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Charlotte Hardouin

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**Habilitation à diriger des recherches.**



Université de Toulouse  
Spécialité MATHÉMATIQUES

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**INSTITUT DE  
MATHEMATIQUES  
DE TOULOUSE**

**Approche galoisienne de la  
transcendance fonctionnelle**

Soutenue le 3 juin 2022  
devant le jury composé de

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# Approche galoisienne de la transcendance fonctionnelle

Charlotte Hardouin

Mémoire présenté pour l'obtention de  
l'habilitation à diriger des recherches de l'Université Toulouse III



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# Introduction

The term *special functions* lacks a formal definition but it can be understood as encompassing all the mathematical functions which arose essentially from functional equations on one hand and from counting problems on the other. The first class of special functions is ubiquitous and connected to many mathematical domains: solutions of dynamical systems in Physics such as the Painlevé transcendents, special functions associated to the uniformization of Riemann surfaces, the Gauss-Manin connection or the Schwarzian differential equations (see for instance the recent work [Sca18]) in Algebraic Geometry, elliptic and modular functions, Gamma function, zeta functions of varieties and exponential sums in Number Theory, solutions of Itô's stochastic differential equations ([Ito51]) in Probability as well as solutions of  $q$ -difference equations associated to representations of quantum groups (see for instance [FR92]). The second class is formed by any power series derived from a counting problem of combinatorial or number theoretical nature. One finds for instance the generating series of automatic sequences which satisfy Mahler equations, the hypergeometric series and their telescoping relations in Zeilberger's algorithm ([Zei91]) or the generating series counting partitions with short sequences and their associated  $q$ -difference equations ([And05]).

We say that a special function is defined over a field  $K$  if either its defining functional equation has coefficients in  $K$  or the sequence of its power series expansion belong to  $K$ . When  $K$  is endowed with a finite set of operators  $\Sigma$ , many of my contributions boil down to the following question:

*Let  $K \subset L$  be a  $\Sigma$ -field extension and let  $f$  be a special function in  $L$  defined over  $K$ , can one determine if  $f$  belongs to  $\text{cl}_\Sigma(K)$ , the closure of  $K$  with respect to  $\Sigma$*

The closure of  $K$  with respect to  $\Sigma$  is formed by elements  $g$ , contained in some  $\Sigma$ -field extension of  $K$ , whose successive transforms with respect to the operators in  $\Sigma$  satisfy a non-trivial polynomial equation with coefficients in  $K$ . This notion can be reformulated in model theoretic as well as difference or differential algebraic terms when  $\Sigma$  is the empty set, a set of commuting derivations or a generic field automorphism (see §3.3.3). For instance, when  $\Sigma$  is empty, the closure of  $K$  is formed by the functions algebraic over  $K$  and when  $\Sigma$  is a derivation  $\delta$ , the closure of  $K$  with respect to  $\delta$  consists in all the functions that satisfy an algebraic differential equation with coefficients in  $K$ , namely, all the functions differentially algebraic over  $K$ . The closure with respect to  $\Sigma$  characterizes a certain level of algebraic complexity which also yields some analytic regularity for the special function: for instance, a certain growth on the sequence of coefficients of its power series expansion as well as some estimate of the size of the gaps in this sequence as well as some bound on the growth of the function itself. The membership of a special function to certain closures gives therefore additional informations on this function.

This memoir provides a synthesis of my research since my PHD. It is organized as follows where the references in Arabic numerals are to my publications's list which is presented at the end of the introductory section.

## **Section 1 : Classical Galois theory for linear difference equations**

This section is a short and partial introduction to some basic notions in differential and difference algebra and to the Galois theory of linear difference equations presented both in the Picard-Vessiot and Tannakian point of view.

## **Section 2 : Galoisian approach to closures with respect to operators ([15, 16, 17, 18, 21]).**

In Section 2, I will present two Galois theories for solutions of linear functional equations, called *parametrized Galois theories* to distinguish them from the Galois theories developed initially by Picard and Vessiot. The first one, developed with Singer in [21] generalizes [CS06] and allows for instance to attach to a linear difference system a Galois group, which is a differential algebraic group and whose defining equations encode the differential algebraic relations satisfied by the solutions of the linear difference system. Via this Galoisian approach, the differential transcendence of the Gamma function is a direct corollary of the classification of the differential algebraic subgroups of the additive group. Mirror to this situation, the second Galois theory established in collaboration with Di Vizio and Wibmer

interprets the difference algebraic relations satisfied by the solutions of a linear differential system as the defining equations of a Galois group, which is a difference group scheme ([17]). This Galois theory required to develop the systematic study of the geometric properties of difference group schemes as well as their classification. Via the Galois correspondence established in [17], one can interpret for instance the contiguity relations satisfied by hypergeometric series as well as the Frobenius structure for  $p$ -adic differential equations in Galois theoretic terms.

**Section 3 : Direct problem in parametrized Galois theories ([11, 13, 14, 19, 21, 23]).**

Section 3 is dedicated to some contributions in the direct problem of the parametrized Galois theories presented above. The direct problem consists in the computation of the parametrized Galois group. My contributions in that domain are not of algorithmic nature but they focus on general criteria for families of functional systems. In §3.1, I will describe how the Galoisian study of diagonal systems is related to certain telescoping problem and yields an algorithmic criteria for difference systems with rational coefficients over curve of genus zero or one. In §3.2, one study the parametrized Galois groups of systems whose classical Galois group is a semi-simple algebraic group. Finally, §3.3.2 is devoted to Tannakian algorithms to compute the unipotent radical of the Galois groups, classical and parametrized of systems that are extension of completely reducible systems. In that case, the group theoretic framework is related to an action of a reductive group on a subgroup of a vector group. Surprisingly, such a situation also appears when one studies the bounded automorphism in models of fields with operators ([13]).

**Section 4 : Arithmetic aspects of difference equations ([3, 7, 9, 10, 12]).**

Section 4.1 is an analogue for  $q$ -difference systems of the Grothendieck conjecture for linear differential equations. The article [10] extends the results of Di Vizio ([DV02]) for  $q$ -difference systems of the form  $Y(qx) = A(x)Y(x)$  with  $A(x) \in \text{Gl}_n(K(x))$ ,  $K$  a number field and  $q$  an element of  $K$  to the general situation of  $K = \mathbb{C}$  and  $q$  a non-zero complex number. The main result of [10] allows to characterize the existence of a full basis of rational solutions to the system  $Y(qx) = A(x)Y(x)$  with  $A(x) \in \text{Gl}_n(\mathbb{C}(x))$  as the vanishing of the curvatures, which are certain specializations of the dynamic  $(A(q^n x) \cdots A(x))_{n \in \mathbb{N}}$  of the  $q$ -difference system. This curvature criteria provides a description *à la Katz* of the Galois group of a  $q$ -difference system as the smallest algebraic group which contains the curvatures.

In §4.2, I shall detail some results concerning the difference or differential algebraic properties of power series, solutions of linear difference equations over the projective line, which were obtained thanks to parametrized Galois theories. I will show how these contributions combine many intermediate works such as the integrability theorem of Schaeffe and Singer ([SS19a]) and its Galois theoretic interpretation by Arreche and Singer ([AS17]) in order to achieve unconditional statements such as the one obtained with Adamczewski and Dreyfus, where we proved that a non-rational formal power series, solution of a linear Mahler equation is differentially transcendental over the field of rational functions ([7]).

**Section 5 : Galoisian approach of walks confined in a cone ([1, 6, 5, 8, 11]).**

Section 5 presents a novel approche to the counting problem of small steps walks in the first quadrant in Combinatorics which is based on the combination of Galois theoretic, geometric and arithmetic tools. The algebraic nature of the generating series  $Q(x, y, t)$  associated to this counting problem is related to the geometry of an algebraic curve, the so called *kernel curve* and to the finiteness of a group of automorphisms of this curve called *the group of the walk*.

In a collaboration with Dreyfus, Roques and Singer, we applied the Galois theory developed in [21] to the dynamical functional equations over elliptic curves satisfied by some analytic prolongation of the generating series counting walks ([11]). This allows to interpret the differential algebraicity of the generating series as a certain telescoping problem. Thanks to some ad hoc criteria, we were able to unravel the telescoping problem and thereby determine all the models with differentially algebraic generating series among the 51 unweighted models of genus one with infinite group.

In [1], we were able to exploit the depth of the geometry of the kernel curve viewing this curve as

a member of a one-parameter family of curves. This allowed us to interpret the telescoping problem in arithmetic terms: namely as a problem of linear dependence between three points in the Mordell-Weil lattice of an elliptic surface. This latter characterization yields an algorithm which given a weighted model of walks determines the algebraic relations among the weights which characterize differentially algebraic generating series.

Finally, with Dreyfus, we proposed a non-archimedean uniformization of the kernel curve, which happens to be a Tate curve in the genus one case. This uniformization is less demanding than the one obtained by Kurkova and Raschel in [KR12] and allows to consider the  $t$ -dependencies of the generating series (cf [6]).

## Section 6 : Perspectives

In this last section, I will propose some future developments and research orientations connected to the research themes detailed in Section 2 to 5.

This memoir presented in view of the graduation to the *habilitation à diriger des recherches* refers to the following papers. Most of these articles are in collaboration and only the article 23 is directly issued from my PhD thesis. All of the articles below except the articles 2 and 6 are published. The publication 16 is a book made of three main chapters, one by each of the three authors.

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# 1 Classical Galois theory for linear difference equations

The Galois theory of linear difference equations was initiated by Franke in [Fra63] and Bialynicki-Birula in [Bia62] in the context of difference field extensions. These constructions were generalized by Singer and van der Put to the context of difference rings in [vdPS97b]. More recently, Michael Wibmer completed the Galois correspondence of Singer and van der Put and build a more schematic Galois theory which no longer relies on strong assumption on the field of difference constants ([Wib10]).

This galoisian approach aims at encoding the algebraic relations satisfied by the solutions of a linear discrete equation in the defining equations of a linear algebraic group. Via this correspondence, one can for instance interpret some properties of sequences defined by linear recurrence relations as structural properties of their ring of solutions or of the associated Galois group. For instance, the structure of a *Picard-Vessiot ring*, that is a minimal ring of solutions, implies that a sequence defined by a linear recurrence relation whose inverse is also defined by a linear recurrence relation is the interlacing of geometric sequences (see [vdPS97b, Prop. 3.5]). Currently, the Galois theory of linear difference equations admits many presentations: a Hopf theoretic approach by [AMT], a scheme theoretic approach by Wibmer, a Tannakian formalism as introduced by André in [And01]. In this section, I will recall some basic notions in difference and differential algebra, the Picard-Vessiot formalism of Wibmer and the Tannakian framework.

## 1.1 Difference algebra

A difference ring or  $\phi$ -ring is a pair  $(R, \phi)$  where  $R$  is a ring and  $\phi$  a ring endomorphism of  $R$ . If  $R$  is a field  $K$ , we say that  $(K, \phi)$  is a difference field or  $\phi$ -field. We say that a difference field  $(K, \phi)$  is *inversive* if  $\phi$  is onto. We denote by  $k$  its field of  $\phi$ -constants, that is, the subfield of  $K$  formed by the elements fixed by  $\phi$ .

**Example 1.1.** *Algebraic geometry provides many example of difference fields. Indeed, given an absolutely irreducible quasi-projective variety  $V$  defined over a field  $k$ , any rational dominant map  $\phi : V \rightarrow V$  yields an endomorphism  $\phi$  of  $k(V)$  defined by  $\phi(f) = f \circ \phi$ . The pair  $(k(V), \phi)$  is a difference field. Below are some examples for  $V$  a curve of genus zero and one:*

- *The field of rational functions  $\mathbb{C}(x) = \mathbb{C}(\mathbb{P}^1)$  over  $\mathbb{C}$  has essentially two structures of inversive difference field that correspond to the following automorphisms of the projective line*
  - $\phi(x) = x + 1$  *the shift or finite difference operator and  $\mathbb{C}(x)^\phi = \mathbb{C}$ ,*
  - $\phi(x) = qx$  *for  $q \neq \{0, 1\}$  not a root of unity. This operator is called  $q$ -difference operator and  $\mathbb{C}(x)^\phi = \mathbb{C}$ .*
  - *For  $p$  a positive integer, the endomorphism of  $\mathbb{P}^1$  given by the exponentiation by  $p$  induces on  $\mathbb{C}(x)$  an endomorphism  $\phi$  that is not onto. However, this endomorphism extends to an automorphism of the field  $\bigcup_{n \in \mathbb{N}^*} \mathbb{C}(x^{1/n})$ . The latter field is an inversive closure of  $(\mathbb{C}(x), \phi)$ , that is, the smallest  $\phi$ -field  $(K, \phi)$  extension of  $(\mathbb{C}(x), \phi)$  such that  $\phi$  is onto on  $K$ .*
- *Let  $E$  be an elliptic curve defined over an algebraically closed field  $k$  of characteristic zero and let  $P \in E(k)$ . The translation  $\phi$  by  $P$  induce a difference field structure on  $k(E)$ . If  $P$  is not torsion then  $k(E)^\phi = k$ . If  $P$  is a torsion point then  $k(E)^\phi$  is the function field of an elliptic curve isogenous to  $E$ .*

The algebraic study of difference rings has been initiated by Ritt and further developed by Cohn [Coh65] and we refer to these textbooks for a more detailed introduction. We quickly recall below some definitions which might be used in this memoir.

Many definitions in difference algebra are straightforward. Algebraic attributes (e.g. Noetherian) are understood to apply to the underlying ring and their compatibility with the difference structure is emphasized via a prefix with  $\phi$  (e.g. finitely  $\phi$ -generated). The expression  $\phi^0$  is understood to be the identity. For instance, a morphism of  $\phi$ -rings is a morphism of rings that commutes with  $\phi$ .

Let  $R$  be a  $\phi$ -ring. The set  $R^\phi$  formed by the elements of  $R$  invariant by  $\phi$  is called the ring of  $\phi$ -constants of  $R$ . By an  *$R$ - $\phi$ -algebra*, we mean a  $\phi$ -ring  $S$  together with a morphism  $R \rightarrow S$  of  $\phi$ -rings.

A  $\phi$ -ideal  $\mathfrak{J}$  of  $R$  is an ideal  $\mathfrak{J} \subset R$  such that  $\phi(\mathfrak{J}) \subset \mathfrak{J}$ . Then  $R/\mathfrak{J}$  is naturally a  $\phi$ -ring. Let  $B$  be a subset of  $R$ . We denote by  $[B]$  the  $\phi$ -ideal generated by  $B$  in  $R$ . As an ideal it is generated by  $B, \phi(B), \dots$ . A  $\phi$ -ideal  $\mathfrak{J}$  of  $R$  is called *reflexive* if  $\phi^{-1}(\mathfrak{J}) = \mathfrak{J}$ , i.e.,  $\phi(r) \in \mathfrak{J}$  implies  $r \in \mathfrak{J}$ . A  $\phi$ -ideal  $\mathfrak{J}$  of  $R$  is called *perfect* if  $\phi^{\alpha_1}(r) \cdots \phi^{\alpha_n}(r) \in \mathfrak{J}$  implies  $r \in \mathfrak{J}$  for all  $r \in R$ ,  $n \geq 1$  and  $\alpha_1, \dots, \alpha_n \geq 0$ . A  $\phi$ -ideal  $\mathfrak{J}$  of  $R$  is called  *$\phi$ -prime* if it is a prime ideal and reflexive. Note that this property is stronger than being a prime  $\phi$ -ideal. One can show that the perfect  $\phi$ -ideals are precisely the intersections of  $\phi$ -prime ideals.

Let  $k$  be a  $\phi$ -field, i.e., a  $\phi$ -ring whose underlying ring is a field. The  $k$ - $\phi$ -algebra  $k\{x_1, \dots, x_n\}$  of  $\phi$ -polynomials over  $k$  in the  $\phi$ -variables  $x_1, \dots, x_n$  is the polynomial ring over  $k$  in the variables  $x_1, \dots, x_n, \phi(x_1), \dots, \phi(x_n), \dots$ , with an action of  $\phi$  as suggested by the names of the variables. Let  $R$  be a  $k$ - $\phi$ -algebra and  $B$  a subset of  $R$ . The smallest  $k$ - $\phi$ -subalgebra of  $R$  that contains  $B$  is denoted with  $k\{B\}_\phi$  and called the  *$k$ - $\phi$ -subalgebra  $\phi$ -generated by  $B$* . As a  $Kk$ -algebra it is generated by  $B, \phi(B), \dots$ . If there exists a finite subset  $B$  of  $R$  such that  $R = k\{B\}_\phi$ , we say that  $R$  is *finitely  $\phi$ -generated over  $k$* . This is equivalent to the fact that, there exists a positive integer  $n$  such that  $R$  is isomorphic as  $k$ - $\phi$ -algebra to the quotient of  $k\{x_1, \dots, x_n\}$  by some  $\phi$ -ideal  $\mathfrak{J}$ . A  $k$ - $\phi$ -algebra  $R$  is called  *$\phi$ -separable over  $k$*  if  $\phi$  is injective on  $R \otimes_k k'$  for every  $\phi$ -field extension  $k'$  of  $k$ .

We say that a  $\phi$ -field  $k$  is  *$\phi$ -closed* if for every finitely  $\phi$ -generated  $k$ - $\phi$ -algebra  $R$  which is a domain and  $\phi$  is injective on  $R$ , there exists a morphism  $R \rightarrow k$  of  $k$ - $\phi$ -algebras. The  $\phi$ -closed fields are the models of ACFA (see for instance [Cha05]). Let  $K \subset L$  a  $\phi$ -field extension. Elements  $a_1, \dots, a_n \in L$  are called *transformally independent over  $K$*  if the elements  $a_1, \dots, a_n, \phi(a_1), \dots, \phi(a_n), \dots$  are algebraically independent over  $K$ . A  $\phi$ -transcendence basis of  $L$  over  $K$  is a maximal transformally independent over  $K$  subset of  $L$ . Any two  $\phi$ -transcendence basis of  $L$  over  $K$  have the same cardinal called the  $\phi$ -transcendence degree of  $L$  over  $K$  and denoted by  $\phi\text{-trdeg}(L|K)$  (see [Lev08, Def. 4.1.7]).

## 1.2 Picard Vessiot theory

We shall consider linear difference systems of the form

$$(1.1) \quad \phi(Y) = AY,$$

where  $A \in \text{Gl}_n(K)$  where  $(K, \phi)$  is a pseudofield. By pseudofield, we mean the following

**Definition 1.2.** *We say that a  $\phi$ -ring  $L$  is a pseudofield if there exist orthogonal idempotent elements  $e_1, \dots, e_r$  such that*

- $L = Le_1 \oplus Le_2 \oplus \dots \oplus Le_r$ ,
- $\phi(e_i) = e_{i+1 \bmod r}$  for any  $i = 1, \dots, r$ ,
- $Le_i$  is a field for any  $i = 1, \dots, r$ .

A linear difference equation  $\mathcal{L}(y)$  over  $K$  is an element of  $K[\phi]$ , that is, an equation of the form

$$\mathcal{L}(y) = \phi^n(y) + a_{n-1}\phi^{n-1}(y) + \dots + a_0y = 0,$$

where  $a_{n-1}, \dots, a_0$  are elements of  $K$  and  $a_0 \neq 0$ . The difference equation  $\mathcal{L}(y) = 0$  can be rewritten into a linear difference system  $\sigma(Y) = A_{\mathcal{L}}Y$  of size  $n$  where  $A_{\mathcal{L}} \in \text{Gl}_n(K)$  is the companion matrix of the difference equation  $\mathcal{L}$ . Conversely, Hendricks and Singer ([HS99, Theorem B.2]) proved that if  $K$  is a difference field with a non-periodic element then any linear difference system  $\phi(Y) = AY$  is *gauge equivalent* to a linear difference system associated to a linear difference equation  $\mathcal{L}$ , that is, there exists  $P \in \text{Gl}_n(K)$  such that  $A = \phi(P)A_{\mathcal{L}}P^{-1}$ . The notion of difference modules presented in §1.4 will give a more intrinsic definition of the notion of linear difference systems up to gauge equivalence.

The following definition introduces the notion of minimal ring of solutions for difference systems over pseudofields. It summarizes in our context [OW15, Definitions 2.2, 2.6, 2.18 and Proposition 2.21].

**Definition 1.3.** *Let  $(K, \phi)$  be a pseudofield and let  $A \in \text{Gl}_n(K)$ . A  $K$ -pseudofield extension  $K_A$  is a Picard-Vessiot extension for  $\phi(Y) = AY$  over  $K$  if there exists a fundamental solution matrix  $U \in$*

$\mathrm{Gl}_n(K_A)$  such that  $\phi(U) = AU$ ,  $K_A = K(U)$ <sup>1</sup> and  $K_A^\phi = k = K^\phi$ . The  $K$ - $\phi$ -algebra  $R_A = K[U, \frac{1}{\det(U)}]$  is called *Picard-Vessiot ring* for  $\phi(Y) = AY$  over  $K$ . In particular,  $R_A$  is  $\phi$ -simple, i.e., it has no proper  $\phi$ -ideal and  $K_A$  is the total ring of quotients of  $R_A$ .

The definition of pseudofield might seem mysterious compared to the classical Picard-Vessiot theory of linear differential equations where the whole theory is based on fields. However it corresponds to a dichotomy that is specific to linear difference systems. Indeed, either one finds a fundamental solution matrix whose coefficients belong to a  $\phi$ -field extension but, in that case, one has to add new  $\phi$ -constants or one has to leave the realm of fields and work with pseudofields to ensure that the field of  $\phi$ -constants will not increase. This was one of the issue of Franke's first attempt to built a Picard-Vessiot field theory for linear difference systems. The following example illustrates this dichotomy.

**Example 1.4.** Let  $\mathbb{C}(x)$  be endowed with  $\phi(f(x)) = f(qx)$  with  $q \in \mathbb{C}^*$  such that  $|q| > 1$ . Note that  $\mathbb{C}(x)^\phi = \mathbb{C}$ . Let us consider the rank 1 difference system given by

$$(1.2) \quad \phi(y) = -y.$$

Suppose that there exists a  $\sigma$ -field extension  $L$  of  $\mathbb{C}(x)$  with  $L^\phi = \mathbb{C}$  and containing a non-zero solution  $z$  of (1.2). Then,  $z^2 \in L^\phi = \mathbb{C}$ . Since  $\mathbb{C}$  is algebraically closed,  $z \in \mathbb{C}$ . A contradiction with  $\phi(z) = -z$ . However, the following hold.

- The field  $\mathbb{C}(x)(e_{q,-1})$  subfield of the field of meromorphic functions  $\mathrm{Mer}(\mathbb{C}^*)$  over  $\mathbb{C}^*$  contains a non-zero solution  $e_{q,-1}$  of (1.2) where  $e_{q,c} = \frac{\theta_q(x)}{\theta_q(cx)}$  and  $\theta_q(x) = \sum_{n \in \mathbb{Z}} q^{-n(n-1)/2} x^n$ . But it contains also the new  $\phi$ -constant  $e_{q,-1}^2$ , which is an elliptic function with respect to the elliptic curve  $\mathbb{C}^*/q^{\mathbb{Z}}$ .
- The polynomial ring  $\mathcal{R} := \mathbb{C}(x)[y, \frac{1}{y}]$  can be endowed with a structure of  $\mathbb{C}(x)$ - $\phi$ -algebra by setting  $\phi(y) = -y$ . If  $\mathfrak{m}$  is a maximal  $\phi$ -ideal of  $\mathcal{R}$  then  $R = \mathcal{R}/\mathfrak{m}$  is a Picard-Vessiot ring for (1.3) which is not an integral domain (see [vdPS97b, Example 1.6 and Lemma 1.8]).

Existence and uniqueness of Picard-Vessiot extension are the content of the following Proposition

**Proposition 1.5** (Lemma 2.13 and Theorem 2.16 in [OW15]). Let  $(K, \phi)$  be a pseudofield and let  $A \in \mathrm{Gl}_n(K)$ . If  $K^\phi$  is an algebraically closed field there exists a unique up to  $\phi$ -isomorphism Picard-Vessiot extension for the system  $\phi(Y) = AY$ .

The algebraic construction behind Proposition 1.5 is as follows. Consider a linear difference system  $\phi(Y) = AY$  with  $A \in \mathrm{Gl}_n(K)$ . One can endow the polynomial ring  $\mathcal{R} = K[X, \frac{1}{\det(X)}]$  with a structure of  $K$ - $\phi$ -algebra by setting  $\phi(X) = AX$ . Modding out  $\mathcal{R}$  by a maximal  $\phi$ -ideal, we find a  $K$ - $\phi$ -algebra  $R$  that is a Picard-Vessiot ring for  $\phi(Y) = AY$  (see [vdPS97b, Lemma 1.8]). The column vectors of the fundamental solution matrix are *abstract solutions* of the initial fundamental system. When one focuses on the algebraic properties of a special function  $f$  satisfying a discrete functional equation, one can wonder if it is possible to construct a Picard-Vessiot extension that contains this particular solution. The following proposition gives a first answer to this question.

**Proposition 1.6** (Lemma 3.7 in [12]). Let  $(K, \phi)$  be an inversive difference field with  $k = K^\phi$  algebraically closed of characteristic zero. Let  $A \in \mathrm{Gl}_n(K)$  and let  $(u_1, \dots, u_n) \in L^n$  a non-zero vector solution of  $\phi(Y) = AY$  with  $L$  a difference field extension of  $K$  such that  $L^\phi = k$ . Then there exists a Picard-Vessiot extension  $K_A$  for  $\phi(Y) = AY$  such that  $u_i \in K_A$  for any  $i = 1, \dots, n$ .

However, for many discrete operators, the difference constants of ring formed by “natural” analytic solutions are no longer algebraically closed. This is the case for any discrete operator on the projective line.

**Example 1.7.** For  $q \in \mathbb{C}^*$  not a root of unity, the field  $\mathrm{Mer}(\mathbb{C}^*)$  of meromorphic functions over  $\mathbb{C}^*$  endowed with the  $q$ -difference operator  $\phi(f) = f(qx)$  for any  $f \in \mathrm{Mer}(\mathbb{C}^*)$  is a difference field extension

<sup>1</sup>The notation  $K_A = K(U)$  is somehow abusive since  $K(U)$  is not the  $\phi$ -field generated by  $U$  but the pseudofield generated by  $U$ . Nonetheless, we prefer to abuse notation rather than introducing one more complicated notation.

of  $(\mathbb{C}(x), \phi)$ . A Theorem of Praagman [Pra83] asserts that any  $q$ -difference system  $\phi(Y) = AY$  with  $A \in \mathrm{Gl}_n(\mathrm{Mer}(\mathbb{C}^*))$  admits a fundamental solution matrix  $U \in \mathrm{Gl}_n(\mathrm{Mer}(\mathbb{C}^*))$ . The field of  $\phi$ -constants  $C_E$  of  $\mathrm{Mer}(\mathbb{C}^*)$  is the field of elliptic functions with respect to the elliptic curve  $\mathbb{C}^*/q^{\mathbb{Z}}$ . This field is not algebraically closed. However, this analytic resolution allows to construct a Picard-Vessiot field extension for any difference system  $\phi(Y) = AY$  with  $A \in \mathrm{Gl}_n(C_E(x))$  as follows. Let  $U \in \mathrm{Gl}_n(\mathrm{Mer}(\mathbb{C}^*))$  be a fundamental solution matrix for  $\phi(Y) = AY$ . The subfield  $C_E(x)(U) \subset \mathrm{Mer}(\mathbb{C}^*)$  is a Picard-Vessiot extension for  $\phi(Y) = AY$  over  $C_E(x)$ .

A similar result holds for the shift operator. Indeed, if one endows  $\mathrm{Mer}(\mathbb{C})$  with the shift operator  $\phi(f(x)) = f(x+1)$  then any difference system  $\phi(Y) = AY$  with  $A \in \mathrm{Mer}(\mathbb{C})$  admits a fundamental solution matrix  $U \in \mathrm{Gl}_n(\mathrm{Mer}(\mathbb{C}))$ . The difference constant field is formed by the one periodic functions and is no longer algebraically closed.

### 1.3 Galois group and correspondence

The Galois group attached to a Picard-Vessiot extension is defined as follows

**Definition 1.8.** Let  $(K, \phi)$  be a pseudofield with  $k = K^\phi$ . Let  $A \in \mathrm{Gl}_n(K)$  and let  $K_A$  be a Picard-Vessiot extension for  $\phi(Y) = AY$ . Let  $R_A$  be an associated Picard-Vessiot ring as in Definition 1.3.

We define  $\mathrm{Gal}(K_A|K)$  to be the functor from the category  $\mathrm{Alg}_k$  of  $k$ -algebras to the category of groups given by

$$\mathrm{Gal}(K_A|K)(S) := \mathrm{Aut}^\phi(R_A \otimes_k S | K \otimes_k S)$$

for every  $k$ -algebra  $S$ . The action of  $\phi$  on  $S$  is trivial, i.e.,  $\phi(r \otimes s) = \phi(r) \otimes s$  for  $r \in R_A$  and  $s \in S$ . On morphisms  $\mathrm{Gal}(K_A|K)$  is given by base extension. We call  $\mathrm{Gal}(K_A|K)$  the Galois group of  $K_A|K$ .

**Lemma 1.9** (Lemma 2.51 in [OW15]). Let  $K, A, K_A$  as above. The Galois group of  $K_A$  over  $K$  is an algebraic group over  $k$ , that is, the functor  $\mathrm{Gal}(K_A|K)$  is represented by a  $k$ -Hopf algebra which is finitely generated over  $k$ .

Now, we are able to state the Galois correspondence for difference systems.

**Theorem 1.10** (Theorem 2.52 in [OW15]). Let  $K, A, K_A$  as above. There is a one to one correspondence between  $k$ -algebraic subgroups of  $\mathrm{Gal}(K_A|K)$  and intermediate pseudofields extension in  $K_A|K$  given by

$$M = K_A^H := \{x \in K_A \mid \text{for all } S \in \mathrm{Alg}_k, g \in H(S) \mid g(x \otimes 1) = x \otimes 1\} \leftrightarrow H := \mathrm{Gal}(K_A|M).$$

The following result shows how the defining equations of the Galois group encode the algebraic relations satisfied by the solutions of the difference system.

**Theorem 1.11** (Theorem 1.13 in [vdPS97b]). Let  $K, A, K_A$  and  $R_A$  as above. Assume that  $K$  is a field of characteristic zero. The affine scheme  $\mathrm{Spec}(R_A)$  is a  $\mathrm{Gal}(K_A|K)$ -torsor and

$$\dim_k(\mathrm{Gal}(K_A|K)) = \mathrm{trdeg}(K_A|K),$$

where  $\mathrm{trdeg}(K_A|K)$  is the transcendence degree of the field extension  $K \subset K_A \cdot e_1$  with  $e_1$  one of the orthogonal idempotents of  $K_A$  as in Definition 1.2.

### 1.4 Tannakian categories for difference modules

Galois theory of linear difference systems can be interpreted in the framework of Tannakian categories. In [And01], André developed a Tannakian framework which allows to encompass difference and differential systems. In this section, we present some of the results of [And01] adapted for linear difference systems.

The category  $\mathrm{Diff}(K, \phi)$  of difference modules over a difference field  $(K, \phi)$  is defined as follows

**Definition 1.12.** A difference module  $\mathcal{M} = (M, \Phi)$  (of rank  $\nu$ ) over  $K$  is a finite dimensional  $K$ -vector space  $M$  (of dimension  $\nu$ ) equipped with an invertible  $\phi$ -semilinear operator  $\Phi : M \rightarrow M$ , i.e., a bijective additive map from  $M$  to itself such that

$$\Phi(fm) = \phi(f)\Phi(m), \text{ for any } f \in K \text{ and } m \in M.$$

We call  $\Phi$  a difference operator on  $\mathcal{M}$

A morphism of difference modules (over  $K$ ) is a morphism of  $K$ -vector spaces, commuting with the difference operators. We denote by  $\text{Diff}(K, \phi)$  the category of difference modules over  $K$ .

Let  $\mathcal{M} = (M, \Phi)$  be a difference module over  $K$  of rank  $\nu$ . We fix a basis  $\underline{e}$  of  $M$  over  $K$ . Let  $A \in \text{GL}_\nu(K)$  be such that:

$$\Phi \underline{e} = \underline{e}A.$$

If  $\underline{f}$  is another basis of  $M$ , such that  $\underline{f} = \underline{e}F$ , with  $F \in \text{GL}_\nu(K)$ , then  $\Phi \underline{f} = \underline{f}B$ , with  $B = F^{-1}A\phi(F)$ . Such an action on  $A$  is called a *gauge transform*. Conversely, given an invertible matrix  $A \in \text{GL}_\nu(K)$ , one constructs a difference module  $\mathcal{M}_A$  as follows:  $M = K^\nu$  and  $\Phi \underline{e} = \underline{e}A^{-1}$  with  $\underline{e}$  the canonical basis of  $K^\nu$ .

The elements  $m \in M$  such that  $\Phi(m) = m$  are called horizontal. If a horizontal element  $m$  corresponds to a vector  $\vec{y} \in K^\nu$  with respect to the basis  $\underline{e}$ , we have:  $\underline{e}\vec{y} = \Phi(\underline{e}\vec{y}) = \underline{e}A\phi(\vec{y})$ . Therefore  $\vec{y}$  verifies the linear difference system  $\phi(\vec{y}) = A^{-1}\vec{y}$ , that we call the linear difference system associated to  $\mathcal{M}$  with respect to the basis  $\underline{e}$ .

The constructions of linear algebra, i.e., direct sums, duals and tensor products of the underlying vector spaces of difference modules over  $K$  can be endowed with a structure of difference modules (see for instance [vdPS97b, Chapter 12], [DV02, Part I] or [DVRSZ03]). One can then show that  $\text{Diff}(K, \phi)$  is a tensor category. Denoting by  $\mathbf{1} = (K, \phi)$  the unit object for the tensor product,  $\text{Diff}(K, \phi)$  is a rigid category, i.e., it has internal Homs and each object is canonically isomorphic to its bidual. The category  $\text{Diff}(K, \phi)$  is therefore a Tannakian category in the sense of [Del90].

In the Tannakian context, the notion of vector space of solutions is replaced by the notion of fiber functor.

**Definition 1.13.** Let  $\mathcal{S}$  be a  $k$ -algebra with  $k = K^\phi$ . Let  $\mathcal{M}$  be a difference module over  $K$  and let  $\langle \mathcal{M} \rangle$  be the Tannakian subcategory of  $\text{Diff}(K, \phi)$  generated by  $\mathcal{M}$ . We say that a functor  $\omega : \langle \mathcal{M} \rangle \rightarrow \text{Proj}_{\mathcal{S}}$  is a fiber functor over  $\mathcal{S}$  if it is

- exact,
- faithful,
- $k$ -linear,
- tensor-compatible

We say moreover that  $\omega$  is a neutral fiber functor if  $\mathcal{S} = k$ .

The Tannakian Galois group associated to a neutral fiber functor is defined as follows

**Proposition 1.14** (Théorème 3.2.1.1 in [And01]). Let  $\mathcal{M}$  be a difference module over  $K$  and let  $\omega$  be a neutral fiber functor for  $\langle \mathcal{M} \rangle$ . The Galois group  $\text{Gal}(\mathcal{M}, \omega)$  of  $\mathcal{M}$  with respect to  $\omega$  is the group scheme over  $k$  which represents the functor  $\text{Aut}^{\otimes}(\omega)$  of tensor automorphisms of the functor  $\omega$ . Moreover, one can show that  $\text{Gal}(\mathcal{M}, \omega)$  coincides with the functor which associates to any  $k$ -algebra  $S$  the stabilizer inside  $\text{Gl}(\omega(\mathcal{M}))(S)$  of the vector-spaces  $\omega(\mathcal{N})$  for any sub-object  $\mathcal{N}$  of a construction of linear algebra of  $\mathcal{M}$ .

