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Contributions to the linear programming approach for mean field games and its applications to electricity markets

Marcos Leutscher de las Nieves

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Marcos Leutscher de las Nieves. Contributions to the linear programming approach for mean field games and its applications to electricity markets. Optimization and Control [math.OC]. Institut Polytechnique de Paris, 2022. English. NNT : 2022IPPAG010 . tel-03921273

HAL Id: tel-03921273

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INSTITUT
POLYTECHNIQUE
DE PARIS

NNT : 2022IPPAG010

Thèse de doctorat



Contributions to the linear programming approach for mean field games and its applications to electricity markets

Thèse de doctorat de l'Institut Polytechnique de Paris
préparée à l'École nationale de la statistique et de l'administration économique

École doctorale n°574 École doctorale de mathématiques Hadamard (EDMH)
Spécialité de doctorat : Mathématiques appliquées

Thèse présentée et soutenue à Palaiseau, le 8 décembre 2022, par

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Acknowledgements

This thesis is the result of several years of academic training and would not have been possible without the contribution of many people, to whom I would now like to address a few words of gratitude.

I would like to start by thanking my thesis supervisors Roxana Dumitrescu and Peter Tankov. You have introduced and guided me into the wonderful world of Mean Field Games, I could not be more grateful for this choice of topic. Your help through your mathematical rigor, insightful ideas and encouragement has been crucial to my training in academic research. I will be eternally grateful for your trust during this period.

I address my thanks to Rainer Buckdahn and Markus Fischer for accepting to be the referees of my thesis. I am indebted to them for the time they took to carefully read the thesis and review it. Many thanks to Clémence Alasseur, Pierre Cardaliaguet and Nizar Touzi for accepting to be the examiners of my thesis.

I have had the privilege of writing my thesis at the CREST Finance-Insurance group and I could not imagine a better environment. I have been surrounded by great researchers and doctoral students who have undoubtedly contributed to my research interests. Many of them have become close friends and I have been able to rely on them at all times.

I am very grateful to all my professors in Paris and Alicante who have guided me during my academic training. In particular, I want to thank Eric Dubon for encouraging me to study mathematics even if I was not pursuing a scientific *baccalauréat* at the time.

Finally, I would like to address a few words of thanks to my family and friends. My friends from Alicante and Paris have been always supportive and for that I am thankful to them. I think of my family who has allowed me to be the person I am today. Without the continuous support of my parents, my brother and my grandparents, this thesis would not have been possible. Thank you Léa for your patience and your continuous encouragement.

During this doctoral stage, I especially think of my grandmother Alida who taught me to be a brave and strong person. I believe that a beautiful way to immortalize someone is to thank them for their unconditional love through a text that will last throughout time.

Abstract

This thesis presents three main contributions related to the linear programming approach for mean field games (MFGs).

The first part of the thesis is concerned with the theoretical aspects of MFGs allowing simultaneously for optimal stopping, stochastic control and absorption. Using the linear programming formulation for this type of MFGs, a general existence result for MFG Nash equilibria is derived under mild assumptions by means of Kakutani-Fan-Glicksberg's fixed point theorem. This relaxation method is shown to be equivalent to the controlled/stopped martingale approach for MFGs, another relaxation method used in earlier papers in the pure control case. Furthermore, under appropriate conditions, we show that our notion of solution satisfies a partial differential equation (PDE) system, allowing to compare our results with the PDE literature.

The second part focuses on a numerical algorithm for approximating the MFG Nash equilibrium taking advantage of the linear programming approach. The convergence of this algorithm is shown for two classes of MFG, MFGs with optimal stopping and absorption, and MFGs with stochastic control and absorption. The numerical scheme belongs to the class of learning procedures. In particular, we apply the Fictitious Play algorithm where the best response at each iteration is computed by solving a linear programming problem.

The last part of the thesis deals with an application of MFGs to the long term dynamics of the electricity industry. Different macroeconomic and climate policy scenarios are possible for the coming years, and the exact scenario remains uncertain. Therefore, conventional or renewable producers aiming to exit or enter the market, respectively, are facing uncertainty about the future carbon price and climate policies. Both classes of producers interact through the electricity market price. Nash equilibrium strategies over stopping times are considered and the problem is analyzed through a MFG model. To this end, we develop the linear programming approach for MFGs of optimal stopping with common noise and partial information in discrete time. We show the existence of an MFG Nash equilibrium and the uniqueness of the equilibrium market price. Finally, we extend the numerical algorithm developed in the second part of the thesis to illustrate the model with an empirical example inspired by the UK electricity market.

Keywords: Mean field games, optimal stopping, stochastic control, absorption, infinite-dimensional linear programming, fictitious play, electricity market, energy transition, scenario uncertainty, common noise.

Résumé

Cette thèse présente trois contributions principales liées à l'approche de programmation linéaire pour les jeux à champ moyen (MFGs).

La première partie de la thèse traite les aspects théoriques des MFGs permettant simultanément arrêt optimal, contrôle stochastique et absorption. En utilisant la formulation de programmation linéaire pour ce type de MFGs, un résultat général d'existence pour les équilibres de Nash MFG est dérivé sous des hypothèses faibles à travers du théorème de point fixe de Kakutani-Fan-Glicksberg. Nous montrons que cette méthode de relaxation est équivalente à l'approche par martingales contrôlées/arrêtées pour les MFG, une autre méthode de relaxation utilisée dans des articles précédents dans le cas du contrôle. De plus, sous des conditions appropriées, nous montrons que notre notion de solution satisfait un système d'équations différentielles partielles (EDP), ce qui permet de comparer nos résultats avec la littérature sur les EDP.

La deuxième partie se concentre sur un algorithme numérique pour l'approximation de l'équilibre de Nash MFG en tirant profit de l'approche par programmation linéaire. La convergence de cet algorithme est démontrée pour deux classes de MFG, les MFG avec arrêt optimal et absorption, et les MFG avec contrôle stochastique et absorption. Le schéma numérique appartient à la classe des procédures d'apprentissage. En particulier, nous appliquons l'algorithme Fictitious Play où la meilleure réponse à chaque itération est calculée en résolvant un problème de programmation linéaire.

La dernière partie de la thèse porte sur une application des MFGs à la dynamique long terme de l'industrie de l'électricité. Différents scénarios macroéconomiques et de politique climatique sont possibles pour les années à venir, or le scénario exact reste incertain. Par conséquent, les producteurs conventionnels ou renouvelables visant à sortir du marché ou à y entrer, respectivement, sont confrontés à l'incertitude concernant le prix du carbone et les politiques climatiques à venir. Les deux classes de producteurs interagissent par le biais du prix de l'électricité. Des stratégies d'équilibre de Nash sur des temps d'arrêt sont considérées et le problème est analysé à travers d'un modèle MFG. À cette fin, nous développons l'approche de programmation linéaire pour les MFG d'arrêt optimal avec bruit commun et information partielle en temps discret. Nous montrons l'existence d'un équilibre de Nash MFG et l'unicité du prix de marché en équilibre. Enfin, nous étendons l'algorithme numérique développé dans la deuxième partie de la thèse pour illustrer le modèle avec un exemple empirique inspiré du marché de l'électricité britannique.

Mots clés : Jeux à champ moyen, arrêt optimal, contrôle stochastique, absorption, programmation linéaire en dimension infinie, fictitious play, marché de l'électricité, transition énergétique, incertitude du scénario, bruit commun.

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Chapter 1

Introduction

Abstract

We present the main problems that we intend to study throughout the thesis. In the first part we introduce the stochastic control and optimal stopping mean field game problems separately. In the second part we introduce some general notations and concepts which will be used throughout the thesis. In particular, we give a first overview of the linear programming formulation, which is the core of the thesis. In a third section, we describe the outline of the thesis with emphasis on the associated literature and our contributions. Finally, we propose some general perspectives related to the thesis.

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1 Introduction to Mean Field Games

In this section we present a general setting for stochastic differential mean field games without common noise in the case of stochastic control and in the case of optimal stopping, separately. We use here the term *stochastic differential* since the expression mean field games encompasses many models which are not necessarily stochastic or in continuous time. However, since most of the literature on mean field games deals with the stochastic case in continuous time, we will use this setting to present the generalities of the model. In the following we will discuss the mathematical concepts underlying these models. We will not impose sufficient conditions for the well-posedness of the problems since we aim to explain only the intuition behind.

Mean field games (MFGs) were introduced around 2006 by Lasry and Lions [LL06a; LL06b; LL07] and simultaneously, by Huang, Caines and Malhamé [HMC06; HCM07] under the name of Nash Certainty Equivalence principle. In the recent years MFGs have been extensively studied theoretically and have been applied to different domains. MFGs represent the *limit version* of certain type of *symmetric games* as the number of players goes to infinity. Classical game theory for non cooperative games deals with settings of multiple players aiming to maximize a specific gain functional. Since the optimizations are coupled and solved simultaneously, the well known concept of Nash equilibrium arises. Mathematically, Nash equilibria are often characterized by systems of coupled equations where the number of equations equals the number of players. When the number of players is large, in general, there are no analytical solutions or numerical methods to solve these problems. The main idea of MFGs is to exploit the symmetry of the game and the smoothing effect of large numbers, leading to mathematical simplifications and numerical methods.

The mathematical theory surrounding these models has generally concentrated on the following three areas: existence and uniqueness of mean field solutions, limit theory relating the finite player game with the MFG and numerical methods for approximating mean field solutions. For the reader interested in applications of MFGs in economics and finance we refer to the recent survey [Car20].

This thesis will focus on several aspects of MFGs. The first one is the existence theory for MFGs of optimal stopping (with or without stochastic control), which in comparison to the stochastic control case was not developed in full generality. One of the main difficulties here is the time irregularity, which is a consequence of how the stopping times of the players act on the empirical distributions. The second aspect is concerned with the study of the linear programming approach for MFGs through the existence theory and the construction of a numerical scheme. Numerical algorithms for MFGs are crucial since explicit solutions are the exception rather than the rule. In particular we will extend the analysis of the linear programming formulation for MFGs initiated in [BDT20] and we will derive a numerical algorithm allowing to approximate the MFG solution. We will also provide a probabilistic representation of our linear programming constraints, which is important for comparing with other existent methods, providing an interpretation of our constraints and allowing to derive crucial estimates for the existence theory. Another key component in our analysis is that we allow for MFGs with ab-

sorption, meaning that the players exit the game when their state process exits some domain. This is not only essential for many applications but also important for numerical reasons since we are constrained to solve the problem in a bounded domain. Finally, we will illustrate how the linear programming approach applies to a concrete example concerning the electricity industry undergoing a major energy transition due to climate change. In this application we will extend our approach to discrete time MFGs with common noise and partial information, showing that our methodology is flexible enough to adapt to new settings. In addition, we compute the solution numerically by solving a sequence of linear programming problems for which nowadays fast solvers are available.

1.1 Stochastic Control

The N -player game. Let $N \in \mathbb{N}^*$ denote the number of players and $T > 0$ denote the time horizon of the game. Each player/agent $k \in \{1, \dots, N\}$ has a private state process $X^{k,N} = (X_t^{k,N})_{t \in [0, T]}$, whose dynamics are given by the stochastic differential equation (SDE)

$$dX_t^{k,N} = b\left(t, X_t^{k,N}, m_t^N, \alpha_t^k\right) dt + \sigma\left(t, X_t^{k,N}, m_t^N, \alpha_t^k\right) dW_t^k,$$

where W^1, \dots, W^N are independent Brownian motions, the initial conditions X_0^1, \dots, X_0^N are i.i.d. of law m_0^* and independent from the Brownian motions, $\alpha^k = (\alpha_t^k)_{t \in [0, T]}$ is the control process chosen by agent k , taking values in a closed subset $A \subset \mathbb{R}$, and m_t^N is the empirical occupation measure of the states of the players:

$$m_t^N(dx) = \frac{1}{N} \sum_{k=1}^N \delta_{X_t^{k,N}}(dx).$$

Each agent k wishes to choose an optimal control α^k to maximize the expected reward J^k which depends on the controls used by all players (only through the empirical measures):

$$J^k(\alpha^1, \dots, \alpha^N) := \mathbb{E} \left[\int_0^T f\left(t, X_t^{k,N}, m_t^N, \alpha_t^k\right) dt + g\left(X_T^{k,N}, m_T^N\right) \right].$$

The objective functionals and the dynamics of the agents are coupled through the flow of empirical measures $(m_t^N)_{t \in [0, T]}$, so that it is natural to look for a Nash equilibrium. We say that $\alpha = (\alpha^1, \dots, \alpha^N)$ is a Nash equilibrium if for all k and all $\bar{\alpha}$,

$$J^k(\bar{\alpha}, \alpha^{-k}) \leq J^k(\alpha),$$

where we use the usual notation $(\bar{\alpha}, \alpha^{-k}) := (\alpha^1, \dots, \alpha^{k-1}, \bar{\alpha}, \alpha^{k+1}, \dots, \alpha^N)$. In an *equilibrium situation*, each player has no incentive to *deviate unilaterally* from its control.

If N is large enough, each agent has small influence on the empirical measures and hence on the optimizations of the other players. Moreover, by the symmetry of the problem we can assume that a Nash equilibrium writes in the form $((\alpha^*(t, X_t^{1,N}, m_t^N))_t, \dots, (\alpha^*(t, X_t^{N,N}, m_t^N))_t)$ (the assumption that the map α^* does not depend on N is a strong assumption, but due

to weakness of the coupling this becomes a quite reasonable heuristic argument). When the number of players N goes to infinity and α^* is regular enough, we can expect by a *propagation of chaos* type result that the SDEs converge (in distribution) to the solution of the corresponding McKean–Vlasov SDE associated to α^* ,

$$dX_t^* = b(t, X_t^*, \mathcal{L}(X_t^*), \alpha^*(t, X_t^*, \mathcal{L}(X_t^*))) dt + \sigma(t, X_t^*, \mathcal{L}(X_t^*), \alpha^*(t, X_t^*, \mathcal{L}(X_t^*))) dW_t,$$

and that the empirical occupation measures converge to the *deterministic flow of probability measures* $(m_t^*)_{t \in [0, T]} = (\mathcal{L}(X_t^*))_{t \in [0, T]}$ associated to the solution of the McKean–Vlasov SDE. Here we used the notation $\mathcal{L}(\cdot)$ for the law of a random variable. In the limit, the problems become decoupled and it is enough to consider the case of a representative agent. Finally, if the representative player deviates from its control α^* , this change will not affect the limiting measures since the weight associated to a single agent is $1/N$. As a consequence, the Nash equilibrium property translates as fixing the measure flow that matches with α^* and verifying the optimality property of α^* among all the controls. This last point is the key difference between MFGs and the *control of McKean–Vlasov systems* (also called *Mean Field Control problem*) where the measures are not fixed but vary with the control (global planner problem).

MFG problem. The MFG problem reads as follows:

- (i) Fix a flow of probability measures $m = (m_t)_{t \in [0, T]}$ and find the solution to the stochastic control problem

$$\begin{aligned} \sup_{\alpha} \quad & \mathbb{E} \left[\int_0^T f(t, X_t^{\alpha, m}, m_t, \alpha_t) dt + g(X_T^{\alpha, m}, m_T) \right], \\ \text{s.t.} \quad & dX_t^{\alpha, m} = b(t, X_t^{\alpha, m}, m_t, \alpha_t) dt + \sigma(t, X_t^{\alpha, m}, m_t, \alpha_t) dW_t, \\ & X_0^{\alpha, m} \sim m_0^*. \end{aligned} \tag{1.1}$$

- (ii) Given a solution α^m of the problem (1.1) for the agent facing a mean-field m , find m such that $m_t = \mathcal{L}(X_t^{\alpha^m, m})$ for all $t \in [0, T]$.

If one can find such a flow of probability measures m , then it will be called a *mean field solution* (or *MFG equilibrium*). A practical way to see the problem is by defining a set-valued map

$$\Phi(m) := \left\{ (\mathcal{L}(X_t^{\alpha^m, m}))_{t \in [0, T]} : \alpha^m \text{ optimal for } m \right\}.$$

Then m is a mean field solution if and only if it is a fixed point of Φ in the sense of set-valued mappings, i.e. $m \in \Phi(m)$. The map Φ is usually called the *best response mapping*.

1.2 Optimal Stopping

One of the principal contributions of this thesis is the study of *MFGs of Optimal Stopping* (also known as *MFGs of Timing*). These models arise in the context of games between agents who decide when to exit (or enter) the game by choosing a stopping time.

The N -player game. As before, let $N \in \mathbb{N}^*$ denote the number of players and $T > 0$ denote the time horizon of the game. In the N -player game of optimal stopping, agent $k \in \{1, \dots, N\}$ chooses a stopping time τ^k valued in $[0, T]$, at which to exit the game and has a state process given by the solution of the SDE

$$dX_t^{k,N} = b\left(t, X_t^{k,N}, m_t^N\right) dt + \sigma\left(t, X_t^{k,N}, m_t^N\right) dW_t^k,$$

where W^1, \dots, W^N are independent Brownian motions, the initial conditions X_0^1, \dots, X_0^N are i.i.d. of law m_0^* and independent from the Brownian motions and m_t^N is the empirical occupation measure of the states of the players *still in the game*:

$$m_t^N(dx) = \frac{1}{N} \sum_{k=1}^N \delta_{X_t^{k,N}}(dx) \mathbb{1}_{t < \tau^k}.$$

Each agent k wishes to choose an optimal stopping time τ^k to maximize the expected reward J^k which depends on the stopping times chosen by all players (only through the empirical measures):

$$J^k(\tau^1, \dots, \tau^N) := \mathbb{E} \left[\int_0^{\tau^k} f\left(t, X_t^{k,N}, m_t^N\right) dt + g\left(\tau^k, X_{\tau^k}^{k,N}, \mu^N\right) \right],$$

where μ^N is the empirical joint distribution of the stopping times and the state processes at the time of stopping:

$$\mu^N(dt, dx) = \frac{1}{N} \sum_{k=1}^N \delta_{(\tau^k, X_{\tau^k}^{k,N})}(dt, dx).$$

Since the problems are coupled through the empirical measures, we will look for Nash equilibria. We say that $\boldsymbol{\tau} = (\tau^1, \dots, \tau^N)$ is a Nash equilibrium if for all k and all $\bar{\tau}$,

$$J^k(\bar{\tau}, \boldsymbol{\tau}^{-k}) \leq J^k(\boldsymbol{\tau}).$$

We remark that even if some of the players exit the game, the players still in the game will have a weight of $1/N$, instead of dividing by the size of this subclass of players. As a consequence, in MFGs of optimal stopping the empirical measure m_t^N is not a probability measure but a subprobability measure. We note also that because of the indicator function, the empirical measure m_t^N is irregular as a function of time, which is in contrast with the usual MFGs of stochastic control presented previously. As we will show later, this time irregularity adds an important difficulty when showing existence of equilibria or developing a numerical algorithm.

Since the game is symmetric, we can expect as before that a Nash equilibrium strategy is given by symmetric strategies and that the problem simplifies in the limit. We present directly the limit problem.

MFG problem. The MFG problem for optimal stopping will be the following:

- (i) Fix a flow of subprobability measures $m = (m_t)_{t \in [0, T]}$ and a probability measure μ , and

find the solution to the optimal stopping problem

$$\begin{aligned} & \sup_{\tau} \mathbb{E} \left[\int_0^{\tau} f(t, X_t^m, m_t) dt + g(\tau, X_{\tau}^m, \mu) \right], \\ \text{s.t.} \quad & dX_t^m = b(t, X_t^m, m_t) dt + \sigma(t, X_t^m, m_t) dW_t, \\ & X_0^m \sim m_0^*. \end{aligned} \tag{1.2}$$

- (ii) Given a solution $\tau^{m,\mu}$ of the problem (1.2) for the agent facing a mean-field (m, μ) , find m such that $m_t = \mathbb{P}(X_t^m \in \cdot, t < \tau^{m,\mu})$ for all $t \in [0, T]$ and $\mu = \mathcal{L}(\tau^{m,\mu}, X_{\tau^{m,\mu}}^m)$.

If (m, μ) verify these properties, then the tuple will be called a mean field solution. As in the stochastic control case we can characterize a mean field solution as a fixed point of a set-valued mapping.

2 Notation and Preliminaries

2.1 General notation

For a given set \mathcal{X} we denote by $2^{\mathcal{X}}$ the power set of \mathcal{X} . If we are given another set \mathcal{Y} we denote by $\mathcal{Y}^{\mathcal{X}}$ the set of functions from \mathcal{X} to \mathcal{Y} . We denote by \mathbb{N} the set of nonnegative integers and by \mathbb{N}^* the set $\mathbb{N} \setminus \{0\}$. The set \mathbb{R}^+ denotes the interval $]0, \infty[$ and the set \mathbb{R}_+ denotes the interval $[0, \infty[$. For any two real numbers x and y we use the following notation: $x \wedge y := \min(x, y)$, $x \vee y := \max(x, y)$, $x^+ := x \vee 0$ and $x^- := (-x) \vee 0$. If we are given a set \mathcal{O} in \mathbb{R}^d , we denote the interior, the closure and the boundary respectively by $\text{int}(\mathcal{O})$, $\bar{\mathcal{O}}$ and $\partial\mathcal{O}$.

2.2 Notations for probability theory

Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a sub-sigma-algebra \mathcal{G} , we define the \mathbb{P} -null sets in \mathcal{G} as

$$\mathcal{N}_{\mathbb{P}}(\mathcal{G}) := \{B \subset \Omega : B \subset C \text{ for some } C \in \mathcal{G} \text{ with } \mathbb{P}(C) = 0\}.$$

We call a complete probability space, a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that $\mathcal{N}_{\mathbb{P}}(\mathcal{F}) \subset \mathcal{F}$. Any probability space $(\Omega, \mathcal{F}, \mathbb{P})$ can be completed by the following procedure: we set

$$\mathcal{F}^{\mathbb{P}} := \mathcal{F} \vee \mathcal{N}_{\mathbb{P}}(\mathcal{F}) = \{B \cup C : B \in \mathcal{F}, C \in \mathcal{N}_{\mathbb{P}}(\mathcal{F})\},$$

$$\bar{\mathbb{P}}(B \cup C) := \mathbb{P}(B), \quad B \in \mathcal{F}, C \in \mathcal{N}_{\mathbb{P}}(\mathcal{F}),$$

then $(\Omega, \mathcal{F}^{\mathbb{P}}, \bar{\mathbb{P}})$ is a complete probability space. With some abuse of notation, we will still denote the extended probability measure $\bar{\mathbb{P}}$ by \mathbb{P} . We recall the following two properties of $\mathcal{F}^{\mathbb{P}}$:

- (i) $B \in \mathcal{F}^{\mathbb{P}}$ if and only if there exist $B_1, B_2 \in \mathcal{F}$ such that $B_1 \subset B \subset B_2$ and $\mathbb{P}(B_2 \setminus B_1) = 0$.
- (ii) Let (E, \mathcal{E}) be a measurable space. $X : \Omega \rightarrow E$ is $\mathcal{F}^{\mathbb{P}}$ -measurable if and only if there exists $Y : \Omega \rightarrow E$ \mathcal{F} -measurable such that $X = Y$ a.s.

Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a random variable X taking values in a measurable space, we denote by $\mathbb{P} \circ X^{-1}$ or $\mathcal{L}(X)$ the law of X .

2.3 Spaces of measures and functions

Consider a topological space E . We denote by $\mathcal{B}(E)$ the Borel σ -algebra on E . Define the set $\mathcal{M}(E)$ of finite signed measures on $(E, \mathcal{B}(E))$, with subsets, $\mathcal{M}^+(E)$ of finite nonnegative measures, $\mathcal{P}^{sub}(E)$ of subprobability measures and $\mathcal{P}(E)$ of probability measures. We denote by $M(E)$ the set of Borel measurable functions from E to \mathbb{R} , by $M_b(E)$ the subset of Borel measurable and bounded functions, by $C(E)$ the subset of continuous functions, and by $C_b(E)$ the subset of continuous and bounded functions. The set $M_b(E)$ is endowed with the supremum norm $\|\varphi\|_\infty = \sup_{x \in E} |\varphi(x)|$.

We assume now that (E, d) is a complete and separable metric space. For any measurable function $w : E \rightarrow [1, \infty[$, called the weight function, define the real vector space $\mathcal{M}_w(E)$ consisting on all finite signed measures μ on $(E, \mathcal{B}(E))$ such that

$$\int_E w(x) |\mu|(dx) < \infty,$$

where $|\mu|$ is the variation of μ . Now, let $C_w(E)$ be the real vector space consisting of continuous functions $\varphi : E \rightarrow \mathbb{R}$ satisfying

$$\sup_{x \in E} \frac{|\varphi(x)|}{w(x)} < \infty.$$

Define the bilinear form

$$\langle \mu, \varphi \rangle := \int_E \varphi(x) \mu(dx), \quad \mu \in \mathcal{M}_w(E), \varphi \in C_w(E).$$

It can be shown that $(\mathcal{M}_w(E), C_w(E), \langle \cdot, \cdot \rangle)$ is a dual pair, i.e.

$$\forall \mu \in \mathcal{M}_w(E) \setminus \{0\}, \quad \exists \varphi \in C_w(E) : \quad \langle \mu, \varphi \rangle \neq 0,$$

$$\forall \varphi \in C_w(E) \setminus \{0\}, \quad \exists \mu \in \mathcal{M}_w(E) : \quad \langle \mu, \varphi \rangle \neq 0.$$

On $\mathcal{M}_w(E)$, we define the weak topology $\tau_w := \sigma(\mathcal{M}_w(E), C_w(E))$ as the coarsest topology on $\mathcal{M}_w(E)$ making continuous the functions $\{\langle \cdot, \varphi \rangle : \varphi \in C_w(E)\}$. In particular, $(\mathcal{M}_w(E), \tau_w)$ is a Hausdorff locally convex topological vector space. Define the subsets $\mathcal{M}_w^+(E)$, $\mathcal{P}_w^{sub}(E)$ and $\mathcal{P}_w(E)$ of $\mathcal{M}_w(E)$ to be the intersections of $\mathcal{M}_w(E)$ with $\mathcal{M}^+(E)$, $\mathcal{P}^{sub}(E)$ and $\mathcal{P}(E)$, respectively.

When $w = 1 + d(\cdot, x_0)^p$ for some $p \geq 1$ and some reference point $x_0 \in E$, we denote $\tau_p := \tau_w$ (which does not depend on x_0). The topology τ , defined by setting $w = 1$, will denote the topology of weak convergence of measures (even if we use the same notation for stopping times, there will be no confusion due to the context). We will use sometimes the standard notation $\mu_n \rightarrow \mu$ to say that $(\mu_n)_{n \geq 1}$ converges in τ to μ . If $p \geq 1$, we denote $\mathcal{M}_p(E) := \mathcal{M}_w(E)$, $\mathcal{M}_p^+(E) := \mathcal{M}_w^+(E)$, $\mathcal{P}_p^{sub}(E) := \mathcal{P}_w^{sub}(E)$, $\mathcal{P}_p(E) := \mathcal{P}_w(E)$ and $C_p(E) := C_w(E)$.

In the thesis we will also work with measures on product spaces for which the marginal measure on the first space is absolutely continuous with respect to the Lebesgue measure. In particular, we expect this type of measures to converge for weak topologies that do not impose

continuity of the test functions with respect to the first argument. This type of topology was studied in [JM81] for bounded test functions and by [Lac15] in the case of probability measures admitting test functions with polynomial growth. Since we will work with unbounded functions and subprobability measures, we give the definitions for general weight functions and general signed measures. Consider two complete and separable metric spaces (E, d_E) and (F, d_F) and let w be a weight function on $E \times F$. Consider the following set of functions:

$$M_{mc,w}(E \times F) = \left\{ \phi \in M(E \times F) : \forall x \in E, \phi(x, \cdot) \in C(F); \sup_{(x,y) \in E \times F} \frac{|\phi(x,y)|}{w(x,y)} < \infty \right\}.$$

The subscript ‘‘mc’’ refers to measurable-continuous, and in fact this set of functions belongs to the class of *Carathéodory functions*, also known as *Carathéodory integrands* (see Definition 4.50 in [AB07]). Carathéodory functions are used in Calculus of Variations together with *Young measures* to relax control problems (see e.g. [CFV04]). This leads to the relaxed control methods used in the stochastic control case, which we will also use in this thesis. With the same type of bilinear form as before

$$\langle \mu, \varphi \rangle := \int_{E \times F} \varphi(x, y) \mu(dx, dy), \quad \mu \in \mathcal{M}_w(E \times F), \varphi \in M_{mc,w}(E \times F),$$

the tuple $(\mathcal{M}_w(E \times F), M_{mc,w}(E \times F), \langle \cdot, \cdot \rangle)$ forms a dual pair. On $\mathcal{M}_w(E \times F)$, we define the weak topology $\bar{\tau}_w := \sigma(\mathcal{M}_w(E \times F), M_{mc,w}(E \times F))$ as the coarsest topology on $\mathcal{M}_w(E \times F)$ making continuous the functions $\{\langle \cdot, \varphi \rangle : \varphi \in M_{mc,w}(E \times F)\}$. Again, $(\mathcal{M}_w(E \times F), \bar{\tau}_w)$ is a Hausdorff locally convex topological vector space. The topology $\bar{\tau}_w$ will be called the topology of stable convergence with weight w .

Consider now some $p \geq 1$ and endow the product space $E \times F$ with the metric

$$d((x, y), (x', y')) = (d_E(x, x')^p + d_F(y, y')^p)^{1/p}, \quad (x, y), (x', y') \in E \times F.$$

Let $(x_0, y_0) \in E \times F$ be an arbitrary reference point and define the weight function $w = 1 + d(\cdot, (x_0, y_0))^p$. We denote $\bar{\tau}_p := \bar{\tau}_w$ (which does not depend on (x_0, y_0)) the topology of stable convergence with p -growth and $M_{mc,p}(E \times F) := M_{mc,w}(E \times F)$. When $w = 1$ we denote $\bar{\tau} := \bar{\tau}_w$ the usual topology of stable convergence and $M_{mc,b}(E \times F) := M_{mc,w}(E \times F)$.

We recall now the disintegration theorem for nonnegative measures (which is deduced via a standard normalization procedure from the disintegration theorem for probability measures, see Theorem 1.1 in [CD18b]).

Theorem 2.1 (*Disintegration Theorem*). *Let (E, \mathcal{E}) be a measurable space and let F be a Polish space. If μ is a finite nonnegative measure on $(E \times F, \mathcal{E} \otimes \mathcal{B}(F))$, then there exists a family of kernels $(\nu(x, C))_{x \in E, C \in \mathcal{B}(F)}$ such that:*

- (1) *For any $x \in E$, the mapping $\mathcal{B}(F) \ni C \mapsto \nu(x, C)$ is a probability measure on $(F, \mathcal{B}(F))$.*
- (2) *For any $C \in \mathcal{B}(F)$, the mapping $E \ni x \mapsto \nu(x, C)$ is measurable from (E, \mathcal{E}) into $([0, 1], \mathcal{B}([0, 1]))$.*

(3) For any $B \times C \in \mathcal{E} \otimes \mathcal{B}(F)$,

$$\mu(B \times C) = \int_B \nu(x, C) \mu^E(dx),$$

where $\mu^E := \mu(\cdot \times F)$.

2.4 Introduction to the linear programming formulation

The main goal of this thesis will be to prove the existence of mean field solutions for different settings under general assumptions and develop numerical methods to approximate them. The linear programming formulation provides a powerful tool for showing existence results under weak assumptions and lends itself well to numerical implementation. We are going to present formally the linear programming (LP) methodology in the case of a single agent optimal stopping problem and then show that it extends in a very natural way to the MFG case. The control case or mixed control/stopping case will be presented in the respective chapters of the thesis.

Consider the following optimal stopping problem for a single agent:

$$\begin{aligned} \sup_{\tau} \quad & \mathbb{E} \left[\int_0^{\tau} f(t, X_t) dt + g(\tau, X_{\tau}) \right], \\ \text{s.t.} \quad & dX_t = b(t, X_t) dt + \sigma(t, X_t) dW_t, \\ & X_0 \sim m_0^*, \end{aligned} \tag{1.3}$$

where W is a standard Brownian motion independent of X_0 .

Define the flow of subprobability measures

$$m_t(B) := \mathbb{P}(X_t \in B, t < \tau), \quad B \in \mathcal{B}(\mathbb{R}), \quad t \in [0, T],$$

and the probability measure $\mu := \mathbb{P} \circ (\tau, X_{\tau})^{-1}$. Applying Itô's formula between time 0 and time τ and taking the expectation, we get for any $u \in C_b^{1,2}([0, T] \times \mathbb{R})$

$$\mathbb{E}[u(\tau, X_{\tau})] = \mathbb{E}[u(0, X_0)] + \mathbb{E} \left[\int_0^{\tau} (\partial_t u + \mathcal{L}u)(t, X_t) dt \right]$$

where

$$\mathcal{L}u(t, x) = b(t, x) \partial_x u(t, x) + \frac{\sigma^2}{2}(t, x) \partial_{xx} u(t, x),$$

and $C_b^{1,2}([0, T] \times \mathbb{R})$ denotes the set of functions $u \in C_b([0, T] \times \mathbb{R})$ such that $\partial_t u, \partial_x u, \partial_{xx} u \in C_b([0, T] \times \mathbb{R})$. In particular (m, μ) satisfies

$$\langle \mu, u \rangle = \langle m_0^*, u(0, \cdot) \rangle + \langle m_t(dx) dt, \partial_t u + \mathcal{L}u \rangle, \quad u \in C_b^{1,2}([0, T] \times \mathbb{R}). \tag{1.4}$$

Moreover the objective function can be reformulated as

$$\mathbb{E} \left[\int_0^{\tau} f(t, X_t) dt + g(\tau, X_{\tau}) \right] = \langle m_t(dx) dt, f \rangle + \langle \mu, g \rangle.$$

The LP formulation consists in maximizing this quantity over (m, μ) satisfying (1.4), i.e.

$$\begin{aligned} & \sup_{(m, \mu)} \langle m_t(dx)dt, f \rangle + \langle \mu, g \rangle, \\ \text{s.t.} \quad & \langle \mu, u \rangle = \langle m_0^*, u(0, \cdot) \rangle + \langle m_t(dx)dt, \partial_t u + \mathcal{L}u \rangle, \quad u \in C_b^{1,2}([0, T] \times \mathbb{R}), \\ & m_t \in \mathcal{P}^{sub}(\mathbb{R}), \quad t \in [0, T], \\ & \mu \in \mathcal{P}(\mathbb{R}). \end{aligned} \tag{1.5}$$

We note that both the reformulation of the objective function and the derivation of the constraint (1.4), do not depend on the specific probability space or the Brownian motion taken to define the problem, but on the joint law of the state process and the stopping time.

The optimization problem (1.5) is a linear programming problem in infinite dimension (space of measures) since the objective function and the constraints are linear with respect to m and μ . Depending on the assumptions on the data m_0^* , b , σ , f and g , one can usually work with some weak topology for measures and show the compactness of the set of constraints and the continuity of the objective function, proving the existence of a maximizer. To approximate the solution of this problem, one can approximate the measures (m, μ) with measures on a discrete time-state space and use a discretization scheme for the derivatives of the test functions u . Moreover on a discrete time-state space the test functions are spanned by the indicator functions, and since the constraint is linear with respect to u , we obtain a finite set of constraints. Nowadays there are fast LP solvers available¹, allowing to solve the discretized LP problem in a very competitive time.

An important question arising in the linear programming formulation concerns the interpretation of the constraints (1.4). In fact, we know that for each admissible joint law of (X, τ) we can associate a couple (m, μ) satisfying (1.4), but, is the converse true? If it is not the case, it is possible that the problem (1.5) is too strongly relaxed and that the optimal stopping interpretation is lost. However, the converse is true under general conditions, meaning that from every admissible couple (m, μ) satisfying the LP constraint (1.4), we can recover a probabilistic set-up such that the measures are defined as occupation measures associated to the state process and the stopping time (see e.g. [CS02] in the time homogeneous case; the time dependent extension allowing for irregularity in time and unbounded coefficients is proved in this thesis). Apart from implying the equivalence between the optimization problems, having a probabilistic representation of the constraints is very useful for characterizing their regularity via the regularity of the SDE.

Illustration: Optimal exercise of an American put option. We illustrate the numerical methodology in the single agent case by solving numerically the LP problem for the optimal exercise of an American put option.

We place ourselves in the context of the Black-Scholes model for a single asset and define the initial price of the underlying asset $S_0 = 100$ (i.e. $m_0^* = \delta_{100}$), the maturity $T = 1$, the strike $K = 100$, the interest rate $r = 0.05$ and the volatility $\sigma = 0.3$. The buyer of the American put

1. see <http://plato.asu.edu/ftp/lpsimp.html> for a comparison of different LP solvers.

option wishes to find the optimal time to exercise the option, i.e. she solves

$$\begin{aligned} & \sup_{\tau \in \mathcal{T}} \mathbb{E}[e^{-r\tau}(K - S_\tau)^+], \\ \text{s.t.} \quad & dS_t = rS_t dt + \sigma S_t dW_t, \\ & S_0 = 100, \end{aligned} \tag{1.6}$$

where \mathcal{T} is the set of $[0, T]$ -valued stopping times with respect to the completed filtration of W . The LP formulation of the problem writes

$$\begin{aligned} & \sup_{(m, \mu)} \int_{[0, T] \times \mathbb{R}} e^{-rt}(K - x)^+ \mu(dt, dx), \\ \text{s.t.} \quad & \int_{[0, T] \times \mathbb{R}} u(t, x) \mu(dt, dx) = u(0, 100) + \int_0^T \int_{\mathbb{R}} (\partial_t u + \mathcal{L}u)(t, x) m_t(dx) dt, \\ & u \in C_b^{1,2}([0, T] \times \mathbb{R}), \\ & m_t \in \mathcal{P}^{sub}(\mathbb{R}), \quad t \in [0, T], \\ & \mu \in \mathcal{P}(\mathbb{R}). \end{aligned} \tag{1.7}$$

Computing the associated discretized solution, we obtain an approximation of the optimal solution (m, μ) , illustrated in Figure 1.1. More precisely, in the left graph we plot the optimal flow of subprobability measures m so that the color intensity at some time t and price s represents the mass of the discretized subprobability measure m_t at the point s . On the right graph we show the optimal μ which is represented by the mass of the discretized probability measure on the time-state space. To compare our results with the classical approach we plot in Figure 1.2 the Optimal Stopping Boundary (OSB) for the American Put, computed using the Cox-Ross-Rubinstein (CRR) binomial tree. We verify numerically that the probability measure μ is supported on the OSB.

Formally speaking, if we take a smooth function φ supported on the continuation set and evaluate the constraint (1.4) with such a test function, we see that

$$\langle m_0^*, \varphi(0, \cdot) \rangle + \langle m_t(dx) dt, \partial_t \varphi + \mathcal{L}\varphi \rangle = 0,$$

since numerically $\mu = 0$ on the continuation region. This means that the Fokker-Planck equation is satisfied on the continuation region. However, when the OSB is reached, there is a loss of mass of the flow of measures and the equality in the above equation breaks down for test functions which are nonzero on the OSB. The equality is restored when we add the probability measure μ , which can be interpreted, in the linear programming jargon, as a *slack variable*.

Linear programming formulation for MFGs. The LP formulation for the single agent problem relaxes the optimization by working with the occupation measures, meaning that we work with laws instead of stopping times. When we defined the MFG problem of optimal stopping, we saw that a mean field solution writes exactly as the occupation measure of some stopping time. In particular, this means that the object of interest in MFGs is exactly the

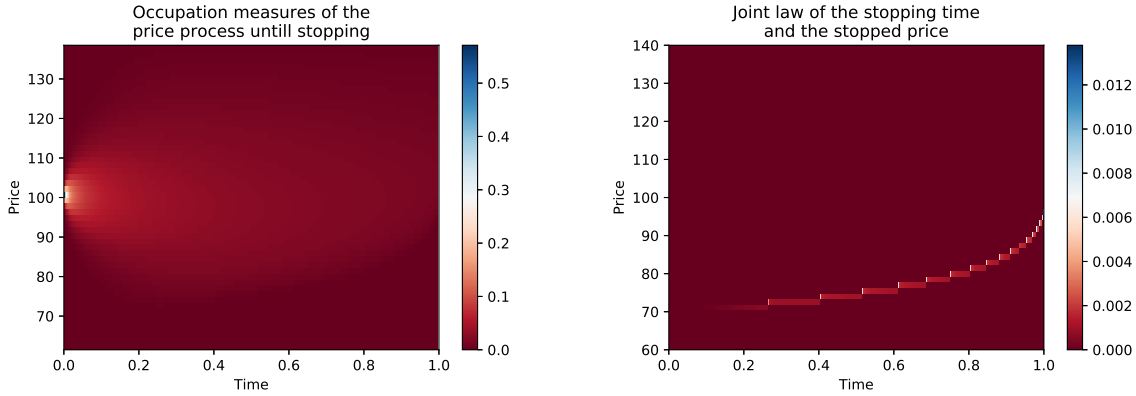


Figure 1.1 – Optimal couple of m (left graph) and μ (right graph).

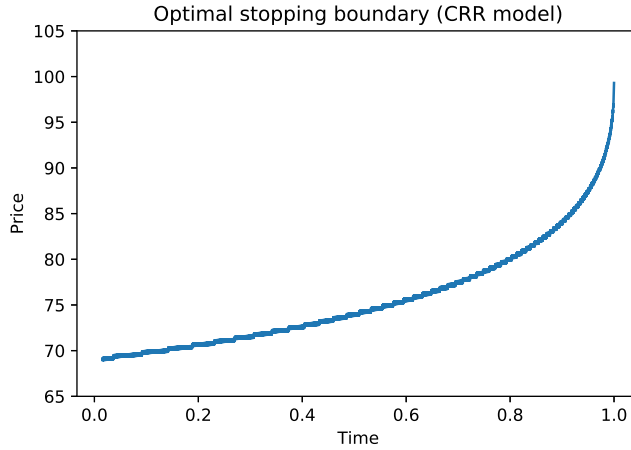


Figure 1.2 – Optimal Stopping Boundary for the American Put using a CRR model.

decision variable in the LP formulation and as a consequence the problem can be fully defined in terms of measures.

Assume that the mean-field interactions occur only through the reward functions f and g . A natural definition for the MFG Nash equilibrium in the LP formulation will be as follows: the couple (m^*, μ^*) is an LP MFG Nash equilibrium if it satisfies the constraint (1.4) and for every (m, μ) satisfying (1.4) we have

$$\langle m_t(dx)dt, f(\cdot, m^*) \rangle + \langle \mu, g(\cdot, \mu^*) \rangle \leq \langle m_t^*(dx)dt, f(\cdot, m^*) \rangle + \langle \mu^*, g(\cdot, \mu^*) \rangle.$$

Here the measures (m^*, μ^*) not only can be interpreted as relaxed stopping times but can be also understood as the distribution of the population of agents still in the game and the joint distribution of stopping times and stopped states of the agents. Furthermore, this approach captures the nature of the MFG since it models how distributions of states and stopping times respond to fixed distributions of the same kind. The definition is very simple and is not based on a value function. As a consequence, numerical methods can be built without computing the value function. For example, we will develop a fictitious play algorithm which consists in

solving a sequence of linear programming problems (in the same fashion as in the American put illustration) to approximate the LP MFG Nash equilibrium.

3 Outline of the thesis and literature review

This section provides a brief description of the main developments of the thesis as well as a detailed overview of the corresponding literature. Chapter 3 focuses on the theory of existence for MFGs in which players can control their dynamics and choose a time to exit the game. The existence theory is based on the linear programming approach for MFGs. Chapter 4 studies the convergence of the Fictitious Play algorithm in which the best responses are computed by solving a linear programming problem at each iteration. Chapter 5 presents an application of the linear programming approach for MFGs of optimal stopping with common noise and partial information to the modeling of energy transition in the electricity industry under scenario uncertainty arising from future climate policies.

3.1 Control and optimal stopping Mean Field Games: a linear programming approach (Chapter 3)

This chapter is based on the paper “Control and optimal stopping Mean Field Games: a linear programming approach” [DLT21], joint work with R. Dumitrescu and P. Tankov, *Electronic Journal of Probability*, 2021. Some modifications are made in view of the techniques developed in [DLT22], which allow to show existence results under weaker assumptions (unbounded coefficients and general nonlocal dependence on the mean-field terms).

Overview of the literature. In the MFG literature, there are two main approaches to prove the existence of a Nash equilibrium in the pure control case: the *PDE approach* and the *probabilistic approach*. The first method was developed initially by Lasry and Lions [LL06a; LL06b; LL07]. The idea behind the PDE approach is that for a fixed flow of probability measures $m = (m_t)_{t \in [0, T]}$ representing the distribution of the players, the stochastic control problem (1.1) faced by the representative agent is characterized by the Hamilton-Jacobi-Bellman (HJB) equation satisfied by the value function

$$\begin{cases} -\partial_t v(t, x) - \sup_{a \in A} [\mathcal{L}[m]v(t, x, a) + f(t, x, m_t, a)] = 0, & \text{on } [0, T) \times \mathbb{R}, \\ v(T, \cdot) = g(\cdot, m_T), \end{cases} \quad (1.8)$$

where the operator $\mathcal{L}[m]$ is defined for $\varphi \in C_b^{1,2}([0, T] \times \mathbb{R})$ by

$$\mathcal{L}[m]\varphi(t, x, a) := b(t, x, m_t, a)\partial_x \varphi(t, x) + \frac{\sigma^2}{2}(t, x, m_t, a)\partial_{xx} \varphi(t, x).$$

Now, assuming that there exists a unique smooth function $\alpha^* : [0, T] \times \mathbb{R} \rightarrow A$ such that for each $(t, x) \in [0, T] \times \mathbb{R}$, $\alpha^*(t, x)$ is the solution of the maximization problem

$$\sup_{a \in A} [\mathcal{L}[m]v(t, x, a) + f(t, x, m_t, a)],$$

we should have the consistency condition $m_t = \mathcal{L}(X_t^{\alpha^*, m})$ for all $t \in [0, T]$. Assuming that the distribution of the players $m_t(dx)dt$ admits a smooth density $(t, x) \mapsto m(t, x)$, this condition can be written as a Fokker-Planck-Kolmogorov equation for this density :

$$\begin{cases} \partial_t m(t, x) + \partial_x (b(t, x, m_t, \alpha^*(t, x))m(t, x)) \\ \quad - \frac{1}{2} \partial_{xx} (\sigma^2(t, x, m_t, \alpha^*(t, x))m(t, x)) = 0, \text{ on } (0, T] \times \mathbb{R}, \\ m(0, \cdot) = m_0^*, \end{cases} \quad (1.9)$$

where $x \mapsto m_0^*(x)$ is the density of the measure m_0^* describing the initial distribution of the players. Coupling the equations (1.8) and (1.9) we obtain the so called MFG PDE system. The main difficulty of these coupled PDEs comes from the opposite directions of time in the two equations. The assumptions required to solve this system are usually strong, e.g. $b(t, x, m, a) = a$, σ is constant and the running gain function f is separable, meaning that $f(t, x, m, a) = f_1(t, x, a) + f_2(t, x, m)$. Nevertheless, one can obtain more insights about the optimal control regularity and even work with local mean-field dependence (i.e. f depending on $m(t, x)$ instead of the whole measure m_t). We refer to [Car12; GS14; Lac18] for more details on the PDE approach and to [GLL11] for some applications using this approach.

The probabilistic approach was introduced by Carmona and Delarue [CD13a; CD13b; CD18a; CD18b] and is based on the Pontryagin stochastic maximum principle which reduces the problem to a system of coupled forward-backward stochastic differential equations (FBSDEs) of McKean-Vlasov type. More precisely, define the generalized Hamiltonian (assuming that σ does not depend on the control):

$$\mathcal{H}(t, x, m, y, z, a) := b(t, x, m, a)y + \sigma(t, x, m)z + f(t, x, m, a),$$

for $(t, x, m, y, z, a) \in [0, T] \times \mathbb{R} \times \mathcal{P}(\mathbb{R}) \times \mathbb{R} \times \mathbb{R} \times A$. Assume that there exists a unique smooth function $(t, x, m, y) \mapsto \hat{\alpha}(t, x, m, y)$ maximizing $a \mapsto \mathcal{H}(t, x, m, y, z, a)$. The probabilistic approach states that a sufficient condition to solve the MFG problem is to solve the following FBSDE of McKean-Vlasov type:

$$\begin{cases} dX_t = b(t, X_t, \mathcal{L}(X_t), \alpha_t^*)dt + \sigma(t, X_t, \mathcal{L}(X_t))dW_t, \\ dY_t = -\partial_x \mathcal{H}(t, X_t, \mathcal{L}(X_t), Y_t, Z_t, \alpha_t^*)dt + Z_t dW_t, \\ X_0 = \xi, \quad Y_T = \partial_x g(X_T, \mathcal{L}(X_T)), \quad \alpha_t^* = \hat{\alpha}(t, X_t, \mathcal{L}(X_t), Y_t), \end{cases} \quad (1.10)$$

where ξ is some \mathcal{F}_0 -measurable random variable with law m_0^* . More precisely, if (X, Y, Z) is a solution of (1.10), then $(\mathcal{L}(X_t))_{t \in [0, T]}$ is a mean field solution with associated optimal control α^* . Both in the PDE approach or in the probabilistic approach, Schauder's fixed point theorem (applied to the distribution of players) is usually used to solve the MFG problem.

Apart from these two main approaches, other methods can be used to study the existence of MFG solutions. In the paper [HMC06], the authors use a contraction fixed point argument to find a solution, however this method requires strong assumptions on the coefficients as well as defining the game over a short time interval. Another technique based on Tarski's fixed point theorem has been applied to solve supermodular MFGs (submodular in the case of a minimization) in the papers [AJ13; CDL17; Wię17; Dia+21; Dia+22]. Unlike Schauder's fixed point theorem, Tarski's fixed point theorem does not use topological arguments, but is based on lattice theory. Moreover, supermodular MFGs are characterized by the property of increasing differences (decreasing differences in the case of a minimization) on the reward functions and one can usually relax the continuity assumptions on the mean-field terms.

The methodology applied in this thesis belongs to the class of relaxation methods, also known as compactification methods. The main idea is to relax the optimization problem by allowing to optimize over measures associated to weak solutions of the SDE and relaxed controls (also known as mixed strategies). The new feasible set will be compact and convex under general assumptions, allowing the application of topological fixed point theorems. In the compactification methods, the uniqueness of the optimal control for fixed mean-field terms is not usually required, implying that Schauder's fixed point theorem is replaced by Kakutani-Fan-Glicksberg's fixed point theorem for set-valued maps.

A probabilistic weak formulation for MFGs was developed in [CL15] based on the weak formulation of stochastic optimal control problems (also known as the martingale approach). In this paper the authors do not use relaxed controls as they work with convexity assumptions but allow for weak solutions to the SDE by means of a Girsanov's change of measure. Moreover, they use a BSDE approach to solve the control problem, together with a fixed point argument. In particular, to use this approach, the authors take the volatility to be independent of the control and the mean-field term. In the paper [Lac15], the author applies the *controlled martingale problem approach* to compactify and convexify the original MFG problem and shows the existence of a mean field solution under general assumptions. Further developments have been made in the case of mean field games with branching [CRT19] or mean field games with singular controls [FH17]. In the standard stochastic control theory, this technique embeds the initial stochastic control problem into an optimization over the possible joint laws of the SDE and the relaxed control, for which tightness criteria can be applied to show the existence of a maximizer (see e.g. [KNJ87], [HL90] and [KT15]). More precisely, the set of decision variables is a subset of the set of probability measures over the infinite-dimensional space $C([0, T]) \times \mathcal{P}(A)^{[0, T]}$.

In this thesis we will use a second relaxation technique, known as *linear programming formulation* for MFGs, which has been introduced in the paper [BDT20] for MFGs of optimal stopping. In the single-agent case this technique is described in various papers, e.g. by R. H. Stockbridge, T. Kurtz and coauthors (e.g. [KS98c; CS02; KS17]) and the papers [BBG97; BB96; Las+08; BGQ11]. As explained in Section 2.4, the idea is to work with measures on a finite dimensional space satisfying a linear constraint. Compactness of the set of admissible measures can be achieved under the right estimates. While the existence theorems of MFG equilibria obtained through the controlled martingale problem approach are in general rather abstract

(since they work with laws in infinite-dimensional spaces) and provide little insight into the computation of MFG solutions, the linear programming approach leads to a tractable method for computing the MFG equilibria as we will show in Chapter 4. For applications of the linear programming approach to MFGs we refer to [ADT21] and [BDT22], as well as Chapter 5 in this thesis. In the single-agent case, a similar approach has been introduced in the works [LV80] (deterministic case) and [FV88] (stochastic case) and in the context of *Aubry-Mather theory* in the deterministic and stochastic cases (see e.g. [Mat91; Mañ96], [Gom10] and [Gom02]). In the mean-field game setting, Aubry-Mather theory has been applied in the recent paper [Ash+21] for studying a price formation MFG model. In the recent paper [GHZ22] (appeared after the article [DLT21] on which this chapter is based), the authors study the linear programming approach for MFGs with stochastic control in discrete time and finite state, allowing for a mean field depending on the law of the control. In this framework, the authors provide an existence result and reformulate the problem in such a way that finding MFG solutions is equivalent to solving a single optimization problem capturing both the best-response condition and the constraints.

The vast majority of the literature on MFGs has focused on pure control MFGs and very few papers have studied optimal stopping MFGs, i.e. games where players can decide when to exit the game. Optimal stopping MFGs arise in a natural way since it is too restrictive to impose that all players have to stay until the end of the game as is done in the classical MFG setting. In the article [GP15], the authors study first-order mean-field game obstacle problems in the stationary periodic setting, which model a certain type of stochastic control/optimal stopping MFGs. They show the existence of a solution to their system via the penalization method under strong assumptions on the coefficients. The paper [CDL17] shows the existence of a mean field solution in two different set-ups involving a common noise. In the first one, the authors work with supermodular MFGs of optimal stopping with common noise and show the existence of “strong equilibria” (meaning that the mean-field term is adapted to the filtration of the common noise) under restrictive assumptions. These assumptions are relaxed in the second setting, where they show the existence of “weak equilibria” in which the mean-field term can be adapted to a larger filtration than the one of the common noise and the stopping can be externally randomized with respect to the initial sources of noise (see also [CDL16]). A specific model of optimal stopping MFGs, studied in [Nut18], allows for a continuum of heterogeneous players as well as common noise. We emphasize that in the papers [CDL17; Nut18] the mean-field term corresponds to the distribution of stopping times chosen by the players (which is the first marginal of the probability measure μ described in Section 1.2). A PDE approach is used in [Ber17], where the author introduces the notion of “mixed solution” for both the optimal stopping MFG and the mixed control/stopping MFG. Mixed solutions represent a PDE version of optimal stopping times which are between the so called minimal and maximal optimal stopping times, meaning that players do not stop immediately when they exit the continuation region. Existence and uniqueness of a solution to this PDE system is shown under quite strong assumptions on the state process and the domain. In a more general framework, the paper [BDT20] introduces a linear programming formulation for optimal stopping MFGs. The authors show existence results and prove that their equilibrium solution is a mixed solution in the terminology of [Ber17]

under appropriate assumptions. Finally, in the recent paper [Dia+22], appeared after the paper [DLT21] on which this chapter is based, the authors show the existence of a strong equilibrium (i.e. adapted to the common noise filtration) for optimal stopping MFGs with common noise in the supermodular setting. In the three papers [Ber17; BDT20; Dia+22], the authors consider interactions with respect to the full distribution of players still in the game (which is the flow of subprobability measures m described in Section 1.2) and not the first marginal of μ as in [CDL17; Nut18].

Another class of MFGs studied in this thesis are MFGs with absorption, where if a player's state process leaves a specific domain, the player is forced to exit the game. This property is important for many applications (e.g., when modeling a game in which players may default) and can be found in both the control setting or the stopping setting. The paper [CF18] introduces MFGs with absorption in the control case and shows the existence of a solution using a similar approach to [CL15] but with technical difficulties arising from the exit time. The assumptions of the article [CF18] are further relaxed in [CGL21] using weak convergence arguments, controlled martingale problems and the notion of relaxed controls. The much more intricate case allowing for common noise is studied in [BC21], where weak equilibria (i.e. adapted to a larger filtration than the one of the common noise) are shown to exist. In the papers [Ber17; BDT20], optimal stopping MFGs with absorption are considered and existence of a solution is shown, the second paper allowing for more general domains (e.g. the unbounded case).

Contributions. In this chapter, based on the paper [DLT21] together with some extensions, we prove existence results in a general framework of MFGs allowing for optimal stopping, stochastic control and absorption.

The lack of explicit solutions for general settings makes natural the use of relaxation techniques allowing to show existence results under weak assumptions. Moreover, since equilibria will not be recovered in explicit form, it is important to relax the problem in such a way that we can apply numerical algorithms. Taking into account those aspects, we choose to show the existence of mean field solutions by using the linear programming approach for MFGs introduced in [BDT20]. Firstly, the assumptions for applying this approach are very weak and, secondly, as the name suggests, the optimization allowing to compute the best response consists in solving a linear programming problem, which nowadays can be done fast numerically.

We extend the techniques used in [BDT20] to a more general setting involving mixed optimal stopping/control and allowing for measure-dependent coefficients. We use the same definition of MFG equilibria as in [DLT21], but the methodology for showing existence is different. More precisely, we use the topology of convergence in measure applied to the flow of subprobability measures with polynomial moments (which was used in [DLT22]). This new topological structure allows to weaken several assumptions imposed in [BDT20]. Firstly, we allow for diffusion coefficients depending on the mean-field terms, implying a much deeper analysis of the linear programming approach. More precisely, we use the probabilistic representation of the measures satisfying the constraint in order to overcome some technical difficulties arising from the mean-field dependence of the coefficients. Secondly, we allow for unbounded coefficients and running

reward functions, as well as a general final reward function. We also relax the continuity assumptions in time and some differentiability assumptions on the rewards. Finally, we show existence for general nonlocal dependence on the mean-field terms.

As explained before, in [Ber17; BDT20; Dia+22], the authors consider interactions with respect to the distribution of players still in the game, and in [CDL17; Nut18], the mean-field term corresponds to the distribution of stopping times chosen by the players. Both of these features are included in our set up.

Concerning the stochastic control part of MFGs, our assumptions compare to the ones in [Lac15] with the difference that we impose that the controls are bounded, the dimension of the state is one and the initial distribution should admit second order moments. Since our techniques and the ones used in [Lac15] are related, as we shall explain in a moment, we should be able to relax these assumptions. In addition, we recover directly the existence of a relaxed Markovian equilibrium shown in [Lac15], where the author makes use of the Mimicking Theorem (or Markovian projection theorem) which can be found in Corollary 3.7. in [BS13]. In our case, the relaxed Markovian control is obtained directly by disintegrating the flow of measures $(m_t(dx, da))_{t \in [0, T]}$, modeling the joint distribution of the players and their controls, with respect to the state marginal.

We extend a uniqueness result for the Nash value given in [BDT20] to our setting in the case where the diffusion coefficients do not depend on the measure, meaning that even if various equilibria exist, the players will obtain the same expected reward, independently on the selected equilibrium. Moreover, if the diffusion coefficients are allowed to depend on the measure, we show that there exists a Nash equilibrium maximizing the Nash value.

The compactness of the admissible measure flows is derived from their probabilistic representation, which allows to do precise estimates. Furthermore, the probabilistic representation allows to establish an equivalence between the linear programming formulation for MFGs and MFGs via the controlled/stopped martingale problem approach, which was the approach used in [Lac15] in the control case. To prove this result, we extend some of the existing results in the literature [KS98c; CS02; DS12; KS17] to mixed optimal stopping/stochastic control in finite horizon. In addition, the probabilistic representation allows an interpretation of the measures in terms of the well known concept of randomized stopping times (see e.g. [BC77; EMS82; TV02; CDL17]).

Finally, in a similar fashion to [BDT20], we show under appropriate conditions (which in particular allow to solve the Hamilton-Jacobi-Bellman Variational Inequality in a Sobolev sense for fixed mean-field terms) that any linear programming MFG Nash equilibrium forms a mixed solution in the sense of [Ber17].

3.2 Linear Programming Fictitious Play algorithm for Mean Field Games with optimal stopping and absorption (Chapter 4)

This chapter is based on the paper “Linear Programming Fictitious Play algorithm for Mean Field Games with optimal stopping and absorption” [DLT22], joint work with R. Dumitrescu and P. Tankov, submitted for publication.

Overview of the literature. As already explained, obtaining explicit solutions to MFG problems is rarely possible because the existence of Nash equilibrium is usually obtained through the application of abstract fixed point theorems rather than in a constructive way. For this reason it is important to construct fast numerical algorithms that can approximate MFG Nash equilibria and therefore be used in applications. As well as in the existence theory for MFGs, there are different approaches to construct numerical algorithms which, in most of the literature, are based on PDE or probabilistic methodologies.

The paper [AC10] introduces for the first time a finite-difference scheme for solving numerically stationary and evolutive MFGs on the torus. Finite-difference schemes for MFGs are further studied in [ACC13; AP16]. More precisely, in [AC10; ACC13] the convergence of the discrete problem to the MFG system is provided under the assumption that the MFG system has a unique classical solution and the Lasry-Lions monotonicity condition holds. In contrast, the paper [AP16] allows for weaker assumptions and shows that the discrete problem converges to a weak solution of the MFG system. A finite-difference scheme has also been applied in the papers [BKS18; Bri+19] for solving numerically MFGs with local couplings. Semi-Lagrangian schemes are studied in [CS14] and [CS15] for first order MFGs and second order MFGs, respectively. In [AP12; Gué12] numerical methods based on Newton’s algorithm are described. Potential MFGs (sometimes called Variational MFGs), i.e. MFGs where the solutions coincide with the maximizers of some functional, are studied in the papers [LST10; BC15; AL16; CH17; BLP21a; BLP21b] for stochastic control MFGs and in [BDT20] for optimal stopping MFGs. For more details on these methodologies we refer to the survey [AL20]. On the other hand, numerical methods for solving FBSDEs of McKean-Vlasov type arising in the probabilistic approach for MFGs are developed in [CCD19; Ang+19]. To avoid the curse of dimensionality in the above methods, stochastic methods based on approximations by neural networks have been built for e.g. in [CL21; CL19; GMW22].

Our methodology is based on the *Fictitious Play Algorithm* which goes back to Brown [Bro51] in the classical setting of game theory. It was extended to MFGs for the first time in the paper [CH17]. The fictitious play is a learning procedure in which players update their strategies based on the past strategies and the best response mapping. The idea behind this algorithm is that equilibrium configurations may arise by repeating the game many times. The fictitious play for MFGs has been applied in the settings of potential MFGs with regular controls in [CH17] and first order MFGs with regular controls in [Had17; HS19; Eli+20]. A theoretical version of the fictitious play (the steps of the algorithm are indexed by the continuum) has been studied in [Per+20] for finite MFGs with common noise. In this last paper, the authors provide theoretically a convergence rate of $O(t^{-1})$ for the exploitability (i.e. the gain increase from switching to the best response). The fictitious play has also been used in the recent paper [DV21] for a specific linear-quadratic MFG with common noise. We refer to the recent survey [Lau+22] for more details on learning algorithms for MFGs.

As explained in the above subsection, the paper [GHZ22] (appeared after the paper [DLT22] on which this chapter is based) develops a linear programming approach for MFGs with stochastic control in discrete time and finite state, allowing for a mean field depending on the law of

the control, and shows the convergence of a projected gradient descent algorithm. They reformulate the problem in such a way that finding MFG solutions becomes equivalent to solve a single optimization problem capturing both the best-response condition and the constraints, and provide numerical methods exploiting the linear structure of the problem. The advantage of this approach is that it does not assume uniqueness of the best response or monotonicity conditions on the rewards.

Most of the numerical literature for MFGs has focused on the stochastic control case, and only very few papers study the optimal stopping case. In the paper [BDT20], a learning algorithm for potential MFGs is developed and shown to converge. Uzawa’s algorithm is studied in [Ber20] for the (possibly non-potential) MFG system introduced in [Ber17] in the stationary case, under the assumption of strict Lasry-Lions monotonicity of the reward map. Recently, supermodular MFGs of optimal stopping (with common noise) have been analyzed in [Dia+22]. The authors provide an approximation of the minimal equilibria (with respect to a specific order structure) by starting with the minimal measure flow and iterating the minimal best response. The results hold under the opposite inequality to the Lasry-Lions monotonicity condition, which we will use in our results. Numerical computations combining the fictitious play and the linear programming approach have been provided in [BDT22] for an application to water management and in [ADT21] for an application to electricity markets. However, in both papers the convergence of the algorithm has not been established. In this chapter we aim to fill this gap.

Contributions. The objective of this chapter is to build a numerical algorithm for MFGs of optimal stopping and MFGs with control and absorption and show its convergence to the MFG solution (which under our assumptions will be unique). More specifically, we will study the fictitious play learning procedure in which the best responses are computed by solving linear programs. The framework will be fully based on the linear programming approach for MFGs, introduced in [BDT20] and further studied in [DLT21], in the case of second-order possibly non-potential games, under general assumptions on the coefficients and a non-strict monotonicity condition on the reward functions with respect to the measures.

The *linear programming fictitious play algorithm* (LPFP) starts with an initial belief of the equilibrium and iterates the following two steps:

- (i) *Optimization step*: Compute the best response corresponding to the belief by solving a linear program.
- (ii) *Learning step*: Update the belief via a convex combination (with well chosen parameters) of the previous belief and the best response computed in Step (i).

While the linear problem to be solved in Step (i) is infinite dimensional, one can approximate it by a finite dimensional linear program (see e.g. [MS98; CS02]), for which fast and accurate algorithms exist². This is in contrast with other relaxation techniques for which existence results follow under weak assumptions but numerical schemes are difficult to build due to the abstract definitions. To be more precise on the discretization procedure, the idea is to approximate the

2. see <http://plato.asu.edu/ftp/lpsimp.html> for a comparison of different LP solvers.

infinite dimensional linear programming problem³ (1.5)

$$\begin{aligned} & \sup_{(m,\mu)} \langle m_t(dx)dt, f \rangle + \langle \mu, g \rangle, \\ \text{s.t.} \quad & \langle \mu, u \rangle = \langle m_0^*, u(0, \cdot) \rangle + \langle m_t(dx)dt, \partial_t u + \mathcal{L}u \rangle, \quad u \in C_b^{1,2}([0, T] \times \mathbb{R}), \\ & m_t \in \mathcal{P}^{sub}(\mathbb{R}), \quad t \in [0, T], \\ & \mu \in \mathcal{P}(\mathbb{R}) \end{aligned}$$

by the finite dimensional linear programming problem:

$$\begin{aligned} & \sup_{(m,\mu)} \sum_{t=t_0}^{t_{n-1}} \langle m_t(dx), f(t, \cdot) \rangle + \langle \mu, g \rangle, \\ \text{s.t.} \quad & \langle \mu, u \rangle = \langle \bar{m}_0^*, u(0, \cdot) \rangle + \sum_{t=t_0}^{t_{n-1}} \langle m_t(dx), \bar{\mathcal{L}}u(t, \cdot) \rangle, \quad u \in \mathcal{D}, \\ & m_t \in \mathcal{P}^{sub}(E), \quad t \in \{t_0, \dots, t_{n-1}\}, \\ & \mu \in \mathcal{P}(\{t_0, \dots, t_n\} \times E), \end{aligned}$$

where $\{t_0 = 0 < t_1 < \dots < t_n = T\}$ is the time grid, $E = \{x_0 < x_1 < \dots < x_k\} \subset \mathbb{R}$ is the state grid, \bar{m}_0^* is a discretization of m_0^* on $E \setminus \{x_0, x_k\}$, $\bar{\mathcal{L}}$ is a discretization of the generator of the process $(t, X_t)_{t \in [0, T]}$ and \mathcal{D} is a finite set of functions from $\{t_0, \dots, t_n\} \times E$ to \mathbb{R} , usually taken to be the set of indicator functions on the time-state grid.

The principal difficulty in analyzing the fictitious play algorithm for MFGs of optimal stopping is the time irregularity of the measure flows. This lack of regularity is also present in MFGs with absorption and to the best of our knowledge, no algorithm has been proposed so far in that case. To be more specific, the instantaneous reward function is irregular in time as a consequence of the mean-field dependence (recall that for MFGs of optimal stopping and MFGs with absorption an indicator function of time appears on the measure flows, implying a lack of continuity in time). Moreover, to show the convergence of the algorithm, it will be important to prove, for a well chosen metric d , the existence of a constant C such that for all $N \geq 1$

$$\langle m^{(N+2)} - m^{(N+1)}, f[\bar{m}^{(N+1)}] - f[\bar{m}^{(N)}] \rangle \leq \frac{C}{N} d(m^{(N+1)}, m^{(N+2)}), \quad (1.11)$$

where $\bar{m}^{(N)}$ is the belief at step N and $m^{(N+1)}$ its best response. This estimate will be obtained using a duality result for a 1-Wasserstein type metric d . However, the lack of time regularity in $f[\bar{m}^{(N+1)}] - f[\bar{m}^{(N)}]$ makes it impossible to use the usual duality result for Wasserstein metrics if the measures have a time component. As a consequence, we will use the topology of the convergence in measure for mappings from time (endowed with the Lebesgue measure) to the space of subprobability measures (endowed with some Wasserstein type metric). For this topology we can choose an appropriate metric d allowing to show (1.11). This topology is used in Chapter 3 of this thesis to generalize the existence results from [DLT21], but it was introduced

3. We take the optimal stopping case as an illustration, but the control case is analogous.

in the paper [DLT22] on which this chapter is based.

In contrast with many algorithms in the MFG literature, including the fictitious play algorithms presented before [CH17; Had17; HS19; Eli+20; Per+20], our methodology for computing the best response is not based on the value function, but on the linear programming approach. While the value function provides important information on the optimal control or optimal stopping time, the main objects in MFGs are the occupation measures induced by the optimal controls or stopping times and describing the population dynamics. These quantities can be recovered in our setting without finding the value function and by solving a linear programming problem, which allows for fast computations. Additionally, we emphasize that the distribution of the optimal controls or stopping times can be recovered from the solution to the linear program. This will be shown in numerical implementations for both cases. In the illustrations we will see empirically that the algorithm converges at the rate $O(N^{-1})$, N being the number of learning steps.

3.3 Energy transition under scenario uncertainty: a mean-field game approach (Chapter 5)

Given the climate emergency, it is urgent to decarbonize the electricity industry. Increasing the penetration of renewable energies is a key factor to limit the global temperature increase but it has a strong impact on the operation of electricity markets. More precisely, the increased penetration of renewable energies, due to their intermittent nature, may lead to higher and more volatile peak prices in the electricity market. In addition, the pace of decarbonization is affected by many uncertainties, in particular related to future climate policies. For all these reasons it is important to introduce flexible and tractable models of electricity market dynamics under energy transition. Our goal will be to apply MFGs of optimal stopping with common noise and partial information to study the exit dynamics of conventional producers as the renewable penetration increases.

This chapter is based on the paper “Energy transition under scenario uncertainty: a mean-field game approach”, joint work with R. Dumitrescu and P. Tankov, submitted for publication.

Overview of the literature. The paper [CCZ15] studies the merit order effect (decrease of the average electricity price due to the renewable penetration) in the Italian day-ahead wholesale electricity market. Game theory with optimal stopping is applied in [Tak+08] in a situation in which two firms try to decide whether to build a new power plant. However, the price is exogenous and does not depend on firms’ decisions. Other papers like [HG13; LB15; RPR15] study the existing market structures proposing some alternatives to improve certain aspects concerning the renewable penetration, but do not present models for simulating their evolution.

An important methodology for studying complex interactions arising in economics is based on Agent-Based Computational Economics (ACE). In general, agent-based modeling aims to simulate a system with complex dynamics involving various agents and study their self emerging behavior. While the global behaviour of a system is hard to model, the local behaviour of the agents making up the system is easier to understand. This micro-macro point of view is similar

to the MFG perspective, however in MFGs, the players are optimizing a long term criterion through the notion of Nash Equilibrium; in contrast, in agent-based models, the agents take simpler optimization decisions at each time allowing to reduce some assumptions with respect to MFGs. ACE models have been used to understand electricity market dynamics in several papers. We refer the reader to the survey [WV08] for references and further details on this topic. In general these models offer a good picture of the market, but are computationally very intensive and the outcome of the simulation may heavily depend on the initial configuration of the agents.

In this chapter we will develop an extension of the MFG framework developed in [ADT21] by allowing for common noise and partial information. This paper uses the linear programming approach introduced in [BDT20] for solving a MFG of optimal stopping with two classes of players modeling the long term dynamics of the electricity industry. More precisely, the conventional producers aim to choose an optimal time to exit the market and the renewable producers an optimal time to enter the market. The state of the conventional producers is represented by their baseline marginal cost per unit of energy produced and they bid a proportion of their installed capacity depending on the market price and their cost. On the other hand, renewable energy producers always bid their total production because they have almost zero marginal cost, however, due to the intermittency of the renewable resource, they will not always be able to bid their maximum installed capacity. Their state is given by their stochastic capacity factor. The paper assumes that the demand is inelastic and deterministic. The two classes of agents interact through the electricity price which is determined by a merit order mechanism similar to the day-ahead electricity market: the renewable producers bid their entire (stochastic) capacity at zero price and if the renewable supply is not sufficient to match the demand, the conventional suppliers fill the residual demand. All participants maximize their expected profit by choosing a stopping time and since the problems are coupled by the price, the authors look for Nash equilibria. The mean-field form of the interactions allow them to reduce the problem to a MFG problem of optimal stopping with two classes of players. They show the existence of an equilibrium through the application of the linear programming approach and prove the uniqueness of the equilibrium price process. Moreover they provide numerical illustrations based on the UK electricity market by combining the fictitious play algorithm with the linear programming approach (see the previous subsection for more details). The authors conclude from their numerical computations that “renewable subsidies lead to higher renewable penetration but can entail a cost to the consumer in terms of higher peak-load prices”. Furthermore, they claim that “in order to avoid rising prices, the renewable subsidies must be combined with mechanisms ensuring that sufficient conventional capacity remains in place to meet the energy demand during peak periods”. In contrast to this reference, where the future parameters of the market are assumed to be deterministic and known to the agents, in our setting we will suppose that the electricity demand and the costs of the conventional producers depend on the random carbon price (common noise) which is influenced by the government announcements / regulatory changes (scenarios leading to partial information for the agents). Allowing for common noise and partial information will be a major technical difficulty when proving the existence of MFG solutions,

which will be addressed in this chapter. In the related paper [CDL22], the authors apply MFGs and mean field control to study the behavior of electricity producers in the presence of a carbon tax.

Partial information arises naturally in stochastic control/optimal stopping problems. In these settings, players optimize an expected reward functional by choosing a control or stopping time with respect to a filtration which does not contain the full information of all the random processes involved in the state dynamics or/and reward functions. For example, in discrete time finite horizon optimal stopping, the partial information problem is as follows: consider a Markov chain $X = (X_t)_t$ valued in some Polish space E called the signal process (which is not necessarily fully observed) and a process $Y = (Y_t)_t$ valued in some other Polish space F , such that the couple $(X_t, Y_t)_t$ is a Markov chain. The process Y will be called the observation process. The goal is to choose a $\{0, 1, \dots, T\}$ -valued stopping time τ with respect to the filtration generated by the observation $\mathbb{F}^Y = (\mathcal{F}_t^Y)_t$, to maximize

$$J(\tau) := \mathbb{E} \left[\sum_{t=0}^{\tau-1} f(X_t, Y_t) + g(X_\tau, Y_\tau) \right].$$

This problem can be reformulated as a complete observation optimal stopping problem at the cost of augmenting the state space. A usual transformation to recover the full information structure is to introduce the posterior distribution process $\Pi = (\Pi_t)_t$ taking values in $\mathcal{P}(E)$ and given by

$$\Pi_t(B) = \mathbb{P}[X_t \in B | \mathcal{F}_t^Y], \quad t \in \{0, 1, \dots, T\}, \quad B \in \mathcal{B}(E).$$

If we consider now the functions

$$\bar{f}(\pi, y) := \int_E f(x, y) \pi(dx), \quad \bar{g}(\pi, y) := \int_E g(x, y) \pi(dx), \quad \pi \in \mathcal{P}(E), \quad y \in F,$$

we can rewrite the expected reward as

$$J(\tau) = \mathbb{E} \left[\sum_{t=0}^{\tau-1} \bar{f}(\Pi_t, Y_t) + \bar{g}(\Pi_\tau, Y_\tau) \right],$$

where we used that τ is an \mathbb{F}^Y -stopping time. Moreover, one can show that $(\Pi_t, Y_t)_t$ is an \mathbb{F}^Y -Markov process, implying that we are facing an optimal stopping problem under complete observation. We refer to the surveys [Mon82; Pha05] and the references therein for more details on stochastic control with partial information and to [PRS05; Lud09; Zho12; EV16; FT22] for the optimal stopping case. Partial information problems have been studied in the stochastic control MFG literature (see e.g. [HW16; BLM17; CJ18; SC19; BFH21; BCL21]) and in the case of supermodular MFGs of optimal stopping in the paper [CDL17].

Mean Field Games with common noise describe a type of symmetric games where all players face a common source of randomness. While the MFG approach without common noise relies on the asymptotic independence of the equilibrium state dynamics of different agents (i.e. when the agents use the Nash equilibrium strategy) as the number of players goes to infinity, in the com-

mon noise setting the limit distribution of the equilibrium state dynamics is not deterministic. In fact, the mean-field terms become adapted functionals of the common noise. More precisely, in the control case the mean field flow of measures $m = (m_t)_t$ writes $m_t(\cdot) = \mathbb{P}(X_t \in \cdot | \mathcal{F}_t^B)$, where X is the state of the representative agent and B the common noise process. Mathematically speaking, the common noise augments the space for the mean field term from $C([0, T], \mathcal{P}(\mathbb{R}^d))$ (in the control case) to $[C([0, T], \mathcal{P}(\mathbb{R}^d))]^\Omega$, where Ω is the space of trajectories of the common noise. When Ω is finite the fixed point problem can be solved in a similar way to the case without common noise (see [BHS21]). However, when Ω is infinite, the space $[C([0, T], \mathcal{P}(\mathbb{R}^d))]^\Omega$ is too large to use topological arguments leaving invariant the transformations of interest. As pointed out in [CDL16], the operation of conditioning fails to be continuous in any useful sense, which complicates the study of fixed points via topological arguments. MFGs with common noise were introduced in [Lio07] for the stochastic control case. They were studied from a PDE point of view via the so called *master equation*. For the treatment of the master equation we refer to the books [Car+19; CD18b]. This methodology for studying MFGs with common noise usually imposes strong assumptions on the diffusion coefficients and reward functions such as conditions for the uniqueness of the optimal control for a fixed mean-field flow and the Lasry-Lions monotonicity conditions. We point out that these conditions can be relaxed at the cost of considering weak equilibria in the sense of [CDL16]. More precisely, weak equilibria do not impose the equilibrium mean-field flow to be adapted to the common noise filtration, but rather to a possibly larger filtration. This notion was used in [CDL16], where the authors show the existence of weak equilibria under very general conditions by discretizing the common noise filtration and passing to the limit. In their paper, under appropriate assumptions related to the ones ensuring the well-posedness of the master equation, the authors show that the unique weak equilibrium is in fact a strong equilibrium (i.e. adapted to the common noise filtration). This notion of weak equilibria was also applied in [Lac16] for characterizing the limit points of the sequence of N -player Nash equilibria (both with and without common noise) as N goes to infinity. In the case without common noise, weak equilibria are not necessarily deterministic contrary to the usual definition used in the MFG literature. This general result underlines the usefulness of considering weak equilibria instead of strong ones. However, in order to build numerical schemes for applications with or without common noise, the notion of weak equilibrium seems too abstract.

MFGs of optimal stopping with common noise were studied in [CDL17; Nut18; Dia+21]. In the paper [CDL17], two cases are considered: the first one is a supermodular MFG setting allowing to show existence of strong equilibria with non-randomized stopping times under quite restrictive assumptions; the second one is more general and allows to show existence of weak equilibria. We emphasize that, in both cases, a general state for the players and a general dependence on the distribution of players still in the game are not considered. Moreover, in the first case the authors provide an approximation result for finite player games where they can construct approximate Nash equilibria in distributed form via the MFG solution and in the more general setting they give the same result by adding the assumption that strong equilibria with non-randomized stopping times exist. A specific model for bank runs is developed in [Nut18].

Here the agents interact through the proportion of players that have already stopped and the equilibrium can be computed quite explicitly. Finally, a general supermodular MFG of optimal stopping with common noise is studied in [Dia+21] providing an existence theorem for strong solutions and a numerical scheme for computing it. We highlight that in the three previous articles, the MFG settings are of *preemption* type, meaning that the departure of other players creates an incentive for the representative player to leave as well. This is in contrast with the so called *war of attrition* games, where the departure of other players creates an incentive for the representative player to stay in the game.

Contributions. The main goal of this chapter is to extend the model introduced in [ADT21] by including common noise and partial information. This extension allows to understand the impact of the uncertainty in the carbon price on the electricity market dynamics. More precisely, we make the demand process and the baseline cost of the conventional producers depend explicitly on the carbon price. The carbon price is modeled by a stochastic process, which is Markovian conditionally on fixing the scenario. In our case the scenarios are plausible trajectories of macroeconomic variables, parameterized by specific assumptions on future climate and policy. We place ourselves in the general setting where the scenario is not observable by the producers, leading us to a partial information setting. Since the carbon price will affect all the producers, it will not be averaged out in the mean-field limit, producing a MFG of optimal stopping with non-Markovian common noise (the non-Markovianity coming from the partial information setting) and partial information.

The game developed here corresponds to the class of war of attrition games. In particular, we do not have a supermodular structure allowing to apply monotonicity theorems to show existence of strong equilibria. As the notion of weak equilibrium is too abstract to build an algorithm for our application, we have chosen to work with a common noise taking a finite number of possible paths. Moreover, to ensure that each state process (conventional or renewable) together with some transformation of the common noise is a Markov process (thus allowing to use the linear programming formulation), we set up the model in discrete time, in contrast to [ADT21]. Inspired by the papers [Kur98; KN11] which study the filtered martingale problem, we construct a Markov process, denoted by U , encoding the history of the carbon price and taking values in a finite set of matrices. This transformation allows for two things, first the mean-field terms at time t will be functions of U_t which allows to use topological arguments for showing the existence of fixed points, second the full information structure is recovered since the posterior distribution process is simply a function of U .

More broadly, we develop a general theory for MFGs of optimal stopping in discrete time with finite valued common noise. We allow the state variable X to take values in an arbitrary compact metric space assuming that the joint process (X, U) is Markovian. Using the generator of this Markov process we get a linear constraint for the underlying occupation measures and furthermore we can rewrite the expected reward in terms of these occupation measures. We therefore develop the linear programming approach for discrete time MFGs of optimal stopping and absorption with common noise, for which we can build a numerical algorithm. Under weak

assumptions, we show the existence of a linear programming Nash equilibrium using Kakutani-Fan-Glicksberg's fixed-point theorem for set valued mappings, both in the case of one population of players and in the case of several populations of players. Furthermore we show that our relaxation set is equal to the closed convex hull of the set of occupation measures induced by non-randomizing stopping times (i.e. functions of the underlying processes (X, U)). A consequence of this result is that a probabilistic representation of our constraint can be recovered, allowing to interpret the measures in terms of randomized stopping times depending on the underlying processes. Since any linear programming MFG Nash equilibrium is associated to a randomized stopping time depending on the underlying processes, we are able to construct an approximation result for finite player games allowing for randomized stopping times.

We show that our initial model for the electricity industry dynamics fits into the theoretical framework described before by verifying the assumptions. This enables us to state a general existence result. Furthermore, using similar techniques to [ADT21], we show that even in the common noise case the electricity price process at equilibrium is unique. The linear programming fictitious play algorithm is extended to our common noise setting without proof of convergence. We take the same UK electricity market example of [ADT21] and calibrate the carbon price parameters using the NGFS (Network for Greening the Financial System) scenarios. In our example we see that empirically the algorithm converges in $O(N^{-1})$, N being the number of learning steps, and that scenario uncertainty leads to significant changes in the speed of replacement of conventional generators by renewable production.

4 General perspectives

Throughout this thesis we focus on the linear programming approach for MFGs allowing for stochastic control, optimal stopping or absorption. The techniques we develop here could be also useful for the theoretical study of control/optimal stopping of McKean-Vlasov systems and their numerical resolution. For example, consider the following optimal stopping problem of the McKean-Vlasov type:

$$\begin{aligned}
& \sup_{\tau} \mathbb{E} \left[\int_0^{\tau} f(t, X_t, m_t^{\tau}) dt + g(\tau, X_{\tau}, \mu^{\tau}) \right], \\
& \text{s.t.} \quad dX_t = b(t, X_t, m_t^{\tau}) dt + \sigma(t, X_t, m_t^{\tau}) dW_t, \\
& \quad m_t^{\tau}(B) = \mathbb{P}[X_t \in B, t < \tau], \quad B \in \mathcal{B}(\mathbb{R}), \quad t \in [0, T], \\
& \quad \mu^{\tau} = \mathcal{L}(\tau, X_{\tau}), \\
& \quad X_0 \sim m_0^*.
\end{aligned} \tag{1.12}$$

The occupation measure formulation of this optimal stopping problem of the McKean-Vlasov type will be:

$$\begin{aligned}
& \sup_{(m, \mu)} \quad \langle m_t(dx)dt, f[m] \rangle + \langle \mu, g[\mu] \rangle, \\
\text{s.t.} \quad & \langle \mu, u \rangle = \langle m_0^*, u(0, \cdot) \rangle + \langle m_t(dx)dt, \partial_t u + \mathcal{L}[m]u \rangle, \quad u \in C_b^{1,2}([0, T] \times \mathbb{R}), \\
& m_t \in \mathcal{P}^{sub}(\mathbb{R}), \quad t \in [0, T], \\
& \mu \in \mathcal{P}(\mathbb{R}),
\end{aligned} \tag{1.13}$$

where

$$\begin{aligned}
\mathcal{L}[m]u(t, x) &:= b(t, x, m_t)\partial_x u(t, x) + \frac{\sigma^2}{2}(t, x, m_t)\partial_{xx}u(t, x), \\
f[m](t, x) &:= f(t, x, m_t), \quad g[\mu](t, x) := g(t, x, \mu).
\end{aligned}$$

Note that this optimization problem is not linear. However, using the techniques applied in this thesis, one can show, under an appropriate topology and some weak assumptions, that the feasible set is compact and the objective function is continuous, allowing to show the existence of a maximizer. Finally, by discretizing the admissible measures, we can get an optimization in finite dimension, and numerical algorithms can be used to approximate the solutions. We refer to the papers [TTZ21; TTZ22; AO22; DM22] for more details on the optimal stopping problem of the McKean-Vlasov type. Similar settings can be considered for stochastic control or absorption and combinations of the three.

We propose now two extensions for the convergence of the linear programming fictitious play algorithm. In this thesis we assume that the diffusion coefficients are independent of the mean-field term. Extending the algorithm to measure-dependent coefficients may be of interest for some applications. In particular, if two input mean-field terms are close enough and the coefficients are Lipschitz with respect to the measure, we can expect some proximity between the best response measures. In addition the results for the uniqueness of the best response in the optimal stopping case should be extended, relaxing the assumptions on the coefficients. Another related problem would be to show a convergence rate for the linear programming fictitious play. In order to achieve this, it should be important to have precise estimates on the distance between two consecutive best responses.

Finally, an extension concerning the last chapter of the thesis would be to relax the compactness of the state space by considering a locally compact separable metric space and allow for Markov chains admitting polynomial moments. This seems to be possible by using the probabilistic representation of the constraints (since we would also have a similar duality result), the techniques of the first part of the thesis and assuming some moment conditions on the Markov chain.

Chapter 2

Introduction (Français)

Abstract

Nous présentons les principaux problèmes que nous avons l'intention d'étudier tout au long de la thèse. Dans la première partie, nous introduisons séparément les problèmes de jeux à champ moyen avec contrôle stochastique et avec arrêt optimal. Dans la deuxième partie, nous introduisons quelques notations et concepts généraux qui seront utilisés tout au long de la thèse. En particulier, nous donnons un premier aperçu de la formulation par programmation linéaire, qui est le cœur de la thèse. Enfin, nous présentons le plan de la thèse en mettant l'accent sur la littérature associée et nos contributions.

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1 Introduction aux jeux à champ moyen

Dans cette section, nous présentons un cadre général pour les jeux à champ moyen stochastiques différentiels sans bruit commun dans le cas du contrôle stochastique et dans le cas de l'arrêt optimal, séparément. Nous utilisons ici le terme *stochastique différentiel* puisque l'expression jeux à champ moyen englobe de nombreux modèles qui ne sont pas nécessairement stochastiques ou en temps continu. Cependant, étant donné que la plupart de la littérature sur les jeux à champ moyen traite le cas stochastique en temps continu, nous utiliserons ce cadre pour présenter les généralités du modèle. Dans ce qui suit, nous discuterons des concepts mathématiques qui sous-tendent ces modèles. Nous n'imposerons pas de conditions suffisantes pour que les problèmes soient bien posés, car notre objectif est d'expliquer uniquement l'intuition sous-jacente.

Les jeux à champ moyen (MFG) ont été introduits vers 2006 par Lasry et Lions [LL06a; LL06b; LL07] et simultanément, par Huang, Caines et Malhamé [HMC06; HCM07] sous le nom de "Nash Certainty Equivalence principle". Dans les dernières années, les MFG ont fait l'objet d'études théoriques approfondies et ont été appliqués à différents domaines. Les MFGs représentent la *version limite* de certains types de *jeux symétriques* lorsque le nombre de joueurs tend vers l'infini. La théorie classique des jeux non coopératifs traite des situations où plusieurs joueurs cherchent à maximiser une fonction de gain spécifique. Comme les optimisations sont couplées et résolues simultanément, le concept bien connu d'équilibre de Nash apparaît. Mathématiquement, les équilibres de Nash sont souvent caractérisés par des systèmes d'équations couplées où le nombre d'équations est égal au nombre de joueurs. Lorsque le nombre de joueurs est important, il n'existe généralement pas de solutions analytiques ou de méthodes numériques pour résoudre ces problèmes. L'idée principale des MFG est d'exploiter la symétrie du jeu et l'effet de lissage des grands nombres, ce qui conduit à des simplifications mathématiques et à des méthodes numériques.

La théorie mathématique entourant ces modèles s'est généralement concentrée sur les trois domaines suivants : l'existence et l'unicité de solutions, la théorie limites reliant le jeu avec un nombre fini de joueurs au MFG et les méthodes numériques d'approximation des solutions. Pour le lecteur intéressé par les applications des MFGs en économie et en finance, nous renvoyons à la récente étude [Car20].

Cette thèse se concentre sur plusieurs aspects des MFGs. Le premier est la théorie d'existence pour les MFGs d'arrêt optimal (avec ou sans contrôle stochastique), qui, par rapport au cas du contrôle stochastique, n'a pas été développée dans toute sa généralité. L'une des principales difficultés ici est l'irrégularité temporelle, qui est une conséquence de la façon dont les temps d'arrêt des joueurs agissent sur les distributions empiriques. Le second aspect concerne l'étude de l'approche par programmation linéaire pour les MFGs à travers de la théorie d'existence et la construction d'un schéma numérique. Les algorithmes numériques pour les MFGs sont cruciaux puisque les solutions explicites sont l'exception plutôt que la règle. En particulier, nous étendrons l'analyse de la formulation par programmation linéaire pour les MFGs initiée dans [BDT20] et nous dériverons un algorithme numérique permettant d'approcher la solution du

MFG. Nous fournirons également une représentation probabiliste de nos contraintes de programmation linéaire, ce qui est important pour la comparaison avec d'autres méthodes existantes, en fournissant une interprétation de nos contraintes et en permettant de dériver des estimées cruciales pour la théorie d'existence. Un autre élément clé de notre analyse est que nous autorisons les MFG avec absorption, ce qui signifie que les joueurs quittent le jeu lorsque leur processus d'état sort d'un certain domaine. Ceci est non seulement essentiel pour de nombreuses applications mais également important pour des raisons numériques puisque nous sommes contraints de résoudre le problème dans un domaine borné. Enfin, nous illustrerons comment l'approche par programmation linéaire s'applique à un exemple concret concernant l'industrie de l'électricité qui subit une transition énergétique majeure en raison du changement climatique. Dans cette application, nous étendrons notre approche aux MFGs en temps discret avec un bruit commun et information partielle, montrant ainsi que notre méthodologie est suffisamment flexible pour s'adapter à de nouveaux contextes. En outre, nous calculons la solution numériquement en résolvant une séquence de problèmes de programmation linéaire pour lesquels des algorithmes rapides sont aujourd'hui disponibles.

1.1 Contrôle Stochastique

Le jeu à N joueurs. Soit $N \in \mathbb{N}^*$ le nombre de joueurs et $T > 0$ l'horizon temporel du jeu. Chaque joueur/agent $k \in \{1, \dots, N\}$ possède un processus d'état privé $X^{k,N} = (X_t^{k,N})_{t \in [0,T]}$, dont la dynamique est donnée par l'équation différentielle stochastique (EDS)

$$dX_t^{k,N} = b\left(t, X_t^{k,N}, m_t^N, \alpha_t^k\right) dt + \sigma\left(t, X_t^{k,N}, m_t^N, \alpha_t^k\right) dW_t^k,$$

où W^1, \dots, W^N sont des mouvements Browniens indépendants, les conditions initiales X_0^1, \dots, X_0^N sont i.i.d. de loi m_0^* et indépendantes des mouvements Browniens, $\alpha^k = (\alpha_t^k)_{t \in [0,T]}$ est le processus de contrôle choisi par l'agent k , prenant des valeurs dans un sous-ensemble fermé $A \subset \mathbb{R}$, et m_t^N est la mesure d'occupation empirique des états des joueurs :

$$m_t^N(dx) = \frac{1}{N} \sum_{k=1}^N \delta_{X_t^{k,N}}(dx).$$

Chaque agent k souhaite choisir un contrôle optimal α^k pour maximiser le gain espéré J^k qui dépend des contrôles utilisés par tous les joueurs (uniquement à travers des mesures empiriques) :

$$J^k(\alpha^1, \dots, \alpha^N) := \mathbb{E} \left[\int_0^T f\left(t, X_t^{k,N}, m_t^N, \alpha_t^k\right) dt + g\left(X_T^{k,N}, m_T^N\right) \right].$$

Les fonctions objectif et la dynamique des agents sont couplées par le flux de mesures empiriques $(m_t^N)_{t \in [0,T]}$, de sorte qu'il est naturel de rechercher un équilibre de Nash. Nous disons que $\alpha = (\alpha^1, \dots, \alpha^N)$ est un équilibre de Nash si pour tout k et tout $\bar{\alpha}$,

$$J^k(\bar{\alpha}, \alpha^{-k}) \leq J^k(\alpha),$$

où nous utilisons la notation habituelle $(\bar{\alpha}, \alpha^{-k}) := (\alpha^1, \dots, \alpha^{k-1}, \bar{\alpha}, \alpha^{k+1}, \dots, \alpha^N)$. Dans une *situation d'équilibre*, chaque joueur n'a aucune incitation à *dévier unilatéralement* de son contrôle.

Si N est assez grand, chaque agent a une faible influence sur les mesures empiriques et donc sur les optimisations des autres joueurs. De plus, par la symétrie du problème, nous pouvons supposer qu'un équilibre de Nash s'écrit sous la forme $((\alpha^*(t, X_t^{1,N}, m_t^N))_t, \dots, (\alpha^*(t, X_t^{N,N}, m_t^N))_t)$ (l'hypothèse selon laquelle la fonction α^* ne dépend pas de N est une hypothèse forte, mais en raison de la faiblesse du couplage, cela devient un argument heuristique tout à fait raisonnable). Lorsque le nombre de joueurs N tend vers l'infini et que α^* est suffisamment régulier, on peut s'attendre par un résultat de type *propagation du chaos* que les EDS convergent (en distribution) vers la solution de l'EDS de McKean-Vlasov correspondante associée à α^* ,

$$dX_t^* = b(t, X_t^*, \mathcal{L}(X_t^*), \alpha^*(t, X_t^*, \mathcal{L}(X_t^*))) dt + \sigma(t, X_t^*, \mathcal{L}(X_t^*), \alpha^*(t, X_t^*, \mathcal{L}(X_t^*))) dW_t,$$

et que les mesures d'occupation empiriques convergent vers le *flux déterministe de mesures de probabilité* $(m_t^*)_{t \in [0, T]} = (\mathcal{L}(X_t^*))_{t \in [0, T]}$ associé à la solution de l'EDS de McKean-Vlasov. Nous avons utilisé ici la notation $\mathcal{L}(\cdot)$ pour la loi d'une variable aléatoire. A la limite, les problèmes se découplent et il suffit de considérer le cas d'un agent représentatif. Enfin, si le joueur représentatif dévie de son contrôle α^* , ce changement n'affectera pas les mesures limites puisque le poids associé à un seul agent est de $1/N$. En conséquence, la propriété d'équilibre de Nash se traduit par la fixation du flux de mesures qui correspond à α^* et la vérification de la propriété d'optimalité de α^* parmi tous les contrôles. Ce dernier point constitue la différence essentielle entre les MFG et le *contrôle des systèmes de McKean-Vlasov* (également appelé *problème de contrôle de type champ moyen*) où les mesures ne sont pas fixées mais varient avec le contrôle (problème du planificateur global).

Problème MFG. Le problème MFG est le suivant :

- (i) Fixez un flux de mesures de probabilité $m = (m_t)_{t \in [0, T]}$ et trouvez la solution du problème de contrôle stochastique

$$\begin{aligned} & \sup_{\alpha} \mathbb{E} \left[\int_0^T f(t, X_t^{\alpha, m}, m_t, \alpha_t) dt + g(X_T^{\alpha, m}, m_T) \right], \\ \text{s.t.} \quad & dX_t^{\alpha, m} = b(t, X_t^{\alpha, m}, m_t, \alpha_t) dt + \sigma(t, X_t^{\alpha, m}, m_t, \alpha_t) dW_t, \\ & X_0^{\alpha, m} \sim m_0^*. \end{aligned} \tag{2.1}$$

- (ii) Étant donné une solution α^m du problème (2.1) pour l'agent faisant face à un champ moyen m , trouver m tel que $m_t = \mathcal{L}(X_t^{\alpha^m, m})$ pour tout $t \in [0, T]$.

Si l'on peut trouver un tel flux de mesures de probabilité m , on l'appellera une *solution MFG* (ou *équilibre MFG*). Une manière pratique de voir le problème est de définir une fonction multivaluée

$$\Phi(m) := \left\{ (\mathcal{L}(X_t^{\alpha^m, m}))_{t \in [0, T]} : \alpha^m \text{ optimal for } m \right\}.$$

Alors m est une solution MFG si et seulement si c'est un point fixe de Φ au sens des fonc-

tions multivaluées, c'est-à-dire $m \in \Phi(m)$. La fonction Φ est généralement appelée *fonction de meilleure réponse*.

1.2 Optimal Stopping

L'une des principales contributions de cette thèse est l'étude des *MFGs avec arrêt optimal* (également connus sous le nom de "*MFGs of Timing*"). Ces modèles apparaissent dans le contexte de jeux entre agents qui décident l'instant auquel sortir (ou entrer) du jeu en choisissant un temps d'arrêt.

