end{tikzcd}$$

and call $C_N(V, L)$ the *generalized cone of dimension* N on (V, L). As ξ is big, the scheme $C_N(V, L)$ has dimension N. Set $\xi_L := \mathcal{O}_{\mathbb{P}(H^0(V,E\oplus L))}(1)|_{C_N(V,L)}$, then ξ_L is ample.

2.6.2. Albanese morphism. We give a sketch on how to construct Albanese morphism for a singular complex variety X. We first recall the Albanese morphism for smooth compact varieties.

Definition 2.33 ([Uen75, Definition 9.6.])**.** Let V be a compact complex manifold. The *Albanese torus* of V is a complex torus $A(V)$ together with a morphism $\mathrm{Alb}_V : V \to A(V)$ such that the universal property is satisfied:

For a morphism $g : V \to T$ from V to a complex torus T, there exists a unique Lie group morphism $h : A(V) \to T$ and a unique element $a \in T$ such that for all $x \in V$ we have

$$g(x) = h(\mathrm{Alb}_V(x)) + a.$$

The universal property characterizes $(A(V), \mathrm{Alb}_V)$ up to isomorphisms.

THEOREM 2.34 ([Uen75, Theorem 9.7.]). *For any smooth compact complex variety V, its Albanese torus $(A(V), \mathrm{Alb}_V)$ exists.*

When X is smooth and projective or more generally compact Kähler, we may construct $(A(X), \mathrm{Alb}_X)$ by

$$A(X) = H^0(X, \Omega_X)^\star / H_1(X, \mathbb{Z})$$
$$\mathrm{Alb}_X : x \longmapsto (\alpha \in H^0(X, \Omega_X) \mapsto \textstyle\int_{x_0}^{x} \alpha)$$

where $x_0 \in X$ is a point that we fix and the integral is defined over any path that connects x_0 to x.

Now, suppose that X is a normal projective variety, not necessarily smooth. Let $r : Y \to X$ be a projective resolution of singularities of X. We consider the following diagram

$$\begin{tikzcd} Y \arrow[r, "\mathrm{Alb}_Y"] \arrow[d, "r"'] & A(Y) \\ X \arrow[ur, dashed] & \end{tikzcd} .$$

If the rational map $\mathrm{Alb}_Y \circ r^{-1}$ has no points of indeterminacy, we can show $(\mathrm{Alb}_Y \circ r^{-1}, A(Y))$ is independent of the choice of the resolution r. We have the following

THEOREM 2.35 ([Rei83, Proposition 2.3.]). *Let X be a normal projective variety and $r : Y \to X$ be a projective resolution of singularities of X. If $R^1 f_*(\mathcal{O}_Y) = 0$, then the rational map $\mathrm{Alb}_Y \circ r^{-1}$ has no points of indeterminacy.*

CHAPTER 3

Polarized varieties with high nef value

3.1. Canonical polarized varieties

In this section, we assume that X is a projective normal variety with canonical singularities.

First we recall the definition of polarized and quasi-polarized varieties.

Definition 3.1. Let (X, L) be a pair consisting of a projective variety X and a line bundle L over X. we call it

(1) A *quasi-polarized variety* if L is nef and big;
(2) A *polarized variety* if L is ample.

Definition 3.2. Let X be a normal projective variety such that K_X is $\mathbb{Q}$-cartier and K_X is not nef. For a big and nef line bundle L on X, we define

$$\mathrm{r}(L) =: \sup\{t \in \mathbb{R} : tK_X + L \text{ is nef}\}.$$

And set $\tau(L) = \dfrac{1}{\mathrm{r}(L)}$ with the convenction $\tau(L) = +\infty$ when $\mathrm{r}(L) = 0$. We call this number $\tau(L)$ the nefvalue of L.

By Theorem 2.18, we know that r is a rational number and that there exists a K_X-negative extremal ray R such that $(rK_X + L) \cdot R = 0$.

Lemma 3.3. *Let X be a normal projective variety with canonical singularities and L a big and nef line bundle on X. Suppose that $\tau(L)$ is finite. If $K_X + (n-1)L \notin \mathrm{Pseff}(X)$, we have that $\tau(L) > n - 1$.*

Proof. We know that $\mathrm{Pseff}(X') = \overline{\mathrm{Big}(X)}$ is a closed cone. Hence there exists an ample $\mathbb{Q}$-divisor A, such that $K_X + (n-1)L + A$ is not pseudo-effective. If $\tau(L) \leq n-1$, we have that

$$K_X + (n-1)L + A = (K_X + \tau(L)L) + (n - 1 - \tau(L))L + A.$$

That is, $K_X + (n-1)L + A$ is a sum of a nef and an ample divisor, which is ample, a contradiction. □

Definition 3.4. Let (X_1, L_1), (X_2, L_2) be two pairs consisting of a normal variety X_i and a line bundle L_i on X_i.

(1) We say that(X_1, L_1) is *isomorphic* to (X_2, L_2), if there exists an isomorphism $\phi : X_1 \to X_2$ such that $\phi^*(L_2)$ is isomorphic to L_1. We denote this by $(X_1, L_1) \cong (X_2, L_2)$.
(2) We say that (X_1, L_1) and (X_2, L_2) are *birationally equivalent*, if there exists a variety X and two birational morphism $\phi_i : X \to X_i$ such that $\phi_1^*(L_1)$ is isomorphic to $\phi_2^*(L_2)$. We denote this by $(X_1, L_1) \sim_{\mathrm{bir}} (X_2, L_2)$.

Lemma 3.5. *Let X be a variety with canonical $\mathbb{Q}$-factorial singularities and L a big and nef line bundle on X. Suppose that $K_X + (n-1)L \notin \mathrm{Pseff}(X)$. Then (X, L) is birationally equivalent to (X', L'), where X' is a normal projective variety with canonical $\mathbb{Q}$-factorial singularities, $K_{X'} + (n-1)L' \notin \mathrm{Pseff}(X')$ and*

(1) Either $\tau(L')$ is finite;
(2) or there is a Mori fiber space structure $\phi : X' \to W$ and a rational number $\tau > (n-1)$ such that L' is ϕ-ample and $K_{X'} + \tau L' \sim_{\mathbb{Q},\phi} 0$.

Proof. By Theorem 2.17, we know that

$$\overline{\mathrm{NE}}(X) = \overline{\mathrm{NE}}(X)_{K_X \geq 0} + \sum \mathbb{R}_{\geq 0}[C_j]$$

where C_j are K_X-negative rational curves and the sum is over countably many j.

(1) For every K_X-negative extremal ray $R = \mathbb{R}_{\geq 0}[C]$, we have that $L \cdot C > 0$. By Theorem 2.18, there exists a K_X-negative extremal C_0 such that $r(L) = -\dfrac{L \cdot C_0}{K_X \cdot C_0} > 0$. Hence $r = 0$ only if there exists an L-trivial K_X-negative extremal ray.
(2) There exists a K_X-negative extremal ray R such that $L \cdot R = 0$. By Theorem 2.17-(3), we consider the contraction with respect to R, $\mathrm{cont}_R : X \to Z$. Note that there exists a line bundle L_Z on Z such that $L \cong \mathrm{cont}_R^*(L_Z)$.

Hence a basic idea is to run a MMP to contract every L-trivial extremal rays to get a (X', L') satisfying case (1). However we may encounter the problem with termination. We now show how to circumvent this problem.

By Theorem 2.24 and Lemma 2.23, we can find a modification $f : Y \to X$ such that Y has $\mathbb{Q}$-factorial terminal singularities. Set $L_Y = f^*L$. We have that L_Y is nef and big and $K_Y + (n-1)L_Y \notin \mathrm{Pseff}(Y)$. By [And13, Lemma 4.1.], we can find an effective $\mathbb{Q}$-divisor Δ on Y such that

$$\Delta \sim_{\mathbb{Q}} (n-1)L_Y \text{ and } (Y, \Delta) \text{ is klt.}$$

Now consider the pair (Y, Δ). We have that $K_Y + \Delta \notin \mathrm{Pseff}(Y)$. By Theorem 2.22, we can run a $(K_Y + \Delta)$-MMP to get

$$(Y, \Delta) = (Y_0, \Delta_0) \dashrightarrow (Y_1, \Delta_1) \dashrightarrow \cdots \dashrightarrow (Y_s, \Delta_s),$$

with Y_s a Mori fiber space. Suppose that the map $\phi_i : Y_i \dashrightarrow Y_{i+1}$ is associated with a $(K_{Y_i} + \Delta_i)$-negative extremal ray R_i. By [And13, Proposition 4.2.], for every $i = 0, 1, \ldots, s$, we have that

(1) Y_i is $\mathbb{Q}$-factorial terminal;
(2) $\Delta_i \cdot R_i = 0$;
(3) There exists nef and big line bundles L_i on Y_i and $\Delta_i \sim_{\mathbb{Q}} (n-1)L_i$.

It is then obvious $K_{Y_i} + (n-1)L_i \notin \mathrm{Pseff}(Y_i)$.

We then set $(X', L') := (Y_s, L_s)$.

(1) If (Y_s, Δ_s) has no K_{Y_s}-negative extremal ray R such that $L_s \cdot R = 0$, take C_0 a $K_{X'}$-negative extremal curve such that $(r(L')K_{X'} + L') \cdot C_0 = 0$. Then $r(L') = -\dfrac{L' \cdot C_0}{K_{X'} \cdot C_0} > 0$. Hence the nefvalue of L', $\tau(L') = \dfrac{1}{r(L')}$ is finite.

(2) Otherwise, we consider the Mori fiber space $\phi_s : Y_s \to W$ obtained in the above $(K_Y + \Delta)$-MMP. Let $R_s := \mathrm{NE}(\phi_s)$ be the extremal ray of ϕ_s. We claim that $L_s \cdot R_s > 0$. Suppose in contrary that $L_s \cdot R_s = 0$. Then by Theorem 2.17-(3), there exists L_W such that $\phi_s^*(L_W) = L_s$. As $\dim(W) < \dim(Y_s)$, we have that $L_s^n = \phi_s^*(L_W^n) = 0$ contradicting L_s to be nef and big. As R_s is a $(K_{Y_s} + \Delta_s)$-negative extremal ray, we have that $(K_{Y_s} + (n-1)L_s) \cdot R_s < 0$. Hence the $\tau > 0$ such that $K_{Y_s} + \tau L_s \sim_{\mathbb{Q},\phi} 0$ satisfies that $\tau > (n-1)$.

□

Before we move on to further classification, we first introduce the definition of Δ-genus.

Definition 3.6. Let (X, L) be a quasi-polarised variety of dimension n. We define its Δ-*genus* to be

$$\Delta(X, L) := n + L^n - h^0(X, L)$$

We will use the relative Kobayashi-Ochiai criterion by Andreatta.

THEOREM 3.7 ([And95, Theorem 2.1.]). *Let X be a projective variety with klt singularities and let L be a line bundle on X. Let $\phi : X \to Z$ be a surjective morphism with connected fibers between normal varieties. Suppose that L is ϕ-ample and $K_X + \tau L \sim_{\mathbb{Q},\phi} 0$ for some $\tau \in \mathbb{Q}^+$. Let $F_1 = \phi^{-1}(z)$ be a non-trivial fiber, $F \subset F_1$ be one of its irreducible components, F' be the normalization of F and let L' be the pullback of L on F. Let $\lfloor \tau \rfloor$ be the integral part of τ and $\tau' = \lceil \tau \rceil = -\lfloor -\tau \rfloor$.*

(I,1) $\dim(F) \geq \tau - 1$;
(I,2) If $\dim(F) < \tau$, *then* $F \cong \mathbb{P}^{\tau'-1}$ *and* $L|_F = \mathcal{O}_{\mathbb{P}^{\tau'-1}}(1)$;
(I,3) If $\dim(F) < \tau + 1$, *then* $\Delta(F', L') = 0$,

If moreover $\dim(F) > \dim(X) - \dim(Z)$, *then*

(II,1) $\dim(F) \geq \tau$;
(II,2) If $\dim(F) = \tau$, *then* $F \cong \mathbb{P}^{\tau}$ *and* $L|_F = \mathcal{O}_{\mathbb{P}^{\tau}}(1)$;
(II,3) If $\dim(F) < \tau + 1$, *then* $\Delta(F', L') = 0$,

If all components of the fiber satisfy $\dim(F) < \tau$, *in case (I.2) or* $\dim(F) \leq \tau$ *in case (II.3), then the fiber is actually irreducible.*

Lemma 3.8. *Let (X, L) be a quasi-polarised variety of dimension n. Suppose that X has canonical $\mathbb{Q}$-factorial singularities and $K_X + (n-1)L \notin \mathrm{Pseff}(X)$. Suppose that there exists a K_X-negative extremal ray $R = \mathbb{R}_{\geq 0}[C_0]$ such that $L \cdot C_0 > 0$. Then (X, L) is the one of the following*

(1) $(X, L) \cong (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$, *and* $\tau = n + 1$;
(2) (X, L) *is isomorphic to a* $(\mathbb{P}^{n-1}, \mathcal{O}_{\mathbb{P}^{n-1}}(1))$*-bundle over a smooth curve C and* $\tau = n$;
(3) $\Delta(X, L) = 0$, $K_X + \tau L \sim_{\mathbb{Q}} \mathcal{O}_X$ *and* $n - 1 < \tau \leq n$.

PROOF. Let $\phi : X \to Z$ be the Mori contraction of the extremal ray R. Set $t > 0$ to be the rational number such that $(K_X + tL) \cdot C_0 = 0$. Let F be a general fiber of ϕ, then $(K_X + (n-1)L)|_F \notin \mathrm{Pseff}(F)$. As $\overline{\mathrm{NE}}(F) = \mathbb{R}_{\geq 0}[C_0]$, we have that $(K_X + (n-1)L) \cdot C_0 < 0$. Thus $t > (n-1)$.

Let m be a divisible enough integer such that mK_X is a Cartier divisor and mt is an integer. The line bundle $mK_X + mtL$ is ϕ-numerically trivial. By Theorem 2.17-(3), we know that $K_X + tL \sim_{\mathbb{Q},\phi} 0$. As $\mathrm{NE}(X/Z) = R$, we have that L is ϕ-ample. Thus we are in the situation of Theorem 3.7.

We first show that ϕ is not birational. Suppose by contradiction that $\phi : X \to Z$ is birational. Let F be a component of a non trivial fiber $F_1 = \phi^{-1}(z)$. By Theorem 3.7 (II,1), we have that $\dim(F) \geq t > n-1$. Thus $\phi(X)$ is a singleton, a contradiction.

By Theorem 3.7, we know that $\dim(F) \geq t-1 > n-2$. Thus either $\dim(F) = n$ or $\dim(F) = n-1$.

(1) If $\dim(F) = n$, we have that $F = X$ and $Z = \{z\}$. Then $K_X + tL \sim_{\mathbb{Q}} \mathcal{O}_X$ and $\tau = t$. If $t > n$, Theorem 3.7 (I.2) implies that $(X, L) = (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$ and $\tau = n+1$. If $n-1 < t \leq n$, we have that $\dim(F) = n < t+1$. By Theorem 3.7 (I.3), we know that $\Delta(X, L) = 0$.

(2) $\dim(F) = n-1$. Let $F' \subset F_1$ be another component of F_1. Then Theorem 3.7 implies $\dim(F') \geq n-1$. On the other hand we could not have that $\dim(F') = n$, for this would imply that $F = F' = X$ which has dimension n, a contradiction. Hence by Theorem 3.7 agian, we know that F_1 is irreducible and $F = F_1$. As ϕ is not birational, by semi-continuity of dimensions of fibers (*cf.* for example [Sta22, Tag 02FZ]) , for any point z', $\phi^{-1}(z')$ has positive dimension. By Theorem 3.7 and repeating the argument for F and F_1, we know that $\phi^{-1}(z')$ is irreducible with dimension $n-1$. Then Theorem 3.7 (I,2) implies that for every fiber $\phi^{-1}(z')$, we have that $(\phi^{-1}(z'), L_{\phi^{-1}(z')}) \cong (\mathbb{P}^{n-1}, \mathcal{O}_{\mathbb{P}^{n-1}}(1))$. Thus we know that (X, L) is isomorphic to a $(\mathbb{P}^{n-1}, \mathcal{O}_{\mathbb{P}^{n-1}}(1))$-bundle over a smooth curve C and $\tau = n$.

□

Thus we are left in the case (3) of Lemma 3.8. In this case, we have:

Lemma 3.9. *Let (X, L) be a quasi-polarized variety of dimension n with $\Delta(X, L) = 0$. Suppose that X has canonical $\mathbb{Q}$-factorial singularities, and that the nefvalue $\tau = \tau(L)$ of L satisfies $n-1 < \tau(L) \leq n$. If $K_X + \tau L \sim_{\mathbb{Q}} \mathcal{O}_X$, then there exists a birational morphism $\mu : X \to Y$ such that*

(1) Y has canonical singularities, $\mu^*(K_Y) = K_X$;
(2) There exists an ample line bundle A on Y such that $\mu^*(A) = L$;
(3) $\Delta(Y, A) = 0$ and $K_Y + \tau A \equiv_{\mathrm{num}} \mathcal{O}_Y$.

Proof. The divisor $L - K_X \sim_{\mathbb{Q}} 2\tau L$ is nef and big. Hence we may apply the basepoint-free theorem for L ([KM98, Theorem 3.3.]), to get that for a sufficient large integer b, $|bL|$ has no basepoints. Now consider the graded algebra

$$R(X, L) =: \bigoplus_{n \geq 0} H^0(X, nL).$$

We have a canonical rational map $\mu : X \to \mathrm{Proj}(R(X, L)) =: Y$. As $\mathrm{Bs}(|bL|) = \emptyset$, we know that μ has no indeterminacy and $R(X, L)$ is finite generated (*cf.* [Deb01, Proposition 7.6.]). Hence the ring $R(X, bL)$ is integral and normal. As L is big, the morphism μ is birational and $L := \mu^*(\mathcal{O}_Y(1))$(*cf.* [Deb01, Lemma 7.10.]). By setting $A := \mathcal{O}_Y(1)$, we get (2).

We now take a divisible enough m such that mK_X is Cartier, the number $m\tau$ is an integer and $mK_X + m\tau L \sim_{\mathbb{Z}} 0$. Denote the exceptional locus of μ by E and $\nu : Y \setminus \mu(E) \to X \setminus E$ the inverse of μ. We have that

$$\mathcal{O}_Y(mK_Y)|_{Y\setminus\mu(E)} \sim \nu^*(\mathcal{O}_X|_{X\setminus E}) \sim \nu^*(-m\tau L|_{X\setminus E}) \sim -m\tau A|_{Y\setminus\mu(E)}$$

The rank one reflexive sheaf $\mathcal{O}_Y(mK_Y)$ and the line bundle $-m\tau A$ agree outside a subset whose codimension is at least 2. Hence $\mathcal{O}_Y(mK_Y)$ is a line bundle and K_Y is $\mathbb{Q}$-Cartier. We thus have the equalities $K_Y = -\tau A$ and $\mu^*(K_Y) = K_X$. Hence μ is crepant and Y has canonical singularities. We get (1). By projection formula, we have that $K_Y + \tau A = \mu_*(K_X + \tau L) = \mathcal{O}_Y$ and $\Delta(Y, A) = n + A^n - h^0(Y, A) = n + L^n - h^0(X, L) = 0$. □

Hence it rest for us to classify the polarized variety (X, L) with L ample, $n - 1 < \tau(L) \leq n$, $\Delta(X, L) = 0$ and $K_X + \tau(L)L \cong \mathcal{O}_X$. We have the following:

Lemma 3.10. *Let (X, L) be a polarized variety with L ample, $n - 1 < \tau(L) \leq n$, $\Delta(X, L) = 0$ and $K_X + \tau(L)L \equiv_{\mathrm{num}} \mathcal{O}_X$. Suppose that X has canonical singularities. Then one of the following occurs:*

(1) $(X, L) \cong (Q, \mathcal{O}_{\mathbb{P}^{n+1}}(1))$, where $Q \subset \mathbb{Q}^{n+1}$ is a hyperquadric;
(2) (X, L) is a $\mathbb{P}^{n-1}$-bundle over $\mathbb{P}^1$ and the restriction of L to each fiber is $\mathcal{O}_{P^{n-1}}(1)$;
(3) $(X, L) \cong (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$;
(4) $(X, L) \cong C_n(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$ is a generalised cone over $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$

Proof. If $\tau(L) = n$, the divisor $K_X + nL$ is numerically trivial. Then Theorem 1.2 implies that $(X, L) \cong (Q, \mathcal{O}_{\mathbb{P}^{n+1}}(1))$, where $Q \subset \mathbb{P}^{n+1}$ is a hyperquadric. Hence we are in case (1).

From now on we may assume that $\tau(L) < n$. As L is ample, we have that $K_X + nL \equiv_{\mathrm{num}} (n - \tau(L))L$ is ample.

By Fujita's classification theorem for polarized varieties with Δ-genus zero (*cf.* [Fuj90, Theorem 5.10 and Theorem 5.15] [BS11, Proposition 3.1.2.]), besides the four cases given above in Lemma 3.10, there are two more possibilities for (X, L):

(i) Either $(X, L) \cong (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$,
(ii) or (X, L) is a generalized cone over (V, L_V), where $V \subset X$ is a smooth submanifold, $L|_V = L_V$ is very ample and $\Delta(V, L_V) = 0$.

Case (i) is impossible, since $\tau(\mathcal{O}_{\mathbb{P}^n}(1)) = n + 1$. Hence we need to investigate case (ii). Set $r := n - \dim(V)$, we have by definition of the generalized cone the following diagram

$$\begin{tikzcd}
 & \mathbb{P}(\mathcal{O}_V^{\oplus r}) = V \times \mathbb{P}^{r-1} \arrow[r, "\mathrm{pr}_2"] \arrow[d, hook, "i"] & \mathbb{P}^{r-1} \arrow[d, hook] \\
\mathbb{P}(L_V) \arrow[r] \arrow[dr, "\cong"'] & \mathbb{P}(\mathcal{O}_V^{\oplus r} \oplus L_V) \arrow[r, "\psi_{|\xi|}"] \arrow[d, "\pi"] & C_n(V, L_V) = X \\
 & V &
\end{tikzcd}$$

where $\xi = \mathcal{O}_{\mathbb{P}(\mathcal{O}_V^{\oplus r} \oplus L_V)}(1)$ is the tautological bundle. The identification of $V \cong \mathbb{P}(L_V)$ is given by the quotient morphism $\mathcal{O}_V^{\oplus r} \oplus L_V \twoheadrightarrow L_V$.

We claim that outside $\mathbb{P}(\mathcal{O}_V^{\oplus r})$ the morphism $\psi_{|\xi|}$ induces an isomorphism onto its image. Take $z \in C_n(V, L)$ such that $\psi_{|\xi|}^{-1}(z)$ has positive dimension. In particular, there exists a curve C_1 such that $\psi_{|\xi|}(C_1) = \{z\}$. Since $\mathcal{O}_V^{\oplus r} \oplus L_V$ is globally generated, the morphism $\psi_{|\xi|}$ restricted to each fiber of π is an embedding. Hence π maps C_1 bijectively to its image C. By generic smoothness, there is an open subset $U \subset C$ such that $\pi : C_0 := \pi^{-1}(U) \to U$ is an isomorphism. We may regard C_0 as a section of π defined over U. That is

$$\begin{tikzcd} \mathbb{P}((\mathcal{O}_V^{\oplus r} \oplus L_V)|_U) \arrow[r] \arrow[d] & \mathbb{P}(\mathcal{O}_V^{\oplus r} \oplus L_V) \arrow[d] \\ U \arrow[r] \arrow[u, bend left, "\sigma"] & V \end{tikzcd}$$

The section σ is defined by a quotient $\rho = (\rho_1, \rho_2) : (\mathcal{O}_V^{\oplus r} \oplus L_V)|_U \twoheadrightarrow M$, with M a line bundle on U. The morphism ρ has a decomposition into $\rho_1 : \mathcal{O}_V^{\oplus r} \to M$ and $\rho_2 : L_V \to M$. As $\psi_{|\xi|} \circ \sigma(U) = \{z\}$, we know that $M \cong \sigma^*(\xi|_{\mathbb{P}((\mathcal{O}_V^{\oplus r} \oplus L_V)|_U)})$ is trivial. As $h^0(\mathrm{Hom}_{\mathcal{O}_U}(L_V|_U, \mathcal{O}_U)) = h^0(U, L_V^\vee|_U) = 0$, we have that $\rho_2 = 0$. Hence the quotient is given by $\rho_1 : \mathcal{O}_U^{\oplus n-1} \to \mathcal{O}_U$. We know that $C_0 = U \subset \mathbb{P}(\mathcal{O}_V^{\oplus r})$ and hence $C = \overline{C_0} \subset \mathbb{P}(\mathcal{O}_V^{\oplus r})$.

As $V = \mathbb{P}(L_V)$ is smooth, we have the short exact sequence

$$0 \to T_{\mathbb{P}(L_V)} \to T_{\mathbb{P}(\mathcal{O}_V^{\oplus r} \oplus L_V)}|_{\mathbb{P}(L_V)} \to N_{\mathbb{P}(L_V)/\mathbb{P}(\mathcal{O}_V^{\oplus r} \oplus L_V)} \to 0.$$

We have thus

$$\omega^\vee_{\mathbb{P}(\mathcal{O}_V^{\oplus r} \oplus L_V)}|_{\mathbb{P}(L_V)} = \omega^\vee_{\mathbb{P}(L_V)} \otimes \wedge^r N_{\mathbb{P}(L_V)/\mathbb{P}(\mathcal{O}_V^{\oplus r} \oplus L_V)}. \tag{4}$$

The canonical bundle formula gives us

$$\omega_{\mathbb{P}(\mathcal{O}_V^{\oplus r} \oplus L_V)} = \pi^*(\omega_V \otimes L_V) \otimes \xi^{\otimes -(r+1)}.$$

With $\xi|_V = L_V$, we know that $\omega_{\mathbb{P}(\mathcal{O}_V^{\oplus r} \oplus L_V)}|_V = \omega_V \otimes L_V^{\otimes -r}$. Thus Equation (4) gives

$$\wedge^r N_{\mathbb{P}(L_V)/\mathbb{P}(\mathcal{O}_V^{\oplus r} \oplus L_V)} = L_V^{\otimes r}.$$

As $\mathbb{P}(L_V)$ is disjoint from the singular locus $\mathbb{P}^{r-1} \subset X$, we also have the exact sequence

$$0 \to T_{\mathbb{P}(L_V)} \to T_X|_{\mathbb{P}(L_V)} \to N_{\mathbb{P}(L_V)/X} \to 0.$$

Hence

$$\omega_X^\vee|_{\mathbb{P}(L_V)} = \omega^\vee_{\mathbb{P}(L_V)} \otimes \wedge^r N_{\mathbb{P}(L_V)/X}.$$

Note that $N_{\mathbb{P}(L_V)/X} = N_{\mathbb{P}(L_V)/\mathbb{P}(\mathcal{O}_V^{\oplus r} \oplus L_V)}$. Hence $\omega_X|_V = \omega_V \otimes L^{\otimes -r}$. Then we have that

$$\omega_X \otimes L^{\otimes n}|_V = \omega_V \otimes L^{\otimes (n-r)}.$$

Hence the divisor $K_V + \dim(\mathrm{V}) L_V$ is ample.

If $\dim(\mathrm{V}) \geq 2$, apply [Fuj90, Theorem 5.10] again for (V, L_V). We know that (V, L_V) is one of the following:

- $(\mathbb{P}^{\dim(V)}, \mathcal{O}_{\mathbb{P}^{\dim(V)}}(1))$; or

- $(Q, \mathcal{O}_Q(1))$, where $Q \subset \mathbb{P}^{\dim(V)+1}$ is a hyperquadric; or
- $(\mathbb{P}(\mathcal{E}), \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1))$ where $\mathcal{E}$ is an ample vector bundle of rank $\dim(V)$ over $\mathbb{P}^1$; or
- $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$

Suppose first that $\dim(V) = 2$. If (V, L) is $(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))$ or $(Q, \mathcal{O}_{\mathbb{P}^3}(1))$, the divisor $K_V + 2L_V$ will not be ample. If (V, L) is a $\mathbb{P}^1$-bundle over $\mathbb{P}^1$, then $K_V + 2L_V$ is trivial on each fiber, contradicting to the fact that $K_V + 2L_V$ is ample. Hence $(V, L) \cong (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$.

If $\dim(V) = 1$, we have that $(V, L_V) \cong (\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(a))$ with $a \geq 3$. By the following Lemma 3.11 we know that for $n \geq 2$, a generalized cone $C_n(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(a))$ has singularities worse than canonical.

Hence when (X, L) is a generalized cone, we have that $(X, L) \cong C_n(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$. □

Lemma 3.11. *Let $(X, L) = C_n(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(a))$ be a generalized cone with $a \geq 3$ and $n \geq 2$. Then*

(1) X has klt singularies and X is not canonical;
(2) the nefvalue of L is $n - \frac{a-2}{a}$;
(3) $K_X + (n-1)L$ is not pseudo-effective.

Proof. Consider the following commutative diagram

$$\begin{tikzcd}
D = \mathbb{P}^1 \times \mathbb{P}^{n-2} \arrow[r, "\mathrm{pr}_2"] \arrow[d, hook, "i"] & \mathbb{P}^{n-2} \arrow[d, hook] \\
T = \mathbb{P}(\mathcal{O}^{\oplus n-1} \oplus \mathcal{O}(a)) \arrow[r, "\psi_{|\xi|}"] \arrow[d, "\pi"] & C_n(\mathbb{P}^1, \mathcal{O}(a)) \\
\mathbb{P}^1 &
\end{tikzcd},$$

where ξ is the tautological bundle $\mathcal{O}_T(1)$. The inclusion $i : D \to T$ is given by the quotient $\mathcal{O}^{\oplus n-1} \oplus \mathcal{O}(a) \to \mathcal{O}^{\oplus n-1}$. The map $\psi_{|\xi|}|_D : D \to \psi_{|\xi|}(D)$ onto its image $\mathbb{P}^{n-2}$ is identified to pr_2. We have that $\xi|_D \cong \mathrm{pr}_2^*(\mathcal{O}_{\mathbb{P}^{n-2}}(1))$ and $\pi \circ i = \mathrm{pr}_1$.

By the canonical bundle formula, we have that

$$K_T = \pi^*(\mathcal{O}_{\mathbb{P}^1}(a-2)) - n\xi, \tag{5}$$

$$K_D = \mathrm{pr}_1^*(\mathcal{O}_{\mathbb{P}^1}(-2)) - (n-1)\xi|_D. \tag{6}$$

We can thus compute

$$\mathcal{O}_T(D)|_D = K_D - K_T|_D \tag{7}$$

$$= \mathrm{pr}_1^*(\mathcal{O}_{\mathbb{P}^1}(-a)) + \mathrm{pr}_2^*(\mathcal{O}_{\mathbb{P}^{n-2}}(1)). \tag{8}$$

Consider $\mathbb{P}^1 \times \{\mathrm{pt}\} \subset D$, which is mapped isomorphically to $\mathbb{P}^1$ by π. We have that

$$\mathcal{O}_T(D) \cdot (\mathbb{P}^1 \times \{\mathrm{pt}\}) = -a < 0,$$
$$K_T \cdot (\mathbb{P}^1 \times \{\mathrm{pt}\}) = a - 2 > 0.$$

We have that

$$K_T = \psi_{|\xi|}^*(K_{C_n(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(a))}) + bD. \tag{9}$$

Intersect both sides with $\mathbb{P}^1 \times \{\mathrm{pt}\}$. We find that $0 > b = -\frac{a-2}{a} > -1$, hence $C_n(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(a))$ has klt singularities and does not have canonical singularities. As $\mathbb{P}^{n-2} \subset C_n(\mathbb{P}^1, \mathcal{O}(a))$ has inverse image D by $\psi_{|\xi|}$, we see that $\psi_{|\xi|} : (T, D) \to (C_n(\mathbb{P}^1, \mathcal{O}(a)), \mathbb{P}^{n-2})$ is a log resolution of $(C_n(\mathbb{P}^1, \mathcal{O}(a)), \mathbb{P}^{n-2})$.

We now compute the nefvalue $\tau = \tau(L)$ of L. As L is very ample, we know that $K_X + \tau L$ is nef but not ample. By Equation (9), we have that

$$K_T - bD + \tau\xi = \psi_{|\xi|}^*(K_X + \tau L).$$

Thus the restriction $(K_T - bD + \tau\xi)|_D$ is nef on D. By Equation (5), Equation (6) and Equation (8), we have that

$$(K_T - bD + \tau\xi)|_D = (\tau - (n+b))\mathrm{pr}_2^*(\mathcal{O}_{\mathbb{P}^{n-2}}(1)).$$

Hence $\tau \geq n + b$. Let $C \subset T$ be a curve. We will consider its intersection with $K_T - bD + \tau\xi = \pi^*(\mathcal{O}_{\mathbb{P}^1}(a-2)) + (\tau - n)\xi - bD$. We have 3 situations:

(1) The curve C is contained in D. Then $(K_T - bD + \tau\xi) \cdot C = (\tau - (n + b))\mathrm{pr}_2^*(\mathcal{O}_{\mathbb{P}^{n-2}}(1)) \cdot C \geq 0$.
(2) The generic point of C is not in D and C is contracted by π. Then C is a curve in a fiber F of π. We have that $F = \mathbb{P}^{n-1}$ and $\xi|_F = \mathcal{O}_{\mathbb{P}^{n-1}}(1)$. If $C \cap D = \emptyset$, then $D \cdot C = 0$ and $(K_T - bD + \tau\xi) \cdot C = (\tau - n)(\xi \cdot C)$; if $C \cap D \neq \emptyset$, we have that $D|_F = \mathbb{P}^{n-2} = \xi|_F$ and $(K_T - bD + \tau\xi) \cdot C = (\tau - n - b)(\xi \cdot C)$.
(3) The generic point of C is not in D and $\pi : C \to \mathbb{P}^1$ is a finite morphism. We identify $\mathbb{P}^1$ with $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(a)) \subset T$. The finite morphism is thus given by

$$\begin{tikzcd} C \arrow[r] \arrow[d] & \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(a)) \arrow[dl] \\ T & \end{tikzcd}$$

As $\xi|_{\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(a))} = \mathcal{O}_{\mathbb{P}^1}(a)$, we have that

$$(\pi^*(\mathcal{O}_{\mathbb{P}^1}(a-2)) + (\tau - n)\xi - bD) \cdot C = \deg(C/\mathbb{P}^1)(a - 2 + (\tau - n)a) = 0.$$

We claim that $\tau = n+b$. By case (2), we know that $\tau \geq n+b$ and $K_T - bD + (n+b)\xi = \phi_{|\xi|}^*(K_X + (n+b)L)$ is nef. On the other hand, any curve $C \subset X$ is dominated by a curve $C' \subset T$. Hence $\tau(L) = n + b$.

Finally we show that $K_X + (n-1)L \notin \mathrm{Pseff}(X)$. In fact, suppose by contradiction that $K_X + (n-1)L$ is pseudo-effective. Then so is $K_T - bD + (n-1)\xi = \phi_{|\xi|}^*(K_X + (n-1)L)$. We restrict the line bundle to $\{\mathrm{pt}\} \times \mathbb{P}^{n-2} \subset D$. The divisor $-(1+b)\mathrm{pr}_2^*(\mathcal{O}_{\mathbb{P}^{n-2}}(1))$ is pseudo-effective. As $-1 < b < 0$, we know that $(1+b)\mathrm{pr}_2^*(\mathcal{O}_{\mathbb{P}^{n-2}}(1)) \in \mathrm{Pseff}(\mathbb{P}^{n-2}) \setminus \{0\}$, a contradiction. □

Lemma 3.12. *Let $(X, L) = C_n(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(a))$ be a generalized cone with $a \leq 2$ and $n \geq 2$. Then $K_X + (n-1)L \in \mathrm{Pseff}(X)$.*

Proof. We use the same notation as in Lemma 3.11. Using the same computation as in Lemma 3.11, we have that

$$K_T = \psi_{|\xi|}^*(K_{C_n(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(a))}) + bD. \tag{10}$$

Intersect with $\mathbb{P}^1 \times \{\mathrm{pt}\}$. We find that $b = -\frac{a-2}{a} \geq 0$. Hence (X, L) has canonical singularity.