A natural fiber functor for  $\text{Diff}(K, \phi)$  is given by the forgetful functor  $\eta$  which “forgets the difference structure” and is defined as follows

$$\eta : \text{Diff}(K, \phi) \rightarrow \text{Proj}_K, (M, \phi) \mapsto M.$$

Given a Picard-Vessiot extension, one can construct a neutral fiber functor as follows.

**Lemma 1.15** (§3.4.1. in [And01]). Let  $A \in \text{GL}_n(K)$  and let  $K_A$  be a Picard-Vessiot extension for the system  $\phi(Y) = AY$  over  $K$ . Let  $\mathcal{M}_A$  be the difference module associated to  $A$  as above. The functor

$$\omega_{K_A} : \langle \mathcal{M}_A \rangle \rightarrow \text{Vect}_k, \mathcal{N} \mapsto \{m \in \mathcal{N} \otimes_K K_A \mid \Phi(m) = m\},$$

is a neutral fiber functor for the Tannakian category  $\langle \mathcal{M}_A \rangle^2$ .

<sup>2</sup>The action of  $\Phi$  on  $\mathcal{N} \otimes_K K_A$  is defined by  $\Phi(m \otimes f) = \Phi(m) \otimes \phi(f)$ .

Conversely, given a difference module  $\mathcal{M}$  over  $K$  and a neutral fiber functor  $\omega$  for  $\langle \mathcal{M} \rangle$ , one can construct a Picard-Vessiot extension as follows.

**Lemma 1.16** (Lemma 3.4.2.1 in [And01]). *Let  $\mathcal{M}$  be a difference module over  $K$  and let  $\omega$  be a neutral fiber functor for  $\langle \mathcal{M} \rangle$ . Then, the functor  $\text{Isom}^{\otimes}(\omega \otimes \text{id}_K, \eta|_{\langle \mathcal{M} \rangle})$  of tensor isomorphisms between  $\omega \otimes \text{id}_K$  and the forgetful functor  $\eta|_{\langle \mathcal{M} \rangle}$  is representable by a  $K$ -scheme whose coordinate ring  $K(\mathcal{M}, \omega)$  is a Picard-Vessiot extension for any difference system associated to  $\mathcal{M}$ .*

Theorem 3.4.2.3 in [And01], which generalizes [Del90, §9] to the context of non-commutative differentials, reduces in our context to the fact that Tannakian categories for difference modules and their fiber functors mirrors Picard-Vessiot theory for linear difference systems.

**Theorem 1.17** (Theorem 3.4.2.3 in [And01]). *Let  $\mathcal{M}$  be a difference module over  $K$ . Assume that there exists a neutral fiber functor for  $\langle \mathcal{M} \rangle$ . Then, the functor  $\omega \mapsto K(\mathcal{M}, \omega)$  and  $K_A \mapsto \omega_{K_A}$  are quasi-inverse equivalence of categories between the category of neutral fiber functors for  $\langle \mathcal{M} \rangle$  and the sub-category of  $K$ - $\phi$ -algebras formed by the Picard-Vessiot extensions for  $\mathcal{M}$ <sup>3</sup>.*

## 2 Galoisian approach of closures with respect to operators ([15, 16, 17, 18, 21])

Hölder's Theorem asserts that the Gamma function  $\Gamma(x) = \int_0^{+\infty} t^{x-1} e^{-t} dt$ , which satisfies the functional equation  $\Gamma(x+1) = x\Gamma(x)$  does not satisfy a polynomial differential equation over the field  $\mathbb{C}(x)$  of rational functions over  $\mathbb{C}$ . The article [DV12] gives many references on the distinct proofs of this statement. Hölder's Theorem is what appears in the literature under the various names of "hypertranscendence", or "differential transcendence" as well as "transcendentally transcendental" results. This characterization defines a new class in the hierarchy of special functions. Indeed, one can classify functions over the field  $\mathbb{C}$  of complex numbers as

- rational functions, that is, the elements of  $\mathbb{C}(x)$ ,
- algebraic functions which satisfy a polynomial equation with coefficients in  $\mathbb{C}(x)$ ,
- and transcendental functions, that is functions which do not satisfy a polynomial equation with coefficients in  $\mathbb{C}(x)$ .

Among the last class, the notion of differential transcendence distinguishes the functions which do not satisfy an algebraic differential equation above  $\mathbb{C}(x)$  from the so called "differential algebraic functions" which form the closure of  $\mathbb{Q}$  with respect to  $\delta$ . For instance, the exponential function is transcendental over  $\mathbb{C}(x)$  but obviously satisfies a linear differential equation over  $\mathbb{C}$ . Such a function is called *holonomic* over  $\mathbb{C}(x)$ . The class of holonomic functions is a ring closed under differentiation and integration. Holonomic functions have also good analytic regularity. For instance, any holonomic power series  $\sum a_n x^n$  has moderate growth ( $a_n = O(n^d)$  for some integer  $d$ ) and has a finite number of singularities. The wider class of differentially algebraic functions does not have such nice analytic characterization. Indeed, the zeta function is differentially transcendental and has one single pole whereas the function counting partition  $P(x) = \prod_{k \in \mathbb{N}^*} \frac{1}{1-x^k}$  is differentially algebraic and has an infinite number of singularities. Moreover, even if Maillet ([Mai03]) proved that the coefficients  $(a_n)$  of a differentially algebraic power series satisfy  $|a_n| \leq K(n!)^\alpha$  for all  $n$  and some positive constants  $K$  and  $\alpha^4$ , these bounds seem too large to characterize the differentially algebraic functions.

Differential transcendence problems appear in various mathematical domains. For instance, in [Ber95] and [BB95], the authors study the hypertranscendence of the local conjugacy in complex dynamics in order to get some informations on the regularity of the dynamic generalizing the work of Ritt on the Schroeder

<sup>3</sup>which means for one (and therefore any) linear difference systems associated to  $\mathcal{M}$ .

<sup>4</sup>Note that these bounds are sharp since the divergent Euler series  $\sum (-1)^n (n!) x^n$  satisfies the linear differential equation  $x^2 \frac{dy}{dx} + y = x$ .



function [Rit26], the works of Denef, Lipschitz on differentially algebraic power series (see for instance [DL84]). In combinatorics, a famous conjecture by Bousquet-Mélou and Mishna [BMM10] was concerned with the holonomicity of the generated functions of walks in the plane. Since a holonomic function satisfies by definition a linear differential equation, this characterization might give linear recurrence relations between the coefficients of the generating function as well as some informations on their growth rate (see for instance [BMM10] or [MM14]). One could also quote the study of the differential transcendence of generating functions of automatic  $p$ -sequence (cf for instance [Ran92] and [12]). More surprising, the combination of results in functional transcendence such as Ax-Schanuel Theorem on independence of logarithm of functions and the  $\mathfrak{o}$ -minimality theorem of Pila-Wilkie paved the way to remarkable proofs of famous conjectures in diophantine geometry (see for instance [Pil09]).

Most of the functions considered above satisfy linear difference equations. The Gamma function satisfies the first order finite difference equation  $\Gamma(x+1) = x\Gamma(x)$ , the incomplete generating series for walks in the quarter plane can be continued so that they satisfied first order non-homogeneous  $q$ -difference equations and the generating series associated to automatic sequences satisfy linear Mahler difference equations. Viewing the difference operator of the functional equation of the special function as a privileged operator  $\theta_1$  and the derivation as a “parametric operator”  $\theta_2$ , the question of the differential transcendence of solutions of discrete equations falls within the following problematic:

Given a linear functional system  $\theta_1(y) = Ay$  with respect to an operator  $\theta_1$  defined over a base field  $K$ , is there a Galoisian approach to the membership of the solutions of  $\theta_1(y) = Ay$  to the closure of  $K$  with respect to  $\theta_2$ ?

The first Galoisian approach to these problems was developed by Landesman in [Lan08] and refined in the linear case by Cassidy and Singer in [CS06]. The Galois theory of Cassidy and Singer allows for instance to consider the  $t$ -dependencies of a special function  $f(x, t)$  satisfying a linear differential equation with respect to the derivation  $\frac{d}{dx}$ . The papers [23, 21] deal with the situation where  $\theta_1$  is a difference operator and  $\theta_2$  a derivation commuting with  $\theta_1$ . In these parametrized Galois theories, the *parametrized Galois group* is a differential algebraic group as introduced by Cassidy ([Cas72]). The Galois theories of [CS06, 21] can be formalized in a Tannakian setting ([Kam12, Ovc09]). The consideration of a difference operator for the parametric operator  $\theta_2$  was far more challenging. On one hand, it required to work with the delicate difference algebraic geometry introduced by [Coh65] and to further develop the geometry of difference group schemes. The papers [17, 15] consider the case where  $\theta_1$  is a derivation and  $\theta_2$  a difference operator in order to avoid the technicalities related to the resolution of difference equations in pseudofields. This difficulty has been overcome by Ovchinnikov and Wibmer who succeeded to develop a Galois theory where  $\theta_1$  and  $\theta_2$  are difference operators ([OW15]). Later these authors have developed a Tannakian framework for parametric action of semi-groups, here  $\theta_2$  encodes the action of the semi-group  $\mathbb{N}$  (see [OW17]).

The following sections detail the parametrized Galois theories constructed in [21, 18, 16] for the differential parameter and in [17, 15] for the discrete one.

## 2.1 Differential parameter ([16, 18, 21])

The article [21] proposes the first Galoisian study of the question of differential transcendence for solutions of linear difference equations over field of characteristic zero. The general framework of [21], is concerned with the parametric action of a finite set of commuting derivations on a consistent system formed by linear differential and difference equations. This framework therefore encompasses the setting of [CS06]. For ease of presentation, I shall only present below the case where one considers the parametric action of a derivation on the solutions of a linear difference equation.

The starting point of [21] was the following naive observation of [23]. Consider a special function  $f$  which satisfy a linear difference equation over a difference field  $(K, \phi)$ , written as a linear difference system of the form  $\phi(Y) = AY$  with  $A \in \text{Gl}_n(K)$ . Assume now that there exists a derivation  $\delta$  on  $K$  commuting with  $\phi$ . The question of the differential transcendence of  $f$  with respect to  $\delta$  amounts to understand the algebraic relations between  $f$  and all its derivatives with respect to  $\delta$ . Since  $\delta$  and  $\phi$  commute, one can derive the system  $\phi(Y) = AY$  and one finds that  $\delta(\phi(Y)) = \phi(\delta(Y)) = \delta(A)Y + A\delta Y$ .

Therefore, the vector  $Y^{(1)} = \begin{pmatrix} \delta Y \\ Y \end{pmatrix}$  satisfies

$$(2.1) \quad \phi(Y^{(1)}) = \begin{pmatrix} A & \delta A \\ 0 & A \end{pmatrix} Y^{(1)}.$$

Note that the difference module corresponding to the latter system is an extension of the difference module attached to  $\phi(Y) = AY$  by itself. Repeating this process, one finds that, for any  $s \in \mathbb{N}$ , the

vector  $Y^{(s)} = \begin{pmatrix} \delta^s(Y) \\ \vdots \\ \delta Y \\ Y \end{pmatrix}$  satisfies a so called “  $s$ -prolongation of the initial difference system ”  
 $\phi(Y^{(s)}) = A^{(s)}Y^{(s)}$  where

$$(2.2) \quad A^{(s)} = \begin{pmatrix} A & 0 & 0 & \cdots & 0 \\ \binom{s}{1}\delta A & A & 0 & \cdots & 0 \\ \binom{s}{2}\delta^2 A & \binom{s}{1}\delta A & A & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \binom{s}{s-1}\delta^{s-1} A & \binom{s}{s-2}\delta^{s-2} A & \binom{s}{s-3}\delta^{s-3} A & \cdots & 0 \\ \binom{s}{s}\delta^s A & \binom{s}{s-1}\delta A^{s-1} & \binom{s}{s-2}\delta^{s-2} A & \cdots & A \end{pmatrix} \in \text{Gl}_{(s+1)n}(K).$$

Therefore, in order to study the algebraic relations between  $f$  and its derivatives, one has to compute the difference Galois group of the system (2.2) for arbitrary positive integers  $s$ . This was done in [23] for systems of size one. But this strategy is not really manageable when the size  $n$  of the system  $\phi(Y) = AY$  is strictly greater than 1 because the size of the  $s$ -prolongation (2.2) is in  $(s+1)n$ .

The Kolchin geometry allows to replace the proalgebraic group obtained as projective limit of the Galois groups of the prolongations by a group with a more exotic geometry namely a differential algebraic group. This idea yields the construction of the parametrized Galois theory of linear difference equations with differential parameter ([21]).

### 2.1.1 Picard-Vessiot formalism

The algebraic formalism of this theory is based on the notion of  $(\phi, \delta)$ -rings, which are commutative rings with unit endowed with an automorphism  $\phi$  and a derivation  $\delta$  such that  $\phi\delta = \delta\phi$ . The notions of  $K$ - $(\phi, \delta)$ -algebra,  $(\phi, \delta)$ -fields,  $(\phi, \delta)$ -ideals are straightforward and only requires the compatibility of the algebraic suffix with the operator prefix. A  $(\phi, \delta)$ -ideal  $I$  of a  $(\phi, \delta)$ -ring  $R$  is an ideal set-wise invariant by  $\phi$  and  $\delta$ . Here are some examples of  $(\phi, \delta)$ -fields:

**Example 2.1.** 1. For  $V$  an irreducible algebraic variety defined over a field of characteristic zero  $k$  together with a dominant self-map  $\phi$ , a  $(\phi, \delta)$ -field structure on  $k(V)$  corresponds to vector field on  $V$  invariant by  $\phi$ . Here are some examples when  $V$  is an algebraic curve of genus 0 and 1.

- Case  $V = \mathbb{P}^1$ : the automorphisms of  $\mathbb{P}^1$  are, up to homographic change of coordinate, of the form  $\sigma(x) = qx$  or  $\phi(x) = x + 1$ . In the first case, the derivation  $\delta = \frac{d}{dx}$  endows  $\mathbb{C}(V) = \mathbb{C}(x)$  with a  $(\phi, \delta)$ -field structure and in the second case, one has to consider the derivation  $\delta = x \frac{d}{dx}$ .
  - $V$  is an elliptic curve and  $\phi$  the addition by a point  $P$  of  $V$ . If  $\delta$  is the derivation of  $\mathbb{C}(V)$  corresponding to the canonical one form on  $V$ , the field  $(\mathbb{C}(V), \phi, \delta)$  is a  $(\phi, \delta)$ -field.
2. The field  $\mathbb{C}(x, t)$  can be endowed with a structure of  $(\phi, \delta)$ -field by setting  $\phi(x, t) = (x + 1, t)$  and  $\delta = \frac{d}{dt}$ .
  3. The field  $\cup_{n \in \mathbb{N}^*} \mathbb{C}(x^{1/n})(\ln(x))$  endowed with  $\phi(x) = x^p$  and  $\phi(\ln(x)) = p \ln(x)$  with  $\delta = x \ln(x) \frac{d}{dx}$  is a  $(\phi, \delta)$ -field.

Initially the parametrized Galois theory of [21] was developed under the assumption that the field  $k$  of  $\phi$ -constants of the base field  $(K, \phi, \delta)$  is a  $\delta$ -closed field of characteristic zero. According to Robinson [Rob59], a field  $(k, \delta)$  of characteristic zero is  $\delta$ -closed if any system of polynomial differential equations with coefficients in  $k$  with a solution in a differential field extension of  $k$  has already a solution in  $k$ . The assumption of a  $\delta$ -closed field of  $\phi$ -constant has the advantage to ensure the existence of  $\delta$ -Picard-Vessiot extension and allows to work directly with the points of the parametrized Galois groups, which are differential algebraic groups over  $k$ . Unfortunately, one of the major drawback of differentially closed field is that they are enormous fields and that most of the special functions satisfy difference equations over difference fields whose field of  $\phi$ -constants are either the field of complex number  $\mathbb{C}$  or a field of elliptic functions which are not differentially closed. This unnatural assumption forced us to apply some back and forth arguments to go from small fields of  $\phi$ -constants to their differential closure. The articles [18] and [Wib12] proved simultaneously the existence of  $\delta$ -Picard-Vessiot extension over a  $(\phi, \delta)$ -field with algebraically closed field of  $\phi$ -constants but didn't push further the construction of a schematic version of [21].

A  $\delta$ -Picard-Vessiot extension is defined as follows

**Definition 2.2.** *Let  $K$  be a  $(\phi, \delta)$ -field and  $A \in \text{Gl}_n(K)$ . A  $\delta$ -Picard-Vessiot extension  $L_A$  for the system  $\phi(Y) = AY$  over  $K$  is a  $K$ - $(\phi, \delta)$ -algebra that is a pseudofield extension of  $K$  such that*

- *there exists  $Z \in \text{Gl}_n(L_A)$  such that  $\phi(Z) = AZ$ ;*
- *$L_A$  is generated as pseudofield by the entries of  $Z$  and all of their derivatives with respect to  $\delta$ ;*
- *$L_A^\phi = K^\phi$ .*

The ring  $S_A = K\{Z, \frac{1}{\det Z}\}_\delta = K[Z, \delta(Z), \dots, \delta^n(Z), \dots, \frac{1}{\det Z}]$  is called a  $\delta$ -Picard-Vessiot ring for the system  $\phi(Y) = AY$ .

The results of [18] and [Wib12] lead to

**Theorem 2.3** ( Cor. 1.19 in [18] and Cor. 9 in [Wib12]). *Let  $K$  be a  $(\phi, \delta)$ -field,  $k = K^\phi$  and  $A \in \text{Gl}_n(K)$ . If  $k$  is algebraically closed then there exists a  $\delta$ -Picard-Vessiot extension  $L_A$  for  $\phi(Y) = AY$  over  $K$ . Moreover, two  $\delta$ -Picard-Vessiot extensions for  $\phi(Y) = AY$  over  $K$  become isomorphic over a finite algebraic  $\delta$ -field extension of  $k$ .*

If one is able to solve analytically the initial difference system, one can sometimes bypass the assumption of an algebraically closed field of  $\phi$ -constants. As in Example 1.7, one can show that for any complex number  $q$  such that  $|q| > 1$  and any  $A \in \text{Gl}_n(C_E(x))$ , the  $(\phi, \delta)$ -field extension of  $C_E(x)$  generated by a meromorphic fundamental matrix of solutions  $U$  of  $\phi(Y) = AY$  is a  $\delta$ -Picard-Vessiot field extension of  $\phi(Y) = AY$  over  $C_E(x)$  though  $C_E$  the field of  $\phi$ -constant is not algebraically closed.

The notion of  $\delta$ -Galois group of a linear difference system is defined as follows.

**Definition 2.4.** *Let  $K$  be a  $(\phi, \delta)$ -field,  $k = K^\phi$  and  $A \in \text{Gl}_n(K)$ . Let  $L_A$  be a  $\delta$ -Picard-Vessiot extension for  $\phi(Y) = AY$  over  $K$  and  $S_A$  an associated  $\delta$ -Picard-Vessiot ring as in Definition 2.2. The  $\delta$ -Galois group of  $L_A$  over  $K$  is the functor*

$$(2.3) \quad \begin{array}{ccc} \delta\text{-Gal}(L_A|K) & : k\text{-}\delta\text{-algebras} & \longrightarrow & \text{Groups} \\ & S & \longmapsto & \text{Aut}^{\phi, \delta}(S_A \otimes_k S | K \otimes_k S) \end{array}$$

where  $\text{Aut}^{\phi, \delta}(S_A \otimes_k S | K \otimes_k S)$  denotes the group of  $K \otimes_k S$ -algebra automorphisms of  $S_A \otimes_k S$  commuting with  $\phi$  and  $\delta$ . Here, we let  $\phi$  acts on  $S$  as the identity.

By [18, Proposition 1.20], the  $\delta$ -Galois group  $\delta\text{-Gal}(L_A|K)$  is a  $\delta$ -algebraic subgroup of  $\text{Gl}_n$  defined over  $(k, \delta)$ . That is, it can be represented by a  $k$ - $\delta$ -algebra which is a quotient of  $k\{\text{Gl}_n\} = k\{X, \frac{1}{\det(X)}\}$ , the  $k$ - $\delta$ -algebra of differential polynomials in the differential indeterminates  $X = (x_{i,j})_{i,j}$  by a radical  $\delta$ -ideal  $\mathfrak{J}$ . The differential polynomials in  $\mathfrak{J}$  are the defining equations of  $\delta\text{-Gal}(L_A|K)$ . If  $k$  is  $\delta$ -closed, the differential nullstellensatz allows to recover the radical  $\delta$ -ideal via the knowledge of its zero in  $k^{n^2}$ . In that case, one can define the  $\delta$ -Galois group of  $L_A$  over  $k$  as  $\text{Aut}^{\phi, \delta}(S_A|K)$ . The following theorem shows that the differential polynomials in  $\mathfrak{J}$  encodes the differential algebraic relations among the solutions of  $\phi(Y) = AY$ .

**Theorem 2.5** (Theorem 2.6 in [21]). *Let  $K$  be a  $(\phi, \delta)$ -field such that  $k = K^\phi$  is  $\delta$ -closed of characteristic zero. Let  $A \in \text{Gl}_n(K)$  and let  $L_A$  be a  $\delta$ -Picard-Vessiot extension for  $\phi(Y) = AY$  over  $K$ . Let  $S_A \subset L_A$  be a  $\delta$ -Picard-Vessiot ring as in Definition 2.2. Then, the  $K$ - $\delta$ -algebra  $S_A$  is the coordinate ring of a  $\delta\text{-Gal}(L_A|K)$ -torsor  $V$  defined over  $K$ . In particular, the differential transcendence degree of  $L_A$  over  $K^5$  equals the differential dimension of the  $\delta$ -algebraic group  $\delta\text{-Gal}(L_A|K)$  over  $k$  (see [16, §4.2 and §7] for the definitions of differential transcendence degree and differential dimension).*

Assuming that the field of  $\phi$ -constants is  $\delta$ -closed, one has the following parametrized Galois correspondence

**Theorem 2.6** (Theorem 2.7 in [21]). *Let  $K$  be a  $(\phi, \delta)$ -field such that  $k = K^\phi$  is  $\delta$ -closed of characteristic zero. Let  $A \in \text{Gl}_n(K)$  and let  $L_A$  be a  $\delta$ -Picard-Vessiot extension for  $\phi(Y) = AY$  over  $K$ . Let  $\mathcal{F}$  denote the set of  $(\phi, \delta)$ -pseudofields  $F$  with  $K \subset F \subset L_A$  and let  $\mathcal{G}$  denote the set of  $\delta$ -algebraic subgroups of  $\delta\text{-Gal}(L_A|K)$  defined over  $k$ . There is a bijective correspondence  $\alpha : \mathcal{F} \rightarrow \mathcal{G}$  given by  $\alpha(F) = G(K|F) = \{\phi \in \delta\text{-Gal}(L_A|K) \mid \phi(u) = u \forall u \in F\}$ . The map  $\beta : \mathcal{G} \rightarrow \mathcal{F}$  given by  $\beta(H) = \{u \in K \mid \phi(u) = u \forall \phi \in H\}$  is the inverse of  $\alpha$ .*

*In particular, an element of  $L_A$  is left fixed by all  $\phi$  in  $\delta\text{-Gal}(L_A|K)$  if and only if it is in  $K$ . Given a  $\delta$ -algebraic subgroup  $H$  of  $\delta\text{-Gal}(L_A|K)$  defined over  $k$ , we have  $H = \delta\text{-Gal}(L_A|K)$  if and only if  $L_A^H = K$ .*

### 2.1.2 Classification of differential algebraic groups and differential transcendence

Theorems 2.6 and Theorem 2.5 allow to reduce the question of the differential algebraic relations among the solutions of a linear difference equation to the classification of  $\delta$ -algebraic groups, which are  $\delta$ -Galois groups. For non-homogeneous rank one linear difference equations, the classification of  $\delta$ -algebraic subgroups of  $\mathbf{G}_m$  and  $\mathbf{G}_a$  leads to the following differential transcendence criteria for non-homogeneous difference equation of order one.

**Proposition 2.7** (Cor. 3.3 and Prop. 3.8 in [21]). *Let  $K$  be a  $(\phi, \delta)$ -field with  $k = K^\phi$  a  $\delta$ -closed field of characteristic zero. Let  $a, b \in K$  with  $a \neq 0$  and let  $z \in L$  a  $(\phi, \delta)$ -field extension of  $K$  such that  $L^\phi = k$ . The following holds:*

- *If there is no non-zero linear differential operator  $L \in k[\delta]$  such that  $L\left(\frac{\delta a}{a}\right) = \phi(a) - a$  for some  $g \in K$  then  $z$  is differentially transcendental over  $K$ .*
- *If  $\delta(a) = 0$  then  $z$  is differentially algebraic over  $K$  if and only if there is a non-zero linear differential operator  $L' \in k[\delta]$  and  $f \in K$  such that  $L'(b) = \phi(f) - af$ .*

For linear difference equation of higher order, one can give a Galoisian characterization of the compatibility of linear system of functional equations as follows

**Proposition 2.8** (Proposition 2.9 in [21]). *In the notation of Theorem 2.6, the  $\delta$ -Galois group  $\delta\text{-Gal}(L_A|K) \subset \text{Gl}_n$  is conjugate over  $k$  to a subgroup of  $\{C \in H \mid \delta(C) = 0\}$  if and only if there exists a  $B \in \mathfrak{gl}_n(K)$  such that*

$$(2.4) \quad \phi(B) = ABA^{-1} + \delta(A)A^{-1}$$

*In this case, there is a solution  $Y = U \in \text{Gl}_n(L_A)$  of the system*

$$\begin{aligned} \sigma(Y) &= AY \\ \delta(Y) &= BY \end{aligned}$$

*If (2.4) holds, we say that the system  $\phi(Y) = AY$  is  $\delta$ -integrable over  $K$ .*

A celebrated result by Phyllis Cassidy asserts that a Zariski dense  $\delta$ -algebraic group  $G$  of an almost simple algebraic group  $H$  both defined over a  $\delta$ -closed field  $k$  is either equal to  $H$  or conjugate to a  $\delta$ -constant differential group, that is, a subgroup of  $H^\delta = \{C \in H \mid \delta(C) = 0\}$ . Since the  $\delta$ -Galois group is a Zariski dense subgroup of the classical Galois group ([21, Proposition 6.21]), one finds the following integrability criteria:

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<sup>5</sup>As in Theorem 1.11, one considers the differential transcendence degree of  $K \subset L_A e_1$  where  $e_1$  is one of the orthogonal idempotents of the pseudofield  $L_A$ .

**Proposition 2.9.** *In the notation of Theorem 2.6, assume that the Galois group of  $\phi(Y) = AY$  is an almost-simple, noncommutative linear algebraic group of dimension  $t$ . The differential transcendence degree of  $L_A$  over  $K$  is less than  $t$  if and only the system  $\phi(Y) = AY$  is  $\delta$ -integrable over  $K$ .*

The schematic approach of Definition 1.3 has been pushed further in the mirror situation of linear differential equation with a difference parameter. Such a schematic presentation of the Parametrized Galois theory of [21] would allow to avoid the unnatural assumption of a  $\delta$ -closed field of constants. Unfortunately, until now there is no available reference (though this reference would be made of straightforward analogues of [Dyc08] and [GGO13, §8.1]). However, relying on some easy descent arguments one can show that Propositions 4.5 and 2.9 hold over a  $(\phi, \delta)$ -field  $K$  with algebraically closed field of  $\phi$ -constants  $k$ .

### 2.1.3 A Tannakian formalism

A differential Tannakian category as introduced in [Kam12, Ovc09] is a Tannakian category over a differential field  $k$  endowed with an endofunctor  $F$  called the prolongation functor satisfying certain diagrammatic properties. For the category  $\text{Diff}(K, \phi)$ , the prolongation functor is defined as follows. We recall that the structure of right  $K$ -module on  $K[\delta]_{\leq 1}$ <sup>6</sup> is defined via the Leibniz rule, i.e.,  $\delta\lambda = \lambda\delta + \delta(\lambda)$  for any  $\lambda \in K$ . The prolongation  $F(\mathcal{M})$  of an object  $\mathcal{M}$  of  $\text{Diff}(K, \phi)$  is  $K[\delta]_{\leq 1} \otimes_K \mathcal{M}$ , the tensor product of the right  $K$ -module  $K[\delta]_{\leq 1}$  with the left  $K$ -module  $\mathcal{M}$ . It can be endowed with a structure of difference modules over  $K$  (see [10, §7.2]). It is easily seen that  $F(\mathcal{M})$  is an extension of  $\mathcal{M}$  by  $\mathcal{M}$  in the category  $\text{Diff}(K, \phi)$ . Moreover, if  $\phi(Y) = AY$  is a difference system associated to  $\mathcal{M}$  then  $\phi(Y) = \begin{pmatrix} A & \delta(A) \\ 0 & A \end{pmatrix} Y$  is a difference system associated to  $F(\mathcal{M})$ . One can show that  $\text{Diff}(K, \phi)$  together with the endofunctor  $F$  is a differential Tannakian category (see [10, §7.2]).

Classical notions for Tannakian categories can be generalized to the differential framework by requiring their compatibility with the prolongation functor. For instance a differential fiber functor  $\omega$  for a differential Tannakian category  $\mathcal{C}$  over  $k$  is a Tannakian functor  $\omega$  from  $\mathcal{C}$  to the category  $\text{Proj}_S$  of finitely generated projective modules over a certain  $k$ - $\delta$ -algebra  $S$  that intertwines with the prolongation functors on  $\mathcal{C}$  and on  $\text{Proj}_S$ . Analogously to the classical case, one can define the group of differential tensor automorphisms  $\text{Aut}^{\otimes, \delta}(\omega)$  of  $\omega$  as a subfunctor of  $\text{Aut}^{\otimes}(\omega)$ . More generally the group of differential tensor automorphisms of two fiber functors is defined as follows.

**Definition 2.10.** *Let  $\omega_1, \omega_2 : \text{Diff}(K, \phi) \rightarrow \text{Proj}_S$  be two differential fiber functors. For any  $S$ - $\delta$ -algebra  $R$ , we define  $\text{Hom}^{\otimes, \delta}(\omega_1, \omega_2)(R)$  as the set of all sequences of the form  $\{\lambda_{\mathcal{X}} | \mathcal{X} \text{ object of } \text{Diff}(K, \phi)\}$  such that:*

- $\lambda_{\mathcal{X}}$  is an  $R$ -linear homomorphism from  $\omega_1(\mathcal{X}) \otimes R$  to  $\omega_2(\mathcal{X}) \otimes R$ ,
- $\lambda_{\mathbf{1}}$  is the identity on  $\mathbf{1} \otimes R$ ,
- for every  $\alpha \in \text{Hom}(\mathcal{X}, \mathcal{Y})$ , we have  $\lambda_{\mathcal{Y}} \circ (\alpha \otimes id_R) = (\alpha \otimes id_R) \circ \lambda_{\mathcal{X}}$ ,
- $\lambda_{\mathcal{X}} \otimes_R \lambda_{\mathcal{Y}} = \lambda_{\mathcal{X} \otimes_K \mathcal{Y}}$ ,
- $F(\lambda_{\mathcal{X}}) = \lambda_{F(\mathcal{X})}$ .

where the  $F$  on the left hand side is the prolongation functor on  $\text{Proj}_R$  whereas the  $F$  on the right hand side is the prolongation functor in  $\text{Diff}(K, \phi)$  (see [Ovc09, §4.3]).

By [GGO13, Prop.4.25], the functor  $\text{Hom}^{\otimes, \delta}(\omega_1, \omega_2)$  is representable by a  $S$ - $\delta$ -Hopf algebra. That is,  $\text{Hom}^{\otimes, \delta}(\omega_1, \omega_2)$  is a  $\delta$ -algebraic group over  $S$ . Via this representation, the prolongation functor  $F$  corresponds to the derivation on the coordinate ring of  $\text{Hom}^{\otimes, \delta}(\omega_1, \omega_2)$ . The existence of a neutral fiber functor  $\omega : \mathcal{C} \rightarrow \text{Vect}_k$  over a  $\delta$ -closed field  $k$  ensures that any differential Tannakian category is equivalent up to some differentially algebraic base change to the category of representations of a linear differential algebraic group, namely  $\text{Aut}^{\otimes, \delta}(\omega)$ .

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<sup>6</sup>It is the ring of differential operator of order less than or equal to one.

For difference modules over a  $(\phi, \delta)$ -field  $K$ , Theorem 3.4.2.3 in [And01] could be generalized to prove the equivalence between the notions of  $\delta$ -Picard-Vessiot extensions and differential fiber functors. In particular, we have the following proposition

**Proposition 2.11** (Propositions 3.4 and 3.5 in [4]). *Let  $K$  be a  $(\phi, \delta)$ -field with  $k = K^\phi$ . Let  $\mathcal{M}$  be a difference module over  $K$  and  $\phi(Y) = AY$  a difference system associated with  $\mathcal{M}$ . Let  $L_A$  be a  $\delta$ -Picard-Vessiot extension for  $\phi(Y) = AY$  over  $K$  and  $S_A \subset L_A$  a  $(\phi, \delta)$ -Picard-Vessiot ring. The following holds:*

- *the functor :*

$$\begin{aligned} \omega_{S_A} : \langle \mathcal{M} \rangle_\delta &\rightarrow \text{Vect}_k, \\ \mathcal{N} &\mapsto \text{Ker}(\Phi - id, \mathcal{N} \otimes_K S_A) \end{aligned}$$

*is a neutral differential fiber functor,*

- *the  $\delta$ -algebraic groups  $\text{Aut}^{\otimes, \delta}(\omega_{S_A})$  and  $\delta\text{-Gal}(L_A|K)$  are isomorphic over  $k$ .*

More intrinsic than the Picard-Vessiot presentation, the Tannakian framework has also many computational advantages. Indeed, any object  $\mathcal{M}$  in  $\text{Diff}(K, \phi)$ , one can define the differential constructions of  $\mathcal{M}$  as the smallest collection of difference modules containing  $\mathcal{M}$  and closed under direct sums, tensor products, symmetric and antisymmetric products, duals and the functor  $F$  (see [10, §7.2]). If  $\langle \mathcal{M} \rangle_\delta$  denotes the differential Tannakian subcategory generated by  $\mathcal{M}$  in  $\text{Diff}(K, \phi)$  and  $\omega : \langle \mathcal{M} \rangle_\delta \rightarrow \text{Vect}_k$  is a differential fiber functor, a differential analogue of [And01, §3.2.2.2] proves that the  $\delta$ -Galois group  $\text{Aut}^{\otimes, \delta}(\mathcal{M})$  of a difference module  $\mathcal{M}$  coincides with the stabilizer of all  $\omega(\mathcal{N})$  with  $\mathcal{N}$  a difference submodule of a differential construction of  $\mathcal{M}$ .

Finally the notion of  $\delta$ -integrability of Proposition 2.8 can be interpreted in the Tannakian framework as follows. Let  $\mathcal{M}$  be an object of  $\text{Diff}(K, \phi)$  and let  $\phi(Y) = AY$  be a difference system associated with  $\mathcal{M}$ . Then, the system  $\phi(Y) = AY$  is  $\delta$ -integrable if and only if the extension  $1 \rightarrow \mathcal{M} \rightarrow F(\mathcal{M}) \rightarrow \mathcal{M} \rightarrow 1$  splits in  $\text{Diff}(K, \phi)$ .

