



# From potential to practical variations in the teaching of functions : contrasting Chinese and French cases in two upper secondary schools

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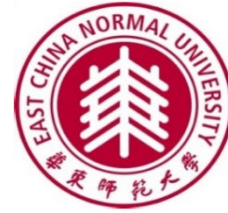
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## From potential to practical variations in the teaching of functions: contrasting Chinese and French cases in two upper secondary schools

Des variations potentielles aux variations effectives dans l'enseignement des fonctions au lycée : étude de cas en France et en Chine

从“潜在”到“实践”：变式教学在函数中的研究

——中法高中案例对比

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This PhD journey has indeed given me pure satisfaction. For me, variation is at the heart of both mathematics teaching and life. I hope I can leave myself open to experience, take a longer-term view on the world, embrace and enjoy variations in my future research career and life.





## Abstract

### **From potential to practical variations in the teaching of functions: contrasting Chinese and French cases in two upper secondary schools**

Teaching mathematics through variation appears an important topic as soon as we study task design, many teachers teaching consciously or unconsciously in varying these tasks. It has been proved to be an effective way of teaching mathematics in China and has a profound theoretical foundation both in China and France. This thesis is situated in between China and France, crossing two traditions and cultural contexts, aiming to think fruitful interactions between them and improve the effectiveness of mathematics teaching by understanding the role and evolution of teaching through variation. At this digital turning point of resources explosion, the development of technologies and the potential variations that it brings are also taken into account.

Variation appears in two traditions of research, the Chinese and the French one. In China, Gu (Gu et al., 2004) situated *BIANSHI jiaoxue* (teaching through variation) as an effective way in mathematics teaching, distinguishing conceptual variation and procedural variation. In France, Brousseau (2002) situated the *didactical variables* as sensitive elements of a problem whose modifications can significantly influence students' behavior.

After presenting and contrasting these two theories, we propose a preliminary model of “teaching mathematics through variation” by combining these two theoretical approaches with the Documentational Approach to Didactics (Trouche, Gueudet & Pepin, 2020). The documentational approach to didactics focuses on the interactions between teachers and resources, named their *documentation work*, which has implications for teachers' professional development. It combines the study of two interrelated processes, *instrumentation* (the way the resources influence teacher activity and knowledge) and *instrumentalization* (the way each teacher adapts each resource s/he integrates in her/his own activity), and considers, for each teacher, the development of her/his *resource system* as a result of these processes.

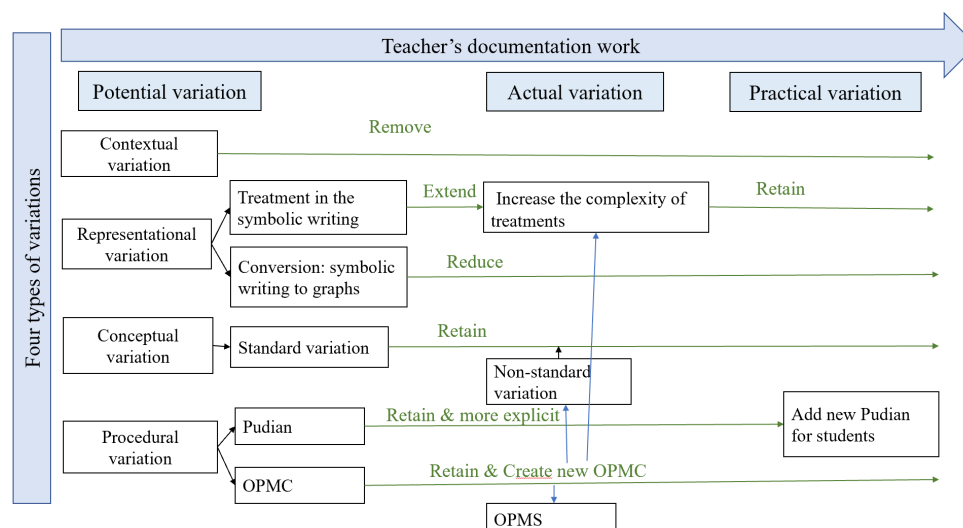
Our preliminary model distinguishes *potential variation* (the changes explicitly proposed by a given task, or series of tasks), *actual variation* (the possible changes made explicit by a teacher when integrating a task as a resource) and *practical variation* (the changes occurring when implementing the resource in class). It is based on the process of transforming potential variation into actual variation, and of using practical variation for rethinking potential and actual variation.

Three research questions are explored: 1) What are the potential variations in the upper secondary school mathematics textbooks and other curriculum resources in China and France? 2) What are the actual variations in the Chinese and French cases to achieve

the teaching goal? 3) What are the practical variations in Chinese and French cases? How do the case teachers take into account these practical variations?

For addressing these issues, an analytical framework based on our initial model is proposed as an analytical tool in our research for studying different types of variations (*contextual, representational, conceptual and procedural*) in each stage (potential, actual, and practical) through teacher's documentation work. It is applied to the analysis of curriculum resources (textbooks and online resources), two case studies (both in France and China) and contrasting of these cases.

As a critical element of our analytical framework, we propose the concept of *variation path*. Variation path is, for a given task, the process of evolution of each type of variations, through the three stages of potential, actual and practical variations, over a teacher documentation work. The evolution could be in the form of *retaining, removing, reducing, extending, or creating variation*. It is applied to the case studies and represented by a schematic representation, see an example below:



We followed in both countries the available resources, the documentation work, and the classroom teaching of two experienced mathematics teachers of upper secondary schools, focusing on the topic of function (due to its interest for varying the way and the form of teaching). We also studied textbooks used in case teachers' classes, and the online platform designed as supplementary resources of textbooks.

This study allows to enlighten our questions of research, to refine our initial model and its corresponding analytical framework. It opens new perspectives for taking into account variation not only for a given task, but also for the enrichment of teachers' resource systems, and for their professional development.

Taking advantage of the two main levels (level of mathematics teaching content, level of education noosphere) proposed by Wang (2019), we discussed the possibilities of extending our reflections from one case teacher to a category of teachers in China or

France.

This study raises new theoretical perspectives, such as thinking variation as a way of moving in a conceptual field (Vergnaud, 2009), opening a window on other mathematical variation traditions, as the Hungarian one (Gosztonyi, 2015), thinking both the evolution of tasks and the evolution of teacher documentation work by linking variation path and *documentational trajectories* (Rocha, 2019). It also opens new practical perspectives, such as integrating a ‘variation spirit’ in the textbooks, in problem-solving, using the digital potentialities of platform for developing living resources, and considering teachers’ collective work and education on variation.

Keywords: teaching through variation; teacher’s documentation work; mathematics teachers; case study; China; France

- Brousseau, G. (2002). *Theory of Didactical Situations in Mathematics*. Netherlands: Springer.
- Gosztonyi, K. (2015). *Traditions et réformes de l’enseignement des mathématiques à l’époque des “mathématiques modernes” : le cas de la Hongrie et de la France*. PhD thesis University of Szeged and University Paris Diderot, <https://hal.archives-ouvertes.fr/tel-01766902/document>
- Gu, L., Huang, R., & Marton, F. (2004). Teaching with variation: A Chinese way of promoting effective mathematics learning. In L. Fan, N. Y. Wong, J. Cai, & S. Li (Eds.), *How Chinese learn mathematics: Perspectives from insiders* (pp. 309-347). Singapore: World Scientific.
- Rocha, K. (2019). *Une étude des effets du travail documentaire et collectif sur le développement professionnel des enseignants de mathématiques : apport des concepts d’expérience et de trajectoire documentaires*. PhD, University of Lyon. Retrieved from <https://tel.archives-ouvertes.fr/tel-02399664/document>
- Trouche, L., Gueudet, G., & Pepin, B. (2020). The documentational approach to didactics. In S. Lerman (Eds.) *Encyclopedia of Mathematics Education* (2<sup>nd</sup> edition, pp. 307-313). Cham, Switzerland: Springer. doi:10.1007/978-3-319-77487-9\_100011-1
- Vergnaud, G. (2009). The theory of conceptual fields. *Human Development*, 52, 83-94.
- Wang, C. (2019). *An investigation of mathematics teachers’ documentation expertise and its development in collectives: two contrasting cases in China and France*. PhD, University of Lyon and ECNU, <https://tel.archives-ouvertes.fr/tel-02275820>



# **Des variations potentielles aux variations effectives dans l'enseignement des fonctions au lycée : étude de cas en France et en Chine**

## **Résumé**

Considérer l'enseignement des mathématiques du point de vue de la « variation » de cet enseignement est une question importante dès lors que l'on s'intéresse à la conception des tâches, car la majorité des enseignants enseignent consciemment ou inconsciemment en prenant en compte la variation de ces tâches. Cette pratique s'est avérée être un moyen efficace pour enseigner les mathématiques en Chine et a une base théorique profonde à la fois en Chine et en France. Cette thèse se situe entre la Chine et la France, bénéficiant de deux traditions et contextes culturels dont elle essaie de penser des interactions dans la perspective d'améliorer l'efficacité de l'enseignement des mathématiques. A un moment d'explosion du nombre des ressources disponibles, l'évolution des technologies et les variations potentielles qu'elle offre sont également prises en compte.

La variation apparaît dans deux traditions de recherche en matière d'enseignement des mathématiques, la chinoise et la française. En Chine, Gu (Gu et al., 2004) situe le BIANSHI jiaoxue (enseignement avec variation) comme un moyen efficace de l'enseignement des mathématiques, distinguant variation *conceptuelle* et variation *procédurale*. En France, Brousseau (2002) situe les variables didactiques comme des éléments sensibles d'un problème dont les modifications peuvent influencer significativement le comportement des élèves.

Après avoir présenté et contrasté ces deux théories, nous proposons un modèle préliminaire « d'enseignement des mathématiques par la variation » en combinant ces deux approches théoriques avec l'Approche documentaire du didactique (Trouche, Gueudet & Pepin, 2020). L'approche documentaire de la didactique est centrée sur les interactions entre les enseignants et les ressources, nommées leur travail documentaire, qui a des implications pour leur développement professionnel. Elle combine l'étude de deux processus interdépendants, *l'instrumentation* (la manière dont les ressources influencent l'activité et les connaissances de l'enseignant) et *l'instrumentalisation* (la manière dont chaque enseignant adapte chaque ressource qu'il intègre dans sa propre activité), et considère, pour chaque enseignant, le développement de son *système de ressources* résultant de ces processus.

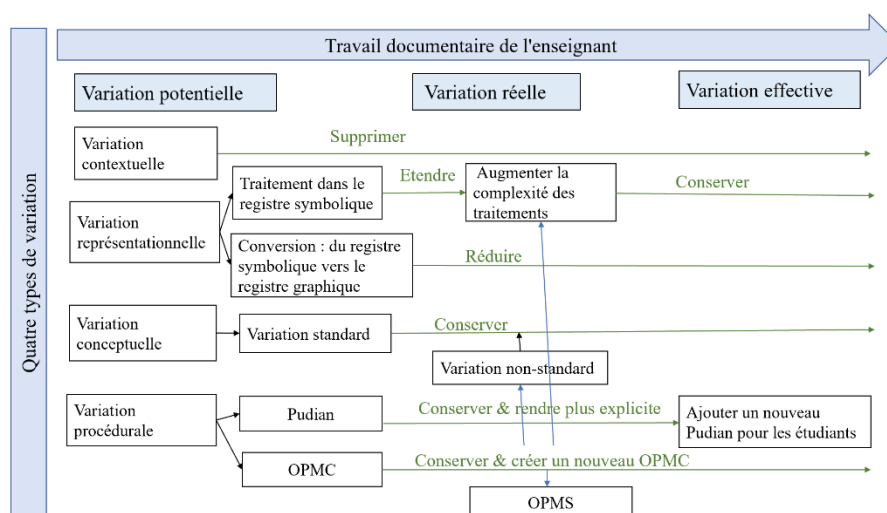
Notre modèle initial distingue les variations *potentielles* (les changements explicitement proposés par une tâche ou une série de tâches), les variations *réelles* (les changements possibles rendus explicites par un enseignant lors de l'intégration d'une tâche comme ressource) et les variations *effectives* (les changements survenant lors de la mise en œuvre la ressource en classe). Il est basé sur le processus de transformation des variations potentielles en variations réelles et sur l'utilisation des variations

effectives pour repenser les variations potentielles et réelles.

Trois questions de recherche sont explorées : 1) Quelles sont les variations potentielles apparaissant dans les manuels de mathématiques et les autres ressources pédagogiques en lycée en Chine et en France ? 2) Quelles sont les variations réelles apparaissant dans les ressources des professeurs visant un enseignement donné ? 3) Quelles sont les variations effectives dans le cas de professeurs chinois et français ? Comment les enseignants de cas prennent-ils en compte ces variations effectives ?

Pour aborder ces questions, un cadre d'analyse basé sur notre modèle initial est proposé comme outil de recherche pour étudier différents types de variations (contextuelles, représentationnelles, conceptuelles et procédurales) à chaque étape (potentielle, réelle et effective) dans le cas de professeurs donnés. Il est appliqué à l'analyse des ressources curriculaires (manuels et ressources en ligne), à l'analyse de cas (en France et en Chine) et au contraste de ces cas.

Comme élément critique de ce cadre d'analyse, nous proposons le concept de *parcours de variation*. Le parcours de variation est, pour une tâche donnée, le processus d'évolution de chaque type de variations, à travers les trois étapes de variations potentielles, réelles et effectives, au cours du travail documentaire d'un professeur donné. Cette évolution peut prendre la forme d'une conservation, d'une suppression, d'une réduction, d'une extension ou de la création d'une variation. Il est appliqué aux études de cas et illustré par une représentation schématique, voir un exemple ci-dessous :



Nous avons suivi dans les deux pays les ressources disponibles, le travail de documentation, et l'enseignement en classe de deux professeurs expérimentés de mathématiques en lycées, en nous concentrant sur le thème des fonctions (qui se prête bien à la variation des modes et des formes de son enseignement). Nous avons également étudié les manuels utilisés dans les cours des enseignants, et des ressources en ligne conçue comme des ressources complémentaires des manuels.

Cette étude permet d'éclairer nos questions de recherche, d'affiner notre modèle initial

et le cadre d'analyse correspondant. Elle ouvre de nouvelles perspectives pour la prise en compte de la variation non seulement pour une tâche donnée, mais aussi pour l'enrichissement des systèmes de ressources des enseignants, et pour leur développement professionnel.

Profitant des deux niveaux principaux (niveau du contenu d'enseignement des mathématiques, niveau de la noosphère) proposés par Wang (2019), nous avons discuté des possibilités d'étendre nos réflexions du cas d'un professeur donné à une catégorie de professeurs en Chine ou en France.

Cette étude soulève de nouvelles perspectives théoriques, telles que penser la variation comme une manière de se déplacer dans un champ conceptuel (Vergnaud, 2009), ouvrir une fenêtre sur d'autres traditions mathématiques de la variation, comme la tradition hongroise (Gosztonyi, 2015), penser à la fois l'évolution des tâches et l'évolution du travail documentaire des enseignants en liant parcours de variation et trajectoires documentaires (Rocha, 2019). Il ouvre également de nouvelles perspectives pratiques, telles que l'intégration d'un « esprit de variation » dans les manuels, dans la résolution de problèmes, l'utilisation des potentialités numériques des plateformes pour développer des ressources vivantes, et la prise en compte du travail collectif des enseignants et de leur formation pour développer, au profit d l'enseignement des mathématiques, les pratiques de variation, et les ressources soutenant ces pratiques.

Mots-clés: enseignement par la variation; travail documentaire de l'enseignant; professeurs de mathématiques; étude de cas; Chine; France

- Brousseau, G. (2002). *Theory of Didactical Situations in Mathematics*. Netherlands: Springer.
- Gosztonyi, K. (2015). *Traditions et réformes de l'enseignement des mathématiques à l'époque des "mathématiques modernes": le cas de la Hongrie et de la France*. PhD thesis University of Szeged and University Paris Diderot, <https://hal.archives-ouvertes.fr/tel-01766902/document>
- Gu, L., Huang, R., & Marton, F. (2004). Teaching with variation: A Chinese way of promoting effective mathematics learning. In L. Fan, N. Y. Wong, J. Cai, & S. Li (Eds.), *How Chinese learn mathematics: Perspectives from insiders* (pp. 309-347). Singapore: World Scientific.
- Rocha, K. (2019). *Une étude des effets du travail documentaire et collectif sur le développement professionnel des enseignants de mathématiques : apport des concepts d'expérience et de trajectoire documentaires*. PhD, University of Lyon. Retrieved from <https://tel.archives-ouvertes.fr/tel-02399664/document>
- Trouche, L., Gueudet, G., & Pepin, B. (2020). The documental approach to didactics. In S. Lerman (Eds.) *Encyclopedia of Mathematics Education*, (2<sup>nd</sup> edition, pp. 307-313). Cham, Switzerland: Springer. doi:10.1007/978-3-319-77487-9\_100011-1
- Vergnaud, G. (2009). The theory of conceptual fields. *Human Development*, 52, 83-94.
- Wang, C. (2019). *An investigation of mathematics teachers' documentation expertise and its development in collectives: two contrasting cases in China and France*. PhD, University of Lyon and ECNU, <https://tel.archives-ouvertes.fr/tel-02275820>





# 从“潜在”到“实践”：变式教学在函数中的研究

## ——中法高中案例对比

### 摘要

变式教学是任务设计中的一个重要课题，许多教师都在有意识或者无意识地对教学任务进行变式。在中国，变式教学被证明是一种行之有效的数学教学方式。中法两国对此领域的研究都具有深厚的理论基础。本文立足于中法两国不同的传统和文化背景，旨在通过探究变式教学在两国的角色及其演变过程，思考不同文化之间的碰撞和互动，提高数学教学的有效性。在这个数字资源高速发展的时间点，技术的发展和它能够带来的潜在变化也在本文中有所体现。

在中国，顾泠沅将变式教学作为一种有效的数学教学方式，他区分了“概念变式”和“过程性变式”（鲍建生等，2003）。在法国，Brousseau (2002) 提出教学变量理论，并将其看作是一个问题中的敏感因素，对这些因素的修改可以明显影响学生的行为。

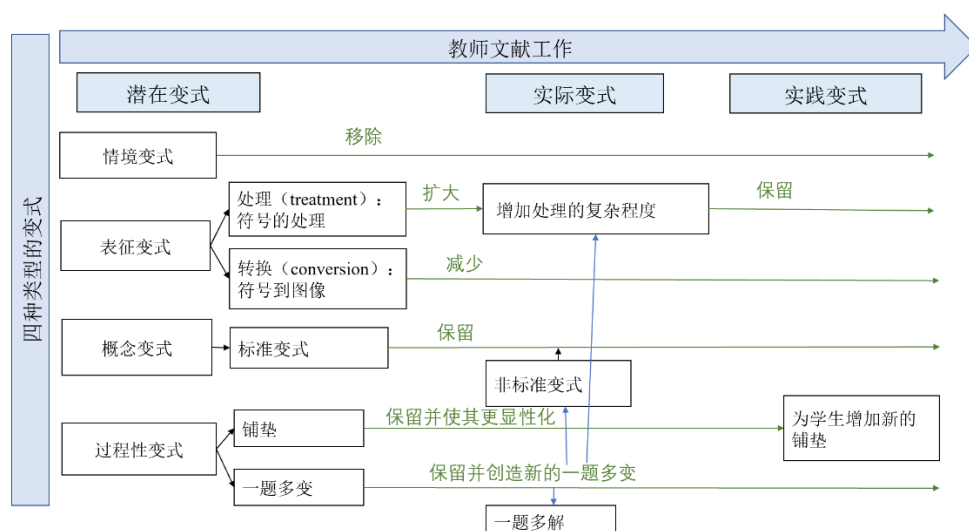
在介绍和对比这两种理论之后，我们通过将这两种理论方法与文献纪录教学论（Trouche, et al., 2020）相结合，提出了一个“通过变式来进行数学教学”的初步模型。文献纪录教学论侧重于教师和资源之间的互动，将其命名为教师的文献工作，这对教师的专业发展有影响。它结合了两个相互关联的过程的研究，工具性（资源影响教师活动和知识的方式）和工具化（每个教师调整她/他在她/他自己的活动中整合每种资源的方式）。本文还探索了此过程的结果，即每位教师的资源系统的发展。

本研究的初始模型区分了潜在变式（给定的任务或一系列任务中明确提出的变式）、实际变式（教师在将任务整合为资源时明确提出的可能的变式）和实践变式（在课堂中使用资源时产生的变式）。这个模型的建立是基于潜在变式到实际变式的转换，并利用实践变式来重新思考潜在变式 and 实际变式的过程。

本文探讨三个研究问题：（1）中法两国高中数学教材和其他课程资源的潜在变式有哪些？（2）为了达到教学目标，中法案例中的实际变式有哪些？（3）中法案例中的实践变式有哪些？案例教师是如何考虑和反思这些实践变式的？

基于以上研究问题，本研究提出了一个基于初始模型的分析框架。此框架旨在研究教师文献工作的每个阶段中（潜在、实际、实践）不同类型的变式（情境变式、表征变式、概念变式、过程性变式）。它被应用于课程资源（教材和在线资源）的分析、两个案例研究（法国和中国）以及案例的对比。

作为分析框架的关键要素之一，我们提出了变式路径的概念。变式路径指的是对于给定的任务，通过教师的文献工作，每一种类型的变式在潜在、实际和实践变式三个阶段的演变。演变的形式可以是保留、移除、减少、扩大或创造变式。变式路径的作用过程如下图所示：



本研究以跟踪观察的方式对中法两国两位经验丰富的高中数学教师，包括他们的可用资源，文献工作和课堂教学进行研究，重点关注函数主题的教学（因为它本身就富含教学方式和形式的变式）。我们还研究了案例教师课堂中使用的教材，以及作为教材补充资源的在线平台。

本研究回答了提出的研究问题，完善了初始模型及其相应的分析框架，并开辟了新的变式视角。变式不再局限于特定任务的变化，而是多个资源相互作用中的变化，教师的资源系统中的变化。新视角下的变式能够丰富教师的资源系统，促进教师的专业发展。

利用 Wang (2019) 提出的两个主要层次（数学教学内容层次、教育智能圈层次），本研究讨论了将研究结果扩展到两国教师群体的可能性。

本研究提出了新的理论研究方向，例如在概念场（Vergnaud, 2009）中讨论变式；考虑其他国家的变式教学传统，例如匈牙利的案例（Gosztonyi, 2015）；将变式路径和文献轨迹（Rocha, 2019）联系起来，同时思考任务和教师文献工作的演变。本研究还开辟了新的实践研究方向，例如将“变式精神”融入教材、问题解决、利用平台的数字潜力开发活跃动态的资源、考虑教师的集体工作和对教师的变式教学的培训。

关键词：变式教学；教师文献工作；数学教师；案例研究；中国；法国

Brousseau, G. (2002). *Theory of Didactical Situations in Mathematics*. Netherlands: Springer.  
 Gosztonyi, K. (2015). *Traditions et réformes de l'enseignement des mathématiques à l'époque des "mathématiques modernes": le cas de la Hongrie et de la France*. PhD thesis University of Szeged and University Paris Diderot, <https://hal.archives-ouvertes.fr/tel-01766902/document>

鲍建生, 黄荣金, 易凌峰, 顾泠沅 (2003). 变式教学研究. 数学教学, 第 1-3 期连载

Rocha, K. (2019). *Une étude des effets du travail documentaire et collectif sur le développement professionnel des enseignants de mathématiques : apport des concepts d'expérience et de trajectoire documentaires*. PhD, University of Lyon. Retrieved from <https://tel.archives-ouvertes.fr/tel-02399664/document>

- Trouche, L., Gueudet, G., Pepin, B., Wang, C., & Xu, B. (2020). 文献纪录教学论, <https://hal.archives-ouvertes.fr/hal-02891008/document>
- Vergnaud, G. (2009). The theory of conceptual fields. *Human Development*, 52, 83-94.
- Wang, C. (2019). *An investigation of mathematics teachers' documentation expertise and its development in collectives: two contrasting cases in China and France*. PhD, University of Lyon and ECNU, <https://tel.archives-ouvertes.fr/tel-02275820>



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# Introduction

This section introduces this thesis, i.e., the origin of the research questions. With an interest on the variation<sup>1</sup> in the field of mathematics education, this research takes the lens of “teachers’ documentation work”<sup>2</sup>, and build a model to analyze teachers’ work on variation.

Developing my PhD with two co-supervisors, in China and France, supported by two institution ENS in Lyon, and ECNU in Shanghai dedicated to education, with a strong interest in mathematics, I wish to seize this opportunity for discussing an effective way of teaching mathematics, i.e., variation in mathematics education.

The “variation” in task design has a profound theoretical foundation in both China and France. In China, Gu et al. (2004) situate BIANSHI jiaoxue (teaching through variation) as an effective way of mathematics teaching and learning, distinguishing *conceptual variation* (varying the connotation of a concept and its instantiations) and *procedural variation* (varying the way of solving a problem and metacognitive strategies). In France, Brousseau (2002) situated the *didactical variables* as sensitive elements of a problem whose modifications (even slight ones) can significantly influence students’ behavior.

From a practical point of view, teaching through variation as a Chinese way of promoting mathematics teaching has been proved to be effective, as the Chinese students have outstanding performance in large scale international comparisons, such as those conducted in the International Assessment of Educational Progress (IAEP)<sup>3</sup>, the Trends International Mathematics and Science Study (TIMSS)<sup>4</sup>, and the Programme for International Student Assessment (PISA)<sup>5</sup>; Chinese education has then attracted much attention all over the world (Fan, Wong, Cai & Li, 2004).

This thesis is situated in between China and France of two traditions and aims to think fruitful interactions between them. Three questions arose based on the discussion above:

- What role does variation play during teacher’s preparation of a lesson? How does teacher interact with a large number of resources from the perspective of

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<sup>1</sup> In general, variation means changing the form of essential or non-essential characteristics of things to highlight the essential characteristics of things. This definition, in the case of mathematics education, will be deepened in Chapter 2 and Chapter 7

<sup>2</sup> This lens has been introduced in the frame of the Documentational Approach to Didactics (DAD Trouche *et al.*, 2020), analyzing teacher professional development through their interactions with resources. It will be presented Chapter 2.

<sup>3</sup> National Research Council (1995), website: <https://www.nap.edu/read/9174/chapter/9#54> (p. 48)

<sup>4</sup> Mullis et al., (2020), <https://timss2019.org/reports/achievement/#math-8>

<sup>5</sup> OCDE (2019a), website: [https://www.oecd.org/pisa/publications/PISA2018\\_CN\\_QCI.pdf](https://www.oecd.org/pisa/publications/PISA2018_CN_QCI.pdf)

variation to reach the teaching goal?

- What role does variation play in classroom teaching? How does teacher interact with students from the perspective of variation?
- What are the characteristics of teaching through variation in China and France respectively according to different cultural contexts?

To figure out these three questions we carried out research. As the first two questions prompted us to model variation, we first build a preliminary model based on the literature review, considering teachers' interaction with resources as a continuous process combining design and usage, and then conducting curriculum resources (textbook and online platform) analysis and cases studies for refining it. For the third question, we adopt case studies in two contrasting contexts (China and France), trying to identify and illustrate the similarities and differences of using variation in mathematics teaching in two different cultural backgrounds. We hope that this research will be useful for thinking about variations used unconsciously by the teacher in teaching, and provide guidance for teachers on how to select, design and reflect on variations, by giving teachers' documentation work a central place, useful also from the point of view of the avenues that it opens to study the cyclical and spiraling way of teachers' documentation work, and the new variation that is created in this process.

This research is made up of four parts (see Figure 1):

- The first part is devoted to setting the scene (Chapter 1), to the literature review, the preliminary model and the analytical framework (Chapter 2), and to the methodological design (Chapter 3);
- The second part is devoted to our first research question. We study the variation in the curriculum resources, mainly the tasks in the textbooks (Hyperbole in France and PEP in China), and two typical online platforms (Sesamath in France and PEP in China), choosing the topic of function for 10<sup>th</sup> grade student (Chapter 4).
- The third part is devoted to our second and third research questions: conducting case studies in China (Chapter 5) and France (Chapter 6). In each case, we study the variation in case teacher's resources system, variation in classroom episodes, and variation in the guiding activities (Chinese case) or in the thematic lessons (French case).
- The fourth part allows us to contrast the two cases, to propose answers to our research questions, to refine our preliminary model based on the analysis of curriculum resources and case studies, to reflect on the issues of our theoretical and methodological design, and to propose some perspectives for further studies (Chapter 7).

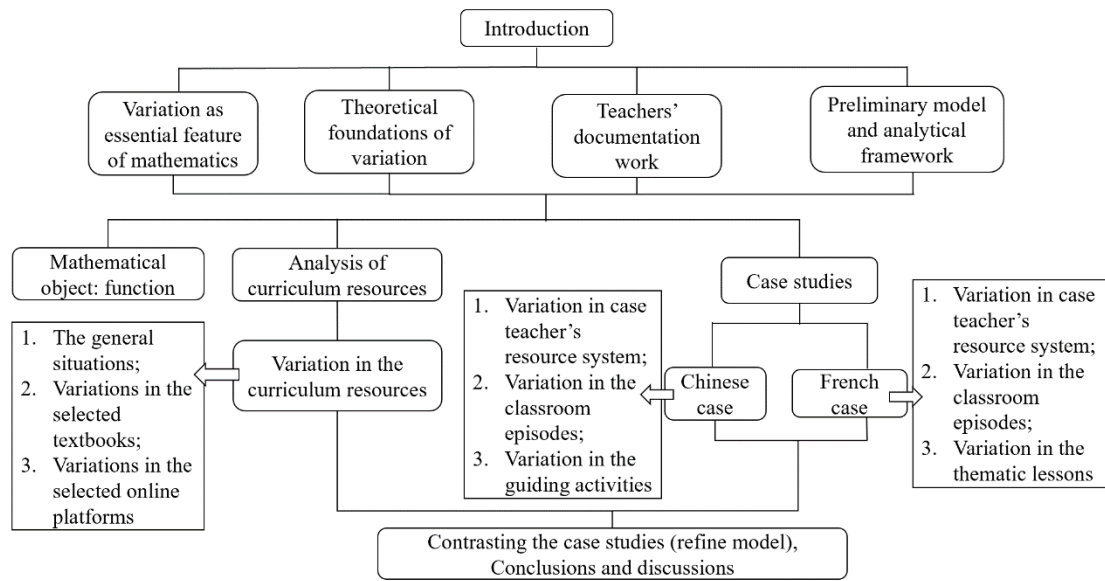


Figure 1 The structure of the thesis

The research interests are being concerned by several projects, such as MaTRiTT project<sup>6</sup> (Trouche, Bao, et al., 2018) in the Program Xu Guangqi cooperated between China and France; the Discussion Group of ICME 14 (Gosztonyi, de Varent, Zhang, & Ramploud, 2021) cooperated among China, France, Hungary, and Italy.

This thesis was developed in a very particular time, during the COVID-19 pandemic, that obliged to continuously adapt the corresponding working timetable, and follow up of Chinese and French cases.

<sup>6</sup> MaTRiTT project (2017) is an answer to the Xu Guangqi call of the French Embassy in Beijing. It is a new step for the collaboration, in the domain of mathematics education, between the French Institute of Education (ENS de Lyon) and ECNU (Shanghai). More information: <http://ife.ens-lyon.fr/ife/recherche/groupe-de-travail/matritt-joriss>





# Chapter 1

## Setting the scene

This chapter aims at presenting the background information of this PhD research, including where this research is situated, what is already known, what should be explored, and what the issues of this research will address. We first present the general situation of mathematics education in China and France (section 1.1). We then situate the “teaching through variation” in mathematics education from different education systems and cultural contexts (section 1.2). At last, we address our preliminary three research questions (section 1.3).

### 1.1 Mathematics education in China and France

In this section, we introduce the general situation of mathematics education in China and France, including the evolution of curriculum and the education system, in particular at the upper secondary school level.

#### 1.1.1 Chinese mathematics education

According to the different development circumstances of Chinese mathematics in different times, Liu, Xu and Zhao (2006) divide the evolution of Chinese mathematics’ curriculum standard into five periods:

- The first period is the budding period of ancient Chinese mathematics curriculum standards, from the emergence of curriculum standards to the eve of the Opium Wars (1840): The ancient Chinese mathematics curriculum germinated at the beginning of the Xia and Shang Dynasties (about 2070 BC-1046 BC), formed in the Western Zhou Dynasty (1046 BC-771 BC), and was finalized in the Spring and Autumn and Warring States Period (770 BC-221 BC). In the Western Zhou Dynasty, mathematics was listed as one of the six arts. Six Arts are the six basic talents that ancient Chinese Confucianism requires students to master, and they also refer to the general term for the disciplines of higher education in ancient China. Six Arts include rites, music, archery, chariot riding, calligraphy, and mathematics.

During the Sui Dynasty (581-618 AD), “mathematics” was established as a discipline. An arithmetic school (equivalent to the department of mathematics) was established in the Guozijian (The Imperial College, equivalent to a university). In the Tang Dynasty (618-907 AD), not only the arithmetic school, but also the

emperor appointed the “ten treatises of mathematical classics” (Suan Jing Shi Shu)<sup>7</sup> as textbooks and examination items. The main purpose of mathematics education during this period was “to use”. That is to say, mathematics knowledge is needed by the society, so it is necessary to learn and learn to use it.

The first curriculum standard of mathematics in China is the “The program of Learning Mathematics” (in Chinese: 习算纲目) in the book “Chengchu tongbian benmo” (乘除通变本末). It was put forward by the mathematician Yang Hui of the Southern Song Dynasty (1127-1279 AD). This “program” includes teaching goals, teaching schedule and teaching methods. It also mentions the requirements of mathematical skills, proposes the available lists of teaching reference books and the advice of studying methods (Zhang, 1993).

- The second period is the initial development period of Chinese mathematics curriculum standards from 1840 to 1912 (the year when the Republic of China was established). The introduction of Western mathematics into China during this period promoted the preliminary development of Chinese mathematics curriculum standards: beginning in 1842, missionaries founded church schools in China, offering mathematics courses, such as geometry, algebra, trigonometry, analytical geometry, and calculus. In this way, modern mathematics education has appeared in China. China founded the new-style school called “Beijing Tongwen Guan” in 1862. In Tongwen Guan, the education system is eight years and mathematics study began from the fourth year<sup>8</sup>. This education system has been in use for 30 years.

In 1902, the Qing government promulgated the “School Charter” - Qinding Regulations for schools (《钦定学堂章程》 in Chinese), also known as “Ren-yin Education System”, which is China’s first legal education system and mathematics education content inside is relatively systematic and comprehensive. However, due to political and other reasons, this school system could not be implemented. In 1904, the Qing government promulgated the “Zouding Regulations for schools” (《奏定学堂章程》 in Chinese), also known as the “Gui-mao Education System”. This is also the first relatively complete, officially announced and implemented national education system in modern China. This education system was determined by

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<sup>7</sup> “Suan Jing Shi Shu” (in Chinese: 算经十书) refers to ten well-known mathematics works from the Han to Tang dynasties of China for more than a thousand years. The names of the ten books are: “Zhoubi Suanjing”, “Nine Chapter on the Mathematical Art”, “Island Suan Jing”, “Zhang Qiuqian Suan Jing”, “Xia Houyang Suanjing”, “Wu Jing”, “Ji Gu”, “Zhui Shu”, “Wu Cao Suanjing”, “Sun Zi Suanjing”. Li Chunfeng and others commented on these ten arithmetical classics, which were completed in 656 AD.

<sup>8</sup> In the fourth year, students began to study “Enlightenment of Mathematics and Physics” (《数理启蒙》) and “Algebra” (《代数学》). In the fifth year, students studied “Euclid’s Elements” (《几何原本》), “Plane trigonometry” (《平面三角》), and “Arc trigonometry” (《弧三角》). In the sixth year, students studied “Calculus” and “Nautical Survey” (《航海测算》).

following the Japanese education system. Under the guidance of the new education system, mathematics education was implemented in kindergartens, elementary schools, middle schools, and higher education schools.

- The third period is a turning point for the mathematics curriculum standards in China, from 1912 to 1949: in September 1912, the education department promulgated the “School System Order” (《学校系统令》 in Chinese). In 1913, the “Renzi-Guichou Education System” was formed and was influenced by Japan and Germany. The content and theoretical level have been greatly improved. To a certain extent, it was influenced by the “modernization” of mathematics education promoted by John Perry (1850-1920) and Klein (F. Klein, 1849-1925).

In 1922, influenced by the American education system and Dewey’s educational ideas, the “School System Reform Order” (《学校系统改革令》 in Chinese), also known as the “Ren-xu Education System” was promulgated. It transplanted the famous “6-3-3” education system from the United States. In 1923, the “Mathematics Curriculum Standards for the New School System for Middle Schools” (《中学新学制算学课程标准》 in Chinese) was promulgated. This curriculum standard integrated the knowledge of arithmetic, algebra, geometry and trigonometry into a comprehensive curriculum; the mastery of mathematical methods is explicitly proposed; reasoning is regarded as a skill that needs to be developed and theory is taken as a competency that needs to be developed.

In 1929, the “Interim Mathematics Curriculum Standards” (《算学暂行课程标准》 in Chinese) was promulgated. In August 1933, the Ministry of Education held a meeting to revise the curriculum standards of various subjects, which was named “Official Curriculum Standards” (《正式课程标准》 in Chinese) and was announced in November of the same year. In 1941, China revised the “Mathematics Curriculum Standards” for junior and senior high schools, and promulgated new junior and senior high school mathematics curriculum standards, which continued until 1949.

- The fourth period is the period exploration of Chinese mathematics curriculum standards after the establishment of People’s Republic of China (PRC) from 1949 to 1976: in 1951, PRC issued the first mathematics curriculum standards, called “the draft of mathematics curriculum of general secondary school” (《普通中学数学课程标准(草案)》 in Chinese), which was revised several times between 1952 and 1956. The new version in 1956 stated that the purpose of mathematics teaching was “to teach students the basics of arithmetic, algebra, geometry and trigonometry, and to develop their ability to think logically and to imagine spatially”.

In September 1958, the Central Committee of the Communist Party of China issued

the “Instruction on Education Work”, which set off a nationwide campaign of educational reform, focusing on the school system and teaching content. In this reform, the content of the mathematics curriculum was increased to include preliminary calculus, preliminary probability statistics, preliminary mathematical logic, vectors, and matrices. This new curriculum reform led to a heavy learning burden for students and a decline in the quality of education.

In October 1961, under the situation of education rectification, the Ministry of Education promulgated the “Full-time Primary and Secondary School Mathematics Teaching program (Draft)” (《全日制中学数学教学大纲(草案)》 in Chinese), which was revised and issued for implementation in May 1963. It indicated the purpose of teaching is to enable students to firmly grasp the knowledge of algebra, plane geometry, three-dimensional geometry, trigonometry, plane analytic geometry, and to develop students’ ability of arithmetic, logical thinking and spatial imagination. This curriculum standard enriched the basic knowledge of mathematics by adding the plane analytic geometry, and put forward the teaching of mathematics to cultivate three kinds of ability (arithmetic ability, logical thinking ability and spatial imagination ability), indicating the basic establishment of the concept of ability in our modern mathematics teaching.

- The fifth period is the period of rapid development of China’s modern mathematics curriculum standards, from 1976 to the present: After the ten-year “Cultural Revolution”, the education is severely damaged and the decline in the quality of education has led to a crisis situation in education. Since 1977, China has restarted to carry out the educational reforms again. In 1978, 1983 and 1986, the mathematics program was successively promulgated.

The “Full-time Mathematics program for General Senior High Schools” (《全日制普通高级中学数学教学大纲》 in Chinese) organized and compiled by the Department of Basic Education of the State Education Commission has been prepared since March 1994. It was officially released in May 1996 (for experimentation). A revised trial version of the program was released in February 2000 by the Ministry of Education. The new high school mathematics program has made great changes in teaching purposes, teaching contents, teaching structure, and teaching methods. In 2003, China promulgated “the Standards of General Senior High School Mathematics Curriculum (Experimental Draft)” (《普通高中数学课程标准(实验)》 in Chinese). The latest curriculum standard of mathematics for upper secondary school is issued in 2017.

China is a centralized country, promulgating national curriculum standards for all disciplines and all class levels from kindergarten (starting at age 3) to upper secondary

school (grade 12). An overview of the current Chinese education system is presented in table 1.1.1.

Table 1.1.1 Overview of the current Education System in China

Age (years)	China
3 to 6-7	kindergarten
6-7 to 12-13	primary school
12-13 to 15-16	lower secondary school
<i>National Assessment: upper secondary school entrance examination</i>	
15-16 to 17-18	general upper secondary school
	technical upper secondary school
	vocational upper secondary school
<i>National Assessment: college entrance examination</i>	

The current school system of primary and secondary schools is “6-3-3” or “5-4-3”, which means that the duration of primary school is 6 or 5 years, the duration of lower secondary school is 3 or 4 years accordingly; and the duration of upper secondary school is 3 years. The age of entry into primary school is 6-7 years old, and the age of entry into lower secondary school is 12-13 years old. The “6-3-3” system is prevailing in most areas in China.

There are three main streams in the upper secondary school in China: general, technical and vocational. In the past, after the first year of studying in the general upper secondary school, the teaching was divided into two streams: liberal arts and sciences. Chinese, mathematics<sup>9</sup> and English as the three main disciplines are taught regardless of arts and sciences. Students of science stream also need to study physics, chemistry and biology, while students of liberal arts study politics, history and geography. In September 2014, marked by the State Council of the People’s Republic of China issuance of the “implementation opinions on deepening the reform of the examination and enrollment system”, China began the most comprehensive, systematic and profound college entrance examination reform since the resumption of the college entrance examination in 1977. Shanghai and Zhejiang province took the lead as education pilot areas to launch this reform of the college entrance examination for students who entered the upper secondary school in the fall of 2014. By September 2021, the new college entrance examination program has been smoothly implemented in 14 provinces and cities, and the reform of the examination and admission system has led to a series of educational changes (Yu, 2021).

There are several different models of the new ways of examination. Zhejiang and Shanghai areas are “3+3” model, that is, Chinese, mathematics and foreign language are three disciplines that must be assessed. Students choose three other disciplines from

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<sup>9</sup> There are differences in the content and difficulty of mathematics taught in the streams of liberal arts and sciences.

six disciplines, including physics, chemistry, biology, politics, history and geography, as optional disciplines for the college entrance examination. Some regions are “3+1+2” model, “3” refers to Chinese, mathematics and one foreign language; “1” is the preferred discipline chosen from physics or history; “2” is the re-choice discipline chosen among chemistry, biology, politics, geography, etc.; the original score of the “3” and “1” disciplines are counted, while the “2” optional disciplines are counted into the total score after assigning points by grade (Yu, 2021).

### 1.1.2 French mathematics education

France has had a national curriculum elaborated by specialized committees for the last two centuries, and associated with textbooks as central for its implementation by teachers. Researchers (Gueudet, Bueno-Ravel, Modeste & Trouche, 2018) compared and contrasted the official and real curriculum. They think that the view of mathematics in France’s curricula grew stronger over time and pointed out three key moments by looking at the official curriculum: 1802, 1902, and 2002.

- 1802: Napoleon’s ordinance of 19 frimaire of Year XI (December 10, 1802) stated: “The Lycées will essentially teach Latin and mathematics.” Gispert notes, “In placing mathematics at the same level as Latin in the male secondary curriculum, [this ordinance] took into account the new situation following the French Revolution, in which mathematics had become a core aspect of an intellectual education combining theory and practice” (Gispert, 2014, p. 230).
- 1902: A new reform, following a survey launched by the French Parliament, reasserted this importance of mathematics education: “It was, for a time, the end of the monopoly on classical humanities by the Lycées, through the creation of a modern curriculum that was on par – at least in theory – with the classical curriculum. It also furthered the development of new disciplines such as the languages, sciences, and mathematics” (Gispert, 2014, p. 233).
- 2002: In 1999, the French education ministry appointed a commission, the CREM (National Commission for Reflection on the Teaching of Mathematics) headed by Jean-Pierre Kahane, for rethinking the teaching of mathematics for the new century. In 2002, a report of the CREM stated “La mathématique est la plus ancienne des sciences et celle dont les valeurs sont les plus permanentes” (Kahane, 2002). It situated mathematics among the other sciences and underlined the necessity of connecting their teaching in combining rigor and imagination.

France has undergone major curricular reforms during this century, and reforms have concerned teaching structures, syllabuses, evaluation methods and teacher training. CFEM (2021) has cited the main reforms in primary and secondary education since 2000, see Figure 1.1.1.

2000	Reform of general high school piloted by the CNP (National council for programs)	2012	Creation of the CSP (Higher Council for Programs)
2002	Reform of elementary school	2015	Peillon law – New common core of knowledge, competences and culture
2005-2006	Fillon law and dissolution of the CNP – Common core of knowledge and competencies – Reform of middle school	2015	Reform of kindergarten
2008	Reform of primary school	2016	Reform of elementary school and middle school
2009-2010	Reform of vocational high school (2009) - Reform of general and technological high school (2010)	2018-2020	Blanquer law – Reform of high school and baccalaureate – Reform of access to tertiary education

Figure 1.1.1 Curricular evolutions of primary and secondary education in France (CFEM, 2021, p.18)

Here, we focus on the reforms at the high school level, which are highlighted in the Figure 1.1.1. As shown in the figure, the century began with a reform of general high school. In this reform, the new importance has been given to the field of statistics, with an introduction to inferential statistics in 10<sup>th</sup> grade, as well as the attention paid to the relationships between scientific disciplines and more broadly to interdisciplinarity. This is illustrated by the introduction of the TPE (Supervised personal work). These projects propose a set of broad national themes and involve at least two disciplines. They are carried out by small groups of students and co-supervised by their teachers. In this reform, the differentiation of mathematics syllabuses between different series has been thoughtfully considered (CFEM, 2021).

A new reform of high school took place in 2009. It confirmed the importance of probability and statistics, and the attention paid to interdisciplinarity with a new optional course MPS (Methods and scientific practices) was proposed in 10<sup>th</sup> grade. Beyond the re-introduction of logic, the main novelty of this reform was the inclusion of algorithmics in the mathematics syllabuses from the 10<sup>th</sup> grade and the creation of the ISN specialty (Informatique et sciences du numérique - Informatics and digital science) in addition to the four existing scientific specialties (Mathematics, Physical Sciences, SVT (Life and Earth Sciences), Engineering sciences) in the 12<sup>th</sup> grade (CFEM, 2021).

The political change in 2017 has led to more radical changes: reform of the Baccalauréat, the abolition of the general high school series (will be discussed in the following), and a complete reorganization of the teaching offer in mathematics. After the reform, students have a unique science course of 2 hours per week in the 11<sup>th</sup> and 12<sup>th</sup> grade of general upper secondary school. This course focus on the cross-cutting



themes<sup>10</sup>, which mobilizes physical and chemical sciences, life and earth sciences, computer science and mathematics. In 12<sup>th</sup> grade, students who have taken the math specialty in 11<sup>th</sup> grade (4h per week) can keep it (6h) and even add a course entitled “Expert mathematics” (3h) (if their high school offers it); if they abandon the math specialty, they may take a course entitled “Complementary mathematics” (3h)<sup>11</sup>.

In general, the main reforms taken place since 2000 have underlined some major trends, such as the increasing importance given to probability and statistics, the increased attention paid to modeling and interdisciplinarity, and the rise of algorithmics and programming, going along with the emergence of computer science as an independent school discipline. These trends are also shared at the international level (CFEM, 2021; Shimizu & Vithal, 2018).

Similar to the situation of China, France also shares a national curriculum for all disciplines and all class levels from kindergarten (starting at age 3) to upper secondary school (end at grade 12). The Ministry of Education is responsible for France’s state schools. An overview of the current French education system is presented in table 1.1.2.

Table 1.1.2 Overview of the current Education System in France<sup>12</sup>

(Adapted from Gueudet et. al, 2018 & CFEM, 2021)

Age (years)	France
3 to 5-6	kindergarten
6 to 10-11	primary school
11 to 14-15	lower secondary school
<i>National Assessment: Brevet des Collèges</i>	
15 to 17-18	upper secondary school (General and Technological)
	upper secondary school (Vocational)
<i>National Assessment: Baccalauréat</i>	

The current school system of primary, lower secondary and upper secondary schools is “5-4-3” in France, which means that the duration of primary school is 5 years, the duration of lower secondary school is 4 years, and the duration of upper secondary school is 3 years; the age of entry into primary school is 6 years old, the age of entry into lower secondary school is 11 years old, and the age of entry into upper secondary school is 15 years old. Different from situation in China, there is no entrance exam for

<sup>10</sup> The titles of the themes for this course are as follows: for 11<sup>th</sup> grade: 1. A long history of matter; 2. The sun, our source of energy; 3. The earth, a singular star, 4. Sound and music as carriers of information ([https://cache.media.education.gouv.fr/file/SP1-MEN-22-1-2019/13/4/spe573\\_annexe\\_1063134.pdf](https://cache.media.education.gouv.fr/file/SP1-MEN-22-1-2019/13/4/spe573_annexe_1063134.pdf)); for 12<sup>th</sup> grade: 1. Science, climate and society; 2. The future of energy; 3. A history of life ([https://cache.media.education.gouv.fr/file/SPE8\\_MENJ\\_25\\_7\\_2019/84/7/spe241\\_annexe\\_1158847.pdf](https://cache.media.education.gouv.fr/file/SPE8_MENJ_25_7_2019/84/7/spe241_annexe_1158847.pdf)).

<sup>11</sup> In Terminale, in 2020-2021, 41.2% of students kept the math specialty and 17.5% chose the Complementary Maths option. Moreover, 14% of those who chose the math specialty also chose the Expert Maths option (CFEM, 2021, p. 20).

<sup>12</sup> For more details, see [https://eduscol.education.fr/2897/systeme-educatif-francais-traduction-de-es?menu\\_id=3590](https://eduscol.education.fr/2897/systeme-educatif-francais-traduction-de-es?menu_id=3590)

state lower secondary schools and they accept all pupils. Students must enroll at their local town hall.

In France, there are three study streams in the upper secondary school (lycée): general (lycée général) upper secondary school for academic training; technological (lycée technologique) for arts or applied sciences or technical training; vocational (lycée professionnel) for vocational training. The structure diversifies at the high school level with two times of separation: A first separation in the first year of high school (seconde) between general and technological high school (71.6% of students), and vocational high school (28.4%). In grade 10 (seconde), stream of general and technological has not been differentiated: regardless of the field of study to which they are destined, students follow the same core curriculum, supplemented if they wish by optional general or technological courses. A second bifurcation takes place in the second year (première) between the general and technological high schools (CFEM, 2021).

Over the last 45 years, there is a voluntary move towards widening access to upper secondary school and a related diversification of the diplomas available at the end of it, especially through the creation of the technological baccalauréat in 1968 and the vocational baccalauréat in 1985 (Ichou & Vallet, 2011). Some upper secondary schools incorporate all three study streams, called “lycée polyvalent (LPO)” in French. All academic streams consist of three levels: 10<sup>th</sup>, 11<sup>th</sup> and 12<sup>th</sup> grade. Students following the technological and vocational streams can leave after two years with a different diploma such as the Le certificat d’aptitude professionnelle (CAP) or Le brevet d’études professionnelles (BEP) and enter work training without having a baccalauréat (Dimitrijevic, 2022).

Until the 2019-2020 school year, in 11<sup>th</sup> grade (première), students in the general upper secondary schools would choose to enter into one of the three series: Literature (L), Economic and Social Sciences (ES) and Sciences (S). Literature focuses on French, languages or arts, philosophy; Economic and Social Sciences focuses on economics, mathematics; Sciences focuses on mathematics, physics/chemistry, earth and life sciences. At the end of high school, students can sit one of several different baccalauréat exams, which grants them entry to a public French university. In addition, a choice of options and specialties in the final year of high school (terminale) complemented this differentiation by series. The technological high school offered 8 distinct series and the vocational high school an even greater diversity (CFEM, 2021).

However, the high school reform, which came into effect in September 2019, has profoundly changed the structure of the general high school. There will no longer be a series in the general stream, but rather paths chosen by each student according to his or her tastes and ambitions. In view of the fact that 60% of students fail to obtain a bachelor’s degree in three years, a total overhaul of the curriculum and baccalaureate

in the upper secondary general or technological school have been implemented in France starting with the 2019 academic year<sup>13</sup>. It is hoped that the strengthening of the links between higher and upper secondary education and the reform of the Bac will make it possible for students to be more successful in higher education. The overhaul mainly concerns the following fields:

- The upper secondary school will offer three types of courses: 1) A broad base of common culture, humanistic and scientific, open to the challenges of the future. All grade 10 students will be taught SES (economic and social sciences) and digital sciences and technology, each for one and a half hours; In grades 11 and 12, students will attend a two-hour science class presenting an integrated approach to the sciences; 2) Specialty subjects chosen by the student and intensified between the first and the last year of upper secondary school. Students will have to choose three specialties (such as physical education, arts, social and economic studies) to better adapt their curriculum to their interests. They will then drop one of these elective subjects in their final year. These disciplines are given significant time to offer ambitious programs and to give students time to learn. Optional courses will also allow students to complete their studies; 3) A time of guidance throughout high school to prepare for career choices and, eventually, entry into higher education. Students will be accompanied according to the timetable provided for within the framework of the autonomy of the schools (small groups, differentiated teaching, Mooc, etc.).
- The technological stream will retain its current organization in series. Adjustments will be made in order to offer a common cultural base articulated with the specialty courses and orientation assistance. The teaching programs<sup>14</sup> have been revised for the first and last year of secondary school, with a view to making them more demanding in terms of subject matter and preparing students for higher education.
- New baccalauréat reform places more emphasis on continuous evaluation (40% of the final note) and less importance on the final examination (60% of the final note). Moreover, there is a positioning test at the start of grade 11 for students to assess their fluency in French and their skills in mathematics. Teachers will use the results of this test as a basis for consolidating the knowledge acquired by pupils in one-to-

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<sup>13</sup> This reform was the result of a consultation with Pierre Mathiot and then with all the trade union organizations and 40,000 high school students. More information in the website: <https://www.education.gouv.fr/upper-secondary-general-or-technological-school-new-grade-10-and-grade-11-classes-5465> More information in the website: <https://www.education.gouv.fr/bac-2021-un-tremplin-vers-la-reussite-1019>

<sup>14</sup> See detailed information: <https://www.education.gouv.fr/les-programmes-du-lycee-general-et-technologique-9812>

one support.

The ministry of national education aims to provide more opportunities for students to choose closest to their desires by offering more specialist subjects at school and offering new combined courses, such as the combination of history-geography, geopolitics, political sciences, and the combination of humanities, literature and philosophy courses and so on. The latest curriculum standard of mathematics for general or technological upper secondary school is issued in 2019<sup>15</sup>. Mathematics curriculum for the 11<sup>th</sup> and 12<sup>th</sup> grade of the vocational Baccalauréat in 2020<sup>16</sup>.

## **1.2 Situating the teaching through variation in mathematics education**

In this section, we first examine and contrast the latest curriculum standards in both countries, mainly from the perspective of variations (section 1.2.1). We then present the current position of teaching through variation both in the field of research and practical teaching from different education systems and cultural backgrounds (section 1.2.2).

### **1.2.1 Exploring the implicit variations in the curriculum standards**

Both China and France have national curriculum standards and have recently issued new curriculum standards of senior high school mathematics. As curriculum standards guide the design of textbooks and reflect nation's expectations of the implementation of the curriculum, we choose one French curriculum standards of grade 10 (Ministère de l'Éducation nationale et de la Jeunesse, 2019a), and one Chinese curriculum standard of high school (MOE, 2017) to explore the mathematical variations proposed or embedded in these two countries.

In this section, we first present the main principles of the curriculum standards in China and France and illustrate the role of variation in the implementation of these principles (section 1.2.1.1). We then explore the possible variations that are implicit in the part of function of grade 10 in the two curriculum standards (section 1.2.1.2).

#### **1.2.1.1 Main principles of the curriculum standards**

The French high school curriculum standards are presented by cycle respectively for

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<sup>15</sup> Programme de mathématiques de seconde générale et technologique:

[https://cache.media.education.gouv.fr/file/SP1-MEN-22-1-2019/95/7/spe631\\_annexe\\_1062957.pdf](https://cache.media.education.gouv.fr/file/SP1-MEN-22-1-2019/95/7/spe631_annexe_1062957.pdf)

Programme d'enseignement scientifique de première Générale:

[https://cache.media.education.gouv.fr/file/SP1-MEN-22-1-2019/13/4/spe573\\_annexe\\_1063134.pdf](https://cache.media.education.gouv.fr/file/SP1-MEN-22-1-2019/13/4/spe573_annexe_1063134.pdf)

Programme d'enseignement scientifique de terminale Générale:

[https://cache.media.education.gouv.fr/file/SPE8\\_MENJ\\_25\\_7\\_2019/84/7/spe241\\_annexe\\_1158847.pdf](https://cache.media.education.gouv.fr/file/SPE8_MENJ_25_7_2019/84/7/spe241_annexe_1158847.pdf)

<sup>16</sup> Programme de mathématiques de la classe de première professionnelle:

[https://cache.media.education.gouv.fr/file/SPE1-MENJ-6-2-2020/84/1/spe003\\_annexe1\\_1239841.pdf](https://cache.media.education.gouv.fr/file/SPE1-MENJ-6-2-2020/84/1/spe003_annexe1_1239841.pdf)

Programme de mathématiques de la classe de terminale professionnelle:

[https://cache.media.education.gouv.fr/file/SPE1-MENJ-6-2-2020/84/3/spe003\\_annexe2\\_1239843.pdf](https://cache.media.education.gouv.fr/file/SPE1-MENJ-6-2-2020/84/3/spe003_annexe2_1239843.pdf)

the general, technological and professional stream. The standards define, for each cycle, the essential knowledge that must be acquired during the cycle as well as the methods that must be assimilated. They constitute the national framework within which teachers organize their teaching taking into account the learning rhythms of each student (Gueudet et al., 2018). The latest French mathematics curriculum standard for grade 10 (Ministère de l'Éducation nationale et de la Jeunesse, 2019a) are based on six mathematics competencies: research (experiment, in particular using software tools); model (make a simulation, validate or invalidate a model); represent (choose a frame, i.e., numeric, algebraic, geometric..., change the register); reason (demonstrate, find partial results and put them into perspective); calculate (apply techniques and implement algorithms); communicate (communicate a result orally or in writing, explain an approach).

Different from the previous curriculum standards, but similar to the latest French curriculum standard, the latest Chinese curriculum standard (MOE, 2017) is written on the basis of six core literacy of mathematics: mathematical abstraction (related to generalization in variation), logical reasoning, mathematical modeling, visual imagination, mathematical operations and data analysis. These core literacies of mathematical science are independent and interact with each other, and they are an organic whole. Therefore, it should be pointed out that in mathematics teaching, we need deconstructive instructional design and practice, so that the core literacy of mathematics can be more clearly reflected. At the same time, from a holistic perspective, we should focus on the specific learning process of students. The student's learning process should not be parsed into one or several aspects of the six aspects because the core literacy of the mathematics discipline is composed of six aspects (Zheng, 2019).

Contrasting the principles in these two curriculum standards, we find that modelling, calculating, and reasoning are mentioned in both countries. The principles proposed by the Chinese curriculum standard are more specific to the teaching content. For example, the Chinese standard focuses more on the geometric aspects of “represent” (in the French standard) and thus proposes “visual imagination”. The same with the principle of “data analysis” in Chinese curriculum standard and “research” in French curriculum standard. There is some overlap between them but they have their own focus.

Mathematical abstraction is unique to the Chinese curriculum standard. It refers to the literacy of getting the mathematical object through the abstraction of quantitative relations and spatial forms. It includes the abstraction of mathematical concepts and the relationships between concepts from quantity relations, graphical relations; the abstraction of general laws and structures from the concrete context of things and represent them in mathematical language (MOE, 2017).

“Communicate” is unique to the French curriculum standard, including oral

communication and writing. It indicates the major role of verbalization and reformulation in the appropriation of mathematical notions and the resolution of problems. It also proposes various situations for students to practice the oral communications: the reformulation by the student of a statement or an approach, interactive exchanges in class, corrections of exercises, group work, individual or group presentations. There is a mobilization in various registers (graphics, formulas, calculation) between natural language and symbolic language in the communication (Ministère de l'Éducation nationale et de la Jeunesse, 2019a).

Regardless of the six competencies proposed in the French curriculum standards or the six core literacies mentioned in the Chinese curriculum standards, for practical teachers, how to implement them in teaching is the top priority. Variation is a way to promote the implementation of students' competencies/literacies (Qiu, 2019; Zheng, 2019). For example, the variation emphasizes the study of the essential and non-essential attributes of mathematical concepts or laws, which means that when learning mathematical concepts, it is easier for students to recognize how to establish mathematical concepts in the process of variation learning (the description of variation teaching from the perspective of students). For senior high school students, this is not only learning a specific mathematical concept or law, but also acquiring a method of establishing a mathematical concept or law, for example, the core literacy of mathematical abstraction.

### **1.2.1.2 Implicit variations in curriculum standards**

In this section, we explore the possible variations that are implicit in Chinese and French curriculum standard, particularly in the topic of function for grade 10:

- Possible variations arising from various representations: given the mathematical properties of the function, both curriculum standards propose the manipulation of different modes of representation, i.e., algebraic expression, table of values, and graphic representation. For example, Chinese curriculum standard indicates “choose appropriate methods to represent functions in practical situations according to different needs, and understand the role of the graph of a function (MOE, 2017, p. 19)”. French curriculum standard indicates “exploit various registers, in particular the algebraic register and the graphic register (Ministère de l'Éducation nationale et de la Jeunesse, 2019a, p.11)”. The rich transformations in/between various registers/representations can bring possible variations.
- Possible variations arising from various digital tools: both curriculum standards propose that digital tools should be integrated into the teaching of functions. For example, Chinese curriculum standard indicates “Students are encouraged to use information technology to learn, explore and solve problems. For example, use calculators, computers to draw graphs of power functions, exponential functions,

logarithmic functions, trigonometric functions, etc., explore and compare their laws of variation, study the properties of functions, find approximate solutions to equations, etc. (MOE, 2017, p. 24)”. French curriculum standard proposes a requirement that “use dynamic geometry or computer algebra software, the calculator or Python to describe the variations of a given function by a formula (Ministère de l’Éducation nationale et de la Jeunesse, 2019a, p. 12)”.

In contrast, French curriculum standard is more specific in describing the tools available. For example, it mentions the digital tools that can be taken into advantage: “dynamic geometry software for graphical representation and the use of cursors; Python, spreadsheet or calculator, to highlight the calculation program aspect” (Ministère de l’Éducation nationale et de la Jeunesse, 2019a, p. 11).

The use of digital tools brings challenges to traditional teaching, such as the variations in the strategies of solving problems, the way of designing tasks, and so on.

- Possible variations arising from the formation of concept: both standards highlight the introduction and formation of the concept of function. French curriculum standard points out “the students gradually discovered the notion of function” (Ministère de l’Éducation nationale et de la Jeunesse, 2019a, p. 10). However, it does not give a specific description of how to “gradually” discover. Chinese curriculum standard proposes more specific suggestions. It indicates “the introduction of the concept of a function can be abstracted by using examples with the context familiar to students. For example, starting with the definition of a function based on the relationship between variables known to students (learnt in the junior high school) and lead them to construct a general concept of a function through problems in their lives or in mathematics (MOE, 2017, p. 23).”

Based on the Chinese curriculum standard, the process of forming new concepts is an opportunity to connect old and new knowledge. The reflection on how to “gradually” discover and form concepts leaves a great deal of room for variation. In addition, how to clarify the specific boundaries of a concepts is also an opportunity for variation.

- Possible variations arising from the situations: French curriculum standard mentions “use functions to model situations from mathematics and other disciplines” (Ministère de l’Éducation nationale et de la Jeunesse, 2019a, p. 12), which leads us to think about the various contexts related to function, and further the possible variations in the situations.

Curriculum standards are the basis for teaching implementers to design and implement practical teaching. Various versions of textbooks are one of the main curriculum

resources that specifically reflect the requirements and content of curriculum standards. Here our exploration of the possible variations in the curriculum standard shows the important role that variation can play in teaching mathematics from the intended curriculum to the implementation of the curriculum.

### **1.2.2 Teaching through variation: cultural phenomena?**

In this section, we present the current situation of teaching through variation in different countries, particularly in China and France.

#### **1.2.2.1 Teaching through variation: fundamental pillar of Chinese mathematics teaching**

In this section, we provide a comprehensive view into teaching through variation in China from three perspectives: its close relationship between theory and practical teaching; variations in the curriculum resources; and variation passed on and improved through teachers' collaborative activities.

As one of the most important teaching approaches proposed by Gu (Gu, 1991; Gu, 1994; Gu, Huang, & Marton, 2004) based on 45-years experiment of mathematics teaching reform, teaching through variation is well-known in Chinese mathematics teaching. Various notions about teaching and learning through variation have been proposed (Gu, Huang, & Gu, 2017; Pang, Bao, & Ki, 2017). There are also extensions of the concept of variation, such as using it in a dynamic learning environment (A. Leung, 2017), combining the theory of variation with learning trajectory (Huang, Zhang, Chang, et al., 2019) and so on. Viewed as an effective pedagogical approach (Huang & Li, 2017), various researches including vivid lesson cases are conducted in China. Different types of lessons are involved: the introduction of new concepts (Huang & Leung, 2017; Mok, 2017); exploration of new propositions (Qi, Wang, Mok et al., 2017); problem solving (Peng, Li, Nie, et al., 2017); and review of knowledge (Huang, Yang, & Zhang, 2017). The topics of algebra and geometry are included in these studies.

Teaching through variation has been used regularly in mathematics classroom instruction and a variety of related books, articles, and teaching materials have been available for mathematics teachers to adapt mathematics problems and methods in their own classes (Huang & Li, 2017). Sun (2011) describes the roles of variation problems in Chinese textbooks. She regards variation problem as “indigenous” Chinese practice aim at discerning and comparing the invariant feature of the relationship between concept and solution. Zhang, Wang, Huang, et al. (2017) examine the tasks in a series of middle school mathematics textbooks and indicate that textbooks introduce new concepts and deepen understanding of concepts through use of variation tasks.

The Chinese concept of teaching through variation is used and passed on in the teachers' collaborative activities between expert and junior teacher, teacher and university



mathematics education experts, teacher and district teaching specialists, etc. Ding, Jones, and Sikko (2017) analyzed an expert mathematics teacher used the idea of teaching through variation to support a junior teacher in the primary school for her professional development. Researchers identified that the expert teacher guided the junior teacher in two sophisticated ways: the use of teaching notions; and the use of teaching frameworks and language that commonly share, understand and practice in China. Han, Gong, and Huang (2017) conducted a designed lesson study designed by a group of sixth grade mathematics teachers on the theoretical basis of learning trajectory<sup>17</sup> (Clements et al., 2011; Daro & Corcoran, 2011) and teaching and learning through variation. The four teachers engaged in the lesson study were supported by university mathematics education experts and district teaching specialists on the development and modification of the learning trajectory as well as observation and debriefing of the lessons. It works out that there is a remarkable shift of the participant teachers' discussions from teacher dominated instruction to an instruction dominated by students' exploration and investigation through the teaching research activities (Krainer, 2017). These researches give an insight of mathematics teacher's reflections on their teaching and their learning for teaching through variation through teachers' collaborative activities, thereby furthering their professional development.

Teaching through variation in China is relatively developed, not only in theory and practice, but also in the transmission from the expert/experienced teacher to the novice teacher as a valuable mathematics teaching experience.

### **1.2.2.2 Teaching through variation in different cultural contexts**

In this section, we first describe several studies on applying Chinese variation theory to mathematics teaching with different cultural backgrounds. Then, we present the other variation theories proposed by western researchers and their corresponding studies in other cultures and education systems.

We can find the practices of Chinese teaching through variation in other education systems. For example, Huang and Li (2017) indicate that U.S. mathematics teachers' instructional designs and classroom instruction of mathematical tasks reflected the idea of variation pedagogy. However, it is hard for the teacher to articulate the rationale. They also make an exploratory study by asking a group of U.S. mathematics teachers to implement a Chinese lesson plan that is designed according to the theory of teaching through variation (Gu, Huang & Marton, 2014). It turns out that the effective teaching practice in China could be adapted in mathematics classroom teaching in other education systems. Hino (2017) connects the Chinese pedagogy of variation with

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<sup>17</sup> Learning trajectories include sequences of tasks and activities aimed at the progressive development of mathematical thinking and skill (Han, Gong, & Huang, 2017).

Japanese problem-solving approaches, which are perceived to be effective practices in the Japanese education system. She concludes that “a structured problem-solving lesson can be viewed from the perspective of variation” (p.317).

Similar to the situation in China, researchers in the west have emphasized the use of variation to improve mathematics teaching and learning, such as deliberately varying mathematical tasks to facilitate students’ learning (Huang & Li, 2017; Rowland, 2008; Watson & Mason, 2006). We can also find the relevant theoretical foundation of variation from the western teaching theory. For example, Marton and his colleagues (Marton & Booth, 1997; Marton, 2015) have developed the variation theory of learning, which also highlights the important role of invariance (similarity) in affording the discernment of differences (Pang et al. 2017). Barlow et al. (2017) present a case that describes a series of tasks whose development was informed by the theory of variation posed by Marton et al. They indicate that variation theory emerges as a powerful vehicle by which “U.S. algebra instruction” might better facilitate student transition from arithmetic to algebraic thinking (Clarke, 2017). Runesson and Kullberg (2017) regard variation theory as a guiding principle in learning study. They present a longitudinal study in Sweden through iterative cycles of teaching and revising.

Teaching without accomplishing a pattern of variation is hardly possible (Runesson and Kullberg, 2017). At the same time, as Sun (2011) suggested that “the idea of variation is almost invisible to teachers” and Runesson and Kullberg (2017) evidence that “despite evident changes in sequence and patterns of variation, the teacher’s discussion of his/her instruction didn’t talk about variation at all”, most of the teachers are using variation in their teaching unconsciously. Helping teachers to realize the variation (what is varied and what is kept invariant) in their teaching can help them to design instruction more effectively according to the teaching goals. Hence, it is necessary for us to provide a pattern or structure or model to make teaching through variation become visible, reflected upon and developed.

### **1.2.2.3 Teaching through variation underlying in French didactics**

In this section, we explore teaching through variation in France based on the related theory and possible variations in teachers’ collaborative activities.

When it comes to French mathematics education, it is inevitable to mention the didactics of mathematics in France. It emerged with the aim of building a genuine field of scientific research and not just a field of application for other scientific fields such as mathematics or psychology. Drawing the lessons of the innovative activism of the New Math period with the disillusion it had generated, French didacticians gave priority to the understanding of the complex interaction between mathematics learning and teaching in didactic systems (Artigue et al., 2019). Therefore, it is essential to build

solid theoretical foundations for mathematics education in tight interaction with empirical research. There are three main theoretical pillars in French didactics: the *theory of didactical situations* due to Brousseau, the *theory of conceptual fields* due to Vergnaud, and *the anthropological theory of the didactic* that emerged from the *theory of didactic transposition*, due to Chevallard. These three theories are complex objects that have been developed and consolidated over decades (Artigue et al., 2019). In Chapter 2, we focus on the theory of didactical situations, which implies the idea of variation.

Two main factors have strongly influenced the development of French didactic research (Artigue et al., 2019). One is the engagement of French mathematicians in educational issues, which was already visible from the time of the French revolution (Gispert, 2014), the other is the creation of the Institutes of Research on Mathematics Teaching (IREMs<sup>18</sup>). In France, the design of mathematical tasks, situations and sequences of situations is essential to didactic research and is controlled by the theoretical frameworks underlying this research. Design as a development activity takes place mostly within the IREMs (Blum, Artigue, Mariotti, Sträßer, & Van den Heuvel-Panhuizen, 2019). IREMs are independent from, but close to mathematics departments, and this structure enables collaborative work part-time in thematic groups among university mathematicians, teachers, teacher educators, didacticians and historians of mathematics. Through their activities and producing materials for teaching and teacher education, IREMs developed the action-research, teacher training sessions, etc., and nurtured institutional and scientific relationships between didacticians and mathematicians (Artigue et al., 2019). It is undoubted that IREMs demonstrate the importance of teachers' documentation work in French mathematics education and provide rich soil for teachers to create variations in designing tasks through collaborative work.

At present there are few studies on teaching through variation in France. We wonder if variation also has its practical experience in France without teachers' consciousness because of the unique French didactics and teachers' collaborative work have provided a good foundation for its emergence.

### 1.3 Preliminary research questions

Our research is in the field of mathematics didactics, seeking to contribute to the improvement of the quality of mathematics teaching and teacher's professional development. We are particularly interested in studying teachers' teaching through

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<sup>18</sup> IREM (Institut de recherche sur l'enseignement des mathématiques), website : <http://www.univ-irem.fr>

variation in different education system and cultural backgrounds. Taking function as an example, we study specific situations in China and France and present three preliminary research questions here:

Q1: What are the potential resources with variation in high school mathematics textbooks and other curriculum resources in China and France? (From the perspective of resources)

Q2: How do case teachers in China and France use teaching resources to design variations? What are the characteristics of case teachers in variation design? (From the perspective of design)

Q3: In the classroom teaching, how do case teachers from China and France use variations to teach? What are the teaching characteristics of case teachers when they are teaching through variation? (From the perspective of practical teaching)

These preliminary research questions will be deepened and formulated in chapter 2 after presenting the theoretical backgrounds and our model of teaching through variation.



## **Chapter 2**

### **Literature review and theoretical considerations**

This chapter is dedicated to construct a preliminary model of variation by illustrating the status of variation in mathematics and mathematics education, studying various different variation theories, and analyzing the documentational approach to didactics, teachers' documentation work and their relations with variation. It proposes four sections: Section 2.1 explains the variation has been inseparable from mathematics and mathematics education, evidencing that variation comes from the essence of mathematics; Section 2.2 describes the theoretical foundations of variation in mathematics education in both China and France and illustrates them with typical examples; Section 2.3 reviews the literature on teachers' documentation work, mainly the Documentational Approach to Didactics and its relation with variation; and Section 2.4 presents two definitions of variation in China and France respectively, proposes a preliminary model of variation through the lens of teachers' documentation work and our research questions, the analytical framework and the concept of "variation path" designed based on the model.

#### **2.1 Variation as one of the essential features of mathematics**

In this section, we evidence that variation is one of the essential features of mathematics through the study of three aspects: the diversity and unity of mathematics (section 2.1.1); The role of diversity and unity of mathematics in mathematics education (section 2.1.2); and to be more specific, the variation in the two classical works -- "Nine Chapters on the Mathematical Art" and "Euclid's Elements" from both mathematics and mathematics education perspectives (section 2.1.3).

##### **2.1.1 The diversity and unity of mathematics**

Mathematicians are trying to determine what is real mathematics and what should be the basis for new mathematical creations. This kind of confusion is increasing day by day (Kline, 1980). In this section, we describe the efforts made by Chinese (section 2.1.1.1) and western mathematicians (section 2.1.1.2) in exploring the nature of mathematics over years. The development of mathematics is dynamic. Mathematics is both diverse and unified and the most important thing in mathematics is active cooperation with other disciplines, as well as mastering mathematical thinking and mathematical methods.

### 2.1.1.1 The development of mathematics in China

The history of Chinese mathematics is divided into two phases: before the 17<sup>th</sup> century, the establishment of traditional Chinese mathematical systems and algorithms centered on “rod calculus”<sup>19</sup>; and after the 17<sup>th</sup> century, the introduction of Western mathematics centered on arithmetic with pen. Further, it can be divided into five periods according to its own characteristics (Li, 2010):

- The rudiments of the traditional mathematics in China (Ancient times-early Eastern Han Dynasty, ca. 4000 BCE-25 CE): According to ancient books<sup>20</sup>, decimal-value notation and rod calculus were already in place.
- The formation and development of the system of traditional mathematics in China (Early Eastern Han Dynasty - Five Dynasties and Ten Kingdoms period, 25-960 CE): The traditional Chinese mathematical system was gradually formed and developed. After a long period of accumulation, by the second half of the 1<sup>st</sup> century CE, a classic of traditional Chinese mathematics, the “Nine Chapters on the Mathematical Art”, was published. It marked the formation of a traditional Chinese mathematical system centered on rod calculus. In the following six to seven hundred years, from the mathematician Zhao Shuang<sup>21</sup>, Liu Hui<sup>22</sup> to Li Chunfeng<sup>23</sup>, there were continuous developments in algorithms, theories and applications. At the beginning of Tang Dynasty (618-907 CE), Li Chunfeng was ordered to annotate “Suanjing Shishu”<sup>24</sup>, which made a relatively systematic arrangement of the mathematical literature of this period. The establishment of various rod calculus algorithms characterized the mathematics of this period.

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<sup>19</sup> Rod calculus: “筹算” in Chinese, is an ancient procedure for calculating in the decimal system using an arithmetic rod in the Chinese cultural sphere. In ancient China, rods were used as a tool to record numbers, list formulas and perform calculations of various numbers and formulas. It was initially a natural object like a small bamboo stick, but later gradually developed into a specialized calculation tool and were increasingly replaced by the more convenient and faster abacus.

<sup>20</sup> The main works are the “Suan Shu Shu (A book on numbers and computation)” (算数书 in Chinese) and “Zhoubi Suanjing (A book on astronomical observation and calculation)” (周髀算经 in Chinese).

<sup>21</sup> Zhao Shuang: The mathematicians of Wu in the Three Kingdoms period. In about 222 CE, Zhao Shuang studied the “Zhoubi Suanjing” in depth and wrote a preface and a detailed commentary, in which there are commentaries on the diagram of the proof of Pythagorean theorem, which is a valuable document in the history of mathematics.

<sup>22</sup> Liu Hui: Mathematician of Wei in the Three Kingdoms period. He not only organized the ancient Chinese mathematical system and constructed its theoretical foundation by annotating the “Nine Chapters on the Mathematical Art”, he also put forward his own original ideas reflected from the classic works.

<sup>23</sup> Li Chunfeng: Astronomers and mathematicians in the early Tang Dynasty. In the Tang Dynasty, he was ordered to annotate the “Suanjing Shishu” and it was completed in 656 CE

<sup>24</sup> “Suanjing Shishu”: “算经十书” in Chinese, ten mathematical manuals were annotated and were used as the teaching material of the academy of arithmetic of “Guozijian” (imperial college) and the examination item of Arithmetic. See the detailed names of these ten books in footnote 7.

- The prosperity of the traditional mathematics in China (From the beginning of the Northern Song Dynasty to the beginning of the Yuan Dynasty, 960-1303 CE): Traditional Chinese mathematics showed a prosperous development. Many new algorithms and new topics appeared one after another within a short period of time, with clear computational procedures and high theoretical abstraction. For example, in “Mathematical Treatise in Nine Sections”<sup>25</sup> (1247 CE), Qin Jiushao<sup>26</sup> introduced a general form of the Chinese remainder theorem that used Da yan shu (大衍术) or algorithms to solve it. Gauss gave this theorem in his book “Arithmetical Investigations” (1801) more than 500 years later. Moreover, he proposed “Qin Jiushao’s algorithm” for finding the positive roots of higher-degree polynomial equations. Horner proposed a similar method in 1819 and named “Horner’s method”<sup>27</sup>. Many theorems were introduced far ahead in the world history of mathematics, thus making traditional Chinese mathematics the peak of the world mathematics at that time,
- The low-tide of the traditional mathematics in China (From the early Yuan to the end of the Ming Dynasty, 1303-1607 CE): Traditional Chinese mathematics was in a state of decline. The so-called declining state means that the algorithm centered on rod calculus failed to reach the level of the previous period and the focus of mathematical development shifted to the popularization of daily use algorithms. This trend led to the creation and development of Chinese Zhusuan<sup>28</sup>. The emergence and development of Zhusuan (including abacus and algorithm) is a manifestation of the development of traditional Chinese mathematics. However, it is an obvious fact that the overall level of mathematics in this period is significantly lower than that of the previous period. The reasons for the decline are a matter of discussion in the history of mathematics. Factors such as the hindrance to the development of mathematics by the examination policy of “Eight-legged essay”<sup>29</sup> are already pointed out. The alternation of rod calculus and Zhusuan is the characteristic of this period of mathematics.
- The confluence of Chinese and Western mathematics (From the end of Ming

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<sup>25</sup> “Mathematical Treatise in Nine Sections”: “数书九章” in Chinese, is a practical mathematical compendium with a wide range of topics drawn from various aspects of Song Dynasty society, including agriculture, astronomy, water conservancy, urban layout, construction engineering, surveying, taxation, armament, military and so on.

<sup>26</sup> Qin Jiushao: Mathematicians of the Southern Song Dynasty. His main work is “Mathematical Treatise in Nine Sections”.

<sup>27</sup> There is a controversy about the right of invention of Horner, refer to Wylie (1852).

<sup>28</sup> Chinese Zhusuan: 珠算 in Chinese, is the knowledge and practices of arithmetic calculation through the Chinese abacus.

<sup>29</sup> The mathematical content was completely eliminated. Talents were selected by the essay which is written in the eight-legged style.



Dynasty to the first year of the Republic of China, 1607-1912 CE): The introduction of western mathematics and research were the mainstream, while the arrangement and study of traditional mathematics was also an aspect that could not be ignored during this period. The emergence of a large number of translations and edited works of western mathematics complemented the traditional Chinese mathematics to some extent. Arithmetic with pen was the characteristic of this period.

In the second decade of the 20th century, Chinese mathematicians began to enter the world of mathematics. After 1949, Chinese mathematics entered a new phase of comprehensive development.

In general, there is no doubt that traditional Chinese mathematics has fruitful outcomes and had made indelible contributions to the development of world mathematics. We can see the trajectory of mathematical works being passed on and developed over generations. However, after the 15<sup>th</sup> century, Chinese mathematics was left far behind, such as the creation of analytic geometry, the invention of calculus, the development of abstract algebra, etc., none of them are related to China. In the next section, we present the development of mathematics in western countries, especially from the period of 18<sup>th</sup> century.

#### **2.1.1.2 The development of mathematics in western countries**

What is mathematics? For the predecessors, mathematics is the most exquisite invention made by people to study the natural world. The main concepts, broad methods, and almost all the important theorems of mathematics are derived in the process of discovering the world. As Kline (1980) proposed, Mathematics can be regarded as a building. In the ancient Greek era (ca. 600 BCE-800 CE)<sup>30</sup>, the structure of mathematics (mainly composed of Euclidean geometry) proved to be stable. Then the Indians and Arabs added a new layer to the building of mathematics: introducing negative numbers, accepting irrational numbers, and formulating rules of operation about them. By expanding the mathematics of various numbers, the mathematics building has been built one after another: complex numbers, various algebras, calculus, differential equations, differential geometry and many subjects. However, the foundation of mathematics began to collapse when its superstructure was strengthened. The invention of non-Euclidean geometry reveals that the axioms of Euclidean geometry are not real and solid ground. In order to prevent the mathematics building from overturning, strong measures must be taken.

With this context, many mathematicians hope to build a modern and leafy mathematics

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<sup>30</sup> We start from the Greek period, being conscious that the starting point of structured mathematics is ordinary situated far before, in the scribal schools, about 2000 BCE (Proust, 2012).

building under a number of concise axiom systems. One of the representatives is the Göttingen school and the idea of unity has experienced three stages of development in this school: specific mathematical problems (for example, fundamental theorem of algebra), sub-disciplines of mathematics (for example, the principle of group theory in geometry), and the whole pure mathematics (for example, the later mathematical axiomatization movement). In 1900, Hilbert gave a famous report on “The Problems of Mathematics” at the Second International Congress of Mathematicians and he pointed out that *“mathematics is an indivisible organic whole, and its vitality is officially in the connection between its parts.”* *“The organic unity of mathematics is an inherent feature of this science, because it is the basis of all precise natural science knowledge.”* (Hilbert, 1902).

The axiomatic method developed by Hilbert and the idea that “mathematics is an organic unity” established by the Göttingen School are important to the schools of mathematics in the United States, the Soviet Union, Britain, France and other countries, and have had a huge impact on the development of mathematics in the 20th century. Among them, the Bourbaki School is particularly eye-catching and influential. Its appearance is even praised by some mathematicians as “the most significant phenomenon in modern mathematics”.

Since the 19th century, especially in the 20th century, mathematics has expanded unprecedentedly, with a large number of new disciplines and new fields emerge. Mathematics has shown unprecedented diversification. Ordinary people may be confined to special things, but scientists need to understand the relationship between them, that is, the unity of nature. Modern mathematicians can only consider their own special problems in a narrow field, while Bourbaki wants to explore the common ground between different fields, which is the unity of mathematics. They emphasized that mathematics is not just a simple summation of various subjects, there are inexhaustible links between various fields of mathematics, and the value of various problems is not the same. The most valuable mathematics are problems that are closely related to various fields, and relatively isolated problems are often of little significance (Bourbaki, 1948). Bourbaki proposed that at the center of the mathematical world there are three main types of structure: algebraic structure (group, ring, field), order structure (partial order, total order), topological structure (neighbourhood, continuity, limit, connectivity, dimension), they can be called parent structure, or core structure, basic structure. Each type of structure has many branches, and there is a certain relationship between them, and they are all determined by axioms. Furthermore, two or more structures can be compounded to form a more complex structure. Each structure maintains its independence, but they can be connected together by mapping, calculation and other relations. The simplest example of a composite structure is a vector space. In

addition, there are multiple structures. If a set has two or more structures at the same time, and these structures have a certain relationship and are compatible with each other, it is called a multiple structure. There are many multiple structures, such as partial order group, total order group, topological group, topological ring, topological field and so on.

The results of these attempts to unify mathematics are now known as failures. Mathematicians gradually realized that attempts to establish a universally acceptable and logically reasonable mathematical expression had failed. Mathematics is a kind of human activity, and it is subject to various weaknesses and shortcomings of human beings (Kline, 1980). However, the ideas put forward by Göttingen and Bourbaki School are not worthless. They bring us enlightenments on the unification of mathematics from two aspects: the necessary combination with other disciplines and the common mathematical methods and the way of thinking in different mathematics branches. Nowadays we still pursue the harmony and unity inherent in the many branches of mathematics, and harmony lies in the unity. But the unity we are talking about does not mean that the entire mathematics can be discussed through a model like “Euclid’s Elements” under one or two axiom systems, but refers to the numerous existing branches of mathematics, including the new fields that may appear in the future. Science has always been the blood that maintains the vitality of mathematics. In the field of science, mathematicians are enthusiastic partners of physicists, astronomers, chemists and engineers. During the 17th, 18th, and 19th centuries, the difference between mathematics and theoretical sciences was rarely noticed, and many outstanding mathematicians have done more work in astronomy, mechanics, dynamics, electricity, magnetism, and elasticity than their work in mathematics. Mathematics is the queen of science and the maid of science at the same time (Kline, 1980). A branch can explode with strong vitality only when it blends with other branches, which is also true for mathematical problems. To be more precise, the objects studied by each branch of mathematics may not be very relevant, but the many mathematical methods and ideas in each field are similar, including some commonly used learning methods and research methods.

F. Klein described the development of mathematics in “Elementary Mathematics from an Advanced Standpoint” (2007) as follows: In fact, mathematics has grown like a tree, but it does not grow from the thinnest root, nor is it just growing upwards; on the contrary, as the branches and leaves expand, its roots get deeper and deeper.....Then, we can see that the foundation in mathematics has no final end. On the other hand, there is no initial starting point. Poincaré (Henri Poincaré, 1854-1912) also expressed a similar view: “There are no solved problems; there are only problems that are more or less solved”.

### **2.1.2 The role of diversity and unity of mathematics in mathematics education**

As we mentioned above, both diversity and unity are embedded in mathematics. Now we turn our attention to how diversity and unity play a role in mathematics education. We describe it from three aspects in mathematics education, from macro to micro: interaction between mathematics and other disciplines (section 2.1.2.1); interactions of different branches of mathematics (section 2.1.2.2); interactions of multiple representations (section 2.1.2.3).

#### **2.1.2.1 Interaction between mathematics and other disciplines in mathematics education**

According to presentation of reports and recommendations proposed by commission for reflection on mathematics education in France (Kahane, 2002), today we should pay more attention to “mathematical sciences” rather than “mathematics”. In addition to its own development, mathematics is continuously influenced by other disciplines (Kahane, 2002).

Mathematics is the oldest science and the one whose values are the most permanent. Over the past fifty years, the perspective on mathematics has changed considerably: mathematics first seemed to have regained its unity on the basis of a solid construction of foundations and structures, then become impoverished; after that, applied mathematics made a breakthrough; currently, the movement of mathematics reveals many sources and influences, as well as constitutes considerable work in mathematics. Mathematics is enriched with problems, methods and concepts from other sciences and practices. At the same time, it creates new concepts and new theories. Mathematics develops through its interactions with other disciplines as well as through interactions within itself. Thus, mathematics is far from being the business of only mathematicians. Today, we must take into account the mathematical activities of mechanics, physicists, computer scientists, engineers, biologists, economists, chemists, and mathematicians in the strict sense of the word. It is good to stop reasoning only on the basis of “mathematics”, “pure and applied mathematics”, and to consider the diversity of participants and users of all “mathematical sciences” (Kahane, 2002).

Moreover, the means of studying mathematics have varied according to the development of technologies. The techniques of printing, navigation, astronomy have helped to form the functions as well as calculus. Today, computer science creates both new means and new disciplines of study, and all sciences advance through mathematical tools and contribute to the creation of new sciences.

Considering the close relation between mathematics and other disciplines, PISA (Program for International Student Assessment) have defined four context categories

and used them to classify assessment items developed for the PISA survey (OCDE, 2017, 2019b):

- **Personal** – Problems classified in the personal context category focus on activities of one’s self, one’s family or one’s peer group. The kinds of contexts that may be considered personal include (but are not limited to) those involving food preparation, shopping, games, personal health, personal transportation, sports, travel, personal scheduling and personal finance.
- **Occupational** – Problems classified in the occupational context category are centred on the world of work. Items categorized as occupational may involve (but are not limited to) such things as measuring, costing and ordering materials for building, payroll/accounting, quality control, scheduling/inventory, design/architecture and job-related decision making. Occupational contexts may relate to any level of the workforce, from unskilled work to the highest levels of professional work, although items in the PISA survey must be accessible to 15-year-old students.
- **Societal** – Problems classified in the societal context category focus on one’s community (whether local, national or global). They may involve (but are not limited to) such things as voting systems, public transport, government, public policies, demographics, advertising, national statistics and economics. Although individuals are involved in all of these things in a personal way, in the societal context category the focus of problems is on the community perspective.
- **Scientific** – Problems classified in the scientific category relate to the application of mathematics to the natural world and issues and topics related to science and technology. Particular contexts might include (but are not limited to) such areas as weather or climate, ecology, medicine, space science, genetics, measurement and the world of mathematics itself. Items that are intra-mathematical, where all the elements involved belong in the world of mathematics, fall within the scientific context.

#### **2.1.2.2 Interaction of different branches of mathematics in mathematics education**

In addition to the interaction with other disciplines mentioned above, there are interactions inside different branches of mathematics in mathematics education. The unity of mathematics is reflected in the unity of the structure of mathematics. Some seemingly different concepts, theorems, and laws in mathematics can be in a unity under certain conditions. For example, after the establishment of set theory, the “operation” in algebra, “transformation” in geometry, and “function” in analysis, these three concepts in different domains can be unified into the concept of “mapping”. Descartes (René Descartes, 1596-1650) unified geometry, algebra, and logic through analytic geometry, which is fully reflected in Chinese high school mathematics. For

example, in analytic geometry of high school, the unity of circle, ellipse, hyperbola, and parabola is manifested in the following four aspects (Jin, 2013):

- From the form of equations, in the rectangular coordinate system, their curve equations can all be written as:  $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ . The feature of the equations is that they are all binary quadratic equations, so they can be called as quadratic curves;
- Except for circles, from the point of view of point sets or trajectories, they are all point sets or trajectories whose ratio of the distance between a fixed point and a fixed line is constant (called eccentricity). This fixed point is the focus, and the fixed line is the directrix. They are divided into ellipse, hyperbola, and parabola due to the difference of eccentricity;
- These four curves can be seen as cross sections obtained by truncating cones in different planes. Therefore, they can be called conic curves, and more subtly: the orbits of celestial bodies are these four curves;
- In the polar coordinates system, we set the distance from the fixed point to the directrix as  $p$ , the eccentricity as  $e$ , the polar angle as  $\theta$ , and the polar radius as  $\rho$ . Then the straight line, circle, ellipse, hyperbola, and parabola can be unified into a simple polar coordinate equation:

$$\rho = \frac{ep}{1 - e\cos\theta}$$

Here, the straight line and the circle are regarded as the extreme cases of hyperbola and ellipse. The unification of these four curves in various aspects proves the conclusions of the mathematician Hilbert: “Although the knowledge of mathematics is very different, we still clearly realize that in mathematics as a whole, the same logical tools are used. There is a conceptual relationship, and at the same time, there are a lot of similarities between its different parts.” “The isolated branches will also reveal unexpected relationships.”

The work of Douady (1986) about “frames” also supported that different branches of mathematics can be subsumed under the concept of “frame”. According to Douady, the word “frame” is to be taken in the usual sense that it has when we speak of algebraic frame, arithmetic frame, geometric frame, etc. The variations of frames (“jeux de cadres” in French) are changes of frames provoked by the teacher, on the occasion of suitably chosen problems, to advance the research phases and evolve the students’ conceptions. Three phases can be distinguished in this procedure: 1) Transfer and interpretation: the students are confronted with a problem formulated within a certain frame. Taking into account their knowledge, practices and habits, their examination of the problem leads them to translate and interpret all or part of questions into another frame. In doing so, they implement correspondences between different frames (between objects and

between relations). 2) Imperfect matches: the correspondences between the frames are imperfect either for mathematical reasons or because of the students' insufficient knowledge. The situation is a source of imbalance. 3) Improving correspondence and advancing knowledge: communication between frames and in particular communication with an auxiliary representation frame is a factor of rebalancing.

### **2.1.2.3 Interaction of multiple representations of mathematics**

Duval (1995) points out "There is no knowledge that can be mobilized by an individual without a representation activity". Multiple representations play an important role in the teaching of mathematics and are one of the manifestations of mathematical diversity. In this section, we first illustrate that multiple representations are a distinguishing feature of mathematics, which many researchers have studied and classified. Then we introduced the symbolic-graphic combination commonly used in high school mathematics education, which fully reflects the diverse representations of mathematics and the flexible use of conversions between different representations.

Goldin (2014) distinguishes *inscription* and *representation*, *internal representation* and *external representation*. According to him, mathematical representations, most commonly interpreted in education, are visible or tangible productions, such as diagrams, number lines, graphs, arrangements of concrete objects or manipulatives, physical models, mathematical expressions, formulas and equations, or depictions on the screen of a computer or calculator -- that encode, stand for, or embody mathematical ideas or relationships. Such a production is sometimes called an inscription when the intent is to focus on a particular instance without referring, even tacitly, to any interpretation. A representation thus includes reference to some meaning or signification it is taken to have. The term representation is also used to refer to a person's mental, cognitive, or brain constructs, concepts, or configurations.

Representation is called external when it is external to the individual who produced them, and accessible to others for observation, discussion, interpretation, and/or manipulation. Representation is called internal to the individual when it includes individuals' visual and/or spatial cognitive representation of geometrical objects or mathematical patterns, operations, or situations; their kinesthetic encoding of operations, shapes, and motions; their internal conceptual models of mathematical ideas; the language that they use internally to describe mathematical situations; their heuristic plans and strategies for problem-solving; and their affective and motivational states in relation to mathematical problems and situations (Goldin, 2014).

Based on the description above, we divide mathematical representations into *internal representations* and *external representations*, where the internal representation of mathematics refers to the meaning and construction of the individual to the object of

mathematics learning; the external representation of mathematics reflects the external form of the object of mathematics learning.

Researchers have proposed different classifications of external representations of mathematics based on different classification standards. Friedlander and Tabach (2001) believe that mathematical representation can be divided into four categories: verbal representation, numerical representation, graphical representation, and algebraic representation. Xu (2001) believes that mathematical representations can be divided into four categories: formal representation, graphical representation, action representation and linguistic representation. Duval (2006) listed several different kinds of registers of semiotic representation: the natural language, the symbolic writing, images, geometrical figures, cartesian graphs and diagrams. Various researches have shown that working with a single register does not allow the students to fully access the concept. One representative of a concept, in one register, can only show part of the concept properties. Therefore, it is necessary for teachers to change the representations of registers of one concept in teaching, which is helpful to comprehend the characteristics of the concept.

Insofar as mathematical activity intrinsically consists in the transformation of representations, Duval (2006) also revealed two types of transformation of semiotic representations that are radically different: *treatment* and *conversion*. Treatments are transformations of representations that happen within the same register. The treatments, which can be carried out, depend mainly on the possibilities of semiotic transformation, which are specific to the register used. Conversions are transformations of representation that consist of changing a register without changing the objects being denoted: for example, passing from the algebraic notation for an equation to its graphic representation, passing from the natural language statement of a relationship to its notation using letters, etc. Conversion is a representation transformation, which is more complex than treatment because any change of register first requires recognition of the same represented object between two representations whose contents have very often nothing in common. It is like a gap that depends on the starting register and the target register. Too often, conversion is classified as translation or encoding. Compared to the conversion between registers, the treatment in one register is easier for students. In order to study the conversions between registers, we need to differentiate different types of registers. Duval (2017) has proposed a two-dimensional table that show the classification of types of semiotic registers, see Figure 2.1.1.



	DISCURSIVE Registers <i>Linearity based on succession to produce and organize sequences of words or symbols</i>	NON-DISCURSIVE Registers <i>Simultaneous apprehension of a two-dimensional organization of nD figural units</i>
MULTIFUNCTIONAL registers : <i>the transformations of the expressions are NON-ALGORITHMIC</i>	<b>NATURAL LANGUAGES.</b> <i>Three hierarchically included operations (naming the objects, enunciation, and reasoning) with their corresponding meaning units</i> <i>Two phenomenological modes of production: oral or written</i>	<b>ICONIC : IMAGES</b> <i>freehand production and internal conservation of topological relations characteristic of parts of the object.</i> <b>NON-ICONIC : GEOMETRICAL FIGURES.</b> <i>Three independent operations: instrumental construction, mereological reconfiguration and deconstruction of the 2D dimensional shapes.</i>
	(Auxiliary transitional representations for free external operations)	
MONOFUNCTIONAL: Registers: <i>the transformations of the expressions are ALGORITHMIC</i>	<b>THE SYMBOLIC WRITING</b> <i>for an unlimited substitution operation: numbering system, algebraic writing, formal languages)</i> <i>A single phenomenological mode of production: writing</i>	<b>CARTESIAN GRAPHS, DIAGRAMS</b> <i>strokes and arrow joining marks or nodes</i> <i>For the graphs :operations of zooming, interpolation, change of axis.</i>

Figure 2.1.1 Classification of types of semiotic registers (Duval, 2017, p85)

This classification provides a tool for analyzing the registers used explicitly in the mathematical activity and those implicitly mobilized, for identifying the root of the troubles with mathematics understanding that many students have (Duval, 2006 & 2017).

Many studies on mathematical representation in education are devoted to how students generate representations, interact and move within various representations, translate between representations, or interpret one representation using another, in a specific domain, such as the concept of algebra, planar and spatial geometry. Researchers seek to characterize students' understandings in terms of multiple representations, to infer students' thinking from the representations they produce and manipulate, to identify the affordances and obstacles associated with particular kinds of representations, and to develop new representational teaching methods using new media (Goldin, 2014).

A large number of studies have evidenced that multiple representations have direct or indirect effects on the process and results of mathematical problem solving. For example, Janvier (1987) pointed out that multiple representations are an inherent part of mathematics learning and appropriate use of multiple representations can make mathematics concrete from multiple perspectives, make mathematics problems easier to solve, and make mathematics learning more interesting. Zhang (1997) indicated that problems with different external representations can cause completely different problem-solving behaviors of students with the same internal representation. Markman and Dietrich (2000) pointed out that the external representation of problems is diverse and rich, and different types of problem representation are suitable for different types

of mental processing.

In high school classroom teaching, symbolic-graphic combination is one of the important strategies to solve the problem and is also a prominent method of applying diversity of mathematical representations. The “symbol” in the symbolic-graphic combination is algebraic representation, which refers to the use of algebraic tools to present mathematical problems and mathematical concepts; at the same time, the “graph” in the symbolic-graphic combination is geometric representation, and intuitive geometric tools are used to analyze mathematical problems. Solving mathematical problems with the symbolic-graphic combination often involves the selection and conversion of various representations of mathematics. Whether the representation can be selected reasonably and whether the conversion can be successfully performed are the key to solving the problem.

The combination process involves two conversion directions: One is from symbol to graph. Converting a symbolic problem to a graphic problem is to first find the “graph” in the “symbol” (Liu, 2018), and then find out the geometric characteristics of the quantitative relationship, and transform the abstract concept into the expression of the concrete graph, making full use of intuitiveness of graph to solve the problem smoothly (Shao, 2009). The other is from graph to symbol, which is the algebraization of geometric problems. Converting a graphic problem to a symbolic problem is to first use the appropriate quantitative relationship to express the characteristics of the graph structure, and then with the advantage of algebraic algorithms, use the algebraic language to give a rigorous explanation and finally solve the problem (Shao, 2009). Therefore, from symbol to graph is to use the intuitiveness of graph to explain algebraic formulas and certain mathematical concepts. Turning abstract into concrete is conducive to the cultivation of imagery thinking. From graph to symbol is to use algebraic language to rigorously explain spatial relations. Converting problems that require logical argumentation into operational problems is conducive to the programming of thinking (Hao, 2019). Through the symbolic-graphic combination, on the one hand, we can concretize abstract problems and simplify complex problems when solving mathematical problems. On the other hand, we can also use this method to help students build connections between knowledge, to promote the harmonious development of students’ imagery thinking and abstract thinking, and to cultivate students’ innovative ability. According to Duval’s two-dimensional types of registers, symbolic-graphic combination is a conversion, i.e., a representation transformation.

Wang (1996) concluded that the combination of symbolic and graphic can mainly solve the following four types of specific problems in high school: 1. Problems related to functions, the graphs and properties of functions; 2. Problems of solving equations and inequality; 3. Problems related to complex numbers (often considered to be solved by

the geometric interpretation of complex numbers); 4. Problems of seeking the extrema of a function.

Considering the complexity of conversion, we divide the conversion mode between symbolic and graphic into two types (Hao, 2019): One is the problem of “corresponding representation”, that is, algebraic representation and geometric representation have an obvious correspondence, such as the one-to-one correspondence between points on the number line and real numbers, points and coordinates in a rectangular coordinate system, and curves and equations in analytical geometric problems. This conversion mode is relatively easy for students, and the way of thinking is mainly logical reasoning; The other is the “constructed representation”, meaning that when there is no obvious correspondence between the two representations of some combination of symbol and graph, students are required to construct another representation based on the problem structure and the known information. This conversion mode has certain requirements for students’ thinking ability. We give an example (example 2.1.1) of function here to show this mode clearer.

#### Example 2.1.1

Find the minimum value of the function  $f(x) = \sqrt{x^2 - 2x + 5} + \sqrt{x^2 - 4x + 13}$  □ □ □

[Solution] :

Because  $f(x) = \sqrt{(x-1)^2 + (0-2)^2} + \sqrt{(x-2)^2 + (0-3)^2}$  □ □ □

Associating the formula for the distance between two points to construct the following Figure 2.1.2. Therefore, the minimum value of  $f(x)$  is the minimum value of the sum of distances between the point  $P(x, 0)$  and  $A(1, 2)$ ,  $P(x, 0)$  and  $B(2, 3)$ .

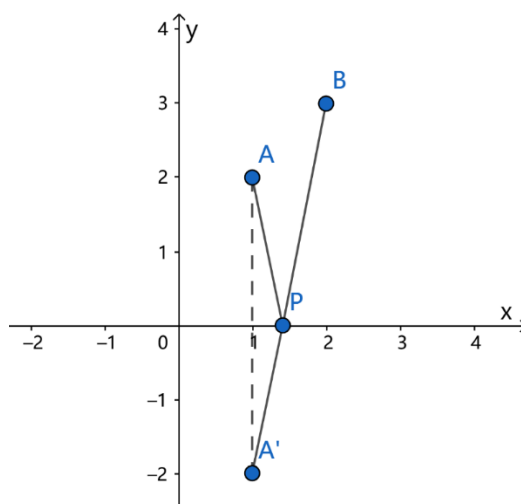


Figure 2.1.2 distances between the point  $P(x, 0)$  and  $A(1, 2)$ ,  $P(x, 0)$  and  $B(2, 3)$

The most important step in this example is to realize the conversion between different representations. Many problems can be repeated in changes and changes can be obtained in repetition. Generally speaking, difficult mathematical problems can be

broken down into relatively basic and easy parts through the appropriate conversion of representations.

When we use the symbolic-graphic combination, we should also pay attention to the following aspects: 1. Equivalence principle: The conversion of algebraic properties and geometric properties should be equivalent. Due to the limitations of graphics, sometimes they cannot fully represent the generality of algebraic properties; 2. Two-way principle: use both geometrically intuitive analysis and algebraic abstract exploration, combining the two aspects. It is naive to only perform geometric analysis on algebraic problems or only perform algebraic analysis on geometric problems; 3. Simplicity principle: After finding the idea of solving a problem, use geometric or algebraic methods, or use both of them depends on which method is simpler and more convenient for teaching purposes (Luo, 2005; Luo, 1997; Lai, 2001; Xu & Chen, 1998).

### **2.1.3 Variation in two classic mathematical works**

In this section, we studied two classic mathematical works, “Nine Chapters on the Mathematical Art” (section 2.1.3.1) and “Euclid’s Elements” (section 2.1.3.2), illustrating that the idea of variation has been rooted in mathematics and mathematics education long time ago.

#### **2.1.3.1 Variation in “Nine Chapters on the Mathematical Art”**

“Nine Chapters on the Mathematical Art” (九章算术, noted as “Nine Chapters” in the following) is a comprehensive work summarizing traditional mathematics thoughts from the Qin Dynasty (221- 206 BC) to the Han Dynasty (202 BC – 220 AD), a mathematical period when using counting rods as a special calculation tool in ancient China. “Nine Chapters” had a profound impact on ancient Chinese mathematics, just as “Euclid’s Elements” had a profound impact on the ancient Greek and Western mathematical systems. It constituted the preliminary model of the ancient mathematical cultural system in China (Zhang, 2019; Wang & Huo, 2014).

There are a total of 246 arithmetic problems in “Nine Chapters”. The calculation problems of the same type are grouped into one chapter and a total of nine chapters, respectively are bounding fields, millet and rice, proportional distribution, reducing dimensions, figuring for construction, equitable taxation, excess and deficit, the two-sided reference (i.e., equations), and base and altitude. “Nine Chapters” have an extremely close relationship with the real world, and each of the problems is a practical problem with the context of daily life, such as farmland measurement and distribution of money and food. The mathematical contents in “Nine Chapters” are equivalent to the current elementary mathematics, focusing on the algebraic content<sup>31</sup> (Zhang &

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<sup>31</sup> The geometry part is also treated algebraically in “Nine Chapters”.

Wang, 1997; Zhang, 2019). In each chapter, the arithmetic problems are shown first, then the answers are given, and the specific calculation method is given in the end.

Affected by the mathematics tradition of I-Ching<sup>32</sup>, “Nine Chapters” contains the variation thought of “When it is exhausted, it mutates; by mutation it achieves continuity; by continuity it endures long.”<sup>33</sup>. Below, we take the ninth chapter of base and altitude (problems involving the principle known in the west as the Pythagoras theorem) as an example to illustrate the variation in “Nine Chapters”.

According to Qi and Wang (2019), there are three types of variations in this chapter as follows:

- (1) One problem multiple changes: Qi and Wang (2019) sorted out the relationship between 24 problems in this chapter and studied multiple changes from four aspects: vary the condition type of problem (change the known conditions of the existing problem through varying the condition type, while keeping the desired goal unchanged), vary specific value of condition (change the known conditions of the existing problem through varying the specific value in the condition, while keeping the desired goal unchanged), vary problem’s goal (change the desired goal of the existing problem, while keeping the known conditions unchanged), exchange between problem’s condition and goal, and chaining between old and new problems (based on the goal of the existing problem to ask new questions). Figure 2.1.3 presents an example of the links among problems in the chapter of base and altitude based on the “one problem multiple changes”.

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<sup>32</sup> I-Ching (易经 in Chinese) is one of the oldest classical Chinese works. It is a book of divination in ancient China. I-Ching used a system of symbolic forms to describe the variations in things, expressing the philosophical and cosmological view of classical Chinese culture.

<sup>33</sup> Translated from Chinese “穷则变，变则通，通则久”

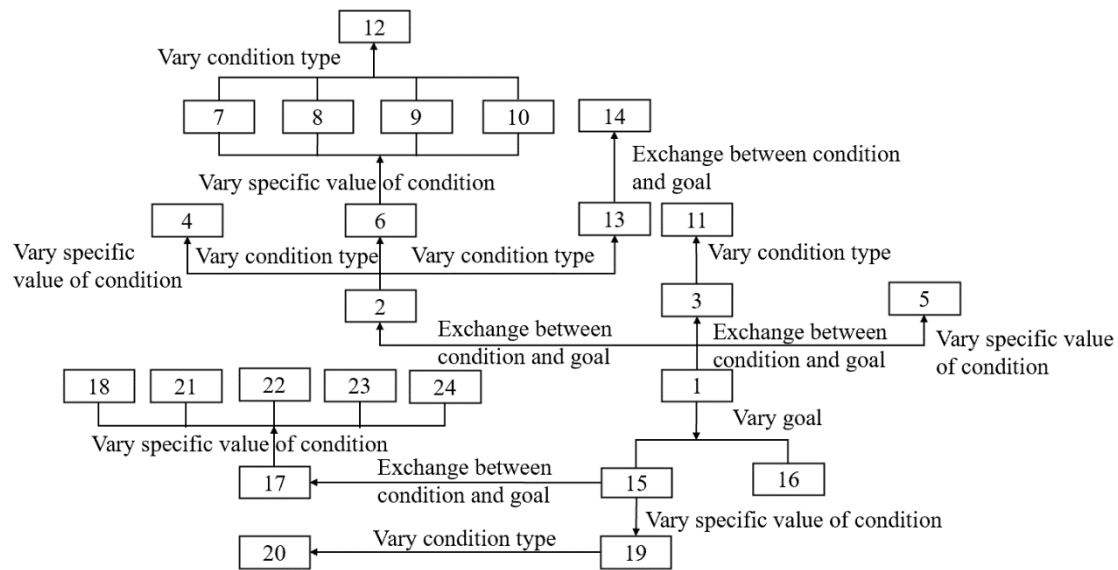


Figure 2.1.3 Links among problems in the chapter of base and altitude (translated from Qi & Wang, 2019)

In Figure 2.1.3, the starting point and foundation of all 24 problems is problem 1. Starting from problem 1, there are two directions to go:

- Problem 5 is obtained by varying specific value of condition of problem 1, problems 2 and 3 are obtained by the exchange between condition and goal. Starting from problem 3, through varying the type of condition, problem 11 is obtained. Starting from problem 2, through varying the specific value of condition or condition type, problem 4, 6 and 13 are obtained respectively. Starting from problem 13, problem 14 is obtained through exchange between condition and goal. Starting from problem 6, through varying the specific value of condition, problems 7-10 are obtained, and for each problem of 7-10, through varying the condition type, problem 12 is obtained.
- Based on problem 1, we vary the goal and obtain problem 15 and 16. Based on problem 15, through varying specific value of the condition, problem 19 is obtained. Based on problem 19, through varying the condition type, problem 20 is obtained. Based on problem 15, problem 17 is obtained through exchange between condition and goal. Based on problem 17, through varying specific value of condition, problems 18, 21-24 are obtained.

Qi and Wang (2019) also pointed out that problem 16 is not merely a variation from problem 1 through varying the goal. The solution of problem 16 is based on the Pythagoras theorem, because the diameter of the inscribed circle of a right-angled triangle needs to be expressed by a right-angle side and a hypotenuse. In other words, from problem 1 to problem 16, chaining between old and new problems is also implied.

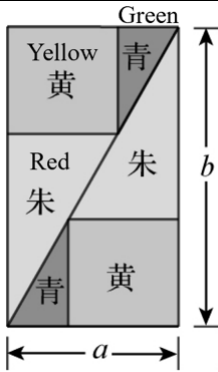
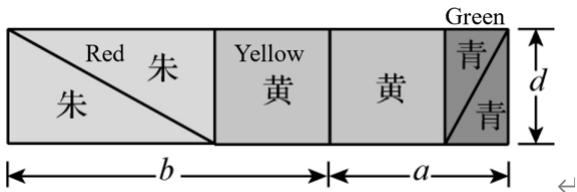
(2) One problem multiple solutions: Chinese mathematician Hui Liu (225-295 CE) paid great attention to one problem with multiple solutions when he annotated “Nine

Chapters”. Qi and Wang (2019) presented a typical example (problem 15 in this chapter) with multiple solutions:  $a$  and  $b$  are the lengths of two sides of the right triangle respectively, then the side length of the inscribed square is  $d = \frac{ab}{a+b}$ . Hui Liu used two different methods to prove this formula in the annotation of “Nine Chapters” (Guo, 2004; Qi & Wang, 2019):

- Method 1: Cut and fill method

Liu Hui called the square inscribed by the right-angled triangle as the Yellow Square, and the remaining two small right-angled triangles as Red Mi, and Green Mi respectively. As shown in Figure 2.1.4 in the table 2.1.1, first make a rectangle with the length of  $a$  and  $b$  as sides, and its area is  $ab$ , which contains 2 Red Mi, 2 Green Mi and 2 Yellow Square. Then, reassemble Red Mi, Yellow Square, and Green Mi into a rectangle with length of  $a + b$  and width of  $d$ . As shown in Figure 2.1.5, because  $ab = (a + b)d$ , so  $d = \frac{ab}{a+b}$ .

Table 2.1.1 Cut and fill method to prove Pythagorean theorem

 <p>Figure 2.1.4</p>	 <p>Figure 2.1.5</p>
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- Method 2: Proportional method

As shown in Figure 2.1.6, assuming that the lengths of right-angle sides are  $a_1$  and  $b_1$ ,  $a_2$  and  $b_2$  respectively. Because  $a:b = a_1:b_1$ , so  $(a + b):b = (a_1 + b_1):b_1$ , and because  $a = a_1 + b_1$ ,  $b_1 = d$ , so  $(a + b):b = a:d$ . Similarly, because  $b:a = b_2:a_2$ , so  $(a + b):a = (a_2 + b_2):a_2$ , and because  $b = a_2 + b_2$ ,  $a_2 = d$ , so  $(a + b):b = a:d$ .

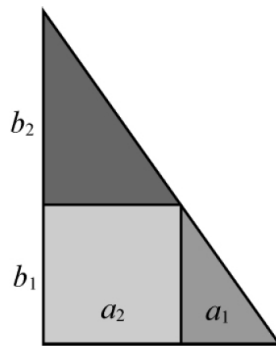


Figure 2.1.6 Proportional method to prove Pythagoras theorem

- (3) Multiple problems one solution: multiple problems one solution is common in “Nine Chapters”. Problems 1-5 in this chapter all can be solved with the formula:  $c = \sqrt{a^2 + b^2}$ . Problem 17-24 are problems about measurement, which can be attributed to the problem of inscribed rectangles in right-angled triangles. Although they have different contexts in the real world, all of them can be solved by using the proportional method above.

