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Sorbonne Université  
École Doctorale Informatique, Télécommunications et Électronique

# Random generation of executions of concurrent systems

THÈSE

pour obtenir le titre de Docteur de Sorbonne Université  
Spécialité: Informatique

présentée par

Yi-Ting Chen

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# Abstract

Concurrency has an important role in modern systems and programming. It reveals the phenomenon that multiple computations run simultaneously. These interleaved executions cause the so-called “state explosion problem”.

In this thesis, we aim at constructing a probabilistic framework on the executions of concurrent systems for the purpose of random generation. The uniform measure of executions is inspired by trace monoids defined on infinite traces. Trace theory has a strong combinatorial foundation around the Möbius polynomial. The irreducibility of trace monoids implies the strong connectivity of the digraph of cliques. Hence, a dominant eigenvalue exists and determines the growth rate of trace monoids.

In our work, we view the abstract concurrent systems as monoid actions on a finite set of states. This setting encompasses 1-bounded Petri nets. We give two interpretations to a uniform measure of executions for concurrent systems. One is constructed by the elementary cylinders in trace monoids. This uniform measure is realized a Markov chain of states-and-cliques. The other is to study the Parry measure on the digraph of states-and-cliques.

The difficulty to extend to concurrent systems is that the Perron-Frobenius theorem is not applicable. To resolve this problem, we found the spectral property of the irreducible concurrent systems. This allows us to distinguish the main components which determine the characteristic root of the system. We also prove the uniqueness of this uniform measure. The transition matrix can be obtained either from the Markov chain of states-and-cliques or from the Parry measure with the spectral radius of the dominant components.



# Resumé

La concurrence joue un rôle important dans les systèmes et la programmation modernes. Il révèle le phénomène selon lequel plusieurs calculs s'exécutent simultanément. Ces exécutions entrelacées entraînent le "problème d'explosion d'états".

Dans cette thèse, nous visons à construire un cadre probabiliste sur les exécutions de systèmes concurrents à des fins de génération aléatoire. La mesure uniforme des exécutions s'inspire des monoïdes de traces définis sur des traces infinies. La théorie des traces a une solide base combinatoire autour du polynôme de Möbius. L'irréductibilité des monoïdes de traces implique la forte connectivité du digraphe des cliques. Par conséquent, une valeur propre dominante existe et détermine le taux de croissance des monoïdes de traces.

Dans notre travail, nous considérons les systèmes concurrents abstraits comme des actions de monoïdes sur un ensemble fini d'états. Ce paramètre englobe les réseaux de Petri à 1-bornés. Nous donnons deux interprétations à la mesure uniforme des exécutions pour les systèmes concurrents. La première interprétation donne la valeur de la mesure uniforme sur les cylindres élémentaires du point de vue algébrique sur le monoïde de traces. Cette mesure uniforme est réalisée par une chaîne de Markov d'états-et-cliques. L'autre interprétation s'intéresse à la mesure de Parry sur le digraphe des états-et-cliques.

La difficulté à étendre aux systèmes concurrents est que le théorème de Perron-Frobenius n'est pas applicable. Pour résoudre ce problème, nous avons trouvé la propriété spectrale des systèmes concurrents irréductibles. Cela nous permet de distinguer les principaux composants qui déterminent la racine caractéristique du système. Nous prouvons également l'unicité de cette mesure uniforme. La matrice de transition peut être obtenue soit à partir de la chaîne de Markov d'états-et-cliques, soit à partir de la mesure de Parry avec le rayon spectral des composantes dominantes.



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# Introduction

*Concurrency* [47, 33, 38] has an important role in modern systems and programming. It reveals the phenomenon that multiple computations run simultaneously. These interleaved executions cause non-deterministic problems which are hard to detect and increase the complexity. This is the so-called *state explosion problem* [43]. The size of a state space of a concurrent system may grow exponentially, whereas the number of states grows linearly for sequential systems.

In reality, few computational systems are sequential. Despite the rich theory for sequential systems, most techniques are inadequate for concurrent systems. Therefore, several mathematical formalisms of concurrency have been developed, see the survey in [47]. Depending on the purpose, a suitable model is needed.

## Models of concurrency

One of the classifications of these models is between that *interleaving* models and *non-interleaving* models .

The *interleaving* semantics describe all the atomic actions by *non-determinism* (see Figure 0.1) and ignore the asynchronous behaviors. For example, the transition systems [24, 5] and acceptance trees [22]. Further, Bodini, Genitrini and Peschanski [8] studied the related combinatorics and the uniform generation for a non-deterministic process.

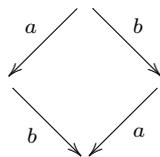


Figure 0.1: Interleaving semantics  $a|b$

However, not all states caused by *interleaving* are significant in the modelization. One of useful technique is to adopt *partial order* reduction. Instead of exploring all possible states and keep all the orderings, we consider only one ordering for each set of concurrent actions. We called it the *noninterleaving* semantics [41] or the *true concurrency* semantics, such as Mazukiewicz traces [32] and event structures [46].

In order to do the manipulations of actions and observe the behavior of the system, *Petri nets* [37, 33, 35, 14, 18] are created as the graphical representation for concurrency. Some typical problems like conflict, concurrency and causality are inherently described in Petri nets. Both semantics mentioned above could be the interpretation of executions in Petri nets [12]. In this thesis, we chose the non-interleaving semantics.

### Random generation and statistical model checking

The *model checking* algorithms [5, 28] help us to ensure the correctness of the system and to convince us that the systems are reliable. The standard numerical methods are based on an exhaustive visit to all reachable states of the specification that we want to check. As the system becomes larger in size, the cost of verification increases and the “*combinatorial explosion problem*” occurs.

Another approach is to adopt a probabilistic point of view to simulate the system for finitely many trials. That is the motivation to study the *random generation* on the desired combinatorial structures, where a uniform measure defined on the structure and an appropriate sampling techniques are required.

### Trace theory and probabilistic framework

In the literature, trace theory [32, 16] has a rich combinatorial structure, which was first considered as *free partially commutative monoids* by Cartier and Foata [10]. Later, Mazukiewicz [31] introduced the name of “trace” and applied to study the behavior of safe Petri nets. Using the same partial order semantics, Diekert [15] extend to more general nets and the synchronizations of nets.

For accomplishing the probabilistic model checking, we need to construct a uniform measure on trace monoids, where we lacked a standard framework until the work of Abbes and Mairesse [2, 3]. Due to the effect of the partial order, the uniform measure on trace monoids was hard to define on finite traces in order to be *consistent* and *memoryless*. The alternative choice is to build the uniform measure on infinite traces, and then consider the weak convergence of finite distributions on trace monoids.

One approach to construct a uniform measure followed the similar idea of the probability measure defined on automata (See overview in Section 1.4). The parameter of a uniform measure is related to the root of Möbius polynomial. By means of the *canonical normal form* of traces on irreducible trace monoids, the sequence of cliques on infinite traces gives rise to an irreducible and aperiodic *Markov chain*. This leads to the study of the *digraph of cliques*, which is strongly connected and aperiodic for irreducible trace monoids. According to the Perron-Frobenius theorem, its adjacency matrix has a dominant eigenvalue, which uniquely defines the uniform measure in a sense of *Parry measure* [34, 30, 25], and determines the growth rate of the trace monoids with fixed length  $n$ .

Hence, the random sampling based on the Markov chain defined on the uniform measure of infinite traces is a good estimate for the larger size of trace monoids. These fundamental results help us to extend the theory to more realistic concurrent systems.

## Abstract concurrent systems

Trace monoids model the concurrency in the form of independence between actions, but lack the feature of the states for real life models. Abbas [1] enrich the trace monoid models by considering a *monoid action* on a finite set of states to define the abstract concurrent systems. This theoretical model encompasses *safe (1-bounded) Petri nets*. With the setting combining the *concurrency* and the *state*, we are able to analyze the future dynamics of the system influenced by the current state. A sink state is adopted by this approach in order to avoid all the trajectories. In other words, a forbidden action in the current state leads to the sink state.

Inherited from trace monoids, the *growth series* of the executions of concurrent systems could be calculated from the *Möbius formula* in a matrix form. Fix an initial state  $\alpha$  and a final state  $\beta$ , all growth series  $G_{\alpha,\beta}(z)$  have the same radius of convergence  $r$  if the system is irreducible. Moreover, with the tools in analytic combinatorics [17], we could deduce that  $r$  is the positive root of the smallest modulus of  $\det \mu(z)$  where  $\mu(z)$  is the Möbius matrix of the system.

## Digraph of states-and-cliques and spectral property

The extension of trace monoids to the concurrent system also realize a Markov chain. Instead of analyzing the digraph of cliques, the Markov chain of executions of the concurrent systems is built on the *digraph of states-and-cliques* (DSC). However, the DSC is not necessarily strongly connected. Therefore, the results for trace monoids and the Perron-Frobenius theorem can't be applied directly. This causes the difficulties to develop th similar framework for concurrent systems.

First, we need to find an adequate notion of *irreducibility* [4] for concurrent systems to understand the dynamics of the system. Then, we analyze the relation between the structure of DSC and the irreducibility of the system. Even though the Perron-Frobenius theorem is not applicable, this allows us to retrieve the dominant eigenvalue of the adjacency matrix of DSC, The result requests some properties for reducible nonnegative matrices [23].

On the other hand, we investigate the spectral radius of the certain submatrices of the system and introduce the *spectral property*. A concurrent system regarded as the action of a trace monoid  $\mathcal{M}(\Sigma, I)$  on a set of states  $X$  has the spectral property, if : for any action  $a \in \Sigma$ , the new concurrent system obtained from removing  $a$  is significantly smaller than the original system. We proved in Theorem 6.2.2 that the spectral property is equivalent to the irreducibility of the concurrent system.

## Probabilistic theory of concurrent systems

With the aid of the spectral property, we prove the uniqueness of the uniform measure defined on the concurrent system. The Markov chain of states-and-cliques goes through only positive nodes, that is to say the node with positive probability. The algorithmic tool are needed in order to compute effectively the probabilistic parameters of the MCSC.

We also have another approach to obtain the transition matrix via the Parry measure on ADSC.

### **Resume of chapter and contribution of the thesis**

Chapter 1 recalls some background in graph theory and the Perron-Frobenius Theorem for primitive matrices and reducible matrices. We also present how to define the probability measure on the words of a finite deterministic automaton. Especially with the Perron right eigenvector, we could obtain a Markovian measure called the Parry measure.

Chapter 2 covers the combinatorics for trace monoids, such as the canonical normal form, the graphical interpretation, the generating series and the Möbius polynomial. We study the augmented digraph of cliques (ADC) where the paths bijects to the traces of fixed-length. We give another point of view to define a uniform measure on trace monoids via the Parry measure on ADC.

Chapter 3 provides a theoretical probabilistic foundation for trace monoids. The results are mainly from [2], but we describes in another way. Some proofs are different from the original article. The probabilistic framework for trace monoids is complete. We define the unique uniform measure on infinite traces and prove the weak convergence of the finite distributions to this uniform measure.

Chapter 4 defines the abstract concurrent systems. The pilot study can be found in [1]. We finish with some interesting examples of the concurrent systems.

Chapter 5 focuses on the digraph of states-and-cliques (DSC). The nodes in DSC are distinguished between positive nodes and null nodes. According to several concurrent models, we find that the DSC are in general not strongly connected and thus the Perron-Frobenius theorem is not applicable.

Chapter 6 introduces the spectral property for concurrent systems. We prove the main theorem in this thesis that a concurrent system with the spectral property is equivalent to the irreducibility of the system. This helps us to retrieve the subgraph which determines the structure.

Chapter 7 defines a uniform measure on concurrent systems previously introduced in [1]. We prove the uniqueness of the uniform measure, which is left open in previous article. This uniform measure can be realized a Markov chain of states-and-cliques.

Chapter 8 gives the effective algorithms to compute the parameters of the uniform measure. We also provides another approach to compute the transition matrix via the Parry measure on ADSC.

At the end of this thesis, we summarize the thesis and indicate several directions for the future work. We also add Appendix for the implementation code in Python and for the output of the random generation with our approach.

# Chapter 1

## Preliminaries on graph theory

Our approach highly depends on the analysis of certain graphs of our mathematical models. In first section, we recall the Perron-Frobenius theorem and the concept of spectral radius. Then we introduce the related notions in graph theory. With the aid of the matrix representation of a graph, the manipulations in linear algebra can be applied. In last section, a uniform measure on the paths of a graph is given by the Perron eigenvalue.

### 1.1 Perron-Frobenius theorem

A matrix or a vector is *positive* (*nonnegative*) if all its entries are positive (nonnegative). Consider a square matrix  $A$ , the maximal modulus of its eigenvalues is called the *spectral radius* of  $A$ , denoted by  $\rho(A)$ .

The Perron-Frobenius theorem can provide us the asymptotic behavior of the powers of a matrix, with exponentially good estimates. The original condition of Perron theorem is for positive matrices. The principal result is that the spectral radius of a positive square matrix is exactly the maximal eigenvalue corresponding to a positive eigenvector.

In this thesis, we will deal with either the adjacency matrices or the probability matrices which are nonnegative in most cases. We restate in Theorem 1.1.3 the main properties of Perron-Frobenius theorem for nonnegative matrices. Some sufficient conditions are needed for this extension in order to get the unique dominant eigenvalue.

A nonnegative square matrix  $A$  is called *irreducible* if for every pair of indices  $i$  and  $j$ , there exists a positive integer  $k$  such that  $(A^k)_{i,j} > 0$ . A nonnegative square matrix  $A$  is called *primitive* if there exists a positive integer  $k$  such that  $A^k > 0$ .

**Proposition 1.1.1.** *A nonnegative matrix  $A$  is irreducible if the related digraph  $G_A$  is strongly connected.*

Let  $A$  be a nonnegative matrix. Fix an index  $i$ . Define the *period of index  $i$*  be the g.c.d. of all natural numbers  $m$  such that  $(A^m)_{ii} > 0$ . We called  $A$  is *aperiodic* if the period of index  $i$  equals 1 for some  $i$ .

**Proposition 1.1.2.** *A nonnegative matrix is primitive if and only if it is irreducible and aperiodic.*

**Theorem 1.1.3** (Perron-Frobenius). *Let  $A$  be a nonnegative matrix with spectral radius  $\rho(A)$ . If  $A$  is primitive, the following statements hold:*

1. *The spectral radius  $\rho(A)$  is an eigenvalue of the matrix  $A$ . And furthermore,  $|\lambda| < \rho(A)$  for all other eigenvalues  $\lambda \in \mathbb{C}$  of  $A$ .*
2.  *$\rho(A)$  is a simple root of the characteristic polynomial of  $A$ .*
3. *The left (right) eigenvector associated with  $\rho(A)$  is the only eigenvector which is strictly positive (up to a scalar factor).*
4. *The eigenvector corresponding to eigenvalues other than  $\rho(A)$  is not positive.*
- 5.

$$\lim_{n \rightarrow \infty} \frac{A^n}{\rho(A)^n} = rl$$

where  $r$  and  $l$  are the right and the left eigenvector associated with  $\rho(A)$  and  $lr = 1$ .

6. *Let  $B$  be a nonnegative matrix with  $0 \leq A < B$ . Then  $\rho(A) \leq \rho(B)$ . Especially if  $B$  is irreducible, then  $\rho(A) < \rho(B)$ .*

The maximal eigenvalue  $\rho(A)$  is called Perron eigenvalue and the corresponding eigenvector is called Perron eigenvector. The fifth statement in Perron-Frobenius theorem is useful to estimate the behavior of the powers of the matrix  $A$ .

## 1.2 Notions of graph theory

A *digraph* or a *directed graph* is a pair  $G = (N, E)$  of disjoint finite sets where  $E \subseteq N \times N$ . The elements of  $N$  are called *nodes* and the elements of  $E$  are called *edges*. A *path* in a digraph  $G$  is a sequence  $x_1 x_2 \dots x_n$  such that  $(x_i, x_{i+1}) \in E$  for  $i = 1, 2, \dots, n-1$ . The positive integer  $n$  is called the *length* of the path. For  $x, y \in N$ , we denote by  $x \xrightarrow{*} y$  if there exists a path from  $x$  to  $y$ .

A digraph  $G$  is *strongly connected* if there exists a path for each ordered pair of nodes in  $N$ . A *subgraph* of  $G$  is a digraph induced by  $(N', E')$  where  $N' \subseteq N$  and  $E' \subseteq E \cap (N' \times N')$ . A *strongly connected component* of  $G$  is a maximal strongly connected subgraph with the maximality in the sense of set inclusion.

An arbitrary digraph can be decomposed into its strongly connected components. Let the collection of sets  $S = \{G_1, \dots, G_p\}$  be the partition of strongly connected components of a digraph  $G$ . For  $i \in \{1, \dots, p\}$ , a component  $G_i$  in  $S$  is a *terminal* strongly connected component if there does not exist an edge pointing to a node outside the component itself.

A digraph  $G = (N, E)$  can be represented by a 0-1 square matrix  $A$  over  $N$  defined by:

$$\forall x, y \in N \quad A_{x,y} = \begin{cases} 1, & \text{if } (x, y) \in E \\ 0, & \text{otherwise} \end{cases} .$$

The matrix  $A$  is called the *adjacency matrix* of the digraph  $G$ . The *spectral radius*  $\rho$  of a digraph is defined to be the spectral radius of its adjacency matrix. A strongly connected component of  $G$  is called *basic* if its own spectral radius equals to the spectral radius of  $G$ .

For  $x, y \in N$ , the number of paths of length  $n$  from  $x$  to  $y$  is  $(A^n)_{x,y}$ . This entry can be calculated with the following formula :

$$(A^n)_{x,y} = \mathbf{1}_x^\top A^{n-1} \mathbf{1}_y$$

where  $\mathbf{1}_x$  is the vector indexed by  $N$ , and defined by  $\mathbf{1}_x(u) = 1$  if  $u = x$  and  $\mathbf{1}_x(u) = 0$  otherwise. Let

$$G_{x,y}(z) = \sum_{n \geq 0} (A^n)_{x,y} z^n$$

be the generating series of the paths from  $x$  to  $y$  with the radius of convergence  $r_{x,y} \in (0, \infty]$ . Define  $r = \min_{(x,y) \in N \times N} r_{x,y}$ . This positive number  $r$  equals the radius of convergence of the matrix series  $\sum_{n \geq 0} A^n z^n$ .

The next proposition describes how to get the radius of convergence of the matrix series  $\sum A^n z^n$ . The first approach comes from the spectral radius of an operator  $A$ ; whereas the second one is obtained after transforming the series into a rational function.

**Proposition 1.2.1.** *Let  $G$  be a digraph with adjacency matrix  $A$ . Then the radius of convergence  $r$  of the series  $\sum_{n \geq 0} A^n z^n$  counting the paths in the graph is:*

- 1)  $r = \rho(A)^{-1}$  if  $\rho(A) \neq 0$ ;
- 2) the smallest positive root of  $\det(I - zA) = 0$ .

**Remark 1.2.2.** *The radius of convergence  $r$  of  $\sum A^n z^n$  is its pole. Recall Pringshiem's theorem in [17], the radius of convergence of a series  $\sum a_n z^n$  with non negative coefficients is its singularity. Moreover, the matrix series is rational which implies that the singularities are poles.*

The framework of paths in graphs have some crucial properties combining graph theory and Perron-Frobenius theorem, see in [17]:

- The strongly-connectedness of the graph corresponds to the irreducibility of its adjacency matrix, which implies that all generating series of paths with fixed initial and final states have the same radius of convergence.

- The irreducibility and aperiodicity of the adjacency matrix implies a dominant eigenvalue, which determines the asymptotics of the coefficients of the generating series of paths.

If the digraph is not strongly connected, we focus on one of its strongly connected component with the largest spectral radius. Then its related submatrix is irreducible and all nodes in this component have the same period. Moreover, the power of irreducible cyclic matrix is of the form of primitive matrix, see in [40]. Hence, we have the following proposition which considers certain submatrix as a primitive matrix.

**Proposition 1.2.3.** *Let  $G$  be a digraph with adjacency matrix  $A$  and let  $r$  be the radius of convergence of  $\sum A^n z^n$ . Then there exists a node  $x$  and a positive integer  $d$  such that, for any multiple  $k$  of  $d$ , the series  $\sum_{n \geq 0} (A^{kn})_{x,x} z^n$  has the radius of convergence  $r^k$ .*

**Remark 1.2.4.** *The integer  $d$  in Proposition 1.2.3 is actually the period of the node  $x$ , and therefore the period of the strongly connected component including  $x$ .*

### 1.3 Properties for reducible matrices

A nonnegative matrix  $A$  is reducible if the related digraph  $G_A$  is not strongly connected. We recall some properties for non negative reducible matrices which is useful for later proofs.

**Proposition 1.3.1.** *Let  $A$  be a nonnegative matrix with spectral radius  $\rho(A)$ . The following statements hold:*

1. [23, Fact 12(b)]  $A$  has a positive right eigenvector corresponding to  $\rho(A)$  if and only if the terminal components of  $G_A$  are precisely its basic components.
2. [23, Fact 6a] For some positive vector  $u$  and for some nonnegative scalar  $\tau$ , such that  $Au = \tau u$ , then  $\tau = \rho(A)$ .

### 1.4 Probability measures on the words of a finite deterministic automaton

This section gives an idea of the proper construction of the uniform measure of words in an automata. We start by defining a probability measure of fixed-length words on a finite automaton in a natural way.

A probability measure on infinite words can be induced by a Markov chain which is consistent and is easy to perform random sampling.

After the introduction of concurrent models in next section, we use similar construction to extend this uniform measure to a general concurrent system. The details are discussed in chapter 7. Our approach will combine the feature of “states” in automata

and the feature of “concurrency” in concurrent models which is more applicable in real systems.

### 1.4.1 Automaton measure

The basics on words can be found in [6].

Let  $\Sigma$  be a finite alphabet. A *one-letter finite automaton*  $\mathcal{A}$  over  $\Sigma$  is a quadruple  $(\Sigma, E, \iota, f)$  where the elements in  $\Sigma$  are called *states*, the pair  $(\Sigma, E)$  is a digraph,  $\iota \subseteq \Sigma$  is the set of initial states and  $f \subseteq \Sigma$  is the set of final states. A word  $x = x_1x_2 \cdots x_n$  over  $\Sigma$  is *accepted* by  $\mathcal{A}$  if and only if 1)  $x_1 \in \iota$ ; 2)  $(x_i, x_{i+1}) \in E$  for  $i = 1, \dots, n-1$ ; 3)  $x_n \in f$ .

Here the labeled edges are not specified as usual since they only depend on the leaving states. The *adjacency matrix* of the finite automaton is defined as the adjacency matrix of the digraph  $(\Sigma, E)$ .

For  $x \in \Sigma$ , let  $\delta_\iota$  be the initial vector or the indicator vector of  $\iota$  where  $\delta_\iota(x) = 1$  if  $x \in \iota$  and  $\delta_\iota(x) = 0$  if  $x \notin \iota$ , for  $x \in \Sigma$  and let  $\delta_f$  be the final vector or the indicator vector of  $f$  where  $\delta_f(x) = 1$  if  $x \in f$  and  $\delta_f(x) = 0$  if  $x \notin f$ ,

The set of all finite words and the set of words with length  $n$  over  $\Sigma$  are denoted by  $\Sigma^*$  and  $\Sigma^n$ . Let  $x = x_1x_2 \cdots x_n \in \Sigma^*$  be a word accepted by the automaton  $\mathcal{A}$ , we define a valuation  $\nu : \Sigma^* \rightarrow \mathbb{R}_{>0}$  by:

$$\nu(x) = \delta_\iota(x_1) \cdot A_{x_1x_2} \cdot A_{x_2x_3} \cdots A_{x_{n-1}x_n} \cdot \delta_f(x_n)$$

where  $(A_{x,y})_{x,y \in \Sigma}$  is the adjacency matrix of the automaton  $\mathcal{A}$ .

By renormalization, define a family of probability measure  $(\nu_n)_{n \geq 0}$  on the  $(\Sigma^n)_n$ :

$$\forall x \in \Sigma^n \quad \nu_n(x) = \nu(x) / \nu(\Sigma^n).$$

We say that  $(\nu_n)_n$  is a family of automaton measures associated with the automaton  $\mathcal{A}$ .

However, we can not apply the Kolmogorov extension theorem to define a probability measure  $\nu_\infty$  on infinite words. In general, the probability measures  $(\nu_n)_n$  are not consistent. That is to say that the equality

$$\nu_n(x) = \sum_{a \in \Sigma} \nu_n(xa)$$

does not hold. In the following two subsections, we construct a subclass of automaton measures which are consistent.

### 1.4.2 Markov measure

A *Markov chain* [29, 9, 21] is a sequence of random variables  $(X_n)_n$  with finite state space  $X$  and *transition matrix*  $P = (P_{i,j})$  if for all  $n$ , for all  $x_i \in X$ , we have

$$P(X_{n+1} = x_j \mid X_0 = x_0, X_1 = x_1, \dots, X_n = x_i) = P(X_{n+1} = x_j \mid X_n = x_i) = P_{i,j}.$$

The transition matrix  $P$  is stochastic satisfying  $P_{i,j} \geq 0$  for all  $i, j$  and  $\sum_j P_{i,j} = 1$  for all  $i$ . The *initial distribution* of a Markov chain is represented by a vector  $v = (v_i)$  with  $v_i = P(X_0 = i)$  and  $\sum_i v_i = 1$ .

Let  $\Sigma^{\mathbb{N}}$  be the set of all infinite words over  $\Sigma$ . The natural prefix ordering  $x \leq y$  between two finite words  $x$  and  $y$  has a natural extension if  $y$  is an infinite word starting with  $x$ , which we still denote by  $x \leq y$ . Let  $x = x_1x_2 \cdots x_n \in \Sigma^*$  be a sequence read by the automaton  $\mathcal{A}$ , the *elementary cylinder of base  $x$*  on  $\mathcal{A}$  is

$$C[x] = \{y = y_1y_2 \cdots \in \Sigma^{\mathbb{N}} \mid y_i = x_i \quad i = 1, 2, \dots, n\} = \{y \in \Sigma^{\mathbb{N}} \mid x \leq y\}.$$

The *Markov measure* [45] induced on  $\Sigma^{\mathbb{N}}$  is defined on the cylinder sets by

$$\nu_{\infty}(C[x]) = v_{x_1}P_{x_1x_2} \cdots P_{x_{n-1}x_n}$$

where  $v = (v_i)$  is the initial distribution of the Markov chain.

### 1.4.3 Parry measure

The original Parry measure is given in [34] which is defined for a shift space or for symbolic dynamic systems. The importance of Parry measure is that it's automatically Markovian.

**Definition 1.4.1** (Parry measure). *Let  $\mathcal{A} = (\Sigma, E, \iota, f)$  be a finite automaton with adjacency matrix  $A$  and initial vector  $\delta_i$ . If  $A$  is primitive, let  $\rho = \rho(A)$  be the Perron eigenvalue of  $A$  corresponding to the right eigenvector  $r = (r_i)$  such that  $\sum_n r_i = 1$ . The Parry matrix  $P$  is the stochastic matrix defined by :*

$$P_{i,j} = A_{i,j} \cdot \frac{r_j}{\rho \cdot r_i}.$$

The Parry measure on the finite automaton  $\mathcal{A}$  is a Markov chain with Parry matrix and initial distribution

$$\forall i \in \Sigma \quad v_i = \frac{r_i \cdot \delta_i(i)}{\sum_{j \in \Sigma} r_j \cdot \delta_i(j)}$$

where  $\delta_i$  is the initial vector.

**Lemma 1.4.2.** *Let  $\mathcal{A}$  be a finite automaton with primitive adjacency matrix. Let  $\nu_{\infty}$  be the Parry measure on  $\mathcal{A}$  and let  $x = x_1x_2 \cdots x_k \in \Sigma^*$  be a sequence read by the automaton  $\mathcal{A}$ . Define*

$$\forall n \geq k \quad C_n[x] = \{y \in \Sigma^n \mid x \leq y\}.$$

The conditional probability  $\nu_n(C_n[x] \mid X_1 = x_1)$  where  $\nu_n$  is defined in Section 1.4.1 converges weakly to the conditional probability  $\nu_{\infty}(C[x] \mid X_1 = x_1)$ .