## 2.2 Discrete parameter ([15, 17])

In [17], we develop a Galois theory for linear differential equations with the parametric action of an endomorphism. Classical examples of such a situation are given by the action of the shift  $\alpha \rightarrow \alpha + 1$  on the family of Bessel linear differential equations  $x^2 \frac{d^2}{dx^2}(y) + x \frac{d}{dx}(y) + (x^2 - \alpha^2)y = 0$  and by the Frobenius action on  $p$ -adic linear differential . The algebraic relations satisfied by the solutions of a linear differential equation are controlled by the Galois group introduced by Picard and Vessiot, which is the analogue of the Galois group introduced in §1.3 for linear differential equations. In [17], we construct a difference Galois theory for linear differential equations which encodes the difference algebraic relations satisfied by the solutions in the defining equations of a difference algebraic group. Thereby, one can interpret the classical difference relation among Bessel functions as well as the existence of a Frobenius structure for a  $p$ -adic linear differential equation as the reflection of the geometric structure of a certain Galois group.

The construction of the difference Galois theory for linear differential equations is analogous to the one described in §2.1 and inverts the role of the derivation and of the difference operator. Therefore, I will not give all the details and technicalities of this Galois theoretic construction but only focus on the specificities related to a difference parameter.

### 2.2.1 Difference algebraic groups

The Galois groups introduced in [17, 15] are difference algebraic groups. While differential algebraic groups (see §1.2) are a classical topic in differential algebra (see e.g. [Cas72], [Kol73]), their difference analogs, that is, groups defined by algebraic difference equations have been essentially studied from a modeltheoretic point of view and motivated by number-theoretic applications( See [CH99], [CHP02], [Hru01], [Cha97], [SV99], [KP07], [CH]). The model companion of the theory of difference field is the theory ACFA. A model  $k$  of ACFA is a  $\sigma$ -closed field. That is any system of difference polynomial equations with coefficients in  $k$  and a solution in a  $\sigma$ -field extension of  $k$  has already a solution in  $k$ .

In order to study the difference algebraic relations satisfied by elements lying in difference rings and not necessarily in difference fields, one needed to develop a schematic approach for the idea of a group defined by algebraic difference equations. A difference algebraic group over a difference field  $(k, \sigma)$  in the sense of [17, Def. A.31]<sup>7</sup> is defined as follows:

**Definition 2.12.** *A  $k$ - $\sigma$ -group scheme is a (covariant) functor  $G$  from the category of  $k$ - $\sigma$ -algebras to the category of groups which is representable by a  $k$ - $\sigma$ -algebra. I.e., there exists a  $k$ - $\sigma$ -algebra  $k\{G\}$  such that  $G \simeq \text{Alg}_k^\sigma(k\{G\}, -)$ , where  $\text{Alg}_k^\sigma$  stands for morphisms of  $k$ - $\sigma$ -algebras. If  $k\{G\}$  is finitely  $\sigma$ -generated over  $k$  (see §1.1), we say that  $G$  is a difference algebraic group, or  $\sigma$ -algebraic group for short, over  $k$ .*

Since ACFA, the model companion of the theory of difference fields, does not (fully) eliminate quantifiers, the definable subgroups in ACFA need not be defined by difference polynomials. For example, the group

$$\{g \in k^\times \mid \exists h \in k^\times : h^2 = g, \sigma(h) = h\} \leq \text{Gl}_1(k)$$

is a definable subgroup of  $\text{Gl}_1(k)$  for  $k$  a  $\sigma$ -closed field but does not correspond to a  $\sigma$ -closed subgroup of  $\text{Gl}_{1,k}$ . On the other hand, the quantifier free definable subgroups of  $\text{Gl}_n(k)$ , i.e., the subgroups of  $\text{Gl}_n(k)$  defined by difference polynomials in the matrix entries, only correspond to a certain subclass of  $\sigma$ -algebraic groups, the perfectly  $\sigma$ -reduced  $\sigma$ -closed subgroups of  $\text{Gl}_{n,k}$  where  $\text{Gl}_{n,k}$  is the  $\sigma$ -algebraic group associated to  $\text{Gl}_n(k)$  as in [17, §A.4]. Perfectly  $\sigma$ -reduced  $\sigma$ -closed subgroups of  $\text{Gl}_{n,k}$  corresponds to  $\sigma$ -algebraic groups  $G$  whose  $\sigma$ -coordinate ring  $k\{G\}$  is *perfectly  $\sigma$ -reduced*, that is, whose zero ideal is perfect (see §1.1 for the definition a perfect ideal). Therefore, none of the notions presented above encompasses the other which doesn't allow to apply directly the classification results for groups definable in ACFA obtained by the logicians to the framework of  $\sigma$ -algebraic groups.

For  $G$  an algebraic group-scheme defined over  $k$ , one can construct a  $\sigma$ -algebraic group scheme  $[\sigma]_k G$  sometimes denoted also  $G$  by abuse of notation such that for any  $k$ - $\sigma$ -algebra  $S$ , one has  $([\sigma]_k G)(S) = G(S^\sharp)$  where  $S^\sharp$  is the underlying  $k$ -algebra of  $S$  (see §A.4 in [17]). This construction is functorial.

For  $G$  a  $\sigma$ -algebraic group over  $k$ , one defines the  $\sigma$ -dimension of  $G$  as follows. Let  $a = (a_1, \dots, a_m)$  be a  $\sigma$ -generating set for  $k\{G\}$  over  $k$ , the  $\sigma$ -dimension of  $G$  over  $k$  is  $\limsup_{i \rightarrow \infty} \dim(k[a, \dots, \sigma^i(a)]) / (i+1)$ . By [17, Prop. A.24], this dimension is well defined and independent from the choice of  $a$ . Many properties of the  $\sigma$ -dimension are contained in [17, §A.7].

Theorem 2.14 below is an analogue of Cassidy's classification for Zariski dense differential algebraic subgroups of almost simple algebraic groups ([Cas89]) in the context of  $\sigma$ -algebraic groups. For groups definable in ACFA, this analogue is due to Z. Chatzidakis, E. Hrushovski and Y. Peterzil.

**Proposition 2.13** (Prop. 7.10, p. 309 in [CHP02]). *Let  $k$  be a  $\sigma$ -closed  $\sigma$ -field of characteristic zero,  $H$  an almost simple algebraic group over  $k$ , and let  $G$  be a Zariski dense definable<sup>8</sup> subgroup of  $H(k)$ . Then, either  $G = H(k)$ , or, there exist an isomorphism  $f: H \rightarrow H'$  of algebraic groups and an integer  $d \geq 1$  such that some subgroup of  $f(G)$  of finite index is conjugate to a subgroup of  $H'(k^\sigma)$ . If  $H$  is defined over the algebraic closure of  $k^\sigma$ , then we may take  $H = H'$  and  $f$  to be conjugation by an element of  $H(k)$ .*

The version of the above proposition in the context of  $\sigma$ -algebraic groups requires that the  $\sigma$ -algebraic group  $G$  is  $\sigma$ -integral which means that  $k\{G\}$  is a domain and that  $\sigma$  is injective on  $k\{G\}$ .

**Theorem 2.14.** *Let  $k$  be an algebraically closed, invertive  $\sigma$ -field of characteristic zero and let  $G$  be a  $\sigma$ -integral,  $\sigma$ -closed subgroup of  $\text{Gl}_{n,k}$ . Assume that the Zariski closure of  $G$  in  $\text{Gl}_{n,k}$  is an almost simple algebraic group, properly containing  $G$ . Then there exists a  $\sigma$ -field extension  $\tilde{k}$  of  $k$  and an integer  $d \geq 1$  such that  $G_{\tilde{k}}$  is conjugate to a  $\sigma^d$ -constant subgroup  $\tilde{H}$  of  $\text{Gl}_{n,\tilde{k}}$  where  $\sigma^d$ -constant means that  $\sigma^d(h) = h$  for all  $h \in \tilde{H}(S)$  and any  $k$ - $\sigma$ -algebra  $S$ .*

The proof of Theorem 2.14 is inspired by the proof of Z. Chatzidakis, E. Hrushovski and Y. Peterzil. Adapting this proof to the context of  $\sigma$ -algebraic groups requires however some care. In particular, Theorem 2.14 is not valid if the group  $G$  is not  $\sigma$ -integral (see [15, Remark A.22]). This subtlety is the main difference with the result of Cassidy in the differential context. Indeed, she proves that

<sup>7</sup>This notion agrees with the notion of a linear  $\mathfrak{M}$ -group in [Kam12], for a suitable choice of  $\mathfrak{M}$  and the notion of affine algebraic  $\sigma$ -group in the sense of [KP07].

<sup>8</sup>In the language of difference rings.

any differential algebraic subgroup  $G$ , Zariski dense in an almost simple algebraic group  $H(k)$  for  $k$  a differentially closed field is either conjugate to a constant subgroup of  $H(k^\delta)$  or equal to  $H(k)$ . This difference between the differential framework and the difference one is essentially due to the fact that the difference field extension generated by an algebraic element might not be finite. For instance, the  $\sigma$ -field  $\mathbb{C}(\sqrt{x}, \sqrt{x+1}, \sqrt{x+2}, \dots)$  is not a finite extension of the difference field  $(\mathbb{C}(x), \sigma(x) = x+1)$  but generated as  $\mathbb{C}(x)$ - $\sigma$ -field extension by the algebraic element  $\sqrt{x}$ . From a group perspective, this situation can be interpreted as follows: the  $\sigma$ -algebraic group  $G$  associated to the algebraic group  $\mu_2$  of the square roots of unity is given by the functor which send a  $k$ - $\sigma$ -algebra  $S$  to the group  $G(S) = \{g \in S^* | g^2 = 1\}$ . Its coordinate ring  $k\{x, \frac{1}{x}\}/\{x^2 = 1\}$  is isomorphic as  $k$ -algebra to  $k[\mu_2^\infty]$ . Thereby, the  $\sigma$ -algebraic group  $G$  does not coincide with the  $\sigma$ -algebraic group  $H$  corresponding to  $H(S) = \{g \in S^* | g^2 = 1 \text{ and } \sigma(g) = g\}$  though  $H(\tilde{k}) = G(\tilde{k})$  for any difference field extension  $\tilde{k}$  of  $k$ . Such a situation cannot appear in the context of differential fields, because the field extension generated by an algebraic element is always stable by the derivation as well as in the context of groups definable in ACFA, essentially since all these groups are perfectly reduced, whence the difference between Theorem 2.14 and proposition 2.13. Some complete classifications for difference algebraic subgroups of tori and vector groups and their semi-direct product are detailed in Lemma A.40 in [17], Theorems A.1 and A.9 in [15].

### 2.2.2 Picard-Vessiot formalism

In [17], we authorize a certain non-commutativity rule for the operators  $\sigma$  and  $\delta$  that is a  $(\delta, \sigma)$ -ring is a ring  $R$  that is simultaneously a  $\delta$  and a  $\sigma$ -ring such that for some unit  $\hbar \in R^\delta$ , one has  $\delta \circ \sigma = \hbar \sigma \delta$ . Of course, the element  $\hbar$  has to be understood as part of the data of the  $(\delta, \sigma)$ -ring so that any algebraic construction for  $(\delta, \sigma)$ -ring must respect this constant. This choice was made in order to consider the  $(\delta, \sigma)$ -field  $\mathbb{C}(x)$  endowed with the derivation  $\delta = x \frac{d}{dx}$ , the operator  $\sigma(x) = x^p$  and the constant  $\hbar = p$ .

Let  $K$  be a  $(\delta, \sigma)$ -field and  $A \in K^{n \times n}$ . A  $(\delta, \sigma)$ -field extension  $L_A$  of  $K$  is called a  $\sigma$ -Picard-Vessiot extension for  $\delta(Y) = AY$  if

1. there exists  $Z \in \text{Gl}_n(L)$  such that  $\delta(Z) = AZ$  and  $L_A = K\langle Z_{ij} | 1 \leq i, j \leq n \rangle_\sigma$  and
2.  $L_A^\delta = K^\delta$ .

A  $K$ - $(\delta, \sigma)$ -algebra  $S_A$  is called a  $\sigma$ -Picard-Vessiot ring for  $\delta(Y) = AY$  if

1. there exists  $Z \in \text{Gl}_n(S_A)$  such that  $\delta(Z) = AZ$  and  $S_A = K\{Z_{ij}, \frac{1}{\det(Z)}\}_\sigma$  and
2.  $S_A$  is  $\delta$ -simple, i.e.,  $S_A$  has no non-trivial  $\delta$ -ideals.

A  $\sigma$ -Picard-Vessiot extension for  $\delta(Y) = AY$  over  $K$  is the smallest field extension of  $K$  generated by a fundamental set of solutions of  $\delta(y) = Ay$  and its successive transforms with respect to  $\sigma$ . By [17, Cor. 1.13], the existence of a  $\sigma$ -Picard-Vessiot extension is granted if one assumes that  $k = K^\delta$  is an algebraically closed field. The uniqueness of the  $\sigma$ -Picard-Vessiot extensions is more subtle and requires both a constrained extension of the field of  $\delta$ -constant  $k = K^\delta$  as well as some extra assumptions on the base field  $K$  (see [17, Theorem 1.16]). One obtains as corollary that, if  $k$  is  $\sigma$ -closed then given  $\sigma$ -Picard-Vessiot extensions  $L_1, L_2$ , there exists  $l \in \mathbb{Z}^{+,*}$  such that  $L_1|K$  and  $L_2|K$  are isomorphic as  $(\sigma^l, \delta)$ -field extensions of  $K$ .

Given a  $\sigma$ -Picard-Vessiot extension  $L_A = K\langle Z \rangle_\sigma$  for  $\delta(Y) = AY$  with  $A \in K^{n \times n}$  and  $S_A = K\{Z, \frac{1}{\det(Z)}\}_\sigma \subset L$ , we define the  $\sigma$ -Galois group  $\sigma\text{-Gal}(L_A|K)$  to be the functor from the category of  $k$ - $\sigma$ -algebras to the category of groups given by

$$\sigma\text{-Gal}(L_A|K)(S) := \text{Aut}^{(\delta, \sigma)}(S_A \otimes_k S | K \otimes_k S)$$

for every  $k$ - $\sigma$ -algebra  $S$ . The action of  $\delta$  on  $S$  is trivial, i.e.,  $\delta(r \otimes s) = \delta(r) \otimes s$  for  $r \in R$  and  $s \in S$ . [17, Prop. 2.5] proves that  $\sigma\text{-Gal}(L_A|K)$  is a  $\sigma$ -algebraic group defined over  $k = K^\delta$ . Moreover, the Zariski closure of  $\sigma\text{-Gal}(L_A|K)$  agrees with the Galois group attached to the linear differential equation  $\delta(Y) = AY$  in the sense of classical Picard-Vessiot theory (see [17, Prop. 2.15]). The  $\sigma$ -Galois group is not necessarily a perfectly  $\sigma$ -reduced even not  $\sigma$ -reduced (which means that  $\sigma$  is injective on the coordinate ring) (see [17, Examples 2.10 and 2.11]). These classical examples yields to  $\sigma$ -Galois groups, which are



not definable in ACFA, provide a strong motivation for the study of difference algebraic groups and their classification.

In this framework, one can prove a Galois correspondence between the  $(\delta, \sigma)$ -subfields of a  $\sigma$ -Picard-Vessiot extension  $L_A|K$  and the  $\sigma$ -closed subgroups of  $\sigma\text{-Gal}(L_A|K)$  (see [17, Theorems 3.2 and 3.3]). Moreover, a difference analogue of the Torsor theorem 2.5 proves that the  $\sigma$ -dimension of  $\sigma\text{-Gal}(L_A|K)$  coincides with the  $\sigma$ -transcendence degree of the field extension  $L_A|K$  (see [17, Lemma 2.7 and Prop. 2.17]). Thereby, the  $\sigma$ -Galois group is a versatile tool for the systematic study of the difference algebraic relations among the solutions of a linear differential equation since it encodes these relations in its defining ideal.

As an illustration, the notion of discrete integrability developed in [15] gives a Galois theoretic flavor to some discrete relations among special functions, solutions of linear differential equations. For  $K$  a  $(\delta, \sigma)$ -field,  $A \in K^{n \times n}$ , for some positive integer  $n$  and  $d \in \mathbb{Z}^{+,*}$ , we say that the system  $\delta(Y) = AY$  is  $\sigma^d$ -integrable over  $K$  if there exists  $B \in \text{Gl}_n(K)$  such that

$$(2.5) \quad \begin{cases} \delta(Y) = AY \\ \sigma^d(Y) = BY \end{cases}$$

is *compatible*, i.e.,

$$(2.6) \quad \delta(B) + BA = \hbar_d \sigma^d(A)B,$$

where  $\hbar_d = \hbar \sigma(\hbar) \cdots \sigma^{d-1}(\hbar)$ . One can interpret the  $\sigma^d$ -integrability in terms of the existence of a fundamental matrix of solutions for  $\delta(Y) = AY$  and  $\sigma^d(Y) = BY$ . The following result is a discrete analogue of Proposition 2.8.

**Theorem 2.15** (Prop. 5.2 in [15]). *Let  $L_A|K$  be a  $\sigma$ -Picard-Vessiot extension for  $\delta(Y) = AY$ , with  $A \in K^{n \times n}$ . Then  $\delta(Y) = AY$  is  $\sigma^d$ -integrable over  $K$  if and only if there exists a  $\sigma$ -separable  $\sigma$ -field extension  $\tilde{k}$  of  $k := K^\delta$ , such that the  $\sigma$ -Galois group  $\sigma\text{-Gal}(L_A|K)$  is conjugate over  $\tilde{k}$  to a  $\sigma^d$ -constant subgroup  $H$  of  $\text{Gl}_{n, \tilde{k}}$ , that is to a  $\sigma$ -algebraic group  $H$  such that  $H(S) = \{h \in \text{Gl}_n(S) \mid \sigma^d(h) = h\}$  for every  $\tilde{k}$ - $\sigma$ -algebra  $S$ .*

Under Theorem 2.15, one can for instance interpret the contiguity relations satisfied by hypergeometric functions in terms of  $\sigma$ -integrability. For the base field  $\mathbb{C}(x, \alpha)$  with  $\sigma(f(x, \alpha)) = f(x, \alpha + 1)$ , the contiguity relations satisfied by the Bessel functions yield to the  $\sigma$ -integrability of the Bessel's equation  $x^2 \delta^2(y) + x \delta(y) + (x^2 - \alpha^2)y = 0$ . One can then easily deduce that the  $\sigma$ -Galois group of the Bessel's equation is  $\text{Sl}_2^\sigma$ , that is, the  $\sigma$ -constant group such that  $\text{Sl}_2^\sigma(S) = \{h \in \text{Sl}_2(S) \mid \sigma(h) = h\}$  for every  $\mathbb{C}$ - $\sigma$ -algebra  $S$  (see [15, §5.1]). On the other hand, Theorem 2.15 combined with Theorem 2.14 gives the following criteria for difference transcendence.

**Theorem 2.16** (Theorem 6.4 in [15]). *Let  $K$  be an inversive  $(\delta, \sigma)$ -field,  $A \in K^{n \times n}$ ,  $k = K^\delta$  an algebraically closed field and  $L_A|K$  a  $\sigma$ -Picard-Vessiot extension for  $\delta(Y) = AY$ . We assume that the Galois group  $H$ , in the sense of Picard-Vessiot theory, of  $\delta(Y) = AY$  over  $K$  is an absolutely almost-simple algebraic group of dimension  $t \geq 1$  over  $k = K^\delta$ . Let  $K'$  be the relative algebraic closure of  $K$  inside  $L$ . Then the following statements are equivalent:*

1.  $\sigma\text{-Gal}(L_A|K')$  is a proper  $\sigma$ -closed subgroup of  $H$ .
2. The  $\sigma$ -transcendence degree of  $L_A|K$  is strictly less than  $t$ .
3. There exists  $d \in \mathbb{Z}_{>0}$  such that the system  $\delta(Y) = AY$  is  $\sigma^d$ -integrable.

Theorem 2.16 is a discrete analogue of Proposition 2.9. The main difference between these two results is that, unlike the case of a differential parameter, one may need to perform an algebraic extension of the base field  $K$  in order to detect the integrability of the initial system. This is essentially due to the fact that differential equations with algebraic solutions are not necessarily  $\sigma^d$ -integrable. For instance,  $z = \sqrt{x}$  satisfies the differential equation  $\delta(y) = \frac{y}{2x}$  but since there is no rational function  $b \in \mathbb{C}(x)$  and

integer  $d$  such that  $\sqrt{x+d} = b(x)\sqrt{x}$ , the equation  $\delta(y) = \frac{y}{2x}$  is not  $\sigma^d$ -integrable for every positive integer  $d$ . This reflects once again the group theoretic considerations of §2.2.1 and the fact that the analogue of Cassidy's result essentially holds for  $\sigma$ -integral  $\sigma$ -algebraic groups.

Thanks to Theorem 2.16, one can prove that the meromorphic solutions  $f, g$  and  $\delta(g)$  of the Airy equation  $\delta^2(y) - xy = 0$  are transformally independent over the  $(\delta, \sigma)$ -field  $\mathbb{C}(x)$  where  $\sigma(f(x)) = f(x+1)$  (see [15, Corollary 6.10]). Indeed, the Galois group of the Airy equation is  $\mathrm{Sl}_2(\mathbb{C})$  and one can easily show that  $\mathbb{C}(x)$  is relatively algebraically closed in the  $\sigma$ -Picard-Vessiot extension  $L$  generated by  $f, g, \delta(f)$  and  $\delta(g)$ . Theorem 2.16 reduces thereby to the following dichotomy: either the  $\sigma$ -transcendence degree of  $L_A|K$  is greater or equal to 3 or the Airy equation is  $\sigma^d$ -integrable over  $\mathbb{C}(x)$  for some positive integer  $d$ . But the discrete integrability is a strong constraints on the coefficients of the initial differential equation, essentially given by the fact that (2.6) must have some rational solutions in  $B$ . A careful study of the poles and order of a putative matrix  $B$  shows that this discrete integrability cannot happen yielding the transformal independence of the  $f, g, \delta(g)$ .

The parametrized theories developed in [21] and in [17] have had many developments since then. On a group theoretic perspective, they were an additional motivation to continue or develop the geometric theory behind their Galois groups. On a Galois theoretic perspective, the Galois theory corresponding to a discrete parameter has been generalized by Wibmer and Ovchinnikov in [OW15]. They developed a Galois theory whose aim is, given two commuting endomorphisms, to study the algebraic relations satisfied by the solutions of a linear  $\phi$ -difference equation and their transforms with respect to  $\sigma$  where  $\phi$  and  $\sigma$  are two commuting ring endomorphisms. The  $\sigma$ -Galois groups of Ovchinnikov and Wibmer are  $\sigma$ -algebraic groups as defined in §2.2.1 to whom one can apply the classification results obtained in [15, Appendix A]. Let me briefly describe their formalism.

Given a field  $K$  endowed with two commuting endomorphism  $\phi$  and  $\sigma$ , one considers here linear difference systems of the form  $\phi(Y) = AY$  where  $A \in \mathrm{Gl}_n(K)$ . A  $\sigma$ -Picard-Vessiot extension  $L_A$  for  $\phi(Y) = AY$  over  $K$  is a  $K$ - $(\phi, \sigma)$ -algebra which is a  $\phi$ -pseudofield such that the following conditions hold

- there exists  $U \in \mathrm{Gl}_n(L_A)$  such that  $\phi(U) = AU$  and  $L_A = K\langle U \rangle_\sigma$ ;
- $L_A^\phi = K^\phi = k$  ([OW15, Def. 2.18]).

Given a  $\sigma$ -Picard-Vessiot extension  $L_A$  for  $\phi(Y) = AY$  over  $K$ , one defines analogously to §2.2.2 its  $\sigma$ -Galois group  $\sigma\text{-Gal}(L_A|K)$  (see [OW15, Def. 2.50]). By [OW15, Lemma 2.51], it is a  $\sigma$ -algebraic group defined over  $k$ . [3, Prop. 2.12] proves that the pseudofield  $K_A = K(U) \subset L_A$  is a Picard-Vessiot extension for  $\phi(Y) = AY$  and that the  $\sigma$ -Galois group  $\sigma\text{-Gal}(L_A|K)$  is Zariski dense in the Galois group  $\mathrm{Gal}(K_A|K)$ .

Ovchinnikov and Wibmer also develop a Tannakian formalism to encompass the theories [17] and [OW15] and characterize the more general action of semigroups on Tannakian categories ([OW17]).

### 3 Direct problem in parametrized Galois theory: criteria and algorithms([11, 13, 14, 19, 21, 23])

The direct problem in the Galois theory of linear difference equations is concerned with the computation of the Galois group. One can understand this problem

- from an algorithmic perspective, that is, given a precise linear functional system, is there an algorithm that will determine its Galois group? And if so, can one implement this algorithm with a reasonable complexity bound?
- from a classification perspective, that is, given a family of linear difference equations, can we find a correspondence between the algebraic relations satisfied by the coefficients of the system and the algebraic defining equations of its Galois group.

The inverse problem in this Galois theory is dedicated to the characterization of the linear algebraic groups that can be realized as Galois group of a linear difference system. This problem is of course interconnected with the direct problem and both questions go hand in hand. The inverse problem has

been partially solved by Etingov for  $q$ -difference systems ([Eti95, Prop.3.4]) and by van der Put and Singer for the shift ([vdPS97b]). Both prove that any connected linear algebraic group defined over  $\mathbb{C}$  can be realized as Galois group of the latter difference systems.

For linear differential systems, Hrushovski produced an algorithm which determines the Galois group of a linear differential equation over the field  $\mathbb{Q}(t)$  of rational functions (see [Hru02]). Hrushovski's algorithm based on a Tannakian formalism and has been detailed and improved by Feng in [Fen15]. Moreover, Feng gave explicit bounds on the complexity of the algorithm. These bounds are unfortunately exponential (several level) in  $n$ , the size of the linear differential system. This algorithm was the first which works for all linear differential equations with rational function coefficients. Till then, the known algorithms were valid only for special types of equations. For instance, Kovacic used the classification of the algebraic subgroups of  $\mathrm{Sl}_2$  to develop an algorithm that computes the Galois group of linear differential equations of order 2 ([Kov86]). This algorithm was generalized to the situation of order three differential equations in [SU93]. In [CS99], the authors compute a bound on the degree the polynomial invariants of a reductive algebraic group in order to develop an algorithm which allows to compute the Galois group of linear differential equations that are completely reducible. Recently, works by Barkatou, Cluzeau, Di Vizio and Weil ([BCDVW16]) for absolutely simple differential modules  $\mathcal{M}$  and by Dreyfus and Weil for completely reducible one's ([DW22]), propose implemented algorithms to compute the Lie algebra of the Galois group. These algorithms are based on the notion of reduced form of a linear differential system and on the conjectural Grothendieck-Katz curvature description of the Lie algebra of the intrinsic Galois group of the differential module. Barkatou-Cluzeau-Di Vizio and Weil's algorithm allows to reach a complexity bound that is polynomial in the size  $n$  of the linear differential module. One has to note however that among the algorithms quoted above, only Kovacic's algorithm allows to work in family since it interprets the algebraic structure of the Galois group in terms of the existence of rational solutions to differential equations depending only on the coefficients of the linear differential module. For instance, for a linear differential equation of the form  $\delta^2(y) - by = 0$  with  $b \in \mathbb{C}(x)$ , the Galois group is triangulisable if and only if the Riccati equation  $\delta(\omega) + \omega^2 = b$  has a rational solution  $\omega \in \mathbb{C}(x)$ .

For linear difference systems, analogues of Kovacic's algorithm have been established by Hendricks in [Hen98] for general difference equations of order 2 and for  $q$ -difference equations of order 2 over  $\overline{\mathbb{Q}}(x)$  or  $\cup_{j \geq 1} \overline{\mathbb{Q}}(x^{\frac{1}{j}})$ . These results have been adapted by Roques for Mahler equations of order 2 in [Roq18] and for difference equations of order 2 on an elliptic curve by Dreyfus and Roques in [DR15]. In [Roq11], Roques used the transcendental description of the Galois group of a  $q$ -difference equation obtained by Sauloy in terms of local monodromy to unravel the computation of the Galois group of the generalized  $q$ -hypergeometric function. More recently, Feng has adapted Hrushovski's algorithm to compute the Galois group of a linear difference equation over  $\overline{\mathbb{Q}}(x)$  ([Fen18]). Unfortunately, as noted by Feng, "one may suspect that the complexity of the algorithm would be very high".

For parametrized Galois theories, the direct problem is only in its infancy. Most of the existing algorithms concern the computation of the parametrized Galois group of a linear differential equation with a parametric action of a finite set of commuting derivations, that is, the Galois theory developed by Cassidy and Singer. Fixing a finite set  $\Delta = \{\partial_1, \dots, \partial_s\}$  of commuting derivations and  $k$  a  $\Delta$ -closed field extension of  $\mathbb{Q}$ , one considers a linear differential equation of the form  $\mathcal{L}(y) = \delta^n(y) + a_{n-1}\delta^{n-1}(y) + \dots + a_0(y) = 0$  where  $a_0, \dots, a_{n-1} \in k(x)$ . Here,  $k(x)$  denote the  $(\delta, \Delta)$ -field of rational fractions where  $x$  is  $\Delta$ -constant,  $k$  is  $\delta$ -constant and  $\delta(x) = 1$ . In [MS12, Dre14a], the authors gave a first answer to the inverse problem in Cassidy and Singer's Galois theory and proved that a linear  $\Delta$ -algebraic group defined over  $k$  is a parametrized Galois group of a linear differential equation  $\mathcal{L}(y) = 0$  if and only if it contains a finitely generated Kolchin dense subgroup. Such a differential algebraic group is called finitely differentially generated. In [Sin13], Singer gave a group theoretic characterization of being finitely differentially generated. In particular, he proved that a linear algebraic group  $G$  defined over  $k$  is finitely differentially generated if and only if there is no differential group homomorphism from  $G$  to the additive group  $\mathbf{G}_a$ . Relying on this characterization, the authors of [MOS14] were able to prove that the unipotent parametrized Galois groups have differential type zero. This numerical characterization allowed them to produce an algorithm to test whether the parametrized Galois group was unipotent and to compute this group when the answer was positive. In [MOS15], Minchenko, Ovchinnikov and Singer combine Cassidy's classification of differential subgroup of semisimple algebraic groups and the theory of differential representations of semisimple linear differential algebraic groups to produce

some bounds on the order of derivatives needed to compute the defining equations of a reductive linear differential algebraic group allowing them to produce an algorithm to compute the parametrized Galois group of a completely reducible differential equation  $\mathcal{L}$ . The algorithms of Minchenko, Ovchinnikov and Singer are unfortunately not easily manageable and their complexity is expected to be quite high with respect to the order of the linear differential equation. For equations of smaller order, one can find some explicit algorithms based on parametrized analogues of Kovacic's algorithm: [Arr16] for second order linear differential equations with parameters, [ADR21] for the parametrized Galois group of second order difference equations given by a translation over an elliptic curve (see Example 2.1).

In this section, we discuss some results obtained in the direct problem for the parametrized Galois theory of linear difference equations. These results concern essentially linear difference equations in diagonal form, completely reducible linear difference equations and extension of two completely reducible equations.

### 3.1 Rank one difference equations ([21, 11])

In this section, we shall show how the differential algebraic relations satisfied by solutions of rank one linear difference equations are intimately related with a discrete analogue of the notion of telescopers. For a bivariate function, a telescoper is a linear differential (resp. difference) operator annihilating the definite integral (resp. sum) of this function. Creative telescoping was popularized by Zeilberger and it plays a key role in the Wilf-Zeilberger theory and algorithm in order to produce hypergeometric identities.

By discrete telescopers, we mean the following.

**Definition 3.1.** *Let  $K$  be a  $(\phi, \delta)$ -field with  $k = K^\phi$  of characteristic zero. Let  $b$  be an element of  $K$ . A discrete telescoper for  $b$  is a pair  $(L, g)$  consisting of a linear differential operator  $L \in k[\delta]$  and  $g \in K$  such that*

$$(3.1) \quad L(b) = \phi(g) - g.$$

The notion of discrete telescopers appears naturally when one studied the differentially algebraic relations satisfied by solutions of a non-homogeneous rank one equations. The following proposition can be proved via the classification of differential algebraic subgroups of vector groups and on the parametrized Galois correspondence but also with an analogue for difference equations of Ostrowski's theorem (see Prop. 2.1 in [23] for the difference analogue and [Kol99, p. 1155] for Ostrowski's theorem)

**Proposition 3.2.** *Let  $K$  be  $(\phi, \delta)$ -field of characteristic zero with  $k = K^\phi$  algebraically closed. Let  $b \in K$  and let  $R$  be a pseudofield extension of  $(K, \phi)$ . Let  $z \in R$  be a solution of  $\phi(y) = y + b$  whose derivatives of all order belong to  $R$ . If  $z$  is differentially algebraic over  $K$  then  $b$  has a discrete telescoper. Conversely if  $b$  has a discrete telescoper and  $R^\phi = k$  then  $z$  is differentially algebraic over  $K$ .*

The assumption on the  $\phi$ -constants of the algebra  $R$  of solutions is crucial to conclude that the existence of a telescoper for  $b$  yields a differential algebraic relation for the solution. Indeed, one can construct some counterexamples where the  $\phi$ -constants are differentially transcendental over the base field. For instance if  $K = \mathbb{C}(x, t)$  endowed with the automorphism  $\phi(x, t) = (x + 1, t)$  and the derivation  $\frac{d}{dt}$ , the function  $\Gamma(t)$  is a  $\phi$ -constant which is differentially transcendental over  $K$ . The control of the ring of  $\phi$ -constants in the ring of solutions is the corner stone of the functional Galois theories developed above. Without any control on these  $\phi$ -constants, there is no hope to deduce any algebraic relation for the solutions from algebraic relations satisfied by the coefficients of the initial functional equation.

Noting that the logarithmic derivative  $z = \frac{\delta u}{u}$  of a solution  $u$  of  $\phi(y) = ay$  satisfies  $\phi(z) = z + \frac{\delta a}{a}$ , one finds the following corollary:

**Corollary 3.3.** *Let  $K$  be  $(\phi, \delta)$ -field of characteristic zero with  $k = K^\phi$  algebraically closed. Let  $a \in K^*$  and let  $R$  be a pseudofield extension of  $(K, \phi)$  with  $R^\phi = k$ . Let  $u \in R$  be an invertible solution of  $\phi(y) = ay$  whose derivatives of all order belong to  $R$ . Then,  $u$  is differentially algebraic over  $K$  if and only if there exists a discrete telescoper for  $\frac{\delta a}{a}$ .*

In order to test the existence of discrete telescopers for difference operators on the projective line, that is, for  $K = \mathbb{C}(x)$ , the article [23] introduced the notion of elliptic divisor for the shift and the  $q$ -difference operator as follows:

**Definition 3.4.** Let  $a = \lambda x^r \prod_{\alpha \in \mathbb{C}^*} (x - \alpha)^{n_\alpha} \in \mathbb{C}(x)^*$  and let  $q \in \mathbb{C}^*$  of modulus distinct from 1. The elliptic divisor  $\text{div}_E(a)$  of  $a$  is the formal sum  $\sum_{\bar{\alpha} \in \mathbb{C}^*/q^{\mathbb{Z}}} (\sum_{\alpha \in \bar{\alpha}} n_\alpha) \bar{\alpha}$ .

This lead to the following characterization for the existence of a discrete telescoper.

**Lemma 3.5** (Lemme 3.5 in [23]). Let  $a \in \mathbb{C}(x)^*$  and  $q \in \mathbb{C}^*$  of modulus distinct from 1. The following are equivalent

- the function  $\frac{\delta(a)}{a}$  has a discrete telescoper;
- $\text{div}_E(a) = 0$ ;
- $a = cz^r \frac{h(qx)}{h(x)}$  for  $c \in \mathbb{C}^*, r \in \mathbb{Z}, h \in \mathbb{C}(x)^*$ .

Combined with Proposition 3.3, Lemma 3.5 gives the following characterization of differentially algebraic solutions of rank one homogeneous  $q$ -difference equations.