Jeu à N joueurs. Comme précédemment, soit $N \in \mathbb{N}^*$ le nombre de joueurs et $T > 0$ l'horizon temporel du jeu. Dans le jeu d'arrêt optimal à N joueurs, l'agent $k \in \{1, \dots, N\}$ choisit un temps d'arrêt τ^k évalué dans $[0, T]$, auquel il quitte le jeu et a un processus d'état donné par la solution de l'EDS

$$dX_t^{k,N} = b(t, X_t^{k,N}, m_t^N) dt + \sigma(t, X_t^{k,N}, m_t^N) dW_t^k,$$

où W^1, \dots, W^N sont des mouvements Browniens indépendants, les conditions initiales X_0^1, \dots, X_0^N sont i.i.d. de loi m_0^* et indépendantes des mouvements Browniens et m_t^N est la mesure d'occupation empirique des états des joueurs qui sont *encore dans le jeu* :

$$m_t^N(dx) = \frac{1}{N} \sum_{k=1}^N \delta_{X_t^{k,N}}(dx) \mathbb{1}_{t < \tau^k}.$$

Chaque agent k souhaite choisir un temps d'arrêt optimal τ^k pour maximiser le gain espéré J^k qui dépend des temps d'arrêt choisis par tous les joueurs (uniquement à travers des mesures empiriques) :

$$J^k(\tau^1, \dots, \tau^N) := \mathbb{E} \left[\int_0^{\tau^k} f(t, X_t^{k,N}, m_t^N) dt + g(\tau^k, X_{\tau^k}^{k,N}, \mu^N) \right],$$

où μ^N est la distribution empirique jointe des temps d'arrêt et des processus d'état au moment de l'arrêt :

$$\mu^N(dt, dx) = \frac{1}{N} \sum_{k=1}^N \delta_{(\tau^k, X_{\tau^k}^{k,N})}(dt, dx).$$

Puisque les problèmes sont couplés par le biais des mesures empiriques, nous rechercherons des équilibres de Nash. Nous dirons que $\boldsymbol{\tau} = (\tau^1, \dots, \tau^N)$ est un équilibre de Nash si pour tout k et tout $\bar{\tau}$,

$$J^k(\bar{\tau}, \boldsymbol{\tau}^{-k}) \leq J^k(\boldsymbol{\tau}).$$

Nous remarquons que même si certains des joueurs quittent le jeu, les joueurs encore dans le jeu auront un poids de $1/N$, au lieu de diviser par la taille de cette sous-classe de joueurs. Par conséquent, dans les MFGs d'arrêt optimal, la mesure empirique m_t^N n'est pas une mesure de probabilité mais une mesure de sous-probabilité. Nous notons également qu'à cause de la fonction indicatrice, la mesure empirique m_t^N est irrégulière vue comme une fonction du temps, ce qui est

en contraste avec les MFGs classiques de contrôle stochastique présentés précédemment. Comme nous le montrerons plus tard, cette irrégularité temporelle ajoute une difficulté importante lors de la démonstration de l'existence d'équilibres ou du développement d'un algorithme numérique.

Puisque le jeu est symétrique, nous pouvons nous attendre comme précédemment à ce qu'une stratégie d'équilibre de Nash soit donnée par des stratégies symétriques et que le problème se simplifie dans la limite. Nous présentons directement le problème limite.

Problème MFG. Le problème MFG avec arrêt optimal est le suivant :

- (i) Fixez un flux de mesures de sous-probabilité $m = (m_t)_{t \in [0, T]}$ et une mesure de probabilité μ , et trouvez la solution au problème d'arrêt optimal

$$\begin{aligned} \sup_{\tau} \quad & \mathbb{E} \left[\int_0^{\tau} f(t, X_t^m, m_t) dt + g(\tau, X_{\tau}^m, \mu) \right], \\ \text{s.t.} \quad & dX_t^m = b(t, X_t^m, m_t) dt + \sigma(t, X_t^m, m_t) dW_t, \\ & X_0^m \sim m_0^*. \end{aligned} \tag{2.2}$$

- (ii) Étant donné une solution $\tau^{m, \mu}$ du problème (2.2) pour l'agent faisant face à un champ moyen (m, μ) , trouver m tel que $m_t = \mathbb{P}(X_t^m \in \cdot, t < \tau^{m, \mu})$ pour tout $t \in [0, T]$ et $\mu = \mathcal{L}(\tau^{m, \mu}, X_{\tau^{m, \mu}}^m)$.

Si (m, μ) vérifient ces propriétés, alors le couple sera appelé une solution MFG. Comme dans le cas du contrôle stochastique, nous pouvons caractériser une solution MFG comme un point fixe d'une fonction multivaluée.

2 Notation et Préliminaires

2.1 Notation générale

Pour un ensemble donné \mathcal{X} , nous désignons par $2^{\mathcal{X}}$ l'ensemble des parties de \mathcal{X} . Si \mathcal{Y} est un autre ensemble, nous dénotons par $\mathcal{Y}^{\mathcal{X}}$ l'ensemble des fonctions de \mathcal{X} dans \mathcal{Y} . Nous dénotons par \mathbb{N} l'ensemble des entiers positifs et par \mathbb{N}^* l'ensemble $\mathbb{N} \setminus \{0\}$. L'ensemble \mathbb{R}^+ désigne l'intervalle $]0, \infty[$ et l'ensemble \mathbb{R}_+ désigne l'intervalle $[0, \infty[$. Pour deux nombres réels quelconques x et y , nous utilisons la notation suivante : $x \wedge y := \min(x, y)$, $x \vee y := \max(x, y)$, $x^+ := x \vee 0$ et $x^- := (-x) \vee 0$. Soit \mathcal{O} un ensemble dans \mathbb{R}^d , on dénote l'intérieur, la fermeture et la frontière respectivement par $\text{int}(\mathcal{O})$, $\bar{\mathcal{O}}$ et $\partial\mathcal{O}$.

2.2 Notations pour la théorie des probabilités

Étant donné un espace de probabilité $(\Omega, \mathcal{F}, \mathbb{P})$ et une sous-tribu \mathcal{G} , nous définissons les ensembles \mathbb{P} -négligeables dans \mathcal{G} comme suit

$$\mathcal{N}_{\mathbb{P}}(\mathcal{G}) := \{B \subset \Omega : B \subset C \text{ pour un } C \in \mathcal{G} \text{ tel que } \mathbb{P}(C) = 0\}.$$

On appelle espace de probabilité complet, un espace de probabilité $(\Omega, \mathcal{F}, \mathbb{P})$ tel que $\mathcal{N}_{\mathbb{P}}(\mathcal{F}) \subset \mathcal{F}$. Tout espace de probabilité $(\Omega, \mathcal{F}, \mathbb{P})$ peut être complété par la procédure suivante : on définit

$$\mathcal{F}^{\mathbb{P}} := \mathcal{F} \vee \mathcal{N}_{\mathbb{P}}(\mathcal{F}) = \{B \cup C : B \in \mathcal{F}, C \in \mathcal{N}_{\mathbb{P}}(\mathcal{F})\},$$

$$\bar{\mathbb{P}}(B \cup C) := \mathbb{P}(B), \quad B \in \mathcal{F}, C \in \mathcal{N}_{\mathbb{P}}(\mathcal{F}),$$

alors $(\Omega, \mathcal{F}^{\mathbb{P}}, \bar{\mathbb{P}})$ est un espace de probabilité complet. Avec quelques abus de notation, nous désignerons toujours la mesure de probabilité étendue $\bar{\mathbb{P}}$ par \mathbb{P} . Nous rappelons les deux propriétés suivantes de $\mathcal{F}^{\mathbb{P}}$:

- (i) $B \in \mathcal{F}^{\mathbb{P}}$ si et seulement si il existe $B_1, B_2 \in \mathcal{F}$ tels que $B_1 \subset B \subset B_2$ et $\mathbb{P}(B_2 \setminus B_1) = 0$.
- (ii) Soit (E, \mathcal{E}) un espace mesurable. $X : \Omega \rightarrow E$ est $\mathcal{F}^{\mathbb{P}}$ -mesurable si et seulement si il existe $Y : \Omega \rightarrow E$ \mathcal{F} -mesurable tel que $X = Y$ a.s.

Étant donné un espace de probabilité $(\Omega, \mathcal{F}, \mathbb{P})$ et une variable aléatoire X prenant des valeurs dans un espace mesurable, on désigne par $\mathbb{P} \circ X^{-1}$ ou $\mathcal{L}(X)$ la loi de X .

2.3 Espaces de mesures et de fonctions

Considérons un espace topologique E . Nous désignons par $\mathcal{B}(E)$ tribu borélienne sur E . Définissons l'ensemble $\mathcal{M}(E)$ de mesures signées finies sur $(E, \mathcal{B}(E))$, avec les sous-ensembles, $\mathcal{M}^+(E)$ de mesures positives finies, $\mathcal{P}^{sub}(E)$ de mesures de sous-probabilité et $\mathcal{P}(E)$ de mesures de probabilité. Nous désignons par $M(E)$ l'ensemble des fonctions boréliennes de E dans \mathbb{R} , par $M_b(E)$ le sous-ensemble des fonctions de boréliennes et bornées, par $C(E)$ le sous-ensemble des fonctions continues, et par $C_b(E)$ le sous-ensemble des fonctions continues et bornées. L'ensemble $M_b(E)$ est muni de la norme du supremum $\|\varphi\|_{\infty} = \sup_{x \in E} |\varphi(x)|$.

Nous supposons maintenant que (E, d) est un espace métrique complet et séparable. Pour toute fonction mesurable $w : E \rightarrow [1, \infty[$, appelée fonction de poids, définissons l'espace vectoriel réel $\mathcal{M}_w(E)$ constitué de toutes les mesures signées finies μ sur $(E, \mathcal{B}(E))$ telles que

$$\int_E w(x) |\mu|(dx) < \infty,$$

où $|\mu|$ est la variation de μ . Soit maintenant $C_w(E)$ l'espace vectoriel réel constitué des fonctions continues $\varphi : E \rightarrow \mathbb{R}$ vérifiant

$$\sup_{x \in E} \frac{|\varphi(x)|}{w(x)} < \infty.$$

Définissons la forme bilinéaire

$$\langle \mu, \varphi \rangle := \int_E \varphi(x) \mu(dx), \quad \mu \in \mathcal{M}_w(E), \varphi \in C_w(E).$$

On peut montrer que $(\mathcal{M}_w(E), C_w(E), \langle \cdot, \cdot \rangle)$ est une paire duale, c'est-à-dire que

$$\forall \mu \in \mathcal{M}_w(E) \setminus \{0\}, \quad \exists \varphi \in C_w(E) : \langle \mu, \varphi \rangle \neq 0,$$

$$\forall \varphi \in C_w(E) \setminus \{0\}, \quad \exists \mu \in \mathcal{M}_w(E) : \quad \langle \mu, \varphi \rangle \neq 0.$$

Sur $\mathcal{M}_w(E)$, nous définissons la topologie faible $\tau_w := \sigma(\mathcal{M}_w(E), C_w(E))$ comme la topologie la plus grossière sur $\mathcal{M}_w(E)$ rendant continues les fonctions $\{\langle \cdot, \varphi \rangle : \varphi \in C_w(E)\}$. En particulier, $(\mathcal{M}_w(E), \tau_w)$ est un espace vectoriel topologique Hausdorff localement convexe. Définissons les sous-ensembles $\mathcal{M}_w^+(E)$, $\mathcal{P}_w^{sub}(E)$ et $\mathcal{P}_w(E)$ de $\mathcal{M}_w(E)$ comme étant les intersections de $\mathcal{M}_w(E)$ avec $\mathcal{M}^+(E)$, $\mathcal{P}^{sub}(E)$ et $\mathcal{P}(E)$, respectivement.

Lorsque $w = 1 + d(\cdot, x_0)^p$ pour un certain $p \geq 1$ et un certain point de référence $x_0 \in E$, nous dénotons $\tau_p := \tau_w$ (qui ne dépend pas de x_0). La topologie τ , définie en fixant $w = 1$, désignera la topologie de convergence faible des mesures (même si nous utilisons la même notation pour les temps d'arrêt, il n'y aura pas de confusion grâce au contexte). Nous utiliserons parfois la notation standard $\mu_n \rightarrow \mu$ pour dire que $(\mu_n)_{n \geq 1}$ converge en τ vers μ . Si $p \geq 1$, on dénote $\mathcal{M}_p(E) := \mathcal{M}_w(E)$, $\mathcal{M}_p^+(E) := \mathcal{M}_w^+(E)$, $\mathcal{P}_p^{sub}(E) := \mathcal{P}_w^{sub}(E)$, $\mathcal{P}_p(E) := \mathcal{P}_w(E)$ et $C_p(E) := C_w(E)$.

Dans cette thèse, nous travaillerons également avec des mesures sur des espaces produits pour lesquelles la mesure marginale sur le premier espace est absolument continue par rapport à la mesure de Lebesgue. En particulier, nous nous attendons à ce que ce type de mesures converge pour des topologies faibles qui n'imposent pas la continuité des fonctions test par rapport au premier argument. Ce type de topologie a été étudié dans [JM81] pour les fonctions test bornées et par [Lac15] dans le cas de mesures de probabilité admettant des fonctions test à croissance polynomiale. Puisque nous allons travailler avec des fonctions non bornées et des mesures de sous-probabilité, nous donnons les définitions pour des fonctions de poids générales et des mesures signées générales. Considérons deux espaces métriques complets et séparables (E, d_E) et (F, d_F) et soit w une fonction de poids sur $E \times F$. Considérons l'ensemble de fonctions suivant :

$$M_{mc,w}(E \times F) = \left\{ \phi \in M(E \times F) : \forall x \in E, \quad \phi(x, \cdot) \in C(F); \quad \sup_{(x,y) \in E \times F} \frac{|\phi(x, y)|}{w(x, y)} < \infty \right\}.$$

L'indice "mc" fait référence à mesurable-continu, et en fait cet ensemble de fonctions appartient à la classe des *fonctions de Carathéodory*, également connues sous le nom de *intégrandes de Carathéodory* (voir la Définition 4.50 dans [AB07]). Les fonctions de Carathéodory sont utilisées dans le Calcul des Variations avec les *mesures de Young* pour relaxer les problèmes de contrôle (voir par exemple [CFV04]). Ceci conduit aux méthodes de contrôle relaxé utilisées dans le cas du contrôle stochastique, que nous utiliserons également dans cette thèse. Avec le même type de forme bilinéaire que précédemment

$$\langle \mu, \varphi \rangle := \int_{E \times F} \varphi(x, y) \mu(dx, dy), \quad \mu \in \mathcal{M}_w(E \times F), \quad \varphi \in M_{mc,w}(E \times F),$$

le uplet $(\mathcal{M}_w(E \times F), M_{mc,w}(E \times F), \langle \cdot, \cdot \rangle)$ forme une paire duale. Sur $\mathcal{M}_w(E \times F)$, on définit la topologie faible $\bar{\tau}_w := \sigma(\mathcal{M}_w(E \times F), M_{mc,w}(E \times F))$ comme la topologie la plus grossière sur $\mathcal{M}_w(E \times F)$ rendant continues les fonctions $\{\langle \cdot, \varphi \rangle : \varphi \in M_{mc,w}(E \times F)\}$. Encore une fois, $(\mathcal{M}_w(E \times F), \bar{\tau}_w)$ est un espace vectoriel topologique Hausdorff localement convexe. La topologie $\bar{\tau}_w$ sera appelée la topologie de la convergence stable avec le poids w .

Considérons maintenant un certain $p \geq 1$ et dotons l'espace produit $E \times F$ de la métrique

$$d((x, y), (x', y')) = (d_E(x, x')^p + d_F(y, y')^p)^{1/p}, \quad (x, y), (x', y') \in E \times F.$$

Soit $(x_0, y_0) \in E \times F$ un point de référence arbitraire et définissons la fonction de poids $w = 1 + d(\cdot, (x_0, y_0))^p$. Nous dénotons par $\bar{\tau}_p := \bar{\tau}_w$ (qui ne dépend pas de (x_0, y_0)) la topologie de convergence stable avec p -croissance et $M_{mc,p}(E \times F) := M_{mc,w}(E \times F)$. Lorsque $w = 1$, on dénote par $\bar{\tau} := \bar{\tau}_w$ la topologie habituelle de la convergence stable et $M_{mc,b}(E \times F) := M_{mc,w}(E \times F)$.

Nous rappelons maintenant le théorème de désintégration pour les mesures positives (qui est déduit via une procédure de normalisation standard du théorème de désintégration pour les mesures de probabilité, voir Théorème 1.1 dans [CD18b]).

Theorem 2.1 (*Théorème de Désintégration*). *Soit (E, \mathcal{E}) un espace mesurable et soit F un espace polonais. Si μ est une mesure positive finie sur $(E \times F, \mathcal{E} \otimes \mathcal{B}(F))$, alors il existe une famille de noyaux $(\nu(x, C))_{x \in E, C \in \mathcal{B}(F)}$ telle que :*

- (1) *Pour tout $x \in E$, la fonction $\mathcal{B}(F) \ni C \mapsto \nu(x, C)$ est une mesure de probabilité sur $(F, \mathcal{B}(F))$.*
- (2) *Pour tout $C \in \mathcal{B}(F)$, la fonction $E \ni x \mapsto \nu(x, C)$ est mesurable de (E, \mathcal{E}) à $([0, 1], \mathcal{B}([0, 1]))$.*
- (3) *Pour tout $B \times C \in \mathcal{E} \otimes \mathcal{B}(F)$,*

$$\mu(B \times C) = \int_B \nu(x, C) \mu^E(dx),$$

où $\mu^E := \mu(\cdot \times F)$.

2.4 Introduction à la formulation par programmation linéaire

L'objectif principal de cette thèse sera de prouver l'existence de solutions MFG pour différents contextes sous des hypothèses générales et de développer des méthodes numériques pour les approcher. La formulation par programmation linéaire fournit un outil puissant pour montrer des résultats d'existence sous des hypothèses faibles et se prête bien à l'implémentation numérique. Nous allons présenter l'intuition de la méthodologie par programmation linéaire (LP) dans le cas d'un problème d'arrêt optimal à un seul agent, puis montrer qu'elle s'étend de manière très naturelle au cas MFG. Le cas du contrôle ou le cas mixte contrôle/arrêt seront présentés dans les chapitres respectifs de la thèse.

Considérons le problème d'arrêt optimal suivant pour un seul agent :

$$\begin{aligned} \sup_{\tau} \quad & \mathbb{E} \left[\int_0^{\tau} f(t, X_t) dt + g(\tau, X_{\tau}) \right], \\ \text{s.t.} \quad & dX_t = b(t, X_t) dt + \sigma(t, X_t) dW_t, \\ & X_0 \sim m_0^*, \end{aligned} \tag{2.3}$$

où W est un mouvement Brownien standard indépendant de X_0 .

Définissons le flux de mesures de sous-probabilité

$$m_t(B) := \mathbb{P}(X_t \in B, t < \tau), \quad B \in \mathcal{B}(\mathbb{R}), \quad t \in [0, T],$$

et la mesure de probabilité $\mu := \mathbb{P} \circ (\tau, X_\tau)^{-1}$. En appliquant la formule d'Itô entre l'instant 0 et l'instant τ et en prenant l'espérance, on obtient pour tout $u \in C_b^{1,2}([0, T] \times \mathbb{R})$

$$\mathbb{E}[u(\tau, X_\tau)] = \mathbb{E}[u(0, X_0)] + \mathbb{E} \left[\int_0^\tau (\partial_t u + \mathcal{L}u)(t, X_t) dt \right]$$

où

$$\mathcal{L}u(t, x) = b(t, x)\partial_x u(t, x) + \frac{\sigma^2}{2}(t, x)\partial_{xx}u(t, x),$$

et $C_b^{1,2}([0, T] \times \mathbb{R})$ désigne l'ensemble des fonctions $u \in C_b([0, T] \times \mathbb{R})$ telles que $\partial_t u, \partial_x u, \partial_{xx}u \in C_b([0, T] \times \mathbb{R})$. En particulier, (m, μ) satisfait

$$\langle \mu, u \rangle = \langle m_0^*, u(0, \cdot) \rangle + \langle m_t(dx)dt, \partial_t u + \mathcal{L}u \rangle, \quad u \in C_b^{1,2}([0, T] \times \mathbb{R}). \quad (2.4)$$

De plus, la fonction objective peut être reformulée comme suit

$$\mathbb{E} \left[\int_0^\tau f(t, X_t) dt + g(\tau, X_\tau) \right] = \langle m_t(dx)dt, f \rangle + \langle \mu, g \rangle.$$

La formulation LP consiste à maximiser cette quantité sur (m, μ) satisfaisant (2.4), soit

$$\begin{aligned} \sup_{(m, \mu)} \quad & \langle m_t(dx)dt, f \rangle + \langle \mu, g \rangle, \\ \text{s.t.} \quad & \langle \mu, u \rangle = \langle m_0^*, u(0, \cdot) \rangle + \langle m_t(dx)dt, \partial_t u + \mathcal{L}u \rangle, \quad u \in C_b^{1,2}([0, T] \times \mathbb{R}), \\ & m_t \in \mathcal{P}^{sub}(\mathbb{R}), \quad t \in [0, T], \\ & \mu \in \mathcal{P}(\mathbb{R}). \end{aligned} \quad (2.5)$$

Nous notons que la reformulation de la fonction objectif et la dérivation de la contrainte (2.4), ne dépendent pas de l'espace de probabilité spécifique ou du mouvement Brownien utilisés pour définir le problème, mais de la loi jointe du processus d'état et du temps d'arrêt.

Le problème d'optimisation (2.5) est un problème de programmation linéaire en dimension infinie (espace des mesures) puisque la fonction objectif et les contraintes sont linéaires par rapport à m et μ . En fonction des hypothèses sur les données m_0^* , b , σ , f et g , on peut généralement travailler avec une certaine topologie faible pour les mesures et montrer la compacité de l'ensemble des contraintes et la continuité de la fonction objectif prouvant l'existence d'un maximiseur. Pour approcher la solution de ce problème, on peut approcher les mesures (m, μ) avec des mesures sur un espace temps-état discret et utiliser un schéma de discrétisation pour les dérivées des fonctions test u . De plus, dans un espace temps-état discret, les fonctions de test sont engendrées par les fonctions indicatrices, et puisque la contrainte est linéaire par rapport à u , nous obtenons un ensemble fini de contraintes. Il existe aujourd'hui des algorithmes de réso-

lution LP rapides disponibles¹, permettant de résoudre le problème LP discrétisé en un temps très compétitif.

Une question importante qui se pose dans la formulation par programmation linéaire concerne l'interprétation des contraintes (2.4). En effet, nous savons que pour chaque loi jointe admissible de (X, τ) on peut associer un couple (m, μ) satisfaisant (2.4), mais, l'inverse est-il vrai ? Si ce n'est pas le cas, il est possible que le problème (2.5) soit trop relaxé et que l'interprétation de l'arrêt optimal soit perdue. Cependant, l'inverse est vrai dans des conditions générales, ce qui signifie qu'à partir de chaque couple admissible (m, μ) satisfaisant la contrainte LP (2.4), nous pouvons récupérer une configuration probabiliste telle que les mesures sont définies comme des mesures d'occupation associées au processus d'état et au temps d'arrêt (voir par exemple [CS02] dans le cas homogène en temps ; l'extension avec dépendance temporelle permettant l'irrégularité en temps et non bornitude des coefficients est prouvée dans cette thèse). En plus d'impliquer l'équivalence entre les problèmes d'optimisation, avoir une représentation probabiliste des contraintes est très utile pour caractériser leur régularité via la régularité de l'EDS.

Illustration : Exercice optimal d'une option put américain. Nous illustrons la méthodologie numérique dans le cas d'un seul agent en résolvant numériquement le problème LP pour l'exercice optimal d'une option put américain.

Nous nous plaçons dans le contexte du modèle de Black-Scholes pour un seul actif et définissons le prix initial de l'actif sous-jacent $S_0 = 100$ (c'est-à-dire $m_0^* = \delta_{100}$), la maturité $T = 1$, le strike $K = 100$, le taux d'intérêt $r = 0.05$ et la volatilité $\sigma = 0.3$. L'acheteur de l'option put américain souhaite trouver l'instant optimal pour exercer l'option, c'est-à-dire qu'il résout le problème suivant

$$\begin{aligned} \sup_{\tau \in \mathcal{T}} \quad & \mathbb{E}[e^{-r\tau}(K - S_\tau)^+], \\ \text{s.t.} \quad & dS_t = rS_t dt + \sigma S_t dW_t, \\ & S_0 = 100, \end{aligned} \tag{2.6}$$

où \mathcal{T} est l'ensemble des temps d'arrêt à valeurs dans $[0, T]$ par rapport à la filtration complétée de W . La formulation LP du problème s'écrit

$$\begin{aligned} \sup_{(m, \mu)} \quad & \int_{[0, T] \times \mathbb{R}} e^{-rt}(K - x)^+ \mu(dt, dx), \\ \text{s.t.} \quad & \int_{[0, T] \times \mathbb{R}} u(t, x) \mu(dt, dx) = u(0, 100) + \int_0^T \int_{\mathbb{R}} (\partial_t u + \mathcal{L}u)(t, x) m_t(dx) dt, \\ & u \in C_b^{1,2}([0, T] \times \mathbb{R}), \\ & m_t \in \mathcal{P}^{sub}(\mathbb{R}), \quad t \in [0, T], \\ & \mu \in \mathcal{P}(\mathbb{R}). \end{aligned} \tag{2.7}$$

En calculant la solution discrétisée associée, nous obtenons une approximation de la solution optimale (m, μ) , illustrée dans la Figure 2.1. Plus précisément, dans le graphique de gauche, nous

1. voir <http://plato.asu.edu/ftp/lpsimp.html> pour une comparaison de différents algorithmes de résolution LP.

traçons le flux optimal de mesures de sous-probabilité m de sorte que l'intensité de la couleur à un certain temps t et à un prix s représente la masse de la mesure de sous-probabilité discrétisée m_t au point s . Sur le graphique de droite, nous montrons le μ optimal qui est représenté par la masse de la mesure de probabilité discrétisée sur l'espace temps-état. Pour comparer nos résultats avec l'approche classique, nous représentons dans la Figure 2.2 la Frontière d'Arrêt Optimal (FAO) pour le put américain, calculée à l'aide d'un arbre binomial dans le modèle de Cox-Ross-Rubinstein (CRR). Nous vérifions numériquement que la mesure de probabilité μ est supportée sur la FAO.

Formellement parlant, si nous prenons une fonction régulière φ supportée sur l'ensemble de continuation et évaluons la contrainte (2.4) avec une telle fonction test, nous voyons que

$$\langle m_0^*, \varphi(0, \cdot) \rangle + \langle m_t(dx)dt, \partial_t \varphi + \mathcal{L}\varphi \rangle = 0,$$

puisque numériquement $\mu = 0$ sur la région de continuation. Cela signifie que l'équation de Fokker-Planck est satisfaite sur la région de continuation. Cependant, lorsque la FAO est atteinte, il y a une perte de masse du flux de mesures et l'égalité dans l'équation ci-dessus est brisée pour les fonctions test qui sont non nulles sur la FAO. L'égalité est rétablie lorsque nous ajoutons la mesure de probabilité μ , qui peut être interprétée, dans le jargon de la programmation linéaire, comme une *variable d'écart*.

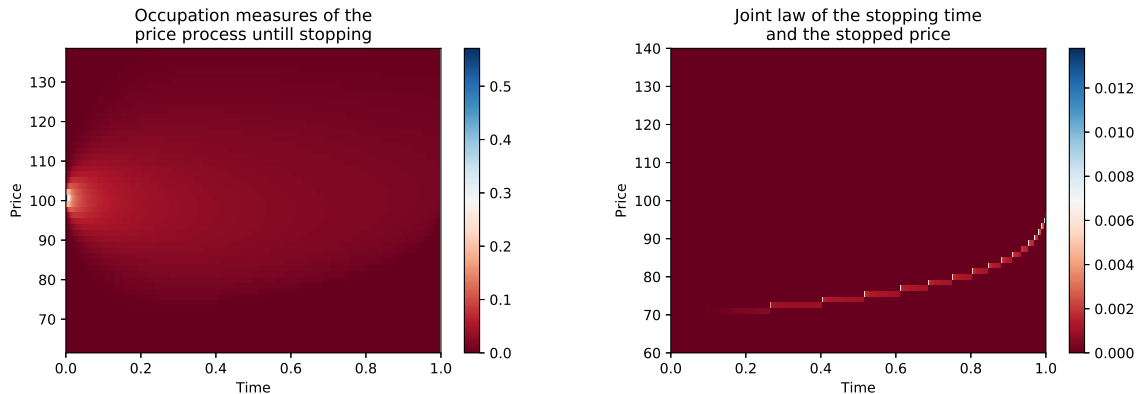


FIGURE 2.1 – Couple optimal de m (graphique de gauche) et μ (graphique de droite).

La formulation par programmation linéaire pour les MFGs. La formulation par programmation linéaire pour le problème du seul agent relaxe l'optimisation en travaillant avec les mesures d'occupation, ce qui signifie que nous travaillons avec des lois au lieu de temps d'arrêt. Lorsque nous avons défini le problème MFG d'arrêt optimal, nous avons vu qu'une solution MFG s'écrit exactement comme la mesure d'occupation d'un certain temps d'arrêt. En particulier, cela signifie que l'objet d'intérêt dans les MFG est exactement la variable de décision dans la formulation LP et, par conséquent, le problème peut être entièrement défini en termes de mesures.

Supposons que les interactions champ-moyen se produisent uniquement par le biais des fonctions de gain f et g . Une définition naturelle de l'équilibre de Nash MFG dans la formulation LP

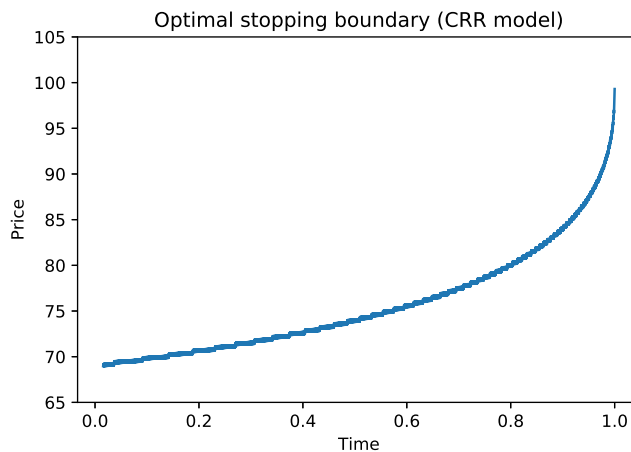


FIGURE 2.2 – Frontière d’Arrêt Optimal pour le Put Américain en utilisant un modèle CRR.

serait la suivante : le couple (m^*, μ^*) est un LP MFG équilibre de Nash s’il satisfait la contrainte (2.4) et pour chaque (m, μ) satisfaisant (2.4) nous avons

$$\langle m_t(dx)dt, f(\cdot, m^*) \rangle + \langle \mu, g(\cdot, \mu^*) \rangle \leq \langle m_t^*(dx)dt, f(\cdot, m^*) \rangle + \langle \mu^*, g(\cdot, \mu^*) \rangle.$$

Ici, les mesures (m^*, μ^*) peuvent non seulement être interprétées comme des temps d’arrêt relaxés mais aussi comme la distribution de la population d’agents encore dans le jeu et la distribution jointe des temps d’arrêt et des états arrêtés des agents. De plus, cette approche capture la nature du MFG puisqu’elle modélise la façon dont les distributions des états et des temps d’arrêt répondent à des distributions fixes du même type. La définition est très simple et ne repose pas sur une fonction valeur. Par conséquent, les méthodes numériques peuvent être construites sans calculer la fonction valeur. Par exemple, nous allons développer un algorithme de Fictitious Play qui consiste à résoudre une séquence de problèmes de programmation linéaire (de la même manière que dans l’illustration du put américain) pour approcher le LP MFG équilibre de Nash.

3 Plan de la thèse et revue de la littérature

Cette section fournit une brève description des principaux développements de la thèse ainsi qu’un aperçu détaillé de la littérature correspondante. Le Chapitre 3 se concentre sur la théorie d’existence pour les MFGs dans lesquels les joueurs peuvent contrôler leur dynamique et choisir un instant pour sortir du jeu. La théorie de l’existence est basée sur l’approche par programmation linéaire pour les MFG. Le Chapitre 4 étudie la convergence de l’algorithme Fictitious Play dans lequel les meilleures réponses sont calculées en résolvant un problème de programmation linéaire à chaque itération. Le Chapitre 5 présente une application de l’approche par programmation linéaire pour les MFGs d’arrêt optimal avec bruit commun et information partielle à la modélisation de la transition énergétique dans l’industrie de l’électricité sous l’incertitude des scénarios découlant des futures politiques climatiques.

3.1 Jeux à champ moyen avec contrôle et arrêt optimal : une approche par programmation linéaire (Chapitre 3)

Ce chapitre est basé sur l'article “Control and optimal stopping Mean Field Games : a linear programming approach” [DLT21], travail en collaboration avec R. Dumitrescu et P. Tankov, *Electronic Journal of Probability*, 2021. Quelques modifications sont apportées en vue des techniques développées dans [DLT22], qui permettent de montrer des résultats d'existence sous des hypothèses plus faibles (coefficients non bornés et dépendance générale non locale des termes champ-moyen).

Revue de la littérature. Dans la littérature MFG, il existe deux approches principales pour prouver l'existence d'un équilibre de Nash dans le cas du contrôle : l'approche *EDP* et l'approche *probabiliste*. La première méthode a été développée initialement par Lasry et Lions [LL06a; LL06b; LL07]. L'idée derrière l'approche EDP est que pour un flux fixe de mesures de probabilité $m = (m_t)_{t \in [0, T]}$ représentant la distribution des joueurs, le problème de contrôle stochastique (2.1) auquel fait face l'agent représentatif est caractérisé par l'équation de Hamilton-Jacobi-Bellman (HJB) satisfaite par la fonction valeur

$$\begin{cases} -\partial_t v(t, x) - \sup_{a \in A} [\mathcal{L}[m]v(t, x, a) + f(t, x, m_t, a)] = 0, & \text{sur } [0, T] \times \mathbb{R}, \\ v(T, \cdot) = g(\cdot, m_T), \end{cases} \quad (2.8)$$

où l'opérateur $\mathcal{L}[m]$ est défini pour $\varphi \in C_b^{1,2}([0, T] \times \mathbb{R})$ par

$$\mathcal{L}[m]\varphi(t, x, a) := b(t, x, m_t, a)\partial_x \varphi(t, x) + \frac{\sigma^2}{2}(t, x, m_t, a)\partial_{xx} \varphi(t, x).$$

Maintenant, en supposant qu'il existe une unique fonction régulière $\alpha^* : [0, T] \times \mathbb{R} \rightarrow A$ telle que pour chaque $(t, x) \in [0, T] \times \mathbb{R}$, $\alpha^*(t, x)$ est la solution du problème de maximisation

$$\sup_{a \in A} [\mathcal{L}[m]v(t, x, a) + f(t, x, m_t, a)],$$

nous devrions avoir la condition de consistance $m_t = \mathcal{L}(X_t^{\alpha^*, m})$ pour tout $t \in [0, T]$. En supposant que la distribution des joueurs $m_t(dx)dt$ admet une densité régulière $(t, x) \mapsto m(t, x)$, cette condition peut être écrite comme une équation de Fokker-Planck-Kolmogorov pour cette densité :

$$\begin{cases} \partial_t m(t, x) + \partial_x (b(t, x, m_t, \alpha^*(t, x))m(t, x)) \\ \quad - \frac{1}{2} \partial_{xx} (\sigma^2(t, x, m_t, \alpha^*(t, x))m(t, x)) = 0, & \text{sur } (0, T] \times \mathbb{R}, \\ m(0, \cdot) = m_0^*, \end{cases} \quad (2.9)$$

où $x \mapsto m_0^*(x)$ est la densité de la mesure m_0^* décrivant la distribution initiale des joueurs. En couplant les équations (2.8) et (2.9), nous obtenons le système d'EDP MFG. La principale difficulté de ces EDP couplées provient des directions opposées du temps dans les deux équations. Les hypothèses requises pour résoudre ce système sont généralement fortes, par exemple $b(t, x, m, a) = a$, σ est constant et la fonction de gain instantanée f est séparable, ce qui signifie

que $f(t, x, m, a) = f_1(t, x, a) + f_2(t, x, m)$. Néanmoins, on peut obtenir plus d'informations sur la régularité du contrôle optimal et même travailler avec une dépendance locale du champ-moyen (c'est-à-dire que f dépend de $m(t, x)$ au lieu de la mesure m_t). Nous renvoyons à [Car12; GS14; Lac18] pour plus de détails sur l'approche EDP et à [GLL11] pour quelques applications utilisant cette approche.

L'approche probabiliste a été introduite par Carmona et Delarue [CD13a; CD13b; CD18a; CD18b] et repose sur le principe du maximum stochastique de Pontryagin qui réduit le problème à un système d'équations différentielles stochastiques couplées progressive rétrograde (EDSPR) de type McKean-Vlasov. Plus précisément, définissons l'Hamiltonien généralisé (en supposant que σ ne dépend pas du contrôle) :

$$\mathcal{H}(t, x, m, y, z, a) := b(t, x, m, a)y + \sigma(t, x, m)z + f(t, x, m, a),$$

pour $(t, x, m, y, z, a) \in [0, T] \times \mathbb{R} \times \mathcal{P}(\mathbb{R}) \times \mathbb{R} \times \mathbb{R} \times A$. Supposons qu'il existe une unique fonction régulière $(t, x, m, y) \mapsto \hat{\alpha}(t, x, m, y)$ maximisant $a \mapsto \mathcal{H}(t, x, m, y, z, a)$. Selon l'approche probabiliste, une condition suffisante pour résoudre le problème MFG est de résoudre la EDSPR de type McKean-Vlasov suivante :

$$\begin{cases} dX_t = b(t, X_t, \mathcal{L}(X_t), \alpha_t^*)dt + \sigma(t, X_t, \mathcal{L}(X_t))dW_t, \\ dY_t = -\partial_x \mathcal{H}(t, X_t, \mathcal{L}(X_t), Y_t, Z_t, \alpha_t^*)dt + Z_t dW_t, \\ X_0 = \xi, \quad Y_T = \partial_x g(X_T, \mathcal{L}(X_T)), \quad \alpha_t^* = \hat{\alpha}(t, X_t, \mathcal{L}(X_t), Y_t), \end{cases} \quad (2.10)$$

où ξ est une variable aléatoire \mathcal{F}_0 -mesurable de loi m_0^* . Plus précisément, si (X, Y, Z) est une solution de (2.10), alors $(\mathcal{L}(X_t))_{t \in [0, T]}$ est une solution MFG avec un contrôle optimal associé α^* . Que ce soit dans l'approche EDP ou dans l'approche probabiliste, le théorème du point fixe de Schauder (appliqué à la distribution des joueurs) est généralement utilisé pour résoudre le problème MFG.

En dehors de ces deux approches principales, d'autres méthodes peuvent être utilisées pour étudier l'existence de solutions MFG. Dans l'article [HMC06], les auteurs utilisent un argument de point fixe par contraction pour trouver une solution, cependant cette méthode nécessite des hypothèses fortes sur les coefficients ainsi que la définition du jeu sur un court intervalle de temps. Une autre technique basée sur le théorème de point fixe de Tarski a été appliquée pour résoudre des MFGs supermodulaires (submodulaires dans le cas d'une minimisation) dans les articles [AJ13; CDL17; Wię17; Dia+21; Dia+22]. Contrairement au théorème de point fixe de Schauder, le théorème du point fixe de Tarski n'utilise pas d'arguments topologiques, mais est basé sur la théorie des treillis. De plus, les MFGs supermodulaires sont caractérisés par la propriété de différences croissantes (différences décroissantes dans le cas d'une minimisation) sur les fonctions de gain et on peut généralement relaxer les hypothèses de continuité sur les termes de champ-moyen.

La méthodologie appliquée dans cette thèse appartient à la classe des méthodes de relaxation, également connues sous le nom de méthodes de compactification. L'idée principale est de relaxer le problème d'optimisation en permettant d'optimiser sur des mesures associées à des solutions

faibles de l'EDS et des contrôles relaxés (également connus sous le nom de stratégies mixtes). Le nouvel ensemble admissible sera compact et convexe sous des hypothèses générales, permettant l'application de théorèmes de point fixe topologiques. Dans les méthodes de compactification, l'unicité du contrôle optimal pour un champ-moyen fixé n'est généralement pas requise, ce qui implique que le théorème de point fixe de Schauder est remplacé par le théorème de point fixe de Kakutani-Fan-Glicksberg pour les fonctions multivaluées.

Une formulation faible probabiliste pour les MFGs a été développée dans [CL15] sur la base de la formulation faible des problèmes de contrôle optimal stochastique (également connue sous le nom d'approche martingale). Dans cet article, les auteurs n'utilisent pas de contrôles relaxés car ils travaillent avec des hypothèses de convexité mais permettent des solutions faibles à l'EDS à travers d'un changement de mesure de type Girsanov. De plus, ils utilisent une approche EDSR pour résoudre le problème de contrôle, ainsi qu'un argument de point fixe. En particulier, pour utiliser cette approche, les auteurs considèrent que la volatilité est indépendante du contrôle et du terme de champ-moyen. Dans l'article [Lac15], l'auteur applique *l'approche du problème martingale contrôlée* pour compactifier et convexifier le problème MFG original et montrer l'existence d'une solution MFG sous des hypothèses générales. D'autres développements ont été faits dans le cas de jeux à champ moyen avec branchement [CRT19] ou de jeux à champ moyen avec contrôles singuliers [FH17]. Dans la théorie standard du contrôle stochastique, cette technique intègre le problème de contrôle stochastique initial dans une optimisation sur les lois jointes possibles de l'EDS et du contrôle relaxé, pour laquelle des critères de tension peuvent être appliqués pour montrer l'existence d'un maximiseur (voir par exemple [KNJ87], [HL90] et [KT15]). Plus précisément, l'ensemble des variables de décision est un sous-ensemble de l'ensemble des mesures de probabilité sur l'espace de dimension infinie $C([0, T]) \times \mathcal{P}(A)^{[0, T]}$.

Dans cette thèse, nous utiliserons une deuxième technique de relaxation, connue sous le nom de *formulation par programmation linéaire* pour les MFGs, qui a été introduite dans l'article [BDT20] pour les MFGs d'arrêt optimal. Dans le cas d'un seul agent, cette technique est décrite dans plusieurs articles, par exemple par R. H. Stockbridge, T. Kurtz et leurs coauteurs (par exemple [KS98c; CS02; KS17]) et les articles [BBG97; BB96; Las+08; BGQ11]. Comme expliqué dans la Section 2.4, l'idée est de travailler avec des mesures sur un espace de dimension finie satisfaisant une contrainte linéaire. La compacité de l'ensemble des mesures admissibles peut être obtenue sous les bonnes estimées. Alors que les théorèmes d'existence d'équilibres MFG obtenus par l'approche du problème martingale contrôlé sont en général plutôt abstraits (puisqu'ils travaillent avec des lois dans des espaces de dimension infinie) et fournissent peu d'informations sur le calcul des solutions MFG, l'approche par programmation linéaire conduit à une méthode tractable pour calculer les équilibres MFG comme nous le montrerons au Chapitre 4. Pour les applications de l'approche par programmation linéaire pour les MFGs, nous nous référons à [ADT21] et [BDT22], ainsi qu'au Chapitre 5 de cette thèse. Dans le cas d'un seul agent, une approche similaire a été introduite dans les articles [LV80] (cas déterministe) et [FV88] (cas stochastique) et dans le contexte de la *théorie d'Aubry-Mather* dans les cas déterministe et stochastique (voir par exemple [Mat91; Mañ96], [Gom10] et [Gom02]). Dans le cadre du jeu à champ moyen, la théorie d'Aubry-Mather a été appliquée dans l'article récent

[Ash+21] pour étudier un modèle MFG de formation de prix. Dans l'article récent [GHZ22] (paru après l'article [DLT21] sur lequel ce chapitre est basé), les auteurs étudient l'approche par programmation linéaire pour les MFGs avec contrôle stochastique en temps discret et espace d'état fini, permettant un champ-moyen dépendant de la loi du contrôle. Dans ce cadre, les auteurs fournissent un résultat d'existence et reformulent le problème de manière à ce que la recherche de solutions MFG soit équivalente à la résolution d'un seul problème d'optimisation capturant à la fois la condition de meilleure réponse et les contraintes.

La grande majorité de la littérature sur les MFGs s'est concentrée sur les MFGs de contrôle pur et très peu d'articles ont étudié les MFGs d'arrêt optimal, c'est-à-dire des jeux où les joueurs peuvent décider quand quitter le jeu. Les MFGs avec arrêt optimal apparaissent de manière naturelle puisqu'il est trop restrictif d'imposer que tous les joueurs doivent rester jusqu'à la fin du jeu comme c'est le cas dans le cadre classique des MFGs. Dans l'article [GP15], les auteurs étudient des problèmes d'obstacles pour des jeux à champ moyen de premier ordre dans le cadre périodique stationnaire, qui modélisent un certain type de MFGs de contrôle stochastique/arrêt optimal. Ils montrent l'existence d'une solution à leur système via la méthode de pénalisation sous de fortes hypothèses sur les coefficients. L'article [CDL17] montre l'existence d'une solution MFG dans deux configurations différentes impliquant un bruit commun. Dans la première, les auteurs travaillent avec des MFGs supermodulaires d'arrêt optimal avec bruit commun et montrent l'existence "d'équilibres forts" (ce qui signifie que le terme de champ-moyen est adapté à la filtration du bruit commun) sous des hypothèses restrictives. Ces hypothèses sont relâchées dans le second cadre, où ils montrent l'existence "d'équilibres faibles" dans lesquels le terme de champ-moyen peut être adapté à une filtration plus grande que celle du bruit commun et le temps d'arrêt peut être randomisé de manière externe par rapport aux sources initiales de bruit (voir également [CDL16]). Un modèle spécifique de MFGs avec arrêt optimal, étudié dans [Nut18], permet un continuum de joueurs hétérogènes ainsi qu'un bruit commun. Nous soulignons que dans les articles [CDL17; Nut18] le terme champ-moyen correspond à la distribution des temps d'arrêt choisis par les joueurs (qui est la première marginale de la mesure de probabilité μ décrite dans la Section 1.2). Une approche EDP est utilisée dans [Ber17], où l'auteur introduit la notion de "solution mixte" à la fois pour le MFG d'arrêt optimal et le MFG mixte contrôle/arrêt. Les solutions mixtes représentent une version EDP des temps d'arrêt optimaux qui se situent entre les temps d'arrêt optimaux dits minimal et maximal, ce qui signifie que les joueurs ne s'arrêtent pas immédiatement lorsqu'ils sortent de la région de continuation. L'existence et l'unicité d'une solution à ce système d'EDPs sont démontrées sous des hypothèses assez fortes sur le processus d'état et le domaine. Dans un cadre plus général, l'article [BDT20] introduit une formulation par programmation linéaire pour les MFGs avec arrêt optimal. Les auteurs montrent des résultats d'existence et prouvent que leur solution d'équilibre est une solution mixte dans la terminologie de [Ber17] sous des hypothèses appropriées. Enfin, dans l'article récent [Dia+22], paru après l'article [DLT21] sur lequel ce chapitre est basé, les auteurs montrent l'existence d'un équilibre fort (c'est-à-dire adapté à la filtration du bruit commun) pour les MFGs avec arrêt optimal et bruit commun dans le cadre supermodulaire. Dans les trois articles [Ber17; BDT20; Dia+22], les auteurs considèrent les interactions par rapport à la distribution des joueurs encore

dans le jeu (qui est le flux de mesures de sous-probabilité m décrit dans la Section 1.2) et non la première marginale de μ comme dans [CDL17 ; Nut18].

Une autre classe de MFGs étudiés dans cette thèse sont les MFGs avec absorption, où si le processus d'état d'un joueur quitte un domaine spécifique, le joueur est obligé de sortir du jeu. Cette propriété est importante pour de nombreuses applications (par exemple, lors de la modélisation d'un jeu dans lequel les joueurs peuvent faire défaut) et peut être trouvée à la fois dans le cadre de contrôle ou dans le cadre d'arrêt. L'article [CF18] introduit les MFGs avec absorption dans le cas de contrôle et montre l'existence d'une solution utilisant une approche similaire à [CL15] mais avec des difficultés techniques provenant du temps de sortie. Les hypothèses de l'article [CF18] sont davantage relaxées dans [CGL21] en utilisant des arguments de convergence faible, des problèmes de martingale contrôlé et la notion de contrôles relaxés. Le cas beaucoup plus complexe permettant un bruit commun est étudié dans [BC21], où l'on montre l'existence d'équilibres faibles (c'est-à-dire adaptés à une filtration plus grande que celle du bruit commun). Dans les articles [Ber17 ; BDT20], des MFGs avec arrêt optimal et absorption sont considérés et l'existence d'une solution est montrée, le second article permettant des domaines plus généraux (par exemple le cas non borné).

Contributions. Dans ce chapitre, basé sur l'article [DLT21] ainsi que sur certaines extensions, nous prouvons des résultats d'existence dans un cadre général de MFGs permettant de l'arrêt optimal, du contrôle stochastique et de l'absorption.

L'absence de solutions explicites dans un cadre général rend naturelle l'utilisation de techniques de relaxation permettant de montrer des résultats d'existence sous des hypothèses faibles. De plus, puisque les équilibres ne seront pas retrouvés sous forme explicite, il est important de relaxer le problème de telle sorte que nous puissions appliquer des algorithmes numériques. En tenant compte de ces aspects, nous choisissons de montrer l'existence de solutions MFG en utilisant l'approche par programmation linéaire pour les MFGs introduite dans [BDT20]. D'une part, les hypothèses d'application de cette approche sont très faibles et, d'autre part, comme son nom l'indique, l'optimisation permettant de calculer la meilleure réponse consiste à résoudre un problème de programmation linéaire, ce qui de nos jours peut être fait rapidement numériquement.

Nous étendons les techniques utilisées dans [BDT20] à un cadre plus général impliquant de l'arrêt optimale/contrôle et permettant des coefficients dépendant de la mesure. Nous utilisons la même définition des équilibres MFGs que dans [DLT21], mais la méthodologie pour montrer l'existence est différente. Plus précisément, nous utilisons la topologie de convergence en mesure appliquée au flux de mesures de sous-probabilité avec des moments polynomiaux (qui a été utilisée dans [DLT22]). Cette nouvelle structure topologique permet d'affaiblir plusieurs hypothèses imposées dans [BDT20]. Premièrement, nous autorisons des coefficients de diffusion dépendant des termes champ-moyen, ce qui implique une analyse beaucoup plus profonde de l'approche par programmation linéaire. Plus précisément, nous utilisons la représentation probabiliste des mesures satisfaisant la contrainte afin de surmonter certaines difficultés techniques découlant de la dépendance des coefficients au champ-moyen. Deuxièmement, nous autorisons des coefficients

et des fonctions de gain instantané non bornés, ainsi qu’une fonction de gain final générale. Nous relaxons également les hypothèses de continuité par rapport au temps et certaines hypothèses de différentiabilité sur les gains. Enfin, nous montrons l’existence avec dépendance non locale générale des termes champ-moyen.

Comme expliqué précédemment, dans [Ber17 ; BDT20 ; Dia+22], les auteurs considèrent des interactions par rapport à la distribution des joueurs encore dans le jeu, et dans [CDL17 ; Nut18], le terme champ-moyen correspond à la distribution des temps d’arrêt choisis par les joueurs. Ces deux caractéristiques sont incluses dans notre cadre.

En ce qui concerne la partie contrôle stochastique des MFGs, nos hypothèses sont comparables à celles de [Lac15] à la différence que nous imposons que les contrôles soient bornés, que la dimension de l’état soit un et que la distribution initiale admette des moments du second ordre. Puisque nos techniques et celles utilisées dans [Lac15] sont liées, comme nous l’expliquerons dans un moment, nous devrions être en mesure de relaxer ces hypothèses. De plus, nous retrouvons directement l’existence d’un équilibre Markovien relaxé montré dans [Lac15], où l’auteur utilise le théorème du mimétisme (ou théorème de projection Markovienne) qui se trouve dans le Corollaire 3.7. de [BS13]. Dans notre cas, le contrôle Markovien relaxé est obtenu directement en désintégrant le flux de mesures $(m_t(dx, da))_{t \in [0, T]}$, modélisant la distribution jointe des joueurs et de leurs contrôles, par rapport à la marginale d’état.

Nous étendons un résultat d’unicité pour la valeur de Nash donné dans [BDT20] à notre cadre dans le cas où les coefficients de diffusion ne dépendent pas de la mesure, ce qui signifie que même si plusieurs équilibres existent, les joueurs obtiendront le même gain espéré, indépendamment de l’équilibre choisi. De plus, si les coefficients de diffusion sont autorisés à dépendre de la mesure, nous montrons qu’il existe un équilibre de Nash maximisant la valeur de Nash.

La compacité des flux de mesures admissibles est dérivée de leur représentation probabiliste, qui permet de faire des estimées précises. De plus, la représentation probabiliste permet d’établir une équivalence entre la formulation par programmation linéaire pour les MFGs et les MFGs via l’approche du problème martingale contrôlée/arrêtée, qui était l’approche utilisée dans [Lac15] dans le cas de contrôle. Pour prouver ce résultat, nous étendons certains des résultats existants dans la littérature [KS98c ; CS02 ; DS12 ; KS17] au cas mixte arrêt/contrôle en horizon fini. De plus, la représentation probabiliste permet une interprétation des mesures en termes du concept bien connu de temps d’arrêt randomisés (voir par exemple [BC77 ; EMS82 ; TV02 ; CDL17]).

Enfin, de manière similaire à [BDT20], nous montrons sous des conditions appropriées (qui permettent notamment de résoudre l’inégalité variationnelle de Hamilton-Jacobi-Bellman au sens Sobolev pour des termes champ-moyen fixés) que tout équilibre de Nash LP MFG forme une solution mixte au sens de [Ber17].

3.2 Algorithme “Linear Programming Fictitious Play” pour les Jeux à Champ Moyen avec arrêt optimal et absorption (Chapitre 4)

Ce chapitre est basé sur l’article “Linear Programming Fictitious Play algorithm for Mean Field Games with optimal stopping and absorption” [DLT22], travail en collaboration avec R. Dumitrescu and P. Tankov, soumis pour publication.

Revue de la littérature. Comme déjà expliqué, il est rarement possible d’obtenir des solutions explicites aux problèmes MFG car l’existence de l’équilibre de Nash est généralement obtenue par l’application de théorèmes abstraits de points fixe plutôt que de manière constructive. Pour cette raison, il est important de construire des algorithmes numériques rapides qui peuvent approcher les équilibres de Nash MFG et donc être utilisés dans les applications. Tout comme dans la théorie de l’existence pour les MFGs, il existe différentes approches pour construire des algorithmes numériques qui, dans la plupart de la littérature, sont basées sur les méthodologies EDP ou probabiliste.

L’article [AC10] introduit pour la première fois un schéma à différences finies pour résoudre numériquement des MFGs stationnaires et évolutifs sur le tore. Les schémas à différences finies pour les MFGs sont étudiés plus en détail dans [ACC13; AP16]. Plus précisément, dans [AC10; ACC13], la convergence du problème discret vers le système MFG est fournie sous l’hypothèse que le système MFG a une solution classique unique et que la condition de monotonicité de Lasry-Lions est vérifiée. En revanche, l’article [AP16] permet des hypothèses plus faibles et montre que le problème discret converge vers une solution faible du système MFG. Un schéma de différences finies a également été appliqué dans les articles [BKS18; Bri+19] pour résoudre numériquement des MFGs avec des couplages locaux. Les schémas semi-lagrangiens sont étudiés dans les articles [CS14] et [CS15] pour les MFGs du premier ordre et les MFGs du second ordre, respectivement. Dans [AP12; Gué12], des méthodes numériques basées sur l’algorithme de Newton sont décrites. Les MFGs potentiels (parfois appelés MFGs variationnels), c’est-à-dire des MFGs dont les solutions coïncident avec les maximiseurs d’une certaine fonctionnelle, sont étudiés dans les papiers [LST10; BC15; AL16; CH17; BLP21a; BLP21b] pour les MFGs de contrôle stochastique et dans [BDT20] pour les MFGs avec arrêt optimal. Pour plus de détails sur ces méthodologies, nous nous référons à la revue [AL20]. D’autre part, des méthodes numériques pour résoudre les EDSPRs de type McKean-Vlasov apparaissant dans l’approche probabiliste pour les MFGs sont développées dans [CCD19; Ang+19]. Pour éviter la malédiction de la dimensionnalité dans les méthodes ci-dessus, des méthodes stochastiques basées sur des approximations par des réseaux de neurones ont été construites, par exemple dans [CL21; CL19; GMW22].

Notre méthodologie est basée sur le *Fictitious Play Algorithm* qui remonte à Brown [Bro51] dans le cadre classique de la théorie des jeux. Il a été étendu aux MFGs pour la première fois dans l’article [CH17]. Le Fictitious Play est une procédure d’apprentissage dans laquelle les joueurs mettent à jour leurs stratégies en fonction des stratégies passées et de la fonction de meilleure réponse. L’idée derrière cet algorithme est que des configurations d’équilibre peuvent apparaître en répétant le jeu de nombreuses fois. Le Fictitious Play pour les MFGs a été appliqué dans le cadre de MFGs potentiels avec contrôles réguliers dans [CH17] et de MFGs de premier ordre avec contrôles réguliers dans [Had17; HS19; Eli+20]. Une version théorique du Fictitious Play (les étapes de l’algorithme sont indexées par le continuum) a été étudiée dans [Per+20] pour les MFGs finis avec bruit commun. Dans ce dernier article, les auteurs fournissent théoriquement un taux de convergence de $O(t^{-1})$ pour l’exploitabilité (i.e. l’augmentation du gain en changeant à la meilleure réponse). Le Fictitious Play a également été utilisé dans le récent article [DV21]

pour un MFG linéaire-quadratique spécifique avec bruit commun. Nous nous référons à la revue récente [Lau+22] pour plus de détails sur les algorithmes d'apprentissage pour les MFGs.

Comme expliqué dans la sous-section précédente, l'article [GHZ22] (paru après l'article [DLT22] sur lequel ce chapitre est basé) développe une approche par programmation linéaire pour les MFGs avec contrôle stochastique en temps discret et état fini, permettant un champ-moyen dépendant de la loi du contrôle, et montre la convergence d'un algorithme de descente de gradient projeté. Ils reformulent le problème de telle sorte que la recherche de solutions MFG devient équivalente à la résolution d'un seul problème d'optimisation capturant à la fois la condition de meilleure réponse et les contraintes, et fournissent des méthodes numériques exploitant la structure linéaire du problème. L'avantage de cette approche est qu'elle ne suppose pas l'unicité de la meilleure réponse ou des conditions de monotonicité sur les gains.

La plupart de la littérature numérique sur les MFGs s'est concentrée sur le cas du contrôle stochastique, et seuls quelques articles étudient le cas de l'arrêt optimal. Dans l'article [BDT20], un algorithme d'apprentissage pour les MFGs potentiels est développé et sa convergence est démontrée. L'algorithme d'Uzawa est étudié dans l'article [Ber20] pour le système MFG (éventuellement non potentiel) introduit dans l'article [Ber17] dans le cas stationnaire, sous l'hypothèse de monotonicité stricte de Lasry-Lions de la fonction de gain. Récemment, les MFG super-modulaires d'arrêt optimal (avec bruit commun) ont été analysés dans [Dia+22]. Les auteurs fournissent une approximation des équilibres minimaux (par rapport à une structure d'ordre spécifique) en commençant par le flux de mesure minimal et en itérant la meilleure réponse minimale. Les résultats sont valables sous l'inégalité opposée à la condition de monotonicité de Lasry-Lions, qui est celle qu'on utilisera dans nos résultats. Des calculs numériques combinant le Fictitious Play et l'approche par programmation linéaire ont été fournis dans [BDT22] pour une application à la gestion de l'eau et dans [ADT21] pour une application aux marchés de l'électricité. Cependant, dans ces deux articles, la convergence de l'algorithme n'a pas été établie. Dans ce chapitre, nous cherchons à combler cette lacune.

Contributions. L'objectif de ce chapitre est de construire un algorithme numérique pour les MFGs avec arrêt optimal et les MFGs avec contrôle et absorption et de montrer sa convergence vers la solution MFG (qui sous nos hypothèses sera unique). Plus spécifiquement, nous étudierons la procédure d'apprentissage Fictitious Play dans laquelle les meilleures réponses sont calculées en résolvant des programmes linéaires. Le cadre sera entièrement basé sur l'approche par programmation linéaire pour les MFGs, introduite dans [BDT20] et étudiée plus en profondeur dans [DLT21], dans le cas de jeux de second ordre possiblement non-potentiels, sous des hypothèses générales sur les coefficients et une condition de monotonicité non stricte sur les fonctions de gain par rapport aux mesures.

Le “*linear programming fictitious play algorithm*” (LPFP) commence avec une supposition initiale de l'équilibre et itère les deux étapes suivantes :

- (i) *Étape d'optimisation* : Calculer la meilleure réponse correspondant à la supposition en résolvant un programme linéaire.
- (ii) *Étape d'apprentissage* : Mettre à jour la supposition via une combinaison convexe (avec des

paramètres bien choisis) de la supposition précédente et de la meilleure réponse calculée à l'étape (i).

Bien que le problème linéaire à résoudre dans l'étape (i) soit en dimension infinie, on peut l'approcher par un programme linéaire en dimension finie (voir par exemple [MS98 ; CS02]), pour lequel il existe des algorithmes rapides et précis². Ceci est en contraste avec d'autres techniques de relaxation pour lesquelles les résultats d'existence suivent sous des hypothèses faibles mais les schémas numériques sont difficiles à construire à cause des définitions abstraites. Pour être plus précis sur la procédure de discrétisation, l'idée est d'approcher le problème de programmation linéaire en dimension infinie³ (2.5)

$$\begin{aligned} & \sup_{(m,\mu)} \quad \langle m_t(dx)dt, f \rangle + \langle \mu, g \rangle, \\ \text{s.t.} \quad & \langle \mu, u \rangle = \langle m_0^*, u(0, \cdot) \rangle + \langle m_t(dx)dt, \partial_t u + \mathcal{L}u \rangle, \quad u \in C_b^{1,2}([0, T] \times \mathbb{R}), \\ & m_t \in \mathcal{P}^{sub}(\mathbb{R}), \quad t \in [0, T], \\ & \mu \in \mathcal{P}(\mathbb{R}) \end{aligned}$$

par le problème de programmation linéaire en dimension finie :

$$\begin{aligned} & \sup_{(m,\mu)} \quad \sum_{t=t_0}^{t_{n-1}} \langle m_t(dx), f(t, \cdot) \rangle + \langle \mu, g \rangle, \\ \text{s.t.} \quad & \langle \mu, u \rangle = \langle \bar{m}_0^*, u(0, \cdot) \rangle + \sum_{t=t_0}^{t_{n-1}} \langle m_t(dx), \bar{\mathcal{L}}u(t, \cdot) \rangle, \quad u \in \mathcal{D}, \\ & m_t \in \mathcal{P}^{sub}(E), \quad t \in \{t_0, \dots, t_{n-1}\}, \\ & \mu \in \mathcal{P}(\{t_0, \dots, t_n\} \times E), \end{aligned}$$

où $\{t_0 = 0 < t_1 < \dots < t_n = T\}$ est la grille de temps, $E = \{x_0 < x_1 < \dots < x_k\} \subset \mathbb{R}$ est la grille d'état, \bar{m}_0^* est une discrétisation de m_0^* sur $E \setminus \{x_0, x_k\}$, $\bar{\mathcal{L}}$ est une discrétisation du générateur du processus $(t, X_t)_{t \in [0, T]}$ et \mathcal{D} est un ensemble fini de fonctions allant de $\{t_0, \dots, t_n\} \times E$ dans \mathbb{R} , généralement considéré comme l'ensemble des fonctions indicatrices sur la grille temps-état.

La principale difficulté de l'analyse de l'algorithme Fictitious Play pour les MFGs avec arrêt optimal est l'irrégularité temporelle des flux de mesures. Ce manque de régularité est également présent dans les MFGs avec absorption et, à notre connaissance, aucun algorithme n'a été proposé jusqu'à présent dans ce cas. Pour être plus précis, la fonction de gain instantané est irrégulière par rapport au temps en raison de la dépendance champ-moyen (rappelons que pour les MFG avec arrêt optimal et les MFG avec absorption, une fonction indicatrice du temps apparaît sur les flux de mesures, impliquant un manque de continuité par rapport au temps). De plus, pour montrer la convergence de l'algorithme, il sera important de prouver, pour une

2. voir <http://plato.asu.edu/ftp/lpsimp.html> pour une comparaison de différents algorithmes de résolution LP.

3. Nous prenons le cas de l'arrêt optimal comme illustration, mais le cas du contrôle est analogue.

métrique d bien choisie, l'existence d'une constante C telle que pour tous les $N \geq 1$.

$$\langle m^{(N+2)} - m^{(N+1)}, f[\bar{m}^{(N+1)}] - f[\bar{m}^{(N)}] \rangle \leq \frac{C}{N} d(m^{(N+1)}, m^{(N+2)}), \quad (2.11)$$

où $\bar{m}^{(N)}$ est la supposition à l'étape N et $m^{(N+1)}$ sa meilleure réponse. Cette estimation sera obtenue en utilisant un résultat de dualité pour une métrique d de type 1-Wasserstein. Cependant, l'absence de régularité temporelle dans $f[\bar{m}^{(N+1)}] - f[\bar{m}^{(N)}]$ rend impossible l'utilisation du résultat de dualité habituel pour les métriques de Wasserstein si les mesures ont une composante temporelle. En conséquence, nous utiliserons la topologie de la convergence en mesure pour les fonctions du temps (doté de la mesure de Lebesgue) vers l'espace des mesures de sous-probabilité (doté d'une certaine métrique de type Wasserstein). Pour cette topologie, nous pouvons choisir une métrique d appropriée permettant de montrer (2.11). Cette topologie est utilisée dans le Chapitre 3 de cette thèse pour généraliser les résultats d'existence de [DLT21], mais elle a été introduite dans l'article [DLT22] sur lequel ce chapitre est basé.

Contrairement à de nombreux algorithmes de la littérature MFG, y compris les algorithmes de Fictitious Play présentés précédemment [CH17; Had17; HS19; Eli+20; Per+20], notre méthodologie pour calculer la meilleure réponse n'est pas basée sur la fonction valeur, mais sur l'approche par programmation linéaire. Alors que la fonction valeur fournit des informations importantes sur le contrôle optimal ou le temps d'arrêt optimal, les objets principaux des MFGs sont les mesures d'occupation induites par les contrôles ou les temps d'arrêt optimaux et décrivant la dynamique de la population. Ces quantités peuvent être récupérées dans notre cadre sans trouver la fonction valeur et en résolvant un problème de programmation linéaire, ce qui permet des calculs rapides. De plus, nous soulignons que la distribution des contrôles optimaux ou des temps d'arrêt peut être récupérée à partir de la solution du programme linéaire. Ceci sera illustré par des implémentations numériques pour les deux cas. Dans les illustrations, nous verrons empiriquement que l'algorithme converge à une vitesse $O(N^{-1})$, N étant le nombre d'étapes d'apprentissage.

3.3 Transition énergétique dans l'incertitude de scénarios : une approche de jeux à champ moyen (Chapitre 5)

Face à l'urgence climatique, il est primordial de décarboner l'industrie électrique. L'augmentation de la pénétration des énergies renouvelables est un facteur clé pour limiter l'augmentation de la température mondiale mais elle a un fort impact sur le fonctionnement des marchés de l'électricité. Plus précisément, la pénétration accrue des énergies renouvelables, en raison de leur nature intermittente, peut entraîner des prix de pointe plus élevés et plus volatils sur le marché de l'électricité. En outre, le rythme de la décarbonisation est affecté par de nombreuses incertitudes, notamment liées aux futures politiques climatiques. Pour toutes ces raisons, il est important d'introduire des modèles flexibles et tractables de la dynamique du marché de l'électricité dans le cadre de la transition énergétique. Notre objectif sera d'appliquer les MFGs avec arrêt optimal, bruit commun et information partielle pour étudier la dynamique de sortie des producteurs conventionnels lorsque la pénétration des énergies renouvelables augmente.

Ce chapitre est basé sur l'article "Energy transition under scenario uncertainty : a mean-field game approach", travail en collaboration avec R. Dumitrescu and P. Tankov, soumis pour publication.

Revue de la littérature. L'article [CCZ15] étudie l'effet de "merit order" (baisse du prix moyen de l'électricité due à la pénétration des énergies renouvelables) sur le marché de l'électricité "day-ahead" de gros italien. La théorie des jeux avec arrêt optimal est appliquée dans l'article [Tak+08] à une situation dans laquelle deux entreprises essaient de décider si elles doivent construire une nouvelle centrale électrique. Cependant, le prix est exogène et ne dépend pas des décisions des entreprises. D'autres articles comme [HG13; LB15; RPR15] étudient les structures de marché existantes en proposant des alternatives pour améliorer certains aspects concernant la pénétration des énergies renouvelables, mais ne présentent pas de modèles pour simuler leur évolution.

Une méthodologie importante pour étudier les interactions complexes survenant en économie est basée sur l'économie computationnelle à base d'agents (ACE). En général, la modélisation à base d'agents vise à simuler un système dynamique complexe impliquant divers agents et à étudier leur comportement auto-émergent. Si le comportement global d'un système est difficile à modéliser, le comportement local des agents qui le composent est plus facile à comprendre. Ce point de vue micro-macro est similaire à la perspective MFG, cependant dans les MFGs, les joueurs optimisent un critère à long terme à travers la notion d'équilibre de Nash ; en revanche, dans les modèles à base d'agents, les agents prennent des décisions d'optimisation plus simples à chaque instant permettant de réduire certaines hypothèses par rapport aux MFGs. Les modèles ACE ont été utilisés pour comprendre la dynamique du marché de l'électricité dans plusieurs articles. Nous renvoyons le lecteur à la revue [WV08] pour des références et plus de détails sur ce sujet. En général, ces modèles offrent une bonne vision du marché, mais ils sont très exigeants en termes de calcul et le résultat de la simulation peut dépendre fortement de la configuration initiale des agents.

Dans ce chapitre, nous développerons une extension du cadre MFG développé dans [ADT21] en y intégrant un bruit commun et information partielle. Cet article utilise l'approche de programmation linéaire introduite dans [BDT20] pour résoudre un MFG avec arrêt optimal et deux classes d'agents modélisant la dynamique à long terme de l'industrie électrique. Plus précisément, les producteurs conventionnels visent à choisir un moment optimal pour sortir du marché et les producteurs de renouvelables un moment optimal pour entrer sur le marché. L'état des producteurs conventionnels est représenté par leur coût marginal de base par unité d'énergie produite et ils offrent une proportion de leur capacité installée en fonction du prix du marché et de leur coût. D'autre part, les producteurs d'énergie renouvelable offrent toujours leur production totale parce que leur coût marginal est presque nul, cependant, en raison de l'intermittence de la ressource renouvelable, ils ne seront pas toujours en mesure d'offrir leur capacité installée maximale. Leur état est donné par leur facteur de capacité stochastique. L'article suppose que la demande est inélastique et déterministe. Les deux catégories d'agents interagissent par le biais du prix de l'électricité, qui est déterminé par un mécanisme "merit order" similaire au marché de

l'électricité "day-ahead" : les producteurs de ressources renouvelables offrent la totalité de leur capacité (stochastique) à un prix nul et si l'offre de ressources renouvelables n'est pas suffisante pour répondre à la demande, les producteurs conventionnels répondent à la demande résiduelle. Tous les participants maximisent leur profit attendu en choisissant un temps d'arrêt et, comme les problèmes sont couplés par le prix, les auteurs recherchent des équilibres de Nash. La forme champ-moyen des interactions leur permet de réduire le problème à un problème MFG avec arrêt optimal et deux classes de joueurs. Ils montrent l'existence d'un équilibre par l'application de l'approche par programmation linéaire et prouvent l'unicité du processus de prix d'équilibre. De plus, ils fournissent des illustrations numériques basées sur le marché de l'électricité britannique en combinant l'algorithme de Fictitious Play avec l'approche par programmation linéaire (voir la sous-section précédente pour plus de détails). Les auteurs concluent de leurs calculs numériques que "les subventions aux énergies renouvelables conduisent à une plus grande pénétration des énergies renouvelables mais peuvent entraîner un coût pour le consommateur en termes de prix de pointe plus élevés". En outre, ils affirment que "pour éviter une hausse des prix, les subventions aux énergies renouvelables doivent être combinées à des mécanismes garantissant qu'une capacité conventionnelle suffisante reste en place pour répondre à la demande d'énergie pendant les périodes de pointe". Contrairement à cette référence, où les paramètres futurs du marché sont supposés être déterministes et connus par les agents, nous supposons dans notre cas que la demande d'électricité et les coûts des producteurs conventionnels dépendent du prix aléatoire du carbone (bruit commun) qui est influencé par les annonces gouvernementales / changements réglementaires (scénarios conduisant à une information partielle pour les agents). La prise en compte du bruit commun et de l'information partielle constitue une difficulté technique majeure pour prouver l'existence de solutions MFG, qui sera abordée dans ce chapitre. Dans l'article connexe [CDL22], les auteurs appliquent les MFGs et le contrôle de type champ moyen pour étudier le comportement des producteurs d'électricité en présence d'une taxe carbone.

L'information partielle apparaît naturellement dans les problèmes de contrôle stochastique/arrêt optimal. Dans ces contextes, les joueurs optimisent une fonction de gain espéré en choisissant un temps de contrôle ou d'arrêt par rapport à une filtration qui ne contient pas l'information complète de tous les processus aléatoires impliqués dans la dynamique d'état et/ou les fonctions de gain. Par exemple, dans le cas de l'arrêt optimal à horizon fini en temps discret, le problème d'information partielle est le suivant : on considère une chaîne de Markov $X = (X_t)_t$ à valeurs dans un espace polonais E appelé processus de signal (qui n'est pas nécessairement entièrement observé) et un processus $Y = (Y_t)_t$ évalué dans un autre espace polonais F , de sorte que le couple $(X_t, Y_t)_t$ est une chaîne de Markov. Le processus Y sera appelé processus d'observation. Le but est de choisir un temps d'arrêt τ à valeurs dans $\{0, 1, \dots, T\}$ par rapport à la filtration engendrée par l'observation $\mathbb{F}^Y = (\mathcal{F}_t^Y)_t$, afin de maximiser

$$J(\tau) := \mathbb{E} \left[\sum_{t=0}^{\tau-1} f(X_t, Y_t) + g(X_\tau, Y_\tau) \right].$$

Ce problème peut être reformulé comme un problème d'arrêt optimal avec observation complète au coût d'augmentation l'espace d'état. Une transformation habituelle pour récupérer la

structure d'information complète consiste à introduire le processus de distribution a posteriori $\Pi = (\Pi_t)_t$ prenant des valeurs dans $\mathcal{P}(E)$ et donné par

$$\Pi_t(B) = \mathbb{P}[X_t \in B | \mathcal{F}_t^Y], \quad t \in \{0, 1, \dots, T\}, \quad B \in \mathcal{B}(E).$$

Si nous considérons maintenant les fonctions

$$\bar{f}(\pi, y) := \int_E f(x, y) \pi(dx), \quad \bar{g}(\pi, y) := \int_E g(x, y) \pi(dx), \quad \pi \in \mathcal{P}(E), \quad y \in F,$$

nous pouvons réécrire le gain espéré comme

$$J(\tau) = \mathbb{E} \left[\sum_{t=0}^{\tau-1} \bar{f}(\Pi_t, Y_t) + \bar{g}(\Pi_\tau, Y_\tau) \right],$$

où nous avons utilisé que τ est un \mathbb{F}^Y -temps d'arrêt. De plus, on peut montrer que $(\Pi_t, Y_t)_t$ est un \mathbb{F}^Y -processus de Markov, ce qui implique que nous sommes confrontés à un problème d'arrêt optimal avec observation complète. Nous renvoyons aux revues [Mon82; Pha05] et aux références qui y figurent pour plus de détails sur le contrôle stochastique avec information partielle et à [PRS05; Lud09; Zho12; EV16; FT22] pour le cas d'arrêt optimal. Les problèmes d'information partielle ont été étudiés dans la littérature sur les MFGs avec contrôle stochastique (voir par exemple [HW16; BLM17; CJ18; SC19; BFH21; BCL21]) et dans le cas des MFGs supermodulaires avec arrêt optimal dans l'article [CDL17].

Les jeux à champ moyen avec bruit commun décrivent un type de jeux symétriques où tous les joueurs sont confrontés à une source commune d'aléa. Alors que l'approche MFG sans bruit commun repose sur l'indépendance asymptotique de la dynamique de l'état d'équilibre des différents agents (c'est-à-dire lorsque les agents utilisent la stratégie d'équilibre de Nash) lorsque le nombre de joueurs tend vers l'infini, dans le cadre du bruit commun, la distribution limite de la dynamique de l'état d'équilibre n'est pas déterministe. En fait, les termes de champ-moyen deviennent des fonctionnels adaptés du bruit commun. Plus précisément, dans le cas du contrôle, le flux champ-moyen de mesures $m = (m_t)_t$ s'écrit $m_t(\cdot) = \mathbb{P}(X_t \in \cdot | \mathcal{F}_t^B)$, où X est l'état de l'agent représentatif et B le processus de bruit commun. D'un point de vue mathématique, le bruit commun augmente l'espace du terme champ-moyen de $C([0, T], \mathcal{P}(\mathbb{R}^d))$ (dans le cas du contrôle) à $[C([0, T], \mathcal{P}(\mathbb{R}^d))]^\Omega$, où Ω est l'espace des trajectoires du bruit commun. Lorsque Ω est fini, le problème de point fixe peut être résolu de manière similaire au cas sans bruit commun (voir [BHS21]). Cependant, lorsque Ω est infini, l'espace $[C([0, T], \mathcal{P}(\mathbb{R}^d))]^\Omega$ est trop grand pour utiliser des arguments topologiques laissant invariants les transformations d'intérêt. Comme souligné dans [CDL16], l'opération de conditionnement ne parvient pas à être continue dans un sens utile, ce qui complique l'étude des points fixes via des arguments topologiques. Les MFGs avec bruit commun ont été introduits dans [Lio07] pour le cas du contrôle stochastique. Ils ont été étudiés du point de vue des EDPs via la "master equation". Pour le traitement de la "master equation" nous nous référons aux livres [Car+19; CD18b]. Cette méthodologie d'étude des MFGs avec bruit commun impose généralement des hypothèses fortes sur les coefficients de diffusion et

les fonctions de gain, telles que les conditions d'unicité du contrôle optimal pour un flux champ-moyen fixe et les conditions de monotonie de Lasry-Lions. Nous soulignons que ces conditions peuvent être relaxées au prix de la considération d'équilibres faibles au sens de [CDL16]. Plus précisément, les équilibres faibles n'imposent pas que le flux champ-moyen d'équilibre soit adapté à la filtration du bruit commun, mais plutôt à une filtration éventuellement plus grande. Cette notion a été utilisée dans [CDL16], où les auteurs montrent l'existence d'équilibres faibles sous des conditions très générales en discrétisant la filtration du bruit commun et en passant à la limite. Dans leur article, sous des hypothèses appropriées liées à celles garantissant que la "master equation" soit bien posée, les auteurs montrent que l'unique équilibre faible est en fait un équilibre fort (c'est-à-dire adapté à la filtration du bruit commun). Cette notion d'équilibre faible a également été appliquée dans [Lac16] pour caractériser les points limites de la séquence des équilibres de Nash à N joueurs (avec et sans bruit commun) lorsque N tend vers l'infini. Dans le cas sans bruit commun, les équilibres faibles ne sont pas nécessairement déterministes contrairement à la définition habituelle utilisée dans la littérature MFG. Ce résultat général souligne l'utilité de considérer les équilibres faibles plutôt que les équilibres forts. Cependant, afin de construire des schémas numériques pour des applications avec ou sans bruit commun, la notion d'équilibre faible semble trop abstraite.

Les MFGs d'arrêt optimal avec bruit commun ont été étudiés dans [CDL17; Nut18; Dia+21]. Dans l'article [CDL17], deux cas sont considérés : le premier est un cadre MFG supermodulaire permettant de montrer l'existence d'équilibres forts avec des temps d'arrêt non randomisés sous des hypothèses assez restrictives ; le second est plus général et permet de montrer l'existence d'équilibres faibles. Nous soulignons que, dans les deux cas, un état général des joueurs et une dépendance générale de la distribution des joueurs encore dans le jeu ne sont pas considérés. De plus, dans le premier cas, les auteurs fournissent un résultat d'approximation pour les jeux avec un nombre fini de joueurs où ils peuvent construire des équilibres de Nash approximatifs sous forme distribuée via la solution MFG et dans le cadre plus général, ils donnent le même résultat en ajoutant l'hypothèse que des équilibres forts associés à des temps d'arrêt non randomisés existent. Un modèle spécifique pour les paniques bancaires est développé dans [Nut18]. Ici, les agents interagissent par le biais de la proportion de joueurs qui se sont déjà arrêtés et l'équilibre peut être calculé de manière assez explicite. Enfin, un MFG supermodulaire général avec arrêt optimal et bruit commun est étudié dans l'article [Dia+21] qui fournit un théorème d'existence pour les solutions fortes et un schéma numérique pour les calculer. Nous soulignons que dans les trois articles précédents, le cadre MFG est de type *préemption*, ce qui signifie que le départ d'autres joueurs incite le joueur représentatif à partir également. Cela contraste avec les jeux dits *guerre d'usure*, où le départ d'autres joueurs incite le joueur représentatif à rester dans le jeu.

Contributions. L'objectif principal de ce chapitre est d'étendre le modèle introduit dans [ADT21] en incluant un bruit commun et de l'information partielle. Cette extension permet de comprendre l'impact de l'incertitude du prix du carbone sur la dynamique du marché de l'électricité. Plus précisément, nous supposons que le processus de demande et le coût de base des

producteurs conventionnels dépendent explicitement du prix du carbone. Le prix du carbone est modélisé par un processus stochastique, qui est Markovien conditionnellement à la fixation du scénario. Dans notre cas, les scénarios sont des trajectoires plausibles de variables macroéconomiques, paramétrées par des hypothèses spécifiques sur le climat et les politiques futures. Nous nous plaçons dans le cadre général où le scénario n'est pas observable par les producteurs, ce qui nous conduit à un cadre d'information partielle. Puisque le prix du carbone affectera tous les producteurs, il ne sera pas moyenné dans la limite de type champ-moyen, produisant un MFG avec arrêt optimal, bruit commun non-Markovien (la non-Markovianité provenant du cadre d'information partielle) et information partielle.

Le jeu développé ici correspond à la classe des jeux de guerre d'usure. En particulier, nous ne disposons pas d'une structure supermodulaire permettant d'appliquer les théorèmes de monotonie pour montrer l'existence d'équilibres forts. Comme la notion d'équilibre faible est trop abstraite pour construire un algorithme pour notre application, nous avons choisi de travailler avec un bruit commun prenant un nombre fini de trajectoires possibles. De plus, pour s'assurer que chaque processus d'état (conventionnel ou renouvelable) avec une certaine transformation du bruit commun est un processus de Markov (permettant ainsi d'utiliser la formulation par programmation linéaire), nous avons établi le modèle en temps discret, contrairement à [ADT21]. Inspirés par les articles [Kur98; KN11] qui étudient le problème de martingale filtrée, nous construisons un processus de Markov, noté U , encodant l'histoire du prix du carbone et prenant des valeurs dans un ensemble fini de matrices. Cette transformation permet deux choses, d'abord les termes champ-moyen au temps t seront des fonctions de U_t ce qui permet d'utiliser des arguments topologiques pour montrer l'existence de points fixes, ensuite la structure d'information complète est récupérée puisque le processus de loi à posteriori est simplement une fonction de U .

Plus largement, nous développons une théorie générale pour les MFGs avec arrêt optimal en temps discret avec un bruit commun à valeur dans un ensemble fini. Nous permettons à la variable d'état X de prendre des valeurs dans un espace métrique compact arbitraire en supposant que le processus joint (X, U) est Markovien. En utilisant le générateur de ce processus de Markov, nous obtenons une contrainte linéaire pour les mesures d'occupation sous-jacentes et, de plus, nous pouvons réécrire le gain espéré en termes de ces mesures d'occupation. Nous développons donc l'approche par programmation linéaire pour les MFGs en temps discret d'arrêt optimal et d'absorption, avec bruit commun, pour lesquels nous pouvons construire un algorithme numérique. Sous des hypothèses faibles, nous montrons l'existence d'un équilibre de Nash LP MFG en utilisant le théorème de point fixe de Kakutani-Fan-Glicksberg pour les fonctions multivaluées, à la fois dans le cas d'une population de joueurs et dans le cas de plusieurs populations de joueurs. De plus, nous montrons que notre ensemble de relaxation est égal à l'enveloppe convexe fermée de l'ensemble des mesures d'occupation induites par des temps d'arrêt non randomisés (c'est-à-dire des fonctions des processus sous-jacents (X, U)). Une conséquence de ce résultat est qu'une représentation probabiliste de notre contrainte peut être obtenue, permettant d'interpréter les mesures en termes de temps d'arrêt randomisés dépendants des processus sous-jacents. Puisque tout équilibre de Nash LP MFG est associé à un temps d'arrêt randomisé dépendant

des processus sous-jacents, nous sommes en mesure de construire un résultat d'approximation pour les jeux avec un nombre fini de joueurs permettant des temps d'arrêt randomisés.

Nous montrons que notre modèle initial pour la dynamique de l'industrie électrique s'inscrit dans le cadre théorique décrit précédemment en vérifiant les hypothèses. Cela nous permet d'énoncer un résultat d'existence général. De plus, en utilisant des techniques similaires à celles de [ADT21], nous montrons que même dans le cas du bruit commun, le processus de prix de l'électricité à l'équilibre est unique. L'algorithme "linear programming fictitious play" est étendu à notre cadre de bruit commun sans preuve de convergence. Nous prenons le même exemple du marché de l'électricité britannique de [ADT21] et calibrons les paramètres du prix du carbone en utilisant les scénarios NGFS (Network for Greening the Financial System). Dans notre exemple, nous constatons qu'empiriquement l'algorithme converge en $O(N^{-1})$, N étant le nombre d'étapes d'apprentissage, et que l'incertitude du scénario entraîne des changements significatifs dans la vitesse de remplacement des générateurs conventionnels par la production renouvelable.

4 Perspectives générales

Tout au long de cette thèse, nous nous concentrons sur l'approche par programmation linéaire pour les MFGs permettant du contrôle stochastique, de l'arrêt optimal ou de l'absorption. Les techniques que nous développons ici pourraient également être utiles pour l'étude théorique du contrôle/arrêt optimal des systèmes de McKean-Vlasov et leur résolution numérique. Par exemple, considérons le problème suivant d'arrêt optimal de type McKean-Vlasov :

$$\begin{aligned}
 & \sup_{\tau} \mathbb{E} \left[\int_0^{\tau} f(t, X_t, m_t^{\tau}) dt + g(\tau, X_{\tau}, \mu^{\tau}) \right], \\
 \text{s.t.} \quad & dX_t = b(t, X_t, m_t^{\tau}) dt + \sigma(t, X_t, m_t^{\tau}) dW_t, \\
 & m_t^{\tau}(B) = \mathbb{P}[X_t \in B, t < \tau], \quad B \in \mathcal{B}(\mathbb{R}), \quad t \in [0, T], \\
 & \mu^{\tau} = \mathcal{L}(\tau, X_{\tau}), \\
 & X_0 \sim m_0^*.
 \end{aligned} \tag{2.12}$$

La formulation par mesures d'occupation de ce problème d'arrêt optimal de type McKean-Vlasov sera :

$$\begin{aligned}
 & \sup_{(m, \mu)} \langle m_t(dx)dt, f[m] \rangle + \langle \mu, g[\mu] \rangle, \\
 \text{s.t.} \quad & \langle \mu, u \rangle = \langle m_0^*, u(0, \cdot) \rangle + \langle m_t(dx)dt, \partial_t u + \mathcal{L}[m]u \rangle, \quad u \in C_b^{1,2}([0, T] \times \mathbb{R}), \\
 & m_t \in \mathcal{P}^{sub}(\mathbb{R}), \quad t \in [0, T], \\
 & \mu \in \mathcal{P}(\mathbb{R}),
 \end{aligned} \tag{2.13}$$

où

$$\begin{aligned}
 \mathcal{L}[m]u(t, x) & := b(t, x, m_t) \partial_x u(t, x) + \frac{\sigma^2}{2}(t, x, m_t) \partial_{xx} u(t, x), \\
 f[m](t, x) & := f(t, x, m_t), \quad g[\mu](t, x) := g(t, x, \mu).
 \end{aligned}$$

Notons que ce problème d’optimisation n’est pas linéaire. Cependant, en utilisant les techniques appliquées dans cette thèse, on peut montrer, sous une topologie appropriée et quelques hypothèses faibles, que l’ensemble admissible est compact et que la fonction objectif est continue, permettant de montrer l’existence d’un maximiseur. Enfin, en discrétisant les mesures admissibles, on peut obtenir une optimisation en dimension finie, et des algorithmes numériques peuvent être utilisés pour approcher les solutions. Nous renvoyons aux articles [TTZ21 ; TTZ22 ; AO22 ; DM22] pour plus de détails sur le problème d’arrêt optimal de type McKean-Vlasov. Des configurations similaires peuvent être considérés pour le contrôle stochastique ou l’absorption et les combinaisons des trois.

Nous proposons maintenant deux extensions pour la convergence de l’algorithme “linear programming fictitious play”. Dans cette thèse, nous supposons que les coefficients de diffusion sont indépendants du terme champ-moyen. L’extension de l’algorithme à des coefficients dépendant de la mesure peut être intéressante pour certaines applications. En particulier, si deux termes champ-moyen d’entrée sont suffisamment proches et que les coefficients sont Lipschitz par rapport à la mesure, nous pouvons nous attendre à une certaine proximité entre les mesures de meilleure réponse. En outre, les résultats concernant l’unicité de la meilleure réponse dans le cas de l’arrêt optimal devraient être étendus, en relâchant les hypothèses sur les coefficients. Un autre problème connexe serait de montrer une vitesse de convergence pour le “linear programming fictitious play”. Pour y parvenir, il serait important d’avoir des estimées précises sur la distance entre deux meilleures réponses consécutives.

Enfin, une extension concernant le dernier chapitre de la thèse serait de relaxer la compacité de l’espace d’état en considérant un espace métrique séparable localement compact et de permettre des chaînes de Markov admettant des moments polynomiaux. Ceci semble être possible en utilisant la représentation probabiliste des contraintes (puisque nous aurions également un résultat de dualité similaire), les techniques de la première partie de la thèse et en supposant certaines conditions sur les moments de la chaîne de Markov.

Chapter 3

Control and optimal stopping Mean Field Games: a linear programming approach

Abstract

We develop the *linear programming approach* to mean field games in a general setting. This relaxed control approach allows to prove existence results under weak assumptions, and lends itself well to numerical implementation. We consider mean-field game problems where the *representative agent* chooses both the optimal control and the optimal time to exit the game, where the instantaneous reward function and the coefficients of the state process may depend on the distribution of the other agents. Furthermore, we establish the equivalence between mean field games equilibria obtained by the linear programming approach and the ones obtained via the controlled/stopped martingale approach, another relaxation method used in earlier papers in the pure control case.

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1 Introduction

Mean Field Games (MFGs) represent the limit version of games with a large number of agents, symmetric interactions and negligible individual influence of each player on the others. This theory has been introduced independently by Lasry and Lions [LL06a; LL06b; LL07] and Huang, Malhamé and Caines [HMC06]. Since the N -player game is rarely tractable, MFGs provide a useful tool for approximating the N -player Nash equilibria.

In this paper, we develop the *linear programming approach* to mean field games in a general setting. The linear programming approach is a control relaxation technique, which allows to prove existence results under weak assumptions and lends itself well to numerical implementation. It is well known in the field of stochastic control and has recently been used in the context of MFGs of optimal stopping in [BDT20]. That paper considers MFGs of optimal stopping (where each agent only decides when to stop) and under restrictive assumptions, in particular the coefficients of the state process of the representative agent do not depend on the distribution of the other agents. The goal of this paper is to present the linear programming approach in a much wider context of control and stopping MFGs, with coefficients depending on the measure and under weaker assumptions than in [BDT20], as well as to establish the equivalence of the linear programming approach with the other control relaxation approaches.

Our aim is to study MFGs in a general setting, including optimal stopping, continuous control and absorption. To explain the concept, assume that we have a “large” number $N \in \mathbb{N}^*$ of players. Each agent $k \in \{1, \dots, N\}$ has a private state process $X^{k,N} = (X_t^{k,N})_{t \in [0, T]}$, whose dynamics are given by the stochastic differential equation (SDE)

$$dX_t^{k,N} = b\left(t, X_t^{k,N}, m_t^N, \alpha_t^k\right) dt + \sigma\left(t, X_t^{k,N}, m_t^N, \alpha_t^k\right) dW_t^k,$$

where W^1, \dots, W^N are independent Brownian motions, $\alpha^k = (\alpha_t^k)_{t \in [0, T]}$ is the control process taking values in a closed subset $A \subset \mathbb{R}$, chosen by agent k and m_t^N is the empirical occupation measure of the players still present in the game and their controls:

$$m_t^N(dx, da) = \frac{1}{N} \sum_{k=1}^N \delta_{(X_t^{k,N}, \alpha_t^k)}(dx, da) \mathbf{1}_{t < \tau^k \wedge \tau_{\mathcal{O}}^k}.$$

Here τ^k is the stopping time, valued in $[0, T]$, chosen by player k , and

$$\tau_{\mathcal{O}}^k = \inf\{t \geq 0 : X_t^{k,N} \notin \mathcal{O}\},$$

denotes the first exit time of agent k from an open subset $\mathcal{O} \subset \mathbb{R}$, with the convention $\inf \emptyset = \infty$.

Each agent k seeks to choose an optimal stopping time τ^k and an optimal control α^k to maximize the reward functional defined as follows:

$$\mathbb{E} \left[\int_0^{\tau^k \wedge \tau_{\mathcal{O}}^k} f\left(t, X_t^{k,N}, m_t^N, \alpha_t^k\right) dt + g\left(\tau^k \wedge \tau_{\mathcal{O}}^k, X_{\tau^k \wedge \tau_{\mathcal{O}}^k}^{k,N}, \mu^N\right) \right],$$

where μ^N is the empirical joint distribution of the stopping time and the state process at the

time of stopping:

$$\mu^N(dt, dx) = \frac{1}{N} \sum_{k=1}^N \delta_{\left(\tau^k \wedge \tau_{\mathcal{O}}^k, X_{\tau^k \wedge \tau_{\mathcal{O}}^k}^{k,N}\right)}(dt, dx).$$

The objective functionals and the dynamics of the agents are coupled through the empirical measures $(m_t^N)_{t \in [0, T]}$ and μ^N , so that it is natural to look for a Nash equilibrium. When the number of players N goes to infinity, we expect by a ‘‘propagation of chaos’’ type result that the empirical occupation measures converge to a *deterministic flow of subprobability measures* $(m_t)_{t \in [0, T]}$, while the empirical joint distributions of the stopping time/state process when each player exits the game (via discretionary stopping or absorption), converges to a *deterministic limiting probability measure* μ . In our setting the MFG problem therefore reads as follows:

- (i) Fix $((m_t)_{t \in [0, T]}, \mu)$ and find the solution to the mixed control / optimal stopping problem

$$\begin{aligned} \sup_{\tau, \alpha} \quad & \mathbb{E} \left[\int_0^{\tau \wedge \tau_{\mathcal{O}}^{\alpha, m}} f(t, X_t^{\alpha, m}, m_t, \alpha_t) dt + g \left(\tau \wedge \tau_{\mathcal{O}}^{\alpha, m}, X_{\tau \wedge \tau_{\mathcal{O}}^{\alpha, m}}^{\alpha, m}, \mu \right) \right], \\ \text{s.t.} \quad & dX_t^{\alpha, m} = b(t, X_t^{\alpha, m}, m_t, \alpha_t) dt + \sigma(t, X_t^{\alpha, m}, m_t, \alpha_t) dW_t, \end{aligned} \quad (3.1)$$

where $\tau_{\mathcal{O}}^{\alpha, m} = \inf\{t \geq 0 : X_t^{\alpha, m} \notin \mathcal{O}\}$.

- (ii) Given the solution $(\tau^{m, \mu}, \alpha^{m, \mu})$ of the problem (3.1) for the agent facing a mean-field $((m_t)_{t \in [0, T]}, \mu)$, find $((m_t)_{t \in [0, T]}, \mu)$ such that

$$m_t(B) = \mathbb{P} \left[(X_t^{\alpha^{m, \mu}, m}, \alpha_t^{m, \mu}) \in B, t < \tau^{m, \mu} \wedge \tau_{\mathcal{O}}^{\alpha^{m, \mu}, m} \right], \quad B \in \mathcal{B}(\bar{\mathcal{O}} \times A), \quad t \in [0, T],$$

and

$$\mu = \mathcal{L} \left(\tau^{m, \mu} \wedge \tau_{\mathcal{O}}^{\alpha^{m, \mu}, m}, X_{\tau^{m, \mu} \wedge \tau_{\mathcal{O}}^{\alpha^{m, \mu}, m}}^{\alpha^{m, \mu}, m} \right).$$

In the literature on MFGs, there are two main approaches to prove existence of an MFG Nash equilibrium. The first approach, developed by Lasry and Lions [LL07], is an analytic one and consists in finding a Nash equilibrium by solving a coupled system of nonlinear partial differential equations: a Hamilton-Jacobi-Bellman equation (backward in time) satisfied by the value function of the representative agent for a given distribution and a Fokker-Planck-Kolmogorov equation (forward in time) describing the evolution of the density when the optimal control is used. The second approach, introduced by Carmona and Delarue [CD13a; CD18a; CD18b], is based on the stochastic maximum principle which reduces the problem to a system of coupled forward-backward stochastic differential equations of McKean-Vlasov type.

In the standard stochastic control theory, the *controlled martingale problem approach* (see e.g. [KNJ87], [HL90] and [KT15]) is a powerful tool allowing to simplify the existence proofs, by compactification of the stochastic control problem. In the original MFG framework (regular control, without optimal stopping), the controlled martingale problem approach was first used in [Lac15] to show the existence of a mean field game equilibrium under general assumptions. Further developments have been made in the case of mean field games with branching [CRT19] or mean field games with singular controls [FH17]. Another relaxation technique used in the clas-

sical stochastic control theory is based on the *linear programming formulation* (see e.g. [KS98c; CS02; KS17]). A similar approach has been introduced in the works [LV80] (deterministic case) and [FV88] (stochastic case) and in the context of *Aubry-Mather theory* in the deterministic and stochastic cases (see e.g. [Mat91; Mañ96], [Gom10] and [Gom02]). In the mean-field game setting, the linear programming formulation has been used in the case of optimal stopping in [BDT20] and Aubry-Mather theory has been applied in the recent paper [Ash+21] for studying a price formation MFG model. We also mention a related approach to the relaxation of MFG developed by [CM20] in the context of mean-field games with linear deterministic dynamics.