Suppose by contradiction that $K_X + (n-1)L$ is not pseudo-effective. As $\rho(X) = 1$, Lemma 3.8 implies that $\Delta(X, L) = 0$ and $K_X + \tau L \sim_{\mathbb{Q}} \mathcal{O}_X$ and $n - 1 < \tau(L) \leq n$. The computation in Lemma 3.10 shows that $K_{\mathbb{P}^1} + \mathcal{O}_{\mathbb{P}^1}(a) = \mathcal{O}_{\mathbb{P}^1}(a-2)$ is ample. This is a contradiction. □

Combining all the precedent results, we have the following:

THEOREM 3.13 (=Theorem 1.6). *Let X be a variety with canonical $\mathbb{Q}$-factorial singularities and L a big and nef line bundle on X. Suppose that $K_X + (n-1)L \notin \mathrm{Pseff}(X)$. Then we have one of the following cases:*

(1) $(X, L) \sim_{\mathrm{bir}} (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$;
(2) (X, L) is birational equivalent to a $(\mathbb{P}^{n-1}, \mathcal{O}_{\mathbb{P}^{n-1}}(1))$-bundle over a smooth curve C;
(3) $(X, L) \sim_{\mathrm{bir}} (Q, \mathcal{O}_{\mathbb{P}^{n+1}}(1))$, where $Q \subset \mathbb{Q}^{n+1}$ is a hyperquadric;
(4) $(X, L) \sim_{\mathrm{bir}} (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$;
(5) $(X, L) \sim_{\mathrm{bir}} C_n(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$, where $C_n(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$ is a generalised cone over $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$

PROOF. By Lemma 3.5, there exists a birational equivalence $(X, L) \sim_{\mathrm{bir}} (X', L')$, where X' is a normal projective variety with canonical $\mathbb{Q}$-factorial singularity, $K_{X'} + (n-1)L' \notin \mathrm{Pseff}(X')$ and

(0-i) Either $\tau(L')$ is finite;
(0-ii) or there is a Mori fiber space structure $\phi : X' \to W$ and a rational number $\tau > (n-1)$ such that L' is ϕ-ample and $K_{X'} + \tau L' \sim_{\mathbb{Q},\phi} 0$.

In the first case, we have that $r(L') = \frac{1}{\tau(L')} > 0$. Hence by Kawamata rationality theorem there exists an K'_X-negative extremal ray $R_0 = \mathbb{R}_{\geq 0}[C_0]$ such that $(r(L')K_{X'} + L') \cdot C_0 = 0$. Hence $L' \cdot C_0 > 0$. In the second case, take $R_0 = \mathbb{R}_{\geq 0}[C_0]$ to be the extremal ray associated to ϕ. Then $L' \cdot C_0 > 0$.

Applying Lemma 3.8 on (X', L'), we get that (X', L') is the one of the following

(1) $(X', L') \cong (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$, and $\tau = n + 1$;
(2-i) (X', L') is isomorphic to a $(\mathbb{P}^{n-1}, \mathcal{O}_{\mathbb{P}^{n-1}}(1))$-bundle over a smooth curve C and $\tau = n$;
($*$) $\Delta(X', L') = 0$, $K_{X'} + \tau L' \equiv_{\mathrm{num}} \mathcal{O}_X$ and $n - 1 < \tau \leq n$.

If we are in case ($*$), apply Lemma 3.9. We have a birational morphism $\mu : X' \to X''$ such that

(a) X'' has canonical singularities, $\mu^*(K_{X''}) = K_{X''}$;
(b) There exists an ample line bundle L'' on X'' such that $\mu^*(L'') = L'$;
(c) $\Delta(X'', L'') = 0$ and $K_{X''} + \tau L'' \equiv_{\mathrm{num}} \mathcal{O}_{X''}$.

In particular $(X', L') \sim_{\mathrm{bir}} (X'', L'')$. Now apply Lemma 3.10 to (X'', L''). We have that (X'', L'') is isomorphic to the following pair:

(3) $(X'', L'') \cong (Q, \mathcal{O}_{\mathbb{P}^{n+1}}(1))$, where $Q \subset \mathbb{Q}^{n+1}$ is a hyperquadric;
(2-ii) (X'', L'') is a $\mathbb{P}^{n-1}$-bundle over $\mathbb{P}^1$ and L restricted to each fiber is $\mathcal{O}_{P^{n-1}}(1)$;
(4) $(X'', L'') \cong (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$;

(5) $(X'', L'') \cong C_n(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$ is a generalised cone over $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$

Thus we get the list stated in Theorem 3.13 □

3.2. Normal polarized varieties

With the help of canonical modification Theorem 2.25, we can give a classification theorem for normal polarized varieties with $\mathbb{Q}$-Gorenstein singularities.

THEOREM 3.14 (=Theorem 1.7). *Let (X, L) be a polarized normal variety of dimension n. Suppose that K_X is $\mathbb{Q}$-Cartier and $K_X + (n-1)L \notin \mathrm{Pseff}(X)$. Then we have one of the following cases :*

(1) $(X, L) \cong (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$;
(2.i) $(X, L) \cong (\mathbb{P}(\mathcal{V}), \mathcal{O}_{\mathbb{P}(\mathcal{V})}(1))$, where $\mathcal{V}$ is a rank n ample vector bundle over a smooth curve C;
(2.ii) $(X, L) \cong C_n(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(a))$ with $a \geq 3$, where $C_n(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(a))$ is a generalized cone;
(3) $(X, L) \cong (Q, \mathcal{O}_{\mathbb{P}^{n+1}}(1))$, where $Q \subset \mathbb{P}^{n+1}$ is a hyperquadric;
(4) $(X, L) \cong (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$;
(5) $(X, L) \cong C_n(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$, a generalised cone over $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$.

PROOF. Apply Theorem 2.25 to the pair $(X, 0)$. We get the canonical modification $f : X' \to X$ with $K_{X'}$ being f-ample. We take a further step, applying Lemma 2.23 to get a small $\mathbb{Q}$-factorial modification $g : Y \to X'$ of X'. We denote the composition $g \circ f$ by μ. As g is small, we have that $K_Y = g^*(K_{X'})$ is μ-nef. Note that $\mu|_{\mu^{-1}(X_{\mathrm{reg}})} : \mu^{-1}(X_{\mathrm{reg}}) \to X_{\mathrm{reg}}$ is an isomorphism. For the canonical sheaves $\omega_Y = \mathcal{O}_Y(K_Y)$ and $\omega_X = \mathcal{O}_X(K_X)$, we know that $\mu_*(\omega_Y)|_{X_{\mathrm{reg}}} \cong \omega_X|_{X_{\mathrm{reg}}}$. Note that $\mu_*(\omega_Y)$ is torsion-free, so we have an injection $\mu_*(K_Y) \rightarrowtail K_X$. By the projection formula we have an injection

$$\mathcal{O}_X(\mu_*(K_Y + (n-1)\mu^* L)) \rightarrowtail \mathcal{O}_X(K_X + (n-1)L).$$

As $K_X + (n-1)L$ is not pseudo-effective, we know that neither is $K_Y + (n-1)\mu^*(L)$. We set $\mu^*(L) := M$. As M is nef and big, we know that $M \in \mathrm{Pseff}(Y)$. Note that K_Y is not pseudo-effective, hence it is not nef.

Let $R = \mathbb{R}_{\geq 0}[C]$ be a K_Y-negative extremal ray, with $C \subset Y$ a rational curve. Then $K_Y \cdot C < 0$. As K_Y is μ-nef, we know that C is not contracted by μ. Hence $\mu(C) \subset X$ has dimension 1. Since L is ample, we have that $M \cdot C = \deg(C/\mu(C))L \cdot \mu(C) > 0$. Thus for any K_Y-negative extremal ray R, we have that $M \cdot R > 0$. Lemma 3.5 implies that $\mathrm{r}(M) > 0$ and $\tau(M) > n-1$. By Lemma 3.8 applied to (Y, M), we have one of the following cases:

(i) $(Y, M) \cong (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$, and $\tau = n+1$, or
(ii) (Y, M) is isomorphic to a $(\mathbb{P}^{n-1}, \mathcal{O}_{\mathbb{P}^{n-1}}(1))$-bundle over a smooth curve C and $\tau = n$, or
(iii) $\Delta(Y, M) = 0$, $K_Y + \tau M \sim_{\mathbb{Q}} \mathcal{O}_Y$ and $n-1 < \tau \leq n$.

In case (i), we have a birational morphism $\mu : \mathbb{P}^n \to X$ with $\mu^*(L) = \mathcal{O}_{\mathbb{P}^n}(1)$. We have that $\overline{\mathrm{NE}}(\mathbb{P}^n/X) = 0$ since both L and $\mathcal{O}_{\mathbb{P}^n}(1)$ are ample. By Lemma 2.16, the morphism μ is an isomorphism. We have case (1) in Theorem 3.14.

In case (ii), we have a birational morphism $\mu : \mathbb{P}(\mathcal{V}) \to X$, such that $K_{\mathbb{P}(\mathcal{V})}$ is μ-nef. We denote by ξ the pull-back $\mathcal{O}_{\mathbb{P}(\mathcal{V})}(1) = \mu^*(L)$. We know that ξ is nef and big.

We first note that ξ is ample if and only if μ is an isomorphism. In fact, if μ is an isomorphism, then ξ is ample. Conversely, if ξ is ample, we have that $\overline{\mathrm{NE}}(\mathbb{P}(\mathcal{V})/X) = 0$ and by Lemma 2.16 μ is an isomorphism. In this case, we have that

$$K_{\mathbb{P}(\mathcal{V})} + (n-1)\xi = \pi^*(K_C + \det\mathcal{V}) - \xi$$

is not pseudo-effective. In fact, the general fiber f is from a covering family and we have that $K_{\mathbb{P}(\mathcal{V})} + (n-1)\xi|_f = \mathcal{O}_f(-1)$. Theorem 2.13 implies that $K_{\mathbb{P}(\mathcal{V})} + (n-1)\xi$ is not pseudo-effective. Thus we get (*2,i*).

Now suppose that ξ is not ample. Then μ is not an ismorphism. We have the following diagram:

$$\begin{tikzcd} (\mathbb{P}(\mathcal{V}), \xi) \arrow[r, "\mu"] \arrow[d, "\pi"'] & (X, L) \\ C & \end{tikzcd} .$$

We know that $\rho(\mathbb{P}(\mathcal{V})) = 2$. As $\mu \neq \pi$, we have that

$$\overline{\mathrm{NE}}(\mathbb{P}(\mathcal{V})) = \mathrm{NE}(\pi) + \mathrm{NE}(\mu).$$

We denote a general fiber of π by f. By [Ful11, Page 450], we know that $\overline{\mathrm{NE}}(\mathbb{P}(\mathcal{V}))$ has as extremal rays $\mathbb{R}_{\geq 0}\xi^{n-2}f$ and $\mathbb{R}_{\geq 0}(\xi^{n-1} + \nu^{(n-1)}\xi^{n-2}f)$ for some $\nu^{(n-1)} \in \mathbb{Q}$. Note that $\mathbb{P}^1 = \xi^{n-2}f$ is contracted by π. Hence $\mathrm{NE}(\pi) = \mathbb{R}_{\geq 0}\xi^{n-2}f$. We have that $K_{\mathbb{P}(\mathcal{V})} = \pi^*(K_C + \det(\mathcal{V})) - n\xi$. Hence $K_{\mathbb{P}(\mathcal{V})} \cdot \xi^{n-2}f = -n$. Thus π is the Mori contraction associated to the extremal ray $\mathbb{R}_{\geq 0}\xi^{n-2}f$. As $\overline{\mathrm{NE}}(\mu)$ is an extremal ray, we know that μ is an extremal contraction and Lemma 2.19 implies that μ is either small or divisorial.

If μ is small, we have that $K_{\mathbb{P}(\mathcal{V})} = \mu^*(K_X)$. As $\rho(X) = 1$, we have that $K_X \equiv_{\mathrm{num}} mL$ for some $m \in \mathbb{Q}$. Hence $K_{\mathbb{P}(\mathcal{V})} \equiv_{\mathrm{num}} m\xi$. We have that

$$m = m\xi \cdot \xi^{n-2}f = K_{\mathbb{P}(\mathcal{V})} \cdot \xi^{n-2}f = -n.$$

Thus we get that $K_X + nL \equiv_{\mathrm{num}} \mathcal{O}_X$. By Theorem 1.2, we have that $(X, L) \cong (Q, \mathcal{O}_Q(1))$ where $Q \subset \mathbb{P}^{n+1}$ is a hyperquadric. Hence we are in case (3) of Theorem 3.14.

If μ is divisorial, we denote the exceptional divisor by $E = \mathrm{exc}(\mu)$. Note that $\mu^*(L) = \xi = \mathcal{O}_{\mathcal{V}}(1)$ is nef, hence $\mathcal{V}$ is nef. We have a unique exact sequence of locally free sheaves:

$$0 \to \mathcal{A} \to \mathcal{V} \to \mathcal{Q} \to 0$$

with $\mathcal{A}$ is an ample vector bundle and $\mathcal{Q}$ is numerically flat. If $l \subset \mathbb{P}(\mathcal{V})$ is a curve such that $\xi \cdot l = 0$, then l is containted in $\mathbb{P}(\mathcal{Q})$. Thus we have that $E \subset \mathbb{P}(\mathcal{Q})$. In particular, $\mathrm{rk}(\mathcal{Q}) = n-1$ and $E = \mathbb{P}(\mathcal{Q})$. We denote the bundle morphism by $\pi' : \mathbb{P}(\mathcal{Q}) \to C$. Now

we compute $E|_E$.

$$\begin{aligned} E|_E &= (K_{\mathbb{P}(\mathcal{Q})} - K_{\mathbb{P}(\mathcal{V})})|_E \\ &= \pi'^*(K_C + \det\mathcal{Q}) - (n-1)\xi|_E - (\pi^*(K_C + \det\mathcal{V}) - n\xi)|_E \\ &= \pi'^*(\det\mathcal{Q} - \det\mathcal{V}) + \xi|_E \\ &= \pi'^*(-\mathcal{A}) + \xi|_E. \end{aligned}$$

Take a rational curve l that is in the fiber of π'. We have that $E \cdot l = E|_E \cdot l = 1$. Now write $K_{\mathbb{P}(\mathcal{V})} = \mu^*(K_X) + \lambda E$. As $\rho(X) = 1$, there exists some $m \in \mathbb{Q}$ such that $K_X \equiv_{\text{num}} mL$. As $K_X + (n-1)L \equiv_{\text{num}} (m+n-1)L \notin \text{Pseff}(X)$, we have that $m+n < 1$. Intersect with l and get

$$-n = K_{\mathbb{P}(\mathcal{V})} \cdot l = (\mu^*(K_X) + \lambda E) \cdot l = (m\xi + \lambda E) \cdot l = m + \lambda.$$

Hence $\lambda = -m - n > -1$ and X has klt singularities. A π'-fiber is isomorphic to $\mathbb{P}^{n-2}$ and is mapped isomorphically onto its image by μ. Hence each non-trivial μ-fiber has dimension 1. As X has klt singularities, *a fortiori* $(X, 0)$ is dlt. Applying [HM07, Corollary 1.5-(1)] to the birational morphism μ, each μ-fiber is rationally chain connected. Hence a non trivial fiber has $\mathbb{P}^1$ as its normalization. We have thus a finite map $\pi'|_{\mathbb{P}^1} : \mathbb{P}^1 \to C$. Thus $C = \mathbb{P}^1$ and $\mathcal{V} = \mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}^{\oplus(n-1)}$.

Consider the morphism $\psi : \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}^{\oplus(n-1)}) \to C_n(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(a))$. We know that ψ does not contract the extremal ray $\overline{\text{NE}}(\pi)$. Hence $\overline{\text{NE}}(\psi) = \overline{\text{NE}}(\mu)$ and by Lemma 2.16 we have $X = C_n(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(a))$. As L and the restriction of $\mathcal{O}_{\mathbb{P}(H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(a)))}$ to $C_n(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(a))$ agree outside a subscheme of codimension at least 2, we have that $(X, L) = C_n(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(a))$. As $K_X + (n-1)L \notin \text{Pseff}(X)$, Lemma 3.12 implies $a \geq 3$. Now Lemma 3.11 shows that for all $a \geq 3$, we have that $K_X + (n-1)L \notin \text{Pseff}(X)$ and X is klt. Thus we get *(2,ii)*

If we are in case (iii), apply Lemma 3.9 to (Y, M). We have a crepant resolution $\nu : Y \to Y_{\text{can}}$ with an ample divisor A on Y_{can} such that $\nu^*(A) = M$, the Δ-genus satisfies $\Delta(Y_{\text{can}}, A) = 0$ and $K_{Y_{\text{can}}} + \tau A \equiv_{\text{num}} \mathcal{O}_{Y_{can}}$. By Lemma 3.10, the polarized variety (Y_{can}, A) is isomorphic to one of the following:

(a) $(Q, \mathcal{O}_{\mathbb{P}^{n+1}}(1))$, where $Q \subset \mathbb{Q}^{n+1}$ is a hyperquadric;
(b) a $\mathbb{P}^{n-1}$-bundle over $\mathbb{P}^1$ and L restricted to each fiber is $\mathcal{O}_{\mathbb{P}^{n-1}}(1)$;
(c) $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$;
(d) a generalised cone $C_n(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$ over $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$.

Case (b) is a special case of (ii) treated above. In case (a),(c) and (d), we have the following diagram

$$\begin{tikzcd} (Y, M) \arrow[r, "\mu"] \arrow[d, "\nu"] & (X, L) \\ (Y_{\text{can}}, A) \arrow[ur, dashed, "h"'] & \end{tikzcd}$$

where h is a birational map *a priori* not necessarily defined on all Y_{can}. We now show that h is indeed an isomorphism and $h^*(L) = A$. Let $C \subset Y$ be a curve. We have that

$$\nu^*(A) \cdot C = M \cdot C = \mu^*(L) \cdot C.$$

As A and L are both ample, we have that $\mathrm{NE}(\mu) = \mathrm{NE}(\nu)$. Lemma 2.16 implies that h is an isomorphism. As $h^*(L)$ agrees with A outside a subscheme of codimension at least 2, we have that $h^*(L) = A$. Hence we get case (3), (4), (5) in Theorem 3.14. $\square$

Using similar methods, we can classify log pairs (X, D) with D a prime Weil divisor. First let us recall:

Definition 3.15. Let $n > 0$ be a integer. A *hyperquadric* Q in the projective space $\mathbb{P}^n = \mathrm{Proj}(\mathbb{C}[x_0, x_1, \ldots, x_n])$ is a subscheme whose ideal sheaf has the form $\mathcal{O}_{\mathbb{P}^n}(-D)$ where $D = \mathrm{div}(s)$ for some $s \in H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(2))$. As $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(2))$ is canonically isomorphic to $(\mathbb{C}[x_0, x_1, \ldots, x_n])_2$, we can write $s = \sum_{0\le i\le n, 0\le j\le n} a_{ij}x_ix_j$ with $a_{ij} \in \mathbb{C}$. We define $\mathrm{rk}(Q)$ the *rank* of the hyperquadric Q to be the rank of the matrix $A + A^t$, where $A_{ij} = a_{ij}$. If s' is another section of $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(2))$ such that $\mathcal{O}_{\mathbb{P}^n}(-\mathrm{div}(s')) = \mathcal{O}_{\mathbb{P}^n}(-D)$, then there exists a unique $r \in \mathbb{C} \setminus 0$ such that $rs = s'$. Hence the rank of Q is well defined.

We have the following classification.

THEOREM 3.16. *Let (X, D) be a log canonical pair, $D \subset X$ a prime divisor, $\dim(X) = n$. Suppose that L is an ample line bundle on X and $(K_X + D) + (n-1)L \notin \mathrm{Pseff}(X)$. Then (X, D, L) is one of the following:*

(1) $(X, L) \cong (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$ and D is a hyperplane in $\mathbb{P}^n$;

(2.i) There is a $(\mathbb{P}^{n-1}, \mathcal{O}_{\mathbb{P}^{n-1}}(1))$-bundle $(\mathbb{P}(E), \mathcal{O}_{\mathbb{P}(E)}(1))$ over a smooth curve C, and a birational morphism $\mu : \mathbb{P}(E) \to X$ such that $\mu^*(L) \cong \mathcal{O}_{\mathbb{P}(E)}(1)$ and $D \cong \mu(\mathbb{P}^{n-1})$ is the image of a general fiber of π by μ and $\mathbb{P}^{n-1} \to D$ has degree 1;

(2.ii) $(X, L) = (\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(1)), \mathcal{O}_{\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(a)\oplus\mathcal{O}_{\mathbb{P}^1}(1))}(1))$ with $a > 1$ and D is the unique section of $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(1)) \to \mathbb{P}^1$ such that

$$D \equiv_{\mathrm{num}} \mathcal{O}_{\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(a)\oplus\mathcal{O}_{\mathbb{P}^1}(1))}(1)) - af,$$

where f is a general fiber;

(3.i) $(X, L) \cong (Q, \mathcal{O}_{\mathbb{P}^{n+1}}(1))$, where $Q \subset \mathbb{P}^{n+1}$ is a $\mathrm{rk}(Q) = 3$ hyperquadric and $D \cong \mathbb{P}^{n-1}$ is a hyperplane in Q with $D \equiv_{\mathrm{num}} \frac{1}{2}\mathcal{O}_Q(1)$;

(3.ii) $(X, L) \cong (Q, \mathcal{O}_{\mathbb{P}^{n+1}}(1))$, where $Q \subset \mathbb{P}^{n+1}$ is a $\mathrm{rk}(Q) = 4$ hyperquadric. If we write $Q = \mathrm{Proj}\left(\frac{\mathbb{C}[x_0, \ldots, x_{n+1}]}{(x_0x_1 - x_2x_3)}\right)$, then D is the cone with vertex $\mathbb{P}^{n-3}$ over $\mathbb{P}^1 \times \mathrm{pt}$ or $\mathrm{pt} \times \mathbb{P}^1$. In particular, $D \cong \mathbb{P}^{n-1}$.

PROOF. As in the proof of Theorem 3.14, we apply Theorem 2.25 then Lemma 2.23 to the normal variety X. We get a birational morphism $\mu : Y \to X$ such that Y has $\mathbb{Q}$-factorial canonical singuarities, the canonical class K_Y is μ-nef and μ is isomorphic over the regular locus of X. We set $D' = \mu_*^{-1}(D)$. If we denote by $\omega_Y = \mathcal{O}_Y(K_Y)$ and by $\omega_X = \mathcal{O}_X(K_X)$ the canonical sheaves, we know that

$$\mu_*(\omega_Y \otimes \mathcal{O}_Y(D'))|_{X_{\mathrm{reg}}} \cong (\omega_X \otimes \mathcal{O}_X(D))|_{X_{\mathrm{reg}}}$$

Note $\mu_*(\omega_Y \otimes \mathcal{O}_Y(D'))$ is torsion-free, so we have an injection $\mu_*(\omega_Y \otimes \mathcal{O}_Y(D')) \rightarrowtail \omega_X \otimes \mathcal{O}_X(D)$. Tensoring with $\mu^*(L^{\otimes n-1})$, we have an injection

$$\mu_*(\omega_Y \otimes \mathcal{O}_Y(D') \otimes \mu^*(L^{\otimes n-1})) \rightarrowtail \omega_X \otimes \mathcal{O}_X(D) \otimes L^{\otimes n-1}.$$

As $(K_X + D) + (n-1)L$ is not pseudo-effective, neither is $(K_Y + D') + (n-1)\mu^*(L)$. We set $\mu^*(L) =: M$. As D' is effective, the divisor $K_Y + (n-1)M$ is not pseudo-effective.

We note that as K_Y is μ-nef and $M = \mu^*(L)$, for any K_Y-negative extremal ray R, we have that $M \cdot R > 0$. Hence we may argue as in Theorem 3.14 by applying Lemma 3.8. We thus get:

(a) Either $(Y, M) \cong (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$ and $\tau = n+1$, or
(b) (Y, M) is isomorphic to a $(\mathbb{P}^{n-1}, \mathcal{O}_{\mathbb{P}^{n-1}}(1))$-bundle over a smooth curve C and $\tau = n$, or
(c) $\Delta(Y, M) = 0$, $K_Y + \tau M \sim_{\mathbb{Q}} \mathcal{O}_Y$ and $n - 1 < \tau \leq n$.

In case (a), μ is an isomorphism. We know that $D \sim \mathcal{O}_{\mathbb{P}^n}(a)$ for some $a \geq 1$. We have that $K_X + (n-1)L + D = \mathcal{O}_{\mathbb{P}^n}(a-2)$. Hence the only possible choice is $a = 1$ and D is a hyperplane and we are in case (1) of Theorem 3.16.

In case (b), we have a diagram

$$\begin{array}{ccc} (\mathbb{P}(\mathcal{V}), \xi) & \xrightarrow{\mu} & (X, L) \\ \pi \big\downarrow & & \\ C & & \end{array}$$

where $\xi = \mathcal{O}_{\mathbb{P}(\mathcal{V})}(1)$ and $K_{\mathbb{P}(\mathcal{V})}$ is μ-nef. As $\tau(\xi) = n$, we know that $K_{\mathbb{P}(\mathcal{V})} + n\xi = \pi^*(K_C + \det(\mathcal{V}))$ is nef.

First we assume that μ is an isomorphism. In this case $\mathcal{V}$ is ample. Let $F = \mathbb{P}^{n-1}$ be a general fiber of π. Suppose that $D|_F = \mathcal{O}_F(d)$ with $d \geq 0$. We have the following equality

$$(K_{\mathbb{P}(\mathcal{V})} + D + (n-1)\xi)|_F = (\pi^*(K_C + \det(\mathcal{V})) + D - \xi)|_F = \mathcal{O}_F(d-1).$$

If $d = 0$, then D is one of the general fiber. In fact, if for a general fiber F, we have that $D \cap F \neq \emptyset$ and $D \not\subseteq F$, there exists a curve $l \subset F \setminus D$ such that $l \cap D \neq \emptyset$. We have that $D \cdot l > 0$, a contradiction. Let l be a rational curve in F. We have that $(K_{\mathbb{P}(\mathcal{V})} + D + (n-1)\xi) \cdot l = -1$. Since F is a member of a covering family, Theorem 2.13 implies that $K_{\mathbb{P}(\mathcal{V})} + (n-1)\xi + F$ is not pseudo-effective. We thus are in case (2) of Theorem 3.16.

If $d > 0$, by Lemma 3.17, we have that $n = \dim(\mathbb{P}(\mathcal{V})) = 2$. We first show that $C = \mathbb{P}^1$. The non pseudo-effective divisor in question $K_X + D + (n-1)L$ thus becomes $K_{\mathbb{P}(\mathcal{V})} + D + \xi$. We have that $(K_{\mathbb{P}(\mathcal{V})} + D + \xi)|_F = \mathcal{O}_F(d-1)$ which is nef. As $K_{\mathbb{P}(\mathcal{V})} + D + \xi$ is not nef, there exists an extremal ray R' which is not generated by the fiber of π, such that $(K_{\mathbb{P}(\mathcal{V})} + D + \xi) \cdot R' < 0$. In particular, we have that R' is $(K_{\mathbb{P}(\mathcal{V})} + D)$-negative. By Theorem 2.17-(4) we know that $R' = \mathbb{R}_{\geq 0}[l]$ for a rational curve l. Note that l maps finitely onto C. Hence $C \cong \mathbb{P}^1$.

Lemma 3.18 implies that (X, D, L) is either in cases $(2.i)$, $(2.ii)$ of Theorem 3.16 or (X, L) is a hyperquadric of rank 4, which will be dealt in the following case $(c1)$.

Assume, from now on, that μ is not an isomorphism. We know that

$$\overline{\mathrm{NE}}(\mathbb{P}(\mathcal{V})) = \overline{\mathrm{NE}}(\mu) + \overline{\mathrm{NE}}(\pi),$$

and π contracts the extremal ray $\mathbb{R}_{\geq 0}\xi^{n-2}f$. The birational morphism μ is either small or divisorial.

If μ is small, by construction, we have that $K_{\mathbb{P}(\mathcal{V})} + D' = \mu^*(K_X + D)$. Let F be a general fiber of π. We have that $D'|_F = \mathcal{O}_F(d)$ for some $d \geq 0$. As $K_X + D$ is $\mathbb{Q}$-Cartier and $\rho(X) = 1$, we have that $K_X + D \equiv_{\text{num}} mL$ for some $m \in \mathbb{Q}$. Hence $K_{\mathbb{P}(\mathcal{V})} + D' \equiv_{\text{num}} \mu^* mL$. Intersect with $\xi^{n-2}f$. We get that $-n + d = m$. Hence $K_X + D + (n-1)L \equiv_{\text{num}} (d-1)L$. Thus we have that $d = 0$ and $D' \sim F$. As $D' = \mu_*^{-1}(D)$ by definition, we get that $D' \to D$ has degree 1. We are thus in case $(2.i)$ of Theorem 3.16.

If μ is divisorial, we denote the exceptional divisor by $E = \text{exc}(\mu)$. [KM98, Proposition 3.36.] implies that X is $\mathbb{Q}$-factorial. In particular K_X is $\mathbb{Q}$-Cartier. We have a unique exact sequence of locally free sheaves:

$$0 \to \mathcal{A} \to \mathcal{V} \to \mathcal{Q} \to 0$$

with $\mathcal{A}$ is an ample vector bundle and $\mathcal{Q}$ is numerically flat. And we know that $E = \mathbb{P}(\mathcal{Q})$ and $E \cdot \xi^{n-2}f = 1$. Let F be a general fiber of π. There exists a $d \geq 0$ such that $D'|_F = \mathcal{O}_F(d)$. As $K_X + D$ is $\mathbb{Q}$-Cartier and $\rho(X) = 1$, there exists an $m \in \mathbb{Q}$ such that $K_X + D \equiv_{\text{num}} mL$. Then $K_X + D + (n-1)L \equiv_{\text{num}} (m + n - 1)L \notin \text{Pseff}(X)$. Hence $m + n < 1$. We now have that

$$K_{\mathbb{P}(\mathcal{V})} + D' = \mu^*(K_X + D) + \lambda E.$$

Intersect both sides with $\xi^{n-2}f$. We get that $-n + d = m + \lambda$. Since $-(m + n) > -1$, we have that $\lambda \geq -1 + d$. Now we claim that $d = 0$. Suppose that by contrary $d \geq 1$. Recall that $(K_{\mathbb{P}(\mathcal{V})} + D' + (n-1)\xi)|_F = \mathcal{O}_F(d-1)$. As $K_{\mathbb{P}(\mathcal{V})} + D' + (n-1)\xi$ is not nef, we know that $\text{NE}(\mu)$ is an $(K_{\mathbb{P}(\mathcal{V})} + D')$-negative extremal ray. By Theorem 2.17-(iii), there is a rational curve l whose class $[l]$ is in $\overline{\text{NE}}(\mu)$. As l maps finitely onto C, we know that $C \cong \mathbb{P}^1$. Hence $(X, L) = C_n(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(a))$. We have by assumption that $K_X + (n-1)L \notin \text{Pseff}(X)$. Hence Lemma 3.12 implies that $a \geq 3$. Lemma 3.11 implies that $K_X \equiv_{\text{num}} (-n + \dfrac{a-2}{a})L$. Suppose that $D \equiv_{\text{num}} m_2 L$ for some $m_2 \in \mathbb{Q}^+$. For $\mathbb{P}^1 \subset F$ mapped isomorphic to its image, we have that $m_2 = m_2\xi \cdot \mathbb{P}^1 = \mu^*(D) \cdot \mathbb{P}^1 = D \cdot \mu(\mathbb{P}^1) \in \mathbb{N}$. Hence $m_2 \geq 1$ and $K_X + D + (n-1)L = (\dfrac{a-2}{a} + (m_2 - 1))L \in \text{Pseff}(X)$, a contradiction. This proves the claim. As $D' = \mu_*^{-1}(D)$ by definition, we get that $D' \to D$ has degree 1. Thus $D \cong \mu(\mathbb{P}^{n-1})$ is the image of a general fiber of π and we are in case $(2.i)$ of Theorem 3.16.

If we are in case (c), apply Lemma 3.9 to (Y, M). We have a crepant resolution $\nu : Y \to Y_{\text{can}}$ with an ample divisor A on Y_{can} such that $\nu^*(A) = M$, $\Delta(Y_{\text{can}}, A) = 0$ and $K_{Y_{\text{can}}} + \tau A \equiv_{\text{num}} 0$. By Lemma 3.10, we have one of the following:

(c1) $(Y_{\text{can}}, A) \cong (Q, \mathcal{O}_{\mathbb{P}^{n+1}}(1))$, where $Q \subset \mathbb{Q}^{n+1}$ is a hyperquadric;
(c2) (Y_{can}, A) is a $\mathbb{P}^{n-1}$-bundle over $\mathbb{P}^1$ and the restriction of A to each fiber is $\mathcal{O}_{\mathbb{P}^{n-1}}(1)$;
(c3) $(Y_{\text{can}}, A) \cong (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$;
(c4) $(Y_{\text{can}}, A) \cong C_n(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$ is a generalised cone over $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$.

We have the following diagram

$$\begin{tikzcd} (Y, M) \arrow[r, "\mu"] \arrow[d, "\nu"] & (X, L) \\ (Y_{\mathrm{can}}, A) \arrow[ur, "h"'] & \end{tikzcd}$$

where h is an isomorphism and $\mu^*(L) = M = \nu^*(A)$, with (Y_{can}, A) being one of the above four pairs.

In case $(c1)$, after an automorphism of $\mathbb{P}^{n+1} = \mathrm{Proj}(\mathbb{C}[x_0, ...x_{n+1}])$, the hyperquadric Q is given by the homogeneous ideal $I_r = (\sum_{0\le i\le r} x_i^2) \subset \mathbb{C}[x_0, ...x_{n+1}]$ for some $r \ge 2$. By [Har77, Exercise II.6.5], the class group $\mathrm{Cl}(Q)$ of Q is the following:

- When $r = 2$, the divisor $\frac{1}{2}[\mathcal{O}_Q(1)]$ is an integral divisor and $\mathrm{Cl}(Q) = \mathbb{Z}\cdot\frac{1}{2}[\mathcal{O}_Q(1)]$. Hence D is numerically equivalent to a hyperplane $\mathbb{P}^{n-1}$ in Q and $K_X + (n-1)L + D = -\frac{1}{2}\mathcal{O}_Q(1)$ is not pseudo-effective. We are thus in case $(3.i)$ of Theorem 3.16.
- When $r = 3$, there is an isomorphism $\mathrm{Cl}(Q) \cong \mathbb{Z} \oplus \mathbb{Z}$. Note that here we can write
$$Q = \mathrm{Proj}\left(\frac{\mathbb{C}[x_0, \dots, x_{n+1}]}{(x_0x_1 - x_2x_3)}\right),$$
which is a cone of vertex $\mathbb{P}^{n-3} = \{x_1 = x_2 = x_3 = 0\} \subset \mathbb{P}^{n+1}$ with base $\mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3 = \{x_4 = \cdots = x_{n+1} = 0\} \subset \mathbb{P}^{n+1}$ (*cf.* [Har77, Exercise I.5.12.(d)]). If we consider the inclusions $\mathbb{P}^3 \subset \mathbb{P}^4 \subset \cdots \subset \mathbb{P}^n \subset \mathbb{P}^{n+1}$, then Q is also obtained by taking projective cone in the sense of [Har77, Exercise I.2.10] of $\mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3$ successively. By [Har77, Exercise II.6.3.(a)], we know that $\mathrm{Cl}(\mathbb{P}^1 \times \mathbb{P}^1) \cong \mathrm{Cl}(Q)$. For a hyperplan $H \subset \mathbb{P}^{n+1}$, $H \cap Q$ has type $(1, 1)$. The cone over $\mathbb{P}^1 \times \mathrm{pt}$ has type $(1, 0)$ and the cone over $\mathrm{pt} \times \mathbb{P}^1$ has type $(0, 1)$. Thus D has type $(1, 0)$ or type $(0, 1)$. We are thus in case $(3.ii)$ of Theorem 3.16.
- When $r \ge 4$, $\mathrm{Cl}(Q) = \mathbb{Z} \cdot [\mathcal{O}_Q(1)]$. Hence $D = d[\mathcal{O}_Q(1)]$, and $K_X + (n-1)L + D \equiv_{\mathrm{num}} \mathcal{O}_Q(d-1)$ is pseudo-effective. Thus this situation is excluded.