*Proof.* Let  $A$  be the adjacency matrix of  $\mathcal{A}$  and let  $x = x_1x_2 \cdots x_k \in \Sigma^*$ . The conditional probability depending on the first letter is

$$\begin{aligned}
& \nu_n(C_n[x] \mid X_1 = x_1) \\
&= \frac{1}{\nu(\Sigma^n \mid X_1 = x_1)} \cdot \sum_{|y|=n: x \leq y} \nu(y) \quad \text{put } y = x \cdot z \text{ and } k' = n - k \\
&= \frac{1}{\nu(\Sigma^n \mid X_1 = x_1)} \cdot \sum_{|z|=k'} \delta_l(x_1) \cdot A_{x_1x_2} \cdot A_{x_2x_3} \cdots A_{x_{k-1}x_k} \cdot A_{x_kz_1} \cdots A_{z_{k'-1}z_{k'}} \cdot \delta_f(z_{k'}) \\
&= \frac{\delta_l(x_1) \cdot A_{x_1x_2} \cdots A_{x_{k-1}x_k}}{\delta_l(x_1) \cdot A^{n-1} \cdot \delta_f} \cdot \sum_{|z|=k'} \delta_l(x_k) \cdot A_{x_kz_1} \cdots A_{z_{k'-1}z_{k'}} \cdot \delta_f(z_{k'}) \\
&= A_{x_1x_2} \cdots A_{x_{k-1}x_k} \cdot \frac{\delta_l(x_k) \cdot A^{n-k} \cdot \delta_f}{\delta_l(x_1) \cdot A^{n-1} \cdot \delta_f}.
\end{aligned}$$

As  $n \rightarrow \infty$  and applying the property 5 of Theorem 1.1,

$$\begin{aligned}
\nu_n(C_n[x] \mid X_1 = x_1) &\sim A_{x_1x_2} \cdots A_{x_{k-1}x_k} \cdot \frac{\delta_l(x_k) \cdot \rho^{n-k} \cdot r \cdot (l \cdot \delta_f)}{\delta_l(x_1) \cdot \rho^{n-1} \cdot r \cdot (l \cdot \delta_f)} \\
&\sim \rho^{-k+1} \cdot A_{x_1x_2} \cdots A_{x_{k-1}x_k} \cdot \frac{r_{x_k}}{r_{x_1}}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\nu_\infty(C[x] \mid X_1 = x_1) &= P_{x_1, x_2} P_{x_2, x_3} \cdots P_{x_{n-1}, x_n} \\
&= \frac{1}{\rho} \cdot A_{x_1x_2} \cdot \frac{r_{x_2}}{r_{x_1}} \cdot \frac{1}{\rho} \cdot A_{x_2x_3} \cdot \frac{r_{x_3}}{r_{x_2}} \cdots \frac{1}{\rho} \cdot A_{x_{k-1}x_k} \cdot \frac{r_{x_k}}{r_{x_{k-1}}} \\
&= \rho^{-k+1} \cdot A_{x_1x_2} A_{x_2x_3} \cdots A_{x_{k-1}x_k} \cdot \frac{r_{x_k}}{r_{x_1}}.
\end{aligned}$$

which completes the proof.  $\square$

**Proposition 1.4.3.** *Let  $\mathcal{A}$  be a finite automaton with primitive adjacency matrix. Let  $\nu_\infty$  be the Parry measure on the finite automaton  $\mathcal{A}$  and let  $x = x_1x_2 \cdots x_k \in \Sigma^*$  be a sequence read by the automaton  $\mathcal{A}$ . Define*

$$\forall n \geq k \quad C_n[x] = \{y \in \Sigma^n \mid x \leq y\}.$$

*The automaton measure  $\nu_n(C_n[x])$  defined in Section 1.4.1 converges weakly to  $\nu_\infty(C[x])$ .*

*Proof.* The proof follows the similar argument as the proof of Lemma 1.4.2. Let  $A$  be

the adjacency matrix of  $\mathcal{A}$  and let  $x = x_1x_2 \cdots x_k \in \Sigma^*$ .

$$\begin{aligned}
\nu_n(C_n[x]) &= \frac{1}{\nu(\Sigma^n)} \cdot \sum_{|y|=n: x \leq y} \nu(y) \\
&= \frac{1}{\delta_l^\top \cdot A^{n-1} \cdot \delta_f} \cdot A_{x_1x_2} \cdots A_{x_{k-1}x_k} \cdot (\delta_l(x_k) \cdot A^{n-k} \cdot \delta_f) \\
&\sim A_{x_1x_2} \cdots A_{x_{k-1}x_k} \cdot \frac{\delta_l(x_k) \cdot \rho^{n-k} \cdot r \cdot (l \cdot \delta_f)}{\delta_l^\top \cdot \rho^{n-1} \cdot r \cdot (l \cdot \delta_f)} \quad \text{As } n \rightarrow \infty \\
&\sim \frac{r_{x_1}}{\delta_l^\top \cdot r} \cdot \rho^{-k+1} \cdot A_{x_1x_2} \cdots A_{x_{k-1}x_k} \cdot \frac{r_{x_k}}{r_{x_1}}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\nu_\infty(C[x]) &= v_{x_1} \cdot P_{x_1,x_2} P_{x_2,x_3} \cdots P_{x_{n-1},x_n} \\
&= v_{x_1} \cdot \frac{1}{\rho} \cdot A_{x_1x_2} \cdot \frac{r_{x_2}}{r_{x_1}} \cdot \frac{1}{\rho} \cdot A_{x_2x_3} \cdot \frac{r_{x_3}}{r_{x_2}} \cdots \frac{1}{\rho} \cdot A_{x_{k-1}x_k} \cdot \frac{r_{x_k}}{r_{x_{k-1}}} \\
&= v_{x_1} \cdot \rho^{-k+1} \cdot A_{x_1x_2} A_{x_2x_3} \cdots A_{x_{k-1}x_k} \cdot \frac{r_{x_k}}{r_{x_1}}.
\end{aligned}$$

By the definition, the initial distribution

$$v_{x_1} = \frac{r_{x_1} \cdot \delta_l(x_1)}{\sum_y r_y \cdot \delta_l(y)} = \frac{r_{x_1}}{\delta_l^\top \cdot r}$$

which concludes the weak convergence of the automaton measure  $\nu_n$ . □

## Chapter 2

# Trace monoids

This chapter is dedicated to the background of trace monoids. Trace monoids were first introduced by Cartier and Foata [10] for proving the MacMahon Master theorem. Later, the trace theory [15, 32] is developed for modeling the parallel operations in the concurrent system.

### 2.1 Definitions

Let  $\Sigma$  be a finite *alphabet*. Elements of  $\Sigma$  are called *letters* or *pieces*. Let  $I \subset \Sigma \times \Sigma$  be an irreflexive and symmetric relation, called an *independence relation*. Let  $D = (\Sigma \times \Sigma) \setminus I$  be the associated *dependence relation*, which is reflexive and symmetric on  $\Sigma$ .

Two letters  $a, b \in \Sigma$  are said to be *parallel* if  $(a, b) \in I$  and denoted by  $a \parallel b$ .

**Definition 2.1.1** (Trace monoid). *The trace monoid  $\mathcal{M} = \mathcal{M}(\Sigma, I)$  is defined as the quotient monoid of the smallest congruence relation on the free monoid  $\Sigma^*$  generated by  $(ab, ba)$  for  $(a, b)$  ranging over  $I$ , presented as follows:*

$$\mathcal{M} = \langle \Sigma \mid ab = ba \text{ for all } (a, b) \in I \rangle.$$

The elements of  $\mathcal{M}$  are called traces which can be viewed as the congruence classes of  $\Sigma^*$ . We called the identity element in a trace monoid the *empty trace*, denoted by  $\varepsilon$ . The concatenation of traces is denoted with the dot “.”. The length of a trace  $x \in \mathcal{M}$  is the length of any word in its congruence class, denoted by  $|x|$ .

For a subset  $H$  of  $\Sigma$ , we denote by  $\langle H \rangle$  the submonoid of  $\mathcal{M}$  generated by  $H$ . In other words,  $\langle H \rangle = \mathcal{M}(H, I_H)$  with  $I_H = (H \times H) \cap I$ .

The left divisibility relation on  $\mathcal{M}$  is defined by  $x \leq y$  if and only if there is  $u \in \mathcal{M}$  such that  $y = x \cdot u$  for  $x, y \in \mathcal{M}$ . The relation  $\leq$  is reflexive, anti-symmetric and transitive, therefore  $(\mathcal{M}, \leq)$  is a partial order set.

Next, we introduce the Coxeter graph which gives us direct information on what letters are parallel, together with the notion of irreducibility which is crucial for the simplification of concurrent systems.

**Definition 2.1.2** (Coxeter graph). *The Coxeter graph of the trace monoid  $\mathcal{M}$  is defined to be the graph of the associated dependence relation  $(\Sigma, D)$ .*

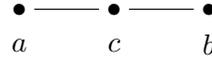


Figure 2.1: Coxeter graph of  $\mathcal{M}_0 = \langle a, b, c \mid a \cdot b = b \cdot a \rangle$

**Definition 2.1.3** (Irreducibility). *The trace monoid  $\mathcal{M} = \mathcal{M}(\Sigma, I)$  is said to be irreducible if the Coxeter graph of  $\mathcal{M}$  is connected.*

*Heap of pieces* is a natural and intuitive graphical interpretation of a trace monoid given by Viennot [44]. A trace is represented by a heap with pieces labeled by letters in  $\Sigma$ . In other words, the elements of the same congruence class of  $\Sigma^*$  will be represented by the same heap. A piece in the heap is blocked by all the letters that are not parallel with it. Otherwise, this piece falls vertically to the ground. The interpretation of a trace in the monoid  $\mathcal{M}_0 = \langle a, b, c \mid a \cdot b = b \cdot a \rangle$  is illustrated in Figure 2.3. In this example, the trace  $abca$  and the trace  $baca$  are the same in  $\mathcal{M}$  presented by a unique heap.



Figure 2.2: Basic pieces of  $\mathcal{M}_0$  according to the associated dependence relation

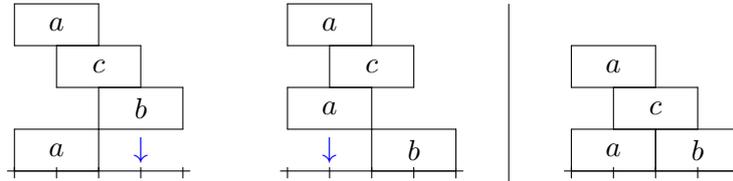


Figure 2.3: Example of the heap interpretation in  $\mathcal{M}_0$ . Left : Two congruent words  $a \cdot b \cdot c \cdot a = b \cdot a \cdot c \cdot a$ . Right : The resulting heap  $abca$ .

## 2.2 Normal form

A *clique*  $c$  is an element in  $\mathcal{M}$  if  $c = a_1 a_2 \dots a_n$  for all  $a_i \in \Sigma$  such that  $i \neq j \implies (a_i, a_j) \in I$ . Let  $\mathcal{C}$  denote the set of cliques, including the empty element  $\epsilon$ . Two cliques  $c, c'$  are said to be *parallel* if  $c \cdot c' \in \mathcal{C}$  and denoted by  $c \parallel c'$ . In other words,  $c \parallel c' \iff c \times c' \subseteq I$ . A clique  $c$  is called *maximal* if  $\nexists a \in \Sigma \setminus \{c\}$  such that  $a \times c \subseteq I$ .

We say that a pair of non-empty cliques  $(c, c')$  is *Cartier-Foata admissible* if  $\forall b \in c', \exists a \in c$  such that  $(a, b) \notin I$ , denoted by  $c \rightarrow c'$ . For every non-empty  $x \in \mathcal{M}$ ,

there exists a unique sequence of non-empty cliques with a positive integer  $n$  such that  $x = c_1 c_2 \dots c_n$  and  $c_i \rightarrow c_{i+1}$  for all  $i$  in  $\{1, \dots, n-1\}$ . This unique sequence of cliques is called the *Cartier-Foata normal form* of  $x$  defined in [10]. The integer  $n$  is called the height of the trace, denoted by  $\tau(x) = n$ . We introduce the notation  $x = c_1 \rightarrow c_2 \rightarrow \dots \rightarrow c_n$  to denote that  $(c_1, c_2, \dots, c_n)$  is the Cartier-Foata normal form of  $x$ .

**Definition 2.2.1** (Digraph of cliques). *Let  $\mathcal{C}$  be the sets of nonempty cliques on a trace monoid. The Cartier-Foata acceptor graph or the digraph of cliques is the directed graph  $(\mathcal{C}, \rightarrow)$ . The edges are constructed by the relation of Cartier-Foata admissible.*

For positive integer  $k$ , the paths of length  $k$  in the graph  $(\mathcal{C}, \rightarrow)$  are in bijection with the Cartier-Foata decomposition of traces with height  $k$ .

Figure 2.4 shows the Cartier-Foata acceptor graph of the monoid  $\mathcal{M}_0 = \langle a, b, c \mid a \cdot b = b \cdot a \rangle$ . Starting from the clique  $a$ , the choices of the next clique are the ones dependent of  $a$ —the clique  $c$  or  $a$  itself. It is straightforward to imagine as the heap interpretation. A letter which is independent of the current clique is not allowed, since it falls down to the ground and has an impact on the current clique. In this case, letter  $b$  is not allowed after the clique  $a$ . The rest of the graph is constructed in a similar way. Moreover, we also observe that after a maximal clique ( the clique  $ab$  in this example ), all letters are executable.

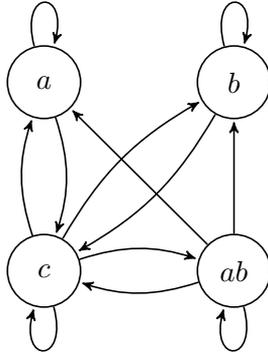


Figure 2.4: Digraph of cliques of  $\mathcal{M}_0 = \langle a, b, c \mid a \cdot b = b \cdot a \rangle$

The following result is known for irreducible trace monoids.

**Proposition 2.2.2.** [2] *A trace monoid is irreducible if and only if its Cartier-Foata acceptor graph is strongly connected.*

## 2.3 Boundary and visual cylinders

We would like to extend  $\mathcal{M}$  to infinite traces. In the heap representation, infinite traces can be imagined as a heap with a countably infinite number of pieces.

By construction, we define a generalized trace as an infinite sequence of cliques  $(c_i)_{i \geq 1}$  such that  $c_i \in \mathcal{C}$  and  $c_i \rightarrow c_{i+1}$  for all  $i$ . Note that  $c_i = \epsilon$  is allowed, but then  $c_j = \epsilon$  for all  $j \geq i$ . For each trace  $x \in \mathcal{M}$ , if  $(c_1, c_2, \dots, c_n)$  is the normal form of  $x$ , its associated generalized trace is obtained by putting  $c_i = \epsilon$  for  $i > n$ . The set of generalized traces form a compactification  $\overline{\mathcal{M}}$  of  $\mathcal{M}$  with the topology induced by the product topology on  $\mathcal{C}^{\mathbb{Z}_{\geq 1}}$ .

Let  $x \in \overline{\mathcal{M}}$  given by  $x = (C_i)_{i \geq 1}$ . We define  $C_i : \overline{\mathcal{M}} \rightarrow \mathcal{C}$  by  $C_i(x) = c_i$ . We obtain the natural projection  $(C_i)_{i \geq 1}$  of mappings  $C_i : \overline{\mathcal{M}} \rightarrow \mathcal{C}$ .

We equip the set  $\overline{\mathcal{M}}$  with the partial order  $\preceq$ , which is extended from the partial order on  $\mathcal{M}$  and defined as follows:

$$\forall x, y \in \overline{\mathcal{M}} \quad x \preceq y \iff \forall k \quad C_k(x) \leq C_k(y).$$

**Proposition 2.3.1.** *Let  $\mathcal{M}$  be a trace monoid. If  $x, y \in \mathcal{M}$ , then*

$$x \leq y \iff x \preceq y.$$

The boundary of  $\mathcal{M}$  is defined as  $\partial\mathcal{M} = \overline{\mathcal{M}} \setminus \mathcal{M}$ . The elements in  $\partial\mathcal{M}$  are called infinite traces. For  $x \in \mathcal{M}$ , the *visual cylinder of base  $x$*  is defined by

$$\uparrow x = \{y \in \partial\mathcal{M} \mid x \leq y\}.$$

The visual cylinders form a semi-algebra which generates the Borel  $\sigma$ -algebra on  $\partial\mathcal{M}$ .

## 2.4 Combinatorial properties

### 2.4.1 Generating series and Möbius polynomial

Let  $G(x)$  be the growth series associated with  $\mathcal{M}$  which is defined by:

$$G(x) = \sum_{u \in \mathcal{M}} x^{|u|} = \sum_{n \geq 0} \#\mathcal{M}_n x^n$$

where  $\mathcal{M}_n = \{x \in \mathcal{M} : |x| = n\}$ . The Möbius polynomial  $\mu_{\mathcal{M}}(x)$  of  $\mathcal{M}$  is defined by:

$$\mu_{\mathcal{M}}(x) = \sum_{c \in \mathcal{C}} (-1)^{|c|} x^{|c|}.$$

On the basis of combinatorial results [10, 44], we know that  $G(x)$  is rational and

$$G(x)\mu_{\mathcal{M}}(x) = 1. \tag{2.1}$$

Furthermore, the Möbius polynomial [19, 27, 13] has a unique root of smallest modulus if  $\Sigma \neq \emptyset$ . This root is real and lies in  $(0, 1]$ . Since the root of smallest modulus of  $\mu_{\mathcal{M}}$  is the dominant singularity [17] of  $G(x)$  by the inversion formula (2.1), we obtain the following asymptotic relation for the number of traces of length  $n$ .

**Proposition 2.4.1** ([2], Thm 4.7). *Let  $\mathcal{M}$  be a trace monoid and let  $p$  be the root of smallest modulus of its Möbius polynomial. The radius convergence of the growth series associated with  $\mathcal{M}$  is  $p$ . Moreover, the following statement satisfies, for some constant  $C > 0$ :*

$$\#\mathcal{M}_n \sim Cp^{-n}.$$

### 2.4.2 Möbius transform and the inversion formula

Some classical results of Möbius formula [39, 42] for trace monoids are introduced.

**Definition 2.4.2.** Let  $\mathcal{M} = \mathcal{M}(\Sigma, I)$  be a trace monoid. Let  $f : \mathcal{C} \rightarrow \mathbb{R}$  be a function. The Möbius transform of  $f$  is the function  $h : \mathcal{C} \rightarrow \mathbb{R}$  defined by:

$$\forall c \in \mathcal{C}, h(c) = \sum_{c' \in \mathcal{C}: c \leq c'} (-1)^{|c'| - |c|} f(c').$$

In particular, if  $f$  is of the form  $f(x) = p^{|x|}$ ,

$$h(\varepsilon) = \sum_{c \in \mathcal{C}} (-1)^{|c|} f(c) = \mu_{\mathcal{M}}(p).$$

**Proposition 2.4.3** (Möbius inversion formula [2]). Let  $\mathcal{M}$  be a trace monoid. Let  $f, h : \mathcal{C} \rightarrow \mathbb{R}$  be two functions. Then  $h$  is the Möbius transform of  $f$  if and only if

$$\forall c \in \mathcal{C} \quad f(c) = \sum_{c' \in \mathcal{C}: c \leq c'} h(c'). \quad (2.2)$$

**Proposition 2.4.4.** Let  $h : \mathcal{C} \rightarrow \mathbb{R}$  be the Möbius transform of a valuation  $f : \mathcal{M} \rightarrow \mathbb{R}_+$ . Let  $g(c) = \sum_{c' \in \mathcal{C}: c \rightarrow c'} h(c')$  for all  $c \in \mathcal{C}$ . If  $h(\varepsilon) = 0$ , then:

$$\forall c \in \mathcal{C} \quad g(c)f(c) = h(c).$$

The formulae introduced previously in this section are defined on the set of cliques  $\mathcal{C}$ . We can extend the domain of  $f$  and  $h$  to the whole trace monoid. These results will be applied in Section 3.3 while constructing the Markov chain.

**Definition 2.4.5** (Graded Möbius transform). Let  $\mathcal{M}$  be a trace monoid and let  $x = c_1 \rightarrow c_2 \rightarrow \dots \rightarrow c_n \in \mathcal{M}$  be the Cartier-Foata normal form. If  $f : \mathcal{M} \rightarrow \mathbb{R}$  is a function, then the graded Möbius transform of  $f$  is the function  $h : \mathcal{M} \rightarrow \mathbb{R}$  defined by:

$$\forall x \in \mathcal{M} \quad h(x) = \sum_{c' \in \mathcal{C}: c_n \leq c'} (-1)^{|c'| - |c_n|} f(c_1 \cdot c_2 \cdots c_{n-1} \cdot c'). \quad (2.3)$$

In particular, if  $f$  is of the form  $f(x) = p^{|x|}$ ,

$$h(x) = f(c_1) \cdot f(c_2) \cdots f(c_{n-1}) \cdot h(c_n). \quad (2.4)$$

**Proposition 2.4.6** (Second Möbius inversion formula [2]). Let  $\mathcal{M}$  be a trace monoid. Let  $f, h : \mathcal{C} \rightarrow \mathbb{R}$  be two functions. Then  $h$  is the graded Möbius transform of  $f$  if and only if

$$\forall x \in \mathcal{M} \quad f(x) = \sum_{y \in \mathcal{M}: x \leq y, \tau(x) = \tau(y)} h(y). \quad (2.5)$$

### 2.4.3 Paths in ADC and Parry measure on trace monoids

We attempt to build a uniform measure on traces as we did for free monoids in Section 1.4. With the aid of the canonical normal form of the traces, there exists a bijection between the traces with height  $k$  for  $k \geq 1$  and the admissible paths of length  $k$  in the digraph of cliques. In other words, the traces with  $k$  layers in heap interpretation.

However, what we desire to do is to generate the trace in length  $k$ , with exactly  $k$  letters. Here we introduce a new graph ADC adjusted by the digraph of cliques. For the clique of length greater than 1, we replace the state by the same amount of the states as the length of the clique. The rigorous definition is given as follows.

**Definition 2.4.7 (ADC).** Let  $\mathcal{M} = (\mathcal{M}, I)$  be a trace monoid. The augmented digraph of cliques (ADC) of  $\mathcal{M}$  is the digraph whose nodes are all pairs of  $(c, i)$  for  $c \in \mathcal{C}$  and  $i = 1, \dots, |c|$ , and whose edges are all pairs of  $((c, i), (d, j))$  satisfying:

- $c \rightarrow d, i = |c|$  and  $j = 1$  or
- $c = d$  and  $j = i + 1$ .

We demonstrate the ADC of  $\mathcal{M}_0 = \langle a, b, c \mid a \cdot b = b \cdot a \rangle$  in Figure 2.5.

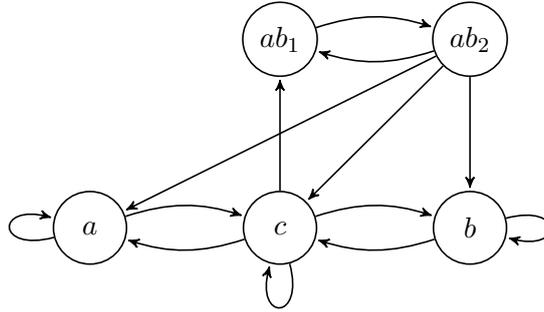


Figure 2.5: The ADC of  $\mathcal{M}_0$

Let  $\iota$  be the set of the initial states in ADC with  $\iota = \{(c, 1) \mid c \in \mathcal{C}\}$  and let  $f$  be the set of the final states in ADC with  $f = \{(c, |c|) \mid c \in \mathcal{C}\}$ . Consider  $A$  be the adjacency matrix of ADC. The number of the traces with length  $n$  can be obtained by (Notion in Sec. 1.2):

$$\#\mathcal{M}_n = \mathbf{1}_\iota^\top A^{n-1} \mathbf{1}_f$$

where  $\mathbf{1}_N$  denote the vector indexed by  $N$  and defined by  $\mathbf{1}_N(x) = 1$  if  $x \in N$ ,  $\mathbf{1}_N(x) = 0$  otherwise.

If a trace monoid  $\mathcal{M}$  is *irreducible*, then the matrix  $A$  is primitive. By the Perron-Frobenius theorem (Theorem 1.1), there exists a unique maximal eigenvalue  $\rho = \rho(A)$  with corresponding strictly positive right and left eigenvectors. It allows us to define the Parry measure on trace monoids.

**Definition 2.4.8** (Parry measure on trace monoids). *Let  $\mathcal{M}$  be an irreducible trace monoid. The Parry measure on  $\mathcal{M}$  is the Parry measure associated with the adjacency matrix  $A$  of ADC. That is :*

$$\forall (c, i), (d, j) \in \text{ADC} \quad P_{(c,i),(d,j)} = A_{(c,i),(d,j)} \cdot \frac{r_{(d,j)}}{\rho \cdot r_{(c,i)}}$$

where  $\rho$  is the Perron eigenvalue of  $A$  and  $r = (r_{(c,i)})$  is the corresponding right eigenvector. The initial distribution is represented as a vector  $(v_{(c,i)})_{(c,i) \in \text{ADC}}$  given by:

$$v_{(c,i)} = \frac{r_{(c,i)} \cdot \delta_{(c,i)}}{\sum_{(d,j)} r_{(d,j)} \cdot \delta_{(d,j)}}$$

where  $\delta$  is the initial vector with  $\delta_{(c,i)} = 1$  if  $i = 1$  and  $\delta_{(c,i)} = 0$  otherwise.

The Parry measure is defined on cliques. In Chapter 3, we will construct the uniform measure based on the visual cylinders from another point of view and figure out how to transform between two probability measures.

In the rest of this subsection, we would like to find the spectral radius and Perron eigenvector of ADC.

**Theorem 2.4.9.** *Let  $\mathcal{M}$  be an irreducible trace monoid and  $\rho$  be the spectral radius of its ADC. Then  $\rho = \frac{1}{p}$  where  $p$  is the root of smallest modulus of the Möbius polynomial. Let  $h : \mathcal{C} \rightarrow \mathbb{R}$  be the Möbius transform of  $f : \mathcal{M} \rightarrow \mathbb{R}$  with  $f(x) = p^{|x|}$ . For  $i \in \{1, 2, \dots, |c|\}$ , put*

$$r_{(c,i)} = \frac{1}{p^{i-1}} h(c).$$

*Then the vector  $r$  is a Perron eigenvector of the adjacency matrix of ADC.*

*Proof.* Let  $A$  be the adjacency matrix of ADC and  $G(x)$  be the generating series associated with  $\mathcal{M}$ . By definition,

$$G(x) = \sum_{n \geq 0} \#\mathcal{M}_n x^n = \sum_{n \geq 0} (\mathbf{1}_l^\top A^{n-1} \mathbf{1}_f) x^n.$$

According to Prop.2.4.1, the radius of convergence of  $G(x)$  is  $p$  which is the root of smallest modulus of  $\mu_{\mathcal{M}}$ . Therefore,  $\rho = 1/p$  is the spectral radius of  $A$ .

Next, we show that  $r$  is an eigenvector associated with  $\rho$ . That is,

$$(Ar)_{(c,i)} = \frac{1}{p} \cdot r_{(c,i)}. \tag{2.6}$$

If  $i < |c|$ , then

$$(Ar)_{(c,i)} = r_{(c,i+1)} = \frac{1}{p^i} \cdot h(c) = \frac{1}{p} \cdot \left( \frac{1}{p^{i-1}} \cdot h(c) \right) = \frac{1}{p} \cdot r_{(c,i)}.$$

If  $i = |c|$ , then

$$(Ar)_{(c,|c|)} = \sum_{c':c \rightarrow c'} r_{(c',1)} = \sum_{c':c \rightarrow c'} h(c') = g(c).$$

On the right hand side of (2.6), we have

$$\frac{1}{p} \cdot r_{(c,|c|)} = \frac{1}{p} \cdot \frac{1}{p^{|c|-1}} h(c) = \frac{h(c)}{f(c)} = g(c)$$

where the last equality is according to Proposition 2.4.4.

Since  $A$  is primitive and  $\rho$  is the spectral radius of  $A$ , by Perron-Frobenius Theorem (Theorem 1.1), its associated eigenvector has three possibilities – strictly positive, strictly negative or 0.