Standard optimal stopping problems have been widely studied in the literature, using different approaches (see e.g. [BL82; Kar81; CGT13; CS02]). Mean field games of optimal stopping are a recent trend in the MFG literature. More generally, only a few papers study mean field games with possible exit of the players leading to a decrease of the total mass of the players still in the game. We refer here to the MFGs with absorption (see e.g. [CF18]) and the MFGs of optimal stopping (see e.g. [GP15; Nut18; CDL17; Ber17; BDT20]).

In this paper, we extend the linear programming approach initiated in [BDT20] to a more general setting including mixed optimal stopping/control, allowing for measure depending coefficients, and involving weaker assumptions. Furthermore, we clarify the relationship between *linear programming MFG equilibria* and *MFG equilibria in the controlled/stopped martingale problem approach* (also known as the *weak formulation*), and state precise conditions of equivalence of the two approaches. In the pure control case, this equivalence enables us to recover directly the result of existence of a Markovian equilibrium shown in [Lac15] by using the Markovian projection technique. In addition, our method allows us to establish the existence of mixed solutions in the sense of [Ber17], under a more general framework (in particular, with coefficients depending on both the control and the measure which was not the case in [Ber17]).

The existence theorems of MFG equilibria obtained through the controlled martingale problem approach are in general rather abstract and provide little insight into the computation of MFG solutions. However, the linear programming approach we develop leads to a tractable method of computing the MFG equilibria, which has been instrumental in several concrete applications (see e.g. [ADT21; BDT22]).

The paper is organized as follows. In Section 2, we first study the single-agent problem under the linear programming formulation and we show the existence of a solution. In Section 3, we solve the MFG problem of optimal stopping/control under mild assumptions. In Section 4 we prove the equivalence of the linear programming formulation in the single-agent case with the various weak formulations, as well as with PDEs. In the same section we relate the notions of linear programming equilibria, weak equilibria and mixed solutions (PDEs). In Section 5 we give some technical results and in particular we make the connection between the linear programming and the weak formulations, extending some of the existing results in the literature to less regular coefficients (see e.g. [KS98c; CS02; DS12; KS17]).

Notation. Let $T > 0$ be a terminal time horizon, \mathcal{O} be an open subset of \mathbb{R} with closure $\bar{\mathcal{O}}$ and A be a compact subset of \mathbb{R} . We denote by $C_b^{1,2}([0, T] \times \bar{\mathcal{O}})$ the set of functions $u \in C_b([0, T] \times \bar{\mathcal{O}})$

such that $\partial_t u, \partial_x u, \partial_{xx} u \in C_b([0, T] \times \bar{\mathcal{O}})$. We denote by \mathbb{R}_+ the set $[0, +\infty[$. For a given real valued process $(Y_t)_{t \in [0, T]}$ on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a Borel subset B of \mathbb{R} , we define the random time

$$\tau_B^Y(\omega) := \inf\{t \geq 0 : Y_t(\omega) \notin B\}, \quad \omega \in \Omega,$$

with the convention $\inf \emptyset = \infty$. For $\varphi \in C([0, T])$ and $t \in [0, T]$ we define the truncated “norms”

$$\|\varphi\|_t = \sup_{s \in [0, t]} |\varphi(s)|.$$

Let U be the set of flows of measures on $\bar{\mathcal{O}} \times A$, $(m_t)_{t \in [0, T]}$, such that: for every $t \in [0, T]$, m_t is a Borel finite signed measure on $\bar{\mathcal{O}} \times A$, for every $B \in \mathcal{B}(\bar{\mathcal{O}} \times A)$, the mapping $t \mapsto m_t(B)$ is measurable, and $\int_0^T |m_t|(\bar{\mathcal{O}} \times A) dt < \infty$, where $|m_t|$ is the total variation measure of m_t .

We define \tilde{U} as the quotient space given by U and the almost everywhere equivalence relation on $[0, T]$, that is, if, dt -almost everywhere on $[0, T]$, the measures m_t^1 and m_t^2 coincide, the measure flows $(m_t^1)_{t \in [0, T]}$ and $(m_t^2)_{t \in [0, T]}$ are considered equivalent. \tilde{U} endowed with the usual sum and scalar multiplication is a vector space, where the zero vector is given by the family of null measures $(\mathbf{0})_{t \in [0, T]}$. To each $(m_t)_{t \in [0, T]} \in \tilde{U}$ we associate a Borel finite signed measure on $[0, T] \times \bar{\mathcal{O}} \times A$ defined by $m_t(dx, da)dt$ and we endow \tilde{U} with the topology of weak convergence of the associated measures, i.e. we say that a sequence of flows of measures $(m^n)_{n \geq 1}$ converges to m in \tilde{U} if $(m_t^n(dx, da)dt)_{n \geq 1} \rightharpoonup m_t(dx, da)dt$. For $p \geq 1$, define the subsets of \tilde{U} ,

$$\tilde{U}_p := \left\{ m \in \tilde{U} : \int_0^T \int_{\bar{\mathcal{O}} \times A} |x|^p |m_t|(dx, da) dt < \infty \right\},$$

endowed with the weak topology with respect to continuous functions with p -growth, denoted by τ_p , of the associated measures (see Chapter 1 Section 2.3 for a definition and Section 5.1 for some properties of this topology). We denote by V (resp. V_p) the set of measure flows $(m_t)_{t \in [0, T]} \in \tilde{U}$ (resp. \tilde{U}_p) such that dt -a.e. m_t is a subprobability measure. We make the convention that when integrating a quantity with respect to dt , the version taken for $(m_t)_{t \in [0, T]}$ is such that, for each $t \in [0, T]$, m_t is a subprobability measure. We note that \tilde{U} is a Hausdorff locally convex topological vector space and V_p is metrizable (we refer to Section 5.2 for more details). We endow the set $\mathcal{P}_p([0, T] \times \bar{\mathcal{O}})$ with the τ_p -topology and we will often work on the product space $V_p \times \mathcal{P}_p([0, T] \times \bar{\mathcal{O}})$ endowed with the product topology, which we will denote $\tau_p \otimes \tau_p$. Since this product space is metrizable, we will often work with sequences. In V_p we will also consider the topology of stable convergence with p -growth, denoted by $\bar{\tau}_p$ (see Chapter 1 Section 2.3 for a definition and Section 5.1 for some properties of this topology), which is the weak topology with respect to test functions that are measurable in t , continuous in (x, a) and with p -growth. For $p \geq 1$, consider the set $M([0, T]; \mathcal{P}_p^{sub}(\bar{\mathcal{O}}))$ of Borel measurable functions from $[0, T]$ to $\mathcal{P}_p^{sub}(\bar{\mathcal{O}})$ identified a.e. on $[0, T]$. This set is endowed with the topology of convergence in measure (see Section 5.5) which is denoted by $\tilde{\tau}_p$, and is well defined since $\mathcal{P}_p^{sub}(\bar{\mathcal{O}})$ is a Polish space. In the case $p = 0$ we make the analogous definitions without the subscript notation.

In the paper we adopt the following terminology: controls of the type α_t with values in A are called *strict controls*; controls of the form $\alpha_t = \alpha(t, X_t)$, with α a given measurable function are called *Markovian strict controls*; controls of the form ν_t (respectively ν_{t, X_t} for some kernel $(\nu_{t,x})$) with values in $\mathcal{P}(A)$ are called *relaxed controls* (respectively *Markovian relaxed controls*). Relaxed controls are related to mixed strategies in game theory and consist in randomizing the action, which allows to embed the controls in a well behaved space. More precisely, instead of choosing an action valued in A , the agent chooses an action in $\mathcal{P}(A)$.

2 Single agent problem

In this section, we will study the linear programming formulation for the mixed optimal stopping/stochastic control problem in the case when there is no interaction. We study the single agent problem before going into the MFG setting since it allows to prove some important statements that apply when we fix the mean field terms in the coefficients (see e.g. Lemma 3.16).

We are given constants $q > p \geq 1 \vee r$, where $r \in [0, 2]$ and $q \geq 2$, and the following functions:

$$(b, \sigma) : [0, T] \times \mathbb{R} \times A \rightarrow \mathbb{R} \times \mathbb{R}, \quad f : [0, T] \times \bar{\mathcal{O}} \times A \rightarrow \mathbb{R}, \quad g : [0, T] \times \bar{\mathcal{O}} \rightarrow \mathbb{R}.$$

Throughout this section, unless specified otherwise, we will impose the following assumption.

Assumption 2.1.

- (1) $m_0^* \in \mathcal{P}_q(\mathcal{O})$.
- (2) The functions $(t, x, a) \mapsto b(t, x, a)$ and $(t, x, a) \mapsto \sigma(t, x, a)$ are jointly measurable and continuous in (x, a) for each $t \in [0, T]$. Moreover, there exists a constant $c_1 > 0$ such that for all $(t, x, y, a) \in [0, T] \times \mathbb{R} \times \mathbb{R} \times A$,

$$|b(t, x, a) - b(t, y, a)| + |\sigma(t, x, a) - \sigma(t, y, a)| \leq c_1 |x - y|,$$

$$|b(t, x, a)| \leq c_1 [1 + |x|], \quad \sigma^2(t, x, a) \leq c_1 [1 + |x|^r].$$

- (3) The function $(t, x, a) \mapsto f(t, x, a)$ is jointly measurable and continuous in (x, a) for each $t \in [0, T]$. The function g is continuous. Moreover, there exists a constant $c_2 > 0$ such that for all $(t, x, a) \in [0, T] \times \bar{\mathcal{O}} \times A$,

$$|f(t, x, a)| \leq c_2 [1 + |x|^p], \quad |g(t, x)| \leq c_2 [1 + |x|^p].$$

- (4) One of the following statements is true:
 - (a) *Unattainable boundary:* For every filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, \mathbb{F} -progressive measurable process ν with values in $\mathcal{P}(A)$, continuous \mathbb{F} -martingale measure M with intensity $\nu_t(da)dt$ and \mathbb{F} -adapted process X such that

$$dX_t = \int_A b(t, X_t, a) \nu_t(da)dt + \int_A \sigma(t, X_t, a) M(dt, da), \quad \mathbb{P} \circ X_0^{-1} = m_0^*,$$

we have $\mathbb{P}(\tau_{\mathcal{O}}^X \geq T) = 1$.

(b) *Attainable boundary*: σ does not depend on the control a and for all $(t, x) \in [0, T] \times \mathbb{R}$, $\sigma^2(t, x) \geq c_\sigma$ for some $c_\sigma > 0$.

We refer to Section 5.3.1 for an overview on martingale measures and their properties. In particular, condition (4)(a) is satisfied for $\mathcal{O} = \mathbb{R}$ or for some uncontrolled processes as for example the Black-Scholes SDE on $\mathcal{O} = \mathbb{R}^+$. Moreover, the martingale measure is replaced with a Brownian motion when σ is uncontrolled.

Let $W = (W_t)_{t \in [0, T]}$ be a standard Brownian motion on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We denote by \mathbb{F}^W the completion of the filtration given by $(\sigma(W_s : s \in [0, t]))_{t \in [0, T]}$. Denote by \mathcal{T} the set of stopping times with respect to this filtration with values in $[0, T]$. Let \mathbb{A} be the set of \mathbb{F}^W -progressively measurable control processes taking values in A . Consider the classical mixed stochastic control/optimal stopping problem

$$\begin{aligned} & \sup_{\tau \in \mathcal{T}, \alpha \in \mathbb{A}} \mathbb{E} \left[\int_0^{\tau \wedge \tau_{\mathcal{O}}^{X^\alpha}} f(t, X_t^\alpha, \alpha_t) dt + g\left(\tau \wedge \tau_{\mathcal{O}}^{X^\alpha}, X_{\tau \wedge \tau_{\mathcal{O}}^{X^\alpha}}^\alpha\right) \right], \\ & \text{s.t.} \quad dX_t^\alpha = b(t, X_t^\alpha, \alpha_t) dt + \sigma(t, X_t^\alpha, \alpha_t) dW_t, \\ & \quad X_0^\alpha \sim m_0^*, \end{aligned} \tag{3.2}$$

which will be called the *strong problem* for the single agent.

We recall that under our assumptions there exists a strong solution to this SDE for each $\alpha \in \mathbb{A}$, satisfying the estimate

$$\mathbb{E} \|X^\alpha\|_T^q \leq C(1 + \mathbb{E}|X_0^\alpha|^q).$$

We shall now provide the *linear programming* formulation which consists in introducing the occupation measures and the forward equation satisfied by them.

Definition 2.2 (*Linear Programming (LP) formulation*). Let \mathcal{R} be the set of pairs $(m, \mu) \in V_p \times \mathcal{P}_p([0, T] \times \bar{\mathcal{O}})$, such that for all $u \in C_b^{1,2}([0, T] \times \bar{\mathcal{O}})$,

$$\langle \mu, u \rangle = \langle m_0^*, u(0, \cdot) \rangle + \langle m_t(dx, da)dt, \partial_t u + \mathcal{L}u \rangle, \tag{3.3}$$

where

$$\mathcal{L}u(t, x, a) := b(t, x, a) \partial_x u(t, x) + \frac{\sigma^2}{2}(t, x, a) \partial_{xx} u(t, x), \quad (t, x, a) \in [0, T] \times \mathbb{R} \times A.$$

Define now the map $\Gamma : \mathcal{R} \rightarrow \mathbb{R}$ as follows:

$$\Gamma(m, \mu) = \langle m_t(dx, da)dt, f \rangle + \langle \mu, g \rangle.$$

The linear programming optimization problem takes the form

$$\sup_{(m,\mu) \in \mathcal{R}} \Gamma(m, \mu). \quad (3.4)$$

The value for the LP formulation is defined by

$$V^{LP} := \sup_{(m,\mu) \in \mathcal{R}} \Gamma(m, \mu). \quad (3.5)$$

In the rest of the paper we will use the shorthand notation $\langle m, \varphi \rangle$ instead of $\langle m_t(dx, da)dt, \varphi \rangle$.

Remark 2.3. The set \mathcal{R} is nonempty. In fact, if we define

$$\mu(B \times C) := \delta_0(B)m_0^*(C \cap \bar{\mathcal{O}}), \quad B \in \mathcal{B}([0, T]), \quad C \in \mathcal{B}(\bar{\mathcal{O}}),$$

then $((\mathbf{0})_t, \mu) \in \mathcal{R}$, where $\mathbf{0}$ denotes the null measure on $\bar{\mathcal{O}} \times A$.

Remark 2.4. By the disintegration theorem (Theorem 2.1 in Chapter 1, Section 2.3), for each $(m_t)_{t \in [0, T]} \in V$, there exists a mapping $\nu_{t,x} : [0, T] \times \bar{\mathcal{O}} \rightarrow \mathcal{P}(A)$ such that for each $B \in \mathcal{B}(A)$, the function $(t, x) \mapsto \nu_{t,x}(B)$ is $\mathcal{B}([0, T] \times \bar{\mathcal{O}})$ -measurable, and

$$m_t(dx, da)dt = \nu_{t,x}(da)m_t^x(dx)dt,$$

where $m_t^x(dx) := m_t(dx, A) := \int_A m_t(dx, da)$.

Remark 2.5. Throughout the paper, solutions of the LP problem taking the form $m_t(dx, da) = \delta_{\alpha(t,x)}(da)m_t^x(dx)$ for some measurable function $\alpha : [0, T] \times \bar{\mathcal{O}} \rightarrow A$ are called strict control LP solutions.

2.1 Existence of a solution for the linear programming problem

Let us first study some preliminary properties of the set \mathcal{R} .

Preliminary properties of the set of constraints \mathcal{R} . We start by showing the following admissibility result.

Proposition 2.6 (Admissibility of the occupation measures). *Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a filtered probability space, τ an \mathbb{F} -stopping time such that $\tau \leq T$ \mathbb{P} -a.s., ν an \mathbb{F} -progressively measurable process with values in $\mathcal{P}(A)$, M a continuous \mathbb{F} -martingale measure such that M^τ has intensity $\nu_t(da)\mathbf{1}_{t < \tau}dt$, X an \mathbb{F} -adapted process such that*

$$dX_t = \int_A b(t, X_t, a)\nu_t(da)dt + \int_A \sigma(t, X_t, a)M(dt, da), \quad t \leq \tau, \quad \mathbb{P} \circ X_0^{-1} = m_0^*.$$

Define now the measures

$$m_t(B \times C) = \mathbb{E}^\mathbb{P} \left[\mathbf{1}_B(X_t)\nu_t(C)\mathbf{1}_{t < \tau \wedge \tau_{\bar{\mathcal{O}}}^X} \right], \quad B \in \mathcal{B}(\bar{\mathcal{O}}), \quad C \in \mathcal{B}(A), \quad t \in [0, T],$$

$$\mu = \mathbb{P} \circ \left(\tau \wedge \tau_{\bar{\mathcal{O}}}^X, X_{\tau \wedge \tau_{\bar{\mathcal{O}}}^X} \right)^{-1}.$$

Then $(m, \mu) \in \mathcal{R}$.

Proof. Let $u \in C_b^{1,2}([0, T] \times \bar{\mathcal{O}})$. Applying Itô's formula,

$$\begin{aligned} u \left(\tau \wedge \tau_{\bar{\mathcal{O}}}^X, X_{\tau \wedge \tau_{\bar{\mathcal{O}}}^X} \right) &= u(0, X_0) + \int_0^{\tau \wedge \tau_{\bar{\mathcal{O}}}^X} \int_A \left(\frac{\partial u}{\partial t} + \mathcal{L}u \right) (t, X_t, a) \nu_t(da) dt \\ &\quad + \int_0^{\tau \wedge \tau_{\bar{\mathcal{O}}}^X} \int_A \left(\sigma \frac{\partial u}{\partial x} \right) (t, X_t, a) M(dt, da). \end{aligned}$$

Now taking the expectation and using the fact that

$$(\sigma \partial_x u)^2(t, X_t, a) \leq C(1 + |X_t|^q),$$

we get $(m, \mu) \in \mathcal{R}$. □

Let $(m, \mu) \in \mathcal{R}$ and consider the Markovian relaxed control $\nu_{t,x}$ coming from the disintegration of m . By Theorem 5.20, there exist a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, an \mathbb{F} -adapted process X , an \mathbb{F} -stopping time τ such that $\tau \leq T \wedge \tau_{\bar{\mathcal{O}}}^X$ \mathbb{P} -a.s., and an \mathbb{F} -martingale measure M with intensity $\nu_{t,X_t}(da) \mathbb{1}_{t \leq \tau} dt$, such that

$$\begin{aligned} X_{t \wedge \tau} &= X_0 + \int_0^{t \wedge \tau} \int_A b(s, X_s, a) \nu_{s, X_s}(da) ds + \int_0^{t \wedge \tau} \int_A \sigma(s, X_s, a) M(ds, da), \quad \mathbb{P} \circ X_0^{-1} = m_0^*, \\ m_t(B \times C) &= \mathbb{E}^{\mathbb{P}} [\mathbb{1}_B(X_t) \nu_{t, X_t}(C) \mathbb{1}_{t < \tau}], \quad B \in \mathcal{B}(\bar{\mathcal{O}}), \quad C \in \mathcal{B}(A), \quad t - a.e., \\ \mu &= \mathbb{P} \circ (\tau, X_\tau)^{-1}. \end{aligned}$$

In particular, if we define

$$m_t^{x,*}(B) = \mathbb{E}^{\mathbb{P}} [\mathbb{1}_B(X_t) \mathbb{1}_{t < \tau}], \quad B \in \mathcal{B}(\bar{\mathcal{O}}),$$

the flows of measures $(m_t(dx, da))_{t \in [0, T]}$ and $(\nu_{t,x}(da) m_t^{x,*}(dx))_{t \in [0, T]}$ are the same element in V . Moreover, the map $t \mapsto m_t^{x,*}$ is càdlàg as function (from $[0, T]$ to $\mathcal{P}^{sub}(\bar{\mathcal{O}})$), implying that it is the unique càdlàg representative of m^x . As a consequence, without loss of generality, we will work always with $m^{x,*}$ instead of m^x , which we will still denote by m^x .

Proposition 2.7 (*Estimates*). *There exist constants $C_1 \geq 0$ and $C_2 \geq 0$ such that for all $(m, \mu) \in \mathcal{R}$ we have the following estimates:*

(1) For all $t \in [0, T]$,

$$\int_{\bar{\mathcal{O}}} |x|^q m_t^x(dx) \leq C_1, \quad \int_{[0, T] \times \bar{\mathcal{O}}} |x|^q \mu(dt, dx) \leq C_1.$$

(2) For all $h \in [0, T]$, we have

$$\int_0^{T-h} d_{BL}(m_{t+h}^x, m_t^x) dt \leq C_2 \sqrt{h},$$

where d_{BL} is the bounded Lipschitz distance (see Section 5.1).

Proof. Let $(m, \mu) \in \mathcal{R}$ and consider their probabilistic representation given by some tuple $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, X, \tau, M)$. Consider the process $\tilde{X}_t := X_{t \wedge \tau}$ and define the sequence of stopping times

$$\tau_n := \inf\{t \in [0, T] : |\tilde{X}_t| > n\}, \quad n \geq 1.$$

We have for all $t \in [0, T]$ and $s \in [0, t]$ (the constant $C \geq 0$ may change from line to line),

$$|\tilde{X}_{s \wedge \tau_n}|^q \leq C \left(|X_0|^q + \left| \int_0^{s \wedge \tau \wedge \tau_n} \int_A b(r, \tilde{X}_r, a) \nu_{r, \tilde{X}_r}(da) dr \right|^q + \left| \int_0^{s \wedge \tau \wedge \tau_n} \int_A \sigma(r, \tilde{X}_r, a) M(dr, da) \right|^q \right)$$

Taking the supremum over $s \in [0, t]$, the expectation and using Jensen's inequality and the Burkholder-Davis-Gundy inequality,

$$\mathbb{E}^{\mathbb{P}} \|\tilde{X}\|_{t \wedge \tau_n}^q \leq C \left(\int_{\mathcal{O}} |x|^q m_0^*(dx) + \mathbb{E}^{\mathbb{P}} \int_0^{t \wedge \tau \wedge \tau_n} \int_A |b(r, \tilde{X}_r, a)|^q \nu_{r, \tilde{X}_r}(da) dr + \mathbb{E}^{\mathbb{P}} \int_0^{t \wedge \tau \wedge \tau_n} \int_A |\sigma^2(r, \tilde{X}_r, a)|^{q/2} \nu_{r, \tilde{X}_r}(da) dr \right).$$

Using the growth assumption on b and σ ,

$$\begin{aligned} \mathbb{E}^{\mathbb{P}} \|\tilde{X}\|_{t \wedge \tau_n}^q &\leq C \left(1 + \int_{\mathcal{O}} |x|^q m_0^*(dx) + \mathbb{E}^{\mathbb{P}} \int_0^{t \wedge \tau \wedge \tau_n} |\tilde{X}_r|^q dr \right) \\ &\leq C \left(1 + \int_{\mathcal{O}} |x|^q m_0^*(dx) + \mathbb{E}^{\mathbb{P}} \int_0^t \|\tilde{X}\|_{r \wedge \tau_n}^q dr \right). \end{aligned}$$

By Gronwall's inequality

$$\mathbb{E}^{\mathbb{P}} \|\tilde{X}\|_{T \wedge \tau_n}^q \leq C \left(1 + \int_{\mathcal{O}} |x|^q m_0^*(dx) \right) =: C_1$$

By Fatou's lemma we obtain $\mathbb{E}^{\mathbb{P}} \|\tilde{X}\|_T^q \leq C_1$. We deduce that

$$\int_{\bar{\mathcal{O}}} |x|^q m_t^x(dx) = \mathbb{E}^{\mathbb{P}}[|X_t|^q \mathbf{1}_{t < \tau}] \leq \mathbb{E}^{\mathbb{P}} \|\tilde{X}\|_T^q \leq C_1, \quad t \in [0, T],$$

and

$$\int_{[0, T] \times \bar{\mathcal{O}}} |x|^q \mu(dt, dx) = \mathbb{E}^{\mathbb{P}}[|X_\tau|^q] \leq \mathbb{E}^{\mathbb{P}} \|\tilde{X}\|_T^q \leq C_1.$$

Let us show the second estimate. For $\phi \in \text{BL}(\bar{\mathcal{O}})$ such that $\|\phi\|_{\text{BL}} \leq 1$ (where the set BL and

the norm $\|\cdot\|_{\text{BL}}$ are defined in Section 5.1), $h \in [0, T]$ and $t \in [0, T - h]$, we have:

$$\begin{aligned} \int_{\bar{\mathcal{O}}} \phi(x)(m_{t+h}^x - m_t^x)(dx) &= \mathbb{E}^{\mathbb{P}}[\phi(\tilde{X}_{t+h})\mathbf{1}_{t+h < \tau} - \phi(\tilde{X}_t)\mathbf{1}_{t < \tau}] \\ &= \mathbb{E}^{\mathbb{P}}[(\phi(\tilde{X}_{t+h}) - \phi(\tilde{X}_t))\mathbf{1}_{t+h < \tau}] + \mathbb{E}^{\mathbb{P}}[\phi(\tilde{X}_t)(\mathbf{1}_{t+h < \tau} - \mathbf{1}_{t < \tau})] \\ &\leq \mathbb{E}^{\mathbb{P}}[|\phi(\tilde{X}_{t+h}) - \phi(\tilde{X}_t)|] + \mathbb{E}^{\mathbb{P}}[|\phi(\tilde{X}_t)|\mathbf{1}_{t < \tau \leq t+h}] \\ &\leq \mathbb{E}^{\mathbb{P}}[|\tilde{X}_{t+h} - \tilde{X}_t|] + \mathbb{E}^{\mathbb{P}}[\mathbf{1}_{t < \tau \leq t+h}]. \end{aligned}$$

Taking the supremum over ϕ ,

$$d_{\text{BL}}(m_{t+h}^x, m_t^x) \leq \mathbb{E}^{\mathbb{P}}[|\tilde{X}_{t+h} - \tilde{X}_t|] + \mathbb{E}^{\mathbb{P}}[\mathbf{1}_{t < \tau \leq t+h}].$$

Moreover, we have

$$\int_0^{T-h} \mathbb{E}^{\mathbb{P}}[\mathbf{1}_{t < \tau \leq t+h}] dt \leq \mathbb{E}^{\mathbb{P}}[\tau \wedge (T - h) - 0 \vee (\tau - h)] \leq h.$$

On the other hand, applying Jensen's inequality and Burkholder-Davis-Gundy inequality (the constant $C \geq 0$ may change from line to line)

$$\begin{aligned} &\mathbb{E}^{\mathbb{P}} \left| \tilde{X}_{t+h} - \tilde{X}_t \right|^2 \\ &\leq \mathbb{E}^{\mathbb{P}} \left| \int_{t \wedge \tau}^{(t+h) \wedge \tau} \int_A b(s, \tilde{X}_s, a) \nu_{s, \tilde{X}_s}(da) ds \right|^2 + \mathbb{E}^{\mathbb{P}} \left| \int_{t \wedge \tau}^{(t+h) \wedge \tau} \int_A \sigma(s, \tilde{X}_s, a) M(ds, da) \right|^2 \\ &\leq C \left(\mathbb{E}^{\mathbb{P}} \int_{t \wedge \tau}^{(t+h) \wedge \tau} \int_A |b(s, \tilde{X}_s, a)|^2 \nu_{s, \tilde{X}_s}(da) ds + \int_{t \wedge \tau}^{(t+h) \wedge \tau} \int_A \sigma^2(s, \tilde{X}_s, a) \nu_{s, \tilde{X}_s}(da) ds \right) \\ &\leq Ch, \end{aligned}$$

where we used that $\mathbb{E}^{\mathbb{P}} \|\tilde{X}\|_T^q \leq C_1$. We deduce that there exists a constant C_2 such that

$$\int_0^{T-h} d_{\text{BL}}(m_{t+h}^x, m_t^x) dt \leq C_2 \sqrt{h}.$$

□

Using the above lemma it is not difficult to show that for $(m, \mu) \in \mathcal{R}$, m^x is càdlàg from $[0, T]$ to $\mathcal{P}_p^{\text{sub}}(\bar{\mathcal{O}})$. In particular m^x is measurable from $[0, T]$ to $\mathcal{P}_p^{\text{sub}}(\bar{\mathcal{O}})$. When viewed as an element of $M([0, T]; \mathcal{P}_p^{\text{sub}}(\bar{\mathcal{O}}))$, we are considering its equivalence class.

Lemma 2.8. *The map $H : \mathcal{P}^{\text{sub}}(\bar{\mathcal{O}}) \rightarrow [0, \infty]$ defined by*

$$H(\eta) = \int_{\bar{\mathcal{O}}} |x|^q \eta(dx), \quad \eta \in \mathcal{P}^{\text{sub}}(\bar{\mathcal{O}}),$$

is a coercive normal integrand in the sense of Definition 5.23.

Proof. To show that H is a coercive normal integrand in the sense of Definition 5.23 it suffices

to prove that H is measurable and has compact level sets. Define the maps

$$\mathcal{P}^{sub}(\bar{\mathcal{O}}) \ni \eta \mapsto H_k(\eta) := \int_{\bar{\mathcal{O}}} [|x|^q \wedge k] \eta(dx), \quad k \geq 1.$$

These functions are continuous for the topology of weak convergence. Moreover by the monotone convergence theorem $H_k(\eta)$ converges to $H(\eta)$ for each $\eta \in \mathcal{P}^{sub}(\bar{\mathcal{O}})$ as $k \rightarrow \infty$, which allows to conclude the measurability of H . Let us show that H has compact level sets. Let $c \geq 0$ and

$$L_c := \{\eta \in \mathcal{P}^{sub}(\bar{\mathcal{O}}) : H(\eta) \leq c\}.$$

By definition

$$\sup_{\eta \in L_c} \int_{\bar{\mathcal{O}}} |x|^q \eta(dx) \leq c, \quad \sup_{\eta \in L_c} \eta(\bar{\mathcal{O}}) \leq 1,$$

which shows that L_c is relatively compact in $\mathcal{P}^{sub}(\bar{\mathcal{O}})$. It remains to show that L_c is closed. Let $(\eta^n)_{n \geq 1} \subset L_c$ converging to some $\eta \in \mathcal{P}^{sub}(\bar{\mathcal{O}})$. We have for each $n \geq 1$ and $k \geq 1$,

$$H_k(\eta^n) \leq H(\eta^n) \leq c,$$

and taking the limit $n \rightarrow \infty$, we get $H_k(\eta) \leq c$. By the monotone convergence theorem we deduce that $\eta \in L_c$. \square

Proposition 2.9. *The set*

$$\{m^x : (m, \mu) \in \mathcal{R}\} \subset M([0, T]; \mathcal{P}_p^{sub}(\bar{\mathcal{O}}))$$

is relatively compact in the topology of convergence in measure $\tilde{\tau}_p$.

Proof. Since $(M([0, T]; \mathcal{P}_p^{sub}(\bar{\mathcal{O}})), \tilde{\tau}_p)$ is metrizable, we will work with sequences. First, we use the relative compactness criterion given in Theorem 5.24 for the topology of convergence in measure $(M([0, T]; \mathcal{P}^{sub}(\bar{\mathcal{O}})), \tilde{\tau})$. Let $(m^n, \mu^n)_{n \geq 1} \subset \mathcal{R}$ and consider the corresponding sequence $(m^{x,n})_{n \geq 1} \subset M([0, T]; \mathcal{P}^{sub}(\bar{\mathcal{O}}))$. Let H be the map defined in Lemma 2.8. Using the first estimates of Lemma 2.7,

$$\sup_{n \geq 1} \int_0^T H(m_t^{x,n}) dt = \sup_{n \geq 1} \int_0^T \int_{\bar{\mathcal{O}}} |x|^q m_t^{x,n}(dx) dt \leq C_1 T.$$

By the second estimate of Lemma 2.7,

$$\limsup_{h \downarrow 0} \sup_{n \geq 1} \int_0^{T-h} d_{BL}(m_{t+h}^{x,n}, m_t^{x,n}) dt = 0.$$

By Theorem 5.24, up to a subsequence, $(m^{x,n})_{n \geq 1}$ converges to some $\eta \in M([0, T]; \mathcal{P}^{sub}(\bar{\mathcal{O}}))$ in measure. Up to another subsequence, $(m_t^{x,n})_{n \geq 1}$ converges weakly to η_t t -a.e. on $[0, T]$. Since for each $t \in [0, T]$,

$$\sup_{n \geq 1} \int_{\bar{\mathcal{O}}} |x|^q m_t^{x,n}(dx) \leq C_1,$$

$(m_t^{x,n})_{n \geq 1}$ converges to η_t in τ_p t -a.e. on $[0, T]$, and in particular $(m^{x,n})_{n \geq 1}$ converges to η in $M([0, T]; \mathcal{P}_p^{sub}(\bar{\mathcal{O}}))$. \square

The topology of convergence in measure will allow us to consider test functions which are only measurable in time, and therefore to use the topology of the stable convergence. More precisely, we say that $(m^n)_{n \geq 1} \subset V_p$ converges in $\bar{\tau}_p$ to $m \in V_p$ if for all test functions $\phi : [0, T] \times \bar{\mathcal{O}} \times A \rightarrow \mathbb{R}$ jointly measurable, continuous in (x, a) for each t and with p -polynomial growth, we have

$$\lim_{n \rightarrow \infty} \int_0^T \int_{\bar{\mathcal{O}}} \phi(t, x, a) m_t^n(dx, da) dt = \int_0^T \int_{\bar{\mathcal{O}}} \phi(t, x, a) m_t(dx, da) dt.$$

We state this result in the next proposition.

Proposition 2.10. *Consider a sequence $(m^n, \mu^n)_{n \geq 1} \subset \mathcal{R}$ converging to some $(m, \mu) \in V_p \times \mathcal{P}_p([0, T] \times \bar{\mathcal{O}})$ in $\tau_p \otimes \tau_p$, then:*

- (1) *The sequence $(m^{x,n})_{n \geq 1}$ converges to m^x in $\tilde{\tau}_p$.*
- (2) *The sequence $(m^n)_{n \geq 1}$ converges to m in $\bar{\tau}_p$.*

Proof. We show statement (1) first. By Proposition 2.9, it suffices to show that any limit point of $(m^{x,n})_{n \geq 1}$ in $\tilde{\tau}_p$ is equal to m^x almost everywhere. Let $\eta \in M([0, T]; \mathcal{P}_p^{sub}(\bar{\mathcal{O}}))$ be a limit point of $(m^{x,n})_{n \geq 1}$ in $\tilde{\tau}_p$. Up to a subsequence we can assume that dt -a.e. on $[0, T]$, $(m_t^{x,n})_{n \geq 1}$ converges to η_t in τ_p . Using that $(m^n)_{n \geq 1}$ converges to m in τ_p , we have that for any $\varphi : [0, T] \rightarrow \mathbb{R}$ and $\psi : \bar{\mathcal{O}} \rightarrow \mathbb{R}$ bounded and continuous functions,

$$\int_0^T \int_{\bar{\mathcal{O}}} \varphi(t) \psi(x) m_t^{x,n}(dx) dt \xrightarrow{n \rightarrow \infty} \int_0^T \int_{\bar{\mathcal{O}}} \varphi(t) \psi(x) m_t^x(dx) dt.$$

We also have that dt -a.e. on $[0, T]$,

$$\lim_{n \rightarrow \infty} \int_{\bar{\mathcal{O}}} \psi(x) m_t^{x,n}(dx) = \int_{\bar{\mathcal{O}}} \psi(x) \eta_t(dx).$$

By the dominated convergence theorem

$$\left| \int_0^T \varphi(t) \int_{\bar{\mathcal{O}}} \psi(x) (m_t^{x,n} - \eta_t)(dx) dt \right| \xrightarrow{n \rightarrow \infty} 0.$$

Finally, we get

$$\int_0^T \int_{\bar{\mathcal{O}}} \varphi(t) \psi(x) m_t^x(dx) dt = \int_0^T \int_{\bar{\mathcal{O}}} \varphi(t) \psi(x) \eta_t(dx) dt,$$

which allows to deduce that m_t^x and η_t are equal almost everywhere.

Let us show now the second statement. First we show that $(m^n)_{n \geq 1}$ converges to m in $\bar{\tau}$ using Corollary 2.9 in [JM81]. We already know that $m_t^n(dx, da) dt \rightarrow m_t(dx, da) dt$, so it suffices to show that the set

$$\left\{ m_t^n(\bar{\mathcal{O}} \times A) dt : n \geq 1 \right\} \subset \mathcal{M}^+([0, T])$$

is relatively compact in $\mathcal{M}^+([0, T])$ endowed with the topology $\sigma(\mathcal{M}^+([0, T]), M_b([0, T]))$, which is equivalent to show that every subsequence of $(m_t^n(\bar{\mathcal{O}} \times A) dt)_{n \geq 1}$ has a convergent subsub-

sequence in this topology. Consider a subsequence still denoted with the same index. By the first part of the proof, we already know that $m^{x,n}$ converges to m^x in $\tilde{\tau}_p$. In particular, up to a subsequence, t -a.e. on $[0, T]$,

$$m_t^n(\bar{\mathcal{O}} \times A) = m_t^{x,n}(\bar{\mathcal{O}}) \xrightarrow[n \rightarrow \infty]{} m_t^x(\bar{\mathcal{O}}) = m_t(\bar{\mathcal{O}} \times A).$$

By the dominated convergence theorem, for each $\phi : [0, T] \rightarrow \mathbb{R}$ bounded and measurable,

$$\lim_{n \rightarrow \infty} \int_0^T \phi(t) m_t^n(\bar{\mathcal{O}} \times A) dt = \int_0^T \phi(t) m_t(\bar{\mathcal{O}} \times A) dt.$$

We deduce that $(m^n)_{n \geq 1}$ converges to m in $\bar{\tau}$. Now, we aim to apply Proposition 5.7. Define the function

$$w(t, x, a) = 1 + |t|^p + |x|^p + |a|^p.$$

We have for each $\phi : [0, T] \rightarrow \mathbb{R}$ bounded and measurable,

$$\begin{aligned} & \int_0^T \int_{\bar{\mathcal{O}} \times A} \phi(t) w(t, x, a) m_t^n(dx, da) dt \\ &= \int_0^T \int_{\bar{\mathcal{O}} \times A} \phi(t) m_t^n(dx, da) dt + \int_0^T \int_{\bar{\mathcal{O}} \times A} \phi(t) |t|^p m_t^n(dx, da) dt \\ & \quad + \int_0^T \int_{\bar{\mathcal{O}} \times A} \phi(t) |x|^p m_t^n(dx, da) dt + \int_0^T \int_{\bar{\mathcal{O}} \times A} \phi(t) |a|^p m_t^n(dx, da) dt. \end{aligned}$$

Again using that $m^{x,n}$ converges to m^x in $\tilde{\tau}_p$, up to a subsequence, we can assume that t -a.e. on $[0, T]$, $(m_t^{x,n})_{n \geq 1}$ converges in τ_p to m_t^x . By the dominated convergence theorem,

$$\lim_{n \rightarrow \infty} \int_0^T \int_{\bar{\mathcal{O}} \times A} \phi(t) |x|^p m_t^n(dx, da) dt = \int_0^T \int_{\bar{\mathcal{O}} \times A} \phi(t) |x|^p m_t(dx, da) dt.$$

The other integrals converging by using of the measures in $\bar{\tau}$, we get

$$\lim_{n \rightarrow \infty} \int_0^T \int_{\bar{\mathcal{O}} \times A} \phi(t) w(t, x, a) m_t^n(dx, da) dt = \int_0^T \int_{\bar{\mathcal{O}} \times A} \phi(t) w(t, x, a) m_t(dx, da) dt.$$

By Proposition 5.7, $(m^n)_{n \geq 1}$ converges to m in $\bar{\tau}_p$. □

A direct consequence of the previous proposition is that in \mathcal{R} , the topologies $\tau_p \otimes \tau_p$ and $\bar{\tau}_p \otimes \tau_p$ coincide. The following theorem states the compactness of the set of constraints.

Theorem 2.11. *The topological space $(\mathcal{R}, \bar{\tau}_p \otimes \tau_p)$ is compact.*

Proof. Since the space $(\mathcal{R}, \bar{\tau}_p \otimes \tau_p)$ is metrizable (see Section 5.1), it suffices to show that it is sequentially compact. Consider a sequence $(m^n, \mu^n)_{n \geq 1} \subset \mathcal{R}$. Using the estimates from Proposition 2.7,

$$\int_0^T \int_{\bar{\mathcal{O}} \times A} |x|^q m_t^n(dx, da) dt \leq C_1 T, \quad \int_{[0, T] \times \bar{\mathcal{O}}} |x|^q \mu^n(dt, dx) \leq C_1,$$

we get by Corollary 5.4 that up to a subsequence, $(\nu^n, \mu^n)_{n \geq 1}$ converges to some (ν, μ) in $\mathcal{M}_p^+([0, T] \times \bar{\mathcal{O}} \times A) \times \mathcal{P}_p([0, T] \times \bar{\mathcal{O}})$, where $\nu^n(dt, dx, da) := m_t^n(dx, da)dt$. Using the test function $u(t, x) = \int_t^T \varphi(s)ds$ in the constraint with φ a nonnegative continuous function, for all $n \geq 1$

$$\int_{[0, T] \times \bar{\mathcal{O}} \times A} \varphi(t) \nu^n(dt, dx, da) \leq \int_0^T \varphi(t) dt.$$

Taking $n \rightarrow \infty$, we conclude that $\int_{\bar{\mathcal{O}} \times A} \nu(dt, dx, da)$ is absolutely continuous with respect to the Lebesgue measure on $[0, T]$, which allows the disintegration $\nu(dt, dx, da) = m_t(dx, da)dt$ for some transition kernel m (which can be taken so that for each $t \in [0, T]$, $m_t \in \mathcal{M}_p^+(\bar{\mathcal{O}} \times A)$). In fact, for any closed set B of $[0, T]$, take $(\varphi_k)_{k \geq 1}$ a sequence of continuous nonnegative functions converging pointwise to $\mathbb{1}_B$, for example

$$\varphi_k(t) := (1 - kd(t, B))^+, \quad k \geq 1.$$

Now by Fatou's Lemma

$$\nu(B \times \bar{\mathcal{O}} \times A) \leq \liminf_{k \rightarrow \infty} \int_{[0, T] \times \bar{\mathcal{O}} \times A} \varphi_k(t) \nu(dt, dx, da) \leq \lambda(B).$$

By the inner regularity of the measures (since they are Radon measures), closed sets are sufficient to characterize absolute continuity. The rest follows by the disintegration theorem (Theorem 2.1 in Chapter 1, Section 2.3).

Let $B = \{t \in [0, T] : m_t(\bar{\mathcal{O}} \times A) > 1\} \in \mathcal{B}([0, T])$ (because $t \mapsto m_t(\bar{\mathcal{O}} \times A)$ is measurable). We define $\varphi : [0, T] \rightarrow \mathbb{R}$ such that $\varphi(t) = \mathbb{1}_B(t)$. Since $C([0, T])$ is dense in $L^1([0, T])$, there exists a sequence $(\varphi_k)_{k \geq 1} \subset C([0, T])$ converging to φ in $L^1([0, T])$. We define a new sequence $(\bar{\varphi}_k)_{k \geq 1} \subset C([0, T])$ as

$$\bar{\varphi}_k(t) := \min \{ \max \{ \varphi_k(t), 0 \}, 1 \}, \quad t \in [0, T].$$

Since for all $k \geq 1$,

$$\int_0^T |\bar{\varphi}_k(t) - \varphi(t)| dt \leq \int_0^T |\varphi_k(t) - \varphi(t)| dt,$$

we conclude that $(\bar{\varphi}_k)_{k \geq 1}$ converges to φ in $L^1([0, T])$. Up to taking a subsequence, we suppose without loss of generality that $(\bar{\varphi}_k)_{k \geq 1}$ converges to φ t -almost everywhere on $[0, T]$. On the other hand, for all $t \in [0, T]$ and all $k \geq 1$,

$$|\bar{\varphi}_k(t)(1 - m_t(\bar{\mathcal{O}} \times A))| \leq 1 + m_t(\bar{\mathcal{O}} \times A).$$

By dominated convergence,

$$\begin{aligned} 0 &\geq \int_0^T \mathbb{1}_B(t)(1 - m_t(\bar{\mathcal{O}} \times A)) dt = \int_0^T \varphi(t)(1 - m_t(\bar{\mathcal{O}} \times A)) dt \\ &= \lim_{n \rightarrow \infty} \int_0^T \bar{\varphi}_k(t)(1 - m_t(\bar{\mathcal{O}} \times A)) dt \geq 0. \end{aligned}$$

Since $t \mapsto \mathbb{1}_B(t)(1 - m_t(\bar{\mathcal{O}} \times A))$ is nonpositive, we conclude that $m_t(\bar{\mathcal{O}} \times A) \leq 1$ t -almost everywhere on $[0, T]$, so that $m \in V_p$. It remains to show that $(m, \mu) \in \mathcal{R}$. By Proposition 2.10, $(m^n)_{n \geq 1}$ converges to m in $\bar{\tau}_p$. Let $u \in C_b^{1,2}([0, T] \times \bar{\mathcal{O}})$. Using that $u \in C_b([0, T] \times \bar{\mathcal{O}})$ and

$$\frac{\partial u}{\partial t} + \mathcal{L}u \in M_{mc,p}([0, T] \times (\bar{\mathcal{O}} \times A)),$$

we can take limits in

$$\langle \mu^n, u \rangle = \langle m_0^*, u(0, \cdot) \rangle + \langle m^n, \partial_t u + \mathcal{L}u \rangle,$$

to obtain

$$\langle \mu, u \rangle = \langle m_0^*, u(0, \cdot) \rangle + \langle m, \partial_t u + \mathcal{L}u \rangle,$$

which shows that $(m, \mu) \in \mathcal{R}$ and hence $(\mathcal{R}, \bar{\tau}_p \otimes \tau_p)$ is compact. \square

The existence result. We now give the main result of this subsection, which consists in showing that there exists an admissible maximizer $(m^*, \mu^*) \in \mathcal{R}$ for Γ .

Theorem 2.12 (*Existence of a solution for the LP problem*). *There exists a solution to the linear programming problem for the single agent.*

Proof. Observe that the mapping

$$\mathcal{R} \ni (m, \mu) \mapsto \Gamma(m, \mu) = \langle m, f \rangle + \langle \mu, g \rangle$$

is continuous in $\bar{\tau}_p \otimes \tau_p$. Since $(\mathcal{R}, \bar{\tau}_p \otimes \tau_p)$ is compact by Theorem 2.11 we get the existence of a maximizer. \square

The following result is well known in the literature (see [KNJ87; HL90; DS12; Lac15]) but we give a proof for sake of completeness.

Proposition 2.13 (*Existence of a strict control LP solution*). *Suppose that for all $(t, x) \in [0, T] \times \bar{\mathcal{O}}$, the subset*

$$K(t, x) := \{(b(t, x, a), \sigma^2(t, x, a), z) : a \in A, z \leq f(t, x, a)\}$$

of $\mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}$ is convex, then there exists a strict control LP solution.

Proof. Let (m^*, μ^*) be a maximizer of the LP problem which exists by Theorem 2.12. Let $\nu_{t,x}^*$ such that

$$m_t^*(dx, da)dt = \nu_{t,x}^*(da)m_t^*(dx, A)dt.$$

Let $(t, x, a) \in [0, T] \times \bar{\mathcal{O}} \times A$ be arbitrary. We have that

$$(b(t, x, a), \sigma^2(t, x, a), f(t, x, a)) \in K(t, x).$$

As in Proposition 3.5 of [HL90] one can prove that $K(t, x)$ is closed. Now, by Theorem I.6.13

(p. 145) in [War72],

$$\left(\int_A b(t, x, a) \nu_{t,x}^*(da), \int_A \sigma^2(t, x, a) \nu_{t,x}^*(da), \int_A f(t, x, a) \nu_{t,x}^*(da) \right) \in K(t, x).$$

By definition of $K(t, x)$ and Theorem A.9 in [HL90] there exists a measurable function $(t, x) \mapsto \alpha(t, x)$ such that

$$\int_A b(t, x, a) \nu_{t,x}^*(da) = b(t, x, \alpha(t, x)), \quad \int_A \sigma^2(t, x, a) \nu_{t,x}^*(da) = \sigma^2(t, x, \alpha(t, x)),$$

$$\int_A f(t, x, a) \nu_{t,x}^*(da) \leq f(t, x, \alpha(t, x)).$$

Define $\bar{m}_t(dx, da) = \delta_{\alpha(t,x)}(da) m_t^*(dx, A)$ for each $t \in [0, T]$. We conclude that $(\bar{m}, \mu^*) \in \mathcal{R}$ and $\Gamma(m^*, \mu^*) \leq \Gamma(\bar{m}, \mu^*)$. \square

3 MFG problem

For the MFG problem we are given constants $q > p \geq 1 \vee r$, where $r \in [0, 2]$ and $q \geq 2$, and the following functions:

$$\begin{aligned} (b, \sigma) &: [0, T] \times \mathbb{R} \times \mathcal{P}_p^{sub}(\bar{\mathcal{O}}) \times A \rightarrow \mathbb{R} \times \mathbb{R}, \\ f &: [0, T] \times \bar{\mathcal{O}} \times \mathcal{P}_p^{sub}(\bar{\mathcal{O}}) \times A \rightarrow \mathbb{R}, \\ g &: [0, T] \times \bar{\mathcal{O}} \times \mathcal{P}_p([0, T] \times \bar{\mathcal{O}}) \rightarrow \mathbb{R}. \end{aligned}$$

Throughout this section, unless specified otherwise, we will impose the following assumption.

Assumption 3.1.

- (1) $m_0^* \in \mathcal{P}_q(\bar{\mathcal{O}})$.
- (2) The functions $(t, x, \eta, a) \mapsto b(t, x, \eta, a)$ and $(t, x, \eta, a) \mapsto \sigma(t, x, \eta, a)$ are jointly measurable and continuous in (x, η, a) for each $t \in [0, T]$. Moreover, there exists a constant $c_1 > 0$ such that for all $(t, x, y, \eta, a) \in [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathcal{P}_p^{sub}(\bar{\mathcal{O}}) \times A$,

$$|b(t, x, \eta, a) - b(t, y, \eta, a)| + |\sigma(t, x, \eta, a) - \sigma(t, y, \eta, a)| \leq c_1 |x - y|,$$

$$|b(t, x, \eta, a)| \leq c_1 \left[1 + |x| + \left(\int_{\bar{\mathcal{O}}} |z|^p \eta(dz) \right)^{1/p} \right],$$

$$\sigma^2(t, x, \eta, a) \leq c_1 \left[1 + |x|^r + \left(\int_{\bar{\mathcal{O}}} |z|^p \eta(dz) \right)^{r/p} \right].$$

- (3) The function $(t, x, \eta, a) \mapsto f(t, x, \eta, a)$ is jointly measurable and continuous in (x, η, a) for each $t \in [0, T]$. The function g is continuous. Moreover, there exists a constant $c_2 > 0$ such that for all $(t, x, \eta, a, \mu) \in [0, T] \times \bar{\mathcal{O}} \times \mathcal{P}_p^{sub}(\bar{\mathcal{O}}) \times A \times \mathcal{P}_p([0, T] \times \bar{\mathcal{O}})$,

$$|f(t, x, \eta, a)| \leq c_2 \left[1 + |x|^p + \int_{\bar{\mathcal{O}}} |z|^p \eta(dz) \right],$$

$$|g(t, x, \mu)| \leq c_2 \left[1 + |x|^p + \int_{[0, T] \times \bar{\mathcal{O}}} |z|^p \mu(ds, dz) \right].$$

(4) One of the following statements is true:

(a) *Unattainable boundary:* For every filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, \mathbb{F} -progressive measurable process ν with values in $\mathcal{P}(A)$, \mathbb{F} -martingale measure M with intensity $\nu_t(da)dt$, $m \in V_p$ and \mathbb{F} -adapted process X such that

$$dX_t = \int_A b(t, X_t, m_t^x, a) \nu_t(da)dt + \int_A \sigma(t, X_t, m_t^x, a) M(dt, da), \quad \mathbb{P} \circ X_0^{-1} = m_0^*,$$

we have $\mathbb{P}(\tau_{\bar{\mathcal{O}}}^X \geq T) = 1$.

(b) *Attainable boundary:* \mathcal{O} is an open interval, σ does not depend on the control a and for all $(t, x, \eta) \in [0, T] \times \mathbb{R} \times \mathcal{P}_p^{sub}(\bar{\mathcal{O}})$, $\sigma^2(t, x, \eta) \geq c_\sigma$ for some $c_\sigma > 0$.

The strong and LP MFG formulations. Let us first provide the *strong formulation of the MFG problem*.

Definition 3.2 (*Strong formulation of the MFG problem*).

1. *First step:* fix $m \in V_p$ and $\mu \in \mathcal{P}_p([0, T] \times \bar{\mathcal{O}})$ and find the solution to the mixed control problem

$$\begin{aligned} \max_{\tau \in \mathcal{T}, \alpha \in \mathbb{A}} \mathbb{E} \left[\int_0^{\tau \wedge \tau_{\bar{\mathcal{O}}}^{\alpha, m}} f(t, X_t^{\alpha, m}, m_t^x, \alpha_t) dt + g \left(\tau \wedge \tau_{\bar{\mathcal{O}}}^{\alpha, m}, X_{\tau \wedge \tau_{\bar{\mathcal{O}}}^{\alpha, m}}^{\alpha, m}, \mu \right) \right], \\ \text{s.t.} \quad dX_t^{\alpha, m} = b(t, X_t^{\alpha, m}, m_t^x, \alpha_t) dt + \sigma(t, X_t^{\alpha, m}, m_t^x, \alpha_t) dW_t, \\ X_0^{\alpha, m} \sim m_0^*, \end{aligned} \quad (3.6)$$

where $\tau_{\bar{\mathcal{O}}}^{\alpha, m} = \inf\{t \geq 0 : X_t^{\alpha, m} \notin \mathcal{O}\}$.

2. *Given the mixed optimal stopping-control $(\tau^{m, \mu}, \alpha^{m, \mu})$ (solution of the problem (3.6)) for the agent with initial distribution m_0^* facing a mean-field (m, μ) , the second step consists in finding the flow of measures $m \in V_p$ and $\mu \in \mathcal{P}_p([0, T] \times \bar{\mathcal{O}})$ such that*

$$m_t(B) = \mathbb{P} \left[(X_t^{\alpha^{m, \mu}, m}, \alpha_t^{m, \mu}) \in B, t < \tau^{m, \mu} \wedge \tau_{\bar{\mathcal{O}}}^{\alpha^{m, \mu}, m} \right], \quad B \in \mathcal{B}(\bar{\mathcal{O}} \times A), \quad t \in [0, T],$$

and

$$\mu = \mathcal{L} \left(\tau^{m, \mu} \wedge \tau_{\bar{\mathcal{O}}}^{\alpha^{m, \mu}, m}, X_{\tau^{m, \mu} \wedge \tau_{\bar{\mathcal{O}}}^{\alpha^{m, \mu}, m}}^{\alpha^{m, \mu}, m} \right).$$

We now give the formulation of the *linear programming MFG problem*. Since for $\bar{m} \in V_p$ we have that t -a.e. on $[0, T]$ $\bar{m}_t^x \in \mathcal{P}_p^{sub}(\bar{\mathcal{O}})$, when integrating with respect to dt we make the convention that $\bar{m}_t^x \in \mathcal{P}_p^{sub}(\bar{\mathcal{O}})$ for all $t \in [0, T]$, so that b , σ and f are well defined when evaluated in \bar{m}_t^x .

Definition 3.3. Fix a pair $(\bar{m}, \bar{\mu}) \in V_p \times \mathcal{P}_p([0, T] \times \bar{\mathcal{O}})$ and define $\mathcal{R}[\bar{m}]$ as the set of pairs

$(m, \mu) \in V_p \times \mathcal{P}_p([0, T] \times \bar{\mathcal{O}})$, such that for all $u \in C_b^{1,2}([0, T] \times \bar{\mathcal{O}})$,

$$\langle \mu, u \rangle = \langle m_0^*, u(0, \cdot) \rangle + \langle m, \partial_t u + \mathcal{L}[\bar{m}]u \rangle$$

where

$$\mathcal{L}[\bar{m}]u(t, x, a) := b(t, x, \bar{m}_t^x, a) \partial_x u(t, x) + \frac{\sigma^2}{2}(t, x, \bar{m}_t^x, a) \partial_{xx} u(t, x), \quad (t, x, a) \in [0, T] \times \mathbb{R} \times A.$$

Let $\Gamma[\bar{m}, \bar{\mu}] : V_p \times \mathcal{P}_p([0, T] \times \bar{\mathcal{O}}) \rightarrow \mathbb{R}$ be defined as

$$\Gamma[\bar{m}, \bar{\mu}](m, \mu) = \langle m, f[\bar{m}] \rangle + \langle \mu, g[\bar{\mu}] \rangle,$$

where

$$f[\bar{m}](t, x, a) := f(t, x, \bar{m}_t^x, a), \quad g[\bar{\mu}](t, x) := g(t, x, \bar{\mu}), \quad (t, x, a) \in [0, T] \times \bar{\mathcal{O}} \times A.$$

We say that $(m^*, \mu^*) \in V_p \times \mathcal{P}_p([0, T] \times \bar{\mathcal{O}})$ is an LP MFG Nash equilibrium if $(m^*, \mu^*) \in \mathcal{R}[m^*]$ and for all $(m, \mu) \in \mathcal{R}[m^*]$,

$$\Gamma[m^*, \mu^*](m, \mu) \leq \Gammam^*, \mu^*.$$

The real number \Gammam^*, μ^* is called Nash value.

The next theorem states the existence of LP MFG Nash equilibria. Its proof is divided in two main steps, similarly to [Lac15]. The first one consists in showing the existence when the coefficients b and σ are bounded and the second step approximates the general coefficients with bounded ones.

Theorem 3.4 (Existence of LP MFG equilibria). *There exists an LP MFG Nash equilibrium.*

Proposition 3.5. *The set of LP MFG Nash equilibria is compact.*

Using an analogous proof to the one in Proposition 2.13, under some convexity assumptions, we get the existence of a strict control LP MFG Nash equilibrium.

Corollary 3.6. *Suppose that for all $(t, x, m) \in [0, T] \times \bar{\mathcal{O}} \times V_p$, the subset*

$$K[m](t, x) := \{(b(t, x, m_t^x, a), \sigma^2(t, x, m_t^x, a), z) : a \in A, z \leq f(t, x, m_t^x, a)\}$$

of $\mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}$ is convex. Then there exists a strict control LP MFG Nash equilibrium.

3.1 Bounded coefficients

Along this subsection suppose that Assumption 3.1 together with the following assumption hold.

Assumption 3.7. *The functions b and σ are bounded.*

Preliminary results. In order to show the existence of an equilibrium we will apply a fixed point theorem. To do this, it is important to construct a compact convex set including all the sets $\mathcal{R}[\bar{m}]$ for $\bar{m} \in V_p$. We will work with the closed convex hull of the union of those sets and show some of its properties. Define the set \mathcal{R}_1 by

$$\mathcal{R}_1 := \bigcup_{\bar{m} \in V_p} \mathcal{R}[\bar{m}].$$

We denote by \mathcal{R}_2 the convex hull of \mathcal{R}_1 and by $\bar{\mathcal{R}}_2$ its closure on $V_p \times \mathcal{P}_p([0, T] \times \bar{\mathcal{O}})$.

Let $(m, \mu) \in \mathcal{R}[\bar{m}]$ for some $\bar{m} \in V_p$ and consider the Markovian relaxed control $\nu_{t,x}$ coming from the disintegration of m . By Theorem 5.20, there exist a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, an \mathbb{F} -adapted process X , an \mathbb{F} -stopping time τ such that $\tau \leq T \wedge \tau_{\bar{\mathcal{O}}}^X$ \mathbb{P} -a.s., and an \mathbb{F} -martingale measure M with intensity $\nu_{t,X_t}(da) \mathbb{1}_{t < \tau} dt$, such that

$$X_{t \wedge \tau} = X_0 + \int_0^{t \wedge \tau} \int_A b(s, X_s, \bar{m}_s, a) \nu_{s, X_s}(da) ds + \int_0^{t \wedge \tau} \int_A \sigma(s, X_s, \bar{m}_s, a) M(ds, da),$$

$$\mathbb{P} \circ X_0^{-1} = m_0^*,$$

$$m_t(B \times C) = \mathbb{E}^{\mathbb{P}} [\mathbb{1}_B(X_t) \nu_{t, X_t}(C) \mathbb{1}_{t < \tau}], \quad B \in \mathcal{B}(\bar{\mathcal{O}}), \quad C \in \mathcal{B}(A), \quad t - a.e.,$$

$$\mu = \mathbb{P} \circ (\tau, X_\tau)^{-1}.$$

In particular we will take always m_t^x as its càdlàg version $\mathbb{E}^{\mathbb{P}}[\mathbb{1} \cdot (X_t) \mathbb{1}_{t < \tau}]$. Clearly, a convex combination of m leads to a convex combination of m^x preserving the càdlàg property, which allow us to take the same convention for $(m, \mu) \in \mathcal{R}_2$. Moreover, using the same proof of Proposition 2.7 (which is even simpler here since the coefficients b and σ are bounded), we get the following result.

Proposition 3.8. *There exist constants $C_1 \geq 0$ and $C_2 \geq 0$ such that for all $(m, \mu) \in \mathcal{R}_1$ we have the following estimates:*

(1) For all $t \in [0, T]$,

$$\int_{\bar{\mathcal{O}}} |x|^q m_t^x(dx) \leq C_1, \quad \int_{[0, T] \times \bar{\mathcal{O}}} |x|^q \mu(dt, dx) \leq C_1.$$

(2) For all $h \in [0, T]$, we have

$$\int_0^{T-h} d_{BL}(m_{t+h}^x, m_t^x) dt \leq C_2 \sqrt{h}.$$

Definition 3.9. Let \mathcal{R}_0 be the set of pairs $(m, \mu) \in V_p \times \mathcal{P}_p([0, T] \times \bar{\mathcal{O}})$, such that

$$\int_0^T \int_{\bar{\mathcal{O}} \times A} |x|^q m_t(dx, da) dt \leq C_1 T, \quad \int_{[0, T] \times \bar{\mathcal{O}}} |x|^q \mu(dt, dx) \leq C_1,$$

and for all $\varphi \in C([0, T]; \mathbb{R}_+)$,

$$\int_0^T \int_{\bar{\mathcal{O}} \times \mathcal{A}} \varphi(t) m_t(dx, da) dt \leq \int_0^T \varphi(t) dt.$$

Lemma 3.10. *The topological space $(\mathcal{R}_0, \tau_p \otimes \tau_p)$ is compact and convex. Moreover, $\mathcal{R}_1 \subset \mathcal{R}_0$, which in particular implies that $\bar{\mathcal{R}}_2 \subset \mathcal{R}_0$ and that $(\bar{\mathcal{R}}_2, \tau_p \otimes \tau_p)$ is compact.*

Proof. Relative compactness of \mathcal{R}_0 in $\tau_p \otimes \tau_p$ follows by the same ideas as in Theorem 2.11, so it suffices to show that it is closed. Consider a sequence $(m^n, \mu^n)_{n \geq 1} \subset \mathcal{R}_0$ converging to $(m, \mu) \in V_p \times \mathcal{P}_p([0, T] \times \bar{\mathcal{O}})$ in $\tau_p \otimes \tau_p$. Let us show that $(m, \mu) \in \mathcal{R}_0$. For all $n \geq 1$ and $k \geq 1$,

$$\int_{[0, T] \times \bar{\mathcal{O}}} [|x|^q \wedge k] \mu^n(dt, dx) \leq C_1.$$

Now taking $n \rightarrow \infty$,

$$\int_{[0, T] \times \bar{\mathcal{O}}} [|x|^q \wedge k] \mu(dt, dx) \leq C_1.$$

By monotone convergence, letting $k \rightarrow \infty$,

$$\int_{[0, T] \times \bar{\mathcal{O}}} |x|^q \mu(dt, dx) \leq C_1.$$

Using the same argument,

$$\int_0^T \int_{\bar{\mathcal{O}} \times \mathcal{A}} |x|^q m_t(dx, da) dt \leq C_1 T.$$

The last constraint follows by passing directly to the limit. The convexity of \mathcal{R}_0 is easy to verify. The inclusion $\mathcal{R}_1 \subset \mathcal{R}_0$ follows by Proposition 3.8 and plugging the test function $u(t, x) = \int_t^T \varphi(s) ds$ in the constraints $\mathcal{R}[\bar{m}]$, $\bar{m} \in V_p$, with $\varphi \in C([0, T]; \mathbb{R}_+)$. \square

Theorem 3.11. *The map $\mathcal{Z} : (\mathcal{R}_2, \tau_p \otimes \tau_p) \rightarrow (M([0, T]; \mathcal{P}_p^{sub}(\bar{\mathcal{O}})), \tilde{\tau}_p)$ defined by*

$$\mathcal{Z}(m, \mu) = m^x, \quad (m, \mu) \in \mathcal{R}_2,$$

is continuous.

Proof. Since we are dealing with metric spaces, it suffices to work with sequences. Consider a sequence $(m^n, \mu^n)_{n \geq 1} \subset \mathcal{R}_2$ converging to some $(m^0, \mu^0) \in \mathcal{R}_2$. In particular we can write for $n \geq 0$

$$(m^n, \mu^n) = \sum_{k=1}^{\ell_n} \lambda_k^n (m^{n,k}, \mu^{n,k}),$$

for some $\ell_n \in \mathbb{N}^*$, $(m^{n,k}, \mu^{n,k})_{1 \leq k \leq \ell_n} \subset \mathcal{R}_1$ and $(\lambda_k^n)_{1 \leq k \leq \ell_n} \subset [0, 1]$ such that $\sum_{k=1}^{\ell_n} \lambda_k^n = 1$. Let us prove that $(m^{x,n})_{n \geq 1}$ converges to $m^{x,0}$ in $\tilde{\tau}_p$. We first show that the sequence $(m^{x,n})_{n \geq 1}$ is relatively compact in $\tilde{\tau}_p$ and then we will identify the set of limit points with the singleton $\{m^{x,0}\}$. To do this, we use the relative compactness criterion given in Theorem 5.24 for the

topology of convergence in measure. Let H be the map defined in Lemma 2.8. Using the first estimates of Proposition 3.8,

$$\sup_{n \geq 1} \int_0^T H(m_t^{x,n}) dt = \sup_{n \geq 1} \int_0^T \int_{\bar{\mathcal{O}}} |x|^q m_t^{x,n}(dx) dt \leq C_1 T.$$

Now we show that

$$\limsup_{h \downarrow 0} \sup_{n \geq 1} \int_0^{T-h} d_{\text{BL}}(m_{t+h}^{x,n}, m_t^{x,n}) dt = 0.$$

For $h \in [0, T]$ and $t \in [0, T-h]$,

$$d_{\text{BL}}(m_{t+h}^{x,n}, m_t^{x,n}) = \sup_{\phi \in \text{BL}(\bar{\mathcal{O}}), \|\phi\|_{\text{BL}} \leq 1} \int_{\bar{\mathcal{O}}} \phi(x) (m_{t+h}^{x,n} - m_t^{x,n})(dx).$$

By the second estimate of Proposition 3.8,

$$\begin{aligned} & \int_0^{T-h} \sup_{\phi \in \text{BL}(\bar{\mathcal{O}}), \|\phi\|_{\text{BL}} \leq 1} \int_{\bar{\mathcal{O}}} \phi(x) (m_{t+h}^{x,n} - m_t^{x,n})(dx) dt \\ & \leq \sum_{k=1}^{\ell_n} \lambda_k^n \int_0^{T-h} \sup_{\phi \in \text{BL}(\bar{\mathcal{O}}), \|\phi\|_{\text{BL}} \leq 1} \int_{\bar{\mathcal{O}}} \phi(x) (m_{t+h}^{x,n,k} - m_t^{x,n,k})(dx) dt \\ & \leq C_2 \sqrt{h}. \end{aligned}$$

We conclude that

$$\limsup_{h \downarrow 0} \sup_{n \geq 1} \int_0^{T-h} d_{\text{BL}}(m_{t+h}^{x,n}, m_t^{x,n}) dt = 0.$$

By Theorem 5.24, up to a subsequence, $(m^{x,n})_{n \geq 1}$ converges to some $\eta \in M([0, T]; \mathcal{P}^{\text{sub}}(\bar{\mathcal{O}}))$ in measure. As in Proposition 2.9 one can show that up to a subsequence $(m^{x,n})_{n \geq 1}$ converges to some η in $\tilde{\tau}_p$. Finally, showing that $m^{x,0}$ is the unique limit point of $(m^{x,n})_{n \geq 1}$ in $\tilde{\tau}_p$ follows by the same arguments used in Proposition 2.10. \square

Using the properties of the topology of convergence in measure to show the existence of an equilibrium needs the extension of the previous theorem to the closure of \mathcal{R}_2 , which will be the domain of the best response mapping.

Corollary 3.12. *The map \mathcal{Z} can be extended to a continuous map on $\bar{\mathcal{R}}_2$, which is still denoted by \mathcal{Z} . Moreover, t -a.e. on $[0, T]$,*

$$\mathcal{Z}(m, \mu)_t = m_t^x.$$

In particular for any $(m, \mu) \in \bar{\mathcal{R}}_2$ we can always take $m^x \in M([0, T]; \mathcal{P}_p^{\text{sub}}(\bar{\mathcal{O}}))$. Finally, for all $(m, \mu) \in \bar{\mathcal{R}}_2$, we have that t -a.e. on $[0, T]$:

$$\int_{\bar{\mathcal{O}}} |x|^q m_t^x(dx) \leq C_1, \quad \int_{[0, T] \times \bar{\mathcal{O}}} |x|^q \mu(dt, dx) \leq C_1.$$

Proof. We prove first that we can extend the map \mathcal{Z} to the closure of \mathcal{R}_2 . Since we are working with metric spaces, by Lemma 2 in [MR12], it suffices to show that for every sequence

$(m^n, \mu^n)_{n \geq 1} \subset \mathcal{R}_2$ converging to some $(m, \mu) \in \bar{\mathcal{R}}_2$ in $\tau_p \otimes \tau_p$, the sequence

$$(\mathcal{Z}(m^n, \mu^n))_{n \geq 1} = (m^{x,n})_{n \geq 1}$$

converges in $\tilde{\tau}_p$. Following the same proof of Theorem 3.11, we get that the sequence $(m^{x,n})_{n \geq 1}$ is relatively compact in $\tilde{\tau}_p$. As in Proposition 2.10, one can show that any limit point in $\tilde{\tau}_p$ of the sequence $(m^{x,n})_{n \geq 1}$ is equal almost everywhere to m^x .

Now, let $(m, \mu) \in \bar{\mathcal{R}}_2$. The second estimate is a consequence of $\bar{\mathcal{R}}_2 \subset \mathcal{R}_0$ which was shown in Lemma 3.10. For the first estimate, consider a sequence $(m^n, \mu^n)_{n \geq 1} \subset \mathcal{R}_2$ converging to (m, μ) in $\tau_p \otimes \tau_p$. By Proposition 3.8 and linearity of the estimate with respect to the measure, for all $n \geq 1$ and $t \in [0, T]$,

$$\int_{\bar{O}} |x|^q m_t^{x,n}(dx) \leq C_1.$$

By continuity of \mathcal{Z} , we get that $m^{x,n} \rightarrow \mathcal{Z}(m, \mu)$ in $\tilde{\tau}_p$, moreover, up to a subsequence, it converges a.e. in τ_p . Using that $\mathcal{Z}(m, \mu) = m^x$ a.e., we obtain by passing to the limit that t -a.e. on $[0, T]$,

$$\int_{\bar{O}} |x|^q m_t^x(dx) \leq C_1.$$

□

Lemma 3.13 (*Stable convergence with polynomial growth*). *In $\bar{\mathcal{R}}_2$, the topologies $\tau_p \otimes \tau_p$ and $\bar{\tau}_p \otimes \tau_p$ coincide.*

Proof. We only need to show that $\bar{\tau}_p \otimes \tau_p \subset \tau_p \otimes \tau_p$. Consider a sequence $(m^n, \mu^n)_{n \geq 1} \subset \bar{\mathcal{R}}_2$ converging to some $(m, \mu) \in \bar{\mathcal{R}}_2$. Using Corollary 3.12, we get that $(m^{x,n})_{n \geq 1}$ converges in $\tilde{\tau}_p$ to m^x . As in Proposition 2.10, this is enough to show that $(m^n)_{n \geq 1}$ converges in $\bar{\tau}_p$ to m . □

Existence of equilibria. We aim to show the existence of equilibria via a fixed point theorem applied to the best response mapping, which is defined as follows.

Definition 3.14. *Define the set valued mapping $\mathcal{R}^* : \bar{\mathcal{R}}_2 \rightarrow 2^{\bar{\mathcal{R}}_2}$ by*

$$\mathcal{R}^*(\bar{m}, \bar{\mu}) = \mathcal{R}[\bar{m}], \quad (\bar{m}, \bar{\mu}) \in \bar{\mathcal{R}}_2.$$

Define the best response mapping $\Theta : \bar{\mathcal{R}}_2 \rightarrow 2^{\bar{\mathcal{R}}_2}$ by

$$\Theta(\bar{m}, \bar{\mu}) = \arg \max_{(m, \mu) \in \mathcal{R}^*(\bar{m}, \bar{\mu})} \Gamma[\bar{m}, \bar{\mu}](m, \mu), \quad (\bar{m}, \bar{\mu}) \in \bar{\mathcal{R}}_2.$$

Remark 3.15. *Note that the set of LP MFG Nash equilibria coincides with the set of fixed points of Θ .*

Lemma 3.16. *\mathcal{R}^* has nonempty compact values and Θ has nonempty convex values.*

Proof. Let $(\bar{m}, \bar{\mu}) \in \bar{\mathcal{R}}_2$ be fixed. The set $\mathcal{R}^*(\bar{m}, \bar{\mu}) = \mathcal{R}[\bar{m}]$ and the function

$$\mathcal{R}[\bar{m}] \ni (m, \mu) \mapsto \Gamma[\bar{m}, \bar{\mu}](m, \mu)$$

have the same structure as the set \mathcal{R} and the function Γ of Section 2, moreover, Assumption 2.1 is satisfied. Henceforth we can apply the results of Section 2 to deduce that $\mathcal{R}^*(\bar{m}, \bar{\mu}) = \mathcal{R}[\bar{m}]$ is nonempty and compact and that there exists a maximizer, in particular, $\Theta(\bar{m}, \bar{\mu})$ is nonempty. By convexity of the set $\mathcal{R}[\bar{m}]$ and the linearity of $\mathcal{R}[\bar{m}] \ni (m, \mu) \mapsto \Gamma[\bar{m}, \bar{\mu}](\mu, m)$, the set $\Theta(\bar{m}, \bar{\mu})$ is convex. \square

We shall first provide some convergence results, which will be useful in the proof of existence of LP MFG Nash equilibria.

Lemma 3.17. *Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a filtered probability space. Let τ be a $[0, T]$ -valued \mathbb{F} -stopping time and let M be an \mathbb{F} -martingale measure with intensity $q_t(da)\mathbb{1}_{t < \tau}dt$, where $(q_t)_{t \in [0, T]}$ is an \mathbb{F} -predictable process with values in $\mathcal{P}(A)$. Consider a sequence $(\bar{m}^n, \bar{\mu}^n)_{n \geq 1} \subset \bar{\mathcal{R}}_2$ converging to some $(\bar{m}, \bar{\mu}) \in \bar{\mathcal{R}}_2$ in $\tau_p \otimes \tau_p$ and let X and $(X^n)_{n \geq 1}$ be \mathbb{F} -adapted processes satisfying,*

$$dX_t = \int_A b(t, X_t, \bar{m}_t^x, a)q_t(da)dt + \int_A \sigma(t, X_t, \bar{m}_t^x, a)M(dt, da), \quad t \leq \tau, \quad X_0 \sim m_0^*.$$

$$dX_t^n = \int_A b(t, X_t^n, \bar{m}_t^{x,n}, a)q_t(da)dt + \int_A \sigma(t, X_t^n, \bar{m}_t^{x,n}, a)M(dt, da), \quad t \leq \tau, \quad X_0^n = X_0.$$

Then, up to a subsequence,

$$\mathbb{E}^{\mathbb{P}} \left[\sup_{t \leq T} |X_{t \wedge \tau}^n - X_{t \wedge \tau}|^2 \right] \xrightarrow{n \rightarrow \infty} 0.$$

Proof. We will denote by $C > 0$ any constant independent from n . To simplify the formulas, in this proof we shall use the following shorthand notation: $b_n(t, x, a) := b(t, x, \bar{m}_t^{x,n}, a)$, $b_0(t, x, a) := b(t, x, \bar{m}_t^x, a)$, $\sigma_n(t, x, a) := \sigma(t, x, \bar{m}_t^{x,n}, a)$ and $\sigma_0(t, x, a) := \sigma(t, x, \bar{m}_t^x, a)$. Let $0 \leq s \leq t \leq T$. We have

$$\begin{aligned} |X_{s \wedge \tau}^n - X_{s \wedge \tau}|^2 &\leq C \left[\left(\int_0^{s \wedge \tau} \int_A (b_n(r, X_r^n, a) - b_0(r, X_r, a))q_r(da)dr \right)^2 \right. \\ &\quad \left. + \left(\int_0^{s \wedge \tau} \int_A (\sigma_n(r, X_r^n, a) - \sigma_0(r, X_r, a))M(dr, da) \right)^2 \right]. \end{aligned}$$

Using Burkholder-Davis-Gundy inequality, we get

$$\begin{aligned} \mathbb{E}^{\mathbb{P}} \left[\sup_{s \leq t} \left(\int_0^{s \wedge \tau} \int_A (\sigma_n(r, X_r^n, a) - \sigma_0(r, X_r, a))M(dr, da) \right)^2 \right] \\ \leq C \mathbb{E}^{\mathbb{P}} \left[\int_0^{t \wedge \tau} \int_A (\sigma_n(r, X_r^n, a) - \sigma_0(r, X_r, a))^2 q_r(da)dr \right]. \end{aligned}$$

Define

$$\xi_n(t) = \mathbb{E}^{\mathbb{P}} \left[\sup_{s \leq t} |X_{s \wedge \tau}^n - X_{s \wedge \tau}|^2 \right].$$

From the above estimates,

$$\begin{aligned} \xi_n(t) \leq C \mathbb{E}^{\mathbb{P}} \left[\int_0^{t \wedge \tau} \int_A (b_n(r, X_r^n, a) - b_0(r, X_r, a))^2 q_r(da) dr \right. \\ \left. + \int_0^{t \wedge \tau} \int_A (\sigma_n(r, X_r^n, a) - \sigma_0(r, X_r, a))^2 q_r(da) dr \right]. \end{aligned}$$

Now, by the Lipschitz assumption on b ,

$$\begin{aligned} & \int_0^{t \wedge \tau} \int_A (b_n(r, X_r^n, a) - b_0(r, X_r, a))^2 q_r(da) dr \\ &= \int_0^{t \wedge \tau} \int_A (b_n(r, X_r^n, a) - b_n(r, X_r, a) + b_n(r, X_r, a) - b_0(r, X_r, a))^2 q_r(da) dr \\ &\leq C \left[\int_0^{t \wedge \tau} \int_A (b_n(r, X_r^n, a) - b_n(r, X_r, a))^2 q_r(da) dr \right. \\ &\quad \left. + \int_0^{t \wedge \tau} \int_A (b_n(r, X_r, a) - b_0(r, X_r, a))^2 q_r(da) dr \right] \\ &\leq C \left[\int_0^t \sup_{r \leq s} |X_{r \wedge \tau}^n - X_{r \wedge \tau}|^2 ds + \int_0^{t \wedge \tau} \int_A (b_n(r, X_r, a) - b_0(r, X_r, a))^2 q_r(da) dr \right]. \end{aligned}$$

Similarly,

$$\begin{aligned} & \int_0^{t \wedge \tau} \int_A (\sigma_n(r, X_r^n, a) - \sigma_0(r, X_r, a))^2 q_r(da) dr \\ &\leq C \left[\int_0^t \sup_{r \leq s} |X_{r \wedge \tau}^n - X_{r \wedge \tau}|^2 ds + \int_0^{t \wedge \tau} \int_A (\sigma_n(r, X_r, a) - \sigma_0(r, X_r, a))^2 q_r(da) dr \right]. \end{aligned}$$

We get finally,

$$\xi_n(t) \leq C \left(\int_0^t \xi_n(s) ds + B_n + S_n \right),$$

where

$$\begin{aligned} B_n &:= \mathbb{E}^{\mathbb{P}} \left[\int_0^T \int_A (b_n(r, X_r, a) - b_0(r, X_r, a))^2 q_r(da) dr \right], \\ S_n &:= \mathbb{E}^{\mathbb{P}} \left[\int_0^T \int_A (\sigma_n(r, X_r, a) - \sigma_0(r, X_r, a))^2 q_r(da) dr \right]. \end{aligned}$$

By Gronwall's inequality,

$$\xi_n(T) \leq C (B_n + S_n) e^{CT}. \quad (3.7)$$

By Corollary 3.12, $\bar{m}^{x,n}$ converges to \bar{m}^x in $M([0, T]; \mathcal{P}_p^{sub}(\bar{\mathcal{O}}))$, which implies that up to a subsequence, dr -a.e. on $[0, T]$ $\bar{m}_r^{x,n} \rightarrow \bar{m}_r^x$ in τ_p . Using the boundedness and the continuity of the coefficients with respect to the measure we get by dominated convergence that B_n and S_n converge to 0 as $n \rightarrow \infty$. Taking $n \rightarrow \infty$ in (3.7) we get the result. \square

We now prove the continuity of the set \mathcal{R}^* in the sense of set-valued mappings.

Proposition 3.18 (Continuity of \mathcal{R}^*). *The set-valued mapping \mathcal{R}^* is continuous (in the sense of Definition 5.32).*

Proof. Step 1. We first prove *the upper hemicontinuity* (in the sense of Definition 5.29). By the Closed Graph Theorem (see Theorem 5.30), it suffices to show that \mathcal{R}^* has closed graph. Let $(m^n, \mu^n) \in \mathcal{R}^*(\bar{m}^n, \bar{\mu}^n) = \mathcal{R}[\bar{m}^n]$ such that $(m^n, \mu^n) \rightarrow (m, \mu) \in \bar{\mathcal{R}}_2$ and $(\bar{m}^n, \bar{\mu}^n) \rightarrow (\bar{m}, \bar{\mu}) \in \bar{\mathcal{R}}_2$ in $\tau_p \otimes \tau_p$. For all $n \geq 1$ and $u \in C_b^{1,2}([0, T] \times \bar{\mathcal{O}})$,

$$\langle \mu^n, u \rangle = \langle m_0^*, u(0, \cdot) \rangle + \langle m^n, \partial_t u + \mathcal{L}[\bar{m}^n]u \rangle.$$

Since $(m^n)_{n \geq 1}$ converges to m in τ_p ,

$$\langle m^n, \partial_t u \rangle \xrightarrow{n \rightarrow \infty} \langle m, \partial_t u \rangle.$$

By Lemma 3.13, we get the convergence of $(m^n)_{n \geq 1}$ to m in $\bar{\tau}_p$ and by Corollary 3.12, $(\bar{m}^{x,n})_{n \geq 1}$ converges to \bar{m}^x in $\tilde{\tau}_p$, in particular, up to a subsequence, it converges a.e. in τ_p . We conclude by Lemma 5.27 that

$$\langle m^n, \mathcal{L}[\bar{m}^n]u \rangle \xrightarrow{n \rightarrow \infty} \langle m, \mathcal{L}[\bar{m}]u \rangle.$$

The above results, together with the convergence

$$\langle \mu^n, u \rangle \xrightarrow{n \rightarrow \infty} \langle \mu, u \rangle,$$

lead to

$$\langle \mu, u \rangle = \langle m_0^*, u(0, \cdot) \rangle + \langle m, \partial_t u + \mathcal{L}[\bar{m}]u \rangle,$$

which means that $(m, \mu) \in \mathcal{R}[\bar{m}] = \mathcal{R}^*(\bar{m}, \bar{\mu})$.

Step 2. We now prove *the lower hemicontinuity* (in the sense of Definition 5.31). Consider a sequence $(\bar{m}^n, \bar{\mu}^n)_{n \geq 1} \subset \bar{\mathcal{R}}_2$ such that $(\bar{m}^n, \bar{\mu}^n) \rightarrow (\bar{m}, \bar{\mu})$ in $\tau_p \otimes \tau_p$ and let $(m, \mu) \in \mathcal{R}^*(\bar{m}, \bar{\mu}) = \mathcal{R}[\bar{m}]$. We need to prove that up to a subsequence, we can find $(m^n, \mu^n)_{n \geq 1} \subset \bar{\mathcal{R}}_2$ such that $(m^n, \mu^n) \in \mathcal{R}^*(\bar{m}^n, \bar{\mu}^n) = \mathcal{R}[\bar{m}^n]$ and $(m^n, \mu^n) \rightarrow (m, \mu)$ in $\tau_p \otimes \tau_p$. Let $\nu_{t,x}(da)$ be such that

$$m_t(dx, da)dt = \nu_{t,x}(da)m_t^x(dx)dt.$$

By Theorem 5.20, there exists a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, an \mathbb{F} -adapted process X , an \mathbb{F} -stopping time τ such that $\tau \leq T \wedge \tau_{\bar{\mathcal{O}}}^X$ \mathbb{P} -a.s., an \mathbb{F} -martingale measure M with intensity $\nu_{t,X_t}(da)\mathbb{1}_{t < \tau}dt$, such that

$$X_{t \wedge \tau} = \int_0^{t \wedge \tau} \int_A b(t, X_t, \bar{m}_t^x, a) \nu_{t,X_t}(da)dt + \int_0^{t \wedge \tau} \int_A \sigma(t, X_t, \bar{m}_t^x, a) M(dt, da), \quad \mathbb{P} \circ X_0^{-1} = m_0^*,$$

$$m_t(B \times C) = \mathbb{E}^{\mathbb{P}} [\mathbb{1}_B(X_t) \nu_{t,X_t}(C) \mathbb{1}_{t < \tau}], \quad B \in \mathcal{B}(\bar{\mathcal{O}}), \quad C \in \mathcal{B}(A), \quad t - a.e.,$$

$$\mu = \mathbb{P} \circ (\tau, X_\tau)^{-1}.$$

On the same filtered probability space, define

$$m_t^n(B \times C) := \mathbb{E}^{\mathbb{P}} [\mathbb{1}_B(X_t^n) \nu_{t,X_t^n}(C) \mathbb{1}_{t < \tau \wedge \tau_{\bar{\mathcal{O}}}^{X^n}}], \quad \mu^n := \mathbb{P} \circ (\tau \wedge \tau_{\bar{\mathcal{O}}}^{X^n}, X_{\tau \wedge \tau_{\bar{\mathcal{O}}}^{X^n}}^n)^{-1},$$

where X^n denotes the unique strong solution of

$$dX_t^n = \int_A b(t, X_t^n, \bar{m}_t^{x,n}, a) \nu_{t, X_t^n}(da) dt + \int_A \sigma(t, X_t^n, \bar{m}_t^{x,n}, a) M(dt, da), \quad X_0^n = X_0.$$

Note that existence and uniqueness follow by the Lipschitz and boundedness condition on the coefficients and the fact that $m_0^* \in \mathcal{P}_q(\mathcal{O})$. We have that $(m^n, \mu^n) \in \mathcal{R}[\bar{m}^n] = \mathcal{R}^*(\bar{m}^n, \bar{\mu}^n)$ by a similar argument as the one in Proposition 2.6. Let us now prove that $m^n \rightarrow m$ in τ_p . Using that

$$\sup_{n \geq 1} \int_0^T \int_{\bar{\mathcal{O}} \times A} |x|^q m_t^n(dx, da) dt < C_1 T,$$

and Proposition 5.3, it suffices to show weak convergence. By Remark 8.3.1 and Exercise 8.10.71 in [Bog07] (Volume 2), it is sufficient to use bounded and Lipschitz functions as test functions. Consider a bounded and Lipschitz function $\phi \in \text{BL}([0, T] \times \bar{\mathcal{O}} \times A)$ with $\|\phi\|_{\text{BL}} \leq 1$. Compute

$$\begin{aligned} & \left| \int_0^T \int_{\bar{\mathcal{O}} \times A} \phi(t, x, a) m_t(dx, da) dt - \int_0^T \int_{\bar{\mathcal{O}} \times A} \phi(t, x, a) m_t^n(dx, da) dt \right| \\ &= \left| \mathbb{E}^{\mathbb{P}} \left[\int_0^\tau \int_A \phi(t, X_t, a) \nu_{t, X_t}(da) dt - \int_0^{\tau \wedge \tau_{\bar{\mathcal{O}}}^{X^n}} \int_A \phi(t, X_t^n, a) \nu_{t, X_t^n}(da) dt \right] \right| \\ &\leq \left| \mathbb{E}^{\mathbb{P}} \left[\int_0^{\tau \wedge \tau_{\bar{\mathcal{O}}}^{X^n}} \int_A (\phi(t, X_t, a) - \phi(t, X_t^n, a)) \nu_{t, X_t}(da) dt \right] \right| \\ &\quad + \left| \mathbb{E}^{\mathbb{P}} \left[\int_0^\tau \int_A \phi(t, X_t, a) \nu_{t, X_t}(da) dt - \int_0^{\tau \wedge \tau_{\bar{\mathcal{O}}}^{X^n}} \int_A \phi(t, X_t, a) \nu_{t, X_t}(da) dt \right] \right| \\ &\leq CT^{\frac{1}{2}} \left(\mathbb{E}^{\mathbb{P}} \left[\sup_{t \leq \tau} |X_t - X_t^n|^2 \right] \right)^{\frac{1}{2}} + C \left(\mathbb{E}^{\mathbb{P}}[\tau] - \mathbb{E}^{\mathbb{P}}[\tau \wedge \tau_{\bar{\mathcal{O}}}^{X^n}] \right). \end{aligned}$$

Now, by Lemma 3.17, we get the convergence of the first term. The convergence of the second one is trivial under the condition (4)(a) of Assumption 3.1. Suppose now condition (4)(b) of Assumption 3.1 holds. Then, by Theorem 5.20, the martingale measure M is replaced by an \mathbb{F} -Brownian motion W and we get

$$X_{t \wedge \tau} = X_0 + \int_0^{t \wedge \tau} \int_A b(t, X_t, \bar{m}_t^x, a) \nu_{t, X_t}(da) dt + \int_0^{t \wedge \tau} \sigma(t, X_t, \bar{m}_t^x) dW_t,$$

$$X_t^n = X_0 + \int_0^t \int_A b(t, X_t^n, \bar{m}_t^{x,n}, a) \nu_{t, X_t^n}(da) dt + \int_0^t \sigma(t, X_t^n, \bar{m}_t^{x,n}) dW_t.$$

Define X^0 as the unique strong solution to

$$X_t^0 = X_0 + \int_0^t \int_A b(t, X_t^0, \bar{m}_t^x, a) \nu_{t, X_t^0}(da) dt + \int_0^t \sigma(t, X_t^0, \bar{m}_t^x) dW_t.$$

By pathwise uniqueness type arguments, we get that $X_t^0 = X_t$ on $t \leq \tau$, which implies that $\tau_{\bar{\mathcal{O}}}^{X^0} \geq \tau$ \mathbb{P} -a.s. We have that for all $\delta > 0$ and $C > 0$, there exists $n_0 \geq 1$ such that for all

$n \geq n_0$,

$$\mathbb{P} \left(\sup_{t \in [0, T]} |X_t^n - X_t^0| \geq C \right) < \delta.$$

We have also that, for all $\delta > 0$, there exists $R > 0$ such that,

$$\mathbb{P} \left(\sup_{t \in [0, T]} |X_t^0| \geq R \right) < \delta,$$

Using these two last properties, we get by Theorem 5.1 and Remark 5.4 in [She15] that $\tau_{\mathcal{O}}^{X^n} \wedge T \xrightarrow[n \rightarrow \infty]{\mathbb{P}} \tau_{\mathcal{O}}^{X^0} \wedge T$. To be more precise, by assumption, $\mathcal{O} =]x_1, x_2[$, $-\infty \leq x_1 < x_2 \leq +\infty$, then one can choose for the assumptions in [She15] the function

$$\varphi(t, x) := \begin{cases} (T-t)(x-x_1)(x-x_2) & \text{if } -\infty < x_1 < x_2 < +\infty, \\ (T-t)(x_1-x) & \text{if } -\infty < x_1 < x_2 = +\infty, \\ (T-t)(x-x_2) & \text{if } -\infty = x_1 < x_2 < +\infty. \end{cases}$$

Therefore, we get $(\tau, \tau_{\mathcal{O}}^{X^n} \wedge T) \xrightarrow[n \rightarrow \infty]{\mathbb{P}} (\tau, \tau_{\mathcal{O}}^{X^0} \wedge T)$ and by the continuous mapping theorem,

$$\tau \wedge \tau_{\mathcal{O}}^{X^n} = \tau \wedge \tau_{\mathcal{O}}^{X^n} \wedge T \xrightarrow[n \rightarrow \infty]{\mathbb{P}} \tau \wedge \tau_{\mathcal{O}}^{X^0} \wedge T = \tau.$$

Since this sequence is uniformly bounded by T we get the convergence in L^1 . Finally, we can conclude that $m^n \rightarrow m$ in τ_p . Now, by the convergence of m^n towards m in τ_p and since $(m^n, \mu^n) \in \mathcal{R}[\bar{m}^n]$, we get that, $\mu^n \rightarrow \mu$ (using the same results as for the upper hemicontinuity). Now, by the estimate

$$\sup_{n \geq 1} \int_{[0, T] \times \bar{\mathcal{O}}} |x|^q \mu^n(dt, dx) \leq C_1,$$

and Proposition 5.3, $(\mu^n)_{n \geq 1}$ converges to μ in τ_p . \square

We prove now the existence for LP MFG Nash equilibria.

Theorem 3.19 (*Existence of LP MFG equilibria: bounded coefficients*). *The set of LP MFG equilibria is compact and nonempty.*

Proof. The proof is based on Kakutani-Fan-Glicksberg's fixed point theorem for set-valued maps (Theorem 5.34). Note that the space $\bar{\mathcal{R}}_2$ is a subset of the locally convex Hausdorff space $\tilde{U} \times \mathcal{M}([0, T] \times \bar{\mathcal{O}})$. Moreover, $\bar{\mathcal{R}}_2$ is nonempty, compact and convex (Lemma 3.10). Remark also that the map Θ has nonempty convex values by Lemma 3.16. Let us show that it has closed graph. To this end, we apply Berge's Maximum Theorem (Theorem 5.33) together with the Closed Graph Theorem (Theorem 5.30). By Proposition 3.18, the correspondence \mathcal{R}^* is continuous, and by Lemma 3.16, it has nonempty compact values. Therefore, it only remains to show that

$$((\bar{m}, \bar{\mu}), (m, \mu)) \in \text{Gr}(\mathcal{R}^*) \mapsto \Gamma[\bar{m}, \bar{\mu}](m, \mu) = \langle m, f[\bar{m}] \rangle + \langle \mu, g[\bar{\mu}] \rangle$$

is continuous. Let $((\bar{m}^n, \bar{\mu}^n), (m^n, \mu^n))_{n \geq 1} \subset \text{Gr}(\mathcal{R}^*)$ converging to $((\bar{m}, \bar{\mu}), (m, \mu)) \in \text{Gr}(\mathcal{R}^*)$.

To show the convergence of the sequence $(\langle m^n, f[\bar{m}^n] \rangle)_{n \geq 1}$ to $\langle m, f[\bar{m}] \rangle$, we are going to use Lemma 5.28: set $\Theta = [0, T]$, $\mathcal{X} = \bar{\mathcal{O}} \times A$, $\mathcal{Y} = \mathcal{P}_p^{sub}(\bar{\mathcal{O}})$, $\varphi = f$, $\psi^n = \bar{m}^{x,n}$, $\psi = \bar{m}^x$, $\eta(dt) = dt$, $\nu^n = m^n$, $\nu = m$. By Lemma 3.13, we get the convergence of $m_t^n(dx, da)dt$ to $m_t(dx, da)dt$ in $\bar{\tau}_p$. By Corollary 3.12, $\bar{m}^{x,n}$ converges to \bar{m}^x in measure, in particular, up to a subsequence, it converges a.e. in $\bar{\tau}_p$. Now, by Corollary 3.12, there exists a constant $C \geq 0$ such that t -a.e. on $[0, T]$,

$$\sup_{n \geq 1} \int_{\bar{\mathcal{O}} \times A} (1 + |x|^p + |a|^p) m_t^n(dx, da) + \sup_{n \geq 1} \int_{\bar{\mathcal{O}}} |x|^p \bar{m}_t^{x,n}(dx) \leq C,$$

Finally, observe that there exists $C' \geq 0$ such that for all $(t, x, \eta, a) \in [0, T] \times \bar{\mathcal{O}} \times \mathcal{P}_p^{sub}(\bar{\mathcal{O}}) \times A$,

$$|f(t, x, \eta, a)| \leq C'(1 + |t|^p + |x|^p + |a|^p + d_{BL,p}(\eta, \mathbf{0})^p),$$

where the definition of $d_{BL,p}$ is given in Remark 5.2. We conclude by Lemma 5.28 that

$$\langle m^n, f[\bar{m}^n] \rangle \xrightarrow{n \rightarrow \infty} \langle m, f[\bar{m}] \rangle.$$

Now, the convergence

$$\langle \mu^n, g[\bar{\mu}^n] \rangle \xrightarrow{n \rightarrow \infty} \langle \mu, g[\bar{\mu}] \rangle$$

follows by Lemma 5.26. We conclude that

$$\Gamma[\bar{m}^n, \bar{\mu}^n](m^n, \mu^n) \xrightarrow{n \rightarrow \infty} \Gamma[\bar{m}, \bar{\mu}](m, \mu),$$

which shows the continuity. □

3.2 Approximation with bounded coefficients

We are ready to prove the main existence theorem, Theorem 3.4. As stated before we will approximate the coefficients b and σ with bounded coefficients for which we know the existence of equilibria (Theorem 3.19).

We recall some results from convex analysis. We denote by $|\cdot|$ the euclidean norm in \mathbb{R}^d . For a set $\emptyset \neq C \subset \mathbb{R}^d$ and a point $x \in \mathbb{R}^d$, any point $x_0 \in C$ verifying

$$|x - x_0| = \inf_{y \in C} |x - y|$$

is called a projection of x onto C . If C is closed and convex, any point $x \in \mathbb{R}^d$ has a unique projection onto C . Therefore we can define the function $p_C : \mathbb{R}^d \rightarrow C$ which associates to each $x \in \mathbb{R}^d$ its unique projection onto C . One can show that p_C is 1-Lipschitz. Now, for $n \geq 1$ we are going to consider the projections $p_n := p_{B_n(0)}$ from \mathbb{R} onto the closed ball of radius n and center 0 and define the coefficients $b_n = p_n \circ b$ and $\sigma_n = p_n \circ \sigma$. The functions b_n, σ_n are jointly measurable and continuous in (x, η, a) for each t . Moreover, for all $n \geq 1$ and $(t, x, y, \eta, a) \in [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathcal{P}_p^{sub}(\bar{\mathcal{O}}) \times A$,

$$|b_n(t, x, \eta, a) - b_n(t, y, \eta, a)| + |\sigma_n(t, x, \eta, a) - \sigma_n(t, y, \eta, a)| \leq c_1|x - y|,$$

$$\begin{aligned}
 |b_n(t, x, \eta, a)| &\leq c_1 \left[1 + |x| + \left(\int_{\bar{\mathcal{O}}} |z|^p \eta(dz) \right)^{1/p} \right], \\
 \sigma_n^2(t, x, \eta, a) &\leq c_1 \left[1 + |x|^r + \left(\int_{\bar{\mathcal{O}}} |z|^p \eta(dz) \right)^{r/p} \right], \\
 |b_n(t, x, \eta, a)| &\leq n, \\
 \sigma_n^2(t, x, \eta, a) &\leq n^2.
 \end{aligned}$$

Applying the results from the previous section, for all $n \geq 1$ there exists at least one LP MFG Nash equilibrium. We denote by $(m^n, \mu^n) \in V_p \times \mathcal{P}_p([0, T] \times \bar{\mathcal{O}})$ an LP MFG Nash equilibrium associated to (b_n, σ_n) . We are going to show that the sequence $(m^n, \mu^n)_{n \geq 1}$ is relatively compact in $V_p \times \mathcal{P}_p([0, T] \times \bar{\mathcal{O}})$ and that any cluster point is an LP MFG Nash equilibrium associated to (b, σ) . This will prove Theorem 3.4.