Case $(c2)$ is already treated in case (b) and we are in case $(2.i)$ of Theorem 3.16.

In case $(c3)$, the divisor D is linearly equivalent to $\mathcal{O}_{\mathbb{P}^2}(1)$. The divisor $K_X + (n-1)L + D$ is numerically trivial and hence is pseudo-effective, a contradiction. Hence case $(c3)$ does not happen.

In case $(c4)$, we proceed as in Lemma 3.11. We consider the following diagram

$$\begin{tikzcd} E = \mathbb{P}^2 \times \mathbb{P}^{n-3} \arrow[r, "\mathrm{pr}_2"] \arrow[d, hook, "i"] & \mathbb{P}^{n-3} \arrow[d, hook] \\ T = \mathbb{P}(\mathcal{O}^{\oplus n-2} \oplus \mathcal{O}(2)) \arrow[r, "\psi_{|\xi|}"] \arrow[d, "\pi"] & C_n(\mathbb{P}^2, \mathcal{O}(2)) = X \\ \mathbb{P}^2 & \end{tikzcd}.$$

Since T is a projective bundle, by [Ful98, Theorem 3.3.(b)] we have that

$$\mathrm{Cl}(T) = \mathbb{Z}[\pi^*(\mathcal{O}_{\mathbb{P}^2}(1))] \oplus \mathbb{Z}[\xi].$$

On the other hand, since $E = \mathrm{exc}(\psi_{|\xi|})$ is contracted, the morphism $(\psi_{|\xi|})_* : \mathrm{Cl}(T) \to \mathrm{Cl}(X)$ is surjective and $\mathrm{rk}(\mathrm{Cl}(X)) = 1$. We have that $\psi^*_{|\xi|}(L) = \xi$. Thus $(\psi_{|\xi|})_*([\xi]) = [L] \neq 0$. To determine $\mathrm{Cl}(X)$, one just need to know the image $(\psi_{|\xi|})_*\pi^*([\mathcal{O}_{\mathbb{P}^2}(1)])$. Let H be a Weil divisor on T such that $\mathcal{O}_T(H) = \pi^*(\mathcal{O}_{\mathbb{P}^2}(1))$. For example, we can take H to be $\pi^{-1}(l)$ where $l \subset \mathbb{P}^2$ is a linear subspace. Then it's easy to see that $H \neq E$. Set $G := (\psi_{|\xi|})_*H$. As L is ample, we know that $[L] \neq 0$ in $\mathrm{Cl}(X) \otimes \mathbb{Q}$. Take $m \in \mathbb{Q}$ such that $[G] = m[L]$ in $\mathrm{Cl(X)} \otimes \mathbb{Q}$. We have that

$$\psi^*_{|\xi|}(G) \sim_{\mathbb{Q}} (\psi_{|\xi|})^{-1}_*(G) + aE, \tag{11}$$

with $(\psi_{|\xi|})^{-1}_*(G) = H$. By the canonical bundle formula, we have

$$K_T = \pi^*(\mathcal{O}_{\mathbb{P}^2}(-1)) - (n-1)\xi \quad \text{and}$$
$$K_E = \mathrm{pr}_1^*(\mathcal{O}_{\mathbb{P}^2}(-2)) - (n-2)\xi|_E.$$

Hence we have

$$\mathcal{O}_E(E) = \mathrm{pr}_1^*(\mathcal{O}_{\mathbb{P}^2}(-2)) \otimes \mathrm{pr}_2^*(\mathcal{O}_{\mathbb{P}^{n-3}}(1)).$$

Let $C_1 = \mathbb{P}^1 \times \{\mathrm{pt}\} \subset E$. Then $E \cdot C_1 = -2$. We intersect both sides of Equation (11) with C_1. As $(\psi_{|\xi|})_*(C_1) = 0$, by the projection formual $(\psi_{|\xi|})^*(G) \cdot C_1 = 0$. By applying the projection formula to the morphism $\pi|_H : H \to \mathbb{P}^2$, we get that $H \cdot C_1 = 1$. Hence $a = \frac{1}{2}$. Thus we have

$$m[\xi] = \pi^*[\mathcal{O}_{\mathbb{P}^2}(1)] + \frac{1}{2}E. \tag{12}$$

Let $F = \mathbb{P}^{n-2}$ be a fibre of π such that $F \cap E \neq \emptyset$. Then $E \cap F = \mathbb{P}^{n-3} \subset \mathbb{P}^{n-2} = F$. Take $C_2 = \mathbb{P}^1 \subset F$ and intersect both side of Equation (12) with C_2. We have that $\xi \cdot C_2 = 1$ and $\pi^*[\mathcal{O}_{\mathbb{P}^2}(1)] \cdot C_2 = 0$ and $E \cdot C_2 = 1$. Thus we get that $m = \frac{1}{2}$. Hence we know the $\mathrm{Cl}(X) \otimes \mathbb{Q} = \mathbb{Q} \cdot \frac{1}{2}[L]$. Suppose $[(\psi_{|\xi|})^{-1}_*(D)] = m_1\pi^*[(\mathcal{O}_{\mathbb{P}^2}(1))] + m_2[\xi]$ with m_1, m_2 natural numbers. We have $D = (\psi_{|\xi|})_*(\psi_{|\xi|})^{-1}_*D \sim_{\mathbb{Q}} (\frac{m_1}{2} + m_2)L$. Being a generalized cone, X is $\mathbb{Q}$-factorial. For $\mathbb{Q}$-Cartier divisor K_X we have

$$K_X = (\psi_{|\xi|})_*(K_T) = (\psi_{|\xi|})_*(\pi^*(\mathcal{O}(-1)) - (n-1)\xi) = -(n - \frac{1}{2})[L].$$

Now $K_X + D + (n-1)L \equiv_{\mathrm{num}} (\frac{m_1 - 1}{2} + m_2)L$ is pseudo-effective. We thus exclude case $(c4)$.

□

Lemma 3.17. *Let $(X, D) = (\mathbb{P}(\mathcal{V}), D)$ be a log canonical pair, where $\pi : \mathbb{P}(\mathcal{V}) \to C$ is a projective bundle over a smooth curve C and $\mathcal{V}$ is an ample vector bunlde of rank n. If for a general fiber F, we have that $D|_F = \mathcal{O}_F(d)$ for some $d > 0$. Then* $\dim(X) = \dim(\mathbb{P}(\mathcal{V}) = 2$.

Proof. We have that

$$(K_{\mathbb{P}(\mathcal{V})} + D + (n-1)\xi)|_F = (\pi^*(K_C + \det(\mathcal{V})) + D - \xi)|_F = \mathcal{O}_F(d-1).$$

We take a *thrifty dlt modification* for $(\mathbb{P}(\mathcal{V}), D)$ as in [Kol13, Corollary 1.36.], *i.e.*, a proper birational morphism $f : \mathbb{P}(\mathcal{V})^{\mathrm{dlt}} \to \mathbb{P}(\mathcal{V})$ with a boundary divisor Δ^{dlt} such that:

(1) $(\mathbb{P}(\mathcal{V})^{\mathrm{dlt}}, \Delta^{\mathrm{dlt}})$ has dlt singularities;
(2) $f^*(K_{\mathbb{P}(\mathcal{V})} + D) \sim_{\mathbb{Q}} K_{\mathbb{P}(\mathcal{V})^{\mathrm{dlt}}} + \Delta^{\mathrm{dlt}}$;
(3) $K_{\mathbb{P}(\mathcal{V})^{\mathrm{dlt}}} + \Delta^{\mathrm{dlt}}$ is f-nef;
(4) $\mathbb{P}(\mathcal{V})^{\mathrm{dlt}}$ is $\mathbb{Q}$-factorial.

Thus we have that

$$\mathbb{P}(\mathcal{V})^{\mathrm{dlt}} \xrightarrow{f} \mathbb{P}(\mathcal{V}) \xrightarrow{\pi} C\ .$$

We set $g = \pi \circ f$ and $\xi' = f^*\xi$. Then we have that

$$f^*(K_{\mathbb{P}(\mathcal{V})} + D + (n-1)\xi) \equiv_{\mathrm{num}} K_{\mathbb{P}(\mathcal{V})^{\mathrm{dlt}}} + \Delta^{\mathrm{dlt}} + (n-1)\xi'.$$

As f is surjective, we have that f_* preserves numerical equivalence. By the projection formula we have that

$$f_*(K_{\mathbb{P}(\mathcal{V})^{\mathrm{dlt}}} + \Delta^{\mathrm{dlt}} + (n-1)\xi') \equiv_{\mathrm{num}} f_*f^*(K_{\mathbb{P}(\mathcal{V})} + D + (n-1)\xi) = K_{\mathbb{P}(\mathcal{V})} + D + (n-1)\xi.$$

Hence $K_{\mathbb{P}(\mathcal{V})^{\mathrm{dlt}}} + \Delta^{\mathrm{dlt}} + (n-1)\xi'$ cannot be pseudo-effective. So there exists an extremal ray R of $\overline{\mathrm{NE}}(\mathbb{P}(\mathcal{V})^{\mathrm{dlt}})$ such that

$$(K_{\mathbb{P}(\mathcal{V})^{\mathrm{dlt}}} + \Delta^{\mathrm{dlt}} + (n-1)\xi') \cdot R < 0.$$

For $0 < \epsilon \ll 1$, we have that

$$(K_{\mathbb{P}(\mathcal{V})^{\mathrm{dlt}}} + (1-\epsilon)\Delta^{\mathrm{dlt}} + (n-1)\xi') \cdot R < 0. \tag{13}$$

We note that $\xi' \cdot R = f^*(L) \cdot R > 0$, for otherwise any curve l such that $[l] \in R$ is contracted by f, which means $(K_{\mathbb{P}(\mathcal{V})^{\mathrm{dlt}}} + \Delta^{\mathrm{dlt}}) \cdot R \geq 0$, a contradiction. Hence R is in fact a $(K_{\mathbb{P}(\mathcal{V})^{\mathrm{dlt}}} + \Delta^{\mathrm{dlt}})$-negative extremal ray.

By Theorem 2.17-(3) we get the contraction morphism $\mathrm{cont}_R : \mathbb{P}(\mathcal{V})^{\mathrm{dlt}} \to Y$ which contracts the ray R. Let $S \subset \mathbb{P}(\mathcal{V})^{\mathrm{dlt}}$ be a fiber of cont_R. If $\dim(S) \geq 2$, there exists a curve $l \subset S$ that is contracted to a point by g. As $K_{\mathbb{P}(\mathcal{V})^{\mathrm{dlt}}} + \Delta^{\mathrm{dlt}}$ is f-nef, the $K_{\mathbb{P}(\mathcal{V})^{\mathrm{dlt}}} + \Delta^{\mathrm{dlt}}$-negative curve l can not be contracted to a point by f. Hence l maps finitely onto a curve $l' \subset F$. Now we have that

$$\begin{aligned}(K_{\mathbb{P}(\mathcal{V})^{\mathrm{dlt}}} + \Delta^{\mathrm{dlt}} + (n-1)\xi') \cdot l &= (f^*(K_{\mathbb{P}(\mathcal{V})} + D + (n-1)\xi)) \cdot l \\ &= \deg(l/l')(K_{\mathbb{P}(\mathcal{V})} + D + (n-1)\xi) \cdot l \\ &= \deg(l/l')\mathcal{O}_F(d-1) \cdot l' \\ &\geq 0,\end{aligned}$$

a contradiction. Hence any fiber of cont_R has dimension at most 1. Let $E \subset \mathrm{exc}(\mathrm{cont}_R)$ be an irreducible component of the exceptional locus of cont_R. We thus have that

$$\dim(E) - \dim(\mathrm{cont}_R(E)) \leq 1.$$

We know that for $0 < \epsilon \ll 1$, the pair $(\mathbb{P}(\mathcal{V})^{\mathrm{dlt}}, (1-\epsilon)\Delta^{\mathrm{dlt}})$ has klt singularities (*cf.* [KM98, Proposition 2.41.]). For sufficiently small ϵ, the divisor $-(K_{\mathbb{P}(\mathcal{V})^{\mathrm{dlt}}} + (1-\epsilon)\Delta^{\mathrm{dlt}})$ is still cont_R-ample. The estimate of the length of extremal ray [Deb01, Theorem 7.46.] for klt pairs shows that the rational curves $l \in R$ cover E and there exists a rational curve $l_\epsilon \in R$ such that

$$0 < -(K_{\mathbb{P}(\mathcal{V})^{\mathrm{dlt}}} + (1-\epsilon)\Delta^{\mathrm{dlt}}) \cdot l_\epsilon \leq 2.$$

For any curve l whose class $[l]$ is in R, we have that $\xi' \cdot l \geq 1$. Combining these two inequalities with Equation (13), we have that

$$0 > (K_{\mathbb{P}(\mathcal{V})^{\mathrm{dlt}}} + (1-\epsilon)\Delta^{\mathrm{dlt}} + (n-1)\xi') \cdot l_\epsilon \geq -2 + (n-1).$$

Hence $n = 2$. □

Lemma 3.18. *Let $\mathcal{V}$ be a rank 2 ample vector bundle over $\mathbb{P}^1$. Set $(X, L) := (\mathbb{P}(\mathcal{V}), \mathcal{O}_{\mathbb{P}(\mathcal{V})}(1))$. Suppose that D is a prime Weil divisor on X and $K_X + D + L$ is not pseudo-effective. Then we have one of the following:*

(1) Either $D \cong \mathbb{P}^1$ is a fiber of the stucture map $\pi : \mathbb{P}(\mathcal{V}) \to \mathbb{P}^1$; or

(2) $(X, L) = (\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(1)), \mathcal{O}_{\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(a)\oplus\mathcal{O}_{\mathbb{P}^1}(1))}(1))$ with $a > 1$ and D is the unique section of $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(1)) \to \mathbb{P}^1$ such that

$$D \equiv_{\mathrm{num}} \mathcal{O}_{\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(a)\oplus\mathcal{O}_{\mathbb{P}^1}(1))}(1)) - af,$$

where f is a general fiber; or

(3) $\mathcal{V} = \mathcal{O}_{\mathbb{P}}(1) \oplus \mathcal{O}_{\mathbb{P}}(1)$.

Proof. As $\mathcal{V}$ is ample, we know that $\mathcal{V} \cong \mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(b)$ with $a, b > 0$. We may suppose that $a \geq b > 0$. Set $e = a - b \geq 0$. Set $\mathcal{W} = \mathcal{V} \otimes \mathcal{O}_{\mathbb{P}^1}(-a)$. We have that $X_e := \mathbb{P}(\mathcal{W}) \cong \mathbb{P}(\mathcal{V})$. We denote by $p : \mathbb{P}(\mathcal{W}) \to \mathbb{P}^1$ the projection. By [Har77, Lemma II.7.9], we know that $(X, L) \cong (X_e, \mathcal{O}_{X_e}(1) \otimes p^*(\mathcal{O}_{\mathbb{P}^1}(a)))$. From now on, we follow the convention in [Har77, Notation V.2.8.1]. We denote the general fiber of p by f'. Note that $\mathcal{W}$ satisfies the assumption in [Har77, Propostion 2.8.]. Hence by [Har77, Propostion 2.8.] there exists a section C_0 of p such that $\mathcal{O}_{X_e}(1) \cong \mathcal{O}_{X_e}(C_0)$. [Har77, Propostion 2.9.] implies $C_0^2 = -e$. Hence if $a \neq b$, we have that C_0 is unique. We know that $L \equiv_{\mathrm{num}} C_0 + af'$ and [Har77, Lemma 2.10.] implies that $K_{X_e} \sim -2C_0 + (-2-e)f'$. Assume that $D \sim xC_0 + d'f'$, with x, d' being integers. We have that

$$K_X + L + D \equiv_{\mathrm{num}} (x-1)C_0 + (d' + b - 2)f'.$$

As D is a prime divisor, [Har77, Corollary V.2.18-(b)] implies one of the following:

i $x = 0, d' = 1$, and $K_X + L + D \equiv_{\mathrm{num}} -C_0 + (b-1)f' \notin \mathrm{Pseff}(X)$;

ii $x = 1, d' = 0$ and $K_X + L + D \equiv_{\mathrm{num}} (b-2)f'$, which is not pseudo-effective iff $b = 1$;

iii $x > 0, d' > xe$. Note $d' + b - 2 \geq 0$. so we have that $K_X + L + D$, being a positive combination of effective divisors, is pseudo-effective;

iv $e > 0, x > 0$, and $d' = xe$. Again $d' + b - 2 \geq 0$. So we have that $K_X + L + D$, being a positive combination of effective divisors, is pseudo-effective.

In case (i), the divisor D is a fiber of p, which maps isomorphically to a fiber of π under the canonical isomorphism $\mathbb{P}(\mathcal{W}) \cong \mathbb{P}(\mathcal{V})$. Hence we are again in case (1) of Lemma 3.18.

In case (ii), if $a > b = 1$, as D is irreducible, [Har77, Proposition V.2.20-(a)] implies that $D = C_0$. Hence D is the unique section of $\pi : \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(1)) \to \mathbb{P}^1$ such that $D \equiv_{\mathrm{num}} \xi - \pi^*(\mathcal{O}_{\mathbb{P}^1}(a))$. Hence we are in case (2) of Lemma 3.18. If $a = b = 1$, then $(X, L) \cong (\mathbb{P}^1 \times \mathbb{P}^1, (1,1))$ which is a rank 4 hyperquadric in $\mathbb{P}^3$. Thus we are in case (3) of Lemma 3.18. □

The proof of Theorem 3.16 can be adapted to the case where the boundary Δ is not irreducible.

Corollary 3.19 (=Proposition 1.8)**.** *Let (X, Δ) be a log canonical pair, with $\Delta \neq 0$ a reduced divisor. Suppose that L is an ample line bundle on X and $(K_X+\Delta)+(n-1)L \notin \mathrm{Pseff}(X)$, where $n = dim(X)$. Then we have one of the following:*

(1) $(X, L) \cong (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$, $\Delta \equiv_{\mathrm{num}} H$ is a prime divisor where H is a hyperplane of $\mathbb{P}^n$;

(2.i) There is a $(\mathbb{P}^{n-1}, \mathcal{O}_{\mathbb{P}^{n-1}}(1))$-bundle $(\mathbb{P}(E), \mathcal{O}_{\mathbb{P}(E)}(1))$ over a smooth curve C, and a birational morphism $\mu : \mathbb{P}(E) \to X$ such that $\mu^*(L) \cong \mathcal{O}_{\mathbb{P}(E)}(1)$ and $\Delta = \sum F_i$ is a finite sum where $F_i \cong \mu(\mathbb{P}^{n-1})$ are images of distinct general fibers of π by μ and for each i, the morphism $\mathbb{P}^{n-1} \to F_i$ has degree 1;

(2.ii) $(X, L) = (\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(1)), \mathcal{O}_{\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(a)\oplus\mathcal{O}_{\mathbb{P}^1}(1))}(1))$ with $a > 1$ and $\Delta = D$ is irreducible, where D is the unique section of $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(a)\oplus\mathcal{O}_{\mathbb{P}^1}(1)) \to \mathbb{P}^1$ such that $D \equiv_{\mathrm{num}} \mathcal{O}_{\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(a)\oplus\mathcal{O}_{\mathbb{P}^1}(1))}(1)) - af$, where f is a general fiber;

(3.i) $(X, L) \cong (Q, \mathcal{O}_{\mathbb{P}^{n+1}}(1))$, where $Q \subset \mathbb{P}^{n+1}$ is a $\mathrm{rk}(Q) = 3$ hyperquadric, the boundary divisor Δ is a hyperplane in Q and $[\Delta] = \frac{1}{2}[H \cap Q]$ where H is a hyperplane in $\mathbb{P}^{n+1}$;

(3.ii) $(X, L) \cong (Q, \mathcal{O}_{\mathbb{P}^{n+1}}(1))$, where $Q \subset \mathbb{P}^{n+1}$ is a $\mathrm{rk}(Q) = 4$ hyperquadirc. If we write $Q = \mathrm{Proj}\left(\frac{\mathbb{C}[x_0, \dots, x_{n+1}]}{(x_0x_1 - x_2x_3)}\right)$, then $\Delta = D$ is prime and D is the cone with vertex $\mathbb{P}^{n-3}$ over $\mathbb{P}^1 \times \mathrm{pt}$ or $\mathrm{pt} \times \mathbb{P}^1$. In particular, $D \cong \mathbb{P}^{n-1}$;

Proof. Let D be a component of Δ.

First we suppose that X is $\mathbb{Q}$-factorial. The pair (X, D) satisfies the assumptions of Theorem 3.16. Hence we have a classification for (X, D, L). If we are in case $(2.i)$ of Theorem 3.16 for (X, D), for a general fiber F, we cannot have another D' such that $D'|_F = \mathcal{O}_F(d)$ with $d \geq 1$. Hence all the components will be fibers. We are thus in case $(2.i)$ of Corollary 3.19. If we are in case $(2.ii)$ of Theorem 3.16 for (X, D), the divisor D is the unique section such that $D \equiv_{\mathrm{num}} \mathcal{O}_{\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(a)\oplus\mathcal{O}_{\mathbb{P}^1}(1))}(1)) - af$. Thus we have that $\Delta = D$ is irreducible. And we are in case $(2.ii)$ of Corollary 3.19. If we are in case (1) or (3) of Theorem 3.16 for (X, D), we cannot add another prime divisor. Hence $\Delta = D$ and we are in case (1) or (3) of Corollary 3.19.

In general, the proof is identical to Theorem 3.16 by adjusting our argument for (X, D) to (X, Δ). We apply Theorem 2.25 and Lemma 2.23 to the normal variety X. We get a birational morphism $\mu : Y \to X$ such that Y has $\mathbb{Q}$-factorial canonical singularities, K_Y is μ-nef and μ is isomorphic over regular points of X. We set $\Delta' = \mu_*^{-1}(\Delta)$. Then Δ' is reduced. Let $\omega_Y = \mathcal{O}_Y(K_Y)$ and $\omega_X = \mathcal{O}_X(K_X)$ be the canonical sheaves. We have an injection

$$\mu_*(\omega_Y \otimes \mathcal{O}_Y(\Delta') \otimes \mu^*(L^{\otimes n-1})) \rightarrowtail \omega_X \otimes \mathcal{O}_X(\Delta) \otimes L^{\otimes n-1}.$$

As $(K_X + \Delta) + (n-1)L$ is not pseudo-effective, neither is $(K_Y + \Delta') + (n-1)\mu^*(L)$. We set $\mu^*(L) =: M$. As Δ' is effective, the divisor $K_Y + (n-1)M$ is not pseudo-effective.

By applying Lemma 3.8, we get:

(a) $(Y, M) \cong (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$, and $\tau = n + 1$;

(b) (Y, M) is isomorphic to a $(\mathbb{P}^{n-1}, \mathcal{O}_{\mathbb{P}^{n-1}}(1))$-bundle over a smooth curve C and $\tau = n$;
(c) $\Delta(Y, M) = 0$, $K_Y + \tau M \sim_{\mathbb{Q}} \mathcal{O}_Y$ and $n - 1 < \tau(M) \leq n$.

If we are in case (a), we know that μ is an isomorphism. The divisor Δ is given by $\mathcal{O}_{\mathbb{P}^n}(a)$ for some $a \geq 1$. We have that $K_X + (n-1)L + D = \mathcal{O}_{\mathbb{P}^n}(a-2)$. Hence the only possible choice is $a = 1$ and $\Delta = D$ is a hyperplane. We are thus in case (1) of Corollary 3.19.

If we are in case (b), we have a diagram

$$\begin{tikzcd} (\mathbb{P}(\mathcal{V}), \xi) \arrow[r, "\mu"] \arrow[d, "\pi"'] & (X, L) \\ C & \end{tikzcd}$$

where $\xi = \mathcal{O}_{\mathbb{P}(\mathcal{V})}(1)$ and $K_{\mathbb{P}(\mathcal{V})}$ is μ-nef.

- If μ is an isomorphism, then X is $\mathbb{Q}$-factorial, and we are in case $(2.i)$ or case $(2.ii)$ of Corollary 3.19;
- If μ is small, the variety X cannot be $\mathbb{Q}$-factorial. We have that $K_{\mathbb{P}(\mathcal{V})} + \Delta' = \mu^*(K_X + \Delta)$. Let F be a general fiber of π. The we have that $\Delta'|_F = \mathcal{O}_F(d)$ for some integer $d \geq 0$. As $K_X + \Delta$ is $\mathbb{Q}$-Cartier and $\rho(X) = 1$, there exists a rational number m such that $K_X + \Delta \equiv_{\text{num}} mL$. Hence $K_{\mathbb{P}(\mathcal{V})} + \Delta' \equiv_{\text{num}} m\mu^*L$. Intersect with $\xi^{n-2}f$ and we get that $-n + d = m$. The divisor $K_X + \Delta + (n-1)L \equiv_{\text{num}} (d-1)L$ is not pseudo-effective. Hence $d = 0$. If we write $\Delta' = \sum D'_i$ with D'_i distinct prime divisors. We have that $D'_i|_F = \mathcal{O}_F(0)$. Thus the D_i's are distinct general fibers. We get that $D_i = \mu(D'i')$ and we are in case $(2.i)$ of Corollary 3.19.
- If μ is divisorial, then X is again $\mathbb{Q}$-factorial (*cf.* [KM98, Corollary 3.18]) and we are in case $(2.i)$ of Corollary 3.19.

If we are in case (c), apply Lemma 3.9 to (Y, M). We have a crepant resolution $\nu : Y \to Y_{\text{can}}$ with an ample divisor A on Y_{can} such that $\nu^*(A) = M$, the Δ-genus $\Delta(Y_{\text{can}}, A) = 0$ and $K_{Y_{\text{can}}} + \tau A \equiv_{\text{num}} \mathcal{O}_{Y_{can}}$. By Lemma 3.10, we have one of the following cases:

(c1) $(Y_{\text{can}}, A) \cong (Q, \mathcal{O}_{\mathbb{P}^{n+1}}(1))$, where $Q \subset \mathbb{Q}^{n+1}$ is a hyperquadric;
(c2) (Y_{can}, A) is a $\mathbb{P}^{n-1}$-bundle over $\mathbb{P}^1$ and the restriction of L to each fiber is $\mathcal{O}_{\mathbb{P}^{n-1}}(1)$;
(c3) $(Y_{\text{can}}, A) \cong (\mathbb{P}^2, \mathcal{O}^2_{\mathbb{P}}(2))$;
(c4) $(Y_{\text{can}}, A) \cong C_n(\mathbb{P}^2, \mathcal{O}^2_{\mathbb{P}}(2))$ is a generalised cone over $(\mathbb{P}^2, \mathcal{O}^2_{\mathbb{P}}(2))$.

We have the following diagram

$$\begin{tikzcd} (Y, M) \arrow[r, "\mu"] \arrow[d, "\nu"] & (X, L) \\ (Y_{\text{can}}, A) \arrow[ur, "h"'] & \end{tikzcd}$$

such that h is an isomorphism and $\mu^*(L) = M = \nu^*(A)$ with (Y_{can}, A) being one of the above four pairs. The case $(c2)$ is treated in case (b). If we are in $(c1), (c3), (c4)$, we have that X is $\mathbb{Q}$-factorial and (X, Δ, L) is classified in the beginning of the proof. Hence we get the list as in Corollary 3.19. □

3.3. Semi-log canonical polarized varieties

Let X be a demi-normal variety. Let $\pi : \bar{X} \to X$ be its normalization. We have that the conductor divisors $\bar{D} \subset \bar{X}$ and $D \subset X$ are reduced. The pair $(\bar{X}, \bar{D})$ is log canonical when X is semi-log canonical (Definition 2.32). Suppose now that we have an ample line bundle L on X. We may consider using the classification results Corollary 3.19 for $(\bar{X}, \bar{D}, \pi^*(L))$ to get a classification for (X, L)

We now state the following:

Proposition 3.20 (=Theorem 1.9)**.** *Let X be a non-normal slc projective variety of dimension n and L an ample line bundle over X. Suppose that $K_X + (n-1)L \notin \operatorname{Pseff}(X)$. Let $\pi : \bar{X} \to X$ be the normalization of X and $D \subset X$, $\bar{D} \subset \bar{X}$ the conductors. Then we have:*

There is a nodal curve C', a rank n-vector bundle E', distinct fibers $F_1, F_2, \ldots, F_m$ of $\mathbb{P}(E')$ and a birational morphism $\mu : \mathbb{P}(E') \to X$ such that

- *$\mu^*(L) = \mathcal{O}_{\mathbb{P}(E')}(1)$ and*
- *$D = \sum_{1 \le i \le m} \mu(F_i)$*

Proof. We know by definition that $(\bar{X}, \bar{D})$ is log canonical. Note that the absolute normalization $\pi : \bar{X} \to X$ is finite (*cf.* [Sta22, Tag 0BXR]). Hence $\pi^*(L)$ is ample. We have by Equation (3) that

$$\pi^*(K_X + (n-1)L) = K_{\bar{X}} + \bar{D} + (n-1)\pi^*(L).$$

Let $C \subset X$ be a movable curve in X such that $(K_X + (n-1)L) \cdot C < 0$. Let $C' \subset \bar{X}$ be a movable curve that dominates C. Then by the projection formula $(K_{\bar{X}} + \bar{D} + (n-1)\pi^*(L)) \cdot C' = \deg(C'/C)(K_X + (n-1)) \cdot C < 0$. Hence by Theorem 2.13, the divisor $K_{\bar{X}} + \bar{D} + (n-1)\pi^*(L)$ is not pseudo-effective.

Note that D and $\bar{D}$ are reduced, and if we denote by $\bar{D}^\nu$, D^ν respectively their normalizations, then π induces a degree 2 map $\nu : \bar{D}^\nu \to D^\nu$ and there is a Galois involution $\tau : \bar{D}^\nu \to \bar{D}^\nu$ which is generically fixed point free (*cf.* [Kol13, 5.2]). Thus we have the following diagram

$$\begin{tikzcd} \tau \circlearrowright \bar{D}^\nu \arrow[r] \arrow[d, "\nu"] & \bar{D} \arrow[r] \arrow[d] & \bar{X} \arrow[d, "\pi"] \\ D^\nu \arrow[r] & D \arrow[r] & X \end{tikzcd} .$$

Since $\nu : \bar{D}^\nu \to D^\nu$ has degree 2, we have by the projection formula that

$$\pi^*(L)|_{\bar{D}^\nu}^{n-1} = \deg(\nu) \cdot (L|_{D^\nu}^{n-1}) \in 2\mathbb{Z}.$$

We now apply Corollary 3.19 to $(\bar{X}, \bar{D}, \pi^*(L))$. We have one of the following:

(1) $(\bar{X}, \pi^*L) \cong (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$, $\bar{D} = H$ is a prime divisor where H is a hyperplane of $\mathbb{P}^n$;

(2.i) There is a $(\mathbb{P}^{n-1}, \mathcal{O}_{\mathbb{P}^{n-1}}(1))$-bundle $(\mathbb{P}(E), \mathcal{O}_{\mathbb{P}(E)}(1))$ over a smooth curve C, and a birational morphism $\mu : \mathbb{P}(E) \to \bar{X}$ such that $\mu^*(\pi^*L) \cong \mathcal{O}_{\mathbb{P}(E)}(1)$ and $\bar{D} = \sum F_i$ is a finite sum where $F_i \cong \mu(\mathbb{P}^{n-1})$ are images of distinct general fibers by μ and $\deg(\mathbb{P}^{n-1}/F_i) = 1$;

(2.ii) $(\bar{X}, \pi^*L) = (\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(1)), \mathcal{O}_{\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(a)\oplus\mathcal{O}_{\mathbb{P}^1}(1))}(1))$ with $a > 1$ and $\bar{D} = C$, where C is the unique section of $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(1)) \to \mathbb{P}^1$ such that $C \equiv_{\text{num}} \mathcal{O}_{\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(a)\oplus\mathcal{O}_{\mathbb{P}^1}(1))}(1)) - af$, where f is a general fiber;

(3.i) $(\bar{X}, \pi^*L) \cong (Q, \mathcal{O}_{\mathbb{P}^{n+1}}(1))$, where $Q \subset \mathbb{P}^{n+1}$ is a $\mathrm{rk}(Q) = 3$ hyperquadric, the divisor $\bar{D}$ is a hyperplane in Q and $[\bar{D}] = \frac{1}{2}[H \cap Q]$ where H is a hyperplane in $\mathbb{P}^{n+1}$;

(3.ii) $(\bar{X}, \pi^*L) \cong (Q, \mathcal{O}_{\mathbb{P}^{n+1}}(1))$, where $Q \subset \mathbb{P}^{n+1}$ is a $\mathrm{rk}(Q) = 4$ hyperquadirc. If we write $Q = \mathrm{Proj}\left(\frac{\mathbb{C}[x_0, \dots, x_{n+1}]}{(x_0x_1 - x_2x_3)}\right)$, then $\bar{D}$ is prime and $\bar{D}$ is the cone with vertex $\mathbb{P}^{n-3}$ over $\mathbb{P}^1 \times \mathrm{pt}$ or $\mathrm{pt} \times \mathbb{P}^1$. In particular, $\bar{D} \cong \mathbb{P}^{n-1}$;

In case (1), we have that $\bar{D}^\nu = \bar{D} \cong \mathbb{P}^{n-1}$ is smooth and $\pi^*(L)|_{\bar{D}^\nu} = \mathcal{O}_{\mathbb{P}^{n-1}}(1)$. As $\pi^*(L)|_{\bar{D}^\nu}^{n-1} = 1$ is odd. We exclude case (1).

In case $(2.i)$, we have that $\bar{D} = \sum_{1\le i\le k} F_i$ for k a natural number and the morphism $\mu : \coprod_{1\le i\le k} \mathbb{P}^{n-1} \to \bar{D}$ factors through $\bar{D}^\nu \to \bar{D}$ [Sta22, Tag 035Q]-(4). Hence $\pi^*(L)|_{\bar{D}^\nu}^{n-1} = k$ and k is even. As $\deg(\mathbb{P}^{n-1}/F_i) = 1$, we have that

$$\pi^*(L)|_{F_i}^{n-1} = \deg(\mathbb{P}^{n-1}/F_i)(\mathcal{O}_{\mathbb{P}(E)}(1)|_{\mathbb{P}^{n-1}})^{n-1} = 1.$$

Thus each irreducible component of D has pre-image consisting of two of the F_i's. We have thus the diagram

$$\begin{tikzcd} \mathbb{P}(E) \arrow[r, "\mu"] \arrow[d, "p"'] & \bar{X} \arrow[r, "\pi"] & X \\ C & & \end{tikzcd} .$$

Set $k = 2m$. We write $D = \sum_{1\le i\le m} D_i$. We denote the two components of $\bar{D}$ that are mapped onto D_i by $F_{i,1}$ and F_{i_2}. Let $x_{i,1}$ (*resp.* $x_{i,2}$) be the point of C such that $\mu(p^{-1}(x_{i,1})) = F_{i,1}$ (*resp.* $\mu(p^{-1}(x_{i,2})) = F_{i,2}$). As C is smooth, we may glue $x_{i,1}$ and $x_{i,2}$. We thus get a nodal curve C' together with a quotient morphism $q : C \to C'$ such that there exists a rank n vector bundle E' on C' satisfying $q^*(E') = E$. The morphism $\pi \circ \mu$ thus factors through $\mathbb{P}(E)$, *i.e.* we have the following commutative diagram:

$$\begin{tikzcd} \mathbb{P}(E) \arrow[r, "\mu"] \arrow[d] & \bar{X} \arrow[d, "\pi"] \\ \mathbb{P}(E') \arrow[r, "\mu'"] & X \end{tikzcd} .$$

The morphism μ' is birational. If we denote $x_i = p(x_{i,1})$ and F_i the fiber of x_i in $\mathbb{P}(E')$, we have that $D_i = \mu'(F_i)$. Thus we have the result of Proposition 3.20.

In case $(2.ii)$, we have that $\pi^*(L) \cdot C = a - e = 1$. Hence we exclude this case.

In case $(3.i)$, $\bar{D}$ is irreducible and $\pi^*(L)|_{\bar{D}^n}^{n-1} = 1$. Hence we also exclude this case.