As we mentioned above, using variations in mathematics education can be traced back to the time when “Nine Chapters” was written thousands of years ago. It is very impressive to see that variation can be used to string together series of problems, and we can design endless problems by varying one fundamental problem, for example problem 1 in the 9<sup>th</sup> chapter. Moreover, the complexity and difficulty of the problems are gradually increasing, and the previous problems often pave the way for the following problems. “Nine Chapters” shows and evidences the great potential of variation in improving the efficiency of mathematics teaching and expanding mathematical thinking.

### 2.1.3.2 Variation in Euclid’s “Elements”

Euclid’s “Elements” (written around in 300 BC) is a masterpiece that uses axioms to establish the earliest example of a deductive system. With the help of logical methods, it organizes the scattered knowledge of the past, classifies and compares them, exposes the internal connections between them, and organizes them in a relatively rigorous system, which has a profound impact on the development of mathematics.

“Elements” consisting of 13 volumes: volumes 1 to 6 are mainly about plane geometry, volumes 7 to 9 mainly explain number theory, volumes 11 to 13 mainly discuss solid geometry. In each volume, it begins with the definitions, postulates, and axioms, and then presents propositions based on them. Propositions in “Elements” are abstract and have almost nothing to do with the real world. The content in “Elements” mainly involves elementary geometry nowadays, and also involves geometric processing in algebraic parts such as ratio calculations. (Zhang & Wang, 1997).



Like the study of “Nine Chapters”, we take the Pythagoras theorem as an example to illustrate the variations in the “Elements”. There are 48 propositions in the first volume, of which the 47th proposition is the Pythagoras theorem, and the 48th proposition is the inverse theorem of the Pythagoras theorem. The first volume is divided into five parts: 1-4 Equilateral triangles and SAS congruent; 5-15 Isosceles triangle and SSS congruent; 16-26 General triangle and ASA congruent; 27-32 Detailed explanation of the fifth postulate (Parallel postulate: If a line segment intersects two straight lines forming two interior angles on the same side that sum to less than two right angles, then the two lines, if extended indefinitely, meet on that side on which the angles sum to less than two right angles.); 33-48 Area transformation and Pythagoras theorem. Series of propositions is carefully arranged in this volume and gradually deepened, which has been paving the way for the proof of the Pythagoras theorem. In “Elements”, compass-and-straightedge construction is an important means to prove the existence of a figure, and then its nature is demonstrated. For the proof of the Pythagoras theorem, it gave the method of constructing vertical line (Proposition 11, 12), parallel line (Proposition 31), and square (Proposition 46) before proceeding to the proof. Before making parallel lines, the approach to constructing the equal angle (Proposition 23) and parallel line decision theorem (Proposition 27) is given.

Many geometric theorems and propositions in “Elements” use the relationship between area or volume to prove (Kline, 1980), which reflects the importance of symbolic-graphic combination. Geometry in the middle school is mainly Euclidean geometry, so many important geometric conclusions can be proved through the ideas of transformation, equal area transformation and symbolic-graphic combination. In solid geometry, the idea of equal volume transformation is often used to solve the problem of line segment length. We take the proof of the Pythagoras theorem in “Elements” as an example to illustrate this feature.

As shown in Figure 2.1.7 in table 2.1.2, set  $\angle ABC=90^\circ$ ,  $AB=c$ ,  $AC=b$ ,  $BC=a$ , to prove that  $a^2 + b^2 = c^2$ . From “Elements”, we construct three squares with side lengths  $a$ ,  $b$ , and  $c$ . Connect  $CD$  and  $BF$ , pass point  $C$  to make  $CL \perp DE$ , intersects  $AB$  to  $O$ , and intersects  $DE$  to  $L$ . It is obvious that  $\triangle CAD \cong \triangle FAB$ . Based on the relationship between the figures with same length and height, we get:  $S_{\triangle CAD} = \frac{1}{2} S_{rectangle ADLO}$ ,

$S_{\triangle FAB} = \frac{1}{2} S_{square FACG}$ , so  $S_{rectangle ADLO} = S_{square FACG}$ , and  $S_{rectangle OLEB} = S_{square CBKH}$ , therefore, we get  $a^2 + b^2 = c^2$ .

The process of proof here is: transforming the side length problem into an area problem; converting the upper two squares into two equal-area rectangles through the relationship of the area of triangles with the same length of base-side and height (Wang, 2016).

Table 2.1.2 The proof of Pythagoras theorem

Figure 2.1.7	Figure 2.1.8

Multiple representations and conversions between different representations provide the opportunity to vary the condition of the problem or use the same method to solve different problems, etc. In “Elements” it proposes the fundamental propositions, which embedded huge potential for varying. For example, if the right-angled triangle in Figure 2.1.7 is turned into a general triangle, and taking each side of a triangle as a square becomes taking two sides separately as a square, we will get such a figure: as shown in Figure 2.1.8. Compared with Figure 2.1.7, the right-angled triangle is turned into a general triangle, which weakens the restriction on the shape of the triangle, and at the same time reduces a square, which is simpler and more generous. In the teaching in middle school, many new, inspiring and challenging tasks have evolved on the basis of Figure 2.1.8 (Yin, 2016).

### 2.1.3.3 Comparison between “Nine Chapters on the Mathematical Art” and “Elements”

“Nine Chapters on the Mathematical Art” and “Elements” are the two major sources of modern mathematics. Although they are both ancient mathematics masterpieces and have been used as textbooks, there are big differences in the style of these two works.

“Nine Chapters” is centered on algorithms and constructs a theoretical system based on algorithms, while “Elements” mainly systematically summarizes logical reasoning and forms a logical deductive system in the mathematical system. The content in the “Nine Chapters” is practical and diverse. It is mainly a summary of the teaching knowledge accumulated in all aspects of the social production process from the Spring and Autumn Period (771-476/403 BCE) to the Qin and Han Dynasties (221BC-220AD). This work involves diverse areas of life, so it is also called the “encyclopedia of mathematics”. However, “Elements” is not as closely related to the real society as “Nine Chapters”. It is algebraic geometry, and the problems are specifically solved through rigorous logical proofs. It has laid a theoretical foundation for the development of geometry. Therefore, the context of tasks in “Nine Chapters” is always about the human activities in real life while the proposition in “Elements” is with the context of pure mathematics. “Nine

Chapters” emphasizes the connection between mathematics and real life, and reflects the practical characteristics of mathematics learning. Through learning, students can combine theory with practice. “Elements” emphasizes that students should pay attention to the internal logical structure, reflecting the abstract and rigorous characteristics of mathematics learning. In addition, the mathematics in “Nine Chapters” is expressed in Chinese characters, and there are no mathematical formulas written in mathematical symbols, which increases the difficulty of learning and applying mathematics, and seriously hinders the popularization and development of mathematics at that time in China.

From the perspective of variation, there are many similarities between these two works: (1) Organizing tasks with the same topic together and arranging series of tasks/propositions step by step, paying attention to the logic and hierarchy of tasks; (2) Having a sense of diversity in mathematical representations and using the idea of symbolic-graphic combination to solve problems; (3) Transforming complex problems that are difficult to solve: when a mathematical problem cannot be solved, transform the problem, decompose a problem into several problems, transform an explicit problem into a recessive problem, and transform algebraic (or geometrical) problems into geometrical (or algebraic) problems are an important method for solving mathematical problems; (4) Providing fundamental problems which have huge potential to vary: taking Pythagoras as an example, more solutions have been proposed on the basis of these two works, and many tasks we used in middle school are the variations from the construction of these basic figures.

“Nine Chapters” focuses on digging out general laws in different mathematical problems, while “Elements” summarizes the general laws and forms theorems before proving them. Liu Hui (Guo, 2004) indicated in the preface of “Notes on Nine Chapters on the Mathematical Art” that “Things are analogous to each other, and each has its own merits. Therefore, although the branches are divided but the roots are the same.”<sup>34</sup>, which means that there are many mathematical problems, on the surface, they are expressed in different ways, but they are the same in mathematical theory and have the common mathematical roots. Liu Hui used the relationship between the branches and roots of trees to compare various algorithms that are derived from the unity of spatial form and quantitative relationship. His statement also supports the diversity and unity of mathematics we explained in section 2.1.1.

#### **2.1.4 Conclusion: Variation comes from the essence of mathematics**

As we have mentioned above, mathematics is both unity and diverse. In this section,

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<sup>34</sup> Translated from Chinese “事类相推，各有攸归，故枝条虽分而同本干知，发其一端而已”

we conclude the previous three sections, and explain why we believe variation comes from the essence of mathematics.

The diversity and unity of mathematics are intertwined. On one hand, mathematics, physics, astronomy, and other disciplines collide with each other, which promotes the vigorous development of mathematics and also provides a rich realistic background for mathematics teaching. On the other hand, multiple representations of mathematics provide possibilities for mathematical transformation, which promotes the emergence of multiple problem-solving methods, inspiring students to think about problems from different perspectives, for example, the symbolic-graphic combination we mentioned in section 2.1.2.3. It is worth noting that the unity of mathematics is embedded in the multiple representations. Here we give an example to illustrate, see example 2.1.2 (proposed by He, 2003).

*Example 2.1.2*

Task A: How many triangles are there in Figure 2.1.9?

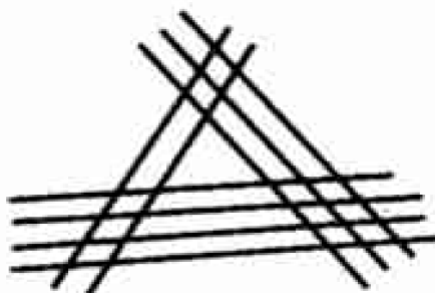


Figure 2.1.9

Task A': How many ways does the director choose 1 mother, 1 father and 1 child from 2 actresses, 3 actors and 4 children? The similarity between these two questions is obvious.

These two tasks seem to be unrelated to each other, but after analysis, it will be found that the two tasks are based on the same mathematical model, and the answer is both  $2 \times 3 \times 4$ . Although some problems in mathematics are presented in different forms/representations, they have the same mathematical model. This requires us to be good at grasping the essence of the problem through phenomena in the process of mathematics teaching and learning. transforming unfamiliar, complex, and unknown problems into familiar, simple, and known problems to solve. This example shows the unity of mathematics embedded in the multiple representations and it is important for teachers to inspire students to reflect and make connections among different mathematical knowledge so as to cope with ever-changing problems. Variation is an effective tool to identify the essence of knowledge through varying mathematical conditions, representations, or solutions, etc.

When we discuss mathematics, we study its history, its present state, and forecast its

future. In recent years, the development of technology has also significantly impacted the outcomes of mathematics. As mathematical education must consider changes that are happening in the mathematical requirements of the workplace, classroom teaching and mathematics design are facing new challenges. How to deal with a large number of resources? How to choose resources? How to use resources to maximize efficiency in teaching? We hope that variation can give us some inspiration.

Unity and diversity are the essence of mathematics. This evidence the rationality of studying variations in different countries with different cultures. “Nine chapters on the mathematical art” and “Elements” proves that variation is widely used in mathematics teaching, whether it is in the East or the West, and its source can be traced back to the teaching in the period of BC. In this study, unity refers to the unity of mathematical methods and mathematical thoughts. As for the diversity we mainly focus on the diversity of the context of mathematical problems and the conversions between multiple representations.

Variation is a powerful weapon. On one hand, variation can be an effective tool to help us highlight these similarities, just as British mathematician Atiyah (Michael Francis Atiyah, 1929-2019) remarked that the inherent similarity of mathematics should be highlighted: *“The aspect of mathematics which fascinates me most is the rich interaction between its different branches, the unexpected links, the surprises...”*. *“...Finding analogies between different phenomena and developing techniques to exploit these analogies is the basic mathematical approach to the physical world. It is therefore hardly surprising that it should also figure prominently internally within mathematics itself...”* (Atiyah, 1978). On the other hand, variation reflects the simplicity (fundamental problems), diversity (context, multiple representations, etc.) and unity (mathematical methods, ideas) of mathematics. As Atiyah (1978) also pointed out *“Both unity and simplicity are essential, since the aim of mathematics is to explain as much as possible in simple basic terms. Mathematics is still after all a human activity, not a computer programme, and if our accumulated experience is to be passed on from generation to generation we must continually strive to simplify and unify.”*

Unity and diversity are the nature of mathematics, which brings the variation in mathematics education. The unity of mathematics ensures the unchangeable and essential part in the variation and the diversity of mathematics (contexts, representations, etc.) provides the possibility of change in the variation.

## **2.2 Theoretical foundations of variation in mathematics teaching**

Variation is an important topic when we study task design, as the majority of teachers teach consciously or unconsciously through variation problems. Besides, variation

appears in two traditions of research, the Chinese and the French one. In China, Gu (Gu et al., 2004) situated BIANSHI jiaoxue (teaching through variation) as an effective way in mathematics teaching, distinguishing between conceptual variation and procedural variation. In France, Brousseau (2002) situated the didactical variables as sensitive elements of a problem whose modifications (even slight ones) can significantly influence students' behavior. Here, we illustrate these two different theoretical foundations of variation, the fruitful interactions between them, as well as the other theories which relate with them.

## 2.2.1 Variation in Chinese theories: Gu's "Bianshi" theory

In this section, we present Gu's "Bianshi" theory, from its origin to the proposition of two kinds of variations, and to the three components in mathematical problems, which can be used to vary tasks (section 2.2.1.1); Then, we focus on Marton's variation theory of learning, which rooted in western culture but has many similarities with Gu's theory (section 2.2.1.2).

### 2.2.1.1 Gu's "Bianshi" theory: Origin, 2 kinds of variations, 3 components

Teaching and learning through variation have been practiced in China for a long time (Gu et al. 2017). As a traditional and typical teaching method of Chinese mathematics, it has not only extensive experience foundation but also has been tested by practice in China.

From 1977 to 1992, Gu (1997a, b) led a team to carry out a large-scale education reform experiment in Qingpu County<sup>35</sup>, which increased the math qualification rate of middle school students from 16% to more than 85%. The National Education Commission set this as the major achievements of the basic education reform and promoted it throughout the country. Based on the systematic and in-depth Qingpu experiment over fifteen years, Gu summarized a set of effective teaching strategies from practice in Qingpu experiment from 1987 to 1997, one of which is "BIANSHI jiaoxue" (teaching through variation)<sup>36</sup>.

"BIANSHI jiaoxue" mainly involves two aspects of work (see Figure 2.2.1): one is to systematically restore and arrange "*conceptual variation*" in traditional teaching; the other is to extend "conceptual variation" to "*procedural variation*", so as to make teaching through variation applies to both the mastery of mathematical concepts and the growth of mathematical activity experience (Bao et al., 2003a, b, c).

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<sup>35</sup> Qingpu County is a county in Shanghai, China. The county covers an area of more than 600 square kilometers and has a population of over 400,000.

<sup>36</sup> 变式教学 in Chinese, where Bian stands for "changing" and shi means "form", can be translated loosely as "teaching through variation" or "teaching with variation" in English. In our research, we use the translation "teaching through variation".

According to Gu et al. (2017), *conceptual variation* refers to the strategies that are used to discern essential features of a concept and to experience connotation of the concept by exploring varying embodiments of the concept (i.e., instances, contexts). Conceptual variation consists of *concept variation* (varying connotation of concept) and *non-concept variation* (varying instances which confuse the connotation of concept, such as illustrating by counterexamples). There are two important aspects of mathematical concepts: *intension* and *extension*. The intension of a concept indicates the essential attributes of the object class reflected by the concept. The extension of a concept indicates the totality of objects in the object class reflected by the concept. The intension defines the extension, and the extension shares the intension. For example, the continuous function defined on the interval  $[a, b]$  is an abstract concept. Its intension is the definition of continuous function given in  $\varepsilon$ - $\delta$  language, and its extension is the whole concrete continuous function, such as trigonometric, polynomial, exponential function, and so on. Therefore, concept variation belongs to the extension set of the concept, while non-concept variation does not belong to the extension set of the concept, but has some non-essential attributes in common with the conceptual object, which needs us to discern. Furthermore, based on the intension and extension aspect of concept, Bao et al. (2003a, b, c) divided the concept variation into *standard variation* and *non-standard variation* according to its function in the teaching.

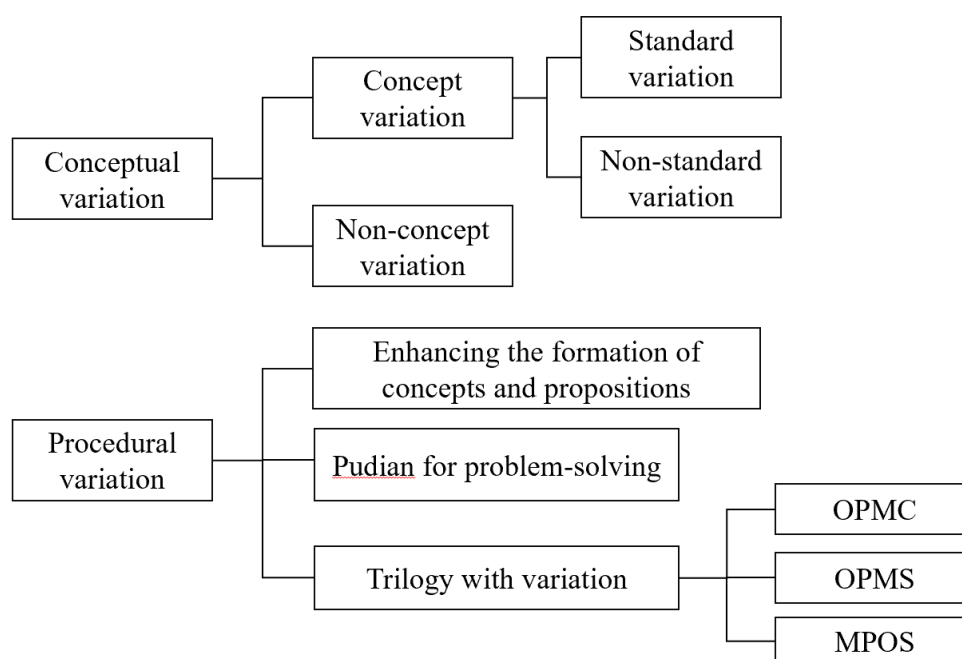


Figure 2.2.1 The structure of BIANSHI jiaoxue (Bao et al., 2003a, b, c)

To make it clearer and easier to understand, we propose an example (example 2.2.1) about the standard, non-standard, and non-concept variation in the teaching of corresponding angles to illustrate. As shown in Figure 2.2.2, 2.2.3, and 2.2.4, two

straight lines intersected by a third line, the pair of angles (Label as  $\angle 1$  and  $\angle 2$ ) are called “corresponding angles”. Figure 2.2.2 is regarded as the standard variation and used to introduce the “informal definition” of corresponding angle, as its form is very “mathematical standard”. Although the standard variation is conducive to students’ accurate grasp of the concept, it is also easy to limit the students’ thinking, thereby artificially narrowing the extension of the concept. One of the ways to solve this problem is to make full use of non-standard variations, by changing the non-essential attributes of the concept to highlight its essential attributes (Bao et al., 2003a, b, c), see Figure 2.2.3. Furthermore, as mathematical concepts are usually not isolated, but exist in a conceptual system composed of multiple concepts, it is necessary to clarify the extension of a concept and draw a clear line between the concept and its surrounding concepts. This is the necessity of using non-concept variation. In this case, we propose some non-concept variations in Figure 2.2.4. There are two main origins of non-concept variation: the logical relationship between concepts and students’ mistakes.

*Example 2.2.1*

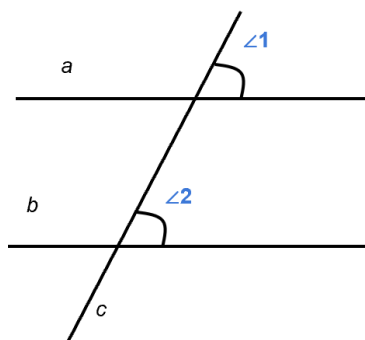


Figure 2.2.2 Standard variation of corresponding angle

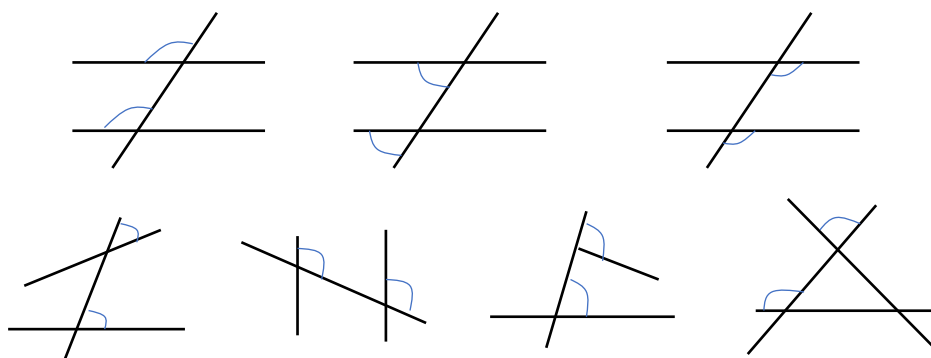


Figure 2.2.3 Non-standard variation of corresponding angle



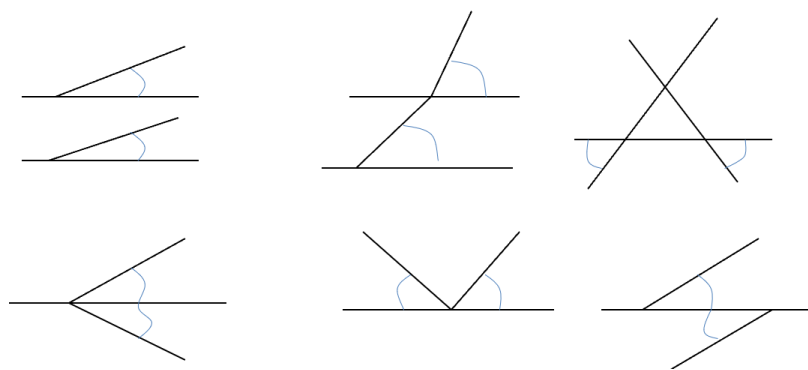


Figure 2.2.4 Non-concept variation of corresponding angle

In summary, conceptual variation aims at providing students with multiple perspectives of experiencing concepts. The functions of it in the practical teaching are as follows: (1) inducing concepts by varying visual and concrete instances; (2) highlighting the essence of concept by contrasting non-standard and standard variation, effectively using non-standard variation can highlight the essential features of the concepts through varying unessential features; (3) clarify the extension of the concept through non-concept variation. (Bao et al., 2003a, b, c). We propose Figure 2.2.5 here to represent the relations among standard, non-standard, concept, and non-concept variation.

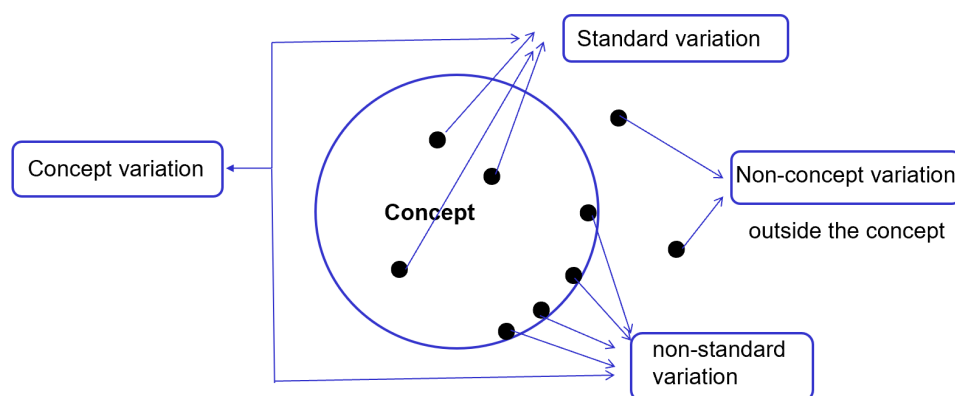


Figure 2.2.5 Relationship among different conceptual variations

*Procedural variation* refers to creating variation problems or situations for students to explore in order to find solutions to problems or develop connections among different concepts step by step or from multiple approaches (Gu et al., 2017). Procedural variation contributes to the progression of mathematics teaching so that students can solve problems step by step and accumulate a variety of activity experiences in the process of mathematical activities. Specifically, procedural variation has three main functions in teaching (Bao et al., 2003a, b, c):

- Enhancing the formation of mathematical concepts and propositions. The essential difference between procedural variation and conceptual variation is that the former aims at providing a process for formation of concept and propositions stage by stage,

while the latter serves as an understanding of concept from multiple perspectives. We take an example (example 2.2.2) of procedural variation that Chinese teacher used in the teaching of “equation” here.

*Example 2.2.2*

Xiao Ming used 2 yuan to buy 3 erasers, and he got 2 dimes back.

Question: How much is each eraser? (10 dimes=1yuan)

We present the teaching process of example 2.2.2 in Figure 2.2.6. In this process, there are two procedural variations to promote students’ learning: Using concrete things to represent the unknowns in the first step and symbolizing the unknowns in the second and third step. Through this process, students experience three stages of development of the algebraic symbol system: iconic algebra, simplified algebra, and symbolic algebra and have a concrete model of the “equation” concept (Bao et al., 2003a, b, c; Gu et al., 2004).

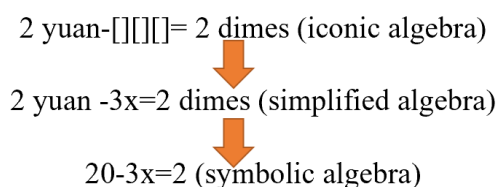


Figure 2.2.6 Process of teaching the concept of “equation”

- “Pudian”<sup>37</sup> for problem solving: When there is no clear relationship between unsolved (complicated) problems and solved (simple) problems, procedural variation can be set as the stages of transferring problems (see Figure 2.2.7).

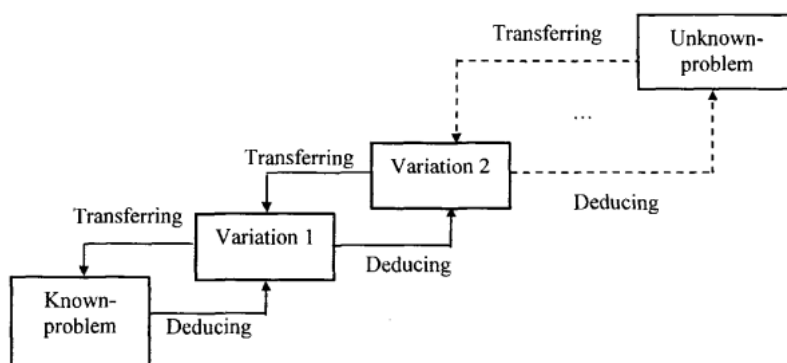


Figure 2.2.7 variation for solving problems

In the 14<sup>th</sup> International Congress on Mathematical Education (ICME-14), Gu (2021) further clarified the two-process shown in Figure 2.2.7. He explained “Teachers deliberately design Pudian for mathematical problems so as to develop the problems from easy to difficult, or reduction from complex to simple, guide

<sup>37</sup> Pudian 铺垫 in Chinese, means “pave certain ways”.

students to solve problems by themselves and avoid mechanical practice”. He concluded through the experimental data: “When students apply their knowledge to solve new problem, it is important for them to find the most essential and transferrable element, which is called ‘*the core connection*’”. According to Gu (2021), this connection could: 1) shorten the cognitive distance between the new problem and the anchoring knowledge; 2) significantly improve the degree of transfer in learning process; 3) simulate students’ constructive thinking in solving mathematical problems.

Bao et al., (2003a, b, c) proposed an example (example 2.2.3) to show the role of procedural variations in problem-solving teaching.

*Example 2.2.3*

As shown in Figure 2.2.8, ABCD is a trapezoid and  $BC \parallel AD$ . If the areas of  $\triangle AOD$  and  $\triangle BOC$  are  $S_1$  and  $S_2$ , prove that the area of the trapezium ABCD

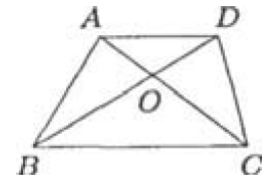


Figure 2.2.8

$$S = S_1 + 2\sqrt{S_1 S_2} + S_2$$

To solve this problem, we must find the relationship between the area of the trapezoid and the area of the two known triangles; since the trapezoid can be divided into four triangles, then the problem is transformed into finding the relationship between the area of  $\triangle AOB$ ,  $\triangle COD$  and the area of the known triangle  $S_1$  and  $S_2$ . In this way, the key to solving the problem is to find a variation problem about the known and unknown area of the triangle. Varying the construction of the figure can pave the way for solving the original problem, see Figure 2.2.9<sup>38</sup> (Bao et al., 2003 a, b, c).

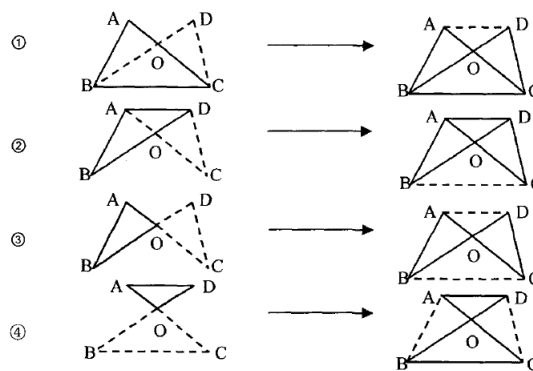


Figure 2.2.9 Multiple analysis of the configuration of the same diagram (Gu, 1981)

<sup>38</sup> Take Figure 2.2.9① as an example, it is easy to find that the area of  $\triangle AOB$  is equal to the area of  $\triangle COD$ . Let  $S^*$  = area of  $\triangle AOB$ , then  $S^*/S_2 = AO/OC$  and  $S^*/S_1 = CO/AO$ . Thus,  $(S^*)^2/S_2 S_1 = 1$ , namely,  $(S^*)^2 = S_2 S_1$ . Considering  $S = S_1 + S_2 + 2S^*$ , the result will be proved immediately (Gu et al., 2004).

- Establishing a system of mathematics experiences (Trilogy with variation): The variation for which are derived from three dimensions of problem solving, “one problem multiple changes” (OPMC, 一题多变, varying conditions and conclusions), “one problem multiple solutions” (OPMS, 一题多解, varying solutions), “multiple problems one solution” (MPOS, 多题一解, varying presentations) (Bao et al., 2003a, b, c).

Compared these two kinds of variations, conceptual variation is static, emphasizing the comparison of different objects, and the essence of concept and its extensions are highlighted and classified by comparing the concept objects and non-concept objects. Procedural variation is dynamic, emphasizing the connections among different processes, the previous knowledge and the new knowledge are connected by providing proper scaffolding based on deliberately separating the mathematics activities into parts progressively (Gu et al., 2004). When we implement variation teaching strategies, conceptual variation strategy is meant the aim of constituting a space of variation which focuses on the critical aspects of the objects of learning. Through procedural variation strategy, students are helped in the establishing of the substantive connections between the new object of learning and the previous knowledge by providing certain stages (Pudian) and further improving their development within ZPD (Zone of proximal development)<sup>39</sup>. These two variation strategies play different roles in teaching and they are complementary and promote each other. Therefore, effective teaching must organize procedural variation with appropriate potential distances and conceptual variation with suitable variation space.

Besides the two kinds of variations above, Gu (1997a, b) had also divided the mathematical problem into three basic components: the initial state A—the condition of the problem; the process of solution B—the application of certain knowledge and experience to change the condition of the problem, and transiting to the conclusion; the final state C—the conclusion of the problem. If the conditions and conclusions of a problem are clear, and the problem-solving process is familiar to the students, then this kind of problem can be called *standard problem*. Making some transformations and changes to the standard problems so that one or two of the three basic components are missing. The students do not know or are not clear about these components. The questions obtained in this way can be called *closed or open variation problems*. These three types of problems have a progressive relationship, as shown in the following Table 2.2.1 (X, Y, Z in the table are unknown components corresponding to A, B, and C). A problem belongs to which type of problem depends on the actual teaching and the

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<sup>39</sup> ZPD: the distance between the actual developmental level as determined by independent problem solving and the level of potential development as determined through problem solving under adult guidance, or in collaboration with more capable peers (Vygotsky, 1978, p. 86)

students' level.

Table 2.2.1. Type of variation problems

Standard problem	Closed problem	Open problem
ABC	ABZ	AYZ
	AYC	XBZ
	XBC	XYC

### 2.2.1.2 Marton's variation theory and its relation with Gu's theory

In the past two decades of cross-cultural research on mathematics education, a key element that has gained popularity is the use of variation and invariance in the teaching and learning of mathematics (e.g., Häggström, 2008; Sun, 2011; Watson & Mason, 2005; Wong et al., 2009). As we have discussed above, Chinese teachers have been found to systematically juxtapose examples, tasks, and problems that differ in important respects to help students develop a deep understanding of the mathematical content to be learned (Gu et al., 2004). Besides *Bianshi Jiaoxue* (teaching through variation; e.g., Bao, et. al, 2003 a, b, c; Gu, 1991; Gu, et al., 2004) developed by Gu's group, the other group of researchers, in Sweden and Hong Kong led by Ference Marton, arrived at a similar insight on the use of variation and invariance in their analysis of mathematics classroom practices, which is known as *the variation theory of learning* (Marton & Booth, 1997; Marton & Tsui, 2004; Marton & Pang, 2006; Marton, 2015; Pang & Ki, 2016).

Variation theory is concerned with how to enable learners to handle novel situations in powerful ways (Marton & Pang, 2006). Above all, learners need to develop the capability to discern which aspects or features must be taken into account when deciding how to achieve the goal in a novel situation. Discerning those critical aspects or features involves noticing the ways in which one specific situation (a problem, task, or instance of the phenomenon) differs from others. For example, a child's ability to solve simple arithmetic word problems is a function of the ability to discern the parts and wholes and the relations between them in the specific problem situation. In this respect, variation theory differs from other learning theories because it focuses on the learning affordance of differences (Pang et al., 2017).

Similar to Bianshi theory, variation theory also highlights the important role of invariance (similarity) in affording the discernment of differences. According to Marton and Tsui (2004), to discern and focus upon the critical aspects (or dimensions of variation) of a phenomenon, the learner must experience variation in these aspects against a background of invariance. They suggest the use of four patterns of variation and invariance, '*contrast*', '*separation*', '*fusion*', and '*generalization*', for organizing the learning instances (Pang et al., 2017). As explained in Pang and Ki (2016), when one critical aspect varies while the other aspects are kept invariant, the pattern is called

'*separation*.' The aspect to be varied is perceptually separated from the other aspects of the phenomenon and becomes salient to the learner. In contrast, when two or more critical aspects of the phenomenon vary simultaneously and are thus brought to the learner's awareness at the same time, this pattern is called '*fusion*'. These two patterns help the learner to discern and attend to the connections between or the joint and/or independent working of the changes in multiple critical aspects. The other two patterns of variation and invariance are more concerned with noticing the sameness and differences across different instances and non-instances of the phenomenon. For example, to grasp the concept of a square, one must '*contrast*' it against something that is not a square. If every shape one has ever experienced is a square, then 'square' will be synonymous with shape and will carry no unique meaning of its own, not to mention the discernment and attention to its necessary aspects. Furthermore, to '*generalize*' the idea of a square, one must see sameness, such as the same set of features (necessary values in its aspects and relations between them), across the various squares that one has experienced (Pang et al., 2017). Out of these four patterns, Marton (2015) further proposed a new logical structure in which only three patterns of variation and invariance exist as the basic ones seen by the learners (i.e., contrast, generalization, and fusion). According to Marton (2015), "Both contrast and generalization separate aspects from what they are aspects of and from each other. Through contrast, we are trying to find necessary aspects of the objects of learning, those that define it. Through generalization, we want to separate the optional aspects from the necessary aspects" (p. 51). Here the 'necessary aspects' refer to the aspects that are necessary for the targeted learning of a particular way of seeing the phenomenon (Pang et al., 2017).

According to variation theory, variation is a necessary condition for effective discrimination, and discrimination is a necessary condition for learning. Without variation, many concepts in the world are meaningless or non-existent. Discrimination, simultaneity, and variability are the three basic concepts that describe learning. Therefore, during teaching, it is vital that the teacher and students can interact to establish various dimensions of variation focused on the key aspects of learning objects, and then build a *space of variation* allows students to experience. Marton (1999) believes that the most important condition for learning is to construct an appropriate space of variation in teaching for students to experience, that is to say, the space of variation (or called learning space) constitutes a necessary condition for learning.

Based on the discussions above, variation theory is intended for learning and teaching powerful ways of seeing in different disciplines (Pang et al., 2017), while Bianshi theory is intended to capture and make explicit good practices in mathematics teaching in China. The variation theory presumes that learning is about developing new ways of seeing or experiencing the object of learning, it is necessary to experience certain

patterns of variation and invariance in order to discern critical features of the object of learning. Teaching then means to construct appropriate spaces of variation (Marton & Booth, 1997; Huang & Li, 2017). If we compare the variation theory with conceptual variation in Bianshi theory, both of them emphasize revealing the essential attributes or key aspects of learning objects by providing appropriate variations. The variation theory clarifies that the experience of proper variation is a necessary prerequisite for learning from the perspective of epistemology. In this way, variation theory provides a theoretical basis for the conceptual variation which is rooted in experience. Furthermore, the space of variation constructed in teaching (composed of various conceptual variations focusing on the key characteristics of the learning objects) constitutes the necessary conditions for learning.

The previous discussions have attempted to discuss the two kinds of variation proposed by Gu, the internal relationship between the two variations, the implications for teaching and the way to category the variation problems, also Marton's variation theory which has a close relation with Bianshi theory. now let's move to the French side, to see French theories through the lens of variation.

### **2.2.2 Variation in French theories: Theory of Didactical Situations and didactical variables**

In this section, we introduce the theory of didactical situations, especially the notion of didactical variables, from Brousseau's original proposition to the evolution and development of this theory by other researchers.

The Theory of Didactical Situations (TDS) has been introduced by Guy Brousseau (1997) and a central characteristic of TDS research is its framing within a deep *a priori* analysis of the underlying mathematics of the topic to be learned, integrating the epistemology of the discipline, and supported by cognitive hypotheses related to the learning of the given topic (Kieran et al., 2015). The three fundamental notions (the a-didactical situation, the didactical situation and the milieu<sup>40</sup> of a situation) proposed by Brousseau seem like an ideal way to introduce a mathematical content in class. According to Ruthven et al. (2009), one of the central task design tools provided by TDS is the a-didactical situation, which mediates the development of students' mathematical knowledge through independent problem-solving. The term a-didactical within TDS refers specifically to that part of the activity "between the moment the student accepts the problem as if it were her own and the moment when she produces her answer, [a time when] the teacher refrains from interfering and suggesting the knowledge that she wants to see appear" (Brousseau, 1997, p. 30). The feedback of the

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<sup>40</sup> Within a situation of action, everything that acts on the student or that he or she acts on is called the "milieu". It can be that it includes neither the teacher nor another student (Brousseau, 2002).

milieu allows the development of the a-didactical situation: a student does not need a validation from the teacher, as the feedback from the milieu is enough to convince him/her that his/her method is wrong. And finally, the targeted knowledge is the necessary way for solving the problem (Trouche, 2016). Through this method, students' interest will be highly motivated and the class teaching effectiveness will be optimized. Like the classic puzzle game introduced by Brousseau (1981), teachers should design a set of exercises and games as an introduction to new mathematics content in class. Brousseau summarizes some MUST HAVE features for this kind of exercises: (1) Students can verify the answer by themselves; (2) The solution to the exercise or game is related to the mathematics knowledge and the objective of class; (3) It is better when the exercise is in the background of game; (4) Every student should be engaged in the problem-solving (Brousseau, 2002).

According to Brousseau (1999), for heuristic reasons, it is assumed that each mathematical knowledge has at least one situation that characterizes it and differentiates it from all others. Moreover, he conjectured that the set of situations that characterize the same notion is structured and that it can be generated from a small number of so-called *fundamental situations*, by operating the variants, variables and the bounds of these variables. Based on his hypothesis, taking advantage of the fundamental situation and the variation can greatly improve teachers' teaching efficiency and reduce students' learning burden at the same time, which also enables students to be better at knowledge transfer. Therefore, in addition to the a-didactical situation, TDS-based design is also informed of the didactical variables. Brousseau deepened the relations between the "situation" and the "variables" and gave a definition of didactical variables as follows: a variable of a situation or a problem is called a didactical variable, when it can be modified by the teacher, and whose modifications (even slight ones) can significantly influence students' behavior and provoke different procedures or types of responses (Brousseau, 1986). The notion of didactical variables reflects the necessity to distinguish, classify and model the situations from a didactical perspective and allows for choices regarding particular aspects of the main task and how it is to be carried out (Kieran et al., 2015). When the properties of a situation capable of justifying (or provoking) the implementation of a specific piece of knowledge are better known, it is possible to study the possibilities of the former for causing the latter to evolve. Didactical variables are those which influence learning and whose value the teacher can choose (Brousseau, 2002).

Brousseau also pointed out that "A field of problems can be generated from a situation by modifying the values of certain variables which, in turn, change the characteristics of solution strategies (cost, validity, complexity, etc.) [...]" "Only the modifications that affect the hierarchy of strategies are to be considered (relevant variables) and among



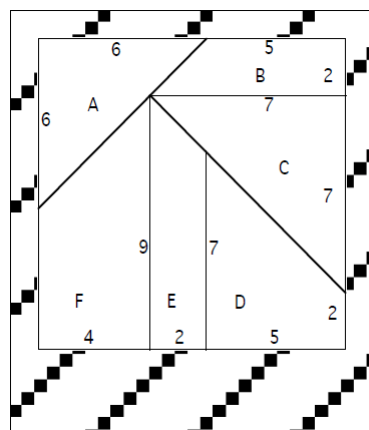
Here we present an example for better understanding of didactical variables (see example 2.2.4). The fundamental problem of this variation is the classic problem “Enlargement of the puzzle” (“agrandissement du puzzle”) proposed by G. and N. Brousseau (1987) and the analysis of didactical variables below is conducted by Bessot (2003).

### Puzzle situation

## Fundamental problem

- 6 to 8 puzzles drawn on rectangular boxes of 20 cm x 15 cm (see Figure 2.2.10). For each of the puzzles, the 6 pieces (A, B, C, D, E, F) cut from the same cardboard.

- 20 cm ruler per child.



Requirement: “Here are puzzles. You are going to make similar ones, larger than the models, respecting the following rule: the segment which measures 4 cm on the model should measure 7 cm on your reproduction.

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have finished, you should be able to reconstruct the same figures as with the model.

Process: The children are divided into teams of 4 or 5. After a brief consultation by team, they are separate to make their part (s). The teacher displays an enlarged representation of the complete puzzle on the board.

#### Didactical variables

##### **V1 = (n, p)**

n and p are the numbers which define the proportionality ratio.

The relevant values being: p multiple of n, or  $p = kn + n/2$  (k integer, and n even), or p/n repeating decimal. For example, here are pairs of integers illustrating each of the categories: pairs (4, 8), (4, 6), (4,7) (3, 7).

An enlargement from 4 to 8 involves multiplication by 2, which is actually the addition  $4 + 4$ . The student remains in the additive model and integers.

An enlargement from 4 to 6 is often analyzed as “x plus half of x”. We therefore have an intermediate model between the additive model and the linear model, more “close” to the additive model because of the particular status of “half”, especially since in this case, half is also an integer.

An enlargement from 4 to 7 does not allow a simple passage, even if it is possible to use divisions by 2: [the enlargement from 4 to 7 could be interpreted as “2 times x minus half of half of x”].

There is a jump when it is necessary to “go from 3 to 7”, because no relation of this type is possible. We are obliged to leave the additive and to leave the integers.

The values of V1 bring into play the direction of the multiplication by a rational, they allow the passage from the multiplication by an integer (additive model, reiterated addition) to the multiplication by a rational (multiplicative model, image by a linear map).

##### **V2 = organization of interactions between students**

The values of this variable refer to the description of different types of phases in time: individual work, cooperation, consultation, confrontation, etc. In the puzzle situation, it is described by: consultation, then individual work, then confrontation. The implementation of the last two phases allows real feedback of the strategy decided in the first phase. If we reduce the phases into one (cooperation to build the enlarged puzzle), we take away from the “milieu” its property of the “milieu” for validation and from the situation it’s a didactic character. From another point of view, for an audience of pupils “closer” to the multiplicative model (college, high school), the first phase (consultation) would risk disseminating the winning strategy from the start of the work and therefore blocking the results. learning at stake for the students who are the most concerned.

##### **V3 = puzzle configuration**

The values of this variable refer to the description of the respective subdivisions of the sides (by the same number or not), the measures of the pieces etc. The relevant values

of this variable are those which relate to the domain of validity of the basic strategies. Here, the measures are chosen in such a way that the “add 3” procedure (which is assumed to be the initial strategy for the great majority of students) leads to constructing non-nestable pieces in a perceptually obvious way.

V1 is on the side of knowledge to vary (multiplication by a repeating decimal, number as an image by a linear map). V2 is constitutive of the a-didactic nature of the situation. V3 is linked to the knowledge of the students, since it is based on the assumptions relating to the basic strategies of the latter. Other “variables” or constraints seem to play a role in the situation. For example, materials available to students, grid paper and a ruler. They are integrated into the situation to facilitate the construction of parts and to measure lengths. They are therefore linked to the measure. Whether we will give the status of didactical variable to this element of the situation depends on the teaching objectives of this course. In the interest of returning to the knowledge in play, that is to say the passage from the additive model to the linear model, the measure is not a question of knowing in this situation. These elements can intervene in the situation, but we make the assumption (which one can argue) that their presence or not does not change it fundamentally.

Brousseau’s Theory of Didactical Situations (TDS) and didactical variables have been developed over the years and has been contributed to by the French mathematical didactique community at large (Kieran et al., 2015)<sup>41</sup>. In summary, it is possible for us to establish a “fundamental” situation corresponding to the knowledge in question; to seek the didactical variables and the different conceptions that they engender—in particular the one which is assumed to be sufficient to explain the error; and then to use factor analysis or more classical statistical methods as a means of identifying groups of students who “separate” these conceptions. If there is no clear discrimination, there is no reason to consider these conceptions as distinct (Brousseau, 2002).

### **2.2.3 Related studies and comparison of Gu’s and Brousseau’s theories**

In this section, we first present several studies that relate with Gu’s and Brousseau’s theories or can lead us to further study the two theories (section 2.2.3.1). Then we make a comparison between Gu’s Bianshi theory and Brousseau’s theory of didactical variables (section 2.2.3.2).

#### **2.2.3.1 Other related studies**

Besides the theories above, the idea of variation is embedded in many researches in

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<sup>41</sup> Brousseau refers to some works in a special section of the bibliography of his article: Balacheff (1982), Berthelot and Berthelot (1983), Bessot and Richard (1979), Brousseau and Brousseau (1987), El Bouazzaoui (1982), Galvez (1985), Katambera (1986), Maudet (1982), Mopondi (1986), Perez (1984), Quevedo de Villegas (1986), Ratsimba-Rajohn (1981), Rouchier (1980).

mathematics education. For example, Dienes (1973) suggested four principles of mathematics learning: (1) Dynamic principle, (2) Constructivity principle, (3) Mathematical variability principle, and (4) Perceptual variability principle. In particular, he further explained the last two principles as follows: According to mathematical variability principle, concepts involving variables should be learnt by experiences involving the largest possible number of variables. To give the maximum amount of experience and structures to encourage the growth of the concept, it seems a priori desirable that all possible variables should be made to vary while keeping the concept intact. Obviously, this view could be related with Brousseau and Gu's ones.

Sun (2011) designed a way to analyze the variation in tasks. She divided the variation into three types: OPMS (one problem multiple methods of solution, varying solutions), OPMC (one problem multiple changes: one problem with multiple representations and one problem with different parameters, varying conditions and conclusions), and MPOS (multiple problems one solution, varying presentations). She situated variation problems as an “indigenous”<sup>42</sup> Chinese practice and carried out a comparison between Chinese and American textbooks. She then focused on the manner in which fraction division is articulated in an important Chinese textbook, clarified the “indigenous roles” (OPMS and OPMC) variation problems play in Chinese textbooks. OPMS, OPMC and MPOS proposed by Sun are exactly match the “trilogy with variations” in the procedural variation proposed by Gu (Gu, et al., 2004, & Bao, et al., 2003 a, b, c), and is a further refinement of the procedural variation.

Other researchers (Silver et al., 1996; Singer et al., 2013) have put forward various ways to generate new problems through the following strategies: (1) Constraint manipulation (i.e., systematic manipulation of the task conditions or implicit assumptions); (2) Goal manipulation (i.e., manipulation of the goal of a given or previously posed problem where the assumptions of the problem are accepted with no change); (3) Symmetry (i.e., a symmetric exchange between the existing problem's goal and conditions); (4) Chaining (i.e., expanding an existing problem so that a solution to the new problem would require solving the existing one first).

OPMC is a variation of the structure of tasks, by changing the conditions or objectives of the tasks, so that the essence of the task can be revealed from different angles and aspects. The four strategies above can be regarded as a further improvement of OPMC in the dimensions of variation proposed by Sun (2011). Therefore, we evidence the “trilogy with variation” in Gu's theory from Sun's work and further develop it by

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<sup>42</sup> “Indigenous” variation practice is mainly applied to mathematics subjects only, which is different from the variation theory applied to all other subjects. This practice refers to the “routine” daily practice commonly accepted by Chinese teachers, the local experience used broadly in example or exercise design to extend the original examples (Sun, 2011).

refining OPMC into four specific strategies: constraint manipulation, goal manipulation, symmetry, and chaining.

In addition, the changes of registers proposed by Duval (see section 2.1.2) is accordance with multi-representation and different variation in the concept understanding which Gu (1981) has proposed (see Figure 2.2.11, 2.2.12). Gu showed the relationship among concept, sample and character and indicated the position of multiple representation in conceptual variation by means of mathematical set, through two directions: one is from conceptual variation, concept understanding, character-based learning to multiple representations; the other is from conceptual variation, concept understanding, sample-based learning, different variation, to conceptual variation. Gu broadly proposed the direct relationship between multiple representations and variation, while Duval's research (the two types of transformation of semiotic representations *treatment* and *conversion*, also the two-dimensional table, see Figure 2.1.1) is much more detailed and provides us with a theoretical framework for further study on multiple representations. With Duval's work, we can understand students' difficulties so that apply or analyze the variation appropriately from the perspective of mathematical representations.

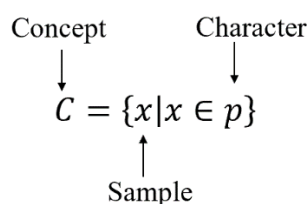


Figure 2.2.11 The relationship among concept, sample and character (Bao et al., 2003a, b, c)

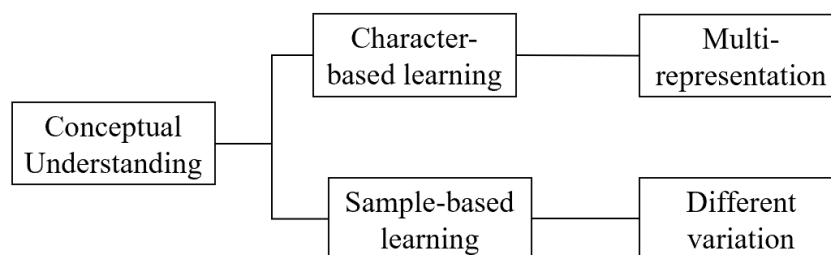


Figure 2.2.12 Conceptual variation ((Bao et al., 2003a, b, c)

### 2.2.3.2 Conclusion: a comparison between Gu's Bianshi theory and Brousseau's theory of didactical variables

We make a comparison between Gu's Bianshi theory and Brousseau's theory of didactical variables. Both theories aim to improve the teaching effect by making changes in teaching. However, Bianshi is proposed on the basis of task design, and it is mainly aimed at the changes to mathematics tasks. Didactical variables are proposed on the basis of task design and didactics, which means in addition to the changes of the tasks, it also includes changes in teacher behavior, classroom organization, teaching

tools, etc. To make it clear, we present the relationship between Gu's and Brousseau's theories in Figure 2.2.13.

As shown in the Figure 2.2.13, there is an intersection between OPMC and Brousseau's theory of didactical variables. Varying the tasks that can influence students' behaviors is the one that in both considerations (the intersection part in the figure). This kind of variation aims to promote the teaching by designing variations to provoke different procedures or types of responses of tasks. To be specific, OPMC, OPMS, Pudian, non-standard variation, non-concept variation, and a-didactical situation can be involved here. In fact, there is a close link between "non-concept variation" in Gu's theory and "a-didactical situation" in Brousseau's theory. Non-concept variation can be regarded as one way to design a-didactical situation by generating the cognitive conflict of students.

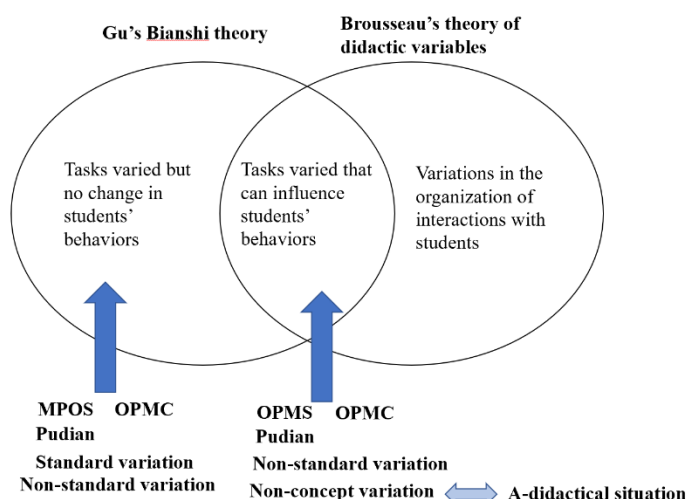


Figure 2.2.13 The relationship between Gu's and Brousseau's theories

The differences between two theories are: 1) Gu's theory includes the variation that not aims to motivate students to change their behaviors, but to reinforce what has been learned by students. For this purpose, OPMC is often accompanied by MPOS to help students master one strategy through variation. To be specific, OPMC, MPOS, Pudian, standard variation, and non-standard variation can be involved; 2) Brousseau's theory includes the variation relates to organization of interaction with students.

In general, from the overall teaching point of view, the meaning of Brousseau's theory is larger than that of Gu's theory, which includes the variation of both teaching contents (tasks as the carrier of contents) and didactical organizations; while from the specific task point of view, Gu's theory is more comprehensive to consider the variation no matter it provokes the changes of students' behaviors or not. Brousseau's theory of didactical variables is more oriented to the design for students' exploration when learning new knowledge, while Gu's Bianshi theory includes both exploration and

consolidation of knowledge.

## **2.3 Teachers' documentation work and its relation with variation**

The massive growth of resources, as well as the Internet potential, offer teachers new possibilities for selectively choosing tasks which address precisely their teaching goals rather than rely on the authority of an unknown author from the web (Watson & Thompson, 2015). Therefore, the role of teachers' documentation work (Gueudet & Trouche, 2009) is becoming more and more important.

In this section, we first introduce the theoretical structure of the documentational approach to didactics to have a general understanding of this approach (section 2.3.1). We then try to have a deeper understanding of documentational approach didactics and its relation with variation (section 2.3.2).

### **2.3.1. Theoretical structure of the documentational approach to didactics**

Gueudet and Trouche (2009) have developed a documentational approach to didactics (DAD), focusing on the documentation work of the teacher outside the class (even if this documentation work goes on in class). The theoretical elements developed in this approach are based on the instrumental approach to mathematics (Guin et al., 2005) and the work of Rabardel (1995) grounding it.

In this section, we first (section 2.3.1.1) introduce in particular the dialectical relationship between resources and documents. Three intertwined components must be considered for the analysis of a resource, or a document: the material component, the mathematical component, and the didactical component. We then (section 2.3.1.2) describe the notion of documentational genesis. Documents are developed throughout these documentational geneses.

#### **2.3.1.1 The distinction between resources and documents**

A crucial distinction in the documentational approach was introduced, that between a *resource* and a *document* (Gueudet & Trouche, 2009). This distinction extended the distinction introduced by Rabardel (1995) between *artifact* and *instrument*.

Rabardel distinguishes between the artifact and the instrument. According to Rabardel, an artifact is a cultural and social means provided by human activity, offered to mediate another human activity. The instrument is built from the *artifact* by a subject through his/her activity. An instrument results from a process, named *instrumental genesis*, through which the subject builds a *scheme* of utilization of the artifact, for a given class of situations. A scheme, as Vergnaud (1998) defined it from Piaget, is an invariant organization of activity for a given class of situations. It comprises goals and subgoals, anticipations, rules of action, of gathering information and exercising control, and

possibilities of inferences. It is structured by operational invariants, which consist of implicit knowledge built through various contexts of utilization of the artifact. The distinction can be represented by the formula:

$$\text{Instrument} = \text{Artifact} + \text{Scheme of Utilization}$$

Instrumental geneses are bidirectional processes: the affordances of the resource/s influence teachers' practice (the *instrumentation* process), and the teachers' dispositions and knowledge guide the choices and transformation processes between different resources (the *instrumentalization* process) (Trouche, et al., 2020).

In the documentational approach to didactics, Gueudet and Trouche (2009) use the term *resources* to emphasize the variety of artifacts: a textbook, a piece of software, a student's sheet, a discussion with a colleague, etc. A resource is never isolated and it belongs to a set of resources. Pepin and Gueudet (2014) defined mathematics curriculum resources as all the resources (e.g., digital interactive, nondigital/traditional text) that are developed and used by teachers and pupils in their interaction with mathematics in/for teaching and learning, inside and outside the classroom. This also includes digital curriculum resources, and Pepin et al (2017a) make a distinction between digital curriculum resources and educational technology. They also distinguish between material curriculum resources (e.g., textbooks, digital curriculum resources, manipulatives, and calculators), social resources (e.g., a conversation on the web/forum), and cognitive resources (e.g., frameworks used to work with teachers). It has been mostly applied to teachers' work, but can also be used to study the work of teacher educators (e.g. Psycharis & Kalogeria, 2018), or students' interaction with resources (e.g. Kock & Pepin, 2018).

The subjects here are teachers. A teacher draws on resource sets for her documentation work. The *document* is built from the resources by a teacher through a process of genesis. In the DAD, the schemes that are considered are schemes of usage of a given resource (or set of resources). The teacher builds schemes of utilization of a set of resources for the same class of situations across a variety of contexts. Similar to the formula that illustrates the relationship between instrument and artifact, the distinction between resource and document can be represented as follows:

$$\text{Document} = \text{Resources} + \text{Scheme of Utilization}$$

During the interaction with a particular resource, or sets of resources, teachers develop their particular schemes of usage of these resources. The concept of scheme (Vergnaud, 1998) is central in the DAD and is closely linked with the concept of class of situations. For a given class of situations, a subject (here a teacher) develops a stable organization of his/her activity, that is a scheme.



### 2.3.1.2 Emphasizing documentational genesis as an ongoing process

As we have mentioned above, the process of documentational genesis works both ways. The instrumentalization dimension conceptualizes the appropriation and reshaping processes, well known by researchers studying the design and diffusion of instructional sequences: “an instructional sequence developed by one group is necessarily reshaped and transformed while others use it” (Cobb et al. 2008, p. 117). The instrumentation dimension conceptualizes the influence on the teacher’s activity of the resources she draws on. Figure 2.3.1 represents the process of documentational genesis.

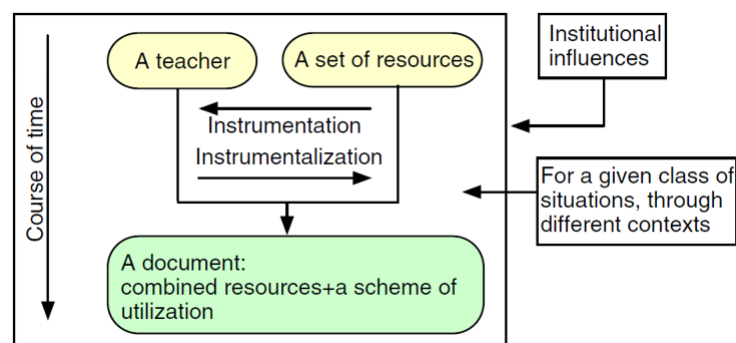


Figure 2.3.1 Schematic representation of a documentational genesis (Gueudet & Trouche, 2009)

It is essential to point out that a documentational genesis must not be considered as a transformation with a set of resources as input, and a document as output. It is an ongoing process. As Rabardel and Bourmaud (2003) claim that the design continues in usage, Gueudet and Trouche (2009) consider accordingly that a document developed from a set of resources provides new resources, which can be involved in a new set of resources, which will lead to a new document etc. Because of this process, the relationship between resources and documents is called as *dialectical* relationship, which is represented by a helix, wrapped around an axis representing time (see Figure 2.3.2).

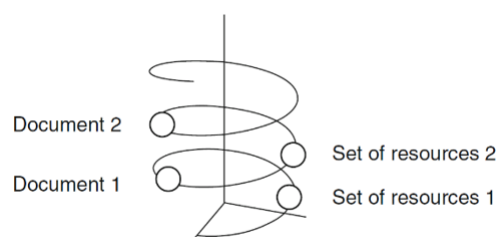


Figure 2.3.2 The resource/document dialectical relationship (Gueudet & Trouche, 2009)

Now we look at the scheme of utilization and usages more precisely. A scheme of utilization of a set of resources entails both an *observable part* and *invisible aspects*. The observable part corresponds to the regularities in the teacher’s action for the same class of situations through different contexts, which is called *usages*. The invisible

aspects are the *operational invariants*, the cognitive structure guiding the action. We distinguish between utilizations: when a teacher draws on a resource one, or a few times, but without developing a stable behavior for a given class of situations; and usages, which correspond to a stable organization of activity, and are part of a scheme. Thus, we can render our formula more precise as follows:

$$\text{Document} = \text{Resources} + \text{Usages} + \text{Operational Invariants}$$

### **2.3.2 Documentational approach to didactics and its relation with variation**

In this section, we study variation based on the documentational approach to didactics, trying to find the link between these two. We first (section 2.3.2.1) reconsider the process of instrumentation and instrumentalization for favoring variation. We then (section 2.3.2.2) focus on the teachers' design, DAD and teachers' professional development, as these three aspects are inextricably linked.

#### **2.3.2.1 Rethinking the concept of instrumentalization and instrumentation and their relation with variation**

We reconsider the concept of instrumentalization and instrumentation on the teachers' side, hoping to have a deeper understanding of the interaction between teacher and resources and discuss its relation with variation.

As mentioned earlier, Gueudet and Trouche (2009) extended the frame of the instrumental approach to a documentational approach to didactics, replacing the notion of "artifact" by the notion of "resources", denoting the wide set of "things" that are supporting teacher activity (Trouche, 2020); replacing the notion of "instrument" by the notion of "document", denoting what is developed by a teacher to do and in doing his or her teaching (Trouche, 2014). The documentational approach maintains two main concepts introduced by Verillon and Rabardel (1995): instrumentation and instrumentalization.

For performing a teaching task, a teacher interacts with a set of resources. This interaction combines two interrelated processes: first, the process of instrumentation, where the selected resources support and influence the teacher's activity, and which can be linked to Brown's (2009) characteristics of curriculum materials, i.e., that they represent an interface between the knowledge, goals, and values of the author and the user. Second, there is the process of instrumentalization, where the teacher adapts the resources for his/her needs, which can also be linked to Brown's (2009) characteristics, i.e., that curriculum materials require craft in their use; they are inert objects that come alive only through interpretation and use by a practitioner (Pepin et al. 2017b). instrumentation and instrumentalization are two intrinsically intertwined processes constituting each instrumental genesis, leading a subject to develop, from a given

artifact, an instrument for performing a particular task; the instrumentation process is the tracer of the artifact on the subject's activity, while the instrumentalization process is the tracer of the subjects' activity on the artifact (Trouche, 2014).

It is not hard to see that instrumentation and instrumentalization can be linked to the variation. As the instrumentalization process means teachers' dispositions and knowledge guide the choices and transformation processes between different resources (Trouche et al., 2020a), which can be regarded that the teacher changing the resources he/she uses (varying teacher's resources). In the process of instrumentalization, there are mainly two ways of teachers' work: personalization (teacher uses resources fit to his/her needs) and transformation (teacher redesigns resources to fit to his/her needs). In line with variation, the teacher can add, remove and even create the variations to resources (mainly mathematical tasks in our research) through different ways of instrumentalization, especially the transformation of the resources. While the instrumentation process means the affordances of the resource/s influence teachers' practice (Trouche et al., 2020a), which can be regarded that the teacher is influenced by the resources that he/she are using (varying teacher's knowledge). In the process of instrumentation, not only textbooks, different software (dedicated, or not, to mathematics), various repertoires of mathematical problems and orchestrations<sup>43</sup>, but also students' reactions, colleagues' comments, are instrumenting a mathematics teacher activity (Trouche, 2014). Moreover, in the thread of digitalization, there are dramatic changes in teachers' interactions through using emails, websites, forum, blogs, etc (Gueudet & Trouche, 2009). This process implies a huge potential for varying teachers' work, through resources, students' feedbacks, others' comments, etc. For example, Soury-Lavergne and Maschietto (2015) evidence the opportunities given to teachers for articulating spatial and geometrical knowledge in using digital resources at primary school. Mercat et al. (2016) show the potential of the digital environments for stimulating teachers' creativity.

As we have emphasized that the documentational genesis is an ongoing process, we try to make this cyclic process explicit. In practical teaching, students are free to invent things when the teacher proposes a task, and afterwards the teacher could benefit from the richness of all these ideas, and can build on it (Gueudet & Trouche, 2009). Meanwhile, the activities and interactions among teachers, the sharing of resources can also promote the teacher to reflect on his/her own teaching. From the perspective of variation, there is no doubt that the new variation has produced in the process of teaching and interactions between teachers and students or between teachers and their

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<sup>43</sup> An orchestration appears thus as a musical score, pinpointing different phases for the problem solving and, in each phase, the monitoring of the various artifacts (how the artifacts could be mobilized by the students and by the teacher) (Trouche, 2014).

colleagues, which leads to a new round of teachers' documentation work. The process of genesis leads to a new document for next time, which will later yield resources for a new elaboration. As time goes on, the scheme of utilization of the teacher would vary.

### **2.3.2.2 Teachers' design (variation), DAD and teachers' professional development**

Teaching is often regarded as design. This is in line with Brown (2009) who explains that the interpretation of teaching as design, and the notion of teachers as designers, is fitting with a range of cognitive theories that "emphasize the vital partnership that exists between individuals and the tools they use to accomplish their goals. Brown (2009) also pointed out that the activity of "designing" is not only dependent on the teacher's competence, but that it is an interrelationship between the teacher/s and the (curriculum) resources, the "teacher-tool relationship," that is at play here, and hence the affordances of the curriculum resources influence this relationship. This is in line with the DAD, emphasizing that any understanding of teacher as designer must include a conscious/deliberate act of designing, of creating "something new" (e.g., combining existing and novel elements) in order to reach a certain (didactical) aim (Trouche et al., 2020a, 2020b).

The DAD proposed a model of the interactions between teachers and resources, which has implications for teachers' professional development (Trouche et al., 2020a). From the perspective of variation, whilst there is an immensity of potentially suitable resources, many teachers feel that it is not easy to design effective tasks with variation. Therefore, it would be necessary to find some effective tools and provide suitable support for instructional design of variation. As Trouche, et al. (2020a) mentioned what is provided for teachers is often "a pile of bricks," without being given guidance on how these bricks might be put together to develop a coherent student's learning trajectory. Focusing on the variation, whether using variation to supplement the selected task in a given learning sequence, or planning learning through tasks with various types of variation, teachers will require professional support to help them develop *teacher design capacity* (Pepin et al. 2017b). This professional support should include a mindfulness/ sensitivity of mathematical and pedagogical aspects of learning resources with variation and the flexibility to use the variation.

In the DAD, the set formed by all the resources used by the teacher is named his/her *resource system*. These resources are associated with schemes of usage, forming documents (the same resource can intervene in several documents). The documents developed by a teacher also form a system, called the *document system* of the teacher. Gueudet and Trouche (2009) proposed that a given teacher develops a structured *documentation system*, and that this *documentation system* and the *teacher's professional practice* evolve together. They also indicated that from a research point of

view, the observation and analysis of the documentation system permits a better understanding of the teacher's professional development, and in particular allows for capturing the evolution introduced by digital resources.

Variation is one of the effective means of teacher's design work. According to what we have mentioned above, we wonder whether we can build a model inspired by the model of DAD to help teachers from the perspective of teachers' documentation work, thereby improving their professional development.

## **2.4 The need for creating a model of variation**

In this section, we will first present definitions of variation in China and France respectively based on the existing literature (section 2.4.1), and then introduce our preliminary model of variation from the perspective of teacher's documentation work (section 2.4.2), zooming out the documentational approach to didactics to focus on the variation. At last, we propose an analytical framework (section 2.4.3) and the concept of "variation path" for the preliminary model to conduct the following research.

### **2.4.1 The definition of "variation"/ "teaching through variation"**

To conduct our study and answer our research questions, we face a number of methodological challenges, one of which is the clear definition of variation.

Qingpu Curriculum reform Experimental Group (Gu, 1991) defined open-ended variation and closed-ended variation. Gu (1994) summarized four instructional principles including: emotional principle, accumulation step by step principle, activity principle, and feedback principle. To implement these principles, Gu (1994) further proposed different variation strategies for learning different types of mathematics knowledge, such as concepts and facts, principles and theorems, and problem solving. He coined the strategies as concept-oriented variation and process-oriented variation. Gu et al. (2004) view the variation in terms of "conceptual variation" and "procedural variation". Developmental variation and multi-faceted variation comprise Park and Leung's definition of variation (2006). Huang (2002) and Huang et al. (2006) stated that variation has to do with explicit variation and implicit variation. Many Chinese researchers defined variation in terms of form variation, problem variation, method variation, and content variation (Li, 2011). Sun (2011) examine tasks from three perspectives: "one problem multiple solutions", "one problem multiple changes", and "multiple problems one solution". Huang and Li (2017) viewed Bianshi (i.e., changing or varying) teaching, or Bianshi exercise, as a basic approach, which has been used regularly in mathematics classroom instruction. They pointed out that seeking multiple solutions to a problem, applying a mathematical method or strategy to solve a set of interconnected problems, and varying a problem into multiple problems are basic skills

valued by all mathematics teachers in China.

The definition of variation shown above is ambiguous and from different point of view, mainly based on categories and examples in the study. The possible reason for the ambiguity is that variation is a wide range of concept and it is difficult to categorize what is not a variation case in learning or teaching, since the pedagogy of every individual is always changing (Sun, 2011). Because according to Marton and Booth (1997), learning is seen as a change in an individual's way of experiencing the object of learning. From this perspective, here we try to narrow the definition of variation according to our research purpose and content. We present two formal and highly recognized definitions of variation in China and France respectively.

In China, traditional Bianshi (teaching through variation) is mainly used to master concepts (Bao et al., 2003a, b, c). For example, the "Dictionary of Education" (Gu, 1999) explains the term "teaching through variation" as follows:

*"An important teaching method through which students can definitely master concepts. It intends to illustrate the essential features by demonstrating different forms of visual materials and instances or highlight the essence of a concept by varying the nonessential features. It aims at understanding the essence of object and forming a scientific concept by putting away nonessential features." (Gu, et al., 2004)*

In France, Brousseau (1986) also gives a definition of didactical variables as we mentioned it in section 2.2.

Both of these two definitions indicated the importance of variation in teaching. However, there are some differences between them. We further explain this from three aspects based on our discussion in section 2.2.3.2: (1) The subject of the variation: Chinese definition pays more attention to distinguishing and classifying, aiming to grasp the essential characteristics. It focuses on the mathematical tasks; while French didactical variables contain both the perspective of task design and the didactics, aiming to inspire students' thinking and creation. It also pays attention to the variables that teacher can manipulate, including the arrangement of classroom teaching, the interactions between teachers and students, etc. Didactical variables involve different kinds of factors that can influence students' behavior; (2) The aim of variation: Chinese definition emphasizes the highlight of essential characteristics, while French definition focuses on stimulating students' active thinking; (3) The intended environment to teach through variation: Chinese definition intends to apply it in the consolidation of basic knowledge, while French definition intends to apply it in the exploration of the new knowledge.

At this point, it is not enough for us to define variation with clear boundaries in consistent ways. Considering different cultural and historical backgrounds in China and France, we retain these two kinds of definition of variation. We will rethink the

relationship between these two and discuss whether it is possible to give a unified definition (Chapter 7), despite the different cultural backgrounds, after having a deeper understanding of variation through the analysis of curriculum resources (Chapter 4) and case studies (Chapter 5, Chapter 6) in our research.

## 2.4.2 Theoretical networking: base for constructing the model and its analytical framework

In this section, we present the theoretical networking strategies that we adopt to construct our preliminary model and its corresponding analytical framework.

Given a diversity of theories of variation or related to variation, we take advantage of the connecting strategies (see Figure 2.4.1) proposed by Prediger, Bikner-Ahsbals and Arzarello (2008) to connect the diversity of theories.

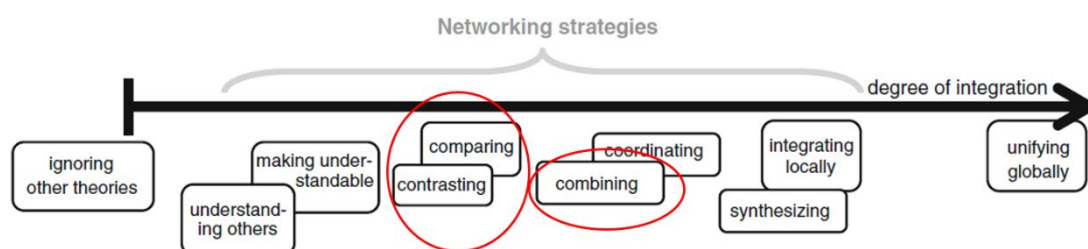


Figure 2.4.1 Networking strategies for connecting theoretical approaches (Prediger, et al., 2008)

First of all, three theories are considered for constructing of our preliminary model:

- Gu's "Bianshi" theory (Bao et al., 2003a, b, c; Gu et al., 2004) provides a Chinese lens to focus on the variation from the perspective of mathematics tasks.
- Brousseau's didactical variables (Brousseau, 1986, 1997, 2002) provides a French lens to focus on the variation from the perspective of the classroom teaching.
- Documentational Approach to Didactics (DAD) provides a lens to structure teacher's work throughout interacting with resources and tasks.

We apply two strategies, i.e., contrasting and comparing, and combining, to connect these theories. Contrasting and comparing the understandings of the variation in these two theories (section 2.2.3.2) prompt us to think about the role of variation in the different stages of teaching. Therefore, we rely on DAD to help us structure the whole process of teacher's work, i.e., combining the DAD and variation theory. The result of this combination is shown in section 2.4.3, i.e., our preliminary model of teaching mathematics through variation, where the interactions between teacher and resources are emphasized.

As for constructing our analytical framework, we dig deeper into these two theories of variation:

- Gu’s “Bianshi” theory (Bao et al., 2003a, b, c; Gu et al., 2004) provides a Chinese lens to analyze the teaching through variation from conceptual and procedural variation. It is a theory formed by long-term practical teaching experiments and focus on the design of mathematical task itself.
- Brousseau’s didactical variables (Brousseau, 1986, 1997, 2002) provides a French lens to analyze the variables in the classroom teaching that can significantly influence students’ behavior. It is a theory formed from a theoretical level and not limited to the mathematical part of a task but also the organization of interactions between students.

Two strategies, i.e., contrasting and comparing (section 2.2.3.2), and combining (section 2.4.4), are applied to connect these theories. In the process of comparing and contrasting, we extract the underlying part of representation from Gu’s theory, and then further refine it through Duval’s theory (see section 2.1.2.3, Duval, 2017). Considering the application of mathematics in daily life and in other disciplines, we add the classification of mathematical contexts proposed by PISA (see section 2.1.2.1, OCDE, 2017, 2019) and construct our analytical framework. The result of this combination is shown in section 2.4.4, i.e., our analytical framework for studying different types of variations.

The outcomes of theoretical networking (model and analytical framework) in our research are further verified by applying them to analyze variations in the curriculum resources (textbook and online platform) and in case studies with Chinese and French cultural contexts.

### 2.4.3 Preliminary model of teaching through variation and research questions

In our research, we take the lens of “teachers’ documentation work” and constructed a preliminary model of “teaching mathematics through variation” (Figure 2.4.2). This model aims to provide guidance to help teachers implement variation in class and reflect their own work after class. We introduce three notions structuring this model, potential, actual and practical variation:

- **Potential variation** represents the changes explicitly proposed by a given task, or series of tasks. To be inferred through an analysis of teacher’s resource system (textbook, resources from teachers’ collaborative work, school-based teaching book, etc.), from both the point of view of tasks and didactics (if there is an instruction of didactics).
- **Actual variation** represents the possible changes made explicit by a teacher when integrating a task as a resource. Teacher takes more or less profit of the potential variation for designing his/her resource. To be inferred through an analysis of



teacher's documentation work (teacher's working sheets, interview about teacher's design intention, students' working sheets designed by the teacher, etc.), from both the point of view of tasks and didactics.

- **Practical variation** represents the changes occurring when implementing the resource in class. To be inferred through the follow-up of the teacher in classroom teaching, from both the point of view of tasks and didactics.

According to the definition of didactical variables (Brousseau, 1986), these variables exist in the entire process of the teachers' documentation work, from potential, actual to practical variation.

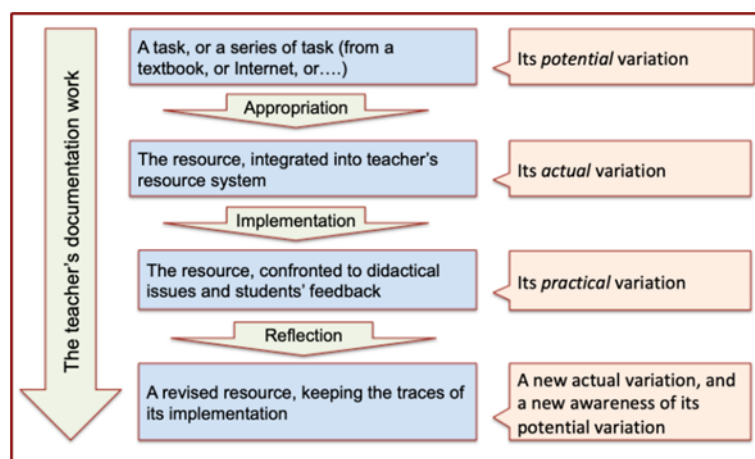


Figure 2.4.2 A preliminary model of teaching mathematics through variation

Gueudet and Trouche (2009) introduced a distinction between available resources and documents developed by teachers through documental genesis process, which inspired us to make a distinction between *potential variation* (the variation embedded in the available resources) and *actual variation* (the variation developed by teachers through their documentation work). In classroom teaching, the interactions between students, or between teacher-student can trigger the production of *practical variation*. The ongoing and spiral process of documental genesis and the relationship between teachers' documentation system and professional development inspired us to visualize the process of teachers' reflection, and thus we added an arrow from practical variation to a new actual variation, and a new awareness of its potential variation. Therefore, in this preliminary model of variation, potential variation is given by the resources, actual variation is the result of teachers' documentation work, and practical variation continues this result when implementing in class, where new variations create.

Teachers' documentation work (Gueudet & Trouche, 2009) plays an important role to promote the design work from potential variation to practical variation and the reflection work from practical variation to a new actual variation or potential variation, through studying variation from "task", "teachers' resources system", to "classroom teaching" and the revised resources. Here, we regard teachers' documentation work as

a thread that runs through the entire process of the model. Moreover, to what extent can a potential variation be transformed into a practical variation is largely related to the teachers' design capacity. We try to use this model to analyze the data in the following sections and to refine the model through case studies in France and China.

This preliminary model to some extent theorizes and forms a logical structure for the research questions we posed in section 1.3. As we proposed the notion of potential, actual and practical variation in our preliminary model, we further specify our preliminary research questions in section 1.3 and three refined research questions are addressed as follows:

Q1: What are the potential variations in the upper secondary school mathematics textbooks and other curriculum resources in China and France?

Q2: What are the actual variations in the Chinese and French cases to achieve the teaching goal?

Q3: What are the practical variations in Chinese and French cases? How does the case teacher respond to the practical variations?

#### **2.4.4 Analytical framework and the concept of variation path corresponding to the model**

Our preliminary model shows both static (potential, actual and practical variation) and dynamic (the process of appropriation, implementation, reflection) aspects of variation. Therefore, in this section, we propose an analytical framework to analyze the static aspect and a concept called “variation path” to have a more global view of the variation through teacher's documentation work.

On the basis of preliminary model presented in section 2.4.2, we further refine the three stages, i.e., potential, actual, and practical variation, by constructing an analytical framework for studying different types of variations, see table 2.4.1.

In table 2.4.1, we categorize variation in tasks into four types: contextual, representational, conceptual, and procedural variations. For each type of variation, several indicators are considered. The indicators for contextual variation are based on the PISA (OCDE, 2017, 2019, detailed information in section 2.1.2.1). The indicators for representational variation are based on the classification of types of semiotic registers (Duval, 2006, detailed information in section 2.1.2.1). Conceptual and Procedural variations are proposed in Gu's Bianshi theory. In the column of conceptual variation, we study it from two perspectives: concept variation (including standard and non-standard variation) and non-concept variation. As for the procedural variation, we study it from three perspectives: enhancing the formation of concepts and propositions, Pudian, and trilogy with variation. Considering Bianshi theory (Gu et al., 2004),

dimensions of variation (Sun, 2011) and their relations, we study trilogy with variation from three aspects: OPMS, OPMC, MPOS. Moreover, as mentioned in section 2.2.3, when we study OPMC, we will use the four strategies (Constraint manipulation, Goal manipulation, Symmetry, and Chaining) to further refine the research.

Table 2.4.1 Analytical framework for studying different types of variations

<b>Contextual variation</b>	<b>Representational variation</b>	<b>Conceptual variation</b>	<b>Procedural variation</b>
Personal; Occupational; Societal; Scientific	Natural languages; The symbolic writing; Iconic (images) and non- iconic (geometrical figures); Cartesian graphs and diagrams	Standard variation ; Non-standard variation ; Non-concept variation	Enhancing the formation of concepts and propositions; Pudian; Trilogy with variation: -OPMS -OPMC (Constraint manipulation, Goal manipulation, Symmetry, and Chaining) - MPOS
PISA (OCDE, 2017, 2019)	Duval (2006)	Gu's Bianshi theory (Gu et al., 2004)	Gu's Bianshi theory (Gu et al., 2004) Sun (2011)

We must clarify that these four types of variations are not completely disconnected and completely independent. For example, as we mentioned in Figure 2.2.12, according to Gu, representational variation relates to conceptual variation. In fact, there are intersections between different types of variations and even a change in one type of the variations may in turn affect another one, producing a chain reaction. We will specify it in the subsequent specific analysis.

To have a more global view, combining the perspective of static and dynamic, we propose the concept of *variation path*. A variation path is, for a given task, the process of evolution of each type of variations, through the three stages of potential, actual and practical variations, over a teacher documentation work. The evolution could be in the form of retaining, removing, reducing, extending, or creating the variation.

To make it more intuitive, we adopt a schematic representation of variation path to present the variation path. Figure 2.4.3 shows a general example of schematic representation of variation path.

As shown in Figure 2.4.3, in the horizontally oriented schematic representation, three stages of variation are presented, i.e., potential, actual and practical variation; in the vertically oriented schematic representation, four different types of variations are presented, i.e., contextual, representational, conceptual, procedural variation. The green arrow in this schematic representation shows the evolution of variation with the note removed, reduced, retained, extended, or create. The blue arrow in this schematic representation indicates the relations between different types of variations. For example,

in Figure 2.4.3, the new OPMC created by the teacher as actual variation leads to the emergence of OPMS and non-standard variation, and the extension of treatments in representational variation.

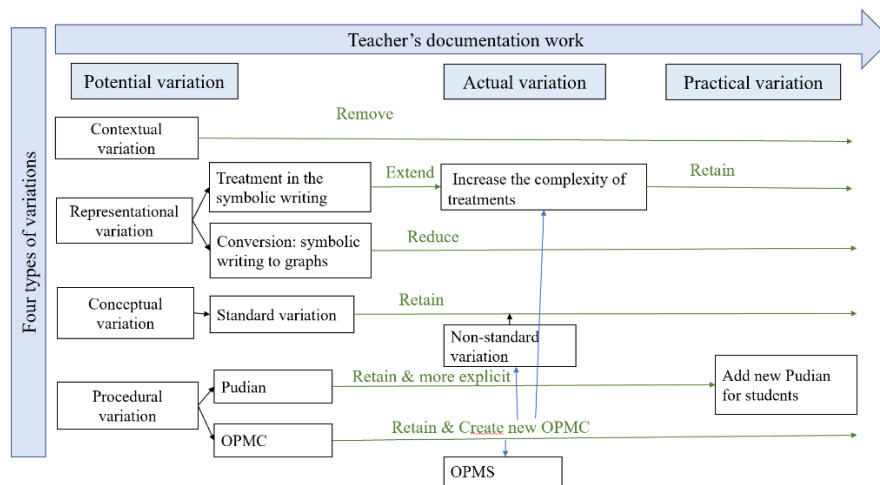


Figure 2.4.3 A general example of schematic representation of variation path



## Chapter 3

### Research methods and design

As we have constructed a preliminary model of variation (2.4.2) based on the literature, through the lens of potential, actual and practical variation, to further refine this model, we will conduct the analysis of curriculum resources and case studies. In this chapter, we will explain the research design from four parts according to our research questions: Section 3.1 presents the mathematical object ‘function’ chosen for our research; Section 3.2 describes the data collection and analysis strategies for studying variations in the tasks of textbooks or online platforms, corresponding to our first research question; Section 3.3 describes the choice of conducting case studies, data collection and analysis in case studies, corresponding to our three research questions; Section 3.4 concludes the overall research trace as well as the way of contrasting Chinese and French cases.

#### 3.1 Mathematical object: function

We choose function as the mathematical object in our study, both in the analysis of curriculum resources and case studies. We explain the reasons for this choice from three aspects: The important position of function in high school mathematics teaching (section 3.1.1); The rich soil that this topic provides for teaching through variation (section 3.1.2); The consistency of the teaching content of function in China and France (section 3.1.3).

##### 3.1.1 The important position of function in high school mathematics teaching

Function is a vital content of high school mathematics teaching, mainly reflected in the following three aspects:

- From the perspective of mathematics, function has a close relation with another knowledge, such as set, equation, inequation, algebra, geometry, solid geometry<sup>44</sup>. In high school mathematics teaching, function and methods of function almost run through the high school mathematics teaching, both in France and China (Ministère de l’Éducation nationale et de la Jeunesse, 2019 & MOE, 2017).
- From the perspective of mathematical education: the method and process of

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<sup>44</sup> Set up appropriate variables in solid geometry, establish functional relations, so as to turn the problem into a problem of function.

studying the basic properties of a function, such as monotonicity, maximum (minimum) value, and parity, are general means that can be widely applied. When studying these basic properties, we generally start with geometric intuition (observing graphs), then use natural language to describe the graphical features of the function at stake, and finally abstract to use mathematical symbols to describe the corresponding quantitative features. To study the properties of a certain function, it is necessary to use definitions such as monotonicity and parity to realize it through reasoning and calculations. This is a gradual process, and it is also a method often used in mathematics learning and research (People's Education Press, 2019).

- From the perspective of solving practical problems, function is an instrumental and necessary tool. Many real situations can be modeled by function, such as the changing trend of things, the most economical material, the greatest profit, the highest efficiency, symmetry. The problem in real-life is transformed into a mathematical function problem by constructing a function model, and the function knowledge and methods are used to deal with it.

### **3.1.2 Function provides rich soil for teaching through variation**

Function itself provides rich soil for teaching through variation. We illustrate these possible variations embedded in function based on four types of variations in our analytical framework (section 2.4.4): contextual, representational, conceptual and procedural variation.

First of all, as function is an important mathematical model to describe the changing law of the objective world (Ministère de l'Éducation nationale et de la Jeunesse, 2019 & People's Education Press, 2004), the situations proposed in this context come from very different fields: plane or space geometry, biology, economy, physics, news etc (Ministère de l'Éducation nationale et de la Jeunesse, 2019). Therefore, in view of the wide application of function, its contextual variation is very fruitful. Secondly, the function itself has three representations: graph, table of values, and formula, which provides huge potential for representational variation and the conversions between different representations. When solving problems, facing different needs, it is very important for students to choose an appropriate representation and conduct some treatments and conversions inside or between different registers (People's Education Press, 2019). Moreover, due to the abstraction of function knowledge, teachers often adopt the method of symbolic-graphic combination to concretize knowledge and improve students' perception. The symbolic-graphic combination inspires students from the geometric characteristics of function and the algebraic characteristics of function properties, and effectively reveals the basic properties of functions and provides a simple and quick way to solve the problem. Thirdly, as the notion of function is abstract, teachers present different kinds of examples for students to distinguish the

concepts, which relates to the conceptual variation. At last, gradually in-depth knowledge, knowledge difficult for students to master, and the close connection between function and other knowledge may also promote the design of Pudian and OPMC. The comprehensiveness of function knowledge allows students to solve problems from different angles, which further develop the OPMS in teaching.

What we discuss above are in line with our exploration of variation in the curriculum standard in section 1.2.1.2. Besides the four aspects we mentioned above, both curriculum standards propose to use digital tools (computer software, calculators, dynamic geometry, etc.) to support the teaching of function, which may bring new possibilities for variation.

### 3.1.3 The consistency of the teaching content of function in China and France

To clarify, in our research, we study the general concept and properties of function, including the concept of function, representation of function, monotonicity of function, parity of function, the extremum (maximum or minimum) of a function, the use of calculator to solve the task of function, solving the task from graphical aspect, and the connection with inequation. We present the specific content of the function in the Chinese and French textbooks<sup>45</sup> and classroom teaching in table 3.1.1. All of the mathematical contents in our research are taught in grade 10 in both China and France.

Table 3.1.1 Mathematical content of function in our research in grade 10

Mathematical content in our research	Textbook analysis				Case studies	
	Chinese (2019)	PEP	France (2014)	Hyperbole	China	France
Concept of function	●		●		●	●
Representation of function	●		●		●	●
Monotonicity of a function	●		●		●	●
Extremum of a function	●		●		●	●
Parity of a function	●		—		●	—
Use of calculator to solve the task of function	●		●		—	●
Solving the task from graphical aspect	○		●		○	●
Connection with inequation	○		○		○	●

In table 3.1.1, symbol “●” means that this content is clear presented in the textbook or in the case. Symbol “○” means that this content is involved in the textbook or case studies, not shown obviously, but embedded in the process of solving tasks. Symbol “—” means that this content does not appear at all.

As shown in table 3.1.1, there is a certain consistency of the teaching content of function between Chinese case and French case. Except the parity of a function and the use of

<sup>45</sup> The textbooks we used are presented in section 3.2.1.



calculator to solve the task of function, the other contents are more or less presented in the Chinese PEP, French Hyperbole textbook and in the Chinese and Chinese case studies. It is important to note that table 3.1.1 only presents the contents involved in our research and there are more contents in teaching function at high school than what is shown in table 3.1.1, such as the periodicity of function, different elementary functions (polynomial, rational, trigonometric, hyperbolic, and exponential functions, etc.). The consistency of functions content in both countries provides a basis for our contrasting research.

In fact, students in both countries have already learnt preliminary knowledge about function in junior high school and have a preliminary understanding of it. They have learnt image, pre-image of function, graphical representation, linear function, and proportional function. In the Chinese case, PEP textbook presents the chapter of sets, quadratic functions, equations and inequalities before the chapter of function to pave the way for teaching function. While in French case, the case teacher arranges the teaching of intervals, inequations, factorization, union and intersection to pave the way for her teaching function.

## **3.2 The analysis of curriculum resources**

Curriculum resources in our research include the different kinds of materials (digital or physical) that teachers use in or for their teaching (textbooks, lesson plans, etc.) and have a significant influence on students' opportunities to learn (Stylianides, 2016).

The analysis of curriculum resources is used to answer research questions 1. In this section, we will introduce the analysis of potential variations in these curriculum resources from two perspectives: data collection (section 3.2.1) and data analysis (section 3.2.2).

### **3.2.1 Data collection of curriculum resources: textbook, online platform**

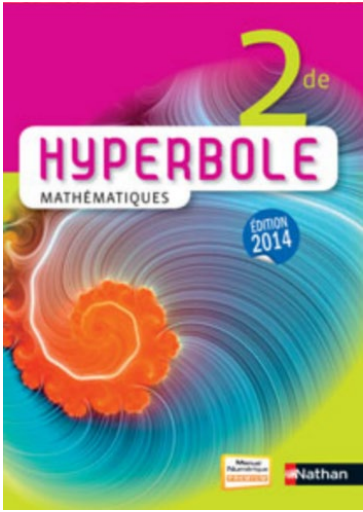

In this section, we present the data collection of curriculum resources we studied in chapter 4, including the textbooks in China and France and their corresponding teachers' guidebooks, and online platforms.

Both China and France have national mathematics curriculum standards<sup>46</sup>, and the textbooks are designed based on the curriculum standards. As mentioned in section 3.1, the content of function in our study is all in grade 10, in this study, we have selected Hyperbole textbook 10<sup>th</sup> grade (2014 edition, see Figure 3.2.1) as French research object and high school mathematics textbook of People's Education Press A version

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<sup>46</sup> See section 1.1 for details.

volume 1(abbreviate as PEP, latest version published in 2019, see Figure 3.2.2) as Chinese research object. There are two main reasons to choose these two textbooks: first and the most important reason is that these textbooks are used by the case teachers in class, which allows us to combine textbook analysis (Chapter 4) with subsequent case studies (Chapter 5 & 6) to have a whole picture of teaching and also the evolution of variations in the resources; Second, the number of teachers who choose these two textbooks for teaching is very considerable.

	
<p>Figure 3.2.1 The cover of Hyperbole textbook of 10<sup>th</sup> grade (2014 edition)</p>	<p>Figure 3.2.2 The cover of PEP textbook of 10<sup>th</sup> grade (2019 edition)</p>

It is important to note that the period during which we collected data coincided with the curriculum reform in both countries, so the choice of textbook versions was to some extent influenced. When we collected French data in 2018, the reform was still under discussion. Therefore, the French case teacher kept the 2014 edition of Hyperbole textbook although there was a more recent one published in 2017. When we collected Chinese data in 2020, the curriculum reform had entered the stage of formal implementation and it was the first round of using the new textbooks after the promulgation of the new curriculum standard in 2017.

Textbook often comes with corresponding teachers' guidebook, and we often refer to the corresponding teachers' guidebook when analyzing the textbook. Figure 3.2.3 and Figure 3.2.4 show the cover of teachers' guidebook of French Hyperbole textbook and Chinese PEP textbook.

	
<p>Figure 3.2.3 The cover of Hyperbole teachers' book of 10th grade (2014 edition)</p>	<p>Figure 3.2.4 The cover of PEP-A teachers' guidebook of 10<sup>th</sup> grade (2019 edition)</p>

Considering the rise of digital resources in recent years, in addition to textbooks and teachers' guidebook, we have also studied two online platforms from different perspectives. One is Chinese PEP platform, that come with PEP textbook, to see the potential variations in the digital curriculum resources. The other is French Sésamath platform, to see the development and future trend of digital curriculum resources and textbooks. We select Sésamath online platform to study as it is a distinctive result of the construction of digital curriculum resource in France today, and it consistent with the concept of textbooks and curriculum standards.

### 3.2.2 Data analysis of curriculum resources

Data analysis of curriculum resources is conducted from two aspects: textbook (including general situation and specific tasks), and the digital curriculum resources (online platform that accompanies the textbook).

For textbook, we analyze it from macro to micro situation. We first describe the situation of textbooks in two countries and draw a general structure of each specific textbook. Then we select tasks based on the structure of textbook we drawn. Tasks are organized in different types and noted by different words in textbooks, such as examples, exercises, and problem-solving. We select one (series of) or two tasks in each type of tasks to illustrate the potential variations embedded in or between them. There are two main criteria for selecting specific tasks: 1) the connection of tasks with the subsequent case studies: we choose tasks that have a connection with the subsequent case studies as we do not want to discuss the textbook in isolation from the subsequent case teachers' teaching. This is also consistent with our idea of studying the textbook that case teachers use; 2) the relevance of variation: we choose tasks that show as much as possible the different types of potential variations in the function part of the textbook, and also to see the connections between the tasks. At last, we study the potential variations in the

specific task of textbook, taking profit of the analytical framework mentioned in section 2.4.4 (present again in table 3.2.1). We analyze each task from four types of variations: contextual, representational, conceptual and procedural variation and illustrate how the textbook designs tasks by manipulating variation.

Table 3.2.1 Analytical framework of potential variations

<b>Contextual variation</b>	<b>Representational variation</b>	<b>Conceptual variation</b>	<b>Procedural variation</b>
Personal; Occupational; Societal; Scientific	Natural languages; The symbolic writing; Iconic (images) and non- iconic (geometrical figures); Cartesian graphs and diagrams	Standard variation ; Non-standard variation ; Non-concept variation	Enhancing the formation of concepts and propositions; Pudian; Trilogy with variation: -OPMS -OPMC (Constraint manipulation, Goal manipulation, Symmetry, and Chaining) - MPOS

For online platform, we study the potential variations embedded or the possible variations that it may bring in the future based on the four types of variations in table 3.2.1.

### 3.3 Case studies

Case studies are used to answer three research questions. In this section, we first generally describe the reasons for choosing case studies in France and China (section 3.3.1). Then we describe the context of two cases, including the information of case teachers, students and schools (section 3.3.2). At last, we present the implementation of case studies in detail through two aspects: data collection (section 3.3.3) and data analysis (section 3.3.4).

#### 3.3.1 The choice of conducting case studies

As Yin (2018) pointed out, case studies are not intended to generalize the population or universes, but to expand or generalize theoretical assumptions. Therefore, this research is not intended to summarize how all teachers use variations in teaching, but to explore the concept of variation from the perspective of teachers' documentation work, to improve our preliminary model of teaching through variation, so as to promote our understanding of variation in mathematics teaching and make better use of it for effective teaching.

This research adopts an analysis of multiple-case embedded design in Chinese and French classroom teaching. (Yin, 2018) proposed four types of potential case study research designs, based on the  $2 \times 2$  matrix in Figure 3.3.1. According to Yin (2018),

the matrix first shows that every type of design will include the desire to analyze contextual conditions in relation to the “case,” with the dotted lines between the two signaling the likely blurriness between the case and its context. The matrix then shows that single-case and multiple-case studies reflect different design situations and that, within these two variants, there also can be unitary or multiple units of analysis. The resulting four types of designs for case studies are (Type 1) single-case (holistic) designs, (Type 2) single-case (embedded) designs, (Type 3) multiple-case (holistic) designs, and (Type 4) multiple-case (embedded) designs.

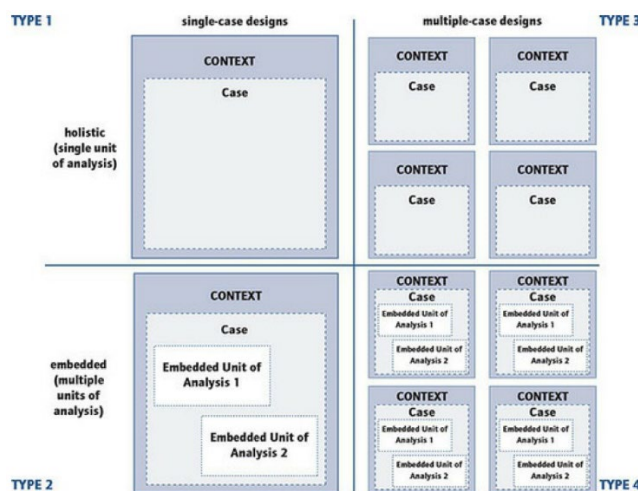


Figure 3.3.1 Basic Types of Designs for Case Studies (Robert K. Yin, 2018, p. 96)

In our research, China and France are two cases. Each case has its own embedded unit of analysis, and we perform multiple case “replication”<sup>47</sup> (Robert K. Yin, 2018) on these analysis units. Replication in this research is reflected in two aspects, i.e., replication of the two cases and replication inside each case. For replication of the two cases in China and France, the data collection methods (section 3.3.3) in both countries are the same<sup>48</sup>, such as conducting two formal interviews, after-class discussions with case teacher, photographs, audio and video recordings. The data analysis methods (section 3.3.4) in both countries are the same, including the teachers’ design, use, and reflection on teaching through variation, focusing on function in grade 10. For replication inside each case, although the mathematics content of each class is different,

<sup>47</sup> The replication logic is directly analogous to that used in multiple experiments (see Barlow, Nock, & Hersen, 2008). For example, upon uncovering a significant finding from a single experiment, an ensuing and pressing priority would be to replicate this finding by conducting a second, third, and even more experiments. Some of the replications might attempt to duplicate the exact conditions of the original experiment. Other replications might alter one or two experimental conditions considered challenges to the original finding, to see whether the finding can still be duplicated. With both kinds of replications, the original finding would be strengthened (Robert K. Yin, 2018).

<sup>48</sup> Impossible to be exactly the same because it involves the differences of observation situations (limitations of the epidemic, teaching tradition, culture, etc.).

our observation methods and the method of selecting teaching episodes<sup>49</sup> for data analysis are replicate. From these two aspects, we ensure the continuity and consistency of our data collection and the reliability of subsequent contrast of two cases. The procedural of multiple-case study in our research is shown in Figure 3.3.2.

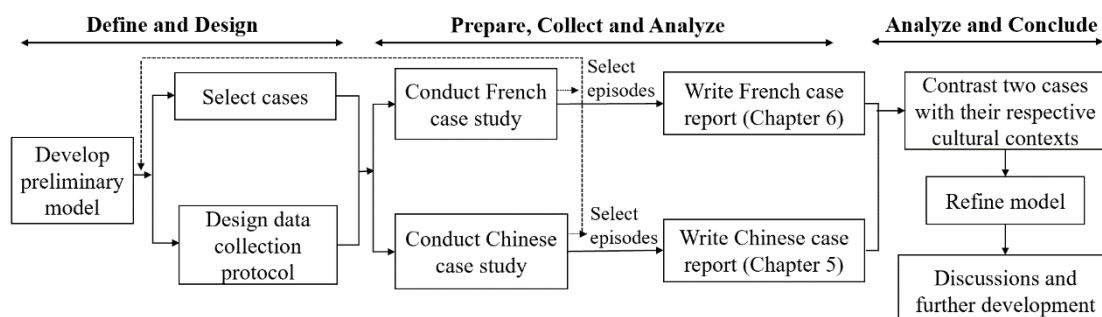


Figure 3.3.2 Procedure of case study in our research (adapted from Robert K. Yin, 2018, p. 105)

The dashed-line feedback loop in Figure 3.3.2 represents the situation where important discovery occurs during the study of one of the individual cases (e.g., one of the cases deviated unexpectedly from the original design) (Robert K. Yin, 2018). Sometimes such a discovery requires us to reconsider our research design and original theoretical propositions. At this point, “redesign” should take place before proceeding further. Therefore, research design in our research is not static, it will be effectively improved according to the actual teaching situation.

For the teacher in each case study (section 3.3.2), we collected data from multiple approaches and using multiple methods. In the following, we will explain case study design in details from two aspects: data collection (section 3.3.3) and data analysis (section 3.3.4). Through these research design and methods, we identified the key features of each case, and finally conducted a contrast between two cases (section 3.4).

### 3.3.2 Case teachers, students and the school

This study involves the observation of classroom teaching of function in 10<sup>th</sup> grade of high school, focusing on teachers’ documentation work on variation, from lesson preparation, classroom teaching, to the reflections after class. Therefore, the choice of teachers is very important.

The criteria for the selection of case teachers in our research are as follows: 1) the teacher should have rich teaching experiences (over 15 years); 2) the teacher gives mathematics lessons of function in grade 10; 3) the teacher is interested and proficient in the task design. These criteria are developed based on the assumption that teacher with rich teaching experience is often associated with well-structured resource systems,

<sup>49</sup> Detailed description of episode is presented in section 3.3.4.3.

which bring possibilities of varying tasks. We also believe that teacher's interest in task design will help us to discover more about variation in teaching, which contributes to our study.

We followed two experienced mathematics teachers in two good secondary schools, one French teacher in Lyon and one Chinese teacher in Hangzhou. Both of them meet the criteria we set with several dimensions of their profiles: rich teaching experiences, giving mathematics lessons of grade 10 during the period of our observation, being interested and proficient in the design and adaptation of the tasks to their own class, which are recognized by their colleagues. We suppose that these two good schools with good students give case teachers a lot of room to make mathematical variations, not tied to the different special needs of students or spending a lot of time on class management. Moreover, both teachers are the core teacher in their own school and are responsible for the training of novice teachers in the school.

During our data collection in China, Chinese case teacher Lin was guiding a preservice teacher Shi and Shi was required to give a class as a demonstration of the internship results. We were fortunate to observe the complete design process of this class and the final presentation. We find that the interaction between case teacher and preservice teacher are valuable for our research to further study Lin's teaching through variation. Therefore, in the case of China, in addition to one experienced teacher, we also added the analysis of a series of guiding activities (section 5.3) about the interactions between the case teacher and the preservice teacher. Basic information of Chinese (including preservice teacher) and French mathematics teachers in our research have shown in table 3.3.1.

Because the main research object in case studies of our research is two mathematics teachers, the class we observe is the 10th grade classes taught by the two teachers at that time (Emma has only one 10th grade class, while the Lin has two classes). Considering that the French classroom has a half-class organization<sup>50</sup> (see introduction to the French case for details), and we also want to analyze how teachers can reflect on teaching and self-feedback in a short time<sup>51</sup> by observing the situation of teachers teaching the same content in different classes, we observed lessons of function given by two case teachers in grade 10.

We give a detailed introduction to the two cases in the following from the school, the students in the observed classes, types of lessons and some collaboration interactions

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<sup>50</sup> Half-class organization: Because students choose different types of other subjects, once a week, the class is split into two classes for teaching. The teaching content is the same in the two classes.

<sup>51</sup> The time interval between the two classes is very short, generally 10 minutes, both in France and in China. In the French case, the time interval between two half-class lessons is 1 hour.

for teachers' professional development.

Table 3.3.1 Basic information of case teachers

	French case teacher	Chinese case teacher	Chinese preservice teacher
Name	Emma	Lin	Shi
Nationality	French	Chinese	Chinese
Work place	High school in Lyon, France (school F)	High school in Hangzhou, China (school C)	Senior student in the university
Gender	Female	Male	Female
Teaching experience (in years)	18	20	0
Professional title	Senior teacher	Senior teacher	
Education background	Master degree	Master degree	Bachelor degree
Observation class	1 class in grade 10 (Natural class <sup>52</sup> )	2 classes in grade 10 (Experimental classes <sup>53</sup> )	Guiding activities with Lin
10th grade math class hours (per week)	5h or 6h <sup>54</sup>	7h30 (3h45 for each class)	

### 3.3.2.1 Introduction to the Chinese case

School C is a school with good geographical condition and high quality of teaching in the area of Hangzhou, China. It has a history of more than 120 years. Over the years, the rate of college entrance examination to key universities has remained above 95%. There are around 1700 students in the school. School C has also established extensive cooperation and exchange relationships with many schools in Japan, Europe, America, and Southeast Asia. In 2014, the school added Sino-US International Classes and hired foreign teachers. In order to ensure the quality of the education, the school carries out the activity of “one hundred teachers listen to a thousand lessons”. The leader of the school, the Teaching Research Group (TRG, 教研组)<sup>55</sup> or the Lesson Preparation Group (LPG, 学科备课组)<sup>56</sup> will open the classroom's door to listen to the lessons, providing guidance and suggestions to teachers, especially young teachers. Besides teachers' practical teaching, to emphasize teachers' scientific research, the school

<sup>52</sup> Natural class: The administrative class is set up by the school for student management and teaching management according to the number of students in each grade at the time of admission, and there is no difference in mathematics achievements from students in other classes.

<sup>53</sup> Experimental class: A class composed of relatively good students selected through examinations. For the introduction of the experimental class in this Chinese case, see section 3.3.2.1 for details.

<sup>54</sup> Students in grade 10 have 4 hours of mathematics in one week. As there is a half-class each week, one of the hours is doubled. So, Emma teaches for 5 hours. And every two weeks, there is one hour of personalized support lesson (AP lesson, see the introduction to the French case for details). So, the teaching hours of grade 10 for Emma each week is 5h or 6h.

<sup>55</sup> The teaching and research group is the primary organization of the schools teaching work and an important part of the teaching command system. At the same time, it is also the basic organization unit for the school to carry out teaching and research activities.

<sup>56</sup> The subject preparation group is the teaching and research organization under the school's teaching and research group and the main organization that implements the “peer mutual assistance” in school-based research.



regularly arranges scientific research lectures, and publishes scientific research magazines within the school to publish teachers' scientific research results.

School C implements hierarchical teaching. Students are mainly divided into 3 levels according to the entrance examination results: 8 classes are parallel classes, 2 classes are experimental classes (specifically one is liberal arts comprehensive class and one is science comprehensive class (observed and noted as class A, class B in our research), and 2 classes are Mathematics Elite Class and Science Elite (Qian Xuesen) Class. This examination is quite difficult designed by the teachers in school C. It includes one examination of mathematics, one examination of physics and chemistry, one examination of Chinese and English. which is equivalent to the examination of five subjects. The classification is based on the total score of all subjects. The difference of the average scores between two adjacent level classes in the entrance examination is about 20 points; the average scores of different classes of the same level are basically the same as the school's intention is to have a competitive situation between the classes in the same level.

In Chinese case, Lin teaches two experimental classes of grade 10 and the math competition class<sup>57</sup>. The two observed classes (noted as class A and class B in the following) respectively have 48 students (31 females, 17 males) and 49 (15 females, 34 males) students. As shown in table 3.1.1, Lin gives lessons of grade 10 for 7 hours 30 minutes each week, 3 hours and 45 minutes for each class.

It should be noted that Lin is also the homeroom teacher<sup>58</sup> of class B. The classification of students in class A and B is based on the students' willingness and the school makes appropriate fine-tuning. The training of class A focuses more on liberal arts, and the training of class B focuses more on science. Lin thinks that this kind of class grouping is reasonable as it is difficult for students of different level to sit together and take a big class. Class grouping is good for the development of students and it is necessary to carry out hierarchical teaching in high school.

However even after this class grouping, there is still a big gap between students in the same class. According to Lin, it is because the entrance examination is all about knowledge in junior high school, where students can spend a lot of time studying and consolidating the basic knowledge. However, the situation changes when students

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<sup>57</sup> The students in the math competition class are selected from the whole grade with excellent mathematics, mainly students in experimental and elite classes. This is a supplementary teaching in addition to regular teaching.

<sup>58</sup> Homeroom teacher (班主任 in Chinese) : A homeroom teacher is a teacher in the school who is fully responsible for the thinking, learning, health and life of a class of students. He / She is the organizer, leader and educator of a class, and is also the coordinator of the teaching of all teachers in the class. The difference between him/her and the class teacher is that the class teacher is only responsible for the teaching of the subject, and the class teacher is in addition to being fully responsible for managing a class.

arrive in high school. The content of high school learning is rich and in-depth, and some students may not be able to keep up. There are seven or eight students in each class of A and B who are not able to keep up. Lin believes that it will be very accurate to conduct the class grouping after learning in the high school for one semester. Lin mentioned that the difficulty and progress of his class are mainly based on the most of the students. For students who are weak in mathematics, he would ask them to make corrections and encourage them to ask questions in private.

In addition to Lin's teaching, we also observe the interactions between the case teacher and the preservice teacher Shi. Part of Shi's background has shown in table 3.3.1. Shi is from a normal school<sup>59</sup> in Hangzhou, majoring in mathematics education, being trained as a future mathematics teacher. Her academic performance is excellent (1/29). Shi's university has a cooperative relationship with School C. Every year, school students are sent to school C for educational internships. Lin is Shi's supervisor in charge in school C. The internship content mainly consists of two parts, one is the teaching of mathematics; the other is the management of the class as a head teacher. Here we mainly focus on the part related to mathematics teaching.

The series of guiding activities we observed between Lin and Shi are arranged at the end of Shi's internship process, and the lesson she gave is as a display of the results of her entire internship. Therefore, the educational internship experience has an important influence on Shi's classroom teaching practice. Shi is quite familiar with the students as she had observed Lin's classroom teaching and corrected the students' homework for 2 months before this lesson. She also answers mathematical questions for some students after class, helps Lin with class management, and participates in the decision-making of class B affairs (Lin's the homeroom teacher of class B), such as the distribution of school uniforms, class preparation for school sports meets, etc.

This series of guiding activities is on the mathematical topic of "operation of logarithmic function". It is chosen by Lin as Shi's guide teacher. According to Lin, this content is easier for Shi to design and implement because it involves calculations and has fewer mathematical knowledge contained than other teaching content. Before this lesson, students have learned the concept of function, the monotonicity of a function, the maximum and minimum value of a function, the parity of a function, and the way to analyze the function from graphical perspective. Lin has taught the concept of logarithms one day before this lesson.

### **3.3.2.2 Introduction to the French case**

According to Emma, school F is a high school with good location and the economic

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<sup>59</sup> Normal school: Colleges and universities mainly aiming at cultivating various types of teachers.

conditions of students' families are generally good. Parents have a relatively high level of education and pay attention to the education of their children. It is a high-quality high school in the area of Lyon, France, with a history of more than 100 years. The pass rate of Baccalauréat<sup>60</sup> in this school is around 95%, which is much higher than the average level. There are a total of 1,070 students in the school. School F think highly of cooperation and exchanges of education, and the innovation in teaching. It organizes study tours for students to Shanghai, Dublin, Paris, Rome and other places. School teachers also actively participate in various activities inside and outside the school. For example, there are Mathematical Rally training for grade 10 and Mathematical Olympiad training for grade 11 organized by school mathematics teachers. Compared with Chinese case, there are relatively few formal activities among teachers in schools but Emma joined IREM<sup>61</sup> DREAM group<sup>62</sup> and Rallye Mathématique de l'Académie de Lyon (RMAL)<sup>63</sup> outside school to improve her own teaching.

In school F, students are not classified into different levels in grade 10, but they go to different series and study different mathematics contents since grade 11 according to the French curriculum<sup>64</sup>. For students who are weak in mathematics, AP (accompagnement personnalisé)<sup>65</sup> lesson is offered. The criteria for entering this class are determined by the teacher.

In French case, Emma teaches one natural class of grade 10, one class of STMG<sup>66</sup> of

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<sup>60</sup> The baccalauréat, often known in France colloquially as the bac, is a French national academic qualification that students can obtain at the completion of their secondary education (at the end of the lycée) by meeting certain requirements. It was introduced by Napoleon I in 1808.

<sup>61</sup> Refer to an international reference related to IREM: Trouche, L. (2016). Didactics of Mathematics: Concepts, Roots, Interactions and Dynamics from France. In J. Monaghan, L. Trouche, & J.M. Borwein, *Tools and mathematics, instruments for learning* (pp. 219-256). Springer

<sup>62</sup> The DREAM (Démarche de Recherche pour l'Enseignement et l'Apprentissage des Mathématiques) group is made up of university professors, trainers and teachers working within a design-driven research paradigm. Website: [https://dreamaths.univ-lyon1.fr/icap\\_website/1324/41153](https://dreamaths.univ-lyon1.fr/icap_website/1324/41153) Some members of the DREAM team participated in the work of the EducTice team which was a research team whose fieldwork methodologies are based on collaborative work and research-driven resource design. Its expertise in this field is attested by scientific publications (Aldon et al., 2013; Nizet, Monod-Ansaldi, Aldon, Prieur, & Criquet, 2019). For several years, they have also participated in the CrAC (Collaborative Research Analysis Crucible, <http://ife.ens-lyon.fr/ife/recherche/groupe-de-travail/creuset-danalyse-des-recherches-collaboratives-crac>) seminar of the Interface and Incubation pole of the IFÉ.