Definition 3.20. For each $n \geq 1$, define \mathcal{R}_n as the set of pairs $(m, \mu) \in V_p \times \mathcal{P}_p([0, T] \times \bar{\mathcal{O}})$, such that for all $u \in C_b^{1,2}([0, T] \times \bar{\mathcal{O}})$,

$$\langle \mu, u \rangle = \langle m_0^*, u(0, \cdot) \rangle + \langle m, \partial_t u + \mathcal{L}_n u \rangle$$

where

$$\mathcal{L}_n u(t, x, a) := b_n(t, x, \bar{m}_t^{x,n}, a) \partial_x u(t, x) + \frac{\sigma_n^2}{2}(t, x, \bar{m}_t^{x,n}, a) \partial_{xx} u(t, x), \quad (t, x, a) \in [0, T] \times \mathbb{R} \times A.$$

Proposition 3.21 (Estimate). *There exist constants $C_3 \geq 0$ and $C_4 \geq 0$ such that for all $n \geq 1$ we have the following estimates:*

(1) For all $t \in [0, T]$,

$$\int_{\bar{\mathcal{O}}} |x|^q m_t^{x,n}(dx) \leq C_3, \quad \int_{[0, T] \times \bar{\mathcal{O}}} |x|^q \mu^n(dt, dx) \leq C_3.$$

(2) For all $h \in [0, T]$, we have

$$\int_0^{T-h} d_{BL}(m_{t+h}^{x,n}, m_t^{x,n}) dt \leq C_4 \sqrt{h}.$$

Proof. Let $n \geq 1$ be fixed and $\nu_{t,x}^n(da)$ be such that

$$m_t^n(dx, da) dt = \nu_{t,x}^n(da) m_t^{x,n}(dx) dt.$$

By Theorem 5.20, there exist a filtered probability space $(\Omega^n, \mathcal{F}^n, \mathbb{F}^n, \mathbb{P}^n)$, an \mathbb{F}^n -adapted process X^n , an \mathbb{F}^n -stopping time τ^n such that $\tau^n \leq T \wedge \tau_{\bar{\mathcal{O}}}^{X^n}$ \mathbb{P}^n -a.s., and an \mathbb{F}^n -martingale measure M^n with intensity $\nu_{t, X_t^n}^n(da) \mathbb{1}_{t < \tau^n} dt$, such that

$$X_{t \wedge \tau^n}^n = X_0^n + \int_0^{t \wedge \tau^n} \int_A b_n(s, X_s^n, m_s^{x,n}, a) \nu_{s, X_s^n}^n(da) ds + \int_0^{t \wedge \tau^n} \int_A \sigma_n(s, X_s^n, m_s^{x,n}, a) M^n(ds, da),$$

$$\begin{aligned}\mathbb{P}^n \circ (X_0^n)^{-1} &= m_0^*, \\ m_t^n(B \times C) &= \mathbb{E}^{\mathbb{P}^n} \left[\mathbf{1}_B(X_t^n) \nu_{t, X_t^n}^n(C) \mathbf{1}_{t < \tau^n} \right], \quad B \in \mathcal{B}(\bar{\mathcal{O}}), \quad C \in \mathcal{B}(A), \quad t - a.e., \\ \mu^n &= \mathbb{P}^n \circ (\tau^n, X_{\tau^n}^n)^{-1}.\end{aligned}$$

As it has been done in all the paper, we choose to work with the càdlàg representative:

$$m_t^{x,n}(B) = \mathbb{E}^{\mathbb{P}^n} [\mathbf{1}_B(X_t^n) \mathbf{1}_{t < \tau^n}], \quad B \in \mathcal{B}(\bar{\mathcal{O}}).$$

Consider the process $\tilde{X}_t^n := X_{t \wedge \tau^n}^n$. We have for all $t \in [0, T]$ and $s \in [0, t]$ (the constant C may change from line to line),

$$\begin{aligned}|\tilde{X}_s^n|^q &\leq C \left(|X_0^n|^q + \left| \int_0^{s \wedge \tau^n} \int_A b_n(r, \tilde{X}_r^n, m_r^{x,n}, a) \nu_{r, \tilde{X}_r^n}^n(da) dr \right|^q \right. \\ &\quad \left. + \left| \int_0^{s \wedge \tau^n} \int_A \sigma_n(r, \tilde{X}_r^n, m_r^{x,n}, a) M^n(dr, da) \right|^q \right).\end{aligned}$$

Taking the supremum over $s \in [0, t]$, the expectation and using Jensen's inequality and the Burkholder-Davis-Gundy inequality,

$$\begin{aligned}\mathbb{E}^{\mathbb{P}^n} \|\tilde{X}^n\|_t^q &\leq C \left(\int_{\mathcal{O}} |x|^q m_0^*(dx) + \mathbb{E}^{\mathbb{P}^n} \int_0^{t \wedge \tau^n} \int_A |b_n(r, \tilde{X}_r^n, m_r^{x,n}, a)|^q \nu_{r, \tilde{X}_r^n}^n(da) dr \right. \\ &\quad \left. + \mathbb{E}^{\mathbb{P}^n} \int_0^{t \wedge \tau^n} \int_A |\sigma_n^2(r, \tilde{X}_r^n, m_r^{x,n}, a)|^{q/2} \nu_{r, \tilde{X}_r^n}^n(da) dr \right).\end{aligned}$$

Using the boundedness of b_n and σ_n , we deduce that

$$\mathbb{E}^{\mathbb{P}^n} \|\tilde{X}^n\|_T^q < \infty, \quad \sup_{t \in [0, T]} \int_{\bar{\mathcal{O}}} |z|^q m_t^{x,n}(dz) < \infty.$$

Using the growth assumption on b_n and σ_n ,

$$\begin{aligned}\mathbb{E}^{\mathbb{P}^n} \|\tilde{X}^n\|_t^q &\leq C \left(1 + \int_{\mathcal{O}} |x|^q m_0^*(dx) + \mathbb{E}^{\mathbb{P}^n} \int_0^{t \wedge \tau^n} |\tilde{X}_r^n|^q dr + \mathbb{E}^{\mathbb{P}^n} \int_0^{t \wedge \tau^n} \int_{\bar{\mathcal{O}}} |z|^q m_r^{x,n}(dz) dr \right) \\ &\leq C \left(1 + \int_{\mathcal{O}} |x|^q m_0^*(dx) + \mathbb{E}^{\mathbb{P}^n} \int_0^t \|\tilde{X}^n\|_r^q dr \right).\end{aligned}$$

By Gronwall's inequality

$$\mathbb{E}^{\mathbb{P}^n} \|\tilde{X}^n\|_T^q \leq C \left(1 + \int_{\mathcal{O}} |x|^q m_0^*(dx) \right) =: C_3.$$

We deduce that

$$\int_{\bar{\mathcal{O}}} |x|^q m_t^{x,n}(dx) = \mathbb{E}^{\mathbb{P}^n} [|X_t^n|^q \mathbf{1}_{t < \tau^n}] \leq \mathbb{E}^{\mathbb{P}^n} \|\tilde{X}^n\|_T^q \leq C_3, \quad t \in [0, T],$$

and

$$\int_{[0,T] \times \bar{\mathcal{O}}} |x|^q \mu^n(dt, dx) = \mathbb{E}^{\mathbb{P}^n} [|X_{\tau^n}^n|^q] \leq \mathbb{E}^{\mathbb{P}^n} \|\tilde{X}^n\|_T^q \leq C_3.$$

Let us show the second estimate. For $\phi \in \text{BL}(\bar{\mathcal{O}})$ such that $\|\phi\|_{\text{BL}} \leq 1$, $h \in [0, T]$ and $t \in [0, T - h]$, we have:

$$\begin{aligned} \int_{\bar{\mathcal{O}}} \phi(x) (m_{t+h}^{x,n} - m_t^{x,n})(dx) &= \mathbb{E}^{\mathbb{P}^n} [\phi(\tilde{X}_{t+h}^n) \mathbf{1}_{t+h < \tau^n} - \phi(\tilde{X}_t^n) \mathbf{1}_{t < \tau^n}] \\ &= \mathbb{E}^{\mathbb{P}^n} [(\phi(\tilde{X}_{t+h}^n) - \phi(\tilde{X}_t^n)) \mathbf{1}_{t+h < \tau^n}] + \mathbb{E}^{\mathbb{P}^n} [\phi(\tilde{X}_t^n) (\mathbf{1}_{t+h < \tau^n} - \mathbf{1}_{t < \tau^n})] \\ &\leq \mathbb{E}^{\mathbb{P}^n} [|\phi(\tilde{X}_{t+h}^n) - \phi(\tilde{X}_t^n)|] + \mathbb{E}^{\mathbb{P}^n} [|\phi(\tilde{X}_t^n)| \mathbf{1}_{t < \tau^n \leq t+h}] \\ &\leq \mathbb{E}^{\mathbb{P}^n} [|\tilde{X}_{t+h}^n - \tilde{X}_t^n|] + \mathbb{E}^{\mathbb{P}^n} [\mathbf{1}_{t < \tau^n \leq t+h}]. \end{aligned}$$

Taking the supremum over ϕ ,

$$d_{\text{BL}}(m_{t+h}^{x,n}, m_t^{x,n}) \leq \mathbb{E}^{\mathbb{P}^n} [|\tilde{X}_{t+h}^n - \tilde{X}_t^n|] + \mathbb{E}^{\mathbb{P}^n} [\mathbf{1}_{t < \tau^n \leq t+h}].$$

We have

$$\int_0^{T-h} \mathbb{E}^{\mathbb{P}^n} [\mathbf{1}_{t < \tau^n \leq t+h}] dt \leq \mathbb{E}^{\mathbb{P}^n} [\tau^n \wedge (T-h) - 0 \vee (\tau^n - h)] \leq h.$$

On the other hand, applying Jensen's inequality and Burkholder-Davis-Gundy inequality (the constant $C \geq 0$ may change from line to line)

$$\begin{aligned} \mathbb{E}^{\mathbb{P}^n} |\tilde{X}_{t+h}^n - \tilde{X}_t^n|^2 &\leq \mathbb{E}^{\mathbb{P}^n} \left| \int_{t \wedge \tau^n}^{(t+h) \wedge \tau^n} \int_A b_n(s, \tilde{X}_s^n, m_s^{x,n}, a) \nu_{s, \tilde{X}_s^n}^n(da) ds \right|^2 \\ &\quad + \mathbb{E}^{\mathbb{P}^n} \left| \int_{t \wedge \tau^n}^{(t+h) \wedge \tau^n} \int_A \sigma_n(s, \tilde{X}_s^n, m_s^{x,n}, a) M^n(ds, da) \right|^2 \\ &\leq C \left(\mathbb{E}^{\mathbb{P}^n} \int_{t \wedge \tau^n}^{(t+h) \wedge \tau^n} \int_A |b_n(s, \tilde{X}_s^n, m_s^{x,n}, a)|^2 \nu_{s, \tilde{X}_s^n}^n(da) ds \right. \\ &\quad \left. + \int_{t \wedge \tau^n}^{(t+h) \wedge \tau^n} \int_A \sigma_n^2(s, \tilde{X}_s^n, m_s^{x,n}, a) \nu_{s, \tilde{X}_s^n}^n(da) ds \right) \\ &\leq Ch, \end{aligned}$$

where we used that $\mathbb{E}^{\mathbb{P}^n} \|\tilde{X}^n\|_T^q \leq C_3$ and $\int_{\bar{\mathcal{O}}} |x|^q m_t^{x,n}(dx) \leq C_3$. We deduce that there exists a constant $C_4 \geq 0$ such that

$$\int_0^{T-h} d_{\text{BL}}(m_{t+h}^{x,n}, m_t^{x,n}) dt \leq C_4 \sqrt{h}.$$

□

Corollary 3.22 (Relative compactness). *The set*

$$\{(m^n, \mu^n) : n \geq 1\}$$

is relatively compact in the topology $\tau_p \otimes \tau_p$.

Proof. This set is included in the set of $(m, \mu) \in V_p \times \mathcal{P}_p([0, T] \times \bar{\mathcal{O}})$ satisfying

$$\int_0^T \int_{\bar{\mathcal{O}} \times A} |x|^q m_t(dx, da) dt \leq C_3 T, \quad \int_{[0, T] \times \bar{\mathcal{O}}} |x|^q \mu(dt, dx) \leq C_3,$$

and for all $\varphi \in C([0, T]; \mathbb{R}_+)$,

$$\int_0^T \int_{\bar{\mathcal{O}} \times A} \varphi(t) m_t(dx, da) dt \leq \int_0^T \varphi(t) dt.$$

In particular, using the same proof as in Lemma 3.10 we can show that the latter set is relatively compact in $\tau_p \otimes \tau_p$. \square

Let (m, μ) be a limit point of the sequence $(m^n, \mu^n)_{n \geq 1}$ and let us show that it is an LP MFG Nash equilibrium. Using Proposition 3.21 and the same ideas of the previous subsection (see Theorem 3.11 and Lemma 3.13), we can show that $m_t^{x, n} \rightarrow m_t^x$ *t*-a.e. on $[0, T]$ in τ_p and $m^n \rightarrow m$ in $\bar{\tau}_p$.

Proposition 3.23 (Forward equation). *We have that $(m, \mu) \in \mathcal{R}[m]$.*

Proof. Let $u \in C_b^{1,2}([0, T] \times \bar{\mathcal{O}})$. For all $n \geq 1$, we have the equality

$$\langle \mu^n, u \rangle = \langle m_0^*, u(0, \cdot) \rangle + \langle m^n, \partial_t u + \mathcal{L}_n u \rangle, \quad (3.8)$$

We get directly,

$$\lim_{n \rightarrow \infty} \langle \mu^n, u \rangle = \langle \mu, u \rangle$$

and

$$\lim_{n \rightarrow \infty} \langle m^n, \partial_t u \rangle = \langle m, \partial_t u \rangle.$$

Now let us show that

$$\lim_{n \rightarrow \infty} \int_0^T \int_{\bar{\mathcal{O}} \times A} (b_n \partial_x u)(t, x, m_t^{x, n}, a) m_t^n(dx, da) dt = \int_0^T \int_{\bar{\mathcal{O}} \times A} (b \partial_x u)(t, x, m_t^x, a) m_t(dx, da) dt.$$

We first compute

$$\begin{aligned} & \left| \int_0^T \int_{\bar{\mathcal{O}} \times A} (\partial_x u b_n)(t, x, m_t^{x, n}, a) m_t^n(dx, da) dt - \int_0^T \int_{\bar{\mathcal{O}} \times A} (\partial_x u b)(t, x, m_t^{x, n}, a) m_t^n(dx, da) dt \right| \\ &= \left| \int_0^T \int_{\bar{\mathcal{O}} \times A} (\partial_x u b)(t, x, m_t^{x, n}, a) \mathbf{1}_{|b(t, x, m_t^{x, n}, a)| > n} m_t^n(dx, da) dt \right| \\ &\leq \|\partial_x u\|_\infty \int_0^T \int_{\bar{\mathcal{O}} \times A} c_1 \left[1 + |x| + \left(\int_{\bar{\mathcal{O}}} |z|^p m_t^{x, n}(dz) \right)^{1/p} \right] \\ &\quad \mathbf{1}_{c_1 \left[1 + |x| + \left(\int_{\bar{\mathcal{O}}} |z|^p m_t^{x, n}(dz) \right)^{1/p} \right] > n} m_t^n(dx, da) dt. \end{aligned}$$

Using Lemma 3.21,

$$\int_{\bar{\mathcal{O}}} |z|^p m_t^{x,n}(dz)$$

is uniformly bounded on $t \in [0, T]$ and $n \geq 1$ and

$$\int_0^T \int_{\bar{\mathcal{O}} \times A} |x|^q m_t^n(dx, da) dt \leq C_3 T.$$

We deduce that

$$\lim_{n \rightarrow \infty} \left| \int_0^T \int_{\bar{\mathcal{O}} \times A} (\partial_x u b_n)(t, x, m_t^{x,n}, a) m_t^n(dx, da) dt - \int_0^T \int_{\bar{\mathcal{O}} \times A} (\partial_x u b)(t, x, m_t^x, a) m_t^n(dx, da) dt \right| = 0.$$

By Lemma 5.28,

$$\lim_{n \rightarrow \infty} \int_0^T \int_{\bar{\mathcal{O}} \times A} (b \partial_x u)(t, x, m_t^{x,n}, a) m_t^n(dx, da) dt = \int_0^T \int_{\bar{\mathcal{O}} \times A} (b \partial_x u)(t, x, m_t^x, a) m_t(dx, da) dt.$$

We conclude finally that

$$\lim_{n \rightarrow \infty} \int_0^T \int_{\bar{\mathcal{O}} \times A} (b_n \partial_x u)(t, x, m_t^{x,n}, a) m_t^n(dx, da) dt = \int_0^T \int_{\bar{\mathcal{O}} \times A} (b \partial_x u)(t, x, m_t^x, a) m_t(dx, da) dt.$$

Using similar techniques,

$$\begin{aligned} \lim_{n \rightarrow \infty} \int_0^T \int_{\bar{\mathcal{O}} \times A} \left(\frac{\sigma_n^2}{2} \partial_{xx} u \right) (t, x, m_t^{x,n}, a) m_t^n(dx, da) dt \\ = \int_0^T \int_{\bar{\mathcal{O}} \times A} \left(\frac{\sigma^2}{2} \partial_{xx} u \right) (t, x, m_t^x, a) m_t(dx, da) dt. \end{aligned}$$

Passing to the limit in (3.8), we conclude that $(m, \mu) \in \mathcal{R}[m]$. \square

We can now show Theorem 3.4.

Proof of Theorem 3.4. It remains to show the optimality of (m, μ) over the set $\mathcal{R}[m]$. Let $(\bar{m}, \bar{\mu}) \in \mathcal{R}[m]$. Let $\bar{\nu}_{t,x}(da)$ be such that

$$\bar{m}_t(dx, da) dt = \bar{\nu}_{t,x}(da) \bar{m}_t^x(dx) dt.$$

By Theorem 5.20, there exist a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, an \mathbb{F} -adapted process X , an \mathbb{F} -stopping time τ such that $\tau \leq T \wedge \tau_{\bar{\mathcal{O}}}^X$ \mathbb{P} -a.s., and an \mathbb{F} -martingale measure M with intensity $\bar{\nu}_{t,X_t}(da) \mathbb{1}_{t \leq \tau} dt$, such that

$$X_{t \wedge \tau} = X_0 + \int_0^{t \wedge \tau} \int_A b(s, X_s, m_s^x, a) \nu_{s, X_s}(da) ds + \int_0^{t \wedge \tau} \int_A \sigma(s, X_s, m_s^x, a) M(ds, da),$$

$$\begin{aligned}\mathbb{P} \circ X_0^{-1} &= m_0^*, \\ \bar{m}_t(B \times C) &= \mathbb{E}^{\mathbb{P}} [\mathbf{1}_B(X_t) \bar{\nu}_{t, X_t}(C) \mathbf{1}_{t < \tau}], \quad B \in \mathcal{B}(\bar{\mathcal{O}}), \quad C \in \mathcal{B}(A), \quad t - a.e., \\ \bar{\mu} &= \mathbb{P} \circ (\tau, X_\tau)^{-1}.\end{aligned}$$

On the same filtered probability space, define

$$\bar{m}_t^n(B \times C) := \mathbb{E}^{\mathbb{P}} \left[\mathbf{1}_B(X_t^n) \bar{\nu}_{t, X_t^n}(C) \mathbf{1}_{t < \tau \wedge \tau_{\bar{\mathcal{O}}}^n} \right], \quad \bar{\mu}^n := \mathbb{P} \circ \left(\tau \wedge \tau_{\bar{\mathcal{O}}}^n, X_{\tau \wedge \tau_{\bar{\mathcal{O}}}^n}^n \right)^{-1},$$

where X^n denotes the unique strong solution of

$$dX_t^n = \int_A b_n(t, X_t^n, m_t^{x,n}, a) \bar{\nu}_{t, X_t^n}(da) dt + \int_A \sigma_n(t, X_t^n, m_t^{x,n}, a) M(dt, da), \quad X_0^n = X_0.$$

Note that existence and uniqueness follow by the Lipschitz and linear growth condition on the coefficients and the fact that $m_0^* \in \mathcal{P}_q(\mathcal{O})$. We have that $(\bar{m}^n, \bar{\mu}^n)$ belongs to \mathcal{R}_n by a similar argument as the one used in Proposition 2.6. One can show that $(\bar{m}^n, \bar{\mu}^n) \rightarrow (\bar{m}, \bar{\mu})$ in $\tau_p \otimes \tau_p$ by using the same techniques as in the proof of the lower hemicontinuity of Proposition 3.18 and by showing that

$$\mathbb{E}^{\mathbb{P}} \left[\sup_{t \leq T} |X_{t \wedge \tau}^n - X_{t \wedge \tau}|^2 \right] \xrightarrow{n \rightarrow \infty} 0.$$

We postpone the proof of the latter convergence result to Lemma 3.24. For all $n \geq 1$

$$\Gamma[m^n, \mu^n](\bar{m}^n, \bar{\mu}^n) \leq \Gammam^n, \mu^n.$$

Applying Lemmas 5.26 and 5.28 and passing to the limit in the inequality we finally get

$$\Gamma[m, \mu](\bar{m}, \bar{\mu}) \leq \Gammam, \mu.$$

Since $(\bar{m}, \bar{\mu})$ was arbitrary in $\mathcal{R}[m]$, we conclude that (m, μ) is an LP MFG Nash equilibrium. \square

Lemma 3.24. *We have*

$$\mathbb{E}^{\mathbb{P}} \left[\sup_{t \leq T} |X_{t \wedge \tau}^n - X_{t \wedge \tau}|^2 \right] \xrightarrow{n \rightarrow \infty} 0.$$

Proof. We will denote by $C > 0$ any constant independent from n . Let $t \in [0, T]$ and define

$$\xi_n(t) = \mathbb{E}^{\mathbb{P}} \left[\sup_{s \leq t} |X_{s \wedge \tau}^n - X_{s \wedge \tau}|^2 \right].$$

Using the Burkholder-Davis-Gundy inequality,

$$\begin{aligned}\xi_n(t) &\leq C \mathbb{E}^{\mathbb{P}} \left[\int_0^{t \wedge \tau} \int_A (b_n(r, X_r^n, m_r^{x,n}, a) - b(r, X_r, m_r^x, a))^2 \bar{\nu}_{r, X_r}(da) dr \right. \\ &\quad \left. + \int_0^{t \wedge \tau} \int_A (\sigma_n(r, X_r^n, m_r^{x,n}, a) - \sigma(r, X_r, m_r^x, a))^2 \bar{\nu}_{r, X_r}(da) dr \right].\end{aligned}$$

We decompose in the following way for the drift (we do the same decomposition for the volatility):

$$\begin{aligned} b_n(r, X_r^n, m_r^{x,n}, a) - b(r, X_r, m_r^x, a) &= [b_n(r, X_r^n, m_r^{x,n}, a) - b_n(r, X_r, m_r^{x,n}, a)] \\ &\quad + [b_n(r, X_r, m_r^{x,n}, a) - b(r, X_r, m_r^x, a)]. \end{aligned}$$

Now, by the Lipschitz property of b_n ,

$$\begin{aligned} &\int_0^{t \wedge \tau} \int_A (b_n(r, X_r^n, m_r^{x,n}, a) - b(r, X_r, m_r^x, a))^2 \bar{\nu}_{r, X_r}(da) dr \\ &\leq C \left[\int_0^t \sup_{r \leq s} |X_{r \wedge \tau}^n - X_{r \wedge \tau}|^2 ds + \int_0^{t \wedge \tau} \int_A (b_n(r, X_r, m_r^{x,n}, a) - b(r, X_r, m_r^x, a))^2 \bar{\nu}_{r, X_r}(da) dr \right]. \end{aligned}$$

Similarly,

$$\begin{aligned} &\int_0^{t \wedge \tau} \int_A (\sigma_n(r, X_r^n, m_r^{x,n}, a) - \sigma(r, X_r, m_r^x, a))^2 \bar{\nu}_{r, X_r}(da) dr \\ &\leq C \left[\int_0^t \sup_{r \leq s} |X_{r \wedge \tau}^n - X_{r \wedge \tau}|^2 ds + \int_0^{t \wedge \tau} \int_A (\sigma_n(r, X_r, m_r^{x,n}, a) - \sigma(r, X_r, m_r^x, a))^2 \bar{\nu}_{r, X_r}(da) dr \right]. \end{aligned}$$

We get finally,

$$\xi_n(t) \leq C \left(\int_0^t \xi_n(s) ds + B_n + S_n \right),$$

where

$$\begin{aligned} B_n &:= \mathbb{E}^{\mathbb{P}} \left[\int_0^T \int_A (b_n(r, X_r, m_r^{x,n}, a) - b(r, X_r, m_r^x, a))^2 \bar{\nu}_{r, X_r}(da) dr \right], \\ S_n &:= \mathbb{E}^{\mathbb{P}} \left[\int_0^T \int_A (\sigma_n(r, X_r, m_r^{x,n}, a) - \sigma(r, X_r, m_r^x, a))^2 \bar{\nu}_{r, X_r}(da) dr \right]. \end{aligned}$$

By Gronwall's inequality,

$$\xi_n(T) \leq C (B_n + S_n) e^{CT}. \quad (3.9)$$

Let us show that $B_n \rightarrow 0$ as $n \rightarrow \infty$. Since $m_r^{x,n} \rightarrow m_r^x$ dr -a.e. in τ_p , and

$$\int_{\bar{\mathcal{O}}} |z|^p m_t^{x,n}(dz)$$

is uniformly bounded on $t \in [0, T]$ and $n \geq 1$, we get that dr -a.e. for all $\omega \in \Omega$ and $a \in A$

$$b_n(r, X_r(\omega), m_r^{x,n}, a) \xrightarrow{n \rightarrow \infty} b(r, X_r(\omega), m_r^x, a).$$

On the other hand,

$$\begin{aligned} &(b_n(r, X_r(\omega), m_r^{x,n}, a) - b(r, X_r(\omega), m_r^x, a))^2 \\ &\leq C \left(1 + |X_r(\omega)|^2 + \int_{\bar{\mathcal{O}}} |z|^q m_r^{x,n}(dz) + \int_{\bar{\mathcal{O}}} |z|^q m_r^x(dz) \right). \end{aligned}$$

We can apply the dominated convergence theorem to get $B_n \rightarrow 0$. The convergence of S_n to 0

follows by the same arguments. Letting $n \rightarrow \infty$ in (3.9) we get the result. \square

Proof of Proposition 3.5. Let \mathcal{N}^* be the set of LP MFG Nash equilibrium. Analogously to Proposition 3.21, there exist constants C_5 and C_6 such that for all $(m, \mu) \in \mathcal{N}^*$ we have the following estimates:

(1) For all $t \in [0, T]$,

$$\int_{\bar{\mathcal{O}}} |x|^q m_t^{x,*}(dx) \leq C_5, \quad \int_{[0,T] \times \bar{\mathcal{O}}} |x|^q \mu(dt, dx) \leq C_5.$$

(2) For all $h \in [0, T]$, we have

$$\int_0^{T-h} d_{\text{BL}}(m_{t+h}^{x,*}, m_t^{x,*}) dt \leq C_6 \sqrt{h}.$$

This shows that \mathcal{N}^* is relatively compact in $\tau_p \otimes \tau_p$. Now let $(m^n, \mu^n)_{n \geq 1} \subset \mathcal{N}^*$ converging to some $(m, \mu) \in V_p \times \mathcal{P}_p([0, T] \times \bar{\mathcal{O}})$ in $\tau_p \otimes \tau_p$. Up to a subsequence, we can show using the estimates that $m_t^{x,n} \rightarrow m_t^x$ t -a.e. on $[0, T]$ in τ_p and $m^n \rightarrow m$ in $\bar{\tau}_p$. Using similar ideas to the ones used for showing Theorem 3.4, we can show that $(m, \mu) \in \mathcal{R}[m]$ and the optimality condition. We conclude that \mathcal{N}^* is compact. \square

3.3 Nash value and selection of Nash equilibria

Case of measure independent coefficients. In the case where the coefficients b and σ do not depend on the measure, we can prove uniqueness of the Nash value, which holds under the well known Lasry-Lions monotonicity condition on f and g .

When the coefficients do not depend on the measure, an LP Nash equilibrium is a pair $(m^*, \mu^*) \in \mathcal{R}$ such that for all $(m, \mu) \in \mathcal{R}$,

$$\Gamma[m^*, \mu^*](m, \mu) \leq \Gammam^*, \mu^*.$$

Here \mathcal{R} is the set defined in Definition 2.2.

Theorem 3.25 (Uniqueness of the Nash value). *Suppose that the coefficients b and σ do not depend on the measure. Suppose also that f and g take the following form*

$$f(t, x, \eta, a) = f_1(t, x) f_2 \left(t, \int_{\bar{\mathcal{O}}} f_1(t, y) \eta(dy) \right) + f_3(t, x, a)$$

$$g(t, x, \mu) = g_1(t, x) g_2 \left(\int_{[0,T] \times \bar{\mathcal{O}}} g_1(s, y) \mu(ds, dy) \right) + g_3(t, x),$$

where $f_1, f_2, f_3, g_1, g_2, g_3$ are such that f and g verify the conditions in Assumption 3.1. Moreover we assume that $(t, x) \mapsto f_2(t, x)$ is nonincreasing in x for each t and g_2 is nonincreasing. If we are given two LP MFG Nash equilibria (m, μ) and $(\bar{m}, \bar{\mu})$, then

$$f_2 \left(t, \int_{\bar{\mathcal{O}}} f_1(t, y) m_t^x(dy) \right) = f_2 \left(t, \int_{\bar{\mathcal{O}}} f_1(t, y) \bar{m}_t^x(dy) \right),$$

almost everywhere on $[0, T]$, and

$$g_2 \left(\int_{[0, T] \times \bar{\mathcal{O}}} g_1(s, y) \mu(ds, dy) \right) = g_2 \left(\int_{[0, T] \times \bar{\mathcal{O}}} g_1(s, y) \bar{\mu}(ds, dy) \right).$$

In particular they lead to the same Nash value, that is

$$\Gammam, \mu = \Gamma\bar{m}, \bar{\mu}.$$

The proof of the above theorem is a slight modification of Theorem 4.4 in [BDT20]. We will detail the proof for sake of clarity.

Proof. Observe that for all $t \in [0, T]$,

$$\begin{aligned} \langle m_t - \bar{m}_t, f[m](t, \cdot) - f[\bar{m}](t, \cdot) \rangle &= [f_2(t, \langle m_t^x, f_1(t, \cdot) \rangle) - f_2(t, \langle \bar{m}_t^x, f_1(t, \cdot) \rangle)] \langle m_t^x - \bar{m}_t^x, f_1(t, \cdot) \rangle \\ &\leq 0. \end{aligned}$$

Analogously, we have $\langle \mu - \bar{\mu}, g[\mu] - g[\bar{\mu}] \rangle \leq 0$. Using the definition of an LP MFG Nash equilibrium we obtain the following two inequalities:

$$\langle m - \bar{m}, f[m] \rangle + \langle \mu - \bar{\mu}, g[\mu] \rangle \geq 0, \quad (3.10)$$

$$\langle \bar{m} - m, f[\bar{m}] \rangle + \langle \bar{\mu} - \mu, g[\bar{\mu}] \rangle \geq 0. \quad (3.11)$$

Adding up these two last inequalities (3.10) and (3.11) we get:

$$\langle m - \bar{m}, f[m] - f[\bar{m}] \rangle + \langle \mu - \bar{\mu}, g[\mu] - g[\bar{\mu}] \rangle \geq 0.$$

By the first inequalities, the left-hand side should be nonpositive, and therefore equal to 0. We deduce that the first inequalities should be in fact equalities. It follows that dt -a.e. on $[0, T]$,

$$f_2(t, \langle m_t^x, f_1(t, \cdot) \rangle) = f_2(t, \langle \bar{m}_t^x, f_1(t, \cdot) \rangle),$$

and

$$g_2(\langle \mu, g_1 \rangle) = g_2(\langle \bar{\mu}, g_1 \rangle).$$

Finally,

$$\Gamma[\bar{m}, \bar{\mu}](m, \mu) = \Gammam, \mu \geq \Gamma[m, \mu](\bar{m}, \bar{\mu}) = \Gamma\bar{m}, \bar{\mu} \geq \Gamma[\bar{m}, \bar{\mu}](m, \mu),$$

which implies the equality of the Nash values. \square

Case of measure dependent coefficients. When the coefficients depend on the measure, we do not prove the uniqueness of the Nash value, but instead we can show that there exists a maximal Nash value. Let \mathcal{N}^* be the set of LP MFG Nash equilibria.

Proposition 3.26. *There exists $(m^*, \mu^*) \in \mathcal{N}^*$ such that for all $(m, \mu) \in \mathcal{N}^*$,*

$$\Gammam, \mu \leq \Gammam^*, \mu^*$$

Proof. By Theorem 3.4 and Proposition 3.5, the set \mathcal{N}^* is nonempty and compact. Consider the functional $\mathcal{V} : \mathcal{N}^* \rightarrow \mathbb{R}$ defined by

$$\mathcal{V}(m, \mu) = \Gammam, \mu.$$

As in the proof of Theorem 3.4, we can show that \mathcal{V} is continuous. By compactness of \mathcal{N}^* and continuity of \mathcal{V} , we conclude the existence of a maximizer. \square

Selection of equilibria In both cases we have not proved uniqueness of Nash equilibria, we study only the Nash value. The natural question arising in this context is how to select the equilibria. In [DT18] the authors propose several ways of choosing equilibria in a particular model of MFGs, one of them is to choose the equilibria by maximizing the Nash value. We have shown in Proposition 3.26 that this method is always possible under our assumptions.

4 Equivalence with the weak formulation and relation with PDEs

4.1 Single agent problem

In this subsection we will suppose that Assumption 2.1 holds.

Following the literature on the linear programming formulation of stochastic control problems for Markov processes, we now prove that solving the linear program allows to construct a solution to the weak problem. The terminology *weak* is introduced in analogy to the notion of *weak solution* of an SDE, the idea being to consider the probabilistic set-up as part of the solution. The weak formulation is of two types, depending on the type of control, either strict control (valued in A) or relaxed control (valued in $\mathcal{P}(A)$).

We now give the weak formulations (with strict control or relaxed control) of the single agent problem.

Definition 4.1 (*Weak formulation with strict control*). *Define \mathcal{U}^W as the set of tuples $U = (\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, W, \alpha, \tau, X)$ such that $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ is a filtered probability space, W is an \mathbb{F} -Brownian motion, α is an \mathbb{F} -progressively measurable process with values in A , τ is an \mathbb{F} -stopping time such that $\tau \leq T$ \mathbb{P} -a.s., X is an \mathbb{F} -adapted process such that*

$$dX_t = b(t, X_t, \alpha_t)dt + \sigma(t, X_t, \alpha_t)dW_t, \quad t \leq \tau, \quad \mathbb{P} \circ X_0^{-1} = m_0^*.$$

Let $\mathcal{H}^W : \mathcal{U}^W \rightarrow \mathbb{R}$ be defined by

$$\mathcal{H}^W(U) = \mathbb{E}^{\mathbb{P}} \left[\int_0^{\tau \wedge \tau_{\mathcal{O}}^X} f(t, X_t, \alpha_t) dt + g(\tau \wedge \tau_{\mathcal{O}}^X, X_{\tau \wedge \tau_{\mathcal{O}}^X}) \right]$$

for all $U = (\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, W, \alpha, \tau, X) \in \mathcal{U}^W$. The value for the weak formulation with strict control is defined by

$$V^W := \sup_{U \in \mathcal{U}^W} \mathcal{H}^W(U). \quad (3.12)$$

Moreover, $U^* \in \mathcal{U}^W$ is a solution of the weak problem with strict control if

$$\mathcal{H}^W(U^*) = V^W.$$

Definition 4.2 (Weak formulation with relaxed control). Define \mathcal{U}^R as the set of tuples $U = (\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, M, \nu, \tau, X)$ such that $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ is a filtered probability space, τ is an \mathbb{F} -stopping time such that $\tau \leq T$ \mathbb{P} -a.s., ν is an \mathbb{F} -progressively measurable process with values in $\mathcal{P}(A)$, M is a continuous \mathbb{F} -martingale measure such that M^τ has intensity $\nu_t(da)\mathbb{1}_{t < \tau}dt$, X is an \mathbb{F} -adapted process such that

$$dX_t = \int_A b(t, X_t, a)\nu_t(da)dt + \int_A \sigma(t, X_t, a)M(dt, da), \quad t \leq \tau, \quad \mathbb{P} \circ X_0^{-1} = m_0^*.$$

Let $\mathcal{H}^R : \mathcal{U}^R \rightarrow \mathbb{R}$ be defined by

$$\mathcal{H}^R(U) = \mathbb{E}^\mathbb{P} \left[\int_0^{\tau \wedge \tau_{\bar{\mathcal{O}}}^X} \int_A f(t, X_t, a)\nu_t(da)dt + g\left(\tau \wedge \tau_{\bar{\mathcal{O}}}^X, X_{\tau \wedge \tau_{\bar{\mathcal{O}}}^X}\right) \right]$$

for all $U = (\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, M, \nu, \tau, X) \in \mathcal{U}^R$. The value for the weak formulation with relaxed control is defined by

$$V^R := \sup_{U \in \mathcal{U}^R} \mathcal{H}^R(U). \quad (3.13)$$

Moreover, $U^* \in \mathcal{U}^R$ is a solution of the weak problem with relaxed control if

$$\mathcal{H}^R(U^*) = V^R.$$

Theorem 4.3 (Existence of a weak solution with Markovian relaxed control). There exists a solution to the weak problem with Markovian relaxed control.

Proof. Let (m^*, μ^*) be a maximizer of the LP problem which exists by Theorem 2.12. Let $\nu_{t,x}^*$ be such that

$$m_t^*(dx, da)dt = \nu_{t,x}^*(da)m_t^{x,*}(dx)dt.$$

By Theorem 5.20, there exist a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, an \mathbb{F} -adapted process X , an \mathbb{F} -stopping time τ such that $\tau \leq \tau_{\bar{\mathcal{O}}}^X \wedge T$ \mathbb{P} -a.s., a continuous \mathbb{F} -martingale measure M with intensity $\nu_{t,X_t}^*(da)\mathbb{1}_{t < \tau}dt$, such that

$$X_{t \wedge \tau} = \int_0^{t \wedge \tau} \int_A b(t, X_t, a)\nu_{t,X_t}^*(da)dt + \int_0^{t \wedge \tau} \int_A \sigma(t, X_t, a)M(dt, da), \quad \mathbb{P} \circ X_0^{-1} = m_0^*,$$

$$m_t^*(B \times C) = \mathbb{E}^\mathbb{P} \left[\mathbb{1}_B(X_t)\nu_{t,X_t}^*(C)\mathbb{1}_{t < \tau} \right], \quad B \in \mathcal{B}(\bar{\mathcal{O}}), \quad C \in \mathcal{B}(A), \quad t - a.e.,$$

$$\mu^* = \mathbb{P} \circ (\tau, X_\tau)^{-1}.$$

Let $(\Omega', \mathcal{F}', \mathbb{F}', \mathbb{P}')$ be another filtered probability space, τ' an \mathbb{F}' -stopping time such that $\tau' \leq T$ \mathbb{P}' -a.s., ν' an \mathbb{F}' -progressively measurable process with values in $\mathcal{P}(A)$, M' a continuous \mathbb{F}' -martingale measure such that $(M')^\tau$ has intensity $\nu'_t(da)\mathbb{1}_{t < \tau'}dt$, X' an \mathbb{F}' -adapted process such that

$$dX'_t = \int_A b(t, X'_t, a)\nu'_t(da)dt + \int_A \sigma(t, X'_t, a)M'(dt, da), \quad t \leq \tau', \quad \mathbb{P}' \circ (X'_0)^{-1} = m_0^*.$$

Define for $t \in [0, T]$

$$m'_t(B \times C) = \mathbb{E}^{\mathbb{P}'} \left[\mathbb{1}_B(X'_t)\nu'_t(C)\mathbb{1}_{t < \tau' \wedge \tau_{\bar{O}}^{X'}} \right], \quad B \in \mathcal{B}(\bar{O}), \quad C \in \mathcal{B}(A),$$

$$\mu' = \mathbb{P}' \circ \left(\tau' \wedge \tau_{\bar{O}}^{X'}, X'_{\tau' \wedge \tau_{\bar{O}}^{X'}} \right)^{-1}.$$

By Proposition 2.6, $(m', \mu') \in \mathcal{R}$. Since (m^*, μ^*) is a maximizer of the LP problem, $\Gamma(m', \mu') \leq \Gamma(m^*, \mu^*)$, which means

$$\begin{aligned} \mathbb{E}^{\mathbb{P}'} \left[\int_0^{\tau' \wedge \tau_{\bar{O}}^{X'}} \int_A f(t, X'_t, a)\nu'_t(da)dt + g \left(\tau' \wedge \tau_{\bar{O}}^{X'}, X'_{\tau' \wedge \tau_{\bar{O}}^{X'}} \right) \right] \\ \leq \mathbb{E}^{\mathbb{P}} \left[\int_0^{\tau \wedge \tau_{\bar{O}}^X} \int_A f(t, X_t, a)\nu_{t, X_t}(da)dt + g \left(\tau \wedge \tau_{\bar{O}}^X, X_{\tau \wedge \tau_{\bar{O}}^X} \right) \right]. \end{aligned}$$

□

Corollary 4.4 (*Existence of a weak solution with markovian strict control*). *Suppose that for all $(t, x) \in [0, T] \times \bar{O}$, the subset*

$$K(t, x) := \{(b(t, x, a), \sigma^2(t, x, a), z) : a \in A, z \leq f(t, x, a)\}$$

of $\mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}$ is convex. Then there exists a weak solution with markovian strict control (which is also optimal among the relaxed controls).

Proof. The proof follows by Proposition 2.13, Theorem 5.20 and the same argument as in Theorem 4.3. □

One important question arising in the linear programming approach is if this relaxation technique preserves the value of the initial problem. We are going to show the equivalence between the different formulations. The values for the linear programming and weak formulations are already defined, so we define now the value for the strong formulation.

Definition 4.5 (*Strong formulation*). *Let $t \in [0, T]$, we denote by \mathbb{F}^t the filtration given by $\mathcal{F}_s^t = \sigma(W_r - W_t : t \leq r \leq s) \vee \mathcal{N}$, $s \geq t$. Denote by \mathcal{T}_t the set of stopping times with respect to this filtration with values in $[t, T]$. Let \mathbb{A}_t be the set of \mathbb{F}^t -progressively measurable control*

processes taking values in A . The value function for the strong formulation is given by

$$v(t, x) = \sup_{\tau \in \mathcal{T}_t, \alpha \in \mathbb{A}_t} \mathbb{E} \left[\int_t^{\tau \wedge \tau_{\mathcal{O}}^{t,x,\alpha}} f(s, X_s^{t,x,\alpha}, \alpha_s) ds + g\left(\tau \wedge \tau_{\mathcal{O}}^{t,x,\alpha}, X_{\tau \wedge \tau_{\mathcal{O}}^{t,x,\alpha}}^{t,x,\alpha}\right) \right], \quad (3.14)$$

with $(t, x) \in [0, T] \times \mathbb{R}$, $\tau_{\mathcal{O}}^{t,x,\alpha} := \inf \{s \geq t : X_s^{t,x,\alpha} \notin \mathcal{O}\}$ and $(X_s^{t,x,\alpha})_{s \in [t, T]}$ is the unique strong solution of the following stochastic differential equation:

$$X_s^{t,x,\alpha} = x + \int_t^s b(r, X_r^{t,x,\alpha}, \alpha_r) dr + \int_t^s \sigma(r, X_r^{t,x,\alpha}, \alpha_r) dW_r, \quad s \in [t, T].$$

We also define

$$V^S := \int_{\mathcal{O}} v(0, x) m_0^*(dx), \quad (3.15)$$

which represents the value for the strong formulation.

The case $\mathcal{O} = \mathbb{R}$. We show that the values at time zero associated to the different formulations (LP, weak and strong) are equal.

Theorem 4.6 (*Equality of the values of the different formulations*). *Assume that $\mathcal{O} = \mathbb{R}$ and the functions b , σ , f and g are bounded. Then the values associated to the formulations (3.5), (3.12), (3.13) and (3.15) are equal:*

$$V^S = V^W = V^R = V^{LP}.$$

Proof. The proof is organized in two steps.

Step 1. We first show that $V^R = V^{LP}$.

By Proposition 2.6, for each $U \in \mathcal{U}^R$, there exists $(m, \mu) \in \mathcal{R}$ such that $\mathcal{H}^R(U) = \Gamma(m, \mu)$. Therefore, we get

$$V^R \leq V^{LP}.$$

Moreover, by Theorem 5.20, for each $(m, \mu) \in \mathcal{R}$ there exists $U \in \mathcal{U}^R$ satisfying $\Gamma(m, \mu) = \mathcal{H}^R(U)$, leading to

$$V^{LP} \leq V^R.$$

Step 2. We will show now that $V^S = V^W = V^R$.

This result follows by Theorem 4.5. in [KT15], which uses an equivalent formulation (see p. 18 in [KT15]), consisting in fixing a canonical space¹ and optimizing on a set of probability measures. To apply Theorem 4.5 in [KT15], we check that the assumptions are satisfied. Define

1. The canonical space used in [KT15] is given by $[0, \infty] \times D(\mathbb{R}_+, \mathbb{R}^d) \times \mathbb{M}$, where $D(\mathbb{R}_+, \mathbb{R}^d)$ is the set of càdlàg paths from \mathbb{R}_+ to \mathbb{R}^d and \mathbb{M} is the set of all σ -finite (Borel) measures on $\mathbb{R}_+ \times A$ whose marginal distribution on \mathbb{R}_+ is the Lebesgue measure. The first space in the product is for the stopping time, the second for the state process, and the third for the relaxed control. Since our state process is continuous, we can replace $D(\mathbb{R}_+, \mathbb{R}^d)$ by $C(\mathbb{R}_+, \mathbb{R}^d)$, which is the space of continuous paths.

for $(t, \mathbf{y}, a) \in \mathbb{R}_+ \times C(\mathbb{R}_+, \mathbb{R}^2) \times A$,

$$\mu(t, \mathbf{y}, a) := \begin{pmatrix} b(t, \mathbf{y}_t^1, a) \\ f(t, \mathbf{y}_t^1, a) \end{pmatrix} \mathbb{1}_{[0, T]}(t),$$

$$\tilde{\sigma}(t, \mathbf{y}, a) := \begin{pmatrix} \sigma(t, \mathbf{y}_t^1, a) \\ 0 \end{pmatrix} \mathbb{1}_{[0, T]}(t).$$

For each $(t, \mathbf{y}, \alpha) \in \mathbb{R}_+ \times C(\mathbb{R}_+, \mathbb{R}^2) \times \mathbb{A}$, there exists a unique strong solution of the SDE

$$Y_s^{t, \mathbf{y}, \alpha} = \mathbf{y}_t + \int_t^s \mu(r, Y_{r \wedge \cdot}^{t, \mathbf{y}, \alpha}, \alpha_r) dr + \int_t^s \tilde{\sigma}(r, Y_{r \wedge \cdot}^{t, \mathbf{y}, \alpha}, \alpha_r) dW_r,$$

with initial condition $Y_s^{t, \mathbf{y}, \alpha} := \mathbf{y}_s$ for all $s \in [0, t]$. In fact, one can find a strong solution for the first component using the assumptions on b and σ , and since the second component is fully determined by the first one, we get the existence. We denote by X the first component and by Z the second component. Therefore, the associated controlled/stopped martingale problem has a solution. Note that the coefficients are continuous in the control variable for any $(t, \mathbf{y}) \in \mathbb{R}_+ \times C(\mathbb{R}_+, \mathbb{R}^2)$. For $(t, \mathbf{y}) \in [0, \infty] \times C(\mathbb{R}_+, \mathbb{R}^2)$ let

$$\Phi(t, \mathbf{y}) = \mathbf{y}_{t \wedge T}^2 + g(t \wedge T, \mathbf{y}_{t \wedge T}^1).$$

Fix $(t, \mathbf{x}) \in \mathbb{R}_+ \times C(\mathbb{R}_+, \mathbb{R})$ and $\mathbf{y} = (\mathbf{x}, \mathbf{0}) \in C(\mathbb{R}_+, \mathbb{R}^2)$, then we have

$$\begin{aligned} \sup_{\tau \in \mathcal{T}_t, \alpha \in \mathbb{A}_t} \mathbb{E} [\Phi(\tau, Y_{\cdot}^{t, \mathbf{y}, \alpha})] &= \sup_{\tau \in \mathcal{T}_t, \alpha \in \mathbb{A}_t} \mathbb{E} \left[\int_t^{\tau \wedge T} f(s, X_s^{t, \mathbf{x}, \alpha}, \alpha_s) ds + g(\tau \wedge T, X_{\tau \wedge T}^{t, \mathbf{x}, \alpha}) \right] \\ &= v(t, x). \end{aligned}$$

Moreover, for each $t \in [0, \infty]$, $\mathbf{y} \mapsto \Phi(t, \mathbf{y}_{t \wedge \cdot})$ is continuous ($C(\mathbb{R}_+, \mathbb{R}^2)$ is endowed with the topology of uniform convergence on compact subsets of \mathbb{R}_+). Since f and g are bounded, the last assumption of Theorem 4.5. in [KT15] is satisfied. Then applying Theorem 4.5 in [KT15] and integrating at time $t = 0$ with respect to m_0^* (see Theorem 3.1 (ii) in [KT15]), we get

$$V^S = V^W = V^R.$$

The result follows. \square

The case \mathcal{O} bounded. In this paragraph, instead of Assumption 2.1, we impose the following assumption:

Assumption 4.7.

- (1) The domain \mathcal{O} is a bounded open domain of class C^2 .
- (2) σ does not depend on the control a and is continuous on $[0, T] \times \bar{\mathcal{O}}$. Moreover, there exists $c_\sigma > 0$ such that for all $(t, x) \in [0, T] \times \bar{\mathcal{O}}$, $\sigma^2(t, x) \geq c_\sigma$.

- (3) The coefficients $b : [0, T] \times \mathbb{R} \times A \rightarrow \mathbb{R}$ and $\sigma : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ are measurable, bounded and Lipschitz in x uniformly on the other variables.
- (4) f is measurable, bounded and continuous on $\bar{\mathcal{O}}$ uniformly with respect to t and a .
- (5) For fixed $(t, x) \in [0, T] \times \bar{\mathcal{O}}$, $a \mapsto b(t, x, a)$ and $a \mapsto f(t, x, a)$ are continuous.
- (6) $g \in C_b^{1,2}([0, T] \times \bar{\mathcal{O}})$ and $g(t, x) = 0$ for $(t, x) \in (0, T) \times \partial\mathcal{O}$.
- (7) If $(m, \mu) \in \mathcal{R}$, then $m_t^x(dx)dt$ admits a square integrable density with respect to the Lebesgue measure on $[0, T] \times \bar{\mathcal{O}}$.

Remark 4.8. In Section 5.8 we give sufficient conditions under which (7) in the above assumption is satisfied.

Let us recall an existence theorem for the strong formulation. The theorem is a particular case of Theorem 3.2, Chapter 4, in [BL82].

Theorem 4.9. Let Assumption 4.7 be satisfied. Let v be the value function defined in (3.14). Then v is the unique solution belonging to $C([0, T] \times \bar{\mathcal{O}}) \cap W^{1,2,2}((0, T) \times \mathcal{O})^2$ satisfying the following Hamilton-Jacobi-Bellman Variational Inequality (HJBVI)

$$\begin{aligned} \min \left(-\partial_t v(t, x) - \sup_{a \in A} [\mathcal{L}v(t, x, a) + f(t, x, a)], v(t, x) - g(t, x) \right) &= 0, \quad (t, x) \in (0, T) \times \mathcal{O}, \\ v(t, x) &= 0, \quad (t, x) \in (0, T) \times \partial\mathcal{O}, \\ v(T, x) &= g(T, x), \quad x \in \mathcal{O}. \end{aligned} \tag{3.16}$$

Moreover, an optimal control and an optimal stopping time for $v(0, x)$ are given by

$$\alpha_t^*(x) := \alpha \left(t, X_t^{x, \alpha^*} \right), \quad \text{where } \alpha(t, x) \in \arg \max_{a \in A} [\mathcal{L}v(t, x, a) + f(t, x, a)], \tag{3.17}$$

$$\tau^*(x) := \inf \left\{ 0 \leq t \leq T : v \left(t, X_t^{x, \alpha^*} \right) = g \left(t, X_t^{x, \alpha^*} \right) \right\}. \tag{3.18}$$

Remark 4.10. Observe that in [BL82], they suppose that b and f are continuous on t . This assumption is used in their proof to establish continuity of the Hamiltonian, however we need only measurability on the Hamiltonian to use the measurable selection theorem.

The next theorem is a slight extension of Theorem 5.2 in [BDT20]. For sake of clarity we give the proof in Section 5.9.

Theorem 4.11. Suppose Assumption 4.7 is in force. Then, the following are true

- (1) $V^S = V^{LP}$.
- (2) Let (m^*, μ^*) be a maximizer of the LP problem. Then m^* satisfies
 - (a)

$$\int_{\mathcal{S} \times A} (f + \partial_t g + \mathcal{L}g)(t, x, a) m_t^*(dx, da) dt = 0,$$

with $\mathcal{S} := \{(t, x) \in [0, T] \times \mathcal{O} : v(t, x) = g(t, x)\}$.

2. The Sobolev space $W^{1,2,2}((0, T) \times \mathcal{O})$ represents the set of functions u such that $u, \partial_t u, \partial_x u, \partial_{xx} u \in L^2((0, T) \times \mathcal{O})$, where the derivatives are understood in the sense of distributions.

(b)

$$- \int_{\mathcal{C} \times A} f(t, x, a) m_t^*(dx, da) dt = \int_{\mathcal{C} \times A} (\partial_t v + \mathcal{L}v)(t, x, a) m_t^*(dx, da) dt,$$

where $\mathcal{C} := ([0, T] \times \mathcal{O}) \setminus \mathcal{S}$.

(c) For all C^∞ functions ϕ such that $\text{supp}(\phi) \subset \mathcal{C}$, the following holds

$$\int_{\mathcal{O}} \phi(0, x) m_0^*(dx) + \int_0^T \int_{\mathcal{O} \times A} (\partial_t \phi + \mathcal{L}\phi)(t, x, a) m_t^*(dx, da) dt = 0. \quad (3.19)$$

Note that (2)(c) holds true if and only if $\mu^*(\mathcal{C}) = 0$, which is also equivalent to $\mu^*(\mathcal{S} \cup ([0, T] \times \partial\mathcal{O})) = 1$.

Proposition 4.12. *Let Assumption 4.7 hold true, and assume that for each $(t, x) \in [0, T] \times \bar{\mathcal{O}}$, the subset*

$$K(t, x) := \{(b(t, x, a), z) : a \in A, z \leq f(t, x, a)\}$$

of $\mathbb{R} \times \mathbb{R}$ is convex. Let (m^*, μ^*) be an LP solution, then, there exists a measurable function $(t, x) \mapsto \alpha^*(t, x)$ such that $(m_t^{x, *})_{t \in [0, T]}$ satisfies the following system:

$$\begin{cases} \int_{\mathcal{S}} (f + \partial_t g + \mathcal{L}g)(t, x, \alpha^*(t, x)) m_t^{x, *}(dx) dt = 0, \\ \alpha^*(t, x) \in \arg \max_{a \in A} [\mathcal{L}v(t, x, a) + f(t, x, a)] \quad m_t^{x, *}(dx) dt - a.e. \text{ on } \mathcal{C}, \\ \int_{\mathcal{O}} \phi(0, x) m_0^*(dx) + \int_0^T \int_{\mathcal{O}} (\partial_t \phi + \mathcal{L}\phi)(t, x, \alpha^*(t, x)) m_t^{x, *}(dx) dt = 0, \\ \text{for all } C^\infty \text{ functions } \phi \text{ such that } \text{supp}(\phi) \subset \mathcal{C}. \end{cases}$$

Proof. Follows by Theorem 4.11 and a similar argument as in the proof of Proposition 2.13. \square

4.2 MFG problem

4.2.1 Relation with MFG equilibria in the weak formulation

In this section we show the equivalence between linear programming MFG equilibria and MFGs in the weak formulation as defined below. We suppose that Assumption 3.1 holds.

Definition 4.13 (*Weak MFG solution with strict control*). For $(m, \mu) \in V_p \times \mathcal{P}_p([0, T] \times \bar{\mathcal{O}})$, define $\mathcal{U}^W[m]$ as the set of tuples $U = (\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, W, \alpha, \tau, X)$ such that $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ is a filtered probability space, τ is an \mathbb{F} -stopping time such that $\tau \leq T$ \mathbb{P} -a.s., α is an \mathbb{F} -progressively measurable process with values in A , W is an \mathbb{F} -Brownian motion, X is an \mathbb{F} -adapted process such that

$$dX_t = b(t, X_t, m_t, \alpha_t) dt + \sigma(t, X_t, m_t, \alpha_t) dW_t, \quad t \leq \tau, \quad \mathbb{P} \circ X_0^{-1} = m_0^*.$$

Let $\mathcal{H}^W[m, \mu] : \mathcal{U}^W[m] \rightarrow \mathbb{R}$ be defined by

$$\mathcal{H}^W[m, \mu](U) = \mathbb{E}^{\mathbb{P}} \left[\int_0^{\tau \wedge \tau_{\bar{\mathcal{O}}}^X} f(t, X_t, m_t, \alpha_t) dt + g(\tau \wedge \tau_{\bar{\mathcal{O}}}^X, X_{\tau \wedge \tau_{\bar{\mathcal{O}}}^X}, \mu) \right]$$

for all $U = (\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, W, \alpha, \tau, X) \in \mathcal{U}^W[m]$. The value of the optimization problem in the weak

formulation with strict control associated to (m, μ) is defined by

$$V^W[m, \mu] := \sup_{U \in \mathcal{U}^W[m]} \mathcal{H}^W[m, \mu](U). \quad (3.20)$$

Moreover, we say that $U^* = (\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, W, \alpha, \tau, X)$ is a weak MFG Nash equilibrium with strict control if $U^* \in \mathcal{U}^W[m^*]$, where

$$m_t^*(B \times C) = \mathbb{E}^{\mathbb{P}} \left[\mathbf{1}_B(X_t) \mathbf{1}_C(\alpha_t) \mathbf{1}_{t < \tau \wedge \tau_{\bar{\mathcal{O}}}^X} \right], \quad B \in \mathcal{B}(\bar{\mathcal{O}}), \quad C \in \mathcal{B}(A), \quad t \in [0, T], \quad (3.21)$$

$$\mu^* = \mathbb{P} \circ \left(\tau \wedge \tau_{\bar{\mathcal{O}}}^X, X_{\tau \wedge \tau_{\bar{\mathcal{O}}}^X} \right)^{-1}, \quad (3.22)$$

and

$$\mathcal{H}^W[m^*, \mu^*](U^*) = V^W[m^*, \mu^*]. \quad (3.23)$$

Definition 4.14 (Weak MFG solution with relaxed control). For $(m, \mu) \in V_p \times \mathcal{P}_p([0, T] \times \bar{\mathcal{O}})$, define $\mathcal{U}^R[m]$ as the set of tuples $U = (\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, M, \nu, \tau, X)$ such that $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ is a filtered probability space, τ is an \mathbb{F} -stopping time such that $\tau \leq T$ \mathbb{P} -a.s., ν is an \mathbb{F} -progressively measurable process with values in $\mathcal{P}(A)$, M is a continuous \mathbb{F} -martingale measure such that M^τ has intensity $\nu_t(da) \mathbf{1}_{t < \tau} dt$, X is an \mathbb{F} -adapted process such that

$$dX_t = \int_A b(t, X_t, m_t, a) \nu_t(da) dt + \int_A \sigma(t, X_t, m_t, a) M(dt, da), \quad t \leq \tau, \quad \mathbb{P} \circ X_0^{-1} = m_0^*.$$

Let $\mathcal{H}^R[m, \mu] : \mathcal{U}^R[m] \rightarrow \mathbb{R}$ be defined by

$$\mathcal{H}^R[m, \mu](U) = \mathbb{E}^{\mathbb{P}} \left[\int_0^{\tau \wedge \tau_{\bar{\mathcal{O}}}^X} \int_A f(t, X_t, m_t, a) \nu_t(da) dt + g \left(\tau \wedge \tau_{\bar{\mathcal{O}}}^X, X_{\tau \wedge \tau_{\bar{\mathcal{O}}}^X}, \mu \right) \right]$$

for all $U = (\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, M, \nu, \tau, X) \in \mathcal{U}^R[m]$. The value of the optimization problem in the weak formulation with relaxed control associated to (m, μ) is defined by

$$V^R[m, \mu] := \sup_{U \in \mathcal{U}^R[m]} \mathcal{H}^R[m, \mu](U). \quad (3.24)$$

Moreover, we say that $U^* = (\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, M, \nu, \tau, X)$ is a weak MFG Nash equilibrium with relaxed control if $U^* \in \mathcal{U}^R[m^*]$, where

$$m_t^*(B \times C) = \mathbb{E}^{\mathbb{P}} \left[\mathbf{1}_B(X_t) \nu_t(C) \mathbf{1}_{t < \tau \wedge \tau_{\bar{\mathcal{O}}}^X} \right], \quad B \in \mathcal{B}(\bar{\mathcal{O}}), \quad C \in \mathcal{B}(A), \quad t \in [0, T], \quad (3.25)$$

$$\mu^* = \mathbb{P} \circ \left(\tau \wedge \tau_{\bar{\mathcal{O}}}^X, X_{\tau \wedge \tau_{\bar{\mathcal{O}}}^X} \right)^{-1}, \quad (3.26)$$

and

$$\mathcal{H}^R[m^*, \mu^*](U^*) = V^R[m^*, \mu^*]. \quad (3.27)$$

The above definition is equivalent to the following formulation of MFG equilibrium via the controlled/stopped martingale problem.

Definition 4.15 (*MFG equilibrium via the controlled/stopped martingale problem*). For $(m, \mu) \in V_p \times \mathcal{P}_p([0, T] \times \bar{\mathcal{O}})$, define $\mathcal{U}^M[m]$ as the set of tuples $U = (\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, \nu, \tau, X)$ such that $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ is a filtered probability space, τ is an \mathbb{F} -stopping time such that $\tau \leq T$ \mathbb{P} -a.s., ν is an \mathbb{F} -progressively measurable process with values in $\mathcal{P}(A)$, X is an \mathbb{F} -adapted process such that $\mathbb{P} \circ X_0^{-1} = m_0^*$ and for all $\varphi \in C_b^2(\mathbb{R})$, the process $(M_{t \wedge \tau}(\varphi))_{t \geq 0}$ is an (\mathbb{F}, \mathbb{P}) -martingale, where

$$M_t(\varphi) := \varphi(X_t) - \int_0^t \int_A \mathcal{L}\varphi(s, X_s, m_s, a) \nu_s(da) ds.$$

Let $\mathcal{H}^M[m, \mu] : \mathcal{U}^M[m] \rightarrow \mathbb{R}$ be defined by

$$\mathcal{H}^M[m, \mu](U) = \mathbb{E}^{\mathbb{P}} \left[\int_0^{\tau \wedge \tau_{\bar{\mathcal{O}}}^X} \int_A f(t, X_t, m_t, a) \nu_t(da) dt + g\left(\tau \wedge \tau_{\bar{\mathcal{O}}}^X, X_{\tau \wedge \tau_{\bar{\mathcal{O}}}^X}, \mu\right) \right]$$

for all $U = (\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, \nu, \tau, X) \in \mathcal{U}^M[m]$. The value of the optimization problem in the controlled/stopped martingale problem associated to (m, μ) is defined by

$$V^M[m, \mu] := \sup_{U \in \mathcal{U}^M[m]} \mathcal{H}^M[m, \mu](U).$$

Moreover, we say that $U^* = (\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, \nu, \tau, X)$ is an MFG equilibrium via the controlled/stopped martingale problem if $U^* \in \mathcal{U}^M[m^*]$, where

$$m_t^*(B \times C) = \mathbb{E}^{\mathbb{P}} \left[\mathbf{1}_B(X_t) \nu_t(C) \mathbf{1}_{t < \tau \wedge \tau_{\bar{\mathcal{O}}}^X} \right], \quad B \in \mathcal{B}(\bar{\mathcal{O}}), \quad C \in \mathcal{B}(A), \quad t \in [0, T],$$

$$\mu^* = \mathbb{P} \circ \left(\tau \wedge \tau_{\bar{\mathcal{O}}}^X, X_{\tau \wedge \tau_{\bar{\mathcal{O}}}^X} \right)^{-1},$$

and

$$\mathcal{H}^M[m^*, \mu^*](U^*) = V^M[m^*, \mu^*].$$

Remark 4.16. This definition is also equivalent to the problem of finding an MFG equilibrium via the controlled/stopped martingale problem on the canonical space (see [Lac15]), where the optimization is considered over a subset of probabilities on the canonical space verifying some conditions, instead of all the tuples $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, \tau, \nu, X)$. We refer to [KT15], p. 18, for more details on this equivalence.

Theorem 4.17 (*Equivalence between LP MFG equilibria and weak MFG equilibria*). The LP MFG problem and the weak MFG problem are equivalent. More specifically,

(i) Given an LP MFG Nash equilibrium (m^*, μ^*) , there exists a weak MFG Nash equilibrium (with Markovian relaxed control) $U^* \in \mathcal{U}^R[m^*]$ such that

$$\Gammam^*, \mu^* = \mathcal{H}^R[m^*, \mu^*](U^*). \quad (3.28)$$

(ii) Given U^* a weak MFG Nash equilibrium, that is $U^* \in \mathcal{U}^R[m^*]$, with m^* (resp. μ^*) given by (3.25) (resp. (3.26)), then (m^*, μ^*) is an LP MFG Nash equilibrium and (3.28) holds.

Proof. Considering measure dependent coefficients, the equivalence follows from Proposition 2.6 and Theorem 5.20. \square

Corollary 4.18. *There exists a weak Nash equilibrium (with Markovian relaxed control).*

Proof. By Theorem 3.4 we get the existence of LP MFG Nash equilibrium, which implies by Theorem 4.17 the existence of a weak Nash equilibrium (with Markovian relaxed control). \square

Remark 4.19. *In the case when there is only control, we recover the existence result of Markovian relaxed controls of [Lac15, Corollary 3.8]. In that paper, this result is shown by using the Mimicking Theorem (or Markovian projection theorem) from Corollary 3.7. in [BS13], while in our case this result follows naturally by the disintegration*

$$m_t(dx, da)dt = \nu_{t,x}(da)m_t^x(dx)dt.$$

Proposition 4.20. *Suppose that $\mathcal{O} = \mathbb{R}$. Let (m^*, μ^*) be an LP Nash equilibrium. Consider the value function given by*

$$v^*(t, x) = \sup_{\tau \in \mathcal{T}_t, \alpha \in \mathbb{A}_t} \mathbb{E} \left[\int_t^{\tau \wedge \tau_{\mathcal{O}}^{t,x,m^*,\alpha}} f(s, X_s^{t,x,m^*,\alpha}, m_s^*, \alpha_s) ds + g\left(\tau \wedge \tau_{\mathcal{O}}^{t,x,m^*,\alpha}, X_{\tau \wedge \tau_{\mathcal{O}}^{t,x,m^*,\alpha}}^{t,x,m^*,\alpha}, \mu^*\right) \right], \quad (3.29)$$

where $(t, x) \in [0, T] \times \mathbb{R}$, $\tau_{\mathcal{O}}^{t,x,m^*,\alpha} := \inf \{s \geq t : X_s^{t,x,m^*,\alpha} \notin \mathcal{O}\}$ and $(X_s^{t,x,m^*,\alpha})_{s \in [t, T]}$ is the unique strong solution of the following stochastic differential equation:

$$X_s^{t,x,m^*,\alpha} = x + \int_t^s b(r, X_r^{t,x,m^*,\alpha}, m_r^*, \alpha_r) dr + \int_t^s \sigma(r, X_r^{t,x,m^*,\alpha}, m_r^*, \alpha_r) dW_r, \quad s \in [t, T].$$

We have the following equality:

$$\int_{\mathcal{O}} v^*(0, x) m_0^*(dx) = V^W[m^*, \mu^*] = V^R[m^*, \mu^*] = \Gammam^*, \mu^*.$$

Proof. Since (m^*, μ^*) is fixed in the functions b , σ , f and g , we can apply Theorem 4.6 noticing that Assumption 2.1 is satisfied. \square

Proposition 4.21. *Suppose that for all $(t, x, m) \in [0, T] \times \bar{\mathcal{O}} \times V_p$, the subset*

$$K[m](t, x) := \{(b(t, x, m_t, a), \sigma^2(t, x, m_t, a), z) : a \in A, z \leq f(t, x, m_t, a)\}$$

of $\mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}$ is convex. Then there exist a strict control LP Nash equilibrium and a weak Nash equilibrium with Markovian strict control.

Proof. The proof is almost the same as the one of Proposition 2.13. \square

4.2.2 Relation with mixed solutions

In this subsection, to establish the link with PDE formulation, we shall need the following assumptions:

Assumption 4.22.

- (1) The domain \mathcal{O} is a bounded open interval.
- (2) The volatility σ does not depend on the control and the measure, and is continuous on $[0, T] \times \bar{\mathcal{O}}$. Moreover, there exists $c_\sigma > 0$ such that for all $(t, x) \in [0, T] \times \bar{\mathcal{O}}$, $\sigma^2(t, x) \geq c_\sigma$.
- (3) The coefficients $b : [0, T] \times \mathbb{R} \times \mathcal{P}^{sub}(\bar{\mathcal{O}}) \times A \rightarrow \mathbb{R}$ and $\sigma : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ are measurable, bounded and Lipschitz in x uniformly on the other variables.
- (4) f is measurable, bounded and continuous in x on $\bar{\mathcal{O}}$, uniformly with respect to t , η and a .
- (5) For fixed $(t, x) \in [0, T] \times \bar{\mathcal{O}}$, $(\eta, a) \mapsto b(t, x, \eta, a)$ and $(\eta, a) \mapsto f(t, x, \eta, a)$ are continuous.
- (6) The function $[0, T] \times \bar{\mathcal{O}} \times \mathcal{P}([0, T] \times \bar{\mathcal{O}}) \ni (t, x, \mu) \mapsto g(t, x, \mu)$ is continuous. For a fixed $\mu \in \mathcal{P}([0, T] \times \bar{\mathcal{O}})$, $(t, x) \mapsto g(t, x, \mu) \in C_b^{1,2}([0, T] \times \bar{\mathcal{O}})$ and $g(t, x, \mu) = 0$ for $(t, x) \in (0, T) \times \partial\mathcal{O}$.
- (7) If $(m, \mu) \in \mathcal{R}[\bar{m}]$ for some $\bar{m} \in V$, then $m_t^x(dx)dt$ admits a square integrable density with respect to the Lebesgue measure on $[0, T] \times \bar{\mathcal{O}}$.

Theorem 4.23 (Relation with mixed solutions). *Suppose Assumption 4.22 holds true. Let (m^*, μ^*) be an LP MFG Nash equilibrium. Consider the value function given by (3.29). We have the following relations.*

- (1) Relation with the strong formulation:

$$\int_{\mathcal{O}} v^*(0, x) m_0^*(dx) = \Gammam^*, \mu^*.$$

- (2) Relation with mixed solutions:

(a)

$$\int_{\mathcal{S}^* \times A} \left(f + \frac{\partial g}{\partial t} + \mathcal{L}g \right) (t, x, m_t^*, \mu^*, a) m_t^*(dx, da) dt = 0,$$

with $\mathcal{S}^* := \{(t, x) \in [0, T] \times \mathcal{O} : v^*(t, x) = g(t, x, \mu^*)\}$.

(b)

$$- \int_{\mathcal{C}^* \times A} f(t, x, m_t^*, a) m_t^*(dx, da) dt = \int_{\mathcal{C}^* \times A} \left(\frac{\partial v}{\partial t} + \mathcal{L}v \right) (t, x, m_t^*, a) m_t^*(dx, da) dt,$$

where $\mathcal{C}^* := ([0, T] \times \mathcal{O}) \setminus \mathcal{S}^*$.

(c) For all C^∞ functions ϕ such that $\text{supp}(\phi) \subset \mathcal{C}^*$, the following holds

$$\int_{\mathcal{O}} \phi(0, x) m_0^*(dx) + \int_0^T \int_{\mathcal{O} \times A} \left(\frac{\partial \phi}{\partial t} + \mathcal{L}\phi \right) (t, x, m_t^*, a) m_t^*(dx, da) dt = 0.$$

Note that (2)(c) holds true if and only if $\mu^*(\mathcal{C}^*) = 0$, which is also equivalent to $\mu^*(\mathcal{S}^* \cup ([0, T] \times \partial\mathcal{O})) = 1$.

Proof. The proof follows by applying Theorem 4.11 taking into account that the inputs (b, σ, f, g) depend now on (m^*, μ^*) but still satisfy the required assumptions. \square

Corollary 4.24. Let Assumption 4.22 hold true and assume that for all $(t, x, m) \in [0, T] \times \bar{\mathcal{O}} \times V$, the subset

$$K[m](t, x) := \{(b(t, x, m_t, a), z) : a \in A, z \leq f(t, x, m_t, a)\}$$

of $\mathbb{R} \times \mathbb{R}$ is convex. Let (m^*, μ^*) be an LP MFG Nash equilibrium, then, there exists a measurable function $(t, x) \mapsto \alpha^*(t, x)$ such that $(m_t^{x,*})_{t \in [0, T]}$ satisfies the following system:

$$\begin{cases} \int_{\mathcal{S}^*} \left(f + \frac{\partial g}{\partial t} + \mathcal{L}g \right) (t, x, m_t^{x,*}, \mu^*, \alpha^*(t, x)) m_t^{x,*}(dx) dt = 0, \\ \alpha^*(t, x) \in \arg \max_{a \in A} [\mathcal{L}v^*(t, x, m_t^{x,*}, a) + f(t, x, m_t^{x,*}, a)] \quad m_t^{x,*}(dx) dt - a.e. \text{ on } \mathcal{C}^*, \\ \int_{\mathcal{O}} \phi(0, x) m_0^*(dx) + \int_0^T \int_{\mathcal{O} \times A} \left(\frac{\partial \phi}{\partial t} + \mathcal{L}\phi \right) (t, x, m_t^{x,*}, \alpha^*(t, x)) m_t^{x,*}(dx) dt = 0, \\ \text{for all } C^\infty \text{ functions } \phi \text{ such that } \text{supp}(\phi) \subset \mathcal{C}^*. \end{cases}$$

Remark 4.25. The above result gives the link with the notion of mixed solution in the case of optimal stopping/continuous control introduced in [Ber17] in a less general framework (in particular, the author considers the drift to be zero and the volatility $\sqrt{2}$).

5 Appendix

5.1 Polynomial growth topologies for measures

Let (E, d) be a complete and separable metric space. The topology in $\mathcal{M}(E)$ is given by the topology of weak convergence $\tau := \sigma(\mathcal{M}(E), C_b(E))$. The relative topology in $\mathcal{M}^+(E)$ is completely metrizable through the bounded Lipschitz distance (see [Bog07], volume II, p.192 and Theorem 8.3.2 p.193)

$$d_{\text{BL}}(\mu^1, \mu^2) := \sup \left\{ \int_E \varphi(x) (\mu^1 - \mu^2)(dx) : \varphi \in \text{BL}(E), \|\varphi\|_{\text{BL}} \leq 1 \right\}, \quad \mu^1, \mu^2 \in \mathcal{M}^+(E),$$

where $\text{BL}(E)$ is the space of all bounded Lipschitzian functions on E endowed with the norm

$$\|\varphi\|_{\text{BL}} := \|\varphi\|_\infty + \sup_{x \neq y} \frac{|\varphi(x) - \varphi(y)|}{|x - y|}.$$

Let x_0 be an arbitrary point in E (in particular, when E is a subset of a vector space we will usually take $x_0 = 0$ if the metric is well defined in this point) and define the weight function

$w : E \rightarrow [1, \infty[$ by

$$w(x) = 1 + d(x, x_0)^p, \quad x \in E.$$

Recall our notation $C_p(E)$ for the set of real valued continuous functions with p -growth,

$$C_p(E) := \left\{ \phi \in C(E) : \sup_{x \in E} \frac{|\phi(x)|}{w(x)} < \infty \right\}$$

and the topology $\tau_p := \sigma(\mathcal{M}_p(E), C_p(E))$ on $\mathcal{M}_p(E)$. Observe that the sets $C_p(E)$ and $\mathcal{M}_p(E)$ do not depend on the choice of x_0 and henceforth the topology τ_p neither.

Lemma 5.1. *The function $F : (\mathcal{M}_p(E), \tau_p) \rightarrow (\mathcal{M}(E), \tau)$ given by*

$$F(\mu) = w(x)\mu(dx)$$

is an homeomorphism. The same function is an homeomorphism between $(\mathcal{M}_p^+(E), \tau_p)$ and $(\mathcal{M}^+(E), \tau)$.

Proof. We start by showing that F is continuous. Consider a net $(\mu_\alpha)_\alpha \subset \mathcal{M}_p(E)$ converging to some $\mu \in \mathcal{M}_p(E)$ in τ_p . Consider a function $\phi \in C_b(E)$, then we have $\phi w \in C_p(E)$. We get

$$\lim_\alpha \int \phi dF(\mu_\alpha) = \lim_\alpha \int \phi w d\mu_\alpha = \int \phi w d\mu = \int \phi dF(\mu),$$

that is, $\lim_\alpha F(\mu_\alpha) = F(\mu)$ in τ . Let us prove that F is injective: given $\mu, \nu \in \mathcal{M}_p(E)$ such that $F(\mu) = F(\nu)$ we must have for all $B \in \mathcal{B}(E)$

$$\int_B \frac{1}{w} dF(\mu) = \int_B \frac{1}{w} dF(\nu),$$

which implies $\mu(B) = \nu(B)$. Now let $\nu \in \mathcal{M}(E)$ and define $\mu(dx) = \frac{1}{w(x)}\nu(dx)$. Then clearly $\mu \in \mathcal{M}_p(E)$ and $\nu = F(\mu)$. We deduce that F is bijective and $F^{-1}(\nu) = \frac{1}{w(x)}\nu(dx)$. Take a net $(\nu_\alpha)_\alpha \subset \mathcal{M}(E)$ converging to some $\nu \in \mathcal{M}(E)$ in τ and consider a function $\phi \in C_p(E)$. Since $\phi/w \in C_b(E)$, we get

$$\lim_\alpha \int \phi dF^{-1}(\nu_\alpha) = \lim_\alpha \int \frac{\phi}{w} d\nu_\alpha = \int \frac{\phi}{w} d\nu = \int \phi dF^{-1}(\nu),$$

that is, $\lim_\alpha F^{-1}(\nu_\alpha) = F^{-1}(\nu)$ in τ_p . It is clear that both F and F^{-1} map nonnegative measures onto nonnegative measures, so that the same function can be used to create an homeomorphism between $(\mathcal{M}_p^+(E), \tau_p)$ and $(\mathcal{M}^+(E), \tau)$. \square

Remark 5.2. *Since $(\mathcal{M}^+(E), \tau)$ is completely metrizable through the bounded Lipschitz distance, we get that $(\mathcal{M}_p^+(E), \tau_p)$ is also completely metrizable by the metric*

$$d_{BL,p}(\mu^1, \mu^2) := \sup \left\{ \int_E \varphi(x)w(x)(\mu^1 - \mu^2)(dx) : \varphi \in BL(E), \|\varphi\|_{BL} \leq 1 \right\}, \quad \mu^1, \mu^2 \in \mathcal{M}_p^+(E).$$

In particular, for all $\mu \in \mathcal{M}_p^+(E)$,

$$d_{BL,p}(\mu, 0) = \int_E w(x) \mu(dx).$$

The following proposition characterizes the convergence in τ_p for nonnegative measures. The proof is analogous to the one from Theorem 7.12 in [Vil03].

Proposition 5.3. *A sequence $(\mu_n)_{n \geq 1} \subset \mathcal{M}_p^+(E)$ converges to $\mu \in \mathcal{M}_p^+(E)$ in τ_p if and only if $(\mu_n)_{n \geq 1}$ converges to μ weakly and*

$$\lim_{r \rightarrow \infty} \limsup_{n \rightarrow \infty} \int_{\{x \in E: d(x, x_0)^p \geq r\}} d(x, x_0)^p \mu_n(dx) = 0. \quad (3.30)$$

Corollary 5.4. *Assume that E is a closed subset of an Euclidean space with norm $|\cdot|$. A set $\mathcal{K} \subset \mathcal{M}_p^+(E)$ is relatively compact in τ_p if there exists $q > p$ such that*

$$\sup_{\mu \in \mathcal{K}} \int_E (1 + |x|^q) \mu(dx) < \infty.$$

Proof. The condition

$$\sup_{\mu \in \mathcal{K}} \int_E (1 + |x|^q) \mu(dx) < \infty,$$

implies that \mathcal{K} is tight (since the map $x \mapsto 1 + |x|^q$ has compact level sets) and uniformly bounded in total variation norm. By Prokhorov's Theorem (Theorem 8.6.2 in [Bog07] Volume II), the set \mathcal{K} is relatively compact in τ . Henceforth, by Proposition 5.3, it suffices to show the uniform integrability condition (3.30) for a given sequence $(\mu_n)_{n \geq 1} \subset \mathcal{K}$. By Hölder's inequality, for all $r \geq 0$

$$\begin{aligned} \int_{\{x \in E: |x|^p \geq r\}} |x|^p \mu_n(dx) &\leq \left[\int_E |x|^q \mu_n(dx) \right]^{p/q} \mu_n(\{x \in E: |x|^p \geq r\})^{(q-p)/q} \\ &\leq C \mu_n(\{x \in E: |x|^p \geq r\})^{(q-p)/q}. \end{aligned}$$

By Markov's inequality,

$$\mu_n(\{x \in E: |x|^p \geq r\}) \leq \frac{1}{r} \int_E |x|^p \mu_n(dx) \leq \frac{C}{r}.$$

This suffices to conclude. \square

Now we are interested in an analogue version of the stable convergence topology for nonnegative measures with the polynomial growth condition. Consider two complete separable metric spaces (E, d_E) and (F, d_F) . The distance on the product space is given by

$$d((x, y), (x', y')) = (d_E(x, x')^p + d_F(y, y')^p)^{1/p}, \quad (x, y) \in E \times F.$$

Let $(x_0, y_0) \in E \times F$ be some arbitrary point and define the weight function

$$w : E \times F \ni (x, y) \mapsto 1 + d((x, y), (x_0, y_0))^p \in \mathbb{R}.$$

Consider the following sets of functions:

$$M_{mc,b}(E \times F) = \{\phi \in M_b(E \times F) : \phi(x, \cdot) \in C(F)\},$$

$$M_{mc,p}(E \times F) = \left\{ \phi \in M(E \times F) : \phi(x, \cdot) \in C(F), \sup_{(x,y) \in E \times F} \frac{|\phi(x, y)|}{w(x, y)} < \infty \right\}.$$

We recall the topology $\bar{\tau} := \sigma(\mathcal{M}(E \times F), M_{mc,b}(E \times F))$ which is known as the topology of stable convergence (see [JM81]) and the topology $\bar{\tau}_p := \sigma(\mathcal{M}_p(E \times F), M_{mc,\bar{w}}(E \times F))$. We are interested in studying the space $\mathcal{M}_p^+(E \times F)$ endowed with the topology $\bar{\tau}_p$. The following lemma is shown as in Lemma 5.1.

Lemma 5.5. *The function $G : (\mathcal{M}_p^+(E \times F), \bar{\tau}_p) \rightarrow (\mathcal{M}^+(E \times F), \bar{\tau})$ given by*

$$G(\mu) = \int w(x, y) \mu(dx, dy)$$

is an homeomorphism.

Remark 5.6. *By Proposition 2.10 in [JM81], the space $(\mathcal{M}^+(E \times F), \bar{\tau})$ is metrizable, henceforth, $(\mathcal{M}_p^+(E \times F), \bar{\tau}_p)$ is also metrizable.*

Proposition 5.7. *Consider a sequence $(\mu_n)_{n \geq 1} \subset \mathcal{M}_p^+(E \times F)$ converging to $\mu \in \mathcal{M}_p^+(E \times F)$ in $\bar{\tau}_p$. If the set of marginal measures*

$$\left\{ \int_F w(x, y) \mu_n(dx, dy) : n \geq 1 \right\} \subset \mathcal{M}^+(E),$$

is relatively compact in $(\mathcal{M}^+(E), \sigma(\mathcal{M}^+(E), M_b(E)))$, then $(\mu_n)_{n \geq 1}$ converges to μ in $\bar{\tau}_p$.

Proof. For $n \geq 1$, let $\nu_n = G(\mu_n)$ and $\nu = G(\mu)$, where G is the homeomorphism given in Lemma 5.5. Since G is $\bar{\tau}_p/\bar{\tau}$ continuous, the sequence $(\nu_n)_{n \geq 1}$ converges to ν in $\bar{\tau}$. By assumption, the set of marginal measures

$$\{\nu_n(\cdot \times F) : n \geq 1\}$$

is relatively compact in $(\mathcal{M}^+(E), \sigma(\mathcal{M}^+(E), M_b(E)))$. By Corollary 2.9 in [JM81], the sequence $(\nu_n)_{n \geq 1}$ converges to ν in $\bar{\tau}$. Finally, since G^{-1} is $\bar{\tau}/\bar{\tau}_p$ continuous, we get that $(\mu_n)_{n \geq 1}$ converges to μ in $\bar{\tau}_p$. \square

5.2 Structure of \tilde{U} and metrizability of V_p

We will show in this section that \tilde{U} is a Hausdorff locally convex topological vector space and V_p is metrizable. We already know that $(\mathcal{M}([0, T] \times \bar{O} \times A), \tau)$ is a Hausdorff locally convex topological vector space. Define the map $\pi : \tilde{U} \rightarrow \mathcal{M}([0, T] \times \bar{O} \times A)$ by

$$\pi(m) = \int m_t(dx, da) dt.$$

The map π is injective since the elements of \tilde{U} are identified t -a.e. We define $\mathfrak{M} := \pi(\tilde{U})$ and with some abuse of notation we still denote by τ the topology $\sigma(\mathfrak{M}, C_b([0, T] \times \bar{\mathcal{O}} \times A))$. The subspace (\mathfrak{M}, τ) is also a Hausdorff locally convex topological vector space since π is linear. Moreover $\pi : \tilde{U} \mapsto \mathfrak{M}$ is a linear bijection. This allows us to endow \tilde{U} with the projective topology $\pi^{-1}(\tau)$ (which is the topology of weak convergence of the associated measures on $[0, T] \times \bar{\mathcal{O}} \times A$). With this definition, π is an isomorphism between topological vector spaces, which implies that $(\tilde{U}, \pi^{-1}(\tau))$ is a Hausdorff locally convex topological vector space.

The relative topology on V_p is metrizable since $\pi(V_p) \subset \mathcal{M}_p^+([0, T] \times \bar{\mathcal{O}} \times A)$ and the p -growth topology on $\mathcal{M}_p^+([0, T] \times \bar{\mathcal{O}} \times A)$ is metrizable (see Remark 5.2), in particular, we can define a natural distance on V_p through π .

5.3 Martingale measures and controlled/stopped martingale problem

5.3.1 Martingale measures

For the sake of clarity we present the definition of martingale measures and some related concepts. This content is taken from [KM90] and [Mél92]. Throughout the section we fix a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ and a Polish space A with Borel σ -algebra $\mathcal{B}(A)$.

Definition 5.8. *We say that $M : \Omega \times \mathbb{R}_+ \times \mathcal{B}(A) \rightarrow \mathbb{R}$ is an (orthogonal) martingale measure if it satisfies the following properties:*

- (1) For all $B \in \mathcal{B}(A)$, $M(\cdot, B)$ ³ is a square integrable martingale and $M(0, B) = 0$.
- (2) For all $t \in \mathbb{R}_+$, $B, C \in \mathcal{B}(A)$ such that $B \cap C = \emptyset$, $M(t, B \cup C) = M(t, B) + M(t, C)$ a.s.
- (3) There exists a nondecreasing sequence of $(A_n)_{n \geq 1} \subset \mathcal{B}(A)$ such that

- $\cup_{n \geq 1} A_n = A$.
- For all $t \in \mathbb{R}_+$, and all $n \geq 1$,

$$\sup_{B \in \mathcal{B}(A_n)} \mathbb{E}[M(t, B)^2] < \infty.$$

- For all $t \in \mathbb{R}_+$, $n \geq 1$ and $(B_k)_{k \geq 1} \subset \mathcal{B}(A_n)$ a decreasing sequence such that $\cap_{k \geq 1} B_k = \emptyset$,

$$\mathbb{E}[M(t, B_k)^2] \xrightarrow[k \rightarrow \infty]{} 0.$$

- (4) For all $B, C \in \mathcal{B}(A)$ such that $B \cap C = \emptyset$, the martingales $(M(t, B))_{t \geq 0}$ and $(M(t, C))_{t \geq 0}$ are orthogonal, i.e. $(M(t, B)M(t, C))_{t \geq 0}$ is a martingale.

A martingale measure M is said to be continuous if for all $B \in \mathcal{B}(A)$, $t \mapsto M(t, B)$ is continuous a.s.

Remark 5.9. *If τ is a stopping time and M a martingale measure, then $M^\tau(t, B) := M(t \wedge \tau, B)$ is also a martingale measure.*

3. Note that we suppress the argument $\omega \in \Omega$ from the notation as usual in probability theory.

Theorem 5.10 (Theorem I-4 in [KM90]). *If M is a martingale measure, then there exists a random σ -finite nonnegative measure ν on $\mathbb{R}_+ \times A$, such that for each $B \in \mathcal{B}(A)$, $(\nu([0, t] \times B))_{t \geq 0}$ is the predictable quadratic variation of $(M(t, B))_{t \geq 0}$. The measure ν is called the intensity of M .*

Let M be a martingale measure with intensity ν and let L_ν^2 be the set of functions $\phi : \Omega \times \mathbb{R}_+ \times A \rightarrow \mathbb{R}$ measurable with respect to the product of the predictable σ -algebra and $\mathcal{B}(A)$, such that

$$\mathbb{E} \left[\int_{\mathbb{R}_+ \times A} \phi^2(s, a) \nu(ds, da) \right] < \infty.$$

Then for any $\phi \in L_\nu^2$ one can construct a stochastic integral of ϕ with respect to M , which is a function from $\Omega \times \mathbb{R}_+ \times \mathcal{B}(A)$ to \mathbb{R} . It is denoted by $\phi \cdot M$. We will also denote

$$\int_0^t \int_B \phi(s, a) M(ds, da) := (\phi \cdot M)(t, B), \quad t \in \mathbb{R}_+, \quad B \in \mathcal{B}(A).$$

The construction is analogous to the one of the Itô integral.

Proposition 5.11 (Proposition I-6 in [KM90]). *Let M be a martingale measure with intensity ν .*

- (1) *If $\phi \in L_\nu^2$, then $\phi \cdot M$ is a martingale measure with intensity $\phi^2(s, a) \nu(ds, da)$. Moreover, if M is continuous, then $\phi \cdot M$ is also continuous.*
- (2) *If $\phi, w \in L_\nu^2$ and $B, C \in \mathcal{B}(A)$, then for all $t \in \mathbb{R}_+$,*

$$\langle \phi \cdot M(\cdot, B), w \cdot M(\cdot, C) \rangle_t = \int_0^t \int_{B \cap C} \phi(s, a) w(s, a) \nu(ds, da).$$

A consequence of this proposition is that

$$\left(\int_0^t \int_A \phi(s, a) M(ds, da) \right)_{t \geq 0}$$

is a martingale with quadratic variation

$$\left(\int_0^t \int_A \phi^2(s, a) \nu(ds, da) \right)_{t \geq 0}.$$

This fact allows the use of Burkholder-Davis-Gundy inequality, which can be applied to prove the existence of strong solutions to SDEs of the type

$$X_t = \xi + \int_0^t \int_A b(s, X_s, a) \nu(ds, da) + \int_0^t \int_A \sigma(s, X_s, a) M(ds, da), \quad t \geq 0.$$

under standard assumptions.

5.3.2 Controlled/stopped martingale problem

Recall that the linear operator \mathcal{L} is given by

$$\mathcal{L}\varphi(t, x, a) = b(t, x, a)\varphi'(x) + \frac{\sigma^2}{2}(t, x, a)\varphi''(x), \quad (t, x, a) \in [0, T] \times \mathbb{R} \times A, \quad \varphi \in C_b^2(\mathbb{R}).$$

Definition 5.12. *The tuple $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, \nu, \tau, X)$ is said to be a solution of the controlled/stopped martingale problem if*

- (1) $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ is a filtered probability space supporting an \mathbb{F} -progressively measurable process ν with values in $\mathcal{P}(A)$, an \mathbb{F} -stopping time τ and an \mathbb{F} -adapted process X .
- (2) For all $\varphi \in C_b^2(\mathbb{R})$, the process $(M_{t \wedge \tau}(\varphi))_{t \geq 0}$ is a martingale, where

$$M_t(\varphi) := \varphi(X_t) - \int_0^t \int_A \mathcal{L}\varphi(s, X_s, a)\nu_s(da)ds.$$

Theorem 5.13. *Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, \nu, \tau, X)$ be a solution of the controlled/stopped martingale problem. Suppose that $X_{\cdot \wedge \tau}$ is continuous, τ is bounded, $\mathcal{L}(X_0) \in \mathcal{P}_q(\mathbb{R})$ and the coefficients b and σ satisfy condition (2) in Assumption 2.1. Then, on an extension of the filtered probability space, there exists a continuous martingale measure M with intensity $\nu_t(da)\mathbb{1}_{t \leq \tau}dt$ such that*

$$dX_t = \int_A b(t, X_t, a)\nu_t(da)dt + \int_A \sigma(t, X_t, a)M(dt, da), \quad t \leq \tau.$$

Moreover, there exists a Brownian motion W such that $M(t, A) = M(t \wedge \tau, A) = W_{t \wedge \tau}$. In particular, if σ is uncontrolled,

$$dX_t = \int_A b(t, X_t, a)\nu_t(da)dt + \sigma(t, X_t)dW_t, \quad t \leq \tau.$$

Proof. Using the same proof as in Lemma 3.2 of [Lac15], there exists an \mathbb{F} -predictable process $\bar{\nu}$ with values in $\mathcal{P}(A)$ such that $\bar{\nu}_t = \nu_t$ t -a.e. on $[0, T]$. In particular, $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, \bar{\nu}, \tau, X)$ is a solution of the controlled/stopped martingale problem. With some abuse of notation we denote $\bar{\nu}$ by ν . For all $u \in C_b^2(\mathbb{R})$,

$$u(X_{t \wedge \tau}) - u(X_0) - \int_0^{t \wedge \tau} \int_A \mathcal{L}u(s, X_s, a)\nu_s(da)ds$$

is an \mathbb{F} -martingale. Define $\tilde{X} := X_{\cdot \wedge \tau}$ and $q_t(da) = \nu_t(da)\mathbb{1}_{t < \tau}$. Then, for all $u \in C_b^2(\mathbb{R})$,

$$u(\tilde{X}_t) - u(\tilde{X}_0) - \int_0^t \int_A \mathcal{L}u(s, \tilde{X}_s, a)q_s(da)ds$$

is an \mathbb{F} -martingale. Moreover, since the processes $(\nu_t)_{t \in [0, T]}$ and $(\mathbb{1}_{t < \tau})_{t \in [0, T]}$ are \mathbb{F} -predictable and the map $\pi : \mathbb{R}_+ \times \mathcal{P}(A) \rightarrow \mathcal{M}^+(A)$ given by $\pi(\lambda, \nu) = \lambda\nu$ is continuous, we get that the process $(q_t)_{t \in [0, T]}$ is \mathbb{F} -predictable. By Theorem IV-2 in [KM90] (more precisely a generalization of the theorem to unbounded coefficients, which holds by a localization procedure), there exists an extension of the filtered probability space, denoted by $(\Omega', \mathcal{F}', \mathbb{F}', \mathbb{P}')$ supporting a martingale

measure M with intensity $q_t(da)dt$ such that

$$\tilde{X}_t = \tilde{X}_0 + \int_0^t \int_A b(s, \tilde{X}_s, a) q_s(da) ds + \int_0^t \int_A \sigma(s, \tilde{X}_s, a) M(ds, da), \quad t \geq 0.$$

Since $\nu_t(A) = 1$ for all $t \geq 0$, we get that $(M(t, A))_{t \geq 0}$ is a continuous square integrable martingale with quadratic variation $(t \wedge \tau)_{t \geq 0}$. Define $M_t := M(t, A)$ for $t \geq 0$ and note that since $(M_{t+\tau} - M_\tau)_{t \geq 0}$ is an $(\mathcal{F}'_{t+\tau})_{t \geq 0}$ martingale,

$$\mathbb{E}^{\mathbb{P}'} [M_{t+\tau} - M_\tau]^2 = \mathbb{E}^{\mathbb{P}'} [\langle M \rangle_{t+\tau} - \langle M \rangle_\tau] = 0, \quad t \geq 0,$$

which means that \mathbb{P}' -a.s., $M_t = M_{t \wedge \tau}$, $t \geq 0$. Consider the filtration \mathbb{G} given by $\mathcal{G}_t := \mathcal{F}'_{t \wedge \tau} \subset \mathcal{F}'_t$. By Theorem 1.7, Chapter V, in [RY99], on an extension of $(\Omega', \mathcal{F}', \mathbb{G}, \mathbb{P}')$ denoted by $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{F}}, \tilde{\mathbb{P}})$, there exists an $\tilde{\mathbb{F}}$ -Brownian motion W such that $W_{t \wedge \tau} = M_{t \wedge \tau} = M_t$, $t \geq 0$. Note that the definition of the stochastic integral

$$\int_0^t \int_A \sigma(s, \tilde{X}_s, a) M(ds, da)$$

depends on the filtration, but since \tilde{X} is \mathbb{G} -progressively measurable, we have that its extension is $\tilde{\mathbb{F}}$ -progressively measurable, and therefore the integrals in both spaces coincide. Analogously to the standard stochastic integral, the stopped integral is equal to the integral with respect to the stopped martingale measure $M^\tau := M_{\cdot \wedge \tau}$, which together with $M = M^\tau$, gives

$$\int_0^t \int_A \sigma(s, \tilde{X}_s, a) M(ds, da) = \int_0^t \int_A \sigma(s, \tilde{X}_s, a) M^\tau(ds, da) = \int_0^{t \wedge \tau} \int_A \sigma(s, X_s, a) M(ds, da).$$

We conclude that

$$X_{t \wedge \tau} = X_0 + \int_0^{t \wedge \tau} \int_A b(s, X_s, a) \nu_s(da) ds + \int_0^{t \wedge \tau} \int_A \sigma(s, X_s, a) M(ds, da), \quad t \geq 0.$$

If σ is uncontrolled, by the construction of the integral with respect to M , one can deduce that

$$\int_0^t \int_A \sigma(s, \tilde{X}_s) M(ds, da) = \int_0^t \sigma(s, \tilde{X}_s) dW_s^\tau = \int_0^{t \wedge \tau} \sigma(s, X_s) dW_s,$$

which allows us to write,

$$X_{t \wedge \tau} = X_0 + \int_0^{t \wedge \tau} \int_A b(s, X_s, a) \nu_s(da) ds + \int_0^{t \wedge \tau} \sigma(s, X_s) dW_s, \quad t \geq 0.$$

□

In the case where the relaxed control ν is replaced by some strict control α , we can also find an SDE representation with respect to a Brownian motion. This result is well known when there is no stopping time (see Remark 1.6. in [KT15]).