In case $(3.ii)$, $\bar{D} \cong \mathbb{P}^{n-1}$ and $\pi^*(L)|_{\bar{D}} = \mathcal{O}_{\mathbb{P}^{n-1}}(1)$ which is not divisible by 2. Hence we exclude this case, too. $\square$

Part 2

Fundamental group of Kähler orbifolds with nef anti-canonical bundle

CHAPTER 4

Introduction

When studying compact Kähler manifolds, we want to understand their topology. In particular, we expect to know what is the fundamental group of a compact Kähler manifold under some geometric assumptions. For a compact Kähler manifold X whose anti-canonical bundle has certain positivity, it turns out that the fundamental group $\pi_1(X)$ of X is quite small.

In [Kob61], Kobayashi proved the following

THEOREM 4.1 ([Kob61, Theorem A]). *A compact Kähler manifold X with positive definitive Ricci tensor is simply connected.*

Kobayashi's proof is straightforward. First apply Myer's theorem (*cf.* [GHL04, 3.85]) to get that the universal covering $\pi : \tilde{X} \to X$ is a k-fold covering with $k \in \mathbb{N}$. Then note, by Kodaira's vanishing theorem, that the Eular characteristics $\chi(\mathcal{O}_X) = \chi(\mathcal{O}_{\tilde{X}}) = 1$. As π is a k-fold covering, we have that $\chi(\tilde{X}) = k\chi(X)$. Hence $k = 1$ and X is simply connected.

By Aubin-Yau's theorem (Theorem 5.15), we know that for a compact Kähler manifold X, its first Chern class $c_1(X) = c_1(-K_X)$ can be represented by $\frac{1}{2\pi}\mathrm{Ricci}_\omega$ for some Kähler form ω on X. Hence we may reformulate Theorem 4.1 as

THEOREM 4.2. *A compact Fano manifold is simply connected.*

When a compact Kähler manifold X has first Chern class $c_1(X) = 0$, we have the following Beauville-Bogomolov decomposition theorem.

THEOREM 4.3 ([Bea83, Théorème 1]). *Let X be a compact Kähler manifold with $c_1(X) = 0$. We have the following*

- *Let $\tilde{X} \to X$ be the universal cover of X. Then we can write $\tilde{X}$ as a product*
$$\tilde{X} \simeq \mathbb{C}^k \times \prod_i Y_i \times \prod_j S_j$$
where Y_i's are irreducible Calabi-Yau manifolds and S_j's are irreducible hyperkähler manifolds. The product is unique up to re-ordering.
- *There exists a finite cover $X' \to X$ such that X' can be written as a product*
$$X' \simeq T \times \prod_i Y_i \times \prod_j S_j$$
where T is a complex torus.

We write separately the direct result of the previous theorem.

Corollary 4.4. *Let X be a compact Kähler manifold with zero first Chern class. Then the fundamental group $\pi_1(X)$ of X is virtually Abelian.*

From the point of view of cones of line bundles, we know that the closure of ample cone $\mathrm{Amp}(X)$ is the nef cone $\mathrm{Nef}(X)$. Hence we expect some restrictions on $\pi_1(X)$ when $-K_X$ is nef. In [Pău97], Mihai Păun proved the following result

THEOREM 4.5 ([Pău97, Theorem 1]). *Let (X, ω) be a compact Kähler manifold whose anti-canonical bundle $-K_X$ nef. Then its fundamental group $\pi_1(X)$ is virtually nilpotent.*

A key material in the proof of Theorem 4.5 is the following geometric Margulis lemma by Cheeger-Colding.

Lemma 4.6 ([CC96, Theorem 8.7.]). *Let $n > 0$ be a natural number. There exists a universal constant $C = C(n)$ depending only on n such that the following holds:*

For any compact manifold M of dimension n and $\mathrm{Ric}_g \geq -(n-1)g$, the morphism induced by the inclusion

$$\pi_1(B_g(p, r), p) \to \pi_1(M, p)$$

has virtually nilpotent image for any $r < C(n)$.

Păun proved Theorem 4.5 by using Aubin-Yau theorem to construct a specific metric ω on X, such that

(1) $\mathrm{Ricci}_\omega \geq -(2\dim_{\mathbb{C}}(X) - 1)\omega$ and
(2) there exists a positive $r_0 < C(2\dim_{\mathbb{C}}(X))$ such that $B_\omega(p, r_0) = X$.

Then he applied Lemma 4.6 directly.

The proof of Theorem 4.2, Theorem 4.3 and Theorem 4.5 use tools from differential geometry. On the other hand, when given a Kähler manifold X, we naturally consider its Albanese morphism $\mathrm{Alb}_X : X \to A(X)$ (see Definition 2.33). Set $Y := \mathrm{Alb}_X(X) \subset A(X)$ the image of X in $A(X)$ and $r : \tilde{Y} \to Y$ a smooth model of Y. Frédéric Campana shows how one could describe the central series of $\pi_1(X)$ by the central series of $\pi_1(\tilde{Y})$ in [Cam95, Théorème 2.2]. In particular, one observes that for a compact Kähler manifold X whose fundamental group $\pi_1(X)$ has polynomial growth, if Alb_X is surjective, we have that $\pi_1(X)$ is virtually Abelian.

When a Kähler manifold X is projective, it had been shown that Alb_X is surjective (*cf.* [Zha96, Theorem 1]) at the time when Păun proved Theorem 4.5. Later, it was proven by Junyan Cao that $\mathrm{Alb}_X : X \to A(X)$ is a fiberation provided X projective and $-K_X$ nef ([Cao19, Theorem 1.2]). The Kähler case is due to Păun:

Proposition 4.7 ([Pău17, Theorem 1.7]). *Let X be a compact Kähler manifold whose anti-canonical bundle $-K_X$ is nef. Then the Albanese morphism*

$$\mathrm{Alb}_X : X \to A(X)$$

is surjective.

Combined with Theorem 4.5, we have:

THEOREM 4.8 (*cf.* [Pău97, Theorem 2]). *Let X be a compact Kähler manifold whose anti-canonical bundle $-K_X$ is nef. Then the fundamental group $\pi_1(X)$ is virtually Abelian.*

The heavy use of tools in differential geometry in the proofs of the various results considering fundamental groups indicates that we may consider the problems of fundamental groups of Kähler spaces with singularities where the tools in differential geometry

can still be applied. A very good candidate is the similar problem for Kähler orbifolds, *i.e.*, Kähler spaces with quotient singularities.

An orbifold is a geometric object whose local model is $\mathbb{C}^n/G$ where G is a finite subgroup of $\mathrm{Aut}(\mathbb{C}^n)$. As a natural generalization of manifolds, since its introduction by Satake in 1956 ([Sat56]), many results in differential geometry of manifolds have been generalized to orbifold case by adapting existing proofs for manifold case.

Theorem 4.2 and Corollary 4.4 has been generalized in the following form.

THEOREM 4.9 ([CC14, Theorem 4.2.]). *Let $\mathcal{X} = (X, \Delta)$ be a compact Kähler orbifold (see Definition 5.37) with non-negative first Chern class $c_1(\mathcal{X})$. Then $\pi_1^{\mathrm{orb}}(\mathcal{X})$ is virtually Abelian. More precisely,*

- *If $c_1(\mathcal{X}) > 0$, then $\pi_1^{\mathrm{orb}}(\mathcal{X})$ is finite;*
- *If $c_1(\mathcal{X}) = 0$, then $\pi_1^{\mathrm{orb}}(\mathcal{X})$ is virtually Abelian of even rank bounded by $2\dim(X)$.*

We mention that there are even more general results for wilder singularities. A recent work by S.Matsumura and J.Wang gives a decomposition theorem ([MW21, Corollary 1.2]) for projective klt pair (X, Δ) with $-(K_X + \Delta)$ nef which can be viewed as a generalization of Theorem 4.3. L.Braun have a deep results showing that for a klt pair (X, Δ) with $-(K_X + \Delta)$ nef and big, if Δ has standard coefficients, then its fundamental group $\pi_1(X, \Delta)$ is finite ([Bra21, Theorem 2]), which is the best result of the generalizations of Theorem 4.2 by far.

In this part of the thesis, we will adapt Păun's arguments in [Pău97] to orbifold case and generalize his results Theorem 4.5 and Theorem 4.8 to Kähler orbifolds. We prove the following

THEOREM 4.10 (=Theorem 8.13, Main Theorem). *Let $(\mathcal{X}, \omega)$ be a compact Kähler orbifold. If the anti-canonical bundle $K_{\mathcal{X}}^{-1}$ is nef, then $\pi_1^{\mathrm{orb}}(\mathcal{X})$ is virtually nilpotent.*

THEOREM 4.11 (=Theorem 9.1). *Let (X, Δ) be a projective orbifold pair with $-(K_X + \Delta)$ nef. The orbifold fundamental group $\pi_1(X, \Delta)$ is virtually Abelian.*

Plan of Part 2. This part is organized as the following. In Chapter 5, we give the necessary materials. In Chapter 6, we examine the metric space structure and Bishop-Gromov theorem (Theorem 6.22) on Riemannian orbifold. In Chapter 7, we use a result in [BGT12] to show a version of orbifold Margulis lemma (Lemma 7.12). In Chapter 8, we adapt Păun's argument by using Lemma 7.12 to show Theorem 4.10. In Chapter 9, we use the pair model for orbifolds to apply the Albanese morphism argument as in the smooth case to obtain Theorem 4.11.

CHAPTER 5

Preliminaries

5.1. Conventions

In Part 2, we deal with complex spaces instead of schemes over $\mathbb{C}$. We use the standard notions as in [GR84]. The conventions that we summarize here are parallel to Section 2.1.

- A *complex model space* is a ringed space $(V(\mathcal{I}), \mathcal{O}_D/\mathcal{I})$, where $D \subset \mathbb{C}^n$ is an open connected subset and $I \subset \mathcal{O}_D$ is an ideal sheaf of finite type.
- A *complex space* is a locally ringed space $(X, \mathcal{O}_X)$ over $\mathbb{C}$ such that for each point $x \in X$, there exists an open subset $U_x \ni x$ and $(U_x, \mathcal{O}_{U_x})$ is isomorphic as locally ringed space over $\mathbb{C}$ to a complex model space. A *complex manifold* of dimension $n \in \mathbb{N}$ is a complex space which is locally isomorphic to the complex model space $\mathbb{C}^n$ at every point.
- A *morphism* between two complex space X and Y is a morphism between X and Y considered as locally ringed space over $\mathbb{C}$.
- We say X is *reduced* (*resp. normal*) is for all $x \in X$ the local ring $\mathcal{O}_{X,x}$ is *reduced* (*resp. normal*). For a reduced complex space X, there is a canonical monomorphism $\mathcal{O}_X \hookrightarrow \mathcal{C}_X$, where $\mathcal{C}_X$ is the sheaf of germs of continuous functions valued in $\mathbb{C}$ over X.
- Let X and Y be two reduced complex spaces. Then f induces a morphism $f^\sharp : \mathcal{C}_Y \to f_*\mathcal{C}_X$ by $\mathcal{C}_Y \ni s \mapsto s \circ f \in \mathcal{C}_X$, which gives a morphism $(f, f^\sharp) : (X, \mathcal{C}_X) \to (Y, \mathcal{C}_Y)$ between locally ringed space over $\mathbb{C}$. We f is *holomorphic* if $f^\sharp(\mathcal{O}_Y) \subset \mathrm{im}(f_*\mathcal{O}_X \to f_*\mathcal{C}_X)$. In this case $(f, f^\sharp) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a morphism between complex spaces.
- Let X be a reduced complex space. A *prime divisor* D of X is an irreducible analytic subspace of codimension 1. A *Weil-divisor* is a formal linear combination
$$F = \sum a_i D_i$$
where $a_i \in \mathbb{Z}$ and D_i is a prime divisor and the sum is required to be locally finite. We denote by $\mathrm{Weil}(X)$ the group of Weil-divisors of a reduced complex space X.
- Let X be a reduced complex space. We denote by $\mathrm{Div}(X) = \Gamma(X, \mathcal{M}_X^*/\mathcal{O}_X^*)$ the group of *Cartier divisors* of X and set $\mathrm{Div}(X)_\mathbb{Q} := \mathrm{Div}(X) \otimes_\mathbb{Z} \mathbb{Q}$.
- Let X be a normal complex space. We say a $\mathbb{Q}$-divisor D of X is $\mathbb{Q}$-Cartier, if there exists an integer m such that $mD \in \mathrm{Weil}(X)$ and $\mathcal{O}_X(mD)$ is invertible. For two $\mathbb{Q}$-Cartier divisors D_1 and D_2, we say that D_1 and D_2 are $\mathbb{Q}$-linearly equivalent, if $D_1 - D_2$ is a $\mathbb{Q}$-combination of principal divisors and we denote

it by $D_1 \sim_{\mathbb{Q}} D_2$; we say that D_1 and D_2 are numerically equivalent if for any irreducible curve $C \subset X$, we have that $D_1 \cdot C = D_2 \cdot C$ and we denote this by $D_1 \equiv_{\text{num}} D_2$.
- We will use Serre's "GAGA principle" [Ser56] and its refinements [Gro03, Exposé XII] in the thesis.

5.2. Metric spaces

In this subsection, we recall basic notions in metric geometry. We follow [BH99, Chapter I.1, Chapter I.3]

Definition 5.1. Let (X, d) be a metric space.

- A *path* is a continous map from a compact interval $[a, b] \subset \mathbb{R}$ to X;
- Let $c : [a, b] \to X$ be a path, the *length* $l(c)$ of c is defined as
$$l(c) := \sup\{\sum_{1\leq i\leq n} d(c(t_{i-1}), c(t_i))\},$$
where the supremum is taken over all partitions $a = t_0 \leq t_1 \leq \cdots \leq t_{n-1} \leq t_n = b$ for $[a, b]$. When $l(c) < \infty$, we say that c is *rectifiable*;
- A *geodesic* is a map $c : I \to X$ where $I \subset \mathbb{R}$ is an interval, such that for any t, $t' \in I$, we have that $d(c(t), c(t')) = |t - t'|$;
- We say (X, d) is a *geodesic space* or *geodesically convex*, if for any $x, y \in X$, we could find a geodesic c joining x and y;
- A *local geodesic* is a map $c : I \to X$ where $I \subset \mathbb{R}$ is an interval, such that for any t, there exists $\epsilon > 0$ such that for all $|t' - t|$, we have that $d(c(t), c(t')) = |t - t'|$

For any metric space (X, d), we may associate a new metric d_i called the *inner metric*.

Definition 5.2. Let (X, d) be a metric space. For any $x, y \in X$, we define $d_i(x, y)$ by
$$d_i(x, y) = \inf l(c)$$
where the infimum is taken over all rectifiable path $c : [a, b] \to X$ such that $c(a) = x$ and $c(b) = y$. We set $d_i(x, y)$ to be ∞ if no such path exists. We call $d_i : X \times X \to [0, \infty]$ the *inner metric* of d.

We have the basic results:

Proposition 5.3 ([BH99, Proposition 3.2.])**.** *Let (X, d) be a metric space. Let d_i be the inner metric of d. Then*

(1) (X, d_i) is a metric space, and $d_i \geq d$;
(2) For any rectifiable path c in (X, d), we have that c is also a path in (X, d_i) and its length with respect to d_i equals to its length with respect to d;
(3) We may consider the inner metric $(d_i)_i$ of d_i. Then $(d_i)_i = d_i$.

Definition 5.4. Let (X, d) be a metric space. We call (X, d) a *length space*, if we have $d = d_i$.

We end this section by recall the Hopf-Rinow theorem.

THEOREM 5.5 ([BH99, Proposition 3.7.]). *Let (X, d) be a length space. If X is complete and locally compact, then*

(1) every closed bounded subset of X is compact, and
(2) X is a geodesic space.

In practice, to show that any two points of a compact metric space (X, d) can be joined by a geodesic, we can try to prove that (X, d) is a length space.

5.3. Differential geometry

In this subsection, we also consider real manifold. For basic notions on Riemannian geometry, the general references are [GHL04][Pet16][Sak96].

Definition 5.6. Let (M, g) be a (real) Riemannian manifold and ∇ its Levi-Civita connection.

(1) The *curvature tensor* R of (M, g) is a $(3, 1)$-tensor given by

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z,$$

where X, Y, Z are local vector fields. We may also use R to denote the associated $(4, 0)$-tensor *i.e.*

$$R(X, Y, Z, W) = g(R(X, Y)Z, W)$$

(2) We define the *Ricci curvature* by contracting R:

$$\mathrm{Ric}_g(v, w) = \mathrm{tr}(x \mapsto R(x, v)w)$$

For any piece-wise C^1 curve $\gamma : I = [a, b] \to M$, there is a length associated to g. We denote it by

$$l_g(c) := \int_a^b \sqrt{g(c'(t), c'(t))}dt.$$

With this length function l_g, for any connected Riemannian manifold (M, g), we define a metric d_g by

$$d_g(x, y) := \inf l_g(\gamma),$$

where the infimum is taken over all piece-wise C^1 curves that join x and y. For a piece-wise C^1 curve $\gamma : I = [a, b] \to M$, regarding it as a path as in Definition 5.1, we may define its length $l_{d_g}(c)$ as in Definition 5.1. We see easily that $l_{d_g}(c) \leq l_g(c)$.

Definition 5.7. Let (M, g) be a Riemmanian manifold. A *geodesic with respect to g* is a smooth curve $c : [a, b] \to M$ such that

$$c^*(\nabla)_{\frac{d}{dt}} \dot{c}(t) = 0,$$

for all $t \in (a, b)$.

The definition is a little confusing with Definition 5.1. However we have:

Proposition 5.8 ([GHL04, 2.91, 2.92])**.** *Let (M, g) be a connected Riemannian manifold and d_g the associated metric. Then*

(1) The topology induced by d_g coincides with the topology of M;

(2) *Let $m_0 \in M$. There exists a neighborhood U of m_0 and $\epsilon > 0$ such that for any $p, q \in U$, there exists a unique geodesic with respect to g, whose image is contained in U, joining p and q and $L_g(c) = d_g(p, q) < \epsilon$.*

The first consequence is the following:

Lemma 5.9. *Let (M, g) be a Riemannian manifold and d_g the associated metric. The metric space (M, d_g) is a length space.*

Proof. Let d_i be the inner metric of d_g. Then $d_i(x, y) = \inf l_{d_g}(c)$ where c is a rectifiable path that joins x and y. By Proposition 5.8, we know that these path are exactly continuous curves in M. Then $d_i(x, y) \leq l_{d_g}(c) \leq l_g(c)$ for any piece-wise C^1-curve that joins x and y. Hence $d_i(x, y) \leq \inf l_g(c) = d_g(x, y)$. □

Suppose that c is a geodesic with respect to g. Then

$$\frac{d}{dt} g(\dot{c}(t), \dot{c}(t)) = 2g(c^*(\nabla)_{\frac{d}{dt}} \dot{c}(t), \dot{c}(t)) = 0$$

shows that we can always reparametrize c such that $l_g(c|_{[t_1,t_2]}) = |t_1 - t_2|$. Then a compactness argument with Proposition 5.8 shows that c is a local geodesic in (M, d_g). Hence we can always say "Let c be a local geodesic in the Riemannian manifold (M, g)" without ambiguity.

Definition 5.10 ([GHL04, 2.84-2.88])**.** Let (M, g) be a Riemannian manifold. Let $p \in M$ and $v \in T_pM$. There exists a unique local geodesic $c_v : I \to M$ such that $c(0) = p$ and $\dot{c}(0) = v$. There exists an open $0 \in U \subset T_pM$ such that for any $w \in U$ there exists $\epsilon > 0$ such that c_w is defined on $(0-\epsilon, 1+\epsilon)$. We define the *exponential map* $\exp_p : U \to M$ by $w \mapsto c_w(1)$. The exponential map is smooth and is a diffeomorphism around $0 \in T_pM$.

5.4. Kähler geometry

We recall some elements of Kähler geometry. The reference for this section is [Dem12][Huy05][Voi02].

Definition 5.11. Let X be a complex manifold. A *Kähler form* is a closed real $(1, 1)$-form ω. A *Kähler manifold* (X, ω) is a complex manifold together with a Kähler form.

Remark 5.12. We will sometimes call a complex manifold X Kähler if it admits a Kähler form. This abuse of language is a common practice among complex geometrists.

For a Kähler manifold (X, ω), we can get the associated Riemannian metric g by $g(x, y) = \omega(x, Jy)$, where $J : TX \to TX$ is the almost complex structure of X and $x, y \in TX$ are real tangent vectors. Let ∇ be the Levi-Civita connection of g. We have that $\nabla J = 0$. Conversely, given a Hermitian manifold (X, g), the associated $(1, 1)$-form $\omega(x, y) := g(Jx, y)$ is Kähler if $\nabla J = 0$ (*cf.* [Huy05, Proposition 4.8.A]).

Definition 5.13. Let (X, ω) be a Kähler manifold and g its associated Riemannian metric. We define the *Ricci form* to be

$$\mathrm{Ricci}_\omega(x, y) := \mathrm{Ric}_g(Jx, y).$$

Locally, taking a holomorphic coordinate $(z^1, z^2, \dots, z^n)$, we can write

$$\omega = g_{i\bar{j}} dz^i \wedge d\bar{z}_j,$$
$$g = g_{i\bar{j}} dz^i \otimes d\bar{z}^j.$$

Here we use the Einstein convention.

Lemma 5.14 (*cf.* [Szé14, Lemma 1.22])**.** *Let (X, ω) be a Kähler manifold. Let $(z^1, z^2, \dots, z^n)$ be a holomorphic coordinate and write $\mathrm{Ric}_g = R_{i\bar{j}} dz^i \otimes d\bar{z}^j$. Then we have that*

$$R_{i\bar{j}} = -\partial_i \partial_{\bar{j}} \log \det(g_{p\bar{q}}).$$

In particular, we get that

$$\mathrm{Ricci}_\omega = -\sqrt{-1}\partial\bar{\partial} \log \det(g_{p\bar{q}}).$$

We finally state the celebrated Aubin-Yau theorem. We will use its orbifold version in the thesis. The statement here is for cultural reasons.

THEOREM 5.15 ([Aub78, Théorème 3][Yau78, Theorem 1]). *Let (X, ω) be a compact Kähler manifold. For any smooth function f on X and real number $\lambda \geq 0$. The equation*

$$\log \mathrm{M}(\phi) = \lambda\phi + f \tag{MA}$$

where $\mathrm{M}(\phi) = \dfrac{(\omega + \sqrt{-1}\partial\bar{\partial}\phi)^n}{\omega^n}$ is the Monge-Ampère operator, has a unique admissible solution if $\lambda > 0$ and has a unique admissible solution up to a constant if $\lambda = 0$.

We remark that the case $\lambda > 0$ is proven by T. Aubin and $\lambda = 0$ is proven by S.-T. Yau. The latter has a much harder C^0-estimates.

5.5. Classical Orbifolds

In this section, we only deal with *effective orbifolds*. The general reference are the original papers and lecture notes by Satake [Sat56][Sat57] and Thurston [Thu79, Chapter 13].

Definition 5.16 (*cf.* [Sat56, 2.Definition])**.** Let X be a topological space. Fix $n \geq 0$.

(1) An n-dimensional real *orbifold chart* on X is a triple consisting of an open subset $\tilde{U} \subset \mathbb{R}^n$, a finite subgroup G of $\mathrm{Aut}(\tilde{U})$ and a homeomorphism $\phi : \tilde{U}/G \to U$, where U is an open subset of X.
(2) Suppose that $U \subset V$ are two open subsets of X. A *chart embedding* $\lambda : (\tilde{U}, G, \phi) \to (\tilde{V}, H, \psi)$ is a smooth embedding $\lambda : \tilde{U} \to \tilde{V}$ such that $\psi \circ \lambda = \phi$
(3) An *orbifold atlas* on X is a family $\mathcal{U} = \{(\tilde{U}, G, \phi)\}$ of orbifold charts such that
   (a) $\{U = \phi(\tilde{U})\}$ covers X; and
   (b) for any $x \in X$ covered by U and V, there exists a third orbifold chart $(\tilde{W}, K, \mu)$ with $x \in W$ and two chart embedding $(\tilde{W}, K, \mu) \to (\tilde{U}, G, \phi)$ and $(\tilde{W}, K, \mu) \to (\tilde{V}, H, \psi)$.
(4) An atlas $\mathcal{V}$ is said to *refine* $\mathcal{U}$ if every chart of $\mathcal{V}$ embeds into some chart of $\mathcal{U}$. Two atlas are said to be *equivalent* if they have a common refinement.
(5) One could proceed with open subsets in $\mathbb{C}^n$ to form analytic charts and atlases.

Note that we don't need that the fixed points of the group G has codimension at least 2, as imposed by Satake. We refer to [MP97, Appendix] for details.

Definition 5.17 ([ALR07, Definition 1.2.])**.** A real (*resp.* complex) *effective orbifold* $\mathcal{X}$ of dimension n is a collection of the following data:

(i) A topological space X which is Hausdorff and second countable;
(ii) An equivalence class $[\mathcal{U}]$ of real (*resp.* complex) n-dimensional orbifold atlas.

We often use $|\mathcal{X}|$ to denote the underlying topological space.

Definition 5.18. Let $\mathcal{X}$ be an orbifold and $x \in |\mathcal{X}|$ a point. Let $(\tilde{U}, G, \phi)$ be a chart around x and $\tilde{x}$ a pre-image of x by ϕ. We define the *local group* at x by

$$G_x := \{g \in G : g \cdot \tilde{x} = \tilde{x}\}.$$

Note that G_x is only defined up to conjugacy.

Definition 5.19. For an effective orbifold $\mathcal{X} = (X, \mathcal{U})$, we define its *singular locus* by

$$X_{\text{sing}} := \{x \in X : G_x \neq 1\}.$$

We set $X_{\text{reg}} := X - X_{\text{sing}}$. It is an open subset of X and the orbifold structure restricted to X_{reg} makes X_{reg} a manifold.

Remark 5.20. It's possible to define *ineffective* orbifolds via atlas, however we have to require more compatibility conditions (*i.e. descent*). See [PST16, Definition 4.10].

Definition 5.21. Let $\mathcal{X} = (X, [\mathcal{U}])$ be an orbifold. We say that $\mathcal{X}$ is *compact* or $\mathcal{X}$ is a *compact orbifold*, if X is compact.

The following notion is due to T. Satake.

Definition 5.22 ([Sat56, 4][ALR07, Definition 1.3.])**.** A *smooth* (*resp. holomorphic*) map f between to orbifolds $\mathcal{X} = (X, \mathcal{U})$ and $\mathcal{Y} = (Y, \mathcal{V})$ is a continuous map $f : X \to Y$ on the underlying spaces, such that the following *local lifting property* is satisfied:
For any $x \in X$ if we denote $y = f(x)$, there exists a chart $(\tilde{U}, G, \phi)$ for x and a chart $(\tilde{V}, H, \psi)$ for y and a smooth (*resp.* holomorphic) map $\tilde{f} : \tilde{U} \to \tilde{V}$ such that the following diagram commutes

$$\begin{tikzcd} \tilde{U} \arrow[r, "\tilde{f}"] \arrow[d] & \tilde{V} \arrow[d] \\ U \arrow[r, "f"] & V \end{tikzcd} .$$

We call $\tilde{f}$ a *local lifting* of f.

We will see later that for a complex orbifold $\mathcal{X}$ we can associate an (analytic) klt pair (X, Δ). However the corresponding morphism are in general not smooth in the sense of smooth morphism between complex spaces. Regretfully, the name and definition are widely used.

Example 5.23 (Frame Bundle)**.** Let $\mathcal{X} = (X, \mathcal{U})$ be a real orbifold of dimension n. By a partition of unity argument, we can construct for each atlas $(\tilde{U}_i, G_i)$ a Riemmanian metric g_i such that if $\lambda : (\tilde{U}_i, G_i) \to (\tilde{U}_j, G_j)$ is an chart embedding. then $\lambda^*(g_j) = g_i$.

We fix such a family $\{g_i\}$ (*cf.* Definition 6.1). In particular, on each chart $(\tilde{U}_i, G_i, U_i)$, we have a metric $g_{i,\tilde{x}}$ on $T_{\tilde{x}}\tilde{U}_i$.

Fix a chart $(\tilde{U}_{i_0}, G_{i_0}, U_{i_0})$. In this paragraph we only deal with this chart. Hence we drop the index for the simplicity of notation. For each $\tilde{x} \in \tilde{U}$, we define the frame $F_{\tilde{x}}$ to be

$$F_{\tilde{x}} := \{p \in \mathrm{Hom}_{\mathbb{R}}(\mathbb{R}^n, T_{\tilde{x}}\tilde{U}) : p \text{ is an isometry between } \mathbb{R}^n \text{ and } (T_{\tilde{x}}\tilde{U}, g_{\tilde{x}})\ \}.$$

Here the metric on $\mathbb{R}^n$ is the standard one. The frame bundle over $\tilde{U}$ is defined to be

$$\mathrm{Fr}(\tilde{U}) := \textstyle\coprod_{\tilde{x}\in\tilde{U}} F_{\tilde{x}},$$

together with the natural projection $p : \mathrm{Fr}(\tilde{U}) \to \tilde{U}$. It's not hard to see that p can be locally trivialized as $V \times O(n, \mathbb{R})$. Hence $p : \mathrm{Fr}(\tilde{U}) \to \tilde{U}$ is a fiber bundle. The compact Lie group $O(n, \mathbb{R})$ acts from right on $\mathrm{Fr}(\tilde{U})$ by $(p.A)(v) := p(Av)$, where $p \in F_{\tilde{x}}$ and $v \in \mathbb{R}^n$. Hence $p : \mathrm{Fr}(\tilde{U}) \to \tilde{U}$ is a right $O(n, \mathbb{R})$-bundle. The group G acts from left on $\mathrm{Fr}(\tilde{U})$ by $\alpha.(\tilde{x}, p) := (\alpha \cdot \tilde{x}, T_{\tilde{x}}\alpha \circ p)$. Note that the actions of G and $O(n, \mathbb{R}^n)$ commutes:

$$\alpha.((\tilde{x}, p).A) = (\alpha.(\tilde{x}, p)).A.$$

As G acts faithfully on $\tilde{U}$, it acts freely on $\mathrm{Fr}(\tilde{U})$. The quotient $G\backslash\mathrm{Fr}(\tilde{U})$ is a manifold sitting in the following diagram

$$\begin{tikzcd} \mathrm{Fr}(\tilde{U}) \arrow[r, "p"] \arrow[d] & \tilde{U} \arrow[d] \\ G\backslash\mathrm{Fr}(\tilde{U}) \arrow[r] & G\backslash\tilde{U} = U \end{tikzcd} .$$

The $O(n, \mathbb{R})$-action on $\mathrm{Fr}(\tilde{U})$ induces an $O(n, \mathbb{R})$-action on $G\backslash\mathrm{Fr}(\tilde{U})$. Let $[\tilde{x}, p]$ be a class in $G\backslash\mathrm{Fr}(\tilde{U})$. We have that $[\tilde{x}, p]A = [\tilde{x}, p.A]$. Let x be the image of $\tilde{x}$ under the map $\tilde{U} \to U$. Then the isotropy group of $O(n, \mathbb{R})$ at $[\tilde{x}, p]$ is G_x. We have that $G\backslash\mathrm{Fr}(\tilde{U})/O(n, \mathbb{R})$ is isomorphic to U.

Let $(\tilde{U}_{i_1}, G_{i_1}, U_{i_1})$ be a second chart. We denote it by $(\tilde{V}, H, V)$. Suppose that we have an chart embedding $\lambda : \tilde{V} \to \tilde{U}$. The embedding λ induces a unique group monomorphism $\lambda_* : H \to G$ such that λ is λ_*-equivariant. From the above construction, we see that λ induces a $O(n, \mathbb{R})$-bundle morphism $\mathrm{Fr}(\lambda) : \mathrm{Fr}(\tilde{V}) \to \mathrm{Fr}(\tilde{U})$ which is λ_*-equivariant. Thus we get a commutative diagram:

$$\begin{tikzcd} & \mathrm{Fr}(\tilde{U}) \arrow[r, "p"] \arrow[d, "p"] & \tilde{U} \arrow[dd] \\ \mathrm{Fr}(\tilde{V}) \arrow[ur, "\mathrm{Fr}(\lambda)"] \arrow[r] \arrow[dd] & \tilde{V} \arrow[ur, "\lambda"] \arrow[d] & \\ & G\backslash\mathrm{Fr}(\tilde{U}) \arrow[r] & U \\ G\backslash\mathrm{Fr}(\tilde{V}) \arrow[ur] \arrow[r] & V \arrow[ur] \arrow[uu] & \end{tikzcd} .$$

By gluing up all the $G_i\backslash\mathrm{Fr}\tilde{U}_i \to U_i$, we get a manifold together with a smooth map $p : \mathrm{Fr}(\mathcal{X}) \to \mathcal{X}$, called the *frame bundle* of $\mathcal{X}$.

5.5.1. Covering maps of orbifolds. In this section we recall Thurston's definition covering map of and its basic properties. The references here are [Thu79, Chapter 13] [Cho04] and [Cho12, Chapter 4].

Definition 5.24 ([Thu79, Definition 13.2.2.])**.** A smooth map $p : \mathcal{X} \to \mathcal{Y}$ between two orbifolds $\mathcal{X}$ and $\mathcal{Y}$ is called an *orbifold covering map*, if for any $y \in Y$, there exists an orbifold chart $(\tilde{V}, G, \psi)$ such that each component U_i of $p^{-1}(V)$ has $(\tilde{V}, G_i, \phi_i)$ as a chart for some subgroup G_i of G and p is lifted with respect to this chart as identity, *i.e.*, we have the following commutative diagram

$$\begin{tikzcd} (\tilde{V}, G_i) \arrow[r, "\mathrm{id}"] \arrow[d] & (\tilde{V}, G) \arrow[d] \\ U_i \arrow[r, "p"] & V \end{tikzcd} .$$

We call such a neighborhood V an *elementary neighborhood* with respect to p at y.

Proposition 5.25 ([Cho04, Proposition 6])**.** *Let $p : \mathcal{X} \to \mathcal{Y}$ be a covering map and $p \in Y$. Let $(\tilde{V}, G)$ be a chart at y. If $\tilde{V}$ is simply connected then $V = \tilde{V}/G$ is an elementary neighborhood with respect to y.*

The above proposition gives the following corollary:

Corollary 5.26. *Let $p : \mathcal{X} \to \mathcal{Y}$ be a covering map and $p \in Y$. The elementary neighborhoods at y with respect to p form a basis of the topology of Y at y.*

Definition 5.27. Let $\mathcal{X}$ be a connected orbifold, $x \in X_{\mathrm{reg}}$. We call a covering map $p : (\tilde{\mathcal{X}}, \tilde{x}) \to (\mathcal{X}, x)$ a *universal covering* if $|\tilde{\mathcal{X}}|$ is connected and for any covering $q : (\mathcal{Y}, y) \to (\mathcal{X}, x)$ there exists a unique smooth map $f : (\tilde{\mathcal{X}}, \tilde{x}) \to (\tilde{Y}, y)$ such that $p = q \circ f$.

The following theorem is due to Thurston.

Theorem 5.28 ([Thu79, Proposition 13.2.4.], *cf.* also [Cho04, Proposition 8]). *For any orbifold $\mathcal{X}$, its universal cover exists and is unique up to an isomorphism.*

Definition 5.29 ([Thu79, Proposition 13.2.5.])**.** Let $p : \tilde{\mathcal{X}} \to \mathcal{X}$ be the universal covering of $\mathcal{X}$. The orbifold *fundamental group* $\pi_1^{\mathrm{orb}}(\mathcal{X})$ is defined to be $\mathrm{Aut}(p)$.

The following proposition is an analogue for the Galois correspondence in the theory of covering spaces for topological spaces.

Proposition 5.30 ([Cho04, Corollary 2])**.** *Let $\mathcal{X}$ be an orbifold and $p : \tilde{\mathcal{X}} \to \mathcal{X}$ be the universal covering of $\mathcal{X}$.*

(1) The orbifold fundamental group $\pi_1^{\mathrm{orb}}(\mathcal{X})$ acts by automorphisms on $\tilde{\mathcal{X}}$. For any $x \in X_{\mathrm{reg}}$, set $F_x := p^{-1}(x) \subset \tilde{X}_{\mathrm{reg}}$. Then $\pi_1^{\mathrm{orb}}(\mathcal{X})$ acts freely and transitively on F_x.