<sup>63</sup> Rallye Mathématique de l'Académie de Lyon, abbreviated as RMAL, Mathematical Rally of the Academy of Lyon in English. Website: <http://rallye-math.univ-lyon1.fr/>

<sup>64</sup> Refer to section 1.1.2 for detailed description of French curriculum.

<sup>65</sup> AP (accompagnement personnalisé) lesson: Every two weeks, Emma has one hour of personalized support lesson of grade 10. She selects the weaker students at this stage for doing a little bit of support according to the class test. According to Emma, she could also do more in-depth work in such lesson. But for the moment, priority is given to support.

<sup>66</sup> STMG: Sciences and technologies of management and administration. Sciences et Technologies du Management et de la Gestion (in French). Students in this class is going to prepare a technological Bac. Therefore, there is a little less general education for these students and they already do subjects such as management, economics, law, etc. In terms of mathematics, these are students who don't have any confidence in themselves. According to Emma, she tries to lead students to get back to the basics and

grade 11, and one class of science series<sup>67</sup> of grade 12. In this research, we only observed one natural class of grade 10 in French case, including the AP lesson taken by students from this natural class. There are 32 students in Emma's mathematics class that we observed. In this class, there is a big gap in the mathematics level of students. In Emma's mathematics class, participants in AP courses are mobile, depending on student's performance on the classroom test. As shown in table 3.3.1, Emma gives lessons of grade 10 for 5 or 6 hours each week, including 1h for AP (every two weeks), 1h for each half-class, and 3h for the whole class.

Corresponding to the hierarchical teaching in Chinese case, AP in French case can be regarded as a kind of hierarchical teaching to some extent. Moreover, there is an organization of half-class in French case. According to the different elective lessons selected by students, once a week, the entire class will be split into two classes for teaching. In this case, we note the split two classes in French case as class C and class D. Students in the first demi-class are better in mathematics than the second, Emma tries to adapt her explanations and go further for the second demi-class after her reflections on the first one. She uses all the same teaching materials in two demi-classes so that everyone can follow suitably the next day.

### **3.3.3 Data collection in case studies**

In this section, we explain what data is collected and how it is collected. We specifically explain data collection in the following sections: the timeline of data collection, and the data collection tools.

#### **3.3.3.1 Timeline of data collection**

As we mentioned in section 3.3.2, we selected one Chinese (including his guiding activity with preservice teacher Shi) and one French mathematics teacher of 10<sup>th</sup> grade for follow-up observation.

The data collection of case studies lasted around four months: two months in France and two months in China. The actual timeline of data collection is shown in Figure 3.3.3. As presented in table 3.3.1, We followed the process of teaching function in one specific class of grade 10 in France, and two specific classes with the Chinese teacher Lin in China. During these two months, the French teacher was followed at least three times (5h) each week and Chinese teacher was followed at least 10 times (5 times, 3h45min for each class) each week, which contained all the mathematics lessons of certain contents of function (mathematical contents shown in table 3.1.1) in the specific classes. In order to have a more in-depth and comprehensive understanding of the role of

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give them back their confidence because, if they listened to themselves, many of them would give up mathematics completely.

<sup>67</sup> Refer to section 1.1.2 for detailed description of different streams in French curriculum.

variation in the teaching of function in the two case teachers, we also observed a series of guiding activities between Lin and one preservice teacher Shi in China, and a series of thematic lessons led by Emma in France. For the more detailed classroom observation time and the content of each lesson, it is presented in Appendix I (Chinese case) and Appendix II (French case).

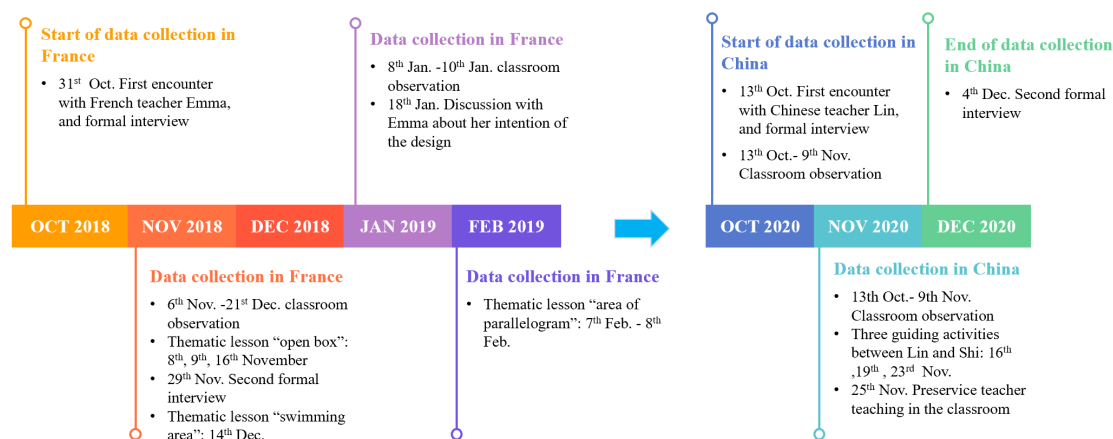


Figure 3.3.3 The timeline of actual data collection

There are mainly three kinds of lessons in each case teacher's classroom teaching: lessons for new knowledge, exercises lessons, guiding activities (Lin and Shi) and thematic lessons (Emma). Every once in a while, Lin will arrange exercises lessons to explain the difficult tasks and the mistakes that students made in exams or their homework. Different from Lin, the exercises explanation is organized in the lesson of new knowledge in Emma's teaching so there is no specific lesson for exercise explanation in French case.

In Chinese case, we collected 24 lessons<sup>68</sup> of Lin's teaching, including 16 lessons of introducing new knowledge of function, 8 lessons of explaining tasks in tests or homework. We conducted two formal interviews. We also collected 3 guiding activities between Lin and preservice teacher Shi, 2 lessons of Shi as a result of the guiding activities, and one formal interview with Shi (See appendix I for details).

In French case, we collected 40 lessons of Emma's teaching. As shown in appendix II, types of lessons in Emma's teaching are various. Some of contents are designed by Emma to present as the pre-knowledge for the teaching of function, such as intervals, factorization, union, intersection, etc.; some lessons are about computer science, such as Python and cryptology; some are mathematical activities as rally. Therefore, we only keep 26 lessons<sup>69</sup> directly related to function to analyze, including 17 lessons of

<sup>68</sup> Here 24 lessons refer to the number of lessons in both class A and B, which means doubles the number of teaching content in appendix I.

<sup>69</sup> Here, 26 lessons=17 lessons of new knowledge (including 3 half-classes of new knowledge) +9 thematic lessons (including 2 half classes of thematic lessons)

introducing new knowledge of function (highlighted in yellow in appendix II) and 9 thematic lessons. Other lessons are used as the background information to support our analysis. We conducted two formal interviews and one discussion with Emma (See appendix II for details).

### **3.3.3.2 Tools of data collection**

In this section, we introduce specific collection tools we use during data collection.

The data collection of this study is based on what Wolcott (1992) called, “watching asking and examining”, that is, teaching observation, interviews, inspection and analysis of curriculum resources, including teachers’ teaching plans, teaching resources, and students’ feedback (feedback in class, students’ homework). We implement the “non-participant observations” in real classroom teaching for two teachers in China and France, and focus their teaching in one (France) or two classes (China).

As teachers’ documentation work is the core of our model of teaching through variation (see section 2.4) and the DAD claims that analyzing teachers’ documentation work requires specific methodology (Trouche et al., 2020a), we refer to the five main principles to organize our data collection. Trouche et al. (2020a) have developed a specific methodology named *reflective investigation of teachers’ documentation work* inspired by the epistemological considerations of the elements related to the analysis of teachers’ activity through their documentation work, such as the variety of resources feeding, and produced by, this work; the variety of interactions (collective, institutional as well as social) influencing this work; the time for developing documentational geneses. As the teacher is the one having access to the teacher’s documentation work, this methodology gives a major role to teachers themselves, and it is underpinned by five main principles: 1) The principle of broad collection of the material resources used and produced in the course of the documentation work; 2) The principle of long-term follow-up; 3) The principle of in- and out-of-class follow-up. The classroom is an important place where the teaching elaborated is implemented, bringing adaptations, revisions and improvisations. However, an important part of teachers’ work takes place beyond the students’ presence - at school (e.g., in staff rooms), at home, in teacher development programs; 4) The principle of reflective follow-up of the documentation work; 5) The principle of permanent confronting the teacher’s views on his or her documentation work, and the materiality of this work (materiality coming, for example, from the collection of material resources; from the teacher’s practices in her classrooms).

We regard these five principles as guidelines for our methodological choices and we adapt them based on our research questions and the actual situations in our classroom observation. In the case study, we mainly collect data through the following activities:

observation of daily classroom function teaching, interviews with case teachers, visiting teachers' resources, informal communications with teachers after class, and observing a series of guiding activities between the case teacher Lin and a preservice teacher Shi in Chinese case and thematic lessons arranged by the case teacher Emma in the French case. We present all the data collection tools we used in the case studies in table 3.3.2.

Table 3.3.2 Data collection tools in China and France

<b>Data</b>		<b>Chinese case</b>	<b>French case</b>
Classroom observation (Daily lessons)		Collect Lin's working sheets Video-recording (camera B) in class Take notes in class Take photos of blackboard	Collect Emma's working sheets Audio-recording in class Take notes in class Take photos of blackboard
Visit teachers' resources		Take photos	Take photos
Interaction with the teacher	Interview (formal)	Audio-record	Audio-record
	Conversation (informal)	Recalled the contents and take notes after communication	Recalled the contents and take notes after communication
Series of guiding activities (Chinese case)		Video-recording in classroom teaching (camera B); Collect working sheets of Shi; Interview with Shi (audio-record); Take notes during the interactions between Lin and Shi; Take photos of blackboard	
Thematic lesson (French case)			Interview with Emma (audio-record); Collect working sheets of Emma; Video-recording (camera A&B); Collect students' group working sheet; Take photos of blackboard

From table 3.3.2, we can see that we mainly use the following tools: collect teacher's working sheets (also students' sheets in thematic lesson), video or audio recording, take photos and notes, and communications with the case teachers. We introduce them one by one:

- Working sheets:

Working sheets refer to the teachers' lesson preparation sheets. Both of the case teachers prepare the corresponding lesson preparation sheet according to the teaching content during class preparation, and bring them to the classroom teaching. We collect these sheets for analyzing case teachers' documentation work (actual variation) and compare it with what actually happened in the classroom (practical variation).

In addition, in the French thematic lesson, we also collected students' working sheets because Emma continues to make teaching adjustments based on students' feedback. The arrangement of teaching is closely related to students' feedback, i.e., teacher's

documentation work (actual variation) relies on the students' reactions.

- Video or audio recording, photos and notes:

During the observation of daily lessons, we collected data by taking pictures of the screen/blackboard, audio-recorded or video-recorded in class, and taking notes in class. Considering the acceptance of the case teacher and students, we adopted video-recording in China and audio-recording in France. The audio and video recordings made during the observation supplement to the notes taken during the classroom observation. This research focuses on how teachers use variations in the classroom, as well as the variations produced by teacher-student interaction. Therefore, we plan to set up two cameras in the research design: one set at the middle of the back of the classroom, facing the blackboard, to observe the overall situation of the class; the other one is on the front side, which takes a detailed picture of the teacher's behavior, writings on the blackboard, etc. When the students conducted group activities and the teacher patrolled the whole class to give guidance to each group, this camera followed the teacher to record the interactions between the teacher and the students. The placement of cameras and recorders in this study is shown in Figure 3.3.4.

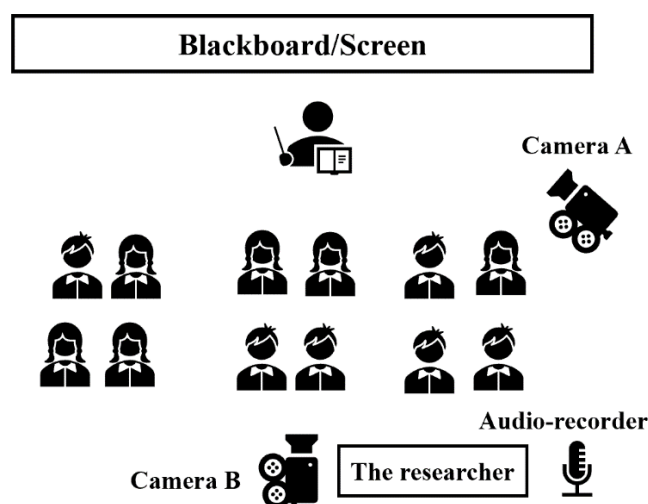


Figure 3.3.4 The design of classroom observation

In practice, since audio-record was used in the daily French classroom, we placed the recording pen next to the researcher so that the researcher can take notes and record the time corresponding to the recording time. In the French thematic lesson, we placed cameras A and B according to our research design above, and the camera A followed the teacher's movement. In the Chinese case, we only placed camera B. As there was no group activity, a single camera also met our recording needs.

- Interviews and communications with the case teacher:

Considering that teachers are very busy, this study used the combination of interviews



and dialogues described by Woods (1985). There are two formal semi-structured interviews in our research, one for realizing the background information, the other for learning more about the teacher's ideas and assuring the data is accurate. The formal interviews usually last from one hour to one and a half hours, and they were audio-recorded. After the data of interview is transcribed, it is delivered to the teacher to confirm that the interpretation is correct. During the period of observation, if case teacher had some time, we discussed the classroom teaching and subsequent lesson arrangements around 10 minutes after class. Sometimes, we used the teacher's tea break or had lunch with the teacher for a more relaxed discussion of mathematics teaching. After that, I recalled the conversation and wrote it down. The discussions that took place in these relaxed atmospheres often inspired us a lot.

In Chinese case, we also conduct one formal semi-structured interview with the preservice teacher Shi about the series of guiding activities. And as the school-based teaching book is a main resource and widely used in Lin's teaching, we have a communication the chef-editor about the design of this book. In French case, we conduct one formal unstructured interview with case teacher Emma about her preparation and her idea of design for the thematic lessons. This interview is conducted before the class.

We adopted progressive focus approach (Woods, 1985) during the interview. During the first encounter (formal interview), the teacher described his/her general situation, including his/her basic information (professional title, academic background, work experience, training activities that teachers participate in, etc.), the overall situation of the school (number of students, overall level, school activities, etc.), the situation of the observed class (students' mathematics level, class atmosphere, etc.), the organization of function lessons (duration, sequence of lessons, pre-knowledge of function, resources used, arrangement of thematic lessons, teacher's classroom design and teaching style, etc.). In order to understand the resources used by teachers in more detail, we also visit case teachers' resources in school and recorded them by taking photos. This is a preparation interview before the first visit to the classroom teaching. Follow-up interviews and conversations focus on issues and topics highlighted from dialogue and classroom observations, conducting retrospective interviews with teachers' thinking, design concepts, and reflections at that time.

It should be pointed out that during data analysis, we do not regard the collected data as static. The dynamics of data are mainly reflected in two aspects: 1) Our model of teaching through variation is dynamic: As we presented in section 2.4, the evolution of data is embedded through teachers' documentation work and their reflections; 2) We try to connect the data obtained in various ways, and hope that there is a comparison between the data to ensure the accuracy of our data analysis. For example, we check whether there is any discrepancy of the data between the interview and the classroom

observation, or there is mutual corroboration between data, or different data complement each other. Through this way, we can see a more comprehensive, accurate and intuitive case displayed before us.

### **3.3.4 Data analysis in case studies**

In response to the three research questions that we put forward, we explain how the data is analyzed in case studies from three perspectives: the analysis of teacher's resource system, the analysis of general situations, qualitative analysis of selected episodes. As we presented in section 3.3.1, we perform multiple case "replication" and adopt the same approach to conduct data analysis in Chinese and French cases.

#### **3.3.4.1 Analysis of case teachers' resources system**

The background of the case mainly involves the following perspectives: case teachers' resources, their resource system and case teachers' documentation work.

We adopt a combination of subjective and objective methods to analyze teachers' resources. From an objective point of view, the analysis of case teachers' resource system is based on the data collected through interview with case teachers, the visit to teachers' office and the observation of case teachers' practical teaching. Considering that there are many types of teachers' resources, we divide case teachers' resources into four types according to different teaching purposes: resources related to tasks; resources related to pure mathematics knowledge; resources related to didactics, and resources of software tools to assist classroom teaching. These four classifications are based on the classification of resources in the inferred mapping proposed by Rocha (2019), see Figure 3.3.5.

From a subjective point of view, we conclude case teachers' understanding of good resources and use of resources through their description in the interview and our observation. We hope that through case teachers' own understanding of good resources, we can see the reason why case teachers choose the resources they use nowadays among the many resources. Table 3.3.3 presents the analytical framework we constructed. Our approach is to analyze these aspects respectively and then try to make out how they interact with each other, and ultimately serve for teaching through variation.

As with the analysis of teacher resources, when analyzing the documentation work of case teachers, we also adopt a combination of subjective and objective methods. From an objective point of view, the analysis of case teachers' documentation work is based on the data collected through interview with case teachers, the visit to teachers' office and the observation of case teachers' practical teaching. We study teachers' documentation work through three perspectives: his/her interaction with resources, interaction with students, and interaction with other teachers. In studying case teacher's interaction with resources, we use the two intrinsically intertwined processes

“instrumentalization” and “instrumentation” in the documentary genesis (section 2.3). Furthermore, three aspects are considered in the analysis of instrumentalization: selection of resources, (re)design of resources, and the track and management of tasks. This is based on the two ways of teachers’ work (personalization and transformation) in the process of instrumentalization proposed by Trouche et al.(2020a), which described in section 2.3.2.1. From a subjective point of view, we describe case teachers’ interpretations of variation in tasks.

Table 3.3.3 Analytical framework of background information

	Objective point of view	Subjective point of view
Teachers’ resources and resource system	Resources related to tasks	Teachers’ understanding of good resources and use of resources
	Resources related to mathematics	
	Resources related to didactics	
	Resources of software tools	
Teachers’ documentation work	Interaction with resources: <u>instrumentalization and instrumentation</u>	Teachers’ interpretation of variation in tasks
	Interaction with students	
	Interaction with other teachers	

At the end of the analysis, we draw a mapping of the resource system of each case teacher through the lens of variation. This inferred mapping is adapted from the inferred mapping of the resource system proposed by Rocha (2019), see Figure 3.3.5.

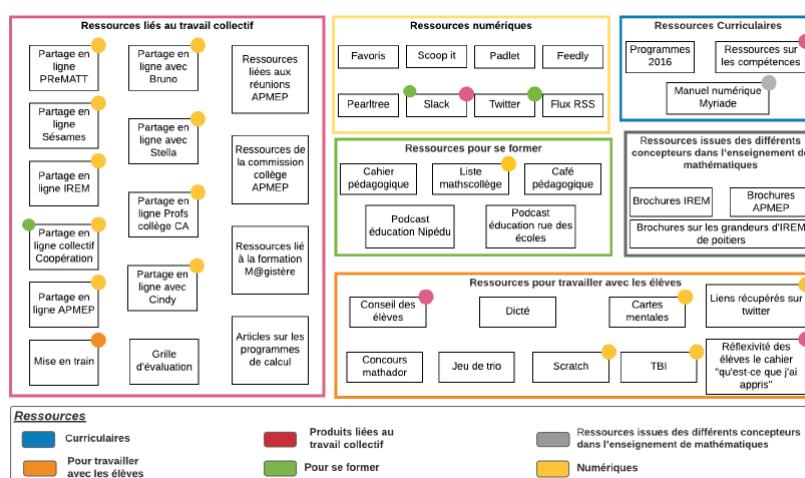


Figure 3.3.5 An inferred mapping of the resource system proposed by Rocha (2019, p. 193)

As we described in section 3.3.3.2, the principles of reflective investigation method are used as guidelines during our data collection. The inferred mapping in this study is made by the researcher reflected on the analysis of teacher resources and their documentation work (see table 3.3.3), which are based on the teacher’s interviews and observations of the teacher’s work. In addition, in order to better serve our research questions, we draw the inferred mapping of case teacher’s resource system through the lens of variation. We focus on the interaction between resources and evolution in the resources based on the variants and invariants in teacher’s resource system and

documentation work, in order to further discuss the potential resources with variation in high school mathematics teaching. We present our inferred mapping of teacher's resource system in Chinese case here as an example, see Figure 3.3.6.

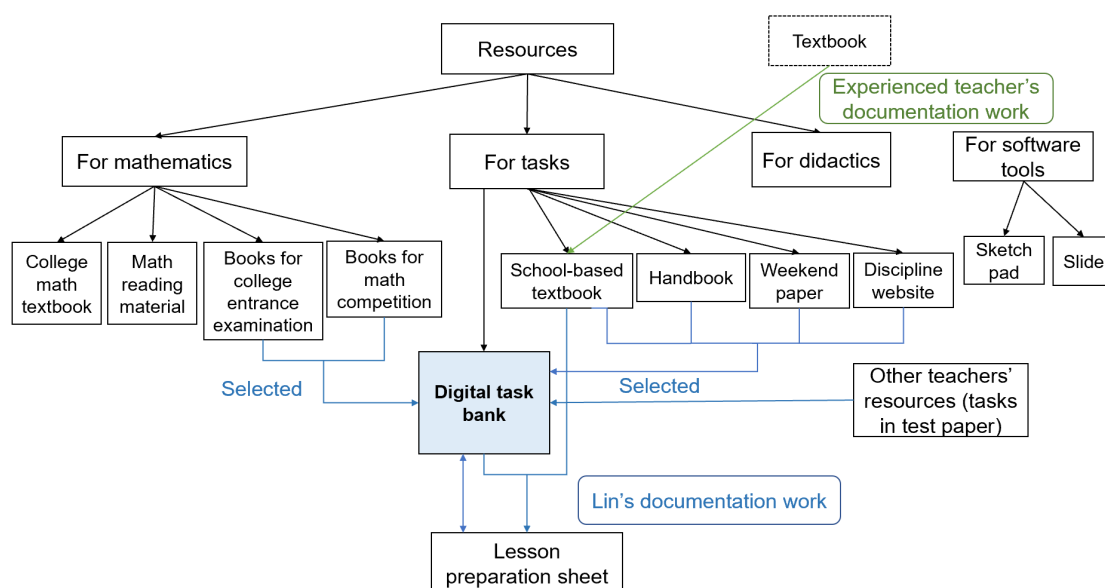


Figure 3.3.6 An inferred mapping of teacher's resource system in Chinese case

### 3.3.4.2 Analysis of General situation in classroom teaching

We study the general situation of case teachers' classroom teaching through variation from four types of variations (contextual, representational, conceptual, and procedural variation) based on our analytical framework (section 2.4.4).

As mentioned in section 3.3.3.1, the general situation in our study is based on the study of 24 lessons in Chinese case, of which 16 are lessons of new knowledge and 8 are lessons for exercise explanation, excluding the classrooms of preservice teacher Shi; and 26 lessons in French case, of which 17 are lessons of new knowledge, 9 are thematic lessons, see table 3.3.4. We study 53 tasks (27 series of tasks and 26 tasks) in Chinese case, and 37 tasks (29 series of tasks and 8 tasks) in French case.

Table 3.3.4 The number of tasks in case studies

		Chinese case	French case
Number of lessons		24 (16+8)	26 (17+9)
Number of tasks	Series of tasks	27	29
	Tasks	26	8
Tasks in total		53	37

Regarding the selection of the lesson and the counting of the tasks, we have the following points to explain:

- Only tasks of function counted: Because of the connection between functions, inequalities, intervals, and other knowledge, there are often review of other knowledge in the teaching of function. We only pay attention to tasks related to the

content of the function.

- The definition of series of tasks: As we need to distinguish series of tasks from tasks, we define a series of tasks in our research: a series of tasks is a list of tasks with a consciously chosen ordering which is relevant for one or several educational purposes (Gosztonyi, 2015a). It is proposed by the curriculum resources or the case teacher<sup>70</sup>. These tasks can be parallel or progressive. Series of tasks often noted with the second-level question number, such as 1(1), (2), or a), b). Sometimes Chinese teacher may use the Chinese characters “变式” (variation) or “知识迁移” (knowledge transfer) to note the tasks to show the relations in this series of tasks.
- We completely retain the task number that case teachers use in their working sheets or on the blackboard when performing case studies (the description of general situation and episodes).

In order to clearly show the number of occurrences of variations, the time of occurrence and the teaching content where they are located, we combine each variation with the timeline and display them in a schematic representation. We give an example of French case here, as shown in Figure 3.3.7.

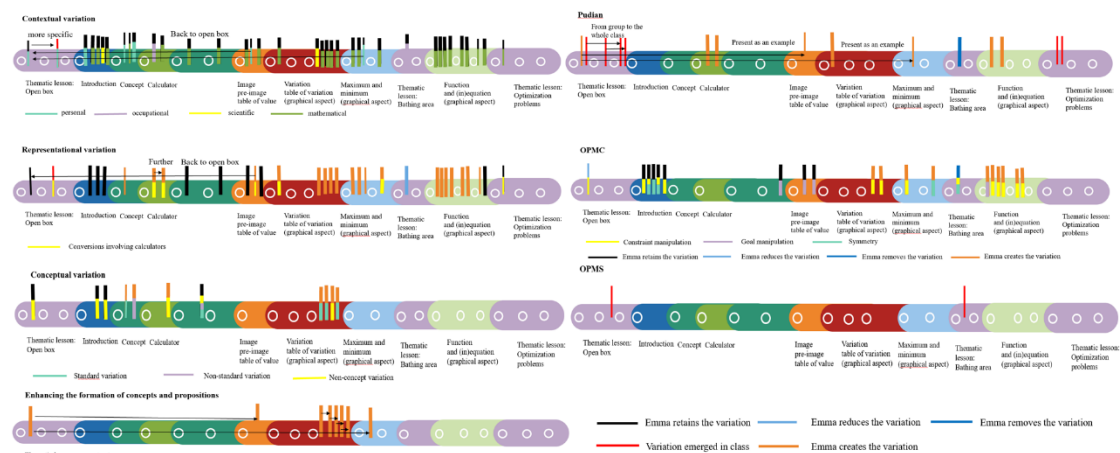


Figure 3.3.7 Schematic representation of occurrences of different types of variations

We make an explanation of this schematic representation:

- Timeline: We draw a timeline of the entire case teachers' teaching of function in grade 10, where each color represents each theme of function marked below the timeline. Circles in the timeline means the start of a lesson. One theme of function may have several lessons. For example, the concept of function is displayed in dark green in our schematic representation with 3 circles, which means that the teacher

<sup>70</sup> This refers to the situation where the tasks are first presented as a series of questions. Because when students encounter difficulties in actual learning, teachers will divide one task into series of tasks for students' understanding, which is especially common in French case. Such kind of situations are still counted as one task here.

has given three lessons for this content.

- Sticks on the timeline: Sticks on the timeline represent the occurrences of the variations. The sticks on the timeline are thick or thin. The thick one represents the variations occur in a series of tasks, and the thin one represents a task. In addition, different colors of the sticks represent different sources of the variations in the tasks: “black” means the variation is already embedded in the tasks and directly taken by the case teacher without changing anything; “orange” means the variation in the task is created by the teacher himself or herself. This may be that the task is created by case teacher, or that case teacher has created a new variation through the arrangement of existing tasks; “green” means there is variation embedded in the tasks and the teacher extends this variation; “blue” means there is variation embedded in the tasks and the teacher reduces this variation; “yellow” means there is variation embedded in the tasks and the teacher removes this variation; “red” means the variation emerges in the practical teaching through the interactions between case teacher and the students.

The main difference between “extend” and “create” here is whether the original task itself contains this type of variation and whether there is a new variation emerged through teachers’ documentation work.

In the process of studying general situations of variations, we first record the different types of variations that appear in the classroom teaching, and then we choose sticks of different colors according to the source (already embedded in the task, extend, reduce, remove or create by the teacher) of the variation in the task.

### **3.3.4.3 Analysis of episodes, guiding activities and thematic lessons**

Episode in our research means a clip or several clips of classroom teaching chosen by the researcher, starting with a given task or series of tasks proposed by the case teacher and end with the explanation and discussion between the teacher and students of this task or this series of tasks.

Analysis of episodes is a further analysis after the analysis of general situations of different types of variations, aiming to dig deeper into the characteristics and reasons of case teacher’s teaching through variation. Guiding activities and thematic lessons can help us to understand case teachers’ work from another dimension. We conduct qualitative analysis in selected episodes, guiding activities and thematic lessons.

In this section we explain the criterion we used for choosing the episodes, the tools we used (analytical framework and variation path) for analyzing each episode, guiding activity or thematic lesson and the schematic representation of variation path we draw for reflecting the evolution of variation through appropriation, implementation and reflection.

### **Criteria for choosing the episodes to zoom in**

We select episodes to study on the basis of the schematic representation of occurrences of variations of the general situation, following the three criteria below:

- We choose the episode with important mathematics knowledge to teach. The “important” here is determined by the national curriculum standard or textbooks, or this content is directly emphasized by the case in class;
- We choose the episodes with multiple processes of teaching. For important knowledge that is not easy for students to understand and master, case teachers often design a series of lessons for this teaching goal. In this process, the variation is more fully and complete, which is often accompanied by the introduction, in-depth, and review of knowledge. We can see the role of variation at different stages of teaching;
- We make sure that every type of variation is studied.

### **Analytical framework and variation path for analyzing each episode, guiding activity and thematic lesson**

We analyze the data based on the preliminary model (section 2.4.3).

Firstly, we study the potential variations embedded in the “original” tasks<sup>71</sup> in the episode, guiding activity, or thematic lesson; the actual variations through case teacher’s documentation work based on teacher’s working sheets, interview with the teacher; and the practical variations based on the classroom teaching observation. The analytical framework (section 2.4.4) is used here to study potential, actual and practical variations from different types of variations. In the analysis process, we retain the title of tasks used by the case teacher, such as the word “example”, “exercise”, or “task”. In our research, we do not make a specific distinction between these words. We use the word “task” to uniformly refer to all of them in the analysis.

Then we focus on the three processes: appropriation, implementation, and reflection. These three processes are to dynamically link the three relatively static variations (potential, actual and practical variations) we analyzed earlier. The schematic representation of variation path (section 2.4.4) is drawn for each episode, guiding activity and thematic lesson to make the process intuitive.

It has to be clear that the word “retained”, “extended”, “reduced” or “created” in the variation path correspond to the meanings of different colors of the sticks we mentioned in the general situation (section 3.3.4.2).

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<sup>71</sup> the “original” here is relative to the case teacher, representing the tasks that teacher gets from the beginning.

### 3.4 Contrasting the French and Chinese case

In our research, we followed the curriculum implementation process from the introduction of the curriculum standard (intended curriculum), to the analysis of the textbook and its associated digital curriculum resources (intended curriculum), to the analysis of specific classroom cases (enacted curriculum), attempting to first draw a picture of teaching through variation in China and France respectively, based on the preliminary model of variation and its corresponding analytical framework (section 2.4.3 & section 2.4.4), and then contrasting the two cases.

Our research is not designed to compare the teaching through variations in two countries, but an investigation with a common goal, i.e., to refine our preliminary model and corresponding analytical framework of teaching mathematics through variation with these two different cultural contexts. To achieve this common goal, three steps are taken in the contrast of two cases:

- Different types of variations in the framework are used to contrast the three stages (potential, actual and practical) of the two cases;
- Variation path (section 2.4.4) is used to contrast the evolution (i.e., retain, remove, reduce, extend, create) of the variation in the two cases;
- A framework with two main levels presented by Wang (2019) is used as a common framework to extend our results in Chinese and French case (see Figure 3.4.1): One is the level of mathematics teaching content and the other is the level of education noosphere.

As shown in Figure 3.4.1, the level of mathematics teaching contents concerns teachers' specific teaching activities, including the discipline, domain, sector, theme and subject (level 1-5) and the level of education "noosphere" (Chevallard, 1985) concerns teachers' teaching environment, including the pedagogy in the curriculum, the decisions or regulations from school level, curriculum developers or politicians, and the potential cultural influences (level 6-9).

According to the specific meaning given by Wang (2019), Artigue and Winsløw (2010), adapted to our own research, the nine detailed levels involved in this two-main-level model were explained and clarified as follows:

1. Subject prescribed by curriculum standards or official evaluations, e.g., operations on logarithmic functions in our research (section 5.3).
2. Theme prescribed by curriculum standards or teachers, or inferred from observation of several subjects within the theme, e.g., extremum of a function in our research.



3. Sector prescribed by curriculum standards or teachers, or inferred from observation of several themes within the sector, e.g., function in our research.
4. Domain within a given discipline, described in curriculum standards or by teachers, or inferred from observation of several sectors, e.g., algebra.
5. Discipline based on curriculum standards or other evidence, such as observation and assertions by teachers, e.g., mathematics.
6. Pedagogy prescribed by schools or curriculum standards, observed or described by teachers, e.g., technologies should be integrated into the teaching of function.
7. School or teaching institution, such as the conditions and characteristics of it and the obligations or autonomy of teachers, e.g., School C and School F in our research.
8. Societies including in which way the school are governed, founded and systemically organized, e.g., The Ministry of Education of China and France in our research.
9. Civilization or cultural contexts, the principles for the role and meaning of teachers' work, or education ideas, e.g., Chinese culture and French culture in our research.

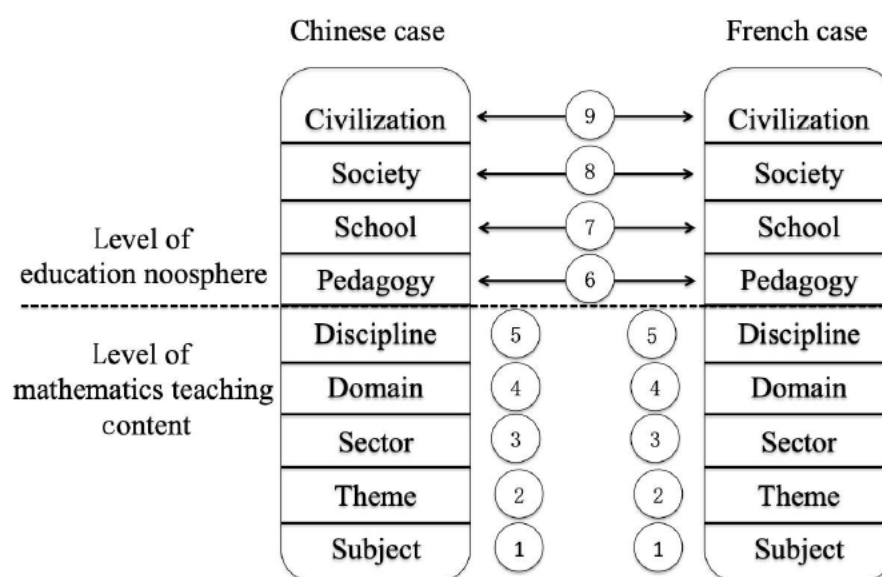


Figure 3.4.1 Two-main-level model for crossing cases (Wang, 2019, p71)

These nine levels run through our cases studies and crossing analysis. The contrasting analysis is based on case studies in their respective contexts with detailed evidences and traces of case teacher's work. It helps us further reflect on the validity of our model and analytical framework, also consider the possibility of situating case teacher's results to a larger context (section 7.1.4).

## Chapter 4

### Potential Variation in the curriculum resources

Textbooks have historically played key roles in providing teaching resources to link the intended curriculum and the enacted curriculum over years. It specifies the content to be taught and provides guidelines about how this content might be taught (Chazan & Yerushalmy, 2014). However, the development of technologies might challenge the traditional role of textbooks and the corresponding role of the teacher in their documentation work for teaching. Therefore, at this turning point, we do not discuss the traditional textbooks apart from digital curriculum resources, but consider them as a dynamic development process of curriculum resources. We analyze the evolution of textbooks, from traditional textbooks to digital curriculum resources, through the lens of potential variation.

In section 4.1, we present the general situation concerning mathematics textbooks, their design and use, for French and Chinese high school. In section 4.2, we analyze the potential variation in Chinese PEP textbooks, focusing on the chapters of function. Section 4.3 is a twin section of section 4.2, aiming at illustrating potential variations in the tasks of function in French Hyperbole textbook. In section 4.4, we focus on the potential variations embedded in the digital resources, taking Sésamath and PEP platform as examples. In section 4.5, we make a conclusion from the traditional textbooks, digital curriculum resources and the role of teachers in this transitional time through the lens of variation. Chapter 4 is also an entrance to our following case studies, visiting the landscape of teachers' work.

#### 4.1 The general situation of the textbooks

Both China and France have national mathematics curriculum standards, and the textbooks are designed based on the curriculum standards. On one hand, international comparative studies of mathematics education (such as TIMSS<sup>72</sup>, PISA<sup>73</sup>, etc.) show that curriculum and textbooks have a significant impact on students' learning and achievement. Studies have shown that “learning opportunities” is a major factor affecting students' mathematics achievement, and curriculum and textbook design are

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<sup>72</sup> TIMSS: Trends in Mathematics and Science Study. Website: <https://timssandpirls.bc.edu/timss-landing.html>

<sup>73</sup> PISA: Programme for International Student Assessment. Website: <https://www.oecd.org/pisa/>

also important factors affecting “learning opportunities” (Wang & Zhang, 2014). On the other hand, in high school, textbooks play a core role in teaching, especially the tasks in the textbooks are used for classroom exercises and homework assignments. Therefore, in order to analyze the mathematics education in China and France, it is essential to analyze the textbooks referenced by teachers.

In this section, we will give a general picture of textbooks in China and France from the following aspects: 1) the evolution of the design of textbooks based on the curriculum reform; 2) the structure and content of the selected textbook; 3) the relationship between textbook and other teaching resources.

#### **4.1.1 The general situation of the textbooks in China**

In 2003, China promulgated “the Standards of General Senior High School Mathematics Curriculum (Experimental Draft)”. This new curriculum reform in senior high school has undergone significant changes in terms of curriculum goals, curriculum structure and content, curriculum implementation and evaluation. In particular, the concept of learning areas and module design has been introduced in curriculum design, the course selection system has been introduced in curriculum management, and the implementation of the credit system in curriculum assessment and evaluation, which not only brings new ideas and methods to ordinary high school education, but also makes Chinese ordinary high school education face great challenges (High School New Curriculum General Training Series Design Group, 2004). This standard promotes the diversification of textbook design. Different regions can appropriately adjust the order of the content in textbook design according to the characteristics of students and teachers, and design their own characteristics, in order to motivate teachers to teach and students to learn (Zhang & Liu, 2020). Taking into account the differences between urban and rural areas and the differences in teaching levels between provinces and cities, different provinces (or regions) will choose the textbooks independently, which is determined by the local education system. For the provinces with advanced education, the difficulty of textbooks can be appropriately increased, and the requirements for students can also be appropriately increased. For backward provinces with underdeveloped education, the choice of textbooks should first consider the popularization of the education, not the difficulty. At the same time, adaptability should be considered in the design of textbooks. For example, the main content and context in the textbook cannot be the content that urban students can understand, while the rural students cannot.

After being reviewed by the National Primary and Middle School Textbooks Evaluation Committee in 2004, there were five sets of experimental textbooks prepared in accordance with the new curriculum standards: People’s Education Press A and B, Beijing Normal University Press, Hunan Normal University Press, Jiangsu Education

Press. These textbooks collectively reflect the basic concepts of the curriculum standards and have entered the National Basic Curriculum Reform Pilot Zone for high school students. In 2007, China comprehensively promoted new textbooks, and since then has formed a new situation of “multiple textbooks with one curriculum standard” (Zhang & Liu, 2020).

Considering the coverage of the textbooks and whether the textbooks conform to the concept of curriculum reform in the country and the overall trend of curriculum reform, we choose Chinese PEP (People’s Education Press A version) textbook to study (as we presented in section 3.1.2).

According to the first-hand information about the suitability of the Standards for General High School Mathematics Curriculum (Experimental) and its teaching materials (People’s Education A version) obtained from classroom observations, sample surveys, interviews between teachers and students, and thematic studies, Zhang (2019) reflected on the existing problems in the previous textbook (published in 2007) and summarize some of the guiding value for textbook revision: (1) Curriculum philosophy and goals: Mathematics curriculum should focus on “basic”. “Selectivity” is needed, but it must be coordinated with the students’ future development tendency. The “three-dimensional goal” of the current mathematics curriculum is very comprehensive, but teachers generally report that “the three-dimensional goal is good, while its operability is not strong.” (2) The structure of the textbook is modular and spiral: First, modularization leads to logical problems in the overall knowledge structure of the curriculum standard textbooks; second, due to the limitation of class hours, some knowledge in a module is deleted, resulting in inconsistent knowledge, and some knowledge which is not directly related is placed in the same module in order to piece together class hours; third, the knowledge is arranged in a spiral, which artificially cut off the logical chain of knowledge, and the teaching requirements become ambiguous and difficult for teachers to grasp; (3) Problems on the content and learning order: Departing from specific subject knowledge and simply training in thinking and proof methods have no effect on improving students’ thinking ability. For example, there are some overlaps between the content of “block diagrams” and “preliminary algorithms”. Besides, many teachers believe that mathematics study in high school is to learn reasoning and proof and it is not necessary to set it as a separate chapter; (4) Heavy academic burden for students: mainly includes the pressure from the college entrance examination, too much new knowledge, structural disorder, unclear teaching requirements, tense class hours, problems with junior high school, the interference of other teaching and learning resources, etc. The above review and reflection on the existing textbooks (PEP, 2007) laid the foundation to design the new textbooks in China. These conclusions thus clarify the direction of the revision of PEP textbooks in the new

round of curriculum reform.

As the new curriculum standard is issued in 2017, the corresponding textbooks are designed and begin to use in September 2020. According to the latest curriculum standards, the new textbook highlights the four main mathematics lines: function, geometry and algebra, probability and statistics, mathematical modeling activities and mathematical inquiry activities. The column “literature reading and mathematical writing” is set up, taking the history of mathematics and culture of mathematics as the basic carrier. Mathematical modeling activities are mainly set in themes such as functions, probability and statistics that are more closely related to reality. Mathematical inquiry activities take the form of daily penetrating teaching and topic setting, mainly set at the intersection of mathematical knowledge.

Besides textbook, PEP published a series of teaching resources to support teachers’ teaching with PEP textbooks, such as teachers’ guidebook, book about research on the design of high school mathematics textbooks, students’ exercise book, students’ reading book, and various digital resources and so on.

Teachers’ guidebook is arranged in five parts: overall design, textbook analysis, exercise solutions, cases of teaching design, evaluation suggestions and tests in the order of the corresponding textbooks. The “overall design” is a general introduction to the whole chapter, focusing on the design ideas of this chapter. It includes the overview of this chapter, the learning objectives of this chapter, the structure diagram of the knowledge, the arrangement of content, the arrangement of lesson hours, the intention of the design of this chapter, and the teaching suggestions of this chapter. “Textbook analysis” takes the section of the textbook as a unit, carry out the analysis according to the sequence of the textbook, focus on presenting and describing the knowledge structure diagram of this section, key points, difficulties, textbook design intention and teaching suggestions, so as to help teachers accurately identify the relevant contents. “Exercise solutions” gives reference answers to textbook exercises. For the comprehensive problems, complete solution steps or problem-solving ideas are given. Some “cases of teaching design” are selected and proposed, especially for the typical, difficult in teaching, and new knowledge. Each teaching design includes teaching content explanation, teaching goal explanation, teaching problem diagnosis and analysis, teaching support condition analysis, teaching process design, examination design and so on. “Evaluation suggestions and tests” are put forward from the aspects of curriculum requirements, evaluation suggestions, task design suggestions, etc., and exemplary evaluation tests are given for reference with the explanations of the design intent of each task. In the follow-up analysis of the tasks in the textbook (section 4.2), we take profit of the explanation in teachers’ guidebook.

Various digital resources for supporting teaching are offered with the PEP textbook,

such as digital textbooks, animations and interactive exercises, online platform and so on. We will discuss the online platform in more detail and in-depth analysis in section 4.4.3.

The structure of each chapter of PEP textbooks (published in 2019) is shown in Figure 4.1.1.

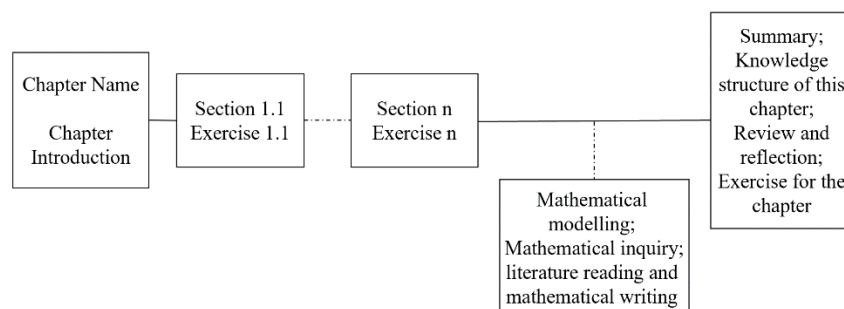


Figure 4.1.1 The structure of each chapter of PEP textbook

Figure 4.1.1 shows that the design of the structure of PEP textbook is consistent. At the beginning of each chapter, the content of this chapter is introduced with the chapter name and chapter introduction reflecting the main content of this chapter, so that students can understand the overview of the content of this chapter, and understand the main ideas and methods of this chapter, which can be used for students to preview, and can also be used as materials for teachers to introduce a new chapter. Teaching content is divided into different sections, and each section is equipped with corresponding exercises. PEP textbook also arranges optional columns (not as an exam requirement), such as “mathematical modelling”, “mathematical inquiry”, “literature reading and mathematical writing”, which provide resources for deepening students’ understanding of relevant content, expanding students’ knowledge, and using information technology to learn. For example, in the chapter of function, PEP textbook arranges “literature reading and mathematical writing” of the formation and development of function. The purpose of the textbook is to understand the history of function formation and development, and to experience the writing process and method of literature review. Four topics are provided for students to choose from: 1) The social background of the function; 2) The historical process of the development of the concept of function; 3) The story of function notation; 4) Mathematicians and functions. Textbook also proposes the structure of a literature review. Each chapter arranges a “summary”, including the knowledge structure of this chapter, the review and reflection on the content of this chapter and exercises for this chapter. “Knowledge structure of this chapter” reflects the main knowledge in this chapter, the development process of knowledge and the interrelation between knowledge in the form of a block diagram. We take the chapter of function as an example and show its block diagram in Figure 4.1.2. “Review and reflection” summarize the main content of this chapter and the ideas

and methods reflected in it to deepen students' understanding of the core content of this chapter and the mathematical ideas and methods reflected in it.

We zoom in on each section in Figure 4.1.1 and then we get Figure 4.1.2, which shows the structure of each section of PEP textbook. As shown in Figure 4.1.2, each section begins with problem analysis. Teachers can arrange mathematical activity such as “observation”, “thinking”, “exploration” and “induction” as needed, or add some open tasks to stimulate students' innovative thinking. For each mathematical statement, there are corresponding examples. The forms of examples are various, including regular, open, exploration and practical. There are also “Posts” and “Thinking” in the textbook. “Posts” introduce background knowledge related to the mathematical content, and “Thinking” contains some questions that help students to understand and to promote their learning. At the end of each section, there is an optional column. Its content is diverse, mainly in four categories: observation and conjecture, reading and thinking, inquiry and discovery, information technology application. According to teachers' guidebook (2019), in the process of content development, emphasis is placed on the use of logical methods such as “analogy”, “induction”, “specialization” and “generalization”, so as to encourage students to carry out active learning. At last, there are exercises designed for the section.

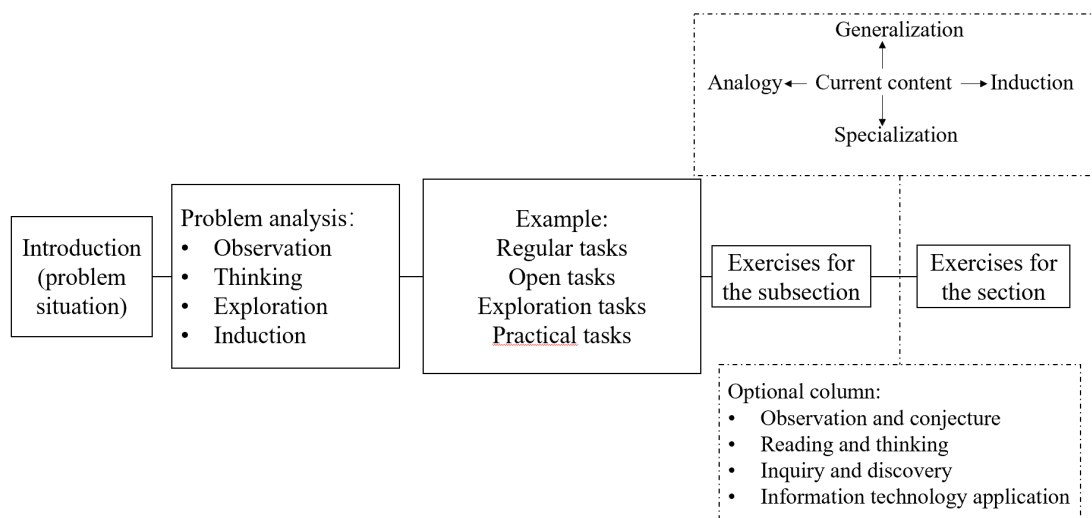


Figure 4.1.2 The structure of each section of PEP textbook

Tasks in PEP textbook can be categorized as problems, examples and exercises. Problems and examples are designed to use in the classroom teaching. The exercises in this textbook are divided into three categories: exercises for the subsection, exercises for the section, and exercises for the whole chapter. Exercises for the subsection are designed to use in class, some are for the practice of what has been learned, and some are for extension of related content; exercises for the section are designed to use as students' homework; exercises for the chapter are designed to use in reviewing the

whole chapter. Tasks of exercises for section and for chapter are divided into three levels according to their functions: “review and consolidation”, “comprehensive application” and “extension and exploration”. We illustrate problems, examples and exercises, and relations between them from the perspective of variation in section 4.2.

#### **4.1.2 The general situation of the textbooks in France**

Publication and use of textbooks are free in France. There is no authority to regulate the publication of textbooks. Textbooks used by students are generally called “classroom textbooks”. In France, textbooks are usually designed by experts (supervisors, teacher educators, etc.). In most cases, classroom textbooks are chosen by a team of teachers and validated by the school board. In elementary and junior high schools, classroom textbooks are purchased by schools, but in high schools, textbooks are purchased by students. In high school, textbooks play a central role in teaching activities, especially the exercises in the textbooks are used for classroom practice and homework assignments (Gueudet, Bueno-Ravel, Modeste & Trouche, 2018). In previous research (Gueudet & Trouche, 2012; Pepin, Gueudet & Trouche, 2013), researchers investigated teachers’ curriculum implementation: selecting resources, converting these resources, creating resources for students, etc., research found that textbooks are still the main resource for math teachers. In addition to the classroom textbooks used for exercises, teachers use four to five other textbooks to prepare lessons, design lesson introduction activities, or design exercises.

In 2016, because of the curriculum reform, the French Ministry of Education selected a series of publishers to provide electronic resources consistent with the new curriculum. Teachers and students can use this series of online platforms to access resources. This evolution naturally also affected the textbook (Gueudet, et al., 2018). Recently, all paper textbooks published in France are accompanied by electronic resources: when a teacher or student purchases this textbook, he/she will get a digital version of the textbook; companion websites related to textbooks will provide resources, such as slides, various software, files, and answers to the exercises in the textbook. This evolution may affect the mode of textbooks (paper and electronic textbooks) design, we further discuss it in section 4.4.

As we explained in section 3.1.2, we choose French Hyperbole textbook (2014 edition) to study in the following sections. According to the designer’s note in this textbook, Hyperbole textbook (2014 edition) is in line with the spirit of the mathematics curriculum standards applicable to the start of the 2014 school year and aims to provide the best possible support for students in learning mathematics and to be a real educational support for teachers. We mainly illustrate its reflection on the curriculum standard from the following aspects: 1) Problem solving occupies an essential place and comes in various forms (discovery activities, practical task, training tasks, in-depth



study tasks); 2) Particular care has been taken in designing tasks. Precise methodological advice helps guide the student in his or her work. In addition, the statements proposed through the various task headings are of a progressive level; 3) Largely based on the use of tools: many tasks are designed in a variety of situations with the note “TICE”, which means solving them by using graphic calculators, dynamic geometry software, computer algebra or spreadsheet software; 4) Algorithms are also covered in all chapters, always with a concern for progressiveness; 5) Tasks of “training in logic” present in each chapter, makes it possible to fix the concepts and the vocabulary of mathematical logic.

The structure of each chapter of Hyperbole textbooks (2014) is shown in Figure 4.1.3.

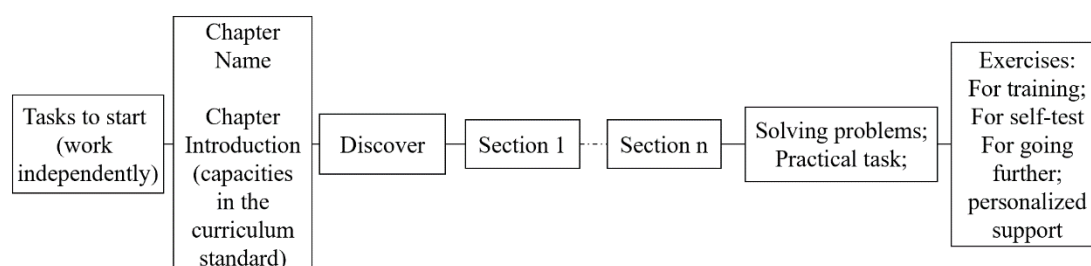


Figure 4.1.3 The structure of each chapter of Hyperbole textbook

As shown in Figure 4.1.3, each chapter begins with several tasks which need to be complete by students themselves. This is more like a small test to realize students’ understanding of related knowledge learnt in the lower secondary school. Then, a photo and a text are presented to make it possible to situate the chapter in the world around us and in the history of science. The task number is proposed in connection with the expected capacities mentioned in the curriculum standard. In the “Discover” part, two pages of activities introduce new notions. Tasks to be solved in the activity is indicated by a frame organized in the textbook and often associated with information and communication technologies for education<sup>74</sup>. After this, textbook presents the contents in each section. At last, there are exercises designed for the whole section. There are many types of exercises in Hyperbole textbook: solving problems, practical task, tasks for training, for auto-test, for going further, and also tasks for providing personalized support. Different types of exercises make it possible to take care of the progression and to diversify the mathematical activity. Similar to the organization of “know-how” part, “solving problems” first propose a problem-solving task with its solution, and then present several problems with the same mathematical knowledge after. Several

<sup>74</sup> Information and communication technologies for education: Often note as “TICE” in French textbook. TICE= Technologies de l’information et de la communication pour l’enseignement (in French). It means digital education refers to the use of information and communication technologies for education in the context of education (TICE = TIC + education).

practical tasks make it possible to experiment using ICT tools<sup>75</sup>. At least one of the practical tasks is designed to use algorithmic notions. In the part of “tasks for training”, there are tasks called “quick questions” to start mathematical topics in each section. There are also tasks called “training the logic” in each section. In the part of “tasks for auto-test”, there are MCQs with progressively and systematically increasing difficulty offered in each section. Tasks for further is to open the students’ horizons, and there are often some open tasks in this part. Two pages of personalized support tasks are dedicated to personalized accompaniment, support students’ classroom learning with specific headlines for each content or deepening their understandings. Not all the answers to the tasks can be found in the textbook, only tasks with the titles in green are corrected at the end of the textbook. In order to promote a progressive and reassuring acquisition of knowledge and methods, the structure of each chapter is identical.

We zoom in on each section in Figure 4.1.3 and then we get Figure 4.1.4, which shows the structure of each section of Hyperbole textbook. In each section, textbook presents mathematical definitions and properties for classroom teaching, which is illustrated with examples and graphs. An exercise called “know-how”<sup>76</sup> is an exercise presented with solution (one or more solving methods). Methodological advice is given to solve the task. Several exercises of the same type are proposed to work on these methods after.

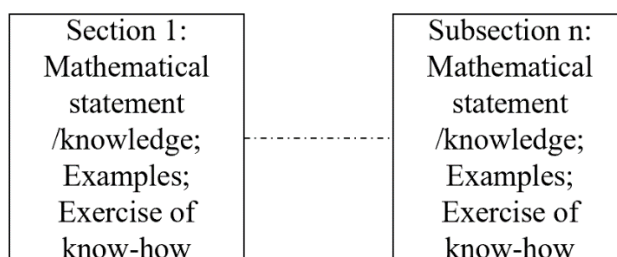


Figure 4.1.4 The structure of each section of Hyperbole textbook

Similar to the situation of Chinese PEP textbook, French Hyperbole textbook also provides various other resources for supporting teaching, such as teachers’ guidebook, digital textbooks, animations and interactive exercises and so on. In Hyperbole teachers’ guidebook, there are only the answers to the tasks in the textbook and the problem-solving process.

### 4.1.3 Comparison of the general situations of Chinese and French textbooks

It is a time of transition of curriculum resources in both sides, China and France,

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<sup>75</sup> ICT tools: Information and Communication Technology tools, such as computer, graphic calculators, specialized software, spreadsheets, etc.

<sup>76</sup> Know-how: Identify, formulate possible responses to a task and apply the solution considered best suited to solve it.

because of the curriculum reform and the development of technology. Textbook is still at the center of curriculum resources in both countries. Besides textbooks, there are numerous resources designed to support teaching with textbooks, such as teachers' guidebook, digital textbook, interactive exercises, online platform and so on. We compare the general situations of Chinese and French textbooks in this situation of transition from the variation point of view:

- Number of tasks and the intended role of the teacher: In Hyperbole textbook, there are great numbers of tasks (around 100 tasks) in each chapter. The intended role of the teacher is to choose which tasks he/she wants to use for supporting his/her teaching. While in PEP textbook, there are much fewer tasks (around 30 tasks) in the same mathematical content. The intended role of the teacher is to adapt, use and extend these tasks.
- Reflective thinking is emphasized in Chinese PEP textbook. In the process of content development, emphasis is placed on the use of logical methods such as “analogy”, “induction”, “specialization” and “generalization”, so as to encourage students to carry out active learning (see table 4.1.2). Students and teachers are given new roles under this design, to think about and reflect on the tasks. Students not only doing tasks but also have reflective understandings about them. Teachers not only teaching or explaining tasks but also guide students to think mathematically. We cannot find this kind of situation in French Hyperbole textbook.
- Diversity of teachers' book of textbook: As the main supporting resources of textbooks, teachers' book in two cases are very different. In French Hyperbole teachers' book, there are only the answers of exercises, while in Chinese PEP teachers' guidebook, there are also the explanation of the intention and the spirit of the design of chapter, section and tasks, etc.

In general, from the point view of variation, in Chinese case, variation is about a given task, for French case, variation is about selecting the sequencing of tasks, not the content of tasks, but the sequencing of tasks.

## **4.2 Potential variations in the tasks in Chinese PEP textbook**

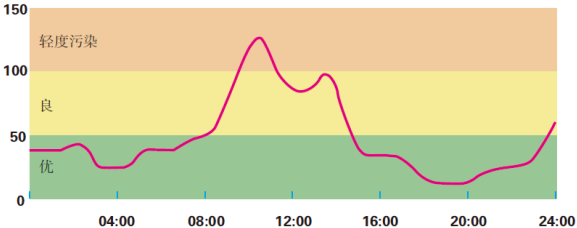
As we present in section 4.1.1, there are mainly three types of tasks in PEP textbook: problem, example, and exercise. In this section, we study the potential variations already embedded in different types of tasks of function in PEP textbook and the relations between them (section 4.2.1, 4.2.2, & 4.2.3). The analysis of potential variations is based on the analytical framework proposed in section 2.4.3: contextual, representational, conceptual, and procedural variation.

### 4.2.1 Tasks as problems for exploration and thinking in classroom teaching

In this section, we describe the four problems of function shown in the PEP textbooks.

Instead of proposing the concept of function to students directly, PEP textbook presents four problems (table 4.2.1) to arouse students' thinking about the function and to help students to generalize the concept.

Table 4.2.1 Series of problems for the concept of function

<p><b>Problem 1:</b> A certain ‘Fuxing’ high-speed train accelerated to 350km/h and kept running at a constant speed for half an hour. During this period, the relationship between the distance traveled by the train (unit: km) and the running time t (unit: h) can be expressed as s=350t.</p>																						
<p><b>Thinking:</b> If someone says: ‘According to the corresponding relationship y=350x, after this train accelerates to 350km/h, it travels 350km for 1h. Do you think this statement is correct?</p>																						
<p><b>Problem 2:</b> An electrical maintenance company requires workers to work at least 1 day a week and no more than 6 days at most. If the company’s wage standard is 350 yuan per person per day, and wages are paid once a week, how do you think a worker’s weekly wage should be determined? Is a worker’s wage w (unit: yuan) a function of his working days d? w=350d, d = {1,2,3,4,5,6}</p>																						
<p><b>Thinking:</b> The functions in problem 1 and problem 2 have the same corresponding relation, do you think they are the same function? Why?</p>																						
<p><b>Problem 3:</b> Figure shows the graph of the variation of Air Quality Index (abbreviated as AQI) on November 23, 2016 in Beijing. How to determine the air quality index (AQI) value Y at any time of the day according to the graph? Do you think Y here is a function of t?</p>																						
<div></div>																						
<p><b>Thinking:</b> Can you find the value of the AQI at 12:00 based on the graph?</p>																						
<p><b>Problem 4:</b> Internationally commonly used Engel coefficient r (<math>r = \frac{\text{Food expenditure}}{\text{Total expenditure}} \times 100\%</math>) reflects the quality of life of the people in a region. The lower the Engel coefficient, the higher the quality of life. Table 3.1-1 shows the variations in Engel’s coefficient of urban residents in a certain province of our country. It can be seen from this that the quality of life of urban residents in this province is getting higher and higher.</p>																						
<table><tr><td>年份 y</td><td>2006</td><td>2007</td><td>2008</td><td>2009</td><td>2010</td><td>2011</td><td>2012</td><td>2013</td><td>2014</td><td>2015</td></tr><tr><td>恩格尔系数 r (%)</td><td>36.69</td><td>36.81</td><td>38.17</td><td>35.69</td><td>35.15</td><td>33.53</td><td>33.87</td><td>29.89</td><td>29.35</td><td>28.57</td></tr></table>	年份 y	2006	2007	2008	2009	2010	2011	2012	2013	2014	2015	恩格尔系数 r (%)	36.69	36.81	38.17	35.69	35.15	33.53	33.87	29.89	29.35	28.57
年份 y	2006	2007	2008	2009	2010	2011	2012	2013	2014	2015												
恩格尔系数 r (%)	36.69	36.81	38.17	35.69	35.15	33.53	33.87	29.89	29.35	28.57												
<p><b>Generalization:</b> What features do the functions in problem 1- problem 4 above have in common? From this, can you generalize the essential characteristics of the concept of function?</p>																						

Problem 1 is selected based on the knowledge of function that students have learned in junior high school. Students are familiar with the analytic formula  $S=350t$ , and it is easy to explain that there is a unique distance  $s$  corresponding to the certain time  $t$ . The formula of problem 2 is the same as that of problem 1, but because the variable range of the independent variable is different, they are two different functions. The purpose of problem 3 and 4 is to provide students with more opportunities to understand the

elements of the function from different angles, especially the importance of the corresponding relation to the function, taking profit of different context and representations of function. In the textbook, these four problems are accurately described by using sets and corresponding language, guiding students to discover the three elements of the function, and prepare for the abstract function concept. After the analysis of these four problems, the textbook arranges a generalization: “What features do the functions in problem 1- problem 4 above have in common? From this, can you generalize the essential characteristics of the concept of function??”

We illustrate potential variations embedded in/among these four problems:

- Contextual variation: The contexts of these four problems are different but all closely related to real life, respectively belong to societal, occupational, scientific, and societal context. Therefore, there is a contextual variation from societal/occupational/scientific context to mathematical context in each problem. On the one hand, problems with various contexts allows students to experience the extensive use of functions in reality, so that students feel the necessity of learning functions. On the other hand, problems related to real life make the acquisition of the corresponding relation and domain of function more direct and easier for students to understand.
- Representational variation: Function in problem 1 and problem 2 are represented in formula, problem 3 is represented in graph, and problem 4 is represented in table of value. Through these 4 problems, three representations of the function are shown, paving the way for the subsequent teaching of the representation of the function. Representational variations here make students realize that the corresponding relation  $f$  can be a formula, a graph, or a table of value. From the graph and a table of value, we can intuitively see the corresponding relation between  $x$  and  $y$ .
- Conceptual variation: About the concept of function, problem 1 is a standard variation, as it is a linear function, simple formula and students are familiar with the context of the problem, which is connected with the content in junior high school. Problem 2-4 are non-standard variations. Problem 2 and problem 4 are discrete functions. Problem 3 uses the graph, and problem 4 uses the table of value to represent functions, which are not familiar ways for students. Various types of functions and representations of various functions help students better grasp the connotation and extension of the concept of function.
- Enhancing the formation of concept is embedded: Teachers’ guidebook (2019) indicates that the discovery of concept of function is difficult for most students, especially to express concepts in set language and corresponding relation, which is very mathematical. It is not necessarily the best way to require students to explore

and discover independently. Therefore, the textbook designed some questions called “Thinking” to guide students. For example, one question is raised in problem 1: “If someone says: ‘According to the corresponding relationship  $y = 350x$ , after this train accelerates to 350km/h, it travels 350km for 1h. Do you think this statement is correct?’” From the reason that this statement is incorrect (not paying attention to the variation range of  $t$ ) leads to the need to define the function in terms of set language and corresponding relation.

Teachers’ guidebook (2019) recommends guiding students to use the same language to describe the corresponding relation and laws for the following three problems, so as to pave the way for generalizing concepts. It also points out it is an important opportunity to implement mathematical abstraction in the teaching of function concepts, and teachers should not replace it. Teachers are suggested to arrange mathematical activities, so that students can experience from sets  $A_1, A_2, A_3, A_4$  to set  $A$ , from sets  $B_1, B_2, B_3, B_4$  to set  $B$ , from specific formulas, graphs, tables to the abstract corresponding relationships  $f$ . On this basis, organize the following table (see Figure 4.2.1) for students to observe the common characteristics: 1) Both contain two non-empty number sets  $A, B$ ; 2) Both have a corresponding relation  $f$ ; 3) Although the representation of the corresponding relation  $f$  is different, they all have the following characteristics: For any number  $x$  in the number set  $A$ , according to the corresponding relation, there is a unique number  $y$  corresponding to it in the number set  $B$ . The general concept of a function can be obtained by using sets and corresponding languages, using a unified notation.

问题情境 Problem	自变量的集合 Set of independent variables	对应关系 relation	函数值所在集合 Set that the function value is in	函数值的集合 Set of function values
问题 1 problem 1	$A_1 = \{t \mid 0 \leq t \leq 0.5\}$	$S = 350t$	$B_1 = \{S \mid 0 \leq S \leq 175\}$	$B_1$
问题 2 problem 2	$A_2 = \{1, 2, 3, 4, 5, 6\}$	$w = 350d$	$B_2 = \{350, 700, 1\ 050, 1\ 400, 1\ 750, 2\ 100\}$	$B_2$
问题 3 problem 3	$A_3 = \{t \mid 0 \leq t \leq 24\}$	graph in problem 3	$B_3 = \{I \mid 0 < I < 150\}$	$C_3$ ( $C_3 \subseteq B_3$ )
问题 4 problem 4	$A_4 = \{2006, 2007, 2008, 2009, 2010, 2011, 2012, 2013, 2014, 2015\}$	table in problem 4 表 3.1-1	$B_4 = \{r \mid 0 < r \leq 1\}$	$C_4 = \{0.366\ 9, 0.368\ 1, 0.381\ 7, 0.356\ 9, 0.351\ 5, 0.335\ 3, 0.338\ 7, 0.298\ 9, 0.293\ 5, 0.285\ 7\}$

Figure 4.2.1 A table for generalization of the concept of function

- Pudian is embedded for the following study of the representation of the function: Teachers’ guidebook (2019) clearly points out that these four examples pave the way for the three representations of learning functions. For example, the curve in problem 3 represents the change of air quality index in Beijing on November 23, 2016. The graph intuitively expresses the trend of the change of the corresponding

function value with the change of the independent variable, which is helpful for us to study some properties of the function.

- OPMC: Problem 2 and Problem 1 have the same formula but different domains. Problem 2 is a discrete function. In “Thinking”, students are required to compare the two, so that students can further realize the importance of paying attention to the domain of a function.

Here, the “inductive” method is adopted in the teaching of the concept of functions. Teachers’ guidebook (2019) states that “In classroom teaching, it is necessary to reflect the intention of the textbook, and arrange a process of ‘specific case-observation, experiment-comparison, classification-analysis, synthesis-abstract, and generalization’ for students, so that students have the opportunity to draw general laws through their own analogy and induction, and obtain the concepts and properties of functions.” It also states that “The emergence of the concepts of functions and mappings is much earlier and more natural than formal definitions. We can and even have to use the concept of functions that arise in practice without having to create or define functions, mappings first. After students have been exposed to many functions and have been able to construct functions, let them generalize what functions are. This is an example of mathematical activities to create this new basic concept.”

We noticed that the series of problems is presented with several thinking questions here. This series of problems are carefully designed with variants and invariants, and questions proposed in thinking make the embedded variations explicit and then trigger students to think, to compare, to generalize, and finally achieve the teaching goal. In this case, concept is explored by students through distinguishing between essential and non-essential features of the variation.

#### **4.2.2 Tasks as examples in classroom teaching**

In this section, we first illustrate the potential variations in one single example (4.2.2.1), and then we use two series of examples to illustrate the potential variations in a series of examples (4.2.2.2).

##### **4.2.2.1 Potential variations in one single example**

Here is an example (table 4.2.2) in the section of representation of the function in PEP textbook. Example 7 uses a table to give four functions, which respectively represent the test scores and class average scores of three students in a certain test in a school year. It is not intuitive to distinguish the performance changes of the three students from this table, which requires students to choose an appropriate method to represent the function for further analysis.

Table 4.2.2 An example of the representation of a function

例7 表 3.1-4 是某校高一（1）班三名同学在高一学年度六次数学测试的成绩及班级平均分表.

表 3.1-4

姓 名	测试序号					
	第 1 次	第 2 次	第 3 次	第 4 次	第 5 次	第 6 次
王 伟	98	87	91	92	88	95
张 城	90	76	88	75	86	80
赵 磊	68	65	73	72	75	82
班级平均分	88.2	78.3	85.4	80.3	75.7	82.6

请你对这三位同学在高一学年的数学学习情况做一个分析.

### Example 7

Table 3.1-4 shows the scores of three students of class1 in six mathematics tests in grade 10 and the class average scores.

Table 3.1-4

Name	Test serial number					
	1	2	3	4	5	6
Wang wei	98	87	91	92	88	95
Zhang cheng	90	76	88	75	86	80
Zhao lei	68	65	73	72	75	82
class average scores	88.2	78.3	85.4	80.3	75.7	82.6

Please make an analysis of the mathematics learning of these three students in the first year of high school.

We illustrate potential variations embedded in this example:

- Contextual variation: The context of this example is personal. Students need to use mathematical knowledge to analyze real-life problems from a mathematical perspective.
- Representational variation: PEP textbook chooses the graph of the function as a representation method to solve the task, see Figure 4.2.2. Teachers' guidebook states that "It is necessary to pay attention to choosing the appropriate method to express the function according to actual needs. In teaching, students should be guided to observe graphs, learn how to obtain useful information from graphs for analyzing the learning situation of each student in the table. Therefore, there is a conversion from table to graphs of the functions, and from graphs to natural languages in this process.

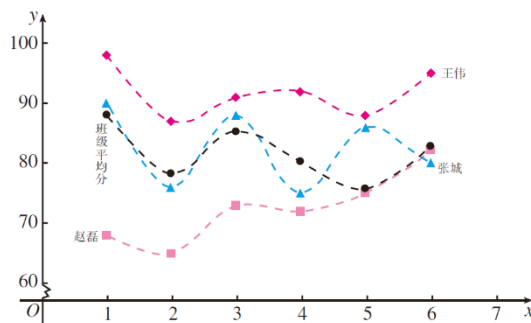


Figure 4.2.2



The context of Example 7 is very close to the life of students. It requires students to use the knowledge they have learned flexibly. After fully understanding the characteristics of three representations of function, students choose the appropriate representation and finally use the mathematical knowledge to analyze the situation and explain it in natural language. This is a comprehensive application task.

#### 4.2.2.2 Potential variations in a series of examples

PEP textbook sometimes designs a series of examples. This series of examples is hierarchical, related, and advanced step by step, and finally achieves the teaching goal. We illustrate two series of examples from the PEP textbook here.

PEP Textbook presents three examples in the section of the monotonicity of a function (table 4.2.3) to illustrate the method of using the definition to prove the monotonicity of a function. Example 1 is to use the definition of monotonicity to study the monotonicity of a linear function  $f(x) = kx + b$  ( $k \neq 0$ ). This is not only to familiarize students with the process of using definitions to study the monotonicity of functions, but also to rigorously prove the conclusions obtained from the graphs at the junior high school stage. Example 2 studies the monotonicity of the inverse proportional function. The function  $f(x) = x + \frac{1}{x}$  in Example 3 is actually the sum of the proportional function and the inverse proportional function, a special case of tick function (when  $a = b = 1$ ).

Table 4.2.3 Three examples of the monotonicity of a function

<p><b>例 1</b> 根据定义, 研究函数 <math>f(x) = kx + b</math> (<math>k \neq 0</math>) 的单调性.</p> <p><b>Example 1</b> According to the definition, we study the monotonicity of the function <math>f(x) = kx + b</math> (<math>k \neq 0</math>)</p>
<p><b>例 2</b> 物理学中的玻意耳定律 <math>p = \frac{k}{V}</math> (<math>k</math> 为正常数) 告诉我们, 对于一定量的气体, 当其体积 <math>V</math> 减小时, 压强 <math>p</math> 将增大. 试对此用函数的单调性证明.</p> <p><b>Example 2</b> Boyle's law in physics <math>p = \frac{k}{V}</math> (<math>k</math> is a positive constant) tells us that for a certain amount of gas, when its volume <math>v</math> decreases, the pressure <math>p</math> will increase. Try to prove this with the monotonicity of the function.</p>
<p><b>例 3</b> 根据定义证明函数 <math>y = x + \frac{1}{x}</math> 在区间 <math>(1, +\infty)</math> 上单调递增.</p> <p><b>Example 3</b> According to the definition, it is proved that the function <math>y = x + \frac{1}{x}</math> increases monotonically in the interval <math>(1, +\infty)</math>.</p>

We illustrate potential variations embedded in/among these three examples:

- Contextual variation: The contexts of example 1 and 3 are mathematical, and the context of example 2 is scientific.
- Representational variation: there are treatments of algebraic registers inside symbolic writing when solving the tasks.

- Pudian: This series of problems is hierarchical. Example 3 is actually the sum of the proportional function and the inverse proportional function.

On the one hand, through examples 1 and 2, students learned how to use definitions to study the monotonicity of functions. We can consolidate students' mastery of strategy as the complexity of the function increases.

On the other hand, example 1 and 2 pave the way for the introduction of tick function<sup>77</sup> in example 3. Rather than introducing the tick function directly, the textbook demonstrates two examples as Pudian, which enables students to quickly discover the properties of this function.

- OPMC: There are constraint manipulation of OPMC in these three examples, by varying the formula of the function. However, the strategies of them are the same, as we called MPOS.

The trace of Pudian in this series of tasks is from studying the monotonicity of a linear function (which link to the function knowledge in junior high school), to a inverse proportional function, and to the sum of the proportional function and the inverse proportional function, the tick function. The study of these three examples also reflects the idea of solving mathematical problems by transferring complex problems into simpler problems, as complex function (tick function) divided into different basic elementary function (proportional and inverse proportional function).

Similarly, PEP textbook also uses a series of examples (table 4.2.4) in the section of representation of the function to help students learn piecewise function since it is a ubiquitous and important type of function, but difficult for students to understand. According to PEP textbook and teachers' guidebook (2019), we introduce the concept and representation of the piecewise function from example 5. Example 6 is to further deepen the understanding of piecewise function and example 8 is an application of piecewise function.

Table 4.2.4 Series of examples of piecewise function

<p><b>例 5</b> 画出函数 <math>y= x </math> 的图象.</p> <p><b>Example 5</b> Draw the graph of the function <math>y= x </math>.</p>
<p><b>例 6</b> 给定函数 <math>f(x)=x+1</math>, <math>g(x)=(x+1)^2</math>, <math>x \in \mathbf{R}</math>,</p> <p>(1) 在同一直角坐标系中画出函数 <math>f(x)</math>, <math>g(x)</math> 的图象;</p> <p>(2) <math>\forall x \in \mathbf{R}</math>, 用 <math>M(x)</math> 表示 <math>f(x)</math>, <math>g(x)</math> 中的较大者, 记为</p> $M(x)=\max\{f(x), g(x)\}.$ <p>例如, 当 <math>x=2</math> 时, <math>M(2)=\max\{f(2), g(2)\}=\max\{3, 9\}=9</math>.</p> <p>请分别用图象法和解析法表示函数 <math>M(x)</math>.</p>

<sup>77</sup> Tick function: a polynomial function that can be represented in the form  $f(x) = ax + \frac{b}{x}$ ,  $ab \geq 0$

## Example 6

Given the function  $f(x) = x + 1$ ,  $g(x) = (x + 1)^2$ ,  $x \in R$

(1) Draw the graphs of functions  $f(x)$  and  $g(x)$  in the same rectangular coordinate system;

(2)  $\forall x \in R$ , use  $M(x)$  to denote the larger one of  $f(x)$ ,  $g(x)$ , denoted as

$M(x) = \max\{f(x), g(x)\}$ .

For example, when  $x = 2$ ,  $M(2) = \max\{f(2), g(2)\} = \max\{3, 9\} = 9$ .

Please use the graph and the analytic formula to represent the function  $M(x)$ .

**例 8** 依法纳税是每个公民应尽的义务，个人取得的所得应依照《中华人民共和国个人所得税法》向国家缴纳个人所得税（简称个税）。2019 年 1 月 1 日起，个税税额根据应纳税所得额、税率和速算扣除数确定，计算公式为

$$\text{个税税额} = \text{应纳税所得额} \times \text{税率} - \text{速算扣除数} \quad ①$$

应纳税所得额的计算公式为

$$\text{应纳税所得额} = \text{综合所得收入额} - \text{基本减除费用} - \text{专项扣除}$$

$$- \text{专项附加扣除} - \text{依法确定的其他扣除} \quad ②$$

其中，“基本减除费用”（免征额）为每年 60 000 元，税率与速算扣除数见表 3.1-5。

表 3.1-5

级数	全年应纳税所得额所在区间	税率（%）	速算扣除数
1	[0, 36 000]	3	0
2	(36 000, 144 000]	10	2 520
3	(144 000, 300 000]	20	16 920
4	(300 000, 420 000]	25	31 920
5	(420 000, 660 000]	30	52 920
6	(660 000, 960 000]	35	85 920
7	(960 000, +∞)	45	181 920

“综合所得”包括工资、薪金，劳务报酬，稿酬，特许权使用费；“专项扣除”包括居民个人按照国家规定的范围和标准缴纳的基本养老保险、基本医疗保险、失业保险等社会保险费和住房公积金等；“专项附加扣除”包括子女教育、继续教育、大病医疗、住房贷款利息或者住房租金、赡养老人等支出；“其他扣除”是指除上述基本减除费用、专项扣除、专项附加扣除之外，由国务院决定以扣除方式减少应纳税的优惠政策规定的费用。

(1) 设全年应纳税所得额为  $t$ ，应缴纳个税税额为  $y$ ，求  $y = f(t)$ ，并画出图象；

(2) 小王全年综合所得收入额为 189 600 元，假定缴纳的基本养老保险、基本医疗保险、失业保险等社会保险费和住房公积金占综合所得收入额的比例分别是 8%，2%，1%，9%，专项附加扣除是 52 800 元，依法确定其他扣除是 4 560 元，那么他全年应缴纳多少综合所得个税？

**分析：**根据个税产生办法，可按下列步骤计算应缴纳个税税额：

## Example 8

Paying taxes in accordance with the law is the obligation of every citizen, and the income obtained by individuals should be paid to the state in accordance with the "Individual Income Tax Law of the People's Republic of China" (referred to as individual tax). From January 1, 2019, the individual tax amount is determined based on the taxable income, the tax rate and the quick deduction. The calculation formula is

Individual tax amount = taxable income  $\times$  tax rate-quick deduction. ①

The formula for calculating taxable income is

Taxable income = comprehensive income income-basic deduction of expenses-special deduction-Special additional deductions-Other deductions determined in accordance with the law. ②

Among them, the "basic deduction of expenses" (exemption amount) is 60000 yuan per year. The tax rates and quick deductions are shown in the table below.

Level	Annual taxable income	Tax rate (%)	quick deduction
1	[0, 36000]	3	0
2	(36000, 144000]	10	2520
3	(144000, 300000]	20	16920
4	(300000, 420000]	25	31920
5	(420000, 660000]	30	52920
6	(660000, 960000]	35	85920
7	(960000, +∞)	45	181920

(1) Set the annual taxable income as  $t$ , and the payable individual tax as  $y$ , find  $y = f(t)$ , and draw the graph; (2) Xiaowang's annual comprehensive income is 189600 yuan. It is assumed that the basic pension insurance, basic medical insurance, unemployment insurance and other social insurance premiums and housing provident funds accounted for 8%, 2%, 1%, and 9% of the total income, respectively. The special additional deduction is 52,800 yuan, and other deductions are determined to be 4560 yuan in accordance with the law, so how much comprehensive income tax should he pay for the whole year?

We illustrate potential variations embedded in/among these three examples:

- Contextual variation: The contexts of example 5 and 6 are mathematical, and the context of example 8 is societal. In example 8, students need to use functional models to express practical problems. According to teachers' guidebook (2019), we should cultivate students' ability to observe, analyze, and solve problems around them with a mathematical vision
- Representational variation: representational variation in this series of examples is rich. There is a conversion from symbolic writing to graph in the solution of example 5. There is a conversion from symbolic writing to graph, and from graph to symbolic writing in the solution of example 6. The solution process of example 6 shows that making full use of the characteristics of the graph can simplify algebraic operations. Moreover, teachers' guidebook (2019) indicates that teachers can also guide students to seek the formula of function from the perspective of pure algebraic operations in teaching to let students feel the simplicity by taking advantage of graphs through comparison. As for example 8, there is a conversion from symbolic writing to graph, also a treatment inside symbolic writing, from algebraic registers to numerical ones. Teachers' guidebook (2019) also points out that technology can be used in the calculation in example 8.
- Pudian: Example 5 is Pudian of example 6 as the form of the piecewise function and the conversions in the solution gradually become more complex. Example 8 is the application of example 5 and 6 with societal context.

This series of examples of piecewise function is designed to be progressive. On the one hand, it gradually deepens the understanding of the piecewise function from the simplest piecewise function  $y=|x|$  to the application of the piecewise function in daily life. On the other hand, in the process of recognizing piecewise functions, representational variations have gradually become richer, not only limited to the conversions from symbolic writing to graphs (example 5), but also extended the conversions from graph to symbolic writing, so that students can realize the importance of the symbolic-graph combination in the process of function learning.

#### **4.2.3 Tasks as exercise: relation between the tasks (problems and examples) in class and the exercises after class**

The relationship between exercises and examples is inseparable. For each exercise in PEP textbook, we can find the corresponding example (teachers' guidebook, 2019). We illustrate the relation of tasks from the perspective of variation in the section of the concept of function in Figure 4.2.3 as an example.

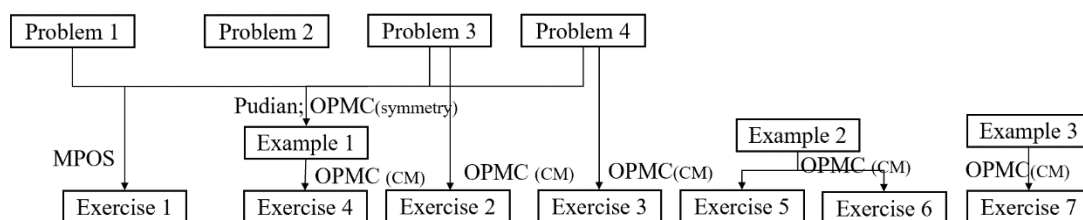


Figure 4.2.3 Relation of tasks from the perspective of variation in the section of the concept of function

Figure 4.2.3 evidences the relations between examples/problems and exercise from the perspective of variation. The most common variation here is constraint manipulation (noted as CM in the figure) of OPMC, by varying the number or the type of the function. In fact, there are mainly two purposes of the design of exercises based on the examples in PEP textbook: 1) to apply the knowledge or strategy learnt from the examples; 2) to further deepen the knowledge or strategy learnt from the examples. We illustrate each of these two goals with an example below.

#### 4.2.3.1 Exercise designed to apply the knowledge or strategy learnt from the examples

Table 4.2.5 shows three widely used mathematical models with contextual variations embedded in each of them. These three tasks have a close relation with the four problems (table 4.2.1). The purpose of these tasks is to let students experience the process of constructing functions and generate functions through different actual situations. Through “restoring” the tasks with actual situations, students can clarify the effects of function elements (domain, corresponding relation, range) on the function, and the same function model can be used in different practical problems, so that students can experience and understand the wide range of applications of functions.

Table 4.2.5 Three tasks related to four problems

<p><b>Example 1 (p63)</b></p> <p>The analytic formula of a function is abstracted away from the actual background of the problem. The corresponding relation between the two quantities can be widely used to describe the relation and law of variables in a class of things. For example, the proportional function <math>y=kx</math> (<math>k \neq 0</math>) can be used to describe the relationship between distance and time in uniform motion, the relationship between mass and volume of a certain density of objects, and the relationship between the circumference and radius of a circle.</p> <p>Try to construct a problem situation so that the relationship between variables can be described by the analytical formula <math>y = x(10 - x)</math>.</p>
<p><b>Exercise 4 (p64)</b></p> <p>Try to construct a problem situation so that the relationship between variables can be described by the analytical formula <math>y = \sqrt{x}</math>.</p>
<p><b>Exercise 14 (p73)</b></p> <p>Try to construct a problem situation so that the relationship between variables can be described by the analytical formula <math>y = \frac{1}{2}ax^2</math> (<math>a &gt; 0</math>).</p>

We illustrate potential variations embedded in/among these tasks:

- Contextual variation: The contextual variation here is special, not proposed in the task, but needs to be given by the students.
- Representational variation: There is a conversion from symbolic writing to natural language in the process of solving the task.
- OPMC: Example 1 is an OPMC of four problems (table 4.2.1), especially problem 1, by using the strategy of symmetry. Problem 1 is to abstract the function from the actual situation, while example 1 is to give practical meaning from the abstract function. When abstracting the concept of function, the corresponding relation is obtained from the problem in real life, such as problem 1-4 showed in section 4.2.1. Example 1 is a reverse, given a functional formula, let students give it practical meaning. This kind of training to “restore” practical problems through formulas is unfamiliar to students. In the process of restoring, students need to clarify the domain, corresponding relation and range of the function. Such symmetry helps to deepen students’ understanding of the concept of function, realize that the same function model can be used to describe different practical problems, and understand the wide range of function applications.

Exercise 4 and exercise 14 are the OPMC of example 1, by using the strategy of constraint manipulation. They keep the goal of the task while varying the type of function. Given the formula of a function, it can often be used to describe many actual problems. For example, teachers’ guidebook (2019) pointed out that  $y = \frac{1}{2}ax^2$

can be used to describe: in free fall motion, the law of the falling body is  $s = \frac{1}{2}gt^2$  (the distance  $s$  is a function of time  $t$ ); the energy  $E$  of a moving object with a certain mass is a function of the speed  $v$  ( $E = \frac{1}{2}mv^2$ ); In a wire with a resistance of  $R$ , there is a definite corresponding relation  $Q = \frac{1}{2}RI^2$  between the heat  $Q$  generated per unit time when the current passes and the current intensity  $I$ ; Given the acute  $Rt\triangle ABC$ , the area  $S$  of the  $Rt\triangle ABC$  is a function of the length  $x$  of the adjacent side of the angle  $A$ ,  $S = \frac{1}{2}x^2\tan A$ ; and so on. The variables  $s$ ,  $t$ ,  $E$ ,  $v$ ,  $Q$ ,  $I$ , and  $S$ , and  $x$  have their own specific physical or geometric meanings. To generalize them, use  $x$  and  $y$  to represent the independent variable and the corresponding function value, then the corresponding relation between them has the form of  $y = \frac{1}{2}ax^2$ .

From the perspective of variation, two exercise are OPMC of example 1, and example 1 is OPMC of the four problems in section 4.2.1. Tasks in PEP textbook are systematic and can be connected. The main goal of these three tasks is to apply the knowledge

students learnt so as to have multi-understanding of the concept of function.

### 4.2.3.2 Exercise designed to further deepen the knowledge or strategy learnt from the examples

Exercise in table 4.2.6 appears in the section of properties of functions in PEP textbook. It has a close relation with example 3, which we proposed in table 4.2.3. The purpose of this exercise is to further deepen of the definition of monotonicity and provide an equivalent form, which can lead students to go back to reflect on example 3.

Table 4.2.6 Exercise related to example 3 of monotonicity of a function

<p>9. 设函数 <math>y=f(x)</math> 的定义域为 <math>I</math>, 区间 <math>D \subseteq I</math>, 记 <math>\Delta x = x_1 - x_2</math>, <math>\Delta y = f(x_1) - f(x_2)</math>. 证明:</p> <p>(1) 函数 <math>y=f(x)</math> 在区间 <math>D</math> 上单调递增的充要条件是: <math>\forall x_1, x_2 \in D, x_1 \neq x_2</math>, 都有 <math>\frac{\Delta y}{\Delta x} &gt; 0</math>;</p> <p>(2) 函数 <math>y=f(x)</math> 在区间 <math>D</math> 上单调递减的充要条件是: <math>\forall x_1, x_2 \in D, x_1 \neq x_2</math>, 都有 <math>\frac{\Delta y}{\Delta x} &lt; 0</math>.</p>
<p>[Translation]</p> <p>9. The domain of the function <math>y = f(x)</math> be <math>I</math>, and the interval <math>D \subseteq I</math>, denoted <math>\Delta x = x_1 - x_2, \Delta y = f(x_1) - f(x_2)</math>, prove:</p> <p>(1) The necessary and sufficient conditions for the function <math>y = f(x)</math> to increase monotonically on the interval <math>D</math> are: <math>\forall x_1, x_2 \in D, x_1 \neq x_2</math>, there is <math>\frac{\Delta y}{\Delta x} &gt; 0</math></p> <p>(2) The necessary and sufficient conditions for the function <math>y = f(x)</math> to decrease monotonically on the interval <math>D</math> are: <math>\forall x_1, x_2 \in D, x_1 \neq x_2</math>, there is <math>\frac{\Delta y}{\Delta x} &lt; 0</math></p>

We illustrate potential variations embedded in/among these tasks:

- Pudian: Exercise 9 enlarges example 3 as it is a further abstraction of example 3. To some extent, we can regard exercise 9 as Pudian of further study in example 3. During teaching of the function  $f(x) = x + \frac{1}{x}$  in example 3, in addition to proving by the definition of function monotonicity in the certain interval, students can also be guided to use the definition to explore the monotonicity of  $f(x) = x + \frac{1}{x}$  in the entire domain of the function. Using the equivalent form of the monotonicity definition provided in exercise 9, we get the following:

$$\frac{\Delta y}{\Delta x} = \frac{f(x_1) - f(x_2)}{x_1 - x_2} = \frac{x_1 x_2 - 1}{x_1 x_2}$$

Therefore, when we reflect on example 3 with the conclusion of exercise 9, we get the following table 4.2.7 for  $\forall x_1, x_2 \in D$ , and  $x_1 \neq x_2$ .

Table 4.2.7 Variation of function  $f(x) = x + \frac{1}{x}$  in example 3

D	$(-\infty, -1]$	$[-1, 0)$	$(0, 1]$	$[1, +\infty)$
The sign of $\frac{\Delta y}{\Delta x}$	+	-	-	+
$f(x)$	Monotonically increasing	Monotonically decreasing	Monotonically decreasing	Monotonically increasing

- OPMC: Example 3 can be regarded as OPMC (chaining) of exercise 9. We can expand the existing problem (example 3) by using the conclusion of exercise 9.

Most of the time, the exercises are derived from examples to let students practice strategies. Sometimes, however, the exercises can be counterproductive to the examples, allowing students to think further and more deeply about the examples, as we have shown here. From the perspective of variation, in this case, example 3 is not only the Pudian of the exercise to a certain extent, but the conclusion drawn in the exercise 9 will further trigger the thinking of the example 3, which makes exercise 9 become the Pudian and chaining of OPMC of the further study of example 3. Exercise designed here is to further deepen the knowledge and strategy learnt from the examples.

#### 4.2.4 Conclusion

Based on the analysis above, we conclude that potential variations in Chinese PEP textbook mainly have the following characteristic:

- The contexts of tasks are rich, from all aspects of life. On the one hand, function becomes less abstract for students through linking functions with real life context. On the other hand, students can experience the wide use of function through applying the knowledge of functions to solving real-world problems.
- Representational variation is various. The conversions are among natural language, auxiliary transitional representation (table of values), symbolic writing and graph. PEP textbook places great emphasis on solving function problems with the help of function graphs, in other words, the conversions between symbolic writing and graph in the process of solving tasks. Because making full use of the characteristics of the graph can simplify algebraic operations in symbolic writing. Although technology is mentioned in the textbook, most of the conversions are completed by students themselves.
- Conceptual variation embedded in a series of tasks: to introduce a new concept, PEP textbook often starts with standard variation, and then propose the non-standard variation. Rather than directly present the definition of the concept, it prefers students themselves to generalize the concept from the standard and non-standard variations under teachers' guidance. In the process of generalization, students are expected to find the essential and non-essential attributes of the concept.
- Procedural variation embedded in a series of tasks: PEP textbook pays great attention to the connections between tasks. Figure 4.2.4<sup>78</sup> shows the procedural

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<sup>78</sup> In section 2.2.3, we indicate that OPMC can be refined into four specific strategies: constraint manipulation, goal manipulation, symmetry, and chaining. In Figure 4.2.4, constraint manipulation is noted as CM, symmetry is noted as S, goal manipulation is noted as GM, chaining is manipulated as C.



variations between tasks in the section of “concept and representation of function”. We relate examples to examples and examples to exercises through procedural variation. From Figure 4.2.4, the most common procedural variation is Pudian and OPMC. Pudian is often found between a series of examples (see section 4.2.2.2) or between a problem and an example. In the PEP textbook, OPMC is mainly implemented through constraint manipulation (noted as CM in the figure), which allows students to master the problem-solving strategy by changing the non-essential attributes of the problem, in which case OPMC often appears simultaneously with MPOS. Moreover, PEP textbook designs a series of tasks to enhance the formation of the concept of function (section 4.2.1).

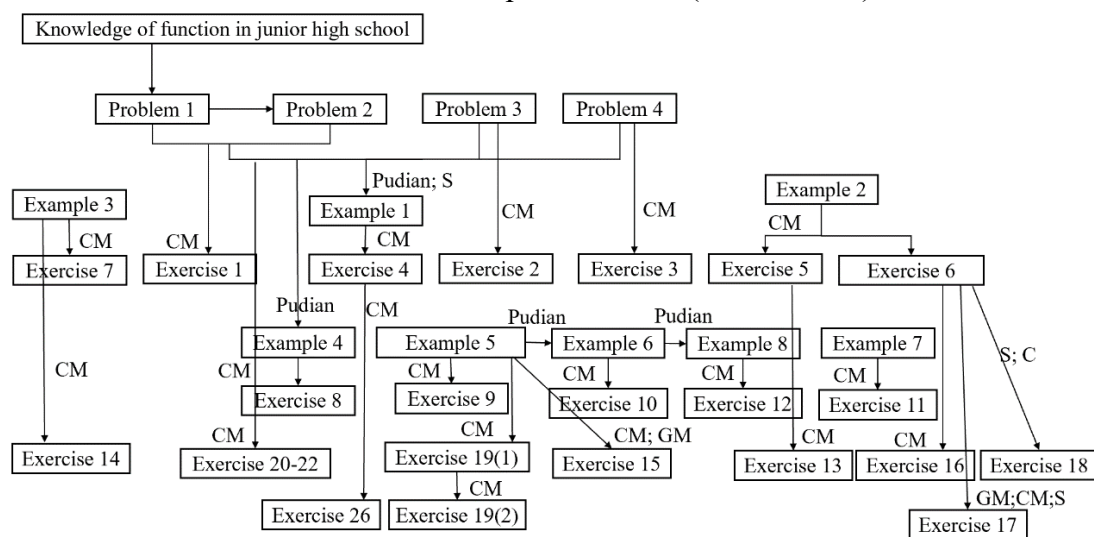


Figure 4.2.4 Procedural variation between tasks in the “concept and representation of function”

The idea of variation is used in the design of PEP textbooks. Variation pedagogy that focuses on providing deliberate mathematical task selection and implementation in classroom instruction (Gu et al. 2004) has direct implications for designing textbooks. Teachers’ guidebook reveals to some extent the variations embedded in or among tasks. For example, in the section of “representation of function”, it explicitly uses the word “Pudian” and mentions that four problems (table 4.2.1) pave the way for learning the three representations of functions. It also indicates the connection between tasks. In each section of the exercises, teachers’ guidebook indicates which examples can be used to complete the exercises. In general, PEP textbook offers the possibility of (re)designing several tasks from one task by manipulating variations. This series of interconnected tasks provides teachers with a systematic and coherent approach to teaching and learning, advancing step by step to achieve the ultimate goal of teaching.

### 4.3 Potential variations in the tasks in French Hyperbole textbook

As we present in section 4.1.2, we classify tasks in Hyperbole into three categories:



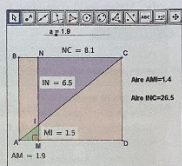




tasks used before class; tasks used in class; tasks used after class. In this section, we study the potential variations already embedded in different types of tasks of function in Hyperbole textbook and the relations between them (section 4.3.1, 4.3.2, & 4.3.3). The analysis of potential variations is based on the analytical framework proposed in section 2.4.3: contextual, representational, conceptual, and procedural variation.

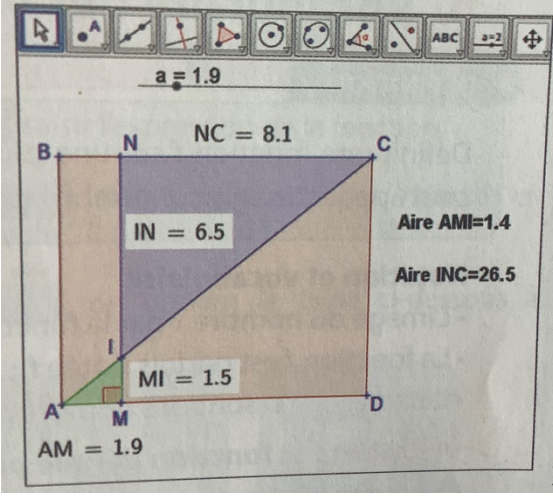
### 4.3.1 Tasks used before class

In Hyperbole textbook, tasks used before class are mainly in the following categories: “Tasks to start” and “Discover”. In this section, we present one task in the discover column and describe the potential variations embedded in it. As for the “Tasks to start”, there is an example in practical teaching presented in section 6.2.2.1.

Task shown in table 4.3.1 is presented in the section of general function in Hyperbole textbook. The aim of this task is to let students experience the process of mathematical modeling (modelling with function in this task), and learn to use geometry software through their explorations.

Table 4.3.1 A task of modeling with functions in “discover”

<div data-bbox="252 952 804 996">  <span>Découvrir</span> </div> <div data-bbox="288 1010 647 1055"> <h2>2 Modélisation avec des fonctions</h2> </div> <div data-bbox="288 1059 790 1144"> <p>Une entreprise découpe des voiles dans des rectangles de tissu de 8 m sur 10 m de côtés pour des petits bateaux de plaisance. Sur la figure ci-dessous, les voiles découpées sont représentées par les triangles AMI et INC.</p> <p>L'entreprise vient de recevoir deux commandes :</p> <ul style="list-style-type: none"> <li>• l'une pour un mât de 3,2 m de haut;</li> <li>• l'autre pour une voile de 12,10 m<sup>2</sup>.</li> </ul> </div> <div data-bbox="300 1149 545 1314">  </div> <div data-bbox="603 1149 785 1314">  </div> <div data-bbox="304 1330 790 1373"> <p><b>Problème</b></p> <p>Déterminer les dimensions des voiles commandées.</p> </div> <div data-bbox="256 1397 804 1785"> <ol style="list-style-type: none"> <li> <p><b>1 Conjecturer avec un logiciel de géométrie dynamique</b> <span style="float: right;">Fascicule p. II</span></p> <ol style="list-style-type: none"> <li>Créer un rectangle ABCD tel que AB = 8 et AD = 10, puis créer le segment [AC].</li> <li>Créer un curseur <math>a</math> allant de 0 à 10 avec un incrément de 0,1 (utiliser  Curseur).</li> <li>Créer le point M du segment [AD] tel que AM = <math>a</math> (utiliser  Créer (points libres)).</li> <li>Créer la perpendiculaire à (AD) passant par M; noter I et N les points d'intersection respectifs avec [AC] et [BC].</li> <li>Créer les triangles AMI et INC. Afficher les longueurs des côtés de l'angle droit de chacun de ces triangles (utiliser  Mesurer ou Longueur) et afficher les aires de ces triangles (utiliser  Aire).</li> <li>Déplacer le point M et conjecturer sa position pour que MI = 3,2, puis pour que l'aire de AMI soit 12,1.</li> </ol> </li> <li> <p><b>2 Satisfaire la première commande</b></p> <ol style="list-style-type: none"> <li>On note <math>a</math> la longueur AM en mètres. Appliquer le théorème de Thalès dans les triangles AMI et ADC pour établir que la longueur IM est donnée par la formule <math>f(a) = 0,8a</math>.</li> <li>Déterminer la ou les valeurs de <math>a</math> pour que le triangle AMI ait un côté de l'angle droit de longueur 3,2 m.</li> <li>Dans chaque cas, préciser les dimensions de la seconde voile représentée par le triangle INC.</li> </ol> </li> <li> <p><b>3 Satisfaire la deuxième commande</b></p> <ol style="list-style-type: none"> <li>Démontrer que l'aire du triangle AMI est donnée par la formule <math>g(a) = 0,4a^2</math>.</li> <li>En déduire la valeur de <math>a</math> pour laquelle l'aire du triangle AMI est 12,10 m<sup>2</sup>.</li> <li>Déterminer alors l'aire de la seconde voile représentée par le triangle INC.</li> </ol> </li> </ol> <p style="text-align: right;">Chapitre 1 Fonctions : généralités 13</p> </div>	<div data-bbox="842 936 1152 969"> <h2>2 Modeling with functions</h2> </div> <div data-bbox="842 972 1359 1137"> <p>A company cuts sails from rectangles of fabric 8 m by 10 m on each side for small pleasure boats. In the figure below, the cut sails are represented by the triangles AMI and INC.</p> </div> <div data-bbox="842 1140 1313 1171"> <p>The company has just received two orders:</p> </div> <div data-bbox="842 1173 1212 1243"> <ul style="list-style-type: none"> <li>• one for a 3.2 m high mast;</li> <li>• the other for a sail of 12.10 m<sup>2</sup>?</li> </ul> </div> <div data-bbox="842 1247 952 1279"> <p><b>Problem</b></p> </div> <div data-bbox="842 1281 1351 1312"> <p>Determine the dimensions of the sails ordered.</p> </div> <div data-bbox="842 1348 1356 1413"> <p><b>1. Conjecture with dynamic geometry software</b></p> </div> <div data-bbox="842 1415 1359 1989"> <ol style="list-style-type: none"> <li>Create a rectangle ABCD as AB=8 and AD = 10, then create the segment [AC].</li> <li>Create a slider <math>a</math> ranging from 0 to 10 with an increment of 0.1.</li> <li>Create the point M in the segment [A D] as AM = <math>a</math>.</li> <li>Create the perpendicular to (AD) passing through M; note I and N the respective intersection points with [AC] and [BC].</li> <li>Create triangles AMI and INC. Display the lengths of the sides of the right angle of each of these triangles and display the areas of these triangles.</li> <li>Move the point M and conjecture its position so that MI = 3.2, then so that the area of AMI is 12.1.</li> </ol> </div>
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	<p><b>2. Satisfy the first order</b></p> <p>a) Let <math>a</math> be the length <math>AM</math> in meters. Apply Thales' theorem in the triangles <math>AMI</math> and <math>ADC</math> to establish that the length <math>IM</math> is given by the formula <math>f(a) = 0.8a</math>.</p> <p>b) Determine the value(s) of <math>a</math> so that triangle <math>AMI</math> has one side of the right angle of length 3.2 m.</p> <p>c) In each case, specify the dimensions of the second sail represented by the triangle <math>INC</math>.</p> <p><b>3. Satisfy the second order</b></p> <p>a) Prove that the area of the triangle <math>AMI</math> is given by the formula <math>g(a) = 0.4a^2</math>.</p> <p>b) Deduce the value of <math>a</math> for which the area of triangle <math>AMI</math> is <math>12.10 \text{ m}^2</math>.</p> <p>c) Then determine the area of the second sail represented by the triangle <math>INC</math>.</p>
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We illustrate potential variations embedded in this task:

- Contextual variation: The contextual variation here is occupational. Students need to solve the problem by transforming real-world context into mathematical models (in this case, modelling with function) to study.
- Representational variation: There are conversions from natural language to symbolic writing, from natural language to geometrical figures, and from geometrical figures to symbolic writing in the process of solving the task.
- OPMC: There is a constraint manipulation of OPMC in the two orders of this task, respectively finding the image and pre-image for the same function  $g(a) = 0.4a^2$ .
- Pudian: On the one hand, this task paves way for the subsequent teaching and learning of constructing function models; On the other hand, the second is how to use the use of geometric software to help us solve the problem, with technical support.

#### 4.3.2 Tasks used in class

In Hyperbole textbook, tasks used in class are mainly the following categories: “Examples”, “Know-how”, “Solving problems”, and “practical task”. In this section, we present one task in each column and describe the potential variations embedded in it.

##### 4.3.2.1 Examples

Task shown in table 4.3.2 is presented in the section of general function in Hyperbole textbook as a series of examples of defining a function.

Table 4.3.2 A series of examples of defining a function

● **EXEMPLE 1 : fonction définie par une courbe**  
 La courbe  $\zeta$ , dans le repère ci-contre, définit une fonction  $f$  sur l'ensemble des nombres compris entre 1 et 5, que l'on appelle **un intervalle** et que l'on note  $D = [1; 5]$ .  
 L'image de 2 par  $f$  est 3 car le point  $A(2; 3)$  appartient à  $\zeta$ . On note  $f(2) = 3$ .

*Remarque :* seuls les points repérés par une croix ou des lignes de rappel permettent une lecture exacte des valeurs.

● **EXEMPLE 2 : fonction définie par un tableau**  
 Au décathlon, l'une des épreuves est le saut en longueur.  
 Le tableau ci-contre définit une fonction  $g$  qui à une longueur (en mètres) associe un score. Son ensemble de définition est un ensemble fini que l'on note  $\{7,96; 7,97; 7,98; 7,99\}$ .  
 L'image de 7,97 par  $g$  est 1053. On note  $g(7,97) = 1053$ .

Longueur en m	Score
7,99	1 058
7,98	1 055
7,97	1 053
7,96	1 050

● **EXEMPLE 3 : fonction définie par une formule**  
 $h$  est la fonction qui, à chaque nombre réel  $x$ , associe le nombre  $h(x) = x^2 - 3$ . Elle est définie sur  $\mathbb{R}$ .  
 Pour calculer l'image de  $-5$ , on remplace  $x$  par  $-5$  dans l'expression :  $h(-5) = (-5)^2 - 3 = 22$ .

Translation

**EXAMPLE 1: function defined by a curve**  
 The curve  $\zeta$ , in the coordinate system aside, defines a function  $f$  on the set of numbers between 1 and 5, which is called **an interval** and we note  $D = [1; 5]$ .  
 The image of 2 by  $f$  is 3 because the point  $A(2; 3)$  belongs to  $\zeta$ . We note  $f(2) = 3$ .  
 Note: only the points marked with a cross or extension lines allow an exact reading of the values.

• **EXAMPLE 2: function defined by a table**  
 In the decathlon, one of the events is the long jump.  
 The table aside defines a function  $g$  which lies in a length (in meters) associating a score. Its definition set is a finite set that we note  $\{7.96; 7.97; 7.98; 7.99\}$ .  
 The image of 7.97 by  $g$  is 1053. We note  $g(7.97) = 1053$ .

• **EXAMPLE 3: function defined by a formula**  
 $h$  is the function which, with each real number  $x$ , associates the number  $h(x) = x^2 - 3$ . It is defined on  $\mathbb{R}$ .  
 To calculate the image of  $-5$ , we replace  $x$  by  $-5$  in the expression:  $h(-5) = (-5)^2 - 3 = 22$ .

We illustrate potential variations embedded in this task:

- **Contextual variation:** The contextual variation of example 2 is personal. It connects real life with functions so as to let students experience the application of functions in daily life.
- **Representational variation:** Function in example 1 is represented in graph, example 2 is represented in table, and example 3 is represented in formula. Through these 3 examples, three representations of the function are shown. There are conversions from graph to symbolic writing (example 1), and from natural language to auxiliary transitional representation (table of values), and to symbolic writing (example 2).
- **Conceptual variation:** Provide students with multi-understanding of function by defining functions in three different forms.



- Pudian: Different representations of three examples pave way for the subsequent teaching and learning of representations of function.

#### 4.3.2.2 Know-how

Three tasks are shown in the “know-how” of the subsection of “variation and table of variation” (table 4.3.3). The first one is shown with the solution, and the following two are derived from the first one through variation. The purpose of these tasks is to let students master the strategy of reading the table of variation.

Table 4.3.3 A series of tasks about reading a table of variation in “Know-how”

<p><b>Savoir-faire</b></p> <p><b>Exercice résolu Lire un tableau de variation</b></p> <p>1. Énoncé</p> <p>Voici ci-contre le tableau de variation d'une fonction <math>g</math> définie sur l'intervalle <math>[-2; 5]</math>.</p> <p>a) Décrire les variations de la fonction <math>g</math>.</p> <p>b) Comparer :</p> <ul style="list-style-type: none"> <li><math>g(-1)</math> et <math>g(0)</math>;</li> <li><math>g(2)</math> et <math>g(4)</math>.</li> </ul> <p>c) Tracer une courbe susceptible de représenter graphiquement la fonction <math>g</math> dans un repère.</p> <p><b>Solution</b></p> <p>a) La fonction <math>g</math> est croissante sur l'intervalle <math>[-2; 1]</math>, décroissante sur l'intervalle <math>[1; 4]</math> et croissante sur l'intervalle <math>[4; 5]</math>.</p> <p>b) <math>-1</math> et <math>0</math> appartiennent à <math>[-2; 1]</math>, or <math>g</math> est croissante sur cet intervalle, donc leurs images sont rangées dans le même ordre : <math>-1 &lt; 0</math> donc <math>g(-1) &lt; g(0)</math>.</p> <p><math>2</math> et <math>4</math> appartiennent à <math>[1; 4]</math>, or <math>g</math> est décroissante sur cet intervalle, donc leurs images sont rangées dans l'ordre contraire : <math>2 &lt; 4</math> donc <math>g(2) &gt; g(4)</math>.</p> <p>c) La courbe ci-dessous est une représentation graphique possible de la fonction <math>g</math>.</p> <p><b>Conseils</b></p> <ul style="list-style-type: none"> <li>Dans le tableau, le sens des flèches indique les variations. On lit les intervalles sur la ligne « <math>x</math> ».</li> <li>On ne connaît pas les valeurs de <math>g(-1)</math> et <math>g(0)</math>, mais comme <math>-1</math> et <math>0</math> appartiennent à un même intervalle sur lequel <math>g</math> est croissante, on peut malgré tout comparer <math>g(-1)</math> et <math>g(0)</math>.</li> <li>On place d'abord les points de coordonnées données dans le tableau de variation.</li> </ul>	<p>Translation:</p> <p>Solving exercise: read a table of variation</p> <p><b>1. Statement</b></p> <p>Here is the variation table of a function <math>g</math> defined on the interval <math>[-2; 5]</math>.</p> <p>a) Describe the variations of the function <math>g</math>.</p> <p>b) Compare:</p> <ul style="list-style-type: none"> <li><math>g(-1)</math> and <math>g(0)</math>; <math>g(2)</math> and <math>g(4)</math>.</li> </ul> <p>c) Draw a curve that graphically represents the function <math>g</math> in a coordinate system.</p>
<p><b>À votre tour</b></p> <p>2. a) Décrire le sens de variation de la fonction <math>h</math> dont le tableau de variation est donné.</p> <p>b) Comparer :</p> <ul style="list-style-type: none"> <li><math>h(0)</math> et <math>h(1)</math>;</li> <li><math>h(2)</math> et <math>h(2.5)</math>.</li> </ul> <p>c) Tracer une courbe susceptible de représenter la fonction <math>h</math> dans un repère.</p> <p><b>3. 1.</b> Décrire le sens de variation de la fonction <math>k</math> dont le tableau de variation est donné.</p> <p>2. Ranger du plus petit au plus grand :</p> <p>a) <math>k(-0.5)</math>, <math>k(-0.6)</math>, <math>k(-\sqrt{2})</math>.</p> <p>b) <math>k(\pi)</math>, <math>k(\sqrt{3})</math>, <math>k(3)</math>.</p>	<p><b>Your turn</b></p> <p>2. a) Describe the variation of the function <math>h</math> whose table of variation is given.</p> <p>b) Compare:</p> <ul style="list-style-type: none"> <li><math>h(0)</math> and <math>h(1)</math>; <math>h(2)</math> and <math>h(2.5)</math>.</li> </ul> <p>c) Draw a curve that represents the function <math>h</math> in a coordinate system.</p> <p>3. 1. Describe the variation of the function <math>k</math> whose table of variation is given.</p> <p>2. Arrange from smallest to largest:</p> <p>a) <math>k(-0.5)</math>, <math>k(-0.6)</math>, <math>k(-\sqrt{2})</math>.</p> <p>b) <math>k(\pi)</math>, <math>k(\sqrt{3})</math>, <math>k(3)</math>.</p>

We illustrate potential variations embedded in/among these tasks:

- Representational variation: There are conversions from auxiliary transitional representations (table of variation) to symbolic writing, and to graphs in the process of solving three tasks.
- OPMC: These three tasks are closely related. Task 2 and task 3 can be regarded as OPMC of task 1. Task 2 is an OPMC of task 1, by using the strategy of constraint

manipulation, varying the specific number in the task. Task3 further expands the constraint manipulation of OPMC, by varying the comparison of the function values from 2 to 3. Mathematical strategy for these three tasks is the same (MPOS) as the essential features of these three tasks are invariant.