Theorem 5.14. *Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, \nu, \tau, X)$ be a solution of the controlled/stopped martingale problem. Suppose that $X_{\cdot \wedge \tau}$ is continuous, $\nu_t = \delta_{\alpha_t}$ a.e. for some \mathbb{F} -progressively measurable process*

α , τ is bounded, $\mathcal{L}(X_0) \in \mathcal{P}_q(\mathbb{R})$ and the coefficients b and σ satisfy condition (2) in Assumption 2.1. Then, on an extension of the filtered probability space, there exists a Brownian motion W such that

$$dX_t = b(t, X_t, \alpha_t)dt + \sigma(t, X_t, \alpha_t)dW_t, \quad t \leq \tau.$$

Proof. Adapting the proof of Theorem 3.3, Chapter 5, in [EK86] to random coefficients, we get the result for the case without stopping time. Using the same techniques as in Theorem 5.13 (i.e. introducing the stopped process $\tilde{X} := X_{\cdot \wedge \tau}$), we get the result. \square

5.4 Link between linear programming and the weak formulation

We have seen in Proposition 2.6 that to any controlled and stopped diffusion we can associate a pair $(m, \mu) \in \mathcal{R}$. In Theorem 5.20 we will prove that any $(m, \mu) \in \mathcal{R}$ can be represented in terms of a controlled and stopped diffusion. We start by a lemma stating that under some conditions the exit time of an Itô process from an open set is equal almost surely from the exit time from its closure.

Lemma 5.15. *Consider a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ supporting an \mathbb{F} -Brownian motion W . Let ξ be an \mathcal{F}_0 -measurable random variable supported in \mathcal{O} , b be an \mathbb{F} -progressively measurable process and σ be an \mathbb{F} -progressively measurable process such that σ^2 is bounded below by a constant $c_\sigma > 0$ and for all $t \geq 0$,*

$$\mathbb{E} \left[\int_0^t (b_s^2 + \sigma_s^2) ds \right] < \infty.$$

Let Y be defined by

$$Y_t = \xi + \int_0^t b_s ds + \int_0^t \sigma_s dW_s.$$

Then $\tau_{\mathcal{O}}^Y = \tau_{\mathcal{O}^c}^Y$ \mathbb{P} -a.s.

Proof. Let $B = \{\tau_{\mathcal{O}}^Y = \infty\}$, then for $\omega \in B$ we obtain $\tau_{\mathcal{O}}^Y(\omega) = \tau_{\mathcal{O}^c}^Y(\omega)$. There exists a measurable set C of probability one such that for all $\omega \in C$,

$$\int_0^t b_s(\omega)^2 ds < \infty, \quad t \in \mathbb{R}_+.$$

On the event $B^c \cap C$, by the Cauchy–Schwarz inequality,

$$\lim_{s \downarrow 0} \frac{\int_{\tau_{\mathcal{O}}^Y}^{\tau_{\mathcal{O}}^Y + s} b_r dr}{\sqrt{s}} = 0.$$

By Dambis-Dubbins-Schwarz theorem, there exists a Brownian motion \tilde{W} such that $\int_0^t \sigma_s dW_s = \tilde{W}_{\int_0^t \sigma_s^2 ds}$ for all $t \geq 0$. Finally, recall the classical result: for a Brownian motion B ,

$$\liminf_{s \downarrow 0} \frac{B_s}{\sqrt{s}} = -\infty \quad \text{and} \quad \limsup_{s \downarrow 0} \frac{B_s}{\sqrt{s}} = +\infty,$$

holds a.s., and by the strong Markov property, this result holds true at any stopping time. We denote by D the event where this result holds true at time $\tau_{\mathcal{O}}^Y$, which has probability one. Therefore, using that $0 < c_\sigma \leq \sigma_t$, on the event $B^c \cap C \cap D$,

$$\begin{aligned} \limsup_{s \downarrow 0} \frac{Y_{\tau_{\mathcal{O}}^Y+s} - Y_{\tau_{\mathcal{O}}^Y}}{\sqrt{s}} &= \limsup_{s \downarrow 0} \frac{\int_{\tau_{\mathcal{O}}^Y}^{\tau_{\mathcal{O}}^Y+s} \sigma_r dW_r}{\sqrt{s}} \\ &= \limsup_{s \downarrow 0} \frac{\widetilde{W}_{\int_0^{\tau_{\mathcal{O}}^Y+s} \sigma_r^2 dr} - \widetilde{W}_{\int_0^{\tau_{\mathcal{O}}^Y} \sigma_r^2 dr}}{\sqrt{\int_{\tau_{\mathcal{O}}^Y}^{\tau_{\mathcal{O}}^Y+s} \sigma_r^2 dr}} \sqrt{\frac{\int_{\tau_{\mathcal{O}}^Y}^{\tau_{\mathcal{O}}^Y+s} \sigma_r^2 dr}{s}} \\ &= +\infty, \end{aligned}$$

and similarly,

$$\liminf_{s \downarrow 0} \frac{Y_{\tau_{\mathcal{O}}^Y+s} - Y_{\tau_{\mathcal{O}}^Y}}{\sqrt{s}} = -\infty.$$

Together, these two results imply that on the event $B^c \cap C \cap D$, $\tau_{\mathcal{O}}^Y = \tau_{\mathcal{O}}^Y$. Since the set $B \cup (B^c \cap C \cap D)$ has probability one we conclude the proof. \square

Let us recall some definitions and results given in [Kur98] Section 2. Let E be a complete, separable metric space. Recall the notation $M_b(E)$ for the set of bounded and measurable functions from E to \mathbb{R} . Let $L \subset M_b(E) \times M_b(E)$ be the graph of an operator L (we abuse of notation as it is usual to identify an operator with its graph). Let L_S be the linear span of an operator L . The domain of an operator L is denoted by $\mathcal{D}(L)$.

Definition 5.16. *Let $L : \mathcal{D}(L) \subset M_b(E) \rightarrow M_b(E)$ be an operator and $\nu_0 \in \mathcal{P}(E)$. We say that a measurable $\mathcal{P}(E)$ -valued function (we endow $\mathcal{P}(E)$ with the Borel σ -algebra generated by the topology of weak convergence) ν on \mathbb{R}_+ is a solution of the forward equation for (L, ν_0) if for all $\phi \in \mathcal{D}(L)$ and $t \in \mathbb{R}_+$,*

$$\int_E \phi(x) \nu_t(dx) = \int_E \phi(x) \nu_0(dx) + \int_0^t \int_E L\phi(x) \nu_s(dx) ds.$$

Definition 5.17. *An operator $L \subset M_b(E) \times M_b(E)$ is dissipative if L_S is dissipative, that is, for $(f, g) \in L_S$ and $\lambda > 0$,*

$$\|\lambda f - g\|_\infty \geq \lambda \|f\|_\infty.$$

Definition 5.18. *An operator $L \subset M_b(E) \times M_b(E)$ is a pre-generator if L is dissipative and there are sequences of functions $\mu_n : E \rightarrow \mathcal{P}(E)$ and $\lambda_n : E \rightarrow [0, \infty)$ such that for each $(f, g) \in L$*

$$g(x) = \lim_{n \rightarrow \infty} \lambda_n(x) \int_E (f(y) - f(x)) \mu_n(x, dy).$$

Proposition 5.19. *If $L \subset C_b(E) \times C_b(E)$ and for each $x \in E$, there exists a solution ν^x of the forward equation for (L, δ_x) that is right-continuous (in the weak topology) at zero, then L is a pre-generator.*

Now, we will show that any $(m, \mu) \in \mathcal{R}$ has a probabilistic representation in terms of a controlled and stopped diffusion. The first part of the proof is based on the works of Stockbridge and coauthors (see e.g. [KS98c; CS02; KS17]) with adaptations to our case. The second part uses the equivalence of the stopped/controlled martingale problem and the diffusions.

Theorem 5.20. *Suppose that (1), (2) and (4) of Assumption 2.1 hold. Let $(m, \mu) \in \mathcal{R}$ and let $\nu_{t,x}(da)$ be such that*

$$m_t(dx, da)dt = \nu_{t,x}(da)m_t^x(dx)dt.$$

Then there exist a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, an \mathbb{F} -adapted process X , an \mathbb{F} -stopping time τ such that $\tau \leq T \wedge \tau_{\bar{O}}^X$ \mathbb{P} -a.s., and an \mathbb{F} -martingale measure M with intensity

$$\nu_{t,X_t}(da)\mathbf{1}_{t < \tau}dt,$$

such that

$$X_{t \wedge \tau} = X_0 + \int_0^{t \wedge \tau} \int_A b(s, X_s, a) \nu_{s, X_s}(da) ds + \int_0^{t \wedge \tau} \int_A \sigma(s, X_s, a) M(ds, da), \quad \mathbb{P} \circ X_0^{-1} = m_0^*,$$

$$m_t(B \times C) = \mathbb{E}^{\mathbb{P}} [\mathbf{1}_B(X_t) \nu_{t, X_t}(C) \mathbf{1}_{t < \tau}], \quad B \in \mathcal{B}(\bar{O}), \quad C \in \mathcal{B}(A), \quad t - a.e.,$$

$$\mu = \mathbb{P} \circ (\tau, X_\tau)^{-1}.$$

Moreover, if σ is uncontrolled or $\nu_{t,x} = \delta_{\alpha(t,x)}$ for some measurable function α , then one can replace the martingale measure by a Brownian motion.

Remark 5.21. *Assume for a moment that there is no control. Then the probabilistic representation of the measures $(m, \mu) \in \mathcal{R}$ can be done directly by stating that on the canonical space for (ξ, W, τ) (where ξ stands for the initial condition) there exists a probability measure P for which, denoting by $\mathbb{F}^{\xi, W, \tau; P}$ the completion under P of the canonical filtration*

$$(\sigma(\xi, W_s, \mathbf{1}_{\tau \leq s} : s \leq t))_{t \in [0, T]},$$

ξ is $\mathcal{F}_0^{\xi, W, \tau; P}$ -measurable, W is an $\mathbb{F}^{\xi, W, \tau; P}$ -Brownian motion, a strong solution X of the SDE can be constructed, $P(\tau \leq \tau_{\bar{O}}^X) = 1$ and a probabilistic representation of (m, μ) holds, i.e.

$$m_t(B) = \mathbb{E}^P [\mathbf{1}_B(X_t) \mathbf{1}_{t < \tau}], \quad B \in \mathcal{B}(\bar{O}), \quad t - a.e.,$$

$$\mu = P \circ (\tau, X_\tau)^{-1}.$$

Let \mathcal{W} denote the Wiener measure on $C([0, T])$. Applying Proposition 1.10 in Chapter 1 in [CD18b] we get that $P(dx, dw, d\theta) = m_0^(dx) \mathcal{W}(dw) \kappa(x, w; d\theta)$, where κ is a transition (probability) kernel from $\mathbb{R} \times C([0, T])$ to $[0, T]$ such that for all $t \in [0, T]$ and $B \in \sigma([0, s] : s \leq t)$, the mapping $(x, w) \mapsto \kappa(x, w; B)$ is $\mathcal{F}_t^{\xi, W; m_0^* \times \mathcal{W}}$ -measurable, where $\mathbb{F}^{\xi, W; m_0^* \times \mathcal{W}}$ is the completed filtration of (ξ, W) under the measure $m_0^* \times \mathcal{W}$. Moreover, the following probabilistic representation holds*

$$m_t(B) = \mathbb{E}^P [\mathbf{1}_B(X_t) \kappa(\xi, W;]t, T)], \quad B \in \mathcal{B}(\bar{O}), \quad t - a.e.,$$

$$\mu(B) = \mathbb{E}^P \left[\int_{[0,T]} \mathbf{1}_B(\theta, X_\theta) \kappa(\xi, W; d\theta) \right], \quad B \in \mathcal{B}([0, T] \times \bar{\mathcal{O}}).$$

In particular, the kernel κ is called a randomized stopping time (see e.g. [BC77; EMS82]). Another equivalent definition can be stated as in [TV02], where the randomized stopping time would be given in our setting directly by the process $(t, \omega) \mapsto \kappa(\xi(\omega), W(\omega); [0, t])$.

Proof. We divide the proof in 4 steps. The first one is the redefinition of the coefficients and measures in order to construct an operator and a measure verifying the stationary equation. The second one contains the verification of the conditions to apply Corollary 1.10 in [KS17]. In the third step we apply the latter corollary to obtain a controlled/stopped martingale problem formulation. Finally, in the fourth step, we go from the controlled/stopped martingale problem to the diffusion representation.

First step: Construction of the operator and the stationary measure. We extend $\nu_{t,x}$ onto $(\mathbb{R}_+ \times \mathbb{R}) \setminus ([0, T] \times \bar{\mathcal{O}})$ with the value δ_{a_0} for an arbitrary $a_0 \in A$. Define the coefficients $\bar{b} : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}$ and $\bar{\sigma} : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}_+$ as follows:

$$\bar{b}(t, x) = \begin{cases} \int_A b(t, x, a) \nu_{t,x}(da), & \text{if } (t, x) \in [0, T] \times \mathbb{R} \\ 0, & \text{otherwise.} \end{cases}$$

$$\bar{\sigma}(t, x) = \begin{cases} \left(\int_A \sigma^2(t, x, a) \nu_{t,x}(da) \right)^{\frac{1}{2}}, & \text{if } (t, x) \in [0, T] \times \mathbb{R} \\ 1, & \text{otherwise.} \end{cases}$$

Note that these coefficients are measurable. Define the measures

$$\tilde{\mu}_\tau(B \times C) := \mu((B \cap [0, T]) \times (C \cap \bar{\mathcal{O}})), \quad B \in \mathcal{B}(\mathbb{R}_+), \quad C \in \mathcal{B}(\mathbb{R}),$$

$$\tilde{\mu}_0(B \times C) := \int_{B \cap [0, T]} m_t((C \cap \bar{\mathcal{O}}) \times A) dt, \quad B \in \mathcal{B}(\mathbb{R}_+), \quad C \in \mathcal{B}(\mathbb{R}).$$

$$\tilde{m}_0(B) := m_0^*(B \cap \mathcal{O}), \quad B \in \mathcal{B}(\mathbb{R}).$$

This implies that $(\tilde{\mu}_\tau, \tilde{\mu}_0, \tilde{m}_0) \in \mathcal{P}(\mathbb{R}_+ \times \mathbb{R}) \times \mathcal{M}^+(\mathbb{R}_+ \times \mathbb{R}) \times \mathcal{P}(\mathbb{R})$. Define the operator

$$\hat{\mathcal{L}}(\gamma\phi)(t, x) = \gamma'(t)\phi(x) + \gamma(t) \left[\bar{b}(t, x)\phi'(x) + \frac{\bar{\sigma}^2}{2}(t, x)\phi''(x) \right],$$

for all $\gamma \in C_b^1(\mathbb{R}_+)$, $\phi \in C_b^2(\mathbb{R})$. Then, by definition of $\tilde{\mu}_\tau$, $\tilde{\mu}_0$ and \tilde{m}_0 ,

$$\int_{\mathbb{R}_+ \times \mathbb{R}} \gamma(t)\phi(x) \tilde{\mu}_\tau(dt, dx) = \gamma(0) \int_{\mathbb{R}} \phi(x) \tilde{m}_0(dx) + \int_{\mathbb{R}_+ \times \mathbb{R}} \hat{\mathcal{L}}(\gamma\phi)(t, x) \tilde{\mu}_0(dt, dx).$$

Let $U = \{0, 1\}$ and define a new operator $\bar{\mathcal{L}}$ by

$$\bar{\mathcal{L}}(\beta\gamma\phi)(r, s, x, u) = u\beta(r)\hat{\mathcal{L}}(\gamma\phi)(s, x) \\ + (1 - u) \left[\beta(0)\gamma(0) \int_{\mathbb{R}} \phi(x) \tilde{m}_0(dx) - \beta(r)\gamma(s)\phi(x) + \beta'(r)\gamma(s)\phi(x) \right],$$

where $\beta \in C_b^1(\mathbb{R}_+)$, $\gamma \in C_b^1(\mathbb{R}_+)$, $\varphi \in C_b^2(\mathbb{R})$ and $(r, s, x, u) \in \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} \times U$. We set

$$\mathcal{D}(\bar{\mathcal{L}}) = \{\beta\gamma\varphi : \beta \in C_b^1(\mathbb{R}_+), \gamma \in C_b^1(\mathbb{R}_+), \varphi \in C_b^2(\mathbb{R})\}.$$

Define $\bar{\mu} \in \mathcal{P}(\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} \times U)$ by

$$\bar{\mu}(dr, ds, dx, du) = K^{-1} [\delta_1(du)\delta_0(dr)\tilde{\mu}_0(ds, dx) + \delta_0(du)e^{-r}\mathbf{1}_{\mathbb{R}_+}(r)dr\tilde{\mu}_\tau(ds, dx)],$$

where $K = \tilde{\mu}_0(\mathbb{R}_+ \times \mathbb{R}) + 1$. The conditional distribution of u given (r, s, x) under $\bar{\mu}$ is

$$\bar{\eta}(r, s, x, du) = \delta_1(du)\mathbf{1}_{\{0\}}(r) + \delta_0(du)e^{-r}\mathbf{1}_{\mathbb{R}_+}(r).$$

As in Theorem 3.3 of [CS02], one can show that $\int \bar{\mathcal{L}}(\beta\gamma\varphi)d\bar{\mu} = 0$ for all $\beta\gamma\varphi \in \mathcal{D}(\bar{\mathcal{L}})$.

Second step: Verification of the conditions to apply Corollary 1.10 in [KS17]. Let $V = \mathbb{R} \times \mathbb{R}_+$. Define the operator

$$\mathcal{L}_0 : \mathcal{D}(\mathcal{L}_0) := \mathcal{D}(\bar{\mathcal{L}}) \subset C_b(\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}) \rightarrow C(\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} \times U \times V)$$

$$\begin{aligned} \mathcal{L}_0(\beta\gamma\varphi)(r, s, x, u, v) &= u\beta(r) \left(\gamma'(s)\varphi(x) + \gamma(s) \left[v^1\varphi'(x) + \frac{v^2}{2}\varphi''(x) \right] \right) \\ &\quad + (1-u) \left[\beta(0)\gamma(0) \int_{\mathbb{R}} \varphi(x)\tilde{m}_0(dx) - \beta(r)\gamma(s)\varphi(x) + \beta'(r)\gamma(s)\varphi(x) \right] \end{aligned}$$

where $v = (v^1, v^2) \in V$. We aim to apply Corollary 1.10 in [KS17] without singular control. Define the transition function η_0 from $\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} \times U$ to V as

$$\eta_0(r, s, x, u, dv) = \delta_{\bar{b}(s,x)}(dv^1)\delta_{\bar{\sigma}^2(s,x)}(dv^2).$$

We have,

$$\bar{\mathcal{L}}(\beta\gamma\varphi)(r, s, x, u) = \int_V \mathcal{L}_0(\beta\gamma\varphi)(r, s, x, u, v)\eta_0(r, s, x, u, dv).$$

Let $w(r, s, x, u, v) = 1 + u(|v^1| + v^2)$. We have that

$$\begin{aligned} &\int_{\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} \times U \times V} w(r, s, x, u, v)\eta_0(r, s, x, u, dv)\bar{\mu}(dr, ds, dx, du) \\ &= \int_{\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}} [1 + u(|\bar{b}(s, x)| + \bar{\sigma}^2(s, x))]\bar{\mu}(dr, ds, dx, du) \\ &= 1 + K^{-1} \int_{\mathbb{R}_+ \times \mathbb{R}} (|\bar{b}(s, x)| + \bar{\sigma}^2(s, x))\tilde{\mu}_0(ds, dx) \\ &\leq 1 + K^{-1} \int_0^T \int_{\bar{\mathcal{O}} \times A} (2 + |x| + |x|^r)\tilde{m}_t(dx, da)dt < +\infty. \end{aligned}$$

Let us check that w and \mathcal{L}_0 verify Condition 1.3 in [KS17]. Taking $\beta = \gamma = \varphi = 1$ we obtain $\mathcal{L}_0(1) = 0$. On the other hand we can verify that there exists a constant $a(\beta, \gamma, \varphi)$ such that for

all (r, s, x, u, v) ,

$$|\mathcal{L}_0(\beta\gamma\varphi)(r, s, x, u, v)| \leq a(\beta, \gamma, \varphi)w(r, s, x, u, v).$$

One can find a countable subset of $C_b^1(\mathbb{R}_+)$ approximating any function of $C_b^1(\mathbb{R}_+)$ under the point-wise convergence of β and β' (the same holds for $C_b^2(\mathbb{R})$ with the pointwise convergence of φ , φ' and φ''). Then, the controlled martingale problem associated with \mathcal{L}_0 is countably generated. Let us prove that for each $(u, v) \in U \times V$, the operator $A_{u,v}(\beta\gamma\varphi)(r, s, x) := \mathcal{L}_0(\beta\gamma\varphi)(r, s, x, u, v)$ is a pre-generator. Suppose first that $u = 1$, then

$$A_{1,v}(\beta\gamma\varphi)(r, s, x) = \beta(r) \left(\gamma'(s)\varphi(x) + \gamma(s) \left[v^1\varphi'(x) + \frac{v^2}{2}\varphi''(x) \right] \right).$$

For $z = (r, s, x) \in \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}$, define the processes $R_t^z = r$, $S_t^z = s+t$ and $X_t^z = x+v^1t+\sqrt{v^2}W_t$. For $t \geq 0$ and $z \in \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}$, define the measures

$$\nu_t^z(B \times C \times D) = \delta_{R_t^z}(B)\delta_{S_t^z}(C)\mathbb{P}(X_t^z \in D), \quad B \in \mathcal{B}(\mathbb{R}_+), \quad C \in \mathcal{B}(\mathbb{R}_+), \quad D \in \mathcal{B}(\mathbb{R}).$$

Since $\nu^z = (\nu_t^z)_{t \geq 0}$ solves the forward equation for $(A_{1,v}, \delta_z)$ and is right continuous at zero by the continuity in time of each process, we get by Proposition 5.19 that $A_{1,v}$ is a pre-generator.

Suppose now that $u = 0$, then

$$A_{0,v}(\beta\gamma\varphi)(r, s, x) = \beta(0)\gamma(0) \int_{\mathcal{O}} \varphi(x)\tilde{m}_0(dx) - \beta(r)\gamma(s)\varphi(x) + \beta'(r)\gamma(s)\varphi(x).$$

We can rewrite the operator as

$$A_{0,v}(h)(z) = \int_{\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}} (h(y) - h(z))\hat{\mu}(dy) + \partial_r h(z), \quad h \in \mathcal{D}(\mathcal{L}_0), \quad z \in \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R},$$

where

$$\hat{\mu}(dr, ds, dx) = \delta_0(dr)\delta_0(ds)\tilde{m}_0(dx).$$

By Proposition 10.2 p. 265 in [EK86], for any initial probability distribution ν on $\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}$, there exists a solution to the martingale problem for $(A_{0,v}, \nu)$ with càdlàg paths. This implies the existence of a right continuous at zero solution to the forward equation for $(A_{0,v}, \delta_z)$, for any z , which in turn entails by Proposition 5.19 that $A_{0,v}$ is a pre-generator.

Finally, the set $\mathcal{D}(\mathcal{L}_0)$ is closed under multiplication and separates points since we can use bump functions.

Third step: Controlled/stopped martingale problem representation. By Corollary 1.10 in [KS17], there exist a complete probability space $(\Omega, \mathcal{F}, \mathbb{Q})$ and a stationary $\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}$ -valued process (R, S, Y) (which we may assume is defined for all $t \in \mathbb{R}$) such that

$$\beta(R_t)\gamma(S_t)\varphi(Y_t) - \int_0^t \int_U \bar{\mathcal{L}}(\beta\gamma\varphi)(R_s, S_s, Y_s, u)\bar{\eta}(R_s, S_s, Y_s, du)ds$$

is an $(\bar{\mathcal{F}}_{t+}^{R,S,Y})_t$ -martingale for all $\beta\gamma\varphi \in \mathcal{D}(\bar{\mathcal{L}})$, where $(\bar{\mathcal{F}}_{t+}^{R,S,Y})_t$ is the complete and right continuous augmentation of the natural filtration $(\mathcal{F}_t^{R,S,Y})_t$.

Following the same proof as Theorem 3.3 in [CS02], we arrive to the existence of a complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, where \mathbb{F} satisfies the usual conditions, an \mathbb{F} -stopping time τ with values in \mathbb{R}_+ , a process \tilde{S} with values in \mathbb{R}_+ such that $\tilde{S}_t \mathbf{1}_{t \leq \tau} = t \mathbf{1}_{t \leq \tau}$, an \mathbb{F} -progressively measurable process X with values in \mathbb{R} such that $\mathbb{P} \circ X_0^{-1} = \tilde{m}_0$. Furthermore,

$$\tilde{\mu}_\tau = \mathbb{P} \circ (\tau, X_\tau)^{-1},$$

$$\tilde{\mu}_0(\Gamma) = \mathbb{E}^{\mathbb{P}} \left[\int_0^\tau \mathbf{1}_\Gamma(t, X_t) dt \right], \quad \forall \Gamma \in \mathcal{B}(\mathbb{R}_+ \times \mathbb{R}),$$

and

$$\gamma(\tilde{S}_{t \wedge \tau}) \varphi(X_{t \wedge \tau}) - \int_0^{t \wedge \tau} \hat{\mathcal{L}}(\gamma \varphi)(\tilde{S}_s, X_s) ds, \quad (3.31)$$

is an \mathbb{F} -martingale for all $\gamma \in C_b^1(\mathbb{R}_+)$, $\varphi \in C_b^2(\mathbb{R})$. Note that

$$1 = m_0^*(\mathcal{O}) = \tilde{m}_0(\mathcal{O}) = \mathbb{P}(X_0 \in \mathcal{O}),$$

which implies that $X_0 \in \mathcal{O}$ \mathbb{P} -a.s. and

$$m_0^* = \mathbb{P} \circ X_0^{-1}.$$

On the other hand, since

$$1 = \mu([0, T] \times \bar{\mathcal{O}}) = \tilde{\mu}_\tau([0, T] \times \bar{\mathcal{O}}) = \mathbb{P}(\tau \in [0, T], X_\tau \in \bar{\mathcal{O}}),$$

we conclude that $\tau \leq T$, $X_\tau \in \bar{\mathcal{O}}$ \mathbb{P} -a.s. and

$$\mu = \mathbb{P} \circ (\tau, X_\tau)^{-1}.$$

Observe also that

$$\int_\Gamma m_t(dx, A) dt = \tilde{\mu}_0(\Gamma) = \mathbb{E}^{\mathbb{P}} \left[\int_0^\tau \mathbf{1}_\Gamma(t, X_t) dt \right], \quad \forall \Gamma \in \mathcal{B}([0, T] \times \bar{\mathcal{O}}).$$

Then, for $B \in \mathcal{B}([0, T])$, $C \in \mathcal{B}(\bar{\mathcal{O}})$, $D \in \mathcal{B}(A)$,

$$\int_B \int_{C \times D} m_t(dx, da) dt = \int_{B \times C} \nu_{t,x}(D) m_t(dx, A) dt = \int_B \mathbb{E}^{\mathbb{P}} [\mathbf{1}_C(X_t) \nu_{t,X_t}(D) \mathbf{1}_{t < \tau}] dt,$$

which implies

$$m_t(B \times C) = \mathbb{E}^{\mathbb{P}} [\mathbf{1}_B(X_t) \nu_{t,X_t}(C) \mathbf{1}_{t < \tau}], \quad B \in \mathcal{B}(\bar{\mathcal{O}}), \quad C \in \mathcal{B}(A), \quad t - a.e.$$

By the definition of $\tilde{\mu}_0$ we have

$$0 = \tilde{\mu}_0(\mathbb{R}_+ \times \bar{\mathcal{O}}^c) = \mathbb{E}^{\mathbb{P}} \left[\int_0^\tau \mathbf{1}_{\bar{\mathcal{O}}^c}(X_t) dt \right],$$

implying that

$$(\mathbb{P} \otimes \lambda)(\{(\omega, t) \in \Omega \times [0, T] : X_t(\omega) \in \bar{\mathcal{O}}^c, t < \tau(\omega)\}) = 0. \quad (3.32)$$

Using that $\tilde{S}_s \mathbb{1}_{s \leq \tau} = s \mathbb{1}_{s \leq \tau}$ and taking $\gamma = 1$ in (3.31), we get that for all $\varphi \in C_b^2(\mathbb{R})$,

$$\varphi(X_{t \wedge \tau}) - \int_0^{t \wedge \tau} \hat{\mathcal{L}}(\varphi)(s, X_s) ds$$

is an \mathbb{F} -martingale. Extending b by 0 and σ by 1 for $t > T$, we obtain that for all $\varphi \in C_b^2(\mathbb{R})$,

$$\varphi(X_{t \wedge \tau}) - \varphi(X_0) - \int_0^{t \wedge \tau} \int_A \left(b(s, X_s, a) \varphi'(X_s) + \frac{\sigma^2}{2}(s, X_s, a) \varphi''(X_s) \right) \nu_{s, X_s}(da) ds$$

is an \mathbb{F} -martingale.

Fourth step: SDE representation of the controlled/stopped martingale problem. Define $\tilde{X}_t := X_{t \wedge \tau}$ for all $t \in \mathbb{R}_+$. Let us show that \tilde{X} is a continuous process. Setting

$$\hat{b}_s(\omega) := \mathbb{1}_{s \leq \tau(\omega)} \int_A b(s, X_s(\omega), a) \nu_{s, X_s(\omega)}(da)$$

$$\hat{c}_s(\omega) := \mathbb{1}_{s \leq \tau(\omega)} \int_A \sigma^2(s, X_s(\omega), a) \nu_{s, X_s(\omega)}(da),$$

we get that for all $\varphi \in C_b^2(\mathbb{R})$,

$$\varphi(\tilde{X}_t) - \varphi(\tilde{X}_0) - \int_0^t \hat{b}_s \varphi'(\tilde{X}_{s-}) ds - \int_0^t \frac{\hat{c}_s}{2} \varphi''(\tilde{X}_{s-}) ds$$

is an \mathbb{F} -martingale. We conclude by Theorem II.2.42 from [JS03] that \tilde{X} is a semimartingale with characteristics $(B, C, 0)$ where

$$B_t = \int_0^t \hat{b}_s ds, \quad C_t = \int_0^t \hat{c}_s ds.$$

This means that the compensator of the random measure defined by

$$\mu^{\tilde{X}}(\omega, dt, dx) = \sum_{s \geq 0} \mathbb{1}_{\{\Delta \tilde{X}_s(\omega) \neq 0\}} \delta_{(s, \Delta \tilde{X}_s(\omega))}(dt, dx),$$

is equal to 0 \mathbb{P} -a.s. Applying Theorem II.1.8 (i) from [JS03] with $W = 1$, we get that $\mu^{\tilde{X}}(\cdot, \mathbb{R}_+ \times \mathbb{R}) = 0$ a.s., which implies that \tilde{X} is a continuous process. Using the continuity and (3.32), we can deduce that \tilde{X} takes values in $\bar{\mathcal{O}}$.

Since $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, (\nu_{t, X_t})_{t \geq 0}, \tau, X)$ is a solution of the controlled/stopped martingale problem, by Theorem 5.13, on an extension of the filtered probability space, there exists a continuous martingale measure M with intensity $\nu_{t, X_t}(da) \mathbb{1}_{t < \tau} dt$ such that

$$dX_t = \int_A b(t, X_t, a) \nu_{t, X_t}(da) dt + \int_A \sigma(t, X_t, a) M(dt, da), \quad t \leq \tau.$$

Moreover, there exists a Brownian motion W such that $M(t, A) = M(t \wedge \tau, A) = W_{t \wedge \tau}$. In

particular, if σ is uncontrolled,

$$dX_t = \int_A b(t, X_t, a) \nu_{t, X_t}(da) dt + \sigma(t, X_t) dW_t, \quad t \leq \tau.$$

Let us prove now that $\tau \leq \tau_{\mathcal{O}}^X$ \mathbb{P} -a.s. If (4)(a) of Assumption 2.1 holds, then $\tau_{\mathcal{O}}^{\tilde{X}} \geq T$ \mathbb{P} -a.s. Since $X_t = \tilde{X}_t$ on $\{t \leq \tau\}$, we get that $\tau \leq \tau_{\mathcal{O}}^X$ \mathbb{P} -a.s. If we suppose now that (4)(b) of Assumption 2.1 holds, then since σ is uncontrolled,

$$X_{t \wedge \tau} = X_0 + \int_0^{t \wedge \tau} \int_A b(s, X_s, a) \nu_{s, X_s}(da) ds + \int_0^{t \wedge \tau} \sigma(s, X_s) dW_s.$$

We define

$$Y_t = X_0 + \int_0^t \int_A b(s, X_s, a) \nu_{s, X_s}(da) \mathbf{1}_{s \leq \tau} ds + \int_0^t \sigma(s, X_s) dW_s.$$

By Lemma 5.15 we get that $\tau_{\mathcal{O}}^Y = \tau_{\mathcal{O}}^X$ \mathbb{P} -a.s. Using that for all $t \geq 0$, $X_{t \wedge \tau} = Y_{t \wedge \tau}$ and $X_{t \wedge \tau}$ is $\bar{\mathcal{O}}$ -valued, we get that $\tau \leq \tau_{\mathcal{O}}^X$ \mathbb{P} -a.s.

The case where $\nu_{t,x} = \delta_{\alpha(t,x)}$ for some measurable function α , follows by the same arguments and replacing Theorem 5.13 by Theorem 5.14. \square

5.5 Topology of convergence in measure

Let (E, d) be a complete and separable metric space endowed with the Borel σ -algebra and let $M([0, T]; E)$ be the space of measurable functions $\phi : [0, T] \rightarrow E$ identified a.e. on $[0, T]$. The topology of convergence in measure in $M([0, T]; E)$ is defined as the topology induced by the metric (see e.g. [RS03])

$$d_M(\phi, \psi) := \int_0^T 1 \wedge d(\phi(t), \psi(t)) dt, \quad \phi, \psi \in M([0, T]; E).$$

A sequence $(\phi_n)_{n \geq 1}$ converges to ϕ in $M([0, T]; E)$ if and only if

$$\lim_{n \rightarrow \infty} \lambda(\{t \in [0, T] : d(\phi_n(t), \phi(t)) \geq \varepsilon\}) = 0, \quad \forall \varepsilon > 0.$$

We recall that convergence in $M([0, T]; E)$ implies convergence of a subsequence in E t -a.e. on $[0, T]$. The topology of convergence in measure remains invariant with respect to any metric inducing the same topology as d on E .

Lemma 5.22. *Let $M_0 \subset M([0, T]; E)$ and assume that there exists a constant $C \geq 0$ such that for all $\phi, \psi \in M_0$ we have $d(\phi(t), \psi(t)) \leq C$, t -a.e. on $[0, T]$. Then*

$$d_{M_0}(\phi, \psi) = \int_0^T d(\phi(t), \psi(t)) dt, \quad \phi, \psi \in M_0$$

metrizes the topology of convergence in measure in M_0 .

Proof. First note that for $(\phi_n)_{n \geq 1} \subset M_0$ and $\phi \in M_0$,

$$\lim_{n \rightarrow \infty} d_{M_0}(\phi_n, \phi) = 0 \Rightarrow \lim_{n \rightarrow \infty} d_M(\phi_n, \phi) = 0.$$

The converse implication follows since the sequence $(d(\phi_n(\cdot), \phi(\cdot)))_{n \geq 1}$ converges in measure to 0 and is bounded a.e. by C , which implies the convergence in $L^1([0, T])$ to 0. \square

We present now a sufficient condition for relative compactness in $M([0, T]; E)$. The following definition and theorem are taken from [RS03]. We denote by \mathcal{L} the Lebesgue measurable sets of $]0, T[$.

Definition 5.23 (Coercive normal integrands). *$H :]0, T[\times E \rightarrow [0, \infty]$ is a coercive normal integrand if*

- (1) *It is measurable with respect to $\mathcal{L} \otimes \mathcal{B}(E)$.*
- (2) *The maps $x \mapsto H_t(x) := H(t, x)$ are lower semicontinuous for a.e. $t \in]0, T[$.*
- (3) *The sets $\{x \in E : H_t(x) \leq c\}$ are compact for any $c \geq 0$ and for a.e. $t \in]0, T[$.*

The following theorem is deduced from Theorem 2 and Extension 1 in [RS03].

Theorem 5.24 (Sufficient condition for relative compactness). *A set $\mathcal{U} \subset M([0, T]; E)$ is relatively compact if there exists a coercive normal integrand H such that*

$$\sup_{\phi \in \mathcal{U}} \int_0^T H(t, \phi(t)) dt < +\infty$$

and

$$\limsup_{h \downarrow 0} \sup_{\phi \in \mathcal{U}} \int_0^{T-h} d(\phi(t+h), \phi(t)) dt = 0.$$

5.6 Technical lemmas

Lemma 5.25. *Let \mathcal{X} and \mathcal{Y} be complete separable metric spaces, and let $\varphi : \mathcal{X} \times \mathcal{Y} \rightarrow \mathbb{R}$ be bounded and continuous. Then, the map*

$$\mathcal{Y} \times \mathcal{M}^+(\mathcal{X}) \ni (y, \nu) \mapsto \int_{\mathcal{X}} \varphi(x, y) \nu(dx)$$

is continuous.

Proof. Let $y^n \rightarrow \bar{y}$ and $\nu^n \rightharpoonup \nu$, and let us prove that

$$\int_{\mathcal{X}} \varphi(x, y^n) \nu^n(dx) \xrightarrow{n \rightarrow \infty} \int_{\mathcal{X}} \varphi(x, \bar{y}) \nu(dx).$$

It suffices to show that $\nu^n \times \delta_{y^n} \rightharpoonup \nu \times \delta_{\bar{y}}$. By Remark 8.3.1 and Exercise 8.10.71 in [Bog07] (Volume 2), it is sufficient to use bounded and Lipschitz functions as test functions. Consider a bounded and Lipschitz function $\phi : \mathcal{X} \times \mathcal{Y} \rightarrow \mathbb{R}$ and denote by L the Lipschitz constant of ϕ .

We have

$$\begin{aligned} & \left| \int_{\mathcal{X} \times \mathcal{Y}} \phi(x, y) \delta_{y^n}(dy) \nu^n(dx) - \int_{\mathcal{X} \times \mathcal{Y}} \phi(x, y) \delta_{\bar{y}}(dy) \nu(dx) \right| \\ &= \left| \int_{\mathcal{X}} \phi(x, y^n) \nu^n(dx) - \int_{\mathcal{X}} \phi(x, \bar{y}) \nu(dx) \right| \\ &\leq \left| \int_{\mathcal{X}} \phi(x, y^n) \nu^n(dx) - \int_{\mathcal{X}} \phi(x, \bar{y}) \nu^n(dx) \right| + \left| \int_{\mathcal{X}} \phi(x, \bar{y}) \nu^n(dx) - \int_{\mathcal{X}} \phi(x, \bar{y}) \nu(dx) \right|. \end{aligned}$$

The second term converges to 0 since $\nu^n \rightarrow \nu$. For the first term we get

$$\left| \int_{\mathcal{X}} \phi(x, y^n) \nu^n(dx) - \int_{\mathcal{X}} \phi(x, \bar{y}) \nu^n(dx) \right| \leq L d_{\mathcal{Y}}(y^n, \bar{y}) \sup_{n \geq 1} \nu^n(\mathcal{X}),$$

which also converges to 0 since $(\nu^n(\mathcal{X}))_{n \geq 1}$ converges to $\nu(\mathcal{X})$, which gives the uniform boundedness of $(\nu^n(\mathcal{X}))_{n \geq 1}$. \square

Lemma 5.26. *Let \mathcal{X} and \mathcal{Y} be complete separable metric spaces, and let $\varphi \in C_p(\mathcal{X} \times \mathcal{Y})$. Then, the map*

$$\mathcal{Y} \times \mathcal{M}_p^+(\mathcal{X}) \ni (y, \nu) \mapsto \int_{\mathcal{X}} \varphi(x, y) \nu(dx)$$

is continuous.

Proof. Consider a sequence $(\nu^n)_{n \geq 1} \subset \mathcal{M}_p^+(\mathcal{X})$ converging to $\nu \in \mathcal{M}_p^+(\mathcal{X})$ in τ_p and a sequence $(y^n)_{n \geq 1} \in \mathcal{Y}$ converging to $\bar{y} \in \mathcal{Y}$. It suffices to show that $\mu^n := \nu^n \times \delta_{y^n} \rightarrow \nu \times \delta_{\bar{y}} =: \mu$ in τ_p . By Lemma 5.25, $\mu^n \rightarrow \mu$ weakly. Using that $\sup_{n \geq 1} d(y_n, \bar{y}) < \infty$ and the direct implication of Proposition 5.3 applied to $(\nu^n)_{n \geq 1}$ and ν , one can show that for any $x_0 \in \mathcal{X}$,

$$\lim_{r \rightarrow \infty} \limsup_{n \rightarrow \infty} \int_{\{(x, y) \in \mathcal{X} \times \mathcal{Y} : d((x, y), (x_0, \bar{y}))^p \geq r\}} d((x, y), (x_0, \bar{y}))^p \mu_n(dx) = 0.$$

By the converse implication of Proposition 5.3 we get that $(\mu^n)_{n \geq 1}$ converges to μ in τ_p . \square

Lemma 5.27 (Stable convergence: the bounded case). *Let Θ , \mathcal{X} , \mathcal{Y} be complete separable metric spaces. Let $\eta \in \mathcal{M}^+(\Theta)$. Let $\varphi : \Theta \times \mathcal{X} \times \mathcal{Y} \rightarrow \mathbb{R}$ be a bounded measurable map and assume that for every $t \in \Theta$, $\varphi(t, \cdot)$ is continuous. Suppose that a sequence of measurable functions $\psi^n : \Theta \rightarrow \mathcal{Y}$ converges η -a.e. in Θ to a measurable function $\psi : \Theta \rightarrow \mathcal{Y}$ and that $(\nu_t^n(dx) \eta(dt))_{n \geq 1} \subset \mathcal{M}^+(\Theta \times \mathcal{X})$ converges to $\nu_t(dx) \eta(dt) \in \mathcal{M}^+(\Theta \times \mathcal{X})$ in $\bar{\tau}$, where $(\nu^n)_{n \geq 1}$ and ν are transition kernels from Θ to \mathcal{X} . Suppose also that there exists a constant $C > 0$ such that η -a.e. $\sup_{n \geq 1} \nu_t^n(\mathcal{X}) \leq C$. Then,*

$$\int_{\Theta} \int_{\mathcal{X}} \varphi(t, x, \psi^n(t)) \nu_t^n(dx) \eta(dt) \xrightarrow{n \rightarrow \infty} \int_{\Theta} \int_{\mathcal{X}} \varphi(t, x, \psi(t)) \nu_t(dx) \eta(dt).$$

Proof. We need to prove that

$$\int_{\Theta \times \mathcal{X} \times \mathcal{Y}} \varphi(t, x, y) \delta_{\psi^n(t)}(dy) \nu_t^n(dx) \eta(dt) \xrightarrow{n \rightarrow \infty} \int_{\Theta \times \mathcal{X} \times \mathcal{Y}} \varphi(t, x, y) \delta_{\psi(t)}(dy) \nu_t(dx) \eta(dt).$$

It suffices to show that $\delta_{\psi^n(t)}(dy) \nu_t^n(dx) \eta(dt)$ converges to $\delta_{\psi(t)}(dy) \nu_t(dx) \eta(dt)$ in the topology

of stable convergence. We are going to use Corollary 2.9 in [JM81]. Since $\delta_{\psi^n(t)}(dy)$ has mass 1, the first condition of Corollary 2.9 in [JM81] follows by stable convergence of $\nu_t^n(dx)\eta(dt)$. Now, we need to show that $\delta_{\psi^n(t)}(dy)\nu_t^n(dx)\eta(dt) \rightarrow \delta_{\psi(t)}(dy)\nu_t(dx)\eta(dt)$. As in the previous lemma, it is sufficient to use bounded and Lipschitz functions as test functions. Moreover, we can take the metric d_Y to be bounded without loss of generality (see Lemma 3.6 p. 75 in [AB07]). Consider a bounded and Lipschitz function $\phi : \Theta \times \mathcal{X} \times \mathcal{Y} \rightarrow \mathbb{R}$ and denote by L the Lipschitz constant of ϕ . We have

$$\begin{aligned} & \left| \int_{\Theta \times \mathcal{X}} \phi(t, x, \psi^n(t)) \nu_t^n(dx) \eta(dt) - \int_{\Theta \times \mathcal{X}} \phi(t, x, \psi(t)) \nu_t(dx) \eta(dt) \right| \\ & \leq \left| \int_{\Theta \times \mathcal{X}} \phi(t, x, \psi^n(t)) \nu_t^n(dx) \eta(dt) - \int_{\Theta \times \mathcal{X}} \phi(t, x, \psi(t)) \nu_t^n(dx) \eta(dt) \right| \\ & \quad + \left| \int_{\Theta \times \mathcal{X}} \phi(t, x, \psi(t)) \nu_t^n(dx) \eta(dt) - \int_{\Theta \times \mathcal{X}} \phi(t, x, \psi(t)) \nu_t(dx) \eta(dt) \right|. \end{aligned}$$

The second term converges to 0 since $\nu_t^n(dx)\eta(dt)$ converges to $\nu_t(dx)\eta(dt)$ in the stable topology. For the first term we get

$$\left| \int_{\Theta \times \mathcal{X}} \phi(t, x, \psi^n(t)) \nu_t^n(dx) \eta(dt) - \int_{\Theta \times \mathcal{X}} \phi(t, x, \psi(t)) \nu_t^n(dx) \eta(dt) \right| \leq CL \int_{\Theta} d_Y(\psi^n(t), \psi(t)) \eta(dt),$$

which, by the dominated convergence theorem also converges to 0. \square

The next Lemma is related to Lemma A.3 of [Lac15].

Lemma 5.28 (Stable convergence: the p -growth case). *Let Θ , \mathcal{X} , \mathcal{Y} be complete, separable metric spaces. Let $\eta \in \mathcal{M}_p^+(\Theta)$. Let $\varphi : \Theta \times \mathcal{X} \times \mathcal{Y} \rightarrow \mathbb{R}$, be a measurable map and assume that for every $t \in \Theta$, $\varphi(t, \cdot)$ is continuous. We assume the following growth condition on φ : there exist $c \geq 0$ and $(t_0, x_0, y_0) \in \Theta \times \mathcal{X} \times \mathcal{Y}$*

$$|\varphi(t, x, y)| \leq c(1 + d_{\Theta}(t, t_0)^p + d_{\mathcal{X}}(x, x_0)^p + d_{\mathcal{Y}}(y, y_0)^p).$$

Suppose that a sequence of measurable functions $\psi^n : \Theta \rightarrow \mathcal{Y}$ converges η -a.e. in Θ to a measurable function $\psi : \Theta \rightarrow \mathcal{Y}$ and that $(\nu_t^n(dx)\eta(dt))_{n \geq 1} \subset \mathcal{M}_p^+(\Theta \times \mathcal{X})$ converges to $\nu_t(dx)\eta(dt) \in \mathcal{M}_p^+(\Theta \times \mathcal{X})$ in $\bar{\tau}_p$, where $(\nu^n)_{n \geq 1}$ and ν are transition kernels from Θ to \mathcal{X} . Suppose also that there exists a constant $C > 0$ such that η -a.e.

$$\sup_{n \geq 1} \int_{\mathcal{X}} (1 + d_{\mathcal{X}}(x, x_0)^p) \nu_t^n(dx) + \sup_{n \geq 1} d_{\mathcal{Y}}(\psi^n(t), y_0) \leq C.$$

Then,

$$\int_{\Theta} \int_{\mathcal{X}} \varphi(t, x, \psi^n(t)) \nu_t^n(dx) \eta(dt) \xrightarrow{n \rightarrow \infty} \int_{\Theta} \int_{\mathcal{X}} \varphi(t, x, \psi(t)) \nu_t(dx) \eta(dt).$$

Proof. We need to prove that

$$\int_{\Theta \times \mathcal{X} \times \mathcal{Y}} \varphi(t, x, y) \delta_{\psi^n(t)}(dy) \nu_t^n(dx) \eta(dt) \xrightarrow{n \rightarrow \infty} \int_{\Theta \times \mathcal{X} \times \mathcal{Y}} \varphi(t, x, y) \delta_{\psi(t)}(dy) \nu_t(dx) \eta(dt).$$

It suffices to show that

$$\mu^n(dt, dx, dy) := \delta_{\psi^n(t)}(dy)\nu_t^n(dx)\eta(dt)$$

converges to

$$\mu(dt, dx, dy) := \delta_{\psi(t)}(dy)\nu_t(dx)\eta(dt)$$

in $\bar{\tau}_p$. For proving this convergence we will use Proposition 5.7. Consider the weight function

$$w(t, x, y) = 1 + d_\Theta(t, t_0)^p + d_{\mathcal{X}}(x, x_0)^p + d_{\mathcal{Y}}(y, y_0)^p.$$

Let $\phi \in M_b(\Theta)$. We have

$$\begin{aligned} \int_{\Theta \times \mathcal{X} \times \mathcal{Y}} \phi(t)w(t, x, y)\mu^n(dt, dx, dy) &= \int_{\Theta \times \mathcal{X}} \phi(t)(1 + d_\Theta(t, t_0)^p + d_{\mathcal{X}}(x, x_0)^p)\nu_t^n(dx)\eta(dt) \\ &\quad + \int_{\Theta \times \mathcal{X}} \phi(t)d_{\mathcal{Y}}(\psi^n(t), y_0)^p\nu_t^n(dx)\eta(dt) \end{aligned}$$

The first integral converges to

$$\int_{\Theta \times \mathcal{X}} \phi(t)(1 + d_\Theta(t, t_0)^p + d_{\mathcal{X}}(x, x_0)^p)\nu_t(dx)\eta(dt)$$

since $(\nu_t^n(dx)\eta(dt))_{n \geq 1}$ converges to $\nu_t(dx)\eta(dt)$ in $\bar{\tau}_p$. Now, using that η -a.e.

$$\sup_{n \geq 1} d_{\mathcal{Y}}(\psi^n(t), y_0) < \infty,$$

and Lemma 5.27, the second integral converges to

$$\int_{\Theta \times \mathcal{X}} \phi(t)d_{\mathcal{Y}}(\psi(t), y_0)^p\nu_t(dx)\eta(dt).$$

We conclude that

$$\lim_{n \rightarrow \infty} \int_{\Theta \times \mathcal{X} \times \mathcal{Y}} \phi(t)w(t, x, y)\mu^n(dt, dx, dy) = \int_{\Theta \times \mathcal{X} \times \mathcal{Y}} \phi(t)w(t, x, y)\mu(dt, dx, dy).$$

In particular, the set of marginal measures

$$\left\{ \int_{\mathcal{X} \times \mathcal{Y}} w(t, x, y)\mu^n(dt, dx, dy) : n \geq 1 \right\} \subset \mathcal{M}^+(\Theta)$$

is relatively compact in $(\mathcal{M}^+(\Theta), \sigma(\mathcal{M}^+(\Theta), M_b(\Theta)))$. Now, we need to show that $\mu^n \rightarrow \mu$ in τ_p . By Lemma 5.27, $\mu^n \rightarrow \mu$ in $\bar{\tau}$ and in particular in the topology of weak convergence. Using that $(\nu_t^n(dx)\eta(dt))_{n \geq 1}$ converges to $\nu_t(dx)\eta(dt)$ in $\bar{\tau}_p$, and in particular converges in τ_p , we get that this sequence satisfies the uniform integrability condition (3.30) in Proposition 5.3. From this and the fact that η -a.e.

$$\sup_{n \geq 1} d_{\mathcal{Y}}(\psi^n(t), y_0) < \infty,$$

we can deduce that the sequence $(\mu^n)_{n \geq 1}$ verifies also the uniform integrability condition (3.30)

and henceforth, by Proposition 5.3, it converges in τ_p to μ . We conclude by Proposition 5.7 that the sequence converges in $\bar{\tau}_p$. \square

5.7 Some results on set-valued analysis

Let us recall some theory about set-valued analysis, which can be found in Chapter 17 of [AB07]. For the next definitions, consider a metric space (X, d) and a set valued map $\varphi : X \rightarrow 2^X$. The graph of φ is defined as the following set:

$$\text{Gr}(\varphi) := \{(x, y) \in X^2 : y \in \varphi(x)\}.$$

Definition 5.29. *The correspondence φ is said to be upper hemicontinuous if for any sequence $(x_n, y_n)_{n \geq 1}$ in the graph of φ such that $x_n \rightarrow x$, the sequence $(y_n)_{n \geq 1}$ has a limit point in $\varphi(x)$.*

Theorem 5.30 (Closed Graph Theorem, Theorem 17.11 in [AB07]). *If X is compact, the following statements are equivalent:*

- (i) $\varphi(x)$ is closed for all $x \in X$ and φ is upper hemicontinuous.
- (ii) The graph of φ is closed.

Definition 5.31. *The correspondence φ is said to be lower hemicontinuous if whenever $x_n \rightarrow x$ and $y \in \varphi(x)$, there exists a subsequence $(x_{n_k})_{k \geq 1}$ of $(x_n)_{n \geq 1}$ and a sequence $(y_k)_{k \geq 1}$, such that $y_k \in \varphi(x_{n_k})$ and $y_k \rightarrow y$.*

Definition 5.32. *We say that φ is continuous if it is both upper hemicontinuous and lower hemicontinuous.*

Theorem 5.33 (Berge's Maximum Theorem, Theorem 17.31 in [AB07]). *Let (X, d) be a metric space. Consider $\mathcal{R}^* : X \rightarrow 2^X$ a continuous correspondence with nonempty compact values and $F : \text{Gr}(\mathcal{R}^*) \rightarrow \mathbb{R}$ a continuous function. Define the function $\Theta : X \rightarrow 2^X$ by*

$$\Theta(x) = \arg \max_{y \in \mathcal{R}^*(x)} F(x, y).$$

Then Θ is upper hemicontinuous and has nonempty compact values.

Theorem 5.34 (Kakutani-Fan-Glicksberg, Corollary 17.55 in [AB07]). *Let K be a nonempty compact convex subset of a locally convex Hausdorff space, and let the correspondence $\Theta : K \rightarrow 2^K$ have closed graph and nonempty convex values. Then the set of fixed points of Θ is compact and nonempty.*

5.8 Sufficient condition for the existence of a square integrable density for $m_t^x(dx)dt$

Proposition 5.35. *Suppose that Assumption 4.7 (1-6) holds true. Moreover, assume that σ^2 is Lipschitz continuous on $[0, T] \times \bar{\mathcal{O}}$ and m_0^* has a bounded density with respect to the Lebesgue measure. If $(m, \mu) \in \mathcal{R}$, then $m_t^x(dx)dt$ admits a square integrable density with respect to the Lebesgue measure on $[0, T] \times \bar{\mathcal{O}}$.*

Proof. We set $\eta(dt, dx) = m_t^x(dx)dt$ and let $\nu_{t,x}(da)$ be such that

$$m_t(dx, da)dt = \nu_{t,x}(da)m_t^x(dx)dt.$$

By Theorem 5.20, there exist a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, an \mathbb{F} -adapted process X , an \mathbb{F} -stopping time τ such that $\tau \leq T \wedge \tau_{\bar{\mathcal{O}}}^X$ \mathbb{P} -a.s., and an \mathbb{F} -Brownian motion W , such that

$$X_{t \wedge \tau} = \int_0^{t \wedge \tau} \int_A b(t, X_t, a) \nu_{t, X_t}(da) dt + \int_0^{t \wedge \tau} \sigma(t, X_t) dW_t, \quad \mathbb{P} \circ X_0^{-1} = m_0^*,$$

$$m_t(B \times C) = \mathbb{E}^{\mathbb{P}} [\mathbf{1}_B(X_t) \nu_{t, X_t}(C) \mathbf{1}_{t < \tau}], \quad B \in \mathcal{B}(\bar{\mathcal{O}}), \quad C \in \mathcal{B}(A), \quad t - a.e.$$

We can rewrite η ,

$$\eta(B \times C) = \int_0^T \mathbb{E}^{\mathbb{P}} [\mathbf{1}_B(t) \mathbf{1}_C(X_t) \mathbf{1}_{t < \tau}] dt, \quad B \in \mathcal{B}([0, T]), \quad C \in \mathcal{B}(\bar{\mathcal{O}}).$$

Since $\tau \leq \tau_{\bar{\mathcal{O}}}^X$, we get that

$$(\lambda \times \mathbb{P})(\{(t, \omega) : X_t(\omega) \in \partial \mathcal{O}, t \leq \tau(\omega)\}) = 0,$$

which means that η puts 0 mass on $\partial \mathcal{O}$ and can thus be treated as a measure on $[0, T] \times \mathcal{O}$. By standard arguments of existence of strong solutions to SDEs, there exists a unique process Y such that

$$Y_t = \begin{cases} X_t & \text{if } t \leq \tau \\ X_\tau + \int_\tau^t \sigma(s, Y_s) dW_s & \text{if } t > (\tau, T]. \end{cases}$$

Let

$$\bar{b}_s := \int_A b(s, X_s, a) \nu_{s, X_s}(da) \mathbf{1}_{s < \tau},$$

and rewrite Y as

$$Y_t = Y_0 + \int_0^t \bar{b}_s ds + \int_0^t \sigma(s, Y_s) dW_s, \quad s \in [0, T].$$

Define

$$\lambda_s = \frac{\bar{b}_s}{\sigma(s, X_s)},$$

$$Z_t = \exp \left[- \int_0^t \lambda_s dW_s - \frac{1}{2} \int_0^t \lambda_s^2 ds \right].$$

Since λ is a bounded process, by Girsanov's Theorem, under \mathbb{Q} ,

$$\widetilde{W}_t = W_t + \int_0^t \lambda_s ds, \quad t \in [0, T],$$

is an (\mathbb{F}, \mathbb{Q}) -Brownian motion, where

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = Z_T.$$

The dynamics of Y under \mathbb{Q} are as follows

$$Y_t = Y_0 + \int_0^t \sigma(s, Y_s) d\widetilde{W}_s, \quad t \in [0, T].$$

By Remark 5.1 in [BDT20],

$$\tilde{\eta}(B) = \mathbb{E}^{\mathbb{Q}} \left[\int_0^T \mathbf{1}_B(t, Y_t) \mathbf{1}_{t < \tau_{\mathcal{O}}^Y} dt \right], \quad B \in \mathcal{B}([0, T] \times \mathcal{O}),$$

admits a bounded density $(t, x) \mapsto \tilde{\eta}(t, x)$ with respect to the Lebesgue measure on $[0, T] \times \mathcal{O}$, i.e. $\tilde{\eta}(dt, dx) = \tilde{\eta}(t, x) dt dx$. Letting $B \in \mathcal{B}([0, T] \times \mathcal{O})$, we have

$$\begin{aligned} \eta(B) &= \int_0^T \mathbb{E}^{\mathbb{P}} [\mathbf{1}_B(t, X_t) \mathbf{1}_{t < \tau}] dt \\ &\leq \int_0^T \mathbb{E}^{\mathbb{P}} [\mathbf{1}_B(t, Y_t) \mathbf{1}_{t < \tau_{\mathcal{O}}^Y}] dt \\ &= \int_0^T \mathbb{E}^{\mathbb{Q}} [\mathbf{1}_B(t, Y_t) \mathbf{1}_{t < \tau_{\mathcal{O}}^Y} Z_T^{-1}] dt \\ &\leq \left(T \mathbb{E}^{\mathbb{Q}} [Z_T^{-2}] \right)^{1/2} \left(\int_0^T \mathbb{E}^{\mathbb{Q}} [\mathbf{1}_B(t, Y_t) \mathbf{1}_{t < \tau_{\mathcal{O}}^Y}] dt \right)^{1/2} \\ &= C (\tilde{\eta}(B))^{1/2}, \end{aligned}$$

where $C = \left(T \mathbb{E}^{\mathbb{Q}} [Z_T^{-2}] \right)^{1/2}$. This allows us to deduce that $\eta(dt, dx) = \eta(t, x) dt dx$ for some nonnegative L^1 function η . Let us show that $\eta \in L^2$. For $n \geq 1$ define $\eta_n = \eta \wedge n$ and compute

$$\begin{aligned} \int_0^T \int_{\mathcal{O}} \eta_n(t, x) \eta(t, x) dt dx &= \int_0^T \mathbb{E}^{\mathbb{P}} [\eta_n(t, X_t) \mathbf{1}_{t < \tau}] dt \\ &\leq \int_0^T \mathbb{E}^{\mathbb{P}} [\eta_n(t, Y_t) \mathbf{1}_{t < \tau_{\mathcal{O}}^Y}] dt \\ &= \int_0^T \mathbb{E}^{\mathbb{Q}} [\eta_n(t, Y_t) \mathbf{1}_{t < \tau_{\mathcal{O}}^Y} Z_T^{-1}] dt \\ &\leq \left(T \mathbb{E}^{\mathbb{Q}} [Z_T^{-2}] \right)^{1/2} \left(\int_0^T \mathbb{E}^{\mathbb{Q}} [\eta_n^2(t, Y_t) \mathbf{1}_{t < \tau_{\mathcal{O}}^Y}] dt \right)^{1/2} \\ &= C \left(\int_0^T \int_{\mathcal{O}} \eta_n^2(t, x) \tilde{\eta}(t, x) dt dx \right)^{1/2} \\ &\leq C' \left(\int_0^T \int_{\mathcal{O}} \eta_n(t, x) \eta(t, x) dt dx \right)^{1/2}, \end{aligned}$$

where $C' = C \|\tilde{\eta}\|_{\infty}^{1/2}$ and we used that $\eta_n \leq \eta$. Thus,

$$\int_0^T \int_{\mathcal{O}} \eta_n(t, x) \eta(t, x) dt dx \leq (C')^2,$$

and by the monotone convergence theorem, we conclude that the density η is in L^2 . \square

5.9 Proof of Theorem 4.11

Proof. (1) We define $\tau_{\mathcal{O}}^{x,\alpha} := \inf \{t \geq 0 : X_t^{x,\alpha} \notin \mathcal{O}\}$. By Theorem 4.9, for all $x \in \mathcal{O}$,

$$v(0, x) = \mathbb{E} \left[\int_0^{\tau^*(x)} f \left(s, X_s^{x,\alpha^*(x)}, \alpha_s^*(x) \right) ds + g \left(\tau^*(x), X_{\tau^*(x)}^{x,\alpha^*(x)} \right) \right]. \quad (3.33)$$

Note that, by definition of the value function v , we have $\tau^*(x) \leq \tau_{\mathcal{O}}^{x,\alpha^*} \wedge T$ a.s. Now consider the measures defined by

$$\bar{m}_t(B) = \int_{\mathcal{O}} \mathbb{P} \left[\left(X_t^{x,\alpha^*(x)}, \alpha_t^*(x) \right) \in B, t < \tau^*(x) \right] m_0^*(dx), \quad B \in \mathcal{B}(\bar{\mathcal{O}} \times A),$$

$$\bar{\mu}(B) = \int_{\mathcal{O}} \mathbb{P} \left[\left(\tau^*(x), X_{\tau^*(x)}^{x,\alpha^*(x)} \right) \in B \right] m_0^*(dx), \quad B \in \mathcal{B}([0, T] \times \bar{\mathcal{O}}).$$

Integrating with respect to m_0^* in (3.33), we derive that

$$V^S = \Gamma(\bar{m}, \bar{\mu})$$

Since $(\bar{m}, \bar{\mu}) \in \mathcal{R}$, we conclude that $V^S \leq V^{LP}$.

We now prove that the converse inequality holds. Fix $(m, \mu) \in \mathcal{R}$. Since $v \in W^{1,2,2}((0, T) \times \mathcal{O})$, there exists a sequence $(u_n)_{n \geq 1} \subset C_b^{1,2}([0, T] \times \bar{\mathcal{O}})$ such that $u_n \rightarrow v$ in $W^{1,2,2}((0, T) \times \mathcal{O}) \cap C([0, T] \times \bar{\mathcal{O}})$. By condition (7) in Assumption 4.7 we get that $m_t^x(dx)dt$ has a square integrable density with respect to the Lebesgue measure. In particular we can change the set $\bar{\mathcal{O}}$ by \mathcal{O} in the integrals with respect to m . Passing to the limit $n \rightarrow \infty$ in the equality

$$\langle \mu, u_n \rangle = \langle m_0^*, u_n(0, \cdot) \rangle + \langle m, \partial_t u_n + \mathcal{L}u_n \rangle,$$

we get

$$\int_{[0, T] \times \bar{\mathcal{O}}} v(t, x) \mu(dt, dx) = \int_{\mathcal{O}} v(0, x) m_0^*(dx) + \int_0^T \int_{\mathcal{O} \times A} \left(\frac{\partial v}{\partial t} + \mathcal{L}v \right) (t, x, a) m_t(dx, da) dt. \quad (3.34)$$

From the above equality, we derive that

$$V^S = \int_{[0, T] \times \bar{\mathcal{O}}} v(t, x) \mu(dt, dx) - \int_0^T \int_{\mathcal{O} \times A} \left(\frac{\partial v}{\partial t} + \mathcal{L}v \right) (t, x, a) m_t(dx, da) dt. \quad (3.35)$$

Now using the HJBVI (3.16), we get $V^S \geq V^{LP}$.

(2) Let (m^*, μ^*) be a maximizer of the LP program. As before, $m_t^{x,*}(dx)dt$ admits a square integrable density with respect to the Lebesgue measure.

(a) By (1) we know that $V^{LP} = V^S$, that is

$$\int_0^T \int_{\mathcal{O} \times A} f(t, x, a) m_t^*(dx, da) dt + \int_{[0, T] \times \bar{\mathcal{O}}} g(t, x) \mu^*(dt, dx) = \int_{\mathcal{O}} v(0, x) m_0^*(dx).$$

Since $g \in C_b^{1,2}([0, T] \times \bar{\mathcal{O}})$,

$$\int_{[0, T] \times \bar{\mathcal{O}}} g(t, x) \mu^*(dt, dx) = \int_{\mathcal{O}} g(0, x) m_0^*(dx) + \int_0^T \int_{\mathcal{O} \times A} \left(\frac{\partial g}{\partial t} + \mathcal{L}g \right) (t, x, a) m_t^*(dx, da) dt. \quad (3.36)$$

Therefore, using the last two equalities

$$\begin{aligned} & \int_{\mathcal{S} \times A} \left(f + \frac{\partial g}{\partial t} + \mathcal{L}g \right) (t, x, a) m_t^*(dx, da) dt = \int_{\mathcal{O}} (v - g)(0, x) m_0^*(dx) \\ & - \int_{\mathcal{C} \times A} \left(f + \frac{\partial g}{\partial t} + \mathcal{L}g \right) (t, x, a) m_t^*(dx, da) dt \\ & \geq \int_{\mathcal{O}} (v - g)(0, x) m_0^*(dx) + \int_{\mathcal{C} \times A} \left(\frac{\partial(v - g)}{\partial t} + \mathcal{L}(v - g) \right) (t, x, a) m_t^*(dx, da) dt \\ & = \int_{\mathcal{O}} (v - g)(0, x) m_0^*(dx) + \int_0^T \int_{\mathcal{O} \times A} \left(\frac{\partial(v - g)}{\partial t} + \mathcal{L}(v - g) \right) (t, x, a) m_t^*(dx, da) dt. \end{aligned} \quad (3.37)$$

The inequality follows from the HJBVI (3.16) and the last equality follows from the fact that for all $a \in A$,

$$\left(\frac{\partial(v - g)}{\partial t} + \mathcal{L}(v - g) \right) (t, x, a) = 0, \quad \text{a.e. on } \mathcal{S}.$$

By (3.34) applied to (m^*, μ^*) and (3.36), we obtain

$$\int_{\mathcal{S} \times A} \left(f + \frac{\partial g}{\partial t} + \mathcal{L}g \right) (t, x, a) m_t^*(dx, da) dt = \int_{[0, T] \times \bar{\mathcal{O}}} (v - g)(t, x) \mu^*(dt, dx) \geq 0.$$

Finally, since for all $a \in A$,

$$\left(f + \frac{\partial g}{\partial t} + \mathcal{L}g \right) (t, x, a) \leq 0 \quad \text{a.e. on } \mathcal{S},$$

we conclude that

$$\int_{\mathcal{S} \times A} \left(f + \frac{\partial g}{\partial t} + \mathcal{L}g \right) (t, x, a) m_t^*(dx, da) dt = 0.$$

(b) The inequality in (3.37) is now an equality, so we have

$$\begin{aligned} & - \int_{\mathcal{C} \times A} f(t, x, a) m_t^*(dx, da) dt = \int_{\mathcal{C} \times A} \left(\frac{\partial v}{\partial t} + \mathcal{L}v \right) (t, x, a) m_t^*(dx, da) dt, \\ & \int_{\mathcal{O}} (v - g)(0, x) m_0^*(dx) + \int_0^T \int_{\mathcal{O} \times A} \left(\frac{\partial(v - g)}{\partial t} + \mathcal{L}(v - g) \right) (t, x, a) m_t^*(dx, da) dt = 0. \end{aligned}$$

Since

$$\int_{[0, T] \times \bar{\mathcal{O}}} g(t, x) \mu^*(dt, dx) = \int_{\mathcal{S}} v(t, x) \mu^*(dt, dx) + \int_{\mathcal{C}} g(t, x) \mu^*(dt, dx) + \int_{[0, T] \times \partial \mathcal{O}} v(t, x) \mu^*(dt, dx),$$

we get

$$\int_{\mathcal{C}} (v - g)(t, x) \mu^*(dt, dx) = 0.$$

We conclude that $\mu^*(\mathcal{C}) = 0$.

(c) The result follows since $\mu^*(\mathcal{C}) = 0$.

□

Chapter 4

Linear programming fictitious play algorithm for mean field games with optimal stopping and absorption

Abstract

We develop the *fictitious play* algorithm in the context of the linear programming approach for mean field games of optimal stopping and mean field games with regular control and absorption. This algorithm allows to approximate the mean field game population dynamics without computing the value function by solving linear programming problems associated with the distributions of the players still in the game and their stopping times/controls. We show the convergence of the algorithm using the topology of convergence in measure in the space of subprobability measures, which is needed to deal with the lack of continuity of the flows of measures. Numerical examples are provided to illustrate the convergence of the algorithm.

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1 Introduction

The goal of this paper is to develop a numerical algorithm for computing Nash equilibria in Mean field games of optimal stopping and Mean field games with regular control and absorption.

Mean Field Games (MFGs) are useful for approximating N -player Nash equilibria, which are rarely tractable. They have been introduced at about the same time by Lasry and Lions [LL06a; LL06b; LL07] and Huang, Malhamé and Caines [HMC06] as limit version of games with a large number of agents, symmetric interactions and negligible individual influence of each player on the others. In the literature, several approaches have been developed to prove existence of an MFG Nash equilibrium. The analytic approach, introduced by Lasry and Lions and Huang, Malhamé and Caines, boils down to solving a coupled system of nonlinear partial differential equations: a Hamilton-Jacobi-Bellman equation (backward in time) satisfied by the value function of the representative agent and a Fokker-Planck-Kolmogorov equation (forward in time) describing the evolution of the density of agents when the optimal control is used. The probabilistic approach, introduced by Carmona and Delarue, based on the stochastic maximum principle, reduces the problem to a system of coupled forward-backward stochastic differential equations of McKean-Vlasov type. Finally, the compactification methods consist in relaxing the optimization problems, and often allow to prove existence under weaker assumptions than the first two approaches. We refer here to the controlled martingale problem approach, introduced in the MFG setting by Lacker [Lac15] and the linear programming approach, developed for MFGs of optimal stopping in [BDT20] and extended to a more general framework in [DLT21].

Fast numerical methods for computing Nash equilibria are very important for applications. Several algorithms exist in the literature for the case of regular control without absorption. These use either analytic or probabilistic approach and iterate over the value function and the mass distribution (see e.g. [AC10; AL20; CCD19; Ang+19]). Another method, based on the fictitious play algorithm which goes back to Brown [Bro51] in the classical setting of game theory, has been introduced in the Mean-field framework in [CH17]. The fictitious play algorithm is a learning procedure, very natural in this setting. Due to the complexity of the game, it is unrealistic to assume that the agents can actually compute the equilibrium configuration. Such a configuration may only arise if the players learn how to play the game. The fictitious play for MFGs has been applied in the settings of potential MFGs with regular controls in [CH17] and first order MFGs with regular controls in [Had17; HS19; Eli+20]. A continuous time version of the fictitious play has been studied in [Per+20] for finite MFGs with common noise.

The compactification methods based on the controlled martingale problem approach, although they simplify the proofs of existence, are very abstract and cannot be exploited for the development of numerical algorithms. On the contrary, the linear programming formulation seems appropriate to build numerical schemes, see e.g. [MS98; CS02].

In this paper we develop and study the *linear programming fictitious play algorithm* (LPFP) for mean field games of optimal stopping and regular control with absorption, in the case of second-order possibly non-potential games, under general assumptions on the coefficients and a non-strict monotonicity condition on the reward function with respect to the measure.

The LPFP algorithm starts with an initial guess of the equilibrium and iterates the following two steps:

- (i) Compute the best response corresponding to the guess by solving a linear program.
- (ii) Update the guess via a convex combination (with well chosen parameters) of the previous guess and the best response computed in Step (i).

While the linear program is infinite dimensional, one can approximate it by a finite dimensional linear program (see e.g. [MS98; CS02]), for which fast and accurate algorithms exist in most computing environments. We provide several numerical examples, which illustrate the convergence of the algorithm.

We emphasize that unlike other algorithms which iterate over both, the value function and the distribution of agents, the LPFP algorithm iterates only over the distribution of agents, which is the main object of interest in the mean field game setting.

Very few papers present numerical algorithms for Mean field games of optimal stopping and discuss their convergence. [BDT20] prove the convergence of fictitious play for potential games and [Ber20] studies the Uzawa algorithm for the (possibly non-potential) MFG system introduced in [Ber17] in the stationary case, under the assumption of strict monotonicity of the reward map. This LPFP algorithm has already been used for applications to water management and electricity markets in [BDT22] and [ADT21], respectively, but no theoretical convergence results have been provided. In the recent paper [Dia+22], the authors study the class of submodular MFGs. In particular, in the case of optimal stopping, they prove the existence of equilibria using Tarski's fixed point theorem and provide an approximation of the minimal equilibria (with respect to a specific order structure) by starting with the minimal measure flow and iterating the minimal best response. The results hold under the opposite inequality to the Lasry-Lions monotonicity condition used in this paper. The theory on mean field games with regular control and absorption has received a lot of interest recently, being developed in [CF18; CGL21; BC21], and using the linear programming approach in [DLT21]. To the best of our knowledge, no algorithm has been proposed so far in this setting and one of our goals is to fill this gap.

One of the principal difficulties of MFGs with stopping/absorption is that the standard weak convergence topology for measures cannot be used due to the possible lack of regularity of the flow of measures. In this paper, we solve this problem and provide general convergence results for LPFP, by finding an appropriate topology, i.e. the topology of the convergence in measure in infinite-dimensional spaces and providing appropriate estimates using well chosen metrics. Furthermore, the use of this topology allows us to prove the existence of an equilibria in a much more general framework compared to [BDT20] and [DLT21].

The paper is organized as follows. In Section 2, we introduce the LPFP algorithm for MFGs of optimal stopping. We study the compactness under the convergence in measure topology of the set of admissible measures and provide several key estimates, which are used to prove the convergence of the algorithm. Numerical illustrations are provided. In Section 3, we propose an LPFP algorithm in the case of regular control with absorption, and show its convergence by using the tools developed in Section 2. In the Section 4 we give some technical results and some examples of sufficient conditions under which the main results hold.

Notation. Let $T > 0$ be a terminal time horizon, \mathcal{O} be an open subset of \mathbb{R} with closure $\bar{\mathcal{O}}$ and A be a compact subset of \mathbb{R} . We denote by $C_b^{1,2}([0, T] \times \bar{\mathcal{O}})$ the set of functions $u \in C_b([0, T] \times \bar{\mathcal{O}})$ such that $\partial_t u, \partial_x u, \partial_{xx} u \in C_b([0, T] \times \bar{\mathcal{O}})$. For a given process $(Y_t)_t$ and a Borel subset B of \mathbb{R} , we define the random time $\tau_B^Y(\omega) := \inf\{t \geq 0 : Y_t(\omega) \notin B\}$, with the convention $\inf \emptyset = \infty$.

Since we will deal with different topologies and distances throughout the paper, we list them here, making reference to the places where they are introduced: the topologies τ (weak convergence), τ_p (weak convergence with p -growth), $\bar{\tau}$ (stable convergence), $\bar{\tau}_p$ (stable convergence with p -growth) and the metric d_{BL} are introduced in Section 2.3 of Chapter 1 and Section 5.1 of Chapter 3. The topology $\tilde{\tau}_p$ of the convergence in measure for flows of subprobabilities in \mathcal{P}_p^{sub} is defined just before Assumption 2.1. The metric W'_1 on \mathcal{P}_1^{sub} is defined just before Proposition 2.14 and the metrics d_M and ρ are defined in Proposition 2.14.

2 Linear programming algorithm for *Optimal stopping MFGs*

2.1 Preliminaries and main result

We describe here the LPFP algorithm for MFGs of optimal stopping, i.e. when players choose the time to exit the game. We present the definition of LP (Linear Programming) MFG Nash equilibrium in this setting and prove the convergence of the LPFP algorithm to the LP MFG Nash equilibrium.

Preliminaries. Let U be the set of flows of measures on $\bar{\mathcal{O}}$, $(m_t)_{t \in [0, T]}$, such that: for every $t \in [0, T]$, m_t is a Borel finite signed measure on $\bar{\mathcal{O}}$, for every $B \in \mathcal{B}(\bar{\mathcal{O}})$, the mapping $t \mapsto m_t(B)$ is measurable, and $\int_0^T |m_t|(\bar{\mathcal{O}}) dt < \infty$, where $|m_t|$ is the total variation measure of m_t .

We define \tilde{U} as the quotient space given by U and the almost everywhere equivalence relation on $[0, T]$, that is, if dt -almost everywhere on $[0, T]$ the measures m_t^1 and m_t^2 coincide, the measure flows $(m_t^1)_{t \in [0, T]}$ and $(m_t^2)_{t \in [0, T]}$ are considered equivalent. \tilde{U} endowed with the usual sum and scalar multiplication is a vector space, where the zero vector is given by the family of null measures $(\mathbf{0})_{t \in [0, T]}$. To each $(m_t)_{t \in [0, T]} \in \tilde{U}$ we associate a Borel finite signed measure on $[0, T] \times \bar{\mathcal{O}}$ defined by $m_t(dx)dt$ and we endow \tilde{U} with the topology of weak convergence of the associated measures, i.e. we say that a sequence of flows of measures $(m^n)_{n \geq 1}$ converges to m in \tilde{U} if $(m_t^n(dx)dt)_{n \geq 1} \rightharpoonup m_t(dx)dt$. For $p \geq 1$, define the subsets of \tilde{U} ,

$$\tilde{U}_p := \left\{ m \in \tilde{U} : \int_0^T \int_{\bar{\mathcal{O}}} |x|^p |m_t(dx)dt < \infty \right\},$$

endowed with the weak topology with respect to continuous functions with p -growth, denoted by τ_p , of the associated measures (see Section 2.3 of Chapter 1 and Section 5.1 of Chapter 3). We denote by V (resp. V_p) the set of measure flows $(m_t)_{t \in [0, T]} \in \tilde{U}$ (resp. \tilde{U}_p) such that dt -a.e. m_t is a subprobability measure. We make the convention that when integrating a quantity with respect to dt , the version taken for $(m_t)_{t \in [0, T]} \in V$ inside the integral is such that, for each $t \in [0, T]$, m_t is a subprobability measure. We note that \tilde{U} is a Hausdorff locally convex topological vector space and V_p is metrizable (see Section 5.2 of Chapter 3). We endow the set

$\mathcal{P}_p([0, T] \times \bar{\mathcal{O}})$ with the τ_p -topology and we will often work on the product space $V_p \times \mathcal{P}_p([0, T] \times \bar{\mathcal{O}})$ endowed with the product topology, which we will denote $\tau_p \otimes \tau_p$. Since this product space is metrizable, we will often work with sequences. In V_p we will also consider the topology of stable convergence with p -growth, denoted by $\bar{\tau}_p$ (see Chapter 1 Section 2.3 for a definition and Section 5.1 of Chapter 3 for some properties of this topology), which is the weak topology with respect to test functions that are measurable in t , continuous in x and with p -growth. For $p \geq 1$, consider the set $M([0, T]; \mathcal{P}_p^{sub}(\bar{\mathcal{O}}))$ of Borel measurable functions from $[0, T]$ to $\mathcal{P}_p^{sub}(\bar{\mathcal{O}})$ identified a.e. on $[0, T]$. This set is endowed with the topology of convergence in measure (see Section 5.5 of Chapter 3) which is denoted by $\tilde{\tau}_p$, and is well defined since $\mathcal{P}_p^{sub}(\bar{\mathcal{O}})$ is a Polish space. In the case $p = 0$ we make the analogous definitions without the subscript notation. Moreover, any $m \in V_p$ admits a representative in $M([0, T]; \mathcal{P}_p^{sub}(\bar{\mathcal{O}}))$. We can thus consider, without loss of generality, the topology $\tilde{\tau}_p$ in V_p .

We are given constants $q > p \geq 1 \vee r$, where $r \in [0, 2]$ and $q \geq 2$, and the following functions:

$$\begin{aligned}
 (b, \sigma) &: [0, T] \times \mathbb{R} \rightarrow \mathbb{R} \times \mathbb{R}, \\
 f &: [0, T] \times \bar{\mathcal{O}} \times \mathcal{P}_p^{sub}(\bar{\mathcal{O}}) \rightarrow \mathbb{R}, \\
 g &: [0, T] \times \bar{\mathcal{O}} \times \mathcal{P}_p([0, T] \times \bar{\mathcal{O}}) \rightarrow \mathbb{R}.
 \end{aligned}$$

The sets $[0, T]$, \mathbb{R} and $\bar{\mathcal{O}}$ are endowed with the usual topology and the sets $\mathcal{P}_p^{sub}(\bar{\mathcal{O}})$ and $\mathcal{P}_p([0, T] \times \bar{\mathcal{O}})$ are endowed with the topology τ_p . As in Chapter 3, we will adopt the following notations

$$\langle m, \varphi \rangle := \int_0^T \int_{\bar{\mathcal{O}}} \varphi(t, x) m_t(dx) dt, \quad \langle \mu, \psi \rangle := \int_{[0, T] \times \bar{\mathcal{O}}} \psi(t, x) \mu(dt, dx),$$

where $(m, \mu) \in V_p \times \mathcal{P}_p([0, T] \times \bar{\mathcal{O}})$, $\varphi \in L^1(m_t(dx)dt)$ and $\psi \in L^1(\mu)$.

In this section, we let the following assumptions hold true.

Assumption 2.1.

- (1) $m_0^* \in \mathcal{P}_q(\bar{\mathcal{O}})$.
- (2) The functions $(t, x) \mapsto b(t, x)$ and $(t, x) \mapsto \sigma(t, x)$ are jointly measurable and continuous in x for each $t \in [0, T]$. Moreover, there exists a constant $c_1 > 0$ such that for all $(t, x, y) \in [0, T] \times \mathbb{R} \times \mathbb{R}$,

$$\begin{aligned}
 |b(t, x) - b(t, y)| + |\sigma(t, x) - \sigma(t, y)| &\leq c_1 |x - y|, \\
 |b(t, x)| &\leq c_1 [1 + |x|], \quad \sigma^2(t, x) \leq c_1 [1 + |x|^r].
 \end{aligned}$$

- (3) The function $(t, x, m) \mapsto f(t, x, m)$ is jointly measurable and continuous in (x, m) for each $t \in [0, T]$. The function g is jointly continuous. Moreover, there exists a constant $c_2 > 0$ such that for all $(t, x, m, \mu) \in [0, T] \times \bar{\mathcal{O}} \times \mathcal{P}_p^{sub}(\bar{\mathcal{O}}) \times \mathcal{P}_p([0, T] \times \bar{\mathcal{O}})$,

$$|f(t, x, m)| \leq c_2 \left[1 + |x|^p + \int_{\bar{\mathcal{O}}} |z|^p m(dz) \right],$$

$$|g(t, x, \mu)| \leq c_2 \left[1 + |x|^p + \int_{[0, T] \times \bar{\mathcal{O}}} |z|^p \mu(ds, dz) \right].$$

(4) One of the following statements is true:

(a) *Unattainable boundary:* We have $\mathbb{P}(\tau_{\bar{\mathcal{O}}}^X \geq T) = 1$ where X is the unique strong solution of

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t, \quad \mathbb{P} \circ X_0^{-1} = m_0^*.$$

(b) *Attainable boundary:* \mathcal{O} is an open interval and for all $(t, x) \in [0, T] \times \mathbb{R}$, $\sigma^2(t, x) \geq c_\sigma$ for some $c_\sigma > 0$.

We now give the formulation of the *linear programming Optimal Stopping MFG problem*, where the set of occupation measures induced by stopping times is replaced by the set of measures satisfying an infinite-dimensional linear constraint. This allows to compactify and convexify the optimization problem. As will be explained later in the paper, the flow of subprobability measures m is not necessarily regular in time. Because of this lack of regularity, MFGs of optimal stopping require a different treatment, and in particular, a different topology than classical MFGs (stochastic control without absorption). For the *strong formulation of Optimal Stopping MFGs*, we refer the reader to Definition 3.2 in Chapter 3 (which is stated in the mixed control/stopping framework but can be easily adapted without control).

The relaxed MFG problem: the linear programming formulation.

Definition 2.2. Let \mathcal{R} be the set of pairs $(m, \mu) \in V_p \times \mathcal{P}_p([0, T] \times \bar{\mathcal{O}})$, such that for all $u \in C_b^{1,2}([0, T] \times \bar{\mathcal{O}})$,

$$\langle \mu, u \rangle = \langle m_0^*, u(0, \cdot) \rangle + \langle m, \partial_t u + \mathcal{L}u \rangle,$$

where

$$\mathcal{L}u(t, x) := b(t, x)\partial_x u(t, x) + \frac{\sigma^2}{2}(t, x)\partial_{xx}u(t, x), \quad (t, x) \in [0, T] \times \mathbb{R}.$$

Definition 2.3 (*LP formulation of the MFG problem*). For $(\bar{m}, \bar{\mu}) \in V_p \times \mathcal{P}_p([0, T] \times \bar{\mathcal{O}})$, let $\Gamma[\bar{m}, \bar{\mu}] : V_p \times \mathcal{P}_p([0, T] \times \bar{\mathcal{O}}) \rightarrow \mathbb{R}$ be the reward functional associated to $(\bar{m}, \bar{\mu})$, defined as

$$\Gamma[\bar{m}, \bar{\mu}](m, \mu) = \langle m, f[\bar{m}] \rangle + \langle \mu, g[\bar{\mu}] \rangle,$$

where

$$f[\bar{m}](t, x) := f(t, x, \bar{m}_t), \quad g[\bar{\mu}](t, x) := g(t, x, \bar{\mu}), \quad (t, x) \in [0, T] \times \bar{\mathcal{O}}.$$

We say that $(m^*, \mu^*) \in V_p \times \mathcal{P}_p([0, T] \times \bar{\mathcal{O}})$ is an LP MFG Nash equilibrium if $(m^*, \mu^*) \in \mathcal{R}$ and for all $(m, \mu) \in \mathcal{R}$, $\Gamma[m^*, \mu^*](m, \mu) \leq \Gammam^*, \mu^*$.

For a finite number of iterations, the LPFP algorithm will produce what we call an ε -LP MFG Nash equilibrium.

Definition 2.4. For a given $\varepsilon \geq 0$, we say that $(m^*, \mu^*) \in V_p \times \mathcal{P}_p([0, T] \times \bar{\mathcal{O}})$ is an ε -LP MFG Nash equilibrium if $(m^*, \mu^*) \in \mathcal{R}$ and for all $(m, \mu) \in \mathcal{R}$, $\Gamma[m^*, \mu^*](m, \mu) - \varepsilon \leq \Gammam^*, \mu^*$.

In the next paragraph, we present the main results of the paper, in particular the linear programming fictitious play algorithm and its convergence.

The Linear Programming Fictitious Play algorithm. In order to show the convergence of the algorithm, we impose the following Assumption.

Assumption 2.5. (1) For each $(\bar{m}, \bar{\mu}) \in \mathcal{R}$, there exists a unique maximizer of $\Gamma[\bar{m}, \bar{\mu}]$ on \mathcal{R} .
 (2) The Lasry–Lions monotonicity condition holds: for all (m, μ) and $(\tilde{m}, \tilde{\mu})$ in $V_p \times \mathcal{P}_p([0, T] \times \bar{\mathcal{O}})$,

$$\langle m - \tilde{m}, f[m] - f[\tilde{m}] \rangle + \langle \mu - \tilde{\mu}, g[\mu] - g[\tilde{\mu}] \rangle \leq 0.$$

(3) There exist constants $c_f \geq 0$ and $c_g \geq 0$ such that for all $t \in [0, T]$, $x, x' \in \bar{\mathcal{O}}$, $m, m' \in \mathcal{P}_p^{\text{sub}}(\bar{\mathcal{O}})$, $\mu, \mu' \in \mathcal{P}_p([0, T] \times \bar{\mathcal{O}})$,

$$|f(t, x, m) - f(t, x, m')| \leq c_f(1 + |x|) \int_{\bar{\mathcal{O}}} (1 + |z|^p) |m - m'| (dz),$$

$$|f(t, x, m) - f(t, x, m') - f(t, x', m) + f(t, x', m')| \leq c_f |x - x'| \int_{\bar{\mathcal{O}}} (1 + |z|^p) |m - m'| (dz),$$

$$|g(t, x, \mu) - g(t, x, \mu')| \leq c_g(1 + |x|) \int_{[0, T] \times \bar{\mathcal{O}}} (1 + |z|^p) |\mu - \mu'| (ds, dz),$$

$$\begin{aligned} & |g(t, x, \mu) - g(t, x, \mu') - g(t', x', \mu) + g(t', x', \mu')| \\ & \leq c_g(|t - t'| + |x - x'|) \int_{[0, T] \times \bar{\mathcal{O}}} (1 + |z|^p) |\mu - \mu'| (ds, dz). \end{aligned}$$

For sufficient conditions on the coefficients under which the above assumptions hold, the reader is referred to Section 4.2.

Note that under Assumption 2.5, the function $\Theta : \mathcal{R} \rightarrow \mathcal{R}$ defined by

$$\Theta(\bar{m}, \bar{\mu}) = \arg \max_{(m, \mu) \in \mathcal{R}} \Gamma[\bar{m}, \bar{\mu}](m, \mu), \quad (\bar{m}, \bar{\mu}) \in \mathcal{R},$$

is well defined, since there is a unique maximizer. Furthermore, we can show the uniqueness of the LP MFG Nash equilibrium.

Proposition 2.6 (Uniqueness of the equilibrium). Under Assumptions 2.1 and 2.5, there exists at most one LP MFG Nash equilibrium.

Proof. Let $(m, \mu) \in \mathcal{R}$ and $(m', \mu') \in \mathcal{R}$ be two LP MFG Nash equilibria. Using the equilibrium property we get the following two inequalities:

$$\langle m - m', f[m] \rangle + \langle \mu - \mu', g[\mu] \rangle \geq 0, \quad \langle m' - m, f[m'] \rangle + \langle \mu' - \mu, g[\mu'] \rangle \geq 0.$$

Adding up these two inequalities we obtain

$$\langle m - m', f[m] - f[m'] \rangle + \langle \mu - \mu', g[\mu] - g[\mu'] \rangle \geq 0.$$

By the Lasry-Lions monotonicity condition we get the equality in the previous inequalities. Using that

$$\Gammam, \mu = \langle m, f[m] \rangle + \langle \mu, g[\mu] \rangle = \langle m', f[m] \rangle + \langle \mu', g[\mu] \rangle = \Gamma[m, \mu](m', \mu'),$$

we deduce by the uniqueness of the best response to (m, μ) that $(m, \mu) = (m', \mu')$. \square

We propose the following algorithm for computing the LP MFG Nash equilibrium.

Algorithm 1: LPFP algorithm (Optimal stopping MFGs)

Data: A number of steps N for the equilibrium approximation; a pair $(\bar{m}^{(0)}, \bar{\mu}^{(0)}) \in \mathcal{R}$;

Result: Approximate LP MFG Nash equilibrium

1 **for** $k = 0, 1, \dots, N - 1$ **do**

2 Compute a linear programming best response $(m^{(k+1)}, \mu^{(k+1)})$ to $(\bar{m}^{(k)}, \bar{\mu}^{(k)})$ by solving the linear programming problem

$$\arg \max_{(m, \mu) \in \mathcal{R}} \Gamma[\bar{m}^{(k)}, \bar{\mu}^{(k)}](m, \mu).$$

3 Set $(\bar{m}^{(k+1)}, \bar{\mu}^{(k+1)}) := \frac{k}{k+1}(\bar{m}^{(k)}, \bar{\mu}^{(k)}) + \frac{1}{k+1}(m^{(k+1)}, \mu^{(k+1)}) = \frac{1}{k+1} \sum_{\nu=1}^{k+1} (m^{(\nu)}, \mu^{(\nu)})$

4 **end**

We state now the main theorem. Its proof is provided in Section 2.3.2.

Theorem 2.7 (*Convergence of the algorithm*). *Let Assumptions 2.1 and 2.5 hold true and consider the sequences $(\bar{m}^{(N)}, \bar{\mu}^{(N)})_{N \geq 1}$ and $(m^{(N)}, \mu^{(N)})_{N \geq 1}$ generated by the Algorithm 1. Then both sequences converge in the product topology $\tilde{\tau}_p \otimes \tau_p$ to the unique LP MFG Nash equilibrium.*

Numerical example. To illustrate the convergence of the algorithm, we solve numerically a simple MFG of optimal stopping. In this game with time horizon $T = 1$, the state of the representative player belongs to the domain $\mathcal{O} = \mathbb{R}$ and has dynamics given by

$$X_t = X_0 + t + W_t,$$

i.e. $b(t, x) = \sigma(t, x) = 1$. The initial states of the players are distributed according to the law $m_0^* = \mathcal{N}(0, 4)$. Before exiting the game, at each time $t \in [0, 1]$, the representative player receives an instantaneous reward given by

$$\int_{\mathbb{R}} (X_t - y) m_t(dy),$$

where m_t is the distribution of players still in the game at time t , and upon exiting the game at time τ , the player receives the terminal reward

$$\int_{[0, T] \times \mathbb{R}} (\tau - s) \mu(ds, dy),$$

where μ is the joint distribution of exit times/states of the players. The functions f and g are therefore defined as follows:

$$f(t, x, m) := \int_{\mathbb{R}} (x - y)m(dy), \quad g(t, x, \mu) := \int_{[0, T] \times \mathbb{R}} (t - s)\mu(ds, dy).$$

The representative player has an incentive to stay in the game if its state is higher than the average state of the other players who are still in the game. It is expected that the players starting with a low state value will exit the game immediately, while the players starting with a high state value will stay until the end of the game.

To apply the LPFP algorithm, we discretize the linear program for the computation of the best response as in [MS98; CS02]. More precisely, we consider a time grid $t_i = i\Delta$ with $\Delta = \frac{T}{n_t}$, for $i \in \{0, 1, \dots, n_t\}$ and a state grid $x_{j+1} = x_j + \delta$, for $j \in \{0, 1, \dots, n_s - 1\}$ with $x_0 \in \mathbb{R}$ and $\delta > 0$. We define

$$Lu(t, x) = \frac{\partial u}{\partial t}(t, x) + b(t, x)\frac{\partial u}{\partial x}(t, x) + \frac{\sigma^2}{2}(t, x)\frac{\partial^2 u}{\partial x^2}(t, x), \quad \forall u \in \mathcal{D}(L) := C_b^{1,2}([0, T] \times \mathbb{R}).$$

We set $\mathcal{D}(\hat{L})$ as the functions in $\mathcal{D}(L) = C_b^{1,2}([0, T] \times \mathbb{R})$ restricted to the time-state discretization grid. For $u \in \mathcal{D}(\hat{L})$, we discretize the derivatives as follows

$$\begin{aligned} \hat{L}_t u(t_i, x_j) &= \frac{1}{\Delta} [u(t_{i+1}, x_j) - u(t_i, x_j)], \\ \hat{L}_x^u u(t_i, x_j) &= \frac{1}{\delta} \max(b(t_i, x_j), 0) [u(t_i, x_{j+1}) - u(t_i, x_j)], \\ \hat{L}_x^d u(t_i, x_j) &= \frac{1}{\delta} \min(b(t_i, x_j), 0) [u(t_i, x_j) - u(t_i, x_{j-1})], \\ \hat{L}_{xx} u(t_i, x_j) &= \frac{1}{\delta^2} \frac{\sigma^2}{2}(t_i, x_j) [u(t_i, x_{j+1}) + u(t_i, x_{j-1}) - 2u(t_i, x_j)]. \end{aligned}$$

The discretized generator has the form:

$$\hat{L}u(t_i, x_j) = \hat{L}_t u(t_i, x_j) + \hat{L}_x^u u(t_i, x_j) + \hat{L}_x^d u(t_i, x_j) + \hat{L}_{xx} u(t_i, x_j).$$

The constraint reads as

$$\sum_{i=0}^{n_t} \sum_{j=0}^{n_s} u(t_i, x_j) \mu(t_i, x_j) - \Delta \sum_{i=0}^{n_t-1} \sum_{j=1}^{n_s-1} \hat{L}u(t_i, x_j) m(t_i, x_j) = \sum_{j=1}^{n_s-1} u(t_0, x_j) m_0^*(x_j),$$

for $u \in \mathcal{D}(\hat{L})$. The set $\mathcal{D}(\hat{L})$ is equal to the linear span of the indicators functions

$$\mathbf{1}_{\{(t_i, x_j)\}}, \quad i \in \{0, 1, \dots, n_t\}, \quad j \in \{0, 1, \dots, n_s\}$$

on the time-state grid. By linearity, it suffices to evaluate the constraint on the set of indicator functions. We obtain a total number of $(n_t + 1) \times (n_s + 1)$ constraints. The discretized reward

associated to a discrete mean field term $(\bar{m}, \bar{\mu})$ is given by

$$\Delta \times \sum_{i=0}^{n_t-1} \sum_{j=1}^{n_s-1} f(t_i, x_j, \bar{m}(t_i, \cdot)) m(t_i, x_j) + \sum_{i=0}^{n_t} \sum_{j=0}^{n_s} g(t_i, x_j, \bar{\mu}) \mu(t_i, x_j).$$

This finite dimensional linear programs are solved with Python Scipy's linear programming algorithm. In order to evaluate the convergence of the algorithm, we compute the (discrete time-space) exploitability (term borrowed from [Per+20]) at each iteration:

$$\begin{aligned} \varepsilon_N = \Delta \sum_{i=0}^{n_t-1} \sum_{j=1}^{n_s-1} f(t_i, x_j, \bar{m}^{(N-1)}(t_i, \cdot)) (m^{(N)}(t_i, x_j) - \bar{m}^{(N-1)}(t_i, x_j)) \\ + \sum_{i=0}^{n_t} \sum_{j=0}^{n_s} g(t_i, x_j, \bar{\mu}^{(N-1)}) (\mu^{(N)}(t_i, x_j) - \bar{\mu}^{(N-1)}(t_i, x_j)). \end{aligned}$$

In Figure 4.1 (left graph), we can observe the evolution of the distribution of the players m_t over time and Figure 4.1 (right) shows the exit distribution μ of the players. As expected, players starting at a low position exit the game immediately and players starting at higher positions exit the game at later dates. Finally, Figure 4.2 illustrates the convergence of the algorithm via a log-log plot (base 10) of the exploitability. We can observe that the convergence is in $O(N^{-1})$.