(2) There is a one-one correspondence between the isomorphism classes of orbifold coverings of $\mathcal{X}$ and the conjugacy classes of subgroups of $\pi_1^{\mathrm{orb}}(\mathcal{X})$.

We conclude this section by the following lemma.

Lemma 5.31. *Let $p : \mathcal{Y} \to \mathcal{X}$ be an orbifold covering map between two n-dimensional real orbifolds. Then there exists an $O(n, \mathbb{R})$-equivariant covering map $\mathrm{Fr}(p) : \mathrm{Fr}(\mathcal{Y}) \to \mathrm{Fr}(\mathcal{X})$, such that the following diagram commutes:*

$$\begin{tikzcd} \mathrm{Fr}(\mathcal{Y}) \arrow[r, "\mathrm{Fr}(p)"] \arrow[d] & \mathrm{Fr}(\mathcal{X}) \arrow[d] \\ \mathcal{Y} \arrow[r, "p"] & \mathcal{X} \end{tikzcd} .$$

Proof. Let $(\tilde{U}, G, U)$ be an elementary neighborhood with respect to p. Let V be a component of $p^{-1}(U)$. Then we have a subgroup $H \subset G$ and a chart $(\tilde{U}, H, V)$ of $\mathcal{Y}$ such that

$$\begin{tikzcd} (\tilde{U}, H) \arrow[r, "\mathrm{id}"] \arrow[d] & (\tilde{U}, G) \arrow[d] \\ V \arrow[r, "p"] & U \end{tikzcd} .$$

It is then immediate that the $O(n, \mathbb{R})$-equivariant map $H\backslash\mathrm{Fr}(\tilde{U}) \to G\backslash\mathrm{Fr}(\tilde{U})$ is a covering map between manifolds and it fits in the commutative diagram

$$\begin{tikzcd} H\backslash\mathrm{Fr}(\tilde{U}) \arrow[r, "\mathrm{Fr}(p)"] \arrow[d] & G\backslash\mathrm{Fr}(\tilde{U}) \arrow[d] \\ V \arrow[r, "p"] & U \end{tikzcd} .$$

Thus we only need to show that the local constructions glue together. Let $U' \subset U$ be an elementary neighborhood with respect to p contained in U and $V' \subset V$ a component of $p^{-1}(U')$ contained in V. Suppose that the charts for V' and U' are $(\tilde{U}', H', V')$ and $(\tilde{U}', H', U')$. If we have charts embeddings $\rho : (\tilde{U}', H', V') \to (\tilde{U}, H, V)$ and $\lambda : (\tilde{U}', H', U') \to (\tilde{U}, G, U)$. Then we have the following commutative diagram

$$\begin{tikzcd} & G'\backslash\mathrm{Fr}(\tilde{U}') \arrow[rr] \arrow[dd] & & G\backslash\mathrm{Fr}(\tilde{U}) \arrow[dd] \\ H'\backslash\mathrm{Fr}(\tilde{U}') \arrow[ur] \arrow[rr] \arrow[dd] & & H\backslash\mathrm{Fr}(\tilde{U}) \arrow[ur] \arrow[dd] & \\ & U' \arrow[rr] & & U \\ V' \arrow[ur] \arrow[rr] & & V \arrow[ur] & \end{tikzcd} .$$

The maps in the upper square do not depend on the choices of embeddings. Now we cover X by elementary neighborhoods. Then Y is covered by components of the inverse images of these elementary neighborhoods by p. Let V_1, V_2 be components of $p^{-1}(U_1), p^{-1}(U_2)$ respectively. As elementary neighborhoods form a base of topology, there exists an elementary neighborhood U_3 and a component V_3 of U_3 such that V_3 embeds into V_1 and V_2, and U_3 embeds into U_1 and U_2. We may thus glue $\mathrm{Fr}(V_1) \to \mathrm{Fr}(U_1)$ and $\mathrm{Fr}(V_2) \to \mathrm{Fr}(U_2)$ over $\mathrm{Fr}(V_3) \to \mathrm{Fr}(U_3)$ which ends the proof. $\square$

5.6. Complex Orbifolds

A good reference on this section is [GK07] and [DM93, Chapter 14].

We begin by recalling Cartan's quotient theorem.

Definition 5.32 (Holomorphic quotient, *cf.* [BM19, Definition 3.8.11])**.** Let M be a reduced complex space. A holomorphic map $\pi : M \to N$ from M to a reduced complex space N is a *holomorphic quotient map* if

(1) The morphism π regarded as a map between topological spaces is a topological quotient;
(2) For any holomophic map $f : M \to N_1$ which is constant on the fiber of π, the natural map $N \to N_1$ is holomorphic.

THEOREM 5.33 (Cartan's quotient theorem, *cf.* [Car54, Théorème 1]). *Let X be a complex manifold. Let $G \subset \mathrm{Aut}(X)$ be a finite automorphism group of X. Then there exists a normal complex space structure on the topological quotient X/G such that $\pi : X \to X/G$ is a holomorphic quotient.*

By the universal property of holomorphic quotient, we know that the complex space X/G is defined up to a unique isomorphism.

Let $\mathcal{X} = (X, \mathcal{U})$ be a complex orbifold of dimension n and $(\tilde{U}, G, \phi)$ an orbifold chart of $\mathcal{X}$. By Theorem 5.33, we have a unique normal complex space structure on U such that $\phi : \tilde{U} \to U$ is a holomorphic quotient. Note that here U and ϕ is fixed, if we choose the complex space structure on U to be that $\mathcal{O}_U \subset \mathcal{C}_U$, then it is unique rather than unique up to a unique isomorphism. Let $\lambda : (\tilde{V}, H, \psi) \to (\tilde{U}, G, \phi)$ be a chart embedding. By Theorem 5.33 again, we know that the inclusion $V \hookrightarrow U$ makes V an open sub complex space of U. We can thus equip a unique normal complex structure on X. The complex dimension of X is n.

Definition 5.34 (*cf.*[GK07, Section 2])**.** Let $\pi : X \to Y$ be a surjective finite morphism between normal complex spaces. The *ramification divisor* $R(\pi)$ of π is defined as following:

(1) If X and Y are both smooth, let $(z_1, z_2, \ldots, z_n)$ and $(w_1, w_2, \ldots, w_n)$ be local coordinates of X and Y respectively. Then $R(\pi)$ is locally given by the equation
$$\det(\frac{\partial \pi_i}{\partial w_j}) = 0,$$
where $\pi_i = z_i \circ \pi$.
(2) For general X and Y, set $X' := \pi^{-1}(Y_{\mathrm{reg}}) \cap X_{\mathrm{reg}}$ and $Y' := \pi(X')$. We have that X' is open and $\mathrm{codim}_X(X') \geq 2$. Set $\pi' := \pi|_{X'} : X' \to Y'$. We define $R(\pi)$ to be the closure of $R(\pi')$ in X.

Definition 5.35 (*cf.*[GK07, Section 2])**.** Let $\pi : X \to Y$ be a surjective finite morphism between normal complex spaces. We may write its ramification divisor as
$$R = R(\pi) = \sum r_j R_j,$$
where $r_j \in \mathbb{N}$ and R_j's are prime divisors. Set $R_{\mathrm{red}} := \sum R_i$.

Set $Y'' := Y' \setminus (\pi_*(R)_{\mathrm{sing}} \cup \pi(R_{\mathrm{sing}}))$ and $X'' := \pi^{-1}(Y'')$. We have that the complements of X'' and Y'' both have codimension ≥ 2. For any $x \in X''$, either x is in

exactly one component R_j of R and we define the *ramification order of* π *at* x to be $\mathrm{ord}_\pi(x) := r_j + 1$; either $x \notin R_{\mathrm{red}}$ and we define the *ramification order of* π *at* x to be $\mathrm{ord}_\pi(x) = 1$.

For a prime divisor D on X, the ramification orders of π are defined for general points of D and they are equal, we call this number *the ramification order of* π *along* D and denote it by $\mathrm{ord}_\pi(D)$.

Definition 5.36 (Branching divisor for a Galois analytic covering)**.** We call a surjective finite morphism $\pi : X \to Y$ a *Galois analytic covering* if $\mathrm{Gal}(\pi) := \{f \in \mathrm{Aut}(X) | f \circ \pi = \pi\}$ acts transitively on any fiber $\pi^{-1}(y)$. Let $R(\pi) = \sum r_t R_t$ be the ramification divisor. Suppose that B_i is an irreducible component of $\pi_* R(\pi)$. Suppose that R_j and R_k are two irreducible component of $R(\pi)$ such that $\pi_*(R_j) = \pi_*(R_k) = B_i$. Then there exists an element $f \in \mathrm{Gal}(\pi)$ such that $f_*(R_j) = R_k$. One sees easily $r_j = r_k$. We may thus assign a *multiplicity*

$$\mathrm{mult}_\pi(B_i) := 1 - \frac{1}{r_j + 1} = 1 - \frac{1}{\mathrm{ord}_\pi(x)}$$

for B_i, where x is a general point in the fiber of π. We define the *branching divisor* to be

$$B := \sum \mathrm{mult}_\pi(B_l) B_l,$$

which is an effective $\mathbb{Q}$-divisor.

For a Galois analytical covering $\pi : Y \to X$, we have the equation of $\mathbb{Q}$-Weil divisors *classes*

$$K_Y = \pi^*(K_X + B). \tag{14}$$

Let $\mathcal{X}$ be a complex orbifold with X its underlying normal complex space. Let $\lambda : (\tilde{U}, G, \phi) \to (\tilde{V}, H, \psi)$ be a chart embedding. We first note that ϕ and ψ are Galois analytic coverings. Hence we may define $B(\phi)$ on U and $B(\psi)$ on V. It's also easy to see that $\lambda^*(R(\psi)) = R(\phi)$. Hence by the rule of assigning multiplicities in Definition 5.36, we have that

$$B(\phi)|_U = B(\psi).$$

Gluing all the $(U, B(\phi))$ together, we get a log pair (X, Δ_X). Conversely, we may use the log pair to encode its orbifold structure.

Definition 5.37 (*cf.*[CC14, Definition 3.1])**.** A log pair (X, Δ) is an *orbifold pair* if Δ is a $\mathbb{Q}$-Weil divisor of the form

$$\Delta = \sum (1 - \frac{1}{m_i}) D_i,$$

where $m_i \geq 2$ are integers and (X, Δ) satisfies the locally uniformizable condition: there exists finite morphisms $\phi_j : U_j \to X$ such that

(1) $\phi_j(U_j) \subset X$ is open and $\bigcup \phi_j(U_j) = X$;
(2) $\phi_j : U_j \to \phi_j(U_j)$ is a Galois analytical cover and it's branching divisor $B(\phi_j) = \Delta|_{\phi_j(U_j)}$

The orbifold pair (X, Δ) uniquely determines a complex orbifold structure on the topological space X. From now on, when dealing with complex orbifold, we will use Definition 5.17 and Definition 5.37 interchangeably.

Let (X, Δ) be a complex orbifold. We know that X has *quotient singularities* (that is X is locally given by U/G for a complex manifold U and a finite group G acting on U). We have the following

THEOREM 5.38 ([KM98, Proposition 5.15]). *Let X be a reduced complex space with quotient singularities only. Then X has rational singularities and X is $\mathbb{Q}$-factorial.*

We say that X has *rational singularities* if for any resolution $r : Y \to X$, one has $\mathcal{O}_X = r_*(\mathcal{O}_Y)$ and $R^i r_*(\mathcal{O}_Y) = 0$ for any $i > 0$. Hence by Theorem 2.35, we have an Albanese morphism $\mathrm{Alb}_X : X \to A(X)$ defined on all X.

For a log pair (X, Δ) with standard coefficients, there is a natural definition of its fundamental group.

Definition 5.39. Let (X, Δ) be a klt pair with $\Delta = \sum(1 - \frac{1}{m_i})D_i$ where $m_i \geq 2$ are integers. We define its *fundamental group* $\pi_1(X, \Delta)$ to be

$$\pi_1(X_{\mathrm{reg}} \setminus |\Delta|)/N,$$

where N is the normal group generated by $\gamma_j^{m_j}$, with γ_j a small loop around the component D_j with multiplicity $1 - \frac{1}{m_j}$ of Δ.

For a complex orbifold $\mathcal{X} = (X, \Delta)$, we thus associate, *a priori*, two fundamental groups to it. We will show in the next section these two groups are canonically isomorphic.

5.7. Orbi-bundles

5.7.1. Orbi-bundles. We will use groupoid theory for orbifolds introduced by Moerdijk and Pronk [MP97] to study the orbi-bundles. We put the basic definitions and the properties of groupoids in Appendix A. The reader can find in [MP97] [Moe02] and [ALR07, Chapters 1 and 2] all the technical results.

Recall we defined smooth morphism in Definition 5.22. Though with the name *smooth*, we note that smooth map does not behaves well. For example, we don't know whether a smooth morphism $f : \mathcal{X} \to \mathcal{Y}$ induces pullback morphism of differential forms. And the author does not know if for $x \in |X|$, the smooth map f induces a morphism of local groups $G_x \to G_{f(x)}$. To overcome this problem, we use the notion of *strong morphism* introduced by Moerdijk and Pronk [MP97].

The other motivation to consider orbifolds as groupoids is to describe the universal covering orbifolds more concretely. Recall that we have defined the orbifold fundamental groups as Galois groups of deck transformations. It is not immediate that we have an orbifold Van-Kampen theorem. On the other hand, the quite mild singularities on orbifolds makes it tempting to realize orbifold fundamental groups by certain "homotopy class of loops" as in the case for fundamental groups of topological space. By considering an orbifold as a groupoid $\mathcal{G}$,we can use the $\mathcal{G}$-paths and $\mathcal{G}$-homotopies to describe its orbifold fundamental group. We refer the reader to [BH99, Chapter $\mathcal{G}$] [Cho12, Chapter 4.7.].

[MP97] deals with orbifolds by identifying them with certain groupoids and define the maps between orbifolds to be the ones induced by morphisms between groupoids.

Here is Moerdijk and Pronk's definition (*cf.* [MP97, Theorem 4.1.] [ALR07, Definition 1.48.]) of orbifolds.

Definition 5.40.

(1) An orbifold structure on a paracompact Hausdorff space X consists of an orbifold groupoid $\mathcal{G}$ and a homeomorphism $f : |\mathcal{G}| \to X$. We say that two orbifold structures $f : |\mathcal{G}| \to X$ and $g : |\mathcal{H}| \to X$ are equivalent, if there exists an equivalence of groupoids $\phi : \mathcal{G} \to \mathcal{H}$ such that $f = g \circ |\phi|$.

(2) An orbifold $\mathcal{X}$ is a space X together with a class of equivalent orbifold structure. An element $f : |\mathcal{G}| \to X$ is called a presentation of the orbifold $\mathcal{X}$.

Let $(X, \mathcal{G}, f : |\mathcal{G}| \to X)$ be an orbifold in the sense of Definition 5.40. Take $x \in X$ and $\tilde{x} \in G_0$ one of its pre-image. By Proposition A.8, we have an orbifold chart $(U_{\tilde{x}}, G_{\tilde{x}}) \to f(U_{\tilde{x}}/G_{\tilde{x}})$ around x. It's easy to see that we can get an atlas consisting of all these charts. Hence we get an orbifold $(X, \mathcal{U})$ in the sense of Definition 5.17. We call $\mathcal{U}$ the orbifold atlas associated to $\mathcal{G}$. If $\mathcal{G}$ is Morita equivalent to $\mathcal{H}$, then their associated atlases are equivalent. In [MP97], Moerdijk and Pronk proved that for any orbifold $(X, \tilde{U})$ in the sense of Definition 5.17, we can get a unique (up to equivalence) groupoid representation of X.

THEOREM 5.41 ([MP97, Theorem 4.1.]). *Let $\mathcal{X} = (X, \mathcal{U})$ be an orbifolds in the sense of Definition 5.17. There exists up to a Morita equivalence, a unique effective orbifold groupoid $\mathcal{G}$ and a homeomorphism $|\mathcal{G}| \to X$, such that the associated atlas $\mathcal{V}$ of $\mathcal{G}$ is equivalent to $\mathcal{U}$.*

Thus we may interchange freely both definitions of orbifolds: in terms of atlas or in terms of groupoid. We now give the defintion of strong maps.

Definition 5.42. Let $\mathcal{X}, \mathcal{Y}$ be two orbifolds. A *strong map* $(f, [[F]])$ from $\mathcal{X}$ to $\mathcal{Y}$ consists of a continuous map $f : |\mathcal{X}| \to |\mathcal{Y}|$ and a $\mathcal{R}$-class $[F]$ of arrows in the category $\mathcal{HS}$ (Definition A.11), satisfying the following conditions:

(1) There are representations $\mathcal{G}, \mathcal{H}$ of $\mathcal{X}$ and $\mathcal{Y}$ respectively;

(2) $F : \mathcal{X} \to \mathcal{Y}$ is a groupoid morphism;

(3) There is a commutative diagram

$$\begin{tikzcd} |\mathcal{G}| \arrow[r, "|F|"] \arrow[d] & |\mathcal{H}| \arrow[d] \\ |\mathcal{X}| \arrow[r, "f"] & |\mathcal{Y}| \end{tikzcd} .$$

Let $\mathcal{X}$, $\mathcal{Y}$ be two orbifolds. From the definition above, we see that the set of strong maps from $\mathcal{X}$ to $\mathcal{Y}$ is canonically bijective to $\mathrm{Hom}_{\mathrm{HS}}(\mathcal{G}_{\mathcal{X}}, \mathcal{G}_{\mathcal{Y}})$, for any representations $|\mathcal{G}_{\mathcal{X}}| \to \mathcal{X}$ of $\mathcal{X}$ and $|\mathcal{G}_{\mathcal{Y}}| \to \mathcal{Y}$ of $\mathcal{Y}$.

The first component of a strong map is clearly smooth. From the above definition, we see that $(f, [[F]])$ induces morphisms between local groups. For simplicity, we will drop the brackets and denote strong maps by (f, F). We note that an orbifold covering maps always comes from a strong map.

Lemma 5.43. *Let $p : \mathcal{Y} \to \mathcal{X}$ be an orbifold covering. There exists a morphism between groupoids $F : \mathcal{G}_{\mathcal{Y}} \to \mathcal{G}_{\mathcal{X}}$ representing p, where $\mathcal{G}_{\mathcal{Y}}$ and $\mathcal{G}_{\mathcal{X}}$ are orbifold groupoids representing $\mathcal{Y}$ and $\mathcal{X}$.*

Proof. [MP97, Theorem 4.1.] implies that we have representations $\mathcal{G}_{\mathcal{Y}}$ of $\mathcal{Y}$ and $\mathcal{G}_{\mathcal{X}}$ of $\mathcal{X}$ such that $\mathcal{G}_{\mathcal{Y}}$ is Morita equivalent to $O(n,\mathbb{R}) \ltimes \mathrm{Fr}(\mathcal{Y})$ and $\mathcal{G}_{\mathcal{X}}$ is Morita equivalent to $O(n,\mathbb{R}) \ltimes \mathrm{Fr}(\mathcal{X})$. Lemma 5.31 implies that we have a Lie groupoid morphism $\mathrm{Fr}(p)$ which induces p on the orbit space. As $\mathrm{Hom}_{\mathcal{HS}}(\mathcal{G}_{\mathcal{Y}}, \mathcal{G}_{\mathcal{X}})$ is isomorphic to $\mathrm{Hom}_{\mathcal{HS}}(O(n,\mathbb{R}) \ltimes \mathrm{Fr}(\mathcal{Y}), O(n,\mathbb{R}) \ltimes \mathrm{Fr}(\mathcal{X}))$, the class $[[\mathrm{Fr(p)}]]$ induces p. □

We finish our comment on strong maps by citing the pathology from [Ler10].

Example 5.44 ([Ler10, Lemma 3.41.])**.** Let $S^1 \subset \mathbb{C}^2$ be the unit circle considered as an orbifold groupoid $\{S^1 \rightrightarrows S^1\}$. Set $U_1 := S^1 \setminus +1$ and $U_2 := S^1 \setminus -1$. Consider the quotient groupoid $\mathbb{Z}/2\mathbb{Z} \ltimes *$, where $*$ is a singleton. There are exactly two distinct strong maps (f_j, F_j), $j = 1, 2$ from $\{S^1 \rightrightarrows S^1\}$ to $\mathbb{Z}/2\mathbb{Z} \ltimes *$. However $(f_1, F_1)|_{U_i} = (f_2, F_2)|_{U_i}$ for $i = 1, 2$.

We now give our definition of orbi-vector bundles on orbifolds.

Definition 5.45. Let $\mathcal{X}$ be an orbifold. A real (resp. complex) vector bundle of rank r is a strong map $p : \mathcal{V} \to \mathcal{X}$ together with the following

(1) A representation $f : \mathcal{G} \to |\mathcal{X}|$ of $\mathcal{X}$;
(2) A real (resp, complex) left-$\mathcal{G}$-vector bundle E (Definition A.12) on G_0;
(3) A homeomorphism $g : \mathcal{G} \ltimes E \to |\mathcal{V}|$, such that g gives a representation of the orbifold $\mathcal{V}$ and $\mathcal{G} \ltimes E \to \mathcal{G}$ represents p

As in the manifold case, we need another definition for holomorphic orbi-vector bundles. Our definition of *holomorphic bundle* is given in Definition 8.4. There is also other issues on how to define sections, see Remark 8.5. A clean definition is to consider a complex orbifolds $\mathcal{X}$ as a stack fibered in the category of complex manifolds CompMan, then define a holomorphic vector bundle as a representable map $V \to X$ covered by $V \times_{U_i} \mathcal{X} \cong \mathbb{C}^n \times U_i$ for some cover $\{U_i \to A\}$ of the atlas A of $\mathcal{X}$. We refer the reader to [Par20].

Let $\mathcal{V} \to \mathcal{X}$ be an orbi-vector bundle represented by $\mathcal{E} = \mathcal{G} \ltimes E \to \mathcal{G}$. For a $x \in G_0$, we take a neighborhood U_x of x such that $G_x \ltimes U_x \cong \mathcal{G}|_{U_x}$ as in Proposition A.8. We may take U_x sufficiently small such that there is a trivialization $E|_{U_x} \cong U_x \times \mathbb{F}^r$. Then there is an isomorphism $\mathcal{E}|_{U_x} \cong G_x \ltimes (U_x \times \mathbb{F}^r)$. The actions of G_x fits into a commutative diagram

$$\begin{tikzcd} G_x \times (U_x \times \mathbb{F}^r) \arrow[r] \arrow[d] & U_x \times \mathbb{F}^r \arrow[d] \\ G_x \times U_x \arrow[r] & U_x \end{tikzcd} .$$

Hence $(U_x \times \mathbb{F}^r)/G_x$ and U_x/G_x are orbifold charts of $\mathcal{V}$ and $\mathcal{X}$ respectively. Note that for any $y \in U_x/G_x$, its fiber $|\mathcal{V}|_y$ is isomorphic to $\mathbb{F}^r/G'$, where G' is a subgroup of G_x.

Definition 5.46. Let $\pi : E \to G_0$ be a left-$\mathcal{G}$-vector bundle. A $\mathcal{G}$-section of E over $U \subset G_0$ is a section $s : U \to E$ of π such that for any $g \in G_1$, we have that $g \cdot s(x) = s(g \cdot x)$. Note that s induces a morphism $\mathcal{G}|_U \to (\mathcal{G} \ltimes E)|_U$.

If the orbi-vector bundle $p : \mathcal{V} \to \mathcal{X}$ is represented by $\mathcal{G} \ltimes E \to \mathcal{G}$, we say that $\sigma : U/\mathcal{G} \to \mathcal{V}$ is a section of p if σ corresponds to a $\mathcal{G}$-section s of E.

Example 5.47. If E and F are left-$\mathcal{G}$-bundles for the topological groupoid $\mathcal{G}$, $E \otimes F$, $\wedge^p E$, $\mathrm{Sym}^n(E)$, $\mathrm{Hom}(E, F)$ and $E^\vee$ have natural left-$\mathcal{G}$-bundle structures, and we denote their associated groupoids by $\mathcal{E} \otimes \mathcal{F}$, $\wedge^p \mathcal{E}$, $\mathrm{Sym}^n(\mathcal{E})$, $\mathrm{Hom}(\mathcal{E}, \mathcal{F})$ and $\mathcal{E}^\vee$ respectively.

Example 5.48. Let $\mathcal{X}$ be a real orbifold of dimension n and $\mathcal{U}$ be its atlas. For each chart $(\tilde{U}_i, G_i, \phi_i)$, we associate a G_i-space $(T\tilde{U}_i, G_i)$, where g_i acts on $T\tilde{U}_i$ by its tangent associated to its action on $\tilde{U}_i$. We identify $[v_i] \in T\tilde{U}_i/G_i$ and $[v_j] \in T\tilde{U}_j/G_j$ if there are orbifold chart embeddings $\rho_i : \tilde{V} \to \tilde{U}_i$ and $\rho_j : \tilde{V} \to \tilde{U}_j$ such that $T\rho_i(v_i) = T\rho_j(v_j)$. The topological space $\{\bigsqcup_i \tilde{U}_i/G_i\}/([v_i] \sim [v_j])$ has an orbifold atlas, whose charts are $(T\tilde{U}_i, G_i)$. We denote this orbifold by $T\mathcal{X}$. By construction there is a natural projection $T\mathcal{X} \to \mathcal{X}$. If $|\mathcal{G}| \to |\mathcal{X}|$ is a representation of $\mathcal{X}$, then $T\mathcal{G} := \mathcal{G} \ltimes TG_0$ is a representation of $T\mathcal{X}$ and the projection $T\mathcal{X} \to \mathcal{X}$ is represented by $T\mathcal{G} \to \mathcal{G}$.

Hence when considering tangent bundles and cotangent bundles of $\mathcal{X}$, we don't distinguish the representation groupoids $\mathcal{G}$ used to construct the representation $T\mathcal{G}$ of $T\mathcal{X}$.

We thus define *tensors* and *forms* to be the orbi-sections of their corresponding bundles (*cf.* Definition 6.9). From the construction of tangent bundle $T\mathcal{X}$ of $\mathcal{X}$, we see that giving a p-form ω over $O \subset X$ an open subset of $X = |\mathcal{X}|$ is equivalent to find a cover of O by orbifold charts (U_i, G_i) and G_i-invariant p-forms ω_i on U_i such that for any chart embedding $\lambda : U_i \to U_j$, we have $\lambda^*(\omega_j) = \omega_i$. We can say the same for p-tensors.

With this description of forms, we now give the following

Lemma 5.49. *Let $p : \mathcal{X} \to \mathcal{Y}$ be an orbifold covering map. Then p induces a morphism between local groups and pullbacks for tensors and forms.*

Proof. Let ω be a differential form on $\mathcal{Y}$. As elementary neighborhoods with respect to p form a base of topology, we can consider ω as a family of invariant forms over a covering by elementary neighborhoods.

Let $(\tilde{V}, H) \to V$ be an elementary neighborhood of $\mathcal{Y}$. For a component $U \subset p^{-1}(V)$, we have a chart $(\tilde{V}, G) \to U$ where G is a subgroup of H and f is lifted by $\mathrm{id}_{\tilde{V}}$. If ω is represented by the H_y-invariant form $\omega_{\tilde{V}}$ on $\tilde{V}$, we set the invariant form over $(\tilde{V}, G) \to U$ to be $\omega_{\tilde{V},U}$. If $[\tilde{V}'/H'] = V' \subset V$ is another elementary neighborhood, after shrinking V', we may assume that there exists an chart embedding $(\tilde{V}', H') \to (\tilde{V}, H)$. Let $U' \subset p^{-1}(V')$ be the component that is contained in U, and $(\tilde{V}', G') \to U'$ be a chart. For any $x \in U'$, there exists a chart $(\tilde{W}, G_x) \to W$ such that $(\tilde{W}, G_x)$ embeds into both $(\tilde{V}', G')$ and $(\tilde{V}, G)$. We may suppose that $\tilde{W} \subset \tilde{V}'$ and $\tilde{x} \in \tilde{W}$ maps to x such that $G_x = \{g \in G' : g\tilde{x} = \tilde{x}\}$. Set $H_x := \{h \in G' : h\tilde{x} = \tilde{x}\}$. Then $G_x = H_x \cap G'$. Shrink $\tilde{W}$ if necessary. We may assume that $\tilde{W}$ is H_x invariant and simply connected. Let O be the image of $\tilde{W}$ in V'. Then O is open and $(\tilde{W}, H_x) \to O$ is an elementary chart of $\mathcal{Y}$ with respect to p. Now $(\tilde{W}, H_x)$ embeds into $(\tilde{V}', H')$ by construction. From the construction, we have that

$$\omega_{\tilde{V}',U'}|_{\tilde{W}} = \omega_{\tilde{V}'}|_{\tilde{W}} = \omega_{\tilde{V}}|_{\tilde{W}} = \omega_{\tilde{V},U}|_{\tilde{W}}.$$

This means that if we associate ω to the family $\{\omega_{\tilde{V},U}\}$, the family will satisfy the compatibility condition hence define a form on $\mathcal{Y}$. We denote this form by $p^*(\omega)$.

From the construction above, we also see that if ω is represented by $\tilde{\omega}$ on an elementary chart $(\tilde{V}, H)$, then on each component $U_i = [\tilde{V}/G_i]$, the pullback $p^*(\omega)$ is also represented by $\tilde{\omega}$. □

5.7.2. Comparison of fundamental groups. For any category C, we can associate functorially a topological space BC, its *classifying space*. When $F : C \to C'$ is an equivalence, the continuous map $BF : BC \to BC'$ is an homotopy equivalence [Seg68, Proposition 2.1.]. If $\mathcal{G}$ and $\mathcal{H}$ are two Morita equivalent groupoids, then $B\mathcal{G}$ and $B\mathcal{H}$ have the same homotopy type. Thus for an orbifold $\mathcal{X}$, we may define its orbifold homotopy group $\pi_n^{\mathrm{orb}}(\mathcal{X})$ to be $\pi_n(B\mathcal{G})$, where $\mathcal{G}$ is any groupoid representation of $\mathcal{X}$. We refer the reader to [ALR07, Section 1.4] for more details. Combining [ALR07, Proposition 2.17] and [BH99, Corollary III.$\mathcal{G}$.3.19], we know that $\pi_1(B\mathcal{G})$ is isomorphic to the group $\pi_1(X, \mathcal{G})$ of $\mathcal{G}$-homotopy classes of $\mathcal{G}$-loops defined in [BH99, Definition III.$\mathcal{G}$.3.6.]. [Cho12, Theorem 4.7.4.] implies that $\pi_1(X, \mathcal{G})$ is isomorphic to the fundamental group defined as deck transformation in Definition 5.29.

If in addition $\mathcal{X} = (X, \Delta)$ is a complex orbifold, we have another fundamental group $\pi_1(X, \Delta)$ as in Definition 5.39. It is well-known among experts that $\pi_1(X, \Delta)$ is isomorphic to the above three groups. As we can not find a reference, we give a proof communicated to us by P.Eyssidieux [Eys].

Proposition 5.50. *Let $\mathcal{X} = (X, \Delta)$ be a complex orbifold. There exists a canonical isomorphism*

$$\pi_1^{\mathrm{orb}}(\mathcal{X}) \cong \pi_1(X, \Delta)$$

Proof. We consider the frame bundle $\mathrm{Fr}(\mathcal{X}) \to \mathcal{X}$. Suppose that $\mathcal{G}$ is an orbifold groupoid representing $\mathcal{X}$. Then [MP97, Theorem 4.1.] implies that $\mathrm{U}(n, \mathbb{C}) \ltimes \mathrm{Fr}(\mathcal{X})$ is Morita equivalent to $\mathcal{G}$. We have thus $\pi_1^{\mathrm{orb}}(\mathcal{X}) = \pi_1([\mathrm{Fr}(\mathcal{X})/\mathrm{U}(n, \mathbb{C})])$, where $[\mathrm{Fr}(\mathcal{X})/\mathrm{U}(n, \mathbb{C})]$ is considered as a topological stack. By [Noo14, Example 5.6.], we have the following exact sequences

$$\pi_1(\mathrm{U}(n, \mathbb{C})) \to \pi_1(\mathrm{Fr}(\mathcal{X})) \to \pi_1^{\mathrm{orb}}(\mathcal{X}) \to 1 \tag{15}$$

Let $U \subset X$ be the snc locus of (X, Δ). We have that $X \setminus U$ has codimension at least 2. As $p : \mathrm{Fr}(\mathcal{X}) \to X$ is equidimensional, we have that $\mathrm{Fr}(\mathcal{X}) \setminus p^{-1}(U)$ has codimension at least 2, too. Note that as a complex variety $\mathrm{Fr}(\mathcal{X})$ is smooth. Hence we have

$$\pi_1(\mathrm{Fr}(\mathcal{X})) = \pi_1(p^{-1}(U)).$$

Set $\mathcal{O}$ to be the open sub-orbifold $(U, \Delta|_U)$. We have the following commutative diagram

$$\begin{tikzcd}
\pi_1(\mathrm{U}(n, \mathbb{C})) \arrow[r] \arrow[d] & \pi_1(p^{-1}(U)) \arrow[r] \arrow[d] & \pi_1^{\mathrm{orb}}(\mathcal{O}) \arrow[r] \arrow[d] & 1 \arrow[d] \\
\pi_1(\mathrm{U}(n, \mathbb{C})) \arrow[r] & \pi_1(\mathrm{Fr}(\mathcal{X})) \arrow[r] & \pi_1^{\mathrm{orb}}(\mathcal{X}) \arrow[r] & 1
\end{tikzcd}.$$

By four lemma, we see that $\pi_1^{\mathrm{orb}}(\mathcal{O}) = \pi_1^{\mathrm{orb}}(\mathcal{X})$. As $U \setminus |\Delta|_U| = X_{\mathrm{reg}} \setminus |\Delta|$, we see from definition that

$$\pi_1(U, \Delta|_U) = \pi_1(X, \Delta).$$

Hence we just need to show that $\pi_1(U, \Delta|_U) = \pi_1^{\mathrm{orb}}(\mathcal{O})$. As $(U, \Delta|_U)$ has as chart

$$(z_1, ..., z_i, z_{i+1}, ..., z_n) \mapsto (z_1^{m_1}, ..., z_i^{m_i}, z_{i+1}, ..., z_n)$$

when $\mathcal{O}$ is covered by such a single chart, we have that $\pi_1(U, \Delta|_U) = \pi_1^{\mathrm{orb}}(\mathcal{O})$. The general case is by applying Van-Kampen theorem. □

5.7.3. Integration and the de Rham cohomology. We recall the integration of orbi-forms and the subsequent de Rham cohomology of orbifolds.

Definition 5.51. Let $\mathcal{X}$ be a real orbifold of dimension n. we say that $\mathcal{X}$ is *orientable* if there exists a non-vanishing n-form α on $\mathcal{X}$. We say that a chart $(\tilde{U}, G, \phi)$ is compatible with this orientation if $\phi^*(\alpha) = \lambda \cdot \mathrm{vol}_{\mathbb{R}^n}$, where λ is a positive function. For ω an n-form supported in U, we define its *integration* by

$$\int_{\mathcal{X}} \omega := \frac{1}{|G|} \int_{\tilde{U}} \phi^* \omega.$$

For general case, we can cover $\mathcal{X}$ by charts $(\tilde{U}_i, G_i, \phi_i)$, take a partition of unity ρ_i with respects to $\{U_i\}$, and define

$$\int_{\mathcal{X}} \omega := \sum_i \int_{\mathcal{X}} \rho_i \omega.$$

We give a quick argument that the integration is well-defined. Suppose first that ω is supported in U, and $(\tilde{U}_i, G_i, \phi_i)$ are charts that are embedded via λ_i to the chart $(\tilde{U}, G, \phi)$ (*cf.* [MP97, page 5 Remark (6)]). Then G_i is a subgroup of G and all the distinct embedding of $(\tilde{U}_i, G_i, \phi_i)$ into $(\tilde{U}, G, \phi)$ will be $g \cdot \lambda_i$, where $g \cdot G_i$ forms the cosets. Thus

$$\begin{aligned}
\int_{\mathcal{X}} \omega &= \frac{1}{|G|} \int_{\tilde{U}} \phi^* \omega \\
&= \frac{1}{|G|} \sum_i \int_{\tilde{U}} (\rho_i \circ \phi) \phi^* \omega \\
&= \sum_i \sum_g \frac{1}{G} \int_{g\lambda(\tilde{U}_i)} (\rho_i \circ \phi) \phi^* \omega \\
&= \sum_i \frac{1}{G_i} \int_{\tilde{U}_i} (\rho_i \circ \phi_i) \phi_i^* \omega
\end{aligned}$$

where in the forth equation g runs through the representatives of cosets G/G_i.