**Corollary 3.6.** A function  $z$  meromorphic over  $\mathbb{C}^*$  solution of  $z(qx) = az(x)$  with  $a \in \mathbb{C}(x)^*$  is differentially algebraic over  $\mathbb{C}(x)$  if and only if  $a = cz^r \frac{h(qx)}{h(x)}$  for  $c \in \mathbb{C}^*, r \in \mathbb{Z}, h \in \mathbb{C}(x)^*$  if and only if  $z = \frac{g}{h} e_{q,c} \theta_q^r$  with  $g \in C_E, r \in \mathbb{Z}, h \in \mathbb{C}(x)$  and  $e_{q,c}$  and  $\theta_q$  are the special functions defined in 1.4. Moreover, if  $z \in \mathbb{C}(\!(x)\!)$  then  $r = 0, c = q^l$  for some  $l \in \mathbb{Z}$  so that  $z$  coincides up to multiplication by a non-zero complex number with the rational function  $\frac{x^l}{h(x)}$ .

The last part of this Lemma shows that a differentially algebraic power series solution of a rank one  $q$ -difference equation is necessary a rational function. Similar results holds for the shift as well as for the Mahler operator (see [7, Proposition 5.1] and [12, Prop.3.1]).

In [21, Lemma 6.4], a first criteria for the existence of discrete telescoper is given.

**Lemma 3.7.** Let  $b \in \mathbb{C}(x)$  and  $q \in \mathbb{C}^*$  of modulus distinct from 1. Then,  $b$  has a discrete telescoper if and only if  $b = h(qx) - h(x) + c$  for some  $c \in \mathbb{C}, h \in \mathbb{C}(x)$ .

Combined with Proposition 3.3, the following lemma shows that differentially algebraic meromorphic solutions of equations of the form  $z(qx) - z(x) = b(x)$  belong to the field extension of  $\mathbb{C}(x)$  generated by the  $q$ -logarithm. In [CS12], the authors define some orbitresidues for rational functions as follows : let  $S = (\beta)_{\beta \in \mathbb{C}^*}$  be a system of represent of the equivalence classes of  $\mathbb{C}^*/q^{\mathbb{Z}}$  and let us write any  $f \in \mathbb{C}(x)$  as follows

$$(3.2) \quad f(x) = c + xp_1(x) + \frac{p_2(x)}{x^s} + \sum_{\beta \in \mathbb{C}^*} \sum_{j \in \mathbb{N}^*} \sum_{l \in \mathbb{Z}} \frac{a_{\beta,j,l}}{(q^{-l}x - \beta)^j},$$

where  $p_1, p_2 \in K[x]$  are not divisible by  $x$ ,  $c \in \mathbb{C}$  and only finitely many of the  $a_{\beta,j,l}$ 's are non-zero complex numbers. Then, the orbit residue  $\text{ores}_{\beta,j}(f)$  of  $f$  at  $\beta$  of order  $j$  is defined by  $\text{ores}_{\beta,j} = \sum_{l \in \mathbb{Z}} a_{\beta,j,l}$  and the orbitresidue at infinity is the constant  $c$ . Combining Lemma 3.7 and [CS12, Proposition 2.10], one can characterize the functions  $b \in \mathbb{C}(x)$  with a discrete telescoper as those having orbit residue zero for any  $\beta \in \mathbb{C}^*, j \in \mathbb{N}^*$ . Similar results hold for the finite difference case (see [23, 21, CS12]) as well as for the Mahler operator.

Only recently, the existence of telescoper has been investigated for the field  $k(E)$  of rational functions on an elliptic curve  $E$  defined over an algebraically closed field of characteristic zero  $k$ , endowed with a structure of  $(\phi, \delta)$ -field where  $\phi$  is the addition by a non-torsion  $k$ -point of the curve and  $\delta$  is the derivation associated to the holomorphic one form on the curve (see Example 2.1). The notion of orbitresidue in that situation is however a little bit more subtle and depends on the choice of a coherent set of local parameters. These notions are defined as follows

**Definition 3.8** (Definition A.5 and A.6 in [11]). A set  $\mathcal{S} = \{u_Q \mid Q \in E\}$  of local parameters at the points of  $E$  is coherent if  $u_{\phi^{-1}(Q)} = \phi(u_Q)$  for any  $Q \in E$ . For  $f \in k(E)$  with a pole at  $Q$  of order  $n$ <sup>9</sup>, write

$$f = \frac{c_{Q,n}}{u_Q^n} + \dots + \frac{c_{Q,2}}{u_Q^2} + \frac{c_{Q,1}}{u_Q} + \tilde{f}$$

<sup>9</sup>Here  $n$  might be equal to 0.

where  $v_Q(\tilde{f}) \geq 0$ . For each  $j \in \mathbb{N}^*$  we define the orbit residue of order  $j$  at  $Q$  to be

$$\text{ores}_{Q,j}(f) = \sum_{i \in \mathbb{Z}} c_{\phi^i(Q),j}.$$

In the definition of the orbit residues over the projective line the set  $(q^{-l}x - \beta)_{\beta \in \mathbb{C}^*, l \in \mathbb{Z}}$  is a coherent set of local parameters over the projective line and the  $q$ -difference operator. These definitions lead to the following criteria for the existence of discrete telescopers.

**Proposition 3.9** (Prop. A.4 and A.7 in [11]). *Let  $E$  be an elliptic curve defined over an algebraically closed field  $k$  of characteristic zero. Let  $k(E)$  be the field of rational functions of  $E$  endowed with a structure of  $(\phi, \delta)$ -field where  $\phi$  is the addition by a non-torsion  $k$ -point of  $E$  and  $\delta$  is the canonical derivation. For  $b \in k(E)$ , the following are equivalent*

- $b$  has a discrete telescoper,
- for any  $Q \in E$  and  $j \in \mathbb{N}^*$ , the orbit residue  $\text{ores}_{Q,j}(b) = 0$ ,
- there exists  $Q \in E, e \in k(E)$  and  $g \in \mathcal{L}(Q + \phi^{-1}(Q))$ <sup>10</sup> such that  $b = \phi(e) - e + g$ .

Let  $E$  be a complex elliptic curve. Viewing  $\mathbb{C}^*$  as the universal cover of  $E = \mathbb{C}^*/q^{\mathbb{Z}}$ , the action of  $\phi$  to  $\mathbb{C}^*$  lifts as the multiplication by some non-zero complex number  $\tilde{q}$ . Endowed with the derivation  $\delta = x \frac{d}{dx}$ , the field  $\text{Mer}(\mathbb{C}^*)$  is a  $(\phi, \delta)$ -field extension of  $\mathbb{C}(E)$  identified with the elements of  $\text{Mer}(\mathbb{C}^*)$  invariant with respect to the multiplication by  $q$ . In that setting, one can combine Proposition 3.2 with Proposition 3.9 to describe all the differentially algebraic meromorphic solutions of non-homogeneous rank one linear difference equation over  $E$ .

**Corollary 3.10.** *Let  $E$  be a complex elliptic curve and  $\phi$  the translation by a non-torsion point of  $E$ . Let  $z \in \text{Mer}(\mathbb{C}^*)$  such that  $\phi(z) = z + b$  for some  $b \in \mathbb{C}(E)$ . Then,  $z$  is differentially algebraic over  $\mathbb{C}(E)$  if and only if there exists some complex numbers  $\alpha, \beta, \gamma$  such that*

$$z = \beta l_q(\alpha x) + \gamma l_{\tilde{q}}(x) + e(x) + h(x)$$

where  $l_q$  (resp.  $l_{\tilde{q}}$ ) is the  $q$ -logarithm (resp.  $\tilde{q}$ -logarithm), that is, the logarithmic derivative of  $\theta_q$  (resp.  $\theta_{\tilde{q}}$ ) and  $e$  (resp.  $h$ ) is an elliptic function with respect to the elliptic curve  $\mathbb{C}/q^{\mathbb{Z}}$  (resp.  $\mathbb{C}/\tilde{q}^{\mathbb{Z}}$ ).

*Proof.* The  $q$ -logarithm satisfies  $l_q(qx) = l_q(x) + 1$  so that its derivative with respect to  $\delta$  is an elliptic function. Similarly the derivative of  $l_{\tilde{q}}$  is an elliptic function with respect to the elliptic curve  $\mathbb{C}/\tilde{q}^{\mathbb{Z}}$ . Since elliptic functions are differentially algebraic over  $\mathbb{C}$ , one direction of the corollary is clear. Conversely, if  $z$  is differentially algebraic over  $\mathbb{C}(E)$  then  $b$  has a discrete telescoper by Proposition 3.2. By Proposition 3.9, there exists  $\alpha \in \mathbb{C}^*, e \in \mathbb{C}(E)$  and  $g$  with at most simple poles on  $\alpha q^{\mathbb{Z}}$  and  $\frac{\alpha}{\tilde{q}} q^{\mathbb{Z}}$  such that  $b = \phi(e) - e + g$ . Since  $l_q$  is a meromorphic function with simple poles on  $q^{\mathbb{Z}}$  such that  $l_q(qx) = l_q(x) + 1$ , the function  $l_q(\frac{\tilde{q}x}{\alpha}) - l_q(\frac{x}{\alpha})$  belongs to  $\mathbb{C}(E)$  and has only simple poles in  $\alpha q^{\mathbb{Z}}$  and  $\frac{\alpha}{\tilde{q}} q^{\mathbb{Z}}$ . Since the sum of the residues of an elliptic function vanish, one can find a complex number  $\beta$  such that the elliptic function  $g - \beta (l_q(\frac{\tilde{q}x}{\alpha}) - l_q(\frac{x}{\alpha}))$  has no poles and is therefore a constant complex number  $\gamma$ . From  $\phi(z) - z = b$ , one finds

$$\phi(z - \beta l_q(\frac{x}{\alpha}) - \gamma l_{\tilde{q}}(x) - e) = z - \beta l_q(\frac{x}{\alpha}) - \gamma l_{\tilde{q}}(x) - e.$$

This allows to conclude that  $z - \beta l_q(\frac{x}{\alpha}) - \gamma l_{\tilde{q}}(x) - e(x)$  is a  $\tilde{q}$ -elliptic function. □

## 3.2 Reductive groups

In classical Picard-Vessiot theory, the algorithm developed by Compoint and Singer in [CS99] allows to compute the Galois group of a completely reducible differential equation by determining its ring of invariants. However Compoint and Singer's algorithm is mostly a theoretical algorithm. Very recently, the authors of [BCDVW16] provide an effective algorithm to compute the Lie algebra of an absolutely irreducible differential module. In the following section, we show how one can rely on these algorithms and their future analogue for difference equations to compute parametrized Galois groups of completely reducible objects.

<sup>10</sup>This denotes the finite dimensional  $k$ -vector space formed by the functions  $h \in k(E)$  whose only poles are of order one and belong to  $\{Q, \phi^{-1}(Q)\}$ .

### 3.2.1 Differential parameter

For  $K$  a  $(\phi, \delta)$ -field with differentially closed field of constants  $k$  of characteristic zero, Proposition 6.21 in [21] asserts that if the Galois group of a system  $\phi(Y) = AY$  with  $A \in \mathrm{GL}_n(K)$  is an almost-simple, noncommutative linear algebraic group  $H$  over  $k$  and  $L_A$  denotes a  $\delta$ -Picard-Vessiot extension for  $\phi(Y) = AY$  over  $K$  then either the  $\delta$ -Galois group  $\delta\text{-Gal}(L_A|K)$  is equal to the  $\delta$ -algebraic group associated to  $H$  or the system  $\phi(Y) = AY$  is  $\delta$ -integrable, that is, there exists  $B \in K^{n \times n}$  such that  $\phi(B) = ABA^{-1} + \delta(A)A^{-1}$  holds. In this latter case, the  $\delta$ -Galois group is conjugate to a subgroup of  $H(k^\delta)$ .

When one restricts the difference framework above to automorphisms of the projective line, the above dichotomy can be drastically simplified. We shall consider the following cases:

- Case  $\delta\mathbf{Q}$ : for  $q$  a nonzero complex number, not a root of unity, one can endow  $K = \mathbb{C}(x)$  with a structure of  $(\phi, \delta)$ -field such that  $\phi(f(x)) = f(qx)$  and  $\delta = x \frac{d}{dx}$ ;
- Case  $\delta\mathbf{S}$ : one can endow the field  $K = \mathbb{C}(x)$  with a structure of  $(\phi, \delta)$ -field with  $\phi(x) = x + 1$  and  $\delta = \frac{d}{dx}$ ;
- Case  $\delta\mathbf{M}$ : for  $p$  a positive integer, one endows  $K = \mathbb{C}(x, \log(x))$  with a structure of  $(\phi, \delta)$ -field by setting  $\phi(f(x, \log(x))) = f(x^p, p \log(x))$  and  $\delta = x \log(x) \frac{d}{dx}$ .

Let  $k$  be a  $\delta$ -closed extension of  $\mathbb{C}$  and let  $\tilde{K}$  be the fraction field of  $k \otimes_{\mathbb{C}} K$  (resp.  $k \otimes K(\log(x))$ ) in the cases  $\delta\mathbf{Q}$  and  $\delta\mathbf{S}$  (resp. in the case  $\delta\mathbf{M}$ ). Then,  $\tilde{K}$  can be endowed with a structure of  $(\phi, \delta)$ -field such that  $\tilde{K}^\phi = k$  (see [12, Lemma 2.3]). In the notation above, the following holds

**Proposition 3.11** (Theorem 3.5 in [12] and Theorem 3.1 in [9]). *Let  $A \in \mathrm{GL}_n(\mathbb{C}(x))$  and assume that the Galois group of the system  $\phi(Y) = AY$  over the  $\phi$ -field  $\mathbb{C}(x)$  is  $\mathrm{Sl}_n(\mathbb{C})$ . Let  $L_A$  be a  $\delta$ -Picard-Vessiot extension for  $\phi(Y) = AY$  over  $\tilde{K}$ . Then,  $\delta\text{-Gal}(L_A|L) = \mathrm{Sl}_n(k)$ .*

Proposition 3.11 was proved in the slightly more general context of “projective isomonodromy”, that is, when the classical Galois group was containing and not equal to  $\mathrm{Sl}_n(\mathbb{C})$ . Theorem 3.1 in [9] also treats the situation where the classical Galois group is an irreducible almost simple algebraic subgroup of  $\mathrm{Sl}_n(\mathbb{C})$ . The proof of Proposition 3.11 combines three ingredients

- Proposition 2.9 which implies that either  $\delta\text{-Gal}(L_A|L) = \mathrm{Sl}_n(k)$  or that the system  $\phi(Y) = AY$  is  $\delta$ -integrable yielding to the existence of a vector solution of  $\phi(Y) = AY$  and of a linear differential system;
- Ramis ([Ram92b]) in the case  $\delta\mathbf{Q}$  and and Bezivin ([Béz94]) in the case  $\delta\mathbf{M}$  proved that a vector with coefficients in the ring of formal power series cannot be simultaneously solution of a linear difference system and a linear differential system unless its coefficients belong to  $K$ ;
- the fact that a solution vector in  $K^n$  is fixed by the Galois group which contradicts the irreducibility of the representation of  $\mathrm{Sl}_n(\mathbb{C})$  on the vector space of solutions of  $\phi(Y) = AY$ .

Ramis and Bézivin’s results have been generalized by Schaefer and Singer in [SS19a]. The crucial ingredient of their proof is the following Theorem.

**Theorem 3.12.** *Let  $(\delta, \phi)$  be as in one of the cases above. Let us consider two compatible systems in the sense of (2.6) given by*

$$(3.3) \quad \begin{cases} \delta(Y) = AY \\ \phi(Y) = BY \end{cases}$$

with  $A \in K^{n \times n}$  and  $B \in \mathrm{GL}_n(K)$ . Then, there exists a gauge transformation  $G \in \mathrm{GL}_n(\tilde{K})$  such that the system (3.3) is equivalent to the almost constant system

$$(3.4) \quad \begin{cases} \delta(Y) = \tilde{A}Y \\ \phi(Y) = \tilde{B}Y \end{cases}$$

with  $\tilde{A} = \delta(G)G^{-1} + GAG^{-1} \in \mathbb{C}^{n \times n}$  and  $\tilde{B} = \phi(G)BG^{-1}$  belongs to  $\mathrm{GL}_n(\mathbb{C})$  except for the case  $M$  where  $B \in \mathrm{GL}_n(\mathbb{C}[x, x^{-1}])$ .

Theorem 3.12 essentially proves that any  $\delta$ -integrable linear difference system over the projective line is gauge equivalent to a linear difference system with constant complex coefficients. In [AS17, Theorem 3.7], Arreche and Singer proved that the Galois group of a linear difference system with constant coefficients is the direct product of a torus, a vector group and some cyclic groups. Combining Cassidy's result, Theorem 3.12 with this description, Arreche and Singer were able to extend Proposition 3.11 to the case of semisimple Galois groups.

**Proposition 3.13** (Lemma 5.1 in [AS17]). *Let  $K$  and  $\tilde{K}$  be as in the cases  $\delta Q, \delta S$  and  $\delta M$  above and let  $A \in \mathbb{C}(x)$ . If the identity component of the Galois group of  $\phi(Y) = AY$  over  $K$  is a semisimple algebraic group  $H(\mathbb{C})$  then the  $\delta$ -Galois group of  $\phi(Y) = AY$  over  $\tilde{K}$  is  $H(k)$ .*

Proposition 3.13 is an important step in the inverse problem in the parametrized Galois theory developed in [21] since it shows that, when the Galois group is semisimple, the parametrized Galois theory doesn't introduce new objects or equations. From the point of view of solutions, this proposition shows that, when the identity component of the Galois group of  $\phi(Y) = AY$  is semisimple then the ideal of differential algebraic relations among the solutions is generated by the polynomial relations among the solutions. In section 4.2.1, we will show how Arreche and Singer's result can be also used to completely solve the problem of the differential algebraic relations satisfied by formal power series, solutions of linear difference equations in cases  $\delta S$ ,  $\delta M$  and  $\delta Q$ .

### 3.2.2 Difference parameter

In that section, we consider the Galois theory of  $\phi$ -difference system with an action of a difference operator  $\sigma$  developed in [OW15]. Adapting a classical argument ([vdPS97b, Cor.1.16]) to this parametrized context, one can show that, assuming that  $k$  is algebraically closed, there exists some positive integers  $r$  and  $s$  such that there exists a  $\sigma^s$ -Picard-Vessiot extension for the system  $\phi^r(Y) = \phi^{r-1}(A) \dots \phi(A)AY$  over the  $(\phi^r, \sigma^s)$ -field  $K$  which is a field. That is, up to iterate the endomorphisms  $\phi$  and  $\sigma$ , one can always assume that the  $\sigma$ -Picard-Vessiot extension is a field. Moreover, under mild assumptions on the base field  $K$ , one can get rid of many complications due to the existence of algebraic difference field extension of infinite degree.

More precisely, we say that the  $(\phi, \sigma)$ -base field  $K$  satisfies *Condition  $\mathcal{H}$*  if

- $\sigma : K \rightarrow K$  is an automorphism;
- $k = K^\phi$  is algebraically closed;
- for every positive integer  $r$ , the  $\phi^r$ -field  $K$  has no finite nontrivial  $\phi^r$ -field extension.

If  $K$  satisfies the condition  $\mathcal{H}$  and  $L_A$  is a  $\sigma$ -Picard-Vessiot field extension for  $\phi(Y) = AY$  over  $K$  then the  $\sigma$ -Galois group  $\sigma\text{-Gal}(L_A|K)$  is  $\sigma$ -integral (see [3, Lemma 3.9]).

This geometric property allows to drastically simplify the classification of  $\sigma$ -Galois groups. Moreover, natural  $(\phi, \sigma)$ -fields attached to endomorphisms of the projective line all satisfy the condition  $\mathcal{H}$ .

- *Case  $2S$* . In this case, we consider  $K = \mathbb{C}(x)$ ,  $\phi(x) = x + h_1$  and  $\sigma(x) = x + h_2$ , where  $h_1, h_2 \in \mathbb{C}$  are  $\mathbb{Z}$ -linearly independent, i.e.  $h_1/h_2 \notin \mathbb{Q}$ .
- *Case  $2Q$* . In this case, we let  $K = \bigcup_{j \geq 1} \mathbb{C}(x^{1/j})$ . We also use the notation  $\mathbb{C}(x^{1/*})$  for this field. We

let  $(\phi, \sigma)$  denote the pair of automorphisms of  $K$  defined by

$$\phi(x) = q_1x \quad \text{and} \quad \sigma(x) = q_2x,$$

where  $q_1$  and  $q_2$  are two *multiplicatively independent* nonzero complex numbers, i.e.  $q_1^{n_1} q_2^{n_2} = 1$  implies  $n_1 = n_2 = 0$  for all  $n_1, n_2 \in \mathbb{Z}$ . Furthermore, we also add the following mild restriction:  $q_1$  and  $q_2$  cannot both be algebraic numbers of modulus one, whose Galois conjugates all have modulus one<sup>11</sup>.

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<sup>11</sup>This assumption can be withdrawn by using a non-archimedean framework ([DSG21])



- *Case 2M.* In this case, we let  $K = \mathbb{C}(x^{1/*})$  and we let  $(\phi, \sigma)$  denote the pair of automorphisms of  $K$  defined by

$$\phi(x) = x^{p_1} \quad \text{and} \quad \sigma(x) = x^{p_2},$$

where  $p_1$  and  $p_2$  are two *multiplicatively independent* natural numbers.

In that context, Schaeffe and Singer proved the discrete analogue of Theorem 3.12.

**Theorem 3.14** (Theorem 13 in [SS19a]). *Let  $(K, \phi, \sigma)$  be as in case 2S, 2M or 2Q. Let  $A, B \in \text{Gl}_n(K)$  such that the systems  $\phi(Y) = AY$  and  $\sigma(Y) = BY$  are compatible in the sense that  $\phi(B)A = \sigma(A)B$  then  $\phi(Y) = AY$  and  $\sigma(Y) = BY$  are simultaneously gauge equivalent to difference systems with coefficients in  $k = \mathbb{C}$ .*

Analogously to Theorem 2.15, the compatibility of a linear difference system  $\phi(Y) = AY$  to some linear  $\sigma$ -system  $\sigma(Y) = BY$  is equivalent to the fact that its  $\sigma$ -Galois group is conjugated to a  $\sigma$ -constant group. Combining Theorem 2.14, Theorem 3.14 with Theorem 3.7 in [AS17], one can reason as in §3.2.1 to obtain the following result.

**Proposition 3.15** (Proposition 4.11 in [3]). *Let  $(K, \phi, \sigma)$  be as in case 2S, 2M or 2Q. Let  $A \in \text{Gl}_n(K)$  and assume that a  $\sigma$ -Picard-Vessiot extension for  $\phi(Y) = AY$  over  $K$  is a field. If the Galois group  $H$  of  $\phi(Y) = AY$  over  $K$  is a simple linear algebraic group then  $\sigma\text{-Gal}(L_A|K)$  coincides with the  $\sigma$ -algebraic group defined over  $\mathbb{C}$  associated to  $H$ .*

The parametrized Galois correspondence allows to compare the defining equations of the  $\sigma$ -Galois group and the algebraic relations satisfied by the solutions and their transforms with respect to  $\sigma$ . Via this correspondence, the above proposition shows that in case 2S, 2M or 2Q, the  $\sigma$ -algebraic relations between the solutions of a linear  $\phi$ -system with a simple Galois group are entirely determined by the algebraic relations among the solutions. For instance, for a linear difference system whose Galois group is  $\text{Sl}_n(\mathbb{C})$ , the ideal of  $\sigma$ -algebraic relations among the coefficients of a fundamental solution matrix  $U$  is generated by a relation of the form  $\det(U) = b$  for some  $b \in K$ .

### 3.3 Unipotent radicals of Galois groups ([13, 14, 19, 23])

In this section, we present some results on the computation of the Galois groups of extensions of completely reducible modules in Tannakian categories. We recall that an object of a neutral Tannakian category  $(\mathbf{T}, \omega)$  over an algebraically closed field  $k$  is completely reducible if it is a direct sum of finitely many irreducible objects in  $\mathbf{T}$ . By Tannakian equivalence, an object  $\mathcal{M}$  is completely reducible if and only if its Tannakian Galois group  $\text{Gal}(\mathcal{M}, \omega)$  is a reductive algebraic group. Now, if  $\mathcal{M}_1, \mathcal{M}_2$  are two completely reducible objects of  $\mathbf{T}$ , an extension  $\mathcal{U}$  of  $\mathcal{M}_1$  by  $\mathcal{M}_2$  is an exact sequence of the form

$$(3.5) \quad 0 \rightarrow \mathcal{M}_2 \rightarrow \mathcal{U} \rightarrow \mathcal{M}_1 \rightarrow 0,$$

in the category  $\mathbf{T}$ .

#### 3.3.1 Tannakian Galois groups

Extensions of completely reducible modules have been initially studied by Berman and Singer in [BS99] in the framework of linear differential equations where the authors produce an algorithm to compute the Galois group of a linear differential equation of the form  $L_1(L_2(y)) = 0$ . Their work can be reproduced in the more general framework of Tannakian categories as follows. The Tannakian Galois group  $\text{Gal}(\mathcal{U}, \omega)$  of  $\mathcal{U}$  is a semidirect product of  $\text{Gal}(\mathcal{M}_1 \oplus \mathcal{M}_2, \omega)$  by the unipotent radical  $W$  of  $\text{Gal}(\mathcal{U}, \omega)$ . The unipotent radical  $W$  of  $\text{Gal}(\mathcal{U}, \omega)$  can be characterized as follows. If  $\mathbf{1}$  denotes the unit object of  $\mathbf{T}$  and  $\underline{\text{Hom}}$  the internal Hom of  $\mathbf{T}$ , one can apply the exact functor  $\underline{\text{Hom}}(\mathcal{M}_1, -)$  to (3.5) and obtain

$$(3.6) \quad 0 \rightarrow \underline{\text{Hom}}(\mathcal{M}_1, \mathcal{M}_2) \rightarrow \underline{\text{Hom}}(\mathcal{M}_1, \mathcal{U}) \rightarrow \underline{\text{Hom}}(\mathcal{M}_1, \mathcal{M}_1) \rightarrow 0.$$

Pulling back (3.6) by the diagonal embedding  $d : \mathbf{1} \rightarrow \underline{\text{Hom}}(\mathcal{M}_1, \mathcal{M}_1), \lambda \mapsto \lambda id_{\mathcal{M}_1}$ , we obtain an exact sequence of the form

$$(3.7) \quad 0 \rightarrow \underline{\mathrm{Hom}}(\mathcal{M}_1, \mathcal{M}_2) \rightarrow \mathfrak{R}(\mathcal{U}) \rightarrow \mathbf{1} \rightarrow 0.$$

If the characteristic of  $k$  is zero, the Kummerian arguments of [Ber01] and [BS99] extends to the Tannakian framework and one can prove that  $W$  equals  $\omega(\mathcal{W}) \subset \mathrm{Hom}(\omega(\mathcal{M}_1), \omega(\mathcal{M}_2))$  where  $\mathcal{W}$  is the smallest sub-object of  $\underline{\mathrm{Hom}}(\mathcal{M}_1, \mathcal{M}_2)$  such that the extension  $\mathfrak{R}(\mathcal{U})/\mathcal{W}$  is a split extension of  $\mathbf{1}$  by  $\underline{\mathrm{Hom}}(\mathcal{M}_1, \mathcal{M}_2)/\mathcal{W}$  (see [19, page 2]). In positive characteristic, the situation is more subtle since the algebraic subgroups of vector groups are given by linear polynomials in the Frobenius and are not necessarily  $k$ -vector spaces. However, one can still prove the following.

**Theorem 3.16** (Theorem 2 in [19]). *Let  $\mathcal{Y}$  be an object of  $\mathbf{T}$ , and let  $\mathcal{U}$  be an extension of  $\mathbf{1}$  by  $\mathcal{Y}$ . Assume that*

1. every  $\mathrm{Gal}(\mathcal{Y}, \omega)$ -module is completely reducible,
2. the center of  $\mathrm{Gal}(\mathcal{Y}, \omega)$  contains  $\mathbf{G}_m$ ,
3. the action of  $\mathbf{G}_m$  on  $\omega(\mathcal{Y})$  is isotypic<sup>12</sup>,
4.  $\mathrm{Gal}(\mathcal{U}, \omega)$  is reduced.

*Then, there exists a smallest sub-object  $\mathcal{V}$  of  $\mathcal{Y}$  such that  $\mathcal{U}/\mathcal{V}$  is a trivial extension of  $\mathbf{1}$  by  $\mathcal{Y}/\mathcal{V}$ . The unipotent radical of the Galois group  $G_{\mathcal{U}}$  is then equal to  $\omega(\mathcal{V})$ .*

Theorem 3.16 has an almost straightforward corollary

**Corollary 3.17** (Corollary 1 in [19]). *Let  $\mathcal{Y}$  be an object of  $\mathbf{T}$ . Let  $\Delta$  be the ring  $\mathrm{End}(\mathcal{Y})$ , and let  $\mathcal{E}_1, \dots, \mathcal{E}_n$  be extensions of  $\mathbf{1}$  by  $\mathcal{Y}$ . Assume that*

1. every  $\mathrm{Gal}(\mathcal{Y}, \omega)$ -module is completely reducible,
2. the center of  $\mathrm{Gal}(\mathcal{Y}, \omega)$  contains  $\mathbf{G}_m$ ,
3. the action of  $\mathbf{G}_m$  on  $\omega(\mathcal{Y})$  is isotypic
4.  $\mathrm{Gal}(\mathcal{E}_1, \omega), \dots, \mathrm{Gal}(\mathcal{E}_n, \omega)$  are reduced.

*Then, if  $\mathcal{E}_1, \dots, \mathcal{E}_n$  are  $\Delta$ -linearly independent in  $\mathrm{Ext}^1(\mathbf{1}, \mathcal{Y})$ , the unipotent radical of  $\mathrm{Gal}(\mathcal{E}_1 \oplus \dots \oplus \mathcal{E}_n, \omega)$  is isomorphic to  $\omega(\mathcal{Y})^n$ .*

As an application of Corollary 3.17, one can compute the Galois group attached to the  $t$ -motive of the Carlitz logarithm and give an alternative proof of the algebraic independence of Carlitz logarithms initially obtained by Papanikolas ([Pap08]). More recently, Green and Ngo Dac determine the algebraic relations among Goss's zeta values by applying Corollary 3.17 (see [GND20]).

### 3.3.2 Parametrized Tannakian Galois groups

The article [14] is devoted to the computation of the parametrized Tannakian Galois group of an extension  $\mathcal{U}$  of two completely reducible objects  $\mathcal{M}_1$  and  $\mathcal{M}_2$ . The parametrized Tannakian category studied in [14] is the category of  $\partial$ -differential modules with coefficients in a  $(\partial, \delta)$ -field  $K$  where  $\partial$  and  $\delta$  are two commuting derivations (see [14, §2.3.1]). However, most of the results of this paper are still valid in the more general context of a parametrized Tannakian category  $(\mathbf{T}, F, \omega)$  over a  $\delta$ -closed field  $k$ . Theorem 3.18 below generalizes the result of §3.3.1 to the parametrized context.

**Theorem 3.18.** *If  $\mathcal{M}_1, \mathcal{M}_2$  are completely reducible in  $\mathbf{T}$  and if  $\mathcal{U}$  is an object of  $\mathbf{T}$ , extension of  $\mathcal{M}_1$  by  $\mathcal{M}_2$ , then*

1.  $\delta\text{-Gal}(\mathcal{U}, \omega)$  is an extension of  $\delta\text{-Gal}(\mathcal{M}_1 \oplus \mathcal{M}_2)$  by a  $\delta$ -algebraic subgroup  $W \subset \mathrm{Hom}(\omega(\mathcal{M}_1), \omega(\mathcal{M}_2))$ .

---

<sup>12</sup>We recall that the action of a group  $G$  on a module  $V$  is isotypic if the module  $V$  is the direct sum of irreducible isomorphic  $G$ -modules.

2.  $W$  is stable under the action of  $\delta\text{-Gal}(\mathcal{M}_1 \oplus \mathcal{M}_2)$  on  $\text{Hom}(\omega(\mathcal{M}_1), \omega(\mathcal{M}_2))$  given by

$$g * \phi = g\phi(g^{-1}) \quad \text{for any } (g, \phi) \in \delta\text{-Gal}(\mathcal{M}_1 \oplus \mathcal{M}_2) \times \text{Hom}(\omega(\mathcal{M}_1), \omega(\mathcal{M}_2)).$$

This theorem identifies the unipotent radical of  $\delta\text{-Gal}(\mathcal{U}, \omega)$  with a  $\delta$ -algebraic subgroup of the vector group  $\text{Hom}(\omega(\mathcal{M}_1), \omega(\mathcal{M}_2))$ . By [Cas72, Proposition 11], such a  $\delta$ -algebraic subgroup is defined by linear homogeneous differential polynomials. In order to illustrate this situation, let me detail a baby example where  $\mathcal{U}$  is an extension of the trivial object  $\mathbf{1}$  by an object  $\mathcal{L}$  of dimension 1 (see [21, Lemma 3.6]). In that case,  $W$  is a  $\delta$ -algebraic subgroup of the one dimensional  $k$ -vector space  $\omega(\mathcal{L})$  and is therefore of the form  $W = \{\beta | L(\beta) = 0\}$  for some linear homogeneous  $\delta$ -polynomial  $L$  in  $k[\delta]$ . The group  $\delta\text{-Gal}(\mathcal{L} \oplus \mathbf{1}) = \delta\text{-Gal}(\mathcal{L})$  is a  $\delta$ -algebraic subgroup of the multiplicative group  $\mathbf{G}_m$  and acts on  $W$ , that is, on the set of solutions of the linear differential equation  $L$ . In that case, one has a dichotomy: either  $\delta\text{-Gal}(\mathcal{L}) \subset \mathbf{G}_m(C)$  where  $C = k^\delta = \{c \in k | \delta(c) = 0\}$  or  $W$  is an algebraic subgroup of  $\omega(\mathcal{L})$ .

One of the main results of [14] is to prove that this dichotomy holds in general. Indeed, one says that a representation  $\rho : H \rightarrow \text{Gl}(V)$  of a  $\delta$ -algebraic group  $H$  defined over  $k$  on a  $k$ -vector space  $V$  is *conjugate to constant* if there exists  $h \in \text{Gl}_n(k)$  such that  $h\rho(H)h^{-1} \subset \text{Gl}_n(C)$ . Then, one can prove the following result:

**Proposition 3.19** (Prop.3.21 in [14]). *Let  $\mathcal{M}_1, \mathcal{M}_2$  be completely reducible objects in  $\mathbf{T}$  and if  $\mathcal{U}$  be an extension in  $\mathbf{T}$  of  $\mathcal{M}_1$  by  $\mathcal{M}_2$ . Assume that  $\delta\text{-Gal}(\mathcal{M}_1 \oplus \mathcal{M}_2, \omega)$  is connected and purely non-constant, that is, there is no non-zero sub-object  $\mathcal{N}$  of  $\text{Hom}(\omega(\mathcal{M}_1), \omega(\mathcal{M}_2))$  such that the representation  $\rho$  of  $\delta\text{-Gal}(\mathcal{M}_1 \oplus \mathcal{M}_2, \omega)$  on  $\omega(\mathcal{N})$  is conjugate to constant, then the unipotent radical of  $\delta\text{-Gal}(\mathcal{U}, \omega)$  is a vector group.*

Proposition 3.21 in [14] is valid for any reductive group but proved in the particular setting of linear differential equations with a parameter. It is not completely obvious that one can get rid of the connectedness assumption in the more general framework of parametrized Tannakian categories. When the group  $\delta\text{-Gal}(\mathcal{M}_1 \oplus \mathcal{M}_2)$  is not necessarily purely non-constant, one can decompose the completely reducible object  $\mathcal{M}_1 \oplus \mathcal{M}_2$  into a purely non-constant sub-object and a constant sub-object and decompose the unipotent radical accordingly. The constant case can be computed thanks to an algorithm of Singer, Ovchinnikov and Minchenko ([MOS14]).