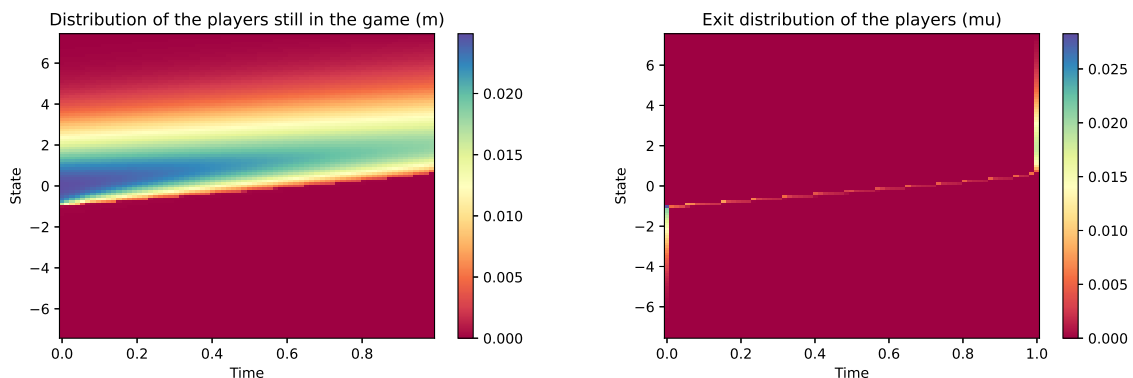


Figure 4.1 – Equilibrium distributions at the final iteration. Left graph: $\bar{m}^{(N)}$, distribution of the players still in the game over time. Right graph: $\bar{\mu}^{(N)}$, exit distribution of the players.

2.2 Compactness of \mathcal{R} under the topology of convergence in measure

In this subsection, we prepare the ground for showing the convergence of our LPFP algorithm, by providing several results which are needed in the proof of the main result. In particular, we show the compactness of the set \mathcal{R} under the topology of the convergence in measure. This result also allows to prove the existence of an equilibrium under Assumption 2.1 only (see Theorem 2.13), improving earlier results in [BDT20] and [DLT21], which did not take into account a general dependence of the map f on m (resp. g on μ) nor coefficients with polynomial growth.

A direct adaptation of Theorem 5.20 in Chapter 3 to the pure stopping case, gives that for

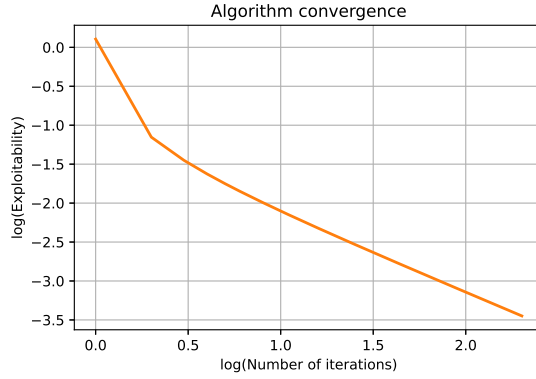


Figure 4.2 – Convergence of the algorithm: log-log plot of the exploitability.

$(m, \mu) \in \mathcal{R}$, there exists $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, W, \tau, X)$, such that $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ is a filtered probability space, W is an \mathbb{F} -Brownian motion, τ is an \mathbb{F} -stopping time such that $\tau \leq T \wedge \tau_{\bar{\mathcal{O}}}^X$ \mathbb{P} -a.s. and X is an \mathbb{F} -adapted process verifying

$$X_t = X_0 + \int_0^t b(s, X_s) ds + \int_0^t \sigma(s, X_s) dW_s, \quad \mathbb{P} \circ X_0^{-1} = m_0^*,$$

$$m_t(B) = m_t^*(B), \quad B \in \mathcal{B}(\bar{\mathcal{O}}), \quad t - a.e., \quad \text{and} \quad \mu = \mathbb{P} \circ (\tau, X_\tau)^{-1},$$

where

$$m_t^*(B) := \mathbb{E}^{\mathbb{P}} [\mathbb{1}_B(X_t) \mathbb{1}_{t < \tau}].$$

Seen as a mapping from $[0, T]$ to $\mathcal{P}^{sub}(\bar{\mathcal{O}})$ endowed with the topology of weak convergence, m^* is càdlàg. Hence, without loss of generality, we can and will always consider the càdlàg representative of m , still denoted by m .

The following estimates will be useful to establish the main result of the subsection, Theorem 2.11.

Lemma 2.8 (Estimates). *There exist constants $C_1 \geq 0$ and $C_2 \geq 0$ such that for all $(m, \mu) \in \mathcal{R}$ we have the following estimates:*

(1) For all $t \in [0, T]$,

$$\int_{\bar{\mathcal{O}}} |x|^q m_t(dx) \leq C_1, \quad \int_{[0, T] \times \bar{\mathcal{O}}} |x|^q \mu(dt, dx) \leq C_1.$$

(2) For all $h \in [0, T]$, we have

$$\int_0^{T-h} d_{BL}(m_{t+h}, m_t) dt \leq C_2 \sqrt{h},$$

where d_{BL} is the bounded Lipschitz distance (see Section 5.1 of Chapter 3).

Proof. Let $(m, \mu) \in \mathcal{R}$ and consider their probabilistic representation. Using standard estimates,

we obtain the existence of a constant $C_1 \geq 0$ such that $\mathbb{E}^{\mathbb{P}} \|X\|_T^q \leq C_1$. We deduce that

$$\begin{aligned} \int_{[0,T] \times \bar{\mathcal{O}}} |x|^q \mu(dt, dx) &= \mathbb{E}^{\mathbb{P}} [|X_\tau|^q] \leq \mathbb{E}^{\mathbb{P}} [\|X\|_T^q] \leq C_1, \\ \int_{\bar{\mathcal{O}}} |x|^q m_t(dx) &= \mathbb{E}^{\mathbb{P}} [|X_t|^q \mathbf{1}_{t < \tau}] \leq \mathbb{E}^{\mathbb{P}} [\|X\|_T^q] \leq C_1, \quad t \in [0, T]. \end{aligned}$$

Let us show the second estimate. For $\phi \in \text{BL}(\bar{\mathcal{O}})$ such that $\|\phi\|_{\text{BL}} \leq 1$ (where $\|\cdot\|_{\text{BL}}$ denotes the bounded Lipschitz norm, see Section 5.1 of Chapter 3), $h \in [0, T]$ and $t \in [0, T - h]$, we have:

$$\begin{aligned} \int_{\bar{\mathcal{O}}} \phi(x) (m_{t+h} - m_t)(dx) &= \mathbb{E}^{\mathbb{P}} [\phi(X_{t+h}) \mathbf{1}_{t+h < \tau} - \phi(X_t) \mathbf{1}_{t < \tau}] \\ &= \mathbb{E}^{\mathbb{P}} [(\phi(X_{t+h}) - \phi(X_t)) \mathbf{1}_{t+h < \tau}] + \mathbb{E}^{\mathbb{P}} [\phi(X_t) (\mathbf{1}_{t+h < \tau} - \mathbf{1}_{t < \tau})] \\ &\leq \mathbb{E}^{\mathbb{P}} [|\phi(X_{t+h}) - \phi(X_t)|] + \mathbb{E}^{\mathbb{P}} [|\phi(X_t)| \mathbf{1}_{t < \tau \leq t+h}] \\ &\leq \mathbb{E}^{\mathbb{P}} [|X_{t+h} - X_t|] + \mathbb{E}^{\mathbb{P}} [\mathbf{1}_{t < \tau \leq t+h}]. \end{aligned}$$

Taking the supremum over ϕ we get $d_{\text{BL}}(m_{t+h}, m_t) \leq \mathbb{E}^{\mathbb{P}} [|X_{t+h} - X_t|] + \mathbb{E}^{\mathbb{P}} [\mathbf{1}_{t < \tau \leq t+h}]$. We also have

$$\int_0^{T-h} \mathbb{E}^{\mathbb{P}} [\mathbf{1}_{t < \tau \leq t+h}] dt \leq \mathbb{E}^{\mathbb{P}} [\tau \wedge (T - h) - 0 \vee (\tau - h)] \leq h.$$

On the other hand, applying Jensen's inequality and Burkholder-Davis-Gundy inequality, we get $\mathbb{E}^{\mathbb{P}} |X_{t+h} - X_t|^2 \leq Ch$, for some constant $C \geq 0$. We deduce that there exists a constant C_2 such that $\int_0^{T-h} d_{\text{BL}}(m_{t+h}, m_t) dt \leq C_2 \sqrt{h}$. \square

Using the above estimates, one can show that for $(m, \mu) \in \mathcal{R}$ the flow of measures m is càdlàg as a map from $[0, T]$ to $\mathcal{P}_p^{\text{sub}}(\bar{\mathcal{O}})$ endowed with the τ_p -topology (weak convergence with p -growth). In particular m is Borel measurable from $[0, T]$ to $\mathcal{P}_p^{\text{sub}}(\bar{\mathcal{O}})$.

Now we study the topology of convergence in measure which is convenient since we will be able to extract subsequences converging in $\mathcal{P}_p^{\text{sub}}(\bar{\mathcal{O}})$ a.e. on $[0, T]$. This topology allows to have a general mean-field dependence on the reward function f , which was not the case in [DLT21]. Moreover, it permits to consider test functions which are only measurable in time, and therefore to use the topology of the stable convergence. More precisely, we say that $(m^n)_{n \geq 1} \subset V_p$ converges in $\bar{\tau}_p$ to $m \in V_p$ if for all test functions $\phi : [0, T] \times \bar{\mathcal{O}} \rightarrow \mathbb{R}$ jointly measurable, continuous in x for each t and with p -polynomial growth, we have

$$\lim_{n \rightarrow \infty} \int_0^T \int_{\bar{\mathcal{O}}} \phi(t, x) m_t^n(dx) dt = \int_0^T \int_{\bar{\mathcal{O}}} \phi(t, x) m_t(dx) dt.$$

We state this result in the next lemma, whose proof follows the same ideas of Proposition 2.10 in Chapter 3.

Lemma 2.9 (*Stable convergence with polynomial growth*). *On the set \mathcal{R} we have the inclusion $\bar{\tau}_p \otimes \tau_p \subset \tilde{\tau}_p \otimes \tau_p$. In other words, the convergence in measure of the flow of measures implies the convergence in the stable topology.*

For sake of clarity, we recall here Lemma 2.8 in Chapter 3, which will be used for showing the compactness of \mathcal{R} under the topology $\tilde{\tau}_p \otimes \tau_p$.

Lemma 2.10. *The map $H : \mathcal{P}^{sub}(\bar{\mathcal{O}}) \rightarrow [0, \infty]$ defined by $H(m) = \int_{\bar{\mathcal{O}}} |x|^q m(dx)$, $m \in \mathcal{P}^{sub}(\bar{\mathcal{O}})$, is a coercive normal integrand in the sense of Definition 5.23 in Chapter 3, where $\mathcal{P}^{sub}(\bar{\mathcal{O}})$ is endowed with the topology of weak convergence.*

Theorem 2.11 (Compactness of \mathcal{R} in $\tilde{\tau}_p \otimes \tau_p$). *The topological space $(\mathcal{R}, \tilde{\tau}_p \otimes \tau_p)$ is compact.*

Proof. Since the space $(\mathcal{R}, \tilde{\tau}_p \otimes \tau_p)$ is metrizable, it suffices to show that it is sequentially compact. Consider a sequence $(m^n, \mu^n)_{n \geq 1} \subset \mathcal{R}$. Using the estimate (1) from Lemma 2.8

$$\int_{[0, T] \times \bar{\mathcal{O}}} |x|^q \mu^n(dt, dx) \leq C_1,$$

we get by Corollary 5.4 in Chapter 3 that up to a subsequence, $(\mu^n)_{n \geq 1}$ converges to some $\mu \in \mathcal{P}_p([0, T] \times \bar{\mathcal{O}})$ in τ_p . Let us show now that we can extract a further subsequence such that $(m^n)_{n \geq 1}$ converges to some m in $\tilde{\tau}_p$. To prove this, we use the relative compactness criterion given in Theorem 5.24 in Chapter 3 for the topology of convergence in measure $(M([0, T]; \mathcal{P}^{sub}(\bar{\mathcal{O}})), \tilde{\tau})$. Let H be the map defined in Lemma 2.10. Using the first estimates of Lemma 2.8,

$$\sup_{n \geq 1} \int_0^T H(m_t^n) dt = \sup_{n \geq 1} \int_0^T \int_{\bar{\mathcal{O}}} |x|^q m_t^n(dx) dt \leq C_1 T.$$

Now, using the second estimate of Lemma 2.8, $\lim_{h \downarrow 0} \sup_{n \geq 1} \int_0^{T-h} d_{BL}(m_{t+h}^n, m_t^n) dt = 0$. By Theorem 5.24 in Chapter 3, up to a subsequence, the sequence $(m^n)_{n \geq 1}$ converges to some $m \in M([0, T]; \mathcal{P}^{sub}(\bar{\mathcal{O}}))$ in measure. Up to another subsequence, $(m_t^n)_{n \geq 1}$ converges weakly to m_t t -a.e. on $[0, T]$. Since for each $t \in [0, T]$,

$$\sup_{n \geq 1} \int_{\bar{\mathcal{O}}} |x|^q m_t^n(dx) \leq C_1,$$

$(m_t^n)_{n \geq 1}$ converges to m_t in τ_p t -a.e. on $[0, T]$, and in particular $(m^n)_{n \geq 1}$ converges to m in $\tilde{\tau}_p$. Finally, by Lemma 2.9 we get $m^n \rightarrow m$ in $\bar{\tau}_p$ and we can pass easily to the limit in the constraint to conclude that $(m, \mu) \in \mathcal{R}$. \square

Recall that if a set is compact and Hausdorff under two comparable topologies, then both topologies coincide (see [Mun00] Chapter 3, Exercise 1.(b) p.168). As a consequence of the above theorem, we get the following result which will be useful to consider different metrics on the space \mathcal{R} in order to show the convergence of the algorithm.

Corollary 2.12. *On the set \mathcal{R} the topologies $\tau \otimes \tau$, $\tau_p \otimes \tau_p$, $\bar{\tau}_p \otimes \tau_p$ and $\tilde{\tau}_p \otimes \tau_p$ coincide.*

The existence of a maximizer of the best response map $\Theta : \mathcal{R} \rightarrow 2^{\mathcal{R}}$ defined by

$$\Theta(\bar{m}, \bar{\mu}) = \arg \max_{(m, \mu) \in \mathcal{R}} \Gamma[\bar{m}, \bar{\mu}](m, \mu), \quad (\bar{m}, \bar{\mu}) \in \mathcal{R}$$

follows as in Theorem 2.12 in Chapter 3. Then, by applying the Kakutani-Fan-Glicksberg's fixed point theorem for set-valued maps (see Theorem 5.34 in Chapter 3) together with an intermediary application of lemmas 5.26 and 5.28 in Chapter 3, we get the existence of an LP MFG Nash equilibrium.

Theorem 2.13 (*Existence of LP MFG Nash equilibria*). *Under Assumption 2.1, there exists an LP MFG Nash equilibrium.*

To show the convergence of the algorithm, we use a precise metric on the set \mathcal{R} , denoted by ρ , which is introduced below. In particular, this metric is used in Corollary 2.18, where we show the convergence to zero of the distance ρ between two successive best responses. We denote by W_1 the 1-Wasserstein metric on $\mathcal{P}_1([0, T] \times \bar{\mathcal{O}})$. We also make use of an analogue of the 1-Wasserstein metric on $\mathcal{P}_1^{sub}(\bar{\mathcal{O}})$, which is denoted by W'_1 and constructed as in Appendix B of [CRT19]. This metric depends on some arbitrary reference point $x_0 \in \bar{\mathcal{O}}$ (which is fixed for the rest of the paper) and metrizes the topology τ_1 in $\mathcal{P}_1^{sub}(\bar{\mathcal{O}})$ described in Section 5.1 in Chapter 3 (see Lemma B.2. in [CRT19]). In particular, we use the following Kantorovich duality type result (Lemma B.1 in [CRT19]):

$$W'_1(m, m') = \sup_{\phi \in \text{Lip}_1(\bar{\mathcal{O}}, x_0)} \int_{\bar{\mathcal{O}}} \phi(x)(m - m')(dx) + |m(\bar{\mathcal{O}}) - m'(\bar{\mathcal{O}})|, \quad m, m' \in \mathcal{P}_1^{sub}(\bar{\mathcal{O}}),$$

where $\text{Lip}_1(\bar{\mathcal{O}}, x_0)$ is the set of all functions $\phi : \bar{\mathcal{O}} \rightarrow \mathbb{R}$ with Lipschitz constant smaller or equal to 1 and such that $\phi(x_0) = 0$.

Proposition 2.14 (*Metric ρ*). *Any of the topologies on the set \mathcal{R} considered in Corollary 2.12 is induced by the metric ρ , given by $\rho((m, \mu), (m', \mu')) = d_M(m, m') + W_1(\mu, \mu')$, with $(m, \mu), (m', \mu') \in \mathcal{R}$, and*

$$d_M(m, m') := \int_0^T W'_1(m_t, m'_t) dt. \quad (4.1)$$

Proof. Let us prove that the metric ρ metrizes the topology $\tilde{\tau}_1 \otimes \tau_1$ on \mathcal{R} . To do so, by Lemma 5.22 in Chapter 3, it suffices to show that there exists a constant $C \geq 0$ such that for each $(m, \mu) \in \mathcal{R}$ we have t -a.e. on $[0, T]$, $W'_1(m_t, \mathbf{0}) \leq C$, where $\mathbf{0}$ denotes the null measure on $\bar{\mathcal{O}}$. Let $(m, \mu) \in \mathcal{R}$. By Lemma 2.8, there exists a constant $C \geq 0$ such that for all $t \in [0, T]$ we have

$$\int_{\bar{\mathcal{O}}} |x| m_t(dx) \leq C.$$

We obtain for all $t \in [0, T]$,

$$W'_1(m_t, \mathbf{0}) = \sup_{\phi \in \text{Lip}_1(\bar{\mathcal{O}}, x_0)} \int_{\bar{\mathcal{O}}} \phi(x) m_t(dx) + m_t(\bar{\mathcal{O}}) \leq \int_{\bar{\mathcal{O}}} |x| m_t(dx) + |x_0| + 1 \leq C + |x_0| + 1.$$

□

2.3 Convergence of the algorithm

In this subsection, Assumption 2.1 and Assumption 2.5 are in force. We recall the quantities computed by Algorithm 1: for $N \geq 0$,

$$(m^{(N+1)}, \mu^{(N+1)}) := \Theta(\bar{m}^{(N)}, \bar{\mu}^{(N)}),$$

$$(\bar{m}^{(N+1)}, \bar{\mu}^{(N+1)}) := \frac{N}{N+1}(\bar{m}^{(N)}, \bar{\mu}^{(N)}) + \frac{1}{N+1}(m^{(N+1)}, \mu^{(N+1)}) = \frac{1}{N+1} \sum_{k=1}^{N+1} (m^{(k)}, \mu^{(k)}).$$

Since all these tuples are in \mathcal{R} , the measures $(m^{(N)})_N$ and $(\bar{m}^{(N)})_N$ admit càdlàg representatives. Without loss of generality, we consider the càdlàg representatives, for which the same notation is used.

We first establish some useful estimates, as well as the continuity of the best response map Θ .

2.3.1 Estimates and regularity of the best response map

Using the definitions of $(\bar{m}^{(N)}, \bar{\mu}^{(N)})$ given by the algorithm procedure, we derive the following estimates between two successive output measures.

Lemma 2.15 (*Estimates between two successive output measures $(\bar{m}^{(N)}, \bar{\mu}^{(N)})$). For all $N \geq 1$, we have the following estimates:*

(i) *There exists a constant $C_3 \geq 0$ (independent of N) such that*

$$d_M(\bar{m}^{(N)}, \bar{m}^{(N+1)}) \leq \frac{C_3}{N}, \quad W_1(\bar{\mu}^{(N)}, \bar{\mu}^{(N+1)}) \leq \frac{C_3}{N}.$$

(ii) *There exists a constant $C_4 \geq 0$ (independent of N) such that for all $t \in [0, T]$,*

$$\int_{\bar{\mathcal{O}}} (1 + |x|^p) |\bar{m}_t^{(N+1)} - \bar{m}_t^{(N)}|(dx) \leq \frac{C_4}{N}, \quad \int_{[0, T] \times \bar{\mathcal{O}}} (1 + |x|^p) |\bar{\mu}^{(N+1)} - \bar{\mu}^{(N)}|(dt, dx) \leq \frac{C_4}{N}.$$

Proof. (i) Note that, $\bar{m}^{(N+1)} - \bar{m}^{(N)} = \frac{1}{N+1} [m^{(N+1)} - \bar{m}^{(N)}]$. Let $\varphi \in \text{Lip}_1(\bar{\mathcal{O}}, x_0)$, i.e. a 1-Lipschitz function with $\varphi(x_0) = 0$. Using Lemma 2.8 (i), for all $t \in [0, T]$,

$$\begin{aligned} \langle \bar{m}_t^{(N+1)} - \bar{m}_t^{(N)}, \varphi \rangle &= \frac{1}{N+1} \langle m_t^{(N+1)} - \bar{m}_t^{(N)}, \varphi \rangle \\ &\leq \frac{1}{N+1} \langle m_t^{(N+1)} + \bar{m}_t^{(N)}, |\varphi| \rangle \\ &\leq \frac{C}{N}, \end{aligned}$$

for some constant $C \geq 0$ independent from N , φ and t . Taking the supremum over φ and integrating over t we get the claimed estimate.

Analogously, $\bar{\mu}^{(N+1)} - \bar{\mu}^{(N)} = \frac{1}{N+1} [\mu^{(N+1)} - \bar{\mu}^{(N)}]$. Let $\varphi \in \text{Lip}_1([0, T] \times \bar{\mathcal{O}})$, i.e. a

1-Lipschitz function. Using again Lemma 2.8 (i),

$$\begin{aligned}
 \langle \bar{\mu}^{(N+1)} - \bar{\mu}^{(N)}, \varphi \rangle &= \frac{1}{N+1} \langle \mu^{(N+1)} - \bar{\mu}^{(N)}, \varphi \rangle \\
 &= \frac{1}{N+1} \langle \mu^{(N+1)} - \bar{\mu}^{(N)}, \varphi - \varphi(0, x_0) \rangle \\
 &\leq \frac{1}{N+1} \langle \mu^{(N+1)} + \bar{\mu}^{(N)}, |\varphi - \varphi(0, x_0)| \rangle \\
 &\leq \frac{C}{N},
 \end{aligned}$$

for some constant $C \geq 0$ independent from N and φ . Taking the supremum over φ we obtain the result.

(ii) By Lemma 2.8, we get for all $t \in [0, T]$:

$$\int_{\bar{\mathcal{O}}} (1 + |x|^p) |\bar{m}_t^{(N+1)} - \bar{m}_t^{(N)}|(dx) \leq \frac{1}{N+1} \int_{\bar{\mathcal{O}}} (1 + |x|^p) (m_t^{(N+1)} + \bar{m}_t^{(N)})(dx) \leq \frac{C}{N}.$$

Similarly, by Lemma 2.8, we get $\int_{[0, T] \times \bar{\mathcal{O}}} (1 + |x|^p) |\bar{\mu}^{(N+1)} - \bar{\mu}^{(N)}|(dt, dx) \leq \frac{C}{N}$.

□

We establish below the following estimates on the reward map.

Lemma 2.16 (*Estimates on the reward map*). *There exist constants C_f and C_g such that for all $N \geq 1$*

$$\langle m^{(N+2)} - m^{(N+1)}, f[\bar{m}^{(N+1)}] - f[\bar{m}^{(N)}] \rangle \leq \frac{C_f}{N} d_M(m^{(N+1)}, m^{(N+2)}),$$

$$\langle \mu^{(N+2)} - \mu^{(N+1)}, g[\bar{\mu}^{(N+1)}] - g[\bar{\mu}^{(N)}] \rangle \leq \frac{C_g}{N} W_1(\mu^{(N+1)}, \mu^{(N+2)}).$$

Proof. Let us first show that the function $\varphi_N : [0, T] \times \bar{\mathcal{O}} \rightarrow \mathbb{R}$ defined by

$$\varphi_N(t, x) := f(t, x, \bar{m}_t^{(N+1)}) - f(t, x, \bar{m}_t^{(N)})$$

is a C/N -Lipschitz continuous function in x uniformly on t , for some constant $C \geq 0$. Indeed, by item (3) in Assumption 2.5 and Lemma 2.15, for each $t \in [0, T]$ and $x, x' \in \bar{\mathcal{O}}$, we have

$$\begin{aligned}
 |\varphi_N(t, x) - \varphi_N(t, x')| &= |f(t, x, \bar{m}_t^{(N+1)}) - f(t, x, \bar{m}_t^{(N)}) - f(t, x', \bar{m}_t^{(N+1)}) + f(t, x', \bar{m}_t^{(N)})| \\
 &\leq c_f |x - x'| \int_{\bar{\mathcal{O}}} (1 + |z|^p) |\bar{m}_t^{(N+1)} - \bar{m}_t^{(N)}|(dz) \leq \frac{C}{N} |x - x'|.
 \end{aligned}$$

The same holds for the function $(t, x) \mapsto \varphi_N(t, x) - \varphi_N(t, x_0)$, which is equal to 0 at x_0 . By definition of W'_1 ,

$$W'_1(m_t^{(N+2)}, m_t^{(N+1)}) = \sup_{\phi \in \text{Lip}_1(\bar{\mathcal{O}}, x_0)} \int_{\bar{\mathcal{O}}} \phi(x) (m_t^{(N+2)} - m_t^{(N+1)})(dx) + |m_t^{(N+2)}(\bar{\mathcal{O}}) - m_t^{(N+1)}(\bar{\mathcal{O}})|.$$

Furthermore, by item (3) from Assumption 2.5 and Lemma 2.15, we get that for $t \in [0, T]$,

$$\begin{aligned} |\varphi_N(t, x_0)| &= |f(t, x_0, \bar{m}_t^{(N+1)}) - f(t, x_0, \bar{m}_t^{(N)})| \\ &\leq c_f(1 + |x_0|) \int_{\bar{\mathcal{O}}} (1 + |z|^p) |\bar{m}_t^{(N+1)} - \bar{m}_t^{(N)}| (dz) \\ &\leq \frac{C'}{N}. \end{aligned}$$

We derive

$$\begin{aligned} &\langle m^{(N+2)} - m^{(N+1)}, f[\bar{m}^{(N+1)}] - f[\bar{m}^{(N)}] \rangle \\ &= \int_0^T \int_{\bar{\mathcal{O}}} (\varphi_N(t, x) - \varphi_N(t, x_0)) (m_t^{(N+2)} - m_t^{(N+1)}) (dx) dt \\ &\quad + \int_0^T \int_{\bar{\mathcal{O}}} \varphi_N(t, x_0) (m_t^{(N+2)} - m_t^{(N+1)}) (dx) dt \\ &\leq \frac{C}{N} \int_0^T W_1'(m_t^{(N+2)}, m_t^{(N+1)}) dt + \frac{C'}{N} \int_0^T W_1'(m_t^{(N+2)}, m_t^{(N+1)}) dt \\ &\leq \frac{C + C'}{N} d_M(m^{(N+2)}, m^{(N+1)}), \end{aligned}$$

where the last inequality follows by definition of the metric d_M given by (4.1).

Let us show the second estimate. To this purpose, we consider the function $\psi_N : [0, T] \times \bar{\mathcal{O}} \rightarrow \mathbb{R}$ defined by

$$\psi_N(t, x) := g(t, x, \bar{\mu}^{(N+1)}) - g(t, x, \bar{\mu}^{(N)})$$

and show that it is a C''/N -Lipschitz continuous function, for some constant $C'' \geq 0$. Indeed, by item (3) in Assumption 2.5 and Lemma 2.15, for each $t, t' \in [0, T]$, $x, x' \in \bar{\mathcal{O}}$, we have

$$\begin{aligned} |\psi_N(t, x) - \psi_N(t', x')| &= |g(t, x, \bar{\mu}^{(N+1)}) - g(t, x, \bar{\mu}^{(N)}) - g(t', x', \bar{\mu}^{(N+1)}) + g(t', x', \bar{\mu}^{(N)})| \\ &\leq c_g(|t - t'| + |x - x'|) \int_{[0, T] \times \bar{\mathcal{O}}} (1 + |z|^p) |\bar{\mu}^{(N+1)} - \bar{\mu}^{(N)}| (ds, dz) \\ &\leq \frac{C''}{N} (|t - t'| + |x - x'|). \end{aligned}$$

By Kantorovich's duality theorem,

$$\langle \mu^{(N+2)} - \mu^{(N+1)}, g[\bar{\mu}^{(N+1)}] - g[\bar{\mu}^{(N)}] \rangle = \langle \mu^{(N+2)} - \mu^{(N+1)}, \psi_N \rangle \leq \frac{C''}{N} W_1(\mu^{(N+2)}, \mu^{(N+1)}).$$

□

In the following lemma, we prove the continuity of the best response map.

Lemma 2.17 (*Continuity of the best response map*). *The function Θ is continuous on \mathcal{R} with respect to all topologies listed in Corollary 2.12.*

Proof. Let $(\bar{m}^n, \bar{\mu}^n)_{n \geq 1} \subset \mathcal{R}$ be a sequence converging to $(\bar{m}, \bar{\mu}) \in \mathcal{R}$. Define $(m^n, \mu^n) := \Theta(\bar{m}^n, \bar{\mu}^n) \in \mathcal{R}$ and let $(m, \mu) \in \mathcal{R}$ be a cluster point of the sequence $(m^n, \mu^n)_{n \geq 1}$ (which exists

since \mathcal{R} is compact). Up to taking a subsequence, we assume that the entire sequence converges to (m, μ) . Let $(\tilde{m}, \tilde{\mu}) \in \mathcal{R}$, we have to show that $\Gamma[\bar{m}, \bar{\mu}](\tilde{m}, \tilde{\mu}) \leq \Gamma[\bar{m}, \bar{\mu}](m, \mu)$. By definition of Θ , $\Gamma[\bar{m}^n, \bar{\mu}^n](\tilde{m}, \tilde{\mu}) \leq \Gamma[\bar{m}^n, \bar{\mu}^n](m^n, \mu^n)$. Taking the limit in the above inequality as $n \rightarrow \infty$ (by an intermediary application of lemmas 5.26 and 5.28 in Chapter 3), we obtain $\Gamma[\bar{m}, \bar{\mu}](\tilde{m}, \tilde{\mu}) \leq \Gamma[\bar{m}, \bar{\mu}](m, \mu)$, which, by uniqueness of the best response, shows that $(m, \mu) = \Theta(\bar{m}, \bar{\mu})$. \square

Using Lemma 2.15 and Lemma 2.17, we derive the following result.

Corollary 2.18 (*Proximity between two successive best responses*). *We have*

$$\lim_{N \rightarrow \infty} d_M(m^{(N)}, m^{(N+1)}) = 0 \quad \text{and} \quad \lim_{N \rightarrow \infty} W_1(\mu^{(N)}, \mu^{(N+1)}) = 0.$$

Proof. Recall the metric ρ on \mathcal{R} defined in Proposition 2.14. Viewing Θ as a function between the metric spaces (\mathcal{R}, ρ) and (\mathcal{R}, ρ) , it is uniformly continuous since it is continuous by Lemma 2.17 and (\mathcal{R}, ρ) is a compact metric space by Theorem 2.11. By Lemma 2.15 (i),

$$\rho((\bar{m}^{(N)}, \bar{\mu}^{(N)}), (\bar{m}^{(N+1)}, \bar{\mu}^{(N+1)})) \xrightarrow{N \rightarrow \infty} 0.$$

We get by the sequential characterization of the uniform continuity that

$$\rho((m^{(N+1)}, \mu^{(N+1)}), (m^{(N+2)}, \mu^{(N+2)})) = \rho(\Theta(\bar{m}^{(N)}, \bar{\mu}^{(N)}), \Theta(\bar{m}^{(N+1)}, \bar{\mu}^{(N+1)})) \xrightarrow{N \rightarrow \infty} 0.$$

\square

2.3.2 Main convergence result

In this section, we prove the convergence of the algorithm. To do so, we first introduce the following sequence of real numbers ε_N , which quantifies how far $(\bar{m}^{(N)}, \bar{\mu}^{(N)})$ is from being the best response when the reward maps depend on $(\bar{m}^{(N)}, \bar{\mu}^{(N)})$. Therefore, ε_N quantifies the proximity of $(\bar{m}^{(N)}, \bar{\mu}^{(N)})$ from an *LP MFG Nash equilibrium*.

Definition 2.19 (*Exploitability*). *We define the sequence of real numbers $(\varepsilon_N)_{N \geq 1}$ by*

$$\varepsilon_N = \langle m^{(N+1)} - \bar{m}^{(N)}, f[\bar{m}^{(N)}] \rangle + \langle \mu^{(N+1)} - \bar{\mu}^{(N)}, g[\bar{\mu}^{(N)}] \rangle \geq 0. \quad (4.2)$$

In particular, $(\bar{m}^{(N)}, \bar{\mu}^{(N)})$ is an ε_N -LP MFG Nash equilibrium and we will show in the next theorem that $\varepsilon_N \rightarrow 0$ as $N \rightarrow \infty$.

Proof of Theorem 2.7. In the proof we will denote by $C \geq 0$ a generic constant which may change from line to line. Recall the expression of ε_N from (4.2). We can rewrite ε_N as

$$\varepsilon_N = \Gamma[\bar{m}^{(N)}, \bar{\mu}^{(N)}](m^{(N+1)}, \mu^{(N+1)}) - \Gamma\bar{m}^{(N)}, \bar{\mu}^{(N)}.$$

Now we have

$$\begin{aligned} \varepsilon_{N+1} - \varepsilon_N &= \Gamma[\bar{m}^{(N+1)}, \bar{\mu}^{(N+1)}](m^{(N+2)}, \mu^{(N+2)}) - \Gamma\bar{m}^{(N+1)}, \bar{\mu}^{(N+1)} \\ &\quad - \Gamma[\bar{m}^{(N)}, \bar{\mu}^{(N)}](m^{(N+1)}, \mu^{(N+1)}) + \Gamma\bar{m}^{(N)}, \bar{\mu}^{(N)}. \end{aligned}$$

Define

$$\begin{aligned} \varepsilon_N^{(1)} &:= \Gamma\bar{m}^{(N)}, \bar{\mu}^{(N)} - \Gamma\bar{m}^{(N+1)}, \bar{\mu}^{(N+1)}, \\ \varepsilon_N^{(2)} &:= \Gamma[\bar{m}^{(N+1)}, \bar{\mu}^{(N+1)}](m^{(N+2)}, \mu^{(N+2)}) - \Gamma[\bar{m}^{(N)}, \bar{\mu}^{(N)}](m^{(N+1)}, \mu^{(N+1)}). \end{aligned}$$

Then $\varepsilon_{N+1} - \varepsilon_N = \varepsilon_N^{(1)} + \varepsilon_N^{(2)}$. Let us make some estimates of these two quantities. Using Lemma 2.15 (ii), for each $t \in [0, T]$ and $x \in \bar{\mathcal{O}}$ we obtain

$$|f(t, x, \bar{m}_t^{(N+1)}) - f(t, x, \bar{m}_t^{(N)})| \leq c_f(1 + |x|) \int_{\bar{\mathcal{O}}} (1 + |z|^p) |\bar{m}_t^{(N+1)} - \bar{m}_t^{(N)}| (dz) \leq \frac{C}{N}(1 + |x|),$$

and therefore we get by Lemma 2.8 (i)

$$\begin{aligned} -\frac{1}{N+1} \langle m^{(N+1)} - \bar{m}^{(N)}, f[\bar{m}^{(N+1)}] - f[\bar{m}^{(N)}] \rangle \\ \leq \frac{1}{N+1} \langle m^{(N+1)} + \bar{m}^{(N)}, |f[\bar{m}^{(N+1)}] - f[\bar{m}^{(N)}]| \rangle \leq \frac{C}{N^2}. \end{aligned}$$

We deduce that

$$\begin{aligned} &\langle \bar{m}^{(N)}, f[\bar{m}^{(N)}] \rangle - \langle \bar{m}^{(N+1)}, f[\bar{m}^{(N+1)}] \rangle = \langle \bar{m}^{(N)}, f[\bar{m}^{(N)}] \rangle \\ &\quad - \langle \bar{m}^{(N)} + \frac{1}{N+1}(m^{(N+1)} - \bar{m}^{(N)}), f[\bar{m}^{(N+1)}] \rangle \\ &= \langle \bar{m}^{(N)}, f[\bar{m}^{(N)}] - f[\bar{m}^{(N+1)}] \rangle - \frac{1}{N+1} \langle m^{(N+1)} - \bar{m}^{(N)}, f[\bar{m}^{(N+1)}] \rangle \\ &\leq \langle \bar{m}^{(N)}, f[\bar{m}^{(N)}] - f[\bar{m}^{(N+1)}] \rangle - \frac{1}{N+1} \langle m^{(N+1)} - \bar{m}^{(N)}, f[\bar{m}^{(N)}] \rangle + \frac{C}{N^2}. \end{aligned}$$

Analogously,

$$\begin{aligned} &\langle \bar{\mu}^{(N)}, g[\bar{\mu}^{(N)}] \rangle - \langle \bar{\mu}^{(N+1)}, g[\bar{\mu}^{(N+1)}] \rangle \\ &\leq \langle \bar{\mu}^{(N)}, g[\bar{\mu}^{(N)}] - g[\bar{\mu}^{(N+1)}] \rangle - \frac{1}{N+1} \langle \mu^{(N+1)} - \bar{\mu}^{(N)}, g[\bar{\mu}^{(N)}] \rangle + \frac{C}{N^2}. \end{aligned}$$

Therefore,

$$\begin{aligned} \varepsilon_N^{(1)} &= \langle \bar{m}^{(N)}, f[\bar{m}^{(N)}] \rangle + \langle \bar{\mu}^{(N)}, g[\bar{\mu}^{(N)}] \rangle - \langle \bar{m}^{(N+1)}, f[\bar{m}^{(N+1)}] \rangle - \langle \bar{\mu}^{(N+1)}, g[\bar{\mu}^{(N+1)}] \rangle \\ &\leq \langle \bar{m}^{(N)}, f[\bar{m}^{(N)}] - f[\bar{m}^{(N+1)}] \rangle + \langle \bar{\mu}^{(N)}, g[\bar{\mu}^{(N)}] - g[\bar{\mu}^{(N+1)}] \rangle - \frac{\varepsilon_N}{N+1} + \frac{C}{N^2}. \end{aligned}$$

On the other hand,

$$\begin{aligned}
 \varepsilon_N^{(2)} &= \Gamma[\bar{m}^{(N+1)}, \bar{\mu}^{(N+1)}](m^{(N+2)}, \mu^{(N+2)}) - \Gamma[\bar{m}^{(N)}, \bar{\mu}^{(N)}](m^{(N+1)}, \mu^{(N+1)}) \\
 &\leq \Gamma[\bar{m}^{(N+1)}, \bar{\mu}^{(N+1)}](m^{(N+2)}, \mu^{(N+2)}) - \Gamma[\bar{m}^{(N)}, \bar{\mu}^{(N)}](m^{(N+2)}, \mu^{(N+2)}) \\
 &= \langle m^{(N+2)}, f[\bar{m}^{(N+1)}] - f[\bar{m}^{(N)}] \rangle + \langle \mu^{(N+2)}, g[\bar{\mu}^{(N+1)}] - g[\bar{\mu}^{(N)}] \rangle \\
 &= \langle m^{(N+1)}, f[\bar{m}^{(N+1)}] - f[\bar{m}^{(N)}] \rangle + \langle \mu^{(N+1)}, g[\bar{\mu}^{(N+1)}] - g[\bar{\mu}^{(N)}] \rangle \\
 &\quad + \langle m^{(N+2)} - m^{(N+1)}, f[\bar{m}^{(N+1)}] - f[\bar{m}^{(N)}] \rangle + \langle \mu^{(N+2)} - \mu^{(N+1)}, g[\bar{\mu}^{(N+1)}] - g[\bar{\mu}^{(N)}] \rangle.
 \end{aligned}$$

Now, by Lemma 2.16,

$$\begin{aligned}
 \langle m^{(N+2)} - m^{(N+1)}, f[\bar{m}^{(N+1)}] - f[\bar{m}^{(N)}] \rangle &\leq \frac{C_f}{N} d_M(m^{(N+1)}, m^{(N+2)}), \\
 \langle \mu^{(N+2)} - \mu^{(N+1)}, g[\bar{\mu}^{(N+1)}] - g[\bar{\mu}^{(N)}] \rangle &\leq \frac{C_g}{N} W_1(\mu^{(N+1)}, \mu^{(N+2)}).
 \end{aligned}$$

Therefore

$$\begin{aligned}
 \varepsilon_N^{(2)} &\leq \langle m^{(N+1)}, f[\bar{m}^{(N+1)}] - f[\bar{m}^{(N)}] \rangle + \langle \mu^{(N+1)}, g[\bar{\mu}^{(N+1)}] - g[\bar{\mu}^{(N)}] \rangle \\
 &\quad + \frac{C}{N} [d_M(m^{(N+1)}, m^{(N+2)}) + W_1(\mu^{(N+1)}, \mu^{(N+2)})].
 \end{aligned}$$

Letting

$$\delta_N = C \left[d_M(m^{(N+1)}, m^{(N+2)}) + W_1(\mu^{(N+1)}, \mu^{(N+2)}) + \frac{1}{N} \right],$$

we get

$$\begin{aligned}
 \varepsilon_{N+1} - \varepsilon_N &= \varepsilon_N^{(1)} + \varepsilon_N^{(2)} \leq \langle \bar{m}^{(N)}, f[\bar{m}^{(N)}] - f[\bar{m}^{(N+1)}] \rangle + \langle \bar{\mu}^{(N)}, g[\bar{\mu}^{(N)}] - g[\bar{\mu}^{(N+1)}] \rangle - \frac{\varepsilon_N}{N+1} \\
 &\quad + \langle m^{(N+1)}, f[\bar{m}^{(N+1)}] - f[\bar{m}^{(N)}] \rangle + \langle \mu^{(N+1)}, g[\bar{\mu}^{(N+1)}] - g[\bar{\mu}^{(N)}] \rangle + \frac{\delta_N}{N} \\
 &= \langle m^{(N+1)} - \bar{m}^{(N)}, f[\bar{m}^{(N+1)}] - f[\bar{m}^{(N)}] \rangle + \langle \mu^{(N+1)} - \bar{\mu}^{(N)}, g[\bar{\mu}^{(N+1)}] - g[\bar{\mu}^{(N)}] \rangle \\
 &\quad - \frac{\varepsilon_N}{N+1} + \frac{\delta_N}{N} \\
 &= (N+1) \left[\langle \bar{m}^{(N+1)} - \bar{m}^{(N)}, f[\bar{m}^{(N+1)}] - f[\bar{m}^{(N)}] \rangle + \langle \bar{\mu}^{(N+1)} - \bar{\mu}^{(N)}, g[\bar{\mu}^{(N+1)}] - g[\bar{\mu}^{(N)}] \rangle \right] \\
 &\quad - \frac{\varepsilon_N}{N+1} + \frac{\delta_N}{N} \\
 &\leq -\frac{\varepsilon_N}{N+1} + \frac{\delta_N}{N},
 \end{aligned}$$

where the last inequality comes from the Lasry–Lions monotonicity condition. Observe that $\delta_N \rightarrow 0$ by Corollary 2.18. By Lemma 3.1 in [Had17], we conclude that $\varepsilon_N \rightarrow 0$ as $N \rightarrow \infty$. Let $((m, \mu), (\bar{m}, \bar{\mu}))$ be a cluster point of the sequence $((m^{(N+1)}, \mu^{(N+1)}), (\bar{m}^{(N)}, \bar{\mu}^{(N)}))_{N \geq 1}$ for the topology $\tilde{\tau}_p \otimes \tau_p$ and let us show that $(m, \mu) = (\bar{m}, \bar{\mu})$, which implies that (m, μ) is an LP MFG Nash equilibrium by continuity of the map Θ (see Lemma 2.17). First note that since $(m^{(N+1)}, \mu^{(N+1)}) = \Theta(\bar{m}^{(N)}, \bar{\mu}^{(N)})$, and Θ is continuous, we obtain $(m, \mu) = \Theta(\bar{m}, \bar{\mu})$. Let

$(\tilde{m}, \tilde{\mu}) \in \mathcal{R}$. We have

$$\Gamma[\bar{m}^{(N)}, \bar{\mu}^{(N)}](m^{(N+1)}, \mu^{(N+1)}) \geq \Gamma[\bar{m}^{(N)}, \bar{\mu}^{(N)}](\tilde{m}, \tilde{\mu}).$$

By definition of ε_N ,

$$\Gamma\bar{m}^{(N)}, \bar{\mu}^{(N)} \geq \Gamma[\bar{m}^{(N)}, \bar{\mu}^{(N)}](\tilde{m}, \tilde{\mu}) - \varepsilon_N.$$

Taking the limit $N \rightarrow \infty$, we obtain $\Gamma\bar{m}, \bar{\mu} \geq \Gamma[\bar{m}, \bar{\mu}](\tilde{m}, \tilde{\mu})$. Since $(\tilde{m}, \tilde{\mu})$ was arbitrary in \mathcal{R} , we get $(\bar{m}, \bar{\mu}) = \Theta(\bar{m}, \bar{\mu}) = (m, \mu)$, i.e. $(\bar{m}, \bar{\mu}) = (m, \mu)$ is the unique LP MFG Nash equilibrium. \square

Remark 2.20. *The proof follows some of the steps given in [Had17], but is based on some new results due to our setting of optimal stopping MFGs (in particular, the flow of measures is discontinuous). More precisely, one needs to establish specific estimates using appropriate distances.*

3 Linear programming algorithm for MFGs with *pure control* and *absorption*

In this section, we illustrate the LPFP algorithm for MFGs with pure control and absorption, its convergence following by the same approach developed in the case of optimal stopping (see Theorem 2.7). In the setting of MFGs with pure control and absorption, the players control their dynamics up to the exit time from a given set \mathcal{O} , when they leave the game. For the reader's convenience, we keep the same notations as in the optimal stopping case with some adaptations of the definitions.

Let U be the set of flows of measures on $\bar{\mathcal{O}} \times A$, $(m_t)_{t \in [0, T]}$, such that: for every $t \in [0, T]$, m_t is a Borel finite signed measure on $\bar{\mathcal{O}} \times A$, for every $B \in \mathcal{B}(\bar{\mathcal{O}} \times A)$, the mapping $t \mapsto m_t(B)$ is measurable, and $\int_0^T |m_t|(\bar{\mathcal{O}} \times A) dt < \infty$, where $|m_t|$ is the total variation measure of m_t . The definitions of \tilde{U} , \tilde{U}_p , V and V_p from the previous section are adapted in a similar way. The topology τ_p denotes, as in the previous section, the weak topology with respect to continuous functions with p -growth. The topology $\bar{\tau}_p$ stands for the stable topology in V_p where test functions of (t, x, a) are allowed to be only measurable with respect to t and have p -growth.

By the disintegration theorem, for each $(m_t)_{t \in [0, T]} \in V$, there exists a mapping $\nu_{t,x} : [0, T] \times \bar{\mathcal{O}} \rightarrow \mathcal{P}(A)$ such that for each $B \in \mathcal{B}(A)$, the function $(t, x) \mapsto \nu_{t,x}(B)$ is $\mathcal{B}([0, T] \times \bar{\mathcal{O}})$ -measurable, and

$$m_t(dx, da) dt = \nu_{t,x}(da) m_t^x(dx) dt,$$

where $m_t^x(dx) := \int_A m_t(dx, da)$. Here $\nu_{t,x}$ is interpreted as a Markovian relaxed control and in the case when $\nu_{t,x}$ is a dirac mass, it is called Markovian strict control (see Chapter 3 for more details).

We define the parabolic boundary as the set $\Sigma = ([0, T] \times \partial\mathcal{O}) \cup (\{T\} \times \bar{\mathcal{O}})$. We are given constants $q > p \geq 1 \vee r$, where $r \in [0, 2]$ and $q \geq 2$, and the following functions:

$$\begin{aligned} b &: [0, T] \times \mathbb{R} \times A \rightarrow \mathbb{R} \times \mathbb{R}, \\ \sigma &: [0, T] \times \mathbb{R} \rightarrow \mathbb{R} \times \mathbb{R}, \\ f &: [0, T] \times \bar{\mathcal{O}} \times \mathcal{P}_p^{sub}(\bar{\mathcal{O}}) \times A \rightarrow \mathbb{R}, \\ g &: \Sigma \times \mathcal{P}_p(\Sigma) \rightarrow \mathbb{R}. \end{aligned}$$

Consider the following assumptions, under which existence of LP MFG Nash equilibria can be shown.

Assumption 3.1.

- (1) $m_0^* \in \mathcal{P}_q(\bar{\mathcal{O}})$.
- (2) \mathcal{O} is an open interval and for all $(t, x) \in [0, T] \times \mathbb{R}$, $\sigma^2(t, x) \geq c_\sigma$ for some $c_\sigma > 0$.
- (3) The functions $(t, x, a) \mapsto b(t, x, a)$ and $(t, x) \mapsto \sigma(t, x)$ are jointly measurable and continuous in (x, a) and x respectively, for each $t \in [0, T]$. Moreover, there exists a constant $c_1 > 0$ such that for all $(t, x, y, a) \in [0, T] \times \mathbb{R} \times \mathbb{R} \times A$,

$$|b(t, x, a) - b(t, y, a)| + |\sigma(t, x) - \sigma(t, y)| \leq c_1 |x - y|,$$

$$|b(t, x, a)| \leq c_1 [1 + |x|], \quad \sigma^2(t, x) \leq c_1 [1 + |x|^r].$$

- (4) The function $(t, x, \eta, a) \mapsto f(t, x, \eta, a)$ is jointly measurable and continuous in (x, η, a) for each $t \in [0, T]$. The function g is continuous. Moreover, there exists a constant $c_2 > 0$ such that for all $(t, x, \eta, a, (\bar{t}, \bar{x}), \mu) \in [0, T] \times \bar{\mathcal{O}} \times \mathcal{P}_p^{sub}(\bar{\mathcal{O}}) \times A \times \Sigma \times \mathcal{P}_p(\Sigma)$,

$$|f(t, x, \eta, a)| \leq c_2 \left[1 + |x|^p + \int_{\bar{\mathcal{O}}} |z|^p \eta(dz) \right], \quad |g(\bar{t}, \bar{x}, \mu)| \leq c_2 \left[1 + |\bar{x}|^p + \int_{\Sigma} |z|^p \mu(ds, dz) \right].$$

We give below the linear programming formulation of the MFG problem with pure control and absorption.

Definition 3.2. Let \mathcal{R} be the set of pairs $(m, \mu) \in V_p \times \mathcal{P}_p(\Sigma)$, such that for all $u \in C_b^{1,2}([0, T] \times \bar{\mathcal{O}})$,

$$\langle \mu, u \rangle = \langle m_0^*, u(0, \cdot) \rangle + \langle m, \partial_t u + \mathcal{L}u \rangle,$$

where

$$\mathcal{L}u(t, x, a) := b(t, x, a) \partial_x u(t, x) + \frac{\sigma^2}{2}(t, x) \partial_{xx} u(t, x).$$

Definition 3.3 (LP formulation of the MFG problem). For $(\bar{m}, \bar{\mu}) \in V_p \times \mathcal{P}_p(\Sigma)$, let $\Gamma[\bar{m}, \bar{\mu}] : V_p \times \mathcal{P}_p(\Sigma) \rightarrow \mathbb{R}$ be defined by

$$\Gamma[\bar{m}, \bar{\mu}](m, \mu) = \langle m, f[\bar{m}] \rangle + \langle \mu, g[\bar{\mu}] \rangle,$$

where

$$f[\bar{m}](t, x) := f(t, x, \bar{m}_t), \quad (t, x) \in [0, T] \times \bar{\mathcal{O}},$$

$$g[\bar{\mu}](t, x) := g(t, x, \bar{\mu}), \quad (t, x) \in \Sigma.$$

We say that $(m^*, \mu^*) \in V_p \times \mathcal{P}_p(\Sigma)$ is an LP MFG Nash equilibrium if $(m^*, \mu^*) \in \mathcal{R}$ and for all $(m, \mu) \in \mathcal{R}$, $\Gamma[m^*, \mu^*](m, \mu) \leq \Gammam^*, \mu^*$.

Observe that, for a given $(\bar{m}, \bar{\mu}) \in V_p \times \mathcal{P}_p(\Sigma)$, the instantaneous reward function f only depends on the marginal in x of \bar{m} , i.e. \bar{m}^x . Henceforth, f does not depend on the distribution of the controls. Note that in this setting, due to the absorption feature of the game, the flow of measures $(m_t)_t$ is not necessarily continuous. Under Assumption 3.1, one can show existence of an LP MFG Nash equilibrium by using the tools developed in the section on optimal stopping, which allow to prove existence in a much more general framework than in [DLT21] (in particular, f, g, b and σ might have polynomial growth with respect to (x, m, μ) and the reward maps f and g are allowed to have a general dependence on the measures m , resp. μ).

Assumption 3.4. *We assume the following:*

- (1) *There exist functions $f_1 : [0, T] \times \bar{\mathcal{O}} \times \mathcal{P}_p^{\text{sub}}(\bar{\mathcal{O}}) \rightarrow \mathbb{R}$ and $f_2 : [0, T] \times \bar{\mathcal{O}} \times A \rightarrow \mathbb{R}$ satisfying the same conditions as f and such that $f = f_1 + f_2$.*
- (2) *For each $(\bar{m}, \bar{\mu}) \in \mathcal{R}$, there exists a unique maximizer of $\Gamma[\bar{m}, \bar{\mu}]$ on \mathcal{R} .*
- (3) *The Lasry–Lions monotonicity condition holds: for all (m, μ) and $(\tilde{m}, \tilde{\mu})$ in $V_p \times \mathcal{P}_p(\Sigma)$,*

$$\langle m^x - \tilde{m}^x, f_1[m^x] - f_1[\tilde{m}^x] \rangle + \langle \mu - \tilde{\mu}, g[\mu] - g[\tilde{\mu}] \rangle \leq 0.$$

- (4) *There exist constants $c_f \geq 0$ and $c_g \geq 0$ such that for all $t \in [0, T]$, $x, x' \in \bar{\mathcal{O}}$, $\eta, \eta' \in \mathcal{P}_p^{\text{sub}}(\bar{\mathcal{O}})$, $(\bar{t}, \bar{x}), (\bar{t}', \bar{x}') \in \Sigma$, $\mu, \mu' \in \mathcal{P}_p(\Sigma)$,*

$$|f_1(t, x, \eta) - f_1(t, x, \eta')| \leq c_f(1 + |x|) \int_{\bar{\mathcal{O}}} (1 + |z|^p) |\eta - \eta'| (dz),$$

$$|f_1(t, x, \eta) - f_1(t, x, \eta') - f_1(t, x', \eta) + f_1(t, x', \eta')| \leq c_f |x - x'| \int_{\bar{\mathcal{O}}} (1 + |z|^p) |\eta - \eta'| (dz),$$

$$|g(\bar{t}, \bar{x}, \mu) - g(\bar{t}, \bar{x}, \mu')| \leq c_g(1 + |\bar{x}|) \int_{\Sigma} (1 + |z|^p) |\mu - \mu'| (ds, dz),$$

$$|g(\bar{t}, \bar{x}, \mu) - g(\bar{t}, \bar{x}, \mu') - g(\bar{t}', \bar{x}', \mu) + g(\bar{t}', \bar{x}', \mu')| \leq c_g(|\bar{t} - \bar{t}'| + |\bar{x} - \bar{x}'|) \int_{\Sigma} (1 + |z|^p) |\mu - \mu'| (ds, dz).$$

For sufficient conditions on the coefficients under which the above assumptions hold, the reader is referred to Section 4.3. Using the same arguments as in Proposition 2.6, one can show that there exists at most one LP MFG Nash equilibrium.

We propose the following algorithm for computing the LP MFG Nash equilibrium.

Algorithm 2: LPFP algorithm (MFGs with pure control and absorption)

Data: A number of steps N for the equilibrium approximation; a pair $(\bar{m}^{(0)}, \bar{\mu}^{(0)}) \in \mathcal{R}$;

Result: Approximate LP MFG Nash equilibrium

1 **for** $k = 0, 1, \dots, N - 1$ **do**

2 Compute a linear programming best response $(m^{(k+1)}, \mu^{(k+1)})$ to $(\bar{m}^{(k)}, \bar{\mu}^{(k)})$ by solving the linear programming problem

$$\arg \max_{(m, \mu) \in \mathcal{R}} \Gamma[\bar{m}^{(k)}, \bar{\mu}^{(k)}](m, \mu).$$

3 Set $(\bar{m}^{(k+1)}, \bar{\mu}^{(k+1)}) := \frac{k}{k+1}(\bar{m}^{(k)}, \bar{\mu}^{(k)}) + \frac{1}{k+1}(m^{(k+1)}, \mu^{(k+1)}) = \frac{1}{k+1} \sum_{\nu=1}^{k+1} (m^{(\nu)}, \mu^{(\nu)})$

4 **end**

Using the topology of the convergence in measure (which is denoted by $\tilde{\tau}_p$) for the marginals m^x given by $m_t^x(dx) = \int_A m_t(dx, da)$ and appropriate estimates (given in terms of well chosen metrics), the convergence of the algorithm follows by similar arguments as in Theorem 2.7.

Theorem 3.5 (*Convergence of the algorithm*). *Let Assumptions 3.1 and 3.4 hold true and consider the sequences $(\bar{m}^{(N)}, \bar{\mu}^{(N)})_{N \geq 1}$ and $(m^{(N)}, \mu^{(N)})_{N \geq 1}$ generated by the Algorithm 2. Then both sequences converge in the product topology $\tilde{\tau}_p \otimes \tau_p$ to the unique LP MFG Nash equilibrium (m^*, μ^*) . Furthermore, the sequences $(\bar{m}^{x, (N)})_{N \geq 1}$ and $(m^{x, (N)})_{N \geq 1}$ converge in the topology $\tilde{\tau}_p$ to $m^{x, *}$.*

Numerical example. We now illustrate the LPFP algorithm for MFGs with pure control and absorption through a numerical example. In this example, let $T = 1$ and assume that the state of the representative player belongs to the domain $\bar{\mathcal{O}}$ with $\mathcal{O} =] - 2, 2[$, and is given by

$$X_t^\alpha = X_0^\alpha + \int_0^t \alpha_s ds + W_t,$$

i.e. $b(t, x, a) = a$ and $\sigma(t, x) = 1$. The control α is assumed to take values in $A = [-1, 1]$. The initial states of the players are distributed according to the law $\mathcal{N}(0, 0.1)$ truncated to \mathcal{O} .

Before exiting the game at time $\tau_{\bar{\mathcal{O}}}^{X^\alpha} \wedge T$, the representative player receives an instantaneous reward

$$-10 \int_{[-2, 2]} e^{-|X_t^\alpha - y|} \eta(dy) - 2||X_t^\alpha| - 1| - \alpha_t^2,$$

and the terminal reward at exit time is given by $-|X_{\tau_{\bar{\mathcal{O}}}^{X^\alpha} \wedge T}^\alpha|$, that is:

$$f(t, x, \eta, a) = -10 \int_{[-2, 2]} e^{-|x - y|} \eta(dy) - 2||x| - 1| - a^2, \quad g(t, x, \mu) = -|x|.$$

During the game, players have an incentive to be near the points -1 or 1 , and converge to the point 0 at the final time, but at the same time the mean-field dependence creates an incentive to be far from other players.

In Figure 4.3, we observe the distribution of the players still in the game over time together

with the exit distributions at the boundary and the distribution of the players at the terminal time. Figure 4.4 shows the Markovian control given by $\bar{\alpha}^{(N)}(t, x) = \int_A a \bar{\nu}_{t,x}^{(N)}(da)$ (which is the optimal control since A is convex, b is affine in a and f is strictly concave in a , see the proof of Theorem 4.10 for more details). We see that players starting in a positive state use a positive control at the beginning to be near the point 1 and switch to a negative control towards the end of the game to be close to 0. On the other hand, the players starting in a negative state use the opposite strategy. Finally, Figure 4.5 illustrates the convergence of the algorithm through the measurement of the exploitability.

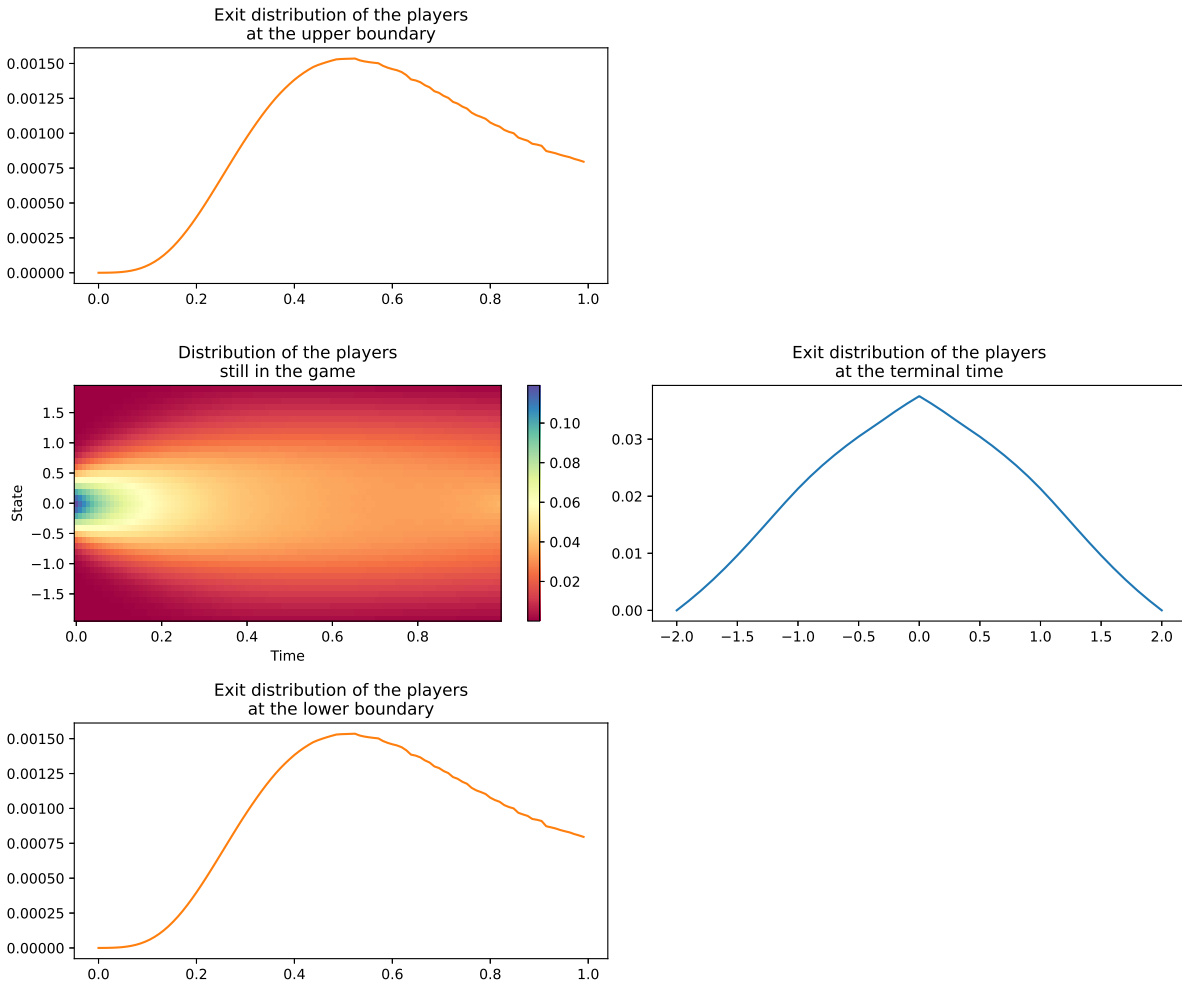


Figure 4.3 – Equilibrium distributions at the final iteration. Top: $\bar{\mu}^{(N)}(\cdot \times \{2\})$. Middle left: $\bar{m}^{x,(N)}$, distribution of the players still in the game over time. Middle right: $\bar{\mu}^{(N)}(\{T\} \times \cdot)$. Bottom: $\bar{\mu}^{(N)}(\cdot \times \{-2\})$.

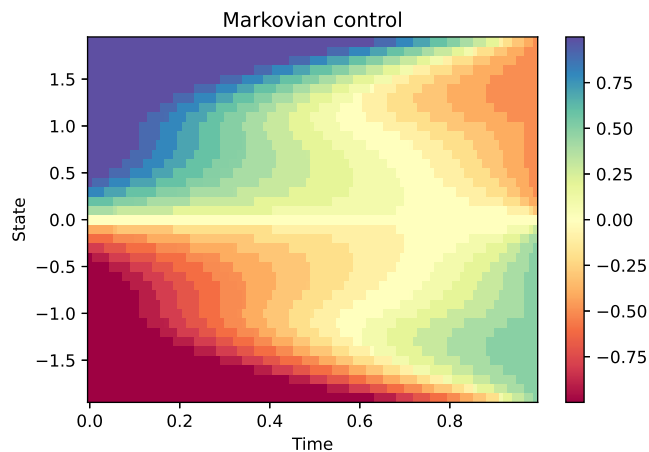


Figure 4.4 – Markovian control at the equilibrium.

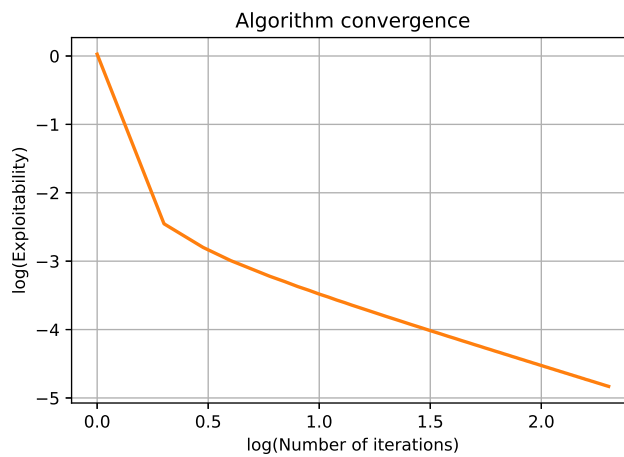


Figure 4.5 – Convergence of the algorithm: log-log plot of the exploitability.

4 Appendix

4.1 Probabilistic representation

In the case when admissible measures $m_t(dx)dt$ (resp. μ) have the support included in some time-dependent domain O (resp. its complement O^c), we obtain the following probabilistic representation.

Theorem 4.1. *Suppose that Assumption 2.1 holds. Let $(m, \mu) \in \mathcal{R}$ and O be an open subset of $[0, T] \times \bar{O}$. Assume that $\mu(O) = 0$ and $\int_{O^c} m_t(dx)dt = 0$. Moreover, suppose that $\tau_O = \tau_{\bar{O}}$ \mathbb{P} -a.s., where*

$$\tau_O = \inf\{t \geq 0 : (t, X_t) \notin O\}, \quad \tau_{\bar{O}} = \inf\{t \geq 0 : (t, X_t) \notin \bar{O}\},$$

and X is the unique strong solution to the SDE

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t, \quad t \in [0, T], \quad \mathbb{P} \circ X_0^{-1} = m_0^*.$$

Then $\tau = \tau_O$ \mathbb{P} -a.s.

Proof. By Theorem 5.20 in Chapter 3, there exists $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, W, \tau, X)$, such that $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ is a filtered probability space, W is an \mathbb{F} -Brownian motion, τ is an \mathbb{F} -stopping time such that $\tau \leq T \wedge \tau_{\bar{O}}^X$ \mathbb{P} -a.s. and X is an \mathbb{F} -adapted process verifying

$$X_t = X_0 + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dW_s, \quad t \in [0, T], \quad \mathbb{P} \circ X_0^{-1} = m_0^*,$$

such that we have the following probabilistic representation of (m, μ) :

$$m_t(B) = \mathbb{E}^{\mathbb{P}} [\mathbf{1}_B(X_t) \mathbf{1}_{t < \tau}], \quad B \in \mathcal{B}(\bar{O}), \quad t - a.e.$$

$$\mu = \mathbb{P} \circ (\tau, X_\tau)^{-1}.$$

Let us show that $\tau = \tau_O$ \mathbb{P} -a.s. Using that μ is supported in O^c we get

$$1 = \mu(O^c) = \mathbb{P}((\tau, X_\tau) \in O^c). \quad (4.3)$$

Now, since $m_t(dx)dt$ is supported in O ,

$$0 = \int_{O^c} m_t(dx)dt = \mathbb{E}^{\mathbb{P}} \left[\int_0^\tau \mathbf{1}_{O^c}(t, X_t)dt \right],$$

which means that

$$(\mathbb{P} \otimes \lambda)(\{(\omega, t) \in \Omega \times [0, T] : (t, X_t(\omega)) \in O^c, t < \tau(\omega)\}) = 0. \quad (4.4)$$

By equality (4.3), we have that $\tau_O \leq \tau$ \mathbb{P} -a.s. Let us show now that with probability 1 we have $\tau \leq \tau_{\bar{O}}$, where $\tau_{\bar{O}} = \inf\{t \geq 0 : (t, X_t) \notin \bar{O}\}$. Assume that $\mathbb{P}(\tau > \tau_{\bar{O}}) > 0$. Using the equality (4.4), we get

$$\mathbb{P}(\lambda(\{t \in [0, T] : (t, X_t) \notin O, t < \tau\}) = 0) = 1.$$

Define the set

$$B = \{\tau > \tau_{\bar{O}}\} \cap \{\lambda(\{t \in [0, T] : (t, X_t) \notin O, t < \tau\}) = 0\} \in \mathcal{F}.$$

Let us show that $B = \emptyset$, which will contradict the fact that $\mathbb{P}(B) > 0$. It is sufficient to prove that for $\omega \in \{\tau > \tau_{\bar{O}}\}$, we have $\omega \notin \{\lambda(\{t \in [0, T] : (t, X_t) \notin O, t < \tau\}) = 0\}$. Let $\omega \in \{\tau > \tau_{\bar{O}}\}$ be fixed. Since $t \mapsto (t, X_t(\omega))$ is continuous, we have $(\tau_{\bar{O}}(\omega), X_{\tau_{\bar{O}}(\omega)}(\omega)) \in \bar{O}$. Moreover, by the definition of the infimum there exists $\tau_1(\omega) \in]\tau_{\bar{O}}(\omega), \tau(\omega)[$ such that $(\tau_1(\omega), X_{\tau_1(\omega)}(\omega)) \notin \bar{O}$. Using again the continuity of $t \mapsto (t, X_t(\omega))$ and the fact that $(\bar{O})^c$ is open, we can find $\tau_1^l(\omega)$

and $\tau_1^r(\omega)$ such that

$$\tau_{\bar{O}}(\omega) < \tau_1^l(\omega) < \tau_1(\omega) < \tau_1^r(\omega) < \tau(\omega) \quad \text{and} \quad \forall t \in]\tau_1^l(\omega), \tau_1^r(\omega)[, \quad (t, X_t(\omega)) \notin \bar{O}.$$

In particular,

$$\lambda(\{t \in [0, T] : (t, X_t(\omega)) \notin O, t < \tau(\omega)\}) \geq \lambda(]\tau_1^l(\omega), \tau_1^r(\omega)[) > 0.$$

This shows that B is empty and contradicts the fact that it has a positive probability. We conclude that \mathbb{P} -a.s. $\tau_O \leq \tau \leq \tau_{\bar{O}}$. Since by assumption we have $\tau_O = \tau_{\bar{O}}$ \mathbb{P} -a.s., we get $\tau = \tau_O$ \mathbb{P} -a.s. \square

4.2 Sufficient conditions for Assumption 2.5

In this section, we provide sufficient conditions on b, σ, f, g, m_0^* and \mathcal{O} such that Assumption 2.5 is verified.

Assumption 4.2. *We assume the following:*

- (1) $\mathcal{O} = \mathbb{R}$.
- (2) m_0^* has a continuous and positive density on $L^2(\mathbb{R})$.
- (3) σ is positive, does not depend on time and satisfies the uniform ellipticity condition. Moreover $\sigma \in C^1(\mathbb{R})$,

$$\sup_{(t,x) \in [0,T] \times \mathbb{R}} \left[\frac{|b(t,x)|}{\sigma(x)} + |\partial_x \sigma(x)| \right] < \infty.$$

- (4) f does not depend on the measure and there exists $\bar{c}_f \geq 0$ such that for all $t \in [0, T]$, $x, x' \in \mathbb{R}$,

$$|f(t, x) - f(t, x')| \leq \bar{c}_f |x - x'|, \quad |f(t, x)| \leq \bar{c}_f (1 + |x|).$$

- (5) g has the form

$$g(t, x, \mu) = g_1(t, x) g_2 \left(\int_{[0,T] \times \mathbb{R}} g_1(s, y) \mu(ds, dy) \right) + g_3(t, x),$$

where g_2 is nonincreasing. There exists $\bar{c}_g \geq 0$ and $\beta \in]0, 1]$ such that for all $t, t' \in [0, T]$, $x, x' \in \mathbb{R}$,

$$|g_1(t, x) - g_1(t', x')| \leq \bar{c}_g (|t - t'| + |x - x'|), \quad |g_1(t, x)| \leq \bar{c}_g (1 + |x|),$$

$$|g_2(x) - g_2(x')| \leq \bar{c}_g |x - x'|,$$

$$|g_3(t, x) - g_3(t', x')| \leq \bar{c}_g (|t - t'|^\beta + |x - x'|), \quad |g_3(t, x)| \leq \bar{c}_g (1 + |x|).$$

- (6) We assume $g_1 \in C^{1,2}([0, T] \times \mathbb{R})$, $g_3 \in C^{1,2}([0, T] \times \mathbb{R})$, $\partial_x g_1 \in C_b([0, T] \times \mathbb{R})$ and $\partial_x g_3 \in C_b([0, T] \times \mathbb{R})$. Moreover, for each $t \in [0, T]$ and $\mu \in \mathcal{P}_p([0, T] \times \mathbb{R})$,

$$x \mapsto (f + \partial_t g + \mathcal{L}g)(t, x, \mu)$$

is increasing. Finally, for each $t \leq t'$, $x \leq x'$ and $\mu \in \mathcal{P}_p([0, T] \times \mathbb{R})$,

$$(f + \partial_t g + \mathcal{L}g)(t', x, \mu) \leq (f + \partial_t g + \mathcal{L}g)(t, x', \mu), \quad b(t', x) \leq b(t, x).$$

The second and third conditions of Assumption 2.5 are easily verified under the above requirements. In the next theorem, we prove that the first condition is also satisfied.

Theorem 4.3. *Under the Assumptions 2.1 and 4.2, for each $(\bar{m}, \bar{\mu}) \in \mathcal{R}$, there exists a unique maximizer of $\Gamma[\bar{m}, \bar{\mu}]$ on \mathcal{R} .*

Proof. Fix $(\bar{m}, \bar{\mu}) \in \mathcal{R}$. Let us show that there exists a unique maximizer of $\Gamma[\bar{m}, \bar{\mu}]$ on \mathcal{R} . Under our assumptions, this map writes

$$\Gamma[\bar{m}, \bar{\mu}](m, \mu) = \int_0^T \int_{\mathbb{R}} f(t, x) m_t(dx) dt + \int_{[0, T] \times \mathbb{R}} g(t, x, \bar{\mu}) \mu(dt, dx).$$

We characterize the maximizer via a probabilistic approach, which allows to deduce the uniqueness result. Consider a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ supporting a Brownian motion $W = (W_t)_{t \in [0, T]}$. Given $t \in [0, T]$, we denote by \mathbb{F}^t the filtration given by $\mathcal{F}_s^t = \sigma(W_r^t, t \leq r \leq s) \vee \mathcal{N}$, $s \geq t$. Here \mathcal{N} is the set of \mathbb{P} -null sets and $W_s^t = W_s - W_t$, $s \geq t$, is the translated Brownian motion. Denote by \mathcal{T}_t the set of stopping times with respect to this filtration with values in $[t, T]$. Consider the value function (to simplify the notation, we omit the dependence on $\bar{\mu}$)

$$v(t, x) = \sup_{\tau \in \mathcal{T}_t} \mathbb{E} \left[\int_t^\tau f(s, X_s^{t, x}) ds + g(\tau, X_\tau^{t, x}, \bar{\mu}) \right]. \quad (4.5)$$

where $X^{t, x}$ is the unique strong solution to the SDE

$$dX_s = b(s, X_s) ds + \sigma(X_s) dW_s; \quad X_t = x.$$

We denote by $\mathcal{C} = \{(t, x) \in [0, T] \times \mathbb{R} : v(t, x) > g(t, x, \bar{\mu})\}$ the continuation region, by $\bar{\mathcal{C}}$ its closure and by $\mathcal{S} = \{(t, x) \in [0, T] \times \mathbb{R} : v(t, x) = g(t, x, \bar{\mu})\}$ the stopping region. We divide now the proof into several steps.

First Step: Properties of the value function and optimal stopping boundary. Under our assumptions, it can be shown by using standard arguments that the value function v is jointly continuous. For sake of clarity we provide this result in Lemma 4.6. In particular, we deduce that the continuation region \mathcal{C} is an open subset of $[0, T] \times \mathbb{R}$ and that the stopping region \mathcal{S} is a closed subset of $[0, T] \times \mathbb{R}$.

By the comparison theorem for SDEs and using that $z \mapsto (f + \partial_t g + \mathcal{L}g)(s, z, \bar{\mu})$ is increasing we can define an extended real-valued function $c : [0, T] \rightarrow [-\infty, \infty]$ by $c(t) = \inf\{x \in \mathbb{R} : v(t, x) > g(t, x, \bar{\mu})\}$, such that

$$\mathcal{C} = \{(t, x) \in [0, T] \times \mathbb{R} : x > c(t)\}, \quad \mathcal{S} = \{(t, x) \in [0, T] \times \mathbb{R} : x \leq c(t)\}.$$

Since \mathcal{S} is the hypograph of c and it is a closed set, we deduce that c is upper semicontinuous.

Consider the filtration $\mathbb{G}^t = (\mathcal{G}_s^t)_{s \in [0, T-t]}$ given by $\mathcal{G}_s^t = \mathcal{F}_{t+s}^t$, $s \in [0, T-t]$. We denote by \mathcal{S}_t the set of stopping times with respect to this filtration with values in $[0, T-t]$. One can show that

$$v(t, x) = \sup_{\tau \in \mathcal{S}_t} \mathbb{E} \left[\int_0^\tau f(t+s, X_{t+s}^{t,x}) ds + g(t+\tau, X_{t+\tau}^{t,x}, \bar{\mu}) \right]. \quad (4.6)$$

Observe that, for a fixed time t , the stopping times in \mathcal{S}_t and the process $X^{t,x}$ are functionals of the translated Brownian motion $(W_s^t)_{s \in [0, T-t]} := (W_{t+s} - W_t)_{s \in [0, T-t]}$. Since $(W_s^t)_{s \in [0, T-t]}$ and $(W_s)_{s \in [0, T-t]}$ have the same law, without loss of generality we can assume that the stopping times in \mathcal{S}_t are with respect to the (completed) filtration of W (we keep the same notation) and replace $X_{t+}^{t,x}$ in (4.6) by the process $Y^{0,x}[t]$ which follows the dynamics

$$Y_s^{0,x}[t] = x + \int_0^s b(t+u, Y_u^{0,x}[t]) du + \int_0^s \sigma(Y_s^{0,x}[t]) dW_u.$$

We have $\mathcal{S}_{t'} \subset \mathcal{S}_t$ for $t \leq t'$. Consider $t, t' \in [0, T]$ such that $t \leq t'$ and let $x \in \mathbb{R}$, we are going to show that $v(t', x) - g(t', x, \bar{\mu}) \leq v(t, x) - g(t, x, \bar{\mu})$. By the comparison theorem for SDEs, we get

$$\mathbb{P} \left(Y_s^{0,x}[t] \geq Y_s^{0,x}[t'], \quad \forall s \in [0, T-t'] \right) = 1.$$

In fact, let $b_1(s, x) = b(t+s, x)$ and $b_2(s, x) = b(t'+s, x)$. For all $(s, x) \in [0, T-t'] \times \mathbb{R}$ we have $b_1(s, x) \geq b_2(s, x)$. Let τ' be an optimal stopping time for

$$v(t', x) = \sup_{\tau \in \mathcal{S}_{t'}} \mathbb{E} \left[\int_0^\tau f(t'+s, Y_s^{0,x}[t']) ds + g(t'+\tau, Y_\tau^{0,x}[t'], \bar{\mu}) \right].$$

Since $\mathcal{S}_{t'} \subset \mathcal{S}_t$, we get $\tau' \in \mathcal{S}_t$, henceforth

$$\begin{aligned} & (v-g)(t', x, \bar{\mu}) - (v-g)(t, x, \bar{\mu}) \\ & \leq \mathbb{E} \left[\int_0^{\tau'} \left[(f + \partial_t g + \mathcal{L}g)(t'+s, Y_s^{0,x}[t'], \bar{\mu}) - (f + \partial_t g + \mathcal{L}g)(t+s, Y_s^{0,x}[t], \bar{\mu}) \right] ds \right] \\ & \leq 0. \end{aligned}$$

We deduce that c is nondecreasing.

Since c is upper semicontinuous and nondecreasing, it is right-continuous. In particular, the set D of discontinuities of c is countable. The optimal stopping boundary writes

$$\partial \mathcal{C} = \{(t, x) \in [0, T] \times \mathbb{R} : x = c(t)\} \cup \bigcup_{t \in D} (\{t\} \times [c(t-), c(t)]).$$

Since the set of discontinuities of c is countable, we get

$$\begin{aligned} \lambda_2(\partial \mathcal{C}) & \leq \lambda_2(\{x = c(t)\}) + \sum_{t \in D} \lambda_2(\{t\} \times [c(t-), c(t)]) = \int_{[0, T] \times \mathbb{R}} \mathbf{1}_{x=c(t)} \lambda_2(dt, dx) \\ & = \int_0^T \lambda(\{c(t)\}) \mathbf{1}_{c(t) \in \mathbb{R}} dt = 0, \end{aligned}$$

where λ_2 denotes the Lebesgue measure in \mathbb{R}^2 .

Second Step: Properties of the maximizers with respect to the continuation and stopping regions. Let (m, μ) be a maximizer of $\Gamma[\bar{m}, \bar{\mu}]$ in \mathcal{R} . We show that $\int_{\mathcal{S}} m_t(dx)dt$ and $\mu(\mathcal{C}) = 0$. By Theorem 5.20 in Chapter 3, there exist a filtered probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{F}}, \tilde{\mathbb{P}})$, an $\tilde{\mathbb{F}}$ -adapted process \tilde{X} , an $\tilde{\mathbb{F}}$ -stopping time $\tilde{\tau}$, and an $\tilde{\mathbb{F}}$ -Brownian motion \tilde{W} , such that

$$\begin{aligned}\tilde{X}_t &= \tilde{X}_0 + \int_0^t b(s, \tilde{X}_s)ds + \int_0^t \sigma(\tilde{X}_s)d\tilde{W}_s, \quad \tilde{\mathbb{P}} \circ \tilde{X}_0^{-1} = m_0^*, \\ m_t(B) &= \mathbb{E}^{\tilde{\mathbb{P}}} \left[\mathbf{1}_B(\tilde{X}_t) \mathbf{1}_{t < \tilde{\tau}} \right], \quad B \in \mathcal{B}(\mathbb{R}), \quad t - a.e., \\ \mu &= \tilde{\mathbb{P}} \circ (\tilde{\tau}, \tilde{X}_{\tilde{\tau}})^{-1}.\end{aligned}$$

Consider the optimal stopping problem in this probabilistic set up:

$$\tilde{v}(t, x) = \sup_{\tau \in \mathcal{T}_t^{\tilde{\mathbb{F}}}} \mathbb{E}^{\tilde{\mathbb{P}}} \left[\int_t^{\tau} f(s, \tilde{X}_s^{t,x})ds + g(\tau, \tilde{X}_{\tau}^{t,x}, \bar{\mu}) \right], \quad (4.7)$$

where $\tilde{X}_s^{t,x}$ satisfies the SDE

$$d\tilde{X}_s = b(s, \tilde{X}_s)ds + \sigma(\tilde{X}_s)d\tilde{W}_s, \quad \tilde{X}_t^{t,x} = x, \quad (4.8)$$

and $\mathcal{T}_t^{\tilde{\mathbb{F}}}$ is the set of stopping times with respect to $\tilde{\mathbb{F}}$ and valued in $[t, T]$. By Chapter I, Section 2, Corollary 2.9 in [PS06], the optimal stopping time is given by

$$\tilde{\tau}_{\tilde{\mathcal{C}}}^{t,x} := \inf \{ s \in [t, T] : \tilde{v}(s, \tilde{X}_s^{t,x}) = g(s, \tilde{X}_s^{t,x}, \bar{\mu}) \}.$$

By uniqueness in law of the solution of the SDE (4.8), we get $v(t, x) = \tilde{v}(t, x)$, for all $(t, x) \in [0, T] \times \mathbb{R}$. Consider the optimal stopping problem (4.7) at time 0, assuming that $\tilde{\mathbb{P}} \circ \tilde{X}_0^{-1} = m_0^*$:

$$\sup_{\tau \in \mathcal{T}_0^{\tilde{\mathbb{F}}}} \mathbb{E}^{\tilde{\mathbb{P}}} \left[\int_0^{\tau} f(s, \tilde{X}_s)ds + g(\tau, \tilde{X}_{\tau}, \bar{\mu}) \right]. \quad (4.9)$$

By measurability arguments (involving the optimal stopping time which is a measurable function of \tilde{X}), we get that

$$\int_{\mathbb{R}} v(0, x)m_0^*(dx) = \mathbb{E}^{\tilde{\mathbb{P}}} [v(0, \tilde{X}_0)] = \sup_{\tau \in \mathcal{T}_0^{\tilde{\mathbb{F}}}} \mathbb{E}^{\tilde{\mathbb{P}}} \left[\int_0^{\tau} f(s, \tilde{X}_s)ds + g(\tau, \tilde{X}_{\tau}, \bar{\mu}) \right].$$

By Theorem 2.21 in [DLT21], we get $\Gamma[\bar{m}, \bar{\mu}](m, \mu) = \int_{\mathbb{R}} v(0, x)m_0^*(dx)$. In particular, using the probabilistic representation of (m, μ) , we deduce that $\tilde{\tau}$ is an optimal stopping time for (4.9). Define the following processes

$$U_t := v(t, \tilde{X}_t) + \int_0^t f(s, \tilde{X}_s)ds,$$

$$Z_t := \int_0^t (f + \partial_t g + \mathcal{L}g)(s, \tilde{X}_s, \bar{\mu}) ds,$$

and

$$\tilde{M}_t := \int_0^t \sigma(\tilde{X}_s) \partial_x g(s, \tilde{X}_s, \bar{\mu}) d\tilde{W}_s.$$

Observe that, since U is the Snell envelope of the process $\left(\int_0^t f(s, \tilde{X}_s) ds + g(t, \tilde{X}_t, \bar{\mu})\right)_{t \in [0, T]}$, by the Doob-Meyer decomposition we get $U_t = M_t - A_t$, where M is an $\tilde{\mathbb{F}}$ -martingale and A is a nondecreasing $\tilde{\mathbb{F}}$ -predictable process with $A_0 = 0$. By Theorem D.13 in [KS98b], since

$$\left(\int_0^t f(s, \tilde{X}_s) ds + g(t, \tilde{X}_t, \bar{\mu})\right)_{t \in [0, T]}$$

is a continuous process, we get that A is continuous (and in particular M is continuous) and

$$\int_0^T \mathbf{1}_{\zeta_t > 0} dA_t = 0, \quad \tilde{\mathbb{P}} - a.s.,$$

where $\zeta_t := v(t, \tilde{X}_t) - g(t, \tilde{X}_t, \bar{\mu}) \geq 0$. By Tanaka's formula,

$$\zeta_t = \max(\zeta_t, 0) = \zeta_0 + \int_0^t \mathbf{1}_{\zeta_s > 0} d\zeta_s + \frac{1}{2} L_t^0(\zeta),$$

where $L^0(\zeta)$ is the local time of ζ in 0. We deduce that

$$\begin{aligned} \zeta_t &= \zeta_0 + \int_0^t \mathbf{1}_{\zeta_s > 0} d(U_s - Z_s - \tilde{M}_s) + \frac{1}{2} L_t^0(\zeta) \\ &= \zeta_0 + \int_0^t \mathbf{1}_{\zeta_s > 0} dM_s - \int_0^t \mathbf{1}_{\zeta_s > 0} d\tilde{M}_s - \int_0^t \mathbf{1}_{\zeta_s > 0} dA_s - \int_0^t \mathbf{1}_{\zeta_s > 0} dZ_s + \frac{1}{2} L_t^0(\zeta) \\ &= \zeta_0 + \int_0^t \mathbf{1}_{\zeta_s > 0} dM_s - \int_0^t \mathbf{1}_{\zeta_s > 0} d\tilde{M}_s - \int_0^t \mathbf{1}_{\zeta_s > 0} dZ_s + \frac{1}{2} L_t^0(\zeta). \end{aligned}$$

We finally get

$$\begin{aligned} U_t &= U_0 + Z_t + \tilde{M}_t + \int_0^t \mathbf{1}_{\zeta_s > 0} dM_s - \int_0^t \mathbf{1}_{\zeta_s > 0} d\tilde{M}_s - \int_0^t \mathbf{1}_{\zeta_s > 0} dZ_s + \frac{1}{2} L_t^0(\zeta) \\ &= U_0 + \int_0^t \mathbf{1}_{\zeta_s > 0} dM_s + \int_0^t \mathbf{1}_{\zeta_s = 0} d\tilde{M}_s + \int_0^t \mathbf{1}_{\zeta_s = 0} dZ_s + \frac{1}{2} L_t^0(\zeta). \end{aligned}$$

Using that

$$\left(U_0 + \int_0^t \mathbf{1}_{\zeta_s > 0} dM_s + \int_0^t \mathbf{1}_{\zeta_s = 0} d\tilde{M}_s\right)_{t \in [0, T]}$$

is a local martingale, by continuity of the processes and uniqueness of the semimartingale decomposition, we get

$$-A_t = \int_0^t \mathbf{1}_{\zeta_s = 0} dZ_s + \frac{1}{2} L_t^0(\zeta).$$

From the above, we deduce that the process

$$\int_0^\cdot \mathbb{1}_{v(t, \tilde{X}_t)=g(t, \tilde{X}_t, \bar{\mu})} (f + \partial_t g + \mathcal{L}g)(t, \tilde{X}_t, \bar{\mu}) dt$$

is nonincreasing. Therefore t -a.e. on $[0, T]$,

$$\mathbb{1}_{v(t, \tilde{X}_t)=g(t, \tilde{X}_t, \bar{\mu})} (f + \partial_t g + \mathcal{L}g)(t, \tilde{X}_t, \bar{\mu}) \leq 0.$$

In particular, $(t, x) \mapsto \mathbb{1}_{\mathcal{S}}(t, x) (f + \partial_t g + \mathcal{L}g)(t, x, \bar{\mu})$ is nonpositive $m_t(dx)dt$ -a.e. Since $\tilde{\tau}$ is optimal, $A_{\tilde{\tau}} = 0$, i.e.

$$\int_0^{\tilde{\tau}} \mathbb{1}_{v(t, \tilde{X}_t)=g(t, \tilde{X}_t, \bar{\mu})} (f + \partial_t g + \mathcal{L}g)(t, \tilde{X}_t, \bar{\mu}) dt = -\frac{1}{2} L_{\tilde{\tau}}^0(\zeta).$$

Since the Lebesgue measure of $\partial\mathcal{C}$ is 0, by Theorem 6 in [Jac93], we have that the local time $L^0(\zeta)$ is indistinguishable from 0. Henceforth, taking the expectation in the last equality, we get

$$\int_{\mathcal{S}} (f + \partial_t g + \mathcal{L}g)(t, x, \bar{\mu}) m_t(dx) dt = 0.$$

To simplify notation, denote by $\nu(dt, dx) = m_t(dx)dt$ (which is absolutely continuous with respect to the Lebesgue measure in $[0, T] \times \mathbb{R}$, since σ satisfies the uniform ellipticity condition). Now there exists a ν -negligible set N such that for all $(t, x) \in N^c$,

$$\mathbb{1}_{\mathcal{S}}(t, x) (f + \partial_t g + \mathcal{L}g)(t, x, \bar{\mu}) \leq 0.$$

In particular, using that for all $t \in [0, T]$, $x \mapsto (f + \partial_t g + \mathcal{L}g)(t, x, \bar{\mu})$ is increasing, if $(t, x) \in \mathring{\mathcal{S}} \cap N^c$, $(f + \partial_t g + \mathcal{L}g)(t, x, \bar{\mu}) < 0$, where $\mathring{\mathcal{S}}$ denotes the interior of \mathcal{S} . Since $\partial\mathcal{C}$ has Lebesgue measure 0, then $\partial\mathcal{S}$ has Lebesgue measure 0 (see [AB07] p. 27), and we obtain

$$\begin{aligned} 0 &= \int_{\mathcal{S}} (f + \partial_t g + \mathcal{L}g)(t, x, \bar{\mu}) \nu(dt, dx) \\ &= \int_{[0, T] \times \mathbb{R}} \mathbb{1}_{\mathring{\mathcal{S}} \cap N^c}(t, x) (f + \partial_t g + \mathcal{L}g)(t, x, \bar{\mu}) \nu(dt, dx). \end{aligned}$$

In other words, $\nu(\mathring{\mathcal{S}} \cap N^c) = 0$, which implies $\int_{\mathcal{S}} m_t(dx)dt = \nu(\mathcal{S}) = \nu(\mathring{\mathcal{S}} \cap N^c) = 0$. Let us show now that $\mu(\mathcal{C}) = 0$. Using the supermartingale property of U ,

$$\mathbb{E}^{\tilde{\mathbb{P}}} \left[v(\tilde{\tau}, \tilde{X}_{\tilde{\tau}}) + \int_0^{\tilde{\tau}} f(t, \tilde{X}_t) dt \right] \leq \mathbb{E}^{\tilde{\mathbb{P}}} [v(0, \tilde{X}_0)]$$

which implies

$$\int_{[0, T] \times \mathbb{R}} v(t, x) \mu(dt, dx) + \int_0^T \int_{\mathbb{R}} f(t, x) m_t(dx) dt \leq \int_{\mathbb{R}} v(0, x) m_0^*(dx).$$

The above inequality, together with $v \geq g$, leads to

$$\int_{[0,T] \times \mathbb{R}} (v - g)(t, x, \bar{\mu}) \mu(dt, dx) = 0$$

and henceforth

$$0 = \int_{\mathcal{C}} (v - g)(t, x, \bar{\mu}) \mu(dt, dx) + \int_{\mathcal{S}} (v - g)(t, x, \bar{\mu}) \mu(dt, dx) = \int_{\mathcal{C}} (v - g)(t, x, \bar{\mu}) \mu(dt, dx).$$

Now, since $(t, x) \mapsto (v - g)(t, x, \bar{\mu}) > 0$ on \mathcal{C} , we must have $\mu(\mathcal{C}) = 0$.