For the general case, suppose that $\{\tilde{U}_i\}$, $\{\tilde{V}_j\}$ are two coverings of $\mathrm{supp}(\omega)$ by orbifold charts. We take a third covering $\{\tilde{W}_k\}$ which refines both $\{\tilde{U}_i\}$ and $\{\tilde{V}_j\}$, *i.e.* each $\tilde{W}_k$ embeds into some $\tilde{U}_{i(k)}$ and $\tilde{V}_{j(k)}$. Then it reduces the argument to a single chart and its refinement. Integration is thus well-defined.

Let $\mathcal{A}^p(\mathcal{X})$ be the global orbi-p-forms on $\mathcal{X}$. We see that the exterior differential d maps $\mathcal{A}^p$ to $\mathcal{A}^{p-1}$. Hence, it makes sense to consider the de Rham cohomology of $\mathcal{X}$. We recall some basic results.

Proposition 5.52. *Let $\mathcal{X}$ be a real n-orbifold and $X = |\mathcal{X}|$ be its underlying topological space. We have a canonical isomorphism $\mathrm{H}^p(X, \mathbb{R}) \cong \mathrm{H}^p_{\mathrm{dR}}(\mathcal{X})$.*

Proof. We give a direct proof. Consider the sheaf $C^\infty_{\mathcal{X}}$ on X, given by $V \mapsto \mathrm{Mor}(V, \mathbb{R})$. Note that $\mathcal{X}$ has a partition of unity by smooth function with respect to any open cover. Thus the sheaf $\mathcal{A}^p$ is fine and acyclic (*cf.* [Voi02, Definition 4.35 and Proposition 4.36]). On the other hand, the complex $\mathcal{A}^\bullet$ is a resolution of $\mathbb{R}_X$. Hence we have the canonical isomorphism $\mathrm{H}^p(X, \mathbb{R}) \cong \mathrm{H}^p_{\mathrm{dR}}(\mathcal{X})$. □

In [Sat56, section 7], Satake showed that there is a canonical morphism

$$\mathrm{H}^{\mathrm{sing}}_p(\mathcal{X}, \mathbb{R}) \to \check{\mathrm{H}}_p(\mathcal{U}, \mathbb{R}),$$

where the latter is the Čech homology group. If we define $\check{\mathrm{H}}_p(X, \mathbb{R}) = \varprojlim \check{\mathrm{H}}_p(\mathcal{U}, \mathbb{R})$, then we have

Proposition 5.53 (*cf.* [Sat56, Theorem 2])**.** *Let $\mathcal{X}$ be a real n-orbifold with $X = |\mathcal{X}|$ its underlying space. We have a canonical isomorphism $\mathrm{H}^{\mathrm{sing}}_p(\mathcal{X}, \mathbb{R}) \to \check{\mathrm{H}}_p(X, \mathbb{R})$*

As $\check{\mathrm{H}}^p(\mathcal{U}, \mathbb{R})$ is dual to $\check{\mathrm{H}}_p(\mathcal{U}, \mathbb{R})$, there is an isomorphism $\mathrm{H}^{n-p}_{\mathrm{dR}}(\mathcal{X}) \cong \mathrm{H}^{\mathrm{sing}}_p(X, \mathbb{R})$. Moreover, we have the Poincaré duality for orbifolds:

Proposition 5.54. *Let $\mathcal{X}$ be a compact real n-orbifold, the natural map*

$$\begin{cases} \mathrm{H}^p_{\mathrm{dR}}(\mathcal{X}) \times \mathrm{H}^{n-p}_{\mathrm{dR}}(\mathcal{X}) & \longrightarrow & \mathbb{R} \\ (\omega, \theta) & \mapsto & \int_{\mathcal{X}} \omega \wedge \theta \end{cases}$$

is a perfect paring.

CHAPTER 6

Riemannian orbifolds

From this chapter onward, we only deal with effective orbifolds..

6.1. Differential calculus on orbifold

Definition 6.1. A *Riemmanian orbifold* is a pair $(\mathcal{X}, g)$ where $\mathcal{X}$ is an orbifold and g is an orbi-section of $(T^2\mathcal{X})^\vee$ satisfying the following equivalent conditions:

(i) If $\mathcal{X}$ is represented by $\mathcal{G}$, and g corresponds to $\sigma : G_0 \to (T^2 G_0)^\vee$ then σ is a Riemannian metric on G_0;

(ii) There exists a family of charts $\{(\tilde{U}_i, G_i)\}$ covering X, with G_i-invariant metrics $\tilde{g}_i$ representing g over $[\tilde{U}_i/G]$.

Most operators on Riemannian manifolds can be generalized to Riemannian orbifolds. We begin to treat some basic results on covariant derivatives on orbifolds.

Let $\mathcal{X}$ be an orbifold and $\{\mathcal{G}, f : |\mathcal{G}| \to |\mathcal{X}|\}$ being a groupoid representation of $\mathcal{X}$. We know from Example 5.48 that $T^p\mathcal{X} \otimes T^q(T\mathcal{X}^\vee)$ is represented by $T^p\mathcal{G} \otimes T^q(T\mathcal{G}^\vee)$. A (p, q)-tensor over an open subset U of $|\mathcal{X}|$ is thus a collection of G_i-invariant (p, q)-tensor over $\tilde{U}_i$ such that $\{U_i = \tilde{U}_i/G_i\}$ cover U. For $T = X_1 \otimes X_2 \otimes \cdots \otimes X_p \otimes S$, we have

$$g \cdot T = g_*(X_1) \otimes g_*(X_2) \otimes \cdots \otimes g_*(X_p) \otimes (g^{-1})^*(S).$$

As all the calculation can be performed locally, in the following we consider a local model (U, H, g) where $U \subset \mathbb{R}^n$ is an open subset, with H a finite subgroup of $\mathrm{Aut}(U)$, and g a Riemannian metric on U which is H-invariant. For any $h \in H$, we note that the action of h on $x \in U$ by $L_h(x)$ or $h \cdot x$. Also for any smooth function f, we define the H-action on f by $h \cdot f = f \circ L_{h^{-1}}$. A easy consequence for this adaption is that for any (p, q)-tensor T, we have that $h \cdot (fT) = (h \cdot f)(h \cdot T)$.

Let T be a $(0, p)$-tensor over U, and X_i vector field over U, where $1 \le i \le p$. We have that

$$\begin{aligned} h^*(T)(X_1, X_2, \ldots, X_p)(x) &= T(hx)(T_x L_h X_1(x), \ldots, T_x L_h X_p(x)) \\ &= T(hx)(h_* X(xh), \ldots, h_* X(xh)) \\ &= T(h_* X_1, \ldots, h_* X_p)(hx). \end{aligned}$$

Thus T being H-invariant is characterized by

$$T(X_1, X_2, \ldots, X_p) = T(h_*(X_1), \ldots, h_*(X_p)) \circ L_h$$

for any $h \in H$ and any vector fields $X_1, ...X_p$.

The metric g being H-invariant, we infer that L_h is an isometry for any $h \in H$ and we have $h_*(\nabla_X Y) = \nabla_{(h_* X)}(h_* Y)$ where ∇ is the Levi-Civita connection of the metric g.

The following easy lemma is an example that differential operators on Riemmanian manifolds have natural generalizations to Riemannian orbifolds.

Lemma 6.2. *Let T be a $(0,p)$-tensor which is H-invariant. Then the $(0,p+1)$-tensor ∇T is H-invariant.*

Proof. Let $X_0,\dots,X_p$ be $p+1$ vector fields. The lemma follows from the following computation:

$$\begin{aligned}
&((\nabla T)(h_*(X_0),h_*(X_1),\dots,h_*(X_p)))\circ L_h\\
&\quad=((\nabla_{h_*(X_0)}T)(h_*(X_1),\dots,h_*(X_p)))\circ L_h\\
&\quad=\Big((h_*(X_0)\cdot T(h_*(X_1),\dots,h_*(X_p)\\
&\qquad\qquad-\sum_i T(h_*(X_1),\dots,\nabla_{h_*(X_0)}h_*(X_i),\dots,h_*(X_p))\Big)\circ L_h\\
&\quad=\Big((X_0\cdot(T(h_*(X_1),\dots,h_*(X_p))\circ L_h))\circ L_{h^{-1}}\\
&\qquad\qquad-\sum_i T(h_*(X_1),\dots,h_*(\nabla_{X_0}X_i),\dots,h_*(X_p))\Big)\circ L_h\\
&\quad=X_0\cdot T(X_1,\dots,X_p)-\sum_i T(X_1,\dots,\nabla_{X_0}X_i,\dots,X_p)\\
&\quad=(\nabla T)(X_0,X_1,\dots,X_p).
\end{aligned}$$

□

As differential commutes with pull-back, we see that if ω is an invariant p-form, so is $d\omega$. In particular, if f is H-invariant smooth function, the df is an invariant 1-form and $\nabla f=(df)^\sharp$ is an invariant vector field.

Let W,V be two orbi-vector fileds over $\mathcal{X}$. Take $(\tilde U,G)$ a chart for $\mathcal{X}$ such that W,V are represented by G-invariant fields $\tilde W,\tilde V$ respectively. Then $h_*(\nabla_{\tilde W}\tilde V)=\nabla_{\tilde W}\tilde V$ for any $h\in G$. If $\lambda:(\tilde U',G')\to(\tilde U,G)$ is an chart embedding, and $\tilde W',\tilde V'$ are the representations of W,V on $\tilde U'$, then $\nabla_{\tilde W'}\tilde V'=\lambda^*(\nabla_{\tilde W}\tilde V)$ as λ is an isometry. Thus all the local representations glue back to an orbi-vector field. We may thus define:

Definition 6.3. Let W,V be two orbi-vector fields over $(\mathcal{X},g)$, represented by $\tilde W_i,\tilde V_i$ on a covering $\{(\tilde U_i,G_i)\}$ respectively. Let ∇_i be the Levi-Civita connection on $\tilde U_i$, then there is a unique vector filed $\nabla_W V$ on $\mathcal{X}$ corresponding to the family $\nabla_{i\tilde W_i}\tilde V_i$. We define the association $\nabla:W,V\mapsto\nabla_W V$ as the *Levi-Civita connection* on $(\mathcal{X},g)$.

If $\tilde R_i$ is the curvature tensor of $(\tilde U_i,\tilde g_i)$, we may glue them to an orbi-tensor R. We call this tensor the curvature of $\mathcal{X}$. Similarly, we can glue all the $\tilde{\mathrm{Ric}}_i$ to get an orbi-tensor Ric_g on $\mathcal{X}$.

6.2. Metric structures on orbifolds

Let $\phi:(\tilde U,H)\to U$ be a chart on $\mathcal{X}$, with $\tilde g$ representing g locally. Let $\tilde p\in\tilde U$ be a pre-image of the point $p\in U$. If $\tilde c:[0,\epsilon)\to\tilde U$ is a local geodesic emanating from $\tilde p$, as H acts by isometry on $\tilde U$, we know that $g\cdot\tilde c=L_g\circ\tilde c$ is a local geodesic emanating

from $g \cdot \tilde{p}$. If $V \in T_{\tilde{p}}\tilde{U} = \tilde{c}'(0)$, then $TL_g(V) = g \cdot V = (L_g \circ \tilde{c})'(0) \in T_{g\tilde{p}}\tilde{U}$. In the orbi-fibre $T_p\mathcal{X} = \mathbb{R}^n/G_p$, the vectors V and $g \cdot V$ represent same orbi-vector. Hence we set $c_{[V]} := \phi \circ \tilde{c}$, and it the *geodesic* emanating from p determined by the orbi-vector $[V]$. It is obvious that the definition does not depend on the choice of orbi-chart.

From the construction, we also note that for $v \in T_p\mathcal{X}$, the geodesic $c_v : I \to \mathcal{X}$ is smooth.

Definition 6.4. Let $(\mathcal{X}, g)$ be a Riemannian orbifold and $p \in X = |\mathcal{X}|$ be a point. Set $O := \{v \in T_p\mathcal{X} : c_v \text{ is defined on } [0,1]\}$. We define the *exponential map* $\exp_p : O \to X$ to be $\exp(v) = c_v(1)$.

As a topological map, the exponential map $\exp_p$ is continous. Note that $[0] \in O$ and $\exp_p[0] = p$. If $(\tilde{U}_p, G_p)$ is a fundamental chart at p, then $\exp_p$ has a local lifting $\exp_{\tilde{p}} : \tilde{\Omega} \to \tilde{U}_p$, where $\tilde{\Omega} \subset T_{\tilde{p}}\tilde{U}$ is an G_p-invariant open subset containing 0, and $\exp_{\tilde{p}}$ is the classical Riemannian exponential map. We know that $\exp_p$ restricts to some $W = [\tilde{W}/G_p]$ gives an open embedding.

We also note that if $p \in X_{\text{reg}}$ is a regular point, then the geodesics are identical to the Riemannian ones around p. Hence so is the exponential map $\exp_p$.

Remark 6.5. With the Levi-Citiva connection defined on $(\mathcal{X}, g)$, one may consider define a covariant connection along a smooth curve $c : I \to \mathcal{X}$. However, even when c is lifted as $\tilde{c} : I \to \tilde{U}$, we don't know if all the lifts are of the form $g \cdot \tilde{c}$. Another hurdle for mere smooth curves is that the definition of orbi-vector fields along them. One of the possible definition is to restrict the curves to be strong curve, i.e. $c : I \to \mathcal{X}$ is strong. In this situation, we could pull the orbi-vector bundle $T\mathcal{X}$ together with ∇ back on I via c. If the strong curve c has image in X_{reg}, the definition coincides with the classical one.

We follow the treatment of [Bor93] for the metric aspects of Riemannian orbifolds.

Let $(\mathcal{X}, g)$ be a Riemannian orbifold and $(\tilde{U}, G)$ a chart for $\mathcal{X}$. Suppose that g is represented over $\tilde{U}$ by $\tilde{g}$. Then $(\tilde{U}, d_{\tilde{g}})$ is a well-defined metric space. If $\lambda : \tilde{V} \to \tilde{U}$ is an orbifold embedding, then λ is an isometry (of metric spaces). If for a continuous curve $c : I \to X$, we have local lifts on charts that cover $c(I)$, we can then define the length of c by adding the lengths of its local liftings. We now precise the definition.

First, we have

THEOREM 6.6 ([Bre72, Chapter 2, Lemma 6.1]). *Let X be a left G-space, with G a compact Lie group. Let $f : I \to X/G$ be any path. Then there exists a lifting $f' : I \to X$ covering f, i.e., we have $p \circ f' = f$.*

Now go back to the Riemannian orbifold $(\mathcal{X}, g)$. By a compactness argument, for a path $c : [0,1] \to X = |\mathcal{X}|$, there exists a partition $0 = t_0 < t_1 < ... < t_k = 1$, and orbifold charts $(\tilde{U}_i, G_i), 1 \le i \le k$ such that $c|_{[t_{i-1}, t_i]}$ has image in U_i and lifting $\tilde{c}_i$ in $\tilde{U}_i$.

As the liftings are not unique, we give the following definition

Definition 6.7. Let $(\mathcal{X}, g)$ be a Riemannian orbifold with underlying space $X = |\mathcal{X}|$. Let $c : [0,1] \to X$ be a path. Let $\mathcal{P}$ be the set of the local liftings that glue back to c, i.e. an element of $\mathcal{P}$ is a triple (A, B, C) where A is a partition $0 = t_0 < t_1 < ... < t_k = 1$,

B is a family of chart $(\tilde{U}_i, G_i), 1 \leq i \leq k$ and C is a family of curves $\tilde{c}_i : [t_{i-1}, t_i] \to \tilde{U}_i$ such that $\tilde{c}_i$ cover $c|_{[t_{i-1},t_i]}$. We define the *length* of c to be

$$\mathrm{L}_g(c) = \inf_{\mathcal{P}} \sum \mathrm{L}_i(\tilde{c}_i).$$

If $c_1 : [0,1] \to X$, $c_2 : [0,1] \to X$ are two curves such that $c_1(1) = c_2(0)$, then we may consider the curve $c_1 * c_2 : [0,1] \to X$ defined by $t \in [0, \frac{1}{2}] \mapsto c_1(2t)$ and $t \in [\frac{1}{2}, 1] \mapsto c_2(2t-1)$. It's obvious that $\mathrm{L}_g(c_1 * c_2) = \mathrm{L}_g(c_1) + \mathrm{L}_g(c_2)$. Hence the definition is coherent with the intuition of the length of a curve.

Lemma 6.8. *Let $(\mathcal{X}, g)$ be a Riemannian orbifold and $(\tilde{U}, G)$ a chart for $\mathcal{X}$. Let $c : [0,1] \to \tilde{U}/G$ and $\tilde{x}$ be a pre-image of $x = c(0)$. If there exists a partition $0 = t_0 < t_1 < \ldots < t_k = 1$, orbifold charts $(\tilde{U}_i, G_i), 1 \leq i \leq k$ such that $c|_{[t_{i-1},t_i]}$ has image in U_i and lifting $\tilde{c}_i$ in $\tilde{U}_i$, then there is a lifting $\tilde{c} : I \to \tilde{U}$ such that $\mathrm{L}(\tilde{c}) = \sum \mathrm{L}_i(\tilde{c}_i)$.*

Proof. Suppose that we have constructed $\tilde{c} : [0, t_i] \to \tilde{U}$ that lift $c|_{[0,t_i]}$ whose length equals to $\sum_{j \leq i} \mathrm{L}_i(\tilde{c}_i)$. We now extend $\tilde{c}$ on $[t_i, t_{i+1}]$. The projections of $\tilde{c}(t_i)$ and $\tilde{c}_i s(t_i)$ on X are both $c(t_i)$. Hence by the definition of orbifolds, there exists a chart $(\tilde{V}, \tilde{y})$ at $c(t_i)$ and two chart embeddings $\lambda : \tilde{V} \to \tilde{U}$ and $\rho : \tilde{V} \to \tilde{U}_i$ such that $\lambda(\tilde{y}) = \tilde{c}(t_i)$ and $\rho(\tilde{y}) = \tilde{c}_i(t_i)$. If $\rho(\tilde{V})$ contains $\tilde{c}_i([t_i, t_i + \epsilon])$, we may then extend $\tilde{c}$ on $[t_i, t_i + \epsilon]$ via $\lambda \circ (\rho)^{-1} \circ \tilde{c}_i$. Note that ρ and λ are isometries. Hence $\mathrm{L}(\tilde{c}|_{[t_i,t_i+\epsilon]}) = \mathrm{L_i}(\tilde{c}|_{[t_i,t_i+\epsilon]})$. A compactness argument shows that we can construct $\tilde{c}$ on $[t_i, t_{i+1}]$ with $\mathrm{L}(\tilde{c}|_{[t_i,t_{i+1}]}) = \mathrm{L_i}(\tilde{c}_i)$. □

Hence for a curve $c : I \to \tilde{U}/G$, we may define its length by only considering its liftings on $\tilde{U}$.

Definition 6.9. Let $(\mathcal{X}, g)$ be a Riemannian orbifold and L_g the length function on paths. Let x, y be two points of X. We define the *distance* $\mathrm{d}_g(x, y)$ of x and y by

$$\mathrm{d_g}(x, y) = \inf \mathrm{L_g}(\gamma)$$

where the infimum is taken over all the curves that join x and y, with the convention $\inf_\emptyset = \infty$.

We see easily that $\mathrm{d}_g : X \times X \to \mathbb{R}_{\geq 0}$, $(x, y) \mapsto \mathrm{d_g}(x, y)$ is a distance on X and $\mathrm{d}(x, y) = \infty$ iff x and y are in different components of X.

Let $(\tilde{U}_x, G_x)$ be a fundamental chart at $x \in X$. As $\tilde{U}_x$ is a Riemannian manifold, Proposition 5.8-(2) implies that there is a $\delta > 0$ such that for any $\tilde{y}, \tilde{z} \in B_\delta(\tilde{x})$ there is a unique geodesic $\tilde{c}$ with endpoints $\tilde{y}$ and $\tilde{z}$, such that $\mathrm{L}(\tilde{c}) = \tilde{\mathrm{d}}(\tilde{y}, \tilde{z})$. By taking the projection $\tilde{U} \to U$ and combine Lemma 6.8, we have

Lemma 6.10. *Let $(\mathcal{X}, g)$ be a Riemannian orbifold with underlying space $X = |\mathcal{X}|$ and d_g the distance on X. The metric topology induced by d_g is the same as the original topology on X. For any $x \in X$, there exists $\delta > 0$ such that for any $y, z \in B_\delta(x) \subset X$, there exists a unique geodesic c with endpoints y and z and $\mathrm{L}_g(c) = \mathrm{d}_g(y, z)$.*

A direct consequence of Lemma 6.10 is:

Corollary 6.11. *Let $(\mathcal{X}, g)$ be a Riemannian orbifold and $c : I \to X$ be a geodesic. Then with respect to the metric d_g, the path c is locally minimizing, i.e. for any $t \in I$*

there exists $\epsilon > 0$ *such that* $\forall t_1, t_2 \in (t-\epsilon, t+\epsilon)$ *we have that*

$$\mathrm{L_d}(c|_{[t_1,t_2]}) = \mathrm{d}_g(c(t_1), c(t_2)).$$

Proposition 6.12. *Let* $(\mathcal{X}, g)$ *be a Riemannian orbifold. For a local geodesic* $c : [0,1] \to X$*, we have that* $\mathrm{L}_g(c) = \mathrm{L_d}(c)$*.*

Proof. Suppose that $c : [0,1] \to X$ be a local geodesic. By compactness, using Corollary 6.11 and Lemma 6.10, we have an $\epsilon > 0$ such that for $s, t \in [0,1], |s-t| < \epsilon$,

$$\mathrm{d}_g(c(s), c(t)) = \mathrm{L_d}(c|_{[s,t]}) \leq \mathrm{L}_g(c|_{[s,t]}) = \mathrm{d_g}(c(s), c(t)).$$

Hence

$$\mathrm{L}_g(c) = \sum \mathrm{L}_g(c|_{[t_i,t_{i+1}]}) = \sum \mathrm{L_d}(c|_{[t_i,t_{i+1}]}) = \mathrm{L_d}(c).$$

□

Proposition 6.13 (*cf.* [Bor93, Page 6])**.** *Let* $(\mathcal{X}, g)$ *be a Riemannian orbifold with underline space* $X = |\mathcal{X}|$*. With the metric* d_g *in Definition 6.9, the metric space* (X, d_g) *is a length space.*

Proof. Let $\mathrm{d_i}$ denote the inner metric associated with d_g. We only need to show $\mathrm{d}_g \geq \mathrm{d_i}$. Suppose that $\mathrm{d}_g(x,y) < \infty$. Let $\epsilon > 0$ be a positive number. By definition, there exists a curve $\gamma : [0,1] \to X$ such that $\gamma(0) = x, \gamma(1) = y$ and $\mathrm{L}_g(\gamma) \leq \mathrm{d}_g(x,y) + \epsilon$. By Lemma 6.10, there exists a $\delta > 0$ such that:

for any $t, t' \in [0,1]$, if $|t-t'| < \delta$, there exits a minimizing geodesic c with endpoints $\gamma(t), \gamma(t')$ whose length $\mathrm{L}_g(c)$ is $\mathrm{d}_g(\gamma(t), \gamma(t'))$.

Let $0 = t_0 < t_1 < \ldots < t_n = 1$ be a partition of $[0,1]$ with $t_{i+1} - t_i < \delta$ and c_i the corresponding geodesic. Set $c := *_i c_i$. We have that

$$\begin{aligned}\mathrm{d}_g(x,y) + \epsilon \geq \mathrm{L_g}(\gamma) = \sum \mathrm{L_g}(\gamma|_{[t_i,t_{i+1}]}) &\geq \sum \mathrm{d_g}([t_i, t_{i+1}]) = \sum \mathrm{L_g}(c_i) \\ &\geq \sum \mathrm{L_d}(c_i) = \mathrm{L_d}(c) \geq \mathrm{d_i}(x,y).\end{aligned}$$

□

As any geodesic is locally minimizing, combining this fact with Corollary 6.11, we have:

Lemma 6.14. *Let* $(\mathcal{X}, g)$ *be a Riemannian orbifold. If* (X, d_g) *is complete, then any two point can be joined by a minimizing geodesic.*

Remark 6.15. In the case of a Riemannian manifold (M, g), one associates a metric d_g on M by defining $d_g(x,y) = \inf \mathrm{L}_g(c)$, where c is a piece-wise smooth curve and $\mathrm{L}_g(c) = \int |c'(t)| dt$. For the metric d_g, we can associate another length L_d as in Definition 5.1. One sees easily $\mathrm{L}_g \geq \mathrm{L}_d$ and it is a classical result that $\mathrm{L}_g = \mathrm{L}_d$. The author does not know if this still holds in the orbifold setting.

For a complete orbifold $(\mathcal{X}, g)$, its geodesics have a good property:

Theorem 6.16 ([Bor93, Proposition 15]). *Let* $(\mathcal{X}, g)$ *be a Riemannian orbifold with underline space* $X = |\mathcal{X}|$*. Let* $\gamma : I = [0,1] \to \mathcal{X}$ *be a minimizing geodesic. Set* $p := \gamma(0)$ *and* $q := \gamma(1)$*. Then we have one of the following mutually exclusive conditions:*

(1) $\gamma(I) \subset X_{\text{sing}}$
(2) $\gamma(I) \cap X_{\text{sing}} \subset \{p, q\}$.

Hence for $p, q \in X_{\text{reg}}$, the minimizing geodesic γ joining p and q lies completely in X_{reg}. In this situation γ is also the minimizing geodesic in the Riemannian manifold X_{reg}.

Corollary 6.17. *Let $(\mathcal{X}, g)$ be a Riemannian orbifold. If (X, d_g) is complete, then (X_{reg}, g) is a convex Riemannian manifold.*

Let (M, g) be a convex Riemannian manifold, $p \in M$. Let $u \in U_pM$ be a unit tangent vector and γ_u be the geodesic emanating from p with $\gamma_u'(0) = u$. We define

$$t(u) := \sup\{t > 0 : \gamma_u \text{ is defined on } [0, t] \text{ and } \gamma_u|_{[0,t]} \text{ is minimal and } t \in [0, \infty]\}.$$

We may define the *cut locus* in the convex situation by

Definition 6.18. Let M be a convex Riemannian manifold and $p \in M$. We define

$$\tilde{C}_p := \{t(u)u : u \in U_pM \text{ such that } t(u) < \infty\} \quad \text{and} \quad C_p := \exp_p(\tilde{C}_p)$$

to be the *tangent cut locus* and *cut locus* of p respectively. We also set

$$\tilde{I}_p := \{tu : u \in U_pM,\ 0 < t < t(u)\} \quad \text{and} \quad I_p := \exp_p(\tilde{I}_p).$$

Then we have similar results as in the complete case:

Lemma 6.19. *Let M be a convex Riemannian manifold and $p \in M$. We use the notations $\tilde{C}_P$, C_P, $\tilde{I}_p$ and I_p as in Definition 6.18. Then we have the following*

(1) I_p is a connected open neighborhood of $0 \in T_pM$.
(2) $I_p \cap C_p = \emptyset$, $M = I_p \cup C_p$, and $\bar{I}_p = M$.
(3) $\exp_p : \tilde{I}_p \to I_p$ is a diffeomorphism.
(4) C_p has measure 0.

Proof. See [Sak96, Proposition III.4.1 and Lemma III.4.4]. Note that though all the statements in *loc. cit.* are for complete manifold M, the proofs only use the fact that M is convex. □

6.3. Volume comparisons

We first recall the classical Bishop-Gromov comparison theorem:

Theorem 6.20. *Let k be a real number. Let (M^n, g) be a convex Riemannian manifold with $\text{Ric}_g \geq (n-1)k$. Let $v(n, k, r)$ be the volume of a ball of radius r in the model space with constant curvature k. The volume ratio*

$$r \mapsto \frac{\text{vol}_g(B(p, r))}{v(n, k, r)}$$

is a non-increasing function whose limit is 1 as $r \to 0$.

Proof. See [Pet16, Lemma 7.1.4]. Note that though all the statement in *loc. cit.* is for complete manifold M, the proof only uses the fact $\exp_p : \tilde{I}_p \to I_p$ is a diffeomorphism and that C_p has measure zero. □

Before we state the orbifold version Bishop-Gromov theorem, we first need to define the measure on X for a Riemannian orbifold $(\mathcal{X}, g)$.

Let $(\mathcal{X}, g)$ be a Riemannian orbifold. Let $(\tilde{U}, \tilde{g})$ be a chart. After taking an orientation of $\tilde{U}$, we have a unique volume form $\mathrm{vol}_{\tilde{g}}$. Hence $\mathrm{vol}_{\tilde{g}}$ is defined up to a sign. If $O \subset X$ is an open subset covered by $\{U_i = \tilde{U}_i/G_i\}$, we can take a partition of unity ρ_i subordinate to $\{U_i\}$ and define

$$\mathrm{vol}_g(O) := \sum_i \left| \int_{\mathcal{X}} \rho_i \, \mathrm{vol}_{\tilde{g}_i} \right|.$$

The integral on orbifold is defined in Definition 5.51. It's easy to see that $\mathrm{vol}_g(O)$ is well-defined.

Lemma 6.21 (*cf.* [Bor93, Lemma 18])**.** *Let $(\mathcal{X}, g)$ be a Riemmanian orbifold and vol_g be its canonical measure on X. Then X_{sing} has measure zero.*

Proof. Note that X_{sing} is a closed subset of X. Hence it is measurable. As X is second countable, we may cover X_{sing} by countable many local charts. Hence it suffices to show for $(\tilde{U}, G)$ the non-free point set has measure zero. As G is finite, it is trivial. □

Now we state the orbifold Bishop-Gromov theorem.

Theorem 6.22 (*cf.* [Bor93, Proposition 20]). *Let k be a real number. Let $(\mathcal{X}, g)$ be a Riemannian orbifold with $\mathrm{Ric}_g \geq (n-1)k$. Let $v(n, k, r)$ be the volume of a ball of radius r in the model space with constant curvature k. The volume ratio*

$$r \mapsto \frac{\mathrm{vol}_g(B(p, r))}{v(n, k, r)}$$

is a non-increasing function whose limit is $|G_p|$ as $r \to 0$.

Proof. For the reader's convenience, we reproduce the proof here.

First suppose that $p \in X_{\mathrm{reg}}$ is a regular point. We have that $\mathrm{vol}_g(B(p, r)) = \mathrm{vol}_g(B(p, r) \cap X_{\mathrm{reg}})$. Set $B'(p, r) = B(p, r) \cap X_{\mathrm{reg}}$ and denote by vol'_g the volume on X_{reg}. We have that

$$\frac{\mathrm{vol}_g(B(p, r))}{v(n, k, r)} = \frac{\mathrm{vol}'_g(B'(p, r))}{v(n, k, r)}.$$

By Theorem 6.20, we get the sought result.

If $p \in X_{\mathrm{sing}}$ is a singular point, let us pick $(p_i)_{i \geq 1} \in X_{\mathrm{reg}}$ a sequence that converges to p. Then $\mathrm{vol}_g(B(p, r)) = \lim_{i \to \infty} \mathrm{vol}_g(B(p_i, r))$. For any $r' > r$, by the results for regular points, we have that

$$\frac{\mathrm{vol}_g(B(p_i, r))}{\mathrm{vol}_g(B(p_i, r')} \geq \frac{v(n, k, r)}{v(n, k, r')}.$$

After taking the limit as $i \to \infty$, we have that

$$\frac{\mathrm{vol}_g(B(p, r))}{\mathrm{vol}_g(B(p, r'))} \geq \frac{v(n, k, r)}{v(n, k, r')}.$$

For r small enough, we have the inclusion $B(p,r) \subset \tilde{U}_p/G_p$, with $(\tilde{U}_p, G_p)$ a fundamental chart at p. If we denote by $\tilde{B}(\tilde{p},r)$ the ball of radius r in $\tilde{U}_p$ centered at $\tilde{p}$, we then have:

$$\mathrm{vol}_{\tilde{g}}\, \tilde{B}(\tilde{p},r) = |G_p| \cdot \mathrm{vol}_g(B(p,r)).$$

The limit $|G_p|$ as $r \to 0$ follows from Theorem 6.20. $\square$

CHAPTER 7

Orbifold coverings and generalized Magulis lemma

7.1. Metric geometery of orbifold coverings

Let $p : \mathcal{Y} \to \mathcal{X}$ be an orbifold covering map. Suppose that we have a Riemannian metric g on $\mathcal{X}$. By Lemma 5.49, we have a pull-back Riemannian metric $p^*(g)$ on $\mathcal{Y}$. We denote by d_Y the metric induced by $p^*(g)$ on $Y = |\mathcal{Y}|$. Then the map p induces a morphism of metric spaces $(Y, d_Y) \to (X, d_X)$ such that $d_Y(a, b) \geq d_X(p(a), p(b))$.

For a Galois covering $p : \mathcal{Y} \to \mathcal{X}$, the distances d_X and d_Y are nicely related:

Proposition 7.1 ([Lan20, Lemma 2.8])**.** *Let $p : \mathcal{Y} \to \mathcal{X}$ be a Galois covering map and $\mathrm{Gal}(p)$ be its Galois group. Then $X = Y/\mathrm{Gal}(p)$ and d_X is the quotient metric of d_Y by $\mathrm{Gal}(p)$.*

For our purpose, we state the following lemma.

Lemma 7.2. *Let $(\mathcal{X}, g)$ be a Riemannian orbifold and $p : \mathcal{Y} \to \mathcal{X}$ an orbifold covering of $\mathcal{X}$. We equip $\mathcal{Y}$ with the Riemannian metric $p^*(g)$. If (X, d_X) is complete, then (Y, d_Y) is complete.*

Proof. Let $\{y_n\}$ be a Cauchy sequence in (Y, d_Y). We may assume that $\{y_n\}$ lies in a compact subset K of a component V of the pre-image $p^{-1}(U)$ of an elementary neighborhood $U = [\tilde{U}/G]$. Hence there exists $H \leqslant G$ such that $V = [\tilde{U}/H]$. Let $U \ni x_n = p(y_n)$ be the image of y_n under p. Then $x_n \in p(K)$. As p does not increase distances, the sequence $\{x_n\}$ is a Cauchy sequence and has a limit $\lim\limits_{n\to\infty} x_n = x \in p(K) \subset U$. Take $\tilde{x} \in \tilde{U}$ be a lifting of x. Then there exist $\tilde{x}_n$ lifting x_n such that $\tilde{x}_n \to \tilde{x}$. Note that for each n there exists $g_n \in G$ such that $g_n\tilde{x}_n$ lifts y_n. As G is finite, there exists a $g \in G$ that appears infinitely many times in $\{g_n\}$. Hence there is a sub-sequence $g \cdot \tilde{x}_{n_i} \to g \cdot \tilde{x}$. If y is the image of $g \cdot \tilde{x}$ in V, we will have $y_n \to y$. □

By Theorem 6.6, we see that for any covering $p : \mathcal{X}' \to \mathcal{X}$ and path $c : [0, 1] \to \mathcal{X}$ with $c(0) = x \in X$ and $x' \in p^{-1}(x)$, there exists a lifting $c' : [0, 1] \to \mathcal{X}'$ starts from x'. However, this will not be unique. Consider $(\mathbb{R}^2, \mathbb{Z}/2\mathbb{Z})$ with the action $(x, y) \to (-x, y)$. The path $t \mapsto (t, 0)$ has liftings $t \mapsto (t, 0)$ and $t \to (-t, 0)$ in $\mathbb{R}^2$. If $\mathrm{id}_{\tilde{U}} : (\tilde{U}, G_{x'}) \to (\tilde{U}, G_x)$ is a local lifting of p and c is a geodesic, then it has at least $[G_x : G_{x'}]$ lifting.

We finish this section by an easy observation.