If the parametrized Tannakian category  $\mathbf{T}$  is the category  $\text{Diff}(K, \sigma)$  for the  $(\delta, \sigma)$ -fields  $K$  corresponding to the cases  $\delta\mathbf{S}$ ,  $\delta\mathbf{Q}$  and  $\delta\mathbf{M}$ , the work of Arreche and Singer shows that if the identity component of the classical Galois group  $\text{Gal}(\mathcal{M}_1 \oplus \mathcal{M}_2)$  is semisimple then the  $\delta$ -Galois group is purely non-constant. (see Proposition 3.13). Under this assumption, Proposition 3.19 will prove that the  $\delta$ -Galois group of a difference module extension of  $\mathcal{M}_1$  by  $\mathcal{M}_2$  is also an algebraic group.

### 3.3.3 Bounded automorphisms of fields with operators

In [13], we studied the bounded automorphisms of fields with operators. We generalize the results of Lascar ([Las92]) for algebraically closed fields and Konnerth [Kon02] for differentially closed fields in characteristic zero by adapting some arguments due to Ziegler. The formalism of fields with operators has been developed by Moosa and Scanlon in [MS14]. A *field with operators* over a base field  $\mathbb{F} \subset K$  is a structure

$$(K, 0, 1, +, -, \cdot, \{\lambda\}_{\lambda \in \mathbb{F}}, F_1, \dots, F_n)$$

such that:

1. the operators  $F_1, \dots, F_n$  are  $\mathbb{F}$ -linear and satisfy for all  $x$  and  $y$  in  $K$ ,

$$F_k(xy) = \sum_{0 \leq i, j \leq n} a_{i, j}^k F_i(x) F_j(y),$$

for some constants  $\{a_{i, j}^k\}_{0 \leq i, j, k \leq n}$  in  $\mathbb{F}$  (with  $F_0$  the identity) ;

2. the  $\mathbb{F}$ -vector space  $\mathbb{F}\epsilon_0 \oplus \dots \oplus \mathbb{F}\epsilon_n$  is a commutative  $\mathbb{F}$ -algebra  $D(\mathbb{F})$ , with

$$\epsilon_i \star \epsilon_j = \sum_{0 \leq k \leq n} a_{i, j}^k \epsilon_k.$$

The operators on the fields considered by Moosa and Scanlon are free in the sense that they do not satisfy any relation ([MS14, Remark 3.8]). In [13] we did not impose this condition for the operators which allowed us to consider differential fields with  $n$  commuting derivations and also the  $(\phi, \delta)$ -fields. Note that for a field with operators, the algebra  $D(\mathbb{F})$  is isomorphic to a product of local  $\mathbb{F}$ -algebras.

We will now list some hypothesis on the field of operators  $K$  which will allow us to study and define its bounded automorphisms. Following [MS14, Assumption 4.1], we assume that the residue fields of the local algebras of  $D(\mathbb{F})$  coincide with  $\mathbb{F}$  (*Hypothesis 1*). Associated to the operators  $F_1, \dots, F_n$ , one find a set of non-trivial field endomorphisms  $\sigma_1, \dots, \sigma_t$  of  $K$  (see [MS14, §4.1]). We shall assume that all these endomorphisms are automorphisms (*Hypothesis 2*). We denote by  $\mathcal{L}$  the language which extends the language of rings by constants for the elements of  $\mathbb{F}$  and by symbols  $F_1, \dots, F_n, \sigma_1, \dots, \sigma_t$  for the operators and their associated automorphisms. We assume that  $K$  is separably closed, sufficiently saturated and homogeneous with respect to the language  $\mathcal{L}$  (*Hypothesis 3*). *Hypothesis 4* requires that the model theoretic algebraic closure of a subset  $A \subset K$  coincides with the intersection of the algebraic closure of the  $\mathcal{L}$ -substructure  $\langle A \rangle$  generated by  $A$  and  $K$ . Moreover, two tuples  $a$  and  $b$  have the same type over  $k = \langle k \rangle$  if and only if there exists a  $\mathcal{L}$ -isomorphism between  $\langle k(a) \rangle^{\text{alg}} \cap K$  and  $\langle k(b) \rangle^{\text{alg}} \cap K$  that sends  $a$  to  $b$  and fixes pointwise  $k$ . Given  $A, B, C$  three subsets of  $K$ , we say that  $A$  is independent from  $B$  over  $C$  if the extensions  $\langle A \cup C \rangle^{\text{alg}}$  and  $\langle B \cup C \rangle^{\text{alg}}$  are linearly disjoint over  $\langle C \rangle^{\text{alg}}$ . This independence relation is symmetric, transitive, of finite and local character and invariant by automorphisms. *Hypothesis 5* imposes that the independence relation also satisfies the extension property so that, by Kim-Pillay, the  $\mathcal{L}$ -theory of  $K$  is simple and that the independence relation corresponds to the non-forking. Moreover, we require that the following holds: given a subfield  $k = \langle k \rangle^{\text{alg}}$  and some tuples  $a$  and  $b$  with  $a \equiv_k b$ , for all subsets  $C$  and  $D$  containing  $k$  such that  $C \downarrow_k D$ , if  $a \downarrow_k C$  and  $b \downarrow_k D$ , then there exists  $e \downarrow_k C \cup D$  such that  $e \equiv_C a$  and  $e \equiv_D b$ . Finally, we require that the theory  $T$  satisfies the following condition: If  $B = \langle B \rangle^{\text{alg}}$  and  $\{a_i\}_{i < \omega}$  is a Morley sequence of  $\text{tp}(a/B)$ , then

$$a \downarrow_X B,$$

where  $X = B \cap \langle \{a_i\}_{i < \omega} \rangle^{\text{alg}}$  (*Hypothesis 6*).

*Hypothesis 1-6* are satisfied by algebraically closed fields, differentially closed fields in characteristic zero with  $n$ -commuting derivations [McG00, Corollary 3.3.2], generic difference fields in any characteristic and differentially closed fields in characteristic zero with a generic automorphism [Med07, Proposition 3.36]. For fields with free operators in characteristic zero, this is the content of [MS14, Theorem 5.12 et Claim 6.17].

For a fixed generic type  $p$  of  $K$  over  $\emptyset$ , one defines the  $p$ -closure  $\text{cl}_p(D)$  of a set  $D$  as the collection of elements  $x$  in  $K$  such that for any  $D \subset D_1$  and any realization  $a$  of  $p$ , generic over  $D_1$ , one has  $a \downarrow_{D_1} x$ . This closure operator doesn't depend from the chosen generic type so that we denote it simply by  $\text{cl}$ . An automorphism  $\tau$  of  $K$  is *bounded* if there exists a finite set  $D$  such that, for all element  $a$  generic over  $D$ , the element  $\tau(a)$  belongs to  $\text{cl}(D, a)$ . This definition extends the original definition of Lascar ([Las92]) for  $K$  an algebraically closed field and the one of Konnerth for  $DCF_0$  ([Kon02]).

The following theorem extends the results obtained by Lascar and Konnerth ([Las92, Théorème 15] and [Kon02, Proposition 2.9]).

**Theorem 3.20** (Théorème 3.1 in [13]). *Let  $\mathbb{F}$  be a base field and  $(K, 0, 1, +, -, \cdot, F_1, \dots, F_n)$  be a field with operators satisfying Hypothesis 1 to 6 above. Then, any bounded automorphism of  $K$  equals a product of entire power of the associated automorphisms (with the Frobenius in positive characteristic).*

The proof of this Theorem follows a proof due to Ziegler and relies heavily on a characterization obtained by Ziegler of the stabilizers of independents elements in stable abelian groups ([Zie06, Theorem 1]) and which was extended to type-definable groups in simple theories by Blossier, Martin-Pizarro and Wagner ([BMW16, Lemme 1.2 et Remarque 1.3]). For  $\tau$  a bounded automorphism and  $a$  in  $K$  sufficiently generic, one can show using Ziegler's result and the fact that  $\tau$  is bounded that the element  $(a, \tau(a))$  belongs to a translate of a type definable subgroup  $H$  of  $\mathbf{G}_a^2(K)$  on one hand and to a translate of a type definable subgroup  $G$  of  $\mathbf{G}_m^2$ . This yields a  $\mathbf{G}_m$ -action on  $H$  entirely similar to the one observed in §3.3.2 and one can conclude that the defining equations of  $G$  must be monomial in the associated automorphisms which allows to conclude.

## 4 Arithmetic aspects of difference equations ([3, 7, 9, 10, 12])

### 4.1 Arithmetic description of the Galois group of a $q$ -difference system ([10])

The Galois theory of difference equations has witnessed a major evolution in the last two decades. In the particular case of  $q$ -difference equations, authors have introduced several different Galois theories. In [10], we consider an arithmetic approach to the Galois theory of  $q$ -difference equations and we use it to establish an arithmetical description of some of the Galois groups attached to  $q$ -difference systems.

Let  $q$  be a non-zero element of the field  $\mathbb{C}$  of complex numbers. A (linear)  $q$ -difference system is a functional equation of the form

$$(4.1) \quad Y(qx) = A(x)Y(x), \text{ with } A(x) \in \text{Gl}_\nu(\mathbb{C}(x)),$$

The *leitmotif* of [10] is the Galoisian properties of the so-called dynamics of the system (4.1), namely the set of maps obtained by iteration of the maps  $(x, X) \mapsto (qx, A(x)X)$  defined over  $U \times \mathbb{C}^\nu$ , where  $U$  is an open subset of  $\mathbb{P}^1(\mathbb{C})$ .

Without loss of generalities one can assume that  $A(x) \in \text{Gl}_\nu(K(x))$  where  $K$  is a finitely generated  $\mathbb{Q}$ -subalgebra of  $\mathbb{C}$  and  $q$  an element of  $K$ . Theorem 5.1 in [10] proves that the algebraic nature of the solutions of the  $q$ -difference system (4.1) is entirely determined by the specialization of certain subsequences of the dynamics  $(A(q^{n-1}x) \dots A(x))_{n \in \mathbb{N}}$ , that are called the curvatures of the  $q$ -difference system. This theorem extends the main result of [DV02], in which the assumption that  $K$  is a number field, and hence that  $q$  is algebraic, is crucial.

In the particular case  $K = \mathbb{Q}(q)$  and under the assumption that  $q$  is a transcendental number, Theorem 5.1 in [10] becomes the following statement.

**Theorem 4.1.** *Let  $A(x) \in \text{Gl}_\nu(\mathbb{Q}(q, x))$ . The  $q$ -difference system  $Y(qx) = A(x)Y(x)$  admits a full set of solutions in  $\mathbb{Q}(q, x)$  if and only if for almost all  $n \in \mathbb{N}$  there exists an  $n$ -th primitive root of unity  $\zeta_n$  such that  $A(q^{n-1}x) \dots A(x)$  specializes to the identity matrix at  $q = \zeta_n$ .*

Relying on the above rationality criteria, one is able to provide an arithmetic set of generators for certain Galois groups attached to  $q$ -difference systems. Indeed, as in §1.4, one attaches to a  $q$ -difference system  $Y(qx) = A(x)Y(x)$ , with  $A(x) \in \text{Gl}_\nu(K(x))$  a  $q$ -difference module  $\mathcal{M}$ . Since  $K(x)$  is a  $(\phi, \delta)$ -field with  $\delta = x \frac{d}{dx}$ , one can define the parametrized intrinsic Galois group  $\delta\text{-Gal}(\mathcal{M}, \eta)$  as the group of differential tensor automorphism of the forgetful functor  $\eta$  from the differential Tannakian category  $\langle \mathcal{M} \rangle_\delta$  generated by  $\mathcal{M}$  to the category of finite dimensional  $K(x)$ -vector spaces. It is a  $\delta$ -algebraic subgroup of the group  $\text{Gl}_{K(x)}(\mathcal{M})$  of linear  $K(x)$ -automorphisms of  $\mathcal{M}$ . Roughly, this differential algebraic group scheme is linked to the differential algebraic relations satisfied by the entries of  $A(x)$ , in the sense that it only relies on constructions of differential algebra of the  $q$ -difference module  $\mathcal{M}$ , and therefore on the associated matrix constructions of  $A(x)$  and its dynamics (see §2.1.3). The advantages of considering the parametrized intrinsic Galois group are its definition in terms of the coefficients of the  $q$ -difference module and its arithmetic description. Indeed, it is the smallest differential group scheme defined over  $K(x)$  that contains the curvatures. This statement is formalized in [10, §7.3]. In the particular case of  $K = \mathbb{Q}(q)$  and  $q$  transcendental over  $\mathbb{Q}$ , one finds the following description. For a  $q$ -difference module  $\mathcal{M}$  of dimension  $\nu$  over  $\mathbb{Q}(q)(x)$ , one can find a  $q$ -difference algebra  $\mathcal{A}$  of the form  $\mathbb{Q}[q] \left[ x, \frac{1}{P(x)}, \frac{1}{P(qx)}, \dots \right]$  for some nonzero polynomial  $P \in \mathbb{Q}[q, x]$  and a  $\Phi$ -stable  $\mathcal{A}$ -lattice  $M_{\mathcal{A}}$  of  $\mathcal{M}$ . Thus, if  $\phi_n$  denotes the  $n$ -th cyclotomic polynomial, one can consider the  $\mathcal{A}/(\phi_n)$ -linear operator  $\Phi^n : M_{\mathcal{A}} \otimes \mathcal{A}/(\phi_n) \rightarrow M_{\mathcal{A}} \otimes \mathcal{A}/(\phi_n)$  which corresponds to the  $n$ -th curvature of  $\mathcal{M}$ . The differential version of Chevalley's theorem (cf [Cas72, Proposition 14], [MO11, Theorem 5.1]) implies that any  $\delta$ -algebraic subgroup  $G \subset \text{Gl}_{K(x)}(\mathcal{M})$  can be defined as the stabilizer of some one dimensional  $K(x)$ -vector space  $\mathcal{L}$  contained in a construction of differential algebra  $\mathcal{W}$  of  $\mathcal{M}$  (see §2.1.3). Up to enlarging  $\mathcal{A}$ , one finds an  $\mathcal{A}$ -lattice  $L_{\mathcal{A}}$  of  $\mathcal{L}$  and an  $\mathcal{A}$ -lattice  $W_{\mathcal{A}}$  of  $\mathcal{W}$ . One says that the  $\delta$ -algebraic group  $G = \text{Stab}(L_{K(x)})$  over  $K(x)$  contains the curvatures  $\Phi^n$  modulo  $\phi_n$ , for almost all  $n \in \mathbb{N}$ , if for almost all (i.e. for almost all and at least one)  $n \in \mathbb{N}$  the operator  $\Phi^n$  stabilizes  $L_{\mathcal{A}} \otimes_{\mathcal{A}} \mathcal{A}/(\phi_n)$  inside  $W_{\mathcal{A}} \otimes_{\mathcal{A}} \mathcal{A}/(\phi_n)$ . In the particular case of  $K = \mathbb{Q}(q)$  and  $q$  transcendental over  $\mathbb{Q}$ , Theorem 7.13 in [10] becomes the following statement.

**Theorem 4.2.** *The parametrized intrinsic Galois group  $\delta\text{-Gal}(\mathcal{M}, \eta)$  is the smallest  $\delta$ -algebraic subgroup of  $\text{Gl}_{K(x)}(\mathcal{M})$  that contains the curvatures  $\Phi^n$  for almost all  $n \in \mathbb{N}$ .*

Theorem 4.2 is a  $q$ -difference analogue of the conjectural description obtained by Katz in [Kat82] for the Lie algebra of the intrinsic Galois group of a linear differential system.

Finally, this arithmetic description of the parametrized intrinsic Galois group allows to understand the connection between the linear and non-linear Galois theory of  $q$ -difference systems. In [Gra], A. Granier introduced a Galois  $D$ -groupoid for non-linear  $q$ -difference equations, in the spirit of Malgrange's work. If  $Y(qx) = F(x, Y(x))$  is a (non-linear)  $q$ -difference system with  $F(x, X) \in \mathbb{C}(x, X)^\nu$ . Denoting by  $M$  the complex analytic variety  $\mathbb{P}^1(\mathbb{C}) \times \mathbb{C}^\nu$ , one can consider the subgroupoid of  $\text{Aut}(M)$  generated by the germs of the application  $(x, X) \mapsto (qx, F(x, X))$  at any point of  $M$  where it is well defined and invertible. We denote this subgroupoid by  $\text{Dyn}(F)$ . The Galois  $D$ -groupoid of the  $q$ -difference system  $Y(qx) = F(x, Y(x))$  is the  $D$ -envelop of  $\text{Dyn}(F)$ , that is the *intersection* of the  $D$ -groupoids on  $M$  whose set of solutions contains  $\text{Dyn}(F)$ .

Via the curvature characterization of the parametrized intrinsic Galois group, one can prove that the Malgrange-Granier  $D$ -groupoid generalizes the parametrized intrinsic Galois group to the non-linear case (Corollary 9.9 of [10]). Thanks to this comparison result, one is able to compare the Malgrange-Granier  $D$ -groupoid to the parametrized Galois group of Hardouin-Singer. This answered a question of Malgrange ([Mal09, page 2]) on the relation among  $D$ -groupoids and Kolchin's differential algebraic groups.

## 4.2 Special functions on the projective line ([3, 7, 9, 12])

### 4.2.1 Differential transcendence

As remarked by Hilbert in [Hil02], the class of differentially algebraic analytic functions is too poor since it misses many classical functions arising from number theory such as the zeta function  $\zeta(x)$  (see [Sta02]) or the Gamma function (see [Höl87]). Though differential algebraic real analytic functions corresponds to the class of real functions computable in polynomial time (see [BGP17]) which can be seen as the output of Shannon's general-method analog computer by Pour-El's theorem (see [Pe74]), the class of differentially transcendental functions contains any *universal* function  $f$ , where universal means that for any compact set  $K$ , any  $\epsilon > 0$  and any entire function  $g$ , there exists a  $t \in \mathbb{C}$  such that  $|f(z - t) - g(z)| < \epsilon$  for all  $z \in K$  (see [Rub89, page 6]). As noticed by Hilbert in his survey, the investigation of the differential transcendence of a special function such as the zeta function relies heavily on the functional equation satisfied by this function.

In this section, we will focus on the study of the differential transcendence of analytic functions, solutions of linear difference equations over the projective line or equivalently of systems of the form  $\phi(Y) = AY$  where  $A \in (k(x))^{n \times n}$  for  $k$  an algebraically closed field of characteristic zero and pairs of operators  $\delta$  and  $\phi$  acting on  $k(x)$ , which are defined as follows:

- Case  $\delta\mathcal{S}$  : the derivation  $\delta = \frac{d}{dx}$  and  $\phi$  the shift operator defined by  $\phi(x) = x + 1$ ,
- Case  $\delta\mathcal{Q}$  : the derivation  $\delta = x \frac{d}{dx}$  and  $\phi$  the  $q$ -difference operator defined by  $\phi(x) = qx$  for some  $q \in k^*$  not a root of unity,
- Case  $\delta\mathcal{M}$  : the derivation  $\delta = x \frac{d}{dx}$  and  $\phi$  the Mahler operator defined by  $\phi(x) = x^p$  for some integer  $p \geq 2$ .

Since any Moebius transformation of infinite order is of one of the first two types, one can therefore reduce the study of the differential transcendence of any special function satisfying a linear difference equation over the projective line to one of the cases above. The generating function  $B(x)$  of the Bell numbers as well as many related series such as the one associated to the Uppuluri-Carpenter numbers, satisfy functional equations of the form  $B(\frac{x}{x+1}) = a(x)B(x) + b(x)$  (see [Kla03]). The change of variable  $x = \frac{1}{t}$  allows to transform the latter equation into  $C(t+1) = a(1/t)C(t) + b(1/t)$  where  $C(t) = B(\frac{1}{t})$  (see [BDVR] for the differential transcendence of an even larger class of functions).

We shall address two kind of questions. The first one is related to the direct problem in the parametrized Galois theory developed in [21] and devoted to unravel the parametrized Galois correspondence. One would like to describe completely the defining equations of the parametrized Galois groups and thereby describe the closure of  $k(x)$  with respect to  $\delta$  in each of the above cases. The second question focuses on certain kind of solutions of the linear difference system characterized by their analytic

properties. One may therefore restrict our study to power series solutions or to meromorphic functions over  $\mathbb{C}$  or some smaller open subset of  $\mathbb{C}$ . For each of these classes, one could then try to characterize the differentially algebraic elements among this class. Of course, these two questions are interrelated but one has to keep in mind that the first is far more demanding.

Among the class of differentially algebraic functions, one finds the holonomic functions, that is, the ones satisfying linear differential equations. In [Ram92a] for the case  $\delta Q$  and in [Béz92] for the case  $\delta M$ , Ramis and Bezin proved that the only holonomic formal series satisfying a linear  $q$ -difference or Mahler equation over  $\mathbb{C}(x)$  were rational fractions. This result has been generalized by Schaefer and Singer who proved the following result:

**Theorem 4.3** (Cor. 3 and Cor. 5 in [SS19a]). *Consider the operators  $\phi$  and  $\delta$  acting on  $\mathbb{C}(x)$  as in case  $\delta Q$ ,  $\delta M$  or  $\delta S$  with  $k = \mathbb{C}$ . If  $f$  is*

- *a meromorphic function on the Riemann surface of the logarithm over  $\mathbb{C}^*$  in cases  $\delta Q$  and  $\delta M$  ;*
- *a meromorphic function on some horizontal strip  $\{x \in \mathbb{C} \mid \text{Im}(x) \in ]a; b[ \}$ ,*

*and  $f$  satisfies a linear  $\phi$ -equation over  $\mathbb{C}(x)$  then  $f$  is of the following form*

- *Cases  $\delta Q$  and  $\delta M$  :  $f(x) = \sum_{i,j=1}^t r_{i,j}(x)x^{\alpha_i} \log(x)^j$  for  $r_{i,j} \in \mathbb{C}(x), \alpha_i \in \mathbb{C}$ ;*
- *Case  $\delta S$  :  $f(x) = \sum_{i=1}^t r_i(x)e^{\alpha_i x}$  with  $r_i(x) \in \mathbb{C}(x), \alpha_i \in \mathbb{C}$ .*

The analytic conditions on the function  $f$  cannot be avoided since they guarantee that the  $\phi$ -constants in the  $\mathbb{C}(x)$ -vector space generated by  $f$  and its transforms with respect to  $\phi$  are still complex numbers and not for instance elliptic functions in the case  $\delta Q$ . Theorem 4.3 is a direct consequence of Proposition 3.12.

The first results in differential transcendence concerned rank one equation such as the one satisfied by the Gamma function  $\Gamma(x+1) = x\Gamma(x)$  or equivalently of its logarithmic derivative  $\frac{\Gamma'}{\Gamma}(x+1) = \frac{\Gamma'}{\Gamma}(x) + \frac{1}{x}$ . Hölder's strategy to prove the differential transcendence of  $\Gamma$  consisted in performing a differential polynomial elimination among the putative differential algebraic relations satisfied by  $\frac{\Gamma'}{\Gamma}$  to show that the difference equation forces one of these putative relation to be a linear differential polynomial. From the point of view of parametrized Galois theory, this elimination process can be interpreted as the fact that the parametrized Galois group of an equation of the form  $\phi(y) = y + b$  is a  $\delta$ -algebraic subgroup of the additive group and thereby given by linear differential polynomial equations. For equation of the form  $\phi(y) = ay$  or  $\phi(y) = y + b$  for  $(a, b) \in K^* \times K$  with  $K$  a  $(\phi, \delta)$ -field, the complete classification of  $\delta$ -algebraic subgroups of the multiplicative or the additive group allows to interpret the differential algebraicity of the solutions in terms of completely explicit differential algebraic relations for  $a$  and  $b$  via the parametrized Galois correspondence (see Proposition 3.2 and Corollary 3.3). Proposition 3.8 in [21] gives an almost complete criteria for the nonhomogeneous rank one difference equations as follows:

**Proposition 4.4.** *Let  $K$  be a  $(\phi, \delta)$ -field with  $k = K^\phi$  a  $\delta$ -closed field and let  $L_A$  be a  $\delta$ -Picard-Vessiot extension over  $K$  for the equation  $\phi(y) = ay + b$  with  $(a, b) \in K^* \times K$ . Let  $z \in L_A$  and assume that  $z \notin K$ . Then,*

- *If there is no nonzero linear difference operator  $L \in k[\delta]$  and  $g \in K$  such that  $L(\frac{\delta(a)}{a}) = \phi(g) - g$  then  $z$  is differentially transcendental over  $K$ .*
- *If  $\delta(a) = 0$  then  $z$  is differentially algebraic over  $K$  if and only if there is a nonzero linear difference operator  $L \in k[\delta]$  and  $h \in K$  such that  $\phi(h) - ah = L(b)$ .*

For the cases  $\delta S$  and  $\delta Q$ , Proposition 3.9 and 3.10 in [21] allows to give complete criteria for the differential algebraicity of solutions of nonhomogeneous rank one difference equations. For instance, one finds the following criteria.

**Proposition 4.5.** *Let  $(a, b) \in \mathbb{C}(x)^* \times \mathbb{C}(x)$  with  $a$  in standard form (see §6.1 in [21]). Let  $z$  be a meromorphic function on  $\mathbb{C}$  such that  $z(x+1) = az(x) + b$  and assume that  $z \notin k(x)$  where  $k$  is the field of one periodic meromorphic functions. Then,*

- If  $a \notin \mathbb{C}$  then  $z$  is differentially transcendental over  $k(x)$ .
- If  $a \in \mathbb{C}$  then  $z$  is differentially algebraic over  $k(x)$  if and only if  $b = h(x+1) - ah(x)$  for some  $h \in \mathbb{C}(x)$ .

Finally, Propositions 3.9 and 3.10 in [21] for the case  $\delta Q$  and  $\delta M$  and Theorem 5.2 in [Ran92] reduce to an unconditional statement when  $z$  is assumed to be a formal power series. More precisely, if one denotes by  $F_0$  the field of Laurent series  $\mathbb{C}((x^{-1}))$  in the case  $\delta S$  and the field  $\cup_{j \leq 1} \mathbb{C}((x^{\frac{1}{j}}))$  of Puiseux series in the cases  $\delta M$  and  $\delta Q$ , the following holds.

**Proposition 4.6** (Proposition 5.1 in [7]). *Let  $z \in F_0$  be solution of an equation of the form  $\phi(y) = ay + b$  with  $a, b \in \mathbb{C}(x)$  in cases  $Q, S$  and  $M$ . Then  $f$  is differentially transcendental or belongs to  $\cup_{j \in \mathbb{N}} \mathbb{C}(x^{\frac{1}{j}})$ .*

In [7], we proved that Proposition 4.6 is valid for power series solution of linear difference equations of arbitrary order.

**Theorem 4.7** (Theorem 1.2 in [7]). *Let  $z \in F_0$  be a solution of the  $\phi$ -linear difference equation of order  $n$  with coefficients in  $K$ . Then, either  $z \in \cup_{j \in \mathbb{N}} \mathbb{C}(x^{\frac{1}{j}})$  or  $z$  is differentially transcendental over  $\mathbb{C}(x)$ .*

The proof of the above theorem relies on an induction. The statement of order  $n$  is that a differentially algebraic solution  $z \in F_0$  on a linear difference operator of order  $n$  must be in  $\cup_{j \in \mathbb{N}} \mathbb{C}(x^{\frac{1}{j}})$ . The steps of this induction are as follows:

- the case  $n = 1$  is Proposition 4.6,
- the induction step divides in two cases. Either the linear difference operator is irreducible in which case one can show by an irreducibility argument that the differential dimension of the parametrized Galois group of the linear difference operator must be zero if the solution  $z$  is differentially algebraic. Reducing to the case where the classical Galois group is simple, Proposition 3.13 implies that the differential dimension of the parametrized Galois group equals the dimension of the Galois group, which must therefore be also zero. A contradiction with the fact that the Galois group is simple and connected. If the linear difference operator is reducible then Proposition 4.5 allows to conclude via the inductive hypothesis.

Theorem 4.7 is an unconditional statement for the differential transcendence of formal power series, which completes and in fact relies on a whole variety of partial results: for equation of order one ([Höl87, Moo96, Mah30, Nis84, Ran92, Ish98, 23, 21, Ngu12]), for equations whose Galois group is simple or semi-simple ([21, 9, 12, AS17, ADR21]), for nonholonomicity (see for instance [Ram92a, BG93, Béz94, SS19a]). Though, Theorem 4.7 is valid for formal power series solution of linear difference equations on the base field  $\mathbb{C}(x)$ , the architecture of its proof as well as many of its central ingredients are essentially Galois theoretic. In fact, only Schaefer-Singer's result and the characterization of differentially algebraic solutions of rank one equations depend on the base field  $\mathbb{C}(x)$  and on the ring of solution  $F_0$ . In §6.2.3, we shall discuss on some generalization of this result to other classes of equations.

Kumiko Nishioka proved that, given a linear  $p$ -Mahler equation  $\mathcal{L}$  with coefficients in  $\mathbb{Q}(x)$ , there exists a precise set  $S \subset \overline{\mathbb{Q}}$  such that the transcendence degree of the field generated by the solutions of  $\mathcal{L}$  over  $\mathbb{Q}(x)$  coincides with the transcendence degree of the field generated by their values at an algebraic number  $\alpha$  over  $\mathbb{Q}$  providing that  $\alpha$  does not belong to  $S$  ([Nis96]). In [Phi15], Philippon proved that any algebraic relation between the values of functions solutions of a Mahler equation at an algebraic number  $\alpha$ , not in  $S$ , can be lifted as an algebraic relation between the functions themselves. Using this transfer property, one finds the following diophantine result.

**Theorem 4.8** (Theorem 1.5 in [7]). *Let  $f(x) \in \overline{\mathbb{Q}}[[x]]$  be a Mahler function that is not rational, let  $r$  be a positive integer, and let  $\mathcal{K}$  be a compact subset of the open unit disc. Then, for all but finitely many algebraic numbers  $\alpha \in \mathcal{K}$ , the complex numbers  $f(\alpha), f'(\alpha), \dots, f^{(r)}(\alpha)$  are algebraically independent over  $\overline{\mathbb{Q}}$ .*



### 4.2.2 Difference transcendence

In this section, we present some discrete analogues of §4.2.1 in the context of  $(\phi, \sigma)$ -fields developed at the end of §2.2.2. The driving conjecture of this section is that a formal power series, solution of a linear  $\phi$ -equation, cannot satisfy a nontrivial  $\sigma$ -algebraic relation providing that the two operators  $\sigma$  and  $\phi$  are *sufficiently independent* and that  $f$  is not too *simple*. Schaefer and Singer obtained the following result for pairs of automorphisms on the projective line.

**Theorem 4.9** (Cor 14 to 16 in [SS19a]). *Let  $K, \phi, \sigma$  be as in case **2S**, **2M** or **2Q** as in §3.2.2. Let  $F$  be the field  $\cup_{j \geq 1} \mathbb{C}((x^{\frac{1}{j}}))$  (resp.  $\mathbb{C}((x^{-1}))$ ) of Puiseux series in case **2M** or **2Q** (resp. in case **2S**). Then, an element  $f \in F$  cannot satisfy both a linear  $\phi$ -difference equation and linear  $\sigma$ -difference equation unless it belongs to  $K$ .*

For case **2M**, the above theorem is initially due to Adamczewski and Bell ([AB17]) and was conjectured by Loxton and van der Poorten in the Eighties. The first results concerning the  $\sigma$ -transcendence for solutions of rank one  $\phi$ -equations were obtained in [9, Prop. 5.3] for case **2Q**. For the other cases, one obtains the following result.

**Proposition 4.10** (Prop. 4.4 in [3]). *Let  $(K, \phi, \sigma)$  as in case **2S**, **2M** or **2Q**. Let  $a \in K^*$ . Let  $L$  be a  $\sigma$ -Picard-Vessiot in the sense of [OW15] for  $\phi(y) = ay$  and assume that  $L$  is a field. Let  $u \in L^*$  be such that  $\phi(u) = au$ . Then, the following statements are equivalent.*

- *The element  $u$  is  $\sigma$ -algebraic over  $K$ .*
- *There exists  $c \in \mathbb{C}^*$ ,  $n \in \mathbb{Q}$  ( $n = 0$  in Cases **2S**, **2M**) and  $b \in K^*$  such that  $a = cx^n \frac{\phi(b)}{b}$ .*

In the case of non-homogeneous equations, one can adapt the proof of [17, Prop. 3.16] to obtain the following lemma.

**Lemma 4.11** (Lemma 4.6 in [3]). *Let  $K \subset L$  be a  $(\phi, \sigma)$ -field extension such that  $C = K^\phi = L^\phi$ . We fix  $u, f \in L, u \neq 0$ , such that  $\phi(f) = af + b$  and  $\phi(u) = au$  with  $a, b \in K$ . Assume that  $f \notin K$  and that  $\sigma: K \rightarrow K$  is surjective. If  $u$  is  $\sigma$ -transcendental over  $K$ , then  $f$  is  $\sigma$ -transcendental over  $K$ .*

When one considers a solution  $f$  in  $F$  of an affine  $\phi$ -difference equations, Proposition 4.10 and Lemma 4.11 allow to conclude that  $f$  is either  $\sigma$ -transcendental over  $K$  or belong to  $K$ . This dichotomy holds indeed for  $\phi$ -difference equations of arbitrary order and one can prove the following theorem.

**Theorem 4.12** (Theorem 4.1). *Let  $(K, \phi, \sigma)$  as in case **2S**, **2M** or **2Q**. Let  $f \in F$  be a solution of the linear  $\phi$ -equation of order  $n$*

$$(4.2) \quad \phi^n(y) + a_{n-1}\phi^{n-1}(y) + \cdots + a_1\phi(y) + a_0y = 0,$$

where  $a_0, \dots, a_{n-1} \in K$  and  $a_0 \neq 0$ . Then either  $f$  belongs to  $K$  or  $f$  is  $\sigma$ -transcendental over  $K$ .

The proof of Theorem 4.12 is analogous to the proof of Theorem 4.7. It relies on an induction process, on the results for rank one equations detailed above and on the inverse problem for  $\phi$ -difference equations with simple algebraic groups (Proposition 3.15). One has to note however that the dichotomy of Theorem 4.12 might be no longer valid if one replaces the field  $F$  by a field of meromorphic functions. Indeed, one can easily show that, in the case **2Q**, the functions  $\theta_{q_1}, e_{q_1, c}$  as well as the  $q_1$ -logarithm  $\ell_{q_1}$  are  $\sigma$ -algebraic over the field of elliptic functions with respect to the elliptic curve  $\mathbb{C}^*/q_1^{\mathbb{Z}}$ . It is not at all obvious to determine if these functions are  $\sigma$ -algebraic over  $\mathbb{C}(x)$ .

As a corollary of Theorem 4.12, one gets a result of algebraic independence as follows.