Third Step: Uniqueness of the maximizer. Assume that $(\mu^1, m^1) \in \mathcal{R}$ and $(\mu^2, m^2) \in \mathcal{R}$ are two maximizers. By the previous step, for $i = 1, 2$, $\int_{\mathcal{S}} m_t^i(dx) dt = 0$ and $\mu^i(\mathcal{C}) = 0$. By Theorem 4.1 (the assumption being verified using the same ideas as in Proposition 2 of [Che+11] with some modifications adapted to our framework, for sake of clarity we give this result in Lemma 4.7 above), there exist, for each $i = 1, 2$, a filtered probability space $(\Omega^i, \mathcal{F}^i, \mathbb{F}^i, \mathbb{P}^i)$, an \mathbb{F}^i -adapted process X^i and an \mathbb{F}^i -Brownian motion W^i such that

$$X_t^i = X_0^i + \int_0^t b(s, X_s^i) ds + \int_0^t \sigma(X_s^i) dW_s^i, \quad \mathbb{P}^i \circ (X_0^i)^{-1} = m_0^*,$$

$$m_t^i(B) = \mathbb{E}^{\mathbb{P}^i} \left[\mathbf{1}_B(X_t^i) \mathbf{1}_{t < \tau_{\mathcal{C}}^i} \right], \quad B \in \mathcal{B}(\mathbb{R}), \quad t - a.e.,$$

$$\mu^i = \mathbb{P}^i \circ \left(\tau_{\mathcal{C}}^i, X_{\tau_{\mathcal{C}}^i}^i \right)^{-1},$$

where $\tau_{\mathcal{C}}^i = \inf\{t \geq 0 : (t, X_t^i) \notin \mathcal{C}\}$. By the pathwise uniqueness of the following SDE,

$$dX_t = b(t, X_t) dt + \sigma(X_t) dW_t,$$

we get the uniqueness in law. This implies that $\mathbb{P}^1 \circ (X^1)^{-1} = \mathbb{P}^2 \circ (X^2)^{-1} =: P$ on $C([0, T])$. For $i = 1, 2$, using that $\tau_{\mathcal{C}}^i$ is $\sigma(X^i)$ -measurable, there exists a measurable map $\varphi^i : C([0, T]) \rightarrow [0, T]$ such that $\tau_{\mathcal{C}}^i = \varphi^i(X^i)$. In particular, for any bounded and measurable function $\psi : C([0, T]) \rightarrow \mathbb{R}$, $\mathbb{E}^{\mathbb{P}^1}[\psi(X^1) \varphi^1(X^1)] = \mathbb{E}^{\mathbb{P}^2}[\psi(X^2) \varphi^2(X^2)]$, that is

$$\int_{C([0, T])} \psi(x) [\varphi^1(x) - \varphi^2(x)] P(dx) = 0.$$

Taking $\psi = \varphi^1 - \varphi^2$ we deduce that $\varphi^1 = \varphi^2$ P -a.e. This is sufficient to conclude that $m^1 = m^2$ and $\mu^1 = \mu^2$. \square

Remark 4.4. *Note that the above theorem gives sufficient conditions which guarantee the representation of the unique best response as a pure solution. Furthermore, the stopping time involved in the probabilistic representation is a Markov stopping time.*

Lemma 4.5. *Under the Assumption 4.2, there exists a constant $C \geq 0$ such that for all $t \in [0, T]$ and $x, x' \in \mathbb{R}$,*

$$|g(t, x, \bar{\mu}) - g(t, x', \bar{\mu})| \leq C|x - x'|.$$

Proof. We have

$$|g(t, x, \bar{\mu}) - g(t, x', \bar{\mu})| \leq |\langle \bar{\mu}, g_1 \rangle| |g_1(t, x) - g_1(t, x')| + |g_3(t, x) - g_3(t, x')| \leq C|x - x'|.$$

□

Lemma 4.6. *Under the Assumptions 2.1 and 4.2, the value function v is jointly continuous.*

Proof. Here the value of the generic constant C may vary from line to line. We first show that the value function is Lipschitz in x uniformly in t . Let $x, y \in \mathbb{R}$ and $t \in [0, T]$.

$$\begin{aligned} |v(t, x) - v(t, y)| &= \left| \sup_{\tau \in \mathcal{T}_t} \mathbb{E} \left[\int_t^\tau f(s, X_s^{t,x}) ds + g(\tau, X_\tau^{t,x}, \bar{\mu}) \right] \right. \\ &\quad \left. - \sup_{\tau \in \mathcal{T}_t} \mathbb{E} \left[\int_t^\tau f(s, X_s^{t,y}) ds + g(\tau, X_\tau^{t,y}, \bar{\mu}) \right] \right| \\ &\leq \sup_{\tau \in \mathcal{T}_t} \left| \mathbb{E} \left[\int_t^\tau [f(s, X_s^{t,x}) - f(s, X_s^{t,y})] ds + g(\tau, X_\tau^{t,x}, \bar{\mu}) - g(\tau, X_\tau^{t,y}, \bar{\mu}) \right] \right| \\ &\leq \sup_{\tau \in \mathcal{T}_t} \mathbb{E} \left[\int_t^\tau |f(s, X_s^{t,x}) - f(s, X_s^{t,y})| ds + |g(\tau, X_\tau^{t,x}, \bar{\mu}) - g(\tau, X_\tau^{t,y}, \bar{\mu})| \right]. \end{aligned}$$

By Lemma 4.5 and classical estimates for SDEs,

$$|v(t, x) - v(t, y)| \leq \sup_{\tau \in \mathcal{T}_t} \mathbb{E} \left[\int_t^\tau C |X_s^{t,x} - X_s^{t,y}| ds + C |X_\tau^{t,x} - X_\tau^{t,y}| \right] \leq C|x - y|.$$

To get the joint continuity it is sufficient to show that the value function is continuous in t for each x . Let $0 \leq t < t' \leq T$ and $x \in \mathbb{R}$, by the Dynamic Programming Principle,

$$v(t, x) = \sup_{\tau \in \mathcal{T}_t} \mathbb{E} \left[\int_t^{\tau \wedge t'} f(s, X_s^{t,x}) ds + \mathbf{1}_{\tau < t'} g(\tau, X_\tau^{t,x}, \bar{\mu}) + \mathbf{1}_{\tau \geq t'} v(t', X_{t'}^{t,x}) \right].$$

Taking $\tau = t'$, we get

$$v(t, x) - \mathbb{E}[v(t', X_{t'}^{t,x})] \geq \mathbb{E} \left[\int_t^{t'} f(s, X_s^{t,x}) ds \right] \geq -C(t' - t),$$

where we used that $|f(s, X_s^{t,x})| \leq C(1 + |X_s^{t,x}|)$ and $\sup_{s \in [t, t']} \mathbb{E}[|X_s^{t,x}|] < \infty$. On the other hand,

$$\begin{aligned} v(t, x) - \mathbb{E}[v(t', X_{t'}^{t,x})] &= \sup_{\tau \in \mathcal{T}_t} \mathbb{E} \left[\int_t^{\tau \wedge t'} f(s, X_s^{t,x}) ds + \mathbf{1}_{\tau < t'} [g(\tau, X_\tau^{t,x}, \bar{\mu}) - v(t', X_{t'}^{t,x})] \right] \\ &\leq \sup_{\tau \in \mathcal{T}_t} \mathbb{E} \left[\int_t^{\tau \wedge t'} f(s, X_s^{t,x}) ds + \mathbf{1}_{\tau < t'} [g(\tau, X_\tau^{t,x}, \bar{\mu}) - g(t', X_{t'}^{t,x}, \bar{\mu})] \right] \\ &\leq C(t' - t)^{\beta \wedge (1/2)}. \end{aligned}$$

Using that v is Lipschitz in x uniformly in t ,

$$\begin{aligned} |v(t, x) - v(t', x)| &\leq |v(t, x) - \mathbb{E}[v(t', X_{t'}^{t,x})]| + |\mathbb{E}[v(t', X_{t'}^{t,x})] - v(t', x)| \\ &\leq C(t' - t)^{\beta \wedge (1/2)} + C\mathbb{E}[|X_{t'}^{t,x} - x|] \\ &\leq C(t' - t)^{\beta \wedge (1/2)}. \end{aligned}$$

□

Lemma 4.7. *Suppose that Assumptions 2.1 and 4.2 hold. Consider a filtered probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{F}}, \tilde{\mathbb{P}})$, an $\tilde{\mathbb{F}}$ -adapted process \tilde{X} and an $\tilde{\mathbb{F}}$ -Brownian motion \tilde{W} such that*

$$\tilde{X}_t = \tilde{X}_0 + \int_0^t b(s, \tilde{X}_s) ds + \int_0^t \sigma(\tilde{X}_s) d\tilde{W}_s, \quad \tilde{\mathbb{P}} \circ \tilde{X}_0^{-1} = m_0^*,$$

then we have $\tilde{\tau}_C = \tilde{\tau}_{\bar{C}}$ $\tilde{\mathbb{P}}$ -a.s., where $\tilde{\tau}_C = \inf\{t \geq 0 : (t, \tilde{X}_t) \notin C\}$ and $\tilde{\tau}_{\bar{C}} = \inf\{t \geq 0 : (t, \tilde{X}_t) \notin \bar{C}\}$.

Proof. By Step 1 in the proof of Theorem 4.3, we obtain an upper semicontinuous free boundary c and we can rewrite the exit times $\tilde{\tau}_C$ and $\tilde{\tau}_{\bar{C}}$ as $\tilde{\tau}_C = \inf\{t \geq 0 : \tilde{X}_t \leq c(t)\}$ and $\tilde{\tau}_{\bar{C}} = \inf\{t \geq 0 : \tilde{X}_t < c(t)\}$. Define the set $\Pi = [0, T] \cap \mathbb{Q}$.

$$\{\tilde{\tau}_C \neq \tilde{\tau}_{\bar{C}}\} = \{\tilde{\tau}_C < \tilde{\tau}_{\bar{C}}\} = \bigcup_{(s,t) \in \Pi^2, s < t} C(s, t),$$

where for $0 \leq s < t \leq T$,

$$C(s, t) = \left\{ \omega \in \tilde{\Omega} : \min_{u \in [s, t]} [\tilde{X}_u(\omega) - c(u)] = 0 \right\}.$$

This sets are measurable. Let us prove that for all $0 \leq s < t \leq T$, $C(s, t)$ has null probability. Assume that there exist $s < t$ such that $\tilde{\mathbb{P}}(C(s, t)) > 0$. Recall that

$$\tilde{X}_u = \tilde{X}_0 + \int_0^u b(r, \tilde{X}_r) dr + \int_0^u \sigma(\tilde{X}_r) d\tilde{W}_r, \quad \tilde{\mathbb{P}} \circ \tilde{X}_0^{-1} = m_0^*.$$

We reduce to the case $\sigma = 1$ by using the Lamperti transformation. Define the function $F : \mathbb{R} \rightarrow \mathbb{R}$ by

$$F(x) = \int_0^x \frac{1}{\sigma(z)} dz \quad x \in \mathbb{R},$$

and define the process $Y_u = F(\tilde{X}_u)$. The function F is bijective from \mathbb{R} to \mathbb{R} since σ satisfies the uniform ellipticity condition. We denote by G the inverse of F . Moreover, since $\sigma \in C^1(\mathbb{R})$, we have that $F \in C^2(\mathbb{R})$, with derivatives:

$$\frac{\partial F}{\partial x}(x) = \frac{1}{\sigma(x)}, \quad \frac{\partial^2 F}{\partial x^2}(x) = -\frac{\partial_x \sigma(x)}{\sigma^2(x)}.$$

By Itô's formula,

$$\begin{aligned} Y_u &= F(\tilde{X}_0) + \int_0^u \left(\frac{b(r, \tilde{X}_r)}{\sigma(\tilde{X}_r)} - \frac{1}{2} \partial_x \sigma(\tilde{X}_r) \right) dr + \tilde{W}_u \\ &= Y_0 + \int_0^u \bar{b}(r, Y_r) dr + \tilde{W}_u, \end{aligned}$$

where

$$\bar{b}(r, x) := \frac{b(r, G(x))}{\sigma(G(x))} - \frac{1}{2} \partial_x \sigma(G(x)).$$

Define $\bar{c} : [0, T] \rightarrow [-\infty, \infty]$ by $\bar{c}(u) = F(c(u))$, $u \in [0, T]$, which is also upper-semicontinuous. Indeed, for each $u \in [0, T]$, consider the set $L_u := \{x \in \mathbb{R} : x \leq \bar{c}(u)\} = \{x \in \mathbb{R} : G(x) \leq c(u)\}$. If $(x_n)_{n \geq 1} \subset L_u$ converges to some $x \in \mathbb{R}$, $G(x_n) \rightarrow G(x)$. Since c is upper semicontinuous we get $x \in L_u$.

On the other hand we have

$$C(s, t) = \left\{ \omega \in \tilde{\Omega} : \min_{u \in [s, t]} [Y_u(\omega) - \bar{c}(u)] = 0 \right\},$$

since F and G are increasing. Define the process

$$Z_u = \exp \left[- \int_0^u \bar{b}(r, Y_r) d\tilde{W}_r - \frac{1}{2} \int_0^u \bar{b}^2(r, Y_r) dr \right], \quad u \in [0, T],$$

which is a martingale by Novikov's condition which can be seen using the assumptions on the coefficients. We can define a probability measure \mathbb{Q} equivalent to $\tilde{\mathbb{P}}$ on $\tilde{\mathcal{F}}_T$ given by $\frac{d\mathbb{Q}}{d\tilde{\mathbb{P}}} = Z_T$. Applying Girsanov's Theorem, under \mathbb{Q} , $W_u^{(1)} = \tilde{W}_u + \int_0^u \bar{b}(r, Y_r) dr$ is an $(\tilde{\mathbb{F}}, \mathbb{Q})$ -Brownian motion. The dynamics of Y under \mathbb{Q} are $Y_u = Y_0 + W_u^{(1)}$, $u \in [0, T]$. Now, since $F \in C^1(\mathbb{R})$ with positive derivative and invertible, we obtain by the inverse function theorem that F is a C^1 -diffeomorphism. By the change of variables theorem we deduce that $Y_0 = F(\tilde{X}_0)$ admits a density given by $\bar{m}_0^*(x) = m_0^*(G(x))\sigma(G(x))$, which is positive and continuous. Therefore Y is a Brownian motion with initial law \bar{m}_0^* under \mathbb{Q} . We have

$$0 < \mathbb{Q}(C(s, t)) = \int_{\mathbb{R}} \mathbb{Q}(C(s, t) | Y_0 = x) \bar{m}_0^*(x) dx.$$

In particular, there exists a bounded interval $I \subset \mathbb{R}$ such that $\int_I \mathbb{Q}(C(s, t) | Y_0 = x) \bar{m}_0^*(x) dx > 0$. Defining for $h \in \mathbb{R}$ the event

$$C^h(s, t) = \left\{ \omega \in \Omega : \min_{u \in [s, t]} [Y_u(\omega) - \bar{c}(u)] = h \right\},$$

we get $\mathbb{Q}(C^h(s, t) | Y_0 = x + h) = \mathbb{Q}(C(s, t) | Y_0 = x)$ dx -a.e. on \mathbb{R} . In fact, under $\mathbb{Q}(\cdot | Y_0 = x)$, Y is a Brownian motion starting at x and under $\mathbb{Q}(\cdot | Y_0 = x + h)$, Y is a Brownian motion starting at $x + h$, hence, $\mathbb{Q}(Y - h \in B | Y_0 = x + h) = \mathbb{Q}(Y \in B | Y_0 = x)$, $B \in \mathcal{B}(C([0, T]))$. Taking

$B = \{z \in C([0, T]) : \min_{u \in [s, t]} [z(u) - \bar{c}(u)] = 0\}$, we get the result. Now, for all $h \in \mathbb{R}$,

$$\begin{aligned}
\mathbb{Q}(C^h(s, t)) &= \int_{\mathbb{R}} \mathbb{Q}(C^h(s, t) | Y_0 = x) \bar{m}_0^*(x) dx \\
&= \int_{\mathbb{R}} \mathbb{Q}(C^h(s, t) | Y_0 = x + h) \bar{m}_0^*(x + h) dx \\
&\geq \int_I \mathbb{Q}(C^h(s, t) | Y_0 = x + h) \bar{m}_0^*(x + h) dx \\
&= \int_I \mathbb{Q}(C(s, t) | Y_0 = x) \bar{m}_0^*(x) \frac{\bar{m}_0^*(x + h)}{\bar{m}_0^*(x)} dx \\
&\geq \min_{z \in I} \frac{\bar{m}_0^*(z + h)}{\bar{m}_0^*(z)} \int_I \mathbb{Q}(C(s, t) | Y_0 = x) \bar{m}_0^*(x) dx \\
&> 0.
\end{aligned}$$

Therefore the events $\{C^h(s, t) : h \in \mathbb{R}\}$ have positive probability and are disjoint. Since a probability can not have an uncountable collection of disjoint events with positive probability, we get a contradiction. We conclude that $\tilde{\mathbb{P}}(C(s, t)) = 0$ for all $0 \leq s < t \leq T$, which implies that $\tilde{\mathbb{P}}(\tilde{\tau}_C \neq \tilde{\tau}_{\bar{c}}) = 0$. \square

4.3 Sufficient conditions for Assumption 3.4

In this section, we provide sufficient conditions on b, σ, f, g, m_0^* and \mathcal{O} such that Assumption 3.4 is verified. We only prove the uniqueness of the best response, since it is immediate to observe that the other conditions are satisfied.

Assumption 4.8. *We assume the following:*

- (1) \mathcal{O} is a bounded open interval, A is convex and $\sigma = 1$.
- (2) m_0^* admits a bounded density with respect to the Lebesgue measure.
- (3) $b(t, x, a) = b_1(t, x) + b_2(t, x)a$, with b_1 and b_2 continuous, Lipschitz in x uniformly on t and with linear growth.
- (4) For all $t \in [0, T]$, $x \in \bar{\mathcal{O}}$, $\eta \in \mathcal{P}^{sub}(\bar{\mathcal{O}})$, $a \in A$, $f(t, x, \eta, a) = f_1(t, x, \eta) + f_2(t, x, a)$. The function f_1 has the form

$$f_1(t, x, \eta) = \hat{f}(t, x) \bar{f} \left(t, \int_{\bar{\mathcal{O}}} \hat{f}(t, y) \eta(dy) \right),$$

where \hat{f} and \bar{f} are jointly measurable, bounded and continuous in x for each t , and $z \mapsto \bar{f}(t, z)$ is nonincreasing. Moreover, there exists $\bar{c}_f \geq 0$ such that for all $t \in [0, T]$, $x, x' \in \mathbb{R}$,

$$|\hat{f}(t, x) - \hat{f}(t, x')| \leq \bar{c}_f |x - x'|, \quad |\hat{f}(t, x)| \leq \bar{c}_f (1 + |x|),$$

$$|\bar{f}(t, x) - \bar{f}(t, x')| \leq \bar{c}_f |x - x'|, \quad |\bar{f}(t, x)| \leq \bar{c}_f (1 + |x|).$$

The function f_2 is jointly measurable, continuous in (x, a) for each t , and for each (t, x) , $a \mapsto f_2(t, x, a)$ is strictly concave.

(5) g has the form

$$g(t, x, \mu) = g_1(t, x)g_2 \left(\int_{[0, T] \times \bar{\mathcal{O}}} g_1(s, y) \mu(ds, dy) \right) + g_3(t, x),$$

where g_1 , g_2 and g_3 are continuous and g_2 is nonincreasing. Moreover, there exists $\bar{c}_g \geq 0$ such that for all $t, t' \in [0, T]$, $x, x' \in \mathbb{R}$,

$$|g_1(t, x) - g_1(t', x')| \leq \bar{c}_g(|t - t'| + |x - x'|), \quad |g_2(x) - g_2(x')| \leq \bar{c}_g|x - x'|.$$

Finally, for a fixed $\mu \in \mathcal{P}([0, T] \times \bar{\mathcal{O}})$, $(t, x) \mapsto g(t, x, \mu) \in C^{1,2}([0, T] \times \bar{\mathcal{O}})$ and $g(t, x, \mu) = 0$ for $(t, x) \in (0, T) \times \partial\mathcal{O}$.

Let $(\bar{m}, \bar{\mu}) \in \mathcal{R}$. Consider a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ supporting a Brownian motion $W = (W_t)_{t \in [0, T]}$. For $t \in [0, T]$, we denote by \mathbb{F}^t the filtration given by $\mathcal{F}_s^t = \sigma(W_r^t, t \leq r \leq s) \vee \mathcal{N}$, $s \geq t$. Here \mathcal{N} is the set of \mathbb{P} -null sets and $W_s^t := W_s - W_t$, $s \geq t$, is the translated Brownian motion. Denote by \mathcal{A}_t the set of \mathbb{F}^t -progressively measurable process with values in A . Omitting the dependence on \bar{m} and $\bar{\mu}$, consider the value function

$$v(t, x) = \sup_{\alpha \in \mathcal{A}_t} \mathbb{E} \left[\int_t^{T \wedge \tau_{\mathcal{O}}} f(s, X_s^{t, x, \alpha}, \bar{m}_s^x, \alpha_s) ds + g(T \wedge \tau_{\mathcal{O}}, X_{T \wedge \tau_{\mathcal{O}}}^{t, x, \alpha}, \bar{\mu}) \right]. \quad (4.10)$$

where $X^{t, x, \alpha}$ is the unique strong solution to the SDE

$$dX_s^{t, x, \alpha} = b(s, X_s^{t, x, \alpha}, \alpha_s) ds + dW_s; \quad X_t^{t, x, \alpha} = x.$$

The following theorem is a particular case of Theorems 2.1 and 2.2, Chapter 4, in [BL82].

Theorem 4.9. *Let Assumptions 3.1 and 4.8 be satisfied. The value function v is the unique solution belonging to $C([0, T] \times \bar{\mathcal{O}}) \cap W^{1,2,2}((0, T) \times \mathcal{O})$ ¹, satisfying the following Hamilton-Jacobi-Bellman equation (HJB)*

$$\begin{aligned} \frac{\partial v}{\partial t}(t, x) + \sup_{a \in A} [\mathcal{L}v(t, x, a) + f(t, x, \bar{m}_t^x, a)] &= 0, \quad (t, x) \in (0, T) \times \mathcal{O}, \\ v(t, x) &= 0, \quad (t, x) \in (0, T) \times \partial\mathcal{O}, \\ v(T, x) &= g(T, x, \bar{\mu}), \quad x \in \mathcal{O}. \end{aligned} \quad (4.11)$$

Theorem 4.10. *Under Assumptions 3.1 and 4.8, there is a unique maximizer (m^*, μ^*) of $\Gamma[\bar{m}, \bar{\mu}]$ in \mathcal{R} .*

Proof. First Step: Optimality implies strict Markovian maximizer. We first prove that the set of maximizers is contained in the set of measures associated to strict controls. Let $(m, \mu) \in \mathcal{R}$

1. The Sobolev space $W^{1,2,2}((0, T) \times \mathcal{O})$ represents the set of functions u such that u , $\partial_t u$, $\partial_x u$, $\partial_{xx} u \in L^2((0, T) \times \mathcal{O})$, where the derivatives are understood in the sense of distributions.

and consider the transition kernel $(\nu_{t,x})_{t,x} \subset \mathcal{P}(A)$ such that

$$m_t(dx, da)dt = \nu_{t,x}(da)m_t^x(dx)dt.$$

Let $D(A) = \{\delta_a : a \in A\}$ be the set of Dirac masses on A which is in $\mathcal{B}(\mathcal{P}(A))$ since it is closed (recall that A is compact). Consider the Borel set $B = \{(t, x) \in [0, T] \times \bar{\mathcal{O}} : \nu_{t,x} \in D(A)\}$. Assume that the measure m is not associated to a strict control, i.e. $\int_{B^c} m_t^x(dx)dt > 0$, and let us show that we can construct a measure \tilde{m} associated to a strict control which leads to a strictly higher reward. Define $\alpha(t, x) := \int_A a\nu_{t,x}(da)$ and $\tilde{m}_t(dx, da) := \delta_{\alpha(t,x)}(da)m_t^x(dx)$. The function α is measurable and takes values in A since this set is convex and $\nu_{t,x}(\cdot) \in \mathcal{P}(A)$. Let $u \in C_b^{1,2}([0, T] \times \bar{\mathcal{O}})$, we obtain

$$\begin{aligned} & \int_{\mathcal{O}} u(0, x)m_0^*(dx) + \int_0^T \int_{\bar{\mathcal{O}} \times A} \left(\frac{\partial u}{\partial t} + \mathcal{L}u \right) (t, x, a)\tilde{m}_t(dx, da)dt \\ &= \int_{\mathcal{O}} u(0, x)m_0^*(dx) + \int_0^T \int_{\bar{\mathcal{O}} \times A} \left(\frac{\partial u}{\partial t} + \mathcal{L}u \right) (t, x, a)m_t(dx, da)dt \\ &= \int_{\Sigma} u(t, x)\mu(dt, dx) \end{aligned}$$

Therefore, $(\tilde{m}, \mu) \in \mathcal{R}$. Now, since $a \mapsto f(t, x, a)$ is strictly concave, by Jensen's inequality we get

$$\begin{aligned} \Gamma[\bar{m}, \bar{\mu}](m, \mu) &= \int_0^T \int_{\bar{\mathcal{O}}} \int_A f(t, x, \bar{m}_t^x, a)\nu_{t,x}(da)m_t^x(dx)dt + \int_{\Sigma} g(t, x, \bar{\mu})\mu(dt, dx) \\ &< \int_0^T \int_{\bar{\mathcal{O}}} f(t, x, \bar{m}_t^x, \alpha(t, x))m_t^x(dx)dt + \int_{\Sigma} g(t, x, \bar{\mu})\mu(dt, dx) \\ &= \Gamma[\bar{m}, \bar{\mu}](\tilde{m}, \mu), \end{aligned}$$

the inequality being strict since $\text{supp}(\nu_{t,x})$ contains more than one element on B^c and

$$\int_{B^c} m_t^x(dx)dt > 0.$$

We have shown that for every $(m, \mu) \in \mathcal{R}$ there exists a strict admissible control with corresponding strictly higher reward, henceforth, the set of maximizers is contained on the set of LP solutions with strict control.

Second Step: Uniqueness of the Markovian strict control. Let (m, μ) be a maximizer. By Step 1, we can write

$$m_t(dx, da) = \delta_{\alpha(t,x)}(da)m_t^x(dx),$$

for some measurable functions α . Using Theorem 4.9 and applying an analogue proof as in Theorem 5.9 in Chapter 3 we get

$$\int_0^T \int_{\bar{\mathcal{O}} \times A} F_v(t, x, \bar{m}_t^x, a)m_t(dx, da)dt = 0,$$

where

$$F_v(t, x, \bar{m}_t^x, a) := \frac{\partial v}{\partial t}(t, x) + (\mathcal{L}v)(t, x, a) + f(t, x, \bar{m}_t^x, a).$$

From the HJB equation, a.e. for all $a \in A$, we get $F_v(t, x, \bar{m}_t^x, a) \leq 0$. We deduce that $m_t^x(dx)dt$ -a.e.,

$$\alpha(t, x) \in \arg \max_{a \in A} F_v(t, x, \bar{m}_t^x, a).$$

Since for each $(t, x) \in [0, T] \times \bar{\mathcal{O}}$, $b(t, x, \cdot)$ is affine and $f(t, x, \bar{m}_t^x, \cdot)$ is strictly concave, there exists a unique maximizer $\alpha^*(t, x) \in A$ of

$$A \ni a \mapsto F_v(t, x, \bar{m}_t^x, a). \quad (4.12)$$

Therefore we get $\alpha(t, x) = \alpha^*(t, x)$ $m_t^x(dx)dt$ -a.e. Without loss of generality we can assume $\alpha = \alpha^*$ because α^* is measurable (it is a particular case of Theorem 18.19 in [AB07]).

Third Step: Uniqueness. If (m^1, μ^1) and (m^2, μ^2) are two maximizers, by the second step,

$$m_t^k(dx, da)dt = \delta_{\alpha^*(t, x)} m_t^{x, k}(dx)dt, \quad k = 1, 2.$$

Now, by Theorem 5.20 in Chapter 3, for $k = 1, 2$, there exist a filtered probability space $(\Omega^k, \mathcal{F}^k, \mathbb{F}^k, \mathbb{P}^k)$, an \mathbb{F}^k -adapted process X^k , an \mathbb{F}^k -Brownian motion W^k such that

$$\begin{aligned} X_{t \wedge \tau_{\bar{\mathcal{O}}}^k}^k &= X_0^k + \int_0^{t \wedge \tau_{\bar{\mathcal{O}}}^k} b(s, X_s^k, \alpha^*(s, X_s^k)) ds + W_{t \wedge \tau_{\bar{\mathcal{O}}}^k}^k, \quad \mathbb{P}^k \circ (X_0^k)^{-1} = m_0^*, \\ m_t^k(B \times C) &= \mathbb{E}^{\mathbb{P}^k} \left[\mathbf{1}_B(X_t^k) \mathbf{1}_{\alpha^*(t, X_t^k)}(C) \mathbf{1}_{t < T \wedge \tau_{\bar{\mathcal{O}}}^k} \right], \quad B \in \mathcal{B}(\bar{\mathcal{O}}), \quad C \in \mathcal{B}(A), \quad t - a.e., \\ \mu^k &= \mathbb{P}^k \circ (T \wedge \tau_{\bar{\mathcal{O}}}^k, X_{T \wedge \tau_{\bar{\mathcal{O}}}^k}^k)^{-1}. \end{aligned}$$

By a similar proof as in Chapter 4, Proposition 3.10 in [KS98a], we get

$$\mathbb{P}^1 \circ \left(X_{\cdot \wedge \tau_{\bar{\mathcal{O}}}^1}^1, T \wedge \tau_{\bar{\mathcal{O}}}^1 \right)^{-1} = \mathbb{P}^2 \circ \left(X_{\cdot \wedge \tau_{\bar{\mathcal{O}}}^2}^2, T \wedge \tau_{\bar{\mathcal{O}}}^2 \right)^{-1},$$

in $C([0, T]) \times [0, T]$. As a consequence $m^1 = m^2$ and $\mu^1 = \mu^2$. \square

Corollary 4.11 (*Pure solution representation of the best response*). *Under the assumptions of Theorem 4.10, the unique best response (m^*, μ^*) can be represented as a pure solution, i.e.*

$$\begin{aligned} m_t^*(B \times C) &= \mathbb{E}^{\mathbb{P}} \left[\mathbf{1}_B(X_t) \mathbf{1}_{\alpha^*(t, X_t)}(C) \mathbf{1}_{t < T \wedge \tau_{\bar{\mathcal{O}}}^X} \right], \quad B \in \mathcal{B}(\bar{\mathcal{O}}), \quad C \in \mathcal{B}(A), \quad t - a.e., \\ \mu^* &= \mathbb{P} \circ \left(X_{T \wedge \tau_{\bar{\mathcal{O}}}^X}, T \wedge \tau_{\bar{\mathcal{O}}}^X \right)^{-1}, \end{aligned}$$

where $(\Omega, \mathcal{F}, \mathbb{P}, W)$ represents the initial probabilistic set up (see p. 34), α^* is the unique maximizer of the Hamiltonian (4.12) and X is the strong solution of the SDE associated to α^* .

Chapter 5

Energy transition under scenario uncertainty: a mean-field game approach

Abstract

We study the impact of transition scenario uncertainty, and in particular, the uncertainty about future carbon price and electricity demand, on the pace of decarbonization of the electricity industry. To this end, we build a discrete time mean-field game model for the long-term dynamics of the electricity market subject to common random shocks affecting the carbon price and the electricity demand. These shocks depend on a macroeconomic scenario, which is not observed by the agents, but can be partially deduced from the frequency of the shocks. Due to this partial observation feature, the common noise is non-Markovian. We consider two classes of agents: conventional producers and renewable producers. The former choose an optimal moment to exit the market and the latter choose an optimal moment to enter the market by investing into renewable generation. The agents interact through the market price determined by a merit order mechanism with an exogenous stochastic demand. We prove the existence of Nash equilibria in the resulting mean-field game of optimal stopping with common noise, developing a novel linear programming approach for these problems. We illustrate our model by an example inspired by the UK electricity market, and show that scenario uncertainty leads to significant changes in the speed of replacement of conventional generators by renewable production.

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1 Introduction

Given the climate emergency, there is no doubt that in the years and decades to come, the transition to a low-carbon economy will continue to lead to fundamental changes in the energy industry. However, there is a considerable uncertainty about the pace of these changes, which is only slowly resolved through government announcements and regulatory shifts. Economic agents operating in energy markets must therefore make their investment decisions taking into account the uncertainty about the future market shares, costs and profits of various electricity generation technologies. In the face of these uncertainties, the scenario approach has emerged as a means to structure decision making and optimize future actions. Scenarios are plausible trajectories of evolution of macroeconomic variables, parameterized by specific assumptions on future climate and policy, which are typically produced with integrated assessment models (IAM) and maintained by international organizations such as IEA (International Energy Agency), IPCC (Intergovernmental Panel on Climate Change) and NGFS (Network for Greening the Financial System). These databases typically contain many scenarios, which differ both in the underlying IAM and the relative stringency of climate policy assumptions. For example, the NGFS database¹ contains 6 sets of scenarios with varying stringency, from Current Policies (least stringent) to Net Zero 2050 (most stringent). Thus, even if a specific scenario database has been fixed, decision makers still face a considerable scenario uncertainty regarding the future policy choices.

The aim of this paper is therefore to evaluate the impact of this scenario uncertainty on the dynamics of the electrical industry, and, in particular, on the rate at which conventional generation is replaced by renewable plants. To this end, we develop an extension of the game theoretical energy market model of [ADT21] by integrating scenario uncertainty. Following [ADT21], we assume that two types of agents, conventional producers and renewable producers, are interacting through the electricity price. The conventional producers aim to find the optimal time to leave the market, and the potential renewable project owners aim to find the optimal time to invest and enter the market. Unlike [ADT21], where the future parameters of the market are assumed to be deterministic and known to the agents, here, we suppose that the electricity demand and the costs of the conventional producers depend on the random carbon price which is influenced by the government announcements / regulatory changes.

For mathematical tractability, we assume that each of the two classes contains a large number of identical agents, and frame the problem in the mean-field game setting. This requires building a theory of optimal stopping mean field games with common noise and partial information, which was not previously available in the literature. Following, [BDT20; DLT21; ADT21] we employ the *linear programming approach* to mean-field games of optimal stopping, which is a compactification technique that works by reformulating the problem in terms of occupation measures of the agents instead of stopping times. We prove the existence of a *linear programming Nash equilibrium* and the uniqueness of the *equilibrium price process*. Our mathematical approach builds on Markov chains, allowing to show the existence of strong solutions in mean

1. Release 2.2, available at <https://data.ene.iiasa.ac.at/ngfs/>

field games (MFGs) with common noise using topological arguments and also to build numerical algorithms, which are instrumental for practical applications. Here strong solution should be understood in the sense of [CDL16], i.e. the mean-field terms are adapted to the common noise filtration. We give a complete treatment of mean-field games with optimal stopping in a discrete time setting and involving a common noise taking values in a finite set, which also includes an interpretation of the occupation measures in terms of randomized stopping times (see [BC77]), as well as a rigorous construction of approximate Nash equilibria for games with finite number of players. In the context of MFGs of optimal stopping of preemption type, under quite strong assumptions, an approximation result with distributed strategies for finite player games is provided in [CDL17]. To the best of our knowledge, in the setting of MFGs of optimal stopping of war of attrition type, our paper is the first to show such an approximation result. Finally, we propose a numerical algorithm based on the linear programming fictitious play introduced in [DLT22], which is adapted to the framework of common noise. We illustrate our model with an example inspired by the UK electricity market, and show that scenario uncertainty leads to significant changes in the speed of replacement of conventional generators by renewable production, emphasizing the role of reliable information for successful energy transition.

Literature review A variety of approaches have been used to analyse the dynamics of environmental transition of the energy sector depending on carbon price levels and technology policies. Transition scenarios used by international organisms such as IPCC, NGFS or IEA are produced with *integrated assessment models*: partial or general equilibrium macroeconomic optimization models with a detailed representation of the energy sector, such as REMIND [BBL12], IMACLIM [Hou+10], TIMES [LL08] etc., or with energy system optimization models, which focus specifically on the energy sector and determine the lowest cost trajectory under a set of assumptions [DeC+17]. These models in most cases produce a single deterministic scenario, usually by solving the optimization problem of a central planner with perfect foresight. Interaction among agents, imperfect information, and uncertainty about the future values of demand, technology costs and other factors are rarely taken into account. At the other end of the spectrum one finds the computational agent-based models [WV08]. These models allow for heterogeneous agents and a precise description of their interactions and of the market structure, but involve very intensive computations and do not provide any insight about the model (uniqueness of the equilibrium, robustness etc.) beyond what can be recovered from a simulated trajectory.

The mean-field games are a viable compromise between the complexity of computational agent-based models and the tractability of fully analytic approaches. MFGs, introduced in [LL06a; LL06b; LL07; HMC06] are stochastic games with a large number of identical agents and symmetric interactions where each agent interacts with the average density of the other agents (the mean field) rather than with each individual agent. This simplifies the problem, leading to explicit solutions or efficient numerical methods for computing the equilibrium dynamics. In the recent years MFGs have been successfully used to model specific sectors of electricity markets, such as price formation [GS20; FTT22], electric vehicles [Cou+12], demand dispatch [BB14], storage [ATM20; Ala+21] and renewable energy certificates markets [SFJ22].

An important recent development is the introduction of MFGs of optimal stopping [GP15; Ber17; CDL17; Nut18; BDT20; DLT21], which can describe technology switches and entry/exit decisions of players.

In [ADT21], the authors used the linear programming approach of [BDT20] to build a MFG model for the long-term evolution of an electricity market allowing for two classes of agents (conventional and renewable) interacting through the market price. That paper, similarly to most of the literature on MFGs, assumes that the sources of randomness affecting different agents are independent, and averaged out in the mean-field limit. As a result, the model produces a deterministic price trajectory for a given deterministic scenario of electricity demand and carbon price. However, the future climate, climate-related economic policies, and therefore also future energy demand and carbon price are subject to deep uncertainties [CRL19; MRF19; Bol+20; Nor18]. These are related, among other factors, to uncertainty about future availability of mitigation technologies and international cooperation [Ede+06]; uncertainty of climate sensitivity and carbon budgets, which may have to be revised in the future [Mee+20; CHW18], uncertainty about tipping points, which may require bold immediate actions [Kee+22; Wei09], etc. In this context, perfect knowledge of the scenario corresponding to a fixed climate objective is clearly not a valid assumption. In this paper, we are therefore interested in the impact of uncertainty affecting the future carbon price and electricity demand on the pace of energy transition. To this end, we extend the model of [ADT21] by allowing for common noise affecting all agents, which is not averaged out in the mean-field limit.

Although MFGs with common noise were introduced already in [Lio07], and early papers contain examples of solvable settings [GLL11], the general analysis of MFGs with common noise (with regular controls) was only presented in [CDL16]. The main difficulty is due to the much larger dimension of the natural space for the main objects of the MFG problem. For example, in the case of regular controls, the time-dependent occupation measure of the representative agent typically lives in the space of continuous functions from the time interval to the space of probability measures $C([0, T], \mathcal{P}(\mathbb{R}^d))$; in the presence of common noise one needs to consider the space $[C([0, T], \mathcal{P}(\mathbb{R}^d))]^\Omega$, where Ω is the probability space carrying the common noise: this makes it near impossible to apply the usual compactness arguments. As pointed out in [CDL16], the operation of conditioning fails to be continuous in any useful sense, which complicates the study of fixed points via topological arguments. The situation simplifies in the setting of finite probability spaces, see [BHS21]. A promising approach for the special class of *supermodular* mean-field games allowing for a simplified treatment of common noise was recently proposed in [Dia+22].

In the context of MFGs of optimal stopping, which are the object of this paper, introducing common noise is even more difficult, since occupation measures of agents are not continuous as function of time. The models of [CDL17; Nut18] do allow for common noise and establish the existence of an equilibrium using the special structure of the game (e.g., the complementarity property which is characteristic of preemption games, appearing e.g., in bank run problems). Our setting is that of games of war of attrition, and the methods of [CDL17; Nut18] are therefore not applicable. Furthermore, in a general setting with common noise in [CDL16; CDL17], the

authors give abstract existence results using the notion of weak solutions, which do not seem convenient for numerical algorithms. For these reasons, to include common noise into the problem, we place ourselves in a discrete-time and finite probability space framework for the common noise. We thus develop a linear programming formulation for discrete-time optimal stopping MFGs with common noise, where the state process is a Markov chain but the common noise may be non-Markovian, showing the existence of Nash equilibrium using Kakutani-Fan-Glicksberg fixed-point theorem for set valued mappings. Note that a very recent preprint [GHZ22] discusses the linear programming formulation for discrete-time MFG based on controlled Markov chains, but these authors do not consider the setting of optimal stopping, common noise and partial information.

The paper is organized as follows. In Section 2, we describe the electricity market model in the finite player game setting. In Section 3, we define the MFG problem in a more general framework. We show existence of equilibria, give a probabilistic representation of our constraint and provide an approximation result for finite player games. In Section 4, we show how our model fits in the formulation given in Section 3, which allows to derive existence of equilibria and the uniqueness of the equilibrium price process. In Section 5 we describe an adaptation of the algorithm proposed in [DLT22] to the common noise case. Finally, an illustration of the model, inspired by the UK electricity market, is provided in Section 6. In Section 7 we give some complementary results on the linear programming approach and other technical results.

2 Model description

We consider a terminal horizon $T_0 > 0$ (measured in years) and discretize the interval $[0, T_0]$ into T intervals of equal size $\Delta t := T_0/T$. We will denote by $I := \{0, 1, \dots, T\}$ and $I^* := \{1, \dots, T\}$ the sets of time indices. We assume that all random variables considered in this section are defined on the same complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Common noise and demand. The economic agents in our model are subject to common random shocks in the carbon price. We denote by $Z = (Z_t)_{t \in I}$ the stochastic process modeling the carbon price (measured in GBP/tonCO2 in the examples) and suppose that Z takes values in a finite subset $H \subset \mathbb{R}^+$ and has a deterministic initial value $z_0 \in H$. Moreover we assume that these common shocks depend on the economic scenario, which is not observed by the agents. We consider a finite number of possible scenarios or regimes and we denote by \mathcal{S} the set of all scenarios. Let S be a random variable taking values in \mathcal{S} representing the economic scenario. The law $\Pi_0 := \mathcal{L}(S) \in \mathcal{P}(\mathcal{S})$ is called the prior scenario distribution. Conditional on each scenario, the agents know the distribution of the common random shocks. More precisely, we assume that $(S, Z_t)_{t \in I}$ is a Markov chain with initial condition $\Pi_0 \times \delta_{z_0}$ characterized by the transition kernels $(\pi_t^Z)_{t \in I^*}$ from $\mathcal{S} \times H$ to H , i.e. $\mathbb{P}(Z_t = z' | \mathcal{F}_{t-1}^{S, Z}) = \pi_t^Z(S, Z_{t-1}; z')$ a.s. Finally we assume that the instantaneous normalized demand process D (measured in GW) is of the form

$$D_t = d(t) + \beta(Z_t - z_0), \quad t \in I,$$

for some deterministic function $d : I \rightarrow \mathbb{R}^+$ and some parameter $\beta \geq 0$ (in GW \times tonCO2/GBP). This means that the demand is positively correlated with the carbon price, which is justified by the fact that as the carbon price increases, carbon-intensive sectors of the industry are forced to electrify and contribute to electricity demand.

To make our model more realistic for the day-ahead electricity markets, we assume that the demand decomposes as peak demand D^p (Mon-Fri, 7AM-8PM) and off-peak demand D^o and we normalize the demand by a deterministic function $(\lambda_t)_{t \in I} \subset \mathbb{R}^+$ (in our illustration λ_t will represent the seasonal cycle correction). More precisely, for all $t \in I$, we have

$$\lambda_t D_t = c_p D_t^p + c_o D_t^o,$$

where $c_p = 65/168$ is the proportion of peak hours per week and $c_o = 1 - c_p$ is the proportion of off-peak hours per week. Moreover we assume that the ratio peak demand/off-peak demand $(D_t^p/D_t^o)_{t \in I}$ is known and constant in time. Denoting by $r_d \geq 1$ this ratio, we can deduce the expressions $D_t^p = \bar{c}_p \lambda_t D_t$ and $D_t^o = \bar{c}_o \lambda_t D_t$, where $\bar{c}_p := \frac{r_d}{r_d c_p + c_o}$ and $\bar{c}_o := \frac{1}{r_d c_p + c_o}$.

Conventional producers. The state of each conventional producer will be described by its baseline marginal cost per MWh produced (in GBP/MWh). In our setting, for well chosen parameters $\Delta x > 0$ and $C_{\max} > C_{\min} \geq 0$ and for given functions b and σ to be fixed later, we define the baseline marginal cost processes of the conventional producers $(B^i)_{1 \leq i \leq N_C}$ to be independent homogeneous Markov chains on the space $E_1 = \{x_k := C_{\min} + k\Delta x : k = 0, \dots, n_1\}$ (with n_1 such that $C_{\min} + n_1\Delta x = C_{\max}$) with i.i.d. initial conditions $(\xi^i)_{1 \leq i \leq N_C}$ of law m_0^* and with the same transition kernel: if $x \in \{x_1, \dots, x_{n_1-1}\}$, we set

$$\pi^B(x, x + \Delta x) := \frac{\sigma^2(x)/2 + \Delta x \max(b(x), 0)}{\sigma^2(x) + \Delta x |b(x)|}, \quad \pi^B(x, x - \Delta x) := \frac{\sigma^2(x)/2 - \Delta x \min(b(x), 0)}{\sigma^2(x) + \Delta x |b(x)|},$$

otherwise we set $\pi^B(x_0, x_1) := 1$ and $\pi^B(x_{n_1}, x_{n_1-1}) := 1$ (i.e. the Markov chain reflects on the boundary). Here we consider $b(x) = k(\theta - x)$ and $\sigma(x) = \delta\sqrt{x}$ with parameters k , θ and δ . The coefficients b and σ as well as the transition probabilities are defined to approximate a CIR process in continuous time (see p. 96 in [KD01]) which has the adequate properties for the modelling of the baseline marginal cost process (see [ADT21]).

If the capacity utilization (i.e. the ratio of the actual production and the installed productive capacity²) of the conventional producer $i \in \{1, \dots, N_C\}$ is equal to $\alpha \in [0, 1]$, then it has a marginal unit cost (in GBP/MWh) of

$$C_t^i(\alpha) := B_t^i + \tilde{\beta} Z_t + c(\alpha), \quad t \in I,$$

where $\tilde{\beta} \geq 0$ (in tonCO2/MWh) is the emission intensity and $c : [0, 1] \rightarrow \mathbb{R}_+$ is a deterministic C^1 function with $c(0) = 0$ and positive derivative on $]0, 1[$. For each step $t \in I$ the conventional producer i observes a given price per unit $p \geq 0$ (in GBP/MWh) and chooses a capacity

² The installed productive capacity is assumed to be the same for all the conventional producers and fixed over time.

utilization $\alpha \in [0, 1]$ in order to maximize its profit per unit (in GBP/MWh)

$$p\alpha - \int_0^\alpha C_t^i(y)dy = (p - B_t^i - \tilde{\beta}Z_t)\alpha - \int_0^\alpha c(y)dy. \quad (5.1)$$

Introduce the function $\bar{\alpha} : \mathbb{R} \rightarrow [0, 1]$ defined by

$$\bar{\alpha}(y) := \begin{cases} 0 & \text{if } y \leq 0, \\ c^{-1}(y) & \text{if } y \in]0, c(1)], \\ 1 & \text{if } y > c(1). \end{cases}$$

Then $\bar{\alpha}$ is continuous and it is C^1 and increasing on $]0, c(1)[$. Moreover the unique solution of the maximization problem (5.1) is given by $\alpha^* = \bar{\alpha}(p - B_t^i - \tilde{\beta}Z_t)$. The conventional producer i will therefore use the fraction $\bar{\alpha}(p - B_t^i - \tilde{\beta}Z_t)$ of its capacity, making a gain of $G(p - B_t^i - \tilde{\beta}Z_t)$ per unit time, where

$$G(x) := x\bar{\alpha}(x) - \int_0^{\bar{\alpha}(x)} c(y)dy = \int_0^{\bar{\alpha}(x)} (x - c(y))dy = \int_0^x \bar{\alpha}(z)dz,$$

the third equality coming from the change of variables $z = c(y)$ and integration by parts.

Renewable producers. We consider a population of renewable producers who are able to produce with a stochastic capacity factor³. More precisely, for well chosen parameters $\Delta\bar{x} > 0$ and $1 \geq R_{\max} > R_{\min} \geq 0$ and for given functions \bar{b} and $\bar{\sigma}$ to be fixed later, we define the stochastic capacity factors of the renewable producers $(R^i)_{1 \leq i \leq N_R}$ to be independent homogeneous Markov chains on the space $E_2 = \{\bar{x}_k := R_{\min} + k\Delta\bar{x} : k = 0, \dots, n_2\}$ (with n_2 such that $R_{\min} + n_2\Delta\bar{x} = R_{\max}$) with i.i.d. initial conditions $(\bar{\xi}^i)_{1 \leq i \leq N_R}$ of law \bar{m}_0^* and with the same transition kernel: if $\bar{x} \in \{\bar{x}_1, \dots, \bar{x}_{n_2-1}\}$, we set

$$\pi^R(\bar{x}, \bar{x} + \Delta\bar{x}) := \frac{\bar{\sigma}^2(\bar{x})/2 + \Delta\bar{x} \max(\bar{b}(\bar{x}), 0)}{\bar{\sigma}^2(\bar{x}) + \Delta\bar{x}|\bar{b}(\bar{x})|}, \quad \pi^R(\bar{x}, \bar{x} - \Delta\bar{x}) := \frac{\bar{\sigma}^2(\bar{x})/2 - \Delta\bar{x} \min(\bar{b}(\bar{x}), 0)}{\bar{\sigma}^2(\bar{x}) + \Delta\bar{x}|\bar{b}(\bar{x})|},$$

otherwise we set $\pi^R(\bar{x}_0, \bar{x}_1) := 1$ and $\pi^R(\bar{x}_{n_2}, \bar{x}_{n_2-1}) := 1$. Here we consider $\bar{b}(x) = \bar{k}(\bar{\theta} - x)$ and $\bar{\sigma}(x) = \bar{\delta}\sqrt{x(1-x)}$ with parameters \bar{k} , $\bar{\theta}$ and $\bar{\delta}$. The coefficients \bar{b} and $\bar{\sigma}$ as well as the transition probabilities are defined to approximate a Jacobi process in continuous time.

Information and decision variables. We assume that the information available to all economic agents is the same and is represented through the complete natural filtration of $(B^i)_{1 \leq i \leq N_C}$, $(R^i)_{1 \leq i \leq N_R}$ and Z , which is denoted by $\mathbb{F} = (\mathcal{F}_t)_{t \in I}$. We denote by \mathcal{T} the set of \mathbb{F} -stopping times valued in I . The conventional producer $i \in \{1, \dots, N_C\}$ aims to choose a stopping time denoted by $\tau_i \in \mathcal{T}$ to *exit* the market in order to maximize a gain functional to be defined later. In contrast, the (potential) renewable producer $i \in \{1, \dots, N_R\}$ aims to choose

3. As for the conventional producers, the installed productive capacity of the renewable producers is assumed to be the same and fixed over time.

a stopping time denoted by $\bar{\tau}_i \in \mathcal{T}$ to build the renewable power plant and *enter* the market.

Electricity supply and price formation. We assume that the installed productive capacity of each conventional producer is I_C/N_C , where $I_C > 0$ (in GW) is the total installed productive capacity of all conventional producers (who may leave the market) at time 0. We consider a third class of economic agents who are conventional producers that do not leave the market at any time and produce a baseline electricity supply. For a given price $p \geq 0$, the baseline supply of the conventional producers is $S^b(p)$ (in GW), where $S^b : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is an increasing continuous function with $S^b(0) = 0$ and such that there exists some $\tilde{p} \geq 0$ verifying

$$S^b(\tilde{p}) \geq \bar{c}_p \lambda_{max}(d_{max} + \beta(z_{max} - z_0)),$$

with $d_{max} := \max_{t \in I} d(t)$, $\lambda_{max} := \max_{t \in I} \lambda_t$ and $z_{max} := \max_{z \in H} z$. This is a technical assumption which guarantees that for a sufficiently high price, there is enough baseline supply to satisfy the maximum possible demand level. In our price formation mechanism we will however impose a price cap to be able to model electricity shortages. At a given price level $p \geq 0$ at step $t \in I$, the total supply of conventional producers is given by

$$S_t^c(p) = \sum_{i=1}^{N_C} \frac{I_C}{N_C} \bar{\alpha}(p - B_t^i - \tilde{\beta}Z_t) \mathbf{1}_{t < \tau_i} + S^b(p). \quad (5.2)$$

We assume also that there is another class of renewable producers who are already in the market at time 0 and provide a baseline supply. We suppose that there is a total number of \tilde{N}_R of these renewable producers and each one has an installed productive capacity of I_R^b/\tilde{N}_R , where $I_R^b > 0$ (in GW) is the total installed productive capacity of these renewable producers. Furthermore we assume that their stochastic capacity factors are given by i.i.d. processes $(\tilde{R}^i)_{1 \leq i \leq \tilde{N}_R}$ with the same distribution as R^1 and which are independent from the other processes. Finally, the installed productive capacity of each potential renewable producer (i.e. not in the class of baseline renewable producers) aiming to enter the market will be I_R/N_R , where $I_R > 0$ (in GW) is the maximum renewable capacity which can potentially be added until time T . The total supply of renewable producers at step $t \in I$ is

$$S_t^r = \sum_{i=1}^{\tilde{N}_R} \frac{I_R^b}{\tilde{N}_R} \tilde{R}_t^i + \sum_{i=1}^{N_R} \frac{I_R}{N_R} R_t^i \mathbf{1}_{t \geq \bar{\tau}_i}. \quad (5.3)$$

The market price (in GBP/MWh) is determined by the merit order mechanism. The renewable producers bid their entire (stochastic) capacity at zero price. If the renewable supply is not sufficient to match the demand, the conventional suppliers fill the residual demand. We assume that the price is capped to some value $p_{max} > 0$. The peak price at time $t \in I$ is given by the random variable

$$P_t^p := \inf\{p \geq 0 : (D_t^p - S_t^r)^+ \leq S_t^c(p)\} \wedge p_{max}.$$

Observe that the set $\{p \geq 0 : (D_t^p - S_t^r)^+ \leq S_t^c(p)\}$ is nonempty since by assumption \tilde{p} belongs

to it. The off-peak price P^o is defined in the same way, but replacing the peak demand D^p by the off-peak demand D^o .

We remark that the producers impact the price through the empirical measures associated to the state processes and stopping times of the two populations of agents:

$$\frac{1}{N_C} \sum_{i=1}^{N_C} \delta_{B_t^i}(dx) \mathbf{1}_{t < \tau_i}, \quad \frac{1}{N_R} \sum_{i=1}^{N_R} \delta_{R_t^i}(dx) \mathbf{1}_{t \geq \bar{\tau}_i}.$$

This structure will allow us to simplify the expression of the price in the mean-field limit (i.e. when the number of agents in each population goes to infinity). Because of the common random shocks that do not average out in the limit, the price will be an adapted functional of the stochastic carbon price Z .

Expected gain of conventional and renewable producers. The conventional producer $i \in \{1, \dots, N_C\}$ aims to maximize its expected discounted profit (in GBP per kW of installed capacity) over the time period $[0, T_0]$ by choosing when to exit the market, i.e., solves the following problem:

$$\sup_{\tau_i \in \mathcal{T}} \mathbb{E} \left[\sum_{t=0}^{\tau_i-1} e^{-\rho t \Delta t} \left(c_p G(P_t^p - B_t^i - \tilde{\beta} Z_t) + c_o G(P_t^o - B_t^i - \tilde{\beta} Z_t) - \kappa_C \right) \Delta t + K_C e^{-(\gamma_C + \rho) \tau_i \Delta t} \right],$$

where $\kappa_C \geq 0$ is the fixed operating cost (in GBP/kWyr), $e^{-\gamma_C \tau_i \Delta t} K_C$ is the value recovered per kW of installed capacity (in GBP/kW) if the plant is sold at the year $\tau_i \Delta t$ with depreciation rate γ_C and ρ is the discount rate.

On the other hand, the renewable producer $i \in \{1, \dots, N_R\}$ bids its full (intermittent) capacity after entering the market, and wants to maximize its expected profit (in GBP per kW of installed capacity) over the time period $[0, T_0]$ by choosing when to enter the market, i.e.

$$\sup_{\bar{\tau}_i \in \mathcal{T}} \mathbb{E} \left[\sum_{t=\bar{\tau}_i}^{T-1} e^{-\rho t \Delta t} \left([c_p P_t^p + c_o P_t^o] R_t^i - \kappa_R \right) \Delta t + K_R e^{-\rho T_0 - \gamma_R (T_0 - \bar{\tau}_i \Delta t)} - K_R e^{-\rho \bar{\tau}_i \Delta t} \right],$$

where $\kappa_R \geq 0$ is the fixed operating cost (in GBP/kWyr), K_R is the fixed cost per kW of building the plant (in GBP/kW), $e^{-\gamma_R (T_0 - \bar{\tau}_i \Delta t)} K_R$ is the value per kW of the plant at time T_0 (in GBP/kW) with γ_R being the depreciation rate.

Nash equilibria. Since the expected gain functionals depend on the electricity price, which is determined by the decisions of both classes of agents, all optimization problems are coupled and it is natural to look for Nash equilibrium strategies. If the conventional producers choose the stopping times $\boldsymbol{\tau} = (\tau_1, \dots, \tau_{N_C}) \in \mathcal{T}^{N_C}$ and the renewable producers choose the stopping times $\bar{\boldsymbol{\tau}} = (\bar{\tau}_1, \dots, \bar{\tau}_{N_R}) \in \mathcal{T}^{N_R}$, then we denote by $V_i(\boldsymbol{\tau}, \bar{\boldsymbol{\tau}})$ the corresponding expected profit per kW of the conventional producer $i \in \{1, \dots, N_C\}$ and by $\bar{V}_j(\boldsymbol{\tau}, \bar{\boldsymbol{\tau}})$ the corresponding expected profit per kW of the renewable producer $j \in \{1, \dots, N_R\}$. We will say that $(\boldsymbol{\tau}^*, \bar{\boldsymbol{\tau}}^*)$ is a Nash

equilibrium if for all $i \in \{1, \dots, N_C\}$ and $\tau_i \in \mathcal{T}$,

$$V_i((\tau_i, \boldsymbol{\tau}^{*, -i}), \bar{\boldsymbol{\tau}}^*) \leq V_i(\boldsymbol{\tau}^*, \bar{\boldsymbol{\tau}}^*),$$

and for all $j \in \{1, \dots, N_R\}$ and $\bar{\tau}_j \in \mathcal{T}$,

$$\bar{V}_j(\boldsymbol{\tau}^*, (\bar{\tau}_j, \bar{\boldsymbol{\tau}}^{*, -j})) \leq \bar{V}_j(\boldsymbol{\tau}^*, \bar{\boldsymbol{\tau}}^*).$$

Here we used the notation $(y, x^{-i}) := (x_1, \dots, x_{i-1}, y, x_{i+1}, \dots, x_d)$ for some $y \in \mathbb{R}$ and $x \in \mathbb{R}^d$. In other words, a Nash equilibrium is a tuple of strategies (stopping times in our case) where no agent has an incentive to deviate from its individual strategy.

Our main goal is to understand the evolution of the market in equilibrium (i.e. when players choose the Nash equilibrium strategies). In particular, we want to understand the renewable penetration under common random shocks on the carbon price and under scenario uncertainty. To study our model we make use of MFGs with common noise and partial information, which are introduced in the next section. We summarize the variables used in our model and their corresponding unit of measure in Table 5.1.

Variable	Notation	Unit
Terminal horizon	T_0	yr
Carbon price	Z	GBP/tonCO2
Demand/Carbon price factor	β	GW \times tonCO2/GBP
Deterministic demand	d	GW
Seasonal cycle	λ	
Proportion Peak Hours	c_p	
Proportion Off-Peak Hours	c_o	
Baseline marginal cost	B^i	GBP/MWh
Emission intensity	$\tilde{\beta}$	tonCO2/MWh
Capacity cost function	c	GBP/MWh
Stochastic capacity factor	R^i	
Conventional total capacity	I_C	GW
Potential renewable total capacity	I_R	GW
Baseline conventional supply	S^b	GW
Baseline renewable total capacity	I_R^b	GW
Fixed operating costs	κ_C, κ_R	GBP/kWyr
Plant values per kW	K_C, K_R	GBP/kW
Depreciation rates	γ_C, γ_R	/yr
Discount rate	ρ	/yr

Table 5.1 – Variables of the model.

3 Linear programming formulation for discrete time MFGs with common noise and partial information

In this section, we define the MFG problem with common noise and partial information using the linear programming formulation and we show the existence of an equilibrium. For the sake of clarity, we present first the case of a single population and then we extend the results to the case of several populations, needed in our application.

3.1 Probabilistic set-up

As before, let $I = \{0, \dots, T\}$ be the set of time indices with $I^* = \{1, \dots, T\}$. We are given a nonempty compact metric space (E, d) and a nonempty finite set H .

Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a complete filtered probability space supporting the process (X, Z) satisfying:

- (1) The state process $X = (X_t)_{t \in I}$ is an \mathbb{F} -adapted process taking values in E with initial law m_0^* .
- (2) The common noise process $Z = (Z_t)_{t \in I}$ is an \mathbb{F} -adapted process taking values in H and such that $Z_0 = z_0$ is deterministic.

Consider a nonempty compact set $D \subset E$ and the \mathbb{F} -stopping time $\tau_D^X := \inf\{t \in I : X_t \notin D\}$ with the convention $\inf \emptyset = \infty$. The agents will undergo absorption if their state exits the set D . We assume that the initial distribution of the agents m_0^* is supported on D .

Common noise information. In the setting of MFGs with common noise, the mean-field terms will be conditional measures given the information of the common source of randomness which in our case is Z . Consider the associated filtration \mathbb{F}^Z , which models the common noise information conditioning the mean-field terms.

The linear programming formulation without common noise for an optimal stopping problem consists in replacing the expectations of the processes and the stopping times by occupation measures and embedding these measures in a well behaved space (usually compact and convex) (see e.g. [CS02; BDT20; DLT21]). This technique has been used in several works as [KS98c; CS02; BGQ11; BDT20; DLT21] in the case when the underlying processes are Markovian and the rewards depend only on the present states. However, in our setting, the reward functions will depend on the mean-field terms, which depend on the past of the common noise. The idea is then to construct another process summarizing the common noise information and such that together with the state processes of the representative agent, it defines a Markov process. This technique was used in e.g. [Kur98] in the context of continuous time processes and filtered martingale problems.

The information given by the trajectory of the common noise Z up to time t will be summarized in a finite dimensional (matrix-valued) process U_t . Let $\mathcal{I} := \{1, \dots, |H|\}$ and write $H = \{z_j : j \in \mathcal{I}\}$ with $z_j \neq z_{j'}$ for $j \neq j'$, $(j, j') \in \mathcal{I}^2$. We define the process U taking values in $W := \{0, 1\}^{(N+1) \times |H|}$ (i.e. the space of $(N+1) \times |H|$ -dimensional matrices with entries being 0

or 1) by

$$U_t(s+1, j) = \mathbb{1}_{s \leq t} \mathbb{1}_{Z_s = z_j}, \quad (s, j) \in I \times \mathcal{I}, \quad t \in I.$$

Alternatively, for $s \in I$ and $\bar{z} \in H$, let $M[s, \bar{z}]$ be the $(N+1) \times |H|$ -dimensional matrix such that

$$M[s, \bar{z}](r+1, j) = \mathbb{1}_{s=r} \mathbb{1}_{\bar{z} = z_j}, \quad (r, j) \in I \times \mathcal{I}.$$

Then we can check that for $t \in I$, $U_t = \sum_{s=0}^t M[s, Z_s]$. For each $t \in I$ we define the function $\Psi_t : W \rightarrow H$ by

$$\Psi_t(u) := \sum_{j \in \mathcal{I}} z_j \mathbb{1}_{u(t+1, j) = 1}, \quad u \in W. \quad (5.4)$$

This function satisfies $Z_s = \Psi_s(U_t)$ for all $s \leq t$ and $t \in I$, hence the information about the entire trajectory of Z up to time t is encoded in U_t . Moreover, we can conclude that $\sigma(U_t) = \mathcal{F}_t^Z = \mathcal{F}_t^U$, which also shows that U is a Markov chain. We can also compute the transition kernels of U as stated in the following lemma.

Lemma 3.1. *The process U is a Markov chain with transition kernels given for $t \in I^*$, $u, u' \in W$ by*

$$\pi_t^U(u; u') := \begin{cases} \sum_{z \in H} \mathbb{1}_{u+M[t, z] = u'} \mathbb{P}(Z_t = z | U_{t-1} = u) & \text{if } \mathbb{P}(U_{t-1} = u) > 0, \\ \mathbb{1}_{u' = u} & \text{if } \mathbb{P}(U_{t-1} = u) = 0. \end{cases}$$

Proof. For any $\varphi : W \rightarrow \mathbb{R}$ measurable and bounded,

$$\mathbb{E}[\varphi(U_t) | \mathcal{F}_{t-1}^U] = \mathbb{E}[\varphi(U_{t-1} + M[t, Z_t]) | U_{t-1}] = \sum_{z \in H} \varphi(U_{t-1} + M[t, z]) \mathbb{P}(Z_t = z | U_{t-1}).$$

We deduce that a transition kernel for U is given for $t \in I^*$, $u, u' \in W$ by

$$\pi_t^U(u; u') := \begin{cases} \sum_{z \in H} \mathbb{1}_{u+M[t, z] = u'} \mathbb{P}(Z_t = z | U_{t-1} = u) & \text{if } \mathbb{P}(U_{t-1} = u) > 0, \\ \mathbb{1}_{u' = u} & \text{if } \mathbb{P}(U_{t-1} = u) = 0. \end{cases}$$

□

To give the intuition behind the linear programming constraint that we will use to relax our problem, we start by defining for any stopping time the associated occupation measures and then we derive a forward equation they satisfy. Assume for the rest of this subsection that U and $(X_t, U_t)_{t \in I}$ are \mathbb{F} -Markov chains with transition kernels $(\pi_t^U)_{t \in I^*}$ and $(\pi_t)_{t \in I^*}$, respectively. Since U is an \mathbb{F} -Markov chain, a similar proof to Lemma 7.7 implies that $\mathbb{F}^U = \mathbb{F}^Z$ is immersed⁴ in \mathbb{F} . In particular, for any \mathbb{F} -adapted and integrable process $Y = (Y_t)_{t \in I}$, we have $\mathbb{E}[Y_t | \mathcal{F}_t^Z] = \mathbb{E}[Y_t | \mathcal{F}_t^Z]$ a.s. for all $t \in I$.

4. Meaning that for all $t \in I$, \mathcal{F}_t is conditionally independent of \mathcal{F}_t^U given \mathcal{F}_t^U .

Occupation measures. For any \mathbb{F} -stopping time τ taking values in I define the random subprobability measures $\tilde{m}_t : \Omega \rightarrow \mathcal{P}^{sub}(D)$ and $\tilde{\mu}_t : \Omega \rightarrow \mathcal{P}^{sub}(E)$ satisfying

$$\begin{aligned}\tilde{m}_t(B) &:= \mathbb{P}[X_t \in B, t < \tau \wedge \tau_D^X | \mathcal{F}_t^Z], \quad a.s., \quad B \in \mathcal{B}(D), \quad t \in I \setminus \{T\}, \\ \tilde{\mu}_t(B) &:= \mathbb{P}[X_t \in B, \tau \wedge \tau_D^X = t | \mathcal{F}_t^Z], \quad a.s., \quad B \in \mathcal{B}(E), \quad t \in I.\end{aligned}$$

Note that \tilde{m}_t and $\tilde{\mu}_t$ are well defined as random variables since E is a compact metric space and they can also be seen as subprobability kernels from (Ω, \mathcal{F}) to $(E, \mathcal{B}(E))$. These random subprobability measures are the central objects of the MFG theory with optimal stopping and common noise. Here \tilde{m}_t represents the distribution at time t of the players still in the game. On the other hand, $\tilde{\mu}_t$ is the distribution at time t of the players who exit the game at this time.

Moreover using that $\mathcal{F}_t^Z = \sigma(U_t)$ which is finitely generated by the sets $(\{U_t = u\})_u$, we can identify these random subprobabilities with the mappings $m_t : \Omega_t \rightarrow \mathcal{P}^{sub}(D)$ and $\mu_t : \Omega_t \rightarrow \mathcal{P}^{sub}(E)$ given by

$$m_t(u)(B) := \mathbb{P}[X_t \in B, t < \tau \wedge \tau_D^X | U_t = u], \quad B \in \mathcal{B}(D), \quad t \in I \setminus \{T\}, \quad u \in \Omega_t, \quad (5.5)$$

$$\mu_t(u)(B) := \mathbb{P}[X_t \in B, \tau \wedge \tau_D^X = t | U_t = u], \quad B \in \mathcal{B}(E), \quad t \in I, \quad u \in \Omega_t, \quad (5.6)$$

where $\Omega_t := \{u \in W : \mathbb{P}(U_t = u) > 0\}$. We will rather work with the latter quantities which have the same interpretation as the first ones but at each particular trajectory of the common noise up to the corresponding time.

Derivation of the constraint. For any $\varphi \in M_b(I \times E \times W)$ we have

$$\mathbb{E}[\varphi(t+1, X_{t+1}, U_{t+1}) - \varphi(t, X_t, U_t) | \mathcal{F}_t] = \mathcal{L}(\varphi)(t, X_t, U_t) \quad a.s.,$$

where for $(t, x, u) \in I \setminus \{T\} \times E \times W$

$$\mathcal{L}(\varphi)(t, x, u) := \int_{E \times W} [\varphi(t+1, x', u') - \varphi(t, x, u)] \pi_{t+1}(x, u; dx', du').$$

Moreover the process $M(\varphi)$ defined by $M_0(\varphi) = \varphi(0, X_0, U_0)$ and

$$M_t(\varphi) = \varphi(t, X_t, U_t) - \sum_{s=0}^{t-1} \mathcal{L}(\varphi)(s, X_s, U_s), \quad t \in I^*,$$

is an \mathbb{F} -martingale. In particular for the stopping time $\theta := \tau \wedge \tau_D^X$, we have $\mathbb{E}[M_\theta(\varphi) | \mathcal{F}_0] = \varphi(0, X_0, U_0)$ a.s. Taking the expectation,

$$\mathbb{E}[\varphi(\theta, X_\theta, U_\theta)] = \mathbb{E}[\varphi(0, X_0, U_0)] + \mathbb{E}\left[\sum_{t=0}^{\theta-1} \mathcal{L}(\varphi)(t, X_t, U_t)\right].$$

Now observe that

$$\begin{aligned}\mathbb{E}[\varphi(\theta, X_\theta, U_\theta)] &= \sum_{t=0}^T \sum_{u \in \Omega_t} p_t(u) \mathbb{E}[\mathbb{1}_{\theta=t} \varphi(t, X_t, u) | U_t = u] \\ &= \sum_{t=0}^T \sum_{u \in \Omega_t} p_t(u) \int_E \varphi(t, x, u) \mu_t(u)(dx),\end{aligned}$$

where $p_t(u) := \mathbb{P}(U_t = u)$, and

$$\begin{aligned}\mathbb{E} \left[\sum_{t=0}^{\theta-1} \mathcal{L}(\varphi)(t, X_t, U_t) \right] &= \sum_{t=0}^{T-1} \sum_{u \in \Omega_t} p_t(u) \mathbb{E}[\mathcal{L}(\varphi)(t, X_t, u) \mathbb{1}_{t < \theta} | U_t = u] \\ &= \sum_{t=0}^{T-1} \sum_{u \in \Omega_t} p_t(u) \int_D \mathcal{L}(\varphi)(t, x, u) m_t(u)(dx).\end{aligned}$$

We deduce that the occupation measures satisfy the constraint

$$\begin{aligned}\sum_{t=0}^T \sum_{u \in \Omega_t} p_t(u) \int_E \varphi(t, x, u) \mu_t(u)(dx) &= \int_D \varphi(0, x, U_0) m_0^*(dx) \\ &\quad + \sum_{t=0}^{T-1} \sum_{u \in \Omega_t} p_t(u) \int_D \mathcal{L}(\varphi)(t, x, u) m_t(u)(dx),\end{aligned}$$

for any bounded and measurable test function φ . In the sequel, we will restrict the constraint to continuous test functions (they are bounded by compactness), which are more suitable for our topological arguments.

3.2 Linear programming formulation for discrete time MFGs with common noise and partial information

We endow any discrete set with the discrete topology, the space E is endowed with the topology induced by its metric and the set of subprobability measures on some Polish space is endowed with the topology of weak convergence. The topology on any product space is taken to be the product topology.

We are given the following reward functions:

$$f_t : D \times W \times \mathcal{P}^{sub}(D) \rightarrow \mathbb{R}, \quad t \in I \setminus \{T\}, \quad g_t : E \times W \times \mathcal{P}^{sub}(E) \rightarrow \mathbb{R}, \quad t \in I.$$

In this subsection and the next one, we let the following assumptions hold true.

Assumption 3.2.

- (1) m_0^* is supported on D .
- (2) (X, U) is an \mathbb{F} -Markov chain with transition kernels $(\pi_t)_{t \in I^*}$ such that for all $t \in I^*$, $x, \bar{x} \in E$, $u \in W$ and $u' \in W$, $\pi_t(x, u; E \times \{u'\}) = \pi_t(\bar{x}, u; E \times \{u'\})$.
- (3) For each $t \in I^*$, π_t is continuous seen as a function from $E \times W$ to $\mathcal{P}(E \times W)$.

(4) For each t , the functions $(x, u, m) \mapsto f_t(x, u, m)$ and $(x, u, \mu) \mapsto g_t(x, u, \mu)$ are continuous.

Remark 3.3. The second assumption is equivalent to the statement that (X, U) and U are \mathbb{F} -Markov chains. We proof this in Lemma 7.15 in the Appendix.

We start by giving the definition of a strong equilibrium with \mathbb{F} -stopping times. Here, the terminology *strong* means that the flow of measures are adapted to the filtration of the common noise. Denote by \mathcal{T} the set of \mathbb{F} -stopping times valued in I .

Definition 3.4 (Strong MFG equilibrium with \mathbb{F} -stopping times).

(1) Fix for each $t \in I \setminus \{T\}$, $m_t : W \rightarrow \mathcal{P}^{sub}(D)$ and for each $t \in I$, $\mu_t : W \rightarrow \mathcal{P}^{sub}(E)$, and find the solution to the optimal stopping problem

$$\sup_{\tau \in \mathcal{T}} \mathbb{E} \left[\sum_{t=0}^{\tau \wedge \tau_D^X - 1} f_t(X_t, U_t, m_t(U_t)) + g_{\tau \wedge \tau_D^X}(X_{\tau \wedge \tau_D^X}, U_{\tau \wedge \tau_D^X}, \mu_{\tau \wedge \tau_D^X}(U_{\tau \wedge \tau_D^X})) \right]. \quad (5.7)$$

(2) Denoting by $\tau^{m, \mu}$ an optimal stopping time associated to (m, μ) (solution of the problem (5.7)), find (m, μ) such that

$$m_t(u)(B) = \mathbb{P}[X_t \in B, t < \tau^{m, \mu} \wedge \tau_D^X | U_t = u], \quad B \in \mathcal{B}(D), \quad t \in I \setminus \{T\}, \quad u \in \Omega_t,$$

$$\mu_t(u)(B) = \mathbb{P}[X_t \in B, \tau^{m, \mu} \wedge \tau_D^X = t | U_t = u], \quad B \in \mathcal{B}(E), \quad t \in I, \quad u \in \Omega_t.$$

Remark 3.5. One could take f_t to depend on $(m_s)_{s \leq t}$ and g_t to depend on $(\mu_s)_{s \leq t}$. In that case, the proofs will be the same as the ones that we will present by observing that for each $s \leq t$, U_s is a deterministic function of U_t . In order to simplify the notations, we have decided not to do so.

Now, if we are given for each t , $\bar{m}_t : W \rightarrow \mathcal{P}^{sub}(D)$, $\bar{\mu}_t : W \rightarrow \mathcal{P}^{sub}(E)$ and some $\tau \in \mathcal{T}$, we can define the occupation measures (m, μ) associated to τ as in the above subsection and rewrite the objective function:

$$\begin{aligned} & \mathbb{E} \left[\sum_{t=0}^{\tau \wedge \tau_D^X - 1} f_t(X_t, U_t, \bar{m}_t(U_t)) + g_{\tau \wedge \tau_D^X}(X_{\tau \wedge \tau_D^X}, U_{\tau \wedge \tau_D^X}, \bar{\mu}_{\tau \wedge \tau_D^X}(U_{\tau \wedge \tau_D^X})) \right] \\ &= \sum_{t=0}^{T-1} \sum_{u \in \Omega_t} p_t(u) \int_D f_t(x, u, \bar{m}_t(u)) m_t(u)(dx) + \sum_{t=0}^T \sum_{u \in \Omega_t} p_t(u) \int_E g_t(x, u, \bar{\mu}_t(u)) \mu_t(u)(dx). \end{aligned}$$

Having in mind this writing of the objective function and the derivation of the constraint for the occupation measures, we are going to present the definition of an equilibrium under the linear programming formulation.

To ease the notation, we define the sets

$$\mathcal{C}_m := \prod_{t \in I \setminus \{T\}} \mathcal{P}^{sub}(D)^{\Omega_t}, \quad \mathcal{C}_\mu := \prod_{t \in I} \mathcal{P}^{sub}(E)^{\Omega_t},$$

where we use $\mathcal{Y}^{\mathcal{X}}$ to denote the set of functions from \mathcal{X} to \mathcal{Y} . Since the sets Ω_t with $t \in I$ are finite, we identify these functions with vectors.

Definition 3.6 (*Set of constraints*). Let \mathcal{R} be the set of pairs $(m, \mu) \in \mathcal{C}_m \times \mathcal{C}_\mu$ such that for all $\varphi \in C(I \times E \times W)$,

$$\begin{aligned} \sum_{t=0}^T \sum_{u \in \Omega_t} p_t(u) \int_E \varphi(t, x, u) \mu_t(u)(dx) &= \int_D \varphi(0, x, U_0) m_0^*(dx) \\ &+ \sum_{t=0}^{T-1} \sum_{u \in \Omega_t} p_t(u) \int_D \mathcal{L}(\varphi)(t, x, u) m_t(u)(dx). \end{aligned}$$

Definition 3.7 (*LP optimization criteria*). For $(\bar{m}, \bar{\mu}) \in \mathcal{R}$, let $\Gamma[\bar{m}, \bar{\mu}] : \mathcal{R} \rightarrow \mathbb{R}$ be the reward functional associated to $(\bar{m}, \bar{\mu})$, defined by

$$\begin{aligned} \Gamma[\bar{m}, \bar{\mu}](m, \mu) &= \sum_{t=0}^{T-1} \sum_{u \in \Omega_t} p_t(u) \int_D f_t(x, u, \bar{m}_t(u)) m_t(u)(dx) \\ &+ \sum_{t=0}^T \sum_{u \in \Omega_t} p_t(u) \int_E g_t(x, u, \bar{\mu}_t(u)) \mu_t(u)(dx). \end{aligned}$$

Definition 3.8 (*LP MFG Nash equilibrium*). We say that $(m^*, \mu^*) \in \mathcal{R}$ is an LP MFG Nash equilibrium if for all $(m, \mu) \in \mathcal{R}$, $\Gamma[m^*, \mu^*](m, \mu) \leq \Gammam^*, \mu^*$.

The set of measures \mathcal{R} is larger than the set of occupation measures defined by (5.5)–(5.6). However, the following theorem provides a probabilistic interpretation of the measures in \mathcal{R} as *occupation measures associated to randomized stopping times* in the sense of [CDL17], i.e. stopping times not necessarily with respect to the filtration of the underlying Markov chains, X and U , but with respect to larger filtrations for which the Markov property of (X, U) can be extended.

Theorem 3.9 (*Probabilistic representation with randomized stopping times*). If $(m, \mu) \in \mathcal{R}$, then there exists $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}}, \bar{\tau}, \bar{X}, \bar{U})$ verifying

- (1) $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}})$ is a complete probability space endowed with a complete filtration $\bar{\mathbb{F}}$ and supporting the random variables $(\bar{\tau}, \bar{X}, \bar{U})$.
- (2) (\bar{X}, \bar{U}) is an $\bar{\mathbb{F}}$ -Markov chain valued in $E \times W$ with transition kernels $(\pi_t)_{t \in I^*}$ and initial distribution $m_0^* \times \delta_{U_0}$.
- (3) $\bar{\tau}$ is an $\bar{\mathbb{F}}$ -stopping time valued in I such that $\bar{\tau} \leq \tau_D^{\bar{X}}$.
- (4) The measures have the following representation:

$$\begin{aligned} m_t(u)(B) &= \bar{\mathbb{P}}[\bar{X}_t \in B, t < \bar{\tau} | \bar{U}_t = u], \quad B \in \mathcal{B}(D), \quad t \in I \setminus \{T\}, \quad u \in \Omega_t, \\ \mu_t(u)(B) &:= \bar{\mathbb{P}}[\bar{X}_t \in B, \bar{\tau} = t | \bar{U}_t = u], \quad B \in \mathcal{B}(E), \quad t \in I, \quad u \in \Omega_t. \end{aligned}$$

Proof. For $B \in \mathcal{B}(E)$ and $u \in W$, define the following measures on E

$$\begin{aligned} \bar{m}_0(B \times \{u\}) &:= m_0^*(B) \times \delta_{U_0}(\{u\}), \\ \bar{m}_t(B \times \{u\}) &:= \begin{cases} p_t(u)m_t(u)(B \cap D), & \text{if } u \in \Omega_t \\ 0, & \text{if } u \notin \Omega_t, \end{cases} & t \in I \setminus \{T\}, \\ \bar{\mu}_t(B \times \{u\}) &:= \begin{cases} p_t(u)\mu_t(u)(B), & \text{if } u \in \Omega_t \\ 0, & \text{if } u \notin \Omega_t, \end{cases} & t \in I. \end{aligned}$$

This measures satisfy for all all $\varphi \in C(I \times E \times W)$,

$$\sum_{t=0}^T \int_{E \times W} \varphi(t, x, u) \bar{\mu}_t(dx, du) = \int_{E \times W} \varphi(0, x, u) \bar{m}_0(dx, du) + \sum_{t=0}^{T-1} \int_{E \times W} \mathcal{L}(\varphi)(t, x, u) \bar{m}_t(dx, du).$$

By Theorem 7.14, there exists $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{F}}, \bar{\mathbb{P}}, \bar{\tau}, \bar{X}, \bar{U})$ verifying

- (1) $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}})$ is a complete probability space endowed with a complete filtration $\bar{\mathbb{F}}$ and supporting the random variables $(\bar{\tau}, \bar{X}, \bar{U})$.
- (2) (\bar{X}, \bar{U}) is an $\bar{\mathbb{F}}$ -Markov chain valued in $E \times W$ with transition kernels $(\pi_t)_{t \in I^*}$ and initial distribution $\bar{m}_0 = m_0^* \times \delta_{U_0}$.
- (3) $\bar{\tau}$ is an $\bar{\mathbb{F}}$ -stopping time valued in I .
- (4) The measures have the following representation:

$$\bar{m}_t(B \times \{u\}) = \bar{\mathbb{P}}[\bar{X}_t \in B, \bar{U}_t = u, t < \bar{\tau}], \quad B \in \mathcal{B}(E), \quad u \in W, \quad t \in I \setminus \{T\},$$

$$\bar{\mu}_t(B \times \{u\}) := \bar{\mathbb{P}}[\bar{X}_t \in B, \bar{U}_t = u, \bar{\tau} = t], \quad B \in \mathcal{B}(E), \quad u \in W, \quad t \in I.$$

Now, we have that

$$\begin{aligned} \bar{\mathbb{P}}(\tau_D^{\bar{X}} < \bar{\tau}) &= \bar{\mathbb{P}}(\cup_{t=0}^{T-1} \{\tau_D^{\bar{X}} \leq t < \bar{\tau}\}) \leq \sum_{t=0}^{T-1} \bar{\mathbb{P}}((\cup_{s=0}^t \{\bar{X}_s \in D^c\}) \cap \{t < \bar{\tau}\}) \\ &\leq \sum_{t=0}^{T-1} \sum_{s=0}^t \bar{\mathbb{P}}(\bar{X}_s \in D^c, t < \bar{\tau}) \leq \sum_{t=0}^{T-1} \sum_{s=0}^t \bar{m}_s(D^c \times W) = 0. \end{aligned}$$

This computation allows to define the $\bar{\mathbb{F}}$ -stopping time $\bar{\tau} := \bar{\tau} \wedge \tau_D^{\bar{X}}$, which is equal to $\bar{\tau}$ $\bar{\mathbb{P}}$ -a.s. It is straightforward to show that

$$m_t(u)(B) = \bar{\mathbb{P}}[\bar{X}_t \in B, t < \bar{\tau} | \bar{U}_t = u], \quad B \in \mathcal{B}(D), \quad t \in I \setminus \{T\}, \quad u \in \Omega_t,$$

$$\mu_t(u)(B) := \bar{\mathbb{P}}[\bar{X}_t \in B, \bar{\tau} = t | \bar{U}_t = u], \quad B \in \mathcal{B}(E), \quad t \in I, \quad u \in \Omega_t.$$

□

3.3 Existence of LP MFG Nash equilibria

In order to show the existence of an LP MFG Nash equilibrium we first show the compactness of the set of admissible measures and then we show that there exists a fixed point to the best response map via Kakutani-Fan-Glicksberg's fixed point theorem.

Topology. We say that $(m^n)_{n \geq 1}$ converges to m in \mathcal{C}_m if for all $t \in I \setminus \{T\}$ and $u \in \Omega_t$, $(m_t^n(u))_{n \geq 1}$ converges to $m_t(u)$ weakly. We say that $(\mu^n)_{n \geq 1}$ converges to μ in \mathcal{C}_μ if for all $t \in I$ and $u \in \Omega_t$, $(\mu_t^n(u))_{n \geq 1}$ converges to $\mu_t(u)$ weakly. We recall that for a compact metric space K , the set $\mathcal{P}^{sub}(K)$ is compact for the topology of weak convergence.

Theorem 3.10. *The set \mathcal{R} is compact.*

Proof. Relative compactness follows since $\mathcal{P}^{sub}(D)$ and $\mathcal{P}^{sub}(E)$ are compact and henceforth \mathcal{C}_m and \mathcal{C}_μ are also compact. Let us check that \mathcal{R} is closed. Consider a sequence $(m^n, \mu^n)_{n \geq 1} \subset \mathcal{R}$ converging to some (m, μ) and let us show that $(m, \mu) \in \mathcal{R}$. For all $\varphi \in C(I \times E \times W)$ and $n \geq 1$,

$$\begin{aligned} \sum_{t=0}^T \sum_{u \in \Omega_t} p_t(u) \int_E \varphi(t, x, u) \mu_t^n(u)(dx) &= \int_D \varphi(0, x, U_0) m_0^*(dx) \\ &+ \sum_{t=0}^{T-1} \sum_{u \in \Omega_t} p_t(u) \int_D \mathcal{L}(\varphi)(t, x, u) m_t^n(u)(dx). \end{aligned}$$

Since for each $t \in I$ and $u \in \Omega_t$ the function $x \mapsto \varphi(t, x, u)$ is continuous and bounded we get that

$$\lim_{n \rightarrow \infty} \int_E \varphi(t, x, u) \mu_t^n(u)(dx) = \int_E \varphi(t, x, u) \mu_t(u)(dx).$$

Now using that for each $t \in I \setminus \{N\}$ and $u \in \Omega_t$ the function $x \mapsto \mathcal{L}(\varphi)(t, x, u)$ is continuous (by the continuity assumption on the transition kernel) and bounded, we have

$$\lim_{n \rightarrow \infty} \int_D \mathcal{L}(\varphi)(t, x, u) m_t^n(u)(dx) = \int_D \mathcal{L}(\varphi)(t, x, u) m_t(u)(dx).$$

We deduce that $(m, \mu) \in \mathcal{R}$. □

In order to find an equilibrium we define the following best response set valued mapping.

Definition 3.11. *Define the set valued mapping $\Theta : \mathcal{R} \rightarrow 2^{\mathcal{R}}$ as*

$$\Theta(\bar{m}, \bar{\mu}) = \arg \max_{(m, \mu) \in \mathcal{R}} \Gamma[\bar{m}, \bar{\mu}](m, \mu).$$

Remark 3.12. *Note that the set of LP MFG Nash equilibria coincides with the set of fixed points of Θ .*

Before stating the existence theorem, we prove a preliminary lemma concerning the continuity of the reward functional.

Lemma 3.13. *The map $\mathcal{R}^2 \ni ((\bar{m}, \bar{\mu}), (m, \mu)) \mapsto \Gamma[\bar{m}, \bar{\mu}](m, \mu) \in \mathbb{R}$ is continuous.*

Proof. Let $(\bar{m}^n, \bar{\mu}^n)_{n \geq 1} \subset \mathcal{R}$ and $(m^n, \mu^n)_{n \geq 1} \subset \mathcal{R}$ be two sequences converging to $(\bar{m}, \bar{\mu}) \in \mathcal{R}$ and $(m, \mu) \in \mathcal{R}$ respectively. Now, applying Lemma 5.25 in Chapter 3, we get for all $t \in I \setminus \{T\}$ and $u \in \Omega_t$,

$$\int_D f_t(x, u, \bar{m}_t^n(u)) m_t^n(u)(dx) \xrightarrow{n \rightarrow \infty} \int_D f_t(x, u, \bar{m}_t(u)) m_t(u)(dx),$$

and for all $t \in I$ and $u \in \Omega_t$,

$$\int_E g_t(x, u, \bar{\mu}_t^n(u)) \mu_t^n(u)(dx) \xrightarrow{n \rightarrow \infty} \int_E g_t(x, u, \bar{\mu}_t(u)) \mu_t(u)(dx).$$

Since the reward functional is a linear combination of these quantities over t and u , the limit is straightforward. \square

Theorem 3.14. *There exists an LP MFG Nash equilibrium.*

Proof. We aim to apply Kakutani-Fan-Glicksberg's Theorem (Theorem 5.34 in Chapter 3). First note that \mathcal{R} is nonempty, compact (Theorem 3.10), convex and is included in

$$\prod_{t \in I \setminus \{T\}} \mathcal{M}(D)^{\Omega_t} \times \prod_{t \in I} \mathcal{M}(E)^{\Omega_t},$$

which is a locally convex Hausdorff space (endowed with the product topology and the set of finite signed measures endowed with the topology of weak convergence). Moreover, Θ has nonempty values since \mathcal{R} is compact and for each $(\bar{m}, \bar{\mu}) \in \mathcal{R}$, the map $\mathcal{R} \ni (m, \mu) \mapsto \Gamma[\bar{m}, \bar{\mu}](m, \mu) \in \mathbb{R}$ is continuous on \mathcal{R} by Lemma 3.13. This map is also linear which gives that Θ has convex values. Again, by Lemma 3.13, the map $\mathcal{R}^2 \ni ((\bar{m}, \bar{\mu}), (m, \mu)) \mapsto \Gamma[\bar{m}, \bar{\mu}](m, \mu) \in \mathbb{R}$ is continuous on \mathcal{R}^2 , which shows that Θ has closed graph. By Kakutani-Fan-Glicksberg's fixed point theorem we conclude that there exists an equilibrium. \square

3.4 Existence of approximate equilibria in the N -player game

In this subsection, we show that an LP MFG Nash equilibrium can be used to obtain ε -Nash equilibria for games with finite number of players.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space and let $(\mathbb{F}^N)_{N \geq 1}$ be a sequence of complete filtrations modeling the information available to the players in each N -player game. Consider players with state processes $X^n = (X_t^n)_{t \in I}$, $n \in \mathbb{N}^*$. Let Z be the common noise and let U be constructed as before.

Assumption 3.15.

- (1) For each $N \geq 1$ and $n \in \{1, \dots, N\}$, (X^n, U) is an \mathbb{F}^N -Markov chain taking values in $E \times W$ with initial condition $m_0^* \times \delta_{U_0}$ and transition kernels $(\pi_t)_{t \in I^*}$.
- (2) The sequence of random variables $(X^n)_{n \geq 1}$ (valued in E^I) is i.i.d. given U (valued in W^I) in the sense of Definition 7.16.

3. Linear programming formulation for discrete time MFGs with common noise and partial information

We fix $N \geq 1$ and introduce the concepts for the N -player game. Let \mathcal{K}_N denote the set of transition probability kernels κ from Ω to I such that for all $t \in I$ and $B \in \sigma(\{0\}, \dots, \{t\})$ the mapping $\omega \mapsto \kappa(\omega, B)$ is \mathcal{F}_t^N -measurable. We say that \mathcal{K}_N is the set of randomized stopping times. A special class of randomized stopping times are the Markovian randomized stopping times (i.e. the randomness in the kernel comes from observing the underlying state process and common noise process). More precisely, denote by $\Omega_{X,U} := (E \times W)^I$ the canonical space for the process (X, U) , which is endowed with the Borel σ -algebra, and define the filtration $\mathcal{F}_t^{X,U} := \sigma(X_s, U_s : s \leq t)$, for $t \in I$. Let $\nu \in \mathcal{P}(\Omega_{X,U})$ be the unique law of the Markov chain with transition kernels $(\pi_t)_{t \in I^*}$ and initial law $m_0^* \times \delta_{U_0}$. We denote by $\mathbb{F}^{X,U,\nu} = (\mathcal{F}_t^{X,U,\nu})_{t \in I}$ the filtration

$$\mathcal{F}_t^{X,U,\nu} := \mathcal{F}_t^{X,U} \vee \mathcal{N}_\nu(\mathcal{F}_T^{X,U}).$$

Define $\mathcal{K}^{X,U}$ as the set of transition kernels κ from $\Omega_{X,U}$ to I such that for all $t \in I$ and $B \in \sigma(\{0\}, \dots, \{t\})$, the mapping $(x, u) \mapsto \kappa(x, u; B)$ is $\mathcal{F}_t^{X,U,\nu}$ -measurable.

Lemma 3.16. *For all $n \in \{1, \dots, N\}$ and all $\kappa \in \mathcal{K}^{X,U}$, $\kappa(X^n(\cdot), U(\cdot); \cdot) \in \mathcal{K}_N$.*

Proof. It suffices to show that for all $n \in \{1, \dots, N\}$, (X^n, U) is $\mathcal{F}_t^N / \mathcal{F}_t^{X,U,\nu}$ -measurable for each $t \in I$. Take $B \in \mathcal{F}_t^{X,U,\nu}$. We can write $B = C \cup D$ with $C \in \mathcal{F}_t^{X,U}$ and $D \in \mathcal{N}_\nu(\mathcal{F}_T^{X,U})$. Now, we have that $(X^n, U)^{-1}(B) = (X^n, U)^{-1}(C) \cup (X^n, U)^{-1}(D) \in \mathcal{F}_t^N$. In fact, $(X^n, U)^{-1}(C) \in \mathcal{F}_t^N$ using that (X^n, U) is \mathbb{F}^N -adapted, and $(X^n, U)^{-1}(D) \in \mathcal{N}_{\mathbb{P}}(\mathcal{F})$. □

We define the set of admissible strategies for the N -player game as the product set \mathcal{K}_N^N . For an admissible strategy $\boldsymbol{\kappa} := (\kappa^1, \dots, \kappa^N) \in \mathcal{K}_N^N$, define the empirical measures

$$m_t^N[\boldsymbol{\kappa}](B) = \frac{1}{N} \sum_{n=1}^N \delta_{X_t^n}(B) \mathbb{1}_{t < \tau_D^{X^n}} \kappa^n(\{t+1, \dots, T\}), \quad B \in \mathcal{B}(D), \quad t \in I \setminus \{T\},$$

$$\mu_t^N[\boldsymbol{\kappa}](B) = \frac{1}{N} \sum_{n=1}^N \delta_{X_t^n}(B) \mathbb{1}_{\tau_D^{X^n} = t} \kappa^n(\{t\}), \quad B \in \mathcal{B}(E), \quad t \in I.$$

Moreover, define the expected gain for player $n \in \{1, \dots, N\}$ as

$$J_n^N(\boldsymbol{\kappa}) := \mathbb{E} \left[\sum_{t=0}^{T \wedge \tau_D^{X^n} - 1} f_t(X_t^n, U_t, m_t^N[\boldsymbol{\kappa}]) \kappa^n(\{t+1, \dots, T\}) + \sum_{t=0}^{T \wedge \tau_D^{X^n}} g_t(X_t^n, U_t, \mu_t^N[\boldsymbol{\kappa}]) \kappa^n(\{t\}) \right].$$

Remark 3.17. *Let τ^1, \dots, τ^N be \mathbb{F}^N -stopping times and define $\kappa^n = \delta_{\tau^n}$. Then*

$$\boldsymbol{\kappa} := (\kappa^1, \dots, \kappa^N) \in \mathcal{K}_N^N,$$

and we have:

$$m_t^N[\boldsymbol{\kappa}](B) = \frac{1}{N} \sum_{n=1}^N \delta_{X_t^n}(B) \mathbb{1}_{t < \tau^n \wedge \tau_D^{X^n}}, \quad B \in \mathcal{B}(D), \quad t \in I \setminus \{T\},$$

$$\mu_t^N[\boldsymbol{\kappa}](B) = \frac{1}{N} \sum_{n=1}^N \delta_{X_t^n}(B) \mathbb{1}_{\tau^n \wedge \tau_D^{X^n} = t}, \quad B \in \mathcal{B}(E), \quad t \in I,$$

and,

$$J_n^N(\boldsymbol{\kappa}) := \mathbb{E} \left[\sum_{t=0}^{\tau^n \wedge \tau_D^{X^n} - 1} f_t(X_t^n, U_t, m_t^N[\boldsymbol{\kappa}]) + g_{\tau^n \wedge \tau_D^{X^n}} \left(X_{\tau^n \wedge \tau_D^{X^n}}^n, U_{\tau^n \wedge \tau_D^{X^n}}, \mu_{\tau^n \wedge \tau_D^{X^n}}^N[\boldsymbol{\kappa}] \right) \right].$$

Definition 3.18. Given $N \in \mathbb{N}^*$ and $\varepsilon \geq 0$, we say that $\boldsymbol{\kappa} := (\kappa^1, \dots, \kappa^N) \in \mathcal{K}_N^N$ is an ε -Nash equilibrium for the N -player game if $J_n^N(\bar{\kappa}, \boldsymbol{\kappa}^{-n}) - \varepsilon \leq J_n^N(\boldsymbol{\kappa})$, for all $\bar{\kappa} \in \mathcal{K}_N$ and $n = 1, \dots, N$.

Let (m^*, μ^*) be an LP MFG Nash equilibrium, which exists by Theorem 3.14. Let

$$(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}}, \bar{\mathbb{P}}, \bar{\tau}, \bar{X}, \bar{U})$$

be the associated probabilistic representation given by Theorem 3.9. The probability measure $P^* := \bar{\mathbb{P}} \circ (\bar{X}, \bar{U}, \bar{\tau})^{-1} \in \mathcal{P}(\Omega_{X,U} \times I)$ disintegrates as $P^*(dx, du, d\theta) = \nu(dx, du) \kappa^*(x, u; d\theta)$ for some $\kappa^* \in \mathcal{K}^{X,U}$ (see e.g. Corollary 7.8). Now define the Markovian randomized stopping times $\kappa^n := \kappa^*(X^n(\cdot), U(\cdot); \cdot)$, for $n \geq 1$.

Theorem 3.19. Let Assumptions 3.2 and 3.15 hold true. For any $\varepsilon > 0$, there is some $N_0 \geq 1$ such that if $N \geq N_0$, then $\boldsymbol{\kappa} = (\kappa^1, \dots, \kappa^N)$ is an ε -Nash equilibrium for the N -player game.

Proof. Let $\varepsilon > 0$. By the symmetry of the problem, it suffices to show that for N large enough,

$$J_1^N(\bar{\kappa}, \boldsymbol{\kappa}^{-1}) - \varepsilon \leq J_1^N(\boldsymbol{\kappa}),$$

for all $\bar{\kappa} \in \mathcal{K}_N$. In fact, if $n \leq N$, since $(X^n, U)_{n \geq 1}$ is exchangeable, we obtain that $J_n^N(\boldsymbol{\kappa}) = J_1^N(\boldsymbol{\kappa})$. Let us show that

$$\sup_{\bar{\kappa} \in \mathcal{K}_N} J_n^N(\bar{\kappa}, \boldsymbol{\kappa}^{-n}) = \sup_{\bar{\kappa} \in \mathcal{K}_N} J_1^N(\bar{\kappa}, \boldsymbol{\kappa}^{-1}).$$

We observe that the supremum over randomized stopping times with respect to \mathbb{F}^N is equal to the supremum over strict stopping times with respect to \mathbb{F}^N (see Proposition 1.5 in [EMS82]) and the latter supremum is equal to the supremum over strict stopping times with respect to the complete filtration of X^1, \dots, X^N and U . Now, any strict stopping times with respect to the complete filtration of X^1, \dots, X^N and U can be represented a.s. by a function of the underlying processes such that under permutation of the $(X^n)_{n \leq N}$ it is also a strict stopping time with respect to the same filtration. Again using that $(X^n, U)_{n \geq 1}$ is exchangeable, we obtain the equality.