As we mentioned in section 4.1.2, in the part of “know-how”, there are often a series of tasks presented. One with the solution is worked as an example and the others are varying from this one through OPMC (probably constraint manipulation). The strategies of this series of tasks are the same. OPMC and MPOS are embedded and widely used in the design of series of tasks in “Know-how”.

### 4.3.2.3 Solving problems

Five tasks are shown in the “solving problems” of the section of “qualitative study of functions” (table 4.3.4). Similar to the series of tasks presented in table 4.3.3, task 9 is shown with the solution, and the following four can be regarded as the variation of task 9 with the same mathematical strategy. The purpose of these tasks is to let students master the strategy of determining the sign of a function.

Table 4.3.4 A series of tasks of determining the sign of a function

**Problème résolu** Déterminer le signe d'une fonction

**9 Énoncé**

$f$  est une fonction définie sur l'intervalle  $[-5; 5]$ .  
Voici ci-contre le tableau de variation de  $f$ .  
Donner le tableau de signes de  $f(x)$ .

**Solution**

• Conséquences tirées du tableau de variation

Intervalle I	$[-5; -3]$	$[-3; 0]$	$[0; 1]$	$[1; 2]$	$[2; 5]$
Sens de variation de $f$	$f$ croissante	$f$ croissante	$f$ décroissante	$f$ décroissante	$f$ croissante
Conséquence	$-5 \leq x \leq -3$ $f(x) \leq f(-3)$ $f(x) \leq 0$	$-3 \leq x \leq 0$ $f(-3) \leq f(x)$ $0 \leq f(x)$ c'est-à-dire : $f(x) \geq 0$	$0 \leq x \leq 1$ $f(x) \geq f(1)$ $f(x) \geq 0$	$1 \leq x \leq 2$ $f(1) \geq f(x)$ $0 \geq f(x)$ c'est-à-dire : $f(x) \leq 0$	$2 \leq x \leq 5$ $f(x) \leq f(5)$ $f(x) \leq -1$ donc : $f(x) \leq 0$

• Résumer dans un tableau de signes

$x$	-5	-3	1	5	
Signe de $f(x)$	-	0	+	0	-

$x$	-5	-3	0	1	2	5
$f(x)$	-2	0	2	0	-4	-1

Translation

Solve the problem: Determine the sign of a function

**9. Statement**

$f$  is a function defined on the interval  $[-5; 5]$ . Here is the table of variation of  $f$ . Give the table of signs of  $f(x)$ .

**À votre tour**

**10** Voici le tableau de variation d'une fonction  $g$  définie sur  $[-1; 6]$ .

$x$	-1	0	1	3	4	6
$g(x)$	1	3	0	-1	0	2

a) Tracer une courbe susceptible de représenter graphiquement  $g$ .  
b) Donner le tableau de signes de  $g(x)$ .

**11**  $h$  est une fonction définie sur  $[-3; 6]$ . Voici son tableau de variation.

$x$	-3	-2	-1	4	6
$h(x)$	-5	0	2	0	3

Résoudre l'inéquation dans l'intervalle  $I$ .  
a)  $I = [-3; 4]$ ,  $h(x) < 0$ ;  
b)  $I = [-3; 6]$ ,  $h(x) \geq 0$ .

**12**  $k$  est une fonction décroissante sur  $]-\infty; 0]$  et croissante sur  $[0; +\infty[$ . De plus  $k(-2) = k(2) = 1$ . Pour chaque affirmation, dire si elle est vraie ou fausse. Justifier.  
a) Pour tout nombre réel  $x$  de l'intervalle  $[-2; 2]$ ,  $k(x) \leq 1$ .  
b) L'ensemble des nombres réels  $x$  tels que  $k(x) \geq 1$  est l'intervalle  $[2; +\infty[$ .  
c) Pour tout nombre réel  $x$  l'intervalle de  $]-\infty; -2]$ ,  $k(x) > 0$ .

**13**  $\ell$  est une fonction décroissante sur  $[0; +\infty[$  telle que  $\ell(1) = 1$ .  $m$  est une fonction croissante sur  $[0; +\infty[$  telle que  $m(1) = 1$ . Pour tout nombre réel  $x$  de  $[0; +\infty[$ , comparer  $\ell(x)$  et  $m(x)$ .

**Your turn**

**10.** Here is the table of variation of a function  $g$  defined on  $[-1; 6]$ .  
a) Draw a curve that representing  $g$  graphically.  
b) Give the table of signs of  $g(x)$ .

**11.**  $h$  is a function defined on  $[-3; 6]$ . Here is its table of variation.  
Solve the inequation in the interval  $I$ .  
a)  $I = [-3; 4]$ ,  $h(x) < 0$ ;  
b)  $I = [-3; 6]$ ,  $h(x) \geq 0$ .

**12.**  $k$  is a decreasing function on  $]-\infty; 0]$  and increasing on  $[0; +\infty[$ . Moreover  $k(-2) = k(2) = 1$ . For each statement, say whether it is true or false. Justify.  
a) For any real number  $x$  in the interval  $[-2; 2]$ ,  $k(x) \leq 1$ .  
b) The set of real numbers that  $k(x) \geq 1$  is the interval  $[2; +\infty[$ .  
c) For any number real  $x$  in the interval of  $]-\infty; -2]$ ,  $k(x) > 0$ .

**13.**  $L$  is a decreasing function on  $[0; +\infty[$  and  $L(1) = 1$ .  
 $m$  is an increasing function on  $[0; +\infty[$  and  $m(1) = 1$ .  
For any real number  $x$  of  $[0; +\infty[$ , compare  $L(x)$  and  $m(x)$ .

We illustrate potential variations embedded in/among these tasks:

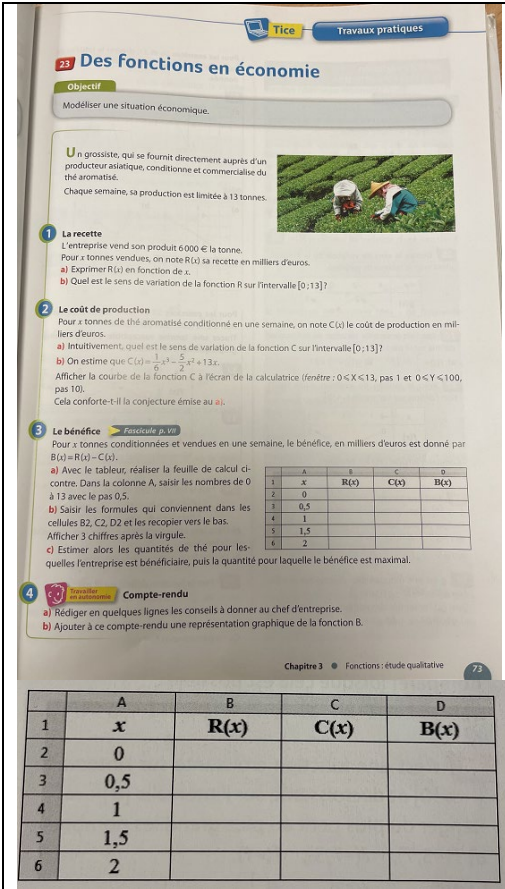
- **Representational variation:** There are conversions from auxiliary transitional representations (table of variation) to symbolic writing (task 9, 11); from auxiliary transitional representations (table of variation) to graph (task 10); from natural language to symbolic writing (task 12, 13).
- **OPMC:** These five tasks are closely related and hierarchical. Task 10 is constraint manipulation of OPMC of task 9, by varying the specific numbers in the table of variation. Task 13 is constraint manipulation of OPMC of task 12. Mathematical strategy for these three tasks is the same (MPOS) as the essential features of these three tasks are invariant.
- **Pudian:** Task 9 is Pudian of task 10. Task 11 makes a connection between functions and inequations based on tasks 9 and 10. Task 12 further abstracts on the basis of task 11, and changes the representation of table of variation to natural language. Task 13 is the Pudian of task 12.

Similar to the design of “know-how” (mentioned in section 4.1.2), there are often a series of tasks presented in the part of “solving problems”. Taking profit of OPMC and Pudian, this series of tasks has gradually increased in difficulty and deepened. Compared to the tasks in “know-how” (see table 4.3.3, since we choose the tasks of the same content of “variation and table of variation”), the tasks in “solving problems” are more complex and are the further application of mathematical methods.

#### 4.3.2.4 Practical task

This task is shown in the “practical task” of the section of “qualitative study of functions” (table 4.3.5). The purpose of this task is to apply function to the real life and also to let students experiment using ICT tools in the process of solving the task.

Table 4.3.5 Practical task of modelling an economic situation



**23. Functions in economy**

**Objective:** Model an economic situation

A wholesaler, who buys directly from an Asian producer, packs and sells flavored tea. Each week, its production is limited to 13 tons.

**1. The recipe**

The company sells its product for 6,000€ per ton. For  $x$  tons sold, we note  $R(x)$  as its revenue in thousands of euros. a) Express  $R(x)$  in terms of  $x$ . b) What is the variation of the function  $R$  on the interval  $[0;13]$ ?

**2. Production cost**

For  $x$  tons of flavored tea packed in one week, we note  $C(x)$  as the production cost in thousands of euros. a) Intuitively, what is the variation of the function  $C$  on the interval  $[0;13]$ ? b) We estimate that  $C(x) = \frac{1}{6}x^3 - \frac{5}{2}x^2 + 13x$ . Display the curve of function  $C$  on the calculator screen. Does this support the conjecture made in a) ?

**3. The profit**

For  $y$  tons packed and delivered in one week, the profit, in thousands of euros, is given by  $B(x) = R(x) - C(x)$ . a) With the table, create the calculation sheet aside. In column A, enter the numbers from 0 to 13 with step 0.5. b) Enter the appropriate formulas in the cells B2, C2, D2 and copy them down. Display 3 decimal places. c) Then estimate the quantities of tea for the which the company is profitable, then the quantity for which the profit is maximum.

**4. Report (work independently)**

a) Write in a few lines the advice to give to the entrepreneur. b) Add to this report a graphical representation of function  $B$ .

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**4. Report (work independently)**

a) Write in a few lines the advice to give to the entrepreneur. b) Add to this report a graphical representation of function  $B$ .

We illustrate potential variations embedded in this task:



- Contextual variation: The contextual variation here is occupational. Students need to solve this task by transforming this economic situation into mathematical models (in this case, modelling with function) to study.
- Representational variation: There are conversions from natural languages to symbolic writing (item 1, 2 and 3); from natural language to auxiliary transitional representations (table of variation, item 1 and 2); from symbolic writing to table of values (item 3); from table of values to graph (item 4). In item 3 of the task, there is also a treatment inside the symbolic writing and this process is carried out by the calculator.
- Pudian is embedded this series of tasks: In order to get the function  $B(x)$ , item 1 and 2 pave the way to discuss the recipe and the cost of production individually.

The context of Practical tasks in Hyperbole textbook is various: personal, occupational, societal, scientific and mathematical. Practical tasks are designed to be combined with algorithms or TICE (see footnote 1 in section 4.1.2). On the one hand, it allows students to experience the application of functions in all aspects of real life, and on the other hand, it offers students opportunities to learn to use mathematical tools to solve mathematical problems.

### **4.3.3 Tasks used after class**

In Hyperbole textbook, tasks used in class are mainly the following categories: “Examples”, “Know-how”, “Solving problems”, and “practical task”. In this section, we present one task in each column and describe the potential variations embedded in it.

#### **4.3.3.1 For training**

As we mentioned in 4.1.2, In the part of “for training”, tasks are divided according to the subsections they belong to. Each subsection starts with the tasks called “quick questions”, and then tasks in different specific knowledge are noted. Tasks shown are progressive and hierarchical in this part. We selected two tasks from “tasks for training” shown in table 4.3.6.

Both of the two tasks are about the variation and table of variation of a function. They are closely related to the tasks in “know-how” and “solving problems” (see table 4.3.3 and table 4.4.4). Task 38 is constraint manipulation of OPMC of tasks in “know-how”. Task 42 is constraint manipulation of OPMC of tasks in “solving problems”. As for the specific potential variations in these two tasks, we don’t repeat them here as it can be referred to section 4.3.2.2 and section 4.3.2.3.

Table 4.3.6 Two tasks of variation and table of variation of a function

**38** Voici le tableau de variation d'une fonction  $h$  définie sur l'intervalle  $[-5; 5]$ .

$x$	-5	0	3	5
$h(x)$	4	-2	0	-1

a) Comparer :

•  $h(-4)$  et  $h(-1)$ ;    •  $h(1)$  et  $h(2)$ ;    •  $h(3)$  et  $h(4)$ .

b) Peut-on comparer  $h(-4)$  et  $h(4)$ ?

Translation

**38.** Here is the table of variation of a function  $h$  defined on the interval  $[-5; 5]$ .

a) compare:

•  $h(-4)$  and  $h(-1)$ ;  
•  $h(1)$  and  $h(2)$ ;  
•  $h(3)$  and  $h(4)$ .

b) Can we compare  $h(-4)$  and  $h(4)$ ?

**42** Voici le tableau de variation d'une fonction  $f$  définie sur  $[-2; 5]$ .

$x$	-2	-1	0	2	3	5
$f(x)$	3	0	-1	0	2	1

a) Déterminer les nombres réels  $x$  de l'intervalle  $[-2; 3]$  tels que  $f(x) \leq 0$ .

b) Quel est le signe des nombres  $f(x)$  lorsque  $x$  appartient à l'intervalle  $[3; 5]$ ?

c) Donner le tableau de signes de  $f(x)$ .

Translation

**42.** Here is the table of variation of a function  $f$  defined on  $[-2; 5]$ .

a) Determine the real numbers  $x$  in the interval  $[-2; 3]$  that  $f(x) \leq 0$ .

b) What is the sign of the numbers  $f(x)$  when  $x$  belongs to the interval  $[3; 5]$ ?

c) Give the table of sign of  $f(x)$ .

### 4.3.3.2 For auto-test

Tasks are design in the form of MCQs in the part of “for auto-test”. We select one task as an example of MCQs in table 4.3.7. The purpose of these tasks is to let students master the strategy of reading the table of variation and the sign of a function.

Table 4.3.7 MCQ task in “tasks for auto-test”

<p><b>65</b> Dans chaque cas, donner la ou les réponses exactes sans justifier.</p> <p>Voici le tableau de variation d'une fonction <math>f</math> définie sur l'intervalle <math>[-8; 6]</math>.</p> <table border="1" style="margin: 10px auto;"> <tr> <td><math>x</math></td> <td>-8</td> <td>-5</td> <td>2</td> <td>6</td> </tr> <tr> <td><math>f(x)</math></td> <td>0</td> <td>2</td> <td>-4</td> <td>4</td> </tr> </table>					$x$	-8	-5	2	6	$f(x)$	0	2	-4	4
$x$	-8	-5	2	6										
$f(x)$	0	2	-4	4										
	A	B	C	D										
1 Pour tout nombre réel $x$ de $[-5; 3]$ , ...	$f(x) \geq -4$	$f(x) \leq 0$	$f(x) \geq -5$	$f(x) \leq 4$										
2 Pour tout nombre réel $x$ de $[-8; 2]$ , ...	$f(x) \geq 0$	$f(x) \leq 3$	$f(x) \geq -4$	$f(x) < 2$										
3 Le nombre $f(0)$ est tel que ...	$f(0) < f(1)$	$f(0) \leq 2$	$f(-5) \geq f(0)$	$f(0) \geq -5$										
4 Le nombre $f(4)$ est tel que ...	$f(4) \leq f(5)$	$f(4) > 4$	$f(3) > f(4)$	$f(4) < 5$										
5 Un nombre supérieur ou égal à 3 est par exemple ...	$f(-7)$	$f(-1)$	$f(0)$	$f(6)$										

 Translation  **65.** In each case, give the exact answer(s) without justifying.  Here is the table of variation of a function  $f$  defined on the interval  $[-8; 6]$ .   |  | |--| | 1. For any real number $x$ of $[-5; 3]$ , ...            | | 2. For any real number $x$ of $[-8; 2]$ , ...            | | 3. The number $f(0)$ is that...                          | | 4. The number $f(4)$ is that...                          | | 5. A number greater than or equal to 3 is for example... | |  |  |  |

We illustrate potential variations embedded in/among these tasks:


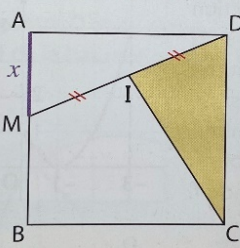
- Representational variation: There are conversions from auxiliary transitional representations (table of variation) to symbolic writing.
- OPMC: MCQs are designed through OPMC. In this task, item 1 and item 2, item 3 and item 4 are two pairs of constraint manipulation of OPMCs, by varying the specific numbers. Mathematical strategy for these five items is the same (MPOS) as the essential features of the task are invariant.

MCQs designed as tasks for auto-test are an efficient way to test whether students have mastered knowledge or mathematical methods as it saves a lot of time in the writing process. Different OPMC can be designed in one MCQ.

#### 4.3.3.3 For going further

Tasks in the part of “for going further” are various: narrative research, group work, TICE, open problem, challenges, etc. One open problem of the section of “qualitative study of functions” is shown in table 4.3.8. The purpose of this problem is to let students explore the variation of a function in a geometrical figure.

Table 4.3.8 Open problem of studying the variations

<p><b>76</b>  <b>Problème ouvert</b> <b>Étudier des variations</b></p> <p>ABCD est un carré de côté 5 cm. M est un point de [AB], on pose <math>x = AM</math> (en cm). I est le milieu du segment [DM].</p>  <p><math>f</math> est la fonction qui à <math>x</math> associe l'aire, en <math>\text{cm}^2</math>, du triangle DCI. Étudier les variations de la fonction <math>f</math>.</p>	<p>Translation</p> <p><b>76.</b> Open problem: Study the variations ABCD is a square with a side length of 5cm. M is a point of [AB], we set <math>x = AM</math> (in cm). I is the midpoint of the segment [DM]. <math>f</math> is the function of <math>x</math>, of the area of the triangle DCI. in <math>\text{cm}^2</math>. Study the variations of the function <math>f</math>.</p>
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We illustrate potential variations embedded in the task:

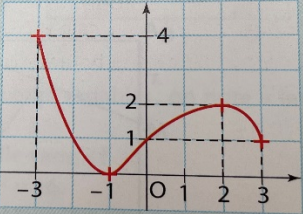
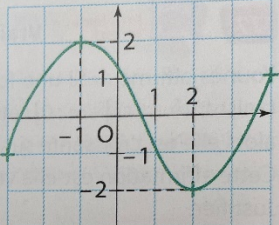
- Contextual variation: the context of this task is mathematics (geometry), making a connection between geometry and functions.
- Representational variation: There are conversions between geometrical figure to symbolic writing in the process of solving this task.

#### 4.3.3.4 Personalized support

There are two pages of personalized support tasks in each section. Tasks are classified and presented according to the specific content in “personalized support” so that students can carry out targeted exercises according to their weakness. At the end of

“personalized support”, there are tasks for in-depth study. We illustrate a series of tasks in the personalized support of describing the variation of a function in table 4.3.9.

Table 4.3.9 A series of tasks in the personalized support of describing the variation of a function

<p><b>Soutien</b> Décrire le sens de variation d'une fonction</p> <p><b>80 Exercice test</b>  <math>f</math> est la fonction définie sur <math>[-3; 3]</math> par la courbe tracée dans le repère ci-contre. Dresser le tableau de variation de <math>f</math>.</p>  <p>Appeler le professeur pour qu'il contrôle vos réponses et qu'il vous indique la suite.</p>	<p>Translation  <b>Support: Describe the variation of a function</b>  <b>80. Test exercise</b>  <math>f</math> is the function defined on <math>[-3; 3]</math> by the curve drawn aside.          Draw the table of variation of <math>f</math>.            Call the teacher to check your answers and tell you what's next.</p>
<p><b>81</b> <math>f</math> est la fonction définie sur l'intervalle <math>[-3; 4]</math> par la courbe tracée dans le repère ci-contre. Recopier et compléter.</p> <p>a) <math>f</math> est croissante sur <math>[-3; \dots]</math>.          b) <math>f</math> est décroissante sur <math>[-1; \dots]</math>.          c) <math>f</math> est croissante sur <math>[\dots; 4]</math>.          d) <math>f</math> est décroissante sur <math>[\dots; 1]</math>.</p> <p><b>82</b> <math>f</math> est une fonction définie sur l'intervalle <math>[-3; 5]</math> telle que :</p> <ul style="list-style-type: none"> <li><math>f</math> est décroissante sur <math>[-1; 2]</math>;</li> <li><math>f</math> est croissante sur <math>[-3; -1]</math> et sur <math>[2; 5]</math>;</li> <li><math>f(-3) = -2</math>; <math>f(2) = 1</math>; <math>f(-1) = 4</math>; <math>f(5) = 5</math>.</li> </ul> <p>Dresser le tableau de variation de <math>f</math>.</p> 	<p><b>81.</b> <math>f</math> is the function defined on the interval <math>[-3; 4]</math> by the curve drawn aside.          Copy and complete.          a) <math>f</math> is increasing on <math>[-3; \dots]</math>.          b) <math>f</math> is decreasing on <math>[-1; \dots]</math>.          c) <math>f</math> is increasing on <math>[\dots; 4]</math>.          d) <math>f</math> is decreasing on <math>[\dots; 1]</math>.</p> <p><b>82.</b> <math>f</math> is a function defined on the interval <math>[-3; 5]</math> that:</p> <ul style="list-style-type: none"> <li><math>f</math> is decreasing on <math>[-1; 2]</math>;</li> <li><math>f</math> is increasing on <math>[-3; -1]</math> and on <math>[2; 5]</math>;</li> <li><math>f(-3) = -2</math>; <math>f(2) = 1</math>; <math>f(-1) = 4</math>; <math>f(5) = 5</math>.</li> </ul> <p>Draw the table of variation of <math>f</math>.</p>

We illustrate potential variations embedded in/among these tasks:

- **Representational variation:** There are conversions from graph to auxiliary transitional representations (table of variation, task 80); from graph to symbolic writing (task 81), and from natural language and symbolic writing to auxiliary transitional representations (table of variation, task 82) in the tasks.
- **Pudian:** These three tasks are closely related. The intention of the design of these series of tasks is that according to the completion of the students' exercise test (task 80 here), the teacher decides the students' next learning. Base on this, task 81 and 82 are the Pudian of task 80. We can reduce the goal of task 80 into small pieces as what is shown in task 81. In other words, task 81 is Pudian of task 80. Task 82 is a step further of task 81 by removing the graph of the function and expressing all the

information of variation in the graph in natural language and symbolic writing.

- OPMC: There are constraint manipulations of OPMC in these three tasks. Mathematical strategy for these three tasks is the same (MPOS) as the essential features of these three tasks are invariant.

According to the designers' note in the textbook, "personalized support" is a central element in this textbook. In each chapter, "test exercises" followed by remedial exercises are proposed as well as deepening exercises, in order to differentiate the students' work. Personalized support and targeted training can be completed by designing appropriate test exercises, and then redesign the exercise test through variation (mainly Pudian and OPMC).

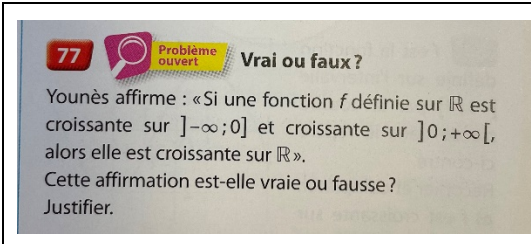
#### **4.3.4 Conclusion**

Based on the analysis above, we conclude that potential variations in French Hyperbole textbook mainly have the following characteristic:

- The contexts of tasks are rich, from all aspects of life. A large number of mathematical context tasks of function are with geometry context, especially in the part of "for training" and "for going further", making a strong connection between geometry and functions.
- Representational variation is various. Besides the conversions among natural language, auxiliary transitional representation (table of values, table of variation, table of signs), symbolic writing and graph, tasks of function with geometrical context in Hyperbole textbook brings the conversions related to geometrical figures. In the process of solving tasks with geometrical context, students need to transform the geometrical figures to symbolic writing based on the geometric properties. Moreover, Hyperbole textbook places great emphasis on the use of technology in problem solving. In terms of function, conversions from symbolic writing to table of values are completed by the calculator.
- Conceptual variation embedded in the tasks for students to explore: To introduce a new concept, Hyperbole textbook often presented several standard variations, and the distinguishment of the concept are organized in the part of exercises as open problem by using non-standard and non-concept variation. For example, table 4.3.10 shows a task noted as open problem in the part of "for going further" to distinguish the concept of monotonicity of a function. The conflict between the non-concept and the concept is designed to explore by the students.



Table 4.3.10 An open problem about the concept of monotonicity of a function

	<p>Translation</p> <p><b>77. Open Problem:</b> True or false?</p> <p>Younès says: “If a function <math>f</math> defined on <math>\mathbb{R}</math> is increasing on <math>] -\infty ; 0]</math> and increasing on <math>] 0 ; +\infty [</math>, then it is increasing on <math>\mathbb{R}</math>”.</p> <p>Is this statement true or false? Justify it.</p>
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- Procedural variation is embedded in certain parts of Hyperbole textbook. Tasks are designed in different forms and with different roles in Hyperbole textbook. There is a trace of variation from tasks in the certain part of “know-how”, “solving problems”, “for training”, “for auto-test” and “personalized support”. Constraint manipulation of OPMC is mainly used. Moreover, OPMC is also embedded in a series of tasks in one part (see an example in section 4.3.2.2 and 4.3.2.3). However, in the part of “discover”, “practical task” and “for going further”, tasks are quite isolated.

In general, Hyperbole textbook, in addition to presenting the basic content, focuses on providing teachers and students with more possibilities by offering a large number of different types of tasks. Possibilities are offered in two aspects: One is in the content of tasks by offering tasks with TICE and algorithm, open problems, practical tasks, etc.; and the other in the organization by offering tasks to work in groups, to work in dependently, to auto-test, and to do narrative research, etc.

## 4.4 Potential variation in the digital curriculum resources

With the development of technologies, textbook is no longer an isolated resource and accompanied with a series of digital curriculum resources. The universe of digital curriculum resources is very wide and very quickly evolving. Therefore, in this section, we first introduce the definition of digital curriculum resources (section 4.4.1). We then choose to analyze the potential variations in the digital curriculum resources from two different lenses: the current digital curriculum resource to support the textbook (section 4.4.2, Chinese PEP platform as an example) and more advanced situation for further evolution of digital curriculum resources (section 4.4.3, French Sésamath platform as an example).

### 4.4.1 Definition of Digital Curriculum Resources

Pepin, Choppin, Ruthven and Sinclair (2017) point out that digital curriculum resources refer to organized, electronic format digital resource systems. These digital resource systems clearly indicate the scope and order of curriculum content. These curriculum resources focus on the order of students’ learning topics based on grade or age level, as well as content related to a specific course (such as algebra) to cover (all or part of) the

curriculum standard. This is also an important indicator to distinguish digital curriculum resources (hereinafter referred to as DCR) from general digital teaching tools or educational software programs. They distinguished between DCR research and digital technology research, thinking that DCR research focuses on the following aspects:

- (1) Objectives and contents of mathematics teaching;
- (2) the role of the teacher in the teaching design process (i.e.: how the teacher selects, revises and improves the curriculum resources);
- (3) Interaction between students and DCR: how to carry out learning in a digital environment and gain learning experience;
- (4) Impact of DCR: how teachers and students determine the scope and order of mathematical topics in a digital environment;
- (5) The possibility of DCR to promote teachers' ability to design teaching activities.

At the same time, Pepin et al. (2017) summarized three basic characteristics of DCR: The first is to change the learning space and enhance the interactivity and customizability of teaching activities; the second is to make embedded evaluation possible, which can provide students and related personnel (such as parents, teachers, administrators, etc.) with data feedback; the third is usually combined with learning management systems (LMS), to achieve unified curriculum and curriculum management, such as assessment reports, attendance records and other curriculum management functions.

#### **4.4.2 The design of Chinese-PEP textbook platform and its potential variations**

In this section, we take Chinese PEP platform as an example to show the potential variations in the current digital curriculum resource to support the textbook. We first introduce the profile of PEP platform (4.5.3.1). Then we study the potential variations in two main resources of this platform: digital resources for high school mathematics textbooks and high school mathematics PEP textbook online training based on the new curriculum standard.

##### **4.4.2.1 Profile of PEP textbook platform**

People's Education Press (PEP) was established on December 1, 1950. It is a large-scale professional press under the Ministry of Education. Since its establishment more than 70 years ago, it is committed to the research, design, publication and distribution of textbooks at all levels of education. At present, the People's Education Press has become the press with the strongest comprehensive strength in China.

In order to strengthen the research of textbooks, Curriculum and Teaching Materials

Research Institute (Abbreviated as CTMRI in the following) was established in 1983 with the approval of the Ministry of Education, which is co-located with the People's Education Press. For more than 30 years, the CTMRI has undertaken more than 30 national-level projects, conducted in-depth research on the curriculum, teaching materials, teaching and evaluation, etc. The in-depth and systematic research on the curriculum, textbooks and practical teaching provides a strong guarantee for the comprehensive improvement of the quality of PEP textbooks. CTMRI also actively expands external contacts, and has established extensive exchanges and cooperation with many educational and scientific research institutions and schools in China and abroad. There are mainly eight functions of Curriculum and Teaching Materials Research Institute: 1) Carry out research on textbook development: In accordance with the national education policy and textbook policy, undertake the research and development of textbooks (including textbooks and supporting educational resources) at all levels; 2) Carry out application research in the textbooks and practical teaching: To support the research and development of textbooks, carry out monographic study in the curriculum, textbooks and practical teaching; 3) Carry out research on the development of digital (electronic) textbooks and teaching resources; 4) Carry out research on experiment and evaluation of textbooks: theoretical research on experiment and evaluation of textbooks; research on experiment and evaluation of textbooks in various disciplines; establish a scientific and systematic feedback and update mechanism for textbooks, and conduct tracking and evaluation of textbooks; 5) Provide comprehensive support and system services for textbook users: provide local education administrative departments, teaching and research departments and practical teachers with consulting services, training services and teaching and research services in terms of textbooks and practical teaching, and provide practical teachers and students with network services; 6) To provide policy consultation and suggestions for the decision-making of national curriculum textbooks construction; 7) To carry out academic exchanges and cooperation on textbooks: organize (including sponsoring, undertaking) or participate in domestic and international seminars and conferences, and conduct exchanges and cooperation with international and domestic institutions and organizations of textbooks; 8) To undertake the training of talents related to the textbooks.

The high school mathematics PEP textbook is jointly designed and published by the People's Education Press, the Curriculum and Teaching Materials Research Institute, and the Middle School Mathematics Curriculum Textbook Research and Development Center.

After the promulgation of new curriculum standards in 2017 and promulgation of new textbooks based on the new curriculum standards in 2019, the PEP website has been



developed. Because the PEP website covers all disciplines in all grades and is not limited to school classroom teaching, here we focus on the digital resources of high school mathematics in this website. Figure 4.4.1 can help us have a holistic view of the large number of resources that PEP platform has in high school mathematics. Most of resources shown here have been developed with the support of the CTMRI. The two main resources are digital resources for high school mathematics teaching and high school mathematics PEP textbook online training. We will introduce these two parts in detail in the next two sections (4.4.3.2 & 4.4.3.3).

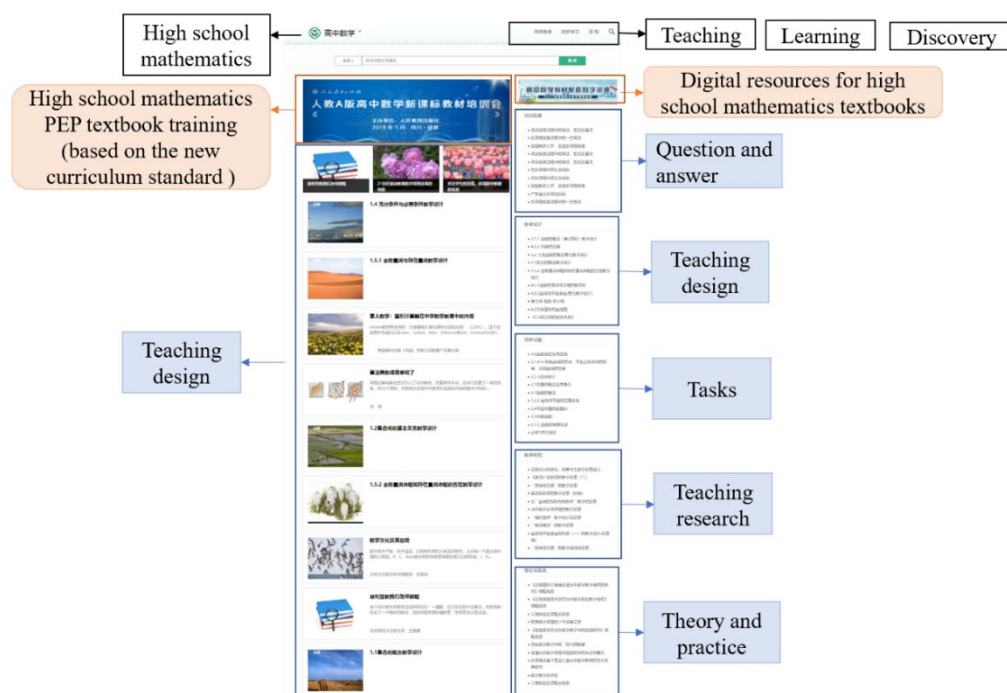


Figure 4.4.1 Homepage of high school mathematics PEP textbook website<sup>79</sup>

#### 4.4.2.2 Digital resources for high school mathematics teaching: Providing supports for teaching

There are mainly the following types of digital resources in this platform: 1) Question and answer; 2) teaching design; 3) tasks; 4) dynamic demonstration of classroom teaching resources; 5) teaching research; 6) theory and practice.

Articles are published in “Question and answer”, mainly for the new curriculum reform, such as some practices and understandings about the use of new textbooks, how to promote the implementation of new curriculum through education research work, the practices, ideas and opinions in the process of curriculum reform implementation, as well as the review of curriculum reform, etc.

<sup>79</sup> From <https://www.pep.com.cn/gzxx/>, English description added by the author. Retrieved on 1<sup>st</sup> march, 2022.

“Teaching design” provides teachers with a complete classroom teaching design of specific mathematical contents for their reference. Each teaching design mainly includes content and content analysis, goal and goal analysis, diagnosis and analysis of teaching problems; analysis of teaching support conditions (technology); teaching process design (tasks, teacher-student activities, preset answers, design intentions of tasks, classroom summaries, homework), design of evaluation, etc. These teaching designs are clear and detailed, and combined with the contents of the PEP textbook, help teachers fully grasp the key points and difficulties of teaching, especially for novice teachers.

“Tasks” offers tasks for a certain mathematical content. For each document, there is an explanation of the key points and difficulties in learning of the content, and more than a dozen related tasks with corresponding solutions presented.

“Dynamic demonstration of classroom teaching resources” mainly provides technical support for teachers’ classroom teaching. The main software used is GeoGebra. For example, figure 4.4.2 shows the resources provided for the teaching of the parity of a function. This resource takes the form of interactive animation, providing users with a material to perceive and understand the definition of function parity. This resource first presents the characteristics of functions with parity through animation demonstrations, and draws preliminary conclusions; then provides a tool that can independently input arbitrary function analytical expressions to explore function parity, and users can explore independently.

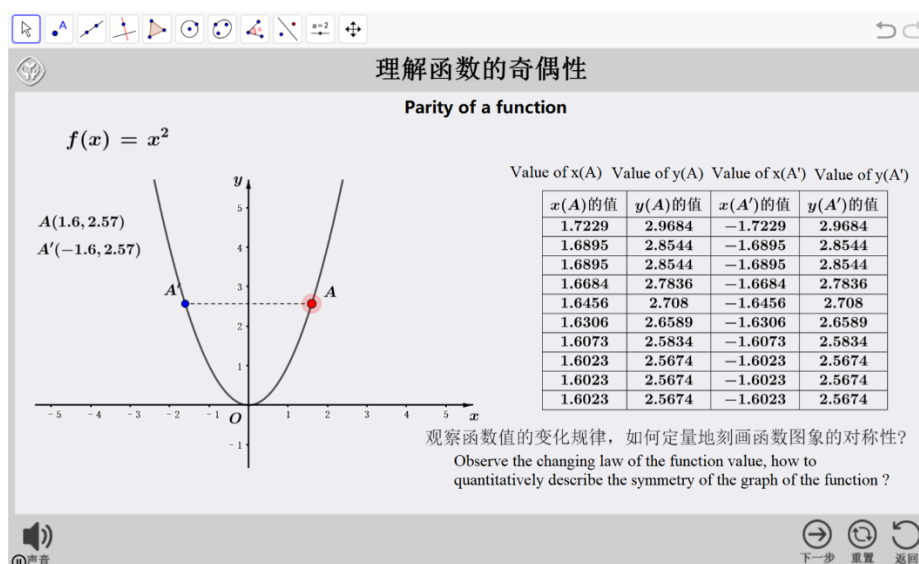


Figure 4.4.2 Dynamic demonstration of classroom teaching resources

Pudian and OPMC is embedded in the design of such activities. As point A moves, guide students step by step to find the invariant. With such technical support, the scope of OPMC in potential variation has been greatly expanded. We are not limited to

changes in a few numbers, but through a series of changes, we can carry out mathematical exploration to discover mathematical laws.

“Teaching research” publishes articles written by practical teachers on the review and reflection of their classroom teaching. “Theory and practice” publish articles written by researchers to interpret textbooks, design of textbooks, classroom teaching, etc., from a theoretical perspective.

In general, PEP platform offers various digital resources to support high school mathematics teaching. Some of resources are designed by the website itself and some are selected from other places. On the one hand, digital resources expand the potential variations. On the other hand, these resources are dedicated to expanding teacher resources from various aspects, and also provide the possibility for teachers to further (re)create variation in the documentation work.

#### **4.4.2.3 High school mathematics PEP textbook training: better understanding the intention of textbooks**

The People’s Education Press organizes teachers and researchers to participate in the activity of “PEP teacher training on the use of teaching materials for primary and secondary schools”, and shared videos and slides of the activity online.

As we mentioned in section 4.1.1, PEP textbook (2019) is designed as core literacy-oriented curriculum based on the general high school mathematics curriculum standards (2017). In order to let teachers and researchers better understand textbooks, grasp the content, system, and characteristics of textbooks, and also to listen to the opinions of all parties, “PEP teacher training on the use of teaching materials for primary and secondary schools” was held. Here, we only focus on the training of high school mathematics textbooks. The theme of this training is the interpretation of the new PEP textbook (2019). In this training, the editor-in-chief of the textbooks introduces the overall situation of textbook design, and the core-authors of the textbooks introduce the compulsory 10 chapters of the textbooks chapter by chapter. We take the chapter of function as an example. The introduction of this chapter in the training mainly includes the learning objectives of this chapter, the knowledge structure diagram, the content arrangement, the status and function of this chapter, suggested schedule of lessons, the intention of design, and teaching suggestions.

PEP platform provides resource support for teachers’ and researchers’ self-development. Teachers and researchers are no longer limited by time and space. They can conduct training at anytime and anywhere, and can watch repeatedly. Teachers can also reflect from practical teaching, which is more conducive to the further improvement of textbooks and teaching materials.

### **4.4.3 The design of French-Sésamath platform and its potential variations**

French researchers have conducted a series of explorations and attempts on the DCR platform, and have made certain progress and results. Among them, the Sésamath platform is a distinctive result of DCR construction in France, which reflects the concepts of DCR in terms of creation, integration, management and maintenance. In this section, we will first introduce the profile of Sésamath platform (4.4.2.1). Then we study the potential variations in three typical products of this platform: 1) Variations in the design and development of Sésamath's teaching materials and electronic teaching materials; 2) variations in the design of Mathenpoche; 3) variations in the design of LaboMEP platforms.

#### **4.4.3.1 Profile of Sésamath**

The Sésamath Association<sup>80</sup> was founded in 2001. It is a French mathematics teachers' association. Its core members are practical teachers in middle schools, and some retired teachers, software engineers, volunteers, etc. are also involved. The association is highly regarded and non-profit. The Association aims to design open-source digital curriculum resources for mathematics teaching: to provide free educational resources and professional tools for mathematics teaching to the public through the Internet, thereby promoting the acquisition of mathematical knowledge. To this end, Sésamath allows everyone to access freely their platform for sharing resources on the Internet; promote as much as possible "open source" resources that are freely available and "open" resources that users can add and modify independently; With the aforementioned open source and open online resources, deepen the combination of teaching and technology and increase the appeal of mathematics.

The Sésamath Association has grown very quickly. In 2001, the Sésamath Association was launched by about 20 math teachers who shared their personal resources website. Subsequently, the association designed a "drill-and-practice" software program, "Math in the pocket" (Mathenpoche). Mathenpoche quickly became successful and was accepted and used by many teachers and students. In 2007, Sésamath stood out from 68 projects from 51 countries and won the third prize for ICT applications from UNESCO. Today, Sésamath has 70 sponsors and is supervised and managed by a board of about 10 people. About 1.3 million people visit its website every month. So far, about 30,000 teachers and about one million students have registered as members of the website (Gueudet, Pepin, Sabra & Trouche, 2016; Gueudet & Trouche, 2012). Gueudet, Pepin and Trouche (2013) believes that the reason why Sésamath can become the main

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<sup>80</sup> The name of the association "Sésamath" is to combine the famous "Open sesame" (Open sesame) in "One Thousand and One Nights" with math (math), in order to reflect the attitude of open source and sharing of resources on the platform.

reference platform in the field of education in France is mainly based on its three advantages: first, it provides opportunities for teachers to collaborate, and second, it is equipped with a series of flexible resources and tools. Third, it builds a platform for discussing and improving the composition and development model of these resources.

Sésamath's resources are voluntarily created by middle school (junior and high school) math teachers or elementary school teachers. The specific research and development process mainly includes the following five steps: 1) determine the structural framework, design concepts and design ideas of the resources; 2) explore and modify the resources created within the design team (small group); 3) place the completed resources in pilot test and edit in a small area; 4) put resources online, open and promote; 5) after resources are applied to real classrooms, resources are further revised and improved according to the classroom situation. It is not difficult to see that the entire research and development process is a reciprocating process. The resources are discussed collaboratively by the design team, and pilot tests were conducted in the classroom. Through the communication within the team, between the team and the classroom teachers, the resources are continuously improved, and publicly released on the Internet after being widely recognized. But this is not the end of the Sésamath resource design. Resource users can submit comments or improvement suggestions in real time. The team will comprehensively consider the feasibility and necessity of making these changes (mainly based on the team's own adjustable power and the importance of the revision of this resource), and ultimately respond to the updates of resources<sup>81</sup>. It should be noted that not only members of Sésamath can participate in the process of creating this resource, but participating members should respect the opinions of others and agree with the concept of "open source" resources of the association. At the same time, Sésamath does not promote any teaching methods and theories.

Figure 4.4.3 can help us have a holistic view of the large number of resources that the Sésamath platform has. Most of the platforms and projects shown here have been developed with the support of the Sésamath Association. The Sésamath website also provides access to these platform projects. Among them, the e-textbooks developed by the association, Mathenpoche, and LaboMEP are the three most distinctive, widely used, and most original projects of the platform. We will introduce these three projects in detail below.

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<sup>81</sup> According to the evaluation results, the association sometimes decides not to modify the old resources in order to concentrate on the development of the new version of the resources and replace the old version with this.

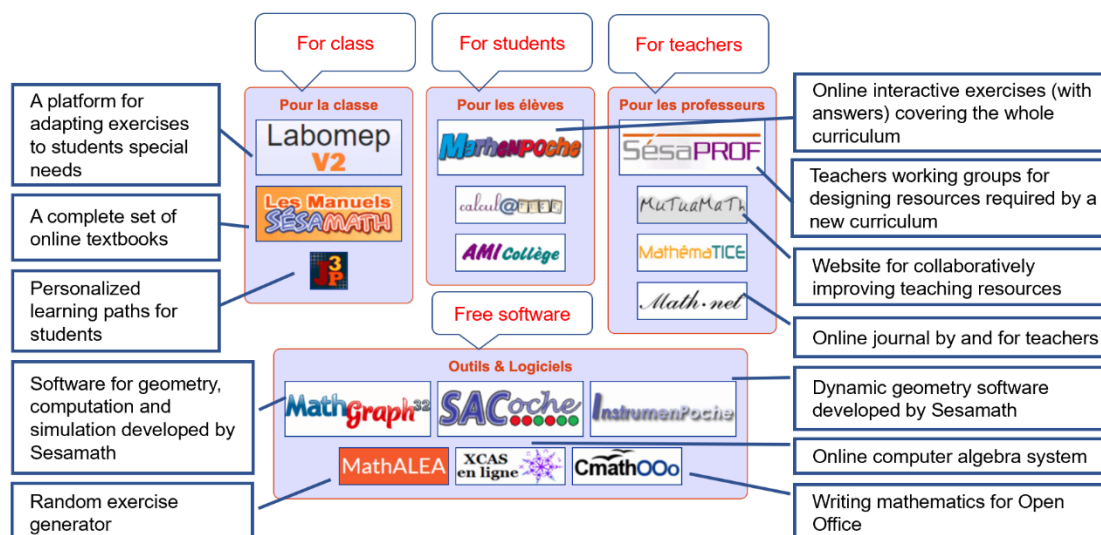


Figure 4.4.3 Homepage of Sésamath website<sup>82</sup>

#### 4.4.3.2 Sésamath e-textbooks: A new way of textbook design and the variation thoughts embedded in it

Since 2005, Sésamath has been providing textbooks<sup>83</sup> for junior and senior high school. Today, Sésamath textbooks are widely used in France. So far, Sésamath textbooks (printed versions of electronic textbooks) have sold more than 1.5 million sets, accounting for about 20% of the French textbook market (Gueudet et al., 2013).

Traditionally, French textbooks and e-textbooks are designed by a small team (four to six) of teacher, trainers and supervisors, and published by commercial publishers. The design of Sésamath textbooks has broken the traditional “top-down” method. The authors of the textbooks are ordinary teachers in the Sésamath association, and adopt the “bottom-up” method (Gueudet et al., 2016). On the other hand, compared with traditional textbooks, Sésamath textbooks have a wider resource platform. When teachers use Sésamath textbooks, they can use related resources on their platform at the same time, thus making their resource system more diversified. So, what methods and means does Sésamath use to ensure the quality of these textbooks?

Gueudet et al., (2013) believe that the concept of "quality" of textbooks is complex, and it includes the concepts of coherence and depth of understanding in textbook analysis. Its complexity is caused by changes brought about by digital means. By comparing Sésamath sixth grade textbooks with traditional sixth grade textbooks<sup>84</sup>, it reveals some

<sup>82</sup> From <http://www.sesamath.net/>, English description added by the author. Retrieved on 1<sup>st</sup> march, 2022.

<sup>83</sup> Comes with free online pdf and odt versions, as well as related supplementary materials.

<sup>84</sup> Helice textbooks: sold by commercial publishers, written by a team of experts (researchers and teacher trainers).

characteristics of Sésamath textbooks that are different from traditional textbooks:

1. Structure of textbooks: Traditional textbooks as a whole have a relatively fixed and closed structure, while Sésamath textbook is an “atomic” system that can be arranged by different users;
2. Textbook content arrangement: Traditional textbooks are based on the advice of an expert team to integrate teaching content and methods, and enjoy a certain degree of freedom in the layout. Sésamath textbooks are arranged in accordance with national curriculum standards, and the connection with curriculum standards is closer.
3. The relationship between textbooks and users: Traditional textbooks are often provided to teachers as a final product, while Sésamath textbooks only provide some basic frameworks for teachers, and teachers can expand and improve the content of textbooks on this basis.

The traditional way of designing textbooks is: there is a fixed team for the design of textbooks. After the textbooks are designed, they are used by teachers. Teachers are only users. Problems found in the process of use need to be improved until the next time the textbook is revised. Compared with this traditional approach to closed textbooks, Sésamath textbooks are designed using an open approach, which appears to have a lower intrinsic quality in its initial version: it provides fewer questions and fewer tasks. In terms of structure, it is just a structure proposed in accordance with national curriculum standards. However, we should note that the content of traditional textbooks will remain **static** in both paper and pdf electronic versions. Sésamath textbook resources are not static and can be based on the needs expressed by users through the content of comments. It has been modified many times and is therefore “**dynamic**”. This new method of textbook modification is closely related to digital development. Sésamath e-textbooks have a large number of teachers involved in resource design, and through continuous modification to achieve the development and improvement of resources. Digital means providing the possibility for this form of textbook modification, and Sésamath integrated this method well with the process of redesigning the textbook.

The impact of digital reform has forced us to rethink the concept of “quality of textbooks.” The Sésamath team set this textbook as a digital textbook from the beginning, and each chapter file is in .odt (OpenDocument) format, which provides teachers with the opportunity to modify the exercises and text of curriculum. This textbook provides a large number of exercises for teachers to choose and apply freely. The Sésamath team believes that digital textbooks are a “brick collection” in which teachers can select the appropriate “bricks” (i.e. resources) to build a coherent curriculum and improve it. In this case, even teachers who are not involved in the design of textbooks can be regarded as “designers” of their own teaching, because it is the

teachers who complete the continuous process of resource selection, resource integration and resource use.

From the perspective of variation, the process that the teacher selects the appropriate “bricks”/ “atom”, adapts it to his/her own class, implements in class, and reflects, improves the tasks can be regarded as the process of identifying, creating, implementing, reflecting, and improving variations, which is consistent with our preliminary model of variation (see figure 2.4.2 in section 2.4.2). An atom or brick is equivalent to a mathematical fundamental problem, and on this basis, teachers make changes to it according to their own curriculum goals to form their own curriculum. On one hand, offering “bricks”/ “atoms” provides more space for teachers to vary and promotes the outcome of variation; on the other hand, this dynamic design of e-textbooks makes the cycle of teachers’ documentation work faster and more frequently.

#### **4.4.3.3 Potential variations in Mathenpoche (Pocket Mathematics) Platform: Variations created through the interactions with computer**

Mathenpoche is the earliest platform built by the Sésamath association, mainly for middle school students. There are two different types of resources in this platform: one is closely related to school teaching and is directly related to teaching content; the other is centered on mathematical games. Below we introduce these two types of resources.

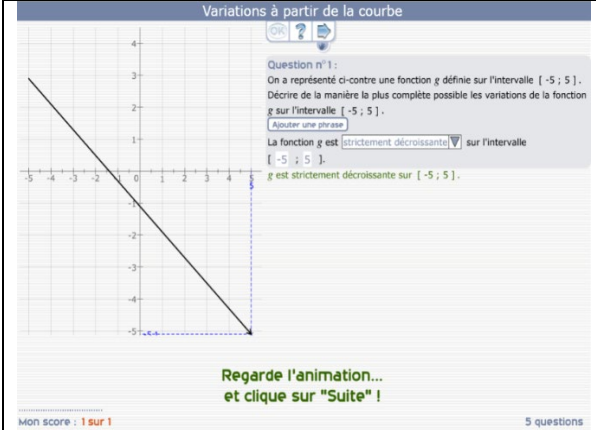
The first type of resources covers the important knowledge points required in the curriculum standard. Each chapter of mathematical knowledge points has three main contents: review, comprehension and application, and self-assessment. Review mainly refers to the review of pre-knowledge before students learn new knowledge. A review of the key concepts in front will help students understand and master new knowledge. The comprehension and application part mainly include the mathematical content, dynamic display of problem-solving methods, and interactive exercises. This part of the interactive exercise is selected from a very large number of exercises, which is completed by the teacher. The self-assessment consists of multiple-choice questions and test papers (with dynamic demonstrations of the answer and exercise analysis process). The content and exercises in each chapter match the school’s teaching content. Dynamic course content display, interactive exercise modification and analysis enable students to learn independently and conduct self-evaluation. It is obvious that this part of Mathenpoche is modeled on the order in which students learn at school (from review, comprehension, application to evaluation), because the designer hopes that students can quickly adapt and use the platform to learn effectively.

The interactive exercises in Mathenpoche are the most popular resources produced by Sésamath (see table 4.4.1). There are more than 2000 interactive exercises and dynamic analysis in the platform. Based on the theme of the exercise, the system randomly generates questions, and each set contains 5 to 10 questions. Students need to answer



each question with two chances to try. Each question is assigned one score. If the answer is wrong twice, the question will not be scored. After answering the questions, the window will display the correct or wrong, and dynamically give or demonstrate the correct solution. The opportunity to answer the question twice gives students a chance to make trial-and-error, and the detailed dynamic explanation and analysis can attract students' attention and deepen their understanding and mastery of knowledge.

Table 4.4.1 Mathenpoche interactive exercise, discerning monotonicity of functions from graph

	<p>Question 1:</p> <p>We represent the function defined on the interval <math>[-5, 5]</math> in the figure. Describe as completely as possible the monotonicity of the function <math>g</math> on the interval <math>[-5, 5]</math>. Click to add description</p> <p>The function <math>g</math> is "strict decrease" ( choose via the drop-down menu) on the interval <math>[ , ]</math> (fill in the blank).</p> <p>Score 1 out of 1: the student answered correctly in the first question</p>
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Mathenpoche gives a string of tasks on one mathematical knowledge point every time and these tasks are randomly selected from the task database of Mathenpoche. Most of the time the string of tasks just vary in the basic value of condition (numerical changes) and the new task can be solved in the same way of the previous task (multiple problems one solution). Sometimes these tasks are designed to be hierarchical according to the difficulty (complexity) of the task. For example, the tasks of multiplication calculation are from two-digit multiplication to three-digit multiplication to four-digit multiplication.

The second type of resource is mainly a series of mathematical games. These fun math activities can provide some support for classroom math teaching: attract students to use the website through games, they may use other resources in the website in this process; at the same time, these fun activities require a certain amount of mathematical knowledge; can make students who are not doing well in mathematics get a sense of accomplishment in the game and settle with mathematics.

DCR provides possibilities and conveniences for variations, and at the same time, it can also provide DCR with new development ideas and improvements through variations. For example, we can intelligently distinguish between students' incorrect answers and correct answers by using DCR. If the student answers incorrectly, after showing the correct solution, we can propose another task (with only minor changes, such as changing the number) to verify whether the student has mastered this knowledge point. The variation here is used as a verification/judgment tool. If the student answers

correctly, then we can increase the difficulty of the task or expand the breadth of the task through variation, thereby expanding the student's learning space and allowing their learning potential to be fully developed. The variation here is used as a tool for students to learn deeply. It can be seen that in different situations, the effects of the variations are different. Therefore, we need to use the variation flexibly, skillfully and ingeniously according to the students' learning situation.

#### **4.4.3.4 Potential variations in LaboMEP Platform: Variations created through sharing**

The Sésamath team is committed to providing teachers with a large selection of resources for teaching, and textbooks are just one of them. "Efficient tools" is also an important resource provided by the team to teachers. LaboMEP is a virtual teaching tool developed by the Sésamath Association. LaboMEP provides teachers with a lot of resources while ensuring that teachers can prepare targeted lessons according to his/her curriculum, class, and individual needs of students. The first version of LaboMEP was developed in 2008 and was put into use in early September 2010. The development of the second version started in 2014. Compared with the first version, the second version has a completely different architecture (more modular and relies on external resource servers that can be developed in parallel) and uses different technologies. The new version was first commissioned in the spring of 2017 and gradually replaced the first version from September 2017. In November 2018, the second version has completely replaced the first version.

The LaboMEP interface (see Figure 4.4.4) contains three main screen areas. The area on the left is mainly "established resources", which are given by the system, including the list of professors' class students, resources shared by teachers from the same school, and resources provided by Sésamath. We can see that LaboMEP follows the exercise resources in Mathenpoche, including interactive exercises, multiple choice questions, interactive dynamic presentations, etc. Similarly, the Sésamath e-textbook is a part of LaboMEP. The right area is mainly "self-built resources", designed and integrated for the teacher himself/herself, including his/her grouping of students, his/her own resources, the courses he/she designed, and the courses of colleagues. Therefore, the teacher can share his/her class schedule with other colleagues through LaboMEP, and colleagues can establish a shared resource library for resource sharing. The middle area is mainly the "teacher work area" and is the active area of LaboMEP. In this area, the teacher can use the left and right area resources to insert it into the document he is designing (the middle area), so as to effectively create and design content such as exercises, student groups, courses, and so on. All of the designed lesson that he/she prepares will be recorded and saved in the "self-built resources". Each document can include excerpts from textbooks, online exercises, geometric construction using

dynamic geometry software, and even external website links. With this technical support, teachers can use this platform to design personalized lessons and exercises for students, and send different documents to different groups of students to achieve the purpose of personalized teaching.

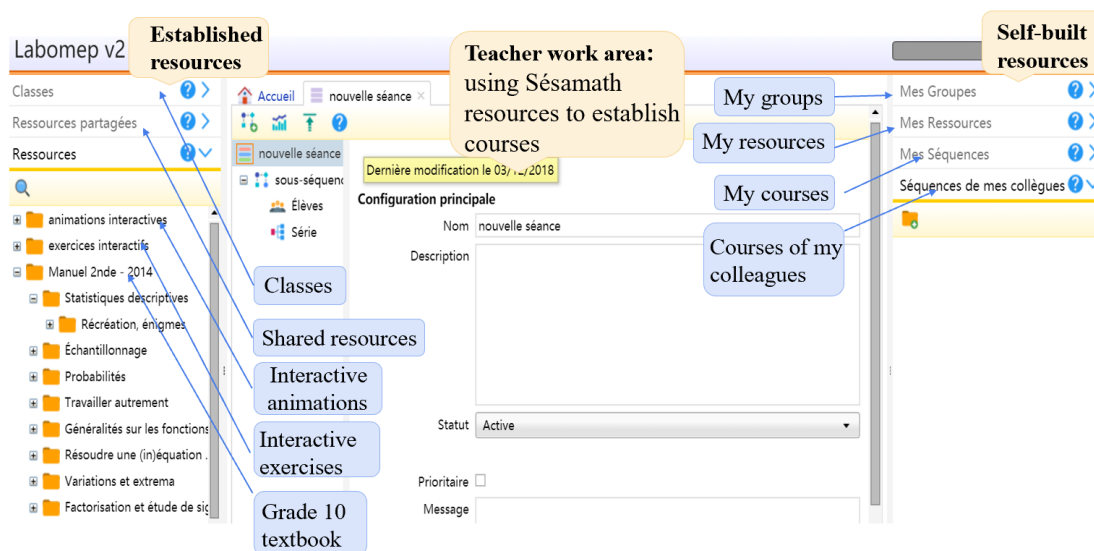


Figure 4.4.4 LaboMEP, a virtual environment designed for teachers

Teachers can also view student practice feedback at LaboMEP (see Figure 4.4.5). This feedback is automatically generated by the system. Teachers only need to click the student's name in the "Classes" menu to view statistics such as how long the student used to complete the problem, the correctness of the answer<sup>85</sup>, and the record of the solution to the problem. These quantified data feedback can help teachers quickly identify the weak points of students and implement more targeted teaching to students.

In addition, teachers can also communicate with members of the Sésamath Association through this platform and make suggestions on the resources of the platform. This possibility of collective collaboration is crucial for the Sésamath Association and is not yet available on other platforms.

<sup>85</sup>In the visual representation of the accuracy rate in Figure 4.4.5, green indicates that the student answered correctly the first time, dark green indicates that the first answer was wrong and the second answer was correct, red indicates that both answers were incorrect, and blue indicates that the student did not answer.



Figure 4.4.5 View student's work in LaboMEP

LaboMEP platform can record the process of teachers' documentation work, including teachers' creation of variations. This way of unifying the resource database, students' situation, and documents shared by other teachers on one platform provides great convenience for teachers to create variations.

#### 4.4.4 Conclusion

Digital trend of textbooks and platforms are irresistible, which inevitably offers new potential variations. In this section, we conclude the changes that digital curriculum resources bring to us and the changes of potential variation in the tasks that comes with it.

From the lens of the current digital curriculum resource of the textbook, it is mainly for resource online sharing. PEP platform offers teaching design, tasks, dynamic demonstration of classroom teaching resources, and research papers. Dynamic demonstration of classroom teaching resources makes Pudian and OPMC easier to implement in class. In fact, the situation is similar in Hyperbole textbook, which also accompanies with digital resources in a platform<sup>86</sup>. Different from PEP platform, Hyperbole platform doesn't present any teaching design of a whole lesson for teachers. In Hyperbole platform, besides the technical support for teachers, it also provides series of MCQs online for students, which are corrected by the computer. In general, at this stage of the DCR of textbooks, there is still a one-way sharing of resources, with the textbook designers providing further instructions and additional resources to teachers and students. The potential variation it brings is also limited.

From the lens more advanced situation, Sésamath is a production of teachers' collaboration work. In this platform, the e-textbooks in the Sésamath platform,

<sup>86</sup> Hyperbole textbook also has its own online platform, unlike PEP textbook, it is not open source and needs to be purchased. Website link for 10<sup>th</sup> grade of Hyperbole textbook (2019 edition): <https://enseignants.nathan.fr/catalogue/hyperbole-2de-manuel-numerique-enseignant-9782091193618.html>

Mathenpoche and LaboMEP are interrelated and complementary. Sésamath's e-textbook and workbook platform provides online viewing of textbooks and workbooks, and a projectable, downloadable version enriches the digital resources of the middle school classroom. Mathenpoche, as a platform to assist school education, is often used for extracurricular tutoring and exercise practice. The interactive exercises can effectively mobilize students' enthusiasm for learning. The dynamic analysis of the exercises enables students to carry out autonomous learning and self-help learning. LaboMEP integrates the resources in electronic textbooks and Mathenpoche to provide a platform for teacher-student communication and teacher-to-teacher communication, enabling personalized teaching. The e-textbooks and Mathenpoche also connect with a large number of external potential resources through the LaboMEP platform. The biggest difference this new intelligent platform brings to variation is to provide customized support for students

In this new type of digital curriculum resources, the sharing of resources gradually turns from one-way to two-way interactions. The biggest difference that this new intelligent platform brings to variation is the customized support for students. This customized support is from two aspects: one is provided by the platform itself, as described in Mathenpoche, where the computer responds to students' responses and further provides tasks appropriate to their levels. This kind of response mechanism requires a large number of tasks and a well-organized categorization of tasks according to mathematical contents and levels. The design and organization of these tasks involves variation with different teaching goals. For example, Pudian should be designed in a series of tasks that continuously go deeper to help students learn step by step. OPMC should be designed in a series of tasks with the same solution to help students master the strategy. When the system recognizes that students are having problems understanding concepts, it automatically mobilizes the explanations of concepts, which are designed with conceptual variations; the second is at the level of teacher-student interaction, as described in LaboMEP, where the teacher follows students' learning by checking their answers to exercises in real time, provides teachers with foundation to design customized curriculum resources.

The development of the digital curriculum resources presents both opportunities and challenges for variation.

#### **4.5 Reflections on the potential variation in curriculum resources**

In this section, we try to answer the first research question "What are the potential resources with variation in high school mathematics textbooks and other curriculum resources in China and France?" from three levels based on the current transition between classical textbook and other totally new teaching environment: The potential

variation in the two textbooks and the comparison between them (section 4.5.1); DCR and relation with traditional textbook (section 4.5.2); the changes of teachers' role (section 4.5.3).

#### **4.5.1 Comparison of PEP and Hyperbole textbook through the lens of potential variations**

The two textbooks are designed with different intentions. The PEP textbook is designed in a systematic way for teachers to follow in teaching while the Hyperbole textbook is meant to provide teachers with more options in teaching based on the curriculum standard. This difference in design intention leads directly to difference in the number of tasks in the two textbooks and the different roles of textbooks in teachers' documentation work. The Chinese PEP textbook has fewer tasks and the connections between the tasks can be clearly seen. The hyperbole textbook has a large number of tasks, which are categorized in detail so that teachers can select them according to their needs. For teachers who use PEP textbook, tasks are already selected for them and variation falls mainly on the tasks themselves. Teachers can design or redesign new tasks by manipulating variations based on the content in the PEP textbook. For teachers who use Hyperbole textbook, it is up to the teacher to select the tasks and reorganize them for different use. Variation falls mainly on the organization of tasks to make them logical and systematic. To show the comparison of potential variations in two textbooks more specific, we explain it from the four aspects (contextual, representational, conceptual, and procedural variations) in the following:

- Both textbooks introduce the concept of function from a realistic context, trying to get students to transform from the concrete real world to the abstract mathematical world and then to apply the knowledge of function to solve realistic problems. In addition, the French Hyperbole textbook also designs a lot of geometric context tasks, which are intended to use functions to solve geometric problems, especially the problem of moving points on geometry.
- The function itself comes with a variety of representations (formula, table of values and graph) that make representational variation very common in the function tasks. Both textbooks focus on conversions between representations. The PEP textbook pays particular attention to the conversions between graph and symbolic writing, trying to simplify the algebraic operations in symbolic writing with the help of graphs of functions from the intuitive point of view. In contrast, the French Hyperbole textbook pays particular attention to the use of TICE in the process of conversions. TICE is an indispensable part in the process of solving tasks in Hyperbole textbook, especially in terms of the conversions between symbolic writing and auxiliary transitional representation (table of values). While in the Chinese PEP textbook, although TICE is mentioned, it is only an auxiliary means

of solving tasks, not a necessary part. The conversions are mainly completed by students themselves through algebraic operations. The geometric context tasks in the Hyperbole textbook also brings new conversions between geometric figures and symbolic writings.

- In teaching the concept of function, rather than directly present the definition of the function (as French Hyperbole designed), Chinese PEP textbook guides students to generalize the concept through standard variations and non-standard variations. The comparison of standard and non-standard variations reveals essential and non-essential attributes, determine the connotation and extension of the concept of function (an example shown in section 4.2.1).
- In the Chinese PEP textbook, you can see the connection between the tasks clearly, especially with the help of the teachers' guidebook, while in the French Hyperbole textbook the tasks are in a very diverse form and we can find the connection in some certain parts. However, the idea of Pudian and OPMC are embedded in a series of tasks or between examples and exercises. OPMC and MPOS are often appear together, and students' mastery of mathematical methods is tested by OPMC. Both textbooks don't show OPMS explicitly, but leave room for the production of OPMS. In the PEP textbook, it mentioned the new method by using the computer to draw the graph of function. In the Hyperbole textbook, there are open problems and tasks for students to work in groups, which offers environment for students to explore and create OPMS by themselves. Moreover, using series of problems to explore and enhance the formation of concept is unique to Chinese PEP textbook (an example shown in section 4.2.1). It can be regarded as a possible approach by using variation to meet the requirement in the curriculum standard (section 1.2.1.2, the importance of the introduction and formation of the concept of function).

It is worth mentioning that sometimes the meaning of the task title is different even though the same words are used in these two textbooks. For example, the word "example" is essentially understood differently in the two textbooks. The task called example in the Chinese PEP textbook is closer to the task called "know-how" in the French textbook and includes the process of solution, while the example in the French textbook is often just an example to illustrate the concept and does not include the solution.

#### **4.5.2 DCR: the future of textbooks**

In addition to the classic textbooks, the advancements in technology promotes the development of DCR. New DCR is not only dedicated to the teacher or the students, but make it possible for one-way communication to become two-way, or even multi-way interaction, bridging the gap between teacher and teacher, teacher and software

engineer, teacher and researcher, and teacher and student. DCR provides new ideas for personalized teaching and learning. The technology breaks the limits of time and space, and the large number of tasks and intelligent choices help students to learn in a more focused way. Variations are implemented by the computer in the process of selecting and presenting the tasks, the correcting and reflecting on the students' answers. We hope that this customized service accumulates data as students and teachers continue to use it, and the more data there is, the more the computer's learning function will allow the computer to better understand the needs of this student and provide a more personalized, targeted and comprehensive service in the long run.

When we compare the resources offered by PEP, Hyperbole, or Sésamath, there are some differences in the type of resources between China and France. In the French resources, whether in the teachers' guidebook or the DCR, we do not see any complete teaching design for one lesson, which is very common in China, or even as part of the teachers' preparation requirements, especially for new teachers. In French resources, most of the design presented is for one specific task. This phenomenon is to some extent consistent with the difference in the design intentions of the textbooks in France and China. As we mentioned above, the PEP textbook is concerned with a holistic, macro design, so it provides teaching design of a certain lesson both in teachers' guidebook and in DCR. While Hyperbole textbook and Sésamath pay attention to the specific tasks, in a more micro level.

#### **4.5.3 Changes of teacher's role**

The traditional role of the teacher is to shape the textbook for use with particular groups and students in particular places on the basis of teacher's own professional judgment (Chazan & Yerushalmy, 2014). However, technological developments change what it means to supplement a textbook. In the current situation, teachers are exposed in an explosion in the accumulation of digital data, traditional teacher roles no longer seem to meet current needs. Then, what should the teacher do?

The evolution of DCR seems to provide a platform for teachers' documentation work in a new way. With new kinds of technologies, supplementation of textbook begins closer to an "open culture" concept of co-authoring a new version, rather than supplementing an existing version that remains unchanged (Chazan & Yerushalmy, 2014). As we mentioned the Sésamath e-textbooks in section 4.4.3.2, particular knowledge products or tasks are not fixed, but developing, altering, or enriching under teachers' collaborative work. The resources are created by teachers' iteratively working and refining. Teachers gain experiences and build their own ways of designing and refining with such kind of platforms. Teachers become significantly more involved in the designing process. Perhaps, in the future, textbook design is no longer the "privilege" of experts (supervisors, teacher educators, etc.), but each teacher will have their own



personalizing digital textbooks based on the common curriculum standard.

To get a fuller picture of the resources currently used by teachers and specify teacher's role in this transitional environment, we conduct case studies of a Chinese teacher and a French teacher in chapter 5 and chapter 6.

## Chapter 5

### Studying teaching through variation in a Chinese case

With an aim of exploring variations in teachers' documentation work and classroom teaching, we have proposed a preliminary model (Section 2.4.3) with its analytical framework (Section 2.4.4) based on the literature. To refine this framework, case studies are presented in the following two chapters. This chapter presents the case study in China in four sections: Section 5.1 focuses on the variation in Lin's resources system through his documentation work, which is the foundation of Lin's classroom teaching through variation; Variation in Lin's classroom teaching with respect to the tasks is presented from two perspectives: analyzing Lin through his own work, directly his documentation work and classroom teaching (Section 5.2), and through the lens of a preservice teacher at the occasion of series of guiding activities for a specific classroom teaching (Section 5.3). Section 5.4 concludes the characteristics of case teacher Lin in design and use variation in teaching to answer our research questions, and balances the specific and generic aspects of this case study for improving our current model.

#### 5.1 Variation in Lin's resources system through his documentation work

In this section, we try to draw a picture of Lin's resource system to understand the factors behind Lin's teaching through variation. We described the variation in Lin's resources system from three perspectives: Lin's resource system (5.1.1); Lin's documentation work (5.1.2); variants and invariants in Lin's resource system and documentation work (5.1.3).

##### 5.1.1 Lin's resource system: Fertile soil for the outcome of variation

Teachers' resources are powerful support for teachers to carry out teaching through variation. In this section, we first present the resources that Lin use (5.1.1.1); we then describe Lin's understanding of 'good resources'<sup>87</sup> and choice of resources (5.1.1.2).

##### 5.1.1.1 Lin's resources

We follow the methodology described in section 3.3.4.1 and present Lin's resources from four types according to different purposes: resources related to tasks; resources related to mathematics knowledge; resources related to didactics, and resources related

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<sup>87</sup> 'Good resources' here are defined by Lin himself, not from the general point of view.

to software tools to assist classroom teaching. The resources presented here are based on the interview with Lin and evidenced by us during the classroom observation.

● **Resources related to tasks:**

Resources for tasks are the main content of our research. There are mainly five resources used in Lin's case (see table 5.1.1): 1) Digital task repository; 2) School-based teaching book; 3) Handbook of Key and Difficulties Points; 4) Classroom test paper; 5) Discipline Website. These resources are used by Lin as resources for lesson preparation, for students' homework and for students' tests or examinations.

Table 5.1.1 Resources for tasks

<b>Naming in Chinese</b>	<b>Translated name in English</b>	<b>Purpose</b>	<b>Designer/Organizer</b>
数字题库	Digital task repository	Supplementary tasks make the classroom teaching more enriched; Provide tasks for test	Lin
校本教材：《重点高中同步精讲精练》	School-based teaching book: "Synchronous Elaborate Lessons and Exercises in Key High School"	Providing learning goal, key points and tasks for the lesson about new knowledge;  Daily Homework for students	Experienced teacher(s) in school C
《重难点手册》	"Handbook of Key and Difficulties Points"	Homework for students	Commercial Publishing
课堂测试卷	classroom test paper	Test in class (every two weeks)	Teachers of grade 10 in turn
学科网	Discipline Website	Search information  Design and organize test paper (rarely)	Commercial Website

Digital task repository is dynamic and self-developed by the teacher. Lin is proud of this resource accumulated by himself for over ten years. This is mainly about the storage of tasks and used to provide supplementary tasks to enrich his classroom teaching. According to Lin, this resource can be characterized as:

- Tasks classified by theme: Lin classified different tasks according to different themes. For example, there is a specific folder for functions.
- Wide variety of sources: The sources of tasks here are very rich and we mainly

divide the sources into the following categories: 1) Tasks in the formal large-scale examinations: tasks of college entrance examination, of independent enrollment of college, of mathematical competition in various provinces, of Chinese High School Mathematics League, etc.; 2) Tasks encountered in daily teaching: Lin gave an example here. The teacher would take turns to design the test or exercising papers for students. If there is a task that he thinks good, then he would record it. 3) Electronic version of books: If Lin found the tasks in a certain book are quite good, he would ask a college student to help him type the whole book. Lin pointed out the importance of digitizing tasks in our interview: *“It’s useless for me to memorize the tasks. I have this task, because I have done the task, and I will probably have an impression. I know when and where it is. Only handwritten (of the tasks) is inconvenient for me when I want to use it. For example, if I want to design a test paper or something else, I need to find some tasks to print for the students. It’s useless for you to write by hand. If I have an electronic manuscript, I just copy it. (That’s why) I generally let it be in the form of an electronic manuscript.”*

- Dynamic and develop over time: This digital task repository is continuously enriched with the teaching of Lin. The diverse sources of this repository form a virtuous circle in the long-term teaching process, providing it with rich nutrients.

According to Lin, the main resources for classroom teaching is school-based teaching book “Synchronous Elaborate Lessons and Exercises in Key High School” (see figure 5.1.1). According to the requirements of the Hangzhou Education Bureau<sup>88</sup>, the latest PEP textbooks A version (2019) are used as textbooks. But in actual teaching, teachers often choose school-based teaching books because it tends to be more in line with the level of students in each school after years of practice. The chief editor of this textbook is an experienced teacher of school C who has been teaching for more than 30 years. He explained the source of this book in our conversation<sup>89</sup>: *“From the beginning of the curriculum reform in 2004, I began to try to design teaching plans and learning plans, which is the prototype of school-based teaching books, with other teachers in school C. The intention of designing is to face the contradiction of the new curriculum with more learning content and less teaching time. Through effective teaching design, certain additions, reductions, selections, and reorganization of textbooks are carried out, the content of textbooks is transformed into teaching content, and the academic form of textbooks is transformed into teaching form, so that students’ mathematical performance can be greatly improved. In the fall of 2020, a new round of core literacy-oriented education*

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<sup>88</sup> Hangzhou Education Bureau is the municipal government work department in charge of education.

<sup>89</sup> We introduced this conversation in section 3.3.2 tools of data collection.

*reform was fully launched. We have carefully studied the latest high school mathematics textbook published by People's Education Press A version, comprehensively revised the school-based teaching books, and carefully designed the book called "Synchronous Elaborate Lessons and Exercises in Key High School". This book adopts the mode of combining teaching plan and learning plan, highlighting the three characteristics of Synchronization, Elaborate lesson and Elaborate exercises."* He explained the connotation of these three characteristics as follows:

- Synchronization: On the one hand, it refers to the synchronization with the idea of the latest curriculum standards and PEP textbooks, which can effectively improve the quality of mathematics teaching and develop the core literacy of mathematics of students; on the other hand, it refers to synchronization with teaching and learning, making this book become a high-quality lesson plan and learning plan that combines classroom teaching and after-school training.
- Elaborate lesson: In order to facilitate teachers to use the new textbooks and implement classroom teaching, the following 4 elaborate lesson columns are set up for each lesson in this book: 1) Learning objectives, combined with curriculum standards and textbooks, use "knowing", "understanding" and "mastering" and the corresponding behavioral verbs "experience", "inquiry", etc. to explain the learning objectives of the content of this lesson; 2) Elaborated explanation for the key knowledge. The main concepts, formulas, methods and other key points that should be explained in this lesson are effectively integrated to facilitate students' self-study, and to reduce the teachers' blackboard writings; 3) Example analysis, select 4 classic examples corresponding to the key and difficult content of this lesson. Example 1 and example 2 focus on consolidating new knowledge, example 3 has the requirements of improved ability, and example 4 has a certain degree of difficulty and the book gives the answering process. This is set in consideration of the different level of students. It is also more convenient for teachers to control the teaching time. It can not only allow teachers to explain the example, but also allow students to learn by themselves. Immediately after Example 4, set up a "knowledge and ability transfer" task, which is equivalent to a spare task. The teacher can decide whether to explain or not according to the actual situation and the teaching time. It can also be used as a students' after-school thinking task. By setting various and flexible examples, teachers have more choices and makes this book suitable for students of different levels. For the setting of examples, designers have carried out in-depth and meticulous considerations. The book also provides series of tasks and tasks with variations. The designer also mentioned that if teachers have better examples, they can also make appropriate supplements and adjustments; 4)

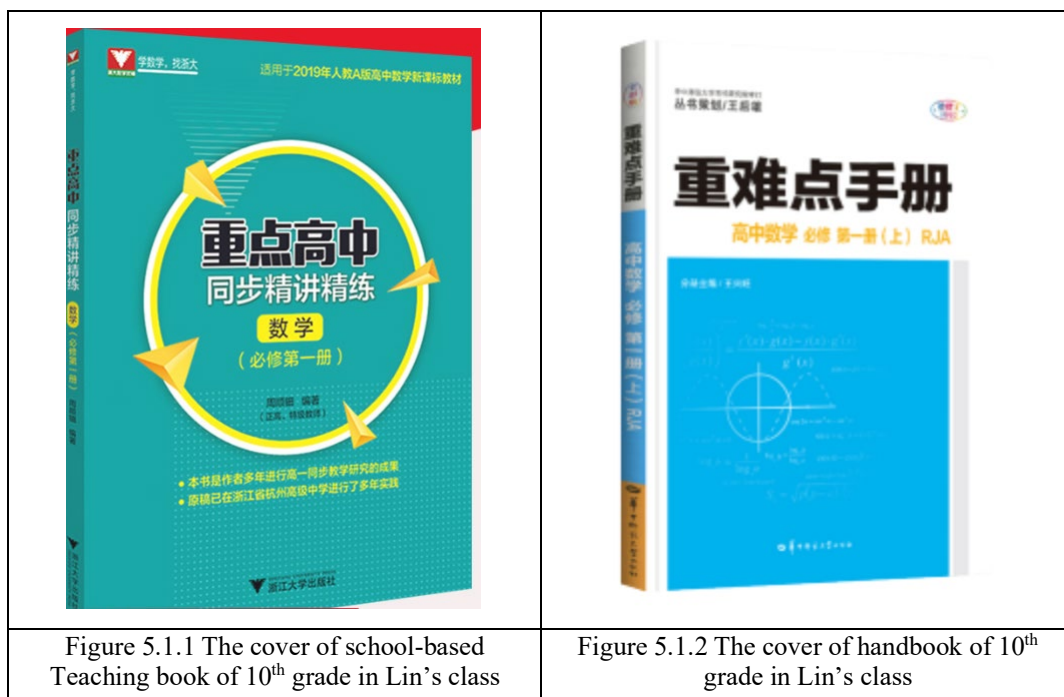
Summary of mathematical laws. Summarizing conclusions, methods, and thinking strategies that have universal guiding significance so as to facilitate students' review after class, and strive to enable students to have a firm grasp of what they have learned and apply them flexibly. In addition, at the end of each chapter, there is a summary and review lesson, which is divided into three parts: knowledge structure, key review, and review and improvement. The examples are set around the mathematical thinking method, which are roughly divided into function and equation thought, symbolic-graphic combination thought, classification discussion thought, reduction and transformation thought, special and general thought, etc., equipped with 5 examples.

- Elaborate exercises: The tasks are targeted and hierarchical, consisting of two parts: "basic training" and "ability improvement". The selection of exercises follows the principles of basicity, applicability and gradual progress, taking into account classics and novelties. Considering the learning burden of students, the number and difficulty of the exercises are controlled. "Basic training" consists of 5 multiple-choice exercises, 3 filling-in-the-blank exercises, and 2 answering questions. It is a basic requirement that all students must complete and it pays attention to the discrimination of concepts, the deformation of formulas and their flexible application. "Ability improvement" consists of 1 multiple-choice or fill-in-the-blank exercise, and 1 answering question. The designers choose the exercises with the form of inquiry as much as possible, and provide more detailed reference answers.

The chief editor further emphasized that the design of this book is based on a long time of exploration in the district of Hangzhou: *"It has been 17 years since the curriculum reform in 2004. This book condensed the teaching experience and teaching attempts of these 17 years. The selection of typical examples and the setting of after-class exercises in this book are determined after a series of topic research and teaching practice, so that the maximum benefit of classroom teaching can be brought into play, and the topics are in line with students' actual level and also comprehensively cover the learning content."*

"Handbook of Key and Difficulties Points" (see Figure 5.1.2) is a supplement to the homework exercises in school-based teaching books, because Lin thinks that the number of exercises in one book (12 exercises per day for homework) is not enough to guarantee students' learning, and two books are appropriate. Different from "Synchronous Elaborate Lessons and Exercises in Key High School", which is used for the everyday homework and corrected by the teacher, the handbook is used by students to learn independently. Lin mentioned in the interview that *"Handbook of Key and Difficulties Points' is for students to do (tasks) by themselves. I am not in charge of it."*

students control their own time. It is not said that one section should be done every day. It is stipulated that it should be completed within a certain period of time. Generally, the inspection would be carried out after the end of a chapter. I only check them the revision of tasks. My requirement (for students) is to do tasks, correct by yourself, and ask questions if you don't understand.”



School-based teaching book and “Handbook of Key and Difficulties Points” above are unified for the entire grade, determined by the LPG<sup>90</sup>, and generally by the leader of LPG<sup>91</sup>. Because the leader of LPG is responsible for the arrangement of the teaching progress, the use of classroom teaching resources, and the design of test, etc. It should be noticed that during the actual operation, we found that although all the teachers in grade 10 used the same classroom teaching resources at the beginning, as the interaction between teachers and students gradually deepened, especially after the mid-term examination, most teachers would vary their classroom teaching resources to varying degrees in order to better adapt to the level of students. Teachers in 8 parallel classes generally think that some tasks in the school-based teaching book are too difficult for students. Most teachers chose part of the tasks for students to do instead of doing all the tasks. Teachers added tasks from their own library of tasks or other places for classroom teaching. One of the teachers directly chose a new book to replace it.

As for the “Classroom test paper”, it is a test conducted in class every two weeks and

<sup>90</sup> The notion of LPG (Lesson Preparation Group) in section 3.1.3.

<sup>91</sup> Since Lin was the leader of LPG, he made the decision on the classroom teaching resources in this case.

lasts around one hour every time. It is not a formal test and only used to detect students' learning in the past two weeks so that teachers can adjust their teaching strategies in time according to students' conditions and students themselves can also know about their own learning situation. The test paper is designed by the school's 10th grade teachers in turn. This test paper usually contains 6-8 multiple choice questions, 4 fill-in-the-blank questions, and 3-4 answer questions. Sometimes teachers will set additional questions for higher-level students.

Lin used Discipline Website (in Chinese: 学科网, <https://www.zxxk.com/>) for two reasons: 1) to search some information from time to time; 2) to help him organize a test paper (rarely). Discipline Website is relatively large and comprehensive. It has various resources of different provinces, different levels of school, including elementary, junior high, and high schools. According to Lin, the school bought an account for the teacher to use. He described that the Discipline Website was an online sharing platform: *"Some teachers uploaded resources, and they would be rewarded for uploading it. Otherwise, how can it have so many resources, and its resources depended on the sharing by others. It is like a platform. If you store all the information in this way, then others can take out the information. It serves as a platform, a platform with maintenance, and it makes a profit from it, which is convenient for teachers."* He also mentioned that there is a test paper composition system in the subject network, which can help teachers compose test papers. For example, the teacher entered the mathematical contents and the difficulty coefficient of the tasks, it will automatically compose the test paper. But Lin didn't use it very much.

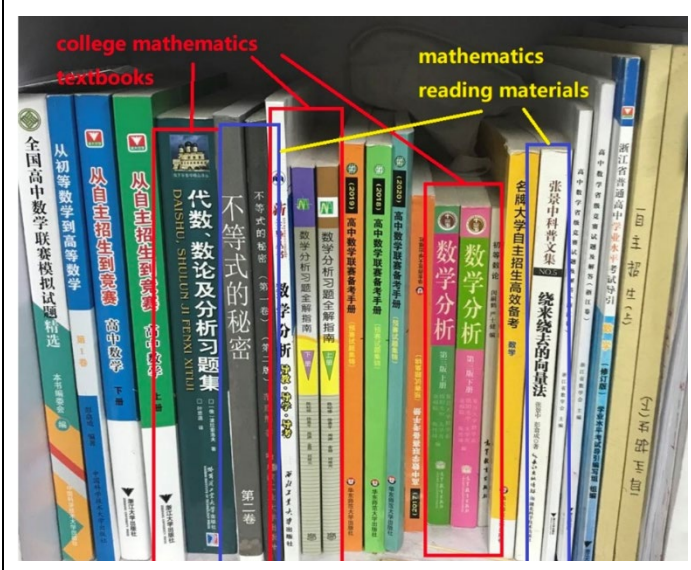
- **Resources related to mathematics knowledge:**

As a teacher, Lin attaches great importance to improving his own understanding of mathematics and his ability to solve problems. In his spare time, he bought many books by himself to study the tasks.

For improving his understanding of mathematics, he mainly used the college mathematics textbooks and mathematics reading materials, see table 5.1.2. According to Lin, he often does this during his summer and winter vacations.



Table 5.1.2 Books for improving his understanding of mathematics

	<p>[Translation]</p> <p>College mathematics textbooks:</p> <ol style="list-style-type: none"> <li>1. Algebra, Number Theory and Analysis Problem Set;</li> <li>2. Mathematical analysis;</li> <li>3. Mathematical analysis exercises;</li> </ol> <p>Mathematics reading materials:</p> <ol style="list-style-type: none"> <li>1. The secret of inequality;</li> <li>2. The method of Vector</li> </ol>
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As for maintaining his ability to solve problems, he used two kinds of resources: books for college entrance examination and books for mathematical competitions. He mentioned that he would study the two types of books at different times because the task types are different. During the interview, he said that he could not remember the specific books he had studied before, but he was currently studying three books (see figure 5.1.3): “Mathematics in the heart” (《心中有数》), “Key high school second-round review books” (《重点高中二轮复习用书》), and “the most concise mathematical thinking method” (《至精至简的数学思想方法》).

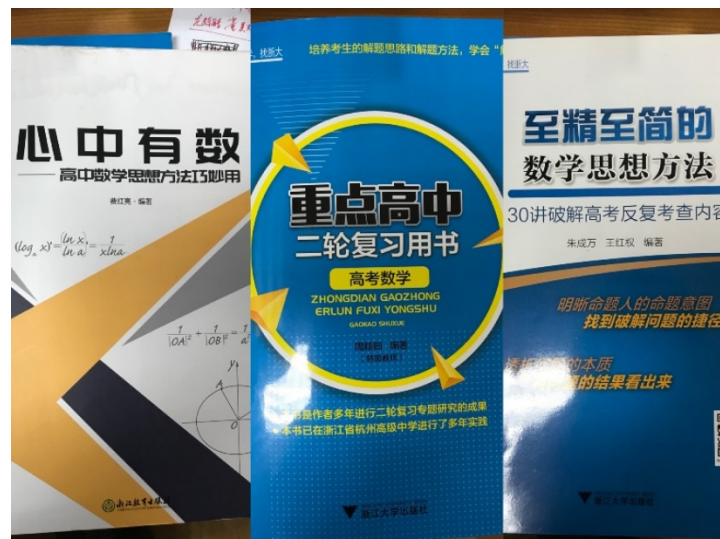


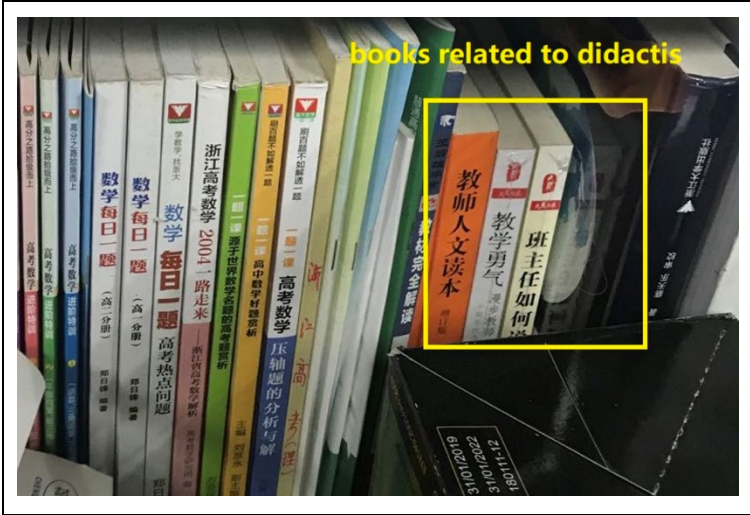
Figure 5.1.3 Current books used by Lin for problem-solving

- **Resources related to didactics:**

Besides the resources related to mathematics and mathematical tasks, Lin also had a

collection of books about didactics, especially about organizing a class as a homeroom teacher<sup>92</sup>. A picture of books taken in Lin’s office seat is presented here, see table 5.1.3. Such books are not his current main resources, because he is already an experienced teacher. As a mathematics teacher, he has his own set of classroom teaching methods; as a class teacher, he has a wealth of practical experience to help him manage the class and deal with the emergencies in the class.

Table 5.1.3 Books related to didactics

	<p>[Translation]</p> <p>Books related to didactics:</p> <ol style="list-style-type: none"> <li>1. Teacher reading materials;</li> <li>2. The courage of teaching;</li> <li>3. How does the homeroom teacher speak</li> </ol>
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• **Resources related to software tools to assist classroom teaching:**

Lin rarely used software tools to assist his classroom teaching. He mentioned two main resources: Sketchpad and the slides. He pointed out the cumbersome use of these technologies and questioned their effectiveness in teaching: “*I don’t know how to do these things. If I really need them, I can ask someone to help me. And I don’t like to use slides, it is very troublesome to do, and the effect of teaching is definitely not as good as mine. Unless there are some dynamic problems, and I can’t draw it, (I will use the Sketchpad), just let it go and let students understand. Apart from this, I will not use anything else.*” “*If you really use slides in class, students will learn worse, showing slides is just like playing a movie, students can feel nothing.*”

**5.1.1.2 Lin’s understanding of good resources and choice of resources**

Lin believed that good resources are the result of long-term accumulation and time-tested. He pointed out “*The best resources for teaching are continuously accumulated in teaching. There are some frequently used tasks, we will have an accumulation. In the process of accumulating, we would consider which tasks are the best for this lesson.*” He also mentioned the importance of teachers accumulating classic tasks and refining

<sup>92</sup> Homeroom teacher (班主任): refer to section 3.3.2.1.

them in the process of teaching over years: *“We must accumulate these tasks in the process of teaching. When we accumulate lots of tasks, we can design tasks by ourselves. If you accumulate more, you can make up your own tasks temporarily in class. There are some classic tasks. We are like this: In the process of teaching, we have to accumulate some particularly classic tasks. Then we are using these tasks in the process for a long time. We think which tasks are more likely to be more refined, better to be able to explain the content of this lesson, better to realize the conceptual method. These tasks have been accumulated over a long period of time. There are many tasks. We have to choose carefully. How to choose carefully? Actually, mainly based on the experiences of our own use of tasks.”* Obviously, Lin’s opinion on the quality of resources and the resources he used (described in section 5.1.1.1) are mutually corroborating. Here, the continuous accumulation in teaching emphasized by Lin happens to be fully reflected in his digital task repository resources.

Lin believes that the most important thing for the teacher is not to accurately memorize good tasks, but to have a preliminary impression and judgment on each task, and to be clear about which level and teaching situation the task is suitable for. As for the accurate information of task, the teacher can check it by looking up his/her own digital task repository: *“In fact, during the teaching process, our teacher must have tasks in his/her mind, so some tasks can be designed temporarily. Some tasks may not always be remembered clearly for me, but I have a concept in my mind. For this kind of understanding and mastering, I will form a network for these tasks. What level can we achieve with these tasks? What tasks can be used to help students reach what level. We teach according to students’ situations, and we teach according to students’ aptitude. (If) I teach parallel classes, what tasks I probably use, what tasks I use in experimental classes. The choice of tasks is very positively related to the students you teach.”*

Moreover, he emphasized that teachers should adapt the tasks flexibly according to their own teaching needs, but the adaptation must be moderate: *“There are some tasks that are already very good, so there is no need to broaden(vary) it. According to your actual teaching needs, you can flexibly vary it yourself. For some topics, I will directly use the original tasks, and for some topics, I will vary the original tasks. I transform the representation of tasks appropriately, but as a new lesson of high school, excessive variation is unnecessary. Appropriate variations can be made, because our main aim is still lesson with new knowledge, what is the main purpose of passing the examples? To master and consolidate some concepts and methods, right?”* We will further introduce Lin’s design and redesign of resources in his documentation work in the following section 5.1.2.

### **5.1.2 Lin’s documentation work: Consciously leaving room for variation**

In the previous section, we introduced four kinds of Lin’s resources from different

purposes of resources. In this section, we focus on Lin's documentation work from three kinds of interactions: his interaction with resources, with students, and with other teachers. At last, we present Lin's own interpretation of variation.

#### **5.1.2.1 Lin's interaction with resources: instrumentalization and instrumentation**

In this section, we describe Lin's interaction with resources from two interrelated processes: instrumentalization and instrumentation (section 3.3.4.1).

From the perspective of instrumentalization, we present from three aspects: Lin's selection, (re)design, and the track and management of tasks during his documentation work.

Regarding the selection of tasks during documentation work, Lin didn't go through the resources directly and select tasks from them. Instead, he used the school-based teaching book "Synchronous Elaborate Lessons and Exercises in Key High School" to prepare lessons first, and then went to find the tasks, usually in his digital task repository, which he came up with during the lesson preparation process. These tasks were used as a supplement. Lin mentioned: *"During my teaching process, I focus on 'Synchronous Elaborate Lessons and Exercises in Key High School', but I will not say that I only use it. I will add something according to the situation of this book and according to what need to master in my own lessons. In this way, the lesson is fuller. So, the content of each class I teach is very rich, not just this book."* According to Lin, the added content is based on his accumulated experience. As shown in Lin's resources, sometimes Lin took a look at some extracurricular tasks in the books purchased by himself. He mentioned: *"In general, I try to squeeze time as much as possible. In addition to the normal preparation of lessons, I will also do tasks in some extracurricular books. The teacher always has to do some tasks frequently. If I have time during the winter and summer vacations, I will go and read the books of advanced mathematics. In the process of this, I have a whole consideration in my heart. As I read a lot of tasks, there are lots of tasks in my mind."* Lin pointed out that the significance of teachers' frequently doing tasks is to broaden their horizons and improve their own level. It does not mean that all the tasks will be used in teaching.

Regarding the (re)design of tasks during documentation work, Lin mentioned that he would design or redesign some tasks in the lesson preparation. If the original task does not satisfy his teaching well, he will mainly make the following adjustments:

- Set "Pudian" tasks: Reduce the tasks or design new tasks to pave the way for students' learning. In this way, tasks are shown from easy to difficult, which is accordance with the students' cognitive rules and easier for them to understand. According to Lin, he will reduce the solution process of the task: *"Decompose the*

*task. How to decompose it? I can make a “Pudian” for the knowledge to be used in this task, to prepare the knowledge in advance, which is equivalent to giving students a step. In the process of teaching, we said from easy to difficult, from shallow to in-depth, how does this process reflect? It can be reflected in the process of explaining tasks, or you can design tasks beforehand so that students can easily get into it. Then when the teacher comes to the task, it will come naturally. In my class, it is possible to design series of tasks to make a “Pudian”, paving the way for this lesson, or even for the next few lessons. In my class, I will construct a system. I’m not blindly talking about this thing now. Some tasks may be purely about this task, but there are also some tasks that may be used to prepare for the later tasks, and it may even be possible to prepare for the next few lessons.”;*

- Emphasize the connections between knowledge and symbolic-graphic combination based on the original tasks: Lin will adapt the original tasks based on his own teaching experience to connect the old knowledge with the new knowledge. In addition, he pays special attention to the methods of symbolic-graphic combination and algebraic transformations, which he believes are the focus of high school mathematical learning;
- Set challenge tasks: During the teaching process, if the situation is good, Lin will then design some more challenging tasks for the students. Lin explained: *“Setting some challenging tasks can strengthen students’ enthusiasm for learning, especially some good students. Because basic task is difficult to inspire the students with better mathematical thinking. So how to stimulate it? We use some challenging tasks as the carrier. For this kind of tasks, students need to think in order to answer, they may not be able to solve it, the solution may not be completely accurate, so that the students will find it very challenging, and they will feel interested in the process of listening to the class. The process of thinking this kind of tasks can also improve students’ ability to do research.”*

Lin also mentioned that the learning of students should not only include passive acceptance of knowledge, but also know how to research problems. He emphasized: *“Research is how to learn independently. This research ability can also be reflected in the tasks. It can be appropriate to put one or two challenge tasks. The level of challenge should be appropriate. For example, 1/3 of the students in class still have the opportunity to solve it, and 1/3 of the students will have some ideas, maybe 1/3 of students may not have any ideas at all, and they really can’t figure it out at all, then I think this topic is a good one. In this case, at least the enthusiasm of the students at first stage can be stimulated. It’s equivalent to giving people a meal. If you always eat what you usually eat, it is tasteless. You need a stimulation. That’s what it meant to be something that also had a stimulating taste. If the task is too hard and nobody has an*

*idea, then the task also loses its value.”*

Lin thinks that it is relatively simple to redesign but it is difficult to design a task. In actual teaching, Lin rarely designs tasks totally by himself for three reasons: 1) There are many good tasks for him to choose; 2) There is not enough time and much energy for him to design tasks in daily teaching; 3) He is more willing to use the original task, because he may make some mistakes when adapting the task.

As for tracking and managing his documentation work, Lin has his own set of concepts: Consciously leaving room for varying his own teaching. Lin prepares a sheet for each lesson (see an example sheet of monotonicity in Figure 5.1.4) during his documentation work. While preparing the lesson, he records the main points of the lesson on the sheet. For example, as shown in Figure 5.1.4, the right side of this sheet is the mathematics knowledge related to this lesson, the concepts and properties of mathematics, and on the left are the three examples with variation tasks and generalization. The complete problem-solving process was not written, but the key points of the strategies was written. Lin brought this sheet to class. He thinks that when he finishes writing this sheet, he has a complete understanding of the content of this lesson. It's worth mentioning that Lin doesn't keep these lesson preparation sheets or record the tasks on it, he throws them away after class. According to him, there is no need to save these sheets, because everything is changing. He already has the most basic and important tasks in his digital task repository. He does not want to unnecessarily restrict his teaching, as he should continue to change according to the teaching situation. He said: *“The students are changing, everything is changing, and I am also changing. I also already kept a record of the tasks that I added. It's not that I have to follow this pattern next time, I am also changing, why should I restrain myself? That's not necessary. Because you can't guarantee that every class will be the same as the previous one. Maybe the pace is slower, and I may adjust the rhythm. Unrestricted, they are all temporarily prepared, but I have the basic resources in my hand. As soon as I take them, I know how to teach.”* “No matter what you teach, the task is the carrier. If there is no carrier, then all will be empty. Improve through the task, explain it thoroughly to understand, and understand the process better.”

According to Lin, his process of instrumentalization is influenced by class time constraints and the pressure of the college entrance examination. In his daily teaching, he would not consider interdisciplinary and scientific contexts when designing tasks. He only uses it in demonstration classes. He emphasized *“The time of class is so tight, it's impossible. No way, the college entrance examination is right in front of you”*.



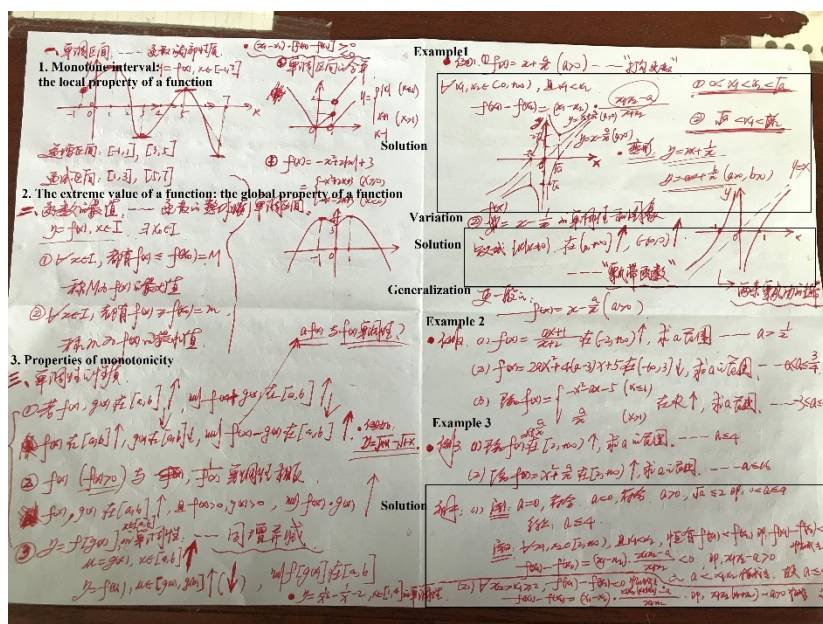


Figure 5.1.4 Lin's lesson preparation sheet about monotonicity of a function

From the perspective of instrumentation, school-based teaching book offers Lin a great deal of support as a guidance of his lesson preparation while his digital task repository works as a supplementary resource for teaching.

### 5.1.2.2 Lin's interaction with students in two classes

The interaction between Lin and students has two major characteristics: one is the sharing of resources between the students in the two classes, Lin acts as a bridge for the transmission of new ideas between the students in the two classes; and the other is Lin's practice of what he believes in teaching, the philosophy of "Teaching others is teaching yourself"<sup>93</sup>.

As Lin teaches two 10th grade classes, Lin pays attention to the sharing of teaching content, information and timely feedback between the two classes. For example, sometimes students would put forward some new strategies to solve the problem or propose some meaningful questions in class, and these contents would be supplemented in another class in time.

Lin emphasized that teaching cannot be separated from students. Teachers and students grow together: "Teachers must teach in the front line so that the teacher's peak state will not change. Once you leave front-line teaching, it may be okay for one or two years, but it may not work after three years. You have students from this school who

<sup>93</sup> "Teaching others is teaching yourself" (in Chinese 教学相长): Both teaching and learning influence and promote each other. It is originally from "Book of Rites" (《礼记》).

accompany you to help you grow up. If you leave this platform, your ability may be greatly reduced. My understanding is that ‘Teaching others is teaching yourself’. I let the students improve, and the students will also improve me. What have the students improved me? First, they can provide some new strategies through daily teaching; second, they also asked questions. The process of helping students answer various questions is also an improvement for the teacher.” Lin also took the extracurricular tutoring organization as an example. He mentioned that teaching that is out of the front line will make teachers only pursue new and difficult tasks in designing tasks, and cannot understand the needs of students well.

In Lin’s case, there are two ways to evaluate students: homework and exams. There is a regular examination every two weeks, and the more formal one, such as mid-term or final examination. Lin thinks that one examination every two weeks is appropriate for three reasons: First, once a week is a bit too often, and students will be bored; Second, there is not enough time to arrange classroom examinations for students; and the third is that there is not enough classroom time to explain the tasks in the examination for students. Lin believes that regular examination is very necessary in teaching mathematics to help teachers keep track of students’ learning and their ability to solve problems independently. Because students may communicate with each other and search the Internet for their homework, but the examination is done independently.

### **5.1.2.3 Lin’s interaction with other teachers**

Lin participated several formal and informal collaborative activities and trainings for teachers’ development inside or outside the school.

Inside school C, Lin mainly participated in two formal activities: the Teaching Research Group (TRG) and the Lesson Preparation Group (LPG). The TRG activities are the most formal. The mathematics teachers of the entire school will participate, usually once a month, 2 to 3 hours every time, and there will be some topic discussions. The LPG is composed of all mathematics teachers of the same grade, mainly discussing the teaching progress and teaching content, usually once every two weeks, 1 hour for each time. Sometimes the teaching methods of certain content will be discussed. But it will not be very detailed, it will be discussed from a macro perspective. For specific issues, teachers will discuss and communicate in the office. As all the math teachers in the school are arranged to sit in the same large office, it is convenient for them to communicate or share ideas in time. The LPG has a WeChat group, but the group is mainly for notifications, not for discussing issues. According to Lin, there is no need to discuss via Wechat as it can be discussed face-to-face in their shared office.

According to Lin, the most direct collaborative activity that has the greatest impact on his teaching is LPG. As Lin is the leader of LPG of grade 10, he is mainly engaged in



the following tasks: (1) At the beginning of each semester, choose supporting teaching resources besides textbooks, develop a detailed, specific and feasible subject teaching plan, and arrange the teaching schedule; (2) Carry out collective lesson preparation (every two weeks in this case) in LPG, modify the teaching plan according to the actual situation and ensure that the progress of all classes is roughly the same, jointly determine the key knowledge and difficulties in teaching, and be responsible for implementation; (3) After the examination/test, conduct teaching quality analysis, regularly summarize teaching experience, analyze the problems in teaching, jointly study methods and measures to improve teaching, and implement relevant measures to improve teaching quality.

Moreover, as Lin is an experienced teacher, he is also responsible for giving guidance to novice teachers and preservice teachers. During the period of our classroom observation, Lin was instructing one novice teacher and one preservice teacher. He believes that the most important thing for the instruction is to communicate with each other. The way of instructing novice teacher and preservice teacher is quite different. For novice teachers, case teachers mainly adopt the mode of listening to her lessons, instructing, and answering questions. Novice teacher would go to Lin's class when she was free, and asked the case teacher questions about the teaching. Lin will occasionally listen to the lesson of the novice teacher and give her some guidance. For preservice teacher, she would go to Lin's class to listen to the lesson every day, and work with him to correct the students' homework. Lin also asked her to complete the tasks in the school-based teaching book before each lesson, and also to prepare each lesson by herself according to her own thinking. He believes that it would be rewarding for the preservice teacher by comparing the lesson prepared by herself with the actual lesson that he given in class. Lin pointed out that *"If the preservice teacher is just listening to the lesson in the classroom, it is the same as the students, but he/she had listened to it for three years in high school. Listen again will not let her gain much. Studying how to teach is the most important thing."* We introduced a collaboration work between Lin and the preservice teacher Shi for a certain lesson in section 5.3.

Outside the school, Lin participated the teacher training mainly in Hangzhou. The Hangzhou Bureau of Education Teaching and Research organized some training and activities for teachers. These training and activities have various forms of organization, including the invitation of expert teachers or researchers to give lectures, or the editors of textbook to illustrate the intention in designing the textbook, or the experts to explain the ideas of examination design, etc. It was generally once or twice a semester. For the teacher who taught the third year of high school, it would be relatively more. This type of seminar focused on the review for students' college entrance examination. Some expert teachers were invited to introduce the review methods for specific mathematics

topics.

Regarding the current tendency to encourage teachers to combine teaching and scientific research (MOE, 2019), Lin insists that his main interest lies in mathematics teaching, including daily teaching and mathematics Olympiad teaching. In his opinion, classroom teaching and the work of the homeroom teacher have taken up most of his time, and he does not have more energy for scientific research.

#### **5.1.2.4 Lin's interpretation of variation in the tasks**

According to Lin, he adheres to the principle of “centering on knowledge, serving the teaching goal” when he varies tasks. In this section, we present Lin's understanding of variations in the tasks based on the interview from several perspectives: the meaning of varying the tasks; the relation between the variation task and the original task; the relationship between the difficulty of the task and the variation.

For Lin, varying tasks is actually centering on knowledge to enable students to master mathematical thinking and methods. He used an example to illustrate: *“For example, in constant establishment, when  $x \in [1,2]$ , we always have  $x^2 - x + a > 0$ . I can vary it through putting the parameter  $a$  as the constant term, as the monomial coefficient, or as quadratic coefficient. My main idea here is to talk about the method of separation of parameters and variables. If the parameter  $a$  falls into a constant term, then its axis of symmetry is given, then the maximum value will come out; If you put parameter  $a$  as the monomial coefficient, then you need to classify the discussion, you should consider the axis of symmetry when you classify the discussion; if you put parameter  $a$  in the quadratic term, you have to consider the opening and also the axis of symmetry. From the process of transforming this task, I told the students to compare two methods, one is called classification discussion, and the other is parameter-variable separation. Through the comparison, students can feel the convenience of using the method of parameter-variable separation.”* In general, the main purpose of the variation is to explain a certain task, a certain method, concept, and serve the purpose of teaching. Variations are not random changes. Random change is boring and leads the students into the trench. We need to pursue the effectiveness of classroom teaching. The most important thing is that the selection of the tasks of effectiveness should serve the teaching goal and teacher's central idea.

When it comes to the relationship between the variation task and the original task, Lin believes that the two are closely related, and he divides the variation into two types: small variation and big variation. According to him, small variation is a slight modification of the original task, by varying the range, or varying the place of the parameters, or varying the number, etc. Big variation refers to the variation of the presentation of the task, while the substance of the task has not changed, at least the

method has not changed, and it still serves the mathematical method.

Lin also pointed out that both the variations and the difficulty of the tasks serve the teaching goals, the arrangement of the whole class, and the situation of students. As the class time is limited, the teacher must improve the efficiency of the class and the difficulty of tasks should be arranged appropriately. If we are pursuing too difficult tasks, we can only teach few contents in a single class, and students may not be able to master them well. The task can be made more difficult or simpler through variation, which depends on the content of the teaching. For example, if the knowledge is difficult in itself, then Lin will arrange the variation task less difficult; if the original task is mainly for generating mathematical methods, and it is not difficult for the students to complete it quickly, then Lin will arrange a variation after class for students, which gives the students a certain degree of challenge. Lin stressed that the tasks in a class should combine the difficult tasks and the easy ones, because a class has to have a climax and a low point to mobilize the enthusiasm of the students.

Moreover, Lin proposed several ways to add the difficulty of tasks through variation: 1) Increase the comprehensiveness of the task: Need to use multiple knowledge to solve one task at the same time; 2) Set trap in the task; 3) Add steps that need to be transformed and deformed in the task: For example, adding parameters or introducing dynamic points in the task.

### **5.1.3 Conclusion: Variants and invariants in Lin's resource system and documentation work -- the foundation of teaching through variation**

In this section, we discuss the invariants and variants in Lin's resource system and his documentation work within a relatively long period of time from the macro and micro perspectives.

From a macro point of view, Lin believes that his resource system has not changed a lot in the past ten years as the core of mathematics teaching has not changed. The following will explain the impact of external changes on Lin's resources from three aspects: curriculum reform (mainly textbook changes for teachers), the emergence of new technologies, and the development of digital resources.

According to Lin, the curriculum reform only changes textbooks in form (The presentation of the content and the order of presentation lead to the adjustment of the corresponding teaching methods), and the nature of mathematics teaching content has not changed. Mathematics teaching still relies on the mathematical tasks, so he still uses the tasks he selected from his resource system. He mentioned: *"In fact, no matter how the textbooks change, the core things and things that students need to master remain unchanged at all. The explanation of the content in textbooks is different, how to introduce it, how to explain it, which is mainly used to pave the way for the explanation*

*of lessons for new knowledge, etc. As for specific examples, they are not relevant (to the change of textbooks). As my examples can already explain it (the knowledge) well, and students can master the concepts and methods through these examples. Then why should I not use them? Why should I change them? In fact, the curriculum reform is a change in form, and the essence has not changed. The external form has changed a bit, and the content and core things have not changed. So, I think the teaching methods will be adjusted appropriately, but the actual content will not change much. I have used several versions of textbooks. I have used the version of Beijing Normal University before, but the essential things have not changed. The layout system of teaching contents may be changed, and the explanation may be changed a bit. For example, some schools emphasize the introduction and appropriate adjustment of teaching methods, but the core content has not changed.”*

Lin believes that technology is an auxiliary tool for mathematics teaching that not only increases students’ understanding of abstract and dynamic mathematics, but also helps teachers’ task design. He also points out that the use of teaching technology should be closely related to teaching content. He mentioned: *“As for teaching technology, in fact, it should be said that for mathematics, we are now pursuing technology, before we were encouraged to use slides. These techniques are an aid to mathematics teaching. In fact, the teaching of mathematics, even for those great mathematicians, the bridge is still a piece of chalk, and the core is a piece of chalk. Because of the characteristics of this subject, it emphasizes a kind of logical reasoning. You can say that other things can help. For example, some things do require other techniques, such as some drawing, spatial dynamics, and three-dimensional geometric spatial things. You can show them through technology. Some things that are difficult to express through blackboard or chalk, and those things that need a dynamic expression can use computer technology. But more things are still based on traditional teaching. In fact, it has a lot to do with your teaching content. For example, some functions, more complex functions, such as functions with a parameter; when the parameter changes, the graph of the function will change, drawing tools can explain this change well. For example, when experts design tasks for the college entrance examination, they also change the parameters and draw the graphics on the geometry sketchpad. Some techniques are needed, but they are all auxiliary functions, this is the characteristics of mathematics teaching.”*

As for the digital resources, Lin pointed out the necessity of its existence. He used his commonly used *Discipline Website* (mentioned above) as an example to illustrate: *“Discipline website is definitely necessary. We can search for information, and we can find some of the latest information. Because we have to look at some new tasks. It is precisely because of the development of these things that the students are more tired, and it is convenient for the teacher to look for the tasks. As soon as it is convenient to*

*obtain resources, students will get tired if there are more tasks.”*

The main structure of Lin’s resource system remains invariant over time. However, from a micro point of view, the content of Lin’s resource system is dynamic and constantly developed by teacher’s documentation work, in the processes of instrumentalization and instrumentation. In order to clarify the diverse resources in Lin’s resources system and the relationship between them, we drew an inferred mapping of Lin’s resource system (section 3.3.4.1), see Figure 5.1.5.