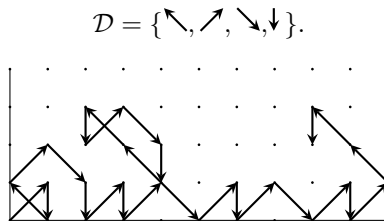
**Corollary 4.13** (Theorem 1.3 in [3]). *Let  $K, F$  and  $(\phi, \sigma)$  as in Cases **2S**, **2Q**, and **2M**. Let  $f \in F$  be a solution to a linear  $\phi$ -difference equation over  $K$  and let  $g \in F$  be a solution to an algebraic  $\sigma$ -difference equation over  $K$ . Then  $f$  and  $g$  are algebraically independent over  $K$ , unless  $f \in K$  or  $g$  is algebraic over  $K$ .*

This Corollary answers a conjecture of Loxton and van der Poorten ([vdP87]) about the algebraic independence of Mahler functions associated to multiplicatively independent integers  $p_1$  and  $p_2$  and yields also to the algebraic independence of  $q$ -hypergeometric functions associated to multiplicatively independent nonzero complex numbers  $q_1, q_2$  (see [3, Theorem 1.7]). Recently, in [AF20, Theorem 1.1], Adamczewski and Favre developed a Mahler’s method in several variables which allowed them to prove the algebraic independence over  $\overline{\mathbb{Q}}$  of values of several Mahler functions associated to pairwise multiplicatively independent integers. As a corollary of their result, one finds the algebraic independence of the Mahler functions themselves. This result underlines the strength of Mahler’s method which allows to treat a finite set of functions and not only two of them. However, from the purely functional point of view, one may hope that the Galois theoretic approach will give more details on the type of relations among the functions themselves if one enlarge the field of solutions.

## 5 Galoisian approach of Walks confined in a cone ([1, 6, 5, 8, 11])

The enumeration of planar lattice walks confined to the first quadrant has attracted a considerable amount of interest over the past fifteen years. For the lattice  $\mathbb{Z}^2$ , a lattice path model is comprised of a finite set  $\mathcal{D}$  of lattice vectors called the *step set* together with a starting point  $P \in \mathbb{Z}^2$ . The step set is called the *model of the walk*.

**Example 5.1.** *A walk confined in the first quadrant with model*



The combinatorial question boils down to the count of  $n$ -step walks, i.e., of polygonal chains, that remain in the first quadrant, starting from  $P$ , ending at  $(i, j)$  and consisting of  $n$  oriented line segments whose associated translation vectors belong to  $\mathcal{D}$ . This question is ubiquitous since lattice walks encode several classes of mathematical objects, in discrete mathematics (permutations, trees, planar maps), in probability theory (lucky games, sums of discrete random variables), statistics (non-parametric tests). We refer to the introduction of [BF02] for more details on these applications as well as [Hum10] for applications in other scientific areas.

*Weighted models* of walks with small steps or multiple steps model arose more recently in the classification project for three dimensional octant models (Bostan et al., 2014a), as it turns out that some models in three dimensions can be reduced by projection to two dimensional models with multiplicities. Weighted models correspond to a certain probabilistic weighting of the set of directions. More explicitly, a weighted model is given by a family  $(d_{i,j})_{(i,j) \in \{0, \pm 1\}^2}$  of elements of  $\mathbb{Q} \cap [0, 1]$  such that  $\sum_{i,j} d_{i,j} = 1$ . Walks attached to the weighted model  $(d_{i,j})_{(i,j) \in \{0, \pm 1\}^2}$  confined in the quadrant  $\mathbb{Z}_{\geq 0}^2$  are lattice walks that go to the direction  $(i, j) \in \{0, \pm 1\}^2 \setminus \{(0, 0)\}$  (resp. stays at the same position) with probability  $d_{i,j}$  (resp.  $d_{0,0}$ ). The direction set  $\mathcal{D}$  of the weighted model is then given by the directions  $(i, j)$  whose weight is non-zero. The  $d_{i,j}$  are called the weights of the walk. The walk is unweighted if  $d_{0,0} = 0$  and if the nonzero  $d_{i,j}$  all have the same value. Unweighted models correspond to the classical counting problem of walks confined in a quadrant. Letting  $q_{i,j}(n)$  be the probability for the walk confined in the quarter plane to reach the position  $(i, j)$  from the initial position  $P = (0, 0)$  after  $n$  steps, one introduces the trivariate generating series  $Q(x, y, t) = \sum_{i,j,n \geq 0} q_{i,j}(n) x^i y^j t^n$ . A step by step induction on the length of the walk allows to prove that the generating series satisfies the following function equation

$$(5.1) \quad K(x, y, t)Q(x, y, t) = xy + K(x, 0, t)Q(x, 0, t) - K(0, y, t)Q(0, y, t) + td_{-1,-1}Q(0, 0, t)$$

where  $K(x, y, t) = xy(1 - t \sum_{(i,j)} d_{i,j} x^i y^j)$  ([BMM10]).

More recently, similar equations, called in this memoir *kernel functional equations*, appear in the context of Krewara's walks with interacting boundaries [BOX20] as well as in the study of the Laplace transform of the stationary distribution for semimartingale Brownian motion confined in a cone ([2]).

The study of these combinatorial and probabilistic objects divides in two general directions which are of course connected. The first one is concerned with finding explicit expressions for the generating series using special functions or their integrals (see for instance [FR19]) and with the computation of their asymptotics (see for instance the multivariate analysis approach of [MM16]). The second strategy consists in the characterization of the generating series in terms of their algebraic properties, that is, as

- algebraic functions over the field  $k(x, y)$  of rational functions in  $x$  and  $y$ ,
- holonomic or  $D$ -finite functions over  $k(x, y)$ , that is, functions that satisfy linear differential equation with respect to the derivation  $\frac{d}{dx}$  and  $\frac{d}{dy}$ ,
- differentially algebraic functions over  $k(x, y)$ .

Each level of this classification encodes a certain analytic or algebraic regularity of the generating series. For instance, a  $D$ -finite function has only a finite number of singularities and one has some precise estimates on the growth of the coefficients of its Taylor series expansion near a regular point (see [FS09, Theorem VII.10]).

Recently, the works of many authors led to a complete classification of the holonomic generating series associated to unweighted models of walks with small steps, that is, with step set  $\mathcal{D} \subset \{-1, 0, 1\}^2 \setminus \{(0, 0)\}$ . Indeed, out of the  $2^8 - 1$  possible choices of unweighted models it was shown in [BMM10] that taking symetries into account and eliminating trivial sets, one need only consider 79 of these models. Of these, 23 models have  $D$ -finite (in all variables) generating series ([BMM10, BvHK10]) of which 4 are algebraic. The remaining 56 models were shown to have non- $D$ -finite generating series with respect to various variables in [KR12, MR09, MM14, BRS14].

The aforementioned works combine a wide variety of technics: singularity analysis via the kernel Method, probabilistic method, guess and proof strategies. Most of these technics rely on the functional equation satisfied by the generating series. In the genus one case, Kurkova and Raschel adapted the strategy of analytic uniformization employed by Fayolle, Iasnogorodski, Malyshev ([FIM99]) to walks with genus one kernel curve. They were then able to produce out of (5.1) a dynamical functional equation (see [KR12]). Their strategy of analytic uniformization is based on the celebrated *kernel method*, which was initiated by Knüth in his book (see [Knu11, Ex. 2.2.1.4 and 2.2.1.11]) and systematized and generalized by several authors (see [FS09, p.508] for references). Let me describe it briefly. By (5.1), the function  $Q(x, y, t)$  is entirely determined by the *sectional series*  $Q(x, 0, t)$  and  $Q(0, y, t)$ . If the polynomial  $K(x, y, t)$  is reducible, the kernel method allows to conclude directly that the generating series  $Q(x, y, t)$  is algebraic. If  $K(x, y, t)$  is irreducible, the *kernel curve*  $E$  is the compactification of the zero set of  $K(x, y, t)$  in  $\mathbb{P}^1 \times \mathbb{P}^1$ .  $E$  is an algebraic curve, which is either of genus one or zero. This algebraic curve is endowed with two canonical involutions  $\iota_1, \iota_2$  corresponding to the projections of  $E \subset \mathbb{P}^1 \times \mathbb{P}^1$  on each of the two  $\mathbb{P}^1$ . That is,  $\iota_1(x, y) = (x, y')$  and  $\iota_2(x, y) = (x', y)$ . These involutions generate a group of automorphisms of  $E$ , called *the group of the walk*. The kernel curve and the group of the walk were initially introduced by Bousquet-Mélou and Mishna who conjectured in [BMM10] that the the holonomy of the generating series was entirely characterized by the finiteness of the group of the walk. Evaluating (5.1) on genus one kernel curves, Kurkova and Raschel were able to prove that one can analytically prolong the sectional series  $Q(x, 0, t)$  and  $Q(0, y, t)$  as meromorphic functions so that they satisfy non-homogeneous linear rank one difference equations over the difference field  $(\mathbb{C}(E), \phi = \iota_2 \circ \iota_1)$ . These difference equations allowed them via a singularity analysis to characterize entirely the  $D$ -finite unweighted models with genus one kernel curve via the finiteness of their group. This was the final step of the proof of the holonomy conjecture of Bousquet-Mélou and Mishna.

As detailed above, one can deduce a linear difference equation from (5.1). The articles [8, 11, 1, 6] apply the parametrized difference Galois theory of [21] to this equation in order to produce criteria or algorithm to test the differential algebraicity of the generating series. We present the results obtained in the following sections.

## 5.1 Geometric properties of the kernel curve ([5])

Fixing a value  $t_0 \in ]0; 1[$  transcendental over  $\mathbb{Q}$ , one can identify the field  $\mathbb{Q}(t)$  with a subfield of  $\mathbb{C}$ , via the morphism sending  $t$  on  $t_0$ . This allows to consider the compactification  $E_{t_0} \subset \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$  of the zero locus of the kernel polynomial  $K(x, y, t_0)$ . The choice of a transcendental value for the parameter  $t$  allows to work with the general fiber of the family of algebraic curves  $E_t$  and thereby to find a classification of the geometric properties (irreducibility, smoothness) of the kernel curve depending only on the weights and not on pathological algebraic values of the parameter  $t$ .

Let me first recall the notion of *degenerate walks introduced in [FIM99]*.

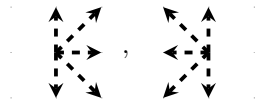
**Definition 5.2.** *A model of walk is called degenerate if one of the following holds:*

- $K(x, y, t)$  is reducible as an element of the polynomial ring  $\mathbb{C}[x, y]$ ,
- $K(x, y, t)$  has  $x$ -degree less than or equal to 1,
- $K(x, y, t)$  has  $y$ -degree less than or equal to 1.

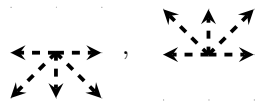
One can entirely characterize the degenerate models via their set of steps. More precisely, one finds the following proposition

**Proposition 5.3** ( Prop 1.2 in [5]). *A model of walk is degenerate if and only if at least one of the following holds:*

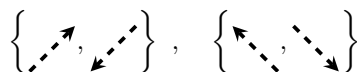
1. *There exists  $i \in \{-1, 1\}$  such that  $d_{i,-1} = d_{i,0} = d_{i,1} = 0$ . This corresponds to the following families of models of walks*



2. *There exists  $j \in \{-1, 1\}$  such that  $d_{-1,j} = d_{0,j} = d_{1,j} = 0$ . This corresponds to the following families of models of walks*



3. *All the weights are 0 except maybe  $\{d_{1,1}, d_{0,0}, d_{-1,-1}\}$  or  $\{d_{-1,1}, d_{0,0}, d_{1,-1}\}$ . This corresponds to the following families of models of walks*



Degenerate models correspond to one dimensional problems and walks in the half-plane restricted to the quarter plane which are more easy to study, as explained in [BMM10, Section 2.1]. They all correspond to algebraic generating series. Discarding degenerate models, one can show that the kernel curve of a non-degenerate weighted model is smooth if and only if direction set is not contained in a half plane ([5, Lemma 1.5]). If the kernel curve is smooth then it is a curve of genus one and if it is singular, it is a curve of genus zero with one singular point ([5, Prop.2.1]).

The choice of a transcendental value for  $t$  though very natural makes impossible the study of the differential dependencies of the generating series with respect to  $t$ . Another geometric approach was developed in [6] where the kernel curve becomes a rigid analytic curve defined over  $\mathbb{Q}(t)$ . More precisely, let us consider the field  $C$  of Hahn series or Malcev-Neumann series with coefficients in  $\overline{\mathbb{Q}}$ , and monomials from  $\mathbb{Q}$ . We recall that a Hahn series  $f$  is a formal power series  $\sum_{\gamma \in \mathbb{Q}} c_\gamma t^\gamma$  with coefficients  $c_\gamma$  in  $\overline{\mathbb{Q}}$  and such that the subset  $\{\gamma | c_\gamma \neq 0\}$  is a well ordered subset of  $\mathbb{Q}$ . The valuation  $v_0(f)$  of  $f$  is the smallest element of the subset  $\{\gamma | c_\gamma \neq 0\}$ . The field  $C$  is an algebraically closed and complete field extension of  $\mathbb{Q}(t)$  with respect to the valuation at  $t$  equal zero, see [AvdDvdH17, Ex. 3.2.23 and p. 151]. One

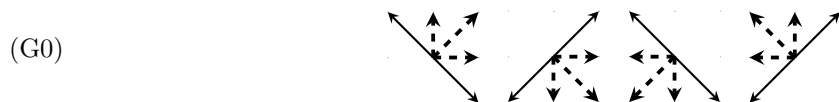
can then consider the compactification  $E \subset \mathbb{P}^1(C) \times \mathbb{P}^1(C)$  of the zero set of  $K(x, y, t)$ . If the model is non-degenerate, the results of [5] still hold and the algebraic curve  $E$  is either of genus zero or of genus one. In that context, one can prove the following additional result

**Proposition 5.4** (Lemma 1.11 in [6]). *Assume that direction set of the weighted model is not contained in a half-plane. Then, the algebraic curve is an elliptic curve with split multiplicative reduction at zero, that is, the  $j$ -invariant of  $E$  has strictly negative valuation at  $t$  equal zero.*

Over non-archimedean fields, there is no good notion of lattices. The period lattice of an elliptic curve is then replaced by a discrete multiplicative group of the form  $q^{\mathbb{Z}}$ . The quotient of  $C^*$  by  $q^{\mathbb{Z}}$  is the so called Tate curve. By Theorem 5.1.18 in [FvdP04], elliptic curves with split multiplicative reduction correspond to Tate curves and can be therefore uniformized by non-archimedean analogues of theta functions (see [Roq70] for a complete introduction to Tate curves over function fields and [6, Theorem 3.2] for an explicit uniformization of the kernel curves of genus one).

## 5.2 Genus zero walks ([8])

By [5, Lemma 1.5], the *nondegenerate genus zero models of walks* are the *nondegenerate models* whose step set is included in an half space whose boundary passes through  $(0, 0)$ . More precisely, they are *nondegenerate models* belonging to one of the following families



By [5, Remark 2.8], using symmetries and discarding the models never entering in the first quadrant, one can focus on the following models



for whom the singular point is  $\Omega = (0, 0)$  the only common fixed point of the two involutions  $\iota_1, \iota_2$ . For these five models, one can uniformize the kernel curve by rational function and analytically prolong the generating series so that it satisfies a linear  $q$ -difference equation. More precisely, the following holds

**Proposition 5.5** (Prop. 2.6 and §3.1 in [8]). *For any model in Figure 5.2, there exist two rational functions  $x(s), y(s)$  such that the morphism  $\psi : \mathbb{P}^1(\mathbb{C}) \rightarrow E_{t_0}, s \mapsto (x(s), y(s))$  is a parametrization of the kernel curve  $E_{t_0}$  and satisfies  $\phi \circ \psi(s) = \psi(qs)$  where one of the two complex numbers  $\{q, q^{-1}\}$  is equal to*

$$(5.3) \quad \frac{-1 + d_{0,0}t_0 - \sqrt{(1 - d_{0,0}t_0)^2 - 4d_{1,-1}d_{-1,1}t_0^2}}{-1 + d_{0,0}t_0 + \sqrt{(1 - d_{0,0}t_0)^2 - 4d_{1,-1}d_{-1,1}t_0^2}}.$$

Moreover, the functions  $K(x, 0, t_0)Q(x, 0, t_0)$  (resp.  $K(0, y, t_0)Q(0, y, t_0)$ ) as in (5.1) can be lifted via  $\psi$  and analytically continued as a meromorphic function  $F^1(s)$  (resp.  $F^2(s)$ ) over  $\mathbb{C}$  satisfying

$$(5.4) \quad F^i(qs) - F^i(s) = b_i(s),$$

where  $b_i$  is an explicit rational fraction for  $i = 1, 2$ . For instance,  $b_2 = x(\iota_1(y) - y) \circ \psi(s)$ .

Since composition by rational functions doesn't change the differential algebraic properties, one can easily prove that the generating series  $Q(x, 0, t)$  (resp.  $Q(0, y, t)$ ) is  $x$ -differentially algebraic over  $\mathbb{C}(x)$  (resp.  $\mathbb{C}(y)$ ) if and only if  $F^1(s)$  (resp.  $F^2(s)$ ) is  $s$ -differentially algebraic over  $\mathbb{C}(s)$ . Using the fact that the model is non-degenerate and that  $t_0 \in ]0, 1[$ , one can show that  $q \in \mathbb{R} \setminus \{\pm 1\}$  proving that for any weighted model associated with a genus zero curve, the group of the walk is always infinite.

One can show that  $F^1(s)$  and  $F^2(s)$  have the same differential behaviour. A result of Ishizaki, see [Ish98], implies that if  $F^2(s)$  is  $s$ -differentially algebraic then  $F^2(s)$  must be the expansion of a rational

fraction. In that case, there exists  $f_2 \in \mathbb{C}(s)$  such that  $b_2(s) = f_2(qs) - f_2(s)$ . By Lemma 3.5, this is equivalent to the fact that  $\text{div}_E(b_2) = 0$  for  $E = \mathbb{C}^*/q^{\mathbb{Z}}$ . Using the parametrization  $\psi$ , one can prove that if one of the poles of  $x(\iota_1(y) - y) \in \mathbb{C}(x, y) = \mathbb{C}(E_{t_0})$  is isolated in its  $\phi$ -orbit then the elliptic divisor of  $b_2$  will never be zero yielding to the  $s$ -differential transcendence of the function  $F^2(s)$  and thereby to the  $y$ -differential transcendence of  $Q(0, y, t_0)$ . Using the transcendence of  $t_0$ , one can show that, for any model in Figure 5.2, one of the poles of  $x(\iota_1(y) - y) \in \mathbb{C}(x, y) = \mathbb{C}(E_{t_0})$  is isolated in its  $\phi$ -orbit yielding to the following Theorem.

**Theorem 5.6** (Theorem 4.1 in [8]). *For any weighted model in Figure 5.2, the generating series  $Q(x, 0, t_0)$  (resp.  $Q(0, y, t_0)$ ) is  $x$ -differentially transcendental (resp.  $y$ -differentially transcendental) over  $\mathbb{C}(x)$  (resp.  $\mathbb{C}(y)$ ).*

### 5.3 Genus one walks ([1, 11] )

Unlike the case of genus zero walks, it might happen that weighted models associated with genus one curve have a finite group of the walk. In these cases, one can show that the generating series are elliptic zeta functions over curves isogeneous to the kernel curve (see [DR17, Theorem 42] for the weighted generalization of [KR15, §9.2]). Therefore, the generating series of a weighted model associated to a genus one curve with a finite group of the walk is always  $D$ -finite. In [BMM10], Bousquet-Mélou and Mishna conjectured that any unweighted model with genus one kernel curve and infinite group of the walk was not  $D$ -finite. Using a transcendental uniformization of the kernel curve, Kurkova and Raschel were able to prove that for unweighted models, the generating series has one singularity whose orbit with respect to the action of the automorphism  $\phi$  gives birth to an infinite amount of singularities. Since a  $D$ -finite function has only a finite number of singular points, Kurkova and Raschel proved the conjecture of Bousquet-Mélou and Mishna in [KR12]. The singularity analysis is however not accurate when one tries to find more general differential algebraic relations since a differentially algebraic function may have an infinite number of singularities.

In [11], we use the parametrized Galois theory of [21] to investigate the question of the differential algebraicity for genus one walks with infinite group of the walks. More precisely, Kurkova and Raschel proved that, for any transcendental value of  $t_0 \in ]0, 1[$ , one can construct a transcendental uniformization  $\psi : \mathbb{C} \rightarrow E_{t_0}$  such that the application  $\phi$  induces an isomorphism between  $E_{t_0}$  and  $\mathbb{C}/(\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2)$  for two explicit  $\mathbb{Z}$ -linearly independent periods  $\omega_1, \omega_2$ . Via the uniformization  $\psi$ , the automorphism  $\phi$  lifts to an automorphism of  $\mathbb{C}$  corresponding to the translation by some non-zero complex number  $\omega_3$ . The group of the walks is then finite if and only if a non-zero integer multiple of  $\omega_3$  belongs to the lattice  $\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ . In [KR12], Kurkova and Raschel proved that the sectional generating series  $K(x, 0, t_0)Q(x, 0, t_0)$  (resp.  $K(0, y, t_0)Q(0, y, t_0)$ ) can be analytically continued via composition by  $\phi$  as a meromorphic function  $F^1(s)$  (resp.  $F^2(s)$ ) over  $\mathbb{C}$  so that

$$(5.5) \quad F^i(s + \omega_3) = F^i(s) + b_i(s),$$

where  $b_2(s) = x(\iota_1(y) - y) \circ \phi(s)$  belongs to  $\mathbb{C}(E_{t_0})$ , the field of rational functions over the kernel curve  $E_{t_0}$  identified via  $\psi$  with the field of  $(\omega_1, \omega_2)$ -periodic meromorphic functions. Since the uniformization map  $\psi$  is made of elliptic functions which are differentially algebraic, one can easily show that the differential algebraicity of  $F^1(s)$  (resp.  $F^2(s)$ ) with respect to  $\frac{d}{ds}$  is equivalent to the differential algebraicity of  $Q(x, 0, t)$  (resp.  $Q(0, y, t)$ ) with respect to  $\frac{d}{dx}$  (resp.  $\frac{d}{dy}$ ). One can therefore apply Proposition 3.9 to the equations (5.5) and the  $(\phi, \delta)$ -base field  $(\mathbb{C}(E_{t_0}), \delta, \phi)$  (as in Examples 1.1) and find the following result:

**Proposition 5.7** (Prop. 3.8 and Prop. 6.2 in [11]). *Assume that the kernel curve  $E_{t_0}$  is of genus one and that the group of the walk is infinite. The following statements are equivalent*

- *the generating series  $Q(x, 0, t)$  is  $x$ -differentially algebraic,*
- *the generating series  $Q(0, y, t)$  is  $y$ -differentially algebraic,*
- *there exists  $(L, g)$  consisting of a linear differential operator  $L \in \mathbb{C}[\frac{d}{ds}]$  and  $g \in \mathbb{C}(E_{t_0})$  such that*

$$L(b_2) = \phi(g) - g.$$

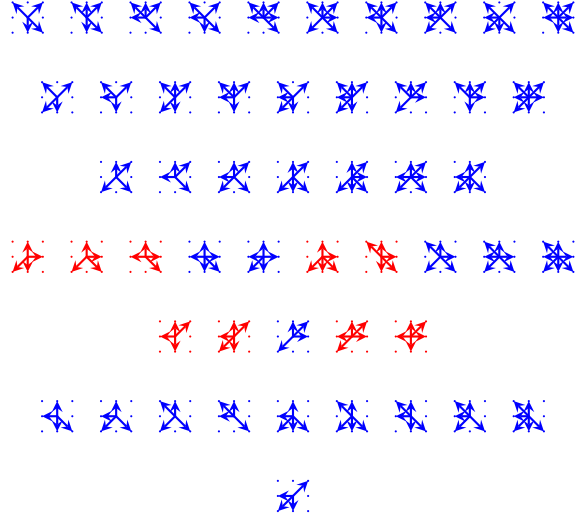


Figure 1: Unweighted models with genus one curve and infinite group

- the orbit residues of  $b_2$  with respect to  $\phi$  are all zero.

Thus, one can reduce the question of the differential algebraicity of the generating series to the computation of some orbit residues. In full generality, this computation might be quite difficult since it requires to expand a function at all its poles with respect to a certain family of local parameters. Using some symmetries arguments on the function  $b_2$  and the fact the sum of the classical residues of an elliptic function is zero, one can without heavy computations find the following classification of the 51 unweighted walks of genus one with infinite group.

In the 42 blue cases, the generating series are differentially transcendental and in the nine red cases, the generating series are differentially algebraic (see Theorem in [11]). This result is unfortunately not explicit and in parallel, the authors of [BBMR] proved that the nine cases were differentially algebraic by giving an explicit differential equation satisfied by the generating series associated to these nine unweighted models. Let me briefly describe their approach. For the nine models, they exhibit a *decoupling pair*, that is, a pair of functions  $(f(x), g(y)) \in \mathbb{Q}(t)(x) \times \mathbb{Q}(t)(y)$  such that

$$(5.6) \quad xy = f(x) + g(y) \text{ in } \mathbb{C}(E_{t_0}).$$

Using the functional equation (5.1), one can show that the function of  $K(x, 0, t)Q(x, 0, t) - f(x)$  is a *weak invariant*, that is,  $K(x, 0, t)Q(x, 0, t) - f(x)$  is analytic over some open region of the complex plane and satisfies some symmetry over some ‘trace’ of the kernel curve. This formulation allows the authors to restate the problem of a complete characterization of the generating series as a boundary value problem. Thus, they prove that the weak invariants under some growth assumption are explicit rational functions of a single canonical invariant  $w(x)$  which satisfies an explicit differential algebraic equation (Theorem 5.7 in [BBMR]). Combining the equations (5.5) and (5.6) composed via  $\phi(s) = (x(s), y(s))$ , one can easily show that the existence of a decoupling pair implies that the meromorphic function  $F^1(s)$  can be written as follows

$$(5.7) \quad F^1(s) = f(x(s)) + G(\wp_{1,3}(s), \wp'_{1,3}(s)),$$

where  $G$  is a rational fraction in  $\mathbb{C}(X, Y)$  and  $\wp_{1,3}$  is the Weierstrass function associated to the elliptic curve  $\mathbb{C}/(\mathbb{Z}\omega_1 + \omega_3)$ . In that setting  $w(x(s)) = \wp_{1,3}(s)$ . The subtlety of the approach of [?] is that the authors gives an explicit expression of the function  $G$  and prove that  $G \in \mathbb{Q}(t)(X)$ .

The notion of a decoupling pair seemed deep and intrinsic and it was intriguing to see if this notion was also accurate for the weighted situation. In [1], we refined the results of [11] and unified them with results of [BBMR] to get the following theorem.

**Theorem 5.8.** *For any weighted model associated with a genus one kernel curve and an infinite group of the walk, the following statements are equivalent*

- *the generating series are  $x, y$ -differentially algebraic,*
- *there exists a decoupling pair,*
- *the orbit residues of  $b_2(s)$  with respect to  $\phi$  are all zero,*
- *there exist  $g \in \mathbb{C}(E_{t_0})$  such that*

$$b_2 = \phi(g) - g$$

The fact that one can lower the order of the differential operator  $L$  in Proposition 5.7 to zero as well as the existence of a decoupling pair are intrinsically related to the fact that the coefficients of the functional equations (5.5) satisfy some nice properties with respect to the Galoisian extension  $\mathbb{C}(x) \subset \mathbb{C}(E_{t_0})$  (resp.  $\mathbb{C}(y) \subset \mathbb{C}(E_{t_0})$ ). For instance, the element  $b_2$  has trace zero with respect to the extension  $\mathbb{C}(y) \subset \mathbb{C}(E_{t_0})$ . A case by case arguments on the pole configuration of  $b_2$  taking into account these galosian symmetries allows to reduce the computation of the orbit residues at all order to the consideration of two precise points on  $E_{t_0}$  and their orbit with respect to  $\phi$ . These points are in fact base points of the family of elliptic curves  $(E_t)_{t \in \mathbb{P}^1}$  that are the compactification in  $\mathbb{P}^1 \times \mathbb{P}^1$  of the zero set of the kernel polynomial  $K(x, y, t)$ .

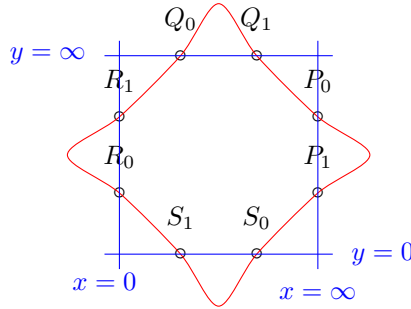


Figure 2: Position of the base points

The points  $P_i, S_i, R_i, Q_i$  for  $i = 0, 1$  are the eight base points of the pencil of elliptic curves defined by the equation  $K(x, y, t) = xy - tS(x, y)$ . The red curve corresponds to the curve  $E_\infty$ , that is, to the compactification of the zero set of  $S(x, y)$  whereas the blue curve corresponds to the compactification of the zero set of  $xy$ . In the notation of Figure 2, one finds the following criteria for the differential algebraicity of the series.

**Theorem 5.9.** *Assume that  $E_{t_0}$  is a curve of genus one and that the automorphism of the walk is not of finite order. Then,*

- *if one of the  $P_i$ 's and one of the  $Q_i$ 's is fixed by an involution then the generating series  $Q(x, 0, t)$  and  $Q(0, y, t)$  are  $x$  and  $y$ -differentially transcendental over  $\mathbb{Q}(x, y)$ ,*
- *if one of the  $P_i$ 's and the  $Q_j$ 's are not simultaneously fixed by an involution, the following holds*
  - *Case  $d_{1,1} \neq 0$  : the generating series  $Q(x, 0, t)$  and  $Q(0, y, t)$  are  $x$  and  $y$ -differentially algebraic over  $\mathbb{Q}(x, y)$  if and only if there exists  $j, k$  such that  $P_j = \phi^n(Q_k)$  for some integer  $n \in \mathbb{Z}$ ;*
  - *Case  $d_{1,1} = 0$  : the generating series  $Q(x, 0, t)$  and  $Q(0, y, t)$  are  $x$  and  $y$ -differentially algebraic over  $\mathbb{Q}(x, y)$  if and only if there exists  $j, k$  such that  $P_0 = \phi^n(P_1)$  for some integer  $n \in \mathbb{Z}$ . Moreover, this last condition is automatically fulfilled if  $(0, \infty)$  belongs to the curve  $E_{t_0}$  and is fixed by  $\iota_1$ .*

One gets easily some concrete examples for the first case of Theorem 5.9



**Example 5.10.** *Any weighted model*



with an infinite group of the walk is  $x, y$ -differentially transcendental.

In the second case of Theorem 5.9, the question of the  $x$  and  $y$ -differential algebraicity is reduced to the fact that two points  $P, Q$  in  $E_{t_0}$  are such that  $P = \phi^n(Q)$ . In [1], we develop an algorithm which answers this last question. This algorithm assigns to each weighted model with genus one curve and infinite group of the walk a set of polynomial conditions on the weights which constitutes the necessary and sufficient conditions to guarantee the differential algebraicity of the series.

Our algorithm relies on a height computation on the elliptic surface corresponding to the pencil of curves and on the classification of Mordell-Weil lattices for rational elliptic surfaces obtained by Oguiso and Shioda. Let me briefly describe the procedure. Choosing  $Q$  as the zero for the group law on the curve  $E_{t_0}$ , one sees that the condition  $P = \phi^n(Q)$  is equivalent to  $P = n\phi(O)$ . Then,  $h(P) = n^2h(\phi(O))$  where  $h : E_{t_0} \rightarrow \mathbb{Q}$  is the Néron-Tate height. Our algorithm boils down to compute up to a finite amount of possibilities the heights of  $P$  and  $\phi(O)$  and thereby a finite set of potential integer  $n$ . For each of those values of  $n$ , the equality  $P = \phi^n(Q)$  yields a polynomial condition on the weights guaranteeing the differential algebraicity of the series. One computes the heights of the points  $P$  and  $\phi(O)$  via intersection theory. Indeed, one considers the Kodaira-Néron model  $\mathcal{S}$  of  $E_t$  that is the smooth projective rational surface defined over  $\mathbb{C}$  with a surjective morphism  $\pi : \mathcal{S} \rightarrow \mathbb{P}^1(\mathbb{C})$  whose generic fiber is isomorphic to  $E_t$ . There is a one to one correspondence between the  $\mathbb{C}(t)$ -points of  $E_t$  and the rational sections of the fibration  $\pi$ . In that setting the heights of the points  $P$  and  $\phi(O)$  can be computed via some numerical invariants attached to the intersection of the corresponding sections with the singular fibers of the fibration  $\pi$ . The singular fibers of  $\pi$  are associated to a certain subroot lattice  $T$  of the root lattice  $E_8$  (see [SS19b] for a complete presentation of Mordell-Weil lattices). Table 8.1 in [SS19b] lists the 70 possible root lattices  $T$ . This classification together with some refinements allows to compute the heights of  $P$  and  $\phi(O)$  up to a finite amount of possibilities. In [1], we were able to apply this algorithm almost by hand to the nine unweighted cases and we found the following polynomial conditions on the weights yielding to  $x$ - $y$ -differentially algebraic generating series (see Figure 3).

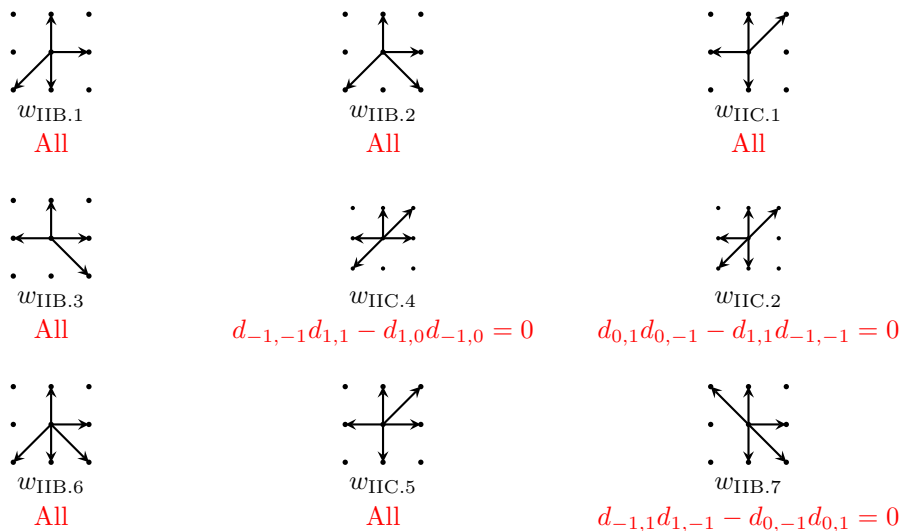


Figure 3: The weighted nine cases

One can easily note that for all the nine cases of Figure 3, the polynomial conditions are trivially satisfied when all the weights are equal, that is, in the unweighted case. Figure 4 shows that the polynomial conditions are not necessarily homogeneous in the weights. It corresponds to a differentially transcendental unweighted model.

$$d_{-1,1}d_{0,1}^2 - d_{0,1}d_{-1,-1}d_{0,-1} + d_{1,1}d_{-1,-1}^2 = 0$$

Figure 4:  $w_{\text{IB.6}}$

One of the first conclusion which can be drawn from this characterization of the differential algebraicity of the walks is that this notion cannot be entirely captured by the set of directions. Moreover, there are a priori no simple algebraic conditions on the weights which can lead to a straightforward combinatorial interpretation in terms of these quantities. At a first glance, the only combinatorial interpretation which can be made out of this study goes through the relative position of the base points of the pencil and their orbits with respect to  $\phi$ .

#### 5.4 Length derivative ([6])

In [11], the parameter  $t$  counting the length of the walks is considered as a complex number  $t_0$  transcendental over  $\mathbb{Q}$  and all the geometric objects, the kernel curve and its involutions, are defined over the field  $\mathbb{C}$  extension of  $\mathbb{Q}(t_0)$ . Such a specialization does not allow to study the  $t$ -dependencies of the generating series. In [6], we choose to uniformize the kernel curve over the the field  $C$  of Hahn series or Mal'cev-Neumann series with coefficients in  $\mathbb{Q}$ , an algebraic closure of  $\mathbb{Q}$ , and monomials from  $\mathbb{Q}$ . The field  $C$  is a complete algebraically closed extension of the valued field  $\mathbb{Q}(t)$  endowed with the valuation at  $t$  equal zero. It is moreover a differential field extension of the differential field  $(\mathbb{Q}(t), \frac{d}{dt})$  ([AvdDvdH17, Ex.(2), §4.4]). In that framework, the kernel curve  $E_t$  is the compactification in  $\mathbb{P}^1(C) \times \mathbb{P}^1(C)$  of the zero set of  $K(x, y, t)$ . Using Lefschetz Principle, one finds that  $E_t$  and  $E_{t_0}$  as in §5.1 share the same geometric properties.