Lemma 7.3. *Let $(\mathcal{X}, g)$ be a Riemannian orbifold. Let $\pi : \mathcal{X}' \to \mathcal{X}$ be an orbifold covering with Galois group $\mathrm{Gal}(p)$. If we endow $\mathcal{X}'$ with the pullback metric $\pi^*(g)$, then $\mathrm{Gal}(p)$ acts on $|\mathcal{X}'|$ by isometries.*

Proof. Let x'_1, x'_2 be two points of $|\mathcal{X}'|$. Suppose that $c : [0, 1] \to |\mathcal{X}'|$ is a path such that $c(0) = x'_1$ and $c(1) = x'_2$. Let γ be an element of $\mathrm{Gal}(p)$. We have that

$L_{\pi^*(g)}(\gamma c) = L_{\pi^*(g)}(c)$. Hence

$$\begin{aligned} d(x_1', x_2') &= \inf_{c:[0,1]\to|\mathcal{X}'|,\ c \text{ connects } x_1' \text{ and } x_2'} L_{\pi^*(g)}(c) \\ &= \inf_{c:[0,1]\to|\mathcal{X}'|,\ c \text{ connects } x_1' \text{ and } x_2'} L_{\pi^*(g)}(\gamma c) \\ &= d(\gamma x_1', \gamma x_2'). \end{aligned}$$

□

7.2. Dirichlet domains and generalized Margulis lemma

Let $(\mathcal{X}, g)$ be an Riemannian orbifold. Let $\pi : \mathcal{X}' \to \mathcal{X}$ be the universal covering of $(\mathcal{X}, g)$ and Γ be its Galois group. We endow $\mathcal{X}'$ with the pullback metric $p^*(g)$. Then (X', d') is a complete length space. Lemma 7.3 shows that Γ acts by isometries on X'.

Suppose now that $\operatorname{diam}(X) \leq 1$. Let $x_0 \in \pi^{-1}(X_{\text{reg}})$ be a regular point. We now show some basic properties of the *Dirichlet domain* of Γ based at x_0:

$$F := \{p \in X' : \mathrm{d}'(x_0, p) \leq \mathrm{d}'(\gamma \cdot x_0, p) \text{ for all } \gamma \in \Gamma\}.$$

Lemma 7.4. $\operatorname{diam}(F) \leq 2$.

Proof. Let p be a point in F. As X' is complete, there is a minimizing geodesic $c' : [0,1] \to X'$ joining x_0 and p such that $\mathrm{L}_d(c') = \mathrm{d}'(x_0, p)$. In particular, c' is a local geodesic. We consider $c := \pi \circ c'$, which joins $\pi(x_0)$ and $\pi(p)$. If c is not a minimizing geodesic, then there is a c_1 which is a minimizing geodesic that joins $\pi(x_0)$ and $\pi(p)$. We have that $\mathrm{d}_g(\pi(x_0), \pi(p)) < \mathrm{L}_{\mathrm{d}}(c) = \mathrm{L}_g(c)$. As c_1 is a geodesic, it has a lifting c_1' starting at x_0. As $\pi \circ c_1' = c_1 \neq c = \pi \circ c'$, we have that $c_1' \neq c'$. Take $\gamma \in \Gamma$ such that $\gamma \cdot p = c_1'(1)$. Then

$$\mathrm{d}'(x_0, \gamma p) \leq \mathrm{L}_{\mathrm{d}}(c_1') = \mathrm{L}_{\mathrm{d}}(c_1) < \mathrm{L}_{\mathrm{d}}(c) = \mathrm{d}'(x_0, p).$$

Thus c is also a segment and $\mathrm{L}(c') = \mathrm{L}(c) \leq 1$. Thus $\operatorname{diam}(F) \leq 2$. □

We consider the subset of Γ defined by

$$S := \{\gamma \in \Gamma : \mathrm{d}(\gamma \cdot x_0, x_0) \leq 4\}.$$

One sees easily that S is symmetric and contains 1. We have

Lemma 7.5. *S generates Γ.*

Proof. We have that $\cup \gamma \cdot F = X'$. We take a segment $c : [0,1] \to X'$ that joins x_0 and $\gamma_0 \cdot x_0$. As $c(I)$ is compact, the set $c(I)$ is contained in a ball $B_r(x_0)$ that meets only finitely many translates $\gamma \cdot F$ of F. Hence c passes through finitely many $\gamma \cdot F$. We list these elements by $1 = \gamma_1, \ldots, \gamma_k = \gamma_0$, ordered by the time when c enters $\gamma_i \cdot F$. Note that they are not necessarily different. Then $\gamma_i \cdot F \cap \gamma_{i+1} \cdot F \neq \emptyset$ and we thus have that $\mathrm{d}'(\gamma_i x_0, \gamma_{i+1} x_0) \leq 4$. Finally we remark that

$$\gamma_0 = \gamma_k = \prod_{i=1}^{k-1} (\gamma_{i+1} \cdot \gamma_i^{-1}) \in S^{k-1}.$$

□

Lemma 7.6. *Let $r > 0$ be an integer. $B(x_0, r) \subset S^r \cdot F \subset B(x_0, 3r + 2)$.*

PROOF. If $p \in S^r \cdot F$, one can write $p = \gamma_1 ... \gamma_r \cdot q$ with $q \in F$ and $\gamma_i \in S$. It yields

$$d'(x_0, p) \leq d'(\gamma_1 \cdots \gamma_r \cdot x_0, x_0) + d'(\gamma_1 \cdots \gamma_r \cdot x_0, \gamma_1 \cdots \gamma_r \cdot q) \leq 2 + 4r.$$

If $p \in B(x_0, r)$, as $\bigcup \gamma \cdot F = X'$, there exists $\gamma_0 \in \Gamma$ such that $\gamma_0^{-1} \cdot p \in F$. Then $d'(x_0, \gamma \cdot x_0) \leq d'(x_0, p) + d'(\gamma_0 \cdot x_0, p) \leq r + 1$. Hence $\gamma_0 \in S^r$. □

For any $\gamma \neq 1$, $\gamma \cdot F \cap F \subset \partial F$ and we have $\partial F = \bigcup_{\gamma \neq 1} (\gamma \cdot F \cap F)$.

Lemma 7.7. *Let μ be the canonical measure associated with $p^*(g)$. Then $\mu(\partial F) = 0$.*

PROOF. Let $p \in \partial F$ be a point. If $\pi(p) \in X_{\text{reg}}$, then $p \in X'_{\text{reg}}$. Suppose that $p \in F \cap \gamma \cdot F$ for some $\gamma \neq 1$. Then $d'(x_0, p) = d(\gamma \cdot x_0, p)$. The proof of Lemma 7.5 shows that we can then find two distinct segments on X joining $\pi(x_0)$ and p. By Theorem 6.16, these two segments are minimizing geodesics in the convex manifold X_{reg}. Hence $\pi(p)$ lies in the cut locus $C_{\pi(x_0)}$ of x_0 in X_{reg}. Thus we have that $\pi(\partial F) \subset C_{\pi(x_0)} \cup X_{\text{sing}}$. Now cover $\pi(\partial F)$ by elementary neighborhoods with respect to π. By second countability, we may find countably many neighborhoods $(\tilde{U}_i, G_i, \phi_i)$. Note that $\phi_i^{-1}(C_{\pi(x_0)} \cup X_{\text{sing}})$ has measure 0 in $\tilde{U}_i$ and ∂F is covered by countably many $\tilde{U}_i / H_{ij}$. Hence $\mu(\partial F) = 0$. □

With Theorem 6.22 and Proposition 7.1 , the Margulis lemma for fundamental groups of compact manifolds with Ricci curvature bounded below by [BGT12] holds for the orbifold case:

Proposition 7.8 (*cf.* [BGT12, Corollary 11.13])**.** *Given $n \in \mathbb{N}$, there is $\epsilon = \epsilon(n) > 0$ such that:*

for any n-dimensional compact Riemmanian orbifold $(\mathcal{X}, g)$ with underlying space $|X|$, if $\text{Ric}_g \geq -\epsilon$ and $\text{diam}(X) \leq 1$, then $\pi_1^{\text{orb}}(\mathcal{X})$ is virtually nilpotent.

PROOF. Let π, x_0, F, be defined as above. With Lemma 7.6 and Lemma 7.7, we have that

$$\frac{|S^r|}{|S|} \leq \frac{\mu(B(x_0, 4r+2))}{\mu(B(x_0, 1))}.$$

By Theorem 6.22, we have

$$\frac{\mu(B(x_0, r))}{\mu(B(x_0, 1))} \leq \frac{v(n, -\epsilon, r)}{v(n, -\epsilon, 1)}.$$

Let ω_n be the volume of $(n-1)$-dimensional unit sphere in Euclidean space $\mathbb{R}^n$. Then $v(n, -\epsilon, r) = \omega_n \int_0^r (\frac{sinh(\sqrt{\epsilon} t)}{\sqrt{\epsilon}})^{n-1} dt$. The latter tends to $\omega_n r^n / n$ when ϵ tends to 0. Thus for any $R_0 \geq 1$, there exists $\epsilon_0 = \epsilon_0(d, R_0)$ such that

$$\frac{|S^r|}{|S|} \leq 2(4r+2)^n$$

for all $r \leq R_0$, provided that $0 < \epsilon < \epsilon_0$. The existence of $\epsilon = \epsilon(d)$ follows from [BGT12, Corollary 11.5]. □

For the main theorem, we introduce the following notion.

Definition 7.9. Let (X, d) be a metric space. We say that X to have *bounded packing* with *packing constant* K if there exists $K > 0$ such that every ball of radius 4 in X can be covered by at most K balls of radius 1.

Lemma 7.10. *Let $(\mathcal{X}, g)$ be a complete Riemannian orbifold with $\mathrm{Ric}_g > -(n-1)$. Then (X, d_g) has bounded packing with packing constant $K = K(n)$.*

Proof. Let p be a point in X. For the ball $B(p, 5)$ and a ball $B(q, \frac{1}{2}) \subset B(p, 5)$, by Theorem 6.22 we have that

$$\frac{\mathrm{vol}_g(B(p,5))}{\mathrm{vol}_g(B(q,\frac{1}{2})} \leq \frac{\mathrm{vol}_g(B(q,10))}{\mathrm{vol}_g(B(q,\frac{1}{2})} \leq \frac{v(n,-1,10)}{v(n,-1,\frac{1}{2})} = K(n).$$

Let $\{B(q_i, \frac{1}{2})\}$ be a family of disjoint balls that is contained $B(p, 5)$ such that for any $q \neq q_i$, if $B(q, \frac{1}{2}) \subset B(p, 5)$, then $B(q, \frac{1}{2})$ intersects with one of the $B(q_i, \frac{1}{2})$. We know that the family has most $K(n)$ balls. Note that the balls $B(q_i, 1)$ cover $B(p, 4)$. Hence we have the packing constant $K = K(n)$. □

Let $(\mathcal{X}, g)$ be a complete Riemannian orbifold and $p : \tilde{\mathcal{X}} \to \mathcal{X}$ its universal covering. For any point $x \in X = |\mathcal{X}|$, its (topological) fiber $p^{-1}(x) \subset \tilde{X} = |\tilde{\mathcal{X}}|$ is a discrete space. Let us pick $\tilde{x} \in p^{-1}(x)$. We have that

$$\min\{\tilde{d}(\tilde{x}, \tilde{x}')\} > 0,$$

where the minimum is taken for all $\tilde{x}' \in p^{-1}(x) \setminus \{\tilde{x}\}$. We thus see that $\pi_1^{\mathrm{orb}}(\mathcal{X})$ acts on $\tilde{X}$ discretely, *i.e.*, for any $\tilde{x} \in \tilde{X}$, for any bounded set $\Sigma \subset \tilde{X}$, the set

$$\{\gamma \in \pi_1^{\mathrm{orb}}(\mathcal{X}) : \gamma \cdot \tilde{x} \in \Sigma\}$$

is finite.

Finally we recall the generalized Margulis lemma established in [BGT12]

Theorem 7.11 ([BGT12, Corollary 11.17]). *Let $K \geq 1$ be a parameter. There exists $\epsilon(K) > 0$, such that the following is true:*

Suppose that X is a metric space with packing constant K and Γ is a subgroup of isometries of X that acts discretely. Then for every $x \in X$ the "almost stabiliser"

$$\Gamma_\epsilon(x) := \langle\{\gamma \in \Gamma : \mathrm{d}(\gamma \cdot x, x) < \epsilon\}\rangle$$

is virtually nilpotent.

With Lemma 7.10, applying Theorem 7.11 to complete Riemannian orbifolds, we get the following lemma.

Lemma 7.12 (*cf.* [BGT12, Corollary 11.19])**.** *Let $n \geq 1$ be an integer. There exists $\alpha = \alpha(n) > 0$ such that the following holds true:*

Let $\mathcal{X}$ be a complete Riemannian orbifold with its Ricci curvature bounded by $\mathrm{Ric} \geq -(n-1)$ and Γ be a subgroup of $Isom(|\mathcal{X}|)$ acting properly discontinuously by isometries on $|\mathcal{X}|$. Then for every $x \in |\mathcal{X}|$, the "almost stabliser"

$$\Gamma_\alpha(x) := \langle\{\gamma \in \Gamma : \mathrm{d}(\gamma \cdot x, x) < \alpha\}\rangle$$

is virtually nilpotent.

CHAPTER 8

Main theorem

In this section and onwards, we only deal with complex orbifolds.

Definition 8.1. Let $\mathcal{X}$ be an orbifold and $p : \mathcal{E} \to \mathcal{X}$ be a complex orbi-vector bundle over $\mathcal{X}$, represented by the left $\mathcal{G}$-space E such that p is represented by $\mathcal{G} \ltimes E \to \mathcal{G}$ (*cf.* Definition 5.45). An *Hermitian metric* on $\mathcal{E}$ is a map $h : |\mathcal{E}| \times_{|\mathcal{X}|} |\mathcal{E}| \to \mathbb{C}$ such that h lifts to a map $\tilde{h} : E \times_{\mathcal{G}_0} E \to \mathbb{C}$ and $\tilde{h}$ is Hermitian and $\mathcal{G}$-invariant.

One can always get a Hermitian metric on an orbi-vector bundle by partition of unity:

Proposition 8.2 (*cf.* [Par20, Lemma 5.1])**.** *Let $\mathcal{X}$ be an orbifold and $\mathcal{E}$ be a complex orbi-vector bundle on $\mathcal{X}$. Then there exists an Hermitian metric on $\mathcal{E}$.*

For a complex orbifold groupoid $\mathcal{G} = \{G_1 \rightrightarrows G_0\}$, we know that the structure maps between G_i are holomorphic (actually they are étale). In particular, for any arrow $g : x \to y$ in G_1, the induced local diffeomorphism $U_x \to U_y$ is biholomorphic. Its tangent groupoid is $T\mathcal{G} = G_1 \ltimes TG_0$. We see that the almost complex structure J, differential operators d, ∂ and $\bar{\partial}$ are G_1-invariant. Thus for a complex orbifold $\mathcal{X}$, we have the decomposition $T_{\mathbb{C}}\mathcal{X} = T^{1,0}\mathcal{X} \oplus T^{0,1}\mathcal{X}$. We define the *anti-canonical bundle* of $\mathcal{X}$ to be $K_{\mathcal{X}}^{-1} = \det(T^{1,0}\mathcal{X})$.

Definition 8.3. Let $\mathcal{X}$ be a complex orbifold. A *Kähler form* on $\mathcal{X}$ is a closed real $(1,1)$-form $\omega \in \Gamma(X, (T^2\mathcal{X})^\vee \cap (T^{(1,1)}\mathcal{X})^\vee)$ such that $\omega(-, J-)$ defines a Riemannian metric on $\mathcal{X}$.

We give a definition of holomorphic orbi-vector bundles that suits our later discussion.

Definition 8.4. Let $\mathcal{X}$ be a complex orbifold, $\mathcal{E} \to \mathcal{X}$ a complex orbi-vector bundle over $\mathcal{X}$. A holomorphic orbi-vector bundle structure over $\mathcal{E}$ is a representation $\mathcal{G} \ltimes E \to \mathcal{G}$ of $\mathcal{E} \to \mathcal{X}$ together with a holomorphic vector bundle structure on $E \to \mathcal{G}_0$. We call $\mathcal{E}$ together with the holomorphic structure a *holomorphic orbi-vector bundle.*

With the above definition, we can see that $(T\mathcal{X}, J)$ and $K_{\mathcal{X}}$ carry natural holomorphic structure. It is thus considered as holomorphic orbi-vector bundle in the rest of the article.

Remark 8.5. The definition is not optimal. Suppose that $\phi : \mathcal{H} \to \mathcal{G}$ is an equivalence, then one will have a nature holomorphic vector bundle $\phi_0^*(E) \to \mathcal{H}_0$. One should consider $\phi^*(\mathcal{G} \ltimes E) = \mathcal{H} \ltimes \phi_0^*(E)$ gives same holomorphic structure on $\mathcal{E} \to \mathcal{X}$. Hence the right definition will be the

{representation + holomorphic structure on the representation} *mod* "equivalence".

Due to the inability of the author, we can not give a satisfying equivalence relation. However, the given definition suffices for our purpose.

Let $\mathcal{E} \to \mathcal{X}$ be a holomorphic orbi-vector bundle represented by $\mathcal{G} \ltimes E \to \mathcal{G}$, where $\mathcal{G} = [G_1 \rightrightarrows G_0]$ is a groupoid representation of $\mathcal{X}$. The holomorphic structure on E induces a natural complex structure on E, Hence $\mathcal{G} \ltimes E$ is a complex orbifold groupoid. For $x \in G_0$, by Proposition A.8 we can take a neighbourhood $U_x \subset G_0$ of x such that $\mathcal{G}|_{U_x} \cong G_x \ltimes U_x$. Shrink U_x if necessary. We may assume E is trivialized by by a holomorphic frame over U_x, *i.e.*, $E|_{U_x} \cong \mathbb{C}^r \times U_x$. Then $(\mathcal{G} \ltimes E)|_{U_x} \cong G_x \ltimes (\mathbb{C}^r \times U_x)$ as complex orbifold groupoid. In particular, $G_x \times (\mathbb{C}^r \times U_x) \to \mathbb{C}^r \times U_x$ is holomorphic.

For an element $g \in G_x$, the action of g is $(v, y) \mapsto (g(y) \cdot v, y)$, where $g(y) \in GL(r, \mathbb{C})$. Thus $y \mapsto g(y)$ is holomorphic and $g \in G_x$ transfers holomorphic section of E on U_x to a holomorphic section. With the same argument, one could show $g : x \to y$ transfers a local holomorphic section around x to a local holomorphic section around y. We then define the *holomorphic section* of $\mathcal{E}$ to be a $\mathcal{G}$-invariant holomorphic section in E.

Let s be a $\mathcal{G}$-invariant holomorphic section of E, and $g : x \to y$ be an arrow in G_1. Suppose that s is defined around x and y, such that $s = \sum \phi_i e_i$ around x and $s = \sum \psi_j f_j$ around y, where $\{e_i\}$ and $\{f_j\}$ are holomorphic frames. We have that $g \cdot e_i = \sum_j A_{ji} f_j$ for some holomorphic functions A_{ji} around y and A_{ji} is invertible. Now around x, we have that $\bar\partial_E(s) = \sum_i \bar\partial\phi_i \otimes e_i$. Around y, we have that

$$\begin{aligned} g \cdot \bar\partial_E(s) &= \sum_i (g^{-1})^* \bar\partial\phi_i \otimes g \cdot e_i = \sum_{i,j} (g^{-1})^* \bar\partial\phi_i A_{ji} \otimes f_j \\ &= \sum_j \bar\partial\psi_j \otimes f_j = \bar\partial_E(s). \end{aligned}$$

Thus the Dolbeault operator $\bar\partial_E$ passes to an *orbifold Dolbeaut operator* $\bar\partial_{\mathcal{E}}$ on $\mathcal{E}$.

Let $\mathcal{L} \to \mathcal{X}$ be a holomorphic orbi-line bundle and h an Hermitian mectric on $\mathcal{L}$. Let $\mathcal{G} \ltimes L \to \mathcal{G}$ be a representation of $\mathcal{L}$ and $\tilde h$ be the $\mathcal{G}$-invariant metric on L. For an arrow $g : x \to y$, we consider two trivializations by holomorphic sections e and f around x and y respectively. Suppose that $g \cdot e(w) = \phi(g^{-1}w)f(w)$. We have the local matrices for $\tilde h$ to be $h_1 = \tilde h(e, e)$ and $h_2 = \tilde h(f, f)$. From the equality $h(g \cdot e, g \cdot e) = h(e, e)$, we see that

$$h_2(w)\phi(g^{-1}w)\overline{\phi(g^{-1}w)} = h_1(g^{-1}w).$$

Hence

$$\partial\bar\partial(log \circ h_2) = \partial\bar\partial(\log \circ h_1 \circ L_{g^{-1}}) = (g^{-1})^* \partial\bar\partial(\log \circ h_1) = g \cdot \partial\bar\partial(\log \circ h_1),$$

which means that the Chern curvature $\Theta_{\tilde h} = -\frac{\sqrt{-1}}{2\pi}\partial\bar\partial\tilde h$ is $\mathcal{G}$-invariant, hence corresponds to an orbifold section of the complex vector bundle $\mathcal{L}$.

We are now ready to give the definition of nefness.

Definition 8.6. Let $\mathcal{X}$ be a compact Kähler orbifold and $\mathcal{L} \to \mathcal{X}$ a holomorphic line bundle on $\mathcal{X}$. We fix a Kähler form ω on $\mathcal{X}$. We say that $\mathcal{L}$ is *nef*, if for any $\epsilon > 0$, there exists a Hermitian metric h_ϵ on $\mathcal{L}$ such that its Chern curvature Θ_{h_ϵ} satisfies

$$\Theta_{h_\epsilon} \geq -\epsilon\omega.$$

Remark 8.7. Suppose that $\mathcal{L} \to \mathcal{X}$ is represented by $\mathcal{G} \ltimes L \to \mathcal{G}$. If G_0 is compact, it is obvious that L is then a nef line bundle on G_0. However, in general G_0 is not compact, and it makes no sense to say L is nef or not.

Let $\mathcal{X}$ be a compact Kähler orbifold whose anti-canonical bundle $K_{\mathcal{X}}^{-1}$ is nef. We fix a Kähler metric ω on $\mathcal{X}$. We repeat the technique used by in [DPS93] to construct a sequence of Kähler metrics $\{\omega_\epsilon\}$ in the same cohomology class of ω such that the Ricci form $\mathrm{Ricci}_{\omega_\epsilon} \geq -\epsilon\omega_\epsilon$.

For any $\epsilon > 0$, since $K_{\mathcal{X}}^{-1}$ is nef, we have a Hermitian metric h_ϵ on $K_{\mathcal{X}^{-1}}$, such that $u_\epsilon = \Theta_{h_\epsilon} \geq -\epsilon\omega$. It is thus sufficient to search ω_ϵ such that

$$\mathrm{Ricci}_{\omega_\epsilon} = -\epsilon\omega_\epsilon + \epsilon\omega + u_\epsilon. \tag{16}$$

The $\partial\bar{\partial}$-lemma still holds in the orbifold setting:

Lemma 8.8 ([Bai56, Theorem H, Theorem K])**.** *Let $(\mathcal{X}, \omega)$ be a Kähler orbifold such that $X = |\mathcal{X}|$ is compact. If α is a d-exact (p,q)-form, then α is $\partial\bar{\partial}$-exact.*

Hence we may write $u_\epsilon = \mathrm{Ricci}_\omega + \sqrt{-1}\partial\bar{\partial}f_\epsilon$. And to search ω_ϵ is the same as search a potential ϕ_ϵ such that $\omega_\epsilon = \omega + \sqrt{-1}\partial\bar{\partial}\phi_\epsilon$. Equation (16) on ω_ϵ is thus equivalent to

$$\frac{(\omega + \sqrt{-1}\partial\bar{\partial}\phi_\epsilon)^n}{\omega^n} = \exp(\epsilon\phi_\epsilon - f_\epsilon) \tag{17}$$

By the following theorem, Equation (17) has a unique solution.

Theorem 8.9 (Aubin-Yau Theorem, *cf.* [Fau19, Theorem 1.1 and Section 6]). *Let $(\mathcal{X}, \omega)$ be a compact Kähler orbifold. For any smooth function f on $\mathcal{X}$ and $\lambda > 0$. The equation*

$$\log \mathrm{M}(\phi) = \lambda\phi + f, \tag{MA}$$

where $\mathrm{M}(\phi) := \dfrac{(\omega + \sqrt{-1}\partial\bar{\partial}\phi)^n}{\omega^n}$ *is the Monge-Ampère operator, has a unique admissible solution.*

Thus we have

Lemma 8.10. *Let $\mathcal{X}$ be a compact Kähler orbifold with $-K_{\mathcal{X}}$ nef. Fix a Kähler metric ω on $\mathcal{X}$. For $\epsilon > 0$, there exists a Kähler metric ω_ϵ cohomologous to ω, and the Ricci form of ω_ϵ satisfying*

$$\mathrm{Ricci}_{\omega_\epsilon} \geq -\epsilon\omega_\epsilon.$$

To prove our main results, we first note that $\pi_1^{\mathrm{orb}}(\mathcal{X})$ is finitely generated. In fact we have

Lemma 8.11 ([MP99, Corollary 1.2.5.])**.** *Let $\mathcal{X}$ be an orbifold and $\mathcal{U}$ be an atlas of $\mathcal{X}$. There exists an atlas $\mathcal{V}$ for $\mathcal{X}$ such that*

(1) $\mathcal{V}$ refines $\mathcal{U}$;
(2) For every chart $(\tilde{V}, H, \psi)$ in $\mathcal{V}$, both $\tilde{V}$ and $V = \psi(\tilde{V}) \subset |\mathcal{X}|$ are contractible;
(3) The intersection of finitely many chart is empty or again a chart in $\mathcal{V}$.

Let $\mathcal{X}$ be a compact orbifold. We may take a finite atlas $\mathcal{V}$ by Lemma 8.11. Note that each open sub-orbifold $[\tilde{V}/H]$ has fundamental group

$$\pi_1^{\mathrm{orb}}([\tilde{V}/H]) \cong H.$$

By orbifold Van-Kampen theorem (*cf.* [BH99, Exercise III.$\mathcal{G}$.3.10]), we know that $\pi_1^{\mathrm{orb}}(\mathcal{X})$ is finitely generated.

We note that the following lemma, which is proved in manifold case in [DPS93], holds true in the orbifold case with exactly the same proof.

Lemma 8.12 (*cf.* [DPS93, Lemma 1.3.]). *Let $\pi : \tilde{\mathcal{X}} \to \mathcal{X}$ be the universal covering of $\mathcal{X}$. Let U be a connected compact subset of $\tilde{X} = |\tilde{\mathcal{X}}|$. Then for any $\delta > 0$, there exists a closed subset $U_{\epsilon,\delta} \subset U$ such that $\mathrm{vol}_\omega(U \setminus U_{\epsilon,\delta}) < \delta$ and $\mathrm{diam}_{\omega_\epsilon}(U_{\epsilon,\delta}) < C_1\delta^{-\frac{1}{2}}$, where C_1 is a constant independent of ϵ and δ.*

Proof. Suppose first that there exists a chart $(\tilde{V} \subset \mathbb{C}^n, G, V)$ of $\tilde{\mathcal{X}}$, a G-invariant compact subset K such that $K/G = U$ and a G-invariant open subset $W \subset \tilde{V}$ such that its closure $\overline{W} \subset \tilde{V}$ is compact and $K \subset W$. We further suppose that V is a component of $\pi^{-1}(\tilde{V}/G_0)$, where $(\tilde{V}, G_0)$ is an elementary neighborhood of $\mathcal{X}$ with respect to π and G is a subset of G_0.

Let $\tilde{\omega}_\epsilon$ be the G-invariant form on $\tilde{V}$. On $\tilde{V}$, we have two metrics: the Euclidean one ω_{euc} of $\mathbb{C}^n$ and the Riemannian one $\tilde{g}_\epsilon$ corresponding to $\tilde{\omega}_\epsilon$. As W is compact, we have a constant $C_0 = C_0(W)$ such that

$$\frac{1}{C_0}\tilde{\omega} \leq \omega_{\mathrm{euc}} \leq C_0\tilde{\omega}$$

Here, we consider these real $(1,1)$-forms as positive currents. If β is a positive $(1,1)$-form, we have that

$$\frac{1}{C_0}\tilde{\omega} \wedge \tilde{\omega}_{\mathrm{euc}}^{n-2} \wedge \beta \leq \omega_{\mathrm{euc}}^{n-1} \wedge \beta \leq C_0\tilde{\omega} \wedge \tilde{\omega}_{\mathrm{euc}}^{n-2} \wedge \beta$$

Iterating $(n-1)$ times, we have that

$$\frac{1}{C_0^{n-1}}\tilde{\omega}^{n-1} \leq \omega_{\mathrm{euc}}^{n-1} \leq C_0^{n-1}\tilde{\omega}^{n-1}$$

As ω and ω_ϵ are in the same cohomology class, we have that

$$\int_{\mathcal{X}} \omega_\epsilon \wedge \omega^{n-1} = \int_{\mathcal{X}} \omega \wedge \omega^{n-1}$$

Hence we have that

$$\int_W \tilde{\omega}_\epsilon \wedge \omega_{\mathrm{euc}}^{n-1} \leq C_0^{n-1}|G_0| \int_{\mathcal{X}} \omega_\epsilon \wedge \omega^{n-1} = C_0^{n-1}|G_0| \int_{\mathcal{X}} \omega \wedge \omega^{n-1} =: C_1$$

For any $x_1, x_2 \in K \times K$, we consider the path $c_{x_1,x_2} : [0,1] \to \tilde{V}$, $t \mapsto (1-t)x_1 + tx_2$. As W is convex and contains K, the path c_{x_1,x_2} is contained in W. We denote by $< -, - >$ and $< -, - >_\epsilon$ the Riemannian metric on $TW = W \times \mathbb{C}^n$ induced by ω_{euc} and $\tilde{\omega}_\epsilon$ respectively. We also use $|\bullet|$ to denote the norm induced by $< -, - >$.

By Cauchy-Schwartz inequality, we have that

$$\int_{K\times K} l^2_{\tilde{g}_\epsilon}(c_{x_1,x_2})dx_1dx_2 = \int_{K\times K}(\int_0^1 (\tilde{g}_\epsilon(\dot{c}_{x_1,x_2}(t),\dot{c}_{x_1,x_2}(t)))^{\frac{1}{2}}dt)^2 dx_1dx_2$$

$$(*) \qquad \leq |x_2-x_1|\int_0^1 dt \int_{K\times K} < v(t,x_1,x_2), v(t,x_1,x_2) >_\epsilon dx_1dx_2$$

where $v(t,x_1,x_2) := ((1-t)x_1+tx_2, \frac{x_2-x_1}{|x_2-x_1|}) \in W\times\mathbb{C}^n = TW$ and dx_1, dx_2 are the volumes with respect to ω_{euc}.

Now

$$\int_0^{\frac{1}{2}} dt \int_{K\times K} < v(t,x_1,x_2), v(t,x_1,x_2) >_\epsilon (\frac{1}{2})^{2n} dx_1dx_2$$
$$\leq \int_0^{\frac{1}{2}} dt \int_{K\times K} < v(t,x_1,x_2), v(t,x_1,x_2) >_\epsilon (1-t)^{2n} dx_1dx_2$$
$$\leq \int_0^{\frac{1}{2}} dt \int_{W\times K} < u_{x_2}(y), u_{x_2}(y) >_\epsilon dydx_2$$

where $u_{x_2} : y \mapsto (y, \dfrac{x_2-y}{|x_2-y|})$ is a vector field on W. Note that $|u_{x_2}(y)| = 1$ for all $y \in W$.

Let $z_i : \mathbb{C}^n \to \mathbb{C}$ be the i-th projection. We have that

$$\omega_{\text{euc}} = \sum_{1\leq i\leq n} \frac{\sqrt{-1}}{2} dz_i \wedge d\bar{z}_i$$
$$\tilde{\omega}_\epsilon = \sum_{1\leq i,j\leq n} \frac{\sqrt{-1}}{2} h_{ij} dz_i \wedge d\bar{z}_j$$
$$\tilde{\omega}_\epsilon \wedge \omega_{\text{euc}}^{n-1} = (n-1)!\text{Tr}(H)(\frac{\sqrt{-1}}{2})^n \bigwedge_{1\leq i\leq n} dz_i \wedge d\bar{z}_i$$

where $H = (h_{ij})$ is a strictly positive Hermitian matrix. If we consider the $\mathbb{C}$-linear extension of the tensor $\tilde{g}_\epsilon$ in $(T^*_{\mathbb{C}}W)^{\otimes 2}$, we have that

$$\tilde{g}_\epsilon = \sum_{1\leq i,j\leq n} \frac{h_{ij}}{2}(dz_i \otimes d\bar{z}_j + d\bar{z}_j \otimes dz_i)$$

Thus the matrix of $\tilde{g}_\epsilon$ with respect to the basis $(\partial/\partial z_1, \ldots, \partial/\partial z_n, \partial/\partial\overline{z_1}, \ldots, \partial/\partial\overline{z_n})$ is

$$\begin{pmatrix} 0 & H \\ H^t & 0 \end{pmatrix}.$$

Hence the maximal eigenvalue of $\tilde{g}_\epsilon$ is at most the maximal eigenvalue of H, which is less than $\text{Tr}(H)$. Thus for any vector u such that $|u| = 1$, we have that $< u, u >_\epsilon \leq \text{Tr}(H)$.

We have that

$$\begin{aligned}
&\int_0^{\frac{1}{2}} dt \int_{K\times K} < v(t,x_1,x_2), v(t,x_1,x_2) >_\epsilon dx_1 dx_2 \\
&\leq 2^{2n} \int_0^{\frac{1}{2}} dt \int_{W\times K} < u_{x_2}(y), u_{x_2}(y) >_\epsilon dy dx_2 \\
&\leq 2^2 \int_0^{\frac{1}{2}} dt \int_{W\times K} \frac{1}{(n-1)!} \tilde{\omega}_\epsilon \wedge \omega_{\text{euc}}^{n-1} dx_2 \\
&\leq 2^{2n-1} \operatorname{vol}(K) C_1
\end{aligned}$$

Similarly, we have that

$$\int_{\frac{1}{2}}^1 dt \int_{K\times K} < v(t,x_1,x_2), v(t,x_1,x_2) >_\epsilon dx_1 dx_2 \leq 2^{2n-1} \operatorname{vol}(K) C_1$$

Combined with Equation (*), we have that

$$\begin{aligned}
\int_{K\times K} l^2_{\tilde{g}_\epsilon}(c_{x_1,x_2}) dx_1 dx_2 &\leq 2^{2n} \operatorname{diam}(K) \operatorname{vol}(K) C_1 \\
&\leq C_2
\end{aligned}$$

where $\operatorname{diam}(-)$ is the diameter with respect to ω_{euc} and C_2 is a constant independent of ϵ.

We set

$$S := \{(x_1,x_2) \in K \times K \mid l_{\tilde{g}_\epsilon}(c_{x_1,x_2}) > (C_2/\delta)^{\frac{1}{2}}\}$$

The above estimate implies that $\operatorname{vol}(S) < \delta$. Set $S(x_1) := \{x_2 \in K : (x_1,x_2) \in S\}$. Now we consider the set

$$Q := \{x_1 \in K \mid \operatorname{vol}(S(x_1)) \geq \frac{1}{2} \operatorname{vol}(K)\}$$

Then by Fubini, we have that $\operatorname{vol}(Q) \leq 2\delta \operatorname{vol}(K)$. For $x_1, x_2 \in K \setminus Q$, we have that $\operatorname{vol}(S(x_i)) < \frac{1}{2} \operatorname{vol}(K)$. Thus we have that

$$(K \setminus S(x_1)) \cap (K \setminus S(x_2)) \neq \emptyset.$$

If y is an element of the above set, then $(x_1,y) \notin S$ and $(x_2,y) \notin S$. Hence we have that

$$l_{\tilde{g}_\epsilon}(c_{x_1,x_2}) \leq l_{\tilde{g}_\epsilon}(c_{x_1,y}) + l_{\tilde{g}_\epsilon}(c_{y,x_2}) \leq 2(C_2/\delta)^{\frac{1}{2}}$$

By continuity, for any $x_1, x_2 \in \overline{K \setminus Q}$, we have that $l_{\tilde{g}_\epsilon}(c_{x_1,x_2}) \leq 2(C_2/\delta)^{\frac{1}{2}}$.