Choose a maximal clique  $c \in \mathcal{C}$ , then

$$h(c) = \sum_{c \leq c'} (-1)^{|c'|-|c|} f(c') = f(c) = p^{|c|} > 0.$$

Hence,  $r$  is strictly positive and we conclude that  $r$  is a Perron eigenvector.  $\square$

#### 2.4.4 Spectral property

The next proposition which is useful for the recursive algorithm in later chapters shows the property of radii of convergence between a trace monoid and its submonoid.

**Proposition 2.4.10.** *Let  $\mathcal{M}$  be an irreducible trace monoid and let  $p$  be the radius of convergence of its growth series. If  $\mathcal{M}$  is irreducible, then: for every  $\tilde{\Sigma} \subsetneq \Sigma$  and  $\tilde{\mathcal{M}} = \langle \tilde{\Sigma} \rangle$ , the radius of convergence  $\tilde{p}$  of the growth series associated with  $\tilde{\mathcal{M}}$  satisfies  $\tilde{p} > p$  and  $\mu_{\tilde{\mathcal{M}}}(p) > 0$ .*

*Proof.* For every  $\tilde{\Sigma} \subsetneq \Sigma$ , let  $\tilde{\mathcal{M}} = \langle \tilde{\Sigma} \rangle$  be a submonoid of  $\mathcal{M}$ . Let  $G(x)$  be the growth series of  $\mathcal{M}$  with radius convergence  $p$  and  $\tilde{G}(x)$  be the growth series of  $\tilde{\mathcal{M}}$  with radius convergence  $\tilde{p}$ . By definition, we have:

$$\tilde{G}(x) = \sum_{n \geq 0} \#\tilde{\mathcal{M}}_n x^n \leq \sum_{n \geq 0} \#\mathcal{M}_n x^n < +\infty$$

if  $x \in [0, \tilde{p})$ . Hence  $\tilde{p} \geq p$ .

For  $x \in [0, \tilde{p})$ , we can obtain:

$$\mu_{\tilde{\mathcal{M}}}(x) = \frac{1}{\tilde{G}(x)} > 0.$$

To show that  $\mu_{\tilde{\mathcal{M}}}(p) \geq 0$ , it is enough to show that  $\tilde{p} > p$ .

Let  $A$  and  $A'$  be the adjacency matrix of the ADC of  $\mathcal{M}$  and  $\tilde{\mathcal{M}}$ , respectively. The matrix  $A$  is indexed by  $(c, i)$  for  $c \in \mathcal{C}$  and  $i = 1, \dots, |c|$ . Replacing the elements of  $A$  by 0 if  $\forall a \notin \Sigma'$  and  $a \in c$ , we get another matrix  $A''$ . By construction, the matrix  $A''$  has

the same size as  $A$  and  $A'' \leq A$ . Since  $A$  is irreducible, by Perron-Frobenius theorem, we have  $\rho(A'') < \rho(A) = \frac{1}{p}$ . Moreover,  $A''$  is identical to  $A'$  after reducing the zero rows and the zero columns. So,  $\rho(A') = \rho(A'') = \frac{1}{\tilde{p}}$ . We conclude that  $\frac{1}{\tilde{p}} < \frac{1}{p}$  and therefore  $\tilde{p} > p$ . □



## Chapter 3

# Probabilistic framework on trace monoids

This chapter is dedicated to the construction of probability measure on trace monoids. The main idea in this chapter is the starting point to extend to the concurrent systems.

### 3.1 Uniform measure

Let  $\partial\mathcal{M}$  be the boundary of monoid  $\mathcal{M}$ . We wish to transpose to trace monoids the usual notion of uniform measure on infinite words easily constructed for free monoids (given in Sec.1.4). This yields the following definition.

**Definition 3.1.1.** *Let  $\mathcal{M}$  be a trace monoid. A uniform measure on  $\partial\mathcal{M}$  is a probability measure such that:*

$$\forall u \in \mathcal{M} \quad \nu(\uparrow x) = p^{|x|}$$

for some fixed real  $p$ .

Contrary to free monoids, the existence of a uniform measure is non obvious. The question is to decide for which values of  $p$  there exists indeed a corresponding uniform measure. The answer is given by Section 3.3.

### 3.2 Elementary cylinders and law of the first clique

Let  $(C_i)_{i \geq 1}$  be the sequence of cliques for a trace. For  $x \in \mathcal{C}$ , the *elementary cylinder of base  $x$*  is defined as

$$[x] = \{y \in \partial\mathcal{M} \mid C_1(y) = x\}.$$

In general, let  $x = c_1 \rightarrow c_2 \rightarrow \dots \rightarrow c_n \in \mathcal{M}$ , the *elementary cylinder of base  $x$*  is

$$[x] = [c_1, c_2, \dots, c_n] = \{y \in \partial\mathcal{M} \mid C_1(y) = c_1, C_2(y) = c_2, \dots, C_n(y) = c_n\}.$$

The construction of a uniform measure defined in previous section is based on the visual cylinders on  $\partial\mathcal{M}$ . Inspired from the finite cases, we know that the infinite paths in digraph of cliques are in bijection with the points of the boundary of the trace monoid [2]. In order to build a Markov chain in digraph of cliques, we would like to know the probability of elementary cylinders. The following Proposition shows that this probability can be calculated by the Möbius transform of the valuation  $f(x) = p^{|x|}$ .

**Proposition 3.2.1.** [2] *Assume that  $\nu$  is a uniform measure on  $\partial\mathcal{M}$  with  $\nu(\uparrow x) = p^{|x|}$ . Put  $f(x) = \nu(\uparrow x) = p^{|x|}$ . Then the law of the first clique  $C_1$  is the Möbius transform  $h : \mathcal{C} \rightarrow \mathbb{R}$  of  $f$ .*

*Proof.* Put  $f(x) = p^{|x|}$  and  $h(c) = \sum_{c' \in \mathcal{C}: c \leq c'} (-1)^{|c'| - |c|} p^{|c'|}$ . Let  $x \in \mathcal{M}$  and let  $C_1(x) = c$ .

The visual cylinder of base  $c$  can be decomposed into a disjoint union of elementary cylinders depending on the first clique. That is,

$$\uparrow c = \bigcup_{c' \in \mathcal{C}: c \leq c'} [d].$$

Then the probability measure of  $\uparrow c$  is calculated by:

$$\nu(\uparrow c) = \nu\left(\bigcup_{d \in \mathcal{C}: c \leq d} [d]\right) = \sum_{d \in \mathcal{C}: c \leq d} \nu([d]).$$

On the other hand, we have

$$\nu(\uparrow c) = f(c) = \sum_{d \in \mathcal{C}: c \leq d} h(d)$$

where the last equality is according to Möbius inversion formula (Proposition 2.4.3). Therefore, the law of the first clique is  $\nu([c]) = h(c)$  for all  $c \in \mathcal{C}$ .  $\square$

### 3.3 Existence, uniqueness and realization of the uniform measure

We prove in this section that there is only one candidate for a uniform measure on  $\partial\mathcal{M}$ . The first observation is that the parameter of a uniform measure is a root of Möbius polynomial of  $\mathcal{M}$ .

**Lemma 3.3.1.** *Let  $\mathcal{M}$  be a trace monoid. If  $\nu(\uparrow x) = p^{|x|}$  is a uniform measure on  $\partial\mathcal{M}$  for some real  $p > 0$ , then  $p$  is a root of the Möbius polynomial  $\mu_{\mathcal{M}} = \sum_{c \in \mathcal{C}} (-1)^c x^{|c|}$ .*

*Proof.* Put  $f(x) = \nu(\uparrow x) = p^{|x|}$  and  $h(c) = \sum_{c' \in \mathcal{C}: c' \geq c} (-1)^{|c'| - |c|} p^{|c'|}$ . It suffices to show that

$$h(\epsilon) = \sum_{c \in \mathcal{C}} (-1)^c p^{|c|} = 0$$

where  $\epsilon$  is the empty trace. According to Proposition 3.2.1,  $h(c)$  is the probability distribution of the first clique. By the total probability summing all  $c \in \mathcal{C}$ , we obtain:

$$\sum_{c \in \mathcal{C}} h(c) = 1$$

On the other hand, by the Möbius inversion formula, we get:

$$1 = f(\epsilon) = \sum_{c \in \mathcal{C}} h(c) = h(\epsilon) + \sum_{c \in \mathcal{C}} h(c)$$

This implies that  $h(\epsilon) = 0$  and therefore  $p$  is a root of  $\mu_{\mathcal{M}}$ . □

The next lemma shows that the probability of elementary cylinders is always positive.

**Lemma 3.3.2.** *Let  $\mathcal{M}$  be an irreducible trace monoid and  $\nu(\uparrow x) = p^{|x|}$  be a uniform measure on  $\partial\mathcal{M}$ . Let  $f(x) = p^{|x|}$  and  $h$  is the Möbius transform of  $f$ . Then  $h(c) > 0$ , for  $c \neq \epsilon$ .*

*Proof.* According to Proposition 3.2.1,  $h(c)$  is equivalent to the law of first clique  $\nu([c])$ . The idea is to show that there exists at least one trace in the elementary cylinder of base  $c$ . Therefore, its probability is always positive.

Since  $\mathcal{M}$  is irreducible, by Proposition 2.2.2, we know that the associated digraph of cliques is strongly connected. Then there exists a sequence of cliques  $c_1 \rightarrow c_2 \rightarrow \dots \rightarrow c_k$  such that  $c_1 = c$  and  $c_k$  is a maximal clique.

Put  $x = c_1 c_2 \dots c_k$ . We have

$$\uparrow x \subseteq [c_1] = [c].$$

Then

$$\nu([c]) \geq p^{|x|} > 0$$

which implies that  $h(c) > 0$ . □

Now, with the aid of the unique canonical normal form for traces, we would like to construct a Markov chain on the trajectories of a trace monoid. By extending the similar argument of the law of first clique, we can get the transition matrix of the chain of cliques. The details of the construction can be found in [2].

**Theorem 3.3.3.** *[2, Theorem 2.5, 2.6] Let  $\mathcal{M}$  be an irreducible trace monoid. There exists a unique uniform measure  $\nu$  on  $\partial\mathcal{M}$ . The real  $p$  such that  $\nu(\uparrow x) = p^{|x|}$  is the root of smallest modulus of the Möbius polynomial of  $\mathcal{M}$ .*

*Let  $f(x) = p^{|x|}$  and let  $h$  be the Möbius transform of  $f$  associated with  $\nu$ . Furthermore, under the probability  $\nu$ , the sequence  $(C_n)_{n \geq 1}$  of cliques of a random infinite trace  $x \in$*

$\partial\mathcal{M}$  is an irreducible and aperiodic Markov chain, with initial distribution  $h$  and with the transition matrix  $P = (P_{c,c'})_{(c,c') \in \mathcal{C} \times \mathcal{C}}$  given by:

$$P_{c,c'} = \begin{cases} h(c')/g(c), & \text{if } c \rightarrow c' \\ 0, & \text{if } c \not\rightarrow c' \end{cases} \quad \text{with } g(c) = \sum_{c' \in \mathcal{C}: c \rightarrow c'} h(c').$$

In addition, for any integer  $n \geq 1$ , if  $c_1, \dots, c_n$  are non-empty cliques such that  $c_1 \rightarrow c_2 \cdots \rightarrow c_n$  holds, then

$$\nu([c_1, c_2, \dots, c_n]) = f(c_1) \cdots f(c_{n-1})h(c_n).$$

*Proof.* First, we prove the uniqueness of the uniform measure. Suppose that there exists a uniform measure  $\nu$  of the form  $\nu(\uparrow x) = p^{|x|}$ . We need the next lemma in order to apply Perron-Frobenius theorem.

**Lemma 3.3.4.** *Let  $\mathcal{M}$  be an irreducible trace monoid. Assume that  $\nu(\uparrow x) = p^{|x|}$  is a uniform measure on  $\partial\mathcal{M}$ . Let  $\mathcal{C}$  be the set of nonempty cliques and  $B = (B_{c,d})_{(c,d) \in \mathcal{C} \times \mathcal{C}}$  given by:*

$$B_{c,d} = \mathbb{1}_{c,d} \cdot p^{|d|}$$

where  $\mathbb{1}_{c,d} = 1$  if  $c \rightarrow d$ , otherwise  $\mathbb{1}_{c,d} = 0$ . Then:

1.  $B$  is nonnegative and primitive.
2. The spectral radius of  $B$  is 1.

*Proof of Lemma 3.3.4.* It is clear that the matrix  $B$  is nonnegative by the definition. Because  $c \rightarrow c$  for all  $c \in \mathcal{C}$ , we have  $B_{c,c} > 0$  and therefore  $B$  is nonzero and aperiodic. The matrix  $B$  can be viewed as the weighted adjacency matrix of digraph of cliques. So the irreducibility of  $\mathcal{M}$  implies the irreducibility of matrix  $B$ . Since  $B$  is irreducible and aperiodic,  $B$  is primitive.

Next, we want to find an eigenvector  $v$  with respect to eigenvalue 1. Put  $f(c) = p^{|c|}$ ,  $h(c)$  is the Möbius transform of  $f$  and  $g(c) = \sum_{c' \in \mathcal{C}: c \rightarrow c'} h(c')$ . Choose  $v_c = g(c)$  for  $c \in \mathcal{C}$  and put  $v = (v_c)_{c \in \mathcal{C}}$ . Then:

$$\begin{aligned} (Bv)_c &= \sum_{d \in \mathcal{C}} B_{c,d} v_d \\ &= \sum_{d \in \mathcal{C}} \mathbb{1}_{c,d} \cdot p^{|d|} \cdot g(d) \\ &= \sum_{d \in \mathcal{C}: c \rightarrow d} f(d) \cdot g(d) \\ &= \sum_{d \in \mathcal{C}: c \rightarrow d} h(d) && \text{By Proposition 2.4.4} \\ &= g(c) = v_c \end{aligned}$$

According to Lemma 3.3.2,  $h(c) > 0$  and therefore  $g(c) > 0$ . Since the eigenvector  $v = (v_c)_{c \in \mathcal{C}} = (g(c))_{c \in \mathcal{C}}$  is strictly positive and  $B$  is primitive, by Perron-Frobenius theorem, we conclude that its associated eigenvalue 1 is the spectral radius.  $\square$

Going back to the proof of the uniqueness of the uniform measure, assume that  $\nu(\uparrow x) = p^{|x|}$  and  $\nu'(\uparrow x) = q^{|x|}$  are two uniform measures with  $p \leq q$ . Let  $B$  and  $B'$  be the matrices as in Lemma 3.3.4, associated with the parameter  $p$  and with the parameter  $q$ . Then  $B \leq B'$ .

But  $B$  and  $B'$  are primitive and they have the same spectral radius. By Perron-Frobenius Theorem (Theorem 1.1, property 6), we have  $B = B'$ . Hence,  $p = q$  is the unique parameter of the uniform measure.

Let  $P = (P_{c,c'})_{(c,c') \in \mathcal{C} \times \mathcal{C}}$  be the matrix as in the statement where  $f(x) = p^{|x|}$  and  $h$  is the Möbius transform of  $f$ . Next, we prove that  $P$  is well-defined as a stochastic matrix. It suffices to show that 1)  $g(c) \neq 0$ , 2)  $P_{c,c'} \geq 0$ , 3)  $\sum_{c'} P_{c,c'} = 1$ .

According to Proposition 3.4.5, we have the following relation:

$$h(c) = f(c) \cdot \mu_{\mathcal{M}'}(p)$$

where  $p$  is the root of smallest modulus of  $\mu_{\mathcal{M}}$  and  $\mathcal{M}' = \langle \Sigma' \rangle$  is the submonoid of  $\mathcal{M}$  with  $\Sigma' = \{b \in \Sigma \mid b \parallel c\}$ . Since  $\mu_{\mathcal{M}'}(p) > 0$  by Prop. 2.4.10 and  $f(c) > 0$  for  $c \in \mathcal{C}$ , we obtain that  $h > 0$  for  $c \in \mathcal{C}$ . Hence  $g(c) = \sum_{c' \in \mathcal{C}: c \rightarrow c'} h(c') > 0$  and  $P_{c,c'} \geq 0$ .

The function  $g$  is actually the normalization factor such that  $\sum_{c'} P_{c,c'} = \sum_{c' \in \mathcal{C}: c \rightarrow c'} \frac{h(c')}{g(c)} = \frac{1}{g(c)} \sum_{c' \in \mathcal{C}: c \rightarrow c'} h(c') = 1$ . Hence,  $P$  is indeed well-defined and stochastic. Moreover, we have seen in the proof of Lemma 3.3.1 that  $\sum_{c \in \mathcal{C}} h(c) = 1$ . Hence, the initial distribution  $h$  is a probability vector.

Let  $x = c_1 \rightarrow c_2 \cdots \rightarrow c_n \in \mathcal{M}$ , the probability of elementary cylinder of base  $x$  is given by:

$$\begin{aligned} \nu([x]) &= \nu([c_1 \cdots c_n]) \\ &= h(c_1) \cdot \frac{h(c_2)}{g(c_1)} \cdot \frac{h(c_3)}{g(c_2)} \cdots \frac{h(c_n)}{g(c_{n-1})} && \text{by the definition of } P \\ &= f(c_1) \cdot f(c_2) \cdots f(c_{n-1}) \cdot h(c_n) && \text{since } h(c) = f(c) \cdot g(c) \text{ (Prop. 2.4.4)} \\ &= h(x). \end{aligned} \tag{3.1}$$

Finally, we prove the existence of the uniform measure of the form  $\nu(\uparrow x) = p^{|x|}$ . From the construction of the Markov chain on a trace monoid as above (3.1), we have  $\nu([x]) = h(x)$ . With the aid of the second Möbius inversion formula, the uniform measure defined on the visual cylinder of base  $x$  is obtained by:

$$\begin{aligned} \nu(\uparrow x) &= \sum_{y \in \mathcal{M}: x \leq y, \tau(x) = \tau(y)} \nu([y]) \\ &= \sum_{y \in \mathcal{M}: x \leq y, \tau(x) = \tau(y)} h(y) && \text{by (3.1)} \\ &= f(x) && \text{by Prop. 2.4.6} \\ &= p^{|x|} \end{aligned}$$

which completes the proof.  $\square$

**Remark 3.3.5.** *From the observation in Section 3.2, we know that the relation between the probability measure of the elementary cylinders and the visual cylinders depending on the first clique is via the Möbius transform. Thank for the canonical decomposition of traces and the graded Möbius transform, this relation can be extended to the cylinders of a trace. We see in the last part of the proof of Theorem 3.3.3. Starting from the probability measure of an elementary cylinder via the Markov chain, finally we get the probability measure on a visual cylinder as we desired.*

### 3.4 Realization of random generation

Following the framework of previous sections, we start with a simple example to demonstrate the calculation of the transition matrix of the Markov chain on a trace monoid.

**Example 3.4.1.** Let  $\mathcal{M}_0 = \langle a, b, c \mid a \cdot b = b \cdot a \rangle$  be the trace monoid. The Möbius polynomial is  $1 - 3x + x^2$  and the characteristic root  $p$  is  $\frac{3-\sqrt{5}}{2}$ .

Put  $f(x) = p^{|x|}$ . The Möbius transform  $h$  of  $f$  and its normalization function  $g$  are as follows:

$$\begin{aligned} h(a) &= f(a) - f(ab) = p - p^2, & g(a) &= h(a) + h(c) = 2p - p^2 \\ h(b) &= f(b) - f(ab) = p - p^2, & g(b) &= h(b) + h(c) = 2p - p^2 \\ h(c) &= f(c) = p, & g(c) &= h(a) + h(b) + h(c) + h(ab) = 3p - p^2 = 1 \\ h(ab) &= f(ab) = p^2, & g(ab) &= h(a) + h(b) + h(c) + h(ab) = 3p - p^2 = 1 \end{aligned}$$

The transition matrix of the Markov chain of cliques is

$$P = \begin{matrix} & \begin{matrix} a & b & c & ab \end{matrix} \\ \begin{matrix} a \\ b \\ c \\ ab \end{matrix} & \begin{pmatrix} 0.382 & 0 & 0.618 & 0 \\ 0 & 0.382 & 0.618 & 0 \\ 0.236 & 0.236 & 0.382 & 0.146 \\ 0.236 & 0.236 & 0.382 & 0.146 \end{pmatrix} \end{matrix}.$$

Sometimes there exists two roots of the Möbius polynomial in  $(0, 1]$ . Here we present an example to show that the root of smallest modulus is the unique parameter to obtain the uniform measure.

**Example 3.4.2.** Let  $\mathcal{M}_1 = \langle a, b, c, d, e \mid a \cdot b = b \cdot a, b \cdot c = c \cdot b, c \cdot d = d \cdot c, d \cdot e = e \cdot d, e \cdot a = a \cdot e \rangle$  be the trace monoid. The Möbius polynomial is  $1 - 5x + 5x^2$ . Two roots of  $\mu_{\mathcal{M}_1}$  lie in  $(0, 1]$  which are  $\frac{5-\sqrt{5}}{10}$  and  $\frac{5+\sqrt{5}}{10}$ . But only the root of smallest modulus can get a well-defined uniform measure....

We aim at uniformly generating traces in larger models. In application, the recursive formulae are more useful. Some notations are needed here.

Let  $\Sigma$  be a finite alphabet. For every  $H \subseteq \Sigma$ , we denote by  $\langle H \rangle$  the submonoid of  $\mathcal{M}$  generated by  $H$ . We define  $\setminus : \mathcal{M}^{\setminus c} = \langle \Sigma \setminus \{c\} \rangle$ ,  $\parallel^c : \mathcal{M}^{\parallel c} = \langle \Sigma_{\parallel c} \rangle$  with  $\Sigma_{\parallel c} = \{b \in \Sigma \mid c = a_1 \cdots a_p \text{ with } a_i \in \Sigma, (a_i, b) \in I\}$  for every clique  $c \in \mathcal{C}$ .

**Lemma 3.4.3.** *Let  $\mathcal{M} = \mathcal{M}(\Sigma, I)$  be a trace monoid. The Möbius polynomial  $\mu_{\mathcal{M}}$  satisfies the following recursive formula:*

$$\forall a \in \Sigma \quad \mu_{\mathcal{M}}(x) = \mu_{\mathcal{M} \setminus a}(x) - x \cdot \mu_{\mathcal{M} \parallel a}(x)$$

where  $\mu_{\mathcal{M} \setminus a}$  and  $\mu_{\mathcal{M} \parallel a}$  are the Möbius polynomials of the trace monoid  $\mathcal{M} \setminus a$  and the trace monoid  $\mathcal{M} \parallel a$ , respectively.

*Proof.* For  $a \in \Sigma$ ,

$$\begin{aligned} \mu_{\mathcal{M}}(x) &= \sum_{c \in \mathcal{C}} (-1)^{|c|} x^{|c|} \\ &= \sum_{c \in \mathcal{C} : a \notin c} (-1)^{|c|} x^{|c|} + \sum_{c \in \mathcal{C} : a \in c} (-1)^{|c|} x^{|c|} \\ &= \mu_{\mathcal{M} \setminus a}(x) + \sum_{c_1 \in \mathcal{C} : c_1 \parallel a} (-1)^{|a \cdot c_1|} x^{|a \cdot c_1|} \\ &= \mu_{\mathcal{M} \setminus a}(x) - x \cdot \mu_{\mathcal{M} \parallel a}(x). \end{aligned}$$

□

Based on the result of Lemma 3.4.3, the following recursive algorithm (Algorithm 1) computes the Möbius polynomial of a trace monoid. The efficiency of Algorithm 1 is showed in Proposition 3.4.4.

---

**Algorithm 1** Computes the Möbius polynomial  $\mu_{\mathcal{M}}$  of a trace monoid  $\mathcal{M}$

---

**Require:** Trace monoid  $\mathcal{M} = \mathcal{M}(\Sigma, I)$

```

1:  $\Delta \leftarrow \Sigma$  ▷ initialization
2: if  $\Delta = \emptyset$  then
3:    $\mu_{\mathcal{M}} \leftarrow 1$  ▷ initialization of  $\mu_{\mathcal{M}}$ 
4: else
5:   Choose  $a \in \Sigma$ 
6:    $\Delta \leftarrow \Sigma \setminus \{a\}$ 
7:    $\Lambda \leftarrow \Sigma \parallel a$ 
8:    $\theta \leftarrow$  Call Algorithm 1 on  $\Delta$ 
9:    $\pi \leftarrow$  Call Algorithm 1 on  $\Lambda$ 
10:   $\mu_{\mathcal{M}} = \theta - z \cdot \pi$  ▷ Lemma 3.4.3
11: end if
12: return  $\mu_{\mathcal{M}}$ 

```

---

**Proposition 3.4.4.** *Let  $\mathcal{M} = \mathcal{M}(\Sigma, I)$  be a trace monoid with  $|\Sigma| = n$ . Let  $k$  be the length of a maximal clique of  $\mathcal{M}$ . The efficiency of Algorithm 1 is*

$$S_n \leq S_{n-1} + S_k$$

where  $S_n$  is the calculation of the Möbius polynomial of  $\mathcal{M}$ .

Next, we give the recursive formula to compute the Möbius transform  $h$ . The transition matrix of the Markov chain of a trace monoid is obtained after the normalization of the Möbius transform. The associated algorithms are followed immediately.

**Lemma 3.4.5.** *Let  $p$  be a real number. Let  $f(x) = p^{|x|}$  be a function. Then the Möbius transform  $h$  of  $f$  satisfies the following formula:*

$$\forall c \in \mathcal{C} \quad h(c) = p^{|c|} \cdot \mu_{\mathcal{M}^{\parallel c}}(p).$$

*Proof.* For  $c \in \mathcal{C}$ , we compute  $h$  from the definition:

$$\begin{aligned} h(c) &= \sum_{c' \in \mathcal{C} : c \leq c'} (-1)^{|c'| - |c|} f(c') \\ &= \sum_{c_1 \in \mathcal{C} : c_1 \parallel c} (-1)^{|c_1|} \cdot p^{|c \cdot c_1|} \quad \text{put } c' = c \cdot c_1 \\ &= p^{|c|} \cdot \sum_{c_1 \in \mathcal{C} : c_1 \parallel c} (-1)^{|c_1|} p^{|c_1|} \\ &= p^{|c|} \cdot \mu_{\mathcal{M}^{\parallel c}}(p). \end{aligned}$$

□

---

**Algorithm 2** Computes the Möbius transform  $h$  of  $f$  for all  $c \in \mathcal{C}$

---

**Require:**  $\mathcal{M} = \mathcal{M}(\Sigma, I)$  and the clique  $c$

- 1:  $\mu_{\mathcal{M}} \leftarrow$  Call Algorithm 1 on  $\Sigma$  ▷ the Möbius polynomial of  $\mathcal{M}$
  - 2:  $p \leftarrow$  The smallest positive root of  $\mu_{\mathcal{M}}$
  - 3:  $\Lambda \leftarrow \Sigma_{\parallel c}$
  - 4:  $\theta \leftarrow$  Call Algorithm 1 on  $\Lambda$  ▷ the Möbius polynomial of  $\mathcal{M}^{\parallel c}$
  - 5: **return**  $p^{|c|} \cdot \theta(p)$
-

---

**Algorithm 3** Computes the transition matrix of the Markov chain of  $\mathcal{M}$

---

**Require:**  $\mathcal{M} = \mathcal{M}(\Sigma, I)$

```

1:  $M \leftarrow 0$  ▷ Initialization of the transition matrix
2: for all  $c, d \in \mathcal{C}$  do
3:   if  $c \rightarrow d$  then ▷ the normal form relation  $\rightarrow$ 
4:      $M_{c,d} \leftarrow$  Call Algorithm 2 on  $d$ 
5:   end if
6: end for
7:  $l \leftarrow |\mathcal{C}|$ 
8:  $u \leftarrow (1, \dots, 1)_{1 \times l}$ 
9:  $n \leftarrow M \cdot u$  ▷ the parameters of normalization
10: for all  $c, d \in \mathcal{C}$  do
11:    $M_{c,d} = \frac{M_{c,d}}{n_c}$  ▷ normalization
12: end for
13: return  $M$ 

```

---

## 3.5 From Parry measure to the uniform measure

### 3.5.1 Transition matrix of two probability measures

Let us put aside the construction of the unique uniform measure in previous sections. We demonstrate here how to calculate the Parry measure on a trace monoid by a simple example.