In the genus one case, the kernel curve is analytically isomorphic to the Tate curve  $C^*/q^{\mathbb{Z}}$ , for some convenient  $q \in C^*$ , as defined in [Roq70]. This analytification is the ultrametric analogue of the well known uniformization of an elliptic curve over  $\mathbb{C}$  by the quotient of  $\mathbb{C}$  by a lattice. However, over non-archimedean fields, such an uniformization requires that the  $J$ -invariant of the elliptic curve is of modulus strictly greater than 1. Surprisingly, this condition is fulfilled by any genus one kernel curve. Then, via some technical non-archimedean estimates, one can prove the ultrametric analogue of [DR17]. More precisely, the series  $Q(x, 0, t)$  (resp.  $Q(0, y, t)$ ) is defined for  $|x| \leq 1$  (resp.  $|y| \leq 1$ ). The intersection of the latter domain with the kernel curve is non-empty and can be pulled back via the uniformization to a non-empty subdomain of  $C^*$ . Then, one can meromorphically continue the pullback of the generating series  $Q(x, 0, t)$  (resp.  $Q(0, y, t)$ ) as a meromorphic function  $F^1(s, t)$  (resp.  $F^2(s, t)$ ) over  $C^*$  satisfying the following  $\tilde{q}$ -difference equations

$$F^i(\tilde{q}s, t) = F^i(s, t) + b_i(s)$$

where  $\tilde{q} \in C^*$  and the  $b_i(s)$  belong to  $C(s)$  in the genus zero case and to  $C(E_t)$  the field of rational functions over  $E_t = C^*/q^{\mathbb{Z}}$ , in the genus one case. Denoting by  $(K, \phi)$  the difference fields  $(C(s), \phi(s) = \tilde{q}s)$  in genus zero and  $(C_q, C_q^{\sim}(\ell_q, \ell_q^{\sim}), \phi(s) = \tilde{q}s)$ , one can one can construct two derivations  $\partial_s$  and  $\Delta_{q,t}^{\sim}$  of  $K$  commuting with the action of  $\phi$ . These derivations will encode the action of the derivations  $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}$  and  $\frac{\partial}{\partial t}$  on  $Q(x, 0, t)$  and  $Q(0, y, t)$ . Combining Prop. 3.8 in [21] to the  $(\phi, \delta)$ -fields above with some technical descent arguments, we were able to prove the following theorem

**Theorem 5.11.** *Let us consider a weighted model for a non-degenerate walk. If the generating series  $Q(x, 0, t)$  (resp.  $Q(0, y, t)$ ) is  $t$ -differentially algebraic then it is  $x$ -differentially algebraic (resp.  $y$ -differentially algebraic).*

Combining Theorem 5.11 with Theorem 5.6, one concludes that the generating series of any weighted model associated with a genus zero kernel curve is  $t$ -differentially transcendental. For the genus one, Dreyfus used Theorem 5.8 to show that the condition in Theorem 5.11 is necessary and sufficient [Dre21].

## 6 Perspectives

The functional Galois theories of linear functional equations are multifaceted. They combine analytic, geometric as well as arithmetic tools and the implementation of their related direct problems yields difficult questions in computer algebra. In what follows, I will present some research directions which focus on three main directions : the non-linear functional equations, the special functions defined over function fields of positive genus and the applications to combinatorial or probabilistic questions.

The research program for Galois theories of linear functional equations over the projective line seems almost achieved. The inverse problem has been almost completed for  $q$ -difference operator by Singer and van der Put ([vdPS97b]) and for the finite difference operator by Etingov ([Eti95]). In both case, the authors proved that any connected algebraic group over  $\mathbb{C}$  can be realized as the Galois group of a linear difference system over  $\mathbb{C}(x)$ . The implementation of the computation of the Galois group of a linear  $q$ -difference equation is a work in progress by Di Vizio, Dreyfus and Weil in the spirit of the algorithm for differential systems based on reduced forms of [BCDVW16]. One can wonder if such an algorithm could be also obtained for Mahler systems as well as finite difference operators. Arithmetic descriptions of the Galois group of a linear difference system were obtained by Di Vizio and myself for the  $q$ -difference operator and for Mahler operator by Roques in [Roq17]. Such a description fails however to be true for the finite difference operators (see [vdPS97b]). Finally, the transcendental description based on the analytic constructions of monodromy and Stokes matrices is now achieved for  $q$ -difference equations by Ramis, Sauloy and Zhang ([RSZ13]). The Mahler case has been initiated by Roques in [Roq21] and the finite difference case by Duval ([Duv83]). All these descriptions exists for linear differential systems and one can study their behavior for linear  $q$ -difference systems “tending” to the the differential system as  $q$  goes to one. Such process is called *confluence* and is quite well understood. For instance, the work of André in [And01] shows that the Galois group of the limit differential system is contained in the special fiber of the Galois group of a  $q$ -deformation. A question which remains still open is the notion of the  $q$ -deformation of linear differential systems over  $\mathbb{Q}(x)$  especially from an arithmetic point of view. Given a linear differential system, one can construct many  $q$ -deformations. However, one would like to know if one could construct a *canonical*  $q$ -deformation which would preserve some notions such as the Frobenius structure or certain curvatures descriptions. This question is detailed in §6.2.1.

Regarding the parametrized Galois theories, most of the research programs detailed above are at their infancy. Indeed, only the arithmetic description by curvatures obtained for classical Galois groups can be easily extended to the parametrized case via the Tannakian formalism. The transcendental description of the parametrized Galois groups is the work of [MS12] and [Dre14a] in the case of differential systems with parameters but the case of difference equations with a differential or difference parameters is still wide open. Only a few cases of the direct and inverse problems are also known for these parametrized Galois theories. Some lines of inquiry for these questions are developed in §6.1.

The results mentioned above concern essentially linear difference systems above the projective line. Recently, the combinatorial study of lattice paths confined in cones brought to light difference equations whose difference operator is a translation on an elliptic curve. The application of Galois theory to these equations brought out a subtle interplay between the arithmetic and the geometry of the elliptic curve on one hand and the algebraic properties of the generating series on the other. Though the difference equations associated to these counting problems are until now of rank one, some rank two difference equations might appear by imposing to the lattice path some boundary or reflection conditions. The connection between the arithmetic of elliptic curves and the Galois theory of difference equations with respect to a translation as well as their combinatorial applications is a strong motivation for the Galoisian study of this new type of difference equations: local analytic classification, differential algebraic properties of the solutions as well as direct and inverse problem, curvatures description.(see §6.2.2 and 6.2.3 for more details).

The application of the parametrized Galois theories to the combinatorics of walks and to some of their continuous analogues is a full research program on its own. For weighted walks with small steps in the first quadrant, the implementation of [1] and of [BBMR] would be a great tool to investigate the combinatorial nature of the algebraic conditions on the weights appearing in [1]. The comprehension of walks with big steps and walks in the orthant is a real challenge since it requires in a first place to find a better geometric understanding of the combinatorial problem which implies to deal with curves of genus

strictly greater than one or  $K3$  surfaces. We will detail some lines of research in that domain in §6.3.

Living the realm of linear functional equations, the non-linear Galois theory developed in parallel by Malgrange and Umemura for non-linear differential equations offers many new research perspectives. A first step would be to unify the presentations of Malgrange and Umemura and to have a better understanding of the Galois correspondence in that context. As a second step, one would like to initiate the classification of the geometric objects behind the Galois groups of non-linear equations: for instance, for the Malgrange's  $D$ -groupoid only the classification obtained by Cartan up to dimension 2 is available. The direct problem in that domain is also wide open. Some precise questions are proposed in §6.4.

## 6.1 Direct and inverse problem in Galois theories of difference systems

### 6.1.1 Differential parameter

The inverse problem in the Galois theory of linear differential equations with coefficients in  $k(x)$  has been solved by Hartmann, who proved that any linear algebraic group defined over an algebraically closed field  $k$  was the Galois group of a linear differential equation with coefficients in  $k(x)$  (see [Har05]). In [MS12, Cor. 5.2] and [Dre14a, Cor. 2.18 and Theorem 3.10], it is proved that a linear differential algebraic group  $G$  defined over  $k$  is the parametrized Galois group of a linear differential equation over  $k(x)$  if and only if  $G$  is differentially finitely generated over  $k$ , that is, is the Kolchin closure of a finitely generated subgroup. This result relies on a transcendental description of the Galois group of a linear differential equation as the Zariski closure of the group generated by the monodromy and Stokes matrices. It also allows to state parametrized Riemann-Hilbert problem for linear differential systems of the form  $\frac{dY}{dx} = A(x, t)Y$  (see [MS12, Theorem 5.1]). In [Sin13, Proposition 1.2], Singer proved that a linear algebraic group  $G$  defined over  $k$  is a differentially finitely generated differential algebraic group if and only if the identity component of  $G$  has no quotient isomorphic to the additive or the multiplicative group. This yields, at least for linear algebraic groups, a more geometric condition for the inverse problem in parametrized Galois theory of linear differential equation. In [Bac15], Bachmayr used the method of patching developed by Harbater and Hartmann to show that any connected semisimple  $\mathbb{C}((t))$ -split linear algebraic group was the parametrized Galois group of a linear differential equation with coefficient in  $\mathbb{C}((t))(x)$ .

Much less is known for linear difference equations over  $k(x)$  for an algebraically closed field of  $\phi$ -constant  $k$ . For the finite difference operator given by  $\phi(x) = x + 1$ , Singer and van der Put conjectured in [vdPS97a, Chap.3] that a linear algebraic group  $G$  defined over  $k$  was a Galois group for a linear difference system of the form  $Y(x+1) = A(x)Y(x)$  with  $A \in \mathrm{GL}_n(k(x))$  if and only if  $G/G^\circ$  was cyclic<sup>13</sup>. In [vdPS97a, Theorem 3.1], it is proved that any connected linear algebraic group over  $k$  is the Galois group of a linear difference system of the form  $Y(x+1) = A(x)Y(x)$  with  $A \in \mathrm{GL}_n(k(x))$ . In [Eti95, Prop. 3.4], Etingof proved an analogous statement for  $q$ -difference systems, that is, for the difference field  $(k(x), \phi)$  where  $\phi(x) = qx$ . In [RS15], Sauloy and Ramis studied the inverse problem in the light of a transcendental description of the Galois group of a  $q$ -difference system.

Focusing on difference systems over the projective line, one could try to understand the inverse problem in the parametrized Galois theory with a differential parameter of [21] for linear difference equations over the  $(\phi, \delta)$ -fields of rational functions  $(K, \phi, \delta)$  described in §4.2.1. The example of the Gamma function, whose parametrized Galois group is the additive group  $\mathbf{G}_a$ , shows that the characterization of the parametrized Galois group of a linear differential equation with differential parameters is no longer true for difference equations. In other words, the parametrized Galois group of a linear difference system is not necessarily differentially finitely generated. However, the recent works of Arreche and Singer ([AS17]) suggests that the inverse problem for the parametrized Galois groups attached to linear difference equation might still be constrained. Indeed, they proved that a parametrized Galois group  $G$  cannot be conjugated to a  $\delta$ -constant group unless its classical Galois group is a product of vector-groups, tori and cyclic groups. As a corollary of Arreche and Singer's result, one finds for instance that the group  $\mathrm{Sl}_2^\delta$  cannot appear as a parametrized Galois group of a linear difference equation whereas it can be realized as the parametrized Galois group of an isomonodromic linear differential equation of order two (see [Dre14b] for the example of the Schrodinger equation with rational potential of odd degree). More generally, Arreche and Singer proved that, among the Zariski dense  $\delta$ -algebraic subgroups of a semisimple algebraic group  $G$  defined

<sup>13</sup>The condition  $G/G^\circ$  cyclic is necessary by [vdPS97a, Prop. 1.20]

over  $k$ , only  $G$  itself could be realized as the parametrized Galois group of a linear difference equation over a  $(\phi, \delta)$ -field  $K$  as in §4.2.1. A slight adaptation of the arguments contained in [14] should also prove that for an extension of two difference modules with semisimple Galois groups, the parametrized Galois group will coincide with the classical Galois group. In [21, Proposition 4.3], the authors investigate the latter inverse problems for  $\delta$ -algebraic subgroup of the additive group and proved, for instance, that only the trivial group and  $\mathbf{G}_a$  itself could be realized as parametrized Galois groups for difference equations attached to the action of the finite difference operator on  $\mathbb{C}(x)$ . Relying on these two examples, one could think that only algebraic groups can appear as parametrized Galois groups for linear difference systems over the projective line. However, one can show that the parametrized Galois group of the equation  $y(qx) = xy(x)$  over the  $(\phi, \delta)$  field  $\mathbb{C}(x)$  endowed with  $\phi(f(x)) = f(qx)$  and  $\delta = x \frac{d}{dx}$  corresponds to the  $\delta$ -algebraic subgroup  $H$  of  $\mathbf{G}_m$  defined by the equation  $\delta(\frac{\delta c}{c}) = 0$  (see for instance [16, Exercice 12.6]). Therefore, as a next step toward the inverse problem, one would like to understand what are the  $\delta$ -subgroups of tori and of unipotent algebraic groups which can occur as parametrized Galois groups for linear difference equations over the  $(\phi, \delta)$ -fields in §4.2.1.

For the direct problem for linear difference or differential equations with differential parameters, a quite natural idea would be to try to adapt the Tannakian arguments of Hrushovski and Feng to the framework of differential Tannakian categories. If this strategy works, it would have the advantage to ensure the existence of an algorithm for the computation of the parametrized Galois group. Though this approach might produce an algorithm with an extremely high complexity, it might also give a better insight on the different bounds needed for the computation of the defining equations of the parametrized Galois group.

Feng's algorithm might have a very high complexity which makes it quite difficult to implement. The adaptation of the approach of Barkatou-Cluzeau-Di Vizio-Dreyfus and Weil to the difference setting, which is an ongoing project by the aforementioned authors, should produce a more efficient algorithm. Indeed, many of the tools, normal forms, curvature characterization used in [BCDVW16] have difference analogues. For instance, assuming that the base field  $(K, \phi)$  is a  $C^1$ -field of characteristic zero,  $k = K^\phi$  is algebraically closed and  $(K, \phi)$  has no non-trivial finite  $\phi$ -field extensions, there exists a normal form for any difference system over  $K$ , that is, for any  $A \in GL_n(K)$ , there exists  $B \in GL_n(K)$  such that  $\phi(B)AB^{-1} \in \text{Gal}(K_A|K)(K)$ , where  $\text{Gal}(K_A|K)(K)$  is the set of  $K$ -points of the Galois group  $\text{Gal}(K_A|K)$  of a Picard-Vessiot extension  $K_A$  for  $\phi(Y) = AY$  over  $K$ . Any of the difference field extensions of  $\mathbb{C}(x)$  described in §4.2.1 fulfill the criteria above. Moreover, in the case of  $q$ -difference equation, the results of [10] describe the intrinsic Galois group of a  $q$ -difference system,  $Y(qx) = A(x)Y(x)$ , in terms of its curvatures which are certain reductions of the matrices  $(A(q^{n-1}x) \dots A(x))_{n \in \mathbb{N}}$ . Thus, one may hope to get fast algorithms to compute these curvatures and perhaps also the intrinsic Galois group in terms of differential polynomial equations annihilated by the curvatures.

In order to produce an efficient algorithm to compute the parametrized Galois group of a difference system, one could try to generalize to the parametrized setting the algorithms of Barkatou-Cluzeau-Di Vizio-Dreyfus and Weil. Indeed, as proved in [10, Theorem 7.13], the parametrized intrinsic Galois group of a  $q$ -difference system over  $\mathbb{C}(x)$  is the smallest differential algebraic group defined over  $\mathbb{C}(x)$  containing the curvatures. A first step would be to investigate the existence of parametrized normal forms. More precisely, given any difference system of the form  $\phi(Y) = AY$  with  $A \in GL_n(K)$  for  $K$  a  $(\phi, \delta)$ -field, one could ask if there exists a  $(\phi, \delta)$ -field extension  $\tilde{K}$  of  $K$  and  $B \in GL_n(\tilde{K})$  such that  $\phi(B)AB^{-1} \in \delta\text{-Gal}(L_A|K)(\tilde{K})$ , where  $\delta\text{-Gal}(L_A|K)(\tilde{K})$  is the set of  $\tilde{K}$ -points of the parametrized Galois group  $\delta\text{-Gal}(L_A|K)$  of  $\phi(Y) = AY$  over  $K$ . Such a question is connected to the triviality of a certain  $\delta\text{-Gal}(L_A|K)$ -torsor and thereby to the vanishing of the first differential Galois cohomology of the latter differential algebraic group. By [MO19, Theorem 1], the vanishing of the first differential cohomology of any linear differential algebraic group defined over  $\tilde{K}$  is equivalent to the fact that  $\tilde{K}$  is algebraically closed, PV closed, and  $\delta$ -linearly closed. This result should guarantee the existence of parametrized normal forms for a linear difference system over a  $(\phi, \delta)$ -field  $K$  that is algebraically closed, PV closed, and  $\delta$ -linearly closed. However such a field is far too complicated to be interesting in view of an implementation. One could therefore try to adapt the arguments of Minchenko and Ovchinnikov to characterize for which  $(\phi, \delta)$ -fields, the first differential cohomology of any parametrized Galois group vanishes. The action of the difference operator  $\phi$  might perhaps allow to lower the structural complexity in that particular case. The search for parametrized normal forms over the base field is motivated by the following remark. For

a linear difference system defined over a  $(\phi, \delta)$ -field as in 4.2.1, a corollary of Schaefer-Singer's main result (Theorem 3.12) in the cases  $\delta\mathbf{S}$  and  $\delta\mathbf{Q}$  is the following: if the parametrized Galois group  $\delta\text{-Gal}(L_A|K)$  of  $\phi(Y) = AY$  in case  $\delta\mathbf{S}$  or  $\delta\mathbf{Q}$  is conjugate to a constant group, which essentially means that a defining equation for  $\delta\text{-Gal}(L_A|K)$  is  $\delta(C) = 0$ , then there exists  $G \in \text{Gl}_n(K)$  such that  $\phi(G)AG^{-1} \in \text{Gl}_n(\mathbb{C}) = \{B \in \text{Gl}_n(K) | \delta(B) = 0\}$ . By [AS17, Lemma 2.1 and Prop. 3.4], the matrix  $\phi(G)AG^{-1}$  belongs to  $\delta\text{-Gal}(L_A|K)(K)$ .

### 6.1.2 Difference parameter

The direct and inverse problem can of course be investigated in the context of the parametrized Galois theories of [17] and [OW15], that is, for a discrete parameter. In [BW], the authors prove that any algebraic group defined over  $\mathbb{C}$  considered as a  $\sigma$ -algebraic group defined over  $\mathbb{C}$  can be realized as a  $\sigma$ -Galois group for a linear differential equation  $\delta(Y) = A(x)Y$  over the  $(\delta, \sigma)$ -field  $\mathbb{C}(x)$  endowed with the derivation  $\delta = \frac{d}{dx}$  and  $\sigma(x) = x + 1$ . Bachmayr and Wibmer also proved that no proper non-trivial  $\sigma$ -algebraic subgroup of  $\mathbf{G}_a$  is a  $\sigma$ -Galois group over  $\mathbb{C}(x)$ .

In that setting, one could slightly adapt the arguments of [3] to show that the  $\sigma$ -Galois group of a linear differential equation over  $\mathbb{C}(x)$  with a simple Galois group  $G$  is necessarily equal to the  $\sigma$ -algebraic group associated to  $G$ . However, the analogue of Arreche and Singer's result, which corresponds to the case where  $G$  is semisimple might be no longer true. Indeed, Arreche and Singer's Theorem relies on the fact that any proper  $\delta$ -algebraic subgroup of  $G$  has a non-trivial quotient that is conjugate to a  $\delta$ -constant  $\delta$ -algebraic group. This is a consequence of the classification of the Zariski dense  $\delta$ -algebraic subgroups of a connected semisimple algebraic group  $H$  obtained by Cassidy ([Cas89, Theorem 19 and 20]). Denoting by  $H_1, \dots, H_r$  the almost-simple components of  $H$ , Cassidy proved that any Zariski dense  $\delta$ -subgroup of  $H$  is conjugate to an almost product of the  $\delta$ -subgroups  $G_i$  where  $G_i$  is either equal to  $H_i$  or to  $H_i^\delta$ . Cassidy's result is no longer true for  $\sigma$ -algebraic groups. Indeed, if  $H$  is a simple algebraic group defined over a difference field  $(k, \sigma)$  where  $\sigma|_k = id$ , the  $\sigma$ -algebraic group  $H = \{(g, h) \in H \times H | \sigma^d(g) = h\}$  for some positive integer  $d$ , is a Zariski dense  $\sigma$ -algebraic subgroup of  $H \times H$  but is not of the form  $H \times H^{\sigma^l}$  for some non-negative integer  $l$ . This example corresponds to the  $\sigma$ -Galois group of a linear differential system of the form

$$(6.1) \quad \delta(Y) = \begin{pmatrix} A & 0 \\ 0 & \sigma^d(A) \end{pmatrix} Y.$$

This system is obtained as the direct sum of the differential system  $\delta(Y) = AY$  and of its transform  $\delta(Z) = \sigma^d(A)Z$  with respect to the action of  $\sigma^d$ . Such a notion might be interpreted more intrinsically via the notion of group actions on Tannakian categories developed in [OW17]. Ovchinnikov and Wibmer considered difference analogues of the prolongation functor introduced in §2.1.3. In their framework, the differential system (6.1) corresponds to a differential module of the form  $\mathcal{M} \oplus T_1(\mathcal{M})$  where  $\mathcal{M}$  is a differential module associated to  $\delta(Y) = AY$ . It would be interesting to understand if any Zariski dense  $\sigma$ -algebraic subgroup  $G$  of a semisimple group  $H$  is conjugate to an almost product of  $\sigma$ -algebraic groups of the form  $H_i, H_i^{\sigma^l}$  or  $\{(g, \sigma^{l_1}(g), \dots, \sigma^{l_r}(g))\} \subset H_i^r$  where the  $H_i$ 's are the almostsimple components of  $H$ . Such a description might perhaps follow from the combination of some Kolchin-Goursat-Ribet lemma for difference algebraic groups and the characterization of the  $\sigma$ -algebraic subgroups of almostsimple algebraic groups (see Theorem 2.14). If such a characterization holds and if one considers now either linear differential systems  $\delta(Y) = AY$  over a  $(\delta, \sigma)$ -field  $K$  as in §4.2.1 or linear difference systems  $\phi(Y) = AY$  over a  $(\phi, \sigma)$ -field as in §4.2.2, one could adapt the arguments of Arreche and Singer to the context of a difference parameter to prove that the  $\sigma$ -Galois group coincides with the  $\sigma$ -algebraic group associated to the classical Galois group if the latter is a semisimple algebraic group with non-isomorphic almost-simple components. Finally, one could ask if the results of Bachmayr and Wibmer for the  $\sigma$ -algebraic subgroups of  $\mathbf{G}_a$  still holds for the inverse problem in the  $\sigma$ -Galois theory of linear  $\phi$ -difference system. More generally, one could ask what are the  $\sigma$ -algebraic subgroups of unipotent algebraic groups that can be realized as  $\sigma$ -Galois group of a linear difference system  $\phi(Y) = AY$ .

## 6.2 Arithmetic applications

### 6.2.1 $q$ -Deformation of geometric differential equations

In [10], we proved an analogue of the Grothendieck conjecture for  $q$ -difference systems of the form  $Y(qx) = A(q, x)Y(x)$  where  $q$  is transcendental over  $\mathbb{Q}$  and  $A \in \mathrm{GL}_n(\mathbb{Q}(q, x))$ . This result generalizes the results of Di Vizio for an algebraic number  $q$  but it also allows to let  $q$  vary. If the matrix  $\frac{A(q, x) - I_n}{(q-1)x}$  has a limit  $B \in \mathbb{Q}(x)^{n \times n}$  when  $q$  goes to one then the  $q$ -difference system  $Y(qx) = A(x)Y(x)$  “tends” to the differential system  $\frac{d}{dx}(Y) = BY$  and one can wonder if the same holds for their solutions, that is, if a fundamental solution matrix of solutions  $Z(q, x)$  of  $Y(qx) = A(q, x)Y(x)$  converges as  $q$  goes to 1 to a fundamental solution matrix of the differential system. This might not be true for any choice of fundamental matrix  $Z(q, x)$  and any limit path for  $q$ . When the limit process exists, one speaks of *confluence* (see for instance [Sau00]). By [And01, Corollaire 3.3.2.4.], the Galois group of the differential system is contained in the special fiber of the Galois group of  $Y(qx) = A(x)Y(x)$ . Via Theorem 4.2, this gives a characterization of the Galois group of the differential system  $\frac{d}{dx}Y = BY$  in terms of the curvatures of the  $q$ -difference system  $Y(qx) = A(q, x)Y(x)$ . However, it is quite difficult a priori to compare the  $p$ -curvatures of the differential system and the curvatures of the  $q$ -difference one. Indeed, the  $p$ -curvature of  $\frac{d}{dx}(Y) = BY$  is the reduction modulo the prime  $p$  of the matrix  $B_p$  defined inductively by  $B_1 = B$  and  $B_{k+1} = \frac{d}{dx}B_k + B_k B$  and the curvatures of  $Y(qx) = A(q, x)Y(x)$  corresponds to the reduction of  $A(q, q^{p-1}x) \cdots A(q, x)$  modulo  $q^p - 1 = 0$ .

One could still dream of a suitable notion of  $q$ -deformation of a differential system which would preserve the arithmetic properties of the curvatures and therefore try to understand the arithmetic of  $q$ -deformations. In [Pul08], Pulita proved that in the  $p$ -adic setting, one can deform a  $p$ -adic linear differential system into a  $q$ -difference system so that they share the same fundamental solution matrix for small values of  $q$ . This deformation is a priori not so satisfactory since it doesn’t deform the set of solutions.

As first step towards the comprehension of the arithmetic of  $q$ -deformations would be to build a canonical way to  $q$ -deform Picard-Fuchs differential equations. For this class of differential equations, the Grothendieck conjecture is known to be true by the work of Katz ([Kat72]). Moreover, thanks to their geometric origin, these connections carry a natural Frobenius structure. In [Aom90], Aomoto constructs a  $q$ -analogue of the de Rahm complex for algebraic  $d$ -dimensional torus and found some  $q$ -differences systems corresponding to the cohomology of this complex. In [TV97], Tarasov and Varchenko have constructed  $q$ -KZ equations by considering a  $q$ -de Rahm complex for vector groups in the spirit of Aomoto’s construction. In both cases, the definition of the discrete one forms is made possible because of their is a canonical choice for the coordinates which are “ $q$ -deformed”. More recently, the notion of  $q$ -de Rahm complex regain interest with the work of Scholze ([Sch17]). For  $A$  a commutative ring, Morrow and Tsuji relying on André’s formalism ([And01]) defined the  $q$ -de Rahm complex of a commutative  $A$ -algebra  $S$  satisfying the following set-up which they called  $qDR1$ :  $S$  must be endowed with  $d$  commuting  $A$ -algebra endomorphisms  $\gamma_1, \dots, \gamma_d$  and an element  $q \in A$  such that  $(q - 1)$  is a non-zero divisor in  $S$  and so that  $\gamma_i \equiv id$  modulo  $q - 1$  for each  $i$  ([MT21]). In that context, Morrow and Tsuji defined the notion of modules with  $q$ -connections and  $q$ -Frobenius structure where the Frobenius sends  $q$  on  $q^p$ . Their definition of the  $q$ -Frobenius structure is quite similar to the one employed by Vargas-Montoya in his thesis [VM21] where he proved that  $q$ -difference modules endowed with a  $q$ -Frobenius structure sending  $q$  to  $q^p$  tend to differential modules with a Frobenius structure as  $q$  goes to one. In [Shi20], Shirai produces a  $q$ -deformation of the de Rahm cohomology of the Legendre family of elliptic curves by using an equivalence of category obtained by Tsuji ([Tsu17, §6 and §7]). By deforming the unit root of the hypergeometric differential equation, which is an eigenvalue of the Frobenius, he was able to show that the  $q$ -connection associated to this  $q$ -deformation was the  $q$ -hypergeometric equation with parameters  $(q^{\frac{1}{2}}, q^{\frac{1}{2}}, q)$ . This  $q$ -difference equation is a  $q$ -deformation of the hypergeometric differential equation with parameters  $(\frac{1}{2}, \frac{1}{2}, 1)$ , the Picard-Fuchs differential equation of the Legendre family. Moreover, Shirai proved that his  $q$ -deformation comes naturally with a  $q$ -Frobenius structure. In that direction, one could ask if the method of Shirai would also hold for general hypergeometric differential equations with a Frobenius structure. This would give a positive answer to the question of Vargas-Montoya concerning the existence of a  $q$ -Frobenius structure for a larger class of  $q$ -hypergeometric equation (see [VM21, §4.4]). Another approach would consist in trying to endow the Monsky-Washnitzer cohomology of the Legendre

family over  $\mathbb{Z}_p[[q-1]]$  with a discrete  $q$ -action similarly to the lift of the Frobenius detailed in [van86, §7.11]. This is work in progress with L. Di Vizio and J. Roques.

### 6.2.2 Arithmetic description for dynamics over genus one curves

One could ask whether a curvature characterization holds for more general difference equations, that is, for dynamics induced by the action of an algebraic group  $G$  on an algebraic variety  $X$ . It appears that an analogue of Theorem 4.1 fails to be true for  $X = \mathbb{P}^1$  already when one replaces the multiplicative group by the additive group which corresponds to the case of difference equations associated to the difference operator  $x \mapsto x + 1$ . In [vdPS97b, page 58, §5.4], the authors provide a counterexample. The article [Roq17] is also concerned with the action of the multiplicative group on  $\mathbb{P}^1$  via  $\mathbb{Z}^* \times \mathbb{P}^1 \rightarrow \mathbb{P}^1$ ,  $(n, x) \mapsto x^n$ . In that setting, Roques proves that a linear Mahler equation defined over  $\overline{\mathbb{Q}}(z)$  has a  $\overline{\mathbb{Q}}$ -basis of solution in  $\overline{\mathbb{Q}}(z)$  if and only if for almost all  $p \in \mathbb{Z}$ , the reduced equation has an  $\mathbf{F}_p$ -basis of solutions in  $\mathbf{F}_p((z))$  algebraic over  $\mathbf{F}_p(z)$ .

A natural question would be to ask if some similar statements will hold on a genus one curve  $X$ . Writing  $X$  as the multiplicative quotient of  $\mathbb{C}^*$  by some  $\tilde{q}^{\mathbb{Z}}$ , one could try to generalize the curvatures criteria of [10] for  $q$ -difference equations over the field of elliptic functions with respect to the elliptic curve  $\mathbb{C}^*/\tilde{q}^{\mathbb{Z}}$ , that is, for linear difference equations whose difference operator is the translation by a prescribed point of the curve  $X$ . A first step towards any arithmetic characterization in that context would be to develop a classification of the singularities of these  $q$ -difference operators as well as some formal classification of  $q$ -difference operators over elliptic curves (i.e. a discrete analogue of Levelt-Turrittin Theorem).

### 6.2.3 Transcendence of special functions

**Over the projective line** In the context of the  $(\phi, \delta)$ -fields introduced in §4.2.1, the results of [7] show that a non-rational<sup>14</sup> formal power series solution of a linear  $\phi$ -difference equation over the projective line is either rational or  $\delta$ -transcendental. By Praagman's result ([Pra83]), any  $q$ -difference system over  $\mathbb{C}(x)$  has a fundamental solution matrix whose coefficients are meromorphic functions over  $\mathbb{C}^*$  and a similar result holds for the finite difference operator. It is then natural to ask if the dichotomy observed for formal power series solution holds for meromorphic solutions. A counterexample to this dichotomy is given by the function  $\theta_q$  introduced in Example 1.4 which satisfies the  $q$ -difference equation  $\theta_q(qx) = x\theta_q(x)$  and is differentially algebraic over  $\mathbb{C}(x)$ . Indeed  $\delta(\frac{\delta\theta_q}{\theta_q})$  is left invariant by the  $q$ -dilatation and is therefore an elliptic function with respect to the elliptic curve  $\mathbb{C}^*/q^{\mathbb{Z}}$ . The differential algebraicity of  $\theta_q$  over  $\mathbb{Q}$  then follows from the differential algebraicity of any elliptic functions over  $\mathbb{Q}$ . One could then wonder if one could describe the closure of  $\mathbb{C}(x)$  with respect to  $\delta$  inside the ring of meromorphic functions over  $\mathbb{C}^*$  that are solutions of a linear  $q$ -difference system over  $\mathbb{C}(x)$ . The answer to such question is of course intimately connected to the inverse problems described in §6.1.

In [3], it is proved that a non-rational formal power series solution of a linear  $q_1$ -difference equation is  $\sigma$ -transcendental over  $\mathbb{C}(x)$  where  $\sigma(x) = q_2x$  under the assumption that  $q_1$  and  $q_2$  were multiplicatively independent. As corollary, one finds that a non-rational formal power series solution of a linear  $q_1$ -difference equation over  $\mathbb{C}(x)$  is algebraically independent from any solution of a  $q_2$ -difference equations. As above, one can ask if this result still holds for meromorphic solutions. For instance, one should be able to prove that  $\theta_{q_1}$  and  $\theta_{q_2}$  are algebraically independent but to prove that two meromorphic solutions of linear  $q_1$  and  $q_2$ -difference equations are algebraically independent is more challenging. The inverse problem for the corresponding  $\sigma$ -Galois theory should give some hints on the form of  $\sigma$ -algebraic relations which can be satisfied by the meromorphic solutions of  $q_1$ -difference equations and thereby on the type of meromorphic solutions of  $q_1$ -difference equations which might be algebraically independent with solutions of  $q_2$ -difference equations. However in order to prove some statement about the algebraic independence of these two worlds, one will have to use some analytic arguments. It would be perhaps interesting to rephrase the problem of the algebraic independence in more geometric terms. Indeed, a linear  $q_i$ -difference equation corresponds to a vector-bundle over the Tate curve  $\mathbb{C}^*/q_i^{\mathbb{Z}}$ .

<sup>14</sup>Here rational means that the function belongs to  $\mathbb{C}(x^{\frac{1}{n}})$  for some positive integer  $n$ .



One could ask similar questions for non-linear equations. In that case, the  $j$ -function provides an example of function which satisfies non-linear  $q$ -difference equations for any positive integer  $q$ . Indeed, the  $j$ -invariant satisfies  $F_q(j(x), j(qx)) = 0$  where  $F_q(X, Y)$  is the modular polynomial ([Sil94, Theorem 6.3]). As a first step towards the nonlinear case, one could try to describe the nonconstant meromorphic functions  $f$  such that

$$\phi(f) = R(f) \text{ and } \sigma(f) = S(f),$$

for some rational fractions  $R$  and  $S$  and  $\phi$  and  $\sigma$  two automorphisms of the projective line. In the situation where  $\sigma$  is a  $q$ -difference operator and  $\phi$  is a finite difference operator, the works of Ritt ([Rit22]) in the case  $S = id$  and generalized by [Ger47] to arbitrary rational functions  $S$  proved that  $f$  must be a linear function of exponentials, cosinus and Weierstrass functions. Let us also mention the work of Di Vizio and Fernandes which characterizes the differential algebraic solutions of non-linear  $q$ -difference equations of the form  $f(qx) = R(f(x))$  (see [DVF21]).