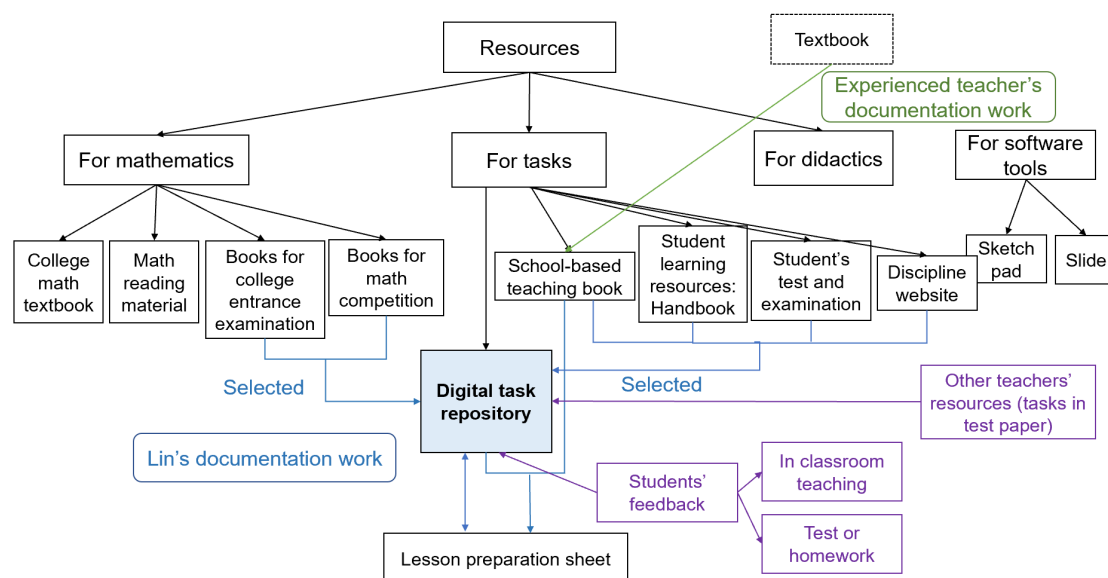


Figure 5.1.5 Inferred mapping of Lin’s resource system

Figure 5.1.5 is drawn by the researcher based on the Lin’s interview and his classroom observation. We centered on the digital task repository, marking out the various resources used by Lin and the connections between the resources. We have used several colors in the figure: notions in black represent the existed resources; notions in purple represent the collaboration work performed by Lin, which can be done with other teachers or with students; notions in blue represent the connection between various resources. In addition, we specially marked the process from textbook to school-based teaching book in green. One is because school-based teaching book is one of main teaching resources for Lin, and the other is because this process takes place inside school C. To a certain extent it reflects the inheritance and development of resources from very experienced teachers to other teachers inside a school.

Taking profit of the figure above and Lin’s interpretations of his process of instrumentalization during his documentation work, we illustrate the variants and invariants in Lin’s resource system through his documentation work:

#### (1) Variants in Lin’s resource system: Update of resources and flexible design

The variants in Lin’s resource system are mainly in two parts: one is that the resource

itself is changing, and the other is Lin's flexible organization of his working sheets.

All resources in Lin's resource system are constantly varying. However, the frequency of changes is different. For example, the update of revised editions of mathematics books is quite long, while school-based teaching books, books for college entrance examination, discipline website, etc. are all affected by the curriculum reform. We mentioned earlier that Lin believes that his teaching has not been affected by the curriculum reform, but in fact the resources he uses (designed by others) have been adjusted and changed accordingly. Here, we use the digital task repository built by Lin himself to illustrate the variants in Lin's resource system. Lin's digital task repository is updating all the time. Lin pays attention to the accumulation of classic tasks in every teaching process. Every one or two weeks, he organizes the recent test papers, weekend exercises, etc., and adds good tasks to his digital task repository. Therefore, the digital task repository is constantly expanding and developing.

Lin consciously leaves space for the variants in his teaching design. Lin believes that the teaching process is actually a dynamic adjustment process. He doesn't save his lesson preparation sheets as he doesn't want these sheets to restrict his minds of teaching. As he said: *"I may want to make a variation, and I still have to prepare for the lesson each time, because in the course of my lesson preparation, I have both the invariants and variants. Each class of students is different, the progress is different, and all aspects are different. I have to adjust my lesson. I don't think there is any need to keep the sheets."* Lin also pointed out the importance of storing classic tasks. He made organizations, variations, and adjustments on the basis of these tasks to construct a systematic teaching.

## (2) Invariants in Lin's resource system: type of resources, classic tasks and instrumentalization

The invariants in Lin's resource system are mainly in three parts: types of resources, classic tasks and the process of instrumentalization for design a new working sheet.

Although the source of digital task repository is various (section 5.1.1), the types of sources are relatively fixed in Lin's resource system: for mathematics, for tasks, for didactics and for software tools.

The connection between some resources is fixed. Variation in the process of multi-person, multi-step relay documentation work There is a routine from textbook to school-based teaching book, and to Lin's lesson preparation sheet. Textbook here is a hidden resources in Lin's resource system as the previous documentation work of textbook was done by an experienced teacher in school C. There are various possibilities in the design and organization of the school-based teaching books from the source of textbooks. This new version is the result of 17 years of different school-based

teaching books. According to the teacher who designed and organized the school-based teaching book, some tasks are all derived from students' mistakes from the past ten years. Combined with the curriculum reform, school-based teaching book is in fact the common results of students and many teachers for many years. Lin continued to combine his own digital task repository to design his lesson preparation sheet based on the school-based teaching books. It is reasonable for us to infer the existence of variation in this process. Multi-person, multi-step relay documentation work provides a good soil for the emergence of variation.

After many years of teaching, Lin has his own mode of instrumentalization. No matter which mathematical topics he designs to teach, he will look for it from the resources shown in figure 5.1.5 in a targeted manner.

Because Lin has been doing this for more than ten years, he has completely formed a habit. He already has his own set of criteria when judging and selecting tasks (section 5.1.1). Through his own personal experience, Lin believes that with the accumulation of tasks and teaching experience, teachers will naturally be able to make certain adjustments and variations in their own teaching according to the actual situation.

Lin believes that the teaching process is actually a dynamic adjustment process. He doesn't save his lesson preparation sheets as he doesn't want these sheets to restrict his minds of teaching. As he said: *"I may want to make a variation, and I still have to prepare for the lesson each time, because in the course of my lesson preparation, I have both the invariants and variants. Each class of students is different, the progress is different, and all aspects are different. I have to adjust my lesson. I don't think there is any need to keep the sheets."* Lin also pointed out the importance of storing classic tasks. He made organizations, variations, and adjustments on the basis of these tasks to construct a systematic teaching.

Lin regards the classic tasks as an invariant, and regards the adaptation of the tasks and the content organization and arrangement as a variant. He pays attention to collecting and storing invariant factors, while leaving enough space for his design of variation in tasks. In fact, from the analysis and discussion above, we can see that it is difficult to distinguish what is varying and what is not. there are invariants in variants, and variants in invariants. These characteristics of Lin resources also deeply affect his classroom teaching.

This kind of instrumentalization provides doubled insurance for Lin's teaching: The invariants of classic tasks ensure that teaching does not deviate from teaching goals and the variants ensure that the teaching adapts to the needs of the students.

## 5.2 Variation of tasks in Lin's own classroom teaching

In this section, we analyze Lin's classroom teaching through variation with respect to the tasks presented in his own teaching of function. We first describe the general situations of different kinds of variations in Lin's teaching (section 5.2.1). We then select seven episodes of classroom teaching to zoom in, analyze in detail from potential, actual to practical variation, and present the schematic representation of variation path for each episode (section 5.2.2). The methodology is presented in section 3.3.4.

### 5.2.1 General situations of Lin's teaching through variation

In this section, we present the general situations of different kinds of variations in Lin's teaching by figure (see section 3.3.3): contextual variation, representational variation, conceptual variation, and procedural variation (including Pudian, OPMC, OPMS, and MPOS). Through the statistical figure of each type of variation, we illustrate the situation of each type of variation in Lin's teaching from the perspective of teaching goals.

#### 5.2.1.1 Contextual variation

We study 53 tasks from Chinese classroom observation and there are few contextual variations in this case. As shown in figure 5.2.1, contextual variation often appears in the lesson of exercise explanation. These tasks are from the examinations or homework. Throughout our classroom observation process, there are 2 societal contextual variations, 2 occupational contextual variations, and 1 personal contextual variation. There are tasks with mathematical context, that is without contextual variation.

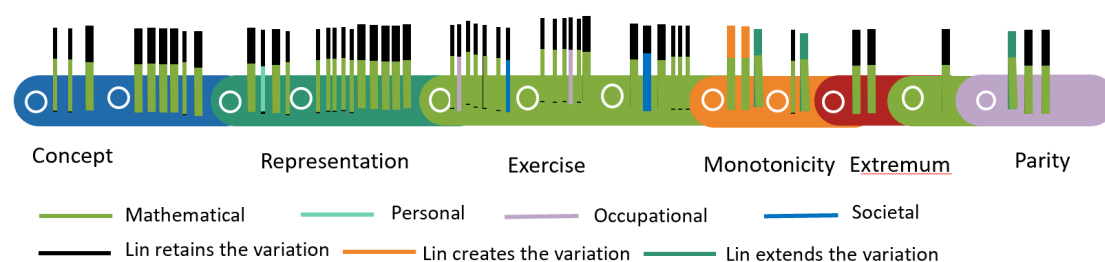


Figure 5.2.1 Contextual variation shown in Lin's teaching

As shown in figure 5.2.1, Lin prefers to retain the context of tasks. In this case, he creates the new tasks with mathematical contexts twice, extends tasks with mathematical contexts three times. In fact, contextual variation is weakened step by step. In the textbook we can see that 41.5% of tasks have contexts (see Chapter 4), in the school-based teaching book 14.29% of tasks have contexts, and in Lin's classroom teaching, there is only one task with contextual variation (personal context) except the exercises from the lessons of exercises explanation. Lin constantly removes the



possibility of contextual variation through his documentation work. According to Lin, the time of a lesson is limited, and the contents for each lesson is very large, which makes it impossible for him to set up tasks with context in daily teaching. Because the college entrance examination puts a lot of pressure on students and teachers, Lin must complete the teaching objectives and teaching contents on time, and for tasks with context, students will take up a lot of time to read the tasks in class.

As we observed in class, Lin prefers to choose the tasks with no context and show the abstract mathematical knowledge for students. However, this does not mean that students are only exposed to tasks without context. We can still see some tasks with context in their homework, tests in class or examinations. And according to the student's test situation, students of class A and B are quite good at solving the math problem with context.

There is no contextual variation emerged as practical variation. As there are no contextual variations (re)designed or organized by Lin, and almost all tasks are conducted in a mathematical context, and there is not enough room for the classroom teaching to generate contextual variations.

### **5.2.1.2 Representational variation**

Lin pays much attention to different forms of representation, and the treatment or conversions inside or between different registers. According to Lin, there are two important representational variations in teaching of function: (1) Treatments inside the symbolic registers, students should be able to appropriately transform the algebraic registers according to the requirements of the task; (2) Conversions between symbolic and graphic registers, visualization of abstract problems and can greatly simplify the problem-solving process. Here, we focus on the conversions<sup>94</sup> occur in Lin's teaching because of the function of the research object, almost all tasks involve treatments inside the symbolic registers.

As shown in Figure 5.2.2<sup>95</sup>, conversion is covered in almost all lessons and more concentrated distributed in the two chapters of monotonicity and parity on the teaching content axis. It occurs 20 times in Lin's teaching of function. However, conversions are

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94 See Duval's theory in section 2.1.2.3.

95 We draw a timeline of the entire Lin's teaching of function in grade 10, and the specific teaching content of each paragraph is marked below. According to the occurrence of the variation, it is marked on the time axis, where each color represents each theme of function and circles in the timeline means the start of a lesson. The sticks on the time axis are thick or thin. The thick one represents the series of tasks, and the thin represents a task. In addition, different colors of the sticks represent Lin's choice for this variation: Retain the variation=black; Create the variation=orange; Extend the variation: green; Reduce the variation: blue; Variation emerged through practical variation: red; Remove the variation =yellow. See section 3.3.3 for details.

relatively single in this case, mainly the conversions from symbolic writing to graphs, except that there is a task which contains the conversion between three types of representation of a function (formula, table, and graph) in the lesson of representation I and Lin retained. In Lin's teaching, there are very few representations of the form of table of a function or the conversions from graphs to symbolic writings.

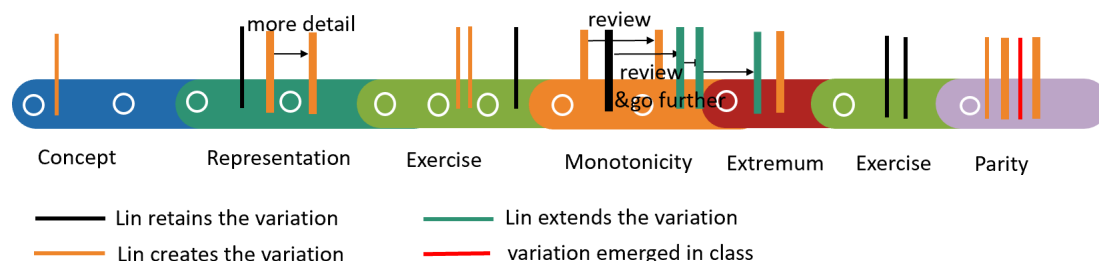


Figure 5.2.2 Representational variation (Conversions) shown in Lin's teaching

In Lin's teaching, representational variation, here is the conversions from symbolic writings to graphs, mainly assumes two roles: 1) The powerful tool of task-solving strategy: Most of these conversions between symbolic writings and graphs are created by Lin in the task-solving strategies. When encountering more complex tasks, Lin often encourages students to use graphs to solve tasks as graph is always more intuitive and can simplify the task (see episode 3); 2. The auxiliary tool of conceptual explanation and understanding: Lin prefers to show conceptual examples, especially non-standard or non-concept variation, through graphs to help students understand concepts, see episode 5.

As shown in Figure 5.2.2, there is one representational variation produced through the interaction between Lin and his students in class. When students were inspired by an example in the class and raised the conclusion that all odd functions of the piecewise function can merge the  $x = 0$  point into one of the piecewise expressions, Lin used the rough graph of a function to quickly and intuitively cite a counter example (see table 5.2.1). Here, the conversion of students' question from symbolic writing to cartesian graph is completed by Lin and serves for the conceptual variation.

Table 5.2.1 Representational and Conceptual Variation produced in classroom

		$f(x) = \begin{cases} x^2 - 2x, & x > 0 \\ 0, & x = 0 \\ -x^2 - 2x, & x < 0 \end{cases}$
<p>Transcription of the interaction between Lin and his three students A, B, C:</p> <p>Lin wrote down the formula of a piecewise function</p> $f(x) = \begin{cases} x^2 - 2x, & x > 0 \\ 0, & x = 0 \\ -x^2 - 2x, & x < 0 \end{cases}$ <p>Student A &amp; B together: How about adding an equal sign at the end? In the domain.</p> <p>Lin: Wouldn't it work if you add a 1 to the end (of the formula)? [<math>f(x) = x^2 - 2x + 1</math>] It's just right for this situation. I can include 0 in these two (piecewise expressions), and they are connected together. But if you add a 2 at the end (of the formula)? [<math>f(x) = x^2 - 2x + 2</math>], it won't work.</p> <p>Student C: Odd functions can all be connected together.</p> <p>Lin: It can be separated, why it is always be connected together? I can let the 0 point isolate, and I draw a graph for you to understand (Lin drew the graph of a counter example, see below), so that it can be like this, 0 is an isolate point here.</p>		

### 5.2.1.3 Conceptual variation

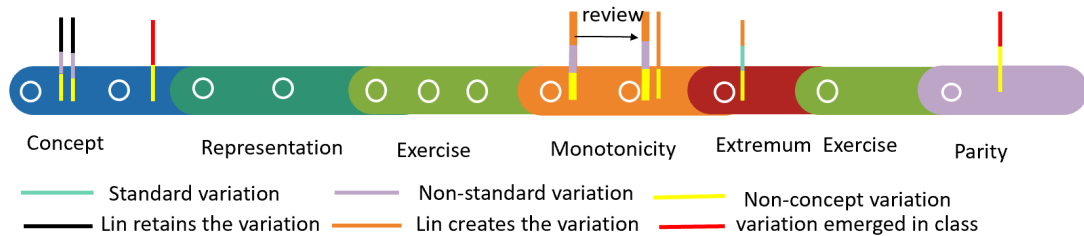


Figure 5.2.3 Conceptual variation shown in Lin's teaching

As shown in figure 5.2.3, conceptual variations mainly occur in the concept-related classroom teaching, such as the concept of function, the monotonicity of a function, etc. It occurs 8 times in Lin's teaching of function, including 8 times of non-concept variation, 1 time of standard variation, and 4 times of non-standard variation. Standard variation is used to illustrate concepts, introduce concepts, while non-concept and non-standard variations are used to distinguish the concept.

There is no conceptual variation in the exercise lessons, which may be related to the emphasis on task-solving strategies in the exercise lessons. The main purpose of conceptual variation is to let students clarify the concept, and the number is not many.

Lin's design of conceptual variation is usually at the beginning of a lesson, and the emergence of new conceptual variation produced by the interaction between Lin and students is generally in the middle or at the end of teaching. Students often encounter problems in the process of solving tasks, and they would use the task as a carrier to raise questions to Lin.

There are two conceptual variations produced through the interaction between Lin and his students in class. In fact, in Lin's classroom teaching, most of the questions raised by students are related to mathematical concepts. When Lin answers these questions, he often uses conceptual variation through graphs (representational variation) to help students distinguish concepts. Therefore, the interactions between students and Lin in the classroom will often produce new conceptual variations. Table 5.2.1 shows one of the examples of conceptual variation produced by Lin in class.

#### 5.2.1.4 Procedural variation

There are various types of Procedural Variations in Lin's teaching and we explain them from four aspects: enhancing the formation of concepts and propositions, Pudian, OPMC and OPMS.

- **Enhancing the formation of concepts and propositions**

In Lin's teaching of functions, all concepts and properties are given directly and further explained by him. Enhancing the formation of the concept of function through a series of problems (Section 4.2.1), which is a unique feature of the Chinese PEP textbook is hard to find in this case. Lin devotes more time to the teaching of problem solving strategies than to the introduction of concepts and properties.

- **Pudian**

As shown in figure 5.2.4, Pudian occurs 10 times in Lin's teaching of function. Most of Pudian are created by Lin through his documentation work.

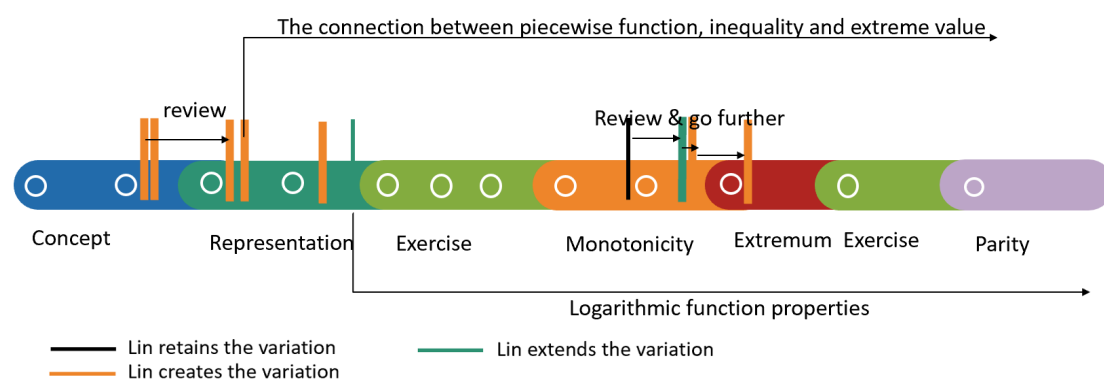


Figure 5.2.4 Pudian shown in Lin's teaching

Pudian is an important means in Lin's teaching, which is mainly reflected in two aspects:

- 1) Let students master the task-solving strategies: Lin pays great attention to the creation and setting of Pudian in order and this type of Pudian often occurs at the beginning of a lesson and paves the way for the follow-up example explanation. It often lasts one lesson or several lessons, such as episode 2 and episode 6;
- 2) Establish core associations with different knowledge: As we have mentioned in section 3.1.1, function is closely related to another mathematical knowledge. Lin uses specific tasks to let students associate function with inequalities, equations, etc. This type of Pudian often runs through different chapters and lasts a long time, see an example in episode 3.

Lin designed Pudian consciously in his class. He sets up different types of Pudian according to the complexity of the learning content. Some Pudian can be completed in one lesson, while some run through the entire function study, even the entire high school study or pave the way for college mathematics. This requires teachers to have a macro understanding of the entire mathematics teaching and learning, have a clear control of the connections between mathematical knowledge, and rely on teachers' long-term teaching experience and self-improvement.

There are no new Pudian produced in class. We think this is mainly because Lin, as an experienced teacher, can predict where students will have difficulties based on the level of the students, and he design Pudian during his documentation work. Another factor may be the limit of classroom teaching time. Students have not had enough time to raise their questions. Therefore, for specific students' needs of Pudian, Lin may conduct it with students after class. Pudian needs to link different knowledge and have an overall overview of the teaching. This is often a challenge for the teacher, especially for novice teachers.

### • OPMC (MPOS)

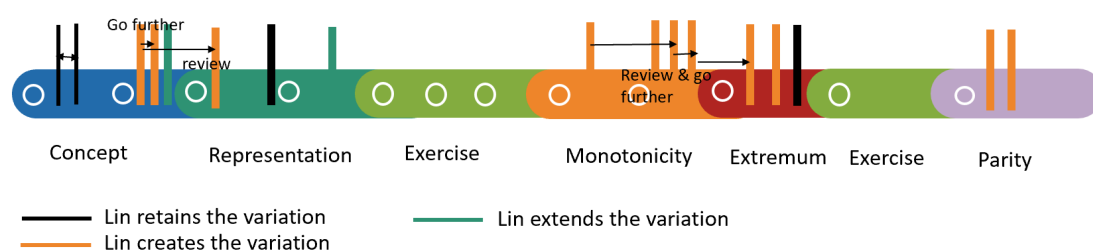


Figure 5.2.5 OPMC shown in Lin's teaching

As shown in figure 5.2.5, OPMC is in every lesson of new knowledge, occurring 17 times in Lin's teaching of function. Most of Pudian are created by Lin through his documentation work. Due to the characteristics of OPMC, it is often associated with

series of tasks. In Lin's teaching process, OPMC mainly serves three teaching purposes:

- 1) To ensure students have mastered the strategy by varying non-essential features<sup>96</sup>, see episode 1, episode 2. This is associated with MPOS, where students use the same strategy to solve tasks with different presentations;
- 2) To ensure students have mastered the strategy by varying the essential features<sup>97</sup>, so that students can better understand the scope of application of the strategy through the contradictions generated, see episode 7;
- 3) In order to go deeper, introduce students new learning content, see episode 6.

## OPMS

OPMS mainly refers to the use of different problem-solving strategies as teachers or students reduce the task from different angles<sup>98</sup>.

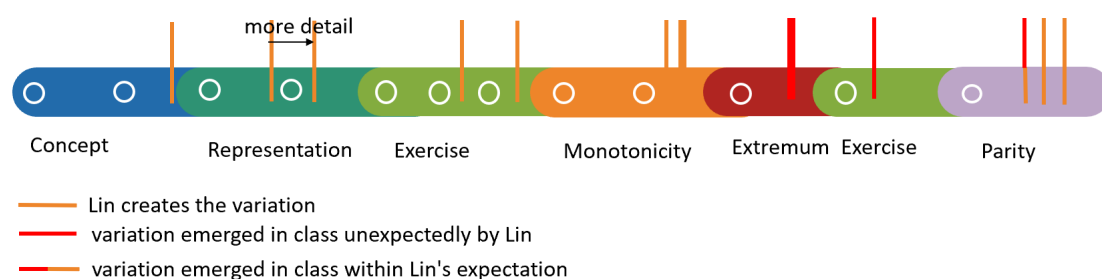


Figure 5.2.6 OPMS shown in Lin's teaching

As shown in figure 5.2.6, OPMS occurs 12 times in Lin's teaching of function, created by Lin through his documentation work or emerged in class through the interactions between Lin and his students. The distribution of OPMS is relatively uniform. In Lin's design, OPMS mainly serves two teaching goals:

- (1) Through the introduction and comparison of different strategies, Lin helps students expand the breadth of thinking, let students understand the condition of application of each strategy (see episode 7 for details), and tries to find the most appropriate strategy to each task (see episode 3 for details).
- (2) Link new and old knowledge through the task, and use learned knowledge as a strategy to solve the new task, see episode 7 for details.

As shown in figure 5.2.6, the emergence of OPMS in the classroom teaching as practical variation is mainly concentrated in the later period of function teaching, such as the

<sup>96</sup> Refer to section 2.2.1

<sup>97</sup> Refer to section 2.2.1

<sup>98</sup> See the detailed description in section 2.2.1 and section 2.2.3

teaching of extreme value, the last exercise lesson, and the parity. We believe that this phenomenon can be attributed to two reasons. One is the difficulty of the task. With the deepening of teaching, the difficulty of the tasks becomes more and more complicated, and there are more and more comprehensive tasks, which gives teachers and students the premise to look at the tasks from different angles, so as to obtain different strategies. And the other is the increase of students' knowledge help them link the knowledge learned before and after and think about tasks from different angles.

Three OPMS produced by students in class. The OPMS produced in class can be divided into two categories:

- 1) OPMS of practical variations that Lin anticipates and expects (happen once): For example, some strategies are proposed by students with the encouragement of Lin, and we can see the different strategies listed in Lin's working sheet during his documentation work, which covers students' propositions of strategies. We think this kind of OPMS is what Lin expects, as shown in figure 5.2.6, the first column of parity noted as red and orange.
- 2) OPMS of practical variations that Lin did not anticipate (happen twice): On two occasions during our observation, students proposed strategies that the teacher did not anticipate. In this case, Lin will write down the problem-solving process on the blackboard based on the student's statement. Lin's understanding and reflection of the new strategies are completed during his blackboard writing and his interactions with students.

It needs to be pointed out that Lin does not necessarily show the new solution produced in one class in his other class. It depends on his views on the new strategy. As an experienced teacher, Lin can quickly judge whether the new strategy is universal and whether it is worth spending time to propose in his other class. To make it clearer, we present an example in episode 7.

#### **5.2.1.5 Conclusion**

Figure 5.2.7 shows the distribution of seven selected episodes and the variation types involved. The use of variation in Lin's teaching is generally very diverse. Except for contextual variation and enhancing the formation of concepts or propositions, the other variations all appear in his teaching. As presented in Figure 5.2.7, the most common types of variation in Lin's teaching of function are representational variations (20 times) and OPMC (17 times). Pudian, OPMC, OPMS, and representational variation are mainly designed by Lin through his documentation work. The main types of variations that appear in the classroom are conceptual variation and OPMS. In addition, in Lin's teaching of function, conceptual variation, Pudian, and OPMC only appear in the lesson of new knowledge, not in the exercise lesson.

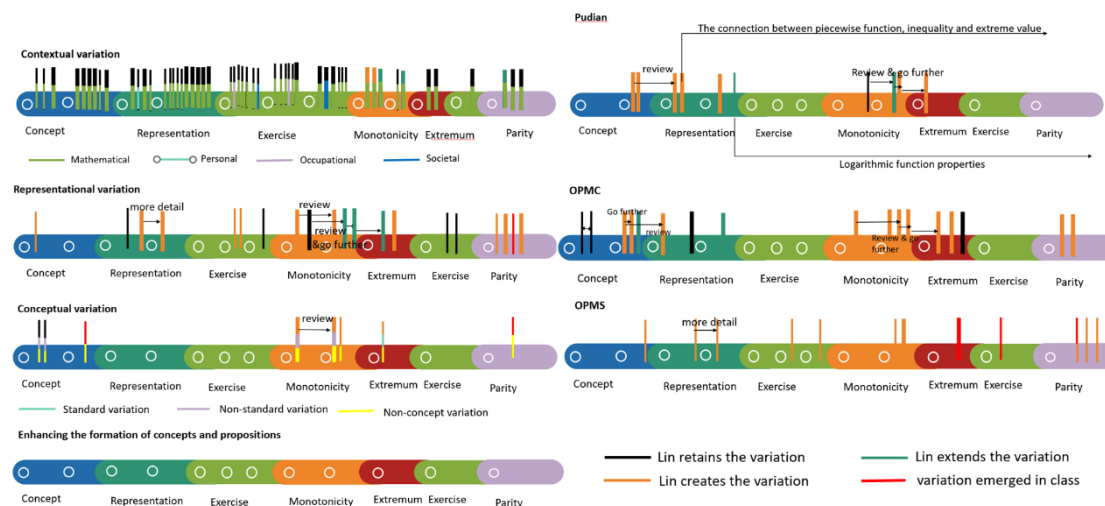


Figure 5.2.7 An overview of the different types of variations in the Chinese case

## 5.2.2 Variations in selected episodes

Based on the graphs presented and general situations described in section 5.2.1, we zoom in seven episodes in this section for further analysis.

What needs to be pointed out is that a task or a (series of) task(s)<sup>99</sup> in one episode often contains more than one kind of variation. For each episode, we first described the background of this episode, mainly the knowledge and teaching goals related to the task. We then analyzed the potential variations already embedded in the tasks, taking profit of the analytical framework of potential variation in tasks (see table 2.4.1). After that we studied teachers' documentation work (actual variation) on the perspective of variation, compared to the potential variation. We distinguished teacher's work between retaining, removing, extending, reducing and creating variations, according to the teaching goal and students' need. At last, we studied the classroom observation (practical variation) of each episode. We also added teacher's reflections on some certain episodes based on the interviews or discussions with the teacher after class.

### 5.2.2.1 Episode 1: Conceptual variation for mastering problem-solving strategies

In episode 1, we analyzed two tasks (shown in table 5.2.2 and table 5.2.3) presented in the teaching of the concept of function, which are originally from school-based teaching book. Lin makes full use of the conceptual variations embedded in these two tasks to help students master the mathematical strategies. We first analyze them separately in detail, and then compare and analyze the relationship between the two tasks.

<sup>99</sup> See the definition of series of tasks in section 3.3.3.



Table 5.2.2 First task of the concept of function

Example 1(source: School-based teaching book, p63):

Determine whether the following correspondence is a function from A to B:

(1)  $A = B = \mathbb{N}^*$ , Corresponding relation  $f: x \rightarrow y = |x - 5|$

(2)  $A = [-1, 1], B = [0, 2]$ , Corresponding relation  $f: x \rightarrow y = -x^2 + 1$

(3)  $A = \mathbb{R}, B = \{-1, 2\}$ , Corresponding relation  $f: x \rightarrow y = \begin{cases} -1, & x \leq 0 \\ 2, & x > 0 \end{cases}$

(4)  $A = [0, +\infty), B = \mathbb{R}$ , Corresponding relation  $f: x \rightarrow y = \pm\sqrt{x}$

Task presented in table 5.2.2 is used in the first lesson to introduce the concept of function and lasted 5 minutes. The aim of this task is to help students understand the concept of function. This task has no context.

Examining the potential variation:

- Conceptual variation is embedded in the task: option 1 and 4 are non-concept variation, while option 2 and 3 are non-standard variation. The standard variation of this situation, which is familiar with students, is  $A = B = \mathbb{R}$ , Corresponding relation  $f: x \rightarrow y = -x^2 + 1$ <sup>100</sup>. For non-standard variations, the teacher regards the extension of the concept of function as the variation space, and the objects (A, B and  $f$ ) that it contains as the variants: based on the standard variation, option 2 limited the domain A and codomain B, keeping the corresponding relation  $f$  as quadratic; Option 3 showed a piecewise constant function, which leads the range of the function to be specific values. Although the exact symbolic representations of these two are different, but students could highlight the essential attributes (for every  $x$  in  $X$ , there is exactly one element  $y$  such that the ordered pair  $(x, y)$  belongs to the set of pairs defining the function  $f$ ) of this concept. The symbolic representations of option 1 and option 4 is like functions, they contradict the essential attributes of the function.
- Procedural variation (OPMC) is embedded in the task: The task covers four common types of function types in high school: absolute value, quadratic, piecewise, and radical function. The task varies the type of condition presented from two aspects: the domain and the formula of function. In this task, the mathematical knowledge of the concept of function is behind the variations of these four options.

<sup>100</sup> The corresponding relationship here can be anything that students are familiar with. Given the level of students in this period (10<sup>th</sup> grade), the functions they are familiar with are mainly: linear function, quadratic function, inverse proportional function, etc.

OPMC provides series of options in which students can understand, reflect and generalize the invariants in the concept of function.

- OPMC in this task is an approach to develop and promote the conceptual variation.

Regarding actual variation, Lin retains the original task with potential variations and creates representational variation through his documentation work:

- New representational variation emerged: The representations of function of four options are all formula, i.e., symbolic representations. In the process of solving this task, there are treatments in the symbolic writing: from an algebraic register to a numerical one<sup>101</sup>. However, Lin adopts different ways to explain option 1 and 4. According to his working sheet, he planned to directly show the counter example of option 1 in class (when  $x = 5, y = 0$ , it doesn't belong to the codomain of  $y$ ) to prove that it is not a function from A to B. But for option 4, besides substitution of different values of  $x$ , he took profit of different representations of function, drawing the graph of  $y = \pm\sqrt{x}$  to prove its contradiction with the concept, see figure 5.2.8.

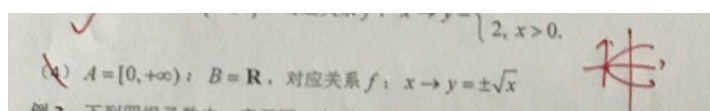


Figure 5.2.8 A counter-example drawn by graph in Lin's working sheet

According to Lin, presenting the graph of a function can be more intuitive for students to understand. Meanwhile it also paved a way (the idea of Pudian) for introducing the method of determining whether it is a graph of a function. Therefore, new representational variations generated through Lin's documentation work. The teacher varied the registers of representing a function, moving from an algebraic register to a graphical one. The aim of the actual variation is to provide multi-angle of the concept for students, with the help of different representations of function.

As for practical variations, the teacher retains the actual variations he designed and no practical variations emerged. He asked students to think 2 minutes and then explained the task by himself based on his working sheet. When he showed the counter example of option 4, he improvised the specific values of  $x$ . According to the teacher, this task is not hard for students to understand, because it is mainly based on the concept, without complex strategies of problem-solving. Moreover, most of students have learned this part of the knowledge during their summer vacation through their own or extracurricular tutoring classes.

In order to make it clearer, we use figure 5.2.9 to illustrate the process from potential,

<sup>101</sup> Based on the solution in school-based teaching book.

actual to procedural variation through Lin's documentation work. It can be seen from the figure that the main teaching goal of this task is to use OPMC as a means of change, through conceptual variation, that is, to show the options of non-standard and non-concept variations so that students can master the strategy of judging the concept of function. The design of drawing graph of non-concept variation in Lin's documentation work is to help students understand the abstract concept of function, which leads to new representational variation (conversions).

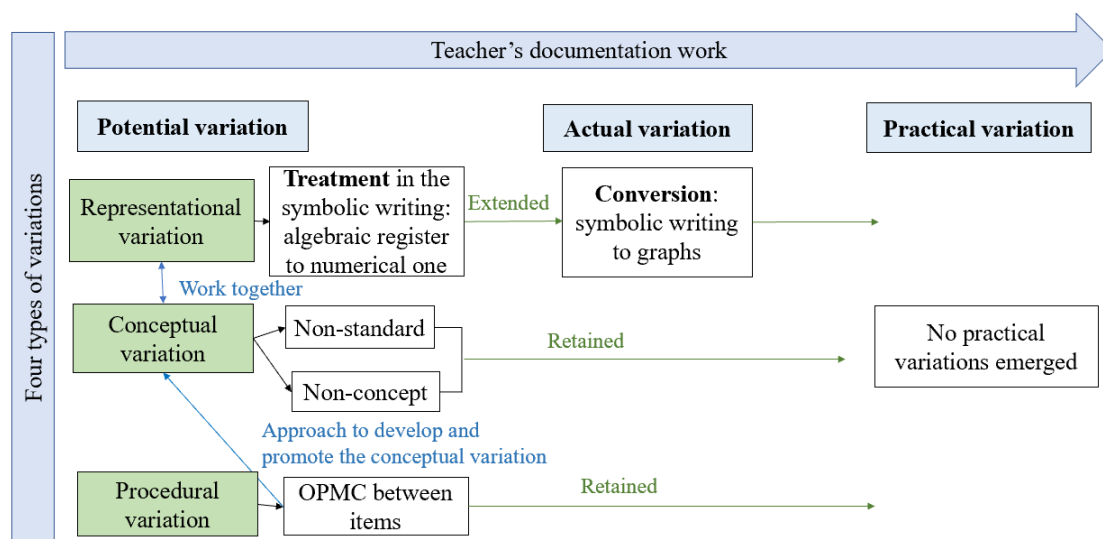


Figure 5.2.9 Schematic representation of variation path of example 1 in episode 1

Example 2 presented in table 5.2.3 was proposed after example 1 in the school-based teaching book, designed as a variation task of example 1, by varying both the condition and the goal of example 1. It relates to the knowledge of the domain and corresponding relationship of a function, which is an extension of the knowledge of function concept. This task has no context. The representations of function of four options are all formula, i.e., symbolic representations. The teaching of example 2 lasted about 5 minutes in class.

Table 5.2.3 Second task of the concept of function

Example 2 (source: School-based teaching book, p63): Among the following four groups of functions, the one representing the same function is ( ).

- A.  $f(x) = |x|, g(x) = \sqrt{x^2}$
- B.  $f(x) = \sqrt{x^2}, g(x) = (\sqrt{x})^2$
- C.  $f(t) = |t - 1|, g(x) = \sqrt{x^2 - 2x + 1}$
- D.  $f(x) = 1, g(x) = \frac{x}{x}$

Examining the potential variation:

- Conceptual variation is embedded in the task. Option B and D are non-concept variation, derived from students' mistakes, while option A and C are non-standard variation. The similarity of non-essential attributes can easily confuse students. For example, the symbolic representation of two options inside B and D are very similar, which will cause students to easily ignore the domain of a function.
- OPMC is embedded in between four options by varying the formula of function. Through the variation, the task covers four common types of function types in high school: absolute value, fractional, radical and constant function. It is expected to use the same solution to four options (MPOS), that is, determining the domain, simplifying the formula of a function and comparing two functions. Similar to example 1, OPMC in four options are based on the mathematical knowledge.

As for the actual variation, Lin retains the original task with potential variations. However, he regarded this task as a new task and did not connect this task with example 1. According to Lin, it was not necessary to make the connection because example 2 has varied too much compared to example 1. The conditions, the goals, and even the solution of example 1 and example 2 are all different. He believes that the weakness of visible invariants between two examples makes the advantages of variations greatly reduced for students. However, example 1 and 2 are worth mentioning separately as he designed in his working sheet because they can help students understand the concept of functions from different angles.

Like example 1, there is no practical variation emerged during the classroom teaching of example 2.

In general, these two tasks (this series of tasks) are designed to allow students to master strategies through the conceptual variation (OPMC) that is embedded in the options of the task. Lin adds the conversions in the function representation in his documentation work to make the non-concept variation more intuitive.

#### **5.2.2.2 Episode 2: Pudian for mastering problem-solving strategies**

This series of tasks (shown in table 5.2.4) is first proposed in the second lesson to introduce the concept of function. Lin designed to review part of these tasks in the teaching of the representation of function the next day. The aim of this task is to apply the strategy to find the domain of a composite function. Example 1 here is directly selected by Lin from the school-based teaching book and the rest of the tasks are designed by Lin himself based on example 1. The tasks arranged here are in the order of appearance in class. This episode lasted 10 minutes in class.

We first analyze the potential variation already embedded in example 1 as it is the

original task from Lin's resources system, and then describe Lin's documentation work of designing Pudian for example 1 and further develop the variation task.

Table 5.2.4 Series of tasks about the domain of composite function

<p>How to find the domain of composite function (source: designed by Lin)</p> <p>The domain of <math>f(x)</math> is <math>[2,3]</math>, Find the domain of <math>f(x^2 + 1)</math>.</p> <p>The domain of <math>f(x^2 + 1)</math> is <math>[1,2]</math>, Find the domain of <math>f(x)</math>.</p>
<p>Example 1 (source: School-based teaching book p66)</p> <p>(1) The domain of function <math>f(x)</math> is <math>[0,1]</math>, Find the domain of <math>f(x + 2)</math></p> <p>(2) The domain of function <math>f(x + 2)</math> is <math>[0,1]</math>, Find the domain of <math>f(x)</math> and <math>f(x^2 + 1)</math></p>
<p>Variation (source: designed by Lin): The domain of <math>f(x)</math> is <math>[0,2]</math>, Find the domain of <math>g(x) = f\left(x + \frac{1}{2}\right) + f\left(x - \frac{1}{2}\right)</math>.</p>

Examining the potential variation of example 1:

- Representational variation: The representations of function of these tasks are symbolic representations. In the process of solving this task, there are treatments in the symbolic writing: from an algebraic register to a numerical one and also the treatments inside the algebraic registers.
- Procedural variation is embedded between two tasks in example 1: 1) The idea of Pudian in two items: Students are unfamiliar with item 2. Compared with the intuition of item 1, item 2 requires more understanding of the definition of the composite function. From this point of view, item 1 can be seen as a Pudian of item 2; 2) The symmetry of OPMC<sup>102</sup> is presented here, interchanging the conditions and goals in item 1 and item 2 of example 1.

Regarding actual variation, Lin redesigns the tasks by providing several variation tasks before and after example 1. New type of variation (OPMC & Pudian) created by Lin through his documentation work:

- Procedural variation extended by using OPMC among tasks: The teacher added several tasks. Before showing the example, he proposed two tasks, which is constraint manipulation I of OPMC, by varying the number of the condition of

<sup>102</sup> Refer to section 2.2.3

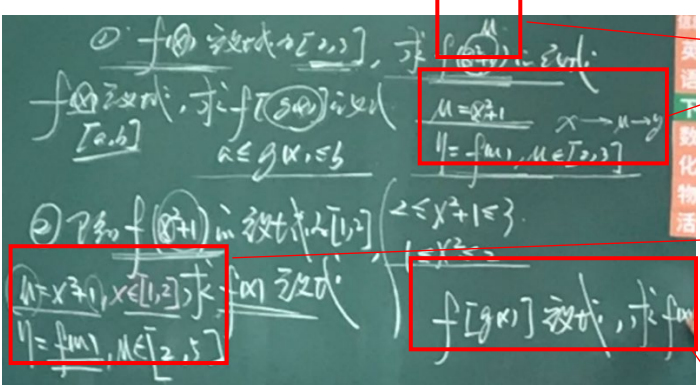
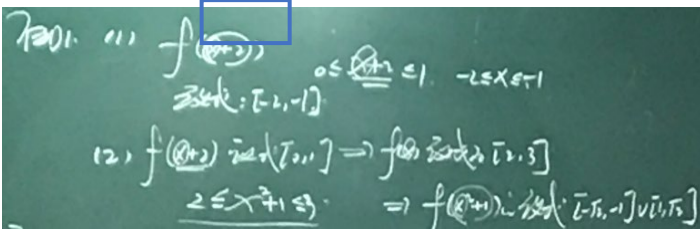
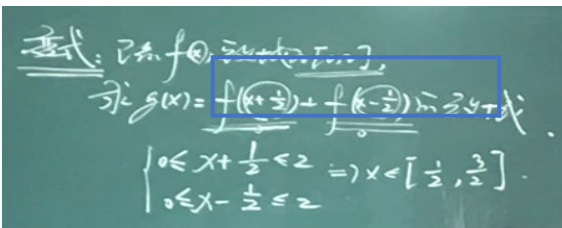
example 1. After example 1, the teacher proposed a variation task, which is constraint manipulation II of OPMC, by varying the type of the condition.

- Procedural variation extended by designing Pudian for problem-solving among tasks: Pudian between example 1 and variation is quite obvious and traditional, as the form of the composite function in variation is more complicated, and also involves the operation of sets (intersection). It is worth noting that task before example 1 is very similar to example 1, but is slightly higher than example 1 in terms of calculations. Then why do teachers adopt such an order of tasks? According to the teacher, these two tasks he redesigned before example 1 are used as examples for students to imitate when they do example 1 by themselves.
- OPMC in this task is an approach to develop and promote Pudian.

As for practical variations, Lin retained the potential and actual variations. Some points are worth noting:

- Classroom teaching time is gradually shortening for the tasks: The directly presented how to solve the first two tasks in class, and then let students do example 1 and the variation by themselves. The teaching of the first two tasks lasted 5min35, example 1 lasted 2min30 (1min for students to think by themselves, and 1min30 for students to explain the solution), and variation task lasted 2 minutes (45 seconds for the teacher analyzed the task, 25 seconds for students think, 50 seconds for the teacher described the solution). Obviously, the teaching time of these three tasks is gradually shortened.
- Lin's explanation of solution and blackboard writing is gradually simplified (see table 5.2.5): This is in accord with the teaching time. When explaining the first two tasks, the teacher connected the problem-solving with the concept of composite function, and reduced the composite function  $f(x^2 + 1)$  to  $u = x^2 + 1$ ,  $y = f(u)$ . He also generalized this specific task by using abstract symbols  $f(g(x))$ , see the red box in table 5.2.5. As for example 1 and the variation tasks, the process of the reduction of composite function is simplified by explaining verbally without written on the blackboard. We can see the circle (blue box in table 5.2.5) drawn by the teacher when explaining it.

Table 5.2.5 Photos of Lin's blackboard writing of series of tasks

 <p>Blackboard writing for the first two tasks</p>	<p>algebraic substitution :</p> $f(x^2 + 1)$ $u = x^2 + 1$ $x \quad u \longrightarrow y \longrightarrow$ $y = f(u), u \in [2, 3]$ <hr/> $u = x^2 + 1$ $x \in [1, 2]$ $y = f(u), u \in [2, 5]$ <hr/> <p>He generalized this type of task as known the definition of <math>f[g(x)]</math> and find the definition of <math>f(x)</math></p>
 <p>Blackboard writing for example 1</p>	<p>He circled <math>x + 2</math> and <math>x</math> inside</p>
 <p>Blackboard writing for variation task</p>	<p>He circled <math>x + \frac{1}{2}</math> and <math>x - \frac{1}{2}</math> to explain the algebraic substitution</p>

In order to make it clearer, we use figure 5.2.10 to illustrate the process from potential, actual to procedural variation through Lin's documentation work in this episode. The teacher's behavior in this episode reflects that the variations can promote teaching and improve teaching efficiency. Design Pudian for students can make them naturally transit from easy to difficult tasks, and the problem-solving strategies are constantly broadened. Here, the original task is placed in the middle, and the variation is set at the beginning and the end. Lin design Pudian for example 1 and then regard example 1 as a Pudian for the further task. It proves dual role of variation: Pudian and development of the further step.

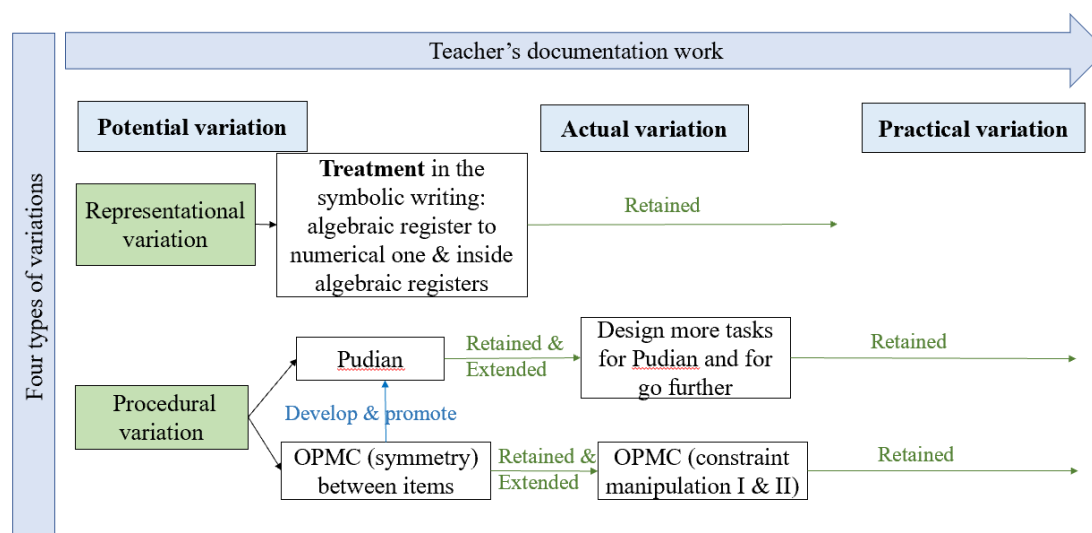


Figure 5.2.10 Schematic representation of variation path in episode 2

Through this series of tasks, teacher can quickly detect whether students have mastered the knowledge and problem-solving strategy. But unfortunately, we don't see a significant difference from the students' reactions in class. There are few responses from the students to the tasks above. We think this is related to the teachers giving students insufficient time to think in class. When the teacher explained the solution, we can still see that half of the students were thinking for themselves with their heads down. The role of variation in this episode is weakened due to time constraints, but it may still play a role when students review the notes after class.

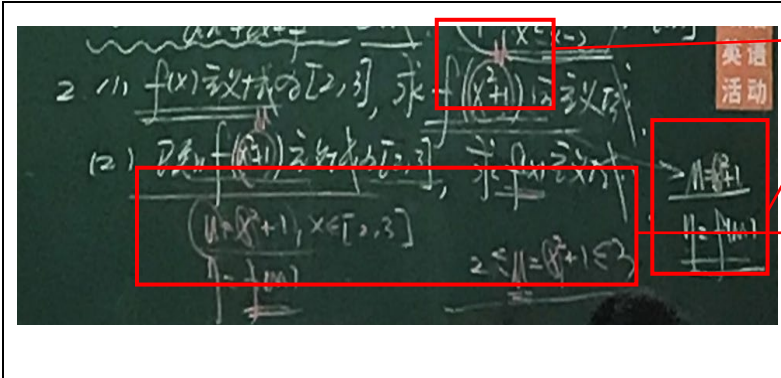
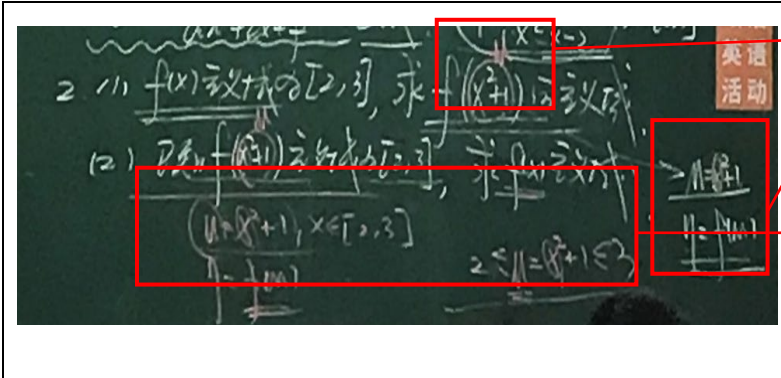
As we have mentioned at the beginning, these tasks do not just appear in one lesson. Since finding the domain of a composite function is one of the important teaching contents and Lin wants to ensure that every student understands it, Lin designed a review in the next day after this class, see table 5.2.6. Obviously, these tasks are almost the same as the tasks that Lin designed in table 5.2.4 in front of example 1. Shown in table 5.2.7, Lin emphasizes the understanding of composite functions, and writes the inner and outer functions of this composite function in detail as the red box in table 5.2.5.

Table 5.2.6 Review about the domain of composite function

Review: How to find the domain of composite function (source: designed by teacher)
The domain of $f(x)$ is $[2,3]$ , Find the domain of $f(x^2 + 1)$ .
The domain of $f(x^2 + 1)$ is $[2,3]$ , Find the domain of $f(x)$ .



Table 5.2.7 Photos of Lin's blackboard writing of the review tasks

	$u = x^2 + 1$ $y = f(u)$
	$u = x^2 + 1$ $y = f(u)$ $x \in [2, 3]$ $2 \leq u = x^2 + 1 \leq 3$

In general, this series of tasks are designed to allow students to master strategies through Pudian and OPMC that created by Lin in his documentation work. All of OPMC created here are designed by varying non-essential features so as to ensure students repeatedly use the same strategy to solve the tasks (MPOS). In addition, when designing the variation task (see the task called “variation” in table 5.2.4), Lin also established a connection between knowledge (sets and function) through OPMC, which achieved two goals with one stone.

### 5.2.2.3 Episode 3: OPMS for mastering problem-solving strategies

This task (shown in table 5.2.8) is proposed in the first and second lesson of the representation of function. The teacher explained it twice in class because the composition of piecewise functions is important knowledge that often appears in the examinations, but is difficult for students to understand. The aim of this task is to apply the strategy to solve the problem of a composite function. This task was directly selected by the teacher from the school-based teaching book, and had no context. This episode lasted 13min55 in total in class, 6min40 for the first time and 7min15 for the second.

Examining the potential variation of example 1 based on school-based teaching book:

- Representational variation is embedded in the task: The representations of function of these tasks are symbolic representations. In the process of solving this task, there are treatments in the symbolic writing: from an algebraic register to algebraic register, and to a numerical one.

Table 5.2.8 Task of composite function

<p>Example 4 (Source: School-based teaching book, p.71)</p> <p>Given function <math>f(x)=\begin{cases} x+2, &amp; x \geq 2 \\ x^2, &amp; 0 \leq x &lt; 2 \end{cases}</math> defined on <math>[0, +\infty)</math>, if <math>f(f(k)) = \frac{17}{4}</math>, find the value of the real number <math>k</math>.</p>	
<p>Strategy 1: Solve the composite function from internal to external</p> <p>When <math>x \geq 2</math>, <math>f(x) \geq 4</math></p> <p>When <math>0 \leq x &lt; 2</math>, <math>0 \leq f(x) &lt; 4</math></p> <p>As <math>f(f(k)) = \frac{17}{4}</math>, so <math>f(k) &gt; 2</math></p> <p><math>f(k) + 2 = \frac{17}{4}</math>, <math>f(k) = \frac{9}{4}</math></p> <p>So <math>k^2 = \frac{9}{4}</math>, <math>0 \leq k &lt; 2</math>, <math>k = \frac{3}{2}</math></p>	
<p>Strategy 2: Solve the composite function from external to internal (symbolic)</p> $f(k) = t$ <p>If <math>t \geq 2</math>, <math>f(t) = t + 2 = \frac{17}{4}</math>, <math>t = \frac{9}{4}</math></p> <p>If <math>0 \leq t &lt; 2</math>, <math>k^2 = \frac{17}{4}</math>, <math>t = \frac{\sqrt{17}}{2} &gt; 2</math> Contradiction</p> <p>So <math>t = \frac{9}{4}</math>, i.e <math>f(k) = \frac{9}{4}</math>, <math>k = \frac{3}{2}</math></p>	
<p>Strategy 3: Solve the composite function from external to internal (symbolic-graphic)</p> $f(k) = t$ <p><math>f(t) = \frac{17}{4}</math>, use the graph, we know <math>t = \frac{9}{4}</math></p> <p>So <math>f(k) = \frac{9}{4}</math>, use the graph, we know <math>k = \frac{3}{2}</math></p>	



Regarding actual variation, Lin retains the task but providing three ways of solutions, which extended the variation. We examine the new variations emerged through teacher's documentation work:

- New Procedural variation (OPMS) emerged: The school-based teaching book only proposed the first strategy of solution, which is most common and straight way. Lin adds strategy 2 and 3 in his working sheet. Both strategy 2 and 3 solve the composite function from external to internal function, but use different ways to solve the equations. Strategy 2 analyzes the equation piecewise while strategy 3

uses the graph of  $f(x)$ . Lin planned to introduce different strategies in order from cumbersome to simple so that students can feel the incredible effect of solving problems with graphs, and the importance of mastering the symbolic-graphic combination method in problem-solving.

- The idea of Pudian is embedded in the arrangement of these three strategies. Lin explains the superiority of the symbolic-graphic combination method by comparing different strategies so as to further encourage students to use this method, paving the way (Pudian) for using this strategy to solve more complicated tasks in the future.
- Representational variation emerged: Strategy 3 introduced the graphical representation of function and the conversions between symbolic writing and cartesian graph.

As for practical variations, Lin retained the potential and actual variations. For the first time, Lin gave the students 1min20 to think, and then used 5min20 to introduce the first and second strategies. Finally, due to the limitation of time, he only briefly mentioned that graph can also be used to solve the task; For the second time, Lin reviewed the two strategies taught in the previous class, and then used graphs to solve the task. The whole process lasted 7min15. When he showed strategy 1 on the blackboard, he commented that this strategy involved classification discussions more than once with a lot of calculation. The cumbersome method of strategy 1 prompted us to find a simpler method, so Lin naturally introduced strategy 2 and strategy 3. At the end, Lin emphasized the importance of graphs in the classroom teaching. As far as this task is concerned, one graph is used twice in strategy 3: for solving both the external and internal function, which reflects the convenience of problem-solving by graphs.

According to the teacher, he arranged this task twice in class because this type of tasks often appears in the exam and is difficult for students to understand. He pointed out the important position of the graph of a function in high school learning:

*“In fact, the real mathematics is a more rigorous concept in our advanced mathematics. For example, continuity, which is guided and defined by the idea of limit. Well, in our high school, it is impossible to do such a detailed description. The knowledge of students is not enough. We may talk about it at a relatively superficial level, mainly by combining graphs. The graph here is a rough graph of a function. And sometimes, we don’t even care about the concavity or convexity of a function at all. Concavity and convexity must be combined with the second derivative of a function, as the first derivative function is about monotonic, and the second derivative of a function is to observe its concavity and convexity. So combined with the graph, students*

*will understand the function very well. How to divide a piecewise function? Studying the features of the graph, then students can have a deep impression on this function at once."*

*"In fact, I found that it is important for the high school students to have such an idea of graph. Because the idea of function for students is too empty, not concrete, and the understanding of the graphs can be of great help to him/her."*

In fact, Lin designed a Pudian at the beginning of this class: He introduced several common piecewise functions in detail (see table 5.2.9). There are two reasons for this design: One is for the teaching goal of this class, i.e., present different representations of a function and pave the way for understanding the piecewise function in example 4; And the other is to pave the way for students' further study in the future: provide multi-angle understanding of piecewise function through presenting different forms; Link the knowledge of the extreme value function, piecewise function, and inequality, etc. For example, Lin combines piecewise functions with inequalities and introduced the symbol of extreme value and simple inequality scaling. Lin pointed out that he will pay attention to overall considerations in teaching "Because we have been teaching for so many years, I know where I might use it later, and I prefer mentioning it a little bit now, give students a step-stone, or give them a little impression. It would be easier for students to accept the knowledge later."

As Lin chose three representative piecewise functions to illustrate: piecewise function consisting of a linear function and a quadratic function, a floor function, and an extreme value function, there is also Pudian designed by Lin in the process of introducing these three kinds of piecewise functions. He first presented the simplest form of piecewise function with linear and quadratic function, and he then take an example of a floor function, whose value is actually divided into countless pieces, different from the usual functions who have limited sections. Furthermore, he introduced the extreme value function, whose requirement is even higher for students as its piecewise form is not directly given, but is to be obtained by solving inequalities. Lin explained that Pudian should also be designed step by step, not just in one lesson. We should not instill too much content at once, and adjust it according to the students' needs. During the interview, Lin pointed out that we can also design some piecewise functions with context, such as one trapezoid, a straight line or a right-angled trapezoid in the process of moving to the right, the length of the area to the right is segmented. At this time, the function is limited to 2~3 pieces.

Table 5.2.9 Example of long period Pudian

The process of introduce piecewise function:

Piecewise function

$$(1) \quad f(x) = \begin{cases} x^2, & 0 \leq x < 2 \\ x + 2, & x \geq 2 \end{cases}$$

(2) Floor function  $f(x) = [x]$  the greatest integer less than or equal to  $x$ .

For example,  $[0.5] = 0$ ,  $[1.2] = 1$ ,  $[-0.5] = -1$

$$f(x) = [x] = \begin{cases} -1, & -1 \leq x < 0 \\ 0, & 0 \leq x < 1 \\ 1, & 1 \leq x < 2 \end{cases}$$

(3) Take the maximum value  $M = \max\{a, b\} = \begin{cases} a & (a \geq b) \\ b & (a < b) \end{cases}$

Combined with absolute value, equivalently,  $\frac{1}{2}[a + b + |a - b|] = \begin{cases} a & (a \geq b) \\ b & (a < b) \end{cases} = \max\{a, b\}$

Take the minimum value  $m = \min\{a, b\} = \begin{cases} b & (a \geq b) \\ a & (a < b) \end{cases}$

Combined with absolute value, equivalently,

$$\frac{1}{2}[a + b - |a - b|] = \begin{cases} b & (a \geq b) \\ a & (a < b) \end{cases} = \min\{a, b\}$$

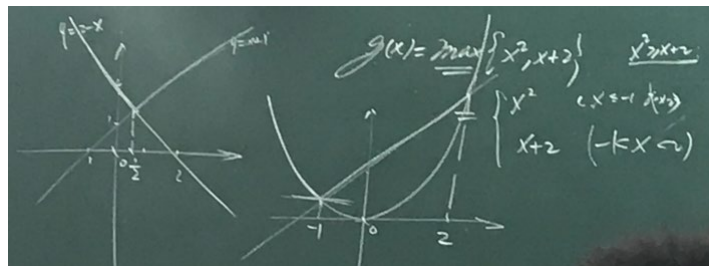
Further, the inequality scales

$$\max\{a, b\} \geq \frac{a+b}{2} \geq \min\{a, b\} \quad \max\{a, b\} \geq \sqrt{ab} \geq \min\{a, b\} \quad (a, b \geq 0)$$

**From the graph point of view**

$$f(x) = \max\{2 - x, x + 1\} = \begin{cases} 2 - x & x \leq \frac{1}{2} \\ x + 1 & x > \frac{1}{2} \end{cases}$$

$$g(x) = \max\{x^2, x + 2\}$$



If we change the function to  $f(x) = \min\{2 - x, x + 1\}$ ,  
 $g(x) = \min\{x^2, x + 2\}$

Conclusion (by teacher): Usually the maximum value function we will take the minimum value; the minimum value function, we will take the maximum value.

We present the evolution of variation in episode 3 in figure 5.2.11. Lin makes full use of the potential variation of the task itself and provides a variety of problem-solving strategies so that students can have a more comprehensive understanding of the problem-solving strategies and understand the advantages and disadvantages of different methods through comparison. In the process of problem-solving, it also strengthened the thought of combining symbolic and graphic representation for students, which brings the representational variation and Pudian.

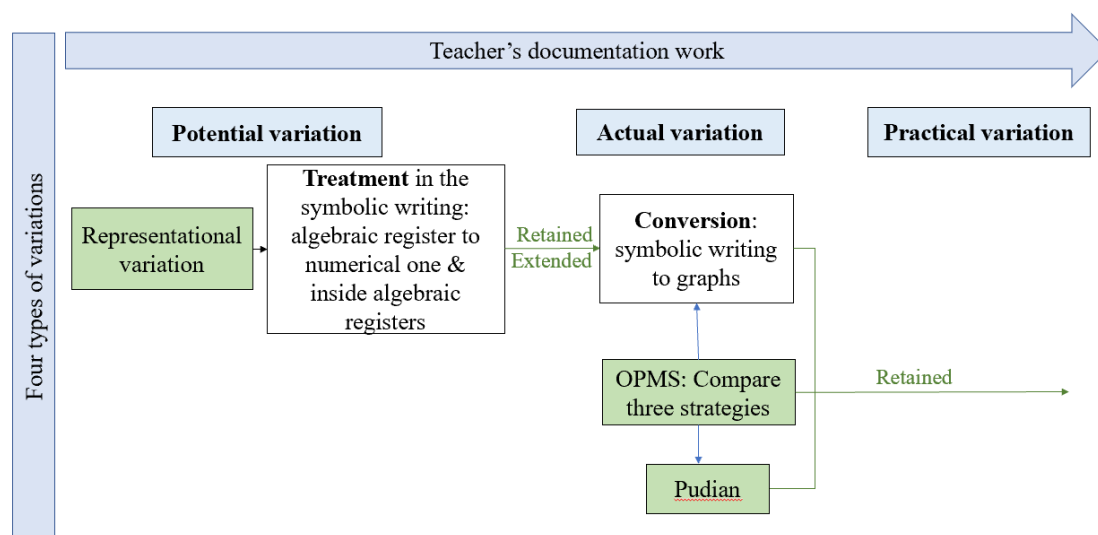


Figure 5.2.11 Schematic representation of variation path in episode 3

In Lin's reflection after class, he indicates that there are some possible variations of example 4 that can be designed for the following teaching. For example, we can vary the  $f(f(k)) = \frac{17}{4}$  to  $f(f(k)) \geq \frac{17}{4}$ , etc. making connections between composite function, composite equation, and the composite inequality.

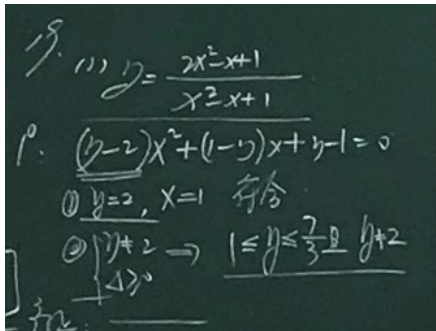
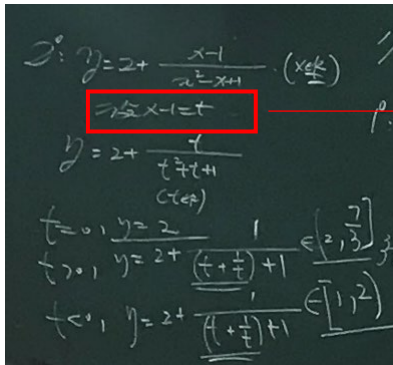
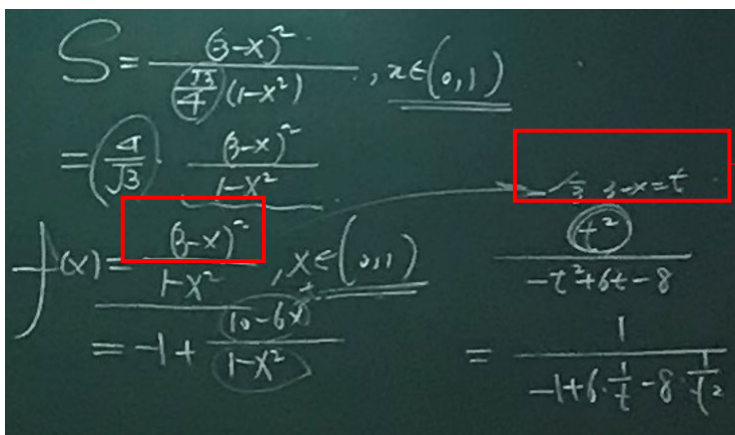
In general, this task is designed to allow students to master strategies through OPMS that created by Lin in his documentation work. All of OPMS created here are carefully designed, arranged coherent step by step, and serving the purpose of teaching, i.e., using symbolic-graphic combination method to solve the task of composite function.

#### 5.2.2.4 Episode 4 OPMS for building core associations and mastering problem-solving strategies

These tasks (shown in table 5.2.10) are proposed in two exercises explanation lessons. The aim of these tasks is to apply the strategy to solve the problem of fractional functions. These tasks are selected by Lin from the mid-term examination and test in class, and had no context. In this episode, our main concern is the way that Lin establishes a connection for the students between the two tasks that appear in isolation. This episode lasted 8min50 in class, of which 4 min50 is to explain the strategy of task

19, and 4 min is to establish a connection with task 25 and review the strategies to solve fractional functions.

Table 5.2.10 Tasks of solving fractional function by separating constant terms

<p>Task 19 Find the range of function <math>y = \frac{2x^2-x+1}{x^2-x+1}</math> (Source: mid-term examination)</p> <p>[Solution] (from Lin's blackboard writing)</p>	
<p>Strategy1: using discriminant</p> 	<p>[Strategy 2] separating constant terms</p>  <p style="color: red; text-align: right;">substitution</p>
<p>Task 15 (Source: test in class, bonus problem)</p> <p>Cut an equilateral triangle sheet with side length 1 along a straight line parallel to the base side into two pieces, one of which is trapezoidal, mark <math>S = \frac{(\text{perimeter of the trapezoid})^2}{\text{Area of trapezoid}}</math>, find the minimum value of <math>S</math>.</p> <p>[Solution] (from Lin's blackboard writing): separating constant terms and using substitution</p>  <p style="color: red; text-align: right;">substitution</p>	

Examining the potential variation of example 1 based on school-based teaching book:

- Representational variation is embedded in the task: The representations of function of these tasks are symbolic representations. In the process of solving this task, there are treatments inside the symbolic writing: from an algebraic register to algebraic register.

Regarding actual variation, Lin creates new variations in the process of establishing core associations between the two tasks. We examine the new variations emerged through teacher's documentation work:

- OPMS emerged: Lin designed two strategies to solve task 19, see table 5.2.10.
- Pudian emerged: The Pudian here is divided into two steps, the first step is to establish the association with task 15, and the further is to review the three methods of finding the range of a function mentioned in the previous lesson (the second lesson of the concept of function).

When Lin explained the task 19 (see table 5.2.10) in an exercise explanation lesson, he asked students to review the task 15 (see table 5.2.10) and compare the similarity and differences between them. As we have mentioned in section 5.2.1, this kind of Pudian is closely related to fundamental problems, OPMC and MPOS. In this episode, Lin established the associations between these two tasks, regarding task 15 as the fundamental problem for finding the range of a function formed as  $y = \frac{ax+b}{cx+d}$ , making use of the method of separating constant terms. There are OPMC between task 19 and task 15 and we can use the same method to solve them (MPOS).

On the other hand, the difference between Task 15 and Task 19 offers Lin opportunity to drive students back to the previous lesson to review the three methods (using discriminant, separating constant terms and method of substitution) of finding the range of a function and their applicable situations.

As for practical variations, Lin retains the potential and actual variations, no new variations emerged. Lin emphasized the importance of fractional functions, especially quadratic fractional functions in high school mathematical learning. He summarized the use of three strategies: the default domain (i.e.,  $x \in R$ ) uses the discriminant method, the given domain of  $x$  uses the separation constant method, and the flexible use of the substitution method.

We present the evolution of variation in episode 4 in figure 5.2.12. Lin creates core associations between different tasks so as to review the problem-solving strategies. Under this kind of teaching, the tasks are not single, but form a system, which not only helps students to master the strategy, but also helps them discover the connections between the tasks, and build their task systems based on the knowledge and mathematical methods (strategies).



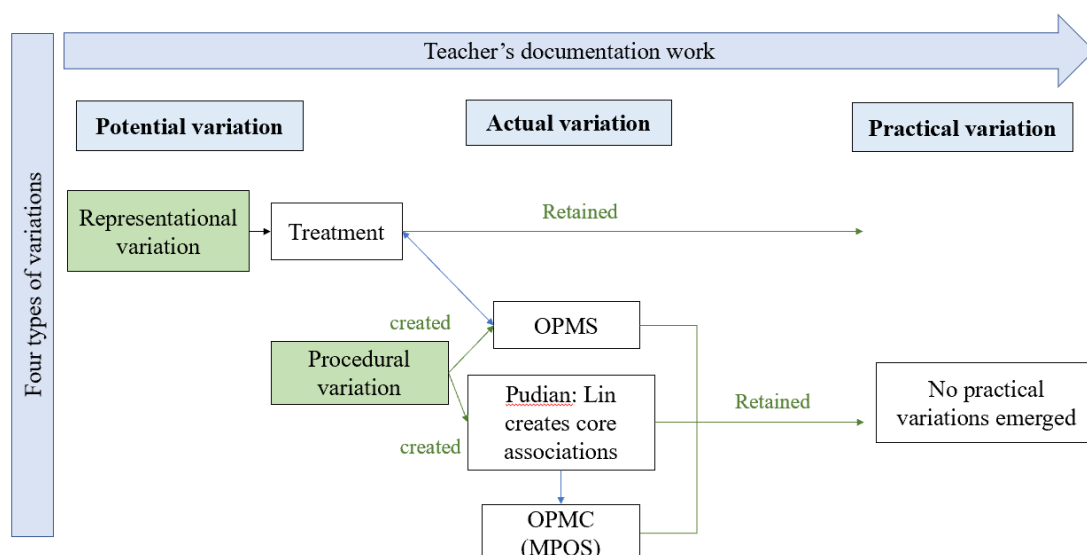


Figure 5.2.12 Schematic representation of variation path in episode 4

In general, this episode is designed to allow students to establish core associations and master strategies through OPMS and Pudians that created by Lin in his documentation work. These different strategies are full of flexible use of treatment in the representation.

#### 5.2.2.5 Episode 5: Conceptual and representational variation for providing multi-angle understanding

This episode is about the merging of monotonic intervals, which involves two series of tasks in two lessons, first lesson as an introduction and second lesson as a review. The teaching of first series of tasks lasts 4 minutes and the second lasts 2 minutes. We focus on the first lesson and the changes in the tasks and explanations in the review lesson and the first one.

Table 5.2.11 shows the blackboard writing in the first lesson. It was presented after the explanation of the definition of monotonicity and aims to help students understand that monotonicity emphasizes the local properties of a function and the properties of monotonic intervals, pointing out the condition of monotonic interval merging.

The creation of the two graphs in table 5.2.11 can be traced back to the textbook. When the textbook presents the definition of monotonicity of a function, it shows the graphs and set up a “thinking” column after, see table 5.2.12.

Examining the potential variation in the textbook:

- Representational variation is embedded in “thinking”: Two representations are shown here: natural languages and symbolic writings.
- Conceptual variation is embedded in “thinking”: Teachers’ guidebook emphasizes that the monotonicity of a function is a local property of a function, which is a

concept that students tend to not fully understand and require to make a distinction. It is common for a function to “increase” in some intervals within its domain and “decrease” in other intervals. “Monotonous increasing” and “increasing function”, “monotonic decreasing” and “decreasing function” need to be distinguished. “Thinking” here is intended to guide students to realize that the monotonicity of a function is for a certain interval in the domain. A function being monotonous in a certain interval does not mean that the function is monotonous in the entire domain. The examples asked to be given here are the standard variation, non-standard variation, and non-concept variation of the concept of the monotonicity of a function.

Table 5.2.11 Lin’s blackboard writing in the first lesson (source: designed by Lin)

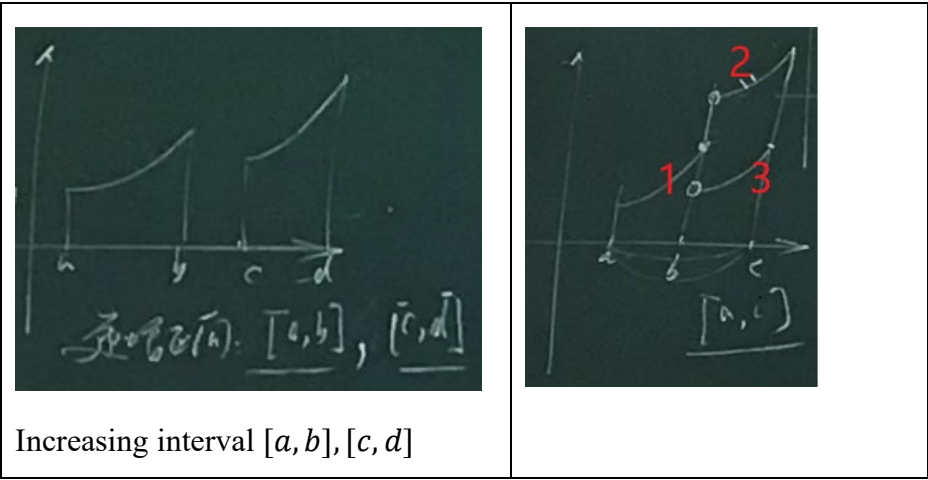


Table 5.2.12 Distinction of the concept of monotonicity of a function in PEP textbook (p77)

思考

(1) 设  $A$  是区间  $D$  上某些自变量的值组成的集合, 而且  $\forall x_1, x_2 \in A$ , 当  $x_1 < x_2$  时, 都有  $f(x_1) < f(x_2)$ , 我们能说函数  $f(x)$  在区间  $D$  上单调递增吗? 你能举例说明吗?

(2) 函数的单调性是对定义域内某个区间而言的, 你能举出在整个定义域内是单调递增的函数例子吗? 你能举出在定义域内的某些区间上单调递增但在另一些区间上单调递减的函数例子吗?

[Translation] Thinking

(1) Let  $A$  be the set of values of some independent variables on the interval  $D$ , and  $\forall x_1, x_2 \in A$ , when  $x_1 < x_2$ , there are  $f(x_1) < f(x_2)$ , can we say that the function  $f(x)$  increases monotonically in the interval  $D$ ? Can you give an example to explain?

(2) The monotonicity of a function refers to a certain interval in the domain. Can you give examples of functions that are monotonically increasing in the entire domain? Can you give examples of functions that monotonically increase in some intervals in the domain but decrease monotonically in other intervals?

Regarding actual variation, the two graphs designed by Lin organically combined the

“thinking” questions with graphs of monotonicity of function in the textbook. New variation emerged through teacher’s documentation work:

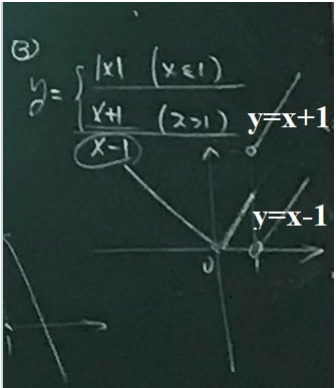
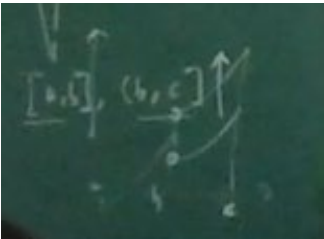
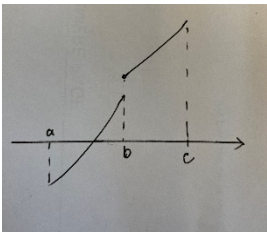
- Representational variation is retained and extended: Lin chose to use rough graphs instead of natural language or symbolic writing to distinguish concepts, allowing students to understand concepts more intuitively. The conversions from natural language (or symbolic writing) to rough graph is completed by Lin through his documentation work and rough graphs are shown to the students as the product of Lin’s documentation work. In this episode, the rough graph, as a tool to help students understand concepts, is a transitional auxiliary representation (Duval, 2017).
- Conceptual variation is extended: Different from letting students give examples in the textbook, Lin directly proposed two graphs to let students judge, which is less difficult and more intuitive for students. The graph on the left (table 5.2.11) implies non-concept variation. Lin organized it to illustrate that even if the two intervals are both increasing, they may not necessarily be merged, breaking the students’ inherent cognition. The non-concept variation he chose here is based on the students’ mistakes according to his experiences. In the graph on the right, the combination of monotonic interval 1, 2 is non-standard variation, while the combination of 1, 3 is non-concept variation.

As for the practical variation, Lin retained the actual variation through his documentation work and new practical variations emerged in the process of teacher-student question-and-answer.

- Representational variation is retained and extended: Conversions from graph to natural language, and to symbolic writings: As Lin proposed the graph on the left, he asked students: “There are two monotonic intervals on the graph, which are respectively.....” “How to write monotonic intervals?” These two questions proposed by Lin guide students to make conversions from the graphic representation to natural language and symbolic writing.
- Various OPMC hidden in the rough graph are revealed: Lin explained the graph on the right through a dynamic process. Lin asked students to think the possibility of merging two monotonic intervals into one monotonic interval. He showed students the possibility of merging with the monotonic interval 1 during the movement of the monotonic interval from 2 to 3. In this dynamic movement, there are actually countless monotonic intervals on  $[c, d]$ , and 2 and 3 are just shown on the graph as two special representatives.
- OPMC in the Representational variation further promotes conceptual variation as the dynamic explanation provides various non-standard and non-concept variations.

When reviewing the merging of monotonic intervals in the second class, Lin designed a new task with the combination of the knowledge of piecewise function and absolute value function, see table 5.2.13. In class, Lin guides students to think about the requirements of merging two interval after he drew the graphs of function  $y = \begin{cases} |x| & (x \leq 1) \\ x+1 & (x > 1) \end{cases}$ . Then he naturally leads students to go back to the basic graph drawn in lesson 1 (table 5.2.11) to review, see the second photo in table 5.2.13. At last, he changes  $x+1$  to  $x-1$  in the function, i.e.,  $y = \begin{cases} |x| & (x \leq 1) \\ x-1 & (x > 1) \end{cases}$ , where the two intervals do not meet the conditions of merging.

Table 5.2.13 Lin's blackboard writing in the second lesson (source: designed by Lin)

<p>Find the monotonic interval of a function</p> 	<p>Find the monotonic interval of the function</p> $y = \begin{cases}  x  & (x \leq 1) \\ x+1 & (x > 1) \end{cases}$ <p>then Lin change <math>x+1</math> to <math>x-1</math>, i.e.,</p> $y = \begin{cases}  x  & (x \leq 1) \\ x-1 & (x > 1) \end{cases}$
<p>Go back to basic graph drawn in the first lesson</p> 	<p><math>[a, b] \uparrow, [b, c] \uparrow</math></p> 

In this review part, Lin does not blindly repeat the original teaching content, but cleverly creates new tasks and flexibly manipulates OPMC to naturally review and apply the previous knowledge.

We present the evolution of variation in episode 5 in figure 5.2.13. Lin used graphs to illustrate the properties of monotonic intervals, making them more concrete and easier for students to understand. As shown in figure 5.2.13, representational variation has gone through two processes: One is that Lin made conversions from natural language/symbolic writings to graphs during his documentation work, and the other is in the classroom where Lin guided the students to make conversions back to the natural

language and symbolic writings by asking questions in class. In this process, the teacher created a transitional auxiliary representation of rough graph to help students understand. As far as conceptual variation is concerned, Lin have designed non-standard and non-concept variation based on the mathematical concept, and with the help of representational variation, the conceptual variation has been further extended to reach the teaching goal.

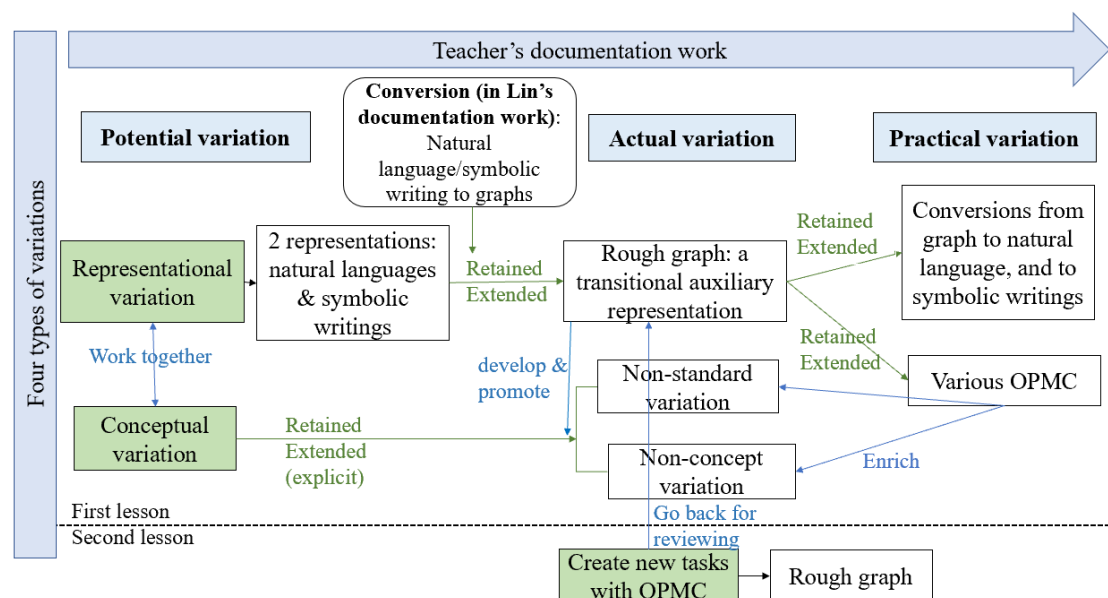


Figure 5.2.13 Schematic representation of variation path in episode 5

In general, this series of tasks are designed to allow students to have a multi-angle understanding of the concept of monotonicity of a function. Lin creates the conceptual and representational variation to achieve this teaching goal. He also used OPMC to create new tasks for students to review.

### 5.2.2.6 Episode 6 Pudian and OPMC for providing multi-angle understanding and mastering problem-solving strategies

This episode is about the monotonicity of the tick type function<sup>103</sup> and streamer type function<sup>104</sup>, which happens three times in three lessons. We summarize and regard these

<sup>103</sup> In mathematics, tick type function, also known as the Nike function in China, is represented as  $f(x) = ax + \frac{b}{x}$ , where  $ab \geq 0$ . The domain of the function is  $(-\infty, 0) \cup (0, +\infty)$ , and the asymptote are the y axis and  $y = ax$ . When  $a \geq 0, b \geq 0$ , In the first quadrant, the turning point is  $(\sqrt{\frac{b}{a}}, 2\sqrt{ab})$ . The graph of the function in the first quadrant is like Nike's brand logo, hence also named Nike function in China.

<sup>104</sup> In mathematics, streamer type function is represented as  $f(x) = ax - \frac{b}{x}$ , where  $ab \geq 0$ . The domain of the function is  $(-\infty, 0) \cup (0, +\infty)$ , and the asymptote are the y axis and  $y = ax$ . Intersection points with the x-axis are  $(\sqrt{\frac{b}{a}}, 0)$  and  $(-\sqrt{\frac{b}{a}}, 0)$ .

lessons as three stages (see table 5.2.14): introduction, in-depth, and review. This episode lasted 40min43 in class in total, of which 13min30 in the first lesson to give an introduction of tick function, 23min13 in the second lesson to go deeper, and 4min in the third lesson to review and generalize. Most of these tasks are selected from the school-based teaching book (see table 5.2.15, which can be originally traced back to the textbook) by Lin and he reorganized these tasks to serve his teaching.

Table 5.2.14 Lin's explanation and blackboard writing in class of episode 6

### Lesson 1 Introduction

#### Example 4 (Source: school-based teaching book)

Study the monotonicity of  $f(x) = x + \frac{1}{x}$

[Solution]

Lin explains first "It's a little earlier to study this function. The domain of this function is  $\{x|x \neq 0\}$ , and it is an odd function (I will explain this later). The graph of the odd function is... (Students replied together: the graph of an odd function has rotational symmetry with respect to the origin). We only need to study half of the odd function or even function. If we study the right or left side of the y-axis, the symmetry of the graph will come out. Therefore, we only need to study the monotonicity on  $(0, +\infty)$ , symmetry about the origin, and the Monotonical graph on  $x < 0$  will also know. During his explanation, he writes down the domain of  $f(x)$  ( $\{x|x \neq 0\}$ ), and part of the domain  $(0, +\infty)$  we need to study, see the yellow box in the blackboard writing below.

Then he writes down the solutions based on the definition of monotonicity of a function.

Handwritten blackboard notes for the monotonicity of  $f(x) = x + \frac{1}{x}$ . The notes include the function definition, domain, and a detailed proof using the definition of monotonicity. A red box highlights the function definition, and a yellow box highlights the domain and the interval  $(0, +\infty)$ .

Handwritten notes on the blackboard:

例4. 研究  $f(x) = x + \frac{1}{x}$  的单调性.

定义域:  $\{x|x \neq 0\}$

研究  $f(x)$  在  $(0, +\infty)$  上的单调性.

设  $x_1, x_2 \in (0, +\infty)$ , 且  $x_1 < x_2$ .

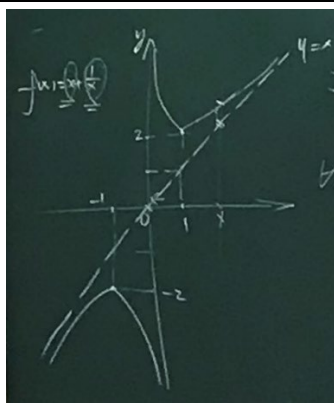
则  $f(x_1) - f(x_2) = x_1 + \frac{1}{x_1} - x_2 - \frac{1}{x_2} = (x_1 - x_2) + \frac{x_2 - x_1}{x_1 x_2} = (x_1 - x_2) \left(1 - \frac{1}{x_1 x_2}\right) = (x_1 - x_2) \frac{x_1 x_2 - 1}{x_1 x_2}$

①  $0 < x_1 < x_2 < 1$ ,  $x_1 x_2 < 1$ ,  $f(x_1) > f(x_2)$

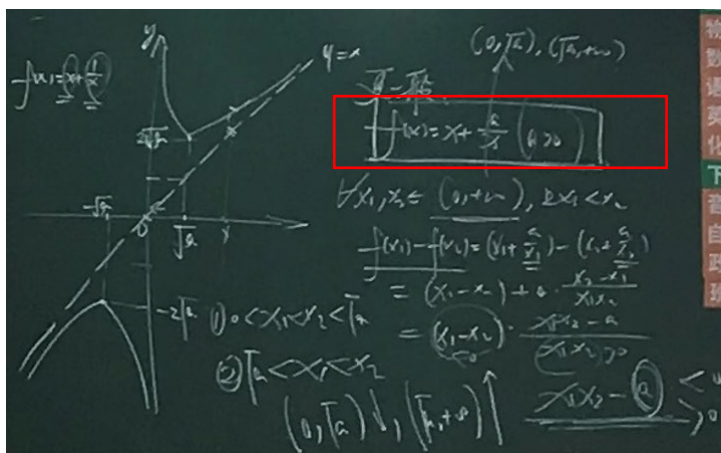
②  $x_2 > x_1 > 1$ ,  $x_1 x_2 > 1$ ,  $f(x_1) < f(x_2)$

③  $x_1 < 1 < x_2$ ,  $x_1 x_2 > 1$ ,  $f(x_1) < f(x_2)$

Present the graph of  $f(x) = x + \frac{1}{x}$



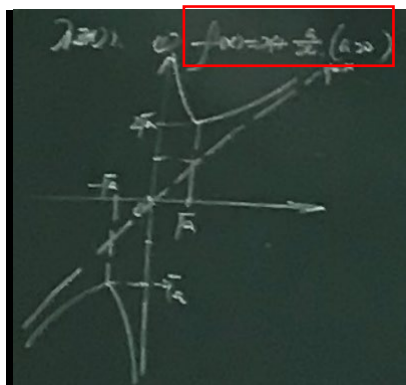
More generally, study the monotonicity of  $f(x) = x + \frac{a}{x}$  ( $a > 0$ ) and draw the graph



In the end, Lin emphasizes that  $f(x) = x + \frac{1}{x}$  is a special case of  $f(x) = x + \frac{a}{x}$  ( $a > 0$ )

## Lesson 2 review and go further

**Example 1 (1)** Study the monotonicity of  $f(x) = x + \frac{a}{x}$  ( $a > 0$ ) and draw the graph of  $f(x)$ . (Source: school-based teaching book)

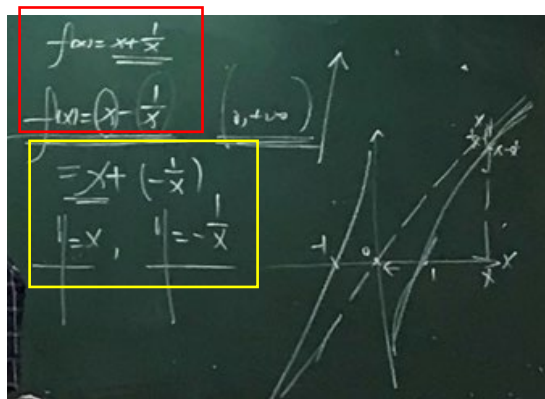


**Thinking:** Study the monotonicity of function  $f(x) = x - \frac{1}{x}$ . (Source: school-based teaching book)

Lin leads students to review the function  $(x) = x + \frac{1}{x}$ , which is the simplest tick function,



and use the same method of studying  $f(x) = x + \frac{1}{x}$  to study  $f(x) = x - \frac{1}{x}$  (the domain, an odd function).



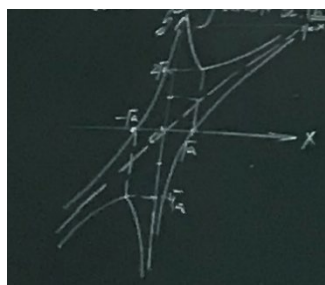
[Solution]

Strategy 1: Use the definition of monotonicity

Strategy 2: regard  $f(x)$  as the addition of two functions:  $y = x$  and  $y = -\frac{1}{x}$  (see the yellow box in the photo above), these two functions are both increasing on  $(0, +\infty)$ . Therefore,  $f(x) = x - \frac{1}{x}$  is monotonically increasing on  $(0, +\infty)$ .

Then, Lin draws the graph of  $f(x) = x - \frac{1}{x}$ , as the graph looks like a streamer, we call it streamer function. Lin leads students to write this function in a general form:

**Example 1(2)** Study the monotonicity of  $f(x) = x - \frac{a}{x}$  ( $a > 0$ ) and draw the graph of  $f(x)$ . (Source: school-based teaching book)



**Example 2** (Source: Lin's digital task repository)

(1)  $f(x) = x + \frac{a}{x}$  increases monotonically in the interval  $[2, +\infty)$ , find the range of  $a$ .

(2)  $f(x) = x^2 + \frac{a}{x}$  increases monotonically in the interval  $[2, +\infty)$ , find the range of  $a$ .

[Solution]

(1) Strategy 1: From the perspective of known functions, use familiar known functions to classify and discuss (see the yellow box below):  $a > 0$ , tick function, so  $\sqrt{a} \leq 2$ , i.e.,  $0 < a \leq 4$ ;  $a = 0$ ,  $f(x) = x$ , linear function, increases monotonically;  $a < 0$ , streamer function, increases monotonically.



Strategy 2: From the perspective of definition

$\therefore 0 < a \leq \sqrt{a^2} \leq 2 \sqrt{a} \leq 4$   
 $\text{② } a = 0 \text{ 时 } f(x) = x^2$   
 $\text{③ } a < 0 \text{ 时 } f(x) = x^2 + \frac{a}{x}$

$\therefore f(x_1) - f(x_2) = (x_1^2 + \frac{a}{x_1}) - (x_2^2 + \frac{a}{x_2}) = (x_1^2 - x_2^2) + a(\frac{1}{x_1} - \frac{1}{x_2})$   
 $= (x_1 - x_2)(x_1 + x_2) - a \frac{x_2 - x_1}{x_1 x_2} = (x_1 - x_2)(x_1 + x_2 - \frac{a}{x_1 x_2})$   
 $\therefore x_1 x_2 - a > 0$

$x_1 x_2 - a > 0$  This inequality is satisfied

(2) Strategy 1: From the perspective of known functions

Cannot tell the monotonicity of the function directly when  $a > 0$

Strategy 2: From the perspective of definition

$\therefore f(x_1) - f(x_2) = (x_1^2 + \frac{a}{x_1}) - (x_2^2 + \frac{a}{x_2}) = (x_1^2 - x_2^2) + a(\frac{1}{x_1} - \frac{1}{x_2})$   
 $= (x_1 - x_2)(x_1 + x_2) - a \frac{x_2 - x_1}{x_1 x_2} = (x_1 - x_2)(x_1 + x_2 - \frac{a}{x_1 x_2})$   
 $\therefore x_1 x_2 - a > 0$

$$\forall 2 \leq x_1 < x_2$$

$\therefore f(x_1) - f(x_2) = (x_1^2 + \frac{a}{x_1}) - (x_2^2 + \frac{a}{x_2}) = (x_1^2 - x_2^2) + a(\frac{1}{x_1} - \frac{1}{x_2})$   
 $= (x_1 - x_2)(x_1 + x_2) - a \frac{x_2 - x_1}{x_1 x_2} = (x_1 - x_2)(x_1 + x_2 - \frac{a}{x_1 x_2})$   
 $\therefore x_1 x_2 - a > 0$

$$(x_1 + x_2)x_1 x_2 - a > 0$$

### Lesson 3 Review and generalization

For tick function, use the definition to tell the monotonicity;

For streamer function, besides the use of definition, can also use observation (see the purple box below).

Some functions are not standard tick function, we call them

Tick type function. For example,

$$y = 2x + \frac{1}{x} = 2\left(x + \frac{1}{2}\right)$$

“\*2” here means the ordinate of the graph is generated twice of the original graph ( $y=(x + \frac{1}{2})$ ), the abscissa remains unchanged, and the ordinate does not change the monotonic interval when the ordinate stretches.

More generally,

Tick type function  $y = ax + \frac{b}{x} (a > 0, b > 0)$

Streamer type function  $y = ax - \frac{b}{x} (a > 0, b > 0)$

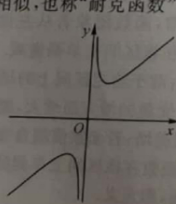
In fact, the arrangements of the teaching of tick function can be traced back to the example 4 and its variations in school-based teaching book, see table 5.2.15. This series of tasks start from the study of the monotonicity of the most basic tick function  $f(x) = x + \frac{1}{x}$ , go further to the general tick function  $f(x) = x + \frac{a}{x}$ , the most basic streamer function  $f(x) = x - \frac{1}{x}$ , and finally to the general streamer function  $f(x) = x - \frac{a}{x}$ .

Examining the potential variation in series of tasks in school-based teaching book:

- Representational variation is embedded in the tasks: The representations of function of these tasks are symbolic representations. In the process of solving this task, there are treatments inside the symbolic writing: from an algebraic register to algebraic register, and also the conversions from symbolic writings to cartesian graphs.
- Conceptual variation is embedded in the series of tasks: Mainly to show the standard variation of tick and streamer functions, and then introduce from special case to general case.

- Pudian is embedded in the series of tasks: There are mainly two aspects of Pudian here: (1) From special to general: Regardless of studying tick function or streamer function, the book selects the most basic and simplest special cases for discussion at first and then comes to the general situation; (2) from tick function to streamer function by changing the arithmetic symbols: imitate the method of studying tick function, from special to general, to study streamer function.
- OPMC is embedded in the series of tasks: Advance the learning process by changing the form of the formula, from  $f(x) = x + \frac{1}{x}$  to  $f(x) = x + \frac{a}{x}$ ,  $f(x) = x - \frac{1}{x}$ , and finally to  $f(x) = x - \frac{a}{x}$ .

Table 5.2.15 series of tasks in the school-based teaching book

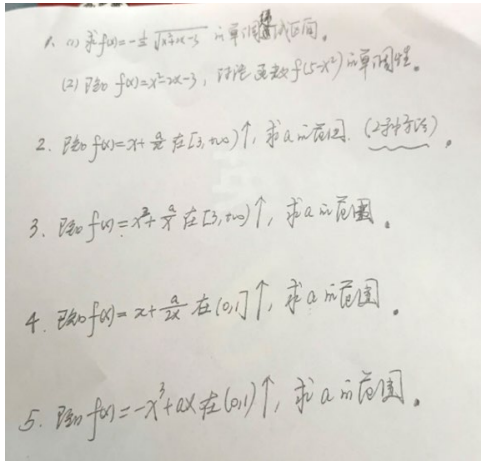
<p>例 4 讨论函数 <math>y = x + \frac{1}{x}</math> 的单调性.</p> <p>解: 函数的定义域为 <math>\{x   x \neq 0\}</math>, 所以在 <math>(-\infty, 0)</math> 和 <math>(0, +\infty)</math> 两个区间上分别讨论.</p> <p><math>\forall x_1, x_2 \in (0, +\infty)</math> 且 <math>x_1 &lt; x_2</math>,</p> <p>则 <math>f(x_2) - f(x_1) = x_2 + \frac{1}{x_2} - x_1 - \frac{1}{x_1}</math></p> $= (x_2 - x_1) + \frac{x_1 - x_2}{x_1 x_2} = \frac{(x_2 - x_1)(x_1 x_2 - 1)}{x_1 x_2}.$ <p>要确定此式的正负只要确定 <math>x_1 x_2 - 1</math> 的正负, 即判断 <math>x_1 x_2</math> 大于 1 还是小于 1. 由于 <math>x_1, x_2</math> 的任意性, 需要将 <math>(0, +\infty)</math> 分为 <math>(0, 1)</math> 与 <math>(1, +\infty)</math> 来讨论 (这是本题的关键).</p> <p>(1) 当 <math>x_1, x_2 \in (0, 1)</math> 时, <math>x_1 x_2 - 1 &lt; 0</math>, 可得 <math>f(x_2) - f(x_1) &lt; 0</math>, 故 <math>f(x)</math> 在 <math>(0, 1)</math> 上为减函数.</p> <p>(2) 当 <math>x_1, x_2 \in (1, +\infty)</math> 时, <math>x_1 x_2 - 1 &gt; 0</math>, 可得 <math>f(x_2) - f(x_1) &gt; 0</math>, 故 <math>f(x)</math> 在 <math>(1, +\infty)</math> 上为增函数.</p> <p>同理可以判断, <math>f(x)</math> 在 <math>(-\infty, -1)</math> 上为增函数, 在 <math>(-1, 0)</math> 上为减函数.</p> <p>评注: 一般地, 函数 <math>y = x + \frac{a}{x} (a &gt; 0)</math> 在 <math>(-\infty, -\sqrt{a}]</math>, <math>(\sqrt{a}, +\infty)</math> 上是增函数, 在 <math>(0, \sqrt{a}]</math>, <math>(-\sqrt{a}, 0)</math> 上是减函数. 此函数在 <math>(0, +\infty)</math> 上的图象与“√”相似, 因此称这个函数为“打勾函数”, 如图. 因其形状与耐克的商标相似, 也称“耐克函数”.</p>  <p>【知能迁移】试讨论函数 <math>f(x) = x - \frac{1}{x}</math> 的定义域、值域、单调性, 并画出图象.</p> <p>【思维启迪】类比“打勾函数”进行研究. 因其图象类似于两条飘动的丝带, 有人称之为“飘带函数”.</p>	<p>[Translation] <b>Example 4</b> Discuss the monotonicity of function <math>f(x) = x + \frac{1}{x}</math></p> <p>Solution: the domain of function is <math>\{x   x \neq 0\}</math>, so we discuss separately on the two intervals of <math>(-\infty, 0)</math> and <math>(0, +\infty)</math>.</p> <p><math>\forall x_1, x_2 \in (0, +\infty)</math> and <math>x_1 &lt; x_2</math>, then</p> $f(x_2) - f(x_1) = x_2 + \frac{1}{x_2} - x_1 - \frac{1}{x_1}$ $= (x_2 - x_1) + \frac{x_1 - x_2}{x_1 x_2}$ $= \frac{(x_2 - x_1)(x_1 x_2 - 1)}{x_1 x_2}$ <p>To determine the positive or negative of this formula, just determine the positive or negative of <math>x_1 x_2 - 1</math>, that is, determine <math>x_1 x_2</math> is greater than 1 or less than 1. Due to the arbitrariness of <math>x_1, x_2</math>, it is necessary to divide <math>(0, +\infty)</math> into <math>(0, 1)</math> and <math>(1, +\infty)</math> to discuss (this is the key to this task)</p> <p><b>Knowledge transfer:</b> Discuss the domain, range, and monotonicity of function <math>f(x) = x - \frac{1}{x}</math>, and draw the graph of <math>f(x)</math>.</p> <p><b>Enlightenment:</b> Analogy “tick function” for studying, because its graph is similar to two floating ribbons, some people call it “streamer function”</p>
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Regarding actual variation, Lin retains the potential variations of representational and OPMC, makes Pudian more explicit, and creates OPMS in the process of establishing core associations between the two tasks. We examine the actual variations through teacher's documentation work:

- Representational variation retained and extended: (1) Treatments inside the symbolic writing extended: Adding the parameter  $a$  increases the complexity of treatments between algebraic registers, see example 1 and example 2 in lesson 2. Through these treatments the original problem transformed into the problem of inequation, see the green box in example 2; (2) Treatments inside the cartesian graphs emerged: Drawing the graphs of  $y = 2x + \frac{1}{x}$  and  $y = ax + \frac{b}{x} (a > 0, b > 0)$  based on the graph of  $y = x + \frac{a}{x}$  in lesson 3 involves the coordinate transformation.
- Conceptual variation retained and extended: As far as potential variation is more about showing standard variation, here Lin has added some non-standard and non-concept variation to promote his teaching. For example,  $f(x) = x^2 + \frac{a}{x}$  in example 2 is non-concept variation of tick function, and  $y = 2x + \frac{1}{x}$  in lesson 3 is non-standard variation of tick function, which promote Lin's further teaching of the generalization of tick type function.
- Pudian is extended and is more explicit: Lin divided the contents into 3 lessons to pave the way for students to understand. These three lessons are hierarchical and deepened step by step, from  $f(x) = x + \frac{1}{x}$  to  $f(x) = x + \frac{a}{x} (a > 0)$  and simulating this exploration process to study function  $f(x) = x - \frac{1}{x}$  and  $f(x) = x - \frac{a}{x} (a > 0)$ , and finally generalize them to  $f(x) = ax + \frac{b}{x} (a > 0, b > 0)$  and  $f(x) = ax - \frac{b}{x} (a > 0, b > 0)$ .
- New OPMC emerged: Besides the tasks Lin takes from school-based teaching book (see table 5.2.15), he added new tasks to deeper the knowledge by introducing a parameter and changing conditions, see example 2 in table 5.2.14. Moreover, Lin designed a students' homework sheet and sent it to students after lesson 2. This homework sheet is full of tasks designed by variation. For example, from task 2 to task 5 are all tasks generated by varying the number or types of function of example 2, see table 5.2.16.

- New OPMS emerged: Lin retains the strategy used in the school-based teaching book, and proposes a new strategy by using the properties of known functions, see example 2 in table 5.2.14.

Table 5.2.16 Students' homework sheet designed by Lin

	<p>Translation:</p> <p>1(1) Find the monotonically decreasing interval of <math>f(x) = -\frac{1}{2}\sqrt{x^2 + 2x - 3}</math>;</p> <p>(2) <math>f(x) = x^2 + 2x - 3</math>, study the monotonicity of function <math>f(5 - x^2)</math>.</p> <p>2. <math>f(x) = x + \frac{a}{x}</math> increases monotonically in the interval <math>[3, +\infty)</math>, find the range of <math>a</math>. (use two methods)</p> <p>3. <math>f(x) = x^2 + \frac{a}{x}</math> increases monotonically in the interval <math>[3, +\infty)</math>, find the range of <math>a</math>.</p> <p>4. <math>f(x) = x + \frac{a}{2x}</math> increases monotonically in the interval <math>(0, 1]</math>, find the range of <math>a</math>.</p> <p>5. <math>f(x) = -x^3 + ax</math> increases monotonically in the interval <math>(0, 1)</math>, find the range of <math>a</math>.</p>
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As for practical variations, Lin retains the potential and actual variations, no new variations emerged. Lin leads students to learn step by step. What we are interested in is that although the new solution proposed by Lin (see strategy 1 in example 2) implies an understanding of the two functions of  $y = x$  and  $y = \frac{1}{x}$ , neither in the school-based teaching book nor in Lin's teaching design, takes profit of the introduction of the composition of a linear function and an inverse proportional function to study  $y = x + \frac{1}{x}$  from the properties of these two functions.

We present the evolution of variation in episode 6 in figure 5.2.14. Lin organizes several hierarchical tasks through OPMC to pave the way for students to understand. These tasks include the standard, non-standard, and bon-concept variations of tick or streamer type function. When solving these tasks, Lin proposes new strategies and there are treatments and conversions in the solution.

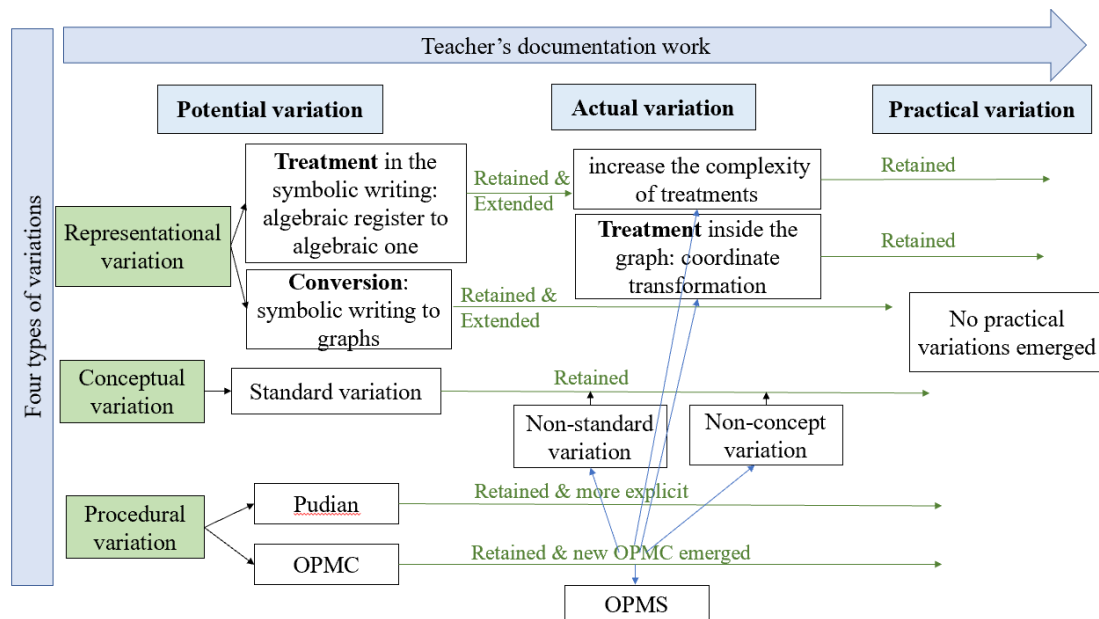


Figure 5.2.14 Schematic representation of variation path in episode 6

In general, this episode is designed to allow students to have multi-angle understanding of tick type function and streamer type function, and also master strategies to solve the tasks. The main variation used here are Pudian and OPMC designed by Lin through his documentation work.

### 5.2.2.7 Episode 7 OPMS for mastering problem-solving strategies and building core associations

This task (shown in table 5.2.17) was proposed in the first lesson of the extreme value of a function. The aim of example 1 here is to let students master the strategy to find the extreme value of a function. There are no contexts in these tasks. This episode lasted 8min15 in class.

Table 5.2.17 Series of tasks of finding the extreme value of a function

<p>Example 1 (source: designed by Lin)</p> <p>Find the extreme value of:</p> $y = \sqrt{x+1} + \sqrt{2-x}$ $y = \sqrt{x+1} - \sqrt{1-x}$
<p>For task (2), there are two strategies to solve this task.</p> <p>Strategy1: domain <math>x \in [-1,1]</math></p> $y^2 = 2 - 2\sqrt{1-x^2}$ $0 \leq y^2 \leq 2$

$$-\sqrt{2} \leq y \leq \sqrt{2}$$

Strategy 2: use the monotonicity

Because  $y = \sqrt{x+1}$  is an increasing function,  $y = \sqrt{1-x}$  is a decreasing function,  $y = \sqrt{x+1} - \sqrt{1-x}$  is an increasing function. So, when  $x = -1$ ,  $y_{\min} = -\sqrt{2}$ ; when  $x = 1$ ,  $y_{\max} = \sqrt{2}$ .

The creation of the tasks here in table 5.2.17 can be traced back to the school-based teaching book, where tasks only cover the quadratic functions ( $f(x) = x^2 - 2x$ ) and fractional functions ( $f(x) = \frac{1}{x}, x \in [1,3]$ ;  $f(x) = \frac{x+2}{x-1}, x \in [-1,0]$ ;  $f(x) = x + \frac{1}{x}, x \in [\frac{1}{2}, 2]$ ), see figure 5.2.15.

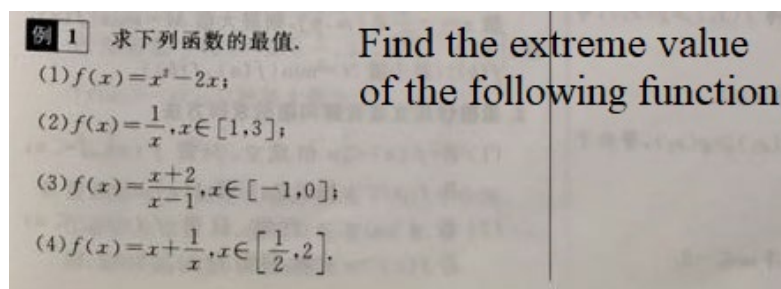


Figure 5.2.15 Original tasks from the school-based teaching book (p81)

Examining the potential variation of original tasks in the school-based teaching book:

- Representational variation is embedded in the task: in the process of solving this task, there are treatments in the symbolic writing and conversions between symbolic writing and cartesian graph.
- OPMC is among the tasks by varying the type of function. The aim of OPMC here is to let students master the methods of finding extreme value of a function.

Regarding actual variation, Lin added the two tasks with radical expression and extended the variation (table 5.2.17). We examine the new variations through his documentation work:

- Representational variation retained and extended: The OPMC tasks designed by the teacher can be regarded as a supplement to the tasks in the school-based teaching book, varying the type of function compared to the tasks in the school-based teaching book.
- OPMS of procedural variation emerged: The teacher prepared two strategies for task 2. one is through algebraic treatment; the other is using the monotonicity of a function. However, strategy 2 cannot be used in task 1, as  $y = \sqrt{x+1}$  is an increasing function,  $y = \sqrt{2-x}$  is a decreasing function, we cannot determine the



monocity of  $y = \sqrt{x+1} + \sqrt{2-x}$ . The teacher deliberately designed these two functions with similar forms but different properties to let students compare so as to master problem-solving strategies.

As for practical variation,

- Representational variation retained: The teacher drew the rough graph (see the red box in Figure 5.2.16) for both two tasks when presenting the first strategy in order to help students to understand the extreme value in a certain interval.

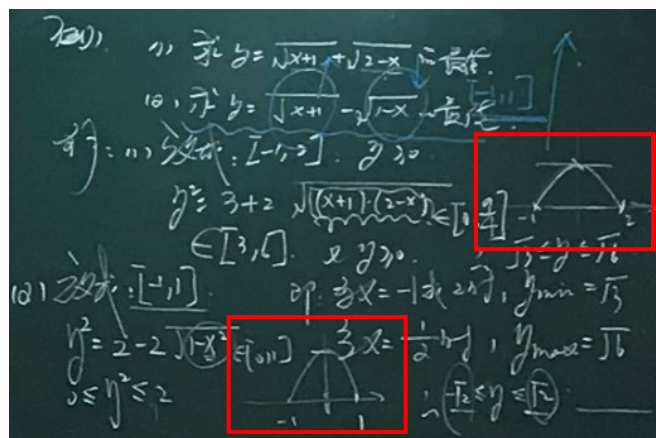


Figure 5.2.16 Lin's handwriting on the blackboard in the teaching of episode 5

- Differences between the two observed class: For class A (first class for this teaching content), the teacher reduced the actual variation as the comparison between the two tasks is omitted, and there is no attempt proposed by the teacher to use strategy 2 to solve task 1. This episode lasted 8min15 (4min15 for task 1 and 4min for task 2). For class B, the teacher retained the actual variations. When presented strategy 2, the teacher went back to task 1 to explain the inadaptability of strategy 2 to task 1. This episode lasted 9min 33 (5min40 for task 1 and 3min53 for task 2).

We present the evolution of variation in episode 7 in Figure 5.2.17. OPMS designed by the teacher in this episode is in line with the teaching goal: 1) To help students master different strategies of finding the extreme value of a function: By comparing different strategies used in different conditions, students can choose the most suitable strategy at the right time; 2) Building core associations with other knowledge: Strategy 1 shows close relation between extreme value and inequalities while strategy 2 is proposed to promote the establishment of connections between knowledge of extreme value and the monotonicity of a function.