**Example 3.5.1.** For the example monoid  $\mathcal{M}_0 = \langle a, b, c \mid a \cdot b = b \cdot a \rangle$ , the ADC of  $\mathcal{M}_0$  is depicted on Figure 2.5. The adjacency matrix of the ADC is the following:

$$A = \begin{matrix} (a, 1) \\ (b, 1) \\ (c, 1) \\ (ab, 1) \\ (ab, 2) \end{matrix} \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix}.$$

The spectral radius  $\rho(A)$  is  $\frac{3+\sqrt{5}}{2} \sim 2.618$  with respect to its right eigenvector  $r = (1, 1, 1.618, 0.618, 1.618)$ . By Definition 2.4.8, the Parry matrix is given by

$$P = \begin{matrix} (a, 1) \\ (b, 1) \\ (c, 1) \\ (ab, 1) \\ (ab, 2) \end{matrix} \begin{pmatrix} 0.382 & 0 & 0.618 & 0 & 0 \\ 0 & 0.382 & 0.618 & 0 & 0 \\ 0.236 & 0.236 & 0.382 & 0.146 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0.236 & 0.236 & 0.382 & 0.146 & 0 \end{pmatrix}$$

with initial distribution  $(0.236, 0.236, 0.382, 0.146, 0)$ .

**Proposition 3.5.2.** *The Parry measure on trace monoids in Definition 2.4.8 coincides to the uniform measure defined in Theorem 3.3.3.*

*Proof.* Let  $P$  be the the transition matrix of Parry measure as in Definition 2.4.8 and let  $P'$  be the the transition matrix of the Markov chain define in Theorem 3.3.3. Replacing the eigenvector that we found in Theorem 2.4.9, we have:

$$\begin{aligned} P_{(c,i),(d,j)} &= A_{(c,i),(d,j)} \cdot \frac{r_{(d,j)}}{\rho \cdot r_{(c,i)}} \\ &= A_{(c,i),(d,j)} \cdot \frac{p^{1-j} \cdot h(d)}{p^{-1} \cdot p^{1-i} \cdot h(c)} \\ &= A_{(c,i),(d,j)} \cdot p^{i-j+1} \cdot \frac{h(d)}{h(c)}. \end{aligned}$$

If  $c \rightarrow d, i = |c|$  and  $j = 1$ ,

$$P_{(c,|c|),(d,1)} = 1 \cdot p^{|c|} \cdot \frac{h(d)}{h(c)} = f(c) \cdot \frac{h(d)}{h(c)} = \frac{h(d)}{g(c)} = P'_{c,d}.$$

If  $c = d$  and  $j = i + 1$ ,

$$P_{(c,i),(d,j)} = 1,$$

which represents the transition between the extended states of a multiple clique.

Otherwise,  $P_{(c,i),(d,j)} = 0$ .

Moreover, the initial distribution is

$$v_{(c,i)} = \frac{r_{(c,i)} \cdot \delta_{(c,i)}}{\sum_{(d,j)} r_{(d,j)} \cdot \delta_{(d,j)}}.$$

If  $i = 1$ ,

$$v_{(c,1)} = \frac{h(c)}{\sum_{d \in \mathcal{C}} h(d)} = h(c).$$

Otherwise,  $v_{(c,i)} = 0$ .

□

**Remark 3.5.3.** *Until now, we have seen two methods to find the uniform measure. One is to construct step by step with the aid of the uniform measure and its Möbius transform. The other is via the graph of ADC and the Parry measure. Both of them are able to prove the existence of the uniform measure on a trace monoid. Nevertheless, it is not clear to have an insight how we build the Markov chain of cliques via the second method. We will see in later chapters the advantage of spectral theory.*

### 3.5.2 Weak convergence of uniform distributions on $\mathcal{M}_n$

Let  $\nu_n$  be the uniform distribution of the finite set  $\mathcal{M}_n = \{x \in \mathcal{M} : |x| = n\}$ . For  $x \in \mathcal{M}_n$ ,

$$\nu_n(x) = \frac{1}{\#\mathcal{M}_n}.$$

**Theorem 3.5.4.** *Let  $(\nu_n)_{n \geq 0}$  be a sequence of uniform distributions on  $\mathcal{M}_n$ . Then  $\nu_n$  converges weakly to the uniform measure  $\nu$  on  $\partial\mathcal{M}$ .*

*Proof.* Consider  $\nu_n$  as a probability distribution on  $\overline{\mathcal{M}} = \mathcal{M} \cup \partial\mathcal{M}$ . We define that  $\uparrow x = \{y \in \overline{\mathcal{M}} : x \leq y\}$ . Let  $x \in \mathcal{M}_k$  for all  $k \geq 1$ . Then, for  $n \geq k$ , we have :

$$\begin{aligned} \nu_n(\uparrow x) &= \frac{\#\{y \in \mathcal{M}_n : x \leq y\}}{\#\mathcal{M}_n} = \frac{\#\mathcal{M}_{n-k}}{\#\mathcal{M}_n} \\ &\sim \frac{C \cdot p^n}{C' \cdot p^{n-k}} \quad \text{by Proposition 2.4.1} \\ &= p^k = p^{|x|} = \nu(\uparrow x). \end{aligned}$$

□



# Chapter 4

## Abstract concurrent systems

In this chapter, we introduce the mathematical models of concurrent systems that we mainly use. The last subsection is dedicated to the examples of concurrent systems.

### 4.1 Notions and monoid actions

Define an action of a monoid  $\mathcal{M} = \mathcal{M}(\Sigma, I)$  over a finite set  $X$  as a mapping  $\phi : X \times \mathcal{M} \rightarrow X$ , denoted by  $\phi(\alpha, x) = \alpha \cdot x$  such that: for  $\alpha \in X$ , 1)  $\alpha \cdot \varepsilon = \alpha$ ; 2)  $\alpha \cdot (x \cdot y) = (\alpha \cdot x) \cdot y$   $\forall x, y \in \mathcal{M}$ .

We adopt the *trace models* or partial order models of concurrency in [15, 32]. The concurrency of actions is encoded into the independent relation  $I$  in a trace monoid  $\mathcal{M}$ . In other words, two actions  $x$  and  $y$  are said to be concurrent if and only if  $(x, y) \in I$ . Under this framework, the abstract concurrent systems are introduced as the actions on a trace monoid.

**Definition 4.1.1.** A concurrent system is a triple  $(\mathcal{M}, X, \perp)$ , where  $X$  is a finite set of states,  $\perp$  is a distinguished symbol not in  $X$  and  $\mathcal{M}$  is a trace monoid acting on  $X \cup \{\perp\}$  such that  $\perp \cdot x = \perp$  for all  $x \in \mathcal{M}$ .

A trace  $x \in \mathcal{M}$  such that  $\alpha \cdot x \neq \perp$  for some state  $\alpha$  is called an *execution* of the system from  $\alpha$ . We could also say that the execution  $x$  leads from the state  $\alpha$  to the state  $\alpha \cdot x$ . The symbol  $\perp$  represents a sink state where all invalid executions go.

To understand more about the structure of concurrent systems, the state graph is a natural object to study sequential executions.

**Definition 4.1.2** (State graph). Let  $S = (\mathcal{M}, X, \perp)$  be a concurrent system, with  $\Sigma$  the associated alphabet of  $\mathcal{M}$ . The state graph of  $S$  is the labeled multigraph whose nodes are the elements in  $X$ , and with an edge from  $\alpha$  to  $\beta$  for every triple  $(\alpha, a, \beta) \in X \times \Sigma \times X$  such that  $\alpha \cdot a = \beta$ . This edge is labeled by  $a$ .

We adapt some terminology from group actions and from Petri nets (for more details see in Section 4.3.2) to general concurrent systems.

**Definition 4.1.3.** A concurrent system  $(\mathcal{M}, X, \perp)$  with  $\mathcal{M} = \mathcal{M}(\Sigma, I)$  is:

- 1) trivial: if  $\forall(\alpha, x) \in X \times \mathcal{M} \quad \alpha \cdot x = \perp$ ; it is non trivial otherwise.
- 2) homogeneous: if its state graph is strongly connected.
- 3) alive: if for every pair  $(\alpha, a) \in X \times \Sigma$ , there exists  $x \in \mathcal{M}_\alpha$  such that  $a \in x$ .
- 4) irreducible: if  $\mathcal{M}$  is an irreducible trace monoid, and if the system is homogeneous and alive.

**Remark 4.1.4.** In our article [4], the terminology of "homogeneous" is written as "accessible". In fact, two concepts are totally the same.

## 4.2 Combinatorics for monoid actions

Given a concurrent system  $S = (\mathcal{M}, X, \perp)$ , we use the following notations defined for the sets of executable traces and cliques. For  $\alpha, \beta \in X$ , for nonnegative integer  $n$ :

$$\begin{aligned} \mathcal{M}_\alpha &= \{x \in \mathcal{M} : \alpha \cdot x \neq \perp\} & \mathcal{M}_{\alpha,\beta} &= \{x \in \mathcal{M}_\alpha : \alpha \cdot x = \beta\} \\ \mathcal{M}_\alpha(n) &= \{x \in \mathcal{M}_\alpha : |x| = n\} & \mathcal{M}_{\alpha,\beta}(n) &= \{x \in \mathcal{M}_{\alpha,\beta} : |x| = n\} \end{aligned}$$

$$\mathcal{C}_\alpha = \mathcal{M}_\alpha \cap \mathcal{C} \quad \mathcal{C}_{\alpha,\beta} = \mathcal{M}_{\alpha,\beta} \cap \mathcal{C} \quad \mathcal{C}_\alpha = \mathcal{C}_\alpha \setminus \{\varepsilon\}.$$

Let  $G_{\alpha,\beta}(z)$  be the generating series associated with  $\mathcal{M}_{\alpha,\beta}$  given by:

$$G_{\alpha,\beta}(z) = \sum_{x \in \mathcal{M}_{\alpha,\beta}} z^{|x|} = \sum_{n \geq 0} \#\mathcal{M}_{\alpha,\beta}(n) z^n.$$

where  $\#\mathcal{M}_{\alpha,\beta}(n)$  is the number of valid executions of length  $n$  leading from the state  $\alpha$  to the state  $\beta$ .

Define the Möbius polynomial leading from  $\alpha$  to  $\beta$  as

$$\mu_{\alpha,\beta}(z) = \sum_{c \in \mathcal{C}_{\alpha,\beta}} (-1)^{|c|} z^{|c|}. \quad (4.1)$$

The matrix  $\mu = (\mu_{\alpha,\beta})_{(\alpha,\beta) \in X \times X}$  is called the *Möbius matrix* of the concurrent system; and the matrix  $G = (G_{\alpha,\beta})_{(\alpha,\beta) \in X \times X}$  is its *growth matrix*.

In trace theory, we have the fundamental combinatorial result (2.1) between generating series and the Möbius polynomial. This formula can be extended to the concurrent system in a matrix form. The result of Prop. 4.2.1 can be found in [1]. Here we provide an alternative proof.

**Proposition 4.2.1.** Let  $(\mathcal{M}, X, \perp)$  be a concurrent system. Consider the growth matrix  $G = (G_{\alpha,\beta})_{(\alpha,\beta) \in X \times X}$  and the Möbius matrix  $\mu = (\mu_{\alpha,\beta})_{(\alpha,\beta) \in X \times X}$ . Then the following identity holds:

$$\mu G = G \mu = I.$$

*Proof.* For  $(\alpha, \beta) \in X \times X$ , suppose that  $G_{\alpha, \beta}(z)$  is the growth series associated with  $\mathcal{M}_{\alpha, \beta}$  and  $\mu_{\alpha, \beta}(z)$  is its related Möbius polynomial.

By definition, we have:

$$\begin{aligned} (\mu G)_{(\alpha, \beta)} &= \sum_{\gamma \in X} \mu_{\alpha, \gamma} G_{\gamma, \beta} \\ &= \sum_{\gamma \in X} \sum_{c \in \mathcal{C}_{\alpha, \gamma}, u \in \mathcal{M}_{\gamma, \beta}} (-1)^{|c|} z^{|c|} z^{|u|}. \end{aligned} \quad (4.2)$$

Our goal is to prove that

$$(\mu G)_{(\alpha, \beta)} = \begin{cases} 1, & \text{if } \alpha = \beta \\ 0, & \text{if } \alpha \neq \beta. \end{cases}$$

We revise the bijective proof of trace monoids in Krattenthaler's paper [26] to fit in with the concurrent system.

Let  $K = \bigcup_{\gamma \in X} \mathcal{C}_{\alpha, \gamma} \times \mathcal{M}_{\gamma, \beta}$ . We construct a bijection  $\varphi_{\alpha, \beta}$  from  $K$  to itself. The idea is to move one piece from  $\mathcal{C}_{\alpha, \gamma}$  to  $\mathcal{M}_{\gamma, \beta}$  or reversely. Then most terms cancel each other pairwise in (4.2). Consider a pair  $(c, u) \in K$ . Let  $E$  be the set of pieces formed by the pieces of  $c$  and the minimal pieces of  $u$  which are parallel with the pieces in  $c$ .

Let  $a$  be the minimal piece of  $E$  depending on an arbitrary given total order on the alphabet  $\Sigma$ . For all pairs  $(c, u) \in K$ , such that  $E \neq \emptyset$ . We put:

$$\varphi_{\alpha, \beta}(c, u) = \begin{cases} (c \setminus a, a \cdot u), & \text{if } a \in c \\ (c \cdot a, u \setminus a), & \text{if } a \in u. \end{cases}$$

With this construction, the bijection  $\varphi_{\alpha, \beta}$  is an involution. Moreover, the only nonvanishing term is that  $E = \emptyset$  which implies that empty clique in  $\mathcal{C}_{\alpha, \beta}$  and empty heap in  $\mathcal{M}_{\alpha, \beta}$ . This completes the proof.  $\square$

**Remark 4.2.2.** Let  $(\mathcal{M}, X, \perp)$  be a concurrent system with  $\mathcal{M} = (\mathcal{M}, I)$ . Let  $\Sigma = \Sigma_1 + \Sigma_2$  be a non trivial partition of  $\Sigma$  such that  $(a, b) \in I$  for all  $(a, b) \in \Sigma_1 \times \Sigma_2$ . So, any two actions of  $\Sigma_1$  and of  $\Sigma_2$  are concurrent. Put  $\mathcal{M} = \mathcal{M}_1 \times \mathcal{M}_2$  with  $\mathcal{M}_1 = \langle \Sigma_1 \rangle$  and  $\mathcal{M}_2 = \langle \Sigma_2 \rangle$ . Let  $\mu_1(z)$  and  $\mu_2(z)$  be the Möbius matrices of the concurrent systems  $(\mathcal{M}_1, X, \perp)$  and  $(\mathcal{M}_2, X, \perp)$ , where the actions of  $\mathcal{M}_1$  and  $\mathcal{M}_2$  is obtained from the restriction of the action of  $\mathcal{M}$  on  $X \cup \{\perp\}$ . Then for any state  $\alpha, \beta \in X$ ,

$$\mathcal{C}_{\alpha, \beta} = \sum_{\gamma \in X} (\mathcal{C}_{\alpha, \gamma} \cap \mathcal{M}_1) \times (\mathcal{C}_{\gamma, \beta} \cap \mathcal{M}_2),$$

and thus

$$\mu_{\alpha, \beta}(z) = \sum_{c \in \mathcal{C}_{\alpha, \beta}} (-1)^{|c|} z^{|c|} = \sum_{\gamma \in X} (\mu_{1\alpha, \gamma}(z) \mu_{2\gamma, \beta}(z)).$$

Therefore, the Möbius matrix  $\mu(z)$  can be calculated as:  $\mu(z) = \mu_1(z) \mu_2(z) = \mu_2(z) \mu_1(z)$ .

Next, we discuss about the radius of convergence of the growth series.

**Definition 4.2.3.** Let  $(\mathcal{M}, X, \perp)$  be a concurrent system. For  $(\alpha, \beta)$  in  $X \times X$ , let  $r_{\alpha, \beta}$  be the radius of convergence of the growth series  $G_{\alpha, \beta}(z)$ . We define that the characteristic root  $r$  of the concurrent system as  $\min\{r_{\alpha, \beta} : (\alpha, \beta) \in X \times X\}$ .

**Proposition 4.2.4.** Let  $S = (\mathcal{M}, X, \perp)$  be a non trivial and homogeneous concurrent system. Then its characteristic root  $r$  satisfies  $r < \infty$ .

*Proof.* Since the system  $S$  is non trivial, there exists a state  $\alpha \in X$  and a trace  $x$  such that  $\alpha \cdot x \neq \perp$ . Let  $\beta = \alpha \cdot x$ . Since  $S$  is homogeneous, there exists a trace  $y$  such that  $\alpha = \beta \cdot y$ . Put  $k = |xy|$ . Then  $\#\mathcal{M}_{\alpha, \alpha}(kn) \geq 1$  for every integers  $k \geq 0$  and we can deduce that

$$\sum_{n \geq 0} \#\mathcal{M}_{\alpha, \alpha}(kn)z^n \geq \sum_{n \geq 0} z^n.$$

Therefore,  $r_{\alpha, \alpha}^k \leq 1$  and thus  $r \leq r_{\alpha, \alpha} < \infty$ .  $\square$

The following proposition shows that the characteristic root can be obtained by solving the Möbius matrix of the system with the aid of the technique in analytic combinatorics.

**Proposition 4.2.5.** Let  $S = (\mathcal{M}, X, \perp)$  be a non trivial and homogeneous concurrent system with characteristic root  $r$ . Then all growth series  $G_{\alpha, \beta}(z)$  are rational with the same radius convergence  $r$  where  $r$  is the positive root of smallest modulus of the polynomial  $\det \mu(z)$ .

*Proof.* Let  $(\alpha, \beta)$  and  $(\alpha', \beta') \in X \times X$ . Since the system  $S$  is homogeneous, there exists  $x, y \in \mathcal{M}$  such that  $\alpha \cdot x = \alpha'$  and  $\beta \cdot y = \beta'$ . Put  $k = |x + y|$ . Then for  $n \geq 0$ ,

$$\mathcal{M}_{\alpha, \beta}(n + k) \supseteq \{xuy : u \in \mathcal{M}_{\alpha', \beta'}\}$$

and thus

$$\#\mathcal{M}_{\alpha, \beta}(n + k) \geq \#\mathcal{M}_{\alpha', \beta'}(n)$$

by cancellation. So the following relation for growth series holds:

$$\frac{1}{z^k} G_{\alpha, \beta}(z) \leq G_{\alpha', \beta'}(z)$$

and therefore  $r_{\alpha, \beta} \geq r_{\alpha', \beta'}$ . Exchanging the roles of  $(\alpha, \beta)$  and of  $(\alpha', \beta')$ , we get  $r_{\alpha, \beta} = r_{\alpha', \beta'} = r$  which implies that all radii of convergence  $r_{\alpha, \beta}$  are equal.

By definition, all growth series  $G_{\alpha, \beta}(z)$  are rational functions with non negative coefficients. According to Remark 1.2.2,  $r = r_{\alpha, \beta}$  is a pole of all series  $G_{\alpha, \beta}(z)$  which implies that  $r$  is a pole of  $G(z)$ . Since the formula  $G(z) = (\mu(z))^{-1}$  holds for  $|z| < r$ ,  $r$  is a root of smallest modulus of  $\det \mu(z)$ .  $\square$

## 4.3 Examples of abstract concurrent systems

### 4.3.1 Trace monoids

The first natural example is trace monoids, which could be considered as the concurrent systems with one state. The properties of combinatorics in previous sections for concurrent systems coincide with the properties for trace monoids discussed in Chapter 2.

For example, the smallest irreducible trace monoid  $\mathcal{M}_0 = \langle a, b, c \mid a \cdot b = b \cdot a \rangle$ . The set of its cliques is  $\mathcal{C} = \{\varepsilon, a, b, c, ab\}$ . The Möbius matrix of the system is exactly the Möbius polynomial of the trace monoid, which is  $\mu(z) = 1 - 3z + z^2$ . The characteristic root  $r$  is  $\frac{3-\sqrt{5}}{2}$ . Moreover, the irreducibility of the trace monoid implies the irreducibility of the system.

### 4.3.2 Safe Petri nets

Petri nets are one of the formalisms for concurrent systems, where its graphical representation intrinsically includes the phenomena of concurrency, causality and non-determinism choices. Moreover, it is easier to understand the behavior of the system and test the performance.

We give the rigorous definition as follows: A net [37] is a triple  $\mathcal{N} = (P, T, F)$ , satisfying:

1.  $P$  and  $T$  are two finite disjoint sets of *places* and of *transitions*, respectively.
2.  $F \subseteq (P \times T) \cup (T \times P)$  is a set of *edges*.

The set  $F$  is called the *flow relation* of  $\mathcal{N}$ . The elements of  $\mathcal{N} = P \cup T$  are called *nodes*. Given a node  $x$  of  $\mathcal{N}$ , the *preset*  $\bullet x$ , the *postset*  $x^\bullet$  of  $x$  and the subset  $\bullet x^\bullet$  are defined by:

$$\bullet x = \{y \in N : (y, x) \in F\} \quad x^\bullet = \{y \in N : (x, y) \in F\} \quad \bullet x^\bullet = \{x\} \cup \bullet x \cup x^\bullet$$

In convention, a net is depicted as a bipartite multigraph where places are represented by circles and transitions are represented by rectangles.

Places can contain *tokens*. A *marking* of a net is a mapping  $\alpha : P \rightarrow \mathbb{Z}_{\geq 0}$  where  $\alpha(p)$  is represented the number of tokens assigned in the place  $p \in P$ .

A transition  $t$  is *enabled* at a marking  $\alpha$  if  $\alpha(p) > 0$  for all  $p \in \bullet t$ . In other words, a transition is enable if all places of its preset contain at least one token. If a transition  $t$  is enabled at  $\alpha$ , the *firing rule* leading to the successor marking  $\beta$  (written  $\alpha \xrightarrow{t} \beta$ ) is defined as follows:

$$\forall p \in P \quad \beta(p) = \begin{cases} \alpha(p) - 1, & \text{if } p \in \bullet t \text{ and } p \notin t^\bullet, \\ \alpha(p) + 1, & \text{if } p \notin \bullet t \text{ and } p \in t^\bullet, \\ \alpha(p), & \text{otherwise.} \end{cases}$$

In this thesis, we assume that  $\bullet t \neq \emptyset$  and  $t \bullet \neq \emptyset$  for every transition  $t \in T$ .

A marking  $\beta$  is *reachable* from  $\alpha$  if there exists a sequence  $s = t_1 t_2 \dots t_n \in T^*$  such that  $\alpha \xrightarrow{t_1} \dots \xrightarrow{t_n} \beta$  (or simply  $\alpha \xrightarrow{*} \beta$ ). The sequence  $s$  is called the *firing sequence* from  $\alpha$ . Let  $\varepsilon$  be the empty sequence. In particular,  $\alpha \xrightarrow{\varepsilon} \alpha$  for every marking  $\alpha$ .

A marking  $\alpha$  is *safe* if  $\alpha(p) \leq 1$  for all  $p \in P$ .

**Definition 4.3.1** (Safe Petri net). *A safe Petri net is a quadruple  $\mathcal{N} = (P, T, F, \alpha_0)$  where  $(P, T, F)$  is a net,  $\alpha_0$  is the initial marking, and all markings of  $\mathcal{N}$  reachable from  $\alpha_0$  are safe.*

Let  $R(\mathcal{N})$  be all reachable markings of  $\mathcal{N}$  from its initial marking and  $\xrightarrow{t}$  be the transition relation for all  $t \in T$ . The *marking graph* of a safe Petri net  $\mathcal{N}$  is the graph of  $(R(\mathcal{N}), \xrightarrow{t})$ .

Note that the execution of Petri nets is *nondeterministic*, since multiple transitions could be enabled at the same time and there is no priority over the firing orders. Due to this property, Petri nets are well suited to model the concurrent behavior in distributed systems.

To cohere with the general concurrent system, we consider the executions understood up to traces, instead of the firing sequences in the following example.

**Example 4.3.2.** The Petri net depicted in Figure 4.1 is a concurrent system  $(\mathcal{M}, X, \perp)$  with  $X = \{M_0, M_1\}$  and  $\mathcal{M} = \langle a, b, c, d, e \mid ad = da, ae = ea, bd = db, be = eb, ce = ec \rangle$ .

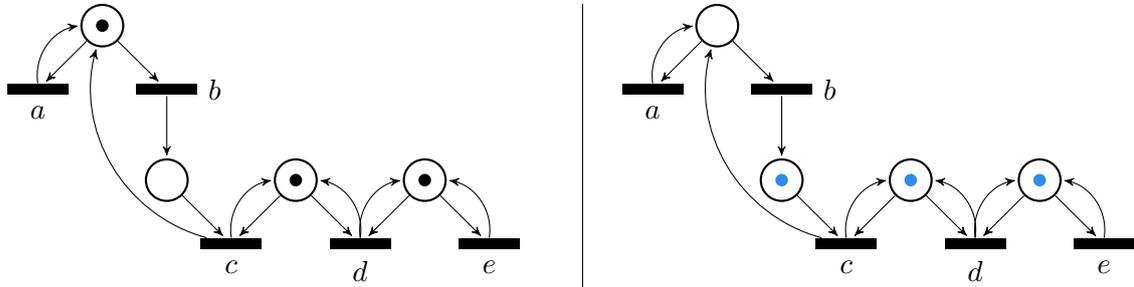


Figure 4.1: Safe Petri nets with two states (Left: The state  $M_0$ . Right: The state  $M_1$ .)

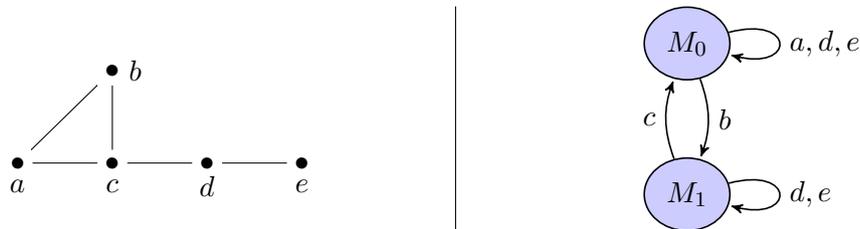


Figure 4.2: Left : Coxeter graph of  $\mathcal{M}$ . Right: Marking graph of the Petri net in Figure 4.1

The Coxeter graph and the marking graph are strongly connected which implies that  $\mathcal{M}$  is irreducible and the system is homogeneous. It is obvious that the Petri net is alive. So, the concurrent system is irreducible.

**Möbius matrix and characteristic root** The Möbius matrix is

$$\mu(z) = \begin{matrix} M_0 \\ M_1 \end{matrix} \begin{pmatrix} 1 - 3z + 2z^2 & -z + 2z^2 \\ -z + z^2 & 1 - 2z \end{pmatrix}.$$

The determinant of  $\mu(z)$  is  $(1 - z) \cdot (2z - 1) \cdot (z^2 + 2z - 1)$  with characteristic root  $\sqrt{2} - 1$ .

### 4.3.3 Tip-top actions

Consider a concurrent system  $(\mathcal{M}, X, \perp)$  with  $\mathcal{M} = \mathcal{M}(\Sigma, I)$ . Let  $X = \mathcal{C}$ , the set of cliques of  $\mathcal{M}$ . We define the *tip-top action of the first kind* as an action  $\psi : (\mathcal{C} \cup \{\perp\}) \times \mathcal{M} \rightarrow (\mathcal{C} \cup \{\perp\})$  satisfying :

$$\forall \gamma \in \mathcal{C} \quad \forall a \in \Sigma \quad \psi(\gamma, a) = \begin{cases} \gamma \cdot a, & \text{if } a \parallel \gamma \\ \gamma \setminus \{a\}, & \text{if } a \leq \gamma \\ \perp, & \text{otherwise} \end{cases}.$$

The action  $\psi$  has the property that  $\varepsilon \cdot a \cdot a = \varepsilon$  for all  $a \in \Sigma$ .