Let $U_{\epsilon,\delta}$ be the image of $\overline{K \setminus Q}$ in V. Then as K is G-variant, we have that

$$\operatorname{vol}_\omega(K \setminus (\bigcup_{\alpha\in G} \alpha \cdot \overline{K \setminus Q})) \leq \operatorname{vol}_\omega(K \setminus \overline{K \setminus Q}) \leq C_3 \operatorname{vol}(Q) < \frac{2C_3\delta}{\operatorname{vol}(K)}$$

where $C_3 = C_0^n$.

Hence we have that

$$\operatorname{vol}_\omega(U \setminus U_{\epsilon,\delta}) < \frac{2C_3\delta}{|G| \operatorname{vol}(K)} .$$

If a_1, a_2 are two points in $U_{\epsilon,\delta}$, we may take $x_1 \in \overline{K \setminus Q}$ and $x_2 \in \overline{K \setminus Q}$ to be one of the inverse images in $\tilde{V}$ of a_1 and a_2 respectively. We have thus that

$$d_{\omega_\epsilon}(a_1, a_2) \leq l_{\tilde{g}_\epsilon}(c_{x_1,x_2}) \leq 2(C_1/\delta)^{\frac{1}{2}}.$$

This implies that $\mathrm{diam}_{\omega_\epsilon}(U_{\epsilon,\delta}) \leq 4(C_1/\delta)^{\frac{1}{2}}$. By scaling δ, we have the result for $U = K/G$.

For a general $U \subset \tilde{X}$, there exists a finite open covering $U \subset \bigcup_{1\leq i\leq N} U_i$ such that

- for each i, the open set U_i is connected;
- for each i, there exists a chart $(\tilde{V}_i, H_i, V_i)$ of $\tilde{\mathcal{X}}$ and an H_i-invariant convex open $W_i \subset \tilde{V}_i$ satisfying that W_i/H_i contains U_i and $\overline{W_i}$ is compact.

As U is connected, after removing all the U_j that do not intersect U, we can arrange U_i such that $U_i \cap U_{i+1} \neq \emptyset$. Now the hypothesis in the first paragraph applies to $\overline{U_i}$. Hence for $\delta > 0$, we get subsets $U_{i,\epsilon,\delta}$ such that

(1) for each i, we have that

$$\mathrm{vol}_\omega(\overline{U}_i \setminus U_{i,\epsilon,\delta}) < \min\left\{\frac{\delta}{N}, \min_{1\leq j\leq N-1}\{\frac{1}{2}\mathrm{vol}_\omega(U_j \cap U_{j+1})\}\right\};$$

(2) for each i, we have that $\mathrm{diam}_{\omega_\epsilon}(U_{i,\epsilon,\delta}) < C_i'\delta^{-\frac{1}{2}}$, where C_i' is independent of ϵ and δ.

Condition (1) implies that $U_{i,\epsilon,\delta} \cap U_{i+1,\epsilon,\delta} \neq \emptyset$. Set

$$C_1 := N^2 \cdot \max_{1\leq i\leq N}\{C_i'\}.$$

Thus $\mathrm{diam}_{\omega_\epsilon}(\bigcup_{1\leq i\leq N} U_{i,\epsilon,\delta}) \leq C_1\delta^{-\frac{1}{2}}$ and

$$\mathrm{vol}_\omega\left(\bigcup_{1\leq i\leq N} U_i \setminus \bigcup_{1\leq i\leq N} U_{i,\epsilon,\delta}\right) \leq \sum_{1\leq i\leq N} \mathrm{vol}_\omega(U_i \setminus U_{i,\epsilon,\delta}) < \delta.$$

Set $U_{\epsilon,\delta} := U \cap (\cup_{1\leq i\leq N} U_{i,\epsilon,\delta})$. We have that

$$\mathrm{vol}_\omega(U \setminus U_{\epsilon,\delta}) \leq \mathrm{vol}_\omega\left(\bigcup_{1\leq i\leq N} U_i \setminus \bigcup_{1\leq i\leq N} U_{i,\epsilon,\delta}\right) < \delta.$$

This finishes the proof. □

THEOREM 8.13 (=Theorem 4.10). *Let $(\mathcal{X}, \omega)$ be a compact Kähler orbifold. If the anti-canonical bundle $K_{\mathcal{X}}^{-1}$ is nef, then $\pi_1^{\mathrm{orb}}(\mathcal{X})$ is virtually nilpotent.*

PROOF. We can reproduce the argument by [DPS93] and [Pău97] in the manifold case. Let ω_ϵ be the sequence of Kähler metrics as in Lemma 8.10, and $\tilde{\mathcal{X}} \to \mathcal{X}$ be the universal covering of $\mathcal{X}$. We fix a finite system of generators $\{\gamma_i\}$ of $\pi_1^{\mathrm{orb}}(\mathcal{X})$. Let $\alpha = \alpha(2n)$ be the constant in Lemma 7.12. It suffices to show that there exists ω' such that all the generators γ_i are contained in the almost stabliser Γ_α with respect to the distance $\mathrm{d}_{\omega'}$ in Lemma 7.12.

Recall that the fundamental domain of $\mathrm{Gal}(\tilde{\mathcal{X}} \to \mathcal{X})$ is defined to be

$$F := \{p \in X' \mid \mathrm{d}_{\tilde{\omega}}(x_0, p) \leq \mathrm{d}_{\tilde{\omega}}(\gamma \cdot x_0, p) \text{ for all } \gamma \in \pi_1^{\mathrm{orb}}(\mathcal{X})\}.$$

We see easily that F is path connected. Thus we can take a connected compact subset $U \subset \tilde{X}$ which contains F. As $\{\gamma_i\}$ is finite, we may take U large enough, such that $U \cap \gamma_j U \neq \emptyset$ for all j. We choose a sufficiently small $\delta > 0$ such that $\delta < \frac{1}{4}\operatorname{vol}_\omega(U \cap \gamma_j U)$ and $\delta < \frac{1}{2}\operatorname{vol}_\omega F$. By Lemma 8.12, there exists a subset $U_{\epsilon,\delta} \subset U$, such that $\operatorname{diam}_{\omega_\epsilon}(U_{\epsilon,\delta}) < C_1\delta^{\frac{1}{2}} := C$. By the choice of δ, we know that $U_{\epsilon,\delta} \cap \gamma_j U_{\epsilon,\delta} \neq \emptyset$. Fix a $\tilde{x}_0 \in U_{\epsilon,\delta}$. We know that $d_{\omega_\epsilon}(\tilde{x}_0, \gamma_j\tilde{x}_0) < C$. We set $\tilde{\omega_\epsilon} := \frac{\epsilon}{2n-1}\omega_\epsilon$. Then $\operatorname{Ricci}_{\tilde{\omega_\epsilon}} \geq -(2n-1)\tilde{\omega_\epsilon}$ and $\mathrm{d}_{\tilde{\omega_\epsilon}}(\tilde{x}_0, \gamma_j\tilde{x}_0) < \frac{\epsilon}{2n-1}C$. For ϵ sufficient small, we see that $\frac{\epsilon}{2n-1}C < \alpha$. $\square$

CHAPTER 9

Projective case

We will consider complex orbifolds as klt pairs described in Section 5.6.

Let $\mathcal{X} = (X, [G_1 \rightrightarrows G_0])$ be a complex orbifold of dimension n and $(X = |\mathcal{X}|, \Delta)$ its associated orbifold pair. For any $x \in G_0$ by Proposition A.8, there exists open neighborhood $x \ni U_x \subset G_x$, such that $\pi_x : U_x \to U_x/G_x \subset |\mathcal{X}|$ gives an orbifold chart. Theorem 5.33 implies that the topological quotient U_x/G_x has a unique normal complex space structure such that $\pi_x : U_x \to U_x/G_x$ is a Galois analycial covering. Hence $U_x/G_x \subset X$ is an open sub-variety and $\pi_x : U_x \to U_x/G_x$ is a local uniformization as in Definition 5.37. The map $\pi : G_0 \to X$ is holomorphic. As G_0 is Kähler, [Var89, Proposition 3.3.1] implies X is a complex Kähler space. We consider the canonical bundle $K_{\mathcal{X}}$. From Definition 8.4, we know that it is represented by $\det(\Omega_{G_0})$. We denote by K_{G_0} its canonical class. Then by Equation (14), we have that

$$K_{G_0}|_{U_x} = \pi_x^*(K_X + \Delta). \tag{18}$$

If $U_x/G_x \subset U_y/G_y$, we have an element $g \in G_1$ which induces an embedding $\rho_g : U_x \to U_y$ and we have the following commutative diagram:

$$\begin{tikzcd} U_x \arrow[r, "\rho_g"] \arrow[d, "\pi_x"] & U_y \arrow[d, "\pi_y"] \\ U_x/G_x \arrow[r] & U_y/G_y \end{tikzcd} .$$

Thus the local equations Equation (18) glue together to

$$K_{G_0} = \pi^*(K_X + \Delta)$$

We may thus regard $K_X + \Delta$ as the canonical class of (X, Δ). As (X, Δ) has klt singularities, let $a(X)$ be the index of (X, Δ), *i.e.*, the minimal natural number such that $a(X)(K_X + \Delta)$ is a Cartier divisor. Let $\mathcal{O}_X(a(X)(K_X + \Delta))$ be the associated line bundle on X. Then we have that

$$\mathcal{O}_{G_0}(K_{G_0})^{\otimes a(X)} = \pi^*\mathcal{O}_X(a(X)(K_X + \Delta)).$$

Let h be an Hermitian metric on $K_{\mathcal{X}}$. The Hermitian metric $h^{\otimes a(X)}$ on $K_{\mathcal{X}}^{\otimes a(X)}$ induces an Hermitian metric on $\mathcal{O}_X(a(X)(K_X + \Delta))$, since h is G_1-invariant. On the other hand, the pullback of an Hermitian metric on $\mathcal{O}_X(a(X)(K_X + \Delta))$ by π will induce an Hermitian metric on $K_{\mathcal{X}}^{\otimes a(X)}$. In particular, if X is compact, the orbi-bundle $K_{\mathcal{X}}^{-1}$ being nef is equivalent to $-(K_X + \Delta)$ being nef.

We have the following result.

THEOREM 9.1 (=Theorem 4.11). *Let $\mathcal{X}$ be a compact Kähler orbifold with $-K_{\mathcal{X}}$ nef. Let (X, Δ) be the associated orbifold pair of $\mathcal{X}$. If X is projective, then $\pi_1^{\mathrm{orb}}(\mathcal{X})$ is virtually Abelian.*

PROOF. By Theorem 8.13, there exists a nilpotent subgroup $\Gamma < \pi^{\mathrm{orb}}(\mathcal{X})$ such that its index $[\pi^{\mathrm{orb}}(\mathcal{X}) : \Gamma]$ is finite. Let $\tilde{\mathcal{X}}$ be the finite cover of $\mathcal{X}$ such that $\pi_1^{\mathrm{orb}}(\tilde{\mathcal{X}}) = \Gamma$, as indicated in Proposition 5.30. For any $\epsilon > 0$ and the Hermitian metric h_ϵ such that $\Theta_{h_\epsilon} \geq -\epsilon\omega$, we may take the induced metric on $-K_{\tilde{\mathcal{X}}}$. In particular, the orbi-bundle $-K_{\tilde{\mathcal{X}}}$ is nef.

To simplify the notation, we may assume that $\pi_1^{\mathrm{orb}}(\mathcal{X})$ is nilpotent. Let (X, Δ) be the associated orbifold pair. Then its anti-canonical bundle $-(K_X + \Delta)$ is nef. [Zha05, Corollary 2] implies that the Albanese map

$$\alpha_X : X \dashrightarrow \mathrm{Alb}(X)$$

is dominant. Recall that the Albanese map of X is defined by the Albanese morphism of its smooth model. Let $r : Y \to (X, \Delta)$ be a log resolution, we have the following commutative

$$\begin{tikzcd} Y \arrow[r, "r"] \arrow[d, "\alpha_Y"'] & X \arrow[dl, dotted, "\alpha_X"] \\ \mathrm{Alb}(Y) =: \mathrm{Alb}(X) & \end{tikzcd}$$

Recall that X has rational singularities (Theorem 5.38). Theorem 2.35 implies that α_X is defined on all X. Hence α_X is surjective. We have that

$$K_Y + r_*^{-1}(\Delta) = r^*(K_X + \Delta) + \sum a_j E_j,$$

where $E_j \subset \mathrm{exc}(r)$ are the irreducible exceptional divisors. We have also that

$$Y \setminus (r_*^{-1}(\Delta) \cup \mathrm{exc}(r)) = X_{\mathrm{reg}} \setminus |\Delta|. \tag{19}$$

If l_j is a loop in Y around E_j, the composition $r \circ l_j$ will be a loop in X. If l_j is small enough, the loop $r \circ l_j$ will be contained in an open subset U of X such that $(U, \Delta|_U) = [\tilde{U}/G]$ with $\tilde{U}$ simply connected. Then $\pi_1(U, \Delta|_U) = G$ and we can find a divisible enough n_j such that $(r \circ l_j)^{n_j}$ is the unit in $\pi_1(X, \Delta)$. We set

$$\Delta_Y := r_*^{-1}(\Delta) + \sum (1 - \frac{1}{n_j}) E_j.$$

By Equation (19), the choice of n_j and Definition 5.39, we have that

$$\pi_1(Y, \Delta_Y) = \pi_1(X, \Delta).$$

As $r : Y \to X$ is a log resolution, the support $|\Delta_Y|$ of Δ_Y is snc. Assume that $r_*^{-1}(D_i)$, $1 \leq i \leq i_0$ and E_j, $1 \leq j \leq j_0$ are all the components of Δ_Y passing through a point $y \in Y$. There is a holomorphic chart (U, ϕ) centered at y, such that $\phi_*(\Delta_Y|_U)$ is the branching divisor of the following map

$$(z_1, ..., z_{i_0}, z_{i_0+1}, ..., z_{i_0+j_0}, ..., z_n) \mapsto (z_1^{m_1}, ..., z_{i_0}^{m_{i_0}}, z_{i_0+1}^{n_1}, ..., z_{i_0+j_0}^{n_j}, z_{i_0+j_0+1}, ..., z_n)$$

Hence (Y, Δ_Y) is an orbifold pair.

[Cam01, lemme 1.9.9] implies that for the orbifold pair (Y, Δ_Y), there exists a short exact sequence

$$1 \to K \to \pi_1(Y, \Delta_Y) \to \pi_1(Y) \to 1$$

with the group K generated by torsion elements.

As $\pi_1(Y, \Delta_Y) = \pi_1(X, \Delta)$ is nilpotent, the quotient $\pi_1(Y)$ is also nilpotent. By [Hir38], for any nilpotent group N of finite type, the torsion element of N forms a finite normal subgroup $N_{\text{tor}} \trianglelefteq N$ and the nilpotent limit of N is N/N_{tor}. By the above exact sequence, we have that

$$\pi_1(Y, \Delta_Y)/\pi_1(Y, \Delta_Y)_{\text{tor}} = \pi_1(Y)/\pi_1(Y)_{\text{tor}}.$$

As α_Y is surjective, [Cam95, Théorème 2.2] implies that

$$\pi_1(Y)/\pi_1(Y)_{\text{tor}} = \pi_1(\text{Alb}(X)).$$

By [Cla07, Lemme A.0.1], we know that $\pi_1(Y, \Delta_Y)$ is then virtually Abelian. As $\pi_1(Y, \Delta_Y)$ has finite index in $\pi_1^{\text{orb}}(\mathcal{X})$, we know that $\pi_1^{\text{orb}}(\mathcal{X})$ is also virtually Abelian. □

Remark 9.2. As one can see from the proof of Theorem 9.1, the hypothesis X is projective is used only to show that the Albanese morphism is surjective. One may reformulate Theorem 9.1 as following:

If the covering (X', Δ') corresponds to the nilpotent subgroup of $\pi_1(X, \Delta)$ has surjective Albanese morphism, then $\pi_1(X, \Delta)$ is virtually Abelian.

Appendice

APPENDIX A

Groupoids

We recall here some basic facts in Lie groupoids. For the general results on categories, we refer the reader to [ML98]. For the Lie groupoids, we refer the reader to [Mac05][MM03]. In this chapter, all groupoids are small.

1.1. Orbifold Groupoids

Definition A.1. A *topological groupoid* $\mathcal{G}$ consists of a topological space G_0 of objects and a topological space G_1 of arrows, together with five continuous structure maps listed below:

1. The source map $s : G_1 \to G_0$, which assigns to each arrow $g \in G_1$ its source $s(g)$.
2. The target map $t : G_1 \to G_0$, which assigns to each arrow $g \in G_1$ its target $t(g)$. For any two objects $x, y \in G_0$, one writes $g : x \to y$ to indicate that $g \in G_1$ is an arrow with $s(g) = x$ and $t(g) = y$.
3. The composition map $m : G_1 \ {}_s\!\times_t G_1 \to G_1$. If $h : y \to z$, $g : x \to y$, then $hg = m(h, g) : x \to z$. We have m to be associative, that is $m(m(h, g), f) = m(h, m(g, f))$ for any three composable h, g, f.
4. The identity map $u : G_0 \to G_1$ which is a two-sided unit for the composition.
5. The inverse map $i : G_1 \to G_1$. If $g : x \to y \in G_1$, then $g^{-1} = i(g) : y \to x$ is the two sided inverse to g, *i.e.*, we have $g \circ i(g) = u(y)$ and $i(g) \circ g = u(x)$.

One can also consider the topological groupoid as a groupoid (*i.e.* a category whose morphisms are all isomorphisms) equipped with two topological structure on the sets of objects and morphisms such that the structural maps are continuous. We then define the

Definition A.2. A *Lie groupoid* is a topological groupoid $\mathcal{G}$ where G_0 and G_1 are smooth manifolds and all the structure maps s, t, m, u and i are smooth. Furthermore, s and t are required to be submersions (hence $G_1 \ {}_s\!\times_t G_1$ is a manifold).

With the topological/smooth structures on G_0 and G_1, one can define the morphisms between groupoids to be functors with required continuity/smoothness on the sets of objects and morphisms. Similarly, a groupoid natural transformation will require the continuity on the functions assign each objects in source groupoid to the morphisms in the target groupoid.

Example A.3. Let M be a topological space K a topological group acting on M. One defines a topological groupoid $K \ltimes M$, by setting $(K \ltimes M)_0 = M$ and $(K \ltimes M)_1 = K \times M$, with $(g, x) : x \to gx$. This groupoid is called the action groupoid or translation groupoid

associated to the group action. Note that if M is a manifold and K a Lie group, then $K \ltimes M$ becomes a Lie groupoid.

Definition A.4. Let $\mathcal{G}$ be a Lie groupoid. For a point $x \in G_0$, we define the *isotropy group* G_x of $\mathcal{G}$ to be $s^{-1}(x) \cap t^{-1}(x)$. And we define the orbit space $|\mathcal{G}|$ of $\mathcal{G}$ to be the quotient of G_0 by the equivalence relation $x \sim y$ iff $\exists\, g : x \to y$.

Lemma A.5 ([Mac05, Corollary 1.4.11])**.** *Let $\mathcal{G}$ be a Lie groupoid. Set $G_x^y := s^{-1}(x) \cap t^{-1}(y)$. Then G_x^y is a smooth manifold, and the morphism $m : G_y^z \times G_x^y \to G_x^z$ is smooth. In particular, G_x is a Lie group.*

Now we define types of groupoids.

Definition A.6. Let $\mathcal{G}$ be a Lie groupoid.

(a) $\mathcal{G}$ is proper, if $(s,t) : G_1 \to G_0 \times G_0$ is proper.
(b) $\mathcal{G}$ is called a foliation groupoid if each isotropy group G_x is discrete.
(c) $\mathcal{G}$ is étale, if s and t are local diffeomorphisms. In this case, one defines the dimension of $\mathcal{G}$ to be $\dim(\mathcal{G}) := \dim(G_0) = \dim(G_1)$.

For a $\mathcal{G}$ is proper étale, we have that G_x^y is finite. Let $G_1 \ni g : x \to y$ be an arrow of $\mathcal{G}_1$. As s and t are diffeomorphisms around g, x and y, we get (*via* $t \circ s^{-1}$) a local diffeomorphism $\phi_g : U_x \to U_y$. After shrinking U_x and U_y, we get a morphism $\phi_x^y : G_x^y \to \mathrm{Diff}(U_x, U_y)$. One can prove that $\phi(hg) = \phi(h) \circ \phi(g)$. In particular, $\phi : G_x \to \mathrm{Diff}(U_x)$ is a group morphism.

Definition A.7. We define an *orbifold groupoid* to be a proper étale Lie groupoid. An orbifold groupoid $\mathcal{G}$ is *effective* if $\forall\, x \in G_0$, $\phi : G_x \to \mathrm{Diff}(U_x)$ is injective.

It turns out that the local structure of an orbifold groupoid $\mathcal{G}$ around $x \in G_0$ is completely determined by the local group G_x. More precisely, we have the following

Proposition A.8 (*cf.* [ALR07, Proposition 1.44])**.** *Let $\mathcal{G}$ be an orbifold groupoid and x an element in G_0. For any neighborhood $G_0 \supset U$ of x, there exists an open neighborhood $N_x \subset U$, such that the restriction of $\mathcal{G}$ over N_x is isomorphic, as Lie groupoid, to the translation groupoid $G_x \ltimes N_x$ and the quotient space N_x/G_x is an embedded open subsets of $|\mathcal{G}|$ via the natural morphism $\mathcal{G}|_{N_x} \hookrightarrow \mathcal{G}$.*

Definition A.9. A morphism $\phi : \mathcal{H} \to \mathcal{G}$ between Lie groupoids is called an *equivalence* between Lie groupoids if both conditions below are satisfied:

(i) the map
$$t\pi_1 : G_1 \,{}_s\!\times_\phi H_0 \to G_0$$
defined on the fibered product of manifolds is a surjective submersion;
(ii) The commutative diagram
$$\begin{tikzcd} H_1 \arrow[r, "\phi_1"] \arrow[d, "(s,t)"'] & G_1 \arrow[d, "(s,t)"'] \\ H_0 \times H_0 \arrow[r, "\phi\times\phi"] & G_0 \times G_0 \end{tikzcd}$$
is Cartesian.

Note that a homomorphism $\phi : \mathcal{H} \to \mathcal{G}$ induces a continuous map $|\mathcal{H}| \to |\mathcal{G}|$. When ϕ is an equivalence, the induced map on orbit spaces is an homeomorphism.

There this a more subtle equivalent relation between groupoids called *Morita equivalence*.

Definition A.10. Two Lie groupoids $\mathcal{G}$ and $\mathcal{G}'$ are said to be *Morita equivalent*, if there exists a third groupoid $\mathcal{H}$ and two equivalences

$$\mathcal{G} \leftarrow \mathcal{H} \to \mathcal{G}'.$$

We denote the Morita equivalence by $G \sim_{\text{Morita}} G'$.

It's not hard to see that Morita equivalance is an equivalence relation. In fact suppose that $\mathcal{G} \leftarrow \mathcal{H} \to \mathcal{G}'$ and $\mathcal{G}' \leftarrow \mathcal{H}' \to \mathcal{G}''$ are two Morita equivalences. We get a Morita equivalence by considering the following diagram

$$\begin{tikzcd}
(\mathcal{G} \times_{\mathcal{H}} \mathcal{G}') \times_{\mathcal{G}'} (\mathcal{G}' \times_{\mathcal{H}'} \mathcal{G}'') \arrow[r] \arrow[d] & \mathcal{G}' \times_{\mathcal{H}'} \mathcal{G}'' \arrow[r] \arrow[d] & \mathcal{G}'' \arrow[d] \\
\mathcal{G} \times_{\mathcal{H}} \mathcal{G}' \arrow[r] \arrow[d] & \mathcal{G}' \arrow[r] \arrow[d] & \mathcal{H}' \\
\mathcal{G} \arrow[r] & \mathcal{H} &
\end{tikzcd}$$

where the product $\mathcal{G} \times_{\mathcal{H}} \mathcal{G}'$ *etc.* are 2-fiber products of groupoids. As Lie groupoid equivalences are stable under 2-fiber products. We have a Morita equivalence between $\mathcal{G}$ and $\mathcal{G}''$.

Definition A.11. We recall several categories related to groupoids.

(1) We denote by LieGrpd the (strict) 2-category of small Lie groupoids.
(2) We denote by Gp the 1-category defined as follows:
  (a) Gp has objects $\mathrm{Ob}(\mathrm{Gp}) = \mathrm{Ob}(\mathrm{LieGrpd})$
  (b) For G_1, G_2 two groupoids, their morphism is $\mathrm{Hom}_{\mathrm{Gp}}(G_1, G_2) = \{$1-arrow of $\mathrm{Hom}_{\mathrm{LieGrpd}}(G_1, G_2)\}/\sim$, where $f, g \in \mathrm{Hom}_{\mathrm{LieGrpd}}(G_1, G_2)$ are equivalent if there exists a 2-morphism $\alpha : f \Rightarrow g$.
(3) Let $W := \{[e]$ is an arrow in Gp: e is an equivalence of Lie groupoid (Definition A.9) $\}$ be the subset of arrows of Gp. The Hilsum-Skandalis cateogry $\mathcal{HS}$ is the localization $\mathrm{Gp}(W^{-1})$ of Gp by W.

We note that $\mathcal{HS}$ is in fact defined as a category with Lie groupoids as objects and isomorphic bi-bundles as morphisms. However [Ler10, Proposition 3.39.] shows that $\mathrm{Gp}(W^{-1})$ and $\mathcal{HS}$ are equivalent. Let G, H be two groupoids. Let $G \leftarrow K \to H$ be a Morita equivalence. Then its obvious that $\mathrm{Hom}_{\mathcal{HS}}(G, L)$ is bijective to $\mathrm{Hom}_{\mathcal{HS}}(H, L)$. We may thus define a pre-relation $\mathcal{R}_0$ on the arrows of $\mathcal{HS}$. Let G and H be Morita equivalent. We say that two 1-arrows $f \in \mathrm{Hom}_{\mathcal{HS}}(G, L)$ and $g \in \mathrm{Hom}_{\mathcal{HS}}(H, L)$ are $f\mathcal{R}_0 g$ if f is mapped to g in the bijection $\mathrm{Hom}_{\mathcal{HS}}(G, L) \cong \mathrm{Hom}_{\mathcal{HS}}(H, L)$ induced by the Morita equivalence. We denote by $\mathcal{R}$ the relation generated by $\mathcal{R}_0$.

1.2. $\mathcal{G}$-bundles

Let $\mathcal{G} = [G_1 \rightrightarrows G_0]$ be a groupoid. The concept of $\mathcal{G}$-space is a natural generalization of G-bundles. We refer the reader to [ALR07, Chapter 2] for more details. We only define vector bundles.

Definition A.12. Let $\mathcal{G}$ be a topological groupoid, G_1 its arrows and G_0 its objects. A left-$\mathcal{G}$-vector bundle is a triple (E, π, μ), where $\pi : E \to G_0$ is a vector bundle over G_0 and $\mu : G_1 \ {}_s\!\times_\pi E \to E$ is a continuous map, satisfying the following

(1) $\mu(g'g, e) = \mu(g', \mu(g, e))$;
(2) $\mu(1, e) = e$;
(3) $\mu(g, -) : E_{s(g)} \to E_{t(g)}$ is a linear isomorphism.

Let E be a left-$\mathcal{G}$-vector bundle over $\mathcal{G}$. We associate with E a groupoid $\mathcal{G} \ltimes E$ in the following way:

Set E to be the objects and $G_1 \ {}_s\!\times_\pi E$ to be the arrows. The source map is $(g, e) \mapsto e$ and target map is $(g, e) \mapsto \mu(g, e)$. If $e \in E_x$, the identity arrow of e is $(1_x, e)$. The inverse arrow of (g, x) is (g^{-1}, gx). The map $\pi : E \to G_0$ extends to a morphism $\mathcal{G} \ltimes E \to \mathcal{G}$ by taking the objects map π and the arrows map $(g, x) \mapsto g$. We also denote $\mathcal{G} \ltimes E$ by $\mathcal{E}$

Example A.13 (Pullback of vector bundles)**.** Let $\mathcal{H}, \mathcal{G}$ be two groupoids and $\phi : \mathcal{H} \to \mathcal{G}$ a morphism between groupoid. Let E be a left-$\mathcal{G}$-vector bundle over $\mathcal{G}$. Consider the pullback vector bundle $\phi_0^*(E) \to H_0$. It has a natural left-$\mathcal{H}$-vector bundle structure $\nu : H_1 \ {}_s\!\times \phi_0^*(E) \to \phi_0^*(E)$, by defining $\nu(h, (x, e)) = \mu(\phi_1(h), e)$. Its associated groupoid $\mathcal{H} \ltimes \phi_0^*(E)$ fits in the following commutative diagram

$$\begin{tikzcd} \mathcal{H} \ltimes \phi_0^*(E) \arrow[r] \arrow[d] & \mathcal{E} \arrow[d] \\ \mathcal{H} \arrow[r] & \mathcal{G} \end{tikzcd} .$$

Thus it makes sense to call $\mathcal{H} \ltimes \phi_0^*(E)$ the pullback of $\mathcal{E}$ and denote it by $\phi^*(\mathcal{E})$.

Example A.14 (Pushforward by equivalence)**.** Suppose that $\phi : \mathcal{H} \to \mathcal{G}$ is an equivalence between two orbifold groupoids. Then ϕ as a functor is an equivalence. Applying Proposition A.8 to both $\mathcal{H}$ and $\mathcal{G}$, we have that ϕ_0 is a local diffeomorphism. Let E be a left-$\mathcal{H}$-vector bundle. We will now construct a left-$\mathcal{G}$-bundle called the *pushforward* of E.

Let $x \in G_0$ be a point in G_0. The morphism ϕ_0 is an equivalence implies that there exists a $y_1 \in H_0$ and an arrow $G_1 \ni g_1 : x \to x_1$ such that $x_1 = \phi_0(y_1)$. Let ψ_1 be the local inverse of ϕ_0 at y_1 and x_1. We thus get a vector bundle $g_1^*\psi_1^*(E)$ around x. We denote the vector bundle by F_{x,g_1,y_1}. If there is another $y_2 \in H_0$ and another arrow $G_1 \ni g_2 : x \to x_2$ such that $x_2 = \phi_0(y_2)$, denoting by ψ_2 the local inverse of ϕ_0 at y_2 and x_2, we will get another vector bundle $F_{x,g_2,y_2} := g_2^*\psi_2^*(E)$. As ϕ is an equivalence, we have a unique $h \in H_1$ such that $\phi_1(h) = g_1 g_2^{-1}$. There is a canonical isomorphism $F_{x,g_1,y_1} \to F_{x,g_2,y_2}$ induced by h.

In fact, let $H_{y_1}, H_{y_2} \subset H_0$ be two neighbourhoods of y_1 and y_2 respectively such that $h : y_2 \to y_1$ induces a diffeomorphism $H_{y_2} \cong H_{y_1}$. As E is a left-$\mathcal{H}$-bundle, we have the following commutative diagram

$$\begin{tikzcd} E|_{H_{y_2}} \arrow[r, "h"] \arrow[d] & E|_{H_{y_1}} \arrow[d] \\ H_{y_2} \arrow[r, "h"] & H_{y_1} \end{tikzcd}$$

And hence a canonical isomorphism $h : E|_{H_{y_2}} \cong h^*(E|_{H_{y_1}})$ induced by h. Now we have canonical isomorphisms:

$$g_2^*\psi_2^*(E|_{H_{y_2}}) \cong g_2^*\psi_2^*h^*(E|_{H_{y_1}}) \cong g_2^*(g_1g_2^{-1})^*\psi_1^*(E|_{H_{y_1}}) \cong g_1^*\psi_1^*(E|_{H_{y_1}}).$$

Thus we get an open covering $\{U_{x,g,y}\}$ of G_0 and a family of vector bundles $F_{x,g,y}$ over $U_{x,g,y}$. We can verify that they satisfy the cocycle condition and hence glue up to a vector bundle on G_0. From the construction, this vector bundle is actually a left-$\mathcal{G}$-vector bundle. We call it the pushfoward of $\mathcal{E}$ and denote it by $\phi_*(\mathcal{E})$. We have the relation $\phi^*\phi_*(\mathcal{E}) = \mathcal{E}$.

If either $\mathcal{H}$ or $\mathcal{G}$ fails to be orbifold groupoid, the ϕ_0 fails in general to be local diffeomorphism. This happens, for example when we consider the Morita equivalence $e : \mathcal{G} \leftarrow \mathcal{K} \rightarrow \mathcal{H}$. Even $\mathcal{G}$ and $\mathcal{H}$ are orbifold groupoids, $\mathcal{K}$ needs not to be an orbifold groupoid. Hence the above explicit construction can not be used on Morita equivalence. However, we have

Proposition A.15 (*cf.* [MP97, page 11 Remark (4)])**.** *Let $e : \mathcal{G} \leftarrow \mathcal{K} \rightarrow \mathcal{H}$ be a Morita equivalence between two topological groupoids. Then e induces an equivalence $e^* : Sh(\mathcal{H}) \rightarrow Sh(\mathcal{G})$ between two topoi.*

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UNIVERSITÉ DE RENNES 1

Titre : Deux résultats sur la classification des variétés singulières à classe canonique semi-négative

Mots clés : variétés polarisées, singularités slc, géométrie birationelle, orbifoldes, lemme de Margulis, groupes fundamentaux

Résumé : Le sujet de cette thèse est d'étudier le problème de classification des espaces singuliers sous deux hypothèses différentes sur la positivité de la classe anti-canonique des espaces et de leurs singularités dans ces deux conditions différentes. Nous appliquerons des méthodes assez différentes dans ces deux contextes.

Dans la première partie, nous étudions un problème de classification des variétés polarisées. Pour la positivité des classes anti-canoniques, nous supposons que les variétés ont une nefvalue élevée, ou en d'autres termes, leurs classes anti-canoniques sont assez positives. Nous donnons une liste complète des classes d'isomorphisme des variétés polarisées normales avec une nefvalue élevée Cela généralise le travail classique sur le cas lisse de Fujita, Beltramitti et Sommese. En conséquence, nous obtenons que les variétés polarisées avec des singularités slc et une nefvalue élevée sont birationnellement équivalentes à des fibrés projectifs sur des courbes nodales.

Dans la deuxième partie, nous considérons une classe spécifique d'espaces singuliers, à savoir les orbifoldes. Une orbifolde a des singularités quotients. Par conséquence, nous avons des singularités mieux contrôlées dans ce contexte par rapport à celles considérées dans la première partie. Nous supposons également que ces orbifoldes sont kähleriennes compactes avec des classes anti-canoniques nef au sens des orbifoldes. Nous étudierons la topologie de ces orbifoldes à travers leurs groupes fondamentaux orbifoldes. Dans cette partie, nous exploiterons pleinement l'hypothèse orbifolde en appliquant des résultats de géométrie différentielle et de la géométrie métrique sur orbifolds. Nous montrerons qu'une orbifolde kählerienne compacte dont la classe anti-canonique est nef a un groupe fondamental orbifolde virtuellement nilpotent

Title : Two results on the classification of singular spaces with semi-negative canonical class

Keywords : polarized varieties, slc singularities, birational geometry, orbifolds, Margulis lemma, fundamental groups

Abstract : The subject of this thesis is to study the classification problem for singular spaces under two different assumptions on the positivity of the anti-canonical class of the spaces and their singularities in these two different setups . We will apply quite different methods for these two assumptions.

In the first part, we study the classification problem for polarized varieties. For the positivity of the anti-canonical classes, we assume that the varieties have high nefvalue, or in other words, their anti-canonical classes are quite positive. We give a complete list of isomorphism classes for normal polarized varieties with high nefvalue. This generalizes classical work on the smooth case by Fujita, Beltrametti and Sommese. As a consequence we obtain that polarized varieties with slc singularities and high nefvalue, are birationally equivalent to projective bundles over nodal curves.

In the second part, we consider a specific class of singular spaces, namely the orbifolds. An orbifold has quotient singularities. Hence we have milder singularities in this context compared to those considered in first part. We also assume that these orbifolds are compact Kähler with nef anti-canonical classes in the orbifold sense. We will study the topology of these orbifolds by characterizing their orbifold fundamental groups. In this part, we will fully exploit the orbifold assumption by applying results from differential geometry and metric geometry on orbifolds. We will show that a compact Kähler orbifold with nef anti-canonical class has virtually nilpotent orbifold fundamental group.