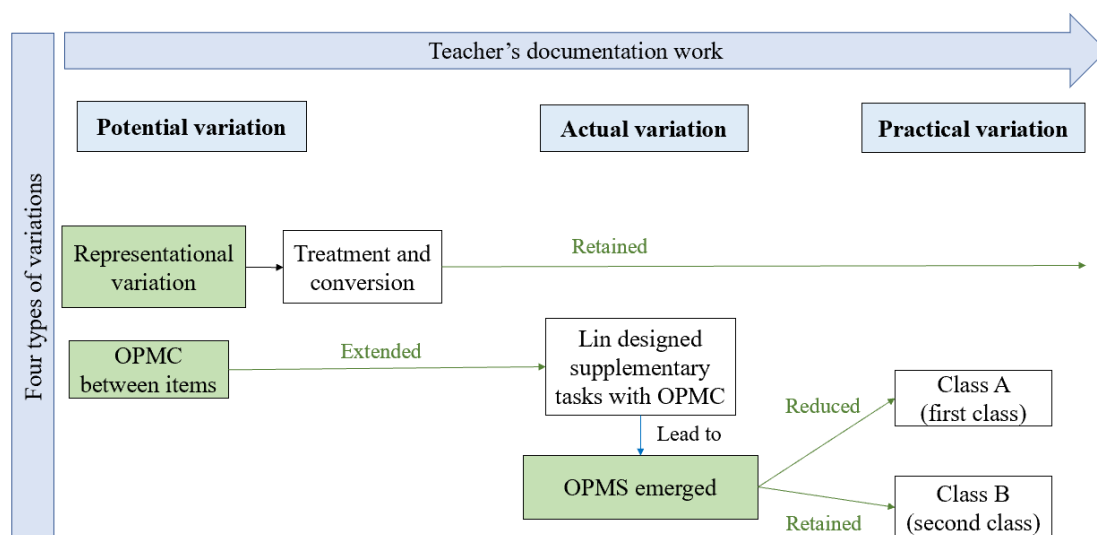


Figure 5.2.17 Schematic representation of variation path in episode 7

Moreover, by comparing the teaching of the two classes, we found that Lin will unconsciously improve their teaching in the process of teaching in the second class. This kind of variation hardly requires the special reflection of an experienced teacher, or we can say that reflection has been rooted in their behaviors and habits. They are monitoring their teaching in real time and making timely improvements.

### 5.2.2.8 Conclusion

Tasks and Series of tasks are both used in Lin's teaching (26 tasks and 27 series of tasks). We summarize the characteristics of Lin's daily classroom teaching through the lens of variation as follows:

- Most of time in daily teaching, Lin retains the variations already organized and embedded in the tasks. He is good at adapting the tasks to his teaching goals based on the original tasks by extending the variation. The most common variation he extends is OPMC. He is devoted to showing students as many types of functions as possible, such as radical, fractional, quadratic, absolute value function and so on. At the same time, new OPMC often leads to a chain reaction of new variation of other types, usually the change of representational variations and Pudian.
- Practical variations emerged in class mainly focus on the OPMS and conceptual variation. Constrained by the limitation of classroom teaching time, there are few variations of OPMS created by students in class. However, Lin shares students' different strategies in class by reviewing students' homework and examinations, which further compensates for the impact of time constraints on the occurrence of new variation. Moreover, Lin creates new conceptual variations in the interaction with students in class. He is good at creating non-concept variations to answer

students' misunderstanding of concept, to help them distinguish the concept.

- In Lin's teaching of function, he pays attention to the conversions between symbolic writing and graph. On the one hand, He believes that intuitive function graphs can help students understand relatively abstract concepts of function. On the other hand, he encourages students to take advantage of the graph of a function to help them simply the tasks of function.

### 5.3 Variation of tasks at the occasion of series of guiding activities

In this section, we analyze Lin's work of variation through a series of guiding activities between Lin and a preservice teacher Shi<sup>105</sup>. We present this series of guiding activities from two perspectives: Shi's resource system and her documentation work in lesson preparation (5.3.1); three guiding activities with different goals (5.3.2).

The aim of these activities is to help the preservice teacher to prepare a lesson for a specific classroom teaching on operation of logarithmic function. Through analyzing these activities, we hope that we can have a better understanding of Lin's teaching through variation and get some enlightenment on how Chinese teaching through variation is passed on between experienced and novice teachers.

The timeline of the guiding activities is presented in figure 5.3.1. We note several important moments on the timeline: Shi's lesson preparation before the guiding activities, three guiding activities (lesson interpretation<sup>106</sup>, demo lesson<sup>107</sup>, and discussions), as well as classroom teaching as the ultimate successful demonstration. At the beginning, it was not clearly stated that there will be three guiding activities. This is Lin to decide whether to schedule the next guiding activity based on each discussion. We will illustrate them one by one in the following sections.

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<sup>105</sup> See the information of preservice teacher Shi in section 3.1.3.

<sup>106</sup> Lesson interpretation (说课): Teacher's verbal presentation of a specific subject's teaching ideas and theoretical basis.

<sup>107</sup> Demo lesson (试讲): teaching activities in which the teacher simulates the situation in class and delivering lessons without students. see Figure 5.3.2

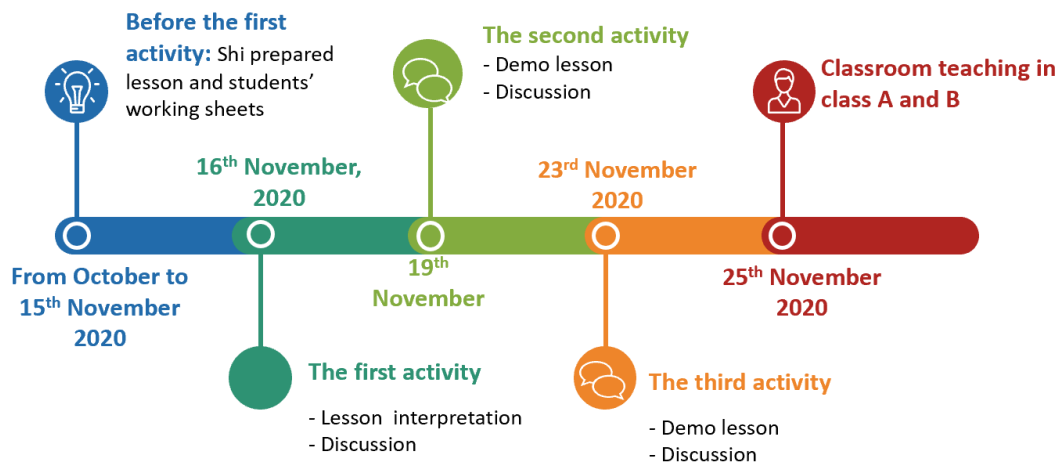


Figure 5.3.1 Timeline of the guiding activities

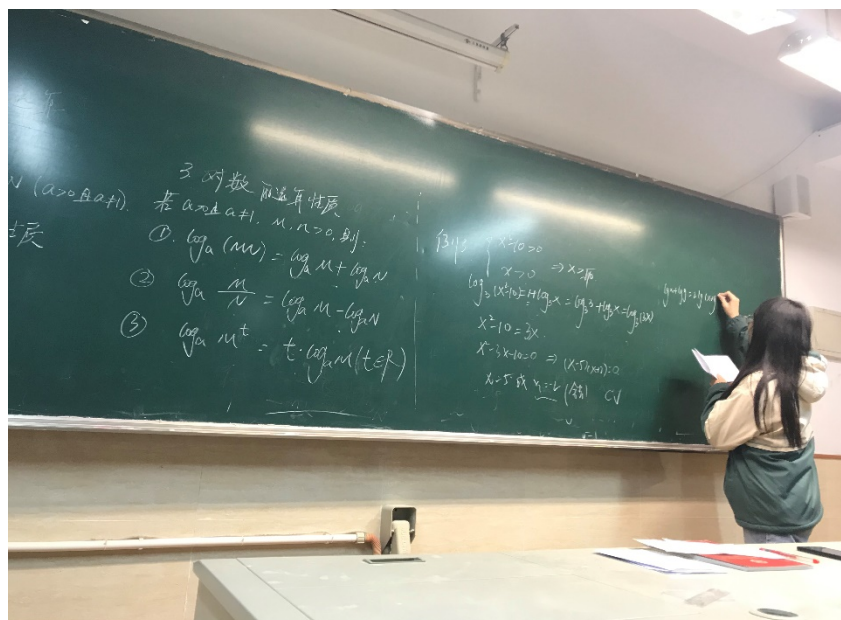


Figure 5.3.2 Shi's delivering her demo lesson

### 5.3.1 Shi's resource system and her documentation work in lesson preparation

In this section, we first tried to draw Shi's resource system and have a general picture of her documentation work based on her lesson preparation before the first guiding activity. Then we analyze the variations in her work.

#### 5.3.1.1 Shi's lesson preparation and her resource system

As a preservice teacher, Shi's resource system is completely different from Lin's. At that moment, Shi's resource system is relatively limited as Shi has not established her own documentation work mode with corresponding resources. We tried to find out the resource Shi used through her lesson preparation process before the first guiding

activity.

Figure 5.3.3 shows the whole process of Shi's lesson preparation with resources. At this step, there is no intervention of Lin. As a preservice teacher, Shi found that it was too difficult for her to write a lesson plan from zero, so she looked for resources that she could take profit of. There are three resources she used during this lesson preparation work: textbook, school-based teaching book, and the resources from QQ group<sup>108</sup>.

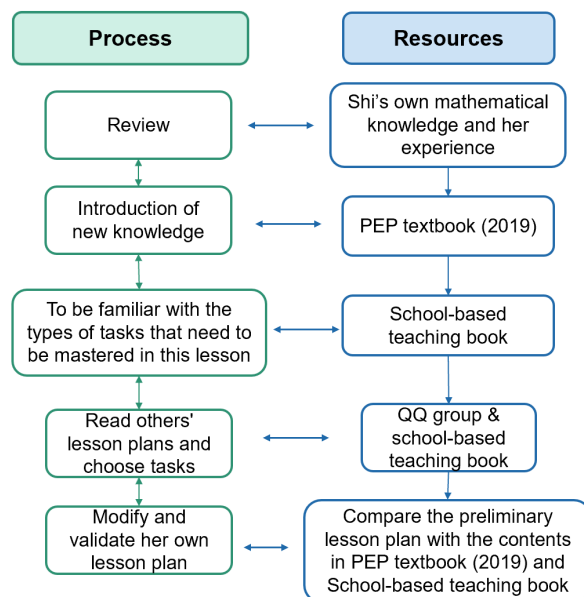


Figure 5.3.3 Shi's lesson preparation process and the resources she used

Shi's process of instrumentalization is very careful and spiral. Figure 5.3.3 shows five stages in Shi's lesson preparation process and that each step of Shi's work progresses is with two-way arrows, which means that in lesson preparation, every time the next step is performed, Shi will look back to consider whether the previous steps are appropriate and confirm it to ensure that the lesson plan covers the required teaching contents and the teaching process is consistent and coherent. Here, we explain the five stages one by one:

- (1) Review: At the beginning, Shi made preliminary thinking on the design of logarithmic operations based on her own knowledge and experience in mathematics. She hopes to combine the knowledge that the students have learned and establish a connection between the new and the old knowledge, so as to transition to the learning content of logarithmic operations more naturally. Therefore, she designed

<sup>108</sup> QQ group is a public platform for multi-person chat exchanges launched by Tencent. After creating a group, group owners can invite friends or people with common interests to chat in a group. In addition to chatting in the group, Tencent also provides group space services. In the group space, users can use group BBS, photo albums, shared files, group videos, etc. to communicate.

the review part, see table 5.3.1.

Table 5.3.1 Review


<p><b>复习回顾</b></p> <p>1.由对数的定义,可以得到对数与指数间的等价关系: 当 <math>a&gt;0</math>, 且 <math>a\neq 1</math> 时, <math>\boxed{\phantom{000}} \Leftrightarrow \boxed{\phantom{000}}</math></p> <p>2.指数幂的运算性质:</p> <p>对 <math>\forall a &gt; 0, r, s \in R</math>, 均有下列运算性质:</p> <p>(1) <math>a^m a^n = \underline{\hspace{2cm}}</math>;</p> <p>(2) <math>\frac{a^m}{a^n} = \underline{\hspace{2cm}}</math>;</p> <p>(3) <math>(a^m)^n = \underline{\hspace{2cm}}</math>.</p>	<p><b>Review</b></p> <p>1. From the definition of logarithm, the equivalence relationship between logarithm and exponent can be obtained:</p> <p>When <math>a&gt;0</math> and <math>a\neq 1</math>, <math>\boxed{\phantom{000}} \Leftrightarrow \boxed{\phantom{000}}</math></p> <p>2. The properties of the operation of exponential power:</p> <p>For <math>\forall a &gt; 0, r, s \in R</math>, all have the following operational properties:</p> <p><math>a^m a^n = \underline{\hspace{2cm}}</math>;</p> <p><math>\frac{a^m}{a^n} = \underline{\hspace{2cm}}</math>;</p> <p><math>(a^m)^n = \underline{\hspace{2cm}}</math>.</p>
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In the review, Shi designed to review the equivalence relationship between logarithm and exponent and three properties of the operation of exponential power. According to Shi, logarithmic operation is based on the concept of logarithm, so it is necessary to first review the concept of logarithm. And the most important thing in the concept of logarithm is the treatment between exponent and logarithm. As three operation properties of logarithm come from the definition and equivalence relations of logarithm, and combined with the three properties of real number exponential power, Shi designed to review the three properties of the exchange equivalence relationship and exponential power.

- (2) Introduction: Shi considered the design of the introduction and wondered whether students can deduce three operation properties of the logarithms by themselves through this equivalence relationship. This idea is derived from the “inquiry” designed in the textbook, using the properties of exponential power operation to obtain the corresponding properties of logarithmic operation. Shi thinks this “inquiry” activity make the introduction of logarithmic operation very natural. Moreover, the whole reasoning process is also shown in the textbook which is helpful for her design. Table 5.3.2 shows the original design of inquiry activity in the textbook and Shi’s redesign based on the textbook.

In addition, this design of inquiry activity also allowed Shi to determine the content of the review in the first phase to ensure the consistency and continuity of the lesson.

Table 5.3.2 Inquiry activity as introduction of the lesson

<p>PEP textbook (2019, pp124)</p> <div style="border: 1px solid #ccc; padding: 10px; margin: 10px 0;"> <p> <b>探究</b></p> <p>我们知道了对数与指数间的关系，能否利用指数幂运算性质得出相应的对数运算性质呢？</p> <p>设 <math>M=a^m, N=a^n</math>，        因为 <math>a^m a^n = a^{m+n}</math>，        所以 <math>MN = a^{m+n}</math>。</p> <p>根据对数与指数间的关系可得</p> <math display="block">\log_a M = m, \log_a N = n,</math> <math display="block">\log_a (MN) = m + n.</math> <p>这样，就得到了对数的一个运算性质：</p> <math display="block">\log_a (MN) = \log_a M + \log_a N.</math> <p>同样地，同学们可以仿照上述过程，由 <math>a^m \div a^n = a^{m-n}</math> 和 <math>(a^m)^n = a^{mn}</math>，自己推出对数运算的其他性质。</p> <p>于是，我们得到如下的对数运算性质。</p> <div style="border: 1px solid #00a0e3; padding: 5px; margin-top: 10px;"> <p>如果 <math>a &gt; 0</math>，且 <math>a \neq 1</math>，<math>M &gt; 0</math>，<math>N &gt; 0</math>，那么</p> <p>(1) <math>\log_a (MN) = \log_a M + \log_a N</math>；</p> <p>(2) <math>\log_a \frac{M}{N} = \log_a M - \log_a N</math>；</p> <p>(3) <math>\log_a M^n = n \log_a M</math> (<math>n \in \mathbf{R}</math>)。</p> </div> </div>	<p>[Inquiry]</p> <p>We know the relationship between logarithm and exponent, can we use the properties of exponential power operation to get the corresponding properties of logarithmic operation?</p> <p>Let <math>M = a^m, N = a^n</math>,</p> <p>Because <math>a^m a^n = a^{m+n}</math>,</p> <p>So <math>MN = a^{m+n}</math></p> <p>According to the relationship between logarithm and exponent, we can get</p> $\log_a M = m, \log_a N = n$ $\log_a (MN) = m + n$ <p>In this way, a property of logarithmic operation is obtained:</p> $\log_a (MN) = \log_a M + \log_a N$ <p>In the same way, students can imitate the above process, from</p> $a^m \div a^n = a^{m-n} \text{ and } (a^m)^n = a^{mn} \text{ to deduce other properties of logarithmic operations.}$ <p>Thus, we get the following properties of logarithmic operations</p> <p>If <math>a &gt; 0</math>, and <math>a \neq 1</math>, <math>M &gt; 0</math>, <math>N &gt; 0</math>, then</p> <p>(1) <math>\log_a (MN) = \log_a M + \log_a N</math>;</p> <p>(2) <math>\log_a \frac{M}{N} = \log_a M - \log_a N</math>;</p> <p>(3) <math>\log_a M^n = n \log_a M</math> (<math>n \in \mathbf{R}</math>).</p>
<p>Shi's redesign based on the textbook</p>	<p><b>Knowledge inquiry</b></p> <p>We know the relationship between logarithm and exponent, can we use the properties of exponential power operation to get the corresponding properties of logarithmic operation? <a href="#">How to deduce?</a></p> <p>If <math>a &gt; 0</math>, and <math>a \neq 1</math>, <math>M &gt; 0</math>, <math>N &gt; 0</math>, then</p>

<p><b>知识探究</b> <small>←←</small></p> <p>我们知道对数与指数间的关系，能否利用指数幂运算性质得出相应的对数运算性质？<u>如何推导</u>？ <small>←</small></p> <p>如果 <math>a &gt; 0</math>，且 <math>a \neq 1</math>，<math>M &gt; 0</math>，<math>N &gt; 0</math>，那么： <small>←</small></p> <p>(1) <math>\log_a(M \cdot N) =</math> _____ (语言表述： ) <small>←</small></p> <p>推导过程： <small>←</small></p> <p>推广形式： <small>←</small></p> <p>如果 <math>a &gt; 0</math>，且 <math>a \neq 1</math>，<math>N_i &gt; 0</math>，<math>i = 1, 2, \dots, n</math>，那么： <math>\log_a(N_1 \cdot N_2 \cdots N_n) =</math> _____</p> <p>特别地，若 <math>N_1 = N_2 = \cdots = N_n = M</math>，则有 _____</p> <p>(2) <math>\log_a \frac{M}{N} =</math> _____ (语言表述： ) <small>←</small></p> <p>推导过程： <small>←</small></p> <p>(3) <math>\log_a M^n =</math> _____ (<math>n \in \mathbf{R}</math>) (语言表述： ) <small>←</small></p> <p>推导过程： <small>←</small></p> <p>知识点归纳 对数运算性质 <small>←</small></p> <p>如果 <math>a &gt; 0</math>，且 <math>a \neq 1</math>，<math>M &gt; 0</math>，<math>N &gt; 0</math>，那么： <small>←</small></p> <p>(1) <math>\log_a(M \cdot N) = \log_a M + \log_a N</math>; <small>←</small></p> <p>(2) <math>\log_a \frac{M}{N} = \log_a M - \log_a N</math>; <small>←</small></p> <p>(3) <math>\log_a M^n = n \log_a M</math> (<math>n \in \mathbf{R}</math>). <small>←</small></p>	<p><math>\log_a(MN) =</math> _____</p> <p>(language expression: )</p> <p>The derivation process:</p> <p>Extension:</p> <p>If <math>a &gt; 0</math>， and <math>a \neq 1</math>， <math>N_i &gt; 0</math>， <math>i = 1, 2, \dots, n</math>， then</p> <p><math>\log_a(N_1 N_2 \cdots N_n) =</math> _____</p> <p>In particular, if <math>N_1 = N_2 = \cdots = N_n = M</math>， then _____.</p> <p><math>\log_a \frac{M}{N} =</math> _____</p> <p>(language expression: )</p> <p>The derivation process:</p> <p><math>\log_a M^n =</math> _____</p> <p>(language expression: )</p> <p>The derivation process:</p> <p><b>Generalization</b></p> <p>If <math>a &gt; 0</math>， and <math>a \neq 1</math>， <math>M &gt; 0</math>， <math>N &gt; 0</math>， then</p> <p>(1) <math>\log_a(MN) = \log_a M + \log_a N</math>;</p> <p>(2) <math>\log_a \frac{M}{N} = \log_a M - \log_a N</math>;</p> <p>(3) <math>\log_a M^n = n \log_a M</math> (<math>n \in \mathbf{R}</math>).</p>
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We represent Shi's redesign in blue, and we can find that Shi mainly problematizes the content displayed in the textbook, to make them more suitable for classroom teaching and more practical. In addition, she added language descriptions in each part, to train students' mathematical expressions, and extended the conclusion to  $N_n$ , hoping to broaden students' horizons.

- (3) To be familiar with the type of tasks in this lesson: according to Shi, she did all the tasks of this lesson in the school-based teaching book to know what types of tasks need to be mastered in this lesson.
- (4) Read others' lesson plans and choose tasks: Shi found the corresponding logarithmic operation lesson plan shared by others in a QQ group<sup>109</sup> about high

<sup>109</sup> More information on Tencent QQ on Wikipedia [https://en.wikipedia.org/wiki/Tencent\\_QQ](https://en.wikipedia.org/wiki/Tencent_QQ)

school knowledge. Then, after analyzing and comparing other people's lesson plans with the existing content she designed, she found that the whole teaching idea is the same, and her own lesson design is more detailed. In addition, other people's lesson plans did not set up inquiry activities as it is in the textbook, but directly gave three properties and then proved them.

Shi selected and arranged the examples in class from these lesson plans and school-based teaching book. She compared the tasks in these lesson plan with the tasks in the school-based teaching book and found that the types of tasks are very similar in these two books. According to Shi, considering that letting students do it on the students' working sheets may help her see clearly, she plans to print students' working sheets. Shi also selected 6 alternative tasks. On the one hand, they are for the following discussions with Lin, and on the other hand, they can be used when there is a surplus in class time.

- (5) Modify and validate lesson plan: This is the last step of Shi's lesson preparation. She compared her preliminary lesson plan with contents in the school-based teaching book and textbook to ensure that her lesson plan covered the necessary knowledge to be learnt and is consistent and coherent.

#### **5.3.1.2 Shi's documentation work through lesson preparation: full of the idea of variation**

Shi's entire lesson preparation is full of ideas of variations. We will analyze it from two aspects: the design of Pudian and the role of variation in the organization of the tasks. Because these two aspects are particularly prominent in Shi's lesson preparation and run through her entire documentation work.

##### **(1) Variation as a tool for organizing tasks**

We noticed an interesting phenomenon in Shi's lesson plan. 3 of the 4 examples (Example 1, Example 2 and Example 4) were arranged with a corresponding variation task. Although all of the tasks in Shi's lesson plan are selected from school-based teaching book or QQ group, the organization of the tasks is carried out by Shi herself. The example 4 and corresponding variations were taken directly from the school-based teaching book. Examples 1 and 2 were reorganized by Shi. According to Shi, she matches the tasks in the lesson plan (including classroom exercises, homework, etc.) according to the same/similar knowledge examined in the tasks, and then sets the more difficult ones into variation tasks.

Shi is organizing the tasks with an idea of variation. When preparing lessons, her understanding of variation is "two similar tasks, the more difficult task can be used as a variation of the less difficult one". We use tasks in the lesson plan with Shi's interpretation of her design intention to further explain, see table 5.3.3.



Table 5.3.3 Tasks in Shi's lesson plan and her interpretations

Task	Shi's interpretation of her design
<p><b>Example 1</b> Which of the following is correct ( ).</p> <p>A. <math>\log_2 3 \cdot \log_2 5 = \log_2 (3 \times 5)</math></p> <p>B. <math>\lg 3 + \lg 4 = \lg (3 + 4)</math></p> <p>C. <math>\log_2 \frac{x}{y} = \log_2 x - \log_2 y</math></p> <p>D. <math>\lg \sqrt[n]{m} = \frac{1}{n} \lg m (m &gt; 0, n &gt; 1, n \in \mathbf{N}^*)</math></p> <p><b>Variation 1</b> If <math>ab &gt; 0</math>, we have the following equations:</p> <p>① <math>\lg(ab) = \lg a + \lg b</math>;</p> <p>② <math>\lg \frac{a}{b} = \lg a - \lg b</math>;</p> <p>③ <math>\frac{1}{2} \lg \left(\frac{a}{b}\right)^2 = \lg \frac{a}{b}</math>;</p> <p>The sequence number of the equation that must be established is _____.</p>	<p>Example 1 is a simple judgment about the form of the property, but in Variation 1, Shi chose a task which gave an addition condition that the antilogarithm is greater than 0.</p> <p>According to Shi, variation 1 is an extension of Example 1: “<i>Although the condition that the antilogarithm is greater than 0 also appeared in the option C of example 1, but in variation 1, there is a condition of <math>ab &gt; 0</math>, which can be equivalent to <math>a &gt; 0, b &gt; 0</math>, or <math>a &lt; 0, b &lt; 0</math>. it is not limited to two antilogarithms greater than 0. At this time, it can be emphasized that there must be a antilogarithm greater than 0 to satisfy the key point of the establishment of the property. So, this is... I don't know if this is a variation.</i>”</p>
<p><b>Example 2</b> The solution of equation <math>\log_3(x^2 - 10) = 1 + \log_3 x</math> is ( ).</p> <p>A. -2 B. -2 or 5 C. 5 D. 3</p> <p><b>Variation 2</b> if <math>\lg x + \lg y = 2\lg(x - 2y)</math>, then <math>\frac{x}{y} =</math> _____.</p>	<p>Example 2 is a task of solving the equation. Shi pointed out that the knowledge of these two tasks is the same, and variation 2 requires more skills in calculation than in example 2. She said: “<i>Why I put variation 2 here? Because I think the mathematical method that this variation 2 may use is a bit more difficult than the original example 2, or it needs you more calculation skills</i>”, “<i>Here I think this may not be a variation. My main idea is to put a task of solving equations.</i>”</p>
<p><b>Example 3</b> Calculate the following formulas:</p> <p>(1) <math>(\lg 5)^2 + 2\lg 2 - (\lg 2)^2</math>;</p> <p>(2) <math>\log_5 \sqrt[3]{625}</math>;</p> <p>(3) <math>\log_2(32 \times 4^2)</math>;</p> <p>(4) <math>\log_5 35 - 2\log_5 \frac{7}{3} + \log_5 7 - \log_5 \frac{9}{5}</math>.</p>	<p>Example 3 is a calculation task. Shi did not design a variation task for it, because she thinks that the goal of this calculation task is to apply the properties, and no need to discuss the variation.</p>
<p><b>Example 4</b> Knowing that <math>x</math> satisfies the inequality <math>2(\log_2 x)^2 - 7\log_2 x + 3 \leq 0</math>, find the maximum and minimum values of the function</p>	<p>Shi believes that example 4 should be regarded as a variation. She explained: “<i>Example 4 was originally a function with an upward opening to find the extreme value. Variation 4 has been</i></p>

$f(x) = \log_2 \frac{x}{2} \cdot \log_2 \frac{x}{4}.$ <p><b>Variation 4</b></p> <p>Knowing that the equation <math>2x^2 + (\log_2 m)x + \log_2 \sqrt{m} = 0</math> about <math>x</math> has real roots, find the range of</p> $f(m) = \log_2 \frac{m}{2} \cdot \log_2 \frac{4}{m}.$	<p><i>transformed into a function with an opening downward, and to find the range. In variation 4, the condition can be transformed to <math>\Delta \geq 0</math>, i.e., a transformation between roots and <math>\Delta</math>. Moreover, in this process, it is not only a similar solution to the inequality, but also an application of the properties of logarithm operation.”</i></p> <p>As a variation task, variation 4 is more complex and difficult than example 4, which is in line with Shi’s design philosophy that more difficult tasks appear as variations.</p>
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It is not difficult to see from Shi’s interpretations that Shi’s understanding of variation is relatively vague. There are certain doubts from her about whether the designed tasks belong to the variation (see the words in green in table 5.3.3). In fact, Shi’s arrangement of tasks is full of ideas of variation. First, Shi uses the essential features (Gu et al., 2004) in the knowledge to determine the types of the four examples and find the corresponding tasks. Then she selects and organizes tasks with the same essential features but different non-essential features (Gu et al., 2004) together, and use the more difficult and more complex tasks as the variation tasks.

## (2) The design of Pudian

Pudian runs through Shi’s entire teaching design. It is mainly reflected in three parts: the design of knowledge introduction, the setting of exercises, and her process of instrumentalization.

As shown in table 5.3.3, Shi pays great attention to make a connection between students’ old and new knowledge, hoping to use old knowledge (the concept of logarithms, the properties of operations of exponents) to naturally introduce new knowledge (operations of logarithms). Here, old knowledge paves the way (Pudian) for learning new knowledge.

In the part of exercise setting, Shi tends to equip examples with corresponding variation tasks, which are more difficult than the original examples with the same teaching goals. Examples here can be regarded as Pudian for the corresponding variation task.

### 5.3.2 Three guiding activities with different goals

For this lesson, three guiding activities were arranged between teacher Lin and preservice teacher Shi (see figure 5.3.1). Each activity originated from a different goal. In this section, we will analyze the role of variation in each of these three guiding activities.

### 5.3.2.1 The first guiding activity: Lesson interpretation and confirm the overall idea of this lesson

The first activity was carried out through the Shi's lesson interpretation, Lin proposed amendments and the two discussed with each other according to the preliminary lesson plans and students' working sheets given by Shi. It lasted around 30 minutes. Table 5.3.4 shows the issues proposed during the first activity and Shi's corresponding revisions.

Table 5.3.4 Issues discussed during the first activity and Shi's revisions

Issues proposed during interactions	Shi's revision
<p>Lin's suggestions on the organization of students' working sheets: Only present tasks in students' working sheets, delete the previous learning objectives, knowledge review, and knowledge inquiry.</p> <p>Lin's reason: (1) The teacher will explain the content on the blackboard, and there is no need to print it out to the students. And students can focus more on the classroom through classroom listening and recording from writings on the blackboard. (2) Don't distribute working sheets with too many pages.</p>	<p>Reorganize students' working sheets. Only the tasks that need to be completed in class are left.</p>
<p>Lin's suggestions on the choice of the tasks:</p> <p>Lin proposes to add a challenge task for students to reflect (for students who are good at math)</p> <p>Lin proposes to extend example 1, add more items in it. He chose the following two from the alternative tasks proposed by Shi:</p> <p>Original task: <math>\frac{\log_2 9}{\log_2 3} = \underline{\hspace{2cm}}</math>. (Students are very easy to make mistakes)</p> <p>Lin revised it to be an item in example 1:</p> $\frac{\log_2 9}{\log_2 3} = \log_2 3$ <p>Use <math>\lg x</math>, <math>\lg y</math>, <math>\lg z</math> to represent the following:</p> <p>(1) <math>\lg(xyz)</math>; (2) <math>\lg \frac{xy^2}{z}</math>; (3) <math>\lg \frac{xy^3}{\sqrt{z}}</math>; (4) <math>\lg \frac{\sqrt{x}}{y^2 z}</math></p>	<p>Lin helped to find one in his digital task repository, and Shi arranged it as the last task of thinking in students' working sheets.</p> <p><b>[Thinking]</b> Calculate: <math>7^{\lg 20} \cdot \left(\frac{1}{2}\right)^{\lg 0.7}</math></p> <p>Shi agreed and accepted</p>
Discussions about the introduction of the properties of	Shi accepted Lin's suggestion and

<p>logarithmic operations. There are two ways: One is using inquiry activity (guide students to explore three properties as Shi designed in her lesson plan); and the other is using the method of proof (first show three properties and then prove it); The other is (for students to explore).</p> <p>In Lin's opinion, these two methods are essentially the same. He suggested to choose the way of proof as it is more direct and clearer for students to understand.</p>	<p>changed to the proof method</p>
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The first discussion focused on the overall design ideas of Shi's lesson plan. The main purpose was to confirm Shi's general idea of this class, so that Lin had a preliminary understanding of Shi's teaching design. Lin was basically satisfied with Shi's overall design, but there was no in-depth discussion on specific tasks and Shi's teaching in this activity. Lin proposed to have another discussion, in this discussion Shi should deliver a demo lesson and simulate a real classroom environment.

Conceptual variation is hidden in Lin's suggestions of adding a task in example 1. Because Lin is committed to providing students with a multi-angle understanding of

the concept, and the newly added task  $\frac{\log_2 9}{\log_2 3} = \log_2 3$  is a very typical non-concept

variation, and students are prone to making mistakes.

### 5.3.2.2 The second guiding activity: Delivering Demo Lesson and discussion about the details of the lesson

The second activity was carried out through Shi's demo lesson. This activity focused on the specific task selection, arrangement, and how the teacher clearly explained the tasks to the students. It lasted around one hour and a half.

Lin solved the tasks and recorded his thoughts on the students' working sheets during Shi's demo lesson, see figure 5.3.4. Words in red were written by Lin.

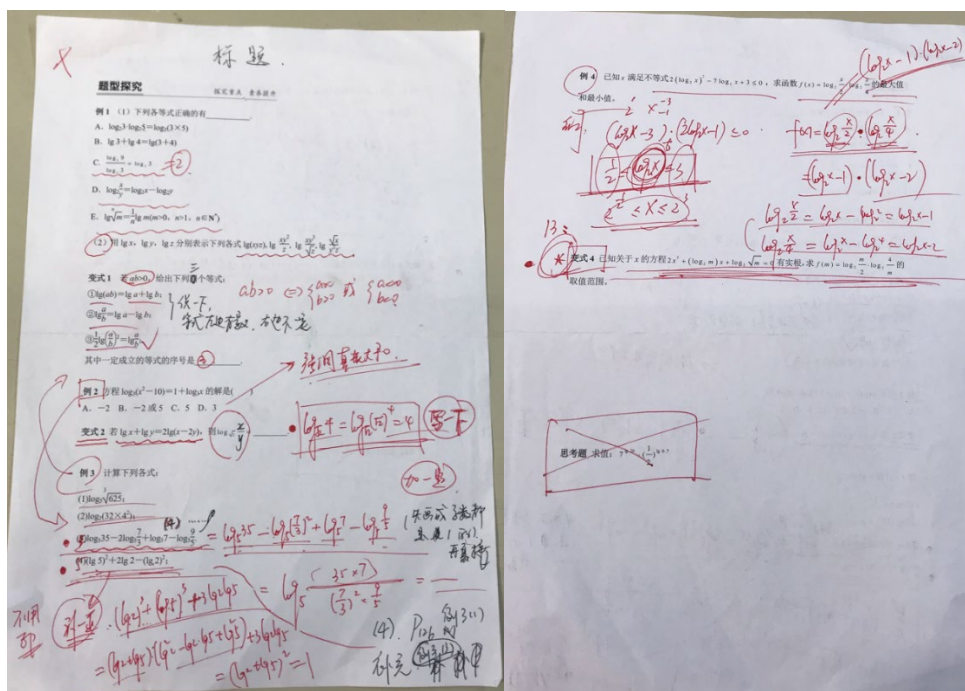
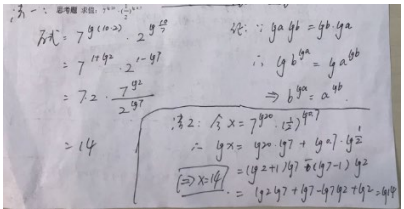


Figure 5.3.4 Lin's notes on students' working sheets during second activity

We organize the contents in figure 5.3.3, combining the dialogue between Lin and Shi to form Table 5.3.5, which shows the issues proposed during the second activity and Shi's corresponding revisions.

Table 5.3.5 Issues discussed during the second activity and Shi's revisions

Issues proposed during interactions	Shi's revision
<p>Lin's suggestion: The order of tasks: change the order of example 2 and example 3. After changing, example 1 is about the basic properties of logarithmic operation, example 2 is calculation application, example 3 is logarithmic calculation and equation, example 4 is a combination of logarithmic calculations, inequalities, and quadratic function (see the original tasks and order in table 5.3.3).</p>	<p>Shi agreed and revised the order.</p>
<p>Lin's suggestion:</p> <p>Different from the previous guiding activity through lesson interpretation, Shi's demo teaching was conducted with the time recorded. Lin found that the time was very tight by using the current lesson plan. Considering the time constraint of classroom teaching, Lin proposed two suggestions:</p> <p>Remove the thinking task</p> <p>Calculate: <math>7^{\lg 20} \cdot \left(\frac{1}{2}\right)^{\lg 0.7}</math></p>	<p>Shi deleted thinking task. However, she thinks it is a pity to remove it, because she has prepared two strategies for it, which she thinks represent different ways of thinking, and she is still working on finding other strategies. As a result, Shi retained this task as a supplementary task and did not show it on the students' working sheets.</p>

<p>Turn variation 4 into a flexible task. If there is not enough time for variation 4, it will be left as homework for students.</p>	 <p>Shi agreed.</p>
<p>Lin's suggestions on the teaching skills:</p> <p>Example 1(2) write down the solutions on the blackboard.</p> <p>Variation 1 Explain that the left side of the two equations 1, 2 is meaningful, but the right side is meaningless when <math>a, b &lt; 0</math>;</p> <p>Example 2: To emphasize the antilogarithm <math>&gt; 0</math></p> <p>The substitution of 1 should be explained and clearly written down: <math>\log_3(x^2 - 10) = 1 + \log_3 x = \log_3 3 + \log_3 x = \log_3 3x</math></p> <p>Variation 2: propose new strategies by using factorization.</p> <p>Need to write down <math>\log_{\sqrt{2}} 4 = \log_{\sqrt{2}} (\sqrt{2})^4 = 4</math>, to be clearer</p> <p>Example 3 (3): write down the process in detail</p> <p>Example 4: link <math>f(x) = \log_2 \frac{x}{2} \cdot \log_2 \frac{x}{4}</math> and <math>\log 2</math></p>	<p>Shi accepted</p>
<p>Lin came up with a new variation task after example 2 (4):</p> <p><math>(\lg 2)^3 + (\lg 5)^3 + 3 \lg 2 \lg 5</math></p> <p>He also suggested not put it in students' sheets, but show directly on the blackboard.</p>	<p>She accepted</p>

We can see from Lin's guidance to Shi that all his suggestions are for his teaching goals. Correspondingly, the variations that Lin proposed in his suggestions are also based on teaching goals. Lin's teaching design mainly considers the following aspects: (1) the logic of mathematics (change the order of tasks); (2) students' needs (set challenge task, ask Shi to be more detailed when explaining the tasks, ask Shi to write down the whole process of the solution); (3) highlight the key knowledge of this lesson ("repeat" the important points, for example the condition of antilogarithm  $> 0$  was emphasized several times in different tasks)

1 is to expand students' horizons and increase the difficulty of the problem through variation topics; 2 is to emphasize the key points

how to highlight, with the help of variation)

### 5.3.2.3 The third guiding activity: Final confirmation

The third activity was carried out through Shi's lesson interpretation. This is the last guiding activity before the classroom teaching. In fact, at that time Lin is quite satisfied with Shi's lesson plan after the second revision. This activity is only the final confirmation for classroom teaching. It lasted around one hour. Table 5.3.6 shows the issues proposed during the third activity and Shi's corresponding revisions.

Table 5.3.6 Issues discussed during the first activity and Shi's revisions

Issues proposed during interactions	Shi's revisions
Delete the task: $\frac{\log_2 9}{\log_2 3} = \underline{\hspace{2cm}}$	accepted
<p>Lin's suggestions on the explanation:</p> <p>After the proof, we need to explain the three operational properties of the logarithmic operation to prove the one-to-one correspondence with the exponent</p> <p>The teaching of Example 4, how to make a transition naturally from language. Focused on how to explain:</p> <p>Link <math>f(x) = \log_2 \frac{x}{2} \cdot \log_2 \frac{x}{4}</math> and <math>\log_2 x</math>. Because there is no logarithmic multiplication rule, it must be converted to <math>(\log_2 x - 1)(\log_2 x - 2)</math> Make a connection between the known and the unknown</p>	Shi find it is quite difficult for her to explain the connections clearly and logically, not bluntly read the strategies.
To add a summary, and to summarize, in this lesson we have learned...the transition from the 3 properties of exponents to the 3 operation properties of logarithms.....	accepted

This activity focused more on specific teaching methods and teaching techniques. It is very interesting that Lin's teaching thinking is also an iterative process. Here, Lin overturned his previous suggestion and deleted the question  $\frac{\log_2 9}{\log_2 3}$  (see table 5.3.4)

because he found that this question is duplicated with the following item  $\log_2 \frac{x}{y} = \log_2 x - \log_2 y$ .

### 5.3.2.4 Classroom teaching and reflection between class A and B

With good preparation, Shi's lesson in both classes went very smoothly. The feedback from the students is very good.

The lesson in two classes (class B and A) are separated by 1 hour. During this hour, Shi

and Lin reflected and communicated the classroom teaching in class B, and adjusted the teaching method in class A: (1) Slow down the rhythm of the class: According to the actual situation of class B, it is better to slow down the overall speaking rhythm and leave more time for students to do the tasks; (2) Shi hopes that students can actively participate in the discussion. Later, she felt that there was not enough interaction with the students in Class B, and did not pay enough attention to the students in the back row. So, she increased the interaction with the students in class A, and asked the students to explain their problem-solving ideas.

### **5.3.3 Conclusion: Chinese teaching through variation: from practice to theory, the efforts of generations**

The three activities are from macro to the micro, and the contents discussed is more and more detailed. In fact, each activity has a clear and different goal in Lin's own mind: The first time is the overall thinking, the second time is the specific task design, and the third time is the detail modification of the teaching method, which makes the teaching more logical and clearer.

As they are guiding activities, the main form is that Lin makes suggestions and Shi learns and revises. From these guiding activities, we can once again confirm the role of variation in Lin's teaching design mentioned in 5.1 and 5.2 through these guiding activities.

#### **Lin's "favorite" variation type**

In this guiding activity, the three most prominent variations are Pudian, conceptual variation and representational variation.

Lin attaches great importance to Pudian. When guiding Shi, he always emphasized that the connection between old and new knowledge should be clearly stated. The treatments of algebraic registers in symbolic writings have brought different problem-solving strategies. When explaining concepts, he strives to show students various types of variations (standard, non-standard, and non-concept variation) to help them understand.

#### **The differences of teaching through variation between experienced and preservice teacher: constraints of his/her own resource system**

Variation requires a strong resource system as support. In Shi's teaching design, we can see that she used variation consciously (although she herself is not sure whether it is a variation). However, due to the constraints of her resource system, she has this awareness, but she does not have enough resources to support her to achieve it. For example, Shi said it is hard for her to find a proper challenge task and asked Lin for help. Lin quickly found one in his digital task repository.

Besides the resources in Lin's resource system, there are potential tasks in his mind.



For example, when he saw the task  $(\lg 5)^2 + 2\lg 2 - (\lg 2)^2$ , he directly came up with a new variation task  $(\lg 2)^3 + (\lg 5)^3 + 3\lg 2 \lg 5$ . This kind of hidden resources come from his years of teaching accumulation and exploration.

## 5.4 Conclusion

In this section, we try to draw a conclusion of teaching through variation in Chinese case study based on the analysis above and answer the three research questions we proposed in section 2.4.3.

### 5.4.1 Lin's resources with potential variation: accumulation and evolution

In this section, we try to answer the first research question “What are the potential variations in the upper secondary school mathematics textbooks and other curriculum resources in China?” we have been proposed, which is from the perspective of resources.

Base on the interview and classroom observation, Lin doesn't look for resources aimlessly when preparing his lessons, but directly refers to two main resources: school-based teaching book and the digital task repository created by himself, which greatly accelerated his documentation work efficiency and ensured the quality of the tasks.

Both of these two resources are formed after years of accumulation and redesign. School-based teaching book is redesigned by several experienced teachers of school C, and Lin's digital task repository is a collection of tasks selected by Lin through his process of teaching. These two resources closely follow the requirements in curriculum standards and textbooks. The biggest difference between these two resources and the original PEP textbook is that these two resources fully take into account the needs of students and adapt to the level of students in school C (appropriately increase the difficulty of the tasks). In Lin's documentation work, school-based teaching book is always used to construct the framework of the entire lesson, clarify the learning objectives and contents, and select the tasks for classroom exercises while digital task repository is used to select tasks as a supplement to the school-based teaching book. Often these tasks are relatively new or more difficult in order to further advance the teaching, as we illustrated in episode 6.

Lin's resource system is evolving all the time. The most representative is his own digital task repository, which likes a flowing container, tasks keep coming in and out. This evolution is influenced by curriculum reform, changes of textbooks, and also influenced by Lin's own professional development, his practical teaching with students, collaboration work with teachers and researchers.

The potential variations in Lin's resources of function are relatively concentrated on representational variations and OPMC accompanied by a series of tasks. Lin tends to

further extend on the basis of retaining these potential variations.

### 5.4.2 Lin's design of variation: Closely related to teaching goal

In this section, we try to answer the second research question “What are the actual variations in the Chinese case to achieve the teaching goal?” we have been proposed, which is from the perspective of teacher's design.

Due to the tight schedule and large contents of mathematics classroom teaching in China, Lin is required to design and organize the teaching efficient. Every time Lin uses variation, we can find the corresponding teaching goal. Based on the variations used by Lin in the teaching of function and their corresponding teaching goals, we associate different teaching goals with different types of variation, shown in table 5.4.1.

Table 5.4.1 variation used by Lin and their corresponding teaching goals

Teaching goal	Variation used
Master problem-solving strategy	Representational variation, Pudian, OPMC (MPOS), OPMS
Build core associations (knowledge)	Pudian
Provide multi-angle understanding	Conceptual variation; representational variation
Introduce new knowledge or go further	OPMC

According to table 5.4.1, we conclude four main teaching goals: mastering problem-solving strategy, building core associations, providing multi-angle understanding and introducing new knowledge or go further. We put “build connection with tasks” in the column of mastering strategy, because the ultimate goal of establishing connections between tasks is to master the problem-solving strategy. For different teaching goals, Lin uses different variations to design teaching. Representational and procedural (Pudian, OPMC, OPMS) variation serve for mastering problem-solving strategies; Pudian serve for building core associations; conceptual and representational variation serve for providing multi-angle understanding; and OPMC serve for introducing new knowledge or go further. Lin's classroom teaching is mainly problem-solving oriented, and he spends a lot of energy on students mastering problem-solving strategies.

As Lin is very experienced and his teaching style is mature, when he is designing his teaching, he has a basic framework for each lesson to be designed in his mind. In Lin's teaching, all of tasks he selected are closed tasks without context, which leave limited space for varying. Lin prefers to retain the potential variations embedded in the resources. Through the teaching of function, Lin doesn't reduce or remove any potential variations. If the potential variations can meet his teaching goals, he will use them

directly. If it is not enough, he will retain, make the potential variation explicit and extend the potential variations or use his digital task repository to create new variations.

Representational variations appear the most in Lin's classroom, which is mainly due to the following reasons: (1) The function itself has a variety of representation forms, which provides possibilities for representational variation; (2) Representational variation is an important means to realize other variations. For example, we can increase the complexity of the task by increasing the conversions between registers of representation and this type of tasks usually require teachers to make appropriate Pudian during teaching. (3) Conversions and treatments of registers have a high status in Chinese high school teaching. However, the application of representational variation in Lin's teaching has certain limitations: The table of value as the auxiliary representation in textbooks is difficult to see in Lin's classroom teaching.

The variations created by Lin in his documentation work is mainly Pudian and OPMS. Lin spends the most time in his documentation work on the design of pudian, so that his classroom is consistent and logical. The idea of Pudian is rooted in Lin's mind and is used by him unconsciously. From the micro perspective, Lin used Pudian to arrange mathematical knowledge and tasks, see the example in episode 6. From the macro perspective, as an experienced teacher, Lin design some long-period Pudians for the entire high school study. As for OPMS, Textbooks or school-based teaching books often only provide one strategy to solve the task, while Lin sometimes provides multiple strategies through his documentation work. Lin emphasizes that we should not be blindly pursuing multiple strategies for one task. The goal of the variation is to promote teaching and serve the teaching goal. OPMS designed should be meaningful, such as think about tasks from different perspectives to build connections between knowledge or display multiple strategies to find the best strategy.

These variations do not exist in isolation, they are interconnected. As we discussed in section 5.2.2, Lin often achieves his teaching goals through combinations of various variations and sometimes the creation of a new variation may lead to the creation or extension of other variations, see episode 3, 5, 6.

#### **5.4.3 Lin's classroom teaching: well-organized and limited space for variations**

In this section, we try to answer the third research question "What are the practical variations in Chinese case? How does the case teacher respond to the practical variations?" we have been proposed, which is from the perspective of practical teaching.

Due to insufficient lesson schedule and relatively difficult tasks, Lin's classroom teaching pace is very fast, and the time allocated to each task is very short (both the time for explanation and thinking is short), so the class is mainly led by Lin. This leads

to a reduction in classroom interaction between Lin and his students, plus the choice of closed tasks, thereby reducing the space left for practical variation. The implementation in class is always exactly what Lin planned in his working sheets (he keeps the actual variations) and there are few practical variations occurring in class.

The practical variation in Lin's class mainly focuses on OPMS and conceptual variation. Conceptual variation raised by students mainly occurs during the explanation of tasks, where involves the understanding of concepts. Lin often uses graphs to help students understand through non-standard and non-concept variation. OPMS generated by students mainly appear in the later periods of learning. After students have a certain learning foundation, they can combine knowledge and apply them comprehensively. Students generate OPMS from two levels in this case: varying the calculating strategy and varying the way of reducing the task.

As Lin is very experienced and his teaching style is mature, he can quickly take corresponding measures to what happens in the classroom, and quickly complete teaching reflection based on his teaching experience. For example, for a conceptual problem raised by students, Lin can immediately propose non-standard or non-concept variations; for a new strategy proposed by students, he can quickly understand the strategy and judge whether it is worth proposing in another class or recording in his book. The trajectory of Lin's reflections is hard to catch as it goes inside his mind. We think it is because the cyclic way of Lin's documentation work, classroom teaching and reflection has been developed in the past, leading to a well-structured lesson.



## Chapter 6

### Studying teaching through variation in a French case

This chapter works as a twin chapter of Chapter 5, aiming at exploring and refining the preliminary model (2.4) through a French case study, where an experienced teacher Emma was involved. This chapter presents the case study in France in four sections: Section 6.1 focuses on the variation in Emma's resources system through her documentation work, which is the foundation of Emma's classroom teaching through variation; Variation in Emma's classroom teaching with respect to the tasks is presented from two perspectives: analyzing Emma through her daily teaching (Section 6.2), and through the lens of three thematic lessons (Section 6.3). Section 6.4 concludes the characteristics of case teacher Emma in design and use variation in teaching to answer our research questions, and balances the specific and generic aspects of this case study for improving our current model.

#### 6.1 Variation in Emma's resources system through her documentation work

In this section, we try to draw a picture of Emma's resource system to understand the factors behind Emma's teaching through variation. We described the variation in Emma's resources system from three perspectives: Emma's resource system (section 6.1.1); Emma's documentation work (section 6.1.2); variations in Emma's resource system and documentation work (section 6.1.3).

##### 6.1.1 Emma's resource system: One resource, various uses

In this section, we follow the methodology described in section 3.3.4.1 and present Emma's resources from four types according to different purposes: resources related to tasks; resources related to mathematics knowledge; resources related to didactics, and resources related to software tools to assist classroom teaching. The resources presented here are based on the interview with Emma and evidenced by us during the classroom observation.

- **Resources related to tasks:**

Resources for tasks are the main content of our research. There are mainly five kinds of resources used in Emma's case (see table 6.1.1): 1) several textbooks; 2) documents from Emma's collaborative work; 3) calculation exercise book; 4) Resources from IREM or APMEP website; 5) Digital document repository. These resources are used by

Emma as resources for lesson preparation, for students' homework or for students' tests or examinations, even for her own professional development.

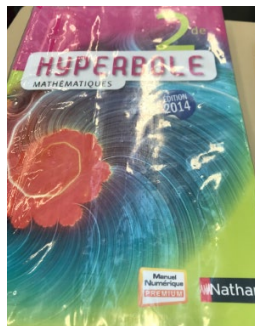
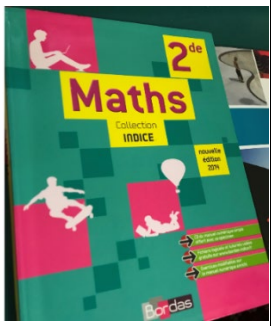
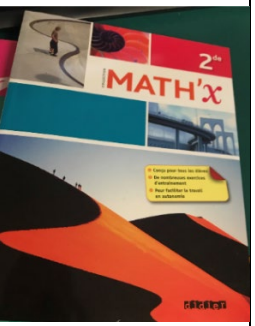
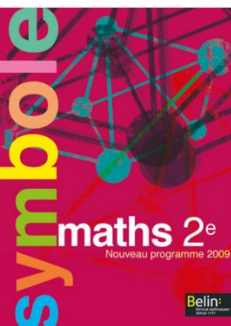
Table 6.1.1 Resources for tasks

Resources		Purpose	Designer/Organizer
Several textbooks	Hyperbole (students' textbook)	Tasks for classroom teaching, for test, and for students' homework	Commercial Publishing
	Indice	Tasks for classroom teaching, for exercise in class, for test	
	Math'x	Tasks for classroom teaching, for test, for exercise in class	
	Symbole	Tasks for classroom teaching, for test, for exercise in class	
Documents from collaborative work		Search information; Tasks for classroom teaching, for test, and for homework; Emma's professional development	Teachers and researchers
Calculation exercise book (cahier de calcul)		Exercise book (homework) for students	Commercial Publishing
Resources from IREM or APMEP website		Tasks for classroom teaching, for test, and for homework	Teachers and researchers
Digital document repository		Restore tasks (with working sheets) used	Emma

Emma mainly uses four textbooks: Hyperbole Mathématiques 2de 2014 (Editions Nathan), Indice Mathématiques 2de 2014 (Editions Bordas), Math'x 2de 2014 (Editions Didier), and Symbole maths 2de 2010 (Editions Belin), see table 6.1.2. Hyperbole textbook is students' textbook and Emma sometimes uses it in classroom teaching or as homework. Emma likes Indice more in the spirit, for interesting situations and modelling. As she explained that she will sometimes find tasks that she took from the rally without knowing that the situations of tasks were in the Indice.

Documents from teachers' collaborative work and teacher training (see section 6.1.2.1 for more information about Emma's collaborative work) are an important part of Emma's teaching resources. She tries to integrate these resources into her own lessons.

Table 6.1.2 Three main textbooks Emma uses for task design

			
Hyperbole Mathématiques 2de 2014 (Nathan)	Indice Maths 2de 2014 (Bordas)	Math'x 2de 2014 (Didier)	Symbole maths 2de 2010 (Belin)

Most of the time, Emma prepares her own working sheets for classroom teaching. Besides students' textbook, Emma refers to the mathematical statements or chooses tasks from the other three textbooks or documents from her collaborative work (see section 6.1.2.1 for details). These resources give Emma new ideas because some of them are not written in the progression. Emma tries to work on the modeling the task from a small one. For example, what function will allow me to solve such a problem, a bit like the “open box” (see section 6.3.1) but sometimes tasks are a little more guided in the textbook.

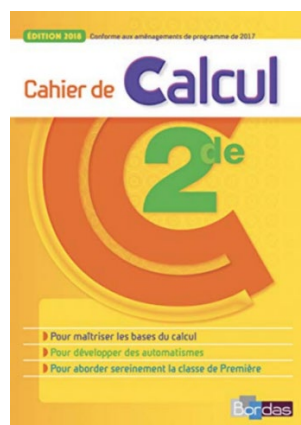


Figure 6.1.1 Cover of calculation exercise book used in Emma's class

Emma also chooses the calculation exercise book (edition Bordas, 2018) for students to use. Calculation exercise book is used to conquer the calculation difficulties of students. It can be used as a complementary of the textbook, during AP (accompagnement personnalisé, personalized support) lessons or in free time. This book is designed to offer exercises quickly and without too much intervention from the



teacher. Diagnostic evaluations (in the form of MCQ<sup>110</sup>) are proposed at the beginning of the notebook to situate the level of the students, from where we see what the level of the students is to calculate fractions, to do a little algebraic calculation. For each sequence, there are mental activities (des activités mentales), schemas-summary (des schéma-bilan), and lesson sheets (des fiches de cours). Emma uses this book for personalized support (accompagnement personnalisé, AP) in particular as it provides different levels of tasks. She gives the lesson of calculation for the whole class and then some of students work more on it because they are in personalized support.

According to Emma, she may search tasks from the Internet, mainly from the website of IREM or APMEP. For example, when she prepares her lesson of function, she likes to with a document that comes from IREM in Clermont Ferrand<sup>111</sup> and is called: Mental calculation and automatisms in 10<sup>th</sup> grade class. This comes in the form of slides with very short questions (quite often True/False).

Emma restored all of the tasks, including working sheets and mathematical statements in her digital document repository. This makes her easier to find the corresponding tasks when she wants to use them again and to trace her own modification process of the tasks.

- **Resources related to mathematical knowledge**

Textbooks and documents from IREM group work are the main resources that Emma uses related to mathematical knowledge. On the one hand, these resources are used as the mathematical statements in her lesson. On the other hand, Emma deepens her mathematical knowledge during her documentation work (also teachers' discussions in IREM) with these resources.

- **Resources related to didactics**

Didactics and tasks are inseparable in Emma's mathematics teaching. How to organize the tasks and better integrate them into teacher's own classrooms involves didactics. Discussions in teachers' associations and teacher training activities are all resources related to didactics.

- **Resources related to software tools to assist classroom teaching**

Emma uses software tools when she can use it. She mainly uses five software tools to assist her classroom teaching at present: Pronote, GeoGebra, calculator, instructional technology support provided by a teacher in school F and Xcas.

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<sup>110</sup> MCQ: Multiple Choice Questions

<sup>111</sup> <http://www.irem.univ-bpclermont.fr/Calcul-Mental-et-Automatismes-en,1306>

“Pronote” is an interface that connects teachers, school, students, and parents online. This is used at the discretion of school F. Teachers, school administrators, students and parents access the interface through a password. Emma mainly uses Pronote to do the following things: 1) To have a trace of things implemented in class as Emma uploads her working sheets, all the notes and reports that given to the students in class; 2) Interaction with students: students can find the homework, classroom working sheets with answers and notes<sup>112</sup>, their marks, and the average of all disciplines in this interface; 3) Interaction with parents: parents can find out about their children’s attendance, what’s being taught in the school, and their children’s performance in the interface; 4) Interaction with school F: To help her verify the students’ situations. When Emma finds that a student does not come to class and does not ask for leave, she sends information to the school management office through Pronote, and the teacher in the management office will contact the student’s parents in time to confirm the student’s situation.

Emma mainly uses Geogebra to display geometric figures, showing students the process of dynamic movements (see examples in section 6.2.2.4 episode 4) or a problem with solid geometry. According to Emma, the role of Geogebra’s in her teaching is to help students understand abstract figures and perceive dynamic variations.

Emma uses calculator a lot in classroom teaching, especially in the lesson of functions. She encourages students to use the calculator to help them solve the tasks (see section 6.2.2.2 episode 2).

Emma also takes profit of the instructional technology support provided by a technology teacher in school F. This teacher is in charge of the technology resources to provide assistance for mathematics teachers. For example, Emma organizes lessons about Python and cryptology in the information technology room from time to time. In this kind of lessons, students try to figure out the solutions and learn by themselves through interactions with the computer, see table 6.1.3. There are detailed instructions and each series of tasks is hierarchical, from easy to difficult. Students can also continue their work at home. Tasks and the online platform here are all designed and organized by an information technology teacher in school F. All these resources are specially designed for students in school F.

Emma is not comfortable enough with Python yet to use it directly. Before the use of Python in programs, she worked from time to time with Algobox<sup>113</sup>. She would also have to work on Xcas<sup>114</sup> to show the use of this type of software to the students. For

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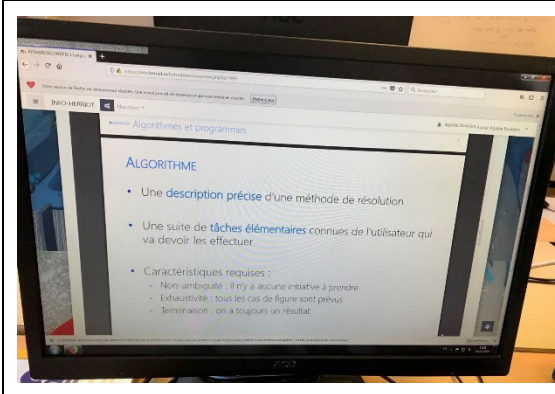
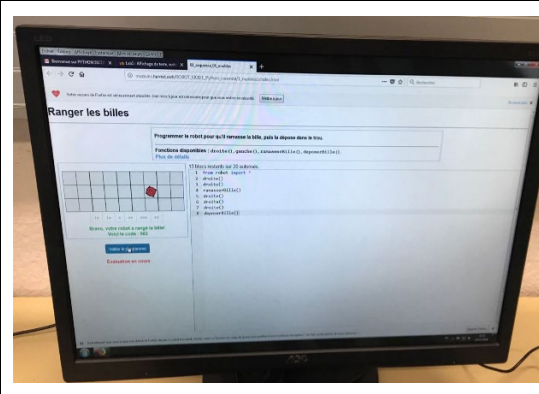
<sup>112</sup> Help students review contents in classroom teaching, especially for absent students.

<sup>113</sup> Algobox is an easy-to-use algorithmic initiation educational software, distributed under the GNU/GPL license. It is available for free for Linux, macOS and Windows platforms and can even run on a simple USB key.

<sup>114</sup> Xcas is an interface of Giac, free and open-source computer algebra software

now, she is content to provide exercises that present images of Xcas.

Table 6.1.3 Interface for students to interact with the computer

	
Slides for students to learn by themselves	A task in Python lesson

Taking functions as an example, Emma believes that the use of software (and the video projector or in the computer room) makes it possible to visualize more quickly different types of representation of the functions, and also makes it possible to conjecture answers to problems posed before solving them.

Emma's resources often do not have only one role, but assume several roles at the same time. For example, documents from DREAM group not include the tasks, but also mathematical knowledge, didactics, and even software support related to classroom teaching. Different resources are interconnected and work together to support the teaching of Emma.

## 6.1.2 Emma's documentation work: Student-Oriented variation

In the previous section, we introduced Emma's resources from different purposes of resources. In this section, we focus on Emma's documentation work from three kinds of interactions: her interaction with resources, with students, and with other teachers.

### 6.1.2.1 Emma's interaction with other teachers: fertile soil for variations in tasks

We describe Emma's collaborative work from two perspectives: inside school F and outside the school.

Inside school F, seven teachers<sup>115</sup> of 10<sup>th</sup> grade try to work together on a common progression (see table 6.1.4) of teaching. According to the common progression, teachers have defined their objectives until the end of January (2019). There are three parts of mathematical contents in their program and progression: statistics, functions, and geometry. Seven teachers work in pairs, two teachers for statistics, two teachers for

[https://www.xcasenligne.fr/giac\\_online/demoGiacPhp.php](https://www.xcasenligne.fr/giac_online/demoGiacPhp.php)

<sup>115</sup> There are eight second classes in school F. Except one teacher who has two classes, one class each for the others. So, there are seven teachers of 10<sup>th</sup> grade trying to work together.



class that year so it is not necessary for him who initiates this common work. In fact, the collaborative work among the 10<sup>th</sup> grade teachers in school F is gradually formed with several interesting coincidence. According to Emma, there was no common progression among teachers in the beginning, and teachers never considered to write something with a common progression. Emma works with one colleague together for the 10<sup>th</sup> and 12<sup>th</sup> grade. Then one day her colleague said: “We are going to try to suggest the others also to work in 10<sup>th</sup> grade.” It happened at that time that one beginning teacher and a new colleague (she had never taught in high school) arrived, and they were all very willing to join this collaborative work. Therefore, just as Emma said: *“There were interesting little situations to initiate this work, I mean that it’s not something that is acquired in all schools. We are encouraged to do common homework, to work together in France but not really compulsory in fact. Common progression is rather individual initiatives of people who said to themselves: ‘Here, we are going to work together.’”* For Emma, she believes that the most important in the collaborative work is to initiate a reflection, to achieve common development.

Teachers may make small variations by themselves of the progression because everyone implements classroom teaching with their own ideas. For example, Emma retained the common progression and made herself a personal version (see table 6.1.4) because there are things that didn’t suit her in the common progression. However, as there is a common objective among teachers, DS (devoir surveillé<sup>119</sup>) is the same for all students in grade 10. Emma’s classroom teaching is not strictly in accordance with her variant. Her variant is not static, we found in the classroom teaching observation that Emma will keep varying it according to the feedback of students.

Outside school F, Emma mainly participates in two teacher organization activities: DREAM<sup>120</sup> and Rallye Mathématique de l’Académie de Lyon (RMAL)<sup>121</sup>.

Emma is a member of IREM Dream group of the 10<sup>th</sup> grade. She doesn’t adapt the statements as it has been worked on by a group that thought about it for a long time. According to Emma, DREAM is about trying to work on research problems to energize teaching and she thinks it is a way of working. In DREAM, researchers pose the problems, and teachers don’t look for what problem to pose but try to think how to integrate the posed problem into their personal practice. Teachers also try to experiment what they do in DREAM in their own classroom teaching.

Since 2006, there is the mathematics rally of the Lyon academy (Each academy in

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<sup>119</sup> Un ‘devoir surveillé’, often abbreviated as DS, is a classroom assessment on one or more topics to be done by students in classroom. Its duration varies according to the difficulty of the subject, the discipline and the level of study.

<sup>120</sup> Refer to section 3.3.2.1 for the description of DREAM.

<sup>121</sup> Refer to section 3.3.2.1 for the description of RMAL.

France has its own mathematics rally, the rally mentioned later refers to the mathematics rally of the Lyon academy). RMAL aimed at students in 10<sup>th</sup> and 9<sup>th</sup> grade. There are several groups of teachers for preparing the rally: a group for designing the topic for the competition, a group for working on logistics, communication, and funding because RMAL tries to give prize to students. What is special at this academy is that the organization of a “final day” (une journée de finale) activity. They select 12 classes to come to the university one day in June and do outdoor math<sup>122</sup>. Most of students are in 10<sup>th</sup> or 9<sup>th</sup> grade and there is one class of vocational high schools, and one ZEP<sup>123</sup> class. For example, Emma at one point had taken large maps (maps of Africa, France with the departments, Europe, etc.) and asked students to color the maps (not really color the maps which are large but put colored pins so that the students could try and manipulate) to do “the four-color map theorem”. Some odd shapes are designed in the maps, the teacher states that “here we are, we have this water, we have this shape, if I immerse it in it, will it be completely covered, finally is it will be completely submerged or not? What will come out? Finally, to try to make calculations of this type and a small manipulation. Students can also be asked, “That building over there, how tall is it or how far does it go around?” Somewhat practical things actually. Students are organized in groups, and everyone should be involved in the activity. Rally activities are small challenges for students. There may sometimes be computer activities but it is something a little bit specific and therefore there is also a group which designs topics and which also organizes the logistics of that day.

The organization of such activities is not easy. RMAL also organizes the training and published books. According to Emma, she is a little around of this group at this moment as she is looking for outside help to make the correction, to find someone help them supervise the “final day”. She doesn’t participate too many meetings anymore and she always offer one or two final stands. Teachers create or redesign tasks, test these tasks and the ideas that they come up with at some point. For example, RMAL had made small bottles with pearls of two different colors (yellow and black), and said to students “Without counting all the pearls, tell us how many there are if I tell you that there are 100 black ones?” In this kind of situation, students had to try to think about the proportion, also about the calculation of the volume. There is either to calculate the volume of what the bottle contained and to see 100 pearls that corresponds to such a volume therefore with the total volume or to use the sampling saying when I take a

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<sup>122</sup> Outdoor mathematics: It takes place on campus La Doua with several stands for students to participate. In outdoor mathematics, teachers try to initiate tasks which are more manipulative.

<sup>123</sup> ZEP (zone d’éducation prioritaire): Priority education zone, set up in 1981 by the government, with the aim of restoring equal opportunities between students, taking into account the social difficulties of the areas concerned. From kindergarten to high school, ZEPs concern all school children living in a rather disadvantaged environment and for whom the government decided to release more resources.

hundred of pearls, I have this number of black ones so the proportion should be that, which is done by manipulating trial. Thanks to her work in RMAL, Emma gets an internship at the PAF<sup>124</sup>.

Emma also tries to organize rally activities in her own class of grade 10. Problems of unequal difficulty will be offered to students in rally activities. In each class, students will need to organize by themselves so that collaborative research leads them to solving as many problems as possible in two hours. The objective of this Rally is to develop curiosity, a taste for research and teamwork and to help students build a positive image of scientific culture, in particular mathematics.

Emma loves to try a lot of things. She also works with a French teacher. They are trying to have a project around argumentation in mathematics and in French. She also takes a competition (*prendre un concours*) called « La racine des mots est-elle carrée » (Are the roots of words square), which is proposed by IREM Montpellier, Apmep, the local education authority of Montpellier and the Media Library Emile Zola Montpellier Métropole. This competition is open to all high schools. This prize proposes to reward a literary work which approaches mathematics. From a selection of 5 books combining mathematical content and literary quality, high school and university students choose the book they liked best.

#### **6.1.2.2 Emma's interaction with students: Sufficient space for students to generate variation**

The interactions between Emma and her students are mainly in two ways: in class and after class.

In Emma's class, there are various types of interactions between Emma and students: 1) One to one guidance: Emma often gives students enough time and space to think (see examples in the episode study in section 6.2 and theme lesson in 6.3) the solution of a task in class while she conducts one-on-one guidance according to the students' individual circumstances. This often happens in the lesson with open task or with the task that are unfamiliar to students; 2) Interaction between Emma and the entire classroom: This often happens in knowledge teaching, when teachers summarize mathematical knowledge or summarize problem solutions; 3) Students' group work: Emma pays attention to the cultivating the interactions between students and encourages students to learn from each other through group work. She also uses this method to understand students' ideas and students' needs. To some extent, she can learn from students through the group work. An example is shown in section 6.3.1.

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<sup>124</sup> PAF (Le plan académique de formation) : The Academic In Service Teacher Education Plan in English. PAF allows the staff of the academy to train as part of lifelong professional training.

Group work with open task offers opportunities for students to create variations, especially OPMS. OPMS creation is one of the expectations of Emma through group work and she also points out “*When we come to the more classic application of the lesson by doing exercises, we will find fewer variations when it comes to the methods of solution. Finding exercises that allow very diverse methods of resolution is not so simple but it is very interesting!*”

As there is half-class teaching<sup>125</sup>, Emma reflects from the performance of the first half-class and make corresponding adjustments in time in the second class. During our classroom observation, the most obvious one was a lesson about tasks in the calculation exercise book. Originally, Emma plan to explain the solution by oral. Later, she found that the students had difficulties, so she wrote on the blackboard, but the effect was not satisfactory. Therefore, after the first half-class, in the interval between two half-classes (1 hour), she reduced two tasks by herself and typed a working sheet for students on the computer. As shown in table 6.1.5, we can see that Emma set up a lot of Pudian to help students learn.

Table 6.1.5 Working sheets designed by Emma after first half-class

CORRECTION DU CAHIER DE CALCUL - EXERCICE 9 P 42

1.  $x$  étant un réel quelconque, quel est le signe de  $2 + x^2$ ?

Un carré est toujours positif.

Donc pour tout  $x$  réel,  $2 + x^2$  est supérieur à .....

Donc pour tout  $x$  réel,  $2 + x^2$  est .....

2. En déduire les solutions de l'inéquation  $\frac{(1+x)(3-7x)}{2+x^2} \geq 0$ .

Comme  $2 + x^2$  est ..... et  $\frac{(1+x)(3-7x)}{2+x^2}$  est  $(1+x)(3-7x)$  ont le même signe.

Les inéquations  $\frac{(1+x)(3-7x)}{2+x^2} \geq 0$  et  $(1+x)(3-7x) \geq 0$  sont donc équivalentes

(elles ont les mêmes ensembles de solutions).

Rappel des règles des signes

La somme de deux nombres positifs est un nombre .....

La somme de deux nombres négatifs est un nombre .....

La somme d'un nombre positif et d'un nombre négatif : .....

.....

Le produit de deux nombres positifs est un nombre .....

Le produit de deux nombres négatifs est un nombre .....

Le produit d'un nombre positif et d'un nombre négatif est un nombre .....

.....

Par conséquent, le produit  $(1+x)(3-7x)$  ne pourra être positif ou nul que si : .....

..... et .....  
ou .....

..... et .....  
.....

.....

On résout :

$1 + x \geq 0$	$3 - 7x \geq 0$	$1 + x \leq 0$	$3 - 7x \leq 0$
.....	.....	.....	.....
.....	.....	.....	.....
.....	.....	.....	.....
.....	.....	.....	.....
.....	.....	.....	.....

Ainsi  $(1+x)$  et  $(3-7x)$  sont tous les deux positifs lorsque  $x \in [-1; +\infty[$  .....

C'est-à-dire  $(1+x)$  et  $(3-7x)$  sont tous les deux positifs lorsque  $x \in$  .....

Ainsi  $(1+x)$  et  $(3-7x)$  sont tous les deux négatifs lorsque  $x \in$  .....

C'est-à-dire  $(1+x)$  et  $(3-7x)$  .....

Conclusion :  $(1+x)(3-7x) \geq 0$  a pour ensemble de solutions : .....

car c'est la réunion des deux ensembles étudiés.

Translation of contents in blue box:

1.  $x$  is any real number, what is the sign of  $2 + x^2$  ?

2. Deduce the solutions of the equation

$$\frac{(1+x)(3-7x)}{2+x^2} \geq 0$$

Students are required to take notes in class, which Emma checks regularly. Each student

<sup>125</sup> Refer to section 3.1.3



used two kinds of notebooks (see table 6.1.6) in Emma's class: one is called "lesson notebook" (cahier de cours), for recording the knowledge, such as mathematical concept, statements, properties, etc.; the other is called "task notebook" (cahier d'exercice), for recording the tasks doing in class. Emma checks and corrects students' notebooks from time to time to ensure students' classroom learning. As shown in table 6.1.6, students cut and paste the distributed working sheets in their notebooks and records what Emma writes on the blackboard.

Table 6.1.6 Photos of student's notebooks in class

Photo of one student's lesson notebook	Photo of one student's exercise notebook

After class, students obtain resources uploaded by Emma through Pronote (section 6.1.1.1), and can ask Emma questions through email. Emma has an express email for her students and she answers students within 24 hours normally. She set it up for her class because she found that many students didn't ask her questions in person at school, but their homework was not good. Therefore, she adds new means of communication with students.

Emma also reflected on students' evaluations. Emma doesn't wait for a chapter to end to evaluate. As presented in table 6.1.7, several ways of evaluations are conducted rhythmic in Emma's case: 1) Small coefficient grades on their research work or in practical task. Simply to value students' involvement in the activities, such as group work in the thematic lesson or in the rally activity, etc.; 2) 20-minute test at the beginning of the chapter: to evaluate students' knowledge of the new chapter. This little test on the basic knowledge aims to realize what students understood, what they didn't understand the beginning of the chapter; 3) DM: to consolidate what has been learned

and to prepare in part for the DS that is coming up. DM is given to students the same day after the 20-minute test, students have a week to work on DM and they are allowed to ask teacher questions. Students hand in DM next week; 4) DS: Tasks in DS are categorized into different levels. Emma organized a bonus exercise for students who performed good and for students who had not understood much of the chapter, they had to redo a basic exercise. Therefore, DS is quite personalized to some extent. On the other hand, students are asked to prepare one summary sheet. It's up to students what they want to write on the paper. They can put in exercises, contents in class, their research, and so on.

Table 6.1.7 Different types of evaluations in Emma's class

	1 <sup>st</sup> week	2 <sup>nd</sup> week	3 <sup>rd</sup> week
Type of evaluation	test	DM	DS
Duration	20min	1h30	1h
Content	- Lesson; - Applications direct from classroom teaching	- Mathematical writing; - Tasks - Preparation of DS	- Preparation of one sheet of summary; - Different levels of tasks;

The variety of assessment methods fully motivates students, focusing not only on students' mathematical problem solving, but also on their thinking about the mathematical learning process, their ability of organizing knowledge, and their independent learning skills. Emma adjusts her teaching and select students to enter AP lesson based on students' evaluations<sup>126</sup>.

### 6.1.2.3 Emma's interaction with resources: instrumentalization and instrumentation

In this section, we describe Emma's interaction with resources from two interrelated processes: instrumentalization and instrumentation (section 3.3.4.1).

From the perspective of instrumentalization, we present Emma's interaction with resources from the selection of tasks or statements, the (re)design of tasks or statements, in the order in which they occurred according to Emma's description. We take the lesson of function as an example to illustrate. One is because that the mathematical topic we discussed here is functions, and the other is because that Emma, who we mentioned earlier, is responsible for the design of the progression of the 10th grade function part this year, which is more conducive to our understanding of her work.

To prepare the lessons on functions, Emma starts to see what she will work on. The first

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<sup>126</sup> Since the mathematics content of each period is different, students in the AP lesson vary from time to time. Emma will select students according to the evaluation results of the students.

resource for her is the curriculum standard<sup>127</sup>, which indicates the intentions of the curriculum on the notion of function and what the students must be able to do on the notion studied. She also works on the progression of the class over the year, see table 6.1.4. Then she looks for activities that can lead her to illustrate the concepts. Emma works on the selection for the following points: 1) The lesson; 2) Exercises; 3) The preparatory activity for the chapter; 4) Student assessment. It should be pointed out that these 4 points are in order.

- The lesson: This is exactly what will be recorded in the students' notebooks (see 6.1.2.2), so it should be very precisely about the notion of function (and in this chapter: General information on functions). This resources for Emma evolved over years. She generally tries to synthesize the lesson to make more room for the examples of the lesson while remaining as rigorous as possible. As Emma said "It remains to choose examples in the lesson that illustrate the concepts as pertinently as possible. A textbook could be a good resource for the lesson for me if it highlights what I know how to do in the chapter and gives examples for it. At the moment, I really like the Indice textbook for their diversity of examples and a good synthesis of the know how in the lesson.
- Tasks: According to Emma, tasks must allow the students to assimilate the know-how of the lesson. She first considers the classroom textbook (Hyperbole 2de 2014) and identifies the tasks that seem to be the most relevant. She looks for the tasks that serve her in classroom teaching (also half-class and in synthesis), for homework or for test or examinations. She looks for tasks from basic (a direct application of the lesson) to gradually more complex.

Emma also looks for tasks to adapt from the statements of the mathematical Rally. This is a part that she developed when she participated in the writing of the book "Le Rallye Mathématique Dans La Classe - Un Jeu Très Sérieux!", and when she led a lesson on mathematical Rally and their uses in class.

- The preparatory activity for the chapter: Emma spends the most time selecting it. It should allow Emma to (re)design the problem to which the chapter will respond. There are few resources that fully satisfy Emma. According to Emma, many of the activities proposed in the textbooks seem poor or too guided, which will not allow students to reflect. For this kind of problems, resources from Dream group helps a

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<sup>127</sup> Curriculum standards which can be found on Eduscol: <https://eduscol.education.fr/1723/programmes-et-ressources-en-mathematiques-voie-gt#link1>

lot. On the website of IREM, there are also several interesting problems.

- Evaluations: Evaluations are mainly in the form of a test, DM or DS. Emma looks for exercises that will allow her to assess what the students have understood. She often invents “basic” tasks for tests or the start of DS. According to Emma, it is not very complicated to propose a simple function in the form of a curve or an expression and to ask about the images, the preimages, the table of variation (for the graphic readings), an extremum (there also with a graphic reading). For DS, she designed tasks of different levels, bonus tasks and also basic tasks. It is personalized for students.

Emma points out that when she works on evaluations, she reflects on the skills (which allowed her to design the lesson), on potential errors: observed during exercise sessions with her students (but also in previous years). For exercises with a more concrete situation, she looks for ideas in textbooks (Index, Symbole, Math’x).

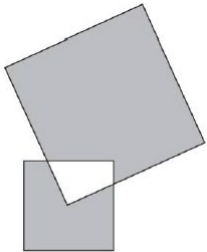
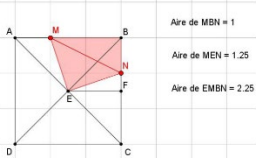
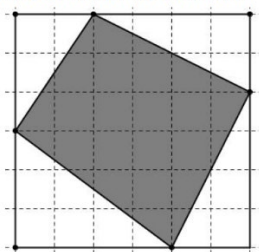
As we discussed above, Emma looks for tasks that she could work on and that will make sense. The lesson can be redesigned according to the tasks that she finds. These tasks can be used to serve as an example or be taken to induce the lesson because it is rich enough in another way (but it is rare according to Emma).

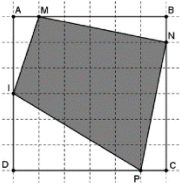
Emma re(design)s the tasks according to the following aspects: 1) The occasion of use: Tasks used at the beginning of a chapter, introduction, end of chapter, homework (DM) or examination (DS) will not be the same. An example of the different ways of adaption is shown in section 6.3.1(task used at the beginning of a chapter) and section 6.3.2 (task used at the end of a chapter); 2) Teaching goals: As shown in table 6.1.8, Emma redesigned on two tasks of Rally to use them in the teaching of function. For the first one, she did not change the statement much but only oriented towards the use of functions (in fact especially towards algebraic calculation); for the second one, she modified the figure so that it becomes a task on the functions and that the students can reuse it in the function; 3) Multiple adaptations of one task: Emma sometimes strengthens the coherence of her lessons by using one task with multiple teaching goals to help students understand. It can be related to the resign of teaching goals in item 2. We illustrate an example in section 6.3.1 in Emma’s documentation work on the “open box”; 4) Reflections after class: Students’ needs: Emma will redesign the tasks based on the students’ classroom performance and students’ needs, see the example of half-class in section 6.1.2.2).

As for tracking and managing her documentation work, Emma doesn’t have a specific online interface. All of her working sheets are generally typed so she has them on file in her computer. For example, the contents that students recorded in their notebooks

are all typed as a document in Emma's computer. However, it is hard for Emma to name the exact source of these contents as she has been teaching for 18 years (at least her 16<sup>th</sup> year of 10<sup>th</sup> grade), the resources have been changed for years.

Table 6.1.8 Emma redesigned two tasks of Rally

<p><b>Original task of Rally:</b></p> <p><i>Deux carrés en équilibre (Niveau 1 - 2016)</i> Deux carrés ont respectivement pour côté 3 cm et 5 cm, et un sommet du grand carré est le centre du petit carré.</p> <p>Quelle est l'aire de la zone commune aux deux carrés ? Donner la réponse en cm<sup>2</sup>, et arrondir si besoin à l'entier le plus proche.</p> 	<p><b>Redesigned task:</b></p> <p><b>Exercice 1</b> Soit ABCD un carré de 3 cm de côté. On appelle E le centre du carré et on place les points M et N respectivement sur les côtés [AB] et [BC] du carré, tels que <math>AM = BN = x</math>. Un dessin illustre la situation ci-dessous. (dessin de gauche) On veut étudier l'évolution de l'aire du quadrilatère MBNE en fonction de x.</p> <ol style="list-style-type: none"> <li>1) Quelles valeurs peut prendre x ?</li> <li>2) Calculer en fonction de x l'aire du triangle MBN.</li> <li>3) On considère le repère orthonormé d'origine D, et tel que dans ce repère, C a pour coordonnées (3;0) et A a pour coordonnées (0;3).</li> <li>a) Donner les coordonnées des points E, M et N dans ce repère.</li> <li>b) Justifier que <math>EN = EM</math></li> <li>c) En déduire que le triangle ENM est rectangle en E.</li> <li>4) Montrer que l'aire du triangle ENM est égale à <math>\frac{1}{2}x^2 - \frac{3}{2}x + \frac{9}{4}</math>.</li> <li>5) En déduire celle du quadrilatère MBNE.</li> </ol>  <p>Aire de MBN = 1 Aire de MEN = 1.25 Aire de EMBN = 2.25</p>
<p>[Translation]</p> <p>Two Squares in Balance (Level 1 - 2016) Two squares have sides 3 cm and 5 cm respectively, and one vertex of the larger square is the center of the smaller square. What is the area of the common area of the two squares? Give the answer in cm, and round off to the nearest integer if necessary.</p>	<p>[Translation]</p> <p>Exercice 1: Let ABCD be a square of side 3 cm. We call E the center of the square and we place points M and N respectively on the sides [AB] and [BC] of the square, <math>AM = BN = x</math>. A drawing illustrates the situation below. We want to study the evolution of the area of the quadrilateral MBNE as a function of x.</p> <ol style="list-style-type: none"> <li>1) What values can x take?</li> <li>2) Calculate the area of the triangle MBN as a function of x.</li> <li>3) Consider making an orthogonal coordinate system with D as the origin, C has coordinates (3;0) and A has coordinates (0;3).</li> <li>a) Give the coordinates of the points E, M and N in this orthogonal coordinate system.</li> <li>b) Justify that <math>EN = EM</math></li> <li>c) Deduce that the triangle ENM is right-angled at E.</li> <li>4) Show that the area of the triangle ENM is equal to <math>\frac{1}{2}x^2 - \frac{3}{2}x + \frac{9}{4}</math></li> <li>5) Deduce the quadrilateral MBNE.</li> </ol>
<p><b>Original task of Rally:</b></p> <p><i>Morceaux de carrés</i> Quelle est la fraction du carré représentée par la partie foncée ?</p> 	<p><b>Redesigned task:</b></p> <p><b>Exercice 2</b> Soit ABCD un carré de côté 6 cm. Soit I le milieu de [AD]. M est un point du segment [AB]. On construit sur [BC] le point N tel que <math>BN = AM</math> puis sur [CD] le point P tel que <math>CP = AM</math>. Vous trouverez une figure avec <math>AM = 1</math> ci-dessus (à droite)</p> <ol style="list-style-type: none"> <li>1. Lorsque <math>AM = 1</math>, calculer l'aire du quadrilatère MNPI.</li> <li>On pose <math>AM = x</math>.</li> <li>a) Quelles valeurs peut prendre x ? On notera l'ensemble de valeurs.</li> <li>b) Calculer l'aire du triangle AMI en fonction de x</li> <li>c) Calculer l'aire du triangle MBN en fonction de x</li> <li>d) Calculer l'aire <math>A(x)</math> du quadrilatère IMNP en fonction de x</li> <li>3. Représenter soigneusement, dans un repère adapté la fonction A associée à l'aire du quadrilatère IMNP.</li> <li>4. A l'aide du graphique, déterminer la valeur de x pour laquelle l'aire de IMNP est minimale.</li> </ol>

	
<p>[Translation]</p> <p>Pieces of squares</p> <p>What is the fraction of the square represented by the dark part?</p>	<p>[Translation]</p> <p>Exercise 2: Let ABCD be a square of side 6 cm. Let I be the midpoint of [AD]. M is a point on segment [AB]. Construct on [BC] the point N so that <math>BN = AM</math> and on [CD] the point P so that <math>CP = AM</math>. You will find a figure with <math>AM = 1</math></p> <ol style="list-style-type: none"> <li>1. When <math>AM = 1</math>, calculate the area of the quadrilateral MNPI. Let <math>AM = x</math>.</li> <li>2. a) What values can x take? b) Calculate the area of the triangle AMI as a function of x c) Calculate the area of the triangle MBN as a function of x d) Calculate the area <math>A(x)</math> of the quadrilateral IMNP as a function of x</li> <li>3. Represent carefully, in a coordinate system, the function A associated with the area of the quadrilateral IMPN.</li> <li>4. Using the graph, determine the value of x for which the area of IMNP is minimal.</li> </ol>

From the perspective of instrumentation, curriculum standard offers Emma support as a guidance while textbook, resources produced in the collective work offers her specific tasks for teaching.

### 6.1.3 Conclusion: Emma's resource system and documentation work

In this section, we make a conclusion of Emma's resource system and documentation work from the perspective of variation. We discuss from three perspectives: Emma's understanding of good resources and choice of resources, Emma's interpretation of variations in the tasks, and variants and invariants in Emma's resource system and documentation work.

#### 6.1.3.1 Emma's understanding of good resources and choice of resources

From the perspective of selection and use of resources, Emma mainly focuses on the following aspects of resources: 1) relevant with the lesson; 2) difficulty of the tasks; 3) resources with good ideas that can be deepen.

For Emma, classroom textbook (Hyperbole 2de 2014) is good resource for her to find the tasks relevant to the lesson. For different levels of difficulties, she looks for several textbooks and also the documents from collaborative work. Indice, Symbole textbook and IREM website offers her interesting problems that she can continue work on and adapt to her own class.

For different teaching periods (at the beginning, in the end, etc.) and different goals of tasks (tasks for teaching, for homework, or for test, etc.), the definition of good

resources is also different.

### **6.1.3.2 Emma's interpretation of variation in the tasks**

In Emma's opinion, variation in the tasks means "the same exercise that can be offered in different forms (and therefore with different objectives) or exercises that focus on the same concepts." For example, for the task from Rally "pieces of squares (morceaux de carrés)" proposed in table 6.1.8, we could imagine a task with greater initiative taken: one could only ask the last question and it is then up to the students to set an unknown, define a function, represent it and answer the question.

### **6.1.3.3 Emma's resource system and documentation work: from varying to unvarying with variants**

We study Emma's resource system and documentation work through the lens of variation. We will first explain the evolution of Emma's resources and her process of instrumentalization over years. And then we focus on her current resource system and documentation work.

According to Emma, her lessons and corresponding resources have evolved over the years. At the beginning her lesson was worked from an organization of the textbooks and when the curriculum standard changed, she changed certain points. What has changed the most in her way of selecting resource is the growing involvement she has had in the teachers' associations, such as RMAL, DREAM, etc. It happens to her more frequently to look for resources on the internet rather than only textbooks. The evolution also comes from the educational teams with whom she worked.

After years of teaching, Emma has constructed her own resources system. In order to clarify the diverse resources in her current resources system, the relationship between them, and the impact of Emma's collaborative work on her resource system, we draw an inferred mapping of Emma's resource system (section 3.3.4.1), see figure 6.1.2.

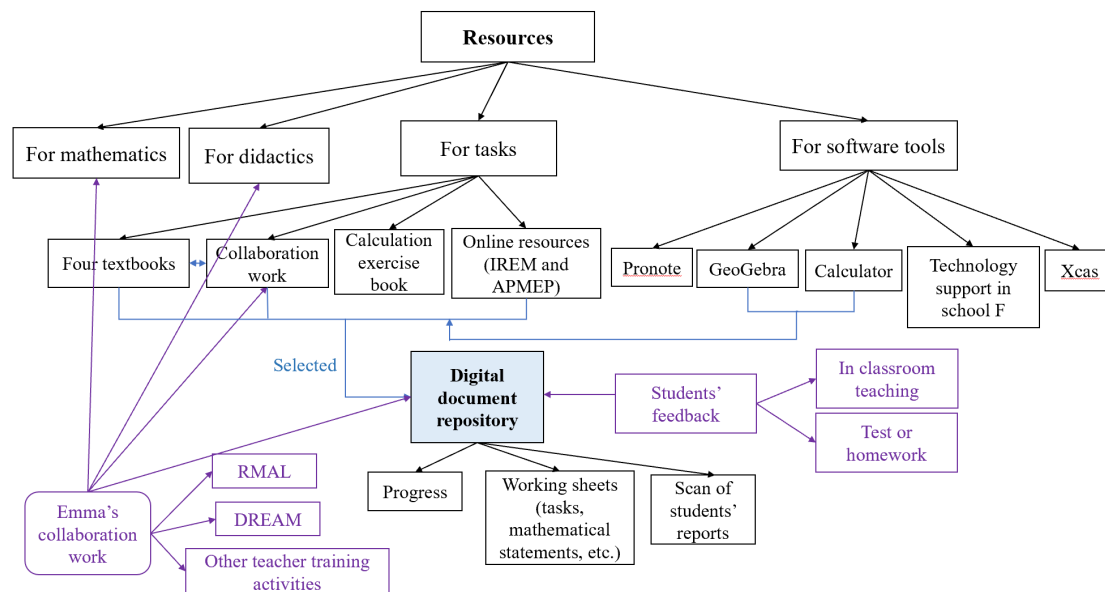


Figure 6.1.2 Inferred mapping of Emma's resource system

Figure 6.1.2 is drawn by the researcher based on the Emma's interview and her classroom observation. We centered on the digital document repository, which are the resources being organized and restored in Emma's own computer, including the progression, her working sheets of tasks and mathematical statements, and the scan of students' reports in the thematic lessons. We have used several colors in the figure to distinguish the various resources used by Emma and the connections between the resources and her collaborative work: black represents the existed resources; purple represents the influence of the collaborative work and the interactions with students; blue represents the connection between various resources. As shown in figure 6.1.2, the main structure of Emma's resource system is relatively stable, but the elements inside it are keeping evolving all the time by the appropriation process. Each appropriation process can be regarded as a combination of instrumentation and instrumentalization.

The invariants in Emma's current resource system are mainly in two parts: the main source of her resource system and her instrumentalization for lesson preparation. According to Emma, these existing sources are sufficient for her teaching needs. The connections between resources are constructed by Emma through her documentation work. For example, she found that many tasks discussed in the IREM group can be traced back to textbooks. Also, software tools are used to support teaching of tasks in her class. After many years of teaching, Emma's mode of instrumentalization (section 6.1.2.3) has been formed. For Emma, she goes through the same process when she prepares her lessons. She has her own set of criteria of selecting tasks for her different teaching goals.

From a micro point of view, the elements inside Emma's resource system are dynamic



and constantly evolving by her documentation work over years. The most prominent is Emma's digital document repository, which is enriched by the continuous collaborative work, Emma's own documentation work and her reflections after classroom teaching.

In general, the establishment of Emma's resource system and interaction with resources (documentational genesis) has undergone a long period of varying. At this point, its structure gradually tends to be stable, but the elements inside is still varying. Therefore, we name Emma's resource system as "from varying to unvarying with variants".

## **6.2 variation of tasks in Emma's classroom teaching**

In this section, we analyze Emma's classroom teaching through variation with respect to the tasks presented in her own teaching of function. We first described the general situations of different kinds of variations in Emma's teaching (section 6.2.1). We then selected five episodes<sup>128</sup> of classroom teaching to zoom in and analyze in detail from potential, actual to practical variation<sup>129</sup> (section 6.2.2).

### **6.2.1 General situations of Emma's teaching through variation**

In this section, we present the general situations of different kinds of variations in Emma's teaching by figure (see section 3.3.3): contextual variation, representational variation, conceptual variation, and procedural variation (including Pudian, OPMC, OPMS, and MPOS). Through the statistical figure of each type of variation, we illustrate the situation of each type of variation in Emma's teaching from the perspective of teaching goals.

#### **6.2.1.1 Contextual variation**

In figure 6.2.1, we list all the contexts of tasks we analyzed in Emma's teaching of function. In Emma's class, various task contexts appeared: personal, occupational, scientific, and mathematical. There are no tasks with societal context presented in class. As shown in figure 6.2.1, the main context of tasks is mathematical, which occurs 26 times (37 times in total, 70,27%) in Emma's teaching of function. Tasks or series of tasks with context (except mathematical context) often appears at the early period of function teaching (the introduction, the concept, etc.) or in thematic lessons.

In Emma's teaching, contextual variation mainly assumes the role of making functions less abstract for students and help students understand functions by linking to students' daily life. Various contexts also make students aware of the wide range of applications of functions so as to have a multi-angle understanding of function modelling. In figure

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<sup>128</sup> For specifying what is an episode, and the criteria to select them, see the definition of episode in section 3.3.3.

<sup>129</sup> See section 3.3.3, where the related method should be introduced.

6.2.1, most of the contextual variations (except mathematical context) are shown at the beginning part of teaching, mainly about the construction of a function from a specific context.

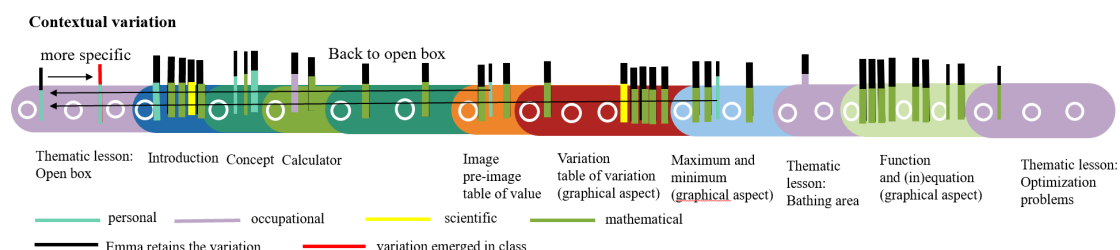


Figure 6.2.1 Contextual variation in Emma's teaching

Almost all contextual variations are embedded in the tasks and Emma retains. It is worth noting that after students' group the personal context of the task used in the first thematic lesson is given by the students to a more specific daily life context in class. Emma also returned to this task with its original context twice in her subsequent teaching, see section 6.3.1 for details.

### 6.2.1.2 Representational variation

Emma pays much attention to different forms of representations, and the treatments or conversions inside or between different registers. As shown in figure 6.2.2, conversion is covered in almost all lessons, which occurs 32 times in Emma's teaching of function. Emma mainly retains the conversions already embedded in the task. For specific content, like function and (in)equation from the graphical aspect, she creates representational variations (see section 6.2.2.5 for details) as she fails to find the proper tasks. At the thematic lesson of conclusion, she reduces representational variations (see section 6.3.2).

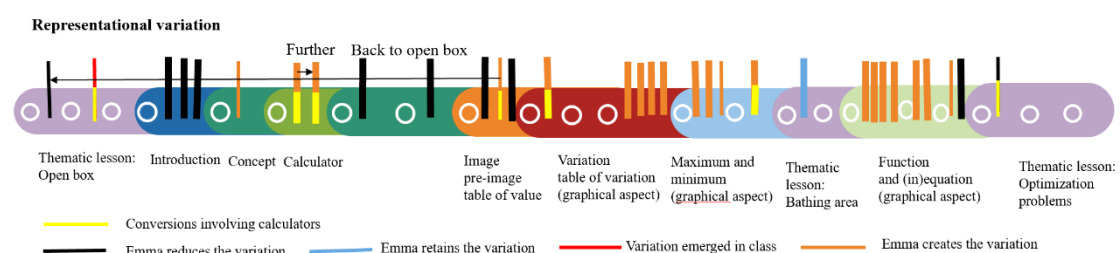


Figure 6.2.2 Representational variation (Conversions) shown in Emma's teaching

Conversions in Emma's teaching are diverse. Conversions among natural languages, transitional auxiliary representation (table of values), symbolic writings, cartesian graphs, and even geometrical figures (see section 6.3.1) are shown in teaching. There are also conversions (conversions from symbolic writings to table of values, or to cartesian graphs) accomplished by using calculator presented by Emma (see the yellow stick in figure 6.2.2). According to the requirements of the curriculum standard, Emma

introduces the way to find the extremum of a function by listing table of values and drawing a graph with a calculator and pointed out the limitations of calculator (see section 6.2.2.2).

In Emma's teaching, representational variation mainly assumes the role as a powerful tool of task-solving strategy. The main conversions in the teaching are from natural languages to symbolic writing (construct a function), and then from symbolic writing to table of values, and to graphs to find the extremum of a function. Treatments inside the symbolic registers are greatly weakened in teaching as students are required to find the answer from the graph of a function. Meanwhile, various representations can make students have a multi-angle understanding of the function.

### 6.2.1.3 Conceptual variation

As shown in figure 6.2.3, conceptual variations mainly occur in the early period of function teachings, such as the introduction and the concept of function. It occurs 8 times, including 5 times of non-concept variation, 2 times of standard variation, and 2 times of non-standard variation. Standard variation is used to illustrate concepts, introduce concepts, while non-concept and non-standard variations are used to distinguish the concept.

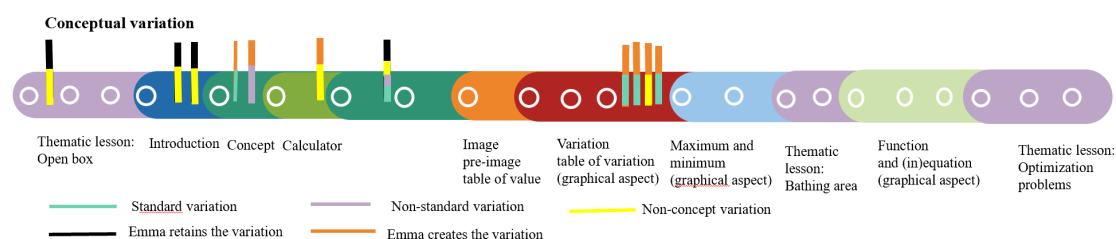


Figure 6.2.3 Conceptual variation shown in Emma's teaching

There are no new conceptual variations produced in classroom teaching. All of the conceptual variations in Emma's teaching are embedded in the tasks and Emma retains the variations. Some non-conceptual variations are designed to make students aware of it by themselves in the process of solving the task. For example, the setting of the domain of the function of the open box will show its necessity in the process of students' handmaking box and subsequent finding of the maximum value of the box, which is what Brousseau called "adidactical situation"<sup>130</sup>.

### 6.2.1.4 Procedural variation

There are various types of Procedural Variations in Emma's teaching and we explain them from three aspects: enhancing the formation of concepts and propositions, pudian,

<sup>130</sup> See section 2.2.2.

OPMC and OPMS.

- **Enhancing the formation of concepts and propositions**

Enhancing the formation of concepts and propositions occurs mainly in the lessons about the concept and the variation of function. As shown in figure 6.2.4, this kind of variation is mainly created by Emma.

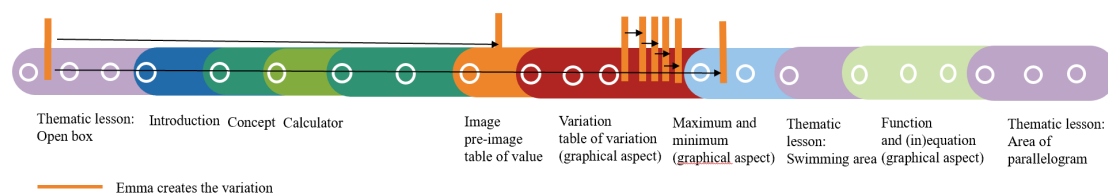


Figure 6.2.4 Enhancing the formation of concepts and propositions shown in Emma's teaching

Emma breaks down the concepts according to the needs of the students and teach them from what is familiar to what is unfamiliar to the students. For example, when she teaches the variation of a function, she begins with a series of tasks with the context of temperature, which is specific and familiar to the students. Then, she proposes a series of tasks with purely mathematical context, but still presented in concrete numbers. Finally, she further abstracts the numbers by replacing them with the abstract symbols  $a, b$  and discusses the different situations.

- **Pudian**

As shown in figure 6.2.5, Pudian occurs 15 times in Emma's teaching of function. 8 Pudians are created and 1 is removed by Emma through her documentation work. There are 6 pudians emerged in classroom teaching. All Pudian are situated at the beginning of the lessons with the same mathematical contents.

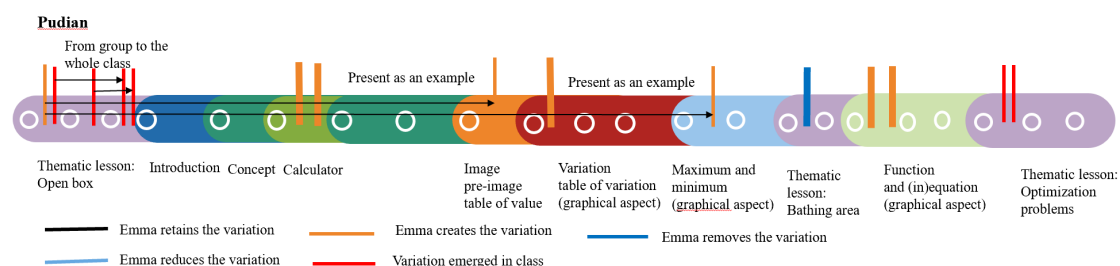


Figure 6.2.5 Pudian shown in Emma's teaching

Emma designs and takes profit of Pudian consciously in her teaching. There are three kinds of Pudian in Emma's teaching:

1) In the series of tasks, the former task is the Pudian of the latter task: Tasks in a series of tasks are designed and organized to be hierarchical, step by step to help students achieve the final goal, an example presented in section 6.2.2.2 and 6.2.2.4;

2) Example is the Pudian of exercise: Emma explains the examples first and then asks students to solve new exercise by imitating the strategy of solving the example, an example presented in section 6.2.2.5;

3) Pudian for a period of learning: this kind of Pudian is not for one knowledge or strategy, but for a more macro level, an example presented in section 6.3.1 of open box.

Corresponding to the above three types, Pudian mainly plays the following roles in Emma's teaching: 1) To let students master the task-solving strategies: no matter the hierarchical series of tasks or example and exercise, these variations are designed for students to master the strategies; 2) To work as a fundamental task: Emma creates Pudian at the beginning of teaching to pave the way for the subsequent teaching of function, see section 6.3.1 for details.

The design of Pudian is closely related to the teaching process. We can see that in the first thematic lesson, the teacher creates variation, and in the second thematic lesson, she removes variation. This is because the task-solving strategies used in these two thematic lessons are the same. In the first time, the students still do not have the ability to solve the task and have not learned the corresponding knowledge, while in the second time, the students have learned the corresponding knowledge, do not need these Pudians anymore.

There are new Pudians produced during the interactions between Emma and her students in class, especially in the thematic lessons. Emma paves the way for students by asking them questions, question their solutions. This kind of interactions is often one (Emma)-to-one (one student or students in one group), based on students' needs, see section 6.3.1 and section 6.3.3 for details.

### • OPMC (MPOS included)

OPMC in this case all appeared together with MPOS. The purpose of OPMC is mainly to let students master a knowledge or a problem-solving strategy to solve the task.

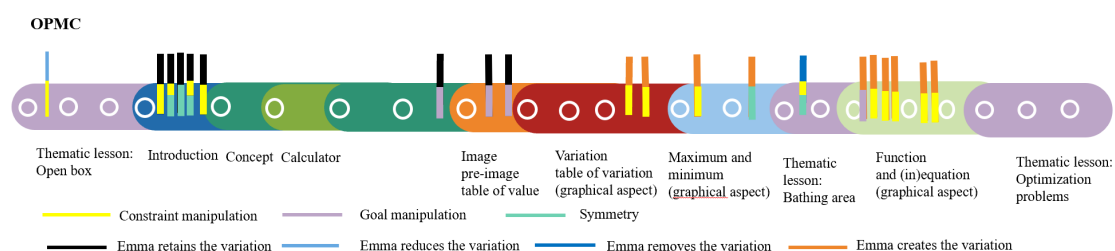


Figure 6.2.6 OPMC shown in Emma's teaching

As shown in figure 6.2.6, OPMC occurs 20 times in Emma's teaching of function. 8 Pudian are retained, 10 Pudian are created, 1 Pudian is reduced, and 1 Pudian is removed by Emma through her documentation work. Several types of OPMC are shown in Emma's teaching: constraint manipulation (14 times), goal manipulation (4 times),

and symmetry (5 times). Emma intends to use constraint manipulation when she creates OPMC in her documentation work.

OPMC here is often associated with series of tasks. In Emma’s teaching process, OPMC mainly serves the teaching goal of ensuring students to master the strategy by varying non-essential features<sup>131</sup>, see section 6.2.2.5. This is associated with MPOS, where students use the same strategy to solve tasks with different presentations.

• **OPMS**

There are few OPMS in Emma’s teaching. As shown in figure 6.2.7, OPMS occurs only 2 times, and both emerged by students in the thematic lesson. We believe that this phenomenon can be attributed to two reasons. One is because of the enough time given to students to think, and the other is the task itself is relatively open.

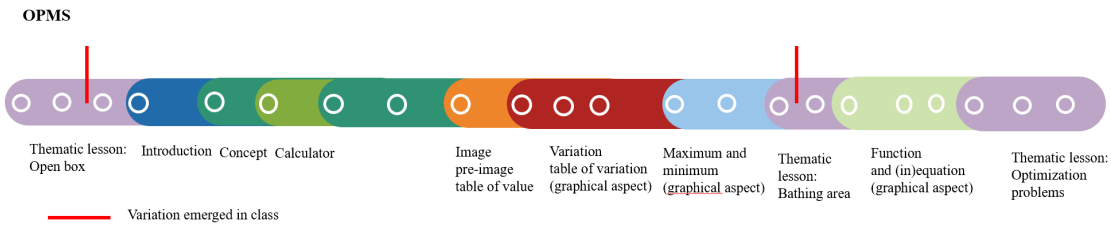


Figure 6.2.7 OPMS shown in Emma’s teaching

In the thematic lesson, the students’ answers sometimes give Emma great surprise. How Emma deal with the new problem-solving ideas and problem-solving methods proposed by students is something worth discussing, see section 6.3.1 for details. Under Emma’s reactions and organizations of OPMS in the classroom, OPMS in this case mainly serves the teaching goal of promoting teaching progress. While encouraging and affirming students’ thinking, Emma will link students’ thinking with subsequent teaching content, and naturally promote teaching through students’ thinking and needs. In this case, the teaching of new knowledge does not appear suddenly, but occurs naturally on the basis that students already have certain thinking and needs.

**6.2.1.5 Conclusion**

Figure 6.2.8 shows the distribution of seven selected episodes and the variation types involved. All types of variations are covered in Emma’s teaching of function, while the frequency of use varies widely. As presented in figure 6.2.8, the most common type of variation in Emma’s teaching is representational variations (32 times) and the least is OPMS (2 times). Conceptual variations appear 8 times, Pudian appears 15 times, and OPMC appear 20 times.

<sup>131</sup> Refer to section 2.2.1

The distribution of different types of variations is closely related to the content of mathematics. For specific mathematical content, there is a corresponding variation that matches more. The distribution of representational variation is very uniform; the distributions of conceptual variation and OPMC are concentrated in some specific lessons; OPMS occurs only in thematic lessons and are all proposed by students; Pudian is situated at the beginning of the lesson with the same mathematical contents; Enhancing the formation of concepts and propositions is designed by Emma.

The main way Emma takes for variation is to retain the potential variations already embedded in the task or to create variations through adaptation or creation. Removing or reducing variations all appear in thematic lessons and can be used to expand or narrow the task. Sometimes because the original task is too open with various OPMC and Emma makes the task become relatively closed through reducing these variations; and sometimes because the original task is too closed with a series of Pudian, Emma makes the task become relatively open through reducing Pudian. Emma also creates and arranges series of tasks to guide the students form the concept or proposition step by step. The design of variations all depends on the teaching goal of the lesson and students' needs.

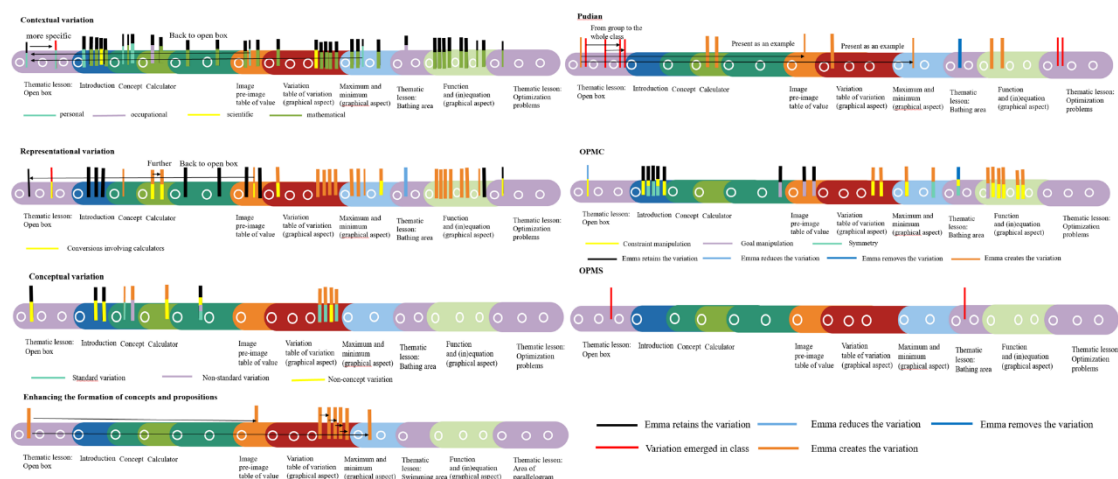


Figure 6.2.8 An overview of the different types of variations in the French case

In general, Emma tends to retain the context of the original task, and then create or retain the variations, especially procedural variations (Pudian, OPMC), through her documentation work according to her teaching goal. Practical variations are all emerged in the thematic lessons.

## 6.2.2 Variations in selected episodes

In this section, we presented 5 episodes with series of tasks in Emma's classroom teaching in the order in which they appear in the class. We illustrated the variation in this episode from several aspects: the general situation of the episode, the potential



### 6.2.2.1 Episode 1: Variation for multi-angle understanding of function

Table 6.2.1 Series of tasks for introducing function






“image”. c) Read graphically: • the image of 5 by  $f$ ; • the preimage of 1 by  $f$ .

**4 Utiliser une fonction définie par un tableau**

Le tableau suivant définit une fonction  $E$  qui à chaque vitesse indiquée  $v$  (en m/s) du vent associe l'énergie  $E(v)$  (en kWh) produite par une petite éolienne en un an.

$v$ (en m/s)	3	4	5	6	7	8	9	10	12	14
$E(v)$ (en kWh)	130	310	570	780	900	890	780	580	240	50

a) Déterminer : • l'image de 7; • les antécédents de 780.  
b) La fonction  $E$  est-elle linéaire?



Translation:

4. Use a function defined by a table

The following table defines a function  $E$  which, at each indicated wind speed (in m/s), associates the energy  $E(v)$  (in kWh) produced by a small wind turbine in one year.

a) Determine: • the image of 7; • the preimages of 780. b) Is the function  $E$  linear?

**5 Utiliser une fonction définie par une formule**

$g$  est la fonction définie par  $g(x) = x(2x - 3)$ .

a) Calculer l'image par la fonction  $g$  de : •  $x = -5$  •  $x = 0,1$   
b) Thomas affirme : « L'antécédent de 0 par la fonction  $g$  est  $\frac{3}{2}$  ». Qu'en pensez-vous?

Translation:

5. Use a function defined by a formula

$g$  is the function defined by  $g(x) = x(2x - 3)$ .

a) Calculate the image by the function  $g$  of: •  $x = -5$  •  $x = 0.1$  b) Thomas asserts: “The preimage of 0 by the function  $g$  is  $\frac{3}{2}$ ”. What do you think?

Examining the potential variation of this series of tasks:

- Contextual variation embedded: The contexts of these five tasks are personal, mathematical, mathematical, scientific, and mathematical. Therefore, there are two contextual variations from realistic context to mathematical context, in task 1 and 4.
- Representational variation embedded: Functions in these 5 tasks appear in the form of graph, mathematical language expressions, graph, table of values and formula. In the process of solving these tasks, conversions between graph and symbolic writing in task 1 and 3; conversions between table of values and symbolic writing in task 4. There are also treatments inside symbolic writing in task 2 and task 5.
- Conceptual variation embedded: item b in task 4 and item b in task 5 are two non-concept variations, respectively about the concept of proportional function and pre-image of function.
- OPMC is embedded in series of tasks: There are constraint manipulation of OPMC in task 1,2,3,5, and symmetry of OPMC in task 2,3,4.

Regarding actual variation, Emma retains the original tasks with potential variations

and chooses these 5 series of tasks to start her formal teaching of function<sup>132</sup>.