Let  $\bar{\Sigma} = \{\bar{a} \mid a \in \Sigma\}$  and let  $I' = I \cup \{(\bar{a}, \bar{b}) \mid (a, b) \in I\} \cup \{(\bar{a}, b) \mid (a, b) \in I\} \cup \{(a, \bar{b}) \mid (a, b) \in I\}$ . Put  $\Sigma' = \Sigma \cup \bar{\Sigma}$  and  $\mathcal{M}' = \mathcal{M}'(\Sigma', I')$ . Consider a concurrent system  $(\mathcal{M}', \mathcal{C}, \perp)$ . Define the *tip-top action of the second kind* as an action  $\bar{\psi} : (\mathcal{C} \cup \{\perp\}) \times \mathcal{M}' \rightarrow \mathcal{C}$  satisfying :

$$\forall \gamma \in \mathcal{C} \quad \forall a \in \Sigma \quad \bar{\psi}(\gamma, a) = \begin{cases} \gamma \cdot a, & \text{if } a \parallel \gamma \\ \perp, & \text{otherwise} \end{cases}$$

$$\forall \gamma \in \mathcal{C} \quad \forall \bar{a} \in \bar{\Sigma} \quad \bar{\psi}(\gamma, \bar{a}) = \begin{cases} \gamma \setminus \{a\}, & \text{if } a \parallel \gamma \\ \perp, & \text{otherwise} \end{cases}.$$

The action  $\bar{\psi}$  has the property that  $\varepsilon \cdot a \cdot \bar{a} = \varepsilon$  for all  $a \in \Sigma$ .

In the concurrent systems of tip-top action, the state space is the same as the set of cliques. Next, we present two different examples where the domino tiling model is of the tip-top action of the first kind, and the dining philosopher model is of the tip-top action of the first kind. For these kinds of models, we have the reduction method to quickly get the characteristic root without the cumbersome calculation of the determinant. The demonstration will give in Section 7.5.2.

#### 4.3.3.1 The domino tiling model

The domino tilings are classical combinatorial models for the purpose of filling a surface with the dominoes of the size  $1 \times 2$  or  $2 \times 1$ .

The  $n$ -Rabati tilings fix the surface in a strip of size  $n \times 2$ . We start with using only the dominoes of the size  $2 \times 1$  as in Figure 4.3 (The case for  $n = 4$ ). Two consecutive dominoes are possible to transform into two dominoes of the size  $1 \times 2$ . We called it the action of a flip.

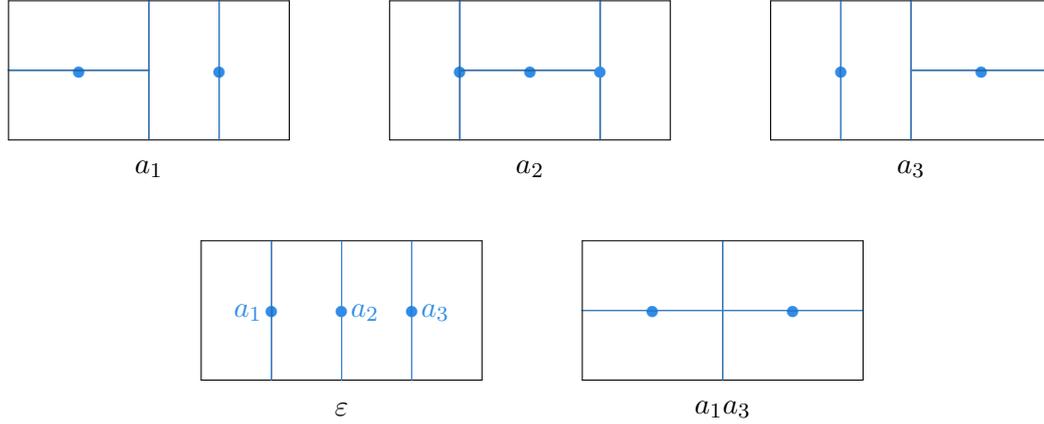


Figure 4.3: The 5 tilings of the 4-Rabati tiling model

The associated trace monoid can be translated as  $\Sigma = \{a_1, a_2, \dots, a_{n-1}\}$  where  $a_i$  correspond to the  $i$ -th position of the flip, and the independent relation

$$I = \{(a_i, a_j) \mid |i - j| \geq 2\}.$$

Consider the 4-Rabati tiling model. Let  $\mathcal{M} = \langle a_1, a_2, a_3 \mid a_1 a_3 = a_3 a_1 \rangle$  with the tip-top action of the first kind. Then this model is the concurrent system  $(\mathcal{M}, X, \perp)$  with  $X = \{\varepsilon, a_1, a_2, a_3, a_1 a_3\}$ .

**Möbius matrix and characteristic root** The Möbius matrix is

$$\mu(z) = \begin{matrix} \varepsilon \\ a_1 \\ a_2 \\ a_3 \\ a_1 a_3 \end{matrix} \begin{pmatrix} 1 & -z & -z & -z & z^2 \\ -z & 1 & 0 & z^2 & -z \\ -z & 0 & 1 & 0 & 0 \\ -z & z^2 & 0 & 1 & -z \\ z^2 & -z & 0 & -z & 1 \end{pmatrix}.$$

The determinant of  $\mu(z)$  is  $(1 - z^2)^2 \cdot (1 - 3z^2 + z^4)$  and the characteristic root  $r$  is  $\sqrt{\frac{\sqrt{5}-1}{2}}$  obtained from the part of  $1 - 3z^2 + z^4$ ; whereas the Möbius polynomial of the associated monoid is  $1 - 3z + z^2$  and its smallest positive root equals to  $r^2$ .

#### 4.3.3.2 The dining philosophers model

We apply our theory to the classical problem for concurrency – “The dining philosophers” which is used as a standard synchronization model in concurrent programming theory.

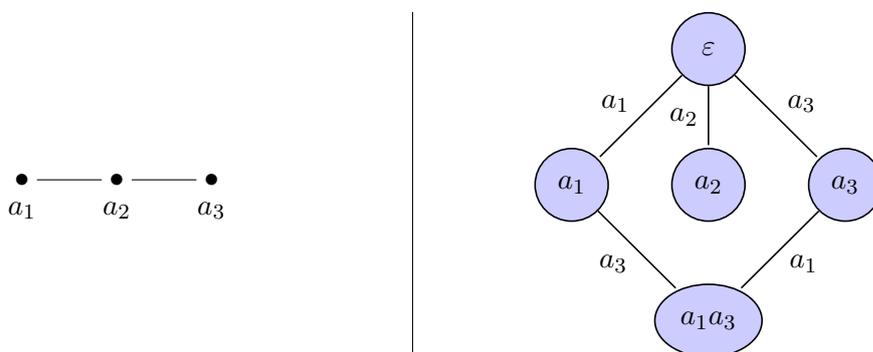


Figure 4.4: Left : Coxeter graph of  $\mathcal{M}$ . Right: State graph of the 4-Rabati tiling

The scenario of the problem is as follows: Five philosophers sit around a circular table. There is one fork between each pair of two adjacent philosophers. The philosophers either think or eat. They allow to eat when both forks on their right and left hand sides are available. After finishing eating, a philosopher should return both two forks on the table.

The Petri net depicted on Figure 4.5 is a model of this problem. The tokens in the places  $0, \dots, 4$  represent the forks. The executions of transitions  $a, b, \dots, e$ , which are executable when both input places with a token, represent the actions of eating. The executions of transitions  $f, g, \dots, j$  represent the actions of returning forks.

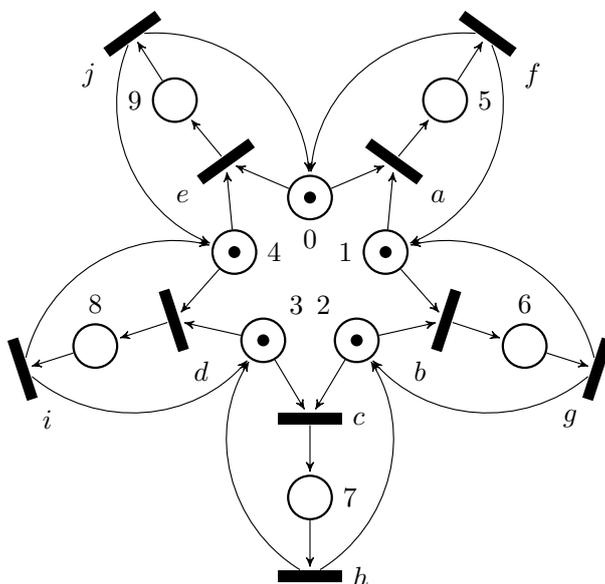


Figure 4.5: The corresponding Petri net of dining philosophers

**Coxeter graph and marking graph** Let  $\Sigma = \{a, b, \dots, e\}$  and let  $I = \{ac, ad, bd, be, ce\}$ . Put  $\mathcal{M} = \mathcal{M}(\Sigma, I)$ . Consider the concurrent system with the tip-top action of the second kind on  $\mathcal{M}$ . The state space  $X$  includes the initial state where no one eat, five possible states where only one philosopher eat and five possible states where exactly two philosophers eat at the same time which corresponds to the set of cliques  $\{\varepsilon, a, b, c, d, e, ac, ad, bd, be, ce\}$ . The marking graph of the concurrent system is depicted on Figure 4.7.

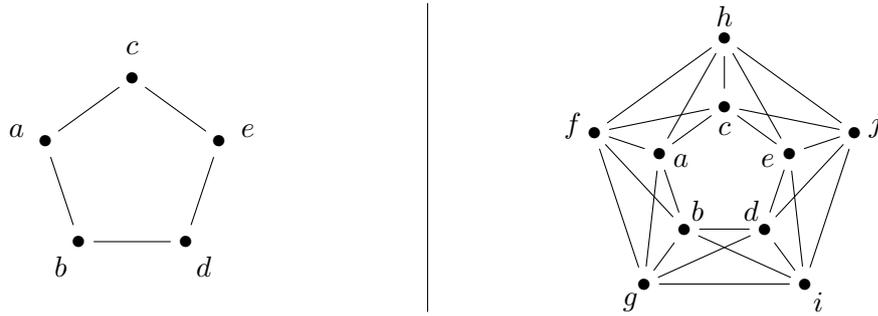


Figure 4.6: Left : Coxeter graph of  $\mathcal{M}$ . Right: Coxeter graph of  $\mathcal{M}'$  of the Petri net in Figure 4.5

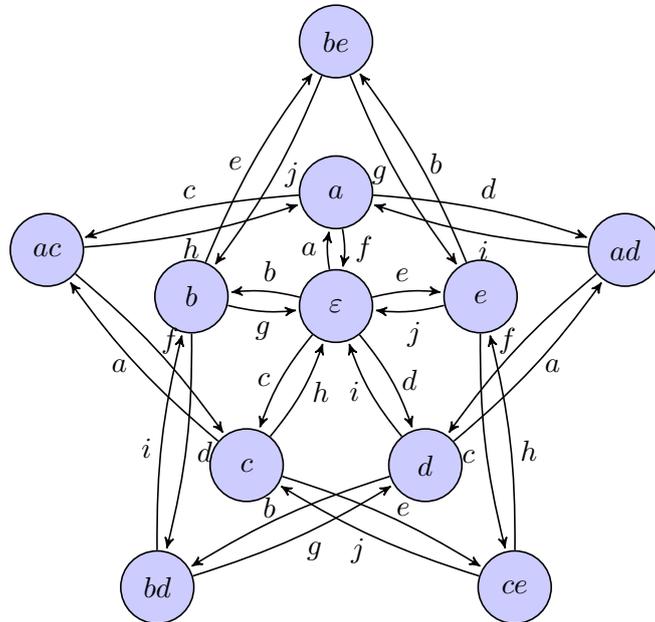


Figure 4.7: Marking graph of the Petri net in Figure 4.5

**Irreducibility** The system is alive and homogeneous since the initial state is homogeneous and all letters are enabled in the initial state. Moreover, the related trace monoid is irreducible which implies the system is irreducible.

**Möbius matrix and characteristic root** The Möbius matrix of five dining philosophers is

$$\mu(z) = \begin{pmatrix} 1 & -z & -z & -z & -z & -z & z^2 & z^2 & z^2 & z^2 & z^2 \\ -z & 1 & 0 & z^2 & z^2 & 0 & -z & -z & 0 & 0 & 0 \\ -z & 0 & 1 & 0 & z^2 & z^2 & 0 & 0 & -z & -z & 0 \\ -z & z^2 & 0 & 1 & 0 & z^2 & -z & 0 & 0 & 0 & -z \\ -z & z^2 & z^2 & 0 & 1 & 0 & 0 & -z & -z & 0 & 0 \\ -z & 0 & z^2 & z^2 & 0 & 1 & 0 & 0 & 0 & -z & -z \\ z^2 & -z & 0 & -z & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ z^2 & -z & 0 & 0 & -z & 0 & 0 & 1 & 0 & 0 & 0 \\ z^2 & 0 & -z & 0 & -z & 0 & 0 & 0 & 1 & 0 & 0 \\ z^2 & 0 & -z & 0 & 0 & -z & 0 & 0 & 0 & 1 & 0 \\ z^2 & 0 & 0 & -z & 0 & -z & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

with determinant  $\theta(z) = -(2z^2 - 1)^5 \cdot (5z^4 - 5z^2 + 1)$ . The characteristic root  $r$  is the smallest positive root of  $\theta(z)$ , where

$$r = \sqrt{\frac{1}{2} - \frac{\sqrt{5}}{10}} \approx 0.526.$$

#### 4.3.4 The mosaic models

Let  $n \geq 3$  be an odd integer and let  $\mathbb{Z}_n = \mathbb{Z}/(n\mathbb{Z})$ . The *mosaic model of order  $n$*  is the concurrent system  $(\mathcal{M}, X, \perp)$  defined as follows. The trace monoid  $\mathcal{M} = \mathcal{M}(\Sigma, I)$  is the ring dimer monoid on  $n$  generators with  $\Sigma = \{a_i : i \in \mathbb{Z}_n\}$ ,  $I = (\Sigma \times \Sigma) \setminus D$  and  $D$  is defined by:

$$D = \{(a_i, a_j) : i, j \in \mathbb{Z}_n, j \in \{i, i-1, i+1\}\}.$$

Let  $\mathcal{B} = \{0, 1\}$  represent a set of two colors, equipped with the mapping  $\bar{\cdot} : \mathcal{B} \rightarrow \mathcal{B}$  defined by  $\bar{0} = 1$  and  $\bar{1} = 0$ . A state  $x = (x_i)_{i \in \mathbb{Z}_n} \in X$  is a mapping  $x : \mathbb{Z}_n \rightarrow \mathcal{B}$  such that  $x(i) = 1$  for an odd number of indices  $i$ . For  $i \in \mathbb{Z}_n$ , the action  $x \cdot a_i$  of  $a_i$  on  $x \in X$  is  $x \cdot a_i = \perp$  if  $x(i) \neq x(i+1)$ . Otherwise,  $y = x \cdot a_i \in X$  is defined by:

$$\forall j \in \mathbb{Z}_n \quad y(j) = \begin{cases} x_j, & \text{if } j \notin \{i, i+1\}, \\ \bar{x}_j, & \text{if } j \in \{i, i+1\}. \end{cases}$$

Referring to Viennot's theory of heaps of pieces, we give a graphical interpretation of the elements of mosaic models of order  $n$  as follows. Let  $W_0, \dots, W_{n-1}$  be  $n$  parallel infinite wires, hanged on the  $x$ -axis. For  $i \in \mathbb{Z}_n$ , the piece  $a_i$  is supported by the

pair  $(W_{i-1}, W_i)$ . Two pieces are  $D$ -related if and only if they share a common wire. Additionally, each wire is labeled with a color in  $\mathcal{B}$ . We say that the piece  $a_i$  can be played in the state  $x$  if  $x_i = x_{i+1}$ . After piling  $a_i$ , we change the color of  $x_i$  and  $x_{i+1}$  into the other. Figure 4.8 illustrates mosaic model of order 5.

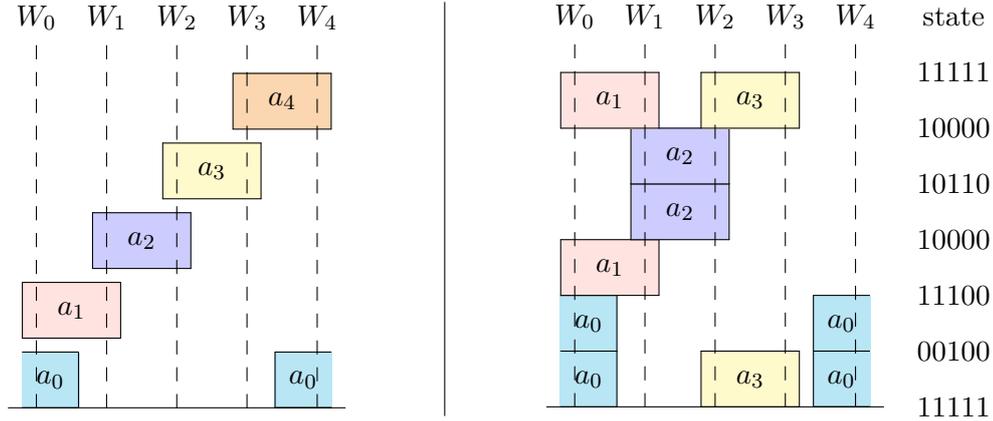


Figure 4.8: Left: Five basic pieces of the mosaic model of order 5. Right: An execution of length 9 from the state 11111 in the mosaic model of order 5.

Mosaic models provide non trivial examples of concurrent systems, interesting for testing the computational methods.

**State graph** We observe first that since the order  $n$  is assumed to be odd, there exists at least a contiguous pair of identical colors in each state  $x$ . In this model, every action is reversible. Hence, the state graph can be viewed as undirected. Therefore, the state space  $X$  is partitioned into its strongly connected components. The following result describes the connectedness.

**Lemma 4.3.3.** *Let  $\iota$  be the state in  $X$  where  $\iota_i = 1$  for all  $i$  in  $\mathbb{Z}_n$  and let  $x \in X$ . Then there exists an executable sequence of actions from state  $x$  to the state  $\iota$ , denoted by  $x \xrightarrow{*} \iota$ .*

*Proof.* For  $x \in X$ , we're going to construct an algorithmic way to reach state  $\iota$  by reducing the number of the occurrences of color 0 in the state  $x$  which is denoted by  $|x|_0$ .

Let  $x \neq \iota$ . Then there exists at least two color 0 in the state  $x$ . If this two color 0 are adjacent (including the first and the final since the indices are in  $\mathbb{Z}_n$ ), we simply change this pair of color 0 into the pair of color 1 and reduce the number of color 0 in the state  $x$ . If not, without loss of generality, suppose  $x_0 = 0$ . Then we get  $x_{n-1} = 1$  and each single color 0 separates consecutive color 1s into two blocks. So the number

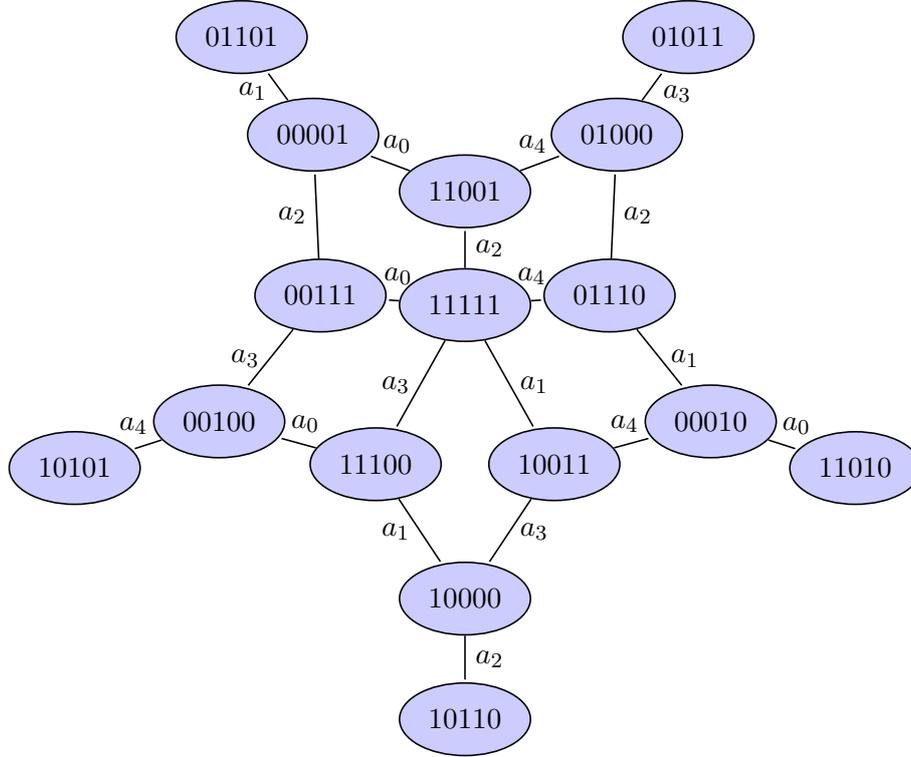


Figure 4.9: State graph of the mosaic model with  $n=5$  (Figure 4.8).

of blocks of color 1 equals  $|x|_0$  which is even. On the other hand, the number of the occurrences of color 1 in state  $x$  is odd. Hence, there exists at least one block of color 1 with an even number of color 1. Focusing on this block, we have the form  $01 \cdots 10$  in a part of the configuration. In this case, we turn all color 1s in this block into color 0 and change again all color 0s (including the two original color 0) into color 1. Then we reduce the number of color 0 in the state  $x$ . Continuing this process, we'll reach the state  $\iota$  and completes the proof.  $\square$

**Proposition 4.3.4.** *The state graph of mosaic models of order  $n$  odd is strongly connected.*

*Proof.* Let  $X$  be the state space of mosaic model of order  $n$  odd and  $S$  be the strongly connected component containing the state  $\iota$  where  $\iota_i = 1$  for all  $i$  in  $\mathbb{Z}_n$ . Because all actions keep the parity of occurrences of colors,  $\iota$  is only connected with states in  $X$  and  $S \subseteq X$ . The other direction is proved by Lemma 4.3.3. Therefore, the state graph is strongly connected and  $X = S$ .  $\square$

Therefore the system of general mosaic model is homogeneous.

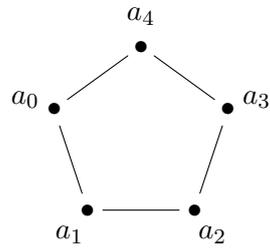


Figure 4.10: Coxeter graph of  $\mathcal{M}$  of the mosaic model of order 5 (Figure 4.8).

**Irreducibility** The ring dimer monoid is irreducible since its Coxeter graph is strongly connected. In every state, there exists an execution back to the initial state. Moreover, all letters in  $\Sigma$  are executable in the initial state. Therefore, the system is alive and irreducible.

## Chapter 5

# Digraph of states-and-cliques of a concurrent system

We see in previous chapter (Def. 4.1.2) that the state graph could be viewed as the automaton of actions. The paths in the state graph correspond to the valid executions in the concurrent system. However, with this approach, the concurrency between the actions in the state graph is not adequately handled. Indeed, we count multiple times the paths in the same congruence class. Therefore, it is necessary to introduce the *digraph of state-and-cliques* of a concurrent system for combinatorics purposes.

The DSC helps us to understand the concurrent executions without losing the information of the current state. The nodes in DSC are classified with positive and null ones. We will revisit this notion in Chapter 7 from a probabilistic point of view.

### 5.1 Notions of DSC and of its augmented version (ADSC)

**Definition 5.1.1** (DSC). *Let  $S = (\mathcal{M}, X, \perp)$  be a concurrent system. The digraph of states-and-cliques (DSC) of  $S$  is the digraph whose nodes are all pairs of  $(\alpha, c)$  for  $\alpha \in X$  and  $c \in \mathcal{C}_\alpha$ , and whose edges are all pairs of  $((\alpha, c), (\beta, d))$  such that  $\alpha \cdot c = \beta$  and  $c \rightarrow d$  holds.*

The DSC can be seen as the automaton of "traces" where both concurrency and states are taken into account. Our goal is to generate the executions of the concurrent systems up to the length rather than traces. So, we need to adjust the DSC into Definition 5.1.2.

**Definition 5.1.2** (ADSC). *Let  $S = (\mathcal{M}, X, \perp)$  be a concurrent system. The augmented digraph of states-and-cliques (ADSC) of  $S$  is the digraph whose nodes are all triples of  $(\alpha, c, i)$  for  $\alpha \in X, c \in \mathcal{C}_\alpha$  and  $i = 1, \dots, |c|$ , whose edges are all triples of  $((\alpha, c, i), (\beta, d, j))$  such that one of the following cases holds:*

- $\alpha \cdot c = \beta, c \rightarrow d, i = |c|$  and  $j = 1$
- $(\alpha, c) = (\beta, d)$  and  $j = i + 1$ .

**Example 5.1.3.** An example of 1-safe Petri nets depicted on the left of Figure 5.1. The associated trace monoid  $\mathcal{M}$  is equipped with alphabet  $\Sigma = \{a, b, c\}$  and the independence relation  $I = \{(a, c)\}$ . The state space  $X$  has two states  $M_0$  and  $M_1$  where  $M_0$  is the initial state and  $M_1 = M_0 \cdot a$ .

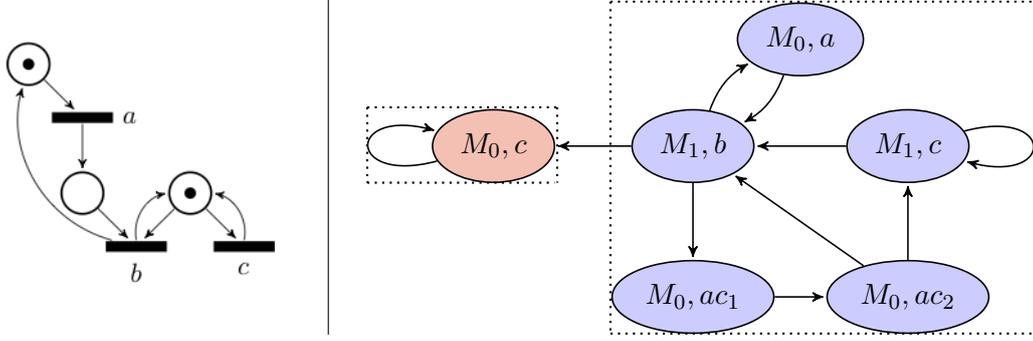


Figure 5.1: Left: 1-safe nets (state  $M_0$ ), Right: The ADSC of the nets.

The ADSC depicted on the right of Figure 5.1 has two strongly connected components which are presented by the dashed boxes. Its associated adjacency matrix  $A$  is

$$A = \begin{matrix} M_0, a \\ M_0, ac_1 \\ M_0, ac_2 \\ M_1, b \\ M_1, c \\ M_0, c \end{matrix} \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \quad (5.1)$$

## 5.2 Paths in DSC and in ADSC

With the aid of DSC and of ADSC, the executions of concurrent systems could be understood as the paths in graphs.

**Proposition 5.2.1.** *Let  $S = (\mathcal{M}, X, \perp)$  be a concurrent system. There exists a bijection between the following correspondences:*

- For integer  $h \geq 1$ , the executions of  $S$  of height  $h$ , and the paths of length  $h$  in DSC.
- For integer  $n \geq 1$ , the executions of  $S$  of length  $n$  leading from a state  $\alpha$  to a state  $\beta$ , and the paths of length  $n$  in ADSC leading from a node of the form  $(\alpha, c, 1)$  to a node of the form  $(\beta, d, |d|)$ .
- For integer  $n \geq 1$ , the executions of  $S$  of length  $n$ , and the paths of length  $n$  in ADSC leading from a node of the form  $(\alpha, c, 1)$  to a node of the form  $(\beta, d, |d|)$  for all  $\alpha, \beta \in X$ .

*Proof.* Consider an initial state  $\alpha$ , and  $x \in \mathcal{M}_\alpha$  be an execution of  $S$  of height  $h$ , for  $h \geq 1$ . The trace  $x$  can be uniquely expressed as the normal form  $c_1 \rightarrow c_2 \rightarrow \cdots \rightarrow c_h$ . Put  $\alpha_0 = \alpha$  and  $\alpha_i = \alpha_{i-1} \cdot c_i$  for  $1 \leq i \leq h$ . Then  $(\alpha_0, c_1) \rightarrow \cdots \rightarrow (\alpha_{h-1}, c_h)$  forms a path of length  $h$  in DSC.