**Over elliptic curves** In [de 21], de Shalit proved an analogue of Schaefer-Singer's Theorem 3.14 for  $\phi$  and  $\sigma$ -difference equations whose coefficients are elliptic functions. More precisely, for a lattice  $\Lambda_0 \subset \mathbb{C}$ , de Shalit considered the field  $K$  defined as the union of the fields of  $\Lambda$ -elliptic functions for any sublattice  $\Lambda$  of  $\Lambda_0$ . The automorphisms  $\sigma$  and  $\phi$  are two independent isogenies. In that context, de Shalit proved that a formal power series solution of a linear  $\phi$ -difference equation and of a linear  $\sigma$ -difference equation must belong to  $\cup_{\Lambda \subset \Lambda_0} K[x, \frac{1}{x}, \zeta(x, \Lambda)]$  where  $\zeta(x, \Lambda)$  is the zeta function associated to the lattice  $\Lambda$ . The proof of de Shalit relies on Atiyah's classification of vector bundles on elliptic curves. It would be interesting to study similar questions by replacing the isogenies by the translation by a point of the lattice. More precisely, if  $\phi$  is the translation by a point  $\omega_3$  of  $\mathbb{C}/\Lambda_0$  and  $\sigma$  is the translation by a point  $\omega'_3$  of  $\mathbb{C}/\Lambda_0$  so that  $\omega_3$  and  $\omega'_3$  are  $\mathbb{Z}$ -linearly independent, can we describe the formal power series solutions simultaneously of

- a linear  $\phi$ -equation and a linear differential equation with coefficients over  $K$ ?
- a linear  $\phi$ -equation and a linear  $\sigma$ -equation?

Such a description could perhaps yield applications to the study of the generating series for walks in the first quadrant since these series are solutions of similar difference equations.

Analogues of Schaefer-Singer's results for difference equations over elliptic curves would then allow us to use the general machinery of the proofs in [3] and [7] to reach some differential or difference transcendence statements.

Finally, it would be interesting to understand if one could prove and generalize the functional transcendence results proved above via a model theoretic approach for instance by using the Zilber's dichotomy obtained by Bustamante for  $DCF A_0$  ([Med20]).

## 6.3 Galoisian approach for walks

### 6.3.1 Two-dimensional biquadratic kernel equations

**The variables  $(x, y)$  for the position of the endpoint** The algebraic nature of weighted walks is now quite well understood thanks to the recent works of many authors but there are some counting problems or probabilistic objects which also yield to kernel equations of the following form

$$(6.2) \quad K(x, y)Q(x, y) = b(x, y) - a_1(x, y)Q(x) - a_2(x, y)Q(y),$$

where  $K$  is a biquadratic polynomial and  $a_1, a_2$  and  $b$  are bivariate polynomials in  $k[x, y]$  for some algebraically closed normed field  $(k, |\cdot|)$ . For instance, in [BOX20], the authors introduced the generating function  $Q(a, b, c, x, y, t) = \sum q(h, v, u, i, j, n) a^h b^v c^u x^i y^j t^n$  of Kreweras walks with interacting boundaries. It counts the number  $q(h, v, u, i, j, n)$  of Kreweras's walk starting at  $(0, 0)$  ending at  $(i, j)$  with  $h$  visits of the horizontal axis,  $v$  visits of the vertical axis and  $u$  visits at the origin. This generating series satisfies

the following equation

$$\begin{aligned} K(x, y)Q(a, b, c, x, y, t) &= \frac{1}{a}((a-1)y - taA(x, y))Q(a, b, c, x, 0, t) + \dots \\ &\dots + \frac{1}{b}((b-1)x - tbB(x, y))Q(a, b, c, 0, y, t) + C(x, y) \end{aligned}$$

where  $A, B, C$  and  $K$  are bivariate polynomials and  $K$  is biquadratic (see Theorem 1 in [BOX20]). Note also that the generating series of the invariant measure for a random walk satisfies an equation of the form (6.2) (see [FIM99, Eq. 1.3.6]).

It seems therefore accurate to develop a Galoisian approach of kernel functional equations of the form (6.2). This Galoisian approach will consist in three steps:

- *Step 1: Uniformization and dynamical functional equation:* this step will consist in connecting the kernel functional equation to a dynamical equation of the form  $\phi(F) = \alpha F + \beta$  where  $\phi$  is the composition of two involutions of a genus zero or one algebraic curve.
- *Step 2: Galoisian approach and invariants:* relying on the dynamical functional equation and on parametrized Galois theory, one would like to construct a theoretical algorithm which will determine the differential algebraic properties of the generating series  $Q(x, y)$  with respect to  $x$  and  $y$  in terms of the combinatorial data of the counting problem (weights, directions).
- *Step 3: Implementation and combinatorial interpretation*

*Step 1: Uniformization and dynamical functional equation*

Assuming that  $K(x, y)$  is an irreducible polynomial, one defines the kernel curve  $E$  as the compactification in  $\mathbb{P}^1 \times \mathbb{P}^1$  of the zero set of  $K(x, y)$ . One can then consider two involutions  $\iota_1, \iota_2$  of  $E$  defined respectively by  $\iota_1(x, y) = (x, y')$  and  $\iota_2(x, y) = (x', y)$ . Let me first describe how one could get easily a linear difference equation over  $k(E)$  out of (6.2) if one does not take into account the convergence domain of the sectional generating series. Evaluating (6.2), for  $(x, y) \in E$  and  $\iota_1(x, y) = (x, \iota_1(y)) \in E$ , one finds

$$(6.3) \quad 0 = \frac{b}{a_1} - Q(x) - \frac{a_2}{a_1}Q(y)$$

$$(6.4) \quad 0 = \iota_1\left(\frac{b}{a_1}\right) - Q(x) - \iota_1\left(\frac{a_2}{a_1}\right)\iota_1(Q(y)).$$

Subtracting (6.4) from (6.3) and noting that  $\iota_2(y) = y$ , we deduce the following equation

$$(6.5) \quad \phi(Q(y)) - \alpha Q(y) = \beta,$$

where  $\phi = \iota_2 \circ \iota_1$ ,  $\alpha = \frac{a_2}{\iota_1(\frac{a_2}{a_1})}$  and  $\beta = \frac{\iota_1(\frac{b}{a_1}) - \frac{b}{a_1}}{\iota_1(\frac{a_2}{a_1})}$ . The equation (6.2) is now a non-homogeneous linear difference equation of rank one over the difference field  $(k(E), \phi)$ . However, since the sectional series converge only on the unit disk, the analytic uniformization is much more technical and tedious than the small computations above though it would lead to the same difference equation (see for instance [KR12] and [DR80] in the archimedean setting and [6] in the non-archimedean framework). Indeed these transcendental uniformizations require some precise norm estimates on the transcendental local coordinates on the kernel curve  $E$  which are quite difficult to obtain with arbitrary combinatorial parameters (weights or directions).

However, it might be possible to bypass this long and technical uniformization by a formal algebraic construction which would lead to a  $(\phi, \delta)$ -ring of solution  $R$ . Applying the parametrized Galois theory of [21] to  $R$ , one could find some necessary conditions to guarantee the differential algebraicity of the series but in order to show that these conditions will be sufficient, we would need to understand the differential algebraic properties of the  $\phi$ -constants of  $R$ . Let me illustrate this discussion on the following example.

**Example 6.1.** Let  $f$  be a meromorphic solution of the linear difference equation

$$(6.6) \quad f(x+1) = f(x) + 1,$$

It is easily seen that  $\frac{df}{dx}$  is 1-periodic by differentiating (6.6). Clearly,  $b = 1$  has a telescoper as in Proposition 3.2 since  $\frac{d1}{dx} = \phi(0) - 0$ . But, a 1-periodic function is not necessarily differentially algebraic over  $\mathbb{Q}$  (for instance,  $\frac{1}{\Gamma}(\sin(2\pi x))$  is 1-periodic and is differentially transcendental over  $\mathbb{Q}$  by an easy corollary of Hölder's theorem). However, if one knows that  $f$  is  $\tau$ -periodic for some non-zero complex number  $\tau$  which is linearly independent from 1 then the function  $\frac{df}{dx}$  is  $(1, \tau)$ -periodic and thereby an elliptic function for the elliptic curve  $E = \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$ . Using the differential algebraic properties for elliptic functions, one conclude that  $f$  is differentially algebraic over  $\mathbb{Q}$ . In that example  $R = \mathbb{Q}[f, f', \dots, f^{(n)}, \dots]$  and  $R^\phi \subset \mathbb{C}(E)$ .

In order to prove that the necessary conditions for differential algebraicity are sufficient, one would like to adapt the Tutte's invariants method developed in [BBMR] to this more general setting. For genus one walks, the works of [BBMR] in the unweighted case and of [1] shows that if the generating series is differentially algebraic then there exists a decoupling pair  $(f(x), g(y)) \in \mathbb{Q}(t)(x) \times \mathbb{Q}(t)(y)$  (see (5.6)). Using the functional equation (5.1), one can show that the function of  $K(x, 0, t)Q(x, 0, t) - f(x)$  is a *weak invariant* that is analytic over some open region of the complex plane in the variable  $x$  and satisfies some symmetry over some 'trace' of the kernel curve. This allows to reformulate the functional equation satisfied by  $Q(x, 0, t)$  as a boundary value problem (see [Lit00, Chap. 3, Lemma 1 and 2]) and thereby to express the sectional generating series as a rational function in some canonical invariant  $w(x)$ . The advantages of this approach are numerous. The first one is that one only needs to work in the initial complex plane in  $x$  and not with a universal cover of the curve and thereby with the transcendental uniformization of the generating series. The second asset of this method is that it produces an explicit differential algebraic equation for the series  $Q(x, 0, t)$  from the one satisfied by the canonical invariant  $w(x)$ . Therefore, it is quite crucial to interpret the necessary conditions for differential algebraicity in terms of decoupling pairs and invariants, which lead us to the second step of the algorithm.

*Step 2: Galoisian approach and invariants*

In [2], we proved that the algebraic nature of the Laplace transform of a semimartingale reflected Brownian motion (SRBM) in a two-dimensional wedge is entirely captured by the algebraic properties of a meromorphic solution of an equation of the form

$$\phi(y) = \alpha y,$$

where  $\phi$  is the composition of two involutions  $\iota_1, \iota_2$  of the Riemann sphere and  $\alpha = \frac{\gamma}{\iota_1(\gamma)}$  for  $\gamma$  a nonzero rational function. This equation is of the form (6.2) with  $\beta$  equal to zero. In that particular setting, we develop a multiplicative version of the notion of invariants and decoupling introduced in [BBMR] and succeed to classify entirely the differential algebraic behavior of the Laplace transform with respect to the reflection angles  $\beta, \delta, \epsilon$  of the motion (see Figure 5) More precisely, we proved that, for an automorphism  $\phi$  of infinite order, the Laplace transform is differentially algebraic if and only if  $\alpha$  or  $\alpha^2$  is decoupled, that is, of the form  $\frac{\eta}{\iota_1(\eta)}$  for some nonzero rational function  $\eta$  fixed by  $\iota_2$ .

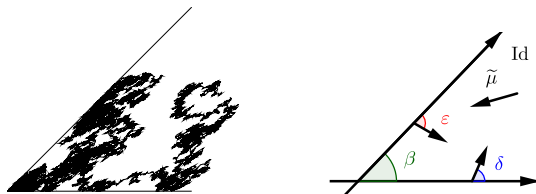


Figure 5: Semimartingale reflected Brownian motion in a cone

This particular case seems to be more general and intrinsically connected to the fact that the norm of  $\alpha$  with respect to the automorphism  $\iota_1$  is equal to 1. We would like to develop a general criteria to compute the parametrized Galois group of an equation of the form

$$(6.7) \quad \phi(y) = \alpha y + \beta,$$

where  $\iota_1, \iota_2$  are two involutions of a function field  $K$  of genus zero or one,  $\phi = \iota_2 \circ \iota_1$  and  $\alpha, \beta \in K$ . [21, Proposition 3.8] classifies almost all the differential algebraic relations which can be satisfied by the solutions of difference equations of the form (6.7) in terms of the coefficients of the equation  $\alpha$  and  $\beta$ . Some of these general conditions on the coefficients have been already specialized to the case of a genus zero base field (see for instance [21, Corollary 3.4]) and partially in the elliptic situation described above (see [ADR21, §4]). Therefore, one should investigate the two following questions:

- could one find an explicit Galoisian criteria for the differential transcendence of the solutions of (6.5) combining the general approach of [21] and some results in [ADR21].
- assuming that  $\alpha \iota_1(\alpha) = 1$  and  $\iota_1(\beta \cdot \iota_1(\gamma)) + \beta \iota_1(\gamma) = 0$ , could we simplify the Galoisian criteria and merge the invariant approach of [BBMR] for  $\alpha = 1$  and [2] for  $\beta = 0$  to develop a suitable notion of invariants and decoupling for (6.7)?
- could one develop an algorithm as in [1] in order to test the decoupling of the equation (6.7)?

*Step 3: Implementation and combinatorial interpretation*

In [1], we proved that the differential algebraicity of the generating series is not entirely determined by the set of directions but by some polynomial conditions on the weights. This interpretation is however not completely satisfactory from a combinatorial point of view. In [CMMR17], the authors have adapted some probabilistic notions such as the drift to define subfamilies of weighted models, which they call *universality classes* since they met common algebraic behaviour. According to [CMMR17], a weighting is central if all paths with the same length, start and end points have the same probability. This combinatorial notion can be reinterpreted in a more geometrical way via [CMMR17, Theorem 13] as follows: A weighting  $\mathcal{W}$  of an unweighted model is central if and only if there exist some non-zero complex numbers  $\alpha_1, \alpha_2, \beta$  such that

$$(6.8) \quad K_{\mathcal{W}}(x, y, t) = K(\alpha_1 x, \alpha_2 y; \beta t),$$

where  $K_{\mathcal{W}}$  is the kernel polynomial of the weighted model  $\mathcal{W}$  and  $K$  the Kernel of the unweighted one. If (6.8) is satisfied, one can construct an isomorphism  $\phi$  from  $E_{\mathcal{W}}$  the kernel curve attached to  $K_{\mathcal{W}}(x, y, t)$  to  $E$  the kernel curve corresponding to  $K(x, y, \beta t)$  which will commute with the canonical involutions on both sides. If a weighting  $\mathcal{W}$  is central then the generating series  $Q_{\mathcal{W}}(x, y, t)$  of the weighted model coincides with  $Q(\alpha_1 x, \alpha_2 y, \beta t)$  where  $Q(x, y, t)$  is the generating series of the unweighted model (see [CMMR17, Prop. 19]). A central weighting therefore preserves the algebraic nature of the generating series. One might therefore wonder if the combinatorial notion of *central weighting* would be sufficient to characterize the differential algebraicity of a weighted model from the differential algebraicity of the unweighted one. For the Gouyou-Beauchamps model given by Figure 6, the condition for central weighting (Example p 24 in [CMMR17]) coincides with the condition for differential algebraicity  $d_{1,0}d_{-1,0} - d_{-1,1}d_{1,-1} = 0$  given in [1, Proposition 6.13].

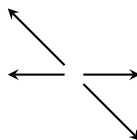


Figure 6: Gouyou-Beauchamps model

However, the weighted model  $w_{II.B.6}$  of Figure 4 seems to be out the realm of the notion of central weighting. It might therefore be interesting to implement the algorithm proposed in [1] to see if one cannot find a combinatorial interpretation of the polynomial conditions on the weights which will encompass the notion of central weighting. This would be a good research topic for a masters student.

**Specialization of the variables** The article [6] is concerned with the study of the differential algebraicity of the generating series  $Q(x, y, t)$  with respect to  $t$ . By Theorem 5.11, the differential algebraicity in  $x$  implies the differential algebraicity in  $t$ . The converse is also true as a consequence of [1] (see [Dre21]). The differential behaviour of the full generating series is therefore almost completely understood. One can therefore consider now the algebraic nature of the specializations of the generating series. They are essentially of two types: the series of the form  $Q(x, y, t_0)$  for  $t_0 \in \overline{\mathbb{Q}}$  and those of the form  $Q(x_0, y_0, t)$  for  $(x_0, y_0) \in \overline{\mathbb{Q}}^2$ . Among these specializations, one finds  $Q(0, 0, t)$  the generating series counting the number of excursions, the generating series  $Q(1, 1, t)$  of nearest-neighbour walks and  $Q(1, 0, t)$  the generating series of walks ending on the horizontals.

*Specialization of the variables  $(x, y)$*  It seems a priori difficult to relate the algebraic nature of the complete generating series  $Q(x, y, t)$  to the nature of its specialization  $Q(x_0, y_0, t)$ . For instance, for the five unweighted genus zero models whose complete generating series  $Q(x, y, t)$  is differentially transcendental in the  $t$ -variables, the generating series  $Q(0, 0, t)$  are equal to 1 and therefore holonomic but the generating series  $Q(1, 1, t)$  are non-holonomic by [MR09, MM14]. These non-holonomicity results were proved via a delicate singularity analysis showing that the specialized generating series had an infinite number of singularities. For the 51 unweighted genus one models, the authors of [BRS14] proved that the series  $Q(0, 0, t)$  was non-holonomic. Their proof relies on the fact that the sequence of coefficients  $(a_n)$  of an holonomic power series  $\sum_n a_n t^n$  with integer coefficients must have a certain asymptotic behavior. More precisely, if  $a_n$  behaves asymptotically as  $K\rho^n n^\alpha$  with  $\rho$  transcendental and  $\alpha$  irrational then  $\sum_n a_n t^n$  is not holonomic. Adapting some asymptotical analysis coming from local limit theorems in probability, the authors of [BRS14] were able to conclude that the generating series counting excursions of the 51 unweighted models were not holonomic. The methods quoted above cannot be adapted to study the differential algebraicity of the specialized series (because a differentially algebraic function may have an infinite number of singularities). The Galois theory fails however to give an answer in that context too. Indeed, if one specialize the variables  $x$  and  $y$ , one destroys the dynamical structure behind (6.2) and there is almost no hope to find a dynamical equation for  $Q(x_0, y_0, t)$  by lack of symmetries in the variable  $t$ . This problem is therefore close to the question of the relation between the transcendence of a special function and the transcendence of its values. Thus, one could try to apply some Mahler method to this context. Using the non-archimedean uniformization of [6], one can analytically prolong  $Q(x, 0, t)$  as well as  $Q(0, y, t)$  as meromorphic solutions of functional equations of the form  $\phi(y) = y + b$  where  $b$  is in  $C(E)$  where  $C$  is an algebraically closed completion of  $\mathbb{Q}(t)$  and  $E$  is the kernel curve. It might be interesting to try to adapt Mahler's method over the function field  $\mathbb{Q}(t)$  to study the differential transcendence of the series  $Q(x_0, 0, t)$  or  $Q(0, y_0, t)$  for  $(x_0, y_0) \in \mathbb{Q}(t)$ . The differential transcendence of the complete generating series  $Q(x_0, y_0, t)$  for  $x_0, y_0 \neq 0$  seems however very difficult to obtain via this process.

*Specialization of the variable  $t$ :*

For  $t_\alpha \in \mathbb{C} \setminus \overline{\mathbb{Q}}$ , the irreducibility and the genus of the kernel curve  $E_{t_\alpha}$  are entirely encoded by the set of directions. In [DR80], the authors proved that for any weighted model and  $t_0 \in [0, 1]$ , the kernel curve  $E_{t_0}$  is isomorphic over  $\mathbb{C}$  with the generic member  $E_t$  of the pencil of curve. Moreover, Dreyfus and Raschel proved that the generating series  $Q(x, 0, t_0)$  and  $Q(0, y, t_0)$  satisfy equations of the form  $\phi(y) = y + b$  for some  $b$  in  $\mathbb{C}(E_{t_0})$ . These equations are nothing but the specialization of the functional equations satisfied by  $Q(x, 0, t_\alpha)$  and  $Q(0, y, t_\alpha)$  by a  $\mathbb{Q}$ -morphism sending  $t_\alpha$  to  $t_0$ . Therefore, one can still apply the parametrized Galois theory of [21] to study the differential algebraic properties of  $Q(x, 0, t_0)$  and  $Q(0, y, t_0)$ . In the genus zero case, there is some ongoing work by Bostan, Di Vizio and Raschel. In order to study this question for the genus one case, one has to be able to answer to the following arithmetical questions:

1. Given a non-torsion point  $P \in E_t(\mathbb{Q}(t))$  and the corresponding section  $\mathcal{P}$  of the elliptic surface  $\mathcal{S}$  as in §5.3, is it possible to determine for which values of  $t_0 \in [0, 1] \cap \overline{\mathbb{Q}}$  the point  $\mathcal{P}.E_{t_0}$  intersection of the section  $\mathcal{P}$  and the fiber  $E_{t_0}$  is torsion?

2. Given two linearly independent points  $P, Q \in E_t(\mathbb{Q}(t))$  and the corresponding section  $\mathcal{P}, \mathcal{Q}$  of the elliptic surface  $\mathcal{S}$ , is it possible to determine for which values of  $t_0 \in [0, 1] \cap \overline{\mathbb{Q}}$  the points  $\mathcal{P}.E_{t_0}, \mathcal{Q}.E_{t_0}$  are linearly dependent?

A recent publication by Corvaja, Demeio, Masser and Zannier ([CDMZ19]) gives precise answers to the first question. In particular, the authors study for an elliptic scheme  $\mathcal{E}$  over an affine algebraic curve  $B$  and a non-torsion section  $\phi : B \rightarrow \mathcal{E}$  the distribution of the set of points in  $B$  where  $\phi$  takes a value which is torsion, that is,  $T_\phi := \{b \in B \mid \exists n. \phi(b) = 0\}$ . If the elliptic scheme  $\mathcal{E}$  is given by a Weierstrass equation  $y^2 = (x - \alpha_1)(x - \alpha_2)(x - \alpha_3)$  and  $\phi = (x_\phi, y_\phi)$ , the search of the values of  $b$  for which  $n.\phi(b)$  is torsion amounts to finding the poles in  $B$  of  $x_{n\phi}$  where  $n.\phi = (x_{n\phi}, y_{n\phi})$ . The set  $T_\phi$  is always infinite but its points are sparse. Indeed, Silverman's bounded height theorem asserts that given independent points  $P(t), Q(t)$  on a non-isotrivial elliptic curve  $E(t)$  defined over  $K(t)$  with  $K$  a number field, there is a constant  $c$  such that the absolute logarithmic height  $h(\tau) < c$  for any algebraic number  $\tau$  such that  $P(\tau), Q(\tau)$  become dependent in  $E(\tau)$  (see [Sil83] and [Zan12, Appendix C]). The results of [DR17], connecting the finiteness of the group of the walk to the holonomy of the generating series do not depend on the fact that the length parameter  $t$  is transcendental or not. Therefore, if one could adapt in an effective way the strategy of [CDMZ19] to the genus one models with an infinite group, one would be able to determine for which values of the parameter  $t_0 \in [0, 1] \cap \overline{\mathbb{Q}}$ , the specialized generating series  $Q(x, y, t_0)$  will become  $x$ - $y$ -holonomic.

The second arithmetical question seems more demanding and I was not able to find some references on the subject outside Silverman's bounded height theorem. Since the characterization of the differential algebraicity of [1] by the existence of certificates is valid for algebraic values of the length parameter  $t$ , this arithmetical question would be the first step in order to find for which values  $t_0$ , a differentially transcendental generating series  $Q(x, y, t)$  of a weighted model of genus one will specialize onto a differentially algebraic generating series  $Q(x, y, t_0)$ .

### 6.3.2 Three dimensional walks with small steps and two dimensional walks with big steps

In [BBMM21, §2], Bousquet-Mélou, Bostan and Melczer consider a positive integer  $d$  and a finite subset  $\mathcal{S} \subset \mathbb{Z}^d$  and walks that take their steps in  $\mathcal{S}$ , start from the origin and are confined in  $\mathbb{N}^d$ . Denoting by  $q(i_1, \dots, i_d, n)$  the number of such walks with  $n$  steps and ending at  $(i_1, \dots, i_d)$ , they consider the generating series

$$Q(x_1, \dots, x_d, t) = \sum_{(i_1, \dots, i_d, n) \in \mathbb{N}^{d+1}} q(i_1, \dots, i_d, n) x_1^{i_1} \dots x_d^{i_d} t^n.$$

Under the condition that the model of walks has only *small backward steps*, they proved that the following functional equation holds

$$(6.9) \quad (1 - tS(x_1, \dots, x_d, t))Q(x_1, \dots, x_d, t) = 1 + t \sum_{\emptyset \neq I \subset \llbracket 1, d \rrbracket} \left( (-1)^{|I|} Q_I(x_1, \dots, x_d, t) \sum_{s \in \mathcal{S}, s_i = -1 \forall i \in I} x_1^{s_1} \dots x_d^{s_d} \right),$$

where  $S(x_1, \dots, x_d, t) = \sum_{(s_1, \dots, s_d) \in \mathcal{S}} x_1^{s_1} \dots x_d^{s_d}$  is the step polynomial and  $Q_I$  is the specialization of  $Q(x_1, \dots, x_d, t)$  where each  $x_i, i \in I$  is set to 0. The equation (6.9) is a generalization of (5.1) to the case of a  $d$ -dimensional walk with arbitrary large forward steps. In [BBMM21, §3], Bousquet-Mélou, Bostan and Melczer introduced the notion of orbit associated to a model of walks and were able to find explicit expression for the generating series when the orbit was finite and some orbit sums were non-zero.

**Two dimensional walks with small backward steps** When  $d = 2$ , the step polynomial  $S$  is of the form  $S(x, y) = \sum_{(i,j) = (-n_1, -n_2)}^{(m_1, m_2)} d_{i,j} x^i y^j$  where the  $d_{i,j}$  belong to  $\{0, 1\}$ ,  $n_1, n_2 \in \{0, 1\}$  and  $d_{-n_1, -n_2} d_{m_1, m_2} \neq 0$ . In that situation the kernel polynomial  $K(x, y, t)$  can be defined as  $K(x, y, t) = x^{n_1} y^{n_2} - t x^{n_1} y^{n_2} S(x, y)$ . The corresponding Kernel curve is an algebraic curve whose genus would be strictly greater than one if the model of the walks has some large forward step and the curve is non-singular (see [5, (2.2)] for a precise formula for the genus  $g$  of the kernel curve). Moreover, if  $K$  is not biquadratic, there is a priori no straightforward analogue of the fundamental involutions attached to a model of walks with small steps (see §5.1). However, the definition of orbit in [BBMM21] suggests that there should be

some algebraic dynamics involved. A first step towards the study of two dimensional walks with large forward steps consists in finding a suitable geometric interpretation of the orbit of [BBMM21]. Such an orbit could perhaps be interpreted as the action of a certain group of endomorphisms in the Jacobian variety of the kernel curve. Even if such a nice geometric framework exists, the uniformization of the generating series would be much more challenging since it would involve passing from an open set on the kernel curve to a  $g$ -dimensional torus. This is work in progress with Dreyfus, Roques and Singer.

**Three dimensional walks with small steps** The study of 3-dimensional walks confined in the octant begins with the work of Bostan and Kauers where the authors study walks with step sets up to five elements via Guessing methods ([BK09]). In [BBKM16], Bostan, Bousquet-Mélou, Kauers and Melczer initiated an intensive and systematic study of the 35 548 models of 3D-walks confined in the octant with step sets up to 6 elements. Via symmetry and analysis of trivial models, they were able to reduce to 20804 models. In [BBKM16, Def.2], they define the dimension of a model which is an integer belonging to  $\{0, 1, 2, 3\}$  as follows. For a walk  $w$  of length  $n$  taking its step in  $\mathcal{S} \subset \{-1, 0, 1\}^3 \setminus \{0, 0, 0\}$ , let  $a_s$  be the number of occurrences of  $s \in \mathcal{S}$  in  $w$ . The walk  $w$  ends in the positive octant if and only if the following three linear inequalities hold:

$$(6.10) \quad \sum_{s \in \mathcal{S}} a_s s_x \geq 0, \quad \sum_{s \in \mathcal{S}} a_s s_y \geq 0, \quad \sum_{s \in \mathcal{S}} a_s s_z \geq 0,$$

where  $s = (s_x, s_y, s_z)$ . A model is of dimension at most  $d \in \{0, 1, 2, 3\}$  if there exists  $d$  inequalities in (6.10) such that any  $|\mathcal{S}|$ -tuple  $(a_s)_{s \in \mathcal{S}}$  of non-negative integers satisfying these  $d$  inequalities satisfies the three ones. Then, Bousquet-Mélou, Bostan and Melczer show how 0 and 1-dimensional models have algebraic generating series. They also prove that the behavior of models with dimension 2 can be understood via models of  $2D$ -walks with repeated steps, that is, weighted models of  $2D$ -walks ([BBKM16, §7]). Among the models of dimension 3, they also introduce the class of Hadamard models whose generating series can be computed as the Hadamard product of generating series of walks with small steps of smaller dimension ([BBKM16, §5]). Finally, relying on methods developed in [BMM10], they were able to characterize situations where the group of the walk was finite or infinite. For most of the finite cases (170 cases), they apply some algebraic kernel methods in order to conclude to the  $D$ -finiteness of the generating series. For 19 models with finite group, they were not able to conclude to the  $D$ -finiteness of the generating series. In [DHW16], the authors proved the non- $D$ -finiteness of the 409 models of dimension 2 with infinite group by showing that the asymptotic of the generating series counting excursion of length  $n$  was of the form  $K\rho^n n^{-\lambda}$  for some irrational exponent  $\lambda$ . These asymptotics were obtained thanks to the work of Denisov and Wachtel ([DW15]) and the algorithmic irrational proof developed in [BRS14, §2.4]. More recently, the authors of [BPRT20] have computed the parameter  $\lambda$  for models of dimension 3 yielding the non- $D$ -finiteness of the generating series for some of these models.

If one considers three dimensional walks with small steps, the equation (6.9) becomes of the following form

$$\begin{aligned} K(x, y, z, t)Q(x, y, z, t) = & \quad xyz + t(a_1(y, z)Q(0, y, z, t) + a_2(x, z)Q(x, 0, z, t) + a_3(x, y)Q(x, y, 0, t) \\ & + b_1(x)Q(x, 0, 0, t) + b_2(y)Q(0, y, 0, t) + b_3(z)Q(0, 0, 0, z, t) \\ & + c(t)Q(0, 0, 0, t)), \end{aligned}$$

where  $K(x, y, z, t)$  is a polynomial of degree less than or equal to 2 in each of the variables. Analogously to the dimension 2, one can fix a transcendental value  $t_0 \in \mathbb{C}$  of  $t$  and consider the Zariski closure  $\mathcal{S}$  in  $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$  of the zero set of  $K(x, y, z, t_0)$ .

If  $\mathcal{S}$  is smooth and irreducible then it is a  $K3$ -surface. Such  $K3$  surfaces were studied by Wehler and thereby called *Wehler surfaces*. Let  $\pi_x : \mathcal{S} \rightarrow \mathbb{P}^1 \times \mathbb{P}^1, (x, y, z) \mapsto (y, z)$ . This is a 2 to 1 cover. The involution which permutes the two points in a general fiber yields an involutive automorphism  $\sigma_x$  of  $\mathcal{S}$  since  $\mathcal{S}$  is a  $K3$ . For a very general  $\mathcal{S}$ , that is, with Picard number 3, the divisor classes  $D_x = \{x = \text{constante}\}, D_y = \{y = \text{constante}\}, D_z = \{z = \text{constante}\}$  are generators of the Néron-Severi group. Using the action of the automorphism  $\sigma_x, \sigma_y, \sigma_z$  on the sub-lattice generated by  $D_x, D_y$  and  $D_z$ , one can show that these three automorphisms of  $\mathcal{S}$  generates a subgroup of the group of automorphisms of  $\mathcal{S}$  which is

isomorphic to the free group  $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$ . For two-dimensional walks, an irreducible and smooth kernel curve might have a finite group of the walk. In three dimension, the group of the walk will always be infinite if the kernel surface is smooth and irreducible. In that situation, the general fibers of the projection of  $\mathcal{S}$  to  $\mathbb{P}^1$  given by the projection on the  $z$ -variables are elliptic curves. Fixing a transcendental value  $z_0$ , one could try to use the non-archimedean approach of [6] to uniformize the fiber  $F_{z_0}$  above  $z_0$  as well as the corresponding sectional generating series  $Q(0, y, z_0, t_0), Q(x_0, 0, z_0, t_0), Q(x, y, 0, t_0), Q(x, 0, 0, t_0)$  and  $Q(0, y_0, 0, t_0)$  in order to obtain difference equations with coefficients in the function field of  $F_{z_0}$  and try to apply the Galoisian approach of [21] to study the differential algebraic properties of the solutions of these difference equations.

It would be also interesting to try to understand several of the notions developed in [BBKM16] from a geometric perspective. We list several questions in that direction:

- is the notion of the dimension of a model connected to some geometric dimension?
- is the notion of Hadamard model connected to the existence of some projection of the kernel surface to some varieties of smaller dimension? Are Hadamard models always associated with finite group of the walk?
- can we apply the algorithm developed in [1] to characterize the models of dimension 2 whose generating series is differentially algebraic ?

## 6.4 Non-linear Galois theories for functional equations

The non-linear Galois theory for functional equations is still at its infancy. Relying on Malgrange's original construction ([Mal01]), Casale defines the Galois groupoid of a vector field  $X$  on an algebraic affine, smooth connected variety  $M$  (resp. of a birational endomorphism  $f$  of  $M$ ) as the  $\mathcal{D}$ -envelop of the flow (resp. of the dynamic of  $f$ ), that is, informally the ideal of partial differential equations satisfied by the flow (resp. local diffeomorphisms of the form  $f^k$  with  $k \in \mathbb{N}$ ). In [Ume10], Umemura defined a Galois theory for algebraic differential equations which was extended later on by Morikawa to the case of algebraic difference equations ([Mor09]).

Until now, the comparison between the *algebraic* constructions of Umemura and Morikawa and the *geometric* theory of Malgrange and Casale is mysterious. This comparison is a highly non-trivial task.

There is also a lack of categorical framework in both of these Galois theories. Such a categorical framework, which would extend the Tannakian formalism, would also allow to investigate the direct problem in these non-linear Galois theories. For Malgrange's construction, one would like to understand the relation between Malgrange's groupoid of a rational vector field on an algebraic variety  $M$  and the collection of all the Galois groups of the linear differential equations associated to order  $k$  frame variational equations along an integral curve  $\mathcal{C}$  of  $X$  for any positive integer  $k$  (see [MRS07, §3.4]). In [Cas09, Theorem 2.3], Casale proved that the direct limit of the Galois groups of variational equations along an algebraic integral curve  $\mathcal{C}$  is a subgroup of Malgrange's groupoid. It would be interesting to see if one would not have equality if one replaces  $\mathcal{C}$  by some generic integral curve. Such a result might perhaps allow to construct a categorical framework for the Malgrange's groupoid of vector fields as limit of the Tannakian categories of connections on the order  $k$  frame bundle. Since Malgrange's groupoid is finite dimensional, there should be a uniform bound on the dimension of the Galois groups of the variational equations. The computation of such a bound in terms of the initial data would yield the existence of a, at least theoretical, algorithm for the computation of Malgrange's groupoid in the spirit of Hrushovski's algorithm [Hru02].

Malgrange's and Umemura constructions do not seem to see the algebraic solutions. This is a true obstacle for the existence of a complete Galois correspondence for non-linear functional equations. The question of the existence of algebraic solutions to a non-linear differential equations seems also quite open from an algorithmic point of view. Indeed, given a non-linear differential equation, one doesn't know if there exists a bound on the degree of its algebraic solutions. Some partial results can be found for instance in [ACFG05]. A first step would be to try understand these questions for strongly normal extensions.



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