In classroom teaching, Emma retains this series of tasks with potential variations. As for practical variations,

- Contextual, representational variation, and OPMC retained.
- Conceptual variation becomes more explicit: Emma gave the non-concept examples with specific values when explaining item b in task 4 and in task 5. For task 4, she presents that “ $f(4) - f(3) \neq f(5) - f(4)$ ”; for task 5, 0 is also the pre-image of 0 of function g.
- Pudian emerged: When Emma explains this series of tasks, she leads the students to recall what the function is. She explains that “a function is a rule that assigns to each input exactly one output. The graph of a function is the set of ordered pairs consisting of an input and the corresponding output.” She mentions the function as in-out machines (see figure 6.2.9) and various representations of function, which undoubtedly paved the way for the further teaching of the concept and representations of function.

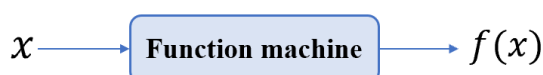


Figure 6.2.9 function machine drawn by Emma in class

We present the evolution of episode 1 in figure 6.2.10. Emma has retained potential variations and created Pudian in her classroom teaching.

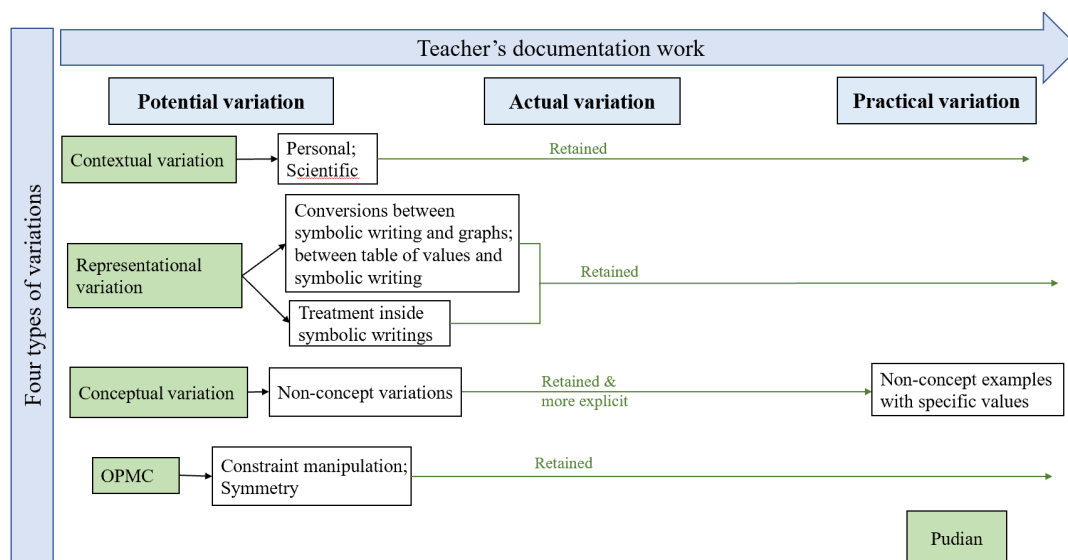


Figure 6.2.10 Schematic representation of variation path in episode 1

<sup>132</sup> Before this, Emma organizes the thematic lesson of function as a mathematical activity, and after these five series of tasks, she gives the concept of function, the properties of function, etc.

#### 6.2.2.2 Episode 2: Pudian in a series of tasks

Table 6.2.2 Working sheet of series of tasks of function and calculator presented in class

Translation :

### Exercise 1: Harvesting potatoes

On May 31, a producer can harvest 1,200 kg of potatoes from his field and sell them for one euro per kilogram. If he delays his harvest, each day the harvest increases by 60 kg. But the selling price drops by €0.02 per kg and per day.

The objective of this work is to determine which day the producer will have to harvest and sell his potatoes in order to obtain the maximum revenue.

### Part A: First calculations

- 1) Calculate the revenue obtained if the producer harvests and sells his potatoes on May 31.
- 2) If the producer harvests and sells his potatoes on June 1, give:
  - a) The quantity harvested:
  - b) The price per kilogram of potatoes:
  - c) The revenue obtained by the sale of this crop:
- 3) Calculate the revenue obtained if the producer harvests and sells on June 2:

## Part B: Table of values

1) Express as a function of the day (noted  $x$ ):

a) The quantity of potatoes harvested:  $q(x) =$

b) The price per kilo:  $p(x) =$

2) Express the recipe obtained as a function of  $x$ :  $r(x) =$

3) Show that  $r(x)$  can also be written  $r(x) = -1,2x^2 + 36x + 1200$

4) We want to present the values of these functions in a table (limited to the month of June)

Date	Harvest (kg)	Price per kg (€)	Recipe (€)
------	--------------	------------------	------------

The table will have 4 columns:

And 31 lines on the day varying from 1 to 30.

To complete this table, we will use the calculator.

5) Represent the function  $r$  in the orthogonal frame given below

To do this, first visualize the curve to be plotted on the calculator.

6) By studying the table of values and the curve, determine the day in June that the producer must choose to obtain the maximum revenue, as well as the value of this revenue.

## Exercise 2 Limit of the calculator

Let  $f$  be the function defined on  $\mathbb{R}$  by  $f(x) = 0.01x^3 - 0.32x^2 + 0.36x + 7.2$ .

- Plot its representative curve on the calculator, with  $x_{\min} = -6$  and  $x_{\max} = 7$ .
- Indicate using the graph, the number of preimages of 0.
- Are we sure that these are the only preimages?
- Calculate  $f(x)$ . (Have the calculator do it using a table of values.) What can be deduced from this?
- Show that for all real  $x$ :  $f(x) = 0.01(x - 30)(x - 6)(x + 4)$ . Then give all the preimages of 0 by the function  $f$ .

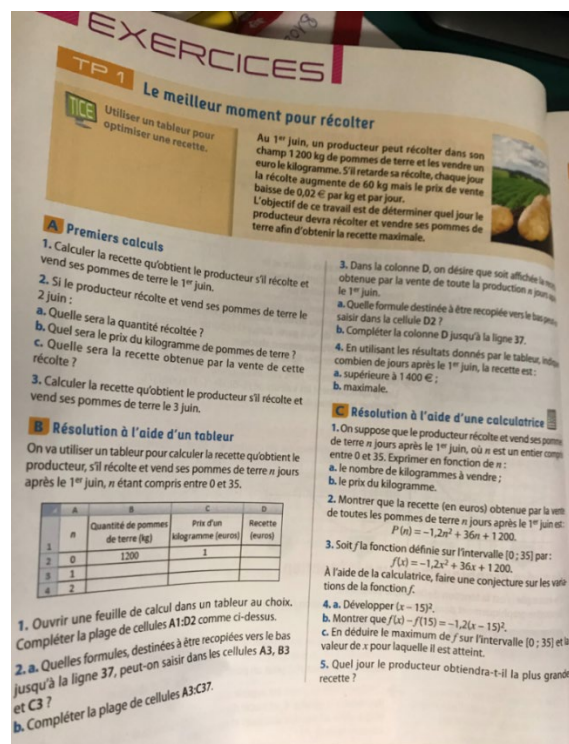


Figure 6.2.11 Original tasks from Math' x textbook

This series of tasks is originally from one textbook (Math'x, 2014) designed as an exercise. Examining the potential variations in the original series of tasks:

- Contextual variation embedded: The context of this series of tasks is occupational. Students need to abstract math problem from occupational context.
- Representational variation embedded: There are conversions from natural languages to symbolic writing (part A and C); from natural language to table of values (part B); from symbolic writing to table of values (part C). In part C of the task, there is also a treatment inside the symbolic writing and this process is carried out by the calculator.
- OPMC is embedded in this series of tasks: There is a constraint manipulation of OPMC. Through varying the date of harvesting potatoes in part A of tasks, students can find which factors have changed and which have not changed in this process, so as to help them build functions.
- Pudian is embedded this series of tasks: Within either part A, part B or part C, the tasks are hierarchically progressive. The former task is the Pudian of the latter. Part A is also the Pudian of part C, from using the concrete date in part A to the arbitrary date ( $x$ ) in part C. The design of this series of tasks helps students complete the transition from concrete questions to abstract thinking.
- OPMC serves for Pudian, OPMC is a means of paving the way in this episode.

Emma retained part A, removed part B and redesigned C of the original tasks through her documentation work. She mainly made three redesigns for the original series of tasks: 1) As the tasks are combined with the operation of calculator, Emma adds specific instructions of calculator operations, which is easier for students to conduct self-learning; 2) Add the graphical representation of a function; 3) Organize exercise 2 after exercise 1, which further deepen the teaching of function with calculator. Emma not only wants students to realize that calculators can be used to solve problems of function, but also wants them to realize the limitations of calculators. Through her redesign and organization, the teaching is made more comprehensive. Regarding actual variation,

- Contextual variation retained.
- Representational variation removed and emerged: Through Emma's documentation work, the conversions from natural language to table of values are removed as part B is deleted. New conversions among symbolic writings, table of values, and graphs emerged. The process of new conversions is all carried out by means of the operation of the calculator.
- OPMC retained.
- Pudian retained for exercise 1 and new Pudian emerged in exercise 2: Although

Emma removed several tasks when she redesigned part C of the original tasks, Pudian is retained. Series of tasks in exercise 2 designed by Emma are linked together, step by step. When students feel that they have found all the answers through the calculator, they then suddenly bump into the trouble when calculating  $f(30)$ , which is inconsistent with their previous knowledge. Conflict arose, which naturally led to the topic of calculator limitations.

In fact, in Emma's original arrangement, the teaching of function and calculator is not designed in this period of teaching (see section 6.2.1.1). It is added here because the idea of using calculators to solve function problems has been proposed in students' previous studies (see section 6.3.1 thematic lesson open box), reflecting their current learning needs. As this content is also included in the curriculum standard, Emma organizes the teaching in advance.

As for practical variations, Emma retained the potential and actual variations. The teaching of this series of tasks lasts in two lessons. In the first lesson, Emma leaves 30 minutes for students their own to think and she tours the classroom and gives individual guidance to students. She never gave direct answers but led the students to think through her interactions with students. For example, she encouraged students to check their previous work of the income of date 1<sup>st</sup> June, 2<sup>nd</sup> June, etc. when students encounter the problems of constructing a function, just as what shown in the working sheet. In the second lesson, Emma explained the tasks and made the correction for 35 minutes. During the classroom teaching, there are no practical variations emerged.

Emma's teaching is dynamic, constantly adjusting based on interactions with students. During her tour in class, she paid attention to the completion progress of the students and give them timely feedback according to the progress. For example, the first lesson took place in form of half-class, and because most of the students of the first half-class completed exercise 1 successfully, Emma gave the answers of part A 10 minutes after handing out the working sheet, and gave all the answers in exercise 1 before the end of the lesson; while the completion of the second half-class did not ideally, Emma wrote all the answers in exercise 1 on the blackboard just before the end of the lesson.

We present the evolution of episode 1 in figure 6.2.9. Emma has retained potential variations in her documentation work and created Pudian in her practical variations. Variations in this episode are embedded in the series of tasks before classroom teaching, carefully redesigned and reorganized by Emma through her documentation work.

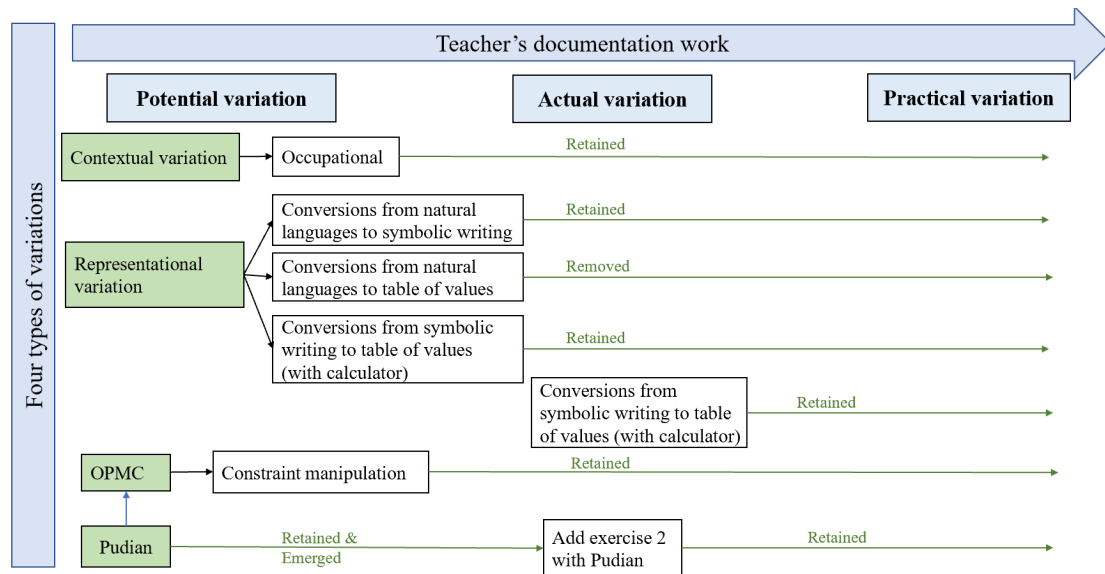


Figure 6.2.12 Schematic representation of variation path in episode 2

In general, this episode is designed to allow students to master strategies of using calculator to solve the tasks of function. In the process of teaching, Emma adopts Pudian embedded in the tasks to help students could discover and learn on their own. Finally, she acts as a mentor to help students improve their self-learning.

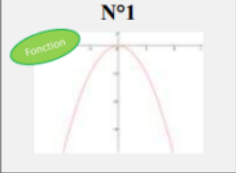
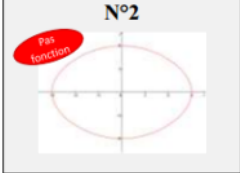
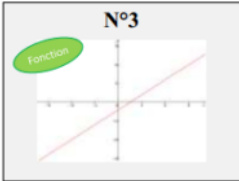
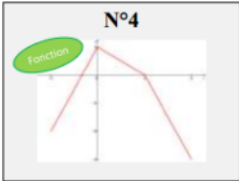
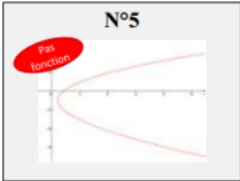
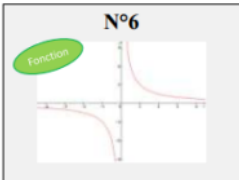

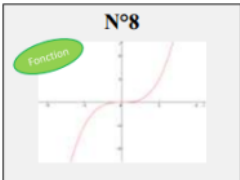
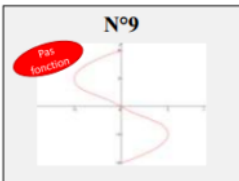

### 6.2.2.3 Episode 3: Series of tasks related to conceptual variation

This series of tasks (shown in table 6.2.3) is proposed in the second lesson of the concept of function. The purpose of these tasks is to further help students understand the concept of function through graphs. This series of tasks is selected by Emma from the resources of APMEP. This episode lasted 20 minutes in class. The context of this episode is mathematical.



Table 6.2.3 series of tasks about the concept of function

**CONSIGNE :** Fonction ou pas fonction ? Pour chaque courbe, dire si elle correspond à la représentation graphique d'une fonction ou non.

CORRECTION	N°1 	N°2 	
	N°3 	N°4 	N°5 
	N°6 	N°7 	N°8 
	N°9 	N°10 	FIN

Translation  
INSTRUCTIONS: Function or not function? For each curve, say whether it corresponds to the graphical representation of a function or not.

Examining the potential variation of this series of tasks:

- Representational variation embedded: There are conversions among natural languages, symbolic writing and graphs. When students use the definition to determine whether it is a function, they need to go through the conversions from mathematical languages (including symbolic writings and natural languages here) to graphs of function.
- Conceptual variation embedded: Standard, non-standard, and non-concept variations are all involved in this series of tasks. For students, the graphs of number 1 and 3 are standard variations. The graphs of number 4,6,7,8 are non-standard variations. The graphs of number 2, 5, 9, 10 are non-concept variations.
- OPMC is embedded in series of tasks: There are constraint manipulation of OPMC, taking profit of graphs of different types of function and common curves in high



school. Number 1 is the graph of a quadratic function, number 2 is the graph of an ellipse, number 3 is the graph of a linear function, number 4 is a piecewise function consists of linear functions, number 5 is the graph of a parabola, number 6 is the graph of a hyperbola, number 7 is the graph of a constant function, and number 8 is the graph of an odd function.

Regarding actual variation, Emma retains the original tasks with potential variations in her teaching with two reasons: 1) This series of tasks are basic tasks designed by a teacher from IREM and have been used by many teachers in class for years<sup>133</sup>. That is to say the task is verified in practical teaching; 2) This series of tasks fully meet her teaching needs.

As for practical variations, Emma retains this series of tasks with its potential variations. She leaves 5 minutes for students to think and then corrected with the students (15 minutes). During the process of correction, there are practical variations emerged:

- Representational variation retained.
- Conceptual variation retained.
- OPMC retained.
- Pudian emerged: Emma makes the best use of the OPMC that existed and pave the way for the following teaching. Because the graphs shown here are all common types in high school, in addition to helping students determine whether the graph is a function, Emma gives a brief introduction to each graph. She also allows students to find the domain of definition of the relatively unfamiliar function types (number 4 and number 6). These all pave the way for the later high school mathematics teaching.

We present the evolution of episode 2 in figure 6.2.13. Emma has retained potential variations in her documentation work and created Pudian in her practical variations.

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<sup>133</sup> This task comes in the form of a slideshow with very short questions (quite often True/False). <http://www.irem.univ-bpclermont.fr/Calcul-Mental-et-Automatismes-en,1306>

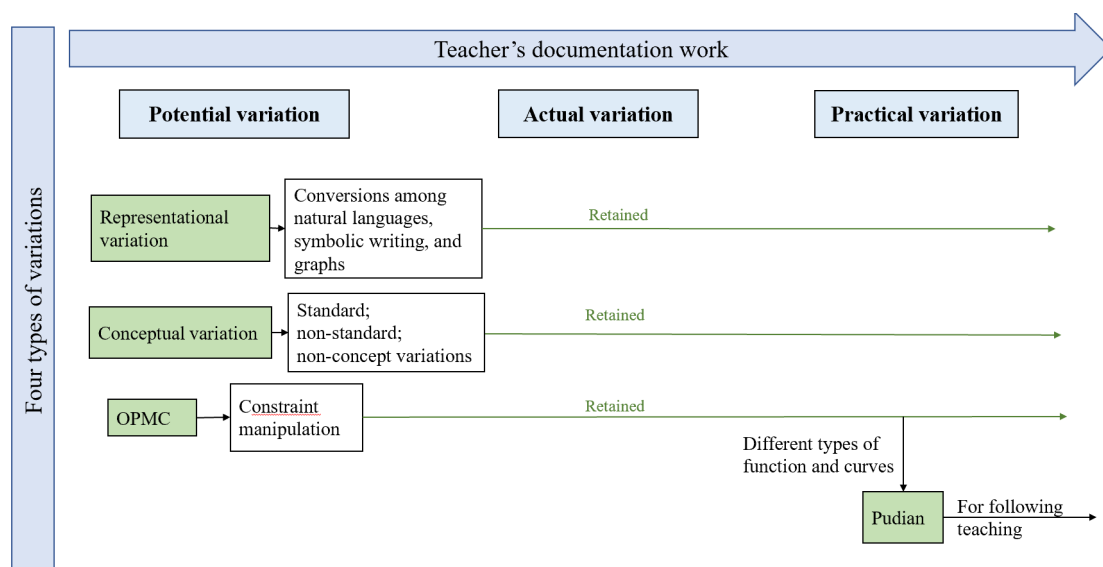


Figure 6.2.13 Schematic representation of variation path in episode 3

In fact, Emma used this kind of series of tasks (originally designed from APMEP) several times in the classroom teaching. This kind of series of tasks is mainly used for students' training on a specific concept or problem-solving strategy, see appendix 2 for more series of tasks.

#### 6.2.2.4 Episode 4: Series of tasks for building connections

This series of tasks (shown in table 6.2.4) is proposed in the lesson on the notion of variation of a function. It was presented to introduce this new topic. This series of tasks is selected by Emma from one textbook. This episode lasted 120 minutes in class. The context of this episode is mathematical.

This series of tasks is originally from one textbook (Indice, 2014) designed as an activity of digital education<sup>134</sup>, see figure 6.2.14.

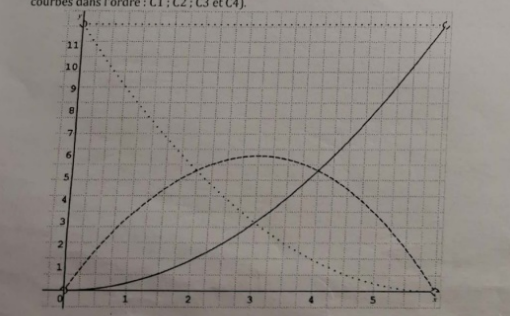
Examining the potential variation of this series of tasks:

- Representational variation embedded: Conversions here are various. There are conversions from natural languages to geometrical figures; between geometrical figures and graphs. The conversions from natural languages to geometrical figures can be verified with the help of geometric software. The place of symbolic writing in this task is not very clear.
- Pudian is embedded in series of task: From determining the range of  $x$  (task a), to imagining (rough) the variation of area (task b), and finally to presenting graphs

<sup>134</sup> Noted as "TICE" in figure 6.2.14, TICE= Technologies de l'information et de la communication pour l'enseignement (in French). It means digital education refers to the use of information and communication technologies for education in the context of education (TICE = TIC + education).

(accurate) of 4 functional changes about  $x$ .

Table 6.2.4 Working sheet of variation of a function

<p><b>Exercice 1</b>  ABC est un triangle isocèle en C tel que <math>AB = 6</math> et <math>AC = 5</math>.  Un point N se déplace sur le segment [AB] en restant différent de A et de B.  M et P sont les points respectivement sur [BC] et [AC] tels que ANMP est un parallélogramme.  On pose <math>AN = x</math>.</p> <ol style="list-style-type: none"> <li>Dans quel intervalle varie le réel <math>x</math> ?</li> <li>En imaginant que le point N se déplace de A vers B et donc que <math>x</math> augmente, décrire l'évolution de l'aire : <ul style="list-style-type: none"> <li>- du triangle ABC</li> <li>- des triangles PMC et NBM</li> <li>- du parallélogramme ANMP</li> </ul> </li> <li>On donne ci-dessous les représentations graphiques des quatre fonctions qui à <math>x</math> appartenant à I, associent chacune des aires de la question précédente. (noter les courbes dans l'ordre : C1 ; C2 ; C3 et C4).</li> </ol>  <p>Vérifier les résultats à la calculatrice en déterminant l'expression algébrique de chacune de ces fonctions.</p>	<p>Translation: Exercise 1</p> <p>ABC is an isosceles triangle in C. <math>AB = 6</math> and <math>AC = 5</math>.</p> <p>A point N moves on the segment [AB] while remaining different from A and from B. M and P are the points respectively on [BC] and [AC]. ANMP is a parallelogram.</p> <p>We set <math>AN = x</math></p> <ol style="list-style-type: none"> <li>In what interval does the real number <math>x</math> vary?</li> <li>By imagining that the point N moves from A to B and therefore that <math>x</math> increases, describe the evolution of the area: <ul style="list-style-type: none"> <li>- the triangle ABC; - the triangles PMC and NBM;</li> <li>- the parallelogram ANMP</li> </ul> </li> <li>We give below the graphic representations of the four functions to <math>x</math> belonging to I, associate each of the areas of the previous question. (Note the curves in the order: C1; C2; C3 and C4).</li> </ol> <p>Check the results on the calculator by determining the algebraic expression of each of these functions.</p>
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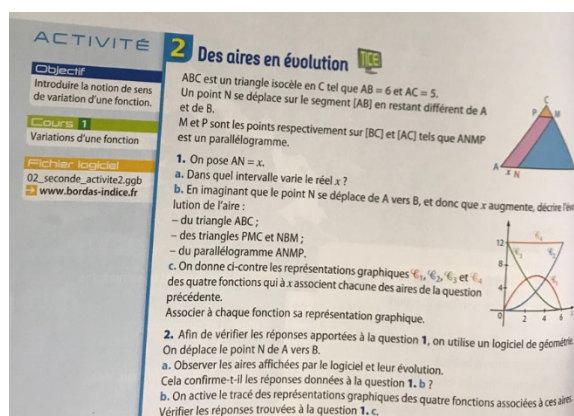


Figure 6.2.14 Original task in the textbook (Indice, 2014)

During her documentation work, Emma only retained the part 1 of the original tasks in her prepared students' working sheet and embedded part 2 of the original tasks in the figure presentation with GeoGebra software designed by herself (see Figure 6.2.15). According to Emma, she selected this series of tasks for two reasons: 1) Relate function with the problem of geometry; 2) The changes of the four areas caused by the moving point N can help her introduce new knowledge of variation of a function. Regarding actual variation, all the potential variations are retained.

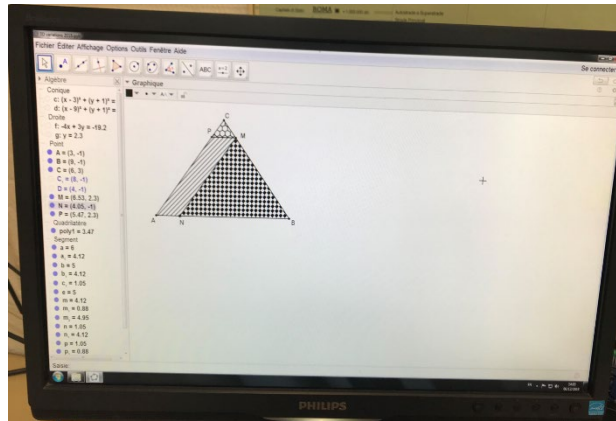


Figure 6.2.15 Presentation with GeoGebra software designed by Emma

As for practical variations, Emma retained the potential and actual variations. The teaching of this series of tasks lasts in two lessons. In the first lesson, Emma leaves 25 minutes for students their own to think and she tours the classroom and gives individual guidance to students. Then she uses GeoGebra to show students the moving trace of point N and explained the area of triangle ABC and NMB in the form of algebra (35 minutes). In the second lesson, Emma led students to express the area of parallelogram ANMP in the form of algebra and at last, she asked students to verify the answer through calculator (60 minutes). During the classroom teaching, there are practical variations emerged:

- Representational variation retained and extended: In classroom teaching, Emma clarified the role of symbolic writing in this series of tasks. The conversions between geometrical figures and symbolic writings are greatly extended. Emma led the students to use the geometric properties of the parallelogram ANMP and the isosceles triangle ABC to represent NB, AP in symbolic writings. There are also treatments inside symbolic writing, see red box in table 6.2.5.

Table 6.2.5 Emma's blackboard writing of episode 4

NBM et ABC sont des triangles semblables  
car  $(AC) \parallel (NM)$ .

Leurs longueurs de côté sont donc proportionnelles  
et on a donc

dans ABC	$AB = 6$	$AC = 5$	$AM = 4$
----------	----------	----------	----------

dans NMB	$NB = 6-x$	$AP = \frac{5(6-x)}{6}$	$h$ ↑
----------	------------	-------------------------	----------

hauteur issue  
de M

$$\frac{6}{6-x} = \frac{5}{5-x} \quad AP =$$

$$6(5-x) = 30-6x$$

$$5(6-x) = 30-5x$$

Translation:

NBM and ABC are similar triangles because  
 $AC \parallel NM$ .

Therefore, their side lengths are proportional,  
so we have

ABC	$AB = 6$	$AC = 5$	$AM = 4$
NMB	$NB = 6-x$	$AP = \frac{5(6-x)}{6}$	$h$ ↑

- Pudian extended: Emma adds Pudian in classroom teaching. In order to analyze the variation of the area of the four figures, Emma first led the students to try to express the length of the base and height of each figure in algebraic form, which paved the way for determining the variation of the area.
- OPMS emerged: Emma is not only satisfied that students make judgments through imagination and their intuitive feelings (with the help of geometric software) as shown in the textbook, she also quantitatively characterizes the areas of four kinds of figures from the perspective of functions, and finally she also had students review the use of calculators. Emma studied this series of tasks by three different means.

We present the evolution of episode 4 in figure 6.2.16. Emma has retained potential variations and reorganized the tasks in her documentation work. She extended representational variation and Pudian, and created OPMS in her classroom teaching.

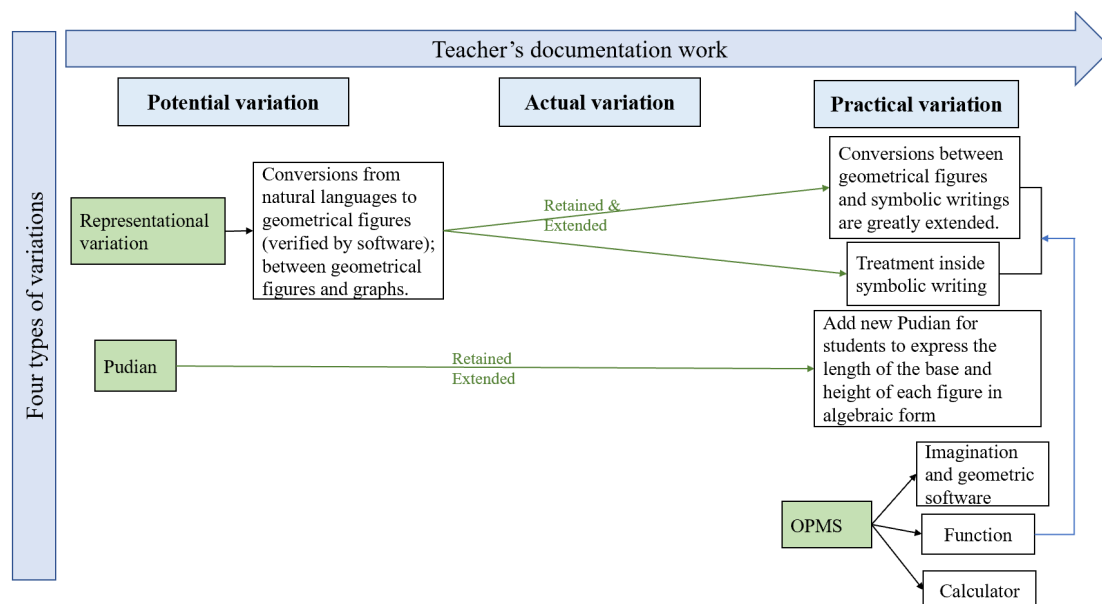


Figure 6.2.16 Schematic representation of variation path in episode 4

In general, this episode is designed to allow students to establish connections between geometry and function, also to master strategies of determining the variation of a function. In the process of teaching, Emma used different tools to solve tasks. She not only builds functions from geometric properties, which led to new conversions of representations, but also led students to review the use of calculators by verifying the answer.

### 6.2.2.5 Episode 5: Series of tasks of representational variation

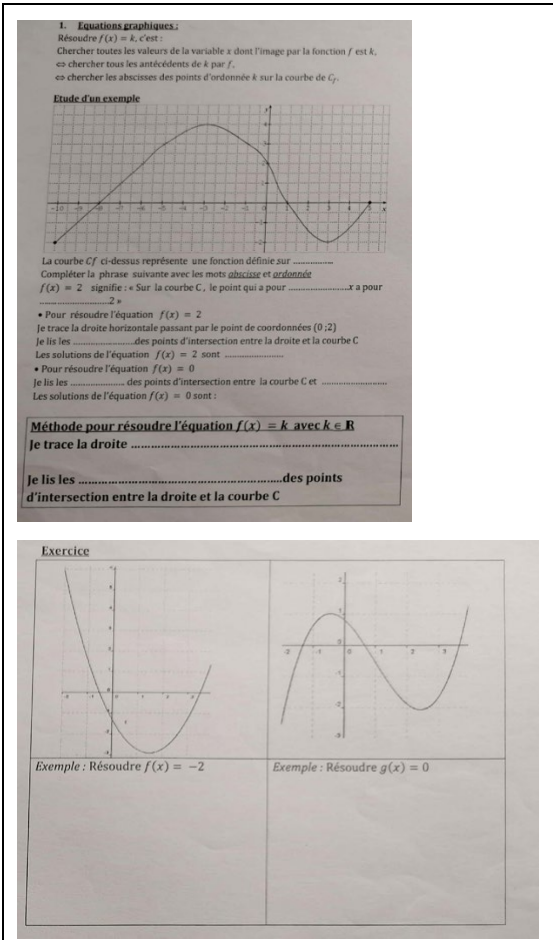
This episode includes two series of tasks (shown in table 6.2.6), one series shown as an example, and one series shown as exercise. They are proposed in the lesson of function and (in)equation from the graphical aspect. The purpose of these tasks is to further help

students understand the concept of function through graphs. This series of tasks is designed by Emma. This episode lasted 14 min20 in class. The context of this episode is mathematical.

Examining the potential variation of this series of tasks:

- Representational variation embedded: Mathematical languages are involved, which include the natural languages and symbolic writing here. Therefore, there are conversions from cartesian graph to mathematical languages (symbolic writing and natural languages).
- OPMC (MPOS) is embedded in two series of tasks: There is a goal manipulation of OPMC in the example by varying the value of  $f(x)$  (solve the equation  $f(x) = 2$  and solve the equation  $f(x) = 0$ ). There is a constraint manipulation of OPMC in the exercise by varying the graph of function. And no matter how these tasks vary, the method of solving them remains the same (MPOS).
- Pudian is embedded in these two series of tasks: The study of example paves the way for the exercise.

Table 6.2.6 Series of tasks of function and (in)equation from the graphical aspect

 <p><b>1. Equations graphiques:</b> Résoudre <math>f(x) = k</math>, c'est: Chercher toutes les valeurs de la variable <math>x</math> dont l'image par la fonction <math>f</math> est <math>k</math>, c'est chercher tous les antécédents de <math>k</math> par <math>f</math>, c'est chercher les abscisses des points d'ordonnée <math>k</math> sur la courbe de <math>C_f</math>.</p> <p><b>Etude d'un exemple</b></p> <p>La courbe <math>C_f</math> ci-dessous représente une fonction définie sur ..... Compléter la phrase suivante avec les mots <u>abscisse</u> et <u>ordonnée</u> <math>f(x) = 2</math> signifie : « Sur la courbe <math>C</math>, le point qui a pour ..... <math>x</math> a pour ..... 2 »</p> <p>• Pour résoudre l'équation <math>f(x) = 2</math> Je trace la droite horizontale passant par le point de coordonnées (0;2) Je lis les ..... des points d'intersection entre la droite et la courbe <math>C</math> Les solutions de l'équation <math>f(x) = 2</math> sont .....</p> <p>• Pour résoudre l'équation <math>f(x) = 0</math> Je lis les ..... des points d'intersection entre la courbe <math>C</math> et ..... Les solutions de l'équation <math>f(x) = 0</math> sont :</p> <p><b>Méthode pour résoudre l'équation <math>f(x) = k</math> avec <math>k \in \mathbb{R}</math></b> Je trace la droite ..... Je lis les ..... des points d'intersection entre la droite et la courbe <math>C</math></p> <p><b>Exercice</b></p> <p>Exemple : Résoudre <math>f(x) = -2</math>      Exemple : Résoudre <math>g(x) = 0</math></p>	<p>Translation:</p> <p><b>Study of an example</b></p> <p>The curve <math>C_f</math> above represents a function defined on .....</p> <p>Complete the following sentence with the words <u>abscissa</u> and <u>ordinate</u> <math>f(x) = 2</math> means: "On the curve <math>C</math>, the point which has for .... <math>x</math> has for ..... 2"</p> <p>• To solve the equation <math>f(x) = 2</math> I draw the horizontal line passing through the point of coordinates (0;2) I read the ..... of the points of intersection between the line and the curve <math>C</math> The solutions of the equation <math>f(x) = 2</math> are.....</p> <p>• To solve the equation <math>f(x) = 0</math> I read the .... of the points of intersection between the curve <math>C</math> and ..... The solutions of the equation <math>f(x) = 0</math> are:</p> <p><b>Method for solving the equation <math>f(x) = k</math> with <math>k \in \mathbb{R}</math></b> I draw the line..... I read the..... of the points of intersection between the line and the curve <math>C</math></p> <p><b>Exercise</b> Example: Solve <math>f(x) = -2</math> Example: Solve <math>g(x) = 0</math></p>
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According to Emma, this working sheet is designed by herself through her documentation work. Therefore, actual variations and potential variations are the same in this episode. Emma's main idea is to summarize the method of solving an equation from the graph of a function through the study of an example, and then use exercise to let students practice by themselves. Emma designed a series of such tasks by herself to help students read the graphs of function and to make a connection between function and (in)equations. According to Emma, after doing so, the students' mastery of the corresponding methods has been greatly improved.

The teaching of this series of tasks lasts 14 min 20: Emma first leaves 2 minutes for students their own to think. Then she explains the example and summarized the method (9 minutes20). At last, she asked students to answer the two exercises orally (3 minutes). As for practical variations,

- Transformation of subjects who implement representational variation: Through the organization of the classroom, the main body of this conversion has changed from the teacher (Emma) to students. In the teaching of example, Emma leads the students to find and summarize methods in mathematical language; while in the exercise stage, students are asked to imitate the description in the example and describe the process of solving the task in mathematical language. At this time, the conversions from cartesian graph to mathematical languages is completely implemented by students themselves.
- OPMC (MPOS) retained.
- Pudian retained and extended: In classroom teaching, before giving this working sheet, Emma also led the students to review the task "area of swimming" in thematic lesson 2. In the solution of that task, students got the maximum value from referring to the graph of the function in the calculator, instead of calculating in algebra. This is actually similar to solving the equation through the graph here, as both of them involve the conversions from the graph to the symbolic writing. Therefore, Pudian is extended through Emma's organization of tasks.

We present the evolution of episode 5 in figure 6.2.17. Emma has retained potential variations and actual variations in her own documentation work. New practical variations emerged through her didactical organizations.



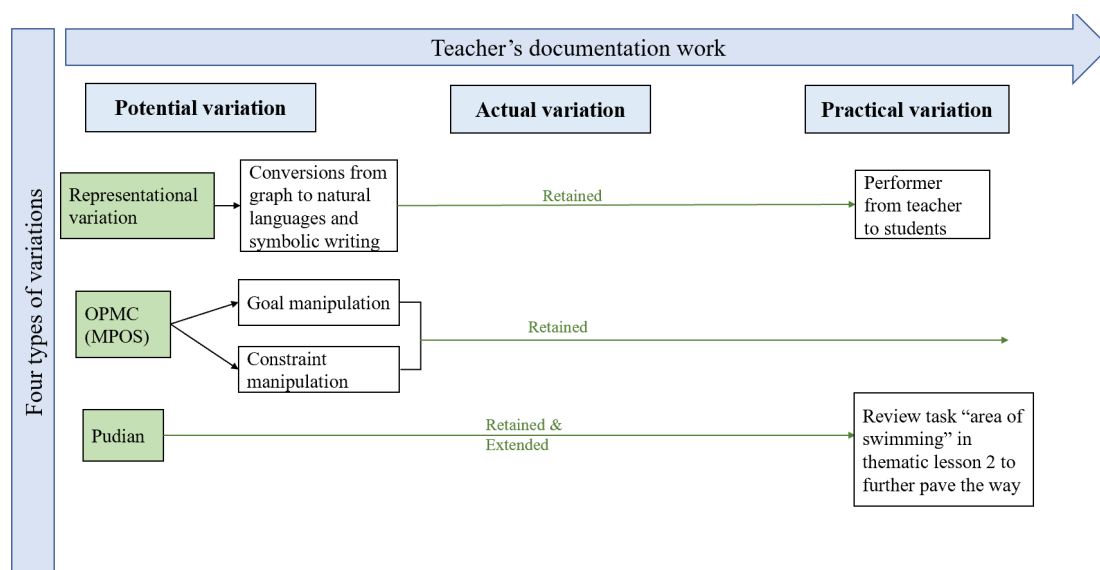


Figure 6.2.17 Schematic representation of variation path in episode 5

In general, these two series of tasks are designed to allow students to master strategies through Pudian and OPMC that created by Emma in her documentation work. All of OPMC created here are designed by varying non-essential features so as to ensure students repeatedly use the same strategy to solve the tasks (MPOS). In addition, Emma is good at making the teaching process smoother by using tasks that students are already familiar with to pave the way for new teaching content.

### 6.2.2.6 Conclusion

Emma tends to use series of tasks in classroom teaching and 78% of tasks in class are series of tasks. We summarize the characteristics of Emma's daily classroom teaching through the lens of variation as follows:

- Most of time in daily teaching, Emma retains the variations already organized and embedded in the tasks. If she can't find a suitable task for teaching, she will design by herself, and in such tasks, she will often use OPMC and representational variations.
- As Emma provides enough time for students to think, new variations emerge in class. There are mainly two types of practical variation: OPMS produced by students, and non-concept variations or Pudian produced by Emma. Non-concept variations are used to distinguish the concept and Pudian are used to lead students to master strategies.
- Emma's teaching is tightly connected to technology. She uses GeoGebra and calculator in the teaching of function. GeoGebra is used to help students understand dynamic tasks, and calculator is a tool that students need to master to solve tasks of function.



- In Emma's teaching of function, she pays attention to the natural languages. She encourages students to express the strategies in natural languages by using mathematical words. On the one hand, this meets the requirements in French curriculum standards; on the other hand, this may relate to Emma's collaborative work with one French teacher about the argumentation in mathematics and in French (description in section 6.1.2.1).

## 6.3 Variations in the thematic lessons

In the teaching of function, Emma arranged three thematic lessons with different teaching goals: to begin a new chapter, to summarize a chapter, and to deepen the knowledge. In this section, we first introduce three thematic lessons respectively, and then compare the thematic lessons from the perspective of variation.

### 6.3.1 Open box: practical variation generated in the interaction between teachers and students in open tasks

In the first thematic lesson, Emma presents an open task to begin the new chapter of function (before the teaching of function). This open task is selected and redesigned by Emma from series of problems proposed by DREAM group. It allows Emma to have an understanding of the students' previous function learning and to design tasks according to the students' needs for the subsequent teaching. This thematic lesson lasts 3 hours and 10minutes, in three lessons.

According to Emma, we analyze the potential variations based on the form of this task that Emma first saw in the DREAM group, organized and written by Antoine Guise (table 6.3.1).

Table 6.3.1 Original task of open box

Enoncé	[Translation]
À partir d'une feuille de papier au format A4, on veut contruire le patron d'une boîte sans couvercle qui a la forme d'un parallélépipède rectangle et qui a le plus grand volume possible. Quelles sont ses dimensions?	Statement: Taking a sheet of paper in A4 format, we want to build a box without lid, having the shape of a cuboid and the largest possible volume. What are its dimensions?

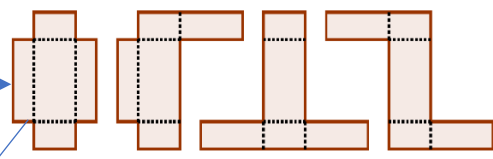
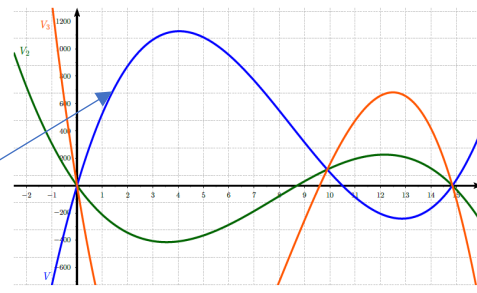
There are many **potential variations** already embedded in this task:

- Contextual variation: This open task is shown with personal context.
- Conceptual variation is embedded in the task: non-concept variations are not shown directly to students in this task. In fact, non-concept variation is produced by the students themselves in this problem and they would meet the contradiction later to force them to correct their earlier work. For example, the background of this task leads to the need to pay attention to the domain of the function created, which relates

to the definition of function. If the students didn't define the domain at the time of constructing a function, they would probably find this problem when they tried different numbers to find out the maximum. From Brousseau's view, this can be regarded as an *adidactical situation*<sup>135</sup>;

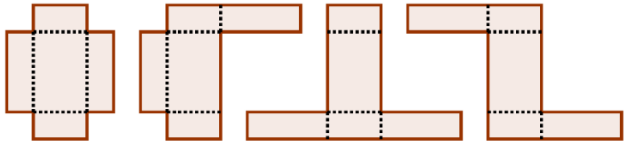
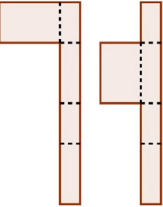
- Representational variation of this task is very rich: It involves various types of representations and conversions between them. According to the solution given in the resources, students will mainly experience the conversions of several representations, see table 6.3.2: (1) In the beginning, the problem is posed in natural languages, and students need to make conversions from natural languages to geometrical figures to build the box by their own hands. More detailed different types of patterns of box are in table 6.3.3; (2) When students find the way to fold the box, they need to make conversions from geometrical figures to symbolic writings to construct functions for different types of patterns of boxes; (3) In the process of finding the largest possible volume of box, students experience the conversions between symbolic writings and cartesian graphs. We can also find that many of the conversions are not one-time. For example, for each pattern, we have to build one function.

Table 6.3.2 Representations and conversions between representations in open box

	DISCURSIVE representations	NON-DISCURSIVE representations
MULTIFUNCTIONAL registers: Representations of NON-ALGORITHMIC TRANSFORMATIONS	Taking a sheet of paper in A4 format, we want to build a box without lid, having the shape of a cuboid and the largest possible volume. What are its dimensions?	 Pattern1   Pattern 2   Pattern 3   Pattern 4 ...
MONOFUNCTIONAL registers: Representations whose TRANSFORMATIONS are determined BY AN ALGORITHM	<p>Note <math>V(x)</math> is the function giving the volume of the box without lid in this pattern, and the height of the box is <math>x</math>.</p> <p>Pattern 1:  <math>x \in [0, 10.5]</math>, <math>V_1(x) = (29.7 - 2x)(21 - 2x)x</math>.</p> <p>Pattern 2: <math>x \in [8.7, 14.85]</math>,  <math>V_2(x) = (29.7 - 2x)(x - 8.7)x</math>.</p> <p>Pattern 3 and 4: <math>x \in [9.6, 14.85]</math>,  <math>V_3(x) = (29.7 - 2x)(4x - 38.4)x</math>.</p>	

<sup>135</sup> See section 2.2.2

Table 6.3.3 Different types of patterns of box (Antoine Guise, 2017)

3 × 3 type of patterns	
2 × 4 type of patterns	

- OPMC is proposed in the document written by Antoine Guise (see appendix 2): Resources proposed at the end that we can make changes to the task. For example, from an A4 sheet of paper, construct the pattern of a solid (closed or open) that has the largest possible volume.

This thematic lesson contains two different ways of documentation work: Emma's collaborative documentation work in the DREAM group, and Emma's individual documentation work after the group discussions. we analyse the actual variations in these two steps of instrumentalization (specially transformation in the instrumentalization) one by one.

Because there are both high school teachers and junior high school teachers in the DREAM group, and the grades of students they taught are different, the discussion in the group is mainly about how to redesign and adapt one same problem to make it suitable for students of different grades and levels. The analysis of collaborative work is based on the interview and conversations with Emma, and also the organized written report by the members of the group<sup>136</sup>. At that time, this task was redesigned as a problem for 9<sup>th</sup> grade students. Examining the **actual variations during this collaborative work**<sup>137</sup>:

- New OPMC emerged: Teachers proposed to change the conditions of the problem. They gave the new statement of the problem (see table 6.3.4). OPMC in this new statement are as follows: 1) Change A4 sheet to a square sheet of cardboard with a side of 20 cm (an integer value); 2) The folding method is directly determined in

<sup>136</sup> See chapter 3.1.3 and 3.3.2.

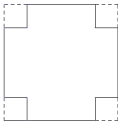
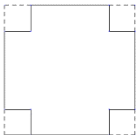
<sup>137</sup> In fact, the discussion on this open box problem in the dream group is ongoing. Through practice in the classroom and reflections after class, teachers share the ideas in the group, further discuss, revise and improve the original task. Our data in this thesis was collected from November 2018 to February 2019, but the discussions among teachers continues. We can find relevant resources of practice from 2015/2016 online. Until 2020, the dream group is still updating the documents about open box. Retrieved from: [https://clarolineconnect.univ-lyon1.fr/icap\\_website/1324/41186](https://clarolineconnect.univ-lyon1.fr/icap_website/1324/41186)

the statement, and there is no need to discuss other folding methods.

In addition to proposing a new statement, teachers have discussed and proposed the following possible extensions (variations): 1) Construction of patterns: We can start with the construction of patterns by the students (each one will take a different dimension) and already begin to frame the solution value; 2) The shape of the box can vary: square base, rectangle, triangle (equilateral or rectangle) for a prism or a pyramid, or even a cylinder. In this case, it will be necessary to propose a starting shape for the pattern, which will be blocked by the dimensions of the sheet; 3) Consider rational numbers (integers, decimals and fractions): Care must be taken that the solution is not a decimal number (When  $x = \frac{10}{3}$ , we obtain a maximum of the function).

- Pudian emerged: teachers designed a question (What volume will the box have if it is 3 cm high?) before the question of the largest possible volume. This is a Pudian designed for students to establish functional relationships.

Table 6.3.4 New statement of open box

<p>Dans une feuille cartonnée carrée de 20 cm de côté, on découpe quatre carrés identiques dans les angles. Avec ce patron, on réalise une boîte parallélépipédique de base carrée (sans couvercle).</p>  <p>Quel volume aura la boîte si elle a une hauteur de 3 cm ? Quelles dimensions donner à ce patron pour que le volume de la boîte soit le plus grand possible ?</p>
<p><b>Translation:</b> In a square sheet of cardboard with a side of 20 cm, we cut out four identical squares in the angles. With this pattern, we make a parallelepiped box with a square base (without lid).</p>  <p>What volume will the box have if it is 3 cm high? What dimensions should be given to this pattern so that the volume of the box is as large as possible?</p>

In general, in the new statement of group discussion, the A4 paper ( $29.7 \times 21$  cm) is changed into a square paper ( $20 \times 20$  cm), the folding method is given, and one question of Pudian is designed. All of these variations are proposed to further reduce the difficulty and better adapt to the level of students in 9<sup>th</sup> grade.

Regarding **Emma's individual documentation work**, Emma takes the original

problem showed in Guise's document<sup>138</sup> (see table 6.3.1) and intended to take full advantage of the variations embedded in it. Meanwhile, she made some variations according to her own class. Emma chose this task as an introduction of the notion of function for several reasons as follows: (1) This problem is in line with the concept of the curriculum standard and solving this problem involves knowledge of constructing functions, conversions between different representations of functions, and maximum values of functions. Through this process of exploration, students can realize the necessity of learning the knowledge of function, paving the way for the introduction of functions later; (2) This problem is very open, which provides students with a lot of space for exploration and for the generation of multiple solutions to one problem; (3) The difficulty of this problem is proper as a beginning task of a new chapter, which means to some extent, this task can be quite difficult for students without help, but with some guidance and support, they can solve it. What they find difficult is exactly what they are about to learn. Moreover, students of different capabilities can handle it in different levels; (4) This is not the first time for her to use this problem, so she is familiar with it and the teaching effect of using this problem in previous years is good.

In fact, Emma used the task from DREAM group in table 6.3.4 last year, taking a square of 20 cm by 20 cm. And all of students used the way of cutting in the table and nobody last year thought of trying another type of cutting. Therefore, Emma wonders if, by giving an A4 sheet that isn't square, students will think of other types of patterns or will they all try to do the same type like what is shown in table 6.3.4. Table 6.3.5 is the problem Emma posed in class with her explanations of organization and evaluations. As it is a lesson to begin a new chapter, Emma pointed out that the most important thing in this activity is not the result is right or not, but the process of thinking.

Table 6.3.5 Working sheets of open box sent to students with requirements

<p>Le problème posé:</p> <p>À partir d'une feuille de papier au format A4, on veut construire le patron d'une boîte sans couvercle qui a la forme d'un parallélépipède rectangle et qui a le plus grand volume possible. Quelles sont ses dimensions?</p> <p>A FAIRE PAR GROUPE DE 3 ou 4:</p> <ul style="list-style-type: none"> <li>- Avant de commencer la recherche, vous pourrez construire quelques boîtes et en calculer le volume.</li> <li>- Chercher à répondre à la question posée.</li> <li>- Rendre un compte-rendu par groupe, écrit sur les feuilles A4 blanches distribuées.</li> <li>- Toute idée est intéressante; même si la démarche n'aboutit pas, notez-la. Expliquez soigneusement tous les raisonnements effectués.</li> </ul>	<p>The problem posed:</p> <p>Taking a sheet of paper in A4 format, we want to build a box without lid, having the shape of a cuboid and the largest possible volume. What are its dimensions?</p> <p>TO DO IN GROUPS OF 3 or 4:</p> <ul style="list-style-type: none"> <li>- Before starting the research, you can build some boxes and calculate their volume.</li> <li>- Seek to answer the question asked.</li> <li>- Make a report by group, written on the white A4 sheets distributed.</li> <li>- Any idea is interesting; even if the process is unsuccessful, write it down. Explain carefully all the reasoning carried out.</li> <li>- Prepare an oral presentation to present your</li> </ul>
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<sup>138</sup> Since Emma is teaching students of 10th grade, she wants to give students enough space to think.

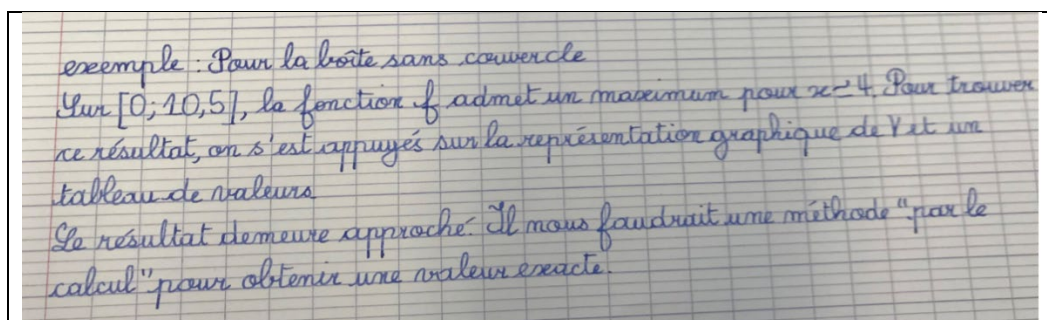
<p>- Préparer un oral pour présenter votre affiche. Sera évalué:</p> <ol style="list-style-type: none"> <li>1) L'originalité des démarches effectuées</li> <li>2) La conjecture d'une réponse</li> <li>3) Le raisonnement général, l'argumentation</li> <li>4) Le soin apporté au compte-rendu, à la construction des boîtes.</li> <li>5) La communication (à l'oral)</li> <li>6) L'écoute et la prise en compte d'autrui lorsque les autres groupes présenteront leur travail.</li> </ol>	<p>poster. Will be evaluated:</p> <ol style="list-style-type: none"> <li>1) The originality of the steps taken</li> <li>2) The conjecture of an answer</li> <li>3) General reasoning, argumentation</li> <li>4) The care taken in the report, in the construction of the boxes.</li> <li>5) Communication (oral)</li> <li>6) Listening to and taking into account others when other groups present their work.</li> </ol>
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Examining the actual variations during Emma's individual documentation work:

- Conceptual variation retained from the potential variation.
- Representational variation retained from the potential variation.
- Pudian created by Emma: As this open task covers almost all of the content to be studied in this chapter of function, Emma consciously adopted this task as a fundamental task in her subsequent teachings of function. For example, when Emma introduced the concept of function, she led students to go back to the task of open box, which students are familiar with, to arouse students' previous study memories and help their follow-up learning of function, see table 6.3.6. By the time this chapter teaching is finished, the students will be able to solve this open task completely by themselves.

Table 6.3.6 open box used as an example

<p>1. Appears when teaching the concept of functions <i>Exemples :</i> La fonction <math>f</math> définie par <math>f(x) = 2x + 3</math> est définie sur <math>\mathbb{R}</math>.</p> <p>La surface d'un disque en fonction de son rayon est définie sur <math>[0 ; +[</math> Le volume de la boîte sans couvercle construit avec le patron : ..... est la fonction définie sur <math>[0 ; 10,5]</math> par <math>V(x) = x(21 - 2x)(29,7 - 2x)</math></p> <p>[Translation] Examples: The volume of the box without cover built with the pattern: ..... is the function defined on <math>[0; 10,5]</math> by <math>V(x) = x(21 - 2x)(29,7 - 2x)</math></p>
<p>2. Appears when teaching table of values</p> <p>Exemple : un tableau de valeurs de la fonction Volume<sup>←1</sup> Obtenu à la calculatrice (partie calculs et touche preced + partie tableau):</p> <p>[Translation] Example: a table of values of the function Volume Obtained at the calculator</p>
<p>1. Appears when making a conclusion of this chapter of function (photo of one student's writing in her notebook)</p>



[translation]

Example: for the box without lid

On  $[0;10,5]$ , the function  $f$  admits a maximum for  $x = 4$ . To find this result, we relied on the graphical representation of  $V$  and a table of values.

The result remains approximate. We would need a method "for the computation" to get an exact value.

- Enhancing the formation of the concept created by Emma: Emma retains the big space in this open task for students to explore by themselves. As we mentioned above, Emma chooses this task as an introduction of the notion of function because every student can give it a try to solve this task and students of different capabilities can handle it in different levels. The problem can be reduced into different parts. According to what students have learned in junior high school before, they can use their previous knowledge of geometry and function<sup>139</sup> to build the box with A4 sheets and construct corresponding functions. Through open box, Emma connects new knowledge of function to the knowledge students have learnt and allows students to independently mobilize their previous learning experience of functions so as to promote their further study of functions, see figure 6.3.1.

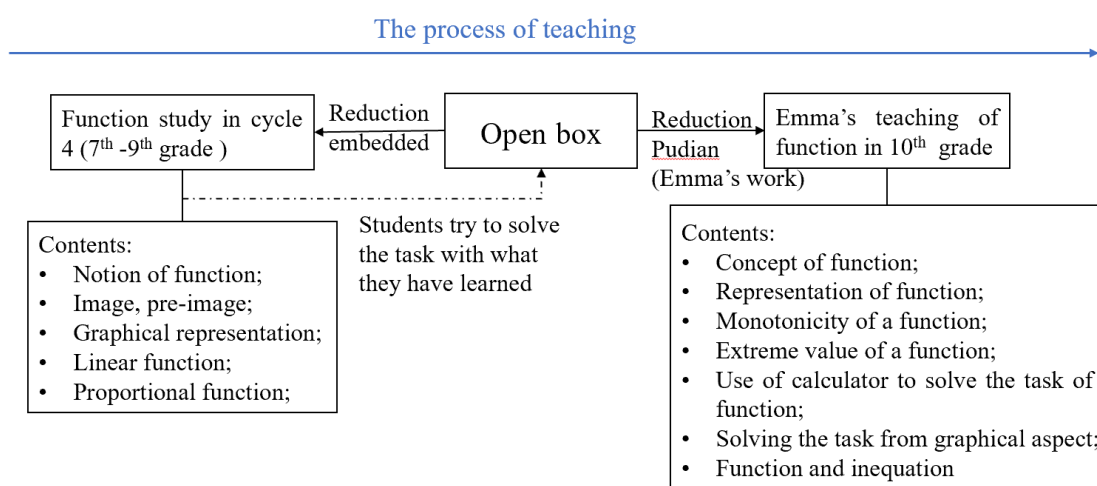


Figure 6.3.1 Open box connects old and new knowledge

<sup>139</sup> Based on the requirements in curriculum standards, see the link <https://eduscol.education.fr/document/17287/download>

- OPMC retained and removed: As various OPMC we discussed in the potential variation and actual variation in Dream group, Emma notices these possible OPMCs and determines which OPMCs to retain and which ones to remove according to the students' mathematical level in her own classroom.

Emma retains the following OPMC from potential variation: 1) The format of the sheet (A4,  $29.7 \times 21$  cm). Emma wants the solution involve decimal calculations, which students resist and are not good at; 2) Different patterns of box retained. Emma designs the lesson starting with the construction of patterns by the students. She hopes that this task could allow students to expand their thinking as much as possible, several ways of folding are points worthy of discussion to train students' ability of spatial imagination. Emma also found a corresponding example in the textbook to support her teaching, see figure 6.3.2.

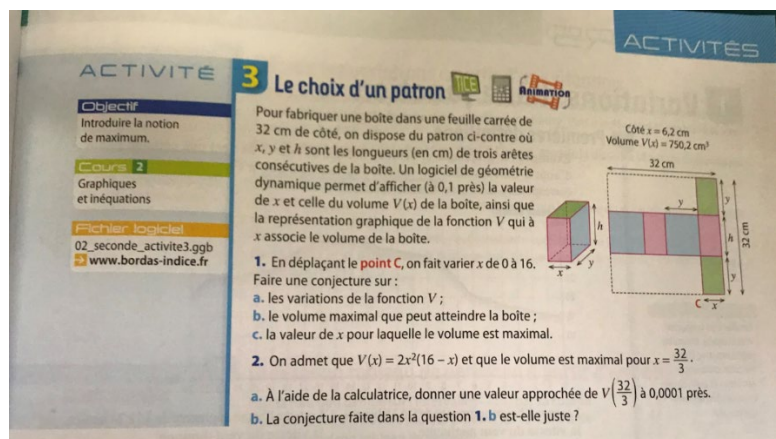


Figure 6.3.2 the choice of a pattern (Indice, 2014)

Emma removes one OPMC proposed in Dream group discussion: The shape of the box. As Emma said, there are a lot of things can vary in this open task, and talking about the shape of the box complicates the task and may deviate from the teaching goals of the function

From the perspective of didactics, Emma organized student-centered classroom under teacher's guidance in these three sessions. As she mentioned (section 6.1.3.3) that there is a specific process for the organization of this type of tasks: group work, written production by the students, and the whole-class summary. She followed this process and made it more specific. As shown in table 6.3.7, Emma organized three sessions with different goals for this task. In the first session, Emma first explained the statements of the task, clarified the requirements (the organization of the group, the group report). Then, students work in groups (3 or 4 persons in each group, 10 groups in total) to understand the problem and try to construct the box. when the students conducted group discussion, the teacher conducted a tour of the whole class. When the students had questions, they could ask the teacher at any time, and the teacher would give some



guidance, not directly tell them how to solve the task, but try to inspire their ideas. In the second session, students continue their previous discussion by group and Emma works as a mentor to give advice according to the needs of the students. Students are required to submit a group report about the solution of task at the end of this session. In the third session, students are asked to present their work in groups. When one group is presenting, the others need to take notes, reflect on the solutions. Emma made a summary of this task at the end of this session and each student hand in their personal reports. According to Emma, she wanted to leave enough time for students to think and discuss how to solve the problem both in class and after class. The teacher intended to be an “invisible” guide in the whole process and does not directly tell students the answer, but will help students when they have questions.

Table 6.3.7 Three sessions of open box

Session	Time	Students	Teacher's role	Teaching goal
1	40 min	Group work (3-4 students in each group), half-class	Explain the statement of task and requirements; Tour the classroom and give guidance	Understand the problem and construct the box
2	1 hour	Group work (Submit group report)	Tour the classroom and give guidance	Construct a function, study the maximum value of a function
Emma's documentation work on students' group reports				
3	1h30min	Oral presentation in groups (Submit personal report)	Make a summary of this task	Communicate and reflect
Emma's documentation work on students' personal reports and arranges her following teaching according to this				

The organization of these three sessions are quite flexible, which leaves enough room for Emma's adjustments of classroom teaching. Emma reflects and revises her own teaching plan based on the feedbacks of students in the practical teaching. As shown in table 6.3.7, there are two process of documentation work throughout these three sessions: 1) Emma worked on the group reports submitted after session 2, classified the strategies of students, and decided the order of presentation (preliminary ideas - partial correct - correct with some improvement - novel method); 2) Emma worked on students' personal reports. In students' personal reports, they need to record the presentations of other groups and provide feedback after the presentation of each group (whether they agree with this approach and what can be improved, etc.). Therefore, we can see students' own comments and comparisons on different strategies.

As for **practical variation**, most of variations are retained. New variations are created by students and some variations are reduced based on students' feedbacks and Emma's

reflections:

- Contextual variation emerged: When constructing the box, most of students think directly from a mathematical point of view, but some students use their experiences in daily life to help them build the box. For example, one group wrote in their report: “We thought of the same tactic to build a casserole (cocotte)” and then they tried to draw its pattern. There is another group of students who gave the box a specific personal context when constructing their open box. They named their open box “box of fine semolina + ketchup”, see figure 6.3.3.

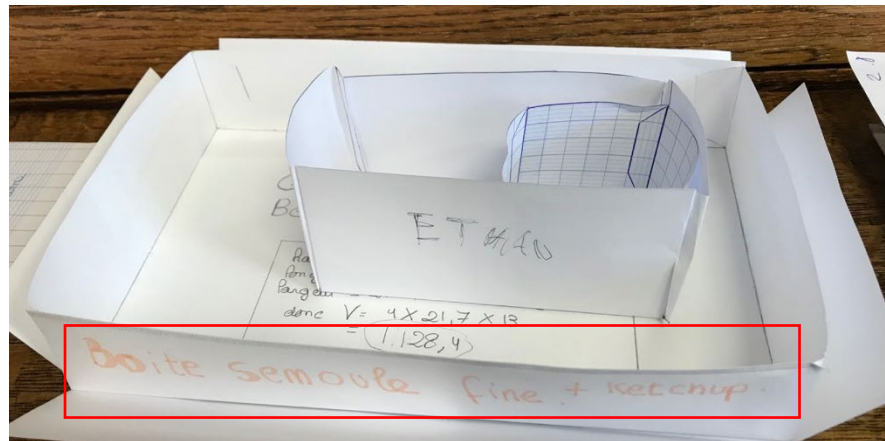
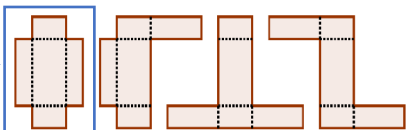
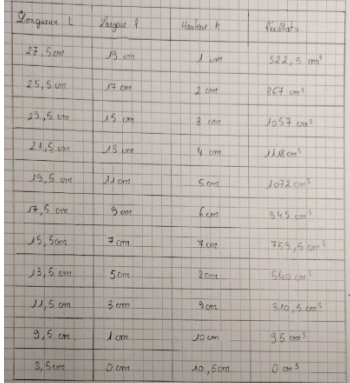
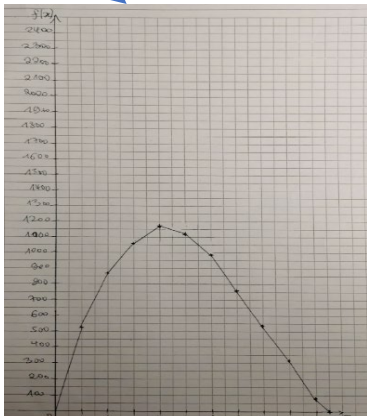


Figure 6.3.3 “Box of fine semolina + ketchup” named by students

- Conceptual variation is retained from potential variation.
- Representation representations retained and new one emerged: After constructing the box, most of the groups will try different numbers to help them find the maximum of the volume. One group of students draws a rough graph of the function by using a table of values (transitional auxiliary representations) to find the answer. In this situation, new representational variation emerged and the process of conversions of registers is: from natural languages, to geometrical figures, to symbolic writings, to transitional auxiliary representations, to graph, and finally go back to symbolic writings, see table 6.3.8.

Table 6.3.8 Conversions of registers in the classroom teaching

	DISCURSIVE representations	NON-DISCURSIVE representations
MULTIFUNCTIONAL registers: Representations of NON-ALGORITHMIC TRANSFORMATIONS	Natural languages: Taking a sheet of paper in A4 format, we want to build a box without lid, having the shape of a cuboid and the largest possible volume. What are its dimensions?	Geometrical figures:  ...
Transitional auxiliary representations	Table of values: 	
MONOFUNCTIONAL registers: Representations whose TRANSFORMATIONS are determined BY AN ALGORITHM	The symbolic writing: Note $V_{L,l}(x)$ is the function giving the volume of the box without lid in this pattern, and the height of the box is $x$ $\forall x \in [0, \min(\frac{l}{2}, \frac{L}{2})], V_{L,l}(x) = (L - 2x)(l - 2x)x$ $0 \leq L \leq 29.7$ $0 \leq l \leq 21$	Cartesian graphs: 

- Pudian emerged through interactions between Emma and her students: As we mentioned, Emma organized student-centered classroom under teacher's guidance and she intended to be an "invisible" guide in the whole process by not directly telling students the answer, but helping students think further (Pudian) through questioning. For example, When the students asked the teacher whether they got the right answer to the largest volume during group discussions, the dialogue between the teacher and student is as follows.

**Emma:** Why are you sure that when you take  $x = 4$ , you get the largest volume?

**Student** (pointed at his table of values and graph of the function he has drawn): because I also took other values,  $x = 2, 3, \dots, 10$ , but none of them were as large as this value.

**Emma:** Don't you plan to try other values? How do you prove that you get the most?

Emma adopted heuristic education<sup>140</sup> in class. During the inspection, the teacher found that except for the group using the calculator, all the other students have the problem to confirm they got the right answer.

- OPMS emerged: there are different strategies emerged through students' group work to find the maximum of the volume. We mainly divide it into 4 categories (table 6.3.9): 1) Number trial: some groups try to vary the height of the box and calculate the corresponding volume to find the maximum volume; 2) Make hypotheses and try to find the relationship between height or bottom area and volume: Some groups make hypotheses and try different values to verify it; 3) Construct the function, table of values, and draw the graph of function based on the points in the table of values. By observing the trends of variation of volume (increase or decrease) to find the maximum from the graph; 4) Use calculator: Two groups of students used calculator to find the maximum after constructing the function.

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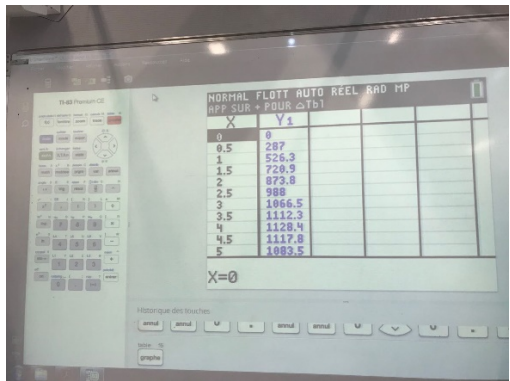
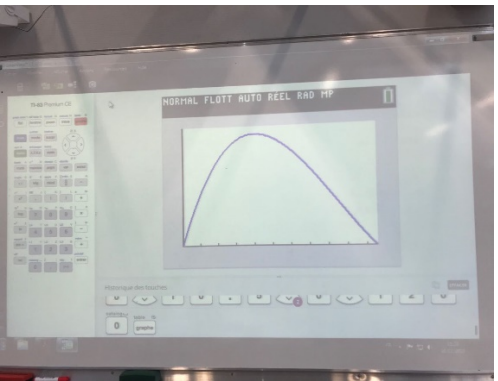
<sup>140</sup> heuristic education

Table 6.3.9 Different strategies shown in group reports

Strategy 1	<p>Number trial:</p> <div><p>Exemple : <math>3 \times 2 = 6</math> <math>21 - 6 = 15</math>; <math>29,7 - 6 = 23,7</math> <math>15 \times 23,7 = 355,5</math> <math>355,5 \times 3 = 1066,5 \text{ cm}^3</math></p><p><math>4 \times 2 = 8</math> <math>21 - 8 = 13</math>; <math>29,7 - 8 = 21,7</math> <math>13 \times 21,7 = 282,1</math> <math>282,1 \times 4 = 1128,4 \text{ cm}^3</math></p><p><math>5 \times 2 = 10</math> <math>21 - 10 = 11</math>; <math>29,7 - 10 = 19,7</math> <math>11 \times 19,7 = 216,7</math> <math>216,7 \times 5 = 1083,5 \text{ cm}^3</math></p></div>	<p><math>h=3\text{cm}, V=1066.5\text{cm}^3</math></p> <p><math>h=4\text{cm}, V=1128.4\text{cm}^3</math></p> <p><math>h=5, V=1083.5\text{cm}^3</math></p>																																												
Strategy 2	<p>Make hypotheses and try to find the relationship between height or base area and volume:</p> <div><p>Hypothèse 1: Plus la hauteur est grande, plus le volume est grand.</p><p><math>N = 7 \times (21 - 2 \times 7) \times (29,7 - 2 \times 7)</math> <math>= 7 \times 7 \times 15,7</math> <math>= 769,3 \text{ cm}^3</math></p><p><math>N = 5 \times (21 - 2 \times 5) \times (29,7 - 2 \times 5)</math> <math>= 5 \times 11 \times 19,7</math> <math>= 1083,5 \text{ cm}^3</math></p><p><math>769,3 &lt; 1083,5</math>, 1<sup>re</sup> hypothèse est fautive.</p><p>Hypothèse 2: Plus la hauteur est petite, plus le volume est grand.</p><p><math>N = 2 \times (21 - 2 \times 2) \times (29,7 - 2 \times 2)</math> <math>= 2 \times 17 \times 25,7</math> <math>= 873,8 \text{ cm}^3</math></p><p><math>N = 0,5 \times (21 - 0,5 \times 2) \times (29,7 - 0,5 \times 2)</math> <math>= 0,5 \times 20 \times 28,7</math> <math>= 287 \text{ cm}^3</math></p><p><math>873,8 &gt; 287</math>, 2<sup>e</sup> hypothèse est fautive.</p><p>Hypothèse 3: Il faut donc essayer un juste milieu en balayant.</p><p><math>V = 3 \times (21 - 2 \times 3) \times (29,7 - 2 \times 3)</math> <math>= 3 \times 15 \times 23,7</math> <math>= 1066,5 \text{ cm}^3</math></p><p><math>V = 4 \times (21 - 2 \times 4) \times (29,7 - 2 \times 4)</math> <math>= 4 \times 13 \times 21,7</math> <math>= 1128,4 \text{ cm}^3</math></p></div>	<p>Hypothesis 1: the greater the height, the greater the volume</p> <p>Hypothesis 2: the greater the bottom (area of the bottom of box), the greater the volume</p> <p>Hypothesis 3: we must therefore find the exact value by exploring</p>																																												
Strategy 3	<p>Use table of values, graph of the function</p> <div><table><tr><th>Longueur L</th><th>Largeur l</th><th>Hauteur h</th><th>Volume V</th></tr><tr><td>21,0 cm</td><td>2,0 cm</td><td>3 cm</td><td>342,1 cm³</td></tr><tr><td>21,0 cm</td><td>2,0 cm</td><td>2 cm</td><td>355,5 cm³</td></tr><tr><td>21,0 cm</td><td>2,0 cm</td><td>1 cm</td><td>355,5 cm³</td></tr><tr><td>21,0 cm</td><td>2,0 cm</td><td>0 cm</td><td>0 cm³</td></tr><tr><td>21,0 cm</td><td>3,0 cm</td><td>2 cm</td><td>355,5 cm³</td></tr><tr><td>21,0 cm</td><td>3,0 cm</td><td>1 cm</td><td>355,5 cm³</td></tr><tr><td>21,0 cm</td><td>3,0 cm</td><td>0 cm</td><td>0 cm³</td></tr><tr><td>21,0 cm</td><td>4,0 cm</td><td>1 cm</td><td>355,5 cm³</td></tr><tr><td>21,0 cm</td><td>4,0 cm</td><td>0 cm</td><td>0 cm³</td></tr><tr><td>21,0 cm</td><td>5,0 cm</td><td>0 cm</td><td>0 cm³</td></tr></table></div>	Longueur L	Largeur l	Hauteur h	Volume V	21,0 cm	2,0 cm	3 cm	342,1 cm³	21,0 cm	2,0 cm	2 cm	355,5 cm³	21,0 cm	2,0 cm	1 cm	355,5 cm³	21,0 cm	2,0 cm	0 cm	0 cm³	21,0 cm	3,0 cm	2 cm	355,5 cm³	21,0 cm	3,0 cm	1 cm	355,5 cm³	21,0 cm	3,0 cm	0 cm	0 cm³	21,0 cm	4,0 cm	1 cm	355,5 cm³	21,0 cm	4,0 cm	0 cm	0 cm³	21,0 cm	5,0 cm	0 cm	0 cm³	
Longueur L	Largeur l	Hauteur h	Volume V																																											
21,0 cm	2,0 cm	3 cm	342,1 cm³																																											
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21,0 cm	5,0 cm	0 cm	0 cm³																																											
Strategy 4	<p>Construct a function, use the calculator:</p> <div><p>Pour déterminer le volume de la boîte en fonction de x en cm, il y a la fonction: <math>f(x) = x \times (21 - 2x) \times (29,7 - 2x)</math></p><p>Pour obtenir toutes les réponses possibles on utilise la fonction.</p><p>Pour la calculatrice: <math>V1 = x \times (21 - 2x) \times (29,7 - 2x)</math> (2nde) (def table) dans TVL=0 ΔTVL=0,5 (2nde) (table)</p><p>Le meilleur résultat obtenu est <math>x=4</math>, avec lequel on obtient <math>V=1128,4</math></p></div>	<p>function</p> <p>calculator</p>																																												

Strategy 4 is beyond teacher’s expectations, and appears as a critical didactical variable: Emma arranged groups using calculator to be the last ones to present in class, which is convenient for class scheduling and can be adjusted at will. After their presentations, the rest of the students expressed great interest in this strategy. Emma let students in these two groups to be the teacher to present the operation process of the calculator on the screen, see table 6.3.10. Moreover, Emma took advantage of the opportunity to advance the teaching of calculators, as it is also required by the curriculum. A few days later she formally introduced how to use the calculator to solve function problems, and the limitations of the calculator, see section 6.2.2.2. With the feedbacks from students and her operation of didactics, Emma has advanced her teaching step by step.

Table 6.3.10 Students present on the screen about the operations on the calculator

	
Use calculator to construct table of values	Use calculator to construct graph of a function

- OPMC reduced based on the students’ feedbacks: Emma finally removed the explanation of different patterns of the box in class. Because when she read students’ group reports, she found that all the students adopted the same pattern of box (see figure 6.3.4, open box constructed by students). Since no students proposed a different folding method, Emma removed this variation after considering the students’ current mathematical level, lesson objectives and teaching schedule. She doesn’t want to add new contents to interfere with the students’ learning of functions and increase the burden on the students when they still have many doubts about functions.

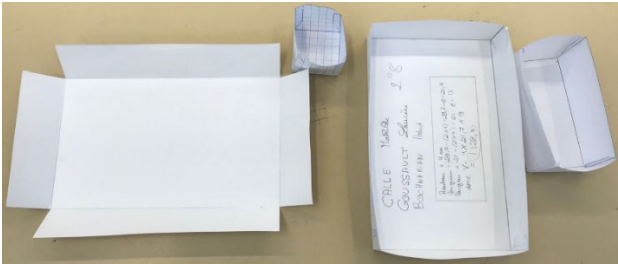


Figure 6.3.4 Open box constructed by students

It is worth mentioning that Emma's documentation work in this process of teaching is cyclic and is constantly modified based on the continuous feedbacks from the students, see figure 6.3.5. New actual variation generated by Emma after organizing, summarizing, and reflecting on the students' responses in class. For example, the reduced OPMC (different patterns of box) we mentioned in practical variations. Actual variation in Emma's documentation work is dynamic and relied on the interactions between teacher and students. Emma contacts with her students through classroom discussions (oral interactions), group reports (written interactions), and individual reports (written interaction).

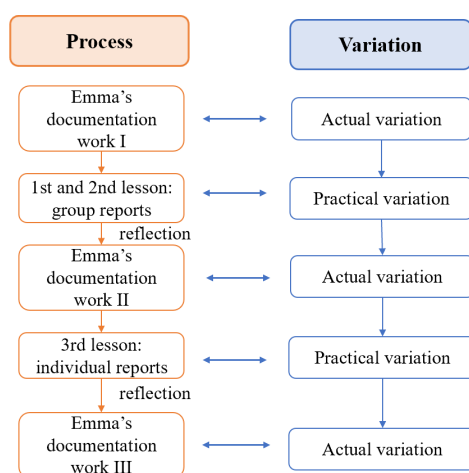


Figure 6.3.5 Emma's cyclic documentation work of this open task

We present the evolution of variation in the teaching of open box in figure 6.3.6. Emma makes full use of the potential variations of this open task and reorganizes it according to her own class. Documentation work in this case is quite complex, not only the involvement of IREM group work, but the evolutions over years.

Teachers' work is very crucial in such open task case. In this case, the potential variation is very broad and it is necessary for the teacher to design specific variation and adaptations as actual variation according to her own teaching goals. Emma's documentation work is inseparable from practical variation in this thematic lesson, as she reflects, modifies, and adapts the task according to students' feedbacks and her own experience.

Emma is quite satisfied with students' performance in the thematic lesson, as she explained: *"The expression of the volume function has been found by many groups. It is necessary to make understand the interest of the formula (and therefore of the definition of a function) without everyone getting bored. I am quite satisfied with the progress of students' personal report this year. There is a good listening, a relevance of the students orally and participation. They went so far as to use the calculator, which made it possible to move towards this goal quite quickly in the chapter."*



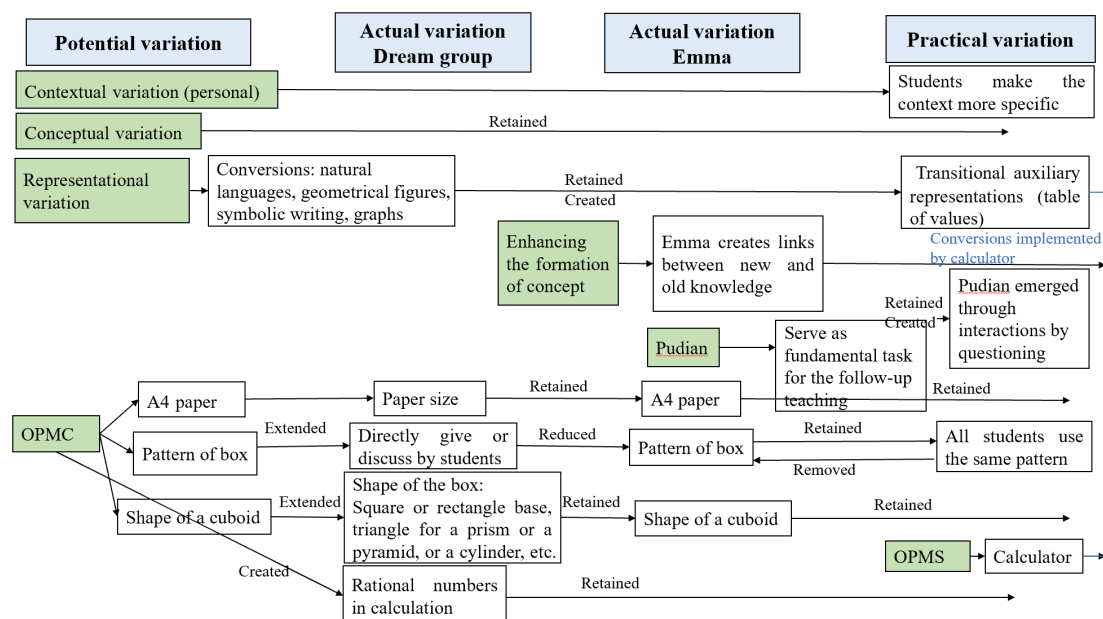


Figure 6.3.6 Schematic representation of variation path of open box

### 6.3.2 Swimming area: greatly remove Pudian

This series of tasks (shown in table 6.3.11) named “swimming area” is proposed at the end of the chapter of generalities of the function (definition, image, pre-image, table of variation, and extremum from the aspect of graph), before introducing the in (equation) and function from graphical aspect. Emma chooses the task from the online resources of IREM. This thematic lesson lasts 60 minutes in class.

Examining the potential variation of this series of tasks:

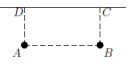
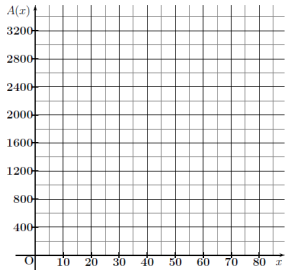
- Contextual variation is in this series of tasks: The context of task here is occupational, which is accessible to students in their daily lives.
- Representational variation is embedded in the series of tasks: There are conversions from natural languages to symbolic writings (task 3), from symbolic writings to transitional auxiliary representations (table of values, task 4), from table of values to natural languages using mathematical words (task 5), from transitional auxiliary representations (table of values) to cartesian graph (task 6), from cartesian graph to natural languages or symbolic writing (task 7, 8, 9). There are also treatments inside the symbolic writing: numerical calculation (task 2, 3).
- Pudian is embedded in the series of tasks: This task is designed with hierarchical tasks, centering on the question “how to place A and B so that the swimming area is as large as possible with this material”. In order to solve this task, students are led by several tasks step by step, from calculation of specific numbers (task 1, 2) to get the inspiration of constructing a function (task 3) to studying the graph of a function (task), and finally solving the task (task 7). Generally speaking, it is



gradually from concrete calculation to abstract study of a function.

- OPMC is embedded: To help students construct a function, there is a constraint manipulation (variation of specific numbers)<sup>141</sup> in tasks 1 and 2. To let students fully master the recognition of the function graph, there is a constraint manipulation (variation of specific numbers) and a symmetry of OPMC in task 8.

Table 6.3.11 A series task of swimming area from IREM

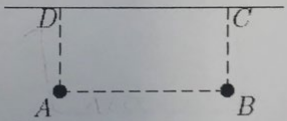
Aire de baignade		Translation																				
<p>Pour aménager une aire de baignade surveillée de forme rectangulaire, on dispose d'un cordon flottant de 160 mètres de longueur, et de deux bouées A et B.</p> <p>On souhaiterait trouver comment placer A et B pour que l'aire de baignade soit la plus grande possible avec ce matériel.</p> 		<p><b>Swimming area</b></p> <p>To create a rectangular swimming area, we have a 160 meters long floating cordon and two buoys A and B. We would like to find how to place A and B so that the swimming area is as large as possible with this material?</p>																				
<ol style="list-style-type: none"> <li>Si la distance de la bouée A à la rive est de 20 mètres, quelle est alors la longueur de la zone de baignade? Quelle est son aire?</li> <li>Mêmes questions si la bouée A est à 35 mètres de la rive.</li> <li>On va maintenant déterminer l'expression <math>A(x)</math> qui donne l'aire de la zone de baignade en fonction de la distance <math>AD</math> notée <math>x</math>.               <ol style="list-style-type: none"> <li>Pourquoi <math>x</math> ne peut-il varier que entre 0 et 80?</li> <li>Pourquoi la longueur <math>AB</math> de la zone de baignade est-elle égale à <math>160 - 2x</math>?</li> <li>Exprimer alors l'aire <math>A(x)</math> de la zone de baignade en fonction de <math>x</math>.</li> </ol> </li> <li>Compléter le tableau suivant à l'aide de la formule trouvée en 3.c) :</li> </ol>		<ol style="list-style-type: none"> <li>If the distance from buoy A to the shore is 20 meters, then what is the length of the zone of swimming? What is its area?</li> <li>Same questions if buoy A is 35 meters from shore.</li> <li>We are now going to determine the expression <math>A(x)</math> which gives the area of the swimming area as a function of</li> </ol>																				
<table border="1"> <thead> <tr> <th><math>x</math></th> <th>0</th> <th>10</th> <th>20</th> <th>30</th> <th>40</th> <th>50</th> <th>60</th> <th>70</th> <th>80</th> </tr> </thead> <tbody> <tr> <td><math>A(x)</math></td> <td>0</td> <td></td> <td></td> <td></td> <td></td> <td></td> <td></td> <td></td> <td></td> </tr> </tbody> </table>		$x$	0	10	20	30	40	50	60	70	80	$A(x)$	0									<ol style="list-style-type: none"> <li>of the distance <math>AD</math> denoted by <math>x</math>.</li> <li>Why can <math>x</math> only vary between 0 and 80?</li> <li>Why is the length <math>AB</math> of the swimming area equal to <math>160 - 2x</math>?</li> </ol>
$x$	0	10	20	30	40	50	60	70	80													
$A(x)$	0																					
<ol style="list-style-type: none"> <li>Écrire une phrase contenant l'expression « image par la fonction <math>A</math> », et une autre contenant l'expression « antécédent par la fonction <math>A</math> », compatibles avec le tableau précédent.</li> <li>Sur le graphique suivant, placer tous les points d'abscisse <math>x</math> et d'ordonnée <math>A(x)</math> lus dans le tableau, puis les relier.</li> </ol> 		<ol style="list-style-type: none"> <li>Then express the area <math>A(x)</math> of the swimming area as a function of <math>x</math>.</li> <li>Complete the following table using the formula found in 3.c)</li> <li>Write a sentence containing the expression « image by the function <math>A</math> », and another containing the expression « preimage by the function <math>A</math> », compatible with the preceding table.</li> </ol>																				
<ol style="list-style-type: none"> <li>En utilisant le graphique, dire pour quelle valeur de <math>x</math> l'aire semble maximale. Conclure.</li> <li>Sur le graphique peut-on trouver la valeur de l'aire de la zone de baignade pour <math>AD = 25</math> mètres? et pour <math>AD = 75</math> mètres? et la valeur de <math>AD</math> pour que l'aire soit de 2000 mètres carrés?</li> <li>Déterminer tous les <math>x</math> pour lesquels l'aire <math>A(x)</math> est plus grande que 2400 mètres carrés.</li> </ol>		<ol style="list-style-type: none"> <li>On the following graph, place all the points of abscissa <math>x</math> and ordinate <math>A(x)</math> read in the table, then connect them.</li> <li>Using the graph, say for which value of <math>x</math> the area seems to be maximal. Conclude.</li> <li>On the graph can we find the value of the area of the swimming zone for <math>AD = 25</math> meters? and for <math>AD = 75</math> meters? and the value of <math>AD</math> so that the area is 2000 square meters?</li> <li>Determine all <math>x</math> for which the area <math>A(x)</math> is greater than 2400 square meters.</li> </ol>																				

<sup>141</sup> See the various strategies of OPMC in section 2.2.3.

Emma placed this task at this time (after teaching the general information about the functions and before specifically introducing reading the graph of a function) for dual effects: to examine whether students have mastered the concepts and properties of the function that they have learned before; and to connect with subsequent teachings. Emma goes back to review the “swimming area” before teaching the relations of graphs and (in)equations (see section 6.2.2.5). Accordingly, she made some changes to the task according to the goal of the lesson. Table 6.3.12 presents the series of tasks of swimming area redesigned by Emma. We examine the actual variations through Emma’s documentation work:

- Contextual variation retained.
- Pudian is removed: The number of tasks has been greatly reduced, from 9 tasks to 3 tasks. Emma removed all the tasks for Pudian (task 1-6), and only kept task 7, part of task 8, and task 9.
- Representational variation retained: Even though the tasks are greatly reduced by Emma, most of the representational variation is retained. The conversions among natural languages, symbolic writing, table of values, and cartesian graphs are retained, except the conversion from table of values to natural languages (task 5) is removed. Because these conversions and treatments are embedded in the process of solving the task.
- OPMC removed: Because of the changes made by Emma, OPMC in task 1 and 2 is removed, and the variation of specific numbers and symmetry of OPMC in task 8 is removed.

Table 6.3.12 Series of tasks of swimming area redesigned by Emma

	<p><b>Exercice 5</b>          Pour aménager une aire de baignade surveillée de forme rectangulaire, on dispose d'un cordon flottant de 160 mètres de longueur, et de deux bouées A et B. On souhaiterait trouver</p> <p>1) Comment placer A et B pour que l'aire de baignade soit la plus grande possible avec ce matériel ?</p> <p>2) Comment placer la bouée A pour que pour que l'aire soit de 2000 mètres carrés ?</p> <p>3) Comment placer la bouée A pour que pour que l'aire soit au moins de 2400 mètres carrés ?</p>	
<p>Translation          Exercice 5</p>	<p>To create a rectangular swimming area, we have a 160 meters long floating cord and two buoys A and B. We would like to find</p> <p>1) How to place A and B so that the swimming area is as large as possible with this material?</p> <p>2) How to place the buoy A so that the area is 2000 square meters?</p> <p>3) How to place the buoy A so that the area is at least 2400 square meters?</p>	

As for practical variations, Emma retained the actual variations and new variations produced by students.

- OPMS emerged: Because Emma removed the setting of  $AD = x$  in her adapted series of tasks, some students considered setting the length of AD to  $x$  or the length of AB to  $x$  when solving the task, see figure 6.3.7.

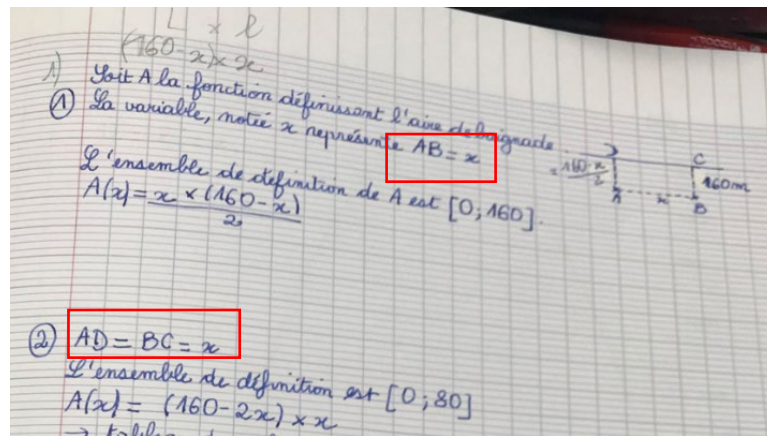


Figure 6.3.7 Two ways of construct a function in one student's sheet

- Students encountered difficulties in the conversions from graph to symbolic writing: Most students have no difficulty in table of values and making graphs by using the table of values. However, students are often not used to using graphs to solve the task. They are used to solving tasks directly through the treatments of algebraic registers inside symbolic writing, see one student's handwriting in blue box in table 6.3.13. Moreover, students always miss information when reading graph of a function. For example, when the student is seeking the value of  $x$  for  $f(x) \geq 2400$ , she only pays attention to the half-branch of the function curve, see the student's handwriting in red box in table 6.3.13.

Table 6.3.13 Photo of one student's notebook

→ tableau de valeurs

x	1	5	10	15	20	25	30	35	40	45	50	55	60	65	70	75	80
f(x)	158	750	1400	1950	2400	2750	3000	3150	3200	3150	3000	2750	2400	1950	1400	750	158

$$160 - 2x \times 40 = 80$$

→ voir graphique derrière

Un maximum pour  $x = 40$ . Ses dimensions sont alors 40 et 80.

Si  $x = 15$  ou  $x = 65$

Vérification:  $f(15) = 15 \times (160 - 2 \times 15) = 1950$

15 et 65. Soient des valeurs approchées des solutions

2)  $A(?) = 2000 \text{ m}^2$

$$(160 - 2x) \times x = 2000$$

$$160x - 2x^2 = 2000$$

3) Il faut qu'elle se trouve à 20m ou plus.

On souhaite résoudre, à l'aide du graphique,  $f(x) \geq 2400$

$f(x) = x(160 - 2x)$ . L'ensemble des solutions est  $[20, 60]$

Translation

Blue box:

$$A(?) = 2000 \text{ m}^2$$

$$(160 - 2x)x = 2000$$

$$160x - 2x^2 = 2000$$

Red box:

Student answer: It should be at 20m or more.

Correction by herself after Emma's explanation: We want to solve, using the graph  $f(x) \geq 2400$ ,  $f(x) = x(160 - 2x)$ . The solution is  $[20, 60]$ .

We present the evolution of variation in swimming area in figure 6.3.8. Emma has greatly reduced potential variations in her documentation work. She removed all the tasks for Pudian, which indirectly leads to the weakening of OPMC and representational variation.

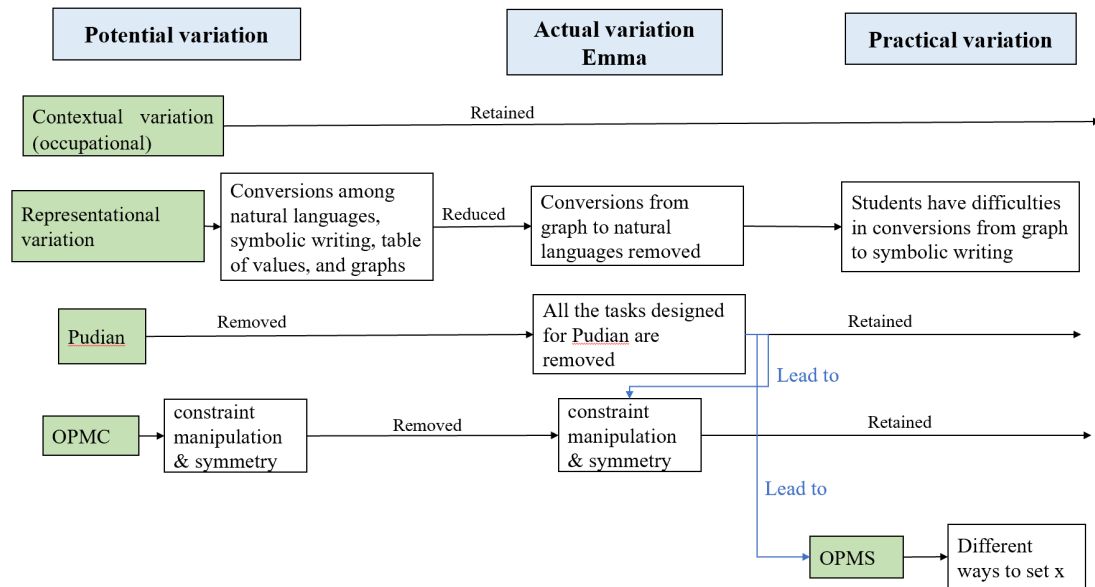


Figure 6.3.8 Schematic representation of variation path of swimming area

Emma's choice of actual variation is closely related to teaching goals. Because this task appeared in lesson of conclusion of this chapter on functions, the students have already learned the corresponding function knowledge, Emma thinks there is no need for Pudian. The (re)design and organization of this task is to test whether the students have mastered the functional knowledge they have learned.

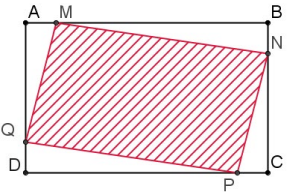
### 6.3.3 Area of parallelogram: create Pudian and make connections

This task (shown in table 6.3.14) named "area of parallelogram" is proposed to deepen students' knowledge of extremum of a function. This content is added by Emma and not in the requirements of curriculum. The difference from the previous study is that the extreme value in this task is obtained by algebraic operations instead of calculator. Emma chooses the task from her digital document repository. This thematic lesson lasts 3 hours in class, 2 hours around this task and 1 hour for the knowledge involved.

Examining the **potential variation** of this task:

- Representational variation is embedded in this task: This task is a combination of functions and geometric figures, the geometric problem needs to be transformed into a function problem, and find the answer through algebraic operations. Therefore, there are conversions from natural languages to geometrical figures, and from geometrical figures to symbolic writing. There are also treatments in the symbolic writing.

Table 6.3.14 task of area of parallelogram from Emma's digital document repository

<p>Exemple 1 :</p> <p>ABCD est un rectangle. <math>AB = 6,5 \text{ cm}</math> ; <math>BC = 4 \text{ cm}</math>.</p> <p>M est un point du segment <math>[AB]</math>, N est un point du segment <math>[BC]</math> ;</p> <p>P est un point du segment <math>[CD]</math>, Q est un point du segment <math>[DA]</math>.</p> <p>De plus, on a <math>AM = BN = CP = DQ</math>.</p> <p>Où faut-il placer le point M pour que l'aire du quadrilatère MNPQ soit la plus petite possible ?</p> 	<p>[Translation]</p> <p>Example 1:</p> <p>ABCD is a rectangle. <math>AB=6.5\text{cm}</math>; <math>BC = 4 \text{ cm}</math>.</p> <p>M is a point of the segment <math>[AB]</math>, N is a point of the segment <math>[BC]</math>;</p> <p>P is a point of the segment <math>[CD]</math>, Q is a point of the segment <math>[DA]</math>.</p> <p>Moreover, we have <math>AM = BN = CP = DQ</math>.</p> <p>Where should point M be placed so that the area of parallelogram MNPQ is as small as possible?</p>
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Emma directly takes this task from her own files which she has worked on for years. For her, this is a classic task that can be proposed as an initiative or guided task. According to Emma, the disadvantages of this task are that the process of modelling may be long for students and the difficulties to find the canonical form for students themselves. The advantages are that the teacher can lead students to guess an approximate result by using GeoGebra, and the exact value of this task cannot be found by graphic reading so interest for the demonstration. Due to these factors, Emma divided the problem into 6 steps and guided students to complete it step by step, see table 6.3.15.

Table 6.3.15 Problems into 6 steps created by Emma

Step 1:

$AM = BN = CP = DQ = x$

Why do we have  $x \in [0; 4]$ ?

What is the area of rectangle ABCD?

Represent AQ and MB with a function of  $x$ .

Explain why the area of triangle AMQ is equal to  $\frac{x(4-x)}{2}$ .

What is the area of triangle MBN with a function of  $x$ ?

Determine the area of the parallelogram MNPQ as a function of  $x$ .

Step 2:

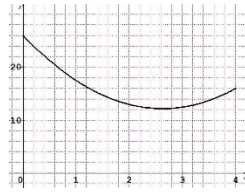
Represent the graph of the function  $f(x) = 2x^2 - 10,5x + 26$  defined on  $[0;4]$

Before representing it, complete the table of values below:

$x$	0	0,2	0,4	0,6	0,8	1	1,2
$f(x)$							
$x$	1,4	1,6	1,8	2	2,2	2,4	2,6
$f(x)$							
$x$	2,8	3	3,2	3,4	3,6	3,8	4
$f(x)$							

Step 3:

Here is the graph of the function  $f$



According to the graph, what is the minimum of the function  $f$  on  $[0;4]$ ? For what value of  $x$  is it reached?

Solve the inequality  $f(x) \geq 12$  from the graph.

Step 4:

We want to demonstrate that the minimum is precisely reached for  $x = \frac{21}{8}$

- Calculate the exact value of  $f\left(\frac{21}{8}\right)$ .
- Find the definition of the minimum of a function.
- Complete the following sentence with positive or negative

$f\left(\frac{21}{8}\right)$  is the minimum of  $f$  on  $[0; 4]$  if and only if  $f(x) - f\left(\frac{21}{8}\right)$  is ..... for all  $x \in [0; 4]$ .

- Calculate  $f(x) - f\left(\frac{21}{8}\right)$  and its variation.

Step 5:

Demonstrate that for all  $x$  belonging to  $[0; 4]$ ,

$$f(x) = 2\left(x - \frac{21}{8}\right)^2 + \frac{391}{32}$$

Deduce the variation of  $f(x) - f\left(\frac{21}{8}\right)$

What can be deduced from this for the problem posed?

Step 6:

Calculate  $f(x) - f\left(\frac{21}{8}\right)$  and try to recognize a remarkable identity.

What can be deduced from this for the problem posed?

We examine the actual variations through Emma's documentation work:

- Pudian created by Emma: Emma reduced the task into several small tasks which students are familiar with, making Pudian to inspire students. By doing so, the difficulty of the problem is greatly reduced.
- Representational variation extended: New representational variation emerged with the six steps. There are new conversions related to graphs and table of values. Conversions from symbolic writing to table of values, and to graph emerge in step 2. Conversion from graph to symbolic writing emerges in step 3.

Emma organized three sessions (3h) for this task in classroom teaching. The first session (1h) is for students to work alone while the teacher tours and give individual guidance. The second session (1h) is for the teacher to explain the strategy to the whole class. And the last one (1h) is to conclude the knowledge in this task.

As for **practical variation**, all of actual variations are retained and new variations are



created by students: One of the students encountered the problem of constructing the function for parallelogram MNPQ. However, he found out this problem is quite familiar. He searched the notes of open box in his notebook and imitate the strategies in open box to solve this problem. This new variation was beyond Emma's expectations, and she asked other students who had difficulties to return to the open box to find inspirations.

We present the evolution of variation in area of parallelogram in figure 6.3.9. Emma has greatly reduced the difficulty of tasks by making Pudian in her documentation work.

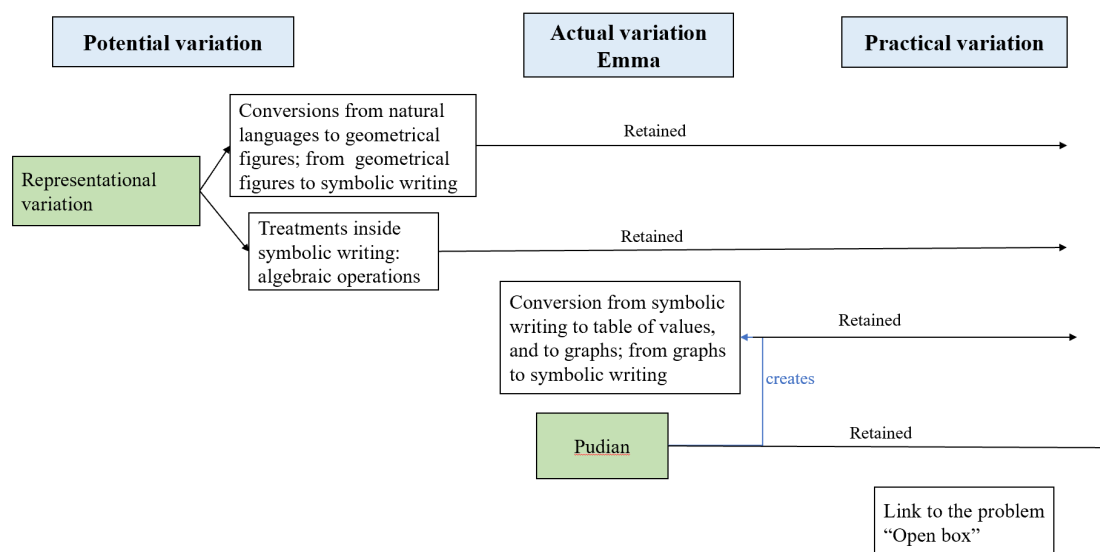


Figure 6.3.9 Schematic representation of variation path of area of parallelogram

The task of area of parallelogram serves in this theme lesson for deepening knowledge. On the one hand, Emma reduces the difficulty and allows students to review the previous knowledge by creating Pudian in her documentation work. On the other hand, the students' performance in the classroom reflects the role of thematic lesson. The task in the thematic lesson works as a fundamental task to help students to come up with inspirations and to help them reduce when they encounter new problems.

### 6.3.4 Conclusion: Unifying problems promotes classroom teaching

In this section, we conclude three thematic lessons from three perspectives : 1) proposition of the word "Unifying problems"; 2) unifying problems and curriculum standards; 3) unifying problems and variations.

#### 6.3.4.1 Unifying problems

Emma chose to use the word "unifying problem" (problème fédérateur in French) to call the tasks used in these three thematic lessons. According to her, a unifying problem is a comprehensive problem, which can illustrate the knowledge of the whole chapter. The key of unifying problem is that you can't solve this problem without using the

target knowledge.

This kind of problem lasts a long time in classroom teaching, it will take almost a week for students to study on this task before the teacher starts the teaching of new chapter. However, the subsequent teaching goes quickly by using this problem as an example. Emma illustrates the concepts and properties with the problem. For example, after open box (section 6.3.1), Emma defines one function in particular, called “the volume function”. When teaching extremum, the variations, the graphic representation of a function, volume function is always a perfect example.

Unifying problem often can be extended continuously. One kind of extension is about the problem itself: we can redesign a series of tasks from the unifying problem and these series of tasks can be used as DM or DS (section 6.3.3). The other kind of extension is that the unifying problem may not be completely solved in teaching, even at the end of this chapter, students can continue to explore it in their following studies. Like the problem of open box, students may get a value close to the answer, they cannot guarantee they got an exact answer. The algebraic way of using basic inequality or derivative method to find the exact value will be presented later.

As Emma mentions that teachers sometimes cannot predict what students will produce while working on the thematic lesson about a unifying problem. For students, studying a unifying problem is a process of mathematical exploration. The effect of teaching largely depends on students’ curiosity and motivation. It’s a challenge but also very interesting, teachers need to interact with students and constantly monitoring them so as to adjust their teachings based on students’ feedback and needs.

#### **6.3.4.2 Unifying problems and curriculum standards**

In fact, when teachers in IREM group was discussing the design or redesign of an unifying problem, they try to ensure that the problem should meet curriculum standards. Here we take the open box as an example to illustrate how mathematical competences embedded in the teaching of this problem:

- 1) Students are expected to construct a mathematical model of function through the context, which is consistent with mathematical competences of modeling (modéliser) proposed in the French curriculum standards to translate a real situation into mathematical language (Guise, 2018).
- 2) This problem contains four ways of representation of function: natural language, symbolic formula, graph, and table of values. In the process of solving problem, it is important for students to manipulate the conversions among these four representations, which is consistent with mathematical competences of representing (représenter) proposed in the French curriculum standards to choose and use suitable representations for dealing with a problem or for studying a mathematical



object (Guise, 2018);

- 3) This problem allows students to mobilize the necessary knowledge, analyze and exploit mistakes, test several solutions, which is consistent with mathematical competences of reasoning (*raisonner*) and searching (*chercher*) proposed in the French curriculum standards. The whole process of solving this problem made students engage in a scientific process, observe, question, manipulate, experiment (on a sheet of paper, with objects, using calculator), formulate hypotheses, look for examples or counter-examples, simplify or particularize a situation, make a guess (Guise, 2018).
- 4) The teacher arranged group work report, oral presentations and discussions for students to use natural language to explain mathematical problems. This organization is consistent with mathematical competences of communicating (*communiquer*) proposed in the French curriculum standards to make the link between natural language and algebraic language, to distinguish the specificities of mathematical language compared to the French language, to explain orally or in writing (his/her approach, reasoning, a calculation, a geometric construction protocol, an algorithm), and to understand the explanations of another and argue in the exchange (Guise, 2018).

#### **6.3.4.3 Unifying problems, variations, and teaching goals**

The Unifying problem often implies a lot of potential variations. The teacher chooses to retain, extend, reduce, remove, or create according to his or her teaching goals. We illustrate the relationship between unifying problems, variations, and teaching goals by comparing the different variation methods adopted by Emma in her documentation work in these three thematic lessons.

All of these three unifying tasks start from a geometric figure to construct a function, but to different degrees. In the Open box, students need to build their own three-dimensional figures, while in the swimming area, students only need to build the function from a plane figure. For the area of parallelogram, the task adds a moving point on the basis of a plane figure. Emma adopts different variation measures, mainly on procedural variations, for three problems in her documentation work. We present the information of these three problems in table 6.3.16.

By using different variations, different teaching effects and teaching goals can be achieved. As shown in table 6.3.16, Emma increases or decreases the difficulty of problems by removing or creating Pudian, adjusts the openness of problems by operating OPMC, and enhancing the formation of concept through establishing connections between tasks and knowledge.

Table 6.3.16 Information of three unifying problems

	Open box	Swimming area	Area of parallelogram
Role of the problem	To begin a chapter	To end a chapter	To deepen the knowledge
Teaching time	3h10	1h	3h
Enhancing the formation of concept	Created by Emma in follow-up teaching		
Pudian	Created by Emma in follow-up teaching	Removed by Emma	Created by Emma
OPMC	Retained some and reduced some by Emma (Select the proper to retain)	Removed by Emma (Chain reaction of remove Pudian)	
OPMS	Created by students	Created by students	

Thematic lessons with unifying problems are typical and iconic in Emma's teaching of function. They work as key points to connect the mathematical contents in the whole chapter, and is the product of Emma's collaborative work. We may be able to learn from these three thematic lessons of Emma, using a relatively open task as a beginning of a chapter to broaden students' horizons, and a relatively closed task as a conclusion to summarize chapter knowledge, and finally use one task to connect old and new knowledge, to help students go further.

## 6.4 Conclusion

In this section, we try to draw a conclusion of teaching through variation in French case study based on the analysis above and answer the three research questions we proposed in section 2.4.3.

### 6.4.1 Emma's resources with potential variation: expanding by absorbing new resources from collaborative work

In this section, we try to answer the first research question "What are the potential variations in the upper secondary school mathematics textbooks and other curriculum resources in France?" we have been proposed, which is from the perspective of resources.

Emma constantly enriches her resource system by seeking external support, especially teachers' collaborative work. She does not directly bring the resources from outside and put it into her digital document repository, but will redesign the tasks based on her students' needs and her teaching goal. She practices the task in class, reflect from the classroom teaching, revise the documents, and use it again in practice. Through such cyclic documentation work, potential variation in the resources keeps evolving.

Emma's collaborative work creates a lot of space for potential variations. She is no longer limited to the relatively closed tasks in the textbook, and tries unifying problems in class, which not only guide students to explore knowledge, but also cultivate students' competences. Because of the characteristics of the unifying problem (section 6.3.4), there are many possibilities for variations in the tasks itself, in the organization, and also raised by the students.

#### **6.4.2 Emma's design of variation: Iteratively evolved in practice**

In this section, we try to answer the second research question "What are the actual variations in the French case to achieve the teaching goal?" we have been proposed, which is from the perspective of teacher's design.

In Emma's teaching of function, all kinds of variations appear. Emma prefers to retain the potential variations embedded in the resources. Emma doesn't reduce or remove any potential variations in the tasks for daily teaching. However, for unifying problems used in the thematic lessons, she sometimes reduces or removes variations that is not very relevant.

Representational variations appear the most in Emma's class. Conversions between different kinds of registers are all involved in Emma's teaching of function. Besides the conversions aroused by several representations of function itself, many tasks related with geometric figures, which leads to the conversions between geometric figures and symbolic writing. Emma also pays attention to students' expressions in natural languages by using mathematical words. The variations created by Emma in her documentation work is mainly Pudian and enhancing the formation of concepts and propositions. Pudian is a flexible way for her to redesign tasks with students' needs. When Emma designs tasks by herself, she prefers to use OPMC to ensure that students master the strategies. Emma rarely directly presents OPMS in class as she prefers to leave enough space for students to produce OPMS and then naturally leads students to the teaching goal with what they produced. Different types of variations do not exist in isolation, they are interconnected and influence each other. A change in one variation may cause a chain reaction in another.

Emma's design of variation is mainly affected by two factors: teaching goal and students' performance and needs.

Based on the variations used by Emma in the teaching of function and their corresponding teaching goals, we associate different teaching goals with different types of variation, shown in table 6.4.1.

Table 6.4.1 variation used by Emma and their corresponding teaching goals

Teaching goal	Variation used
Master problem-solving strategy	Representational variation; Pudian; OPMC; OPMS
Build core associations of knowledge	Pudian; enhancing the formation of concept
Provide multi-angle understanding	Contextual variation; conceptual variation; OPMC

According to table 6.4.1, we conclude four main teaching goals: mastering problem-solving strategy, building connections with tasks, building core associations, providing multi-angle understanding. For different teaching goals, Emma uses different variations to design teaching. Representational and procedural (Pudian, OPMC, OPMS) variation serve for mastering problem-solving strategies; Pudian serves for building core associations of knowledge; conceptual, representational variation and OPMC serve for providing multi-angle understanding.

Emma's quite good at redesigning tasks to meet her teaching goals and students' needs, by making full use of her resources. The collaborative work with other teachers and researchers offers her rich resources and develops her ability of design or redesign tasks. When she prepares her lesson, she doesn't select tasks for only one lesson but for the entire chapter. She selects a series of related tasks as DS and DM after selecting the tasks for classroom teaching. For Emma, tasks are not scattered, but a system. This ensures consistent coherence in her tasks. She usually prepares more tasks so as to have a room to adjust according to the performance of the students.