Observe that a node  $(\alpha, c) \in \text{DSC}$  corresponds to the definitive path of length  $|c|$  in ADSC as

$$(\alpha, c, 1) \rightarrow (\alpha, c, 2) \rightarrow \cdots \rightarrow (\alpha, c, |c|) \quad (5.2)$$

for all valid executions in  $S$ . Fix an initial state  $\alpha$  and a final state  $\beta$ , a valid execution of  $S$  leading from  $\alpha$  to  $\beta$  should start from the first letter of a clique  $c \in \mathcal{C}_\alpha$  and end at the last letter of a clique  $d \in \mathcal{C}_\beta$ . Therefore, the bijection is based on the length of executions.  $\square$

In Section 1.2, we present the framework of paths in graphs. However, DSC is NOT strongly connected in most cases for concurrent systems. Therefore, all growth series of paths in DSC don't have the consistent radius of convergence (without irreducibility), and no dominant eigenvalue matters the coefficients of the series of paths (without aperiodicity). We see more examples in the last section of this chapter.

We still want to figure out similar properties for concurrent systems. In Proposition 4.2.5, we prove that all generating series  $G_{\alpha,\beta}$  have the same radius of convergence which is relied on the accessibility of the concurrent system. Despite the fact that the constraints of executions for each state are different in the stategraph, we could say the accessibility of the system is somehow a weak definition of "strong connectivity" of the stategraph.

From now on, together with next Chapter, we will try to obtain the dominant eigenvalue in a recursive way without adopting the whole ADSC. The following proposition connects the characteristic root of the concurrent system and the spectral radius of ADSC.

**Proposition 5.2.2.** *Let  $S = (\mathcal{M}, X, \perp)$  be a concurrent system of the characteristic root  $r$  and let  $\rho$  be the spectral radius of ADSC. Then  $r = \rho^{-1}$  if  $\rho \neq 0$  and  $r = \infty$  if  $\rho = 0$ .*

*Proof.* Let  $G(z) = (G_{\alpha,\beta})_{(\alpha,\beta) \in X \times X}$  be the growth matrix of the system. Let  $B$  be the adjacency matrix of ADSC with spectral radius  $\rho$ . Consider the matrix series  $F = (F_{(\alpha,c,i),(\beta,d,j)})_{(\alpha,c,i),(\beta,d,j) \in \text{ADSC}}$  of all paths in ADSC where

$$F_{(\alpha,c,i),(\beta,d,j)} = \sum_{n \geq 0} B_{(\alpha,c,i),(\beta,d,j)}^n z^n$$

with radius of convergence  $R_{(\alpha,c,i),(\beta,d,j)}$ . Denote the radius of convergence of  $F$  by  $R = \min\{R_{(\alpha,c,i),(\beta,d,j)} \mid (\alpha, c, i), (\beta, d, j) \in \text{ADSC}\}$ .

By Proposition 1.2.1, we have  $R = \rho^{-1}$ . Then it suffices to show that  $r = R$ . In other words, the radius of convergence  $r$  of  $G(z)$  associated with executions equals to the radius of convergence  $R$  of  $F(z)$  associated with the paths in ADSC.

Referring to (5.2), all paths in ADSC can be extended to be valid executions in  $S$ . So, we have the following relation, for all integers  $n \geq 0$ :

$$B_{(\alpha,c,i),(\beta,d,j)}^n = B_{(\alpha,c,1),(\beta,d,|d|)}^{n+(i-1)+(|d|-j)}$$

and deduce that

$$R = \min\{R_{(\alpha,c,1),(\beta,d,|d|)} \mid (\alpha, c), (\beta, d) \in \text{DSC}\}.$$

Next, we focus on the coefficients of  $G(z)$  and of  $F(z)$ . According to point 2 of Proposition 5.2.1,

$$\#\mathcal{M}_{\alpha,\beta}(n) \geq B_{(\alpha,c,1),(\beta,d,|d|)}^n.$$

Therefore,  $r = r_{\alpha,\beta} \leq R_{(\alpha,c,i),(\beta,d,j)}$  which implies that  $r \leq R$ .

On the other hand, again by point 2 of Proposition 5.2.1,

$$\forall \alpha \in X \quad \#\mathcal{M}_{\alpha,\alpha}(n) = \sum_{(c,d) \in \mathcal{C}_\alpha \times \mathcal{C}_\alpha} B_{(\alpha,c,1),(\alpha,d,|d|)}^n.$$

Since  $\sum \#\mathcal{M}_{\alpha,\alpha}(n)r^n$  is divergent, there exists at least one of the series  $\sum B_{(\alpha,c,1),(\alpha,d,|d|)}^n r^n$  which is divergent. Hence,  $R \leq r$ .  $\square$

### 5.3 Positive and null nodes of DSC and ADSC

Let  $x = c_1 \rightarrow \dots \rightarrow c_n \in \mathcal{M}$ . Define  $C_1 : \overline{\mathcal{M}} \rightarrow \mathcal{C}$  by  $C_1(x) = c_1$ , the natural projection of the first clique in the normal form.

**Definition 5.3.1.** Let  $(\mathcal{M}, X, \perp)$  be a concurrent system and let  $(\alpha, c)$  be a node of DSC. An execution  $x \in \mathcal{M}_\alpha$  is an  $(\alpha, c)$ -protection if

$$\forall y \in \mathcal{M}_{\alpha \cdot x} \quad C_1(xy) = c.$$

**Definition 5.3.2** (Positive nodes, null nodes). A node  $(\alpha, c)$  in DSC is called a positive node, if there exists a  $(\alpha, c)$ -protection ; a node  $(\alpha, c)$  is called a null node, otherwise. We denote the subgraph of DSC including only positive nodes by  $\text{DSC}^+$ .

A node  $(\alpha, c, i)$  in ADSC is called a positive node or a null node depending on whether  $(\alpha, c)$  is positive or null in DSC. We denote the subgraph of ADSC including only positive nodes by  $\text{ADSC}^+$ .

Some related properties are discussed.

**Proposition 5.3.3.** Let  $(\mathcal{M}, X, \perp)$  be a concurrent system with  $\mathcal{M} = \mathcal{M}(\Sigma, I)$ .

1. Let  $\alpha$  be a state such that  $\mathcal{M}_\alpha \neq \{\varepsilon\}$ . For all pairs  $(\alpha, c) \in \text{DSC}$ , if  $c$  is maximal clique of  $\mathcal{M}_\alpha$ , then  $(\alpha, c)$  is a positive node.
2. Let  $((\alpha, c), (\beta, d))$  be an edge in DSC where  $(\beta, d)$  is a positive node. Then  $(\alpha, c)$  is also a positive node.

3. Let  $(\alpha, c)$  be a null node of DSC. Then for every execution  $x \in \mathcal{M}_\alpha$  such that  $C_1(x) = c$ , there exists a letter  $a \in \Sigma$  such that  $x \in \mathcal{M}_\alpha^{\setminus a}$ , where  $\mathcal{M}^{\setminus a} = \langle \Sigma \setminus \{a\} \rangle$ .

*Proof.* Point 1: A maximal clique  $c \in \mathcal{C}_\alpha$  is itself an  $(\alpha, c)$ -protection. Once  $c$  is executed, the execution of other trace doesn't change the first clique. Hence,  $(\alpha, c)$  is a positive node.

Point 2: Since  $(\beta, d)$  is a positive node, there exists  $y \in \mathcal{M}_\beta$  such that  $y$  is a  $(\beta, d)$ -protection. Take an execution  $x = c \cdot y$ , then  $x$  is a  $(\alpha, c)$ -protection which implies that  $(\alpha, c)$  is a positive node.

Point 3: Seeking for contradiction, suppose that there exists an execution  $x \in \mathcal{M}_\alpha$  such that  $C_1(x) = c$  and  $x$  contains every letter in  $\Sigma$ . Then for every execution  $y \in \mathcal{M}_{\alpha \cdot x}$ , the first clique of the execution  $xy$  is still  $c$ . Therefore,  $x$  is an  $(\alpha, c)$ -protection, which contradicts that  $x$  is a null node.  $\square$

**Example 5.3.4** (Example 5.1.3 continued). On the right of Figure 4.2, the positive nodes are filled with purple and the null node is filled with pink. The node  $(M_0, c)$  is the only null node since no  $(M_0, c)$ -protection exists. After the execution of  $(M_0, c)$ , we always stay in  $M_0$ . Moreover, all executions including letter  $a \in \mathcal{M}_{M_0}$  have an influence on the first clique.

## 5.4 Examples of different configurations of DSC

For simplicity, we only discuss the DSC of the concurrent systems without losing the sight of the big picture. It is clear to get an ADSC from a DSC as we did in Example 5.1.3. Both the DSC and the ADSC have the same number of strongly connected components. The advantage of ADSC is especially for the random generation of a fixed-length execution, instead of a fixed-height execution.

### 5.4.1 Example of trace monoids

The examples of trace monoids are rare cases in concurrent systems where the DSC is strongly connected. For the trace monoid  $\mathcal{M}_0 = \langle a, b, c \mid a \cdot b = b \cdot a \rangle$ , the DSC is isomorphic to the digraph of cliques of  $\mathcal{M}_0$  (See Figure 2.4).

### 5.4.2 4-Rabati tiling model

The settings of the domino tiling model as a concurrent system are discussed in Section 4.3.3.1. For instance in 4-Rabati tiling, the associated monoid is  $\mathcal{M} = \langle a_1, a_2, a_3 \mid a_1 a_3 = a_3 a_1 \rangle$  with tip-top action of the first kind.

The DSC in Figure 5.2 has 14 nodes as follows: the state  $\varepsilon$  can be followed by all cliques; the state  $a_1, a_3$  and  $a_1 a_3$  can be followed by the cliques  $a_1, a_3$  and  $a_1 a_3$ ; whereas the state  $a_2$  can be followed by the clique  $a_2$ . Its edges between nodes are constructed according to the Cartier-Foata decomposition.

The four strongly connected components in the DSC are presented in dashed boxes. The positive nodes are filled with blue or purple, where the purple nodes are the ones related to a terminal strongly connected component.

On the other hand, the null nodes are filled with pink. Consider the null node  $(a_1, a_3)$ , which means that the current state is  $a_1$  and the following clique is  $a_3$ . This choice restricts to play a letter  $a_1$  in the following firing sequence due to the Cartier-Foata decomposition. Thus, it is impossible to have an  $(a_1, a_3)$ -protection, which implies that  $(a_1, a_3)$  is a null node and the system will stay infinitely in this component. The other null nodes can be explained in a similar way.

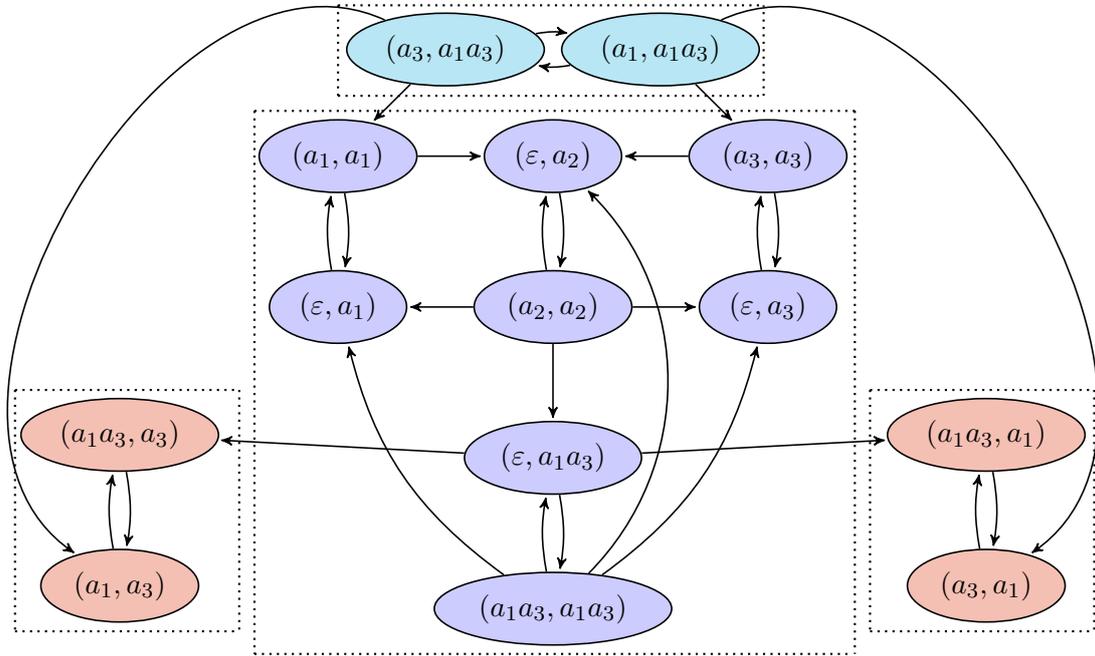


Figure 5.2: The DSC of 4-Rabati tiling

### 5.4.3 The dining philosophers model

The description of the dining philosophers model can be found in Sec 4.3.3.2. For the case of five philosophers, the DSC of the system is not strongly connected which contains seven strongly connected components of size 20, 10, 4,4,4,4 and 4, separately. Figure 5.3 shows the condensation of DSC where each node represents a strongly connected component of the size indicated in the node. The components of size 20 and of size 10 contain positive nodes, and the rest of the components contain null nodes.

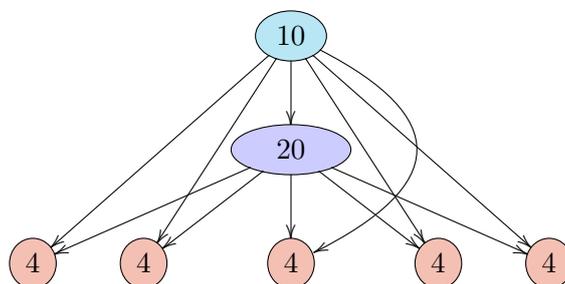
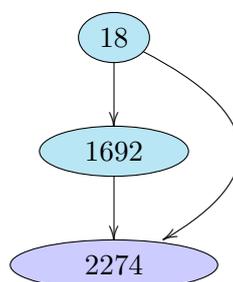


Figure 5.3: The condensation of DSC of 5 dining philosophers model

#### 5.4.4 The mosaic model

The mosaic model is introduced in Section 4.3.4. We show in Figure 5.4 the condensation of the DSC for the case of  $n = 9$ . Each node represents a strongly connected component of the size indicated in the node. The only terminal component is filled with purple.

This model shows an example where the DSC has only positive nodes. That is,  $\text{DSC} = \text{DSC}^+$ . In the following proposition, we prove this property for all mosaic models.

Figure 5.4: The condensation of DSC of the mosaic model with  $n = 9$ 

**Proposition 5.4.1.** *Consider the mosaic model of order  $n$  defined as in Section 4.3.4. Then all mosaic models of order  $n$  contain only positive nodes in its DSC.*

*Proof.* For each node  $(\alpha, c) \in \text{DSC}$ , it suffices to show that there exists an  $(\alpha, c)$ -protection  $x \in \mathcal{M}_\alpha$ .

We shall construct the execution  $x$  as a sequence of cliques  $(c_1, \dots, c_k)$  such that:

- $x = c_1 \rightarrow c_2 \rightarrow \dots \rightarrow c_k \in \mathcal{M}_\alpha$ ,
- $c_1 = c$ ,
- $c_k$  is a maximal clique.

Then  $(\alpha, c_1) \xrightarrow{*} (\alpha \cdot c_1 \cdot \dots \cdot c_{k-1}, c_k)$  is a valid path in the DSC.

We assume first that  $|c| = 1$ . Then  $c$  is a single piece, say  $c = a_i$ . Let  $\beta = \alpha \cdot a_i$ , and put  $\beta = (\beta_1, \dots, \beta_n)$ . Then  $\beta = \alpha \cdot a_i$  implies  $\alpha_i = \alpha_{i+1} = \beta_i = \beta_{i+1}$ . We analyze two disjoint cases according to two colors of  $\beta_{i-1}$  and  $\beta_{i+2}$ .

- Case 1:  $\beta_{i-1} = \beta_{i+2}$ 
  - If  $\beta_{i-1} = \beta_i$ , we have  $\beta_{i-1} = \beta_i = \beta_{i+1} = \beta_{i+2}$ . Since  $a_i \rightarrow a_{i-1} \cdot a_{i+1}$ , we get the valid path  $(\alpha, a_i) \rightarrow (\beta, a_{i-1} \cdot a_{i+1})$ .
  - If  $\beta_{i-1} \neq \beta_i$ , then  $\alpha_{i-1} = \alpha_i = \alpha_{i+1} = \alpha_{i+2}$ . Indeed, we know that  $\alpha_{i+1} = \alpha_i = \beta_i = \beta_{i-1} = \alpha_{i-1}$  and  $\alpha_{i-1} = \beta_{i-1} = \beta_{i+2} = \alpha_{i+2}$  by assumption. Hence, we get the valid path  $(\alpha, a_i) \rightarrow (\beta, a_i) \rightarrow (\alpha, a_{i-1} \cdot a_{i+1})$ .

- Case 2:  $\beta_{i-1} \neq \beta_{i+2}$

The color  $\beta_i$  equals either  $\beta_{i-1}$  or  $\beta_{i+2}$ , say  $\beta_{i+2}$  without loss of generality. Hence  $\beta_i = \beta_{i+1} = \beta_{i+2}$  and thus  $a_{i+1} \in \mathcal{C}_\beta$ . Let  $\gamma = \beta \cdot a_{i+1}$ , corresponding to the state reached by the execution of the valid path  $(\alpha, a_i) \rightarrow (\beta, a_{i+1})$ , and put  $\gamma = (\gamma_1, \dots, \gamma_n)$ . Observe that the colors in positions  $(i-1, i, i+1)$  of  $\gamma$  are alternating. Then, checking the color  $\gamma_{i+3}$ :

- If  $\gamma_{i+3} = \gamma_i$ , we apply the same technique as above in Case 1 to obtain the valid path  $(\alpha, a_i) \rightarrow (\beta, a_{i+1}) \rightarrow (\gamma, a_{i+1}) \rightarrow (\beta, a_i \cdot a_{i+2})$ .
- Otherwise, we may extend the valid path already obtained to  $(\alpha, a_i) \rightarrow (\beta, a_{i+1}) \rightarrow (\gamma, a_{i+2})$ , reaching a state  $\delta = \gamma \cdot a_{i+2}$  with alternating colors in positions  $(i-1, i, i+1, i+2)$ . We continue this process, which will necessarily end due to the fact that the number  $n$  is odd, at which point the technique of Case 1 may be applied.

Now, we consider  $|c| \geq 2$ .

- Assume  $|c| = k$ . If all pieces in  $c$  are adjacent to each other, say  $c = a_i \cdot \dots \cdot a_{i+k}$ , we observe the colors of the positions  $i-1$  and  $i+k+1$  and check if these two colors are the same. Then we can follow a similar procedure of single piece to extend the length of clique piece by piece. Finally, we will arrive the state of configuration  $0 \dots 01$  understood mod  $n$  and followed by a maximal clique.
- If the pieces in  $c$  are not adjacent, we can first execute one of these pieces and continue the process with adjacent pieces.

Therefore, we construct a way to execute a maximal clique no matter we start from an arbitrary clique.

According to Point 1 of Proposition 5.3.3,  $c_k$  is a maximal clique implies that  $(\alpha \cdot c_1 \cdot \dots \cdot c_{k-1}, c_k)$  is a positive node. Then by Point 2 of Proposition 5.3.3,  $(\alpha, c_1)$  is a positive node following the reversed path, which completes the proof.  $\square$

**Remark 5.4.2.** *There exists a unique terminal component which contains the node of state of configuration  $0 \dots 01$  followed by a maximal clique.*

### 5.4.5 An example with several terminal components of $DSC^+$

All previous examples show that there exists only one terminal component in  $DSC$ . However, this may not always be the case. We see in the following example.

Let  $S = (\mathcal{M}, X, \perp)$  be the concurrent system defined as the state graph in Figure 5.5 with  $\mathcal{M} = \langle a, b, c, d, e \mid ab = ba, ad = da, bf = fb, cd = dc, ce = ec, ef = fe \rangle$  and  $X = \{0, 1, \dots, 11\}$ .

We claim that the system  $S$  is well-defined and irreducible. The action of  $\mathcal{M}$  defined on  $X \cup \{\perp\}$  fits with the independence relation. The monoid  $\mathcal{M}$  is irreducible according to its Coxeter graph. Moreover, it is clear that the system is homogeneous and alive.

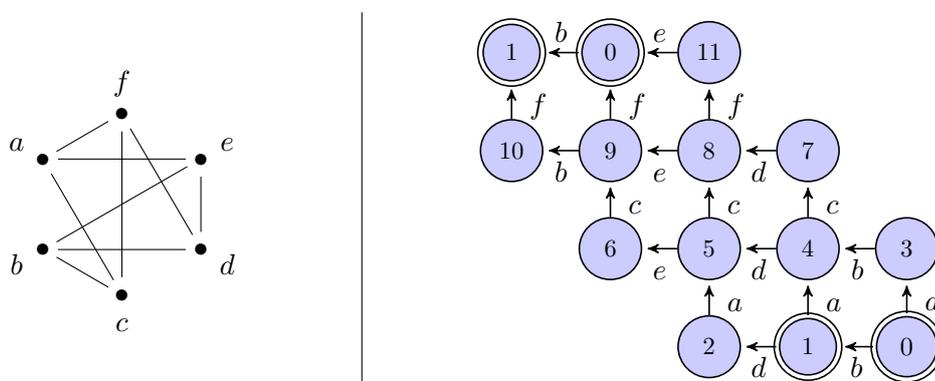


Figure 5.5: Left: Coxeter graph of  $\mathcal{M}$ . Right: State graph of a concurrent system. Two states 0 (1) in double circles are identical.

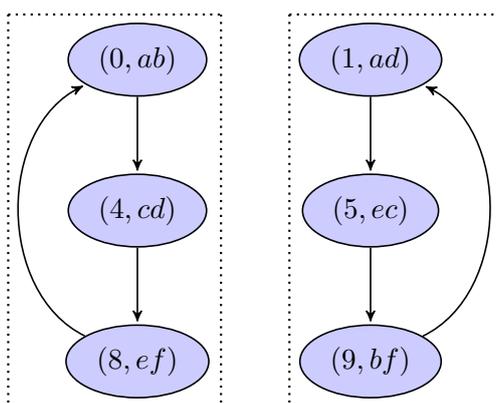


Figure 5.6: The  $DSC^+$  of the concurrent system in Figure 5.5

The  $DSC^+$  illustrated in Figure 5.6 has two terminal strongly connected components presented by dashed boxes.



## Chapter 6

# Spectral property for concurrent systems

The spectral property is related to the characteristic root of certain submonoid. For trace monoids, a direct result from the Perron-Frobenius Theorem shows that the irreducibility implies the spectral property.

However, we see in previous chapter that the DSC of most concurrent systems are not strongly connected. Instead, we focus on a component of ADSC where its adjacency matrix has the property like a primitive matrix. This allows us to retrieve the main components in ADSC which determine the characteristic root of the system.

### 6.1 Definitions and the case of trace monoids

Let  $(\mathcal{M}, X, \perp)$  be a concurrent system with  $\mathcal{M} = \mathcal{M}(\Sigma, I)$ . Let  $a$  be a letter in  $\Sigma$ , define  $\mathcal{M}^{\setminus a} = \langle \Sigma \setminus \{a\} \rangle$  as the submonoid of  $\mathcal{M}$ . Restricting the action of  $\mathcal{M}$  on  $X \cup \{\perp\}$  to  $\mathcal{M}^{\setminus a}$ , then  $(\mathcal{M}^{\setminus a}, X, \perp)$  is still a concurrent system.

**Definition 6.1.1.** *Let  $S = (\mathcal{M}, X, \perp)$  be a concurrent system with  $\mathcal{M} = \mathcal{M}(\Sigma, I)$ . For every letter  $a \in \Sigma$ , let  $r^{\setminus a}$  be the characteristic root of the concurrent system  $(\mathcal{M}^{\setminus a}, X, \perp)$ . We say that  $S$  has the spectral property if  $r^{\setminus a} > r$  for every  $a \in \Sigma$ .*

Let  $\alpha, \beta \in X$ . In general case, we have  $r_{\alpha, \beta}^{\setminus a} \geq r_{\alpha, \beta} \geq r$  since  $\mathcal{M}_{\alpha, \beta}^{\setminus a}(n) \subseteq \mathcal{M}_{\alpha, \beta}(n)$  for all integers  $n$ . The spectral property describes the phenomenon of the strict inequality.

Before getting into the case for concurrent systems, we first take a look at the case for trace monoids. For a trace monoid, the irreducibility of a trace monoid is equivalent to the spectral property. This result is directly from the Perron-Frobenius theorem, where the irreducibility of the adjacency matrix implies the existence of the dominant eigenvalue (Ref. proof of Proposition 2.4.10).

Here we give another proof, which is original in our paper [4] without applying the Perron-Frobenius Theorem. We use the property of the root of the Möbius polynomial. The advantage of this method is that we could retrieve the dominant eigenvalue even

though we don't know the structure of digraph of cliques. This inspires us to extend to the case for concurrent systems

**Proposition 6.1.2.** *Let  $\mathcal{M} = (\mathcal{M}, I)$  be a trace monoid with Möbius polynomial  $\mu(z)$  and characteristic root  $r$ . Then  $\mathcal{M}$  has the spectral property if and only if  $\mathcal{M}$  is irreducible.*

*Proof.* Suppose that  $\mathcal{M}$  doesn't have the spectral property. Let  $\mu_{\mathcal{M}}(z)$  be the Möbius polynomial of  $\mathcal{M}$ . Then, for  $a \in \Sigma$ , we have  $r^{\setminus a} = r$  where  $r^{\setminus a}$  is the characteristic root of the submonoid  $\mathcal{M}^{\setminus a}$ , which implies that

$$\mu_{\mathcal{M}^{\setminus a}}(r) = \mu_{\mathcal{M}^{\setminus a}}(r^{\setminus a}) = 0. \quad (6.1)$$

Recall the recursive formula for the Möbius polynomial in Lemma 3.4.3:

$$\mu_{\mathcal{M}}(z) = \mu_{\mathcal{M}^{\setminus a}}(z) - z \cdot \mu_{\mathcal{M}^{\parallel a}}(z)$$

where  $\mathcal{M}^{\parallel a} = \langle \Sigma_{\parallel a} \rangle$  with  $\Sigma_{\parallel a} = \{b \in \Sigma \mid (a, b) \notin I\}$ . By (6.1), we can deduce that  $\mu_{\mathcal{M}^{\parallel a}}(r) = 0$ . Hence, the characteristic root of  $\mathcal{M}^{\parallel a}$  is  $r$ . Repeating the same reasoning until the subset  $\tilde{\Sigma}$  only containing the letters dependent with  $a$ . Let  $\tilde{\mathcal{M}} = \langle \tilde{\Sigma} \rangle$ . We obtain that the characteristic root of  $\tilde{\mathcal{M}}$  is  $r$ . Since  $\tilde{\Sigma} \neq \emptyset$ ,  $\mathcal{M}$  is not irreducible.

On the other hand, assume that  $\mathcal{M}$  is not irreducible. We want to show that  $\mathcal{M}$  doesn't have the spectral property. Let  $\Sigma = \Sigma_1 + \Sigma_2$  be a non trivial partition of  $\Sigma$  such that  $\Sigma_1 \times \Sigma_2 \subseteq I$  and put  $\mathcal{M} = \mathcal{M}_1 \times \mathcal{M}_2$  with  $\mathcal{M}_1 = \langle \Sigma_1 \rangle$  and  $\mathcal{M}_2 = \langle \Sigma_2 \rangle$ . Let  $\mu(z)$ ,  $\mu_1(z)$  and  $\mu_2(z)$  be the Möbius polynomials of  $\mathcal{M}$ ,  $\mathcal{M}_1$  and  $\mathcal{M}_2$  with the characteristic roots  $r$ ,  $r_1$  and  $r_2$ . According to Remark 4.2.2, as a special case for trace monoids, we know  $\mu(z) = \mu_1(z)\mu_2(z)$ . Then  $r = \min\{r_1, r_2\}$ . Without loss of the generality, assume that  $r_2 \geq r_1$ . Choose a letter  $a \in \Sigma_2$  and let  $\mathcal{M}^{\setminus a} = \mathcal{M}_1 \times \mathcal{M}_2^{\setminus a}$ . Then its characteristic root  $r^{\setminus a} = \min\{r_1, r_2^{\setminus a}\}$ . Since  $r_2^{\setminus a} \geq r_2 \geq r_1$ , hence  $r^{\setminus a} = r_1 = r$ , which completes the proof.  $\square$

Next, we introduce the linking sequences and the linking executions. The linking executions will use for the propose of proving the spectral property for concurrent systems.