Let $t \in I \setminus \{T\}$ and $\varphi \in C(E)$. By Theorem 7.17,

$$\begin{aligned} \int_E \varphi(x) m_t^N[\boldsymbol{\kappa}](dx) &= \frac{1}{N} \sum_{n=1}^N \varphi(X_t^n) \mathbb{1}_{t < \tau_D^{X^n}} \kappa^*(X^n, U; \{t+1, \dots, T\}) \\ &\xrightarrow{N \rightarrow \infty} \mathbb{E}[\varphi(X_t^1) \mathbb{1}_{t < \tau_D^{X^1}} \kappa^*(X^1, U; \{t+1, \dots, T\}) | U] \\ &= \int_E \varphi(x) m_t^*(U_t)(dx). \end{aligned}$$

Taking φ from a countable family characterizing the weak convergence, we get that $m_t^N[\boldsymbol{\kappa}] \rightharpoonup m_t^*(U_t)$ a.s. Analogously, $\mu_t^N[\boldsymbol{\kappa}] \rightharpoonup \mu_t^*(U_t)$ a.s. By dominated convergence,

$$\begin{aligned} J_1^N(\boldsymbol{\kappa}) &= \mathbb{E} \left[\sum_{t=0}^{T \wedge \tau_D^{X^1} - 1} f_t(X_t^1, U_t, m_t^N[\boldsymbol{\kappa}]) \kappa^*(X^1, U; \{t+1, \dots, T\}) \right. \\ &\quad \left. + \sum_{t=0}^{T \wedge \tau_D^{X^1}} g_t(X_t^1, U_t, \mu_t^N[\boldsymbol{\kappa}]) \kappa^*(X^1, U; \{t\}) \right] \\ &\xrightarrow{N \rightarrow \infty} \mathbb{E} \left[\sum_{t=0}^{T \wedge \tau_D^{X^1} - 1} f_t(X_t^1, U_t, m_t^*(U_t)) \kappa^*(X^1, U; \{t+1, \dots, T\}) \right. \\ &\quad \left. + \sum_{t=0}^{T \wedge \tau_D^{X^1}} g_t(X_t^1, U_t, \mu_t^*(U_t)) \kappa^*(X^1, U; \{t\}) \right] \\ &= \Gammam^*, \mu^*. \end{aligned}$$

We also have that

$$\begin{aligned} \int_E \varphi(x) m_t^N[(\bar{\kappa}, \boldsymbol{\kappa}^{-1})](dx) &= \frac{1}{N} \sum_{n=2}^N \varphi(X_t^n) \mathbb{1}_{t < \tau_D^{X^n}} \kappa^*(X^n, U; \{t+1, \dots, T\}) \\ &\quad + \frac{1}{N} \varphi(X_t^1) \mathbb{1}_{t < \tau_D^{X^1}} \bar{\kappa}(\{t+1, \dots, T\}) \\ &\xrightarrow{N \rightarrow \infty} \mathbb{E}[\varphi(X_t^1) \mathbb{1}_{t < \tau_D^{X^1}} \kappa^*(X^1, U; \{t+1, \dots, T\})] \\ &= \int_E \varphi(x) m_t^*(U_t)(dx). \end{aligned}$$

We can deduce that $m_t^N[(\bar{\kappa}, \boldsymbol{\kappa}^{-1})] \rightharpoonup m_t^*(U_t)$ and $\mu_t^N[(\bar{\kappa}, \boldsymbol{\kappa}^{-1})] \rightharpoonup \mu_t^*(U_t)$ a.s. Let $(\bar{m}, \bar{\mu})$ be the measures associated to $\bar{\kappa}$, which by Proposition 7.18 are in \mathcal{R} . Again, by dominated convergence, $\lim_{N \rightarrow \infty} J_1^N(\bar{\kappa}, \boldsymbol{\kappa}^{-1}) = \Gamma[m^*, \mu^*](\bar{m}, \bar{\mu})$. Taking N_0 big enough so that for all $N \geq N_0$

$$J_1^N(\boldsymbol{\kappa}) \geq \Gammam^*, \mu^* - \frac{\varepsilon}{2}, \quad \Gamma[m^*, \mu^*](\bar{m}, \bar{\mu}) \geq J_1^N(\bar{\kappa}, \boldsymbol{\kappa}^{-1}) - \frac{\varepsilon}{2},$$

we obtain for such N ,

$$J_1^N(\bar{\kappa}, \boldsymbol{\kappa}^{-1}) - \varepsilon \leq J_1^N(\boldsymbol{\kappa}).$$

□

3.5 Scenario uncertainty

In this subsection, we show that our application described in Section 2 can be solved using the theory developed above. To fix the ideas, consider a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ supporting (X, S, Z) satisfying:

- (1) The process $X = (X_t)_{t \in I}$ is a Markov chain taking values in E with transition kernels $(\pi_t^X)_{t \in I^*}$ and with initial law m_0^* .
- (2) The random variable S takes values in $\mathcal{S} := \{1, \dots, n_S\}$, $n_S \geq 1$.
- (3) The process $(S, Z_t)_{t \in I}$ is a Markov chain taking values in $\mathcal{S} \times H$, with initial distribution $\Pi_0 \times \delta_{z_0}$ and such that

$$\mathbb{P}(Z_t = z' | \mathcal{F}_{t-1}^{S,Z}) = \pi_t^Z(S, Z_{t-1}; z') \quad a.s., \quad z' \in H, \quad t \in I^*,$$

for some transition kernels $(\pi_t^Z)_{t \in I^*}$ from $\mathcal{S} \times H$ to H .

- (4) The random elements X and (S, Z) are independent.

Remark 3.20. *Our probabilistic set-up can include more general cases of scenario uncertainty in which $(S, Z_t)_{t \in I}$ is not Markovian and X can depend on the past of Z .*

The random variable S represents the random scenario, which is not observable. However the *prior distribution* $\Pi_0 := \mathcal{L}(S)$ is known. In stochastic control and optimal stopping problems with partial information, we usually want to maximize some expected reward depending on X , Z and S in a Markovian way. In the case of partial information, when we do not observe the scenario S , the process Z is usually not Markovian by itself. In that case, the problem can be reformulated by replacing S_t with the conditional law of S_t given \mathcal{F}_t^Z , which is also called the posterior distribution, and obtain a new Markovian setting (e.g. [Rie75; PRS05]). However in our framework it will not be necessary to introduce this conditional law process into the state process since we will add the process U presented before, which encompasses it (since it fully characterizes the filtration of Z).

Consider the filtration $\mathbb{F} := \mathbb{F}^{X,U} = \mathbb{F}^{X,Z}$, then (X, U) is an \mathbb{F} -Markov chain with transition kernels $(\pi_t)_{t \in I^*}$ given by

$$\pi_t((x, u), B \times \{u'\}) := \pi_t^X(x; B) \pi_t^U(u; u'), \quad x \in E, B \in \mathcal{B}(E), u, u' \in W.$$

In particular, if the corresponding assumptions on the initial distribution and the transition kernels are satisfied, we can apply Theorem 3.14.

Using Lemma 3.1, we obtain that U is a Markov chain with transition kernels given for $t \in I$, $u, u' \in W$ by

$$\pi_t^U(u; u') := \begin{cases} \sum_{z \in H} \mathbb{1}_{u+M[t,z]=u'} \mathbb{P}(Z_t = z | U_{t-1} = u) & \text{if } \mathbb{P}(U_{t-1} = u) > 0, \\ \mathbb{1}_{u'=u} & \text{if } \mathbb{P}(U_{t-1} = u) = 0. \end{cases}$$

We observe that $\mathbb{P}(Z_t = z | U_{t-1} = u) = \mathbb{P}(Z_t = z | Z_1 = \Psi_1(u), \dots, Z_{t-1} = \Psi_{t-1}(u))$. Moreover, when $\mathbb{P}(Z_1 = z_1, \dots, Z_t = z_t) > 0$, setting $z_0 = 0$, we have

$$\begin{aligned} \mathbb{P}(Z_1 = z_1, \dots, Z_t = z_t) &= \sum_{s \in \mathcal{S}} \mathbb{P}(Z_0 = z_0, \dots, Z_t = z_t | S = s) \mathbb{P}(S = s) \\ &= \sum_{s \in \mathcal{S}} \left[\prod_{k=1}^t \mathbb{P}(Z_k = z_k | Z_{k-1} = z_{k-1}, S = s) \right] \mathbb{P}(S = s) \\ &= \sum_{s \in \mathcal{S}} \left[\prod_{k=1}^t \pi_t^Z(s, z_{k-1}; z_k) \right] \Pi_0(s), \end{aligned}$$

which allows to fully determine $\mathbb{P}(Z_t = z | U_{t-1} = u)$.

3.6 Extension to several populations

We extend the above results to the case of several populations which will be the situation in our application. We assume that there is a total number of $K \in \mathbb{N}^*$ different populations of agents.

We are given K compact metric spaces (E_k, d_k) , $k \in \{1, \dots, K\}$, and compact subsets $D_k \subset E_k$ for each $k \in \{1, \dots, K\}$.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space together with a complete filtration \mathbb{F} and supporting (\mathbf{X}, Z) where $\mathbf{X} := (X^1, \dots, X^K)$ satisfying:

- (1) For $k \in \{1, \dots, K\}$, the state process $X^k = (X_t^k)_{t \in I}$ is an \mathbb{F} -adapted process taking values in E_k with initial law $m_0^{*,k}$.
- (2) The common noise process $Z = (Z_t)_{t \in I}$ is an \mathbb{F} -adapted process taking values in H and such that $Z_0 = z_0$ is deterministic.

Let U , W and $(\Omega_t)_{t \in I}$ be defined as in the previous subsection. We are given the following reward functions for $k \in \{1, \dots, K\}$:

$$f_t^k : D_k \times W \times \prod_{i=1}^K \mathcal{P}^{sub}(D_i) \rightarrow \mathbb{R}, \quad t \in I \setminus \{T\}, \quad g_t^k : E_k \times W \times \prod_{i=1}^K \mathcal{P}^{sub}(E_i) \rightarrow \mathbb{R}, \quad t \in I.$$

In this subsection, we let the following assumptions hold true.

Assumption 3.21.

- (1) $m_0^{*,k}$ is supported on D_k for $k \in \{1, \dots, K\}$.
- (2) For each $k \in \{1, \dots, K\}$, (X^k, U) is an \mathbb{F} -Markov chain with transition kernels $(\pi_t^k)_{t \in I^*}$ such that for all $t \in I^*$, $x, \bar{x} \in E_k$, $u \in W$ and $u' \in W$, $\pi_t^k(x, u; E_k \times \{u'\}) = \pi_t^k(\bar{x}, u; E_k \times \{u'\})$.
- (3) For each $t \in I^*$, and $k \in \{1, \dots, K\}$, π_t^k is continuous seen as a function from $E_k \times W$ to $\mathcal{P}(E_k \times W)$.
- (4) For each t and $k \in \{1, \dots, K\}$, the functions $(x, u, \mathbf{m}) \mapsto f_t^k(x, u, \mathbf{m})$ and $(x, u, \boldsymbol{\mu}) \mapsto g_t^k(x, u, \boldsymbol{\mu})$ are continuous.

For $k \in \{1, \dots, K\}$, denote by \mathcal{L}_k the operator

$$\mathcal{L}_k(\varphi)(t, x, u) := \int_{E_k \times W} [\varphi(t+1, x', u') - \varphi(t, x, u)] \pi_{t+1}^k(x, u, dx', du'),$$

where $\varphi \in M_b(I \times E_k \times W)$ and $(t, x, u) \in I \setminus \{T\} \times E_k \times W$. As in the previous section we define the sets

$$\mathcal{C}_m^k := \prod_{t \in I \setminus \{T\}} \mathcal{P}^{sub}(D_k)^{\Omega_t}, \quad \mathcal{C}_\mu^k := \prod_{t \in I} \mathcal{P}^{sub}(E_k)^{\Omega_t}, \quad k \in \{1, \dots, K\}.$$

Definition 3.22. For $k \in \{1, \dots, K\}$, let \mathcal{R}_k be the set of pairs $(m, \mu) \in \mathcal{C}_m^k \times \mathcal{C}_\mu^k$ such that for all $\varphi \in C(I \times E \times W)$,

$$\begin{aligned} \sum_{t=0}^T \sum_{u \in \Omega_t} p_t(u) \int_{E_k} \varphi(t, x, u) \mu_t(u)(dx) &= \int_{D_k} \varphi(0, x, U_0) m_0^{*,k}(dx) \\ &+ \sum_{t=0}^{T-1} \sum_{u \in \Omega_t} p_t(u) \int_{D_k} \mathcal{L}_k(\varphi)(t, x, u) m_t(u)(dx). \end{aligned}$$

With some abuse of notation we will write $(\mathbf{m}, \boldsymbol{\mu}) \in \prod_{k=1}^K \mathcal{R}_k$ instead of

$$((m^1, \mu^1), \dots, (m^K, \mu^K)) \in \prod_{k=1}^K \mathcal{R}_k.$$

Definition 3.23. For $(\bar{\mathbf{m}}, \bar{\boldsymbol{\mu}}) \in \prod_{k=1}^K \mathcal{R}_k$ and $k \in \{1, \dots, K\}$, let $\Gamma_k[\bar{\mathbf{m}}, \bar{\boldsymbol{\mu}}] : \mathcal{R}_k \rightarrow \mathbb{R}$ be the reward functional of the population k associated to $(\bar{\mathbf{m}}, \bar{\boldsymbol{\mu}})$ and defined by

$$\begin{aligned} \Gamma_k[\bar{\mathbf{m}}, \bar{\boldsymbol{\mu}}](m, \mu) &= \sum_{t=0}^{T-1} \sum_{u \in \Omega_t} p_t(u) \int_{D_k} f_t^k(x, u, \bar{\mathbf{m}}_t(u)) m_t(u)(dx) \\ &+ \sum_{t=0}^T \sum_{u \in \Omega_t} p_t(u) \int_{E_k} g_t^k(x, u, \bar{\boldsymbol{\mu}}_t(u)) \mu_t(u)(dx). \end{aligned}$$

Definition 3.24. We say that $(\mathbf{m}^*, \boldsymbol{\mu}^*) \in \prod_{k=1}^K \mathcal{R}_k$ is an LP MFG Nash equilibrium if for all $k \in \{1, \dots, K\}$ and all $(m, \mu) \in \mathcal{R}_k$,

$$\Gamma_k[\mathbf{m}^*, \boldsymbol{\mu}^*](m, \mu) \leq \Gamma_k[\mathbf{m}^*, \boldsymbol{\mu}^*](m^{k,*}, \mu^{k,*}).$$

Theorem 3.25. There exists an LP MFG Nash equilibrium.

Proof. First note that for each $k \in \{1, \dots, K\}$, the set \mathcal{R}_k is compact using the same arguments as in Theorem 3.10. Now consider the best response map $\Theta : \prod_{k=1}^K \mathcal{R}_k \rightarrow 2 \prod_{k=1}^K \mathcal{R}_k$ defined by

$$\Theta(\bar{\mathbf{m}}, \bar{\boldsymbol{\mu}}) = \prod_{k=1}^K \arg \max_{(m, \mu) \in \mathcal{R}_k} \Gamma_k[\bar{\mathbf{m}}, \bar{\boldsymbol{\mu}}](m, \mu).$$

As in the case of only one population, we aim to apply Kakutani-Fan-Glicksberg's Theorem (Theorem 5.34 in Chapter 3). Observe that $\prod_{k=1}^K \mathcal{R}_k$ is nonempty, compact, convex and is included in

$$\prod_{k=1}^K \left[\prod_{t \in I \setminus \{T\}} \mathcal{M}(D_k)^{\Omega_t} \times \prod_{t \in I} \mathcal{M}(E_k)^{\Omega_t} \right],$$

which is a locally convex Hausdorff space. Moreover, Θ has nonempty values since for $k \in \{1, \dots, K\}$, \mathcal{R}_k is compact and for each $(\bar{m}, \bar{\mu}) \in \prod_{j=1}^K \mathcal{R}_j$, by an extension of Lemma 3.13, the map $(m, \mu) \mapsto \Gamma_k[\bar{m}, \bar{\mu}](m, \mu)$ is continuous on \mathcal{R}_k . These maps are also linear which gives that Θ has convex values. For $k \in \{1, \dots, K\}$, again by an extension of Lemma 3.13, the map $(\bar{m}, \bar{\mu}, m, \mu) \mapsto \Gamma_k[\bar{m}, \bar{\mu}](m, \mu)$ is continuous on $(\prod_{j=1}^K \mathcal{R}_j) \times \mathcal{R}_k$, which shows that Θ has closed graph. By Kakutani-Fan-Glicksberg's Fixed Point Theorem we conclude the existence of an equilibrium. \square

4 MFGs for electricity markets

The game theoretic framework for day-ahead electricity markets described in Section 2 is not tractable for a large number of conventional and renewable producers. However, since the interactions of the agents via the price are of the mean-field type and there is symmetry of the agents dynamics inside each class, we can expect that the problem is well approximated by a Mean Field Game with common noise and two populations. Here the common noise is the process Z interpreted as the carbon price. Note that Z is not a Markov process in general due to its dependence on the economic scenario, which is in contrast with most of the literature in MFGs with common noise. From now on we consider the asymptotic formulation. We place ourselves in a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ big enough to carry all the random variables described below.

Probabilistic set-up. In the MFG setting we consider a representative agent for each class. The representative conventional producer is characterized by its baseline marginal cost per MWh B which is a homogeneous Markov chain on E_1 with initial distribution m_0^* and transition kernel π^B . On the other hand, the stochastic capacity factor of the representative renewable producer is given by a homogeneous Markov chain R on E_2 with initial distribution \bar{m}_0^* and transition kernel π^R . We consider the common noise process Z , the economic scenario S and the normalized demand process D as described in Section 2. We assume that B , R and (S, Z) are independent. We endow the probability space with the completed natural filtration of B , R and Z , denoted by $\mathbb{F} = (\mathcal{F}_t)_{t \in I}$. We denote by \mathcal{T} the set of \mathbb{F} -stopping times valued in I . Construct U , W , $(\Omega_t)_{t \in I}$ and $(\Psi_t)_{t \in I}$ as in the previous section.

Electricity supply and price formation. Define the supply mappings $S^c : H \times \mathcal{P}^{sub}(E_1) \times \mathbb{R}_+ \rightarrow \mathbb{R}$ and $S^r : I \times \mathcal{P}^{sub}(E_2) \rightarrow \mathbb{R}$ by

$$S^c(z, m, p) := I_C \int_{E_1} \bar{\alpha}(p - x - \tilde{\beta}z) m(dx) + S^b(p),$$

$$S^r(t, \bar{m}) := (I_R^b + I_R) \int_{E_2} x \eta_t^R(dx) - I_R \int_{E_2} x \bar{m}(dx),$$

where $(t, z, m, \bar{m}, p) \in I \times H \times \mathcal{P}^{sub}(E_1) \times \mathcal{P}^{sub}(E_2) \times \mathbb{R}_+$ and $\eta_t^R = \mathcal{L}(R_t)$. Here m corresponds to the distribution of the baseline cost of the representative conventional producer who is still in the game and \bar{m} corresponds to the distribution of the stochastic capacity factor of the representative potential renewable project who has not yet made the decision to build the plant. Recall that I_C is the total initial capacity of conventional plants which may leave the market, I_R is the total potential capacity of renewable plants which may enter the market but are not yet constructed at time $t = 0$, and I_R^b is the total capacity of renewable plants which are already in the market at time $t = 0$. These mappings should be compared with equations (5.2) and (5.3) in the setting of finite number of agents.

Define the price function $P : I \times H \times \mathbb{R}_+^2 \times \mathcal{P}^{sub}(E_1) \times \mathcal{P}^{sub}(E_2) \rightarrow [0, p_{max}]^2$ by

$$P(t, z, d, m, \bar{m}) = (\bar{P}(t, z, d_1, m, \bar{m}), \bar{P}(t, z, d_2, m, \bar{m})),$$

where $\bar{P} : I \times H \times \mathbb{R}_+ \times \mathcal{P}^{sub}(E_1) \times \mathcal{P}^{sub}(E_2) \rightarrow [0, p_{max}]$ is defined by

$$\bar{P}(t, z, d, m, \bar{m}) = \inf\{p \geq 0 : (d - S^r(t, \bar{m}))^+ \leq S^c(z, m, p)\} \wedge p_{max}.$$

The first component of P will correspond to the peak price and the second component to the off-peak price.

Reward mappings. Define the reward functions for the conventional producer $f : I \setminus \{T\} \times E_1 \times H \times \mathbb{R}_+^2 \rightarrow \mathbb{R}$ and $g : I \rightarrow \mathbb{R}$ by

$$f(t, x, z, p) := e^{-\rho t \Delta t} (c_p G(p_1 - x - \tilde{\beta}z) + c_o G(p_2 - x - \tilde{\beta}z) - \kappa_C) \Delta t, \\ (t, x, z, p) \in I \setminus \{T\} \times E_1 \times H \times \mathbb{R}_+^2,$$

$$g(t) := K_C e^{-(\gamma_C + \rho)t \Delta t}, \quad t \in I.$$

Define also the reward functions for the renewable producer $\bar{f} : I \setminus \{T\} \times E_2 \times \mathbb{R}_+^2 \rightarrow \mathbb{R}$ and $\bar{g} : I \rightarrow \mathbb{R}$ by

$$\bar{f}(t, x, p) := -e^{-\rho t \Delta t} ([c_p p_1 + c_o p_2] x - \kappa_R) \Delta t, \quad (t, x, p) \in I \setminus \{T\} \times E_2 \times \mathbb{R}_+^2,$$

$$\bar{g}(t) := K_R e^{-\rho T_0 - \gamma_R(T_0 - t \Delta t)} - K_R e^{-\rho t \Delta t}, \quad t \in I.$$

Observe that we defined \bar{f} with a minus sign. This will be justified in the definition of the strong formulation for the MFG problem.

Strong MFG equilibrium with \mathbb{F} -stopping times. We give here the definition of a strong MFG equilibrium with \mathbb{F} -stopping times.

Definition 4.1 (*Strong MFG equilibrium with \mathbb{F} -stopping times*).

(1) Fix for each $t \in I \setminus \{T\}$, $m_t : W \rightarrow \mathcal{P}^{sub}(E_1)$ and $\bar{m}_t : W \rightarrow \mathcal{P}^{sub}(E_2)$, and find the solution to the optimal stopping problems

$$\sup_{\tau \in \mathcal{T}} \mathbb{E} \left[\sum_{t=0}^{\tau-1} f(t, B_t, Z_t, P(t, Z_t, D_t^p, D_t^o, m_t(U_t), \bar{m}_t(U_t))) + g(\tau) \right], \quad (5.8)$$

$$\sup_{\bar{\tau} \in \mathcal{T}} \mathbb{E} \left[\sum_{t=\bar{\tau}}^{T-1} (-\bar{f})(t, R_t, P(t, Z_t, D_t^p, D_t^o, m_t(U_t), \bar{m}_t(U_t))) + \bar{g}(\bar{\tau}) \right], \quad (5.9)$$

Remark that problem (5.9) is equivalent to the following problem

$$\sup_{\bar{\tau} \in \mathcal{T}} \mathbb{E} \left[\sum_{t=0}^{\bar{\tau}-1} \bar{f}(t, R_t, P(t, Z_t, D_t^p, D_t^o, m_t(U_t), \bar{m}_t(U_t))) + \bar{g}(\bar{\tau}) \right], \quad (5.10)$$

which is the one that we will use later.

(2) Denoting by $\tau^{m, \bar{m}}$ and $\bar{\tau}^{m, \bar{m}}$ some optimal stopping times associated to (m, \bar{m}) (solutions of the problems (5.8) and (5.10) respectively), find (m, \bar{m}) such that

$$m_t(B) = \mathbb{P}[B_t \in B, t < \tau^{m, \bar{m}} | U_t = u], \quad B \in \mathcal{B}(E_1), \quad t \in I \setminus \{T\}, \quad u \in \Omega_t,$$

$$\bar{m}_t(B) = \mathbb{P}[R_t \in B, t < \bar{\tau}^{m, \bar{m}} | U_t = u], \quad B \in \mathcal{B}(E_2), \quad t \in I \setminus \{T\}, \quad u \in \Omega_t.$$

This problem is difficult to solve for general coefficients, so we will work with the linear programming formulation presented in the previous section. This formulation allows to prove existence of equilibria under general assumptions and facilitates their numerical computation.

Linear programming formulation. As explained in the previous section, the linear programming formulation consists in replacing the expectations concerning the processes and stopping times by integrals of occupation measures. To define the constraints, we define the transition kernels of the underlying Markov chains. We denote by $(\pi_t)_{t \in I^*}$ the transition kernels of (B, U) , which are given by $\pi_t(x, u; x', u') = \pi^B(x; x')\pi_t^U(u; u')$, where $t \in I^*$, $x, x' \in E_1$, $u, u' \in W$. Analogously, we denote by $(\bar{\pi}_t)_{t \in I^*}$ the transition kernels of (R, U) , which are given by, $\bar{\pi}_t(x, u; x', u') = \pi^R(x; x')\pi_t^U(u; u')$, where $t \in I^*$, $x, x' \in E_2$, $u, u' \in W$.

We define the generator associated to the conventional producer by

$$\mathcal{L}(\varphi)(t, x, u) := \int_{E_1 \times W} [\varphi(t+1, x', u') - \varphi(t, x, u)] \pi_{t+1}(x, u; dx', du'),$$

where $(t, x, u) \in I \setminus \{T\} \times E_1 \times W$ and $\varphi \in M_b(I \times E_1 \times W)$. In a similar manner, the generator associated to the renewable producer is given by

$$\bar{\mathcal{L}}(\varphi)(t, x, u) := \int_{E_2 \times W} [\varphi(t+1, x', u') - \varphi(t, x, u)] \bar{\pi}_{t+1}(x, u; dx', du'),$$

where $(t, x, u) \in I \setminus \{T\} \times E_2 \times W$ and $\varphi \in M_b(I \times E_2 \times W)$. Define the sets

$$\mathcal{C}_m^k := \prod_{t \in I \setminus \{T\}} \mathcal{P}^{sub}(E_k)^{\Omega_t}, \quad \mathcal{C}_\mu^k := \prod_{t \in I} \mathcal{P}^{sub}(E_k)^{\Omega_t}, \quad k \in \{1, 2\}.$$

Definition 4.2. Let \mathcal{R} be the set of pairs $(m, \mu) \in \mathcal{C}_m^1 \times \mathcal{C}_\mu^1$ such that for all $\varphi \in C(I \times E_1 \times W)$,

$$\begin{aligned} \sum_{t=0}^T \sum_{u \in \Omega_t} p_t(u) \int_{E_1} \varphi(t, x, u) \mu_t(u)(dx) &= \int_{E_1} \varphi(0, x, U_0) m_0^*(dx) \\ &+ \sum_{t=0}^{T-1} \sum_{u \in \Omega_t} p_t(u) \int_{E_1} \mathcal{L}(\varphi)(t, x, u) m_t(u)(dx). \end{aligned}$$

Definition 4.3. Let $\bar{\mathcal{R}}$ be the set of pairs $(\bar{m}, \bar{\mu}) \in \mathcal{C}_m^2 \times \mathcal{C}_\mu^2$ such that for all $\varphi \in C(I \times E_2 \times W)$,

$$\begin{aligned} \sum_{t=0}^T \sum_{u \in \Omega_t} p_t(u) \int_{E_2} \varphi(t, x, u) \bar{\mu}_t(u)(dx) &= \int_{E_2} \varphi(0, x, U_0) \bar{m}_0^*(dx) \\ &+ \sum_{t=0}^{T-1} \sum_{u \in \Omega_t} p_t(u) \int_{E_2} \bar{\mathcal{L}}(\varphi)(t, x, u) \bar{m}_t(u)(dx). \end{aligned}$$

Now define the peak and off-peak demand functions: for $t \in I$ and $u \in W$,

$$d_t^p(u) := \bar{c}_p \lambda_t(d(t) + \beta(\Psi_t(u) - z_0)), \quad d_t^o(u) := \bar{c}_o \lambda_t(d(t) + \beta(\Psi_t(u) - z_0)),$$

where Ψ_t is defined by (5.4), and maps U_t to Z_t .

Definition 4.4. For $(m', \mu', \bar{m}', \bar{\mu}') \in \mathcal{R} \times \bar{\mathcal{R}}$, let $\Gamma[m', \mu', \bar{m}', \bar{\mu}'] : \mathcal{R} \rightarrow \mathbb{R}$ be the reward functional of the conventional producer associated to $(m', \mu', \bar{m}', \bar{\mu}')$ and defined by

$$\begin{aligned} &\Gamma[m', \mu', \bar{m}', \bar{\mu}'](m, \mu) \\ &= \sum_{t=0}^{T-1} \sum_{u \in \Omega_t} p_t(u) \int_{E_1} f(t, x, \Psi_t(u), P(t, \Psi_t(u), d_t^p(u), d_t^o(u), m_t'(u), \bar{m}_t'(u))) m_t(u)(dx) \\ &+ \sum_{t=0}^T \sum_{u \in \Omega_t} p_t(u) \int_{E_1} g(t) \mu_t(u)(dx). \end{aligned}$$

We also define the reward functional of the renewable producer associated to $(m', \mu', \bar{m}', \bar{\mu}')$,

$\bar{\Gamma}[m', \mu', \bar{m}', \bar{\mu}'] : \bar{\mathcal{R}} \rightarrow \mathbb{R}$, by

$$\begin{aligned} & \bar{\Gamma}[m', \mu', \bar{m}', \bar{\mu}'](\bar{m}, \bar{\mu}) \\ &= \sum_{t=0}^{T-1} \sum_{u \in \Omega_t} p_t(u) \int_{E_2} \bar{f}(t, x, P(t, \Psi_t(u), d_t^p(u), d_t^o(u), m_t'(u), \bar{m}_t'(u))) \bar{m}_t(u)(dx) \\ & \quad + \sum_{t=0}^T \sum_{u \in \Omega_t} p_t(u) \int_{E_2} \bar{g}(t) \bar{\mu}_t(u)(dx). \end{aligned}$$

Definition 4.5. We say that $(m^*, \mu^*, \bar{m}^*, \bar{\mu}^*) \in \mathcal{R} \times \bar{\mathcal{R}}$ is an LP MFG Nash equilibrium if for all $(m, \mu) \in \mathcal{R}$, $\Gamma[m^*, \mu^*, \bar{m}^*, \bar{\mu}^*](m, \mu) \leq \Gamma[m^*, \mu^*, \bar{m}^*, \bar{\mu}^*](m^*, \mu^*)$, and for all $(\bar{m}, \bar{\mu}) \in \bar{\mathcal{R}}$, $\bar{\Gamma}[m^*, \mu^*, \bar{m}^*, \bar{\mu}^*](\bar{m}, \bar{\mu}) \leq \bar{\Gamma}[m^*, \mu^*, \bar{m}^*, \bar{\mu}^*](\bar{m}^*, \bar{\mu}^*)$.

We aim to apply Theorem 3.25 with 2 populations in order to show the existence of an LP MFG Nash equilibrium in our framework. Let us verify Assumption 3.21. In our setting we do not have absorption of the state processes so that $D_1 = E_1$ and $D_2 = E_2$. The transition kernels are continuous since we work with finite sets endowed with the discrete topology. Let us show the continuity of the reward functions. The function f is continuous using that G is continuous and the functions \bar{f} , g and \bar{g} are obviously continuous. Now the function $(t, z, d, m, \bar{m}) \mapsto \bar{P}(t, z, d, m, \bar{m})$ is continuous by Lemma 7.19. Assumption 3.21 is verified and by Theorem 3.25 there exists an LP MFG Nash equilibrium. We present now a uniqueness result for the price in an equilibrium situation. The proof follows the same ideas as in [ADT21], but we will give it for sake of clarity.

Proposition 4.6. Assume that the function S^b is C^1 on \mathbb{R}_+ and its derivative is bounded below by some constant $c_b > 0$. If $(m^1, \mu^1, \bar{m}^1, \bar{\mu}^1), (m^2, \mu^2, \bar{m}^2, \bar{\mu}^2) \in \mathcal{R} \times \bar{\mathcal{R}}$ are two LP MFG Nash equilibria, then for all $t \in I \setminus \{T\}$ and $u \in \Omega_t$,

$$P(t, \Psi_t(u), d_t^p(u), d_t^o(u), m_t^1(u), \bar{m}_t^1(u)) = P(t, \Psi_t(u), d_t^p(u), d_t^o(u), m_t^2(u), \bar{m}_t^2(u)).$$

Proof. Set for $t \in I \setminus \{T\}$ and $u \in W$

$$P_t^1(u) := P(t, \Psi_t(u), d_t^p(u), d_t^o(u), m_t^1(u), \bar{m}_t^1(u)),$$

$$P_t^2(u) := P(t, \Psi_t(u), d_t^p(u), d_t^o(u), m_t^2(u), \bar{m}_t^2(u)).$$

By the definition of LP MFG Nash equilibria we have:

$$\sum_{t=0}^{T-1} \sum_{u \in \Omega_t} p_t(u) \int_{E_1} [f(t, x, \psi_t(u), P_t^1(u)) - f(t, x, \psi_t(u), P_t^2(u))] [m_t^1 - m_t^2](u)(dx) \geq 0,$$

$$\sum_{t=0}^{T-1} \sum_{u \in \Omega_t} p_t(u) \int_{E_2} [\bar{f}(t, x, P_t^1(u)) - \bar{f}(t, x, P_t^2(u))] [\bar{m}_t^1 - \bar{m}_t^2](u)(dx) \geq 0.$$

Let $t \in I \setminus \{T\}$ and $u \in \Omega_t$ and write $P_t^1(u) = (p_1^1, p_2^1)$ and $P_t^2(u) = (p_1^2, p_2^2)$. Assume that

$p_1^1 > p_1^2$ and $p_2^1 > p_2^2$. Observe that

$$\begin{aligned} (d_t^p(u) - S^r(t, \bar{m}_t^1(u)))^+ &\geq S^c(\Psi_t(u), m_t^1(u), p_1^1), \quad d_t^p(u) > S^r(t, \bar{m}_t^1(u)), \\ (d_t^p(u) - S^r(t, \bar{m}_t^2(u)))^+ &= S^c(\Psi_t(u), m_t^2(u), p_1^2). \end{aligned}$$

Similarly,

$$\begin{aligned} (d_t^o(u) - S^r(t, \bar{m}_t^1(u)))^+ &\geq S^c(\Psi_t(u), m_t^1(u), p_2^1), \quad d_t^o(u) > S^r(t, \bar{m}_t^1(u)), \\ (d_t^o(u) - S^r(t, \bar{m}_t^2(u)))^+ &= S^c(\Psi_t(u), m_t^2(u), p_2^2). \end{aligned}$$

Using that $G(x) = \int_0^x \bar{\alpha}(y)dy$ and that $\bar{\alpha}$ is nondecreasing, we get for all $0 \leq q \leq p$:

$$(p - q)\bar{\alpha}(q) \leq G(p) - G(q) \leq (p - q)\bar{\alpha}(p).$$

We also have for all $0 \leq q \leq p$:

$$c_b(p - q) \leq S^b(p) - S^b(q).$$

We obtain that

$$\begin{aligned} &I_C \int_{E_1} [G(p_1^1 - x - \tilde{\beta}\Psi_t(u)) - G(p_1^2 - x - \tilde{\beta}\Psi_t(u))] [m_t^1 - m_t^2](u)(dx) \\ &\leq I_C(p_1^1 - p_1^2) \left[\int_{E_1} \bar{\alpha}(p_1^1 - x - \tilde{\beta}\Psi_t(u)) m_t^1(u)(dx) - \int_{E_1} \bar{\alpha}(p_1^2 - x - \tilde{\beta}\Psi_t(u)) m_t^2(u)(dx) \right] \\ &= (p_1^1 - p_1^2) \left[S^c(\Psi_t(u), m_t^1(u), p_1^1) - S^b(p_1^1) - S^c(\Psi_t(u), m_t^2(u), p_1^2) + S^b(p_1^2) \right] \\ &\leq (p_1^1 - p_1^2) \left[d_t^p(u) - S^r(t, \bar{m}_t^1(u)) - (d_t^p(u) - S^r(t, \bar{m}_t^2(u)))^+ \right] - c_b(p_1^1 - p_1^2)^2. \end{aligned}$$

On the other hand,

$$-I_R \int_{E_2} [p_1^1 x - p_1^2 x] [\bar{m}_t^1 - \bar{m}_t^2](u)(dx) = (p_1^1 - p_1^2) [S^r(t, \bar{m}_t^1(u)) - S^r(t, \bar{m}_t^2(u))].$$

Adding up the last two expressions we get

$$\begin{aligned} &I_C \int_{E_1} [G(p_1^1 - x - \tilde{\beta}\Psi_t(u)) - G(p_1^2 - x - \tilde{\beta}\Psi_t(u))] [m_t^1 - m_t^2](u)(dx) \\ &\quad - I_R \int_{E_2} [p_1^1 x - p_1^2 x] [\bar{m}_t^1 - \bar{m}_t^2](u)(dx) \\ &\leq (p_1^1 - p_1^2) \left[(d_t^p(u) - S^r(t, \bar{m}_t^2(u))) - (d_t^p(u) - S^r(t, \bar{m}_t^2(u)))^+ \right] - c_b(p_1^1 - p_1^2)^2 \\ &\leq -c_b(p_1^1 - p_1^2)^2. \end{aligned}$$

In a similar way,

$$\begin{aligned}
 & I_C \int_{E_1} [G(p_2^1 - x - \tilde{\beta}\Psi_t(u)) - G(p_2^2 - x - \tilde{\beta}\Psi_t(u))][m_t^1 - m_t^2](u)(dx) \\
 & \quad - I_R \int_{E_2} [p_2^1 x - p_2^2 x][\bar{m}_t^1 - \bar{m}_t^2](u)(dx) \\
 & \leq -c_b(p_2^1 - p_2^2)^2.
 \end{aligned}$$

We can deduce the inequality

$$\begin{aligned}
 & I_C \int_{E_1} [f(t, x, \Psi_t(u), P_t^1(u)) - f(t, x, \Psi_t(u), P_t^2(u))][m_t^1 - m_t^2](u)(dx) \\
 & \quad + I_R \int_{E_2} [\bar{f}(t, x, P_t^1(u)) - \bar{f}(t, x, P_t^2(u))][\bar{m}_t^1 - \bar{m}_t^2](u)(dx) \\
 & \leq -c_b e^{-\rho t \Delta t} [c_p(p_1^1 - p_1^2)^2 + c_o(p_2^1 - p_2^2)^2].
 \end{aligned}$$

The other cases comparing (p_1^1, p_2^1) and (p_1^2, p_2^2) are done in a similar fashion. This allows us to conclude that for all $t \in I \setminus \{T\}$ and $u \in \Omega_t$,

$$\begin{aligned}
 & I_C \int_{E_1} [f(t, x, \Psi_t(u), P_t^1(u)) - f(t, x, \Psi_t(u), P_t^2(u))][m_t^1 - m_t^2](u)(dx) \\
 & \quad + I_R \int_{E_2} [\bar{f}(t, x, P_t^1(u)) - \bar{f}(t, x, P_t^2(u))][\bar{m}_t^1 - \bar{m}_t^2](u)(dx) \\
 & \leq -c_b e^{-\rho t \Delta t} [c_p(\bar{P}(t, \Psi_t(u), d_t^p(u), m_t^1(u), \bar{m}_t^1(u)) - \bar{P}(t, \Psi_t(u), d_t^p(u), m_t^2(u), \bar{m}_t^2(u)))^2 \\
 & \quad + c_o(\bar{P}(t, \Psi_t(u), d_t^o(u), m_t^1(u), \bar{m}_t^1(u)) - \bar{P}(t, \Psi_t(u), d_t^o(u), m_t^2(u), \bar{m}_t^2(u)))^2].
 \end{aligned}$$

If we multiply by $p_t(u)$ and sum over $t \in I \setminus \{T\}$ and $u \in \Omega_t$, we get that the left hand side is nonnegative, which implies the equality of the prices as stated. \square

5 Numerical computation via the Linear Programming Fictitious Play algorithm

The Linear Programming Fictitious Play algorithm is presented in detail in Chapter 4 for the case of continuous time MFGs without common noise and with one class of agents. We present here an algorithm (Algorithm 3) for discrete time MFGs in the common noise setting and with two populations.

Algorithm 3: LPFP algorithm

Data: A number of steps N_{iter} for the equilibrium approximation; a pair $(m^{(0),g}, \mu^{(0),g}) \in \mathcal{R}$; a pair $(\bar{m}^{(0),g}, \bar{\mu}^{(0),g}) \in \bar{\mathcal{R}}$

Result: Approximate LP MFG Nash equilibrium

- 1 **for** $\ell = 0, 1, \dots, N_{iter} - 1$ **do**
- 2 Compute a linear programming best response $(m^{(\ell+1)}, \mu^{(\ell+1)})$ to $(m^{(\ell),g}, \mu^{(\ell),g}, \bar{m}^{(\ell),g}, \bar{\mu}^{(\ell),g})$ by solving the linear programming problem

$$\arg \max_{(m,\mu) \in \mathcal{R}} \Gamma[m^{(\ell),g}, \mu^{(\ell),g}, \bar{m}^{(\ell),g}, \bar{\mu}^{(\ell),g}](m, \mu).$$
- 3 Compute a linear programming best response $(\bar{m}^{(\ell+1)}, \bar{\mu}^{(\ell+1)})$ to $(m^{(\ell),g}, \mu^{(\ell),g}, \bar{m}^{(\ell),g}, \bar{\mu}^{(\ell),g})$ by solving the linear programming problem

$$\arg \max_{(\bar{m}, \bar{\mu}) \in \bar{\mathcal{R}}} \bar{\Gamma}[m^{(\ell),g}, \mu^{(\ell),g}, \bar{m}^{(\ell),g}, \bar{\mu}^{(\ell),g}](\bar{m}, \bar{\mu}).$$
- 4 Set $(m^{(\ell+1),g}, \mu^{(\ell+1),g}) := \frac{\ell}{\ell+1}(m^{(\ell),g}, \mu^{(\ell),g}) + \frac{1}{\ell+1}(m^{(\ell+1)}, \mu^{(\ell+1)})$.
- 5 Set $(\bar{m}^{(\ell+1),g}, \bar{\mu}^{(\ell+1),g}) := \frac{\ell}{\ell+1}(\bar{m}^{(\ell),g}, \bar{\mu}^{(\ell),g}) + \frac{1}{\ell+1}(\bar{m}^{(\ell+1)}, \bar{\mu}^{(\ell+1)})$.
- 6 **end**

The computation of the best responses in the algorithm is done by solving the linear programming problems. Observe that by linearity, the constraint defining the set \mathcal{R} (resp. $\bar{\mathcal{R}}$) is satisfied for all $\varphi \in C(I \times E_1 \times W)$ (resp. $C(I \times E_2 \times W)$) if and only if it is satisfied for the subset of indicator functions, which gives us a finite number of constraints. Henceforth, the algorithm consists in iteratively solving finite dimensional linear programming problems, for which there exist fast solvers⁵. We solve the LP problems by using the Gurobi⁶ solver in Python (Version 9.5.1).

6 Illustration

6.1 Parameters

We specify here all the parameters used in our model. We refer to [ADT21] for the calibration of the parameters non-related to the carbon price.

Time: We set the time horizon to $T_0 = 18$ years and the time steps to 3 months, i.e. $T = 72$ and $\Delta t = 0.25$. We set the discount rate to $\rho = 0.086$.

Common noise: To cover the range of carbon prices across the NGFS scenarios, we fix the possible values of the carbon price to be $H = \{50, 150, 250, 350, 450, 550, 650\}$ and denote its elements by $z_0 < z_1 < \dots < z_6$. We shall use six scenarios $\mathcal{S} = \{0, 1, \dots, 5\}$ which are inspired by

5. see <http://plato.asu.edu/ftp/lpsimp.html> for a comparison of different LP solvers.

6. <https://www.gurobi.com/>

the REMIND 2.1 scenarios in the NGFS database. We assume that the carbon price Z starts at z_0 in every scenario, and then, in each scenario $s \in \mathcal{S}$ at each adjustment date $t \in J$ can stay in the same state with probability $p_t(s) \in [0, 1]$ or jump to the next state with probability $1 - p_t(s)$. The adjustment dates of the carbon price are $J = \{10, 20, 30, 40, 50, 60\}$; with $\Delta t = 0.25$ this means that the carbon price adjustment happens every 2.5 years.

More precisely, the dynamics of Z are as follows: for $t \in J$ and $i = 0, 1, \dots, 5$, we set $\pi_t^Z(s, z_i; z_i) = p_t(s)$ and $\pi_t^Z(s, z_i; z_{i+1}) = 1 - p_t(s)$, otherwise we set $\pi_t^Z(s, z_i; z_i) = 1$. To calibrate the probabilities $p_t(s)$ we use the carbon price projections for the UK under the REMIND 2.1 scenarios in the NGFS database. Denoting by $C_t(s)$ the REMIND 2.1 carbon price under scenario s , for each $s \in \mathcal{S}$, we select the $(p_t(s))_{t \in J} \subset [0.3, 0.7]$ minimizing

$$\sum_{t \in J} |\mathbb{E}[Z_t | S = s] - C_t(s)|.$$

The prior scenario distribution is taken to be a uniform distribution.

Conventional dynamics: Following [ADT21], we take $C_{\min} = 0$, $C_{\max} = 70$, $\theta = 33.4 - \tilde{\beta}z_0$, $k = 0.5$, $\delta = \text{std}_C \sqrt{\frac{2k}{\theta}}$ where $\text{std}_C = 11$. In the spirit of [KD01] p. 96, we choose Δx so that

$$\frac{\Delta x^2}{\sigma^2(x) + \Delta x |b(x)|} \approx \Delta t.$$

We evaluate σ and b at $x = \theta$, which is the expectation of the stationary distribution of the CIR process. Since $b(\theta) = 0$, we obtain $\Delta x \approx \sigma(\theta) \sqrt{\Delta t} = \delta \sqrt{\theta \Delta t}$. We define $n_1 = \lfloor (C_{\max} - C_{\min})(\delta \sqrt{\theta \Delta t})^{-1} \rfloor$ and set $\Delta x = (C_{\max} - C_{\min})/n_1$. We finally define m_0^* as the discretization (by evaluating the density on the grid and then normalizing) of a gamma distribution with shape parameter $2\theta k/\delta^2$ and scale parameter $\delta^2/2k$, which is the stationary distribution of the CIR process.

Renewable dynamics: Once again, following [ADT21], we take $R_{\min} = 0.3$, $R_{\max} = 0.6$, $\bar{\theta} = 0.43$, $\bar{k} = 0.5$, $\bar{\delta} = \text{std}_R \sqrt{\frac{2\bar{k}}{\bar{\theta}(1-\bar{\theta}) - \text{std}_R^2}}$ where $\text{std}_R = 0.044$. We define $n_2 = \lfloor (R_{\max} - R_{\min})(\bar{\delta} \sqrt{\bar{\theta}(1-\bar{\theta}) \Delta t})^{-1} \rfloor$ and set $\Delta \bar{x} = (R_{\max} - R_{\min})/n_2$. We define \bar{m}_0^* as the discretization of a beta distribution with shape parameters $2\bar{k}\bar{\theta}/\bar{\delta}^2$ and scale parameter $2\bar{k}(1-\bar{\theta})/\bar{\delta}^2$, which is the stationary distribution of the Jacobi process.

Demand: As in the paper [ADT21], we use the peak/off-peak ratio $r_d = 1.29$ and we take the seasonal cycle λ_t to be equal to 1.10 in the first quarter, 0.93 in the second quarter, 0.91 in the third quarter and 1.06 in the last quarter. Using the NGFS database we obtain the electricity projections demand under the REMIND 2.1 scenarios. We denote by $(D_t(s))_{t \in I}$ the demand in scenario $s \in \mathcal{S}$ and set $\bar{d}(t) := D_t(s^*)$, where s^* is the scenario ‘‘Current Policies’’. We then

calibrate the carbon price sensitivity of the demand $\beta \geq 0$ by minimizing

$$\sum_{s \in \mathcal{S}} \sum_{t \in I} |D_t(s) - \bar{d}_t - \beta(\mathbb{E}[Z_t | S = s] - z_0)|.$$

Finally, we take the baseline demand level d from the British government demand projections⁷.

Capacity levels: As in [ADT21], we use the conventional baseline supply function $S^b(p) = \frac{12.1}{p_{max}} p$ where $p_{max} = 150$, the conventional capacity $I_C = 35.9$, installed renewable capacity $I_R^b = 35.6$ and potential renewable capacity $I_R = 47$.

Conventional rewards: We set the emission intensity to $\tilde{\beta} = 0.429$ ⁸. As in [ADT21], we use the fixed operating cost $\kappa_C = 30$ (in GBP per kW per year), scrap value of the plant $K_C = 0$ and an ad hoc supply function $\bar{\alpha} : \mathbb{R} \rightarrow [0, 1]$ defined by

$$\bar{\alpha}(y) = \mathbb{1}_{y > c_{max}} + \frac{1}{2} \left(1 + \sin \left(-\frac{\pi}{2} + \frac{\pi}{c_{max}} y \right) \right) \mathbb{1}_{0 \leq y \leq c_{max}},$$

where $c_{max} = 0.5$ corresponds to $c(1)$. This choice of $\bar{\alpha}$ leads to

$$G(x) = \int_0^x \bar{\alpha}(y) dy = \left(x - \frac{c_{max}}{2} \right) \mathbb{1}_{x > c_{max}} + \frac{1}{2} \left(x - \frac{c_{max}}{\pi} \cos \left(-\frac{\pi}{2} + \frac{\pi}{c_{max}} x \right) \right) \mathbb{1}_{0 \leq x \leq c_{max}}.$$

Renewable rewards: As in [ADT21], we use the fixed operating cost $\kappa_R = 17.21$ GBP per kW per year, investment cost $K_R = 1377$ GBP per kW of capacity and depreciation rate $\gamma_R = \ln(2)/10$.

6.2 Results

In our setting, there is a total number of $2^6 = 64$ possible trajectories of the carbon price Z . We select the lowest carbon price trajectory $\omega_{min} = (50, \dots, 50)$ and the highest carbon price trajectory $\omega_{max} = (50, 150, \dots, 650)$ to plot the quantities of interest conditionally on Z taking these paths. For comparison, we also solve the MFG model without common noise where the carbon price follows the (deterministic) trajectories ω_{min} and ω_{max} with probability 1. In the latter case, the agents know the exact trajectory of the carbon price which they use for choosing their stopping times, whereas in the stochastic case the realized carbon price trajectory is the same, but, when making their decisions, the agents take into account the risk of a carbon price rise at each date. The difference between the stochastic and deterministic case therefore quantifies the impact of climate policy uncertainty on the agents' behavior.

In Figure 5.1 we represent the log-log plot (base 10) of the exploitability of both the conventional and the renewable producers (i.e. the gain increase from switching to the best response, see [DLT22] for a definition in the LP approach), which shows that, at least empirically, the algorithm converges at the rate $O(N^{-1})$.

7. <https://www.gov.uk/government/publications/updated-energy-and-emissions-projections-2019>, Annex F.

8. see <https://www.rte-france.com/en/eco2mix/co2-emissions>

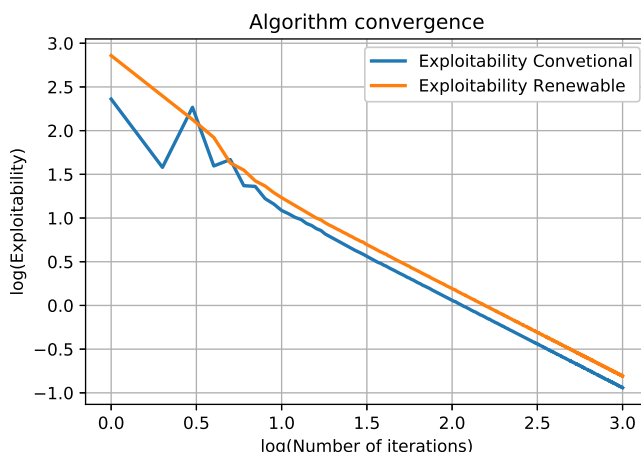


Figure 5.1 – Convergence of the algorithm: log-log plot (base 10) of the exploitability.

Figure 5.2 shows the total installed capacity of conventional and renewable producers as function of time for our sample trajectories ω_{min} and ω_{max} of the carbon price, for the MFG without common noise, as well as the expectation over all possible trajectories of the carbon price. As expected, higher carbon prices lead to faster growth of renewable penetration and faster departures of conventional producers. Besides, in the presence of “upward” scenario uncertainty (when the carbon price is stochastic and follows the lowest carbon price trajectory ω_{min}), the renewable power plants are built much earlier, and the conventional capacity decreases much faster than when the price follows the deterministic trajectory ω_{min} , even if the realized carbon price trajectory is the same in both cases. When there is a credible possibility of a carbon price rise, even if this possibility is not realized, conventional producers exit the market to avoid the risk of paying a high carbon price, and renewable producers enter the market to benefit from potential opportunities created by the departure of the conventional producers. In the presence of “downward” uncertainty (when the carbon price is stochastic and follows the highest carbon price trajectory ω_{max}), the impact of uncertainty is weaker: the conventional producers leave a bit later, but the renewable penetration is almost the same as with deterministic trajectory. We conclude that in all cases, the agents base their decisions on the highest possible carbon price.

Figure 5.3 shows the peak and off-peak prices for our sample trajectories ω_{min} and ω_{max} of the carbon price, for the MFG without common noise, as well as the expectation over all possible trajectories of the carbon price. As expected, higher carbon prices lead to higher peak electricity prices. Interestingly, the impact on off-peak prices is less clear, especially in the beginning of the period when enough conventional capacity is still present in the market. Higher carbon prices increase renewable penetration, which leads to lower electricity prices in summer, when demand is low, but they also increase the costs of conventional generation, which may push winter prices up, even in off-peak periods. Scenario uncertainty, especially that of the “upward” kind, although it does not modify the production costs directly, pushes some conventional producers, anticipating potential carbon price hikes, out of the market, which leads to higher peak prices compared to a deterministic trajectory. On the other hand, off-peak prices in summer are

somewhat lower in the presence of scenario uncertainty than for a deterministic carbon price trajectory. This is due to the fact that in these periods generation is predominantly renewable, and under scenario uncertainty renewable penetration is higher than for the deterministic price path.

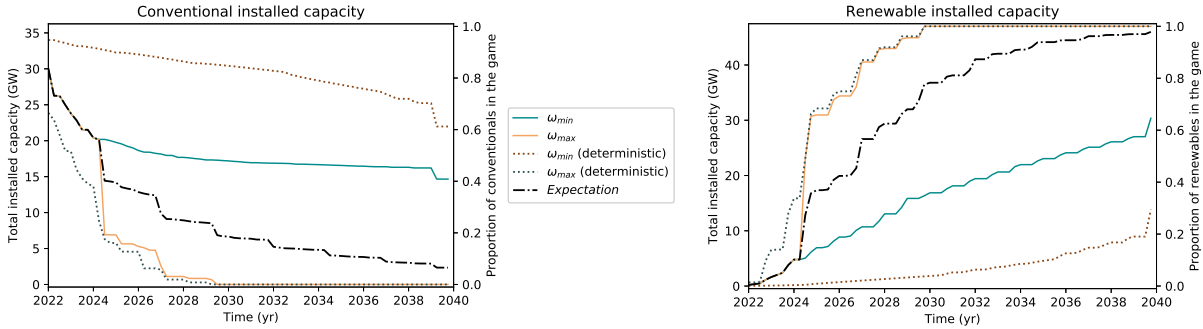


Figure 5.2 – Total installed capacities and proportions of conventional and renewable producers.

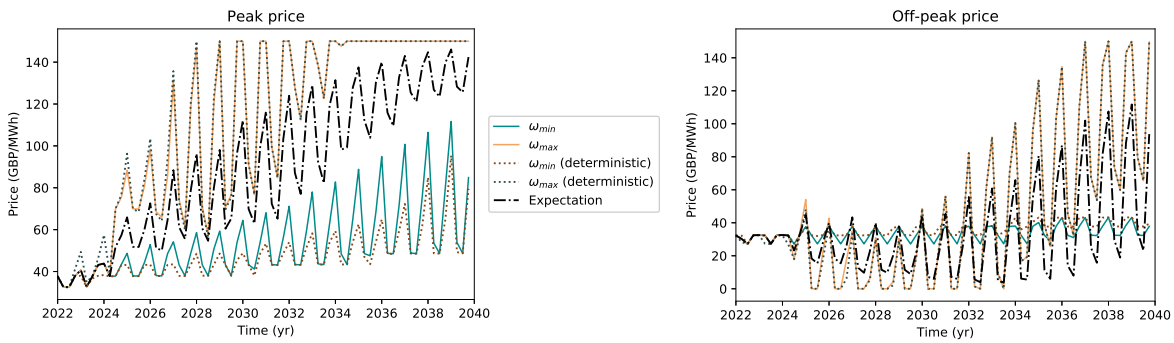


Figure 5.3 – Peak and Off-peak prices.

7 Appendix

7.1 Complementary results on the linear programming approach for MFGs of optimal stopping

In this appendix, we provide a theoretical analysis of the linear programming approach in discrete time in an abstract and general case. We show by a duality argument that the relaxed set is equal to the closed convex hull of the set induced by Markovian stopping times. In particular, this result allows us to derive a probabilistic representation of the relaxed set in terms of randomized stopping times (in the sense of [CDL17]).

Markovian and randomized stopping times in the canonical space We consider a discrete time setting with horizon $T \in \mathbb{N}^*$ and we let $I = \{0, \dots, T\}$ be the set of time indices and $I^* = \{1, \dots, T\}$. We are given a nonempty compact metric space (E, d) and consider two canonical spaces: $\Omega_X = E^I$ and $\bar{\Omega} = E^I \times I$, where E^I is the set of functions from I to E endowed

with the pointwise convergence topology (which coincides with the uniform convergence topology here). Denote by $\mathcal{B}(\Omega_X)$ and $\mathcal{B}(\bar{\Omega})$ the respective Borel σ -algebras. Let X be the identity map on Ω_X and let \bar{X} and $\bar{\tau}$ be the projections on $\bar{\Omega}$, i.e. $\bar{X}(\bar{\omega}) = x$ and $\bar{\tau}(\bar{\omega}) = \theta$ for each $\bar{\omega} = (x, \theta) \in \bar{\Omega}$. On $(\Omega_X, \mathcal{B}(\Omega_X))$ we define the filtration $\mathcal{F}_t^X := \sigma(X_s : s \leq t)$, for $t \in I$, and on $(\bar{\Omega}, \mathcal{B}(\bar{\Omega}))$ we define the filtrations $\mathcal{F}_t^{\bar{X}} := \sigma(\bar{X}_s : s \leq t)$ and $\mathcal{F}_t^{\bar{\tau}} := \sigma(\{\bar{\tau} = 0\}, \dots, \{\bar{\tau} = t\})$, for $t \in I$. Note that $\mathbb{F}^{\bar{\tau}}$ is the smallest filtration for which $\bar{\tau}$ is a stopping time. Define $\bar{\mathcal{F}}_t := \mathcal{F}_t^{\bar{X}} \vee \mathcal{F}_t^{\bar{\tau}}$, for $t \in I$. For any probability $\bar{\mathbb{P}}$ on $(\bar{\Omega}, \mathcal{B}(\bar{\Omega}))$, we denote by $\mathbb{F}^{\bar{X}, \bar{\mathbb{P}}} = (\mathcal{F}_t^{\bar{X}, \bar{\mathbb{P}}})_{t \in I}$ and $\bar{\mathbb{F}}^{\bar{\mathbb{P}}} = (\bar{\mathcal{F}}_t)_{t \in I}$ the filtrations

$$\mathcal{F}_t^{\bar{X}, \bar{\mathbb{P}}} := \mathcal{F}_t^{\bar{X}} \vee \mathcal{N}_{\bar{\mathbb{P}}}(\bar{\mathcal{F}}_T), \quad \bar{\mathcal{F}}_t^{\bar{\mathbb{P}}} := \bar{\mathcal{F}}_t \vee \mathcal{N}_{\bar{\mathbb{P}}}(\bar{\mathcal{F}}_T).$$

We are given transition kernels $(\pi_t)_{t \in I^*}$ on E and an initial law m_0^* on E . We assume that the transition kernels are continuous as maps from E to $\mathcal{P}(E)$. Let $\nu \in \mathcal{P}(E^I)$ be the unique law of the Markov chain with transition kernels $(\pi_t)_{t \in I^*}$ and initial law m_0^* . We denote by $\mathbb{F}^{X, \nu} = (\mathcal{F}_t^{X, \nu})_{t \in I}$ the filtration

$$\mathcal{F}_t^{X, \nu} := \mathcal{F}_t^X \vee \mathcal{N}_\nu(\mathcal{F}_T^X).$$

The following definition defines the set of Markovian stopping times. We call these stopping times Markovian since they are functions of the underlying Markov process.

Definition 7.1. Denote by \mathcal{T} the set of measurable functions $\tau : \Omega_X \rightarrow I$ such that τ is an $\mathbb{F}^{X, \nu}$ -stopping time.

Definition 7.2. Let \mathcal{A}_0 be the set of probabilities $\bar{\mathbb{P}}$ on $(\bar{\Omega}, \mathcal{B}(\bar{\Omega}))$ such that

- (1) Under $\bar{\mathbb{P}}$, \bar{X} is a Markov chain with initial distribution m_0^* and transition kernels $(\pi_t)_{t \in I^*}$.
- (2) $\bar{\tau}$ is an $\mathbb{F}^{\bar{X}, \bar{\mathbb{P}}}$ -stopping time.

Lemma 7.3 (Characterization of \mathcal{A}_0 by Markovian stopping times). We have the equality

$$\mathcal{A}_0 = \{\nu(dx)\delta_{\tau(x)}(d\theta) : \tau \in \mathcal{T}\}.$$

Proof. Let $\bar{\mathbb{P}} \in \mathcal{A}_0$. By the first condition, the first marginal of $\bar{\mathbb{P}}$ coincides with ν . Since $\bar{\tau}$ is $\mathcal{F}_T^{\bar{X}, \bar{\mathbb{P}}}$ -measurable, there exists some measurable function $\tau : \Omega_X \rightarrow I$ such that $\bar{\mathbb{P}}(\bar{\tau} = \tau(\bar{X})) = 1$. In particular we can disintegrate $\bar{\mathbb{P}}$ as $\bar{\mathbb{P}}(dx, d\theta) = \nu(dx)\delta_{\tau(x)}(d\theta)$. Using that $\bar{\tau}$ is an $\mathbb{F}^{\bar{X}, \bar{\mathbb{P}}}$ -stopping time and that $\bar{\tau} = \tau(\bar{X})$ $\bar{\mathbb{P}}$ -a.s., we get that $\tau(\bar{X})$ is an $\mathbb{F}^{\bar{X}, \bar{\mathbb{P}}}$ -stopping time. Moreover, for any $t \in I$ and $B \in \mathcal{F}_t^{\bar{X}, \bar{\mathbb{P}}}$, it is easy to check that $(X, \tau(X))^{-1}(B) \in \mathcal{F}_t^{X, \nu}$. In particular, taking $B = \{\tau(\bar{X}) \leq t\}$, we get $\{\tau(X) \leq t\} \in \mathcal{F}_t^{X, \nu}$, implying that $\tau \in \mathcal{T}$.

Now let $\tau \in \mathcal{T}$ and define $\bar{\mathbb{P}}(dx, d\theta) := \nu(dx)\delta_{\tau(x)}(d\theta)$. Since $\bar{\mathbb{P}} \circ \bar{X}^{-1} = \nu$, the first condition is verified. Define the set $B := \{\bar{\tau} = \tau(\bar{X})\} \in \bar{\mathcal{F}}_T$ which has probability 1 under $\bar{\mathbb{P}}$. We have

$$\{\tau(\bar{X}) \leq t\} = \{x \in \Omega_X : \tau(x) \leq t\} \times I = (C \times I) \cup (D \times I),$$

where $C \in \mathcal{F}_t^X$ and $D \in \mathcal{N}_\nu(\mathcal{F}_T^X)$ are such that $\{x \in \Omega_X : \tau(x) \leq t\} = C \cup D$. We can deduce that $C \times I \in \mathcal{F}_t^{\bar{X}, \bar{\mathbb{P}}}$ and $D \times I \in \mathcal{N}_{\bar{\mathbb{P}}}(\bar{\mathcal{F}}_T)$, so that $\{\tau(\bar{X}) \leq t\} \in \mathcal{F}_t^{\bar{X}, \bar{\mathbb{P}}}$. Finally,

$$\{\bar{\tau} \leq t\} = (\{\tau(\bar{X}) \leq t\} \cap B) \cup (\{\bar{\tau} \leq t\} \cap B^c) \in \mathcal{F}_t^{\bar{X}, \bar{\mathbb{P}}}.$$

□

Let \mathcal{L} be the generator associated to $(\pi_t)_{t \in I^*}$. For any $\varphi \in C(I \times E)$, define the process $M(\varphi)$ by $M_0(\varphi) = \varphi(0, \bar{X}_0)$ and

$$M_t(\varphi) = \varphi(t, \bar{X}_t) - \sum_{s=0}^{t-1} \mathcal{L}(\varphi)(s, \bar{X}_s), \quad t \in I^*.$$

By taking the conditional expectation, one can show the following lemma.

Lemma 7.4 (*Equivalence between the Markov property and the martingale problem*). *Let $\bar{\mathbb{G}}$ be a filtration including $\mathbb{F}^{\bar{X}}$ and $\bar{\mathbb{P}}$ be a probability on $(\bar{\Omega}, \mathcal{B}(\bar{\Omega}))$ such that $\bar{\mathbb{P}} \circ \bar{X}_0^{-1} = m_0^*$. Then, the following are equivalent:*

- (1) *Under $\bar{\mathbb{P}}$, \bar{X} is a $\bar{\mathbb{G}}$ -Markov chain with transition kernels $(\pi_t)_{t \in I^*}$.*
- (2) *Under $\bar{\mathbb{P}}$, for any $\varphi \in C(I \times E)$, the process $M(\varphi)$ is a $\bar{\mathbb{G}}$ -martingale.*

Definition 7.5. *Let \mathcal{A}_1 be the set of probabilities $\bar{\mathbb{P}}$ on $(\bar{\Omega}, \mathcal{B}(\bar{\Omega}))$ such that under $\bar{\mathbb{P}}$, \bar{X} is an $\bar{\mathbb{F}}$ -Markov chain with initial distribution m_0^* and transition kernels $(\pi_t)_{t \in I^*}$. By Lemma 7.4, \mathcal{A}_1 is exactly the set of probability measures $\bar{\mathbb{P}}$ on $(\bar{\Omega}, \mathcal{B}(\bar{\Omega}))$ such that $\bar{\mathbb{P}} \circ \bar{X}_0^{-1} = m_0^*$ and under $\bar{\mathbb{P}}$, for any $\varphi \in C(I \times E)$, the process $M(\varphi)$ is an $\bar{\mathbb{F}}$ -martingale.*

Lemma 7.6. *We have the inclusion $\mathcal{A}_0 \subset \mathcal{A}_1$.*

Proof. Let $\bar{\mathbb{P}} \in \mathcal{A}_0$. Since $\bar{\tau}$ is an $\mathbb{F}^{\bar{X}, \bar{\mathbb{P}}}$ -stopping time, we have that $\bar{\mathbb{F}}^{\bar{\mathbb{P}}} = \mathbb{F}^{\bar{X}, \bar{\mathbb{P}}}$. For any $\varphi \in C(E)$, we obtain

$$\mathbb{E}^{\bar{\mathbb{P}}}[\varphi(\bar{X}_t) | \bar{\mathcal{F}}_{t-1}] = \mathbb{E}^{\bar{\mathbb{P}}}[\varphi(\bar{X}_t) | \bar{\mathcal{F}}_{t-1}^{\bar{\mathbb{P}}}] = \mathbb{E}^{\bar{\mathbb{P}}}[\varphi(\bar{X}_t) | \mathcal{F}_{t-1}^{\bar{X}, \bar{\mathbb{P}}}] = \mathbb{E}^{\bar{\mathbb{P}}}[\varphi(\bar{X}_t) | \bar{X}_{t-1}].$$

□

Lemma 7.7 (*Immersion property*). *If $\bar{\mathbb{P}} \in \mathcal{A}_1$, then $\mathbb{F}^{\bar{X}}$ is immersed in $\bar{\mathbb{F}}$ under $\bar{\mathbb{P}}$.*

Proof. We need to show that for all $t \in I$, under $\bar{\mathbb{P}}$, $\bar{\mathcal{F}}_t$ is conditionally independent of $\mathcal{F}_T^{\bar{X}}$ given $\mathcal{F}_t^{\bar{X}}$. Let $\varphi_t : \bar{\Omega} \rightarrow \mathbb{R}$ be bounded and $\bar{\mathcal{F}}_t$ -measurable, $\psi_t : E^I \rightarrow \mathbb{R}$ be bounded and $\mathcal{F}_t^{\bar{X}}$ -measurable, $\psi_{t+} : E^I \rightarrow \mathbb{R}$ be bounded and $\sigma(\bar{X}_{t+1}, \dots, \bar{X}_T)$ -measurable and $\phi_t : E^I \rightarrow \mathbb{R}$ be bounded and $\mathcal{F}_t^{\bar{X}}$ -measurable. We have

$$\begin{aligned} \mathbb{E}^{\bar{\mathbb{P}}}[\varphi_t(\bar{X}, \bar{\tau}) \psi_t(\bar{X}) \psi_{t+}(\bar{X}) \phi_t(\bar{X})] &= \mathbb{E}^{\bar{\mathbb{P}}}[\varphi_t(\bar{X}, \bar{\tau}) \psi_t(\bar{X}) \mathbb{E}^{\bar{\mathbb{P}}}[\psi_{t+}(\bar{X}) | \bar{\mathcal{F}}_t] \phi_t(\bar{X})] \\ &= \mathbb{E}^{\bar{\mathbb{P}}}[\varphi_t(\bar{X}, \bar{\tau}) \psi_t(\bar{X}) \mathbb{E}^{\bar{\mathbb{P}}}[\psi_{t+}(\bar{X}) | \mathcal{F}_t^{\bar{X}}] \phi_t(\bar{X})] \\ &= \mathbb{E}^{\bar{\mathbb{P}}}[\mathbb{E}^{\bar{\mathbb{P}}}[\varphi_t(\bar{X}, \bar{\tau}) | \mathcal{F}_t^{\bar{X}}] \mathbb{E}^{\bar{\mathbb{P}}}[\psi_t(\bar{X}) \psi_{t+}(\bar{X}) | \mathcal{F}_t^{\bar{X}}] \phi_t(\bar{X})]. \end{aligned}$$

This shows that

$$\mathbb{E}^{\bar{\mathbb{P}}}[\varphi_t(\bar{X}, \bar{\tau}) \psi_t(\bar{X}) \psi_{t+}(\bar{X}) | \mathcal{F}_t^{\bar{X}}] = \mathbb{E}^{\bar{\mathbb{P}}}[\varphi_t(\bar{X}, \bar{\tau}) | \mathcal{F}_t^{\bar{X}}] \mathbb{E}^{\bar{\mathbb{P}}}[\psi_t(\bar{X}) \psi_{t+}(\bar{X}) | \mathcal{F}_t^{\bar{X}}].$$

This is sufficient to prove the claim.

□

Applying a discrete time version of Proposition 1.10 in Chapter 1 in [CD18b], we get the following corollary.

Corollary 7.8. *We have the equality $\mathcal{A}_1 = \{\nu(dx)\kappa(x, d\theta) : \kappa \in \mathcal{K}\}$, where \mathcal{K} is the set of transition kernels from Ω_X to I such that for all $t \in I$ and $B \in \sigma(\{0\}, \dots, \{t\})$, the mapping $x \mapsto \kappa(x, B)$ is $\mathcal{F}_t^{X, \nu}$ -measurable.*

In particular, the set \mathcal{A}_1 can be interpreted as the set of randomized stopping times since each kernel $\kappa \in \mathcal{K}$ gives a probability to stop given the observation of the underlying process (in an adapted way through the filtration of the process). The set \mathcal{A}_0 corresponds to the set of Markovian stopping times in the sense that up to null sets, the stopping rule is determined by the information given by the underlying Markov process and the kernel is given by a Dirac measure (see Lemma 7.3).

Proposition 7.9. *The set \mathcal{A}_1 is compact and convex.*

Proof. The relative compactness of \mathcal{A}_1 follows since $\mathcal{P}(\bar{\Omega})$ is compact. Let us show that \mathcal{A}_1 is closed. Let $(\bar{\mathbb{P}}_n)_{n \geq 1} \subset \mathcal{A}_1$ converging weakly to some $\bar{\mathbb{P}}$. Using the continuity of the map $(x, \theta) \mapsto x_0$ and passing to the limit, we get $\bar{\mathbb{P}} \circ \bar{X}_0^{-1} = m_0^*$. Let $\varphi \in C(E)$ and $\psi \in C(\bar{\Omega})$ an $\bar{\mathcal{F}}_{t-1}$ -measurable function, then

$$\mathbb{E}^{\bar{\mathbb{P}}}[(M_t(\varphi) - M_{t-1}(\varphi))\psi(\bar{X}, \bar{\tau})] = \lim_{n \rightarrow \infty} \mathbb{E}^{\bar{\mathbb{P}}_n}[(M_t(\varphi) - M_{t-1}(\varphi))\psi(\bar{X}, \bar{\tau})] = 0.$$

This shows that \mathcal{A}_1 is closed and henceforth compact. The convexity follows by the linearity of the conditions with respect to the probability measure. □

For any $\bar{\mathbb{P}} \in \mathcal{A}_1$, we define the occupation measures

$$m_t^{\bar{\mathbb{P}}}(B) := \bar{\mathbb{P}}(\bar{X}_t \in B, t < \bar{\tau}), \quad B \in \mathcal{B}(E), \quad t \in I \setminus \{T\},$$

$$\mu_t^{\bar{\mathbb{P}}}(B) := \bar{\mathbb{P}}(\bar{X}_t \in B, \bar{\tau} = t), \quad B \in \mathcal{B}(E), \quad t \in I.$$

Definition 7.10. *Define the sets*

$$\mathcal{R}_0 := \{(m^{\bar{\mathbb{P}}}, \mu^{\bar{\mathbb{P}}}) : \bar{\mathbb{P}} \in \mathcal{A}_0\},$$

$$\mathcal{R}_1 := \{(m^{\bar{\mathbb{P}}}, \mu^{\bar{\mathbb{P}}}) : \bar{\mathbb{P}} \in \mathcal{A}_1\}.$$

Proposition 7.11. *The set \mathcal{R}_1 is compact and convex.*

Proof. For the compactness it suffices to show that $\mathcal{A}_1 \ni \bar{\mathbb{P}} \mapsto (m^{\bar{\mathbb{P}}}, \mu^{\bar{\mathbb{P}}})$ is continuous, since \mathcal{A}_1 is compact. Let $(\bar{\mathbb{P}}_n)_{n \geq 1} \subset \mathcal{A}_1$ converging weakly to $\bar{\mathbb{P}} \in \mathcal{A}_1$. For any $\varphi \in C(E)$ and $t \in I \setminus \{T\}$, we have

$$\int_E \varphi(x) m_t^{\bar{\mathbb{P}}_n}(dx) = \mathbb{E}^{\bar{\mathbb{P}}_n}[\varphi(\bar{X}_t) \mathbf{1}_{t < \bar{\tau}}] \xrightarrow{n \rightarrow \infty} \mathbb{E}^{\bar{\mathbb{P}}}[\varphi(\bar{X}_t) \mathbf{1}_{t < \bar{\tau}}] = \int_E \varphi(x) m_t^{\bar{\mathbb{P}}}(dx),$$

where we used the continuity of the function $\bar{\Omega} \ni (x, \theta) \mapsto \varphi(x_t) \mathbf{1}_{t < \theta}$. This shows that for each $t \in I \setminus \{T\}$, $(m_t^{\bar{\mathbb{P}}^n})_{n \geq 1}$ converges to $m_t^{\bar{\mathbb{P}}}$. By the same argument, for each $t \in I$, $(\mu_t^{\bar{\mathbb{P}}^n})_{n \geq 1}$ converges to $\mu_t^{\bar{\mathbb{P}}}$. We conclude that \mathcal{R}_1 is compact. By the linearity of the same mapping $\mathcal{A}_1 \ni \bar{\mathbb{P}} \mapsto (m^{\bar{\mathbb{P}}}, \mu^{\bar{\mathbb{P}}})$ and the convexity of \mathcal{A}_1 , we get that \mathcal{R}_1 is convex. \square

We define the relaxed set in this framework as follows.

Definition 7.12. *Let \mathcal{R} be the set of pairs $(m, \mu) \in \mathcal{P}^{sub}(E)^I \times \mathcal{P}^{sub}(E)^{I \setminus \{T\}}$ such that for all $\varphi \in C(I \times E)$,*

$$\sum_{t=0}^T \int_E \varphi(t, x) \mu_t(dx) = \int_E \varphi(0, x) m_0^*(dx) + \sum_{t=0}^{T-1} \int_E \mathcal{L}(\varphi)(t, x) m_t(dx).$$

Proposition 7.13. *We have the inclusions $\mathcal{R}_0 \subset \mathcal{R}_1 \subset \mathcal{R}$. In particular $\overline{\text{conv}}(\mathcal{R}_0) \subset \mathcal{R}_1 \subset \mathcal{R}$.*

Proof. The first inclusion is a consequence of the inclusion $\mathcal{A}_0 \subset \mathcal{A}_1$ proved in Lemma 7.6. For the second inclusion, let $\bar{\mathbb{P}} \in \mathcal{A}_1$ and consider the associated measures $(m^{\bar{\mathbb{P}}}, \mu^{\bar{\mathbb{P}}})$. Using that for any $\varphi \in C(I \times E)$, $M(\varphi)$ is an $\bar{\mathbb{F}}$ -martingale under $\bar{\mathbb{P}}$ and $\bar{\tau}$ is an $\bar{\mathbb{F}}$ -stopping time, we get

$$\mathbb{E}^{\bar{\mathbb{P}}}[M_{\bar{\tau}}(\varphi) | \bar{\mathcal{F}}_0] = \varphi(0, \bar{X}_0),$$

which implies by taking the expectation and replacing the expression of $M_{\bar{\tau}}(\varphi)$,

$$\mathbb{E}^{\bar{\mathbb{P}}}[\varphi(\bar{\tau}, \bar{X}_{\bar{\tau}})] = \mathbb{E}^{\bar{\mathbb{P}}}[\varphi(0, \bar{X}_0)] + \mathbb{E}^{\bar{\mathbb{P}}}\left[\sum_{t=0}^{\bar{\tau}-1} \mathcal{L}(\varphi)(t, \bar{X}_t)\right].$$

By the definition of the measures we obtain

$$\sum_{t=0}^T \int_E \varphi(t, x) \mu_t^{\bar{\mathbb{P}}}(dx) = \int_E \varphi(0, x) m_0^*(dx) + \sum_{t=0}^{T-1} \int_E \mathcal{L}(\varphi)(t, x) m_t^{\bar{\mathbb{P}}}(dx).$$

The last inclusions follow by Proposition 7.11. \square

Probabilistic representation We want to show that the sets \mathcal{R} and \mathcal{R}_1 are equal. We will follow similar arguments to [FV88] in order to show that the closed convex hull of \mathcal{R}_0 is equal to \mathcal{R} , which is sufficient to obtain the desired equality according to Proposition 7.13.

Equality of the values. We are given two functions $f \in C(I \setminus \{T\} \times E)$ and $g \in C(I \times E)$. We place ourselves on the filtered probability space $(\Omega_X, \mathcal{B}(\Omega_X), \mathbb{F}^X, \nu)$. For each $(t, x) \in I \times E$, define the probability measure on $\nu_{t,x} \in (\Omega_X, \mathcal{B}(\Omega_X))$ by

$$\nu_{t,x}(dx_0, \dots, dx_T) = \prod_{s=0}^t \delta_x(dx_s) \prod_{\ell=t+1}^T \pi_\ell(x_{\ell-1}; dx_\ell).$$

Denote by $\mathcal{T}_{t,x}$ the set of $\{t, \dots, T\}$ -valued $\mathbb{F}^{X, \nu_{t,x}}$ -stopping times. Define the value function

$$v(t, x) := \sup_{\tau \in \mathcal{T}_{t,x}} \mathbb{E}^{\nu_{t,x}} \left[\sum_{s=t}^{\tau-1} f(s, X_s) + g(\tau, X_\tau) \right].$$

The value function verifies the dynamic programming principle (see Theorem 1.9 in Chapter 1 in [PS06]):

$$v(T, x) = g(T, x), \quad x \in E,$$

$$v(t, x) = \max \left\{ f(t, x) + \int_E v(t+1, x') \pi_{t+1}(x, dx'), g(t, x) \right\}, \quad x \in E, t \in I \setminus \{T\}.$$

By backward induction, since g , f and the transition kernels are continuous, we obtain that $v \in C(I \times E)$. Define the quantities

$$V^S(f, g) := \sup_{(m, \mu) \in \mathcal{R}_0} \sum_{t=0}^{T-1} \int_E f(t, x) m_t(dx) + \sum_{t=0}^T \int_E g(t, x) \mu_t(dx)$$

$$= \sup_{\tau \in \mathcal{T}} \mathbb{E}^\nu \left[\sum_{t=0}^{\tau-1} f(t, X_t) + g(\tau, X_\tau) \right].$$

$$V^{LP}(f, g) := \sup_{(m, \mu) \in \mathcal{R}} \sum_{t=0}^{T-1} \int_E f(t, x) m_t(dx) + \sum_{t=0}^T \int_E g(t, x) \mu_t(dx).$$

By Proposition 7.13, we have $V^S \leq V^{LP}$. The Snell envelope $Y = (Y_t)_{t \in I}$ associated to (f, g) is given by

$$Y_t = v(t, X_t) + \sum_{s=0}^{t-1} f(s, X_s).$$

We say that $\tau^* \in \mathcal{T}$ is an (f, g) -optimal stopping time if

$$V^S(f, g) = \mathbb{E}^\nu \left[\sum_{t=0}^{\tau^*-1} f(t, X_t) + g(\tau^*, X_{\tau^*}) \right].$$

The random variable $\tau_{min} := \inf\{t \in \{0, \dots, T\} : v(t, X_t) = g(t, X_t)\}$ is an (f, g) -optimal stopping time and $(Y_{t \wedge \tau_{min}})_{t \in I}$ is an \mathbb{F}^X -martingale. In particular,

$$V^S(f, g) = \mathbb{E}^\nu[Y_{\tau_{min}}] = \mathbb{E}^\nu[v(0, X_0)] = \int_E v(0, x) m_0^*(dx).$$

Now, for all $(m, \mu) \in \mathcal{R}$, using v as a test function in the constraint, we get

$$V^S(f, g) = \sum_{t=0}^T \int_E v(t, x) \mu_t(dx) - \sum_{t=0}^{T-1} \int_E \mathcal{L}(v)(t, x) m_t(dx)$$

$$\geq \sum_{t=0}^T \int_E g(t, x) \mu_t(dx) + \sum_{t=0}^{T-1} \int_E f(t, x) m_t(dx),$$

where the inequality is a consequence of the dynamic programming principle. Taking the supremum over $(m, \mu) \in \mathcal{R}$, we deduce that $V^S(f, g) \geq V^{LP}(f, g)$, which implies the equality $V^S(f, g) = V^{LP}(f, g)$.

Theorem 7.14. *The set \mathcal{R} is equal to the closed convex hull of \mathcal{R}_0 . As a consequence, $\mathcal{R} = \mathcal{R}_1$, meaning that we can represent any $(m, \mu) \in \mathcal{R}$ with some randomized stopping time.*

Proof. Assume that there exists some $(m^0, \mu^0) \in \mathcal{R} \setminus \overline{\text{conv}}(\mathcal{R}_0)$. Recall that for any compact metric space K , the topological dual of $\mathcal{M}(K)$ endowed with the weak topology $\sigma(\mathcal{M}(K), C(K))$ is $C(K)$. In particular, by Theorem 3.4 (b) p.59 in [Rud91], there exists some $(f^0, g^0) \in C(I \setminus \{T\} \times E) \times C(I \times E)$ and $c \in \mathbb{R}$ such that for all $(m, \mu) \in \overline{\text{conv}}(\mathcal{R}_0)$,

$$\begin{aligned} \sum_{t=0}^{T-1} \int_E f^0(t, x) m_t(dx) + \sum_{t=0}^T \int_E g^0(t, x) \mu_t(dx) &< c \\ &< \sum_{t=0}^{T-1} \int_E f^0(t, x) m_t^0(dx) + \sum_{t=0}^T \int_E g^0(t, x) \mu_t^0(dx) \\ &\leq V^{LP}(f^0, g^0). \end{aligned}$$

Taking the supremum over $(m, \mu) \in \mathcal{R}_0$, we get $V^S(f^0, g^0) < V^{LP}(f^0, g^0)$, which is a contradiction. □

7.2 Other technical results

7.2.1 Equivalence of Markov chains and property on the kernel

We justify here the statement made in Remark 3.3.

Lemma 7.15. *The following statements are equivalent:*

- (1) (X, U) is an \mathbb{F} -Markov chain with transition kernels $(\pi_t)_{t \in I^*}$ such that for all $t \in I^*$, $x, \bar{x} \in E$, $u \in W$ and $u' \in W$, $\pi_t(x, u; E \times \{u'\}) = \pi_t(\bar{x}, u; E \times \{u'\})$.
- (2) (X, U) and U are \mathbb{F} -Markov chains.

Proof. Let us show the direct implication. Let $x_0 \in E$ be arbitrary and define the kernel $\bar{\pi}_t(u; \{u'\}) := \pi_t(x_0, u; E \times \{u'\})$, which is independent of x_0 by assumption. For any $\varphi : W \rightarrow \mathbb{R}$ measurable and bounded,

$$\begin{aligned} \mathbb{E}[\varphi(U_t) | \mathcal{F}_{t-1}] &= \int_{E \times W} \varphi(u) \pi_t(X_{t-1}, U_{t-1}; dx, du) \\ &= \int_W \varphi(u) \bar{\pi}_t(U_{t-1}; du), \end{aligned}$$

which is sufficient to show that U is an \mathbb{F} -Markov chain (with transition kernel $\bar{\pi}$). We show now the converse implication. Since both (X, U) and U are \mathbb{F} -Markov chains, there exist some

associated transition kernels $\tilde{\pi}$ and $\bar{\pi}$. Now, for all $\varphi : W \rightarrow \mathbb{R}$ measurable and bounded

$$\begin{aligned} \int_W \varphi(u) \bar{\pi}_t(U_{t-1}; du) &= \mathbb{E}[\varphi(U_t) | \mathcal{F}_{t-1}] \\ &= \int_{E \times W} \varphi(u) \tilde{\pi}_t(X_{t-1}, U_{t-1}; dx, du), \end{aligned}$$

implying that a.s. for all $u' \in W$,

$$\bar{\pi}_t(U_{t-1}; \{u'\}) = \tilde{\pi}_t(X_{t-1}, U_{t-1}; E \times \{u'\}).$$

By the disintegration theorem we can write for some kernels ν

$$\tilde{\pi}_t(x, u; dy, dv) = \nu_t(x, u, v; dy) \tilde{\pi}_t(x, u; E, dv).$$

We conclude by defining the kernels $\pi_t(x, u; dy, dv) := \nu_t(x, u, v; dy) \tilde{\pi}_t(x, u; E, dv)$. \square

7.2.2 Exchangeable random variables and De Finetti's theorem

We recall a version of De Finetti's theorem adapted to our setting. Let (E, d) and (F, ρ) be two complete and separable metric spaces. Consider a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, a sequence of E -valued random variables $(X_n)_{n \geq 1}$ and an F -valued random variable Y .

Definition 7.16. *We say that $(X_n)_{n \geq 1}$ is i.i.d. given Y if the following two conditions are satisfied.*

(1) *For all $n \geq 1$ and all $(B_k)_{k \leq n} \in \mathcal{B}(E)^n$,*

$$\mathbb{P}(\cap_{k=1}^n \{X_k \in B_k\} | Y) = \prod_{k=1}^n \mathbb{P}(X_k \in B_k | Y) \quad a.s.$$

(2) *For all $n, k \geq 1$ and all $B \in \mathcal{B}(E)$, $\mathbb{P}(X_n \in B | Y) = \mathbb{P}(X_k \in B | Y)$ a.s.*

If $(X_n)_{n \geq 1}$ is i.i.d. given Y , then one can show that the sequence $(X_n, Y)_{n \geq 1}$ is exchangeable. In particular, using the same argument as in [Kin78], we obtain that for all measurable functions $\psi : E \times F \mapsto \mathbb{R}$ such that $\mathbb{E}[|\psi(X_1, Y)|] < \infty$,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \psi(X_k, Y) = \mathbb{E}[\psi(X_1, Y) | \mathcal{G}] \quad a.s. \quad (5.11)$$

where \mathcal{G} is the exchangeable σ -algebra. Using Theorem 3 in [Ols74], one can replace \mathcal{G} in (5.11) by $\sigma(Y)$ in order to derive the following result.

Theorem 7.17. *If $(X_n)_{n \geq 1}$ is i.i.d. given Y , then, for all measurable functions $\psi : E \times F \mapsto \mathbb{R}$ such that $\mathbb{E}[|\psi(X_1, Y)|] < \infty$, we have*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \psi(X_k, Y) = \mathbb{E}[\psi(X_1, Y) | Y] \quad a.s.$$

7.2.3 Dynkin's formula for randomized stopping times

Proposition 7.18. *Let $\varphi \in C(I \times E)$ and κ be a randomized stopping time. Then,*

$$\sum_{t=0}^T \mathbb{E}[\varphi(t, X_t) \kappa(\{t\})] = \mathbb{E}[\varphi(0, X_0)] + \sum_{t=0}^{T-1} \mathbb{E}[\mathcal{L}(\varphi)(t, X_t) \kappa(\{t+1, \dots, T\})].$$

Proof. Let $\bar{\Omega} := \Omega \times I$, $\bar{\mathcal{F}} := \mathcal{F} \otimes 2^I$, $\bar{\mathbb{P}}(d\omega, ds) := \mathbb{P}(d\omega) \kappa(\omega, ds)$. Consider $\bar{X}_t(\omega, s) := X_t(\omega)$ and $\theta(\omega, s) := s$. Define the filtration $\mathbb{F}^\theta := (\mathcal{F}_t^\theta)_t$ with $\mathcal{F}_t^\theta := \sigma(\{\theta = 0\}, \dots, \{\theta = t\})$. Now, construct the filtration $\bar{\mathbb{F}} := (\bar{\mathcal{F}}_t)_t$, with $\bar{\mathcal{F}}_t := \mathcal{F}_t \otimes \mathcal{F}_t^\theta$. By construction θ is an $\bar{\mathbb{F}}$ -stopping time. We are going to show that $\bar{X} := (\bar{X}_t)_t$ is an $\bar{\mathbb{F}}$ -Markov chain. Let $\psi \in C(E)$. We show first that

$$\bar{\mathbb{E}}[\psi(\bar{X}_t) | \bar{\mathcal{F}}_{t-1}] = \mathbb{E}[\psi(X_t) | \mathcal{F}_{t-1}], \quad \bar{\mathbb{P}} - a.s.$$

Let $B \in \mathcal{F}_{t-1}$ and $C \in \mathcal{F}_{t-1}^\theta$. We have

$$\begin{aligned} \bar{\mathbb{E}}[\psi(\bar{X}_t) \mathbf{1}_{B \times C}] &= \mathbb{E}[\psi(X_t) \mathbf{1}_B \kappa(C)] \\ &= \mathbb{E}[\mathbb{E}[\psi(X_t) | \mathcal{F}_{t-1}] \mathbf{1}_B \kappa(C)] \\ &= \bar{\mathbb{E}}[\mathbb{E}[\psi(X_t) | \mathcal{F}_{t-1}] \mathbf{1}_{B \times C}], \end{aligned}$$

where in the second equality we used the measurability property of κ . We finally have

$$\bar{\mathbb{E}}[\psi(\bar{X}_t) | \bar{\mathcal{F}}_{t-1}] = \mathbb{E}[\psi(X_t) | \mathcal{F}_{t-1}] = \int_E \psi(x) \pi_t(\bar{X}_{t-1}; dx).$$

This shows that \bar{X} is an $\bar{\mathbb{F}}$ -Markov chain. Now, by Dynkin's formula,

$$\bar{\mathbb{E}}[\varphi(\theta, \bar{X}_\theta)] = \bar{\mathbb{E}}[\varphi(0, \bar{X}_0)] + \bar{\mathbb{E}} \left[\sum_{t=0}^{\theta-1} \mathcal{L}(\varphi)(t, \bar{X}_t) \right].$$

By applying Fubini's theorem we obtain the desired result. □

7.2.4 A continuity property for the price function

Lemma 7.19. *Let (E, d) be a metric space. Consider the function $P : E \rightarrow \mathbb{R}_+$ defined by*

$$P(x) := \inf\{p \geq 0 : r(x) \leq s(x, p)\} \wedge p_{max}, \quad x \in E,$$

where $r : E \rightarrow \mathbb{R}_+$ and $s : E \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ are continuous, for each $x \in E$, $p \mapsto s(x, p)$ is increasing with $s(x, 0) = 0$ and there exists $\tilde{p} \geq 0$ such that $\inf_{x \in E} s(x, \tilde{p}) \geq \sup_{x \in E} r(x)$. Then there exists for each $x \in E$ a unique $p^*(x) \in [0, \tilde{p}]$ such that

$$r(x) = s(x, p^*(x)), \quad P(x) = p^*(x) \wedge p_{max}, \quad x \in E.$$

Moreover, $x \mapsto p^*(x)$ is continuous and as a consequence P is also continuous.

Proof. For each $x \in E$, since $r(x) \leq s(x, \tilde{p})$, we have that $\tilde{p} \in \{p \geq 0 : r(x) \leq s(x, p)\}$. Using that $s(x, 0) = 0 \leq r(x)$ and $p \mapsto s(x, p)$ is continuous and increasing we must have that there exists a unique $p^*(x) \in [0, \tilde{p}]$ such that $r(x) = s(x, p^*(x))$ and for all $p < p^*(x)$, $r(x) > s(x, p^*(x))$. This allows us to write $P(x) = p^*(x) \wedge p_{max}$. Let us show that $p^*(x)$ is continuous. Let $(x_n)_{n \geq 1} \subset E$ converging to $x \in E$. Since $[0, \tilde{p}]$ is compact, there exists at least one limit point of the sequence $(p^*(x_n))_{n \geq 1}$. Let p_0 be a limit point of the above sequence and assume without loss of generality that the whole sequence converges to that point. Using the continuity of r and s and passing to the limit in the equality $r(x_n) = s(x_n, p^*(x_n))$, we obtain $r(x) = s(x, p_0)$, and henceforth $p_0 = p^*(x)$. □

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Titre : Contributions à l'approche par programmation linéaire pour les jeux à champ moyen et applications aux marchés de l'électricité

Mots clés : Jeux à champ moyen, arrêt optimal, programmation linéaire en dimension infinie, fictitious play, transition énergétique, bruit commun.

Résumé : Cette thèse présente trois contributions principales liées à l'approche de programmation linéaire pour les jeux à champ moyen (MFGs).

La première partie de la thèse traite les aspects théoriques des MFGs permettant simultanément arrêt optimal, contrôle stochastique et absorption. En utilisant la formulation de programmation linéaire pour ce type de MFGs, un résultat général d'existence pour les équilibres de Nash MFG est dérivé sous des hypothèses faibles à travers du théorème de point fixe de Kakutani-Fan-Glicksberg. Nous montrons que cette méthode de relaxation est équivalente à l'approche par martingales contrôlées/arrêtées pour les MFG, une autre méthode de relaxation utilisée dans des articles précédents dans le cas du contrôle. De plus, sous des conditions appropriées, nous montrons que notre notion de solution satisfait un système d'équations différentielles partielles (EDP), ce qui permet de comparer nos résultats avec la littérature sur les EDP.

La deuxième partie se concentre sur un algorithme numérique pour l'approximation de l'équilibre de Nash MFG en tirant profit de l'approche par programmation linéaire. La convergence de cet algorithme est démontrée pour deux classes de MFG, les MFG avec arrêt optimal et absorption, et les MFG avec contrôle stochastique et absorption. Le schéma numérique appartient à la classe des procédures d'apprentissage.

En particulier, nous appliquons l'algorithme Fictitious Play où la meilleure réponse à chaque itération est calculée en résolvant un problème de programmation linéaire.

La dernière partie de la thèse porte sur une application des MFGs à la dynamique long terme de l'industrie de l'électricité. Différents scénarios macroéconomiques et de politique climatique sont possibles pour les années à venir, or le scénario exact reste incertain. Par conséquent, les producteurs conventionnels ou renouvelables visant à sortir du marché ou à y entrer, respectivement, sont confrontés à l'incertitude concernant le prix du carbone et les politiques climatiques à venir. Les deux classes de producteurs interagissent par le biais du prix de l'électricité. Des stratégies d'équilibre de Nash sur des temps d'arrêt sont considérées et le problème est analysé à travers d'un modèle MFG. À cette fin, nous développons l'approche de programmation linéaire pour les MFG d'arrêt optimal avec bruit commun et information partielle en temps discret. Nous montrons l'existence d'un équilibre de Nash MFG et l'unicité du prix de marché en équilibre. Enfin, nous étendons l'algorithme numérique développé dans la deuxième partie de la thèse pour illustrer le modèle avec un exemple empirique inspiré du marché de l'électricité britannique.

Title : Contributions to the linear programming approach for mean field games and its applications to electricity markets

Keywords : Mean field games, optimal stopping, infinite-dimensional linear programming, fictitious play, energy transition, common noise.

Abstract : This thesis presents three main contributions related to the linear programming approach for mean field games (MFGs).

The first part of the thesis is concerned with the theoretical aspects of MFGs allowing simultaneously for optimal stopping, stochastic control and absorption. Using the linear programming formulation for this type of MFGs, a general existence result for MFG Nash equilibria is derived under mild assumptions by means of Kakutani-Fan-Glicksberg's fixed point theorem. This relaxation method is shown to be equivalent to the controlled/stopped martingale approach for MFGs, another relaxation method used in earlier papers in the pure control case. Furthermore, under appropriate conditions, we show that our notion of solution satisfies a partial differential equation (PDE) system, allowing to compare our results with the PDE literature.

The second part focuses on a numerical algorithm for approximating the MFG Nash equilibrium taking advantage of the linear programming approach. The convergence of this algorithm is shown for two classes of MFG, MFGs with optimal stopping and absorption, and MFGs with stochastic control and absorption. The

numerical scheme belongs to the class of learning procedures. In particular, we apply the Fictitious Play algorithm where the best response at each iteration is computed by solving a linear programming problem.

The last part of the thesis deals with an application of MFGs to the long term dynamics of the electricity industry. Different macroeconomic and climate policy scenarios are possible for the coming years, and the exact scenario remains uncertain. Therefore, conventional or renewable producers aiming to exit or enter the market, respectively, are facing uncertainty about the future carbon price and climate policies. Both classes of producers interact through the electricity market price. Nash equilibrium strategies over stopping times are considered and the problem is analyzed through a MFG model. To this end, we develop the linear programming approach for MFGs of optimal stopping with common noise and partial information in discrete time. We show the existence of an MFG Nash equilibrium and the uniqueness of the equilibrium market price. Finally, we extend the numerical algorithm developed in the second part of the thesis to illustrate the model with an empirical example inspired by the UK electricity market.