In classroom teaching, she also leaves enough time for students to work on tasks in class to better understand students' needs. For Emma, the design of the outline of one chapter is given at the beginning, and when it is actually implemented later, the outline remains unchanged, but the contents inside varying according to students' performance in class and in their evaluations. For example, Emma may go through several times of documentation work in one thematic lesson, where the variations are dynamic during this process.

### 6.4.3 Emma's classroom teaching: Student-oriented teaching

In this section, we try to answer the third research question "What are the practical variations in French case? How does the case teacher respond to the practical variations?" we have been proposed, which is from the perspective of practical teaching.

Emma's classroom teaching is student-oriented, and the time allocated to each task is rich. There are many classroom interactions between Emma and her students, and also

between students themselves, which provides space for students to produce practical variations. As we mentioned in section 6.2 and 6.3, many practical variations emerged in Emma's case. Emma respects students' ideas and she believes that the most important thing is not whether the problem-solving strategy is correct, but whether the students' thinking is logical and has the spirit of mathematical exploration, especially in the thematic lesson.

The organization of daily lesson and thematic lesson are quite different. For daily lesson, the implementation in class is always what Emma planned in her working sheets (she keeps the actual variations) and sometimes she reacts to students' questions by creating variations. For thematic lesson, the implementation is more flexible. Emma monitors students in real time to ensure that students' discussions or ideas are in the direction of the teaching goal. Therefore, in thematic lesson we may observe changes in actual variation caused by Emma's reflections on students' feedbacks, through a new round of documentation work. Sometimes Emma will even vary the task (often in the next year's classroom, such as the changes of open box from previous year mentioned in 6.3.1), which further leads to changes in potential variation from classroom teaching.

There are limits to the changes caused by students in the classroom. According to Emma, no matter how it changes according to students' needs, she must guarantee the rigor of knowledge explanation and that teaching goals are met.

As an experienced teacher, Emma continues to learn in the classroom teaching, from her students. For her, collaborative work nourishes her resource system for variation while classroom teaching offers her opportunities to reflect and develop.

## **Chapter 7**

# **Contrasting the case studies, discussion and perspectives**

This chapter, drawing from Chapters 4, 5 and 6, presents the discussions on our three research questions in three parts: Section 7.1 refers to the reflections on these questions by contrasting the potential, actual and practical variation in Chinese and French case. We further refine our preliminary model of variation based on the contrast; Section 7.2 reflects on issues of theoretical and methodological design; Section 7.3 proposes some perspectives, both for deepening this research, and for teacher education.

## **7.1 Enlightening the research questions**

This section reflects on the three research questions proposed in section 2.4.2 based on the analysis of curriculum resources (chapter 4) and cases studies (chapter 5 and 6): seeing from the two contrasting cases, what are the potential variations in the upper secondary school mathematics textbooks and other curriculum resources (section 7.1.1), what are the actual variations in the cases to achieve the teaching goals (section 7.1.2), what are the practical variations in the cases and how do the case teachers respond to the practical variations (section 7.1.3). Then (section 7.1.4), we discuss the possibilities of extending our reflections on three research questions from one case teacher to a category of teachers in China or France.

### **7.1.1 Related to research question 1: potential variation**

This section proposes three elements related to the first research question “What are the potential variations in the upper secondary school mathematics textbooks and other curriculum resources in China and France?” by contrasting the curriculum resources and their roles in Chinese and French case: curriculum standard (section 7.1.1.1); textbook (section 7.1.1.2); and the evolution of variation in tasks from curriculum standard to the textbooks and to the other curriculum resources (section 7.1.1.3). These reflections mainly draw from chapter 1 and chapter 4.

#### **7.1.1.1 Variation in curriculum standard: guiding the design of curriculum resources**

Curriculum standards serve as the pillar of the curriculum resources, guiding the design of textbooks and reflecting nation’s expectations of the implementation of the curriculum. Taking function as an example, we contrast the potential variations in the

Chinese and French curriculum standards from the four aspects, i.e., contextual, conceptual, representational, and procedural variations. This contrast is based on the exploration of the variations implied in the curriculum standards in section 1.2.1.

Regarding contextual variation, both curriculum standards indicate the importance of abstracting function problems from real life contexts and constructing function models.

Regarding representational variation, both curriculum standards indicate the multiple representations of function, which is based on its feature, with particular emphasis on algebraic and geometrical registers. In particular, French curriculum standard highlight the use of digital tools, such as dynamic geometry software or Python, or calculator for describing the variation, drawing the graph or for calculation.

Regarding conceptual variation, neither of the related content appears in two curriculum standards.

Regarding procedural variation, both curriculum standards highlight the importance of enhancing the formation of a concept for students. In particular, it is recommended in the Chinese curriculum standard that the concept of function can be introduced by abstracting examples with familiar contexts to students. It even provides the lesson plan to help teachers design the process of abstracting a mathematics concept. The idea of OPMC is embedded in the formation of a concept because series of related tasks with OPMC are presented at the same time. OPMS and MPOS are not mentioned in both curriculum standards.

In the curriculum standard, variation is still a concept that can only be put into practice with the support of specific mathematical tasks in the textbook or other curriculum resources.

#### **7.1.1.2 Different expectations of textbooks, difference variations**

At this turning point of resources explosion, the position of textbooks seems to be gradually becoming nuanced. Researchers start to discuss the role of textbook in the future mathematics teaching (Chazan & Yerushalmy, 2014). In this section, we contrast the variation design embedded in two textbooks, which can be traced back to the different expectations of textbooks in two countries.

In section 4.5.1, we have contrasted and concluded the potential variations in two textbooks (PEP and Hyperbole), used by students of two cases, based on four types of variations in our analytical framework. Three differences related to variation can be distinguished between the two textbooks:

- The presentation of the concepts: in PEP textbook, tasks are designed to help students experience the formation of concept (an example in section 4.2.1). The trace of concept formation is explicit and concrete. While this kind of trace is hard

to capture in Hyperbole textbook, where concept is usually directly presented.

- We can find the connections (variations) between the tasks in both textbooks (section 4.2 and 4.3). However, teachers' guidebook in Chinese case makes these connections (variations) more explicit and help teachers to capture them. While in Hyperbole textbook, the large number of tasks<sup>142</sup> may obscure the connections between them and French teachers' guidebook only presents the solution of tasks in the textbook.
- Hyperbole textbook offers a variety of tasks, such as open task, group tasks, verbal math problems, while Chinese textbook offers mainly closed tasks. Various types of tasks leave much more room for variation in the French case than in the Chinese case.

These differences of variation design can be traced back to the different expectations of the role of the textbook. In China, the importance of the contents in practice, the rationality of the examples, and the logic of the entire chapter design are highly valued. The textbook is a concrete vehicle to accurately convey the concepts of the curriculum standard. Therefore, Chinese textbook gives the impression of a good balance between variation theory and practice in classroom teaching. Chinese teachers can teach directly with the tasks in the textbook. At the same time, the French Hyperbole textbook offers a wide range of tasks for teachers to choose. French teachers need to select and organize the tasks in the textbook before teaching. For variation, this is a game for textbook design between multiple choices and well-organized structure.

### **7.1.1.3 The evolution of variation from curriculum standard to the textbooks and to the other curriculum resources**

In this section, we first specify the direction of the flow of teaching resource, and then discuss the evolution of variation from curriculum standard to the textbooks, and to the other curriculum resources in Chinese and French case.

In the Chinese case, there is an invisible line of the redesign of resources, especially the mathematical tasks, from national curriculum standard, to textbook (accompanied by teacher's guidebook), to school-based teaching book, and then to teacher's own resources. The textbook in a way serves as a link between the curriculum standard and school-based teaching book. School-based teaching book can be regarded as a secondary resource based on the textbook. Chinese textbook visualizes the abstract requirements in the curriculum standard through concrete mathematical contents and tasks, and different schools then design their own school-based teaching books based

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<sup>142</sup> For the same mathematical content, the number of tasks in the French Hyperbole textbook is about 4 times of the tasks in the Chinese PEP textbook.



on the textbook according to specific student's mathematical level in their own schools. As for the online platform, it is mainly a supplementary resource for teachers to choose.

In the French case, it is difficult to find a linear evolution of resources like Chinese case. Curriculum resources in France is more like a multi-line development at the same time. Besides the textbook, there are resources coming from the teacher's collaborative work, the online platform developed by the teachers. But in fact, many modules of these tasks coming from different sources can be traced back to the textbook.

Regarding the evolution of variation among curriculum standard, textbook, school-based teaching book and online platform, we explain the two cases as follows:

- From curriculum standard to textbook, variation is retained and extended. The idea of variation is mainly made visible, concrete, and actionable by means of tasks. The possible variations arising from various representations, various digital tools, the formation of concepts and situations are all made explicit by PEP and Hyperbole textbooks (section 1.2.1 and chapter 4).
- From textbook to school-based teaching book in Chinese case, different types of variations have different changes. Contextual variation (section 5.2.1.1) and the formation of concepts are simplified and reduced because of the limitation of classroom teaching time. OPMC is further extended, especially by varying the type of function.
- From textbook to online platform, variation is more intelligent and explicit. Textbook is still irreplaceable and current online platform is mainly used as a supplementary resource to textbook. Online platform makes variations, especially Pudian and OPMC easier to implement in class (section 4.4).

In general, textbook as one of the main curriculum resources, its position is still special and irreplaceable. Under the onslaught of resources explosion, teachers who are overwhelmed with resources may overlook the role of textbook and opt for faster web search resources to find the appropriate resources to meet their teaching goals. In fact, many online resources can eventually be returned to the textbook. We believe that new forms of textbook combining with related online platform can lead the teaching more effective.

Finally, we must be clear that resources are always in the process of varying, evolving, in which the task is also varying through the documentation work of teachers, researchers, experts and so on. This is exactly what we called an ongoing process of instrumentalization (section 2.3.1.2).

### **7.1.2 Related to research question 2: actual variation**

This section proposes three elements related to the second research question "What are

the actual variations in the Chinese and French cases to achieve the teaching goal?” by contrasting the two cases from the process of appropriation: Case teacher’s resource system (section 7.1.2.1); the actual variations in two cases (section 7.1.2.2); and the variation path from potential variation to actual variation generated by case teachers (section 7.1.2.3). These reflections are drawn from curriculum standard exploration in chapter 1, curriculum resources analysis in chapter 4, case studies in chapter 5 and 6.

#### **7.1.2.1 Case teacher’s resource system and documental genesis: base of actual variation**

In this section, we first contrast case teachers’ resource systems and then contrast the two cases through the two processes of teachers’ interactions with resources: instrumentalization and instrumentation.

As experienced teachers, both case teachers in our research have their own relatively structured resource systems, where the core of their resource systems is named “digital task repository” in our research (section 5.1 & section 6.1). Both Lin’s and Emma’s digital task repositories are categorized by mathematical topics. In Emma’s digital task repository, she tends to record task and its solution(s), working sheets designed for students, instructional organization of tasks, and records of previous students’ solutions, especially for the resources used in thematic lessons. In other words, the resources in her digital task repository are the resources already (re)designed and manipulated by her. Lin’s digital task repository, on the other hand, only contains the original tasks and the corresponding solution(s). In other words, most of the resources are presented in their original versions. Teachers’ resource systems are formed under the two intrinsically intertwined processes of documental genesis: instrumentation and instrumentalization.

From the perspective of instrumentation, teacher’s digital task repository is influenced by the resources provided to the case teachers, i.e., the sources of resources. Both of case teachers’ source of resources is a combination of the collective work in teachers’ associations/groups and textbooks or school-based teaching book. The sources of Lin’s digital task repository are the resources he obtained during his process of teaching and designed by the other teachers (i.e., school-based teaching book, student’s learning resources, students’ test paper, online resources) and books that he read outside school (i.e., college math textbook, math reading material, books for college entrance examination, and books for math competition). Most of Emma’s resources are from her collective work, relying on her interactions with other teachers.

The role of the resources in the digital task repositories in these two cases are quite different. In Lin’s case, these resources are only used as a supplement to the school-based teaching book. When the tasks in the school-based teaching book do not meet

Lin's needs of teaching, he looks for appropriate tasks to add to it. While in Emma's tasks, most of the resources used in class are taken from her digital task repository. When the tasks in the digital task repository do not meet Emma's needs of teaching, she looks for appropriate tasks from several textbooks and online resources from IREM to add to it. The lack of a school-based teaching book, an important secondary resource of textbook, forced Emma to search for and adapt more tasks on her own and to build a more structured personal resource system.

From the perspective of instrumentalization, case teacher's digital task repository in our research has been refined and reorganized by the case teachers for years, and is still evolving based on the teacher's reflections from their collective work, curriculum resources, and students' reactions in class.

As presented in our preliminary model (section 2.4.3), teacher's resource system can be regarded as a result of appropriation, from where actual variation grows.

#### **7.1.2.2 Actual variations contrasting**

In this section, we contrast the actual variations in Chinese and French cases based on the analytical framework of different types of variation: contextual, representational, conceptual, procedural.

Regarding contextual variation, many contextual variations can be discovered in Emma's case, especially function tasks with the context of geometry, which is very consistent with the characteristics of potential contextual variation in the French textbook. While in Lin's case, very few contextual variations can be observed.

Regarding representational variation, based on the characteristics of the function itself, both Chinese and French cases use a lot of conversions and treatments in teaching functions, but the specific points they focus on are different. Treatments, in particular treatments in the algebraic registers, have many more steps and more complex in the Chinese case than in the French case. However, the conversions in Chinese case are relatively single, mainly between symbolic writings and graphs. In French case, the conversions involve nearly all kinds of registers, such as natural language, symbolic writing, graph and also geometrical figures (comes from the tasks in geometrical context). Moreover, the conversions involving table (auxiliary transitional representation) has a much higher proportion in the process of solving tasks than in the Chinese case. The other difference between Chinese and French case is the use of calculator, we can hardly find this content in the Chinese case while it is an important part in the French case. That is to say, in French case, certain conversions are implemented by the use of calculator. This also explains why there is more treatments in Chinese teaching, because complex algebraic calculations that require mathematical skill are replaced by calculators in French teaching.

Regarding conceptual variation, Emma intends to use various non-standard variations to help students understand new concepts while Lin intends to use non-concept variations to help students distinguish the concepts. In other words, Emma pays more attention to the intension of one concept while Lin pays more attention to the extension.

Regarding procedural variation, both case teachers use Pudian to help students gradually understand mathematical knowledge or acquire mathematical strategies. Teachers adjust their design of Pudian according to the student's reaction in class. For Lin most of the design of Pudian is completed in his lesson preparation and he does not make too many changes. However, in Emma's classroom, especially in thematic lessons using unifying problems, she often needs to make changes based on students' real-time feedback. During the one-on-one instruction in class, she also takes different leads (Pudian) depending on the students' different responses. OPMC is frequently designed during case teachers' lesson preparation. The most common way is the constraint manipulation, by varying the numbers in the task. In Chinese case, Lin pays attention to varying the tasks from different types of function and sometimes use OPMC to create a contradiction between the old strategy and new task, which impels students to think about new strategy. In this way, the new strategy emerges naturally and logically, without being abrupt. The attitudes towards OPMS are quite different between two case teachers. Emma believes that what she needs to do is to provide the space and platform for students to generate OPMS in class, and that the students should be the ones to generate the OPMS. At the same time, we can see the OPMS in Lin's teaching design based on his own teaching, and he will select the OPMS emerged in class and share them with different classes. He believes that the appropriate amount of OPMS is good, but not too much. Last but not the least, in the Chinese case, using variation to enhance the formation of concepts and propositions is greatly reduced compared to the Chinese PEP textbook where there is an obvious emphasis on the concept formation process, especially for the concept of functions. Lin skips this process with two reasons, this knowledge has been learnt by students outside school and the limitation of time in class. As for enhancing the formation of concepts and propositions, Emma creates two (one for the concept of function and the other for the variation of a function) to better achieve teaching goals and meet students' needs, while it is hard to find in the Chinese case.

Both Lin and Emma prefer teaching new knowledge (concept or strategy) by using series of tasks, which inspires us to discover the fundamental task in mathematics teaching. In other words, using as few tasks as possible and learning by varying them.

The classroom organization of the two case teachers is very different. Lin has always adopted a single didactic style of teaching, with a fixed flow for each lesson: review of what was learned in the previous lesson, explanation of new concepts or properties in this lesson, and presentation of examples. He leaves limited room for students in class.

In contrast, Emma's design is very diverse. First of all, the diversity of her lessons (lessons in the information room, thematic lesson, daily lesson, Rally) offers opportunities of variations in didactics. She can choose different organizational forms for different types of lessons. For example, she uses the same approach as Lin for formal concepts and properties teaching, with her taking the lead and students taking notes, and she uses a one (teacher)-to-one (student or one group of students) interaction for group activities or independent problem solving. Secondly, even for one specific task, she will design a series of different ways to organize it, which is mainly reflected in the teaching of unifying problems in the thematic lesson. For example, in the unifying problem, students are asked to work and make a presentation in groups, then reflect by individuals, at last listen to the explanation from the teacher. With such kind of variations in didactics, we can stimulate student motivation, let them experience the process of exploring mathematical problems, and exercises their mathematical language expression. Moreover, variations in the didactics indicate the important points in one specific teaching trajectory. This is further discussed in section 7.2.1 with our refined model.

### **7.1.2.3 Variation path generated by case teacher's appropriation**

Teachers play a crucial role as interpreters of curriculum resources, so there is a complex relationship between curriculum resources and their classroom use (Stylianides, 2016). The differences in the resource system of two case teachers lead to different choices in their documentation work of (re)designing tasks. In this section, we contrast the variation path in two cases and figure out the gap between potential and actual variations through teachers' appropriation of resources.

Chinese teacher tends to choose well-organized task, which is targeted to his teaching goal, while French teacher also chooses unifying problems to extend the initiative of students as much as possible. Most of the time case teachers choose to retain the potential variations embedded in the tasks selected. The variation path contrast is explained from 4 types of variations as follows:

Regarding contextual variation, Lin mainly removes it while Emma retains it. Lin tries to avoid choosing tasks with contexts in his appropriation. He prefers to choose tasks without contextual variation (i.e., tasks in mathematical context) to save time for students to read the contexts in class. Although the contextual variation in Chinese PEP textbook is quite considerable and compared with previous PEP textbooks, textbook designers have consciously added more tasks with different contexts in the new version of PEP textbooks. There are even tasks designed to let students create contextual variation (section 4.2.3.1, examples shown in table 4.2.5) in PEP textbook. Most of the contextual variations are lost during Lin's selection of tasks for classroom teaching. While Emma prefers to choose tasks with contexts and retains the potential contextual

variation already embedded in the tasks. There are many function tasks with the context of geometry shown in Emma's case, which is very consistent with the characteristics of potential contextual variation in the French textbook.

Regarding representational variations, both of teachers retain and extend representational variations from the potential variations already embedded in the tasks, especially conversions between symbolic writings and graphs. Sometimes they also create representational variations by themselves. However, the specific design is quite different. Lin makes this conversion implicit in the solution strategy, mainly by using graph to simplify the process of treatments in the algebraic registers (an example in section 5.2.2.3). Moreover, the complexity of representational variation is hugely increased. Tasks selected by Lin contains more treatments in algebraic registers (an example in section 5.2.2.4). Emma, on the other hand, designs to make this conversion process explicit as a specific strategy teaching (an example in section 6.2.2.5).

Regarding conceptual variations, Lin creates non-standard and non-concept variation for distinguishing concepts (an example in section 5.2.2.5) while Emma prefers to retain the non-standard and non-concept variation in the tasks (an example in section 6.2.2.3).

Regarding procedural variation, Lin retains, extends or creates Pudian and OPMC, creates OPMS, while Emma creates variations to enhancing the formation of concepts or propositions, creates or removes Pudian, retain, reduce, create, or remove OPMC. In the Chinese case, the teaching process of concept formation is gradually simplified, in particular, the process of generalization of concept of function (section 4.2.1, shown in the PEP textbook). Lin tends to directly show and explain the mathematical concepts and statements by himself. In contrast, in the design of Emma, using variation to enhance the formation of concepts and propositions is further strengthened. Although this kind of variation is not obvious in the French Hyperbole textbook, Emma makes it explicit by using unifying problem (section 6.3.1). That is to say, the special part of enhancing the formation of concepts and propositions in Chinese PEP textbooks is greatly weakened in our Chinese case, but strengthened in our French case.

A noteworthy point is that both case teachers intentionally make connections between different tasks when designing. For example, unifying problem "open box" (section 6.3.1) is used as a fundamental task throughout the process of function teaching. There is also an example in section 5.2.2.4 where case teacher Lin helps students make connections between tasks that use the same solution strategy.

In both China and France, the choice of teachers is closely linked to the teaching goal. Table 7.1.1 present different types of variations used by Lin and Emma in the teaching of function and their corresponding teaching goals.

Table 7.1.1 Variation used in two cases and their corresponding teaching goals

Teaching goal	Variation used in Chinese case	Variation used in French case
Master problem-solving strategy	Representational variation, Pudian, OPMC (MPOS), OPMS	Representational variation; Pudian; OPMC; OPMS
Build core associations (knowledge)	Pudian	Pudian; enhancing the formation of concepts or propositions
Provide multi-angle understanding	Conceptual variation; representational variation	Contextual variation; conceptual variation; OPMC
Introduce new knowledge or go further	OPMC	Unifying problem with different variations embedded

As shown in table 7.1.1, teaching goals directly affect the types of variations that teachers use. There are many similarities between the two case teachers' choices of variation for the same teaching goal. For mastering problem-solving strategy, various types of variations are used, such as representational variation, Pudian, OPMC (MPOS), and OPMS. For building core associations, Pudian is used. For providing multi-angle understanding, conceptual variation is used. Besides, representational variation is used in the Chinese case and contextual variation, OPMC are used in the French case. For introducing new knowledge or go further, OPMC is used in the Chinese case while unifying problem with different variations embedded is used in the French case. Unifying problems offer large space for teachers and students to pose OPMC and OPMS. Moreover, OPMC can further reduce, increase, create or remove the other types of variations. The variation here becomes unpredictable, as the teacher can constantly make further changes based on student feedback (see our description of practical variation and teacher's reflection in section 7.1.3).

In general, both Chinese and French case teachers are given the importance and challenge to select, (re)design and organize the tasks to reach the teaching goal and to simulate active student participation. Variation in Chinese case pays more attention to the specific aspect, for mastering specific mathematics knowledge, while French case emphasize on a larger field of view for solving series of practical problems. In Chinese teaching, the variations are mainly designed and dominated by teachers, which guarantees the smooth and complete progress of teaching through variation; while in French teaching, the teacher gives an anchor, and the extent of variations depends on the interactions between teachers and students, students and students, which increases the flexibility of teaching through variation.

### **7.1.3 Related to research question 3: practical variation**

This section proposes two elements related to the third research question “What are the practical variations in Chinese and French cases? How does the case teacher respond to the practical variations?” by contrasting the two cases from the process of implementation and reflection in the practical teaching: implementation and practical variation (section 7.1.3.1), and the reflections from the practical teaching in two cases (section 7.1.3.2). These reflections draw from case studies in chapter 5 and 6.

#### **7.1.3.1 Practical variations contrasting**

In this section, we contrast the actual variations in Chinese and French cases based on the analytical framework of different types of variation: contextual, representational, conceptual, procedural.

Regarding contextual variation, different contexts of tasks are presented in French case while very few contextual variations in the Chinese case. As we mentioned in 7.1.2.3, case teachers tend to retain the original context of the task that they chose. It is worth mentioning that in the French case, the students gave familiar contexts from their daily life to the task in the process of solving it (section 6.3.1). In other words, contextual variation is emerged as practical variation in French case, in particular, in the thematic lesson.

Regarding representational variation, it is mainly the difficulty of conversions between symbolic writing and graph that students presented in class during the teaching of function. In Chinese case, Lin uses rough graph to present counter example so as to remove student’s misunderstandings emerged in class (section 5.2.1.2, an example shown in table 5.2.2). In French case, students take advantage of the transitional auxiliary representation (table of values in section 6.3.1) to help them solve the task. Moreover, the emergence of representational variation is sometimes influenced by other variations. For example, the emergence or extension of representational variation may be caused by OPMS (section 6.2.2.4).

Regarding conceptual variation, it often occurs when Lin answers students’ questions in Chinese cases and no new conceptual variations emerged in French case. In the Chinese case, conceptual variation raised by students mainly occurs during the explanation of tasks, where involves the understanding of concepts. Lin prefers to use graphs to show the non-standard and non-concept variation to help students understand (section 5.2.1.2, an example shown in table 5.2.2).

Regarding procedural variation, OPMS appears in both cases, and Pudian also appears in the French case. In the Chinese case, OPMS is created by students when they try to connect different knowledge to solve the task. It mainly occurs in the later periods of function teaching. In the French case, similar to the Chinese case, OPMS is also created



by students in the process of solving tasks by connecting different mathematical knowledge. Pudian are created by case teacher during the interaction with students, guiding them to solve the task. OPMS and Pudian occur in the thematic lessons in the French case.

In general, contextual, representational, OPMS and Pudian emerge in the French case while representational, conceptual, OPMS emerge in the Chinese case. Practical variation relies on the students' reactions in class. Contextual variation and OPMS are often created by students while conceptual variation and Pudian are often created by the case teacher during the interaction with the students.

We aware that the occurrence of practical variation requires certain conditions. For example, all the practical variation in the French case occurs in the thematic lesson, while the Chinese practical variation frequently occurs in the later period of the teaching of function. Therefore, we believe that the unifying problem in the thematic lesson contributes to the generation of practical variation. With sufficient time and space for students to think, students are motivated to apply what they have learned and make connections between knowledge to solve problems. In turn, the questions that students ask in making connections between knowledge will prompt teachers and students to create new practical variation.

#### **7.1.3.2 Variation generated by case teacher's reflections from classroom teaching**

As shown in our model of teaching through variation, the classroom teaching is not linear, practical variations promote teachers to reflect and refine their teaching. In this section, we contrast the variation path and figure out the process of using practical variation for rethinking potential and actual variation.

During the implementation in class, both case teachers followed their working sheets prepared before class. They may make changes in their following working sheets based on students' feedbacks in class. As we mentioned in section 7.1.2.3, Lin tends to design classroom teaching with well-organized task while Emma also designs thematic lessons with unifying problems to extend the initiative of students as much as possible. Therefore, the reflective work of two case teachers is quite different.

In the Chinese case, Lin's reflections are mainly related to Pudian and OPMS. On the one hand, he would reflect on the OPMS that emerged in the classroom teaching and selectively share them with the students in the other class (based on his judgment of the generalizability of the strategy used). If he thinks the new strategy is good, he would share it with other class and also add it to his own digital task repository. On the other hand, he adjusts the difficulty of the tasks based on students' reaction in class, their performances in homework and exams. He tends to adopt a "new knowledge-review -

go further” strategy (section 5.2.2.6) for important but relatively difficult knowledge that for students to master. Pudian is inevitably therefore involved in this process. In the process of review, Lin sometimes helps students master the strategy with the help of OPMC (section 5.2.2.2).

Compared to Lin, Emma has to make more adjustments because her lesson is more open-ended. These adjustments are mainly reflected in procedural variation. For example, Emma create Pudian for the second half-class according to the interactions with the first half-class (section 6.1.2.2, an example shown in table 6.1.5). Emma would also remove the variation according to students’ feedback. For example, Emma removes OPMC to reduce the complexity of the task (section 6.3.1). On the other hand, Emma collects students’ working sheets, especially the student response and reflections in the thematic lesson. She adds these resources to her own digital task repository.

In general, both case teachers focus on two main things during the reflection: firstly, they adapt the tasks through variation to better suit the students’ needs and serve the teaching goals; secondly, they record the students’ “meaningful”<sup>143</sup> work to enrich their own resource systems.

#### **7.1.4 Discussing the possibilities of extending our reflections**

In this section, we discuss the possibilities of extending our reflections on three research questions from one case teacher to a category of teachers in each country (China or France). The framework with two main levels presented in section 3.4 is used to illustrate.

At the level of mathematics teaching contents, the same mathematics sector is chosen, i.e., function. We try to ensure as much consistency as possible in the mathematical content of the subject as well as the theme related to function, more detailed description in section 3.1.3.

At the level of education “noosphere” (Chevallard, 1985), teachers are unconsciously but strongly influenced by it. We illustrate it from the following four perspectives: pedagogy, school, society and civilization.

From the perspective of pedagogy, our results can be generalized within a certain group of teachers with the following characteristics:

- Rich experience and good educational backgrounds: both case teachers have rich teaching experience (18 years and 20 years) with good educational backgrounds (master degree of mathematics education).
- Active in collective work: both case teachers are involved in the collective work

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<sup>143</sup> “meaningful” here is completely based on case teacher’s own subjective judgment

and play an important role in the collective work, i.e., Lin as the leader of LPG (section 5.1.2.3) and Emma as an active and indispensable role in IREM and the common progression promotion at school F (section 6.1.2.1). The results of their documentation work are a combination of collective work and individual reflections.

- Structured and evolving resource system: both case teachers pay attention to constructing their own digital repository and improving their resource systems in the teaching process (section 5.1.2 and section 6.1.2). For them, each appropriation process is a combination of instrumentation and instrumentalization, influencing by their resource systems.
- Dedicated to self-development: both case teachers spend time outside the classroom to improve themselves. For example, Emma joins IREM groups to improve her task design skills and to solve the problems she encounters in their teaching (6.1.2.1). Lin reads college books and works on mathematical problems to improve his mathematical knowledge and problem-solving skills (section 5.1.1.1).

From the perspective of school, there are many differences between Chinese and French school. Therefore, our results can be generalized within teachers in a certain group of schools respectively with the following corresponding characteristics:

- Available teaching resources offered by school: Besides textbook, school C (Chinese case school) offers school-based teaching book for each grade designed and refined by experienced teachers of school C over years. Compared to textbook, school-based teaching book can better meet the needs of students in school C, such as the difficulty of the task, the extension of mathematical knowledge, etc. Teachers have the autonomy to use or not to use this book. There are no such resources in school F.
- Classroom teaching time: There is a big difference of classroom teaching time in Chinese case and French case. Lin works 7h30 a week (different teachers vary in school C) in front of students in class, two or three lessons a day (one lesson lasts 45 minutes), while Emma works 20 hours in class in front of students, three or four lessons a day (one lesson lasts 1 hour).

The time in the classroom teaching plays a critical role. French teacher leaves much more time for students to think and discuss the tasks than the Chinese teacher, opening a more or less important time to students' initiatives. In China, mathematics teaching is under the baton of the college entrance examination and the class hours are tight for the contents for students to learn, therefore, the task needs to be well-organized.

- Teachers' working space: There is a big difference of working space in Chinese case and French case. Lin has a fixed working place in the teachers' office. This office

is a vast room, shared by 12 mathematics teachers. Lin is supposed to be in school from 7 a.m. to 5 p.m., but most of the time, he stays from 7 a.m. to 6 p.m., and to 9 p.m. each week for the self-study session at night<sup>144</sup>. Except for classroom teaching, he basically stays in the office. In this working place, Lin receives students who have questions, prepares lessons, makes corrections of students' homework and examinations. He also communicates with other mathematics teachers about the problems in the classroom teaching, in the interactions with students, the arrangements of test paper, etc., in this office. Emma doesn't have an office after class, so she sometimes goes to the shared information room to prepare her lesson.

There is also a strong relationship between teachers' office and the classroom (Trouche & Wang, 2015). Teachers' office makes the interactions between teachers, between teacher and students much easier, offering space for the emergence of variation.

- Collective work offered by school: collective work is mandatory in school C, but none in school F. In school C, there are various mandatory collective work, such as the Teaching Research Group (TRG), the Lesson Preparation Group (LPG), open class (section 3.3.2.1). School F doesn't offer any kinds of collective work.

From the perspective of society, our results can be generalized within a certain region:

- Curriculum standards (Chapter 1) and textbooks (Chapter 4) studied in our research are not specially designed for our case teacher. They are shared by all the communities. Moreover, national examination for secondary schools, as one main factor that influences case teachers' choices, should also be considered.
- A basic consensus about teaching through variation in China: Teaching through variation has been promoted by the examination-oriented education system, and can be implemented to promote deep learning of mathematics in classrooms and teachers' professional development (Gu, Huang & Gu, 2017). "Teaching through variation" could serve as a bridge between the "basic skills" and the development of higher-order thinking abilities (Zhang & Dai, 2004). In 1952, the concept of "Two Basics" (namely, basic knowledge and basic skills) has been proposed for the first time by the Ministry of Education to enable students to "get the basic knowledge and skills of modern science and develop a scientific worldview". Because the national curriculum standard is considered as the compulsory "core curricula", all stakeholders of education (e.g., teachers, parents, schools, etc.) are required to work well together to carry out this educational goal. According to Sun

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<sup>144</sup> Self-study session at night is a session organized by the school in addition to the regular classes at school. Students are scheduled to do some intensive self-study and homework at night (usually from 18h-21h) at school.

(2011), variation problems could be regarded as a curriculum implementation tool encircled by the invariable “two basics”, which are characterized by “dispersed and progressive difficulties”. A similar statement has been proposed by Zheng (2004): “Teaching and exercise unfolded by variation problems activities are considered as the pathway to the “Two Basics”.” In 2011, the Ministry of Education explicitly proposed the “Four Basics”, namely the basic knowledge, basic skills, basic ideas and basic activity experience of mathematics (MOE, 2011). “Four basics” separates “mathematics basic ideas” from “Two Basics”, also added “mathematics basic activity experience”, which is the inheritance and development of “Two Basics”. By 2016, Six core mathematical literacy has been proposed, namely mathematical abstraction, logical reasoning, mathematical modeling, intuitive imagination, mathematical operations, and data analysis. The first three of them are the basic ideas of mathematics and inherited from the “Four basics” while the last three are traditional mathematical abilities. From the “Two basics” to the “Four basics” to the “core literacy of mathematics”, it is clear that the goals of mathematical education are the same and there are inheritance and continuation (Liu, 2017). In summary, we combine basic knowledge, basic skills with mathematical abilities, and higher-order thinking to meet the requirements of the curriculum standards by teaching through variation.

From the perspective of civilization, we can find the trace of variation in two traditions.

- In China, theoretical concepts of extending a lesson/exercise with variation practice can be traced back to Chinese traditional philosophy and culture, which could be rooted in the ancient “general consensus” developed from the system of Daoism (Chinese: 道家) and Confucianism (Chinese: 儒家). For example, one of the oldest classic Chinese texts from the Chinese I-ching theory (The Book of Changes, Chinese: 易经), subscribes to “abstracting invariant concepts from a varied situation” (in Chinese: 变中发现不变) and “applying these invariant concepts to the varied situations” (以不变应万变) (Sun, 2011). Education in Chinese traditional culture talked about experiencing rather than indoctrination. The most often quoted saying of Confucius on the process of education also highlights the importance of variation, especially about “multiple problems one solution”: “Enlightenment comes when one is stunned and understanding when one is aroused; if one can’t respond with the other three corner if a corner is shown, then one is not in the proper track (of learning).” (不愤不启，不悱不发；举一隅不以三隅反，则不复也。) (The Analects, 《论语·述而》). Doubt (疑) and realization (悟) form the central theme of learning in Chinese culture, like Zhu Xi stressed on arousing skepticism and reflections of learners: “Reading books [learning] is to arouse doubt when one does not doubt and let those in doubt settle in the state of no doubt. This is how one

grows”（读书无疑者须教有疑、有疑者却要无疑。到这里方是长进。——《朱子语类·卷十一读书法下》）。Arousing doubt in a disciple is also a main theme of Chan initiation. As it is said that “It is essential to arouse a sense of doubt in Chan practice. A little doubt leads to small understanding while strong doubt leads to deep understanding.”（参禅须是起疑情，小疑小悟，大疑大悟。）(progressing through the Chan Gates [禅关策进]) (Wong, 2004). Therefore, the master’s guidance and initiation are of vital importance and variation could be helpful for arousing the students’ doubt. According to the earlier findings in phenomenography that one way of experiencing a phenomenon that are discerned and kept in focal awareness by the learner. As the discernment is an essential element in learning and variation is crucial in bringing about the discernment, repetition by systematically introducing variations could be the key to bringing about learning and understanding (Wong, 2004; Bowden & Marton, 1998; Marton & Booth, 1997; Runesson, 1999).

- In France, the variation in French language can be a symbol of variation. According to F. Leung (2017), language plays a crucial role in the culture since beliefs, ideas, etc., are constructed through the medium of language. The structure of language controls human thought patterns, thought directs human action (human action is the actualization of thought), and human action creates culture. French language is called as “fusional/inflectional language” (Humboldt, 1971), as there are relatively rich inflections, and the relationship between words is expressed through inflections. In French, verbs, adjectives, etc. can express different tenses, voices, and modes through their inflections (variations). Since all experiences (including the experience of mathematics teaching) are mediated by language and language is the medium and tool for mathematics learning (F. Leung, 2017), we think that the idea of variation has been directly represented in French language which influence students’ learning and teachers’ teaching of mathematics.

In general, with this hypothesis, the case teacher can be a representative of a group of teachers in a particular region.

## **7.2 A critical view on the theoretical and methodological design**

This section reflects on the theoretical (section 7.3.1) and methodological (section 7.3.2) aspects of this research.

### **7.2.1 Reflecting on the theoretical aspects of this research and refining the model**

In this section, we first reflect on our theoretical networking and discuss the need to refine our initial model and framework (section 7.2.1.1); we then have a critical view

on our initial model and analytical framework, making efforts to refine them (section 7.2.1.2 and section 7.2.1.3).

### 7.2.1.1 Considerations on theoretical networking

In section 2.4.2, we construct our model and analytical framework at the degree of combining theories. During the curriculum resources analysis and case studies, we apply our model and aware several weak pointes in our model and analytical framework:

- A retrospective view on teachers' view on variation

There is no shared unique definition of variation among mathematics education researchers. Here, we rethink our model of teaching mathematics through variation based on the ideas given by the case teachers from the perspective of practical teaching.

Both case teachers and the preservice teacher in Chinese case mentioned the connections between different tasks when they talked about variation. For example, preservice teacher Shi highlights the ideal variation in her mind:

“My own understanding of variation is that for one concept, we can vary it from different components of this concept. Of course, this variation needs to have a purpose. The purpose of the variation is to make students really understand the essence or importance of the knowledge. The varied task does not have to be more difficult than the original task, but it must be a meaningful variation. I think it's ideal to have a variation that meets the teaching goal and also makes a connection to what we've learned before at the same time.

Taking an example (table 7.1.2) from our guiding activities, I think variation 4 is a good variation. In example 4, it is a condition to solve inequalities, variation 4 varied it to solve the equation with the background of the roots of the function. The properties of operations we are going to learn are embedded in the process of solving inequalities, so this is in line with the teaching goal.”

Table 7.1.2 An example of a good variation according to Shi

<p>例4 已知<math>x</math>满足不等式<math>2(\log_2 x)^2 - 7\log_2 x + 3 \leq 0</math>, 求函数<math>f(x) = \log_2 \frac{x}{2} * \log_2 \frac{x}{4}</math>的最大值和最小值。</p> <p>变式 4 已知关于<math>x</math>的方程<math>2x^2 + (\log_2 m)x + \log_2 \sqrt{m} = 0</math>有实根, 求<math>f(m) = \log_2 \frac{m}{2} * \log_2 \frac{4}{m}</math>的取值范围。</p>
<p>Translation:</p> <p>Example 4 <math>x</math> meets the inequality <math>2(\log_2 x)^2 - 7\log_2 x + 3 \leq 0</math>, find the maximum and minimum values of the function <math>f(x) = \log_2 \frac{x}{2} * \log_2 \frac{x}{4}</math></p> <p>Variation 4 <math>x</math> has real roots in the equation <math>2x^2 + (\log_2 m)x + \log_2 \sqrt{m} = 0</math> and find the range of <math>f(m) = \log_2 \frac{m}{2} * \log_2 \frac{4}{m}</math>.</p>

We can also find the evidence of connecting tasks in case teacher's documentation work. For example, Emma added a lot of prior knowledge (inequalities, intervals, etc.) in her own design before function teaching. Lin mentioned that the new version of PEP

textbook changes the order of mathematics contents, making the inequalities being taught before the functions, which made his class more coherent.

In fact, the connections are based on the mathematical knowledge itself. For example, inequality is often used as a tool to solve function tasks (examples in section 5.2.2.7 and section 6.2.2.5). Therefore, the connections between the mathematical content itself motivate us not to limit our model to variation in one specific task, but to a sequence of related tasks, and furthermore, the interaction between the various resources.

- The need of considering the interactions with students

The crossing of these two theories provides a wider and more comprehensive view on the variation in mathematics teaching. Inspired by the contrast, one type of variation about teacher's interactions with students is missing in our initial analytical framework. Various types of interactions are proposed in French case, such as interactive exchanges, group work, individual, group presentations and so on. It is presented in French curriculum standard, Hyperbole textbook, and Emma's working sheets. While we can find only few words in the Chinese case. For example, Chinese curriculum standard proposes a case in the specific lesson plan example about measuring the height of buildings inside and outside of a school, which illustrates that the organization of students can be in groups, or cooperatively, etc. As for Lin's working sheets, we couldn't find any evidence about the design of interactions with students during our observation period.

This phenomenon seems to be consistent with the different understanding of variation proposed in Gu's theory and Brousseau's theory. From the perspective of variation in the classroom organization, Brousseau's theory extends the concept of variation in Gu's theory by introducing the organization of interactions between students (section 2.2.3.2).

In view of the weakness discovered by our research, we further refine our initial model and analytical framework in the following sections. As we described in section 2.4.2, the strategies of comparing and contrasting are used in our literature review (chapter 2) for a better understanding of typical characteristics of the theoretical approaches in view (Prediger, Bikner-Ahsbahr & Arzarello, 2008). The strategy of combining is used for constructing our initial model and framework. The process of refining is the process of developing from combining to coordinating (Prediger, Bikner-Ahsbahr & Arzarello, 2008) different theoretical approaches, at least locally, see figure 7.2.1.



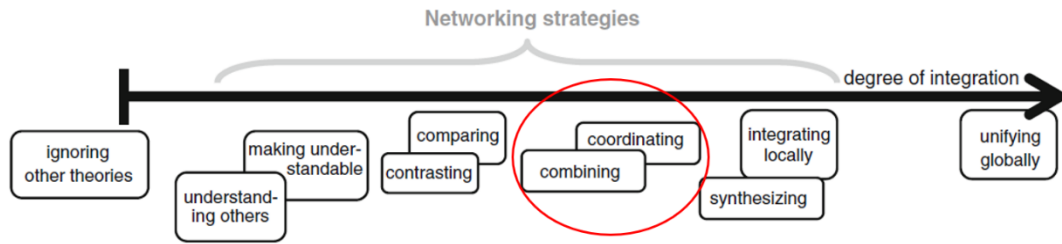


Figure 7.2.1 Networking strategies for connecting theoretical approaches (Prediger, et al., 2008)

### 7.2.1.2 Enlightening the model of teaching mathematics through variation

In this section, we refine our model of teaching mathematics through variation based on the reflections on curriculum resources analysis and case studies, taking teachers' resource system as the structuring element for modelling the variation process.

Cases studies show the necessity of enlarging our preliminary model of teaching mathematics through variation from analyzing the variation of a single resource (task) to also analyzing the connection between different resources (tasks), and get the revised resources based on the structure of mathematics contents and of teacher's knowledge. Figure 7.2.2 shows the refined model of teaching mathematics through variation.

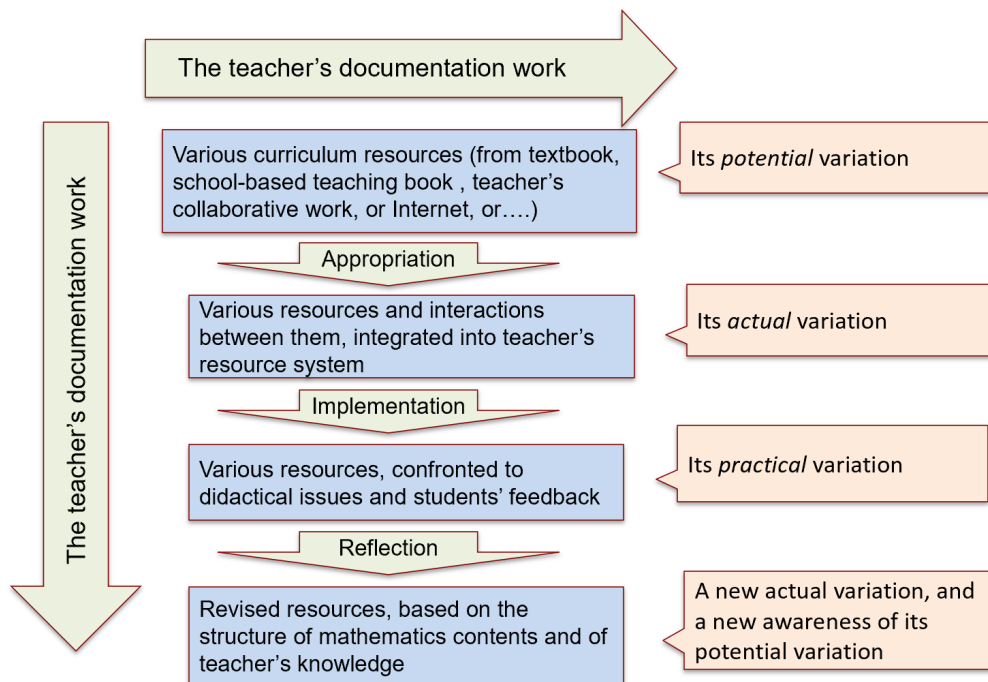


Figure 7.2.2 Refined model of teaching mathematics through variation

The refined model helps us to enlarge the vision of variation. The teacher's documentation work is no longer limited to the vertical level, but is also reflected in the horizontal level. To make it clearer, we enlarge the potential and actual variation part of our model to illustrate, see figure 7.2.3.

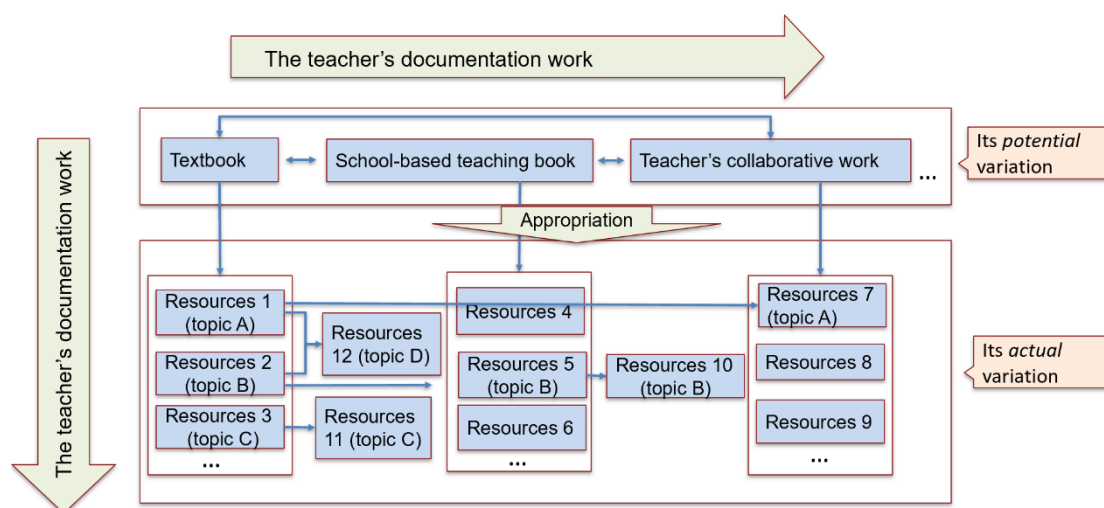


Figure 7.2.3 The enlargements of potential and actual variation part

As shown in figure 7.2.3, some resources themselves are implicitly linked, and the teacher can make these links explicit, or construct links to connect different resources through his/her appropriation.

This new perspective of variation can lead us to two important study fields and evidences of these connections can be found in our studies:

- To support the concept of “variation path”, which includes both the evolution of variation in both horizontal and vertical directions. For example, a typical trace of the evolution of variations from textbook to school-based teaching book and to teacher’s collaborative work have been discussed in section 7.1.1.2 and section 7.1.1.3.
- To study the connections between different tasks so as to form series of tasks. These connections can be built under the same mathematical topics, as the trace from resources 2 to resource 5, and to resource 10 in the figure 7.2.3, examples presented in section 5.2.2.6 and section 5.2.2.7; or under different mathematical topics, as the trace from resource 1 and 2 to new resource 12 with new topic, examples presented in section 6.2.2.5. There is also new resource created directly from one curriculum resource, examples presented in section 6.2.2.2 and 6.2.2.4. The relationships between the resources are intricate and complex.

The prevalence of connections between topics in mathematics teaching motivates us to think about the necessity of noticing and proposing “fundamental task” in mathematics teaching. Inspired by Marton’s “learning space” (described in section 2.2.1.2), we can first construct a “teaching space” by finding the core knowledge and its related connections. Then we find or design a fundamental task and vary tasks based on it. In this way, teaching a fundamental task is equivalent to teaching series of tasks and

mathematics knowledge is not taught in an isolated manner, but in a networked, intertwined manner, which greatly improves teaching efficiency and effectiveness.

Taking quadratic functions as an example, we build a teaching space based on the curriculum standards and textbooks, see figure 7.1.3.

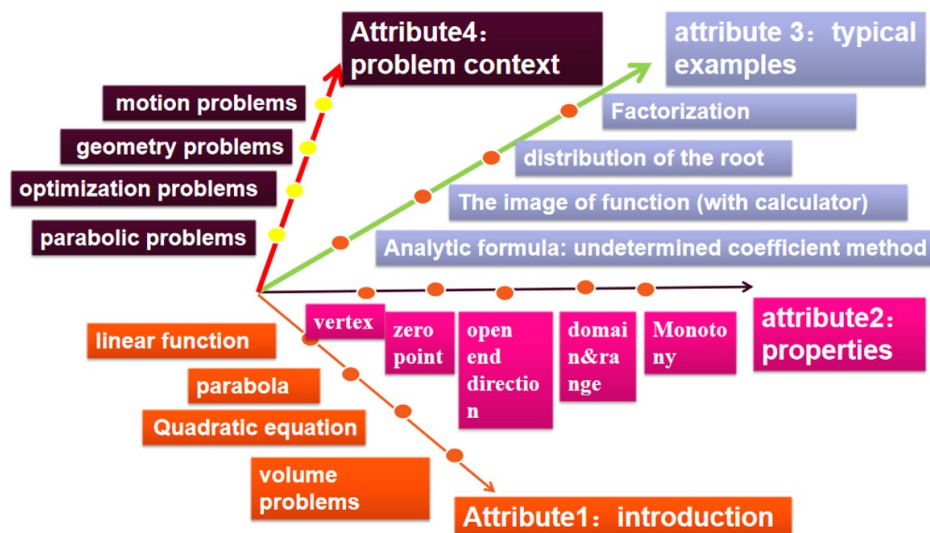


Figure 7.1.3 Teaching space of quadratic functions

Through this teaching space, we can clearly see the important attributes contained in this content and what are supposed to teach at the points of each attribute. The points in each attribute represent the key knowledge or difficult content in teaching. Therefore, studying these points and their possible variations are essential. Teaching space can control the scope of the variation and to the specific teaching goals.

### 7.2.1.3 Enlightening the analytical framework

In this section, we refine our analytical framework (table 7.2.1) based on the reflections on our curriculum standard exploration, curriculum resources analysis and case studies.

As we mentioned in section 7.2.1.1, we have made one main refinement:

- Add “variations in the interactions with students” as one aspect

In the refined model, we examine the variations in each stage (potential, actual and practical) from two aspects: variations in tasks and variations in the interaction with students. We category the interactions into six aspects: individual work, group work, consultation, presentation, confrontation, and lecture method. These six aspects are extracted from our case studies.

There are still several aspects in the refined model that deserve to be discussed and developed further:

- This framework can be further graded and refined. Similar to the further classification of OPMC, OPMS proposed by students can be categorized into

varying the calculation method, varying the way of thinking, etc.

- The relationship between the different factors in our analytical framework should be further clarified. As we mentioned that the different factors are not completely isolated from each other and it is difficult to separate them completely (section 2.4.3), it is hard to systemize the relations of the factors. For example, most of the time, OPMC and MPOS appear together in our study.

Table 7.2.1 Refined Analytical framework for studying different types of variations

Variations in tasks				Variations in the interaction with students
Contextual variation	Representational variation	Conceptual variation	Procedural variation	
Personal; Occupational; Societal; Scientific	Natural languages; The symbolic writing; Iconic (images) and non-iconic (geometrical figures); Cartesian graphs and diagrams	Standard variation ; Non-standard variation ; Non-concept variation	Enhancing the formation of concepts and propositions; Pudian; Trilogy with variation: -OPMS -OPMC (Constraint manipulation, Goal manipulation, Symmetry, and Chaining) - MPOS	Individual work; Group work; consultation; Presentation; confrontation; lecture method
PISA (OCDE, 2017, 2019)	Duval (2006)	Gu's Bianshi theory (Gu et al., 2004)	Gu's Bianshi theory (Gu et al., 2004) Sun (2011)	Extract from our research

At last, we would like to clarify two points of view about our framework design and improvement: holding a dynamic view on our work of framework, and never talk about theories in isolation from their cultural context. As Prediger, Bikner-Ahsbahs and Arzarello (2008) have pointed out that theoretical perspectives in mathematics education evolved independently in different regions of the world and different cultural circumstances, including traditions of typical classroom cultures, values, but also varying institutional settings (like the location of mathematics educators in the mathematics or education department, their involvement in pre-service or in-service training, the real and the intended curriculum, the books, etc.) (Sriraman & English, 2005), and the topic of research itself is complex.

## 7.2.2 Reflecting on the methodological aspects of this research

This section draws reflections on the methodology of data collection and data analysis in this thesis.

### 7.2.2.1 Reflecting on methodological choices

We reflect on our methodological choices from three perspectives:

- Chosen two experienced teachers, who are good at designing tasks. This choice is based on the assumption that experienced teachers can create more variations. More variations can better help us refine the model. In fact, the study of guiding activities (section 5.3) in the Chinese case corroborates this hypothesis to some extent. Novice teachers are often constrained by their own capacity and their current resource system, which in turn affects both the instrumentation and instrumentalization processes of teacher-resource interaction. This is especially true when the teacher's current level of experience does not support him/her in designing or redesigning variations through the process of instrumentalization.

In a follow-up study, we can make more observations by selecting novice teachers and experienced teachers, or by selecting teachers with different levels of experience. This would allow us to understand the extent to which teachers' experience contributes to the design of variation.

- Chosen the subject of teaching function, which is one of the important mathematical topics in the upper secondary school, and function itself has a lot of potential for variation, e.g., various representations, close relation with other mathematics topics, or subjects, etc. Therefore, the variation of functions is more explicit and more easily to grasp for us. Considering it is the first time for us to build and refine a model of variation, collecting more diverse data possible can be very helpful. We image to choose other mathematical topics, such as spatial geometry or probability, we can still find these types of variations, probably the number of variations will decrease, variation path will be less active, and there will be fewer connections between tasks.
- Followed one individual teacher in each country. If we had taken more into account the collective aspect, for example in choosing pairs of teachers working closely together, as the “documentation working mates” (Wang, 2019, p.85), then more possibilities for the interactions between variation and resource are supposed to be revealed, making our model richer, and this can be done in our subsequent studies.

#### **7.2.2.2 Not exactly symmetrical tools for data collection in two cases**

Due to various factors, there are three aspects where we adopted different strategies of data collection in China and France: (1) In France, except for the third thematic lessons, the previous classrooms are all audio-recorded with photos taken in class, while the Chinese classrooms are all video-recorded. Here we consider the acceptance of the case teachers, students and students' parents, which is decided after consulting with the teachers and signature of students' parents; (2) We use two cameras (camera A and B in figure 3.3.4) to record in French third thematic lessons, but one (only camera B) in China. The Chinese case was observed at the time of the epidemic Covid-19, and there

was no condition to borrow two video equipment at that time. In Chinese observation, we found that because the Chinese case teacher mainly adopted the whole class teaching style, there were very few individual exchanges with individuals, so the impact was not significant. (3) The time for second interview is different in two cases, The second interview in France was in the 4th week of classroom observation, while in China it was one week after the end of classroom observation. The main reason for this difference was that the middle period of Chinese observation coincided with the mid-term exams for students and there were many school activities. It was difficult for the case teacher to spare complete time for the interview. In order to ensure the quality of the interview, we chose his relatively free time after communicating with the case teacher.

We collected the data in France from November 2018 to February 2019 with no impact of pandemic, while we collected the data in China when the school discontinued online teaching and restarted school teaching in the new semester (September 2020) because of Covid-19. Some variations may occur in teachers' work after the online teaching for several months, where teachers have to move from face-to-face teaching to distance teaching. However, according to Lin, he believes that nothing has changed in the way he prepares his lessons. His main feeling is that online teaching is ineffective because he cannot get timely feedback from students as opposed to face-to-face teaching, and that correcting homework through the screen has caused him some trouble.

#### **7.2.2.3 Data collected being further analyzed and data be further collected.**

Due to the limitation of observation time at the school and the constraints by the school arrangements, we only present the general situation of teachers' resources. In fact, there are many other learning opportunities for students and for teachers themselves, like WhatsApp group in France and Wechat or QQ group in China. As for the collaborative work, we have observed several times of IREM group activities but unfortunately, we didn't encounter a seminar about the content of function, and during the observation period in China, there were too many school activities during the function teaching period, resulting in TRG activities not being carried out for a while, and there was no opportunity for observation in person. Descriptions of TRG in our research are based on interactions with the case teacher. Otherwise, we believe that lots of valued information about the use of variation in teaching functions can be found in these activities. And it would no longer be limited to one or a few specific teachers, but a collaborative work conducted by a group of teachers.

In the French case, besides the lessons about function, we also observe the mathematical activities of Rally, the lessons about mathematical-related subject, such as Python and cryptology, which are conducted in the information room. We don't analyze them in our thesis because our study mainly focuses on the study of function. However, in our

further studies, we can extend our variation study to such mathematical-related subject and also to the topic of geometry. Since there are substantial differences between algebra and geometry, how should design variation tasks be designed that reflects the differences between them? In geometry, the truthfulness of many propositions can be judged based on visual representations. However, in algebra, a proposition often be justified based on algebra structure and formula. It is an inductive construction process. How can these differences be reflected in design of using variation tasks?

In the Chinese case, we are aware that we did not fully use the data of guiding activity between Lin and Shi. In our study, we regard Lin's guiding activities of preservice teacher Shi as another side to explain and evidence Lin's teaching through variation. In our subsequent study, we can further analyze and compare the difference of teaching through variation between experienced and preservice (or novice) teacher. Furthermore, we can conduct a long-term follow-up observation of a preservice teacher to see the trajectory of becoming an experienced teacher, which show the role of variation in teacher's professional development.

#### **7.2.2.4 Some interactions to be thought between the two cases**

As a Chinese who was born, raised and educated in China, contrasting studies help me think outside the box. For example, sometimes it is hard to find the corresponding English words for some specific words because these words are closely tied to the country's culture, such as "milieu" in Brousseau's theory, "Pudian" in Gu's theory.

It is worth noting that even the same word can have different meanings in different educational contexts, such as the definition of "variation" in our study.

### **7.3 Variation from other perspectives and further studies**

In this section, we propose some research directions related with this research that can go further from two perspectives: theoretical and practical.

#### **7.3.1 Further studies from theoretical perspective**

From the perspective of theory, we propose further studies related to our research from the following perspectives:

- Beyond contrasting Chinese and French traditions, opening a window on other mathematical variation traditions, such as the Hungarian case<sup>145</sup> (Gosztonyi, 2015b).

A discussion group has been organized in ICME-14 on variation activities

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<sup>145</sup> Gosztonyi (2015b) wrote her PhD thesis on the comparison of the Hungarian reform of mathematics education led by Varga (pointing out the importance of structuring problems in series and networks) and the French "mathématiques modernes" reform.

considered as sequencing and networking tasks and problems. The aim of these discussions was to confront different implementations of the “variation” emerging in different cultural contexts and to advance towards the elaboration of a common model, or a diversity of models. Confronting different implementations of the “variation” can help us benefit, on the one hand from the sharing and development of analytical tools for researchers, but also to support teachers’ design work using variations. The study of variation seems to be both very common in international research, but also very relevant for describing teacher activity (Gosztonyi, de Varent, Zhang, & Ramploud, 2021).

- Beyond teachers’ resource system, thinking variation as a way of moving in a conceptual field (Vergnaud, 2009). That is to say, to reveal the structure of mathematics hidden behind a series of tasks. As shown in our refined model in section 7.1.4.2, we zoom in potential and actual variation part (figure 7.1.2) and consider variations connecting series of tasks based on the structure of mathematics contents. From a meta view on variation, these connections can be regarded as moving in a conceptual field (a set of situations, a set of concepts and representations, shown in figure 7.3.1).

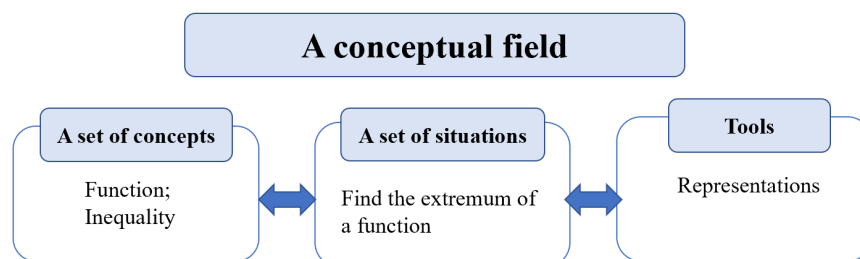


Figure 7.3.1 Initial ideas for linking the conceptual field and variation (example of function)

We can study the variation in a conceptual field from three perspectives: assortment (the choice of problems), orchestration (the choice of the variation), path (the choice of sequency, such as retain, remove, reduce, create, extend the variation).

Moreover, a link can be made between conceptual field and learning space proposed by Marton (section 2.2.1.2) as there are some common ideas between them.

- Thinking both the evolution of tasks and the evolution of teacher documentation work, in linking variation path and documentational trajectories (Rocha, 2019). That is to say, to reveal the structure of teacher’s knowledge. Combining these two evolving aspects (variation path and documentational trajectories) and their interactions with each other could help us to trace teacher’s professional development.



### 7.3.2 Further studies from practical perspective

In this section, we discuss the possible further studies of our interest on the practical way from the following four perspectives:

- From the point of view of resources: integrating a variation spirit in the textbooks (instead of list of exercises, thinking exercises associated with variation paths, the mathematical structure behind), using the digital potentialities of platform for developing living resources, with open variation paths? In addition, variation leads us to think to what extent are the variation tasks conducive to teaching? Less may be not enough for students to explore critical features, while too many variation problems may obscure teaching goals.
- From the point of view of teachers' practices: integrating a variation spirit in problem solving (Trouche et al. 1998). The students solve one problem with different kinds of strategies, relating to OPMS in our study. Teaching goal and mathematics knowledge are designed behind the problem.
- From the point of view of teachers' collective work: developing French-Chinese interactions (TRG - IREM), in school or out of school, which may lead to the idea of complementary traditions to be developed;
- From the point of view of teacher education: considering preservice or in-service teacher, how to develop the necessary knowledge for mastering teaching through variation? How to take advantage of the guiding activities (section 5.3) or TRG or teachers' collective work?

A strong, rich, cross-network resource system can contribute to the mathematics teaching from the following perspectives: 1) Shortening teacher's lesson preparation time, which is the most direct impact; 2) Guaranteeing the quality of the task: after years of use and modification, these tasks have been basically perfected. When the teacher uses it again, she will make some adjustments (variations) to the tasks according to the level of the current students, so that the tasks can better serve the teaching goals. 3) Easier to predict students' reactions: because some classic tasks have been used for many years, teachers can predict students' classroom responses and behaviors more accurately.

Finally, our itinerary, throughout this thesis work, convinced us even more of the interest of variations for the teaching of mathematics, and of the interest of this subject for research in mathematics education. To use a quote from Marton (1999): “**Variatio est mater studiorum**”. This quotation is also the opportunity to add a language to the linguistic variations which constituted the background of our work. After English, French and Chinese, Latin...

## References

- Aldon, G., Arzarello, F., Cusi, A., Garuti, R., Martignone, F. et al. (2013). The meta-didactical transposition: a model for analysing teachers education programs. In Lindmeier, A.M., Heinze, I. (Eds) *Mathematics learning across the life span, proceedings of PME 37*, Jul 2013, Kiel, Germany. 97-124.
- Artigue, M., Bosch, M., Chaachoua, H., Chellougui, F., Chesnais, A., Durand-Guerrier, V., Knipping, C., Maschietto, M., Romo-Vázquez, A., & Trouche, L. (2019). The French Didactic Tradition in Mathematics. In Blum, W., Artigue, M., Mariotti, M., Sträßer, R., & Van den Heuvel-Panhuizen, M. (Eds.), *European Traditions in Didactics of Mathematics* (pp. 11-56). ICME-13 Monographs. Cham: Springer.
- Artigue, M., & Winsløw, C. (2010). International comparative studies on mathematics education: A viewpoint from the anthropological theory of didactics. *Recherches en Didactique des Mathématiques*, 30(1), 48–82. <https://revue-rdm.com/2010/international-comparative-studies/>
- Atiyah, M. F. (1978). The Unity of Mathematics. *Bulletin of the London Mathematical Society*, 10(1), 69–76.
- Balacheff N. (1982) Preuve et démonstration en mathématique. *Recherches en Didactique des Mathématiques*, 3(3) 261–304.
- Bao, J., Huang, R., Yi, L., & Gu, L. (2003a). Study in bianshi teaching [In Chinese]. *Mathematics Teaching [Shuxue Jiaoxue]*, 1, 11–12.
- Bao, J., Huang, R., Yi, L., & Gu, L. (2003b). Study in bianshi teaching [In Chinese]. *Mathematics Teaching [Shuxue Jiaoxue]*, 2, 6–10.
- Bao, J., Huang, R., Yi, L., & Gu, L. (2003c). Study in bianshi teaching [In Chinese]. *Mathematics Teaching [Shuxue Jiaoxue]*, 3, 6–12.
- Barlow, D. H., Nock, M., & Hersen, M. (2008). *Single-case experimental designs: Strategies for studying behavior (3rd ed.)*. New York: Pergamon.
- Barlow, A. T., Prince, K. M., Lischka, A. E., & Duncan, M. D. (2017). Developing algebraic reasoning through variation in the U.S. In Huang, R., & Li, Y. (Eds.). *Teaching and Learning Mathematics through Variation: Confucian Heritage Meets Western Theories* (pp. 321–339). Rotterdam: Sense Publishers.
- Berthelot C., Berthelot R. (1983). *Quelques apports de la théorie des situations à l'introduction de la notion de limite en classe de première*. Mémoire de DEA de Didactique des Mathématiques. Bordeaux : Université de Bordeaux 1.
- Bessot, A. (2003). *Une introduction à la théorie des situations didactiques* (Master “Mathématiques, Informatique” de Grenoble 2003-2004).
- Bessot A., Richard F. (1979). *Commande de variable dans une situation didactique pour provoquer l'élargissement des procédures, rôle du schéma*. PhD. Bordeaux : Université de Bordeaux 1.
- Blum, W., Artigue, M., Mariotti, M., Sträßer, R., & Van den Heuvel-Panhuizen, M. (2019). *European Traditions in Didactics of Mathematics*. ICME-13 Monographs. Cham: Springer.

- Bourbaki, N. (1948). L'architecture de la mathématique. In F. Le Lionnais (Ed.), *Les grands courants de la pensée mathématique* (pp. 35–47). Cahiers du sud.
- Bowden, J., & Marton, F. (1998). *The University of Learning: Beyond Quality and Competence in Higher Education*. London: Kogan Page.
- Brousseau G. (1981) Problèmes de didactique des décimaux. *Recherches en Didactique des Mathématiques*, 2(1) 37–127.
- Brousseau, G. (1982 a). Les objets de la didactique des mathématiques, in *Actes de la Troisième école d'été de didactique des mathématiques*, Olivet.
- Brousseau, G. (1982 b). Ingénierie didactique : d'un problème à l'étude a priori d'une situation didactique, in *Actes de la Deuxième école d'été de didactique des mathématiques*.
- Brousseau, G. (1986). Fondements et méthodes de la didactique des mathématiques, *Recherches en didactique des mathématiques*, 7(2), 33–115.
- Brousseau, G. (1997). *Theory of didactical situations in mathematics*. Dordrecht, The Netherlands: Springer.
- Brousseau, G. (1999). Education et didactique des mathématiques. *Educacion y didactica de las matematicas*, Oct 1999, Mexique.
- Brousseau, G. (2002). *Theory of Didactical Situations in Mathematics*. Dordrecht, The Netherlands: Springer.
- Brousseau, G., & Brousseau, N. (1987). *Rationnels et décimaux dans la scolarité obligatoire*, Publication de l'I.R.E.M. de Bordeaux
- Brown, MW. (2009). The teacher-tool relationship: theorizing the design and use of curriculum materials. In J.T. Remillard, B.A. Herbel-Eisenmann, & G.M. Lloyd (Eds.), *Mathematics teachers at work: connecting curriculum materials and classroom instruction* (pp. 17–36). New York: Routledge.
- CFEM. (2021). *National Presentation of Mathematics Education in France at ICME-14*. [http://www.cfem.asso.fr/icmi/icme-14/NP\\_France\\_ICME14English.pdf](http://www.cfem.asso.fr/icmi/icme-14/NP_France_ICME14English.pdf)
- Chazan, D., & Yerushalmy, M. (2014). The future of mathematics textbooks: Ramifications of technological change. In M. Stocchetti (Ed.), *Media and Education in the Digital Age. Concepts, Assessments, Subversions* (pp.63–76). New York : Peter Lang.
- Chevallard, Y. (1985). *La transposition didactique. Du savoir savant au savoir enseigné*. Grenoble : La Pensée Sauvage.
- Clarke, D. (2017). Introduction: Putting Variation Theory to Work. In Huang, R., & Li, Y. (Eds.). *Teaching and Learning Mathematics through Variation: Confucian Heritage Meets Western Theories* (pp. 297-300). Rotterdam: Sense Publishers.
- Clements, D., Sarama, J., Spitler, M., Lange, A., & Wolfe, C. B. (2011). Mathematics learned by young children in an intervention based on learning trajectories: A large-scale cluster randomized trial. *Journal for Research in Mathematics Education*, 42, 127–166.
- Cobb, P., Zhao, Q., & Visnovska, J. (2008). Learning from and adapting the theory of realistic mathematics education. *Education et Didactique*, 2(1), 105–123.
- Daro, P., Mosher, F., & Corcoran, T. (2011). *Learning trajectories in mathematics*

- (Research Report No. 68). Madison, WI: Consortium for Policy Research in Education.
- Dienes, Z. P. (1973). A theory of mathematics learning. In F. J. Crosswhite, J. L. Highins, A. R. Ostone, & R. J. Shunway (Eds.), *Teaching mathematics: Psychological foundations* (pp. 137–148). Ohio: Charles A. Jones Publishing Company.
- Dimitrijevic, U. (2022). *Moving to France with teenagers? Find out everything you need to know about their education with our guide to secondary schools in France*. Retrieved 10<sup>th</sup> April, 2022, from <https://www.expatica.com/fr/education/children-education/secondary-school-france-286551/>
- Ding, L., Jones, K. & Sikko, S. A. (2017). An expert teacher's use of teaching with variation to support a junior mathematics teacher's professional learning. In Huang, R., & Li, Y. (Eds.). *Teaching and Learning Mathematics through Variation: Confucian Heritage Meets Western Theories* (pp. 241–266). Rotterdam: Sense Publishers.
- Douady, R. (1986). Jeux de cadres et dialectique outil-objet. *Recherches En Didactique Des Mathématiques*, 7(2), 5–31. <https://revue-rdm.com/1986/jeux-de-cadres-et-dialectique/>
- Duval, R. (1995). Sémiosis et pensée humaine. Registres sémiotiques et apprentissages intellectuels. [Semiosis and human thought. Semiotic registers and intellectual learning]. Berne, Switzerland: Peter Lang.
- Duval, R. (2006). A cognitive analysis of problems of comprehension in a learning of mathematics. *Educational Studies in Mathematics*, 61, 103–131.
- Duval, R. (2017). *Understanding the Mathematical Way of Thinking - The Registers of Semiotic Representations*. Cham: Springer.
- El Bouazzaoui H. (1982). *Étude de situations scolaires des enseignements du nombre et de la numération*. PhD. Bordeaux : Université de Bordeaux 1.
- Fan, L., Wong, N. Y., Cai, J. & Li, S. (2004). *How Chinese learn mathematics: Perspectives from insiders*. Singapore: World Scientific.
- Friedlander, A., & Tabach, M. (2001). Promoting multiple representations in algebra. In A. A. Cuoco (Ed.), *2001 Yearbook of the national council of the teachers of mathematics: the roles of representation in school mathematics* (pp. 173–185). Reston, Virginia: The Council.
- Galvez G. (1985). *El aprendizaje de la orientación en el espacio urbano*. PhD. Mexico.
- Gispert, H. (2014). *Mathematics education in France, 1900-1980*. In A. Karp & G. Schubring (Eds.), *Handbook on the history of mathematics education* (pp. 229–240). New York, NY: Springer.
- Goldin, G.A. (2014). Mathematical Representations. In S. Lerman (Ed.), *Encyclopedia of Mathematics Education* (pp. 409–413). Springer, Netherlands.
- Gosztonyi, K. (2015a). Séries de problèmes dans une tradition d'enseignement des mathématiques en Hongrie au 20<sup>e</sup> siècle. *SHS Web of Conferences*, 22, 00013. <https://doi.org/10.1051/shsconf/20152200013>
- Gosztonyi, K. (2015b). *Traditions et réformes de l'enseignement des mathématiques à l'époque des "mathématiques modernes": le cas de la Hongrie et de la France*. PhD

- thesis University of Szeged and University Paris Diderot, <https://hal.archives-ouvertes.fr/tel-01766902/document>
- Gosztonyi, K. (coord.), de Varent, C., Zhang, L., & Ramploud, A. (2021). *Variations and series of tasks, crossing the approaches*. Discussion group at ICME-14. <https://www.icme14.org/static/en/news/39.html?v=1646813795650>
- Gu, F., Huang, R. & Gu, L. (2017). Theory and development of teaching through variation in mathematics in China. In Huang, R., & Li, Y. (Eds.). *Teaching and Learning Mathematics through Variation: Confucian Heritage Meets Western Theories* (pp. 13–41). Rotterdam: Sense Publishers.
- Gu, L. (1981). The visual effect and psychological implications of transformation of figures on teaching geometry [In Chinese 演变图形在几何教学中的直观效果和心理学意义]. Paper presented at annual conference of Shanghai Mathematics Association, Shanghai, China.
- Gu, L. (1991). *Xuehui Jiaoxue [Learning to teach]*. Beijing, China: People's Education Press.
- Gu, L. (1994). Research on Mathematics Teaching Efficiency and Learning Theory. *Curriculum, Teaching Material and Method*, 5, 1–8.
- Gu, L. (1997a). Qingpu Experiment-A report on the reform of mathematics education based on China's contemporary level. *Shanghai Education*, 6, 35–38.
- Gu, L. (1997b). Qingpu Experiment-A report on the reform of mathematics education based on China's contemporary level. *Shanghai Education*, 7, 39–42.
- Gu, L. (2021). 45 years: an experiment of mathematics teaching reform. Plenary lecture in the 14<sup>th</sup> International Congress on Mathematical Education (ICME-14). Shanghai, China.
- Gu, L., Huang, R., & Marton, F. (2004). Teaching with variation: A Chinese way of promoting effective mathematics learning. In L. Fan, N. Y. Wong, J. Cai, & S. Li (Eds.), *How Chinese learn mathematics: Perspectives from insiders* (pp. 309–347). Singapore: World Scientific.
- Gu, M. (1999). *Dictionary of Education*. Shanghai: Shanghai Education Press.
- Gueudet, G., Bueno-Ravel, L., Modeste, S. & Trouche, L. (2018). Curriculum in France: A National Frame in Transition, In D. Thompson, M.A. Huntley, & C. Suurtamm. *International Perspectives on Mathematics Curriculum* (pp.41–70). International Age Publishing.
- Gueudet, G., Pepin, B., Sabra, H., & Trouche, L. (2016). Collective design of an e-textbook: Teachers' collective documentation. *Journal of Mathematics Teacher Education*, 19(2), 187–203.
- Gueudet, G. & Trouche, L. (2012). Communities, documents and professional geneses: interrelated stories. In Gueudet, G., Pepin, B. & Trouche, L. (Eds.), *From Text to 'Lived' resources: Mathematics Curriculum Material and Teacher Development* (pp. 305–322). New York, NY: Springer.
- Gueudet, G., Pepin, B., & Trouche, L. (2013). Textbooks' design and digital resources. In A. Watson & O. Minoru (Eds.), *ICMI study 22-Task design in mathematics education* (pp. 327–338). Springer Nature.

- Gueudet, G., & Trouche, L. (2009). Towards new documentation systems for mathematics teachers? *Educational Studies in Mathematics*, 71(3), 199–218.
- Guin, D., Ruthven, K., & Trouche, L. (Eds.) (2005). *The didactical challenge of symbolic calculators: Turning a computational device into a mathematical instrument*. New York: Springer.
- Guo, S. (2004). Summarizing and Proof-reading on “Nine Chapters on the mathematical Art”. [In Chinese 汇校九章算术] Shenyang: Liaoning Education Press.
- Hägglström, J. (2008). *Teaching systems of linear equations in Sweden and China: What is made possible to learn?* Göteborg: Acta Universitatis Gothoburgensis.
- Hao, L. (2019). *The Study of the Present Situation of High School Students' Understanding and Application of symbolic-graphic combination*. [In Chinese: 高中生对数形结合思想理解及运用现状的研究] Dissertation for master degree. East China Normal University, Shanghai.
- Han, X., Gong, Z., & Huang, R. (2017). Teaching and Learning Mathematics through Variation in Lesson Study. In Huang, R., & Li, Y. (Eds.). *Teaching and Learning Mathematics through Variation: Confucian Heritage Meets Western Theories* (pp. 267-293). Rotterdam: Sense Publishers.
- He, Q. (2003). Unity in mathematics. [In Chinese: 数学中的统一性] *Teaching and Learning in high school mathematics*, 2, 40–41.
- Hilbert, D. (1902). Mathematical Problems. *Bulletin of the American Mathematical Society*. 8 (10), 437–479.
- Hino, K. (2017). Improving Teaching through Variation: A Japanese Perspective. In Huang, R., & Li, Y. (Eds.). *Teaching and Learning Mathematics through Variation. Confucian Heritage Meets Western Theories* (pp. 301–320). Rotterdam: Sense Publishers.
- Huang, R. (2002). *Mathematics teaching in Hong Kong and Shanghai: A classroom analysis from the perspective of variation* (Unpublished doctoral dissertation). The University of Hong Kong, Hong Kong.
- Huang, R. & Leung, F. K. S. (2017). Teaching Geometrical Concepts through Variation: A Case Study of a Shanghai Lesson. In Huang, R., & Li, Y. (Eds.). *Teaching and Learning Mathematics through Variation: Confucian Heritage Meets Western Theories* (pp. 151–168). Rotterdam: Sense Publishers.
- Huang, R. & Li, Y. (2017). Introduction. In Huang, R., & Li, Y. (Eds.). *Teaching and Learning Mathematics through Variation. Confucian Heritage Meets Western Theories* (pp. 3-11). Rotterdam: Sense Publishers.
- Huang, R., Mok, I., & Leung, F. K. S. (2006). Repetition or variation: “Practice” in the mathematics classrooms in China. In D. J. Clarke, C. Keitel, & Y. Shimizu (Eds.), *Mathematics classrooms in twelve countries: The insider's perspective* (pp. 263–274). Rotterdam, The Netherlands: Sense Publishers.
- Huang, R., Yang, X., & Zhang, P. (2017). Teaching geometry review lesson through variation: a case study. In Huang, R., & Li, Y. (Eds.). *Teaching and Learning Mathematics through Variation: Confucian Heritage Meets Western Theories* (pp. 169-186). Rotterdam: Sense Publishers.

- Huang, R., Zhang, Q., Chang, Y-P. & Kimmins, D. (2019). Developing students' ability to solve word problems through learning trajectory-based and variation task-informed instruction. *ZDM Mathematics Education*, 51, 169–181  
<https://doi.org/10.1007/s11858-018-0983-8>
- Humboldt, W. V. (1971). *Linguistic Variability and Intellectual Development*. Coral Gables, FL: University of Miami Press.
- Ichou, M. & Vallet, L-A. (2011). Do all roads lead to inequality? Trends in French upper secondary school analysed with four longitudinal surveys. *Oxford Review of Education*, 37(2), 167–194.
- IEA (2021). *TIMSS: Trends in International Mathematics and Science Study*.  
<https://timssandpirls.bc.edu/timss-landing.html>
- Janvier, C. (1987). *Problems of representation in the teaching and learning of mathematics*. Hillsdale, NJ: Lawrence Erlbaum.
- Jin, M. (2013). Unity and High School Mathematics Teaching. [In Chinese 统一性与高中数学教学] *Mathematics Learning and Research*. 21, 46.
- Kahane, J-P. (2002). *L'enseignement des sciences mathématiques. Rapport au ministre. - Commission de réflexion sur l'enseignement des mathématiques*. Paris : Odile Jacob. [http://www-old.irem.univ-paris-diderot.fr/up/presentation\\_des\\_rapports.pdf](http://www-old.irem.univ-paris-diderot.fr/up/presentation_des_rapports.pdf)
- Katambura I. (1986). *Sur la résolution des problèmes de soustraction au cours élémentaire*. PhD. Bordeaux: Université de Bordeaux 1.
- Krainer, K. (2017). Introduction of part III. In Huang, R., & Li, Y. (Eds.). *Teaching and Learning Mathematics through Variation. Confucian Heritage Meets Western Theories* (pp. 209-212). Rotterdam: Sense Publishers.
- Kieran, C., Doorman, M., & Ohtani, M. (2015). Frameworks and Principles for Task Design. In A. Watson, & M. Ohtani (Eds.), *Task Design in Mathematics Education-ICMI study 22* (pp. 19-81). Switzerland: Springer.
- Klein, F. (2007). *Elementary Mathematics from an Advanced Standpoint: Arithmetic, Algebra, Analysis*. Cosimo Classics.
- Kline, M. (1980). *Mathematics: The Loss of Certainty*. Oxford University Press.
- Kock, Z-J., & Pepin, B. (2018). *Student use of resources in calculus and linear algebra*. In: INDRUM conference, Kristiansand, Apr 2018.
- Lai, H. (2001). Points for Attention in Solving Problems of “Symbolic-Graphic Combination”. [In Chinese: “数形结合”解题的注意事项] *Bulletin of Mathematics (Wuhan)*. 10, 25.
- Leung, A. (2017). Variation in Tool-Based Mathematics Pedagogy: The Case of Dynamic Virtual Tool. In Huang, R., & Li, Y. (Eds.). *Teaching and Learning Mathematics through Variation: Confucian Heritage Meets Western Theories* (pp.69–84). Rotterdam: Sense Publishers.
- Leung, F. K. S. (2017). Making Sense of Mathematics Achievement in East Asia: Does Culture Really Matter? In Kaiser G. (Eds). *Proceedings of the 13th International Congress on Mathematical Education ICME-13 Monographs* (pp. 201–218). Springer, Cham.
- Li, J. (2011). *A study of algebra instruction with variation based on multi-*



- representations in middle school*. [In Chinese 基于多元表征的初中代数变式教学研究] Dissertation for doctor degree. Chongqing, Southwest University.
- Li, Z. (2010). *History of mathematics in China*. Tianjin Education Press, China.
- Liu, X. (2017). 中学化学“双基”教学的继承与超越. *Teacher Education Forum*, 30(12), 48–51.
- Liu, X., Xu, Z., & Zhao, Y. (2006). The development history of Chinese mathematics course standard. *Journal of Yanan University (Natural Science Edition)*, 25(2), 21–25.
- Liu, Y. (2018). Discussion on the Application of Symbolic-Graphic Combination in Junior Middle School Mathematics Teaching. [In Chinese 初中数学教学中数形结合思想的应用探讨] *Children's World*. 5, 41–42.
- Luo, X. (2005). *Study on Problem-solving of Symbolic-Graphic combination: Perspective from representation*. [In Chinese 数形结合的解题研究：表征的视角] Dissertation for doctor degree. East China Normal University, Shanghai.
- Luo, Z. (1997). Introduction to Mathematical Problem-Solving. [In Chinese 数学解题学引论] Xi'an: Shaanxi Normal University Press.
- Malaval, J., Bousseyroux, J-L., Desrousseaux, P-A., Destruhaut, F., Lample, H., Lécole, J-M., Lericque, I., Ternoy, J., & Vedrine, M. (2014). *Hyperbole mathématiques 2de*. Nathan, France.
- Malaval, J., Bousseyroux, J-L., Desrousseaux, P-A., Destruhaut, F., Lample, H., Lécole, J-M., Lericque, I., Ternoy, J., Vedrine, M. (2014). *Hyperbole mathématiques 2de. Livre du professeur*. Nathan, France.
- Markman, A. & Dietrich, E. (2000). In defense of representation. *Trends in cognitive science*. 4, 470–475.
- Marton, F. (1999, August). *Variatio est mater studiorum*. Opening address at the 8th EARLI conference, Göteborg, Sweden.
- Marton, F. (2015). *Necessary conditions of learning*. London: Routledge.
- Marton, F., & Booth, S. (1997). *Learning and awareness*. Mahwah, NJ: Lawrence Erlbaum.
- Marton, F., & Pang, M. F. (2006). On some necessary conditions of learning. *Journal of the Learning Sciences*, 15(2), 193–220.
- Marton, F., & Tsui, A. B. M. (2004). *Classroom discourse and the space of learning*. Mahwah, NJ: Lawrence Erlbaum Associates.
- Maudet C. (1982). *Les situations et les processus de l'apprentissage d'une fonction logique*. PhD. Bordeaux : Université de Bordeaux 1.
- Mercat, M., Filho, P., & El-Demerdash, M. (2016). Creativity and technology in Mathematics: from story telling to algorithmic with Op'Art. *Acta Didactica Napocensia*, Universitas Napocensis - Babes Bolyai University, 10(1), 63–70.
- Ministère de l'Éducation nationale et de la Jeunesse (2019a). *Programme de mathématiques de seconde générale et technologique*. [https://cache.media.education.gouv.fr/file/SP1-MEN-22-1-2019/95/7/spe631\\_annexe\\_1062957.pdf](https://cache.media.education.gouv.fr/file/SP1-MEN-22-1-2019/95/7/spe631_annexe_1062957.pdf)
- Ministère de l'Éducation nationale et de la Jeunesse (2019b). *Programme*



- d'enseignement scientifique de première Générale.*  
[https://cache.media.education.gouv.fr/file/SP1-MEN-22-1-2019/13/4/spe573\\_annexe\\_1063134.pdf](https://cache.media.education.gouv.fr/file/SP1-MEN-22-1-2019/13/4/spe573_annexe_1063134.pdf)
- Ministère de l'Éducation nationale et de la Jeunesse (2019c). *Programme d'enseignement scientifique de terminale Générale.*  
[https://cache.media.education.gouv.fr/file/SPE8\\_MENJ\\_25\\_7\\_2019/84/7/spe241\\_annexe\\_1158847.pdf](https://cache.media.education.gouv.fr/file/SPE8_MENJ_25_7_2019/84/7/spe241_annexe_1158847.pdf)
- Ministère de l'Éducation nationale et de la Jeunesse (2020a). *Programme de mathématiques de la classe de première professionnelle.*  
[https://cache.media.education.gouv.fr/file/SPE1-MENJ-6-2-2020/84/1/spe003\\_annexe1\\_1239841.pdf](https://cache.media.education.gouv.fr/file/SPE1-MENJ-6-2-2020/84/1/spe003_annexe1_1239841.pdf)
- Ministère de l'Éducation nationale et de la Jeunesse (2020b). *Programme de mathématiques de la classe de terminale professionnelle.*  
[https://cache.media.education.gouv.fr/file/SPE1-MENJ-6-2-2020/84/3/spe003\\_annexe2\\_1239843.pdf](https://cache.media.education.gouv.fr/file/SPE1-MENJ-6-2-2020/84/3/spe003_annexe2_1239843.pdf)
- Ministère de l'Éducation nationale et de la Jeunesse (2021a). *Système éducatif français - Traduction DE, EN, ES.* [https://eduscol.education.fr/2897/systeme-educatif-francais-traduction-de-en-es?menu\\_id=3590](https://eduscol.education.fr/2897/systeme-educatif-francais-traduction-de-en-es?menu_id=3590)
- Ministère de l'Éducation nationale et de la Jeunesse (2021b). *Les programmes du lycée général et technologique.* <https://www.education.gouv.fr/les-programmes-du-lycee-general-et-technologique-9812>
- Ministère de l'Éducation nationale et de la Jeunesse. (2022). *[ARCHIVE] Upper secondary general or technological school: new Grade 10 and Grade 11 classes.* <https://www.education.gouv.fr/upper-secondary-general-or-technological-school-new-grade-10-and-grade-11-classes-5465>
- Ministry of National Education (2021). Bac 2021: un tremplin vers la réussite. Retrieved 21<sup>st</sup> April 2022, from <https://www.education.gouv.fr/bac-2021-un-tremplin-vers-la-reussite-1019>
- Ministry of Education of the People's Republic of China (MOE, 2011). *Compulsory Education Mathematics Curriculum Standards.* <http://www.moe.gov.cn/srcsite/A26/s8001/201112/W020220418401382030426.pdf>
- Ministry of Education of the People's Republic of China (MOE, 2017). *General high school mathematics curriculum standards.* People's Education Press: Beijing, China.
- Ministry of Education of the People's Republic of China (MOE, 2019). *Opinions of the Ministry of Education on Strengthening Educational Scientific Research in the New Era.* Retrieved from [http://www.moe.gov.cn/srcsite/A02/s7049/201911/t20191107\\_407332.html](http://www.moe.gov.cn/srcsite/A02/s7049/201911/t20191107_407332.html)
- Mok, I. A. C. (2017). Teaching Algebra through Variations: Contrast, Generalization, Fusion, and Separation. In Huang, R., & Li, Y. (Eds.). *Teaching and Learning Mathematics through Variation: Confucian Heritage Meets Western Theories* (pp. 187–205). Rotterdam: Sense Publishers.
- Mopondi, B. (1986). *Le sens dans la négociation didactique. Notion de proportionnalité au cours moyen.* PhD. Bordeaux: Université de Bordeaux 1.

- Mullis, I. V. S., Martin, M. O., Foy, P., Kelly, D. L., & Fishbein, B. (2020). *TIMSS 2019 International Results in Mathematics and Science*. Retrieved from Boston College, TIMSS & PIRLS International Study Center website: <https://timssandpirls.bc.edu/timss2019/international-results/>
- National Research Council. (1995). *International Comparative Studies in Education: Descriptions of Selected Large-Scale Assessments and Case Studies*. Washington, DC: The National Academies Press. <https://doi.org/10.17226/9174>.
- Nizet, I., Monod-Ansaldi, R., Aldon, G., Prieur, M., & Criquet, A. (2019). L'analyse de valuations dans une démarche collaborative de recherche. *La Revue LEeE*, (1). <https://doi.org/10.48325/rleee.001.04> (Original work published 16 mars 2019)
- OCDE (2017). PISA 2015 Mathematics Framework, in *PISA 2015 Assessment and Analytical Framework: Science, Reading, Mathematic, Financial Literacy and Collaborative Problem Solving*, Éditions OCDE, Paris. <https://doi.org/10.1787/9789264281820-5-en>.
- OCDE (2019a). Program for international student assessment (PISA) result from PISA 2018: China. [https://www.oecd.org/pisa/publications/PISA2018\\_CN\\_QCI.pdf](https://www.oecd.org/pisa/publications/PISA2018_CN_QCI.pdf)
- OCDE (2019b). *PISA 2018 Assessment and Analytical Framework*, PISA, Éditions OCDE, Paris. <https://doi.org/10.1787/b25efab8-en>.
- Pang, M. F., & Ki, W. W. (2016). Revisiting the idea of 'critical aspects'. *Scandinavian Journal of Educational Research*, 60(3), 323–336.
- Pang, M. F., Bao, J., & Ki, W. W. (2017). “Bianshi” and the Variation Theory of Learning: Illustrating Two Frameworks of Variation and Invariance in the Teaching of Mathematics. In Huang, R., & Li, Y. (Eds.). *Teaching and Learning Mathematics through Variation: Confucian Heritage Meets Western Theories* (pp. 43–68). Rotterdam: Sense Publishers.
- Park, K., & Leung, F. K. S. (2006). A comparative study of the mathematics textbooks of China, England, Japan, Korea, and the United States. In F. K. S. Leung, K. D. Graf, & F. J. Lopez-Real (Eds.), *Mathematics education in different cultural traditions--A comparative study of East Asia and the West: The 13th ICMI study*. New York, NY: Springer.
- Peng, A., Li, J., & Nie, B., & Li, Y. (2017). Characteristics of Teaching Mathematical Problem Solving in China: Analysis of a Lesson from the Perspective of Variation. In Huang, R., & Li, Y. (Eds.). *Teaching and Learning Mathematics through Variation: Confucian Heritage Meets Western Theories* (pp. 111–126). Rotterdam: Sense Publishers.
- People's Education Press, Curriculum and Teaching Materials Research Institute, & Middle School Mathematics Curriculum Textbook Research Center (2019). *General High School Textbook: Mathematics (PEP version A) – Teachers' guidebook (Compulsory series I)*. People's Education Press, Beijing.
- People's Education Press, Curriculum and Teaching Materials Research Institute, & Middle School Mathematics Curriculum Textbook Research Center (2019). *General High School Textbook: Mathematics-Compulsory series I (PEP version A)*. People's Education Press, Beijing.

- Pepin, B., Choppin, J., Ruthven, K., & Sinclair, N. (2017a). Digital curriculum resources in mathematics education: foundations for change. *ZDM Mathematics Education*, 49(5), 645–661. <https://doi.org/10.1007/s11858-017-0879-z>
- Pepin, B., Gueudet, G., & Trouche, L. (2013). Investigating textbooks as crucial interfaces between culture, policy and teacher curricular practice: Two contrasted case studies in France and Norway. *ZDM Mathematics Education*, 45(5), 685–698.
- Pepin, B. & Gueudet, G. (2014). Curriculum Resources and Textbooks in Mathematics Education. In Lerman S. (Eds.), *Encyclopedia of Mathematics Education*. Dordrecht: Springer.
- Pepin, B., Gueudet, G., Trouche, L. (2017b). Refining teacher design capacity: mathematics teachers' interactions with digital curriculum resources. *ZDM Mathematics Education*, 49(5), 799–812. <https://rdcu.be/tmXb>
- Perez J. (1984). *Construction d'un code de désignation d'objets*. PhD. Bordeaux : Université de Bordeaux II.
- Prediger, S., Bikner-Ahsbahr, A., & Arzarello, F. (2008). Networking strategies and methods for connecting theoretical approaches: First steps towards a conceptual framework. *ZDM Mathematics Education*, 40(2), 165–178.
- Proust, C. (2012). Teachers' writings and students' writings: school material in Mesopotamia. In G. Gueudet, B. Pepin, & L. Trouche (Eds.), *From text to 'Lived' Resources. Mathematical Curriculum Materials and Teacher Development* (pp. 161–180). New York: Springer Verlag.
- Psycharis, G., Kalogeria, E. (2018). Studying the process of becoming a teacher educator in technology-enhanced mathematics. *J Math Teacher Educ* 21, 631–660. <https://doi.org/10.1007/s10857-017-9371-5>
- Qi, C., Wang, R., Mok, I. A. C., & Huang, D. (2017). Teaching the Formula of Perfect Square through *Bianshi* Teaching. In Huang, R., & Li, Y. (Eds.). *Teaching and Learning Mathematics through Variation. Confucian Heritage Meets Western Theories* (pp. 127–150). Rotterdam: Sense Publishers.
- Qi, C. & Wang, X. (2019). The Pythagoras Chapter of “Nine Chapters on the Mathematical Art” and the Variation Thought in Liu Hui's Annotation. [In Chinese: 《九章算术》勾股章及其刘徽注中的变式思想] *Journal of Mathematics (China)*, 58(5), 5–9.
- Qiu, Q. (2019). Variation teaching is an important way to land core literacy. *Correspondence of the Teaching of Mathematics*, 17, 38–39.
- Quevedo De Villegas B. (1986). *Le rôle de l'énumération dans l'apprentissage du dénombrement*. PhD. Bordeaux : Université de Bordeaux 1.
- Rabardel, P. (1995). *Les hommes et les technologies, approche cognitive des instruments Contemporains*. Paris: Armand Colin. (English version at [http://ergoserv.psy.univ-paris8.fr/Site/default.asp?Act\\_group=1](http://ergoserv.psy.univ-paris8.fr/Site/default.asp?Act_group=1)).
- Rabardel, P., & Bourmaud, G. (2003). From computer to instrument system: A developmental perspective. In P. Rabardel and Y. Waern (eds.), *Special Issue “From Computer Artifact to Mediated Activity”, Part 1: Organisational Issues*,

- Interacting with Computers*, 15(5), 665–691.
- Ratsimba-Rajohn, H. (1981) *Etudes de deux méthodes de mesures rationnelles : la commensuration et le fractionnement de l'unité en vue de l'élaboration des situations didactiques*. PhD. Bordeaux : Université de Bordeaux 1.
- Rocha, K. (2019). *Une étude des effets du travail documentaire et collectif sur le développement professionnel des enseignants de mathématiques : apport des concepts d'expérience et de trajectoire documentaires*. PhD, ENS de Lyon. Retrieved from <https://tel.archives-ouvertes.fr/tel-02399664/document>
- Rouchier, A. (1980). Situations et processus didactiques dans l'étude des nombres rationnels positifs. *Recherches en Didactique des Mathématiques*, 1(2), 225–275.
- Rowland, T. (2008). The purpose, design, and use of examples in the teaching of elementary mathematics. *Educational Studies in Mathematics*, 69, 149–163.
- Runesson, U. (1999). *The pedagogy of variation: Different ways of handling mathematical topics*. Doctoral Dissertation, Gothenburg, Sweden: University of Gothenburg.
- Runesson, U. & Kullberg, A. (2017). Learning to teach with variation: experiences from learning study in Sweden. In Huang, R., & Li, Y. (Eds.). *Teaching and Learning Mathematics through Variation. Confucian Heritage Meets Western Theories* (pp. 355–372). Rotterdam: Sense Publishers.
- Ruthven, K., Laborde, C., Leach, J., & Tiberghien, A. (2009). Design tools in didactical research: Instrumenting the epistemological and the cognitive aspects of the design of teaching sequences. *Educational Researcher*, 38, 329–342.
- Shao, G. (2009). *Mathematical thoughts and methods as educational tasks*. [ In Chinese 作为教育任务的数学思想与方法] Shanghai: Shanghai Education Press.
- Shimizu, Y., Vithal, R. (2018). *School Mathematics Curriculum Reforms: Challenges, Changes and Opportunities. Proceedings of ICMI Study 24 Conference*. Tsukuba : University of Tsukuba. [https://drive.google.com/file/d/1xR0QOjI8X\\_mJb5mmhH-G\\_OqoMaFE54Gy/view](https://drive.google.com/file/d/1xR0QOjI8X_mJb5mmhH-G_OqoMaFE54Gy/view)
- Silver, E. A. Mamona-Downs, J., Leung, S., & Ann, P. (1996). Posing mathematical problems: an exploratory study. *Journal for research in mathematics education*, 27(3), 293–309.
- Singer, F.M., Ellerton, N. & Cai, J. (2013). Problem-posing research in mathematics education: new questions and directions. [\*Educational Studies in Mathematics\*](#), 83, 1–7.
- Soury-Lavergne, S., & Maschietto, M. (2015). Articulation of spatial and geometrical knowledge in problem solving with technology at primary school. *ZDM Mathematics Education*, 47(3), 435–449.
- Sriraman, B. & English, L. D. (2005). Theories of mathematics education: A global survey of theoretical frameworks/trends in mathematics education research. *ZDM*, 37(6), 450–456.
- Stylianides G. J. (2016). *Curricular Resources and Classroom Use: The Case of Mathematics*. Oxford University Press.

- Sun, X. (2011). 'Variation problem' and their roles in the topic of fraction division in Chinese mathematics textbook examples. *Educational Studies in Mathematics*, 76, 65–85.
- Trouche, L. (2014). Instrumentation in Mathematics Education. In S. Lerman (Ed.), *Encyclopedia of Mathematics Education* (pp. 307–313). N.Y.: Springer. doi:10.1007/978-3-319-77487-9\_80-3.
- Trouche L. (2016). Didactics of Mathematics: Concepts, Roots, Interactions and Dynamics from France. In J. Monaghan, L. Trouche, & J.M. Borwein (Eds.), *Tools and Mathematics, instruments for learning* (pp. 219–256). Cham: Springer.
- Trouche L. (2020). Instrumentalization in Mathematics Education. In S. Lerman (Ed.), *Encyclopedia of Mathematics Education* (2<sup>nd</sup> edition, pp. 404–412). Cham: Springer. doi: 10.1007/978-3-030-15789-0\_80
- Trouche, L., Bao, J., Soury-Lavergne, S., Xu, B., Zhang, L., Zhu, F., Wang, C., Gitirana, V., & Miyakawa, T. (2018). *Mathematics Teachers Resources in a Time of Transitions* (MaTRiTT). <http://ife.ens-lyon.fr/ife/recherche/groupe-de-travail/matritt-joriss>
- Trouche, L., Drijvers, P., Gueudet, G., & Sacristan, A. I. (2012). Technology-driven developments and policy implications for mathematics education. In A.J. Bishop, M.A. Clements, C. Keitel, J. Kilpatrick, & F.K.S. Leung (Eds.), *Third international handbook of mathematics education* (pp. 753–789). New York, NY: Springer,
- Trouche, L., Gueudet, G., & Pepin, B. (2019). *The 'Resource' Approach to Mathematics Education*. Springer series Advances in Mathematics Education. Cham: Springer.
- Trouche, L., Gueudet, G., & Pepin, B. (2020a). The documental approach to didactics. In S. Lerman (Eds.) *Encyclopedia of Mathematics Education*, (2<sup>nd</sup> edition, pp. 307–313). Cham, Switzerland: Springer. doi:10.1007/978-3-319-77487-9\_100011-1
- Trouche, L., Gueudet, G., Pepin, B., Wang, C., & Xu, B. (2020b). 文献纪录教学论. DAD Multilingual <https://hal.archives-ouvertes.fr/hal-02891008/document>
- Trouche, L. & Wang, C. (2015). *Le bulletin de la CFEM (Commission française pour l'enseignement des mathématiques) de ce mois-ci rend compte d'une visite dans un collège de Shanghai* (p.10-12). <http://www.cfem.asso.fr/liaison-cfem/lettre-cfem-decembre%202015>
- Trouche, L. et 37 élèves d'une classe de terminale S (1998). *Faire des mathématiques avec des calculatrices symboliques, conjecturer et prouver. 37 variations sur un thème imposé*. Montpellier : IREM, Université Montpellier 2
- Vergnaud, G. (1998). Toward a cognitive theory of practice. In A. Sierpiska, & J. Kilpatrick (Eds.), *Mathematics education as a research domain: A search for identity* (pp. 227–241). Dordrecht: Kluwer.
- Vergnaud, G. (2009). The theory of conceptual fields. *Human Development*, 52, 83–94
- Verillon, P., & Rabardel, P. (1995). Cognition and artifact: A contribution to the study of thought in relation to instrument activity. *European Journal of Psychology in Education*, 10 (1), 77–101.
- Vygotsky, L.S. (1978). *Mind in society: The development of higher psychological*



- processes. Cambridge, MA: Harvard University Press.
- Wang, B. (1996). Examples of several types of problems that are commonly solved by symbolic-graphic combination. [In Chinese 例谈常用数形结合解答的几类问题] *Reference for middle school mathematics teaching*, 6, 24–25.
- Wang, C. (2019). *An investigation of mathematics teachers' documentation expertise and its development in collectives: two contrasting cases in China and France*. PhD, University of Lyon and ECNU, <https://tel.archives-ouvertes.fr/tel-02275820>
- Wang, H. (2016). Thoughts on the teaching value of Pythagoras theorem based on the history of mathematics and the idea of “Elements”. [In Chinese 基于数学史和《原本》思想的勾股定理教学价值思考] *Teaching and Management (middle school edition)*, 4, 44–46.
- Wang, J., & Zhang, J. (2014). An international comparison of core content in high school mathematics textbooks. *Curriculum, teaching material and method*, 34(10), 112–119. (In Chinese)
- Wang, X., & Huo, Y. (2014). Analysis of the cultural style of ancient Chinese mathematics from “Nine Chapters on the Mathematical Art”. [In Chinese 从《九章算术》中分析我国古代数学的文化风格] *Lantai World*, 17, 151–152.
- Watson, A., & Mason, J. (2005). *Mathematics as a constructive activity: Learners generating examples*. Mahwah, NJ: Lawrence Erlbaum.
- Watson, A., & Mason, J. (2006). Seeing an exercise as a single mathematical object: Using variation to structure sense-making. *Mathematical Thinking and Learning*, 8(2), 91–111.
- Watson, A., & Thompson, D. R. (2015). Design issues related to text-based tasks. In Watson, A., & Ohtani, M. (Eds.), *Task Design in Mathematics Education-ICMI study 22* (pp.143–190). Switzerland: Springer.
- Wong, N. Y. (2004). The CHC learner's phenomenon: its implications on mathematics education. In L. Fan, N. Y. Wong, J. Cai, & S. Li (Eds.), *How Chinese learn mathematics: Perspectives from insiders* (pp. 309–347). Singapore: World Scientific.
- Wong, N. Y., Lam, C. C., Sun, X., & Chan, A. M. Y. (2009). From “exploring the middle zone” to “constructing a bridge”: Experimenting in the Spiral bianshi mathematics curriculum. *International Journal of Science and Mathematics Education*, 7, 363–382.
- Woods, P. (1985). Conversations with teachers: Some aspects of the life history method. *British Educational Research Journal*, 2(1), 13–26.
- Wolcott, H. F. (1992). *Writing up Qualitative Research*. Newbury Park: CA Sage.
- Wylie, A. (1852). Jottings on the Science of Chinese. *North China Herald*.
- Xu, B. (2001). *Prospects of Mathematics Education*. [In Chinese 数学教育展望] Shanghai: East China Normal University Press, pp141–142.
- Xu, B. & Chen, Y. (1998). *Graphics and Mathematics Problem Solving*. [In Chinese 图形与数学解题] Changsha: Hunan Education Press.
- Yin, L. (2016). “One graph multiple changes” cleverly solving problems--Starting from the Proof of Pythagoras Theorem in “Elements”. [In Chinese “一图多变”巧解题--从《几何原本》中勾股定理的证明说起] *Shuxue Jiaoxue Tongxun*, 17, 62–64.

- Yin, R. K. (2018). *Case Study Research and Applications: Design and Methods* (6th ed.). Thousand Oaks, CA: Sage.
- Yu, M. (2021). *What are the new changes brought by the new college entrance examination—A look back at the seventh anniversary of the reform of the college entrance examination*. Retrieved 20<sup>th</sup> April 2022, from [http://www.moe.gov.cn/jyb\\_xwfb/s5147/202109/t20210914\\_562841.html](http://www.moe.gov.cn/jyb_xwfb/s5147/202109/t20210914_562841.html)
- Zhang, C., etc. (reprinted in 2019). *The Nine Chapters on the Mathematical Art*. [In Chinese 九章算术] Chongqing: Chongqing Press.
- Zhang, D. & Dai, Z. (2004). “Two basics”: Mathematics teaching approach and open-ended problem solving in China. *Regular lecture delivered at the 10th International Congress of Mathematics Education*, Copenhagen, Denmark, July.
- Zhang, J. (1997). The nature of external representations in problem solving. *Cognitive Science*, 21(2), 179–217.
- Zhang, J. (2019). Core literacy-oriented high school mathematics textbook reform--Research and compilation of “General High School Textbook: Mathematics (PEP version A)”. *Maths Teaching in Middle School*, 31, 9–14. (In Chinese)
- Zhang, J., Wang, R., Huang, R., & Kimmins, D. (2017). Strategies for using variation tasks in selected mathematics textbooks in China. In Huang, R., & Li, Y. (Eds.). *Teaching and Learning Mathematics through Variation. Confucian Heritage Meets Western Theories* (pp. 213–240). Rotterdam: Sense Publishers.
- Zhang, R., & Liu, Y. (2020). Comparative Study of High School Mathematics Teaching Material “Geometry” Part --Taking “Teach A Version” and “Sue Textbooks” as Examples. *Education teaching forum*, 2, 307–309. (In Chinese)
- Zhang, X., & Wang, Z. (1997). Comparisons between "Nine Chapters on the Mathematical Art" and “Elements”. [In Chinese 《九章算术》与《几何原本》的一些比较] *Journal of Mathematics Education*, 3, 109–110.
- Zhang, Y. (1993). “The program of learning mathematics” is Yang Hui’s major contribution to the theory of mathematical curriculum and instruction. *Journal mathematics education*, 2(1), 45–49.
- Zheng, F. (2019). Using variation to cultivate mathematics subject reflections on core literacy. *Correspondence of the Teaching of Mathematics*, 24, 36–37.
- Zheng, Y., & Xie, M. (2004). “双基”与“双基教学”:认知的观点. *Teaching Reference of Middle School Mathematics*, 6, 1–5.

## Appendix I: Schedule and Content of classroom observation in China

Two classes for each lesson in the same day. In the table below, interviews with Lin are noted in light yellow; activities related to preservice teacher Shi are noted in nude; exercise lessons are noted in light blue.

Week	Date (2020)	Content
1	13 <sup>th</sup> Oct.	First interview about some background information
	13 <sup>th</sup> Oct.	The concept of function I
	14 <sup>th</sup> Oct.	The concept of function II
	15 <sup>th</sup> Oct.	The representation of function I
	16 <sup>th</sup> Oct.	The representation of function II
2	19 <sup>th</sup> Oct.	Exercise lesson (homework and test)
	20 <sup>th</sup> Oct.	Exercise lesson (homework and test)
	21 <sup>st</sup> -23 <sup>rd</sup> Oct.	Mid-term examination
3	26 <sup>th</sup> Oct.	Exercise lesson (midterm examination)
	27 <sup>th</sup> Oct.	Monotonicity of function I
	28 <sup>th</sup> – 30 <sup>th</sup> Oct.	School sports meeting
4	2 <sup>nd</sup> Nov.	Monotonicity of function II
	3 <sup>rd</sup> Nov.	Extremum of a function
	4 <sup>th</sup> Nov.	Exercise lesson (homework)
	5 <sup>th</sup> Nov.	Parity of a function
	Lin's classroom teaching observation ends	
6	16 <sup>th</sup> Nov.	First time: Guiding activity (Discuss lesson plans)
	19 <sup>th</sup> Nov.	Second time: Guiding activity (Trial teaching I)
7	23 <sup>rd</sup> Nov.	Third time: Guiding activity (Trial teaching II)
	25 <sup>th</sup> Nov.	Shi's practical teaching in class
	25 <sup>th</sup> Nov.	Interview with Shi
8	4 <sup>th</sup> Dec.	Second interview





## Appendix II: Schedule and Content of classroom observation in France

In the table below, AP lesson is noted; half-class is noted in light blue; thematic lessons are noted in nude; interviews and discussion with Emma are noted in light yellow. The analysis of general situations and episodes are selected from the lessons highlighted in yellow and the thematic lessons. Other lessons are used as the background information to support our analysis.

Week	Date (2018/2019)	Content
0	31 <sup>st</sup> Oct.	First interview about some background information
1	6 <sup>th</sup> Nov.	Chapter 3 <sup>146</sup> : Intervals, inequation
	8 <sup>th</sup> Nov. (AP)	Review: factorization, intervals, inequation
	8 <sup>th</sup> Nov.	Thematic lesson “open box” (1)
	9 <sup>th</sup> Nov. 10h-11h	20mn Test (intervals, inequation); Union and intersection
	9 <sup>th</sup> Nov. 11h-12h	Thematic lesson “open box” (2)
2	13 <sup>th</sup> Nov.	Correct the exercises, make a conclusion of chapter 3 (including intervals, inequation, union and intersection)
	15 <sup>th</sup> Nov.	Chapter 3: System of inequations, product equation
	16 <sup>th</sup> Nov. 10h-11h30	Thematic lesson “open box” (3)
	16 <sup>th</sup> Nov. 11h30-12h	Introduction of function
3	20 <sup>th</sup> Nov.	Review: product equation; Chapter 4: concept of the function (1)
	22 <sup>nd</sup> Nov. (AP)	Review: System of inequations
	22 <sup>nd</sup> Nov.	Review: product equation; Chapter 4: Use the calculator to solve the problem of function (1)
	23 <sup>rd</sup> Nov. 10h-11h	Chapter 4: concept of the function (2)
	23 <sup>rd</sup> Nov. 11h-12h	contrôle
4	27 <sup>th</sup> Nov.	Chapter 4: concept of the function (3)
	29 <sup>th</sup> Nov.	Python: in the information room (Autonomic learning)
	29 <sup>th</sup> Nov.	Second interview
	30 <sup>th</sup> Nov. 10h-11h30	Chapter 4: images, pre-images, table of value

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<sup>146</sup> This chapter is noted by teacher according to her progress of 10th grade mathematics teaching, shown in table 6.1.4.

	30 <sup>th</sup> Nov. 11h30-12h	test
5	6 <sup>th</sup> Dec. (AP)	Review: concept of the fonction, images, pre-images, product equation
	6 <sup>th</sup> Dec.	Chapter 4: Variation, table of variation (graphical aspect) (1)
	7 <sup>th</sup> Dec. 10h-11h	Chapter 4: Variation, table of variation (graphical aspect) (2)
	7 <sup>th</sup> Dec. 11h-12h	Chapter 4: Variation, table of variation (graphical aspect) (3); table of sign (graphical aspect)
6	11 <sup>th</sup> Dec.	Maximum and minimum (graphical aspect) (1)
	14 <sup>th</sup> Dec. 10h-11h	Maximum and minimum (graphical aspect) (2)
	14 <sup>th</sup> Dec. 11h-12h	Thematic lesson “swimming area”
7	18 <sup>th</sup> Dec.	Chapter 5: functions and inequations (graphical aspect) (1)
	20 <sup>th</sup> Dec. (AP)	Exercise review: system of inequations; images, pre-images, solve the problem of inequation (graphical aspect)
	20 <sup>th</sup> Dec.	Cryptology: in the information room (autonomic learning)
	21 <sup>st</sup> Dec. 10h-10h30	test
	21 <sup>st</sup> Dec. 10h30-12h	Rally (work by group)
8,9	22 <sup>nd</sup> Dec. - 6 <sup>th</sup> Jan.	Vacances de Noël
10	8 <sup>th</sup> Jan.	Chapter 5: functions and inequations (graphical aspect) (2)
	10 <sup>th</sup> Jan.	Chapter 5: functions and inequations (graphical aspect) (3)
	18 <sup>th</sup> Jan.	A discussion about Emma’s intention of the design
11	7 <sup>th</sup> Feb. (half)	Thematic lesson “area of parallelogram” (1)
	8 <sup>th</sup> Feb. 10h-11h	Thematic lesson “area of parallelogram” (2)
	8 <sup>th</sup> Feb. 11h-12h	Thematic lesson “area of parallelogram” (2)