**Definition 6.1.3.** *Let  $(\mathcal{M}, X, \perp)$  be a concurrent system with  $\mathcal{M} = \mathcal{M}(\Sigma, I)$  irreducible, and let  $\alpha \in X$ . A linking sequence from  $\alpha$  is a sequence of letters  $a_1, \dots, a_n$  satisfying, for some sequence of integers  $1 \leq j_1 < \dots < j_m \leq n$ :*

- $\alpha \cdot (a_1 \cdots a_n) \neq \perp$ ;
- $(a_{j_k}, a_{j_{k+1}}) \in D$  for  $k = 1, \dots, m - 1$ , where  $D = (\Sigma \times \Sigma) \setminus I$ ;
- each letter of  $\Sigma$  has at least one occurrence in the sequence  $(a_{j_1}, \dots, a_{j_m})$ .

Let  $a \in \Sigma$  be a letter. If the sequence  $j_1, \dots, j_m$  can be chosen such that  $a_{j_1} = a$ , the linking sequence is called  $a$ -rooted.

A *linking execution from  $\alpha$*  is an execution which is the image in  $\mathcal{M}$  of a linking sequence from  $\alpha$ . It is  *$a$ -rooted* if it is the image in  $\mathcal{M}$  of an  *$a$ -rooted* sequence.

An example of an  *$a$ -rooted* linking execution of a concurrent system is depicted in Figure 6.1. The system is defined as the stategraph in the middle of the figure with the associated monoid  $\mathcal{M} = \langle a, b, c, d \mid ab = ba, bc = cb, cd = dc, ad = da \rangle$ . In the heap interpretation of this linking execution, the blue pieces mark the occurrences of each letter.

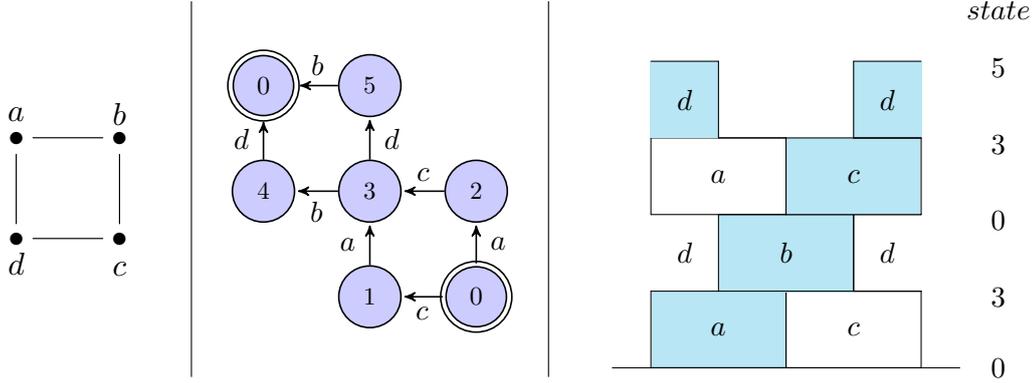


Figure 6.1: Left: Coxeter graph of  $\mathcal{M}$ . Middle: State graph of a concurrent system. Two state 0 in double circles are identical. Right:  *$a$ -rooted linking execution from the state 0* of this system.

**Proposition 6.1.4.** *Let  $S = (\mathcal{M}, X, \perp)$  be an homogeneous concurrent system with  $\mathcal{M} = \mathcal{M}(\Sigma, I)$ . Then  $S$  is irreducible if and only if, for every state  $\alpha \in X$  and for every letter  $a \in \Sigma$ , there exists an  *$a$ -rooted linking execution from  $\alpha$* .*

*Proof.* Assume that the concurrent system  $S$  is irreducible. Then the associated trace monoid  $\mathcal{M}$  is irreducible which guarantees the existence of the linking sequence  $(a_1, \dots, a_k)$ . Since  $S$  is alive, it is possible to insert some traces into  $(a_1, \dots, a_k)$  in order to obtain a valid execution.

On the other hand, if the linking execution exists for  $\alpha \in X$  and for  $a \in \Sigma$ , we can deduce that  $\mathcal{M}$  is irreducible. The accessibility of  $S$  and the existence of linking execution implies that  $S$  is alive and therefore  $S$  is irreducible.  $\square$

**Lemma 6.1.5.** *Let  $(\mathcal{M}, X, \perp)$  be a concurrent system with  $\mathcal{M} = \mathcal{M}(\Sigma, I)$ . Let  $a$  be a letter in  $\Sigma$  and let  $x$  be an  *$a$ -rooted linking execution from some state  $\alpha$* . Define  $\mathcal{M}^{\setminus a} = \langle \Sigma \setminus \{a\} \rangle$ . Fix two integers  $i$  and  $j$  and a state  $\beta$ , then the mapping*

$$\mathcal{M}_{\beta, \alpha}(i) \times \mathcal{M}_{\alpha, x}^{\setminus a}(j) \rightarrow \mathcal{M}_{\beta}, \quad (u, v) \mapsto uxv$$

*is injective.*

*Proof.* Suppose that  $uxv = u'xv'$  for  $u, u' \in \mathcal{M}_{\beta, \alpha}(i)$  and for  $v, v' \in \mathcal{M}_{\alpha, x}^a(j)$ . Imagine these two identical executions  $uxv = u'xv'$  as a heap interpretation. If  $u \neq u'$  and  $v \neq v'$ , then a letter from  $v$  must cross  $x$  to  $u$ . However, it is impossible since the letter will be blocked by the piece of letter  $a$  in the linking execution  $x$ . Therefore,  $u = u'$  and  $v = v'$  which implies that the mapping is injective.  $\square$

With the aid of the  $a$ -rooted linking execution, we define a class of executions uniquely determined by a trace including  $a$  and the other trace without  $a$ . This technique will be used in the proof of Theorem 6.2.2.

## 6.2 Spectral property for concurrent systems

The discussion in Section 1.2 shows that the generating series of paths in a strongly connected graph have the common radius of convergence and there exists a dominant eigenvalue of its adjacency matrix. These two properties are useful to get the asymptotics of the counting of paths.

However, DSC in concurrent systems are not strongly connected in general. Thanks to the structure of concurrent systems, the common radius of convergence is proved in Proposition 4.2.5. Concerning the dominant eigenvalue, we want to prove the spectral property, which helps us to retrieve the dominant eigenvalue in a crucial strongly connected component. We would adopt the weaker property for a general graph as in Proposition 1.2.3. Before getting into the theorem, we prove that this property can be extended from paths in ADSC to the generating series of concurrent systems.

**Lemma 6.2.1.** *Let  $(\mathcal{M}, X, \perp)$  be a concurrent system with characteristic root  $r$ . Then there exists a state  $\alpha \in X$  and an integer  $d$  such that, for every multiple  $k$  of  $d$ , the series  $\sum_{n \geq 0} \mathcal{M}_{\alpha, \alpha}(kn)z^n$  has the radius of convergence  $r^k$ .*

*Proof.* Let  $B$  be the adjacency matrix of ADSC with spectral radius  $\rho = \rho(B)$ . By Proposition 5.2.2, we know that  $\rho = r^{-1}$ . According to Proposition 1.2.3, there exists a node  $u = (\alpha, c, i)$  and an integer  $d > 0$  such that, for every multiple  $k$  of  $d$ , the series

$$\tilde{F}(z) = \sum_{n \geq 0} B_{u, u}^{kn} z^n$$

has the radius of convergence  $r^k$ .

Consider the series

$$\widetilde{G_{\alpha, \alpha}}(z) = \sum_{n \geq 0} \#\mathcal{M}_{\alpha, \alpha}(kn)z^n.$$

Let  $R$  be the radius of convergence of  $\widetilde{G_{\alpha, \alpha}}(z)$ . It suffices to show that  $R = r^k$ . Let

$$G_{\alpha, \alpha}(z) = \sum_{n \geq 0} \#\mathcal{M}_{\alpha, \alpha}(n)z^n$$

with radius of convergence  $r_{\alpha,\alpha}$ . By the definition of the radius of convergence and the characteristic root, we have  $R = r_{\alpha,\alpha}^k \geq r^k$ .

On the other hand, since  $B_{u,u}^{kn} \leq \mathcal{M}_{\alpha,\alpha}(kn)$ , then

$$\tilde{F}(z) \leq \widetilde{G_{\alpha,\alpha}}(z)$$

which implies that  $r^k \geq R$ . This completes the proof.  $\square$

The following theorem is the main result of this chapter.

**Theorem 6.2.2** ([4]). *Let  $S = (\mathcal{M}, X, \perp)$  be a non trivial and homogeneous concurrent system with characteristic root  $r$ . Then  $S$  is irreducible if and only if  $S$  has the spectral property.*

*Proof.* Let  $S = (\mathcal{M}, X, \perp)$  be a non trivial and homogeneous concurrent system of the characteristic root  $r$ . By Proposition 4.2.4,  $r < \infty$ .

Suppose the system  $S$  is irreducible. Let  $a$  be an arbitrary letter in  $\Sigma$  and let  $r^{\setminus a}$  be the characteristic root of the concurrent system  $(\mathcal{M}^{\setminus a}, X, \perp)$ . Our goal is to prove that

$$r^{\setminus a} > r.$$

According to Lemma 6.2.1, there exists a state  $\alpha_0 \in X$  and an integer  $d > 0$  such that, for every multiple  $k$  of  $d$ , the series

$$Q(z) = \sum_{n \geq 0} \#\mathcal{M}_{\alpha_0, \alpha_0}^{\setminus a}(kn) z^n$$

has the radius of convergence  $(r^{\setminus a})^k$ . To apply this property, we turn to show that  $(r^{\setminus a})^k > r^k$ .

For this integer  $d$ , we claim that there exists a family  $(y_\alpha)_{\alpha \in X}$  of  $a$ -rooted linking executions satisfying:

- $y_\alpha \in \mathcal{M}_{\alpha,\alpha}$ ;
- $|y_\alpha| = k$  where  $k$  is a multiple of  $d$  and is independent of  $\alpha$ .

First, by Proposition 6.1.4, there exists an  $a$ -rooted linking execution  $u_\alpha$  for every state  $\alpha \in X$ . Since  $S$  is homogeneous, there exists  $v_\alpha \in \mathcal{M}_{\alpha, u_\alpha, \alpha}$ . Put  $z_\alpha = u_\alpha v_\alpha$ , then  $z_\alpha \in \mathcal{M}_{\alpha,\alpha}$  is an  $a$ -rooted linking execution and  $|z_\alpha| \geq |u_\alpha| > 0$ . Let  $m_\alpha = \prod_{\delta \in X, \delta \neq \alpha} |z_\delta|$ .

Choose  $y_\alpha = (z_\alpha)^{dm_\alpha}$ . Then  $y_\alpha \in \mathcal{M}_{\alpha,\alpha}$  is also an  $a$ -rooted linking execution and  $|y_\alpha| = dm_\alpha |z_\alpha| = d \cdot \prod_{\delta \in X} |z_\delta|$  which is a multiple of  $d$  and is independent of  $\alpha$ .

Together with this family  $(y_\alpha)_{\alpha \in X}$  of  $a$ -rooted linking executions, we would like to construct an auxiliary family  $(\mathcal{H}_{\alpha,\beta}(kn))_{n \geq 0, (\alpha,\beta) \in X \times X}$  of sets of executions such that

$$\mathcal{M}_{\alpha,\beta}^{\setminus a}(kn) \subseteq \mathcal{H}_{\alpha,\beta}(kn) \subseteq \mathcal{M}_{\alpha,\beta}(kn). \quad (6.2)$$

The construction of  $(\mathcal{H}_{\alpha,\beta}(kn))$  defines inductively as follows:

$$\mathcal{H}_{\alpha,\beta}(0) = \mathcal{M}_{\alpha,\beta}^{\setminus a}(0),$$

$$\forall n \geq 0 \quad \mathcal{H}_{\alpha,\beta}(k(n+1)) = (\mathcal{H}_{\alpha,\beta}(kn) \cdot y_\beta) \cup \bigcup_{\gamma \in X} \mathcal{H}_{\alpha,\gamma}(kn) \cdot \mathcal{M}_{\gamma,\beta}^{\setminus a}(k) \quad (6.3)$$

$$= (\mathcal{H}_{\alpha,\beta}(kn) \cdot y_\beta) + \bigcup_{\gamma \in X} \mathcal{H}_{\alpha,\gamma}(kn) \cdot \mathcal{M}_{\gamma,\beta}^{\setminus a}(k). \quad (6.4)$$

In (6.3), we denote  $U \times V = \{u \cdot v \mid (u, v) \in U \times V\}$  for  $U, V \subseteq \mathcal{M}$  and we get (6.4) by using the symbols  $+$  and  $\sum$  to denote unions of pairwise disjoint families of sets. Next, we claim that, for all integers  $n$  and for all  $(\alpha, \beta) \in X \times X$ :

$$\mathcal{H}_{\alpha,\beta}(kn) = \sum_{i=0}^{n-1} \left( \bigcup_{\gamma \in X} \mathcal{H}_{\alpha,\gamma}(k(n-1-i)) \cdot y_\gamma \cdot \mathcal{M}_{\gamma,\beta}^{\setminus a}(ki) \right) + \mathcal{M}_{\alpha,\beta}^{\setminus a}(kn). \quad (6.5)$$

Property (6.2) is obvious by induction on the integer  $n$ . For formula (6.5), it is true for  $n = 0$ . Suppose that (6.5) holds for  $n \geq 0$  and we want to show for  $n + 1$ .

Replacing  $\mathcal{H}_{\alpha,\gamma}(kn)$  in (6.4) by the induction hypothesis, we obtain:

$$\begin{aligned} \mathcal{H}_{\alpha,\beta}(k(n+1)) &= (\mathcal{H}_{\alpha,\beta}(kn) \cdot y_\beta) \\ &+ \bigcup_{\gamma \in X} \left( \sum_{i=0}^{n-1} \left( \bigcup_{\delta \in X} \mathcal{H}_{\alpha,\delta}(k(n-1-i)) \cdot y_\delta \cdot \mathcal{M}_{\delta,\gamma}^{\setminus a}(ki) \right) \cdot \mathcal{M}_{\gamma,\beta}^{\setminus a}(k) \right) \\ &+ \bigcup_{\gamma \in X} \left( \mathcal{M}_{\alpha,\gamma}^{\setminus a}(kn) \cdot \mathcal{M}_{\gamma,\beta}^{\setminus a}(k) \right) \end{aligned}$$

Notice that the following relation holds in every concurrent system, for every state  $\alpha, \beta$  and every integers  $i, j \geq 0$ :

$$\mathcal{M}_{\alpha,\beta}(i+j) = \bigcup_{\gamma \in X} \mathcal{M}_{\alpha,\gamma}(i) \cdot \mathcal{M}_{\gamma,\beta}(j). \quad (6.6)$$

Applying (6.6) and rearranging the terms in the summation, we get:

$$\begin{aligned} \mathcal{H}_{\alpha,\beta}(k(n+1)) &= (\mathcal{H}_{\alpha,\beta}(kn) \cdot y_\beta) + \sum_{i=1}^n \left( \bigcup_{\delta \in X} \mathcal{H}_{\alpha,\delta}(k(n-i)) \cdot y_\delta \cdot \mathcal{M}_{\delta,\beta}^{\setminus a}(ki) \right) + \mathcal{M}_{\alpha,\beta}^{\setminus a}(k(n+1)) \\ &= \sum_{i=0}^n \left( \bigcup_{\gamma \in X} \mathcal{H}_{\alpha,\gamma}(k(n-i)) \cdot y_\gamma \cdot \mathcal{M}_{\gamma,\beta}^{\setminus a}(ki) \right) + \mathcal{M}_{\alpha,\beta}^{\setminus a}(k(n+1)) \quad (6.7) \end{aligned}$$

We finish the induction proof for (6.5).

Consider the series

$$H_{\alpha,\beta}(z) = \sum_{n \geq 0} \#\mathcal{H}_{\alpha,\beta}(kn) z^n.$$

Let  $\rho_{\alpha,\beta}$  be the radius of convergence of  $H_{\alpha,\beta}(z)$ . Since  $\mathcal{H}_{\alpha,\beta}(kn) \subseteq \mathcal{M}_{\alpha,\beta}(kn)$ , one has for non negative real  $z < r^k$ :

$$H_{\alpha,\beta}(z) \leq \sum_{n \geq 0} \#\mathcal{M}_{\alpha,\beta}(kn) (z^{\frac{1}{k}})^{kn} < +\infty.$$

Hence,

$$\forall (\alpha, \beta) \in X \times X \quad \rho_{\alpha,\beta} \geq (r_{\alpha,\beta})^k \geq r^k \quad (6.8)$$

where the last inequality is by the definition of the characteristic root  $r$ .

According to Lemma 6.1.5, we can conclude that

$$\#(\mathcal{H}_{\alpha,\gamma}(kn) \cdot y_\gamma \cdot \mathcal{M}_{\gamma,\beta}^{\setminus a}(ki)) = \#(\mathcal{H}_{\alpha,\gamma}(kn)) \cdot \#(\mathcal{M}_{\gamma,\beta}^{\setminus a}(ki)) \quad (6.9)$$

since  $y_\gamma$  is an  $a$ -rooted linking execution. Focusing on  $(\alpha_0, \alpha_0) \in X \times X$ , the counting of (6.7) has the following relation by (6.9):

$$\#\mathcal{H}_{\alpha_0,\alpha_0}(k(n+1)) \geq \sum_{i=0}^n (\#\mathcal{H}_{\alpha_0,\alpha_0}(k(n-i)) \cdot \#\mathcal{M}_{\alpha_0,\alpha_0}^{\setminus a}(ki)) + \#\mathcal{M}_{\alpha_0,\alpha_0}^{\setminus a}(k(n+1))$$

Multiplying by  $z^n$  and summing over  $n \geq 0$ , the series converge for non negative real  $z$  and we obtain:

$$\frac{1}{z}(H_{\alpha_0,\alpha_0}(z) - 1) \geq H_{\alpha_0,\alpha_0}(z)Q(z) + \frac{1}{z}(Q(z) - 1),$$

and thus:

$$Q(z) \leq \frac{H_{\alpha_0,\alpha_0}(z)}{1 + z \cdot H_{\alpha_0,\alpha_0}(z)}.$$

Therefore,  $Q(z)$  is bounded on the interval  $(0, \rho_{\alpha_0,\alpha_0})$ , and on the interval  $(0, r^k)$  by (6.8).

On the other hand, since  $Q(z)$  is rational series with non negative coefficients, its radius of convergence  $(r^{\setminus a})^k$  is one of its poles. Hence,  $(r^{\setminus a})^k > r^k$  which implies  $S$  has the spectral property.

For the other direction, suppose that the concurrent system  $S = (\mathcal{M}, X, \perp)$  is not irreducible. Our goal is to prove that  $S$  doesn't have the spectral property. Since the system is homogeneous, we discuss two cases as follows:

*First Case: the trace monoid  $\mathcal{M}$  is not irreducible.* The proof is similar to the proof of Proposition 6.1.2 for trace monoids. Let  $\Sigma = \Sigma_1 + \Sigma_2$  be a non trivial partition of  $\Sigma$  such that  $\Sigma_1 \times \Sigma_2 \subseteq I$  and put  $\mathcal{M} = \mathcal{M}_1 \times \mathcal{M}_2$  with  $\mathcal{M}_1 = \langle \Sigma_1 \rangle$  and  $\mathcal{M}_2 = \langle \Sigma_2 \rangle$ . Let  $\mu(z)$ ,  $\mu_1(z)$  and  $\mu_2(z)$  be the Möbius matrices of  $S$ ,  $(\mathcal{M}_1, X, \perp)$  and  $(\mathcal{M}_2, X, \perp)$  with characteristic roots  $r$ ,  $r_1$  and  $r_2$ . According to Remark 4.2.2,  $\mu(z) = \mu_1(z)\mu_2(z)$ . Then  $r = \min\{r_1, r_2\}$ . Without loss of the generality, assume that  $r_2 \geq r_1$ . Choose a letter  $a \in \Sigma_2$  and let  $\mathcal{M}^{\setminus a} = \mathcal{M}_1 \times \mathcal{M}_2^{\setminus a}$ . Then its characteristic root  $r^{\setminus a} = \min\{r_1, r_2^{\setminus a}\}$ . Since  $r_2^{\setminus a} \geq r_2 \geq r_1$ , hence  $r^{\setminus a} = r_1 = r$  and  $S$  doesn't have the spectral property.

*Second Case: the system is not alive.* Then there exists a state  $\alpha_0 \in X$  and a letter  $a \in \Sigma$  such that  $a$  is not executable in the following steps and thus  $\mathcal{M}_{\alpha_0} = \mathcal{M}_{\alpha_0}^{\setminus a}$ . Since  $S$  is homogeneous,  $\mathcal{M}_\alpha = \mathcal{M}_\alpha^{\setminus a}$  for every state  $\alpha \in X$ . Therefore,  $r = r^{\setminus a}$  and  $S$  doesn't have the spectral property.  $\square$

## 6.3 Applications of the spectral property

### 6.3.1 Spectral radius of ADSC<sup>+</sup>

**Corollary 6.3.1.** *Let  $(\mathcal{M}, X, \perp)$  be a non trivial and irreducible concurrent system with characteristic root  $r$ . Then the spectral radius of ADSC<sup>+</sup> equals  $r^{-1}$  which is the spectral radius of ADSC.*

*Proof.* Let  $B$  be the adjacency matrix of ADSC with spectral radius  $\rho = \rho(B)$ . By Proposition 5.2.2, we know that  $\rho = r^{-1}$ . After renumbering the positive nodes in ADSC in order to keep positive nodes in upper left corner of  $B$ , the matrix  $B$  can be written as:

$$B = \begin{pmatrix} B^+ & J \\ 0 & B^0 \end{pmatrix}$$

where  $B^+$  and  $B^0$  are the adjacency matrices of the digraph ADSC<sup>+</sup> and ADSC<sup>0</sup> = ADSC \ ADSC<sup>+</sup>. The block 0 in the matrix  $B$  is because no null nodes are accessible to positive nodes in DSC according to Point 2 of Proposition 5.3.3.

Then  $\rho = \max(\rho^+, \rho^0)$  where  $\rho^+ = \rho(B^+)$  and  $\rho^0 = \rho(B^0)$ . Consider an execution  $x$  starting from a null node in DSC. By Point 3 of Proposition 5.3.3, there exists a letter  $a$  such that  $x \in \mathcal{M}^a$ . Since the concurrent system satisfies the spectral property from Theorem 6.2.2, we have  $\rho^0 < \rho$ . Therefore,  $\rho = \rho^+$ , which completes the proof.  $\square$

**Remark 6.3.2.** *From the proof of Corollary 6.3.1, we know that the spectral radius of the submatrix  $B^0$  including null nodes is less than  $r^{-1}$ . Therefore, every component of null nodes has the spectral radius  $< r^{-1}$ .*

**Example 6.3.3** (Example 5.1.3 continued). The Möbius matrix of the system is

$$\mu(z) = \begin{matrix} M_0 \\ M_1 \end{matrix} \begin{pmatrix} 1-z & -z+z^2 \\ -z & 1-z \end{pmatrix}.$$

The characteristic root  $r$  is  $\frac{-1+\sqrt{5}}{2}$  which is the root of smallest modulus of  $\det \mu(z) = (1-z)(1-z-z^2)$ .

Its associated adjacency matrix  $A$  is

$$A = \begin{matrix} M_{0,a} \\ M_{0,ac_1} \\ M_{0,ac_2} \\ M_{1,b} \\ M_{1,c} \\ M_{0,c} \end{matrix} \left( \begin{array}{cccccc|c} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right)$$

The adjacency matrix  $A^+$  of ADSC<sup>+</sup> is identified with the upper left corner in the matrix  $A$ . Both spectral radii of  $A$  and  $A^+$  are  $\frac{1+\sqrt{5}}{2}$ . According to Proposition 5.2.2 and Corollary 6.3.1, we can also calculate the characteristic root by  $\rho(A)^{-1}$  or  $\rho(A^+)^{-1}$ .

## Chapter 7

# Probabilistic framework for concurrent systems

As we did in Chapter 3 for trace monoids, we would like to define a uniform measure on a concurrent system. A cocycle function is introduced in order to interpret the state change inherent in a concurrent system. This uniform measure defined on infinite executions induces a Markov chain of states-and-cliques. Based on the combinatorics of the abstract concurrent systems and the analysis of the DSC, we also prove the uniqueness of the uniform measure in this form. With this probabilistic framework at hand, we present again the ADSC in a schematic way and discuss the spectral radius of its different components.

### 7.1 Existence of a uniform measure

Our goal is the random generation of larger executions of the concurrent systems. For  $\alpha \in X$  and for a non-negative integer  $n$ , let  $\nu_\alpha(n)$  be the uniform distribution of the finite set  $\mathcal{M}_\alpha(n) = \{x \in \mathcal{M}_\alpha : |x| = n\}$ . In order to obtain arbitrary large executions, it is natural to consider the weak limit  $\nu_\alpha$  of the sequence  $(\nu_\alpha(n))_{n \geq 0}$ .

We will not complete the study of this weak limit. However, in Section 7.1.2, we will study a regularized version of this sequence. At the end, the form obtained for the weak limit of the regularized version justifies to introduce the following definition of a uniform measure.

**Definition 7.1.1** (uniform measure). *Let  $S = (\mathcal{M}, X, \perp)$  be a concurrent system. A uniform measure on  $S$  is a family of probability measures  $(\nu_\alpha)_{\alpha \in X}$  such that  $\nu_\alpha$  is defined on  $\partial\mathcal{M}_\alpha$  for every  $\alpha \in X$ , satisfying the chain condition:*

$$\forall \alpha \in X \quad \forall x \in \mathcal{M}_\alpha \quad \forall y \in \mathcal{M}_{\alpha \cdot x} \quad \nu_\alpha(\uparrow(xy)) = \nu_\alpha(\uparrow x)\nu_{\alpha \cdot x}(\uparrow y),$$

and such that,

$$\forall \alpha \in X \quad \forall x \in \mathcal{M}_\alpha \quad \nu_\alpha(\uparrow x) = t^{|x|}\Delta(\alpha, \alpha \cdot x) \tag{7.1}$$

for some positive real  $t$  and for some positive function  $\Delta : X \times X \rightarrow \mathbb{R}_{>0}$ .

### 7.1.1 Case of a trace monoid

If the system is a trace monoid, then it has a unique state. In the above definition of a uniform measure writes, the function  $\Delta$  reduces to a constant  $k$ , and the measure satisfies:

$$\nu(\uparrow x) = t^{|x|}k.$$

Hence  $k = 1$  and Definition 7.1.1 yields back the notion of uniform measure for trace monoids.

### 7.1.2 Parry cocycle and construction of the uniform measure

**Definition 7.1.2** (uniform valuation). *Let  $S = (\mathcal{M}, X, \perp)$  be a concurrent system. Assume that  $(\nu_\alpha)_{\alpha \in X}$  is a uniform measure on  $S$  as defined in Definition 7.1.1. Then its induced uniform valuation is the family of functions  $f = (f_\alpha)_{\alpha \in X}$  with  $f_\alpha : \mathcal{C} \rightarrow \mathbb{R}$ , defined by:*

$$\forall \alpha \in X \quad \forall x \in \mathcal{M}_\alpha \quad f_\alpha(x) = \nu_\alpha(\uparrow x).$$

Define a *cocycle* on a finite set  $X$  as a real-valued positive function  $\Delta : X \times X \rightarrow \mathbb{R}_{>0}$  such that  $\Delta(\alpha, \gamma) = \Delta(\alpha, \beta)\Delta(\beta, \gamma)$  for all  $\alpha, \beta, \gamma \in X$ .

Assume that  $(\mathcal{M}, X, \perp)$  is an homogeneous concurrent system and  $f = (f_\alpha)_{\alpha \in X}$  is the induced valuation of a uniform measure. Then the function  $\Delta$  in (7.1) should be a *cocycle*. Let  $\alpha, \beta, \gamma \in X$ , and let  $x \in \mathcal{M}_\alpha$  and  $y \in \mathcal{M}_{\alpha \cdot x}$  such that  $\beta = \alpha \cdot x$  and  $\gamma = \beta \cdot y$ . Then  $f_\alpha(xy) = f_\alpha(x)f_\beta(y)$  implies that  $r^{|xy|} \cdot \Delta(\alpha, \gamma) = r^{|x|} \cdot \Delta(\alpha, \beta) \cdot r^{|y|} \cdot \Delta(\beta, \gamma)$ . Hence,  $\Delta(\alpha, \gamma) = \Delta(\alpha, \beta)\Delta(\beta, \gamma)$ .

We follow the construction of a Markov measure over a concurrent system in [1]. The existence of a uniform measure for a concurrent system  $(\mathcal{M}, X, \perp)$  is inspired by the Patterson-Sullivan construction (ref. [11]).

Let  $r$  be the characteristic root of the concurrent system. For each state  $\alpha \in X$ , and for each real  $t \in (0, r)$ , let  $\nu_{\alpha, t}$  denote the probability measure on  $\mathcal{M} \subseteq \overline{\mathcal{M}}$  defined by:

$$\nu_{\alpha, t} = \frac{1}{G_\alpha(t)} \sum_{x \in \mathcal{M}_\alpha} \delta_{\{x\}} t^{|x|} \tag{7.2}$$

where  $G_\alpha(t) = \sum_{x \in \mathcal{M}_\alpha} t^{|x|}$  and  $\delta_{\{x\}}$  denotes the Dirac measure on  $x$ .

Let  $\uparrow x$  denote the full visual cylinder defined by:

$$\uparrow x = \{y \in \overline{\mathcal{M}} \mid x \leq y\}.$$

Then the  $\nu_{\alpha, t}$ -probability is given by:

$$\begin{aligned} \nu_{\alpha, t}(\uparrow x) &= \frac{1}{G_\alpha(t)} \sum_{y \in \mathcal{M}_\alpha : x \leq y} t^{|y|} \\ &= \frac{1}{G_\alpha(t)} \sum_{z \in \mathcal{M}_{\alpha \cdot x}} t^{|x \cdot z|} \quad \text{put } y = x \cdot z \text{ with } z \in \mathcal{M}_{\alpha \cdot x} \\ &= t^{|x|} \cdot \frac{G_{\alpha \cdot x}(t)}{G_\alpha(t)}. \end{aligned}$$

Applying the Portmanteau theorem [7], then for each  $\alpha \in X$ , the family of probability measures  $(\nu_{\alpha,t})_{t \in (0,r)}$  converges weakly, as  $t \rightarrow r$  towards a probability measure  $\nu_\alpha$  only on  $\partial\mathcal{M}_\alpha$ , whereas  $\nu_\alpha(\mathcal{M}_\alpha) = 0$ , such that  $(\nu_\alpha)_{\alpha \in X}$  is a uniform measure.

Therefore, we deduce:

$$\nu_\alpha(\uparrow x) = \lim_{t \rightarrow r, t < r} r^{|x|} \frac{G_{\alpha,x}(t)}{G_\alpha(t)},$$

and the associated cocycle is called the *Parry cocycle*, given by:

$$\Gamma(\alpha, \beta) = \lim_{t \rightarrow r, t < r} \frac{G_\beta(t)}{G_\alpha(t)}. \quad (7.3)$$

In Section 7.4, we will prove that there exists the unique uniform measure for irreducible concurrent systems.

## 7.2 Markov chain of states-and-cliques (MCSC)

In previous sections, we study a uniform measure  $\nu = (\nu_\alpha)_{\alpha \in X}$  at infinity on the system. For each  $\alpha$ , the probability measure  $\nu_\alpha$  is entirely characterized by its values  $\nu_\alpha(\uparrow x)$  on visual cylinders  $\uparrow x$ .

Taking account of concurrency, we'd like to know the probability measure on elementary cylinders  $[x]$ , for  $x$  ranging over  $\mathcal{M}_\alpha$ . We adopt the same approach – the Möbius transform as we did for trace monoids.

**Definition 7.2.1** (Möbius transform). *Let  $S = (\mathcal{M}, X, \perp)$  be a concurrent system. Let  $(f_\alpha)_{\alpha \in X}$  be the fibred valuation induced by a uniform measure  $(\nu_\alpha)_{\alpha \in X}$  on  $S$ . Put  $f_\alpha(x) = \nu_\alpha(\uparrow x)$  for all  $\alpha \in X$  and for all  $x \in \mathcal{M}_\alpha$ . The Möbius transform of  $f_\alpha$  is the function  $h_\alpha : \mathcal{C} \rightarrow \mathbb{R}$ , defined by:*

$$\forall \alpha \in X \quad \forall c \in \mathcal{C} \quad h_\alpha(c) = \sum_{c' \in \mathcal{C} : c \leq c'} (-1)^{|c'| - |c|} f_\alpha(c').$$

By the second Möbius inversion formula in (2.5) and the graded Möbius transform in (2.3), the probability measure of elementary cylinders satisfies:

$$\forall c \in \mathcal{C}_\alpha \quad \nu_\alpha([c]) = h_\alpha(c). \quad (7.4)$$

In order to obtain a Markov measure from the fibred valuation  $f_\alpha$  [1], the following conditions for the Möbius transform  $h_\alpha$  hold:

$$\forall \alpha \in X \quad h_\alpha(\varepsilon) = 0, \quad (7.5)$$

$$\forall \alpha \in X \quad \forall c \in \mathcal{C}_\alpha \quad h_\alpha(c) \geq 0, \quad (7.6)$$

where (7.5) is obtained by looking at the total probability law for the first clique

$$\sum_{c \in \mathcal{C}} h_\alpha(c) = 1$$

and by the Möbius inversion formula (2.2)

$$f_\alpha(\varepsilon) = h_\alpha(\varepsilon) + \sum_{c \in \mathcal{C}} h_\alpha(c).$$

We recall a property of  $h_\alpha$  related to normalized conditions for later proof, which could be deduced from (7.5).

**Proposition 7.2.2.** [1, Lemma 4.7] *Let  $(\mathcal{M}, X, \perp)$  be a concurrent system. Let  $(f_\alpha)_{\alpha \in X}$  be the fibred valuation induced by a uniform measure  $(\nu_\alpha)_{\alpha \in X}$  on the system. For each  $\alpha \in X$ , let  $h_\alpha : \mathcal{C}_\alpha \rightarrow \mathbb{R}$  be the Möbius transform of  $f_\alpha$ , and let  $g_\alpha(c) = \sum_{d \in \mathcal{C}_\beta : c \rightarrow d} h_\beta(d)$  where  $\beta = \alpha \cdot c$ . Assume that  $h_\alpha(\varepsilon) = 0$  for all  $\alpha \in X$ . Then:*

$$\forall \alpha \in X \quad \forall c \in \mathcal{C}_\alpha \quad h_\alpha(c) = f_\alpha(c)g_\alpha(c). \quad (7.7)$$

The Markov chain on a concurrent system can be found in [1]. The study of the probability of elementary cylinders provides us to determine the probability process  $(C_k(\xi))_{k \geq 1}$  where  $\xi$  is an infinite execution drawn at random according to the probability measure  $\nu_\alpha$ . By considering the random process of states-and-cliques rather than the process of cliques only, it has the structure of a Markov chain, where the initial measure and the transition matrix are entirely determined by a Markov measure.

For an effective computation of the Markov chain and its simulation, we need to restrict to irreducible concurrent systems.

**Theorem 7.2.3** (Markov chain of states-and-cliques (MCSC), [1]). *Let  $(\mathcal{M}, X, \perp)$  be an irreducible concurrent system equipped with the uniform measure  $\nu = (\nu_\alpha)_{\alpha \in X}$  on  $\partial\mathcal{M}$ . Fix an initial state  $\alpha \in X$ . Let  $\xi \in \partial\mathcal{M}_\alpha$ , and let  $(\alpha_i)_{i \geq 0}$  be the sequence of states defined by:  $\alpha_i(\xi) = \alpha \cdot (C_1(\xi) \cdot \dots \cdot C_i(\xi))$  for  $i \geq 0$ . Then the process  $(Y_i)_{i \geq 0} = ((\alpha_i, C_{i+1}))_{i \geq 0}$  is a homogeneous Markov chain with values in DSC under  $\nu_\alpha$ .*

*For every initial state  $\alpha \in X$ , the initial distribution of the Markov chain  $(Y_i)_{i \geq 0}$  is  $\delta_{\{\alpha\}} \otimes h_\alpha$ . The transition matrix of the Markov chain  $(Y_i)_{i \geq 0}$  is given by:*

$$M_{(\alpha,c),(\beta,d)} = \mathbf{1}_{\beta=\alpha \cdot c} \cdot \mathbf{1}_{c \rightarrow d} \cdot \frac{h_\beta(d)}{g_\alpha(c)}, \quad \text{if } g_\alpha(c) \neq 0, \quad (7.8)$$

where  $g_\alpha(c)$  is the normalization factor defined by :

$$\forall \alpha \in X \quad \forall c \in \mathcal{C}_\alpha \quad g_\alpha(c) = \sum_{d \in \mathcal{C}_\beta : c \rightarrow d} h_\beta(d), \quad \text{with } \beta = \alpha \cdot c. \quad (7.9)$$

If  $g_\alpha(c) = 0$ , then the expression (7.8) is replaced by:

$$M_{(\alpha,c),(\beta,d)} = 0.$$

*Proof.* Let  $\alpha_0 = \alpha$  be the initial state. According to (7.4) and (7.6),

$$\forall c \in \mathcal{C}_\alpha \quad \nu_\alpha(C_1 = c) = h_\alpha(c) \geq 0.$$

Hence, the law of  $(\alpha_0, C_1)$  under the uniform measure  $\nu_\alpha$  is  $\delta_{\{\alpha\}} \otimes h_\alpha$ .

Let  $M$  be the square matrix indexed by  $(X, \mathcal{C})^2$  and defined by (7.8). For any sequence  $(\alpha_0, c_1), \dots, (\alpha_{n-1}, c_n)$  in  $X \times \mathcal{C}$  with  $n \geq 1$ , let

$$p_n = \mathbf{1}_{\alpha=\alpha_0} \cdot h_{\alpha_0}(c_1) \cdot M_{(\alpha_0, c_1), (\alpha_1, c_2)} \cdots M_{(\alpha_{n-2}, c_{n-1}), (\alpha_{n-1}, c_n)}$$

denote the probability of the sequence calculated by the transition matrix  $M$ .

If the following statements hold: 1)  $\alpha_0 = \alpha$ ; 2)  $\alpha_{i+1} = \alpha_i \cdot c_{i+1}$  and  $g_{\alpha_i}(c_{i+1}) \neq 0$  for  $i \in \{0, \dots, n-2\}$ ; 3)  $c_i \rightarrow c_{i+1}$  for  $i \in \{1, \dots, n-1\}$ , we have :

$$\begin{aligned} p_n &= h_{\alpha_0}(c_1) \frac{h_{\alpha_1}(c_2)}{g_{\alpha_0}(c_1)} \cdots \frac{h_{\alpha_{n-1}}(c_n)}{g_{\alpha_{n-2}}(c_{n-1})} \\ &= f_{\alpha_0}(c_1) \cdot f_{\alpha_1}(c_2) \cdots f_{\alpha_{n-2}}(c_{n-1}) \cdot h_{\alpha_{n-1}}(c_n) \quad \text{using (7.7)} \\ &= h(c_1 c_2 \dots c_n) \quad \text{using (2.4) and the chain rule of } f_\alpha \\ &= \nu_\alpha([c_1 c_2 \cdots c_n]) \\ &= \nu_\alpha(Y_0 = (\alpha_0, c_1), \dots, Y_{n-1} = (\alpha_{n-1}, c_n)) \end{aligned}$$

which implies the probability of a Markov chain in DSC. Otherwise,  $p_n = 0$ .  $\square$

**Remark 7.2.4.** *In previous proof, the nodes  $(\alpha, c)$  such that  $g_\alpha(c) = 0$  corresponds to those nodes such that  $h_\alpha(c) = 0$ . Therefore, they are not reached by the MCSC and the probability vanishes.*

### 7.3 Properties related to DSC and ADSC

The trajectories of the Markov chain of states-and-cliques correspond to the infinite paths in DSC. The following lemma shows the probabilistic interpretation of a positive node in DSC.

**Lemma 7.3.1.** *Let  $(\mathcal{M}, X, \perp)$  be a concurrent system. Assume that  $(f_\alpha)_{\alpha \in X}$  is the fibred valuation induced by some uniform measure  $(\nu_\alpha)_{\alpha \in X}$ . Let  $h_\alpha$  be the Möbius transform of  $f_\alpha$ . Then  $h_\alpha(c) > 0$  for every positive node  $(\alpha, c) \in \text{DSC}^+$ .*

*Proof.* Let  $(\alpha, c)$  be a positive node. By Definition 5.3.2, there exists an  $(\alpha, c)$ -protection, say an execution  $x$ . Consider the infinite trace  $\uparrow x$ , we have the following relation:

$$\{y \in \partial \mathcal{M}_\alpha \mid C_1(y) = c\} \supseteq \uparrow x.$$

Therefore,

$$\nu_\alpha([c]) \geq \nu_\alpha(\uparrow x) > 0.$$

According to the initial measure of the MCSC with initial state  $\alpha$  (Th. 7.2.3), one has:

$$\nu_\alpha([c]) = h_\alpha(c).$$

Hence,  $h_\alpha(c) > 0$ .  $\square$

We try to imitate the Parry measure on trace monoids in Section 2.4.3. Since  $\text{ADSC}^+$  is not strongly connected, we couldn't deduce the existence of the dominant eigenvalue of its adjacency matrix. With the aid of previous lemma, we could still prove the eigenvector corresponding to the parameter  $t$  of a uniform measure is strictly positive.

**Lemma 7.3.2.** *Let  $(\mathcal{M}, X, \perp)$  be a non trivial concurrent system, and let  $B$  be the adjacency matrix of its  $\text{ADSC}^+$ . Assume that  $(f_\alpha)_{\alpha \in X}$  is the fibred valuation induced by some uniform measure  $(\nu_\alpha)_{\alpha \in X}$ . For all  $\alpha \in X$ , for some real positive  $t$  and for some cocycle  $\Delta$ , define  $f_\alpha(x) = t^{|\alpha|} \Delta(\alpha, \alpha \cdot x)$ . Let  $h_\alpha$  be the Möbius transform of  $f_\alpha$ . Fix an arbitrary state  $\alpha_0 \in X$ , let  $u$  be the vector defined by:*

$$u_{(\alpha, c, i)} = \frac{1}{t^{i-1}} \cdot \Delta(\alpha_0, \alpha) \cdot h_\alpha(c),$$

for all  $\alpha \in X, c \in \mathcal{C}_\alpha$  and  $i \in \{1, \dots, |c|\}$  such that  $(\alpha, c, i) \in \text{ADSC}^+$ . Then  $t^{-1}$  is an eigenvalue of  $B$  and  $u$  is its associated positive right eigenvector.

*Proof.* Let  $(\alpha, c, i) \in \text{ADSC}^+$ . According to Lemma 7.3.1,  $h_\alpha(c) > 0$  for every  $(\alpha, c) \in \text{DSC}^+$ . Therefore,  $u$  is a positive vector.

It suffices to show that

$$Bu = \frac{1}{t}u.$$

According to the definition of  $\text{ADSC}$ , we discuss two cases as follows.

If  $i < |c|$ , the row  $B_{(\alpha, c, i)}$  is identically zero, except for the entry of the column indexed by  $(\alpha, c, i + 1)$ . Therefore,

$$(Bu)_{(\alpha, c, i)} = u_{(\alpha, c, i+1)} = \frac{1}{t^i} \Delta(\alpha_0, \alpha) h_\alpha(c) = \frac{1}{t} \cdot u_{(\alpha, c, i)}.$$

If  $i = |c|$ ,

$$\begin{aligned} (Bu)_{(\alpha, c, i)} &= \sum_{d \in \mathcal{C}_\beta: c \rightarrow d} u_{(\beta, d, 1)} && \text{where } \beta = \alpha \cdot c \\ &= \sum_{d \in \mathcal{C}_\beta: c \rightarrow d} \Delta(\alpha_0, \beta) h_\beta(c) && \text{let } g_\alpha(c) \text{ as defined in (7.9)} \\ &= \Delta(\alpha_0, \beta) g_\alpha(c) \\ &= \Delta(\alpha_0, \beta) \frac{h_\alpha(c)}{t^{|\alpha|} \Delta(\alpha, \beta)} && \text{using (7.7)} \\ &= \frac{1}{t} \cdot \frac{1}{t^{|\alpha|-1}} \Delta(\alpha_0, \alpha) h_\alpha(c) && \text{using the cocycle property of } \Delta \\ &= \frac{1}{t} \cdot u_{(\alpha, c, |c|)}. \end{aligned}$$

Hence,  $u$  is the positive right eigenvector associated with  $t^{-1}$ . □

## 7.4 Uniqueness of the uniform measure

The existence of a uniform measure for concurrent systems is proved in Section 7.1. A uniform measure  $\nu = (\nu_\alpha)_{\alpha \in X}$  of the system is defined by:  $\nu_\alpha(\uparrow x) = t^{|x|} \Delta(\alpha, \alpha \cdot x)$  for some real  $t$  and some cocycle  $\Delta : X \times X \rightarrow \mathbb{R}_{>0}$ . The next theorem states the uniqueness of the uniform measure of this form. Our goal is to prove that  $t = r$  and  $\Delta = \Gamma$  where  $r$  is the characteristic root of the system and  $\Gamma$  is the Parry cocycle given in (7.3), under the hypothesis that the system is irreducible.

**Theorem 7.4.1** (uniqueness of the uniform measure). *Let  $(\mathcal{M}, X, \perp)$  be an irreducible concurrent system. Then there exists a unique uniform measure  $\nu = (\nu_\alpha)_{\alpha \in X}$  associated to the system, which is determined by:*

$$\forall \alpha \in X \quad \forall x \in \mathcal{M}_\alpha \quad \nu_\alpha(\uparrow x) = r^{|x|} \Gamma(\alpha, \alpha \cdot x)$$

where  $r$  is the characteristic root of the concurrent system and  $\Gamma : X \times X \rightarrow \mathbb{R}_{>0}$  is the Parry cocycle introduced in (7.3).

*Proof.* Assume that  $\nu = (\nu_\alpha)_{\alpha \in X}$  is a uniform measure for the concurrent system. Let  $(f_\alpha)_{\alpha \in X}$  be its induced fibred valuation defined by:

$$\forall \alpha \in X \quad \forall x \in \mathcal{M}_\alpha \quad f_\alpha(x) = t^{|x|} \Delta(\alpha, \alpha \cdot x)$$

where  $t > 0$  and  $\Delta : X \times X \rightarrow \mathbb{R}_{>0}$  is a cocycle function. For  $\alpha \in X$ , let  $h_\alpha : \mathcal{C} \rightarrow \mathbb{R}$  be the Möbius transform of  $f_\alpha$ , and let  $g_\alpha$  be defined by  $g_\alpha(c) = \sum_{d \in \mathcal{C}_\beta : c \rightarrow d} h_\beta(d)$  with  $\beta = \alpha \cdot c$ .

First, we show that  $t = r$ . Let  $B$  be the adjacency matrix of  $\text{ADSC}^+$ . According to Corollary 6.3.1, the spectral radius of  $B$  is  $r^{-1}$ . Fix an arbitrary state  $\alpha_0 \in X$ , suppose that  $u$  is a vector defined as in Lemma 7.3.2. Then  $Bu = t^{-1}u$  and  $u$  is the associated positive right eigenvector. Therefore, by Proposition 1.3.1 for reducible matrices, one has  $t^{-1} = r^{-1}$ , and thus  $t = r$ .

It remains to prove that  $\Delta = \Gamma$ , where  $\Gamma$  is the Parry cocycle. Let  $\nu' = (\nu'_\alpha)_{\alpha \in X}$  be the uniform measure associated with  $f'_\alpha(x) = r^{|x|} \Gamma(\alpha, \alpha \cdot x)$  for  $\alpha \in X$  and  $x \in \mathcal{M}_\alpha$ , where  $\Gamma$  is the Parry cocycle. Let  $h'_\alpha$  be the Möbius transform of  $f'_\alpha$ . Both uniform measures  $\nu$  and  $\nu'$  induce Markov chains of states-and-cliques on the nodes of  $\text{DSC}$ .

Consider a terminal component  $T$  of  $\text{ADSC}^+$ . Then it corresponds to a terminal component  $\tilde{T}$  of  $\text{DSC}^+$ . We want to claim that :

*The transition matrices of the Markov measures  $\nu$  and  $\nu'$  are equal on the nodes of  $\tilde{T}$ .*

Let  $u'$  be a vector defined as in Lemma 7.3.2, relatively to the uniform measure  $\nu'$ . Since the adjacency matrix of  $T$  is irreducible, both the restriction of  $u$  to  $T$  and the restriction of  $u'$  to  $T$  are Perron eigenvectors, up to a scalar factor. Therefore,  $u$  and  $u'$  are proportional on the nodes of  $T$ , and thus for some positive constant  $k$ ,

$$\forall (\alpha, c) \in \tilde{T} \quad \Delta(\alpha_0, \alpha) h_\alpha(c) = k \Gamma(\alpha_0, \alpha) h'_\alpha(c). \quad (7.10)$$

Since the state  $\alpha_0$  is arbitrary, there exists a clique  $c$  such that  $(\alpha_0, c) \in \tilde{T}$ . In particular, by  $\Delta(\alpha_0, \alpha_0) = \Gamma(\alpha_0, \alpha_0) = 1$ , one has:

$$\forall (\alpha_0, c) \in \tilde{T} \implies h_{\alpha_0}(c) = kh'_{\alpha_0}(c). \quad (7.11)$$

Let  $M$  and  $M'$  be the transition matrices of the MCSC associated with  $\nu$  and with  $\nu'$ . For every  $(\alpha_0, c), (\beta, d) \in \tilde{T}$ ,

$$\begin{aligned} M_{(\alpha_0, c), (\beta, d)} &= \mathbf{1}_{\beta=\alpha_0 \cdot c} \cdot \mathbf{1}_{c \rightarrow d} \cdot \frac{h_\beta(d)}{g_{\alpha_0}(c)} \\ &= \mathbf{1}_{\beta=\alpha_0 \cdot c} \cdot \mathbf{1}_{c \rightarrow d} \cdot r^{|c|} \cdot \Delta(\alpha_0, \beta) \cdot \frac{h_\beta(d)}{h_{\alpha_0}(c)} && \text{using (7.7)} \\ &= \mathbf{1}_{\beta=\alpha_0 \cdot c} \cdot \mathbf{1}_{c \rightarrow d} \cdot r^{|c|} \cdot \Gamma(\alpha_0, \beta) \cdot \frac{h'_\beta(d)}{h'_{\alpha_0}(c)} && \text{using (7.10) and (7.11)} \\ &= M'_{(\alpha_0, c), (\beta, d)} \end{aligned}$$

Since  $\alpha_0$  was arbitrary chosen such that  $(\alpha_0, c) \in \tilde{T}$ , we prove the claim.

Fix  $(\alpha_0, c) \in \tilde{T}$ . Let  $z$  be an  $(\alpha_0, c)$ -protection such that the last clique of  $z$  is maximal, and let  $n = \tau(z)$  be the height of  $z$ . Put  $\beta = \alpha_0 \cdot z$  and let  $((\alpha_0, d_1), \dots, (\alpha_{n-1}, d_n))$  be the path in DSC corresponding to  $z$ . According to Proposition 5.3.3, point 1 and 2, the last node  $(\alpha_{n-1}, d_n)$  is positive, and thus the path in DSC corresponding to  $z$  goes through only positive nodes. Let  $(Y_i)_{i \geq 0}$  denote the Markov chain of states-and-cliques, with  $Y_i = (\alpha_i, C_{i+1})$ . Then the maximality of  $z$  implies:

$$\uparrow z = \{\xi \in \mathcal{M}_{\alpha_0} \mid (Y_0(\xi), \dots, Y_{n-1}(\xi)) = ((\alpha_0, d_1), \dots, (\alpha_{n-1}, d_n))\}$$

Consider an arbitrary state  $\alpha \in X$  and pick  $x \in \mathcal{M}_\beta$  such that  $\beta \cdot x = \alpha$ . Let  $m = \tau(x)$  be the height of  $x$ . Then, for any  $\xi \in \partial \mathcal{M}_{\alpha_0}$ , one has  $zx \leq \xi$  if and only if the truncation of  $\xi$  at height  $n + m$  satisfies  $zx \leq C_1(\xi) \cdots C_{m+n}(\xi)$ . Therefore,  $\uparrow(zx)$  decomposes as the following disjoint unions:

$$\uparrow(zx) = \bigcup_{y \in \mathcal{M}_{\alpha_0} : \tau(y) = n+m \wedge (zx \leq y)} \{\xi \in \partial \mathcal{M}_{\alpha_0} \mid C_1(\xi) \cdots C_{m+n}(\xi) = y\}.$$

As a matter of fact, each of subsets  $\{\xi \in \partial \mathcal{M}_{\alpha_0} \mid C_1(\xi) \cdots C_{m+n}(\xi) = y\}$  is an elementary cylinder of the Markov chain of states-and-cliques. Recall the previous claim, the probabilities with respect to  $\nu_{\alpha_0}$  and with respect to  $\nu'_{\alpha_0}$  are the same, and thus  $\nu_{\alpha_0}(\uparrow(zx)) = \nu'_{\alpha_0}(\uparrow(zx))$ . Then we could deduce that  $\Delta(\alpha_0, \alpha) = \Gamma(\alpha_0, \alpha)$ . Since  $\alpha$  is arbitrary, the cocycle property of  $\Delta$  and of  $\Gamma$  implies that  $\Delta = \Gamma$ .  $\square$

With this uniform measure of the concurrent system, some properties between the vectors of the Parry cocycle and the Möbius matrix are given as follows.

**Proposition 7.4.2.** *Let  $(\mathcal{M}, X, \perp)$  be a concurrent system. For some positive real  $r$  and some cocycle  $\Gamma : X \times X \rightarrow \mathbb{R}_{>0}$ , assume that  $\nu_\alpha(\uparrow x) = r^{|\alpha|} \Gamma(\alpha, \alpha \cdot x)$  defines the*

uniform measure. Let  $\mu(z)$  be the Möbius matrix of the system. Fix an arbitrary state  $\alpha_0 \in X$ . Consider a positive vector  $u = (u_\alpha)_{\alpha \in X}$  defined by  $u_\alpha = \Gamma(\alpha_0, \alpha)$ . Then one has:

$$u \in \ker(\mu(r)).$$

*Proof.* Fix  $\alpha_0 \in X$ , let  $u = (u_\alpha)_{\alpha \in X} = (\Gamma(\alpha_0, \alpha))_{\alpha \in X}$  be a positive vector. Let  $(f_\alpha)_{\alpha \in X}$  be the fibred valuation induced by  $(\nu_\alpha)_{\alpha \in X}$ , and let  $h_\alpha$  be the Möbius transform of  $f_\alpha$  for all  $\alpha \in X$ . We know that  $h_\alpha(\varepsilon) = 0$  from (7.5), which yields

$$\sum_{c \in \mathcal{C}_\alpha} (-1)^{|c|} r^{|c|} \Gamma(\alpha, \alpha \cdot c) = 0. \quad (7.12)$$

Rewriting the equation (7.12) depending on the reaching states, we obtain:

$$\sum_{\beta \in X} \left( \sum_{c \in \mathcal{C}_{\alpha, \beta}} (-1)^{|c|} r^{|c|} \right) \Gamma(\alpha, \beta) = \sum_{\beta \in X} \mu_{\alpha, \beta}(r) \Gamma(\alpha, \beta) = 0. \quad (7.13)$$

Put the vector  $v = \mu(r) \cdot u$ , where:

$$v_\alpha = \sum_{\beta \in X} \mu_{\alpha, \beta}(r) \cdot u_\beta = \sum_{\beta \in X} \mu_{\alpha, \beta}(r) \Gamma(\alpha_0, \beta).$$

Since  $\Gamma(\alpha_0, \alpha) \Gamma(\alpha, \beta) = \Gamma(\alpha_0, \beta)$ ,  $v_\alpha$  is proportional to the middle part of (7.13). Therefore,  $v = 0$  which completes the proof.  $\square$

**Corollary 7.4.3.** *Let  $(\mathcal{M}, X, \perp)$  be an irreducible concurrent system of characteristic root  $r$ . Let  $\mu(z)$  be the Möbius matrix of the system. Then:*

$$\dim(\ker(\mu(r))) = 1.$$

*Proof.* Since  $\det(\mu(r)) = 0$  by Proposition 4.2.5, we know that  $\dim(\ker(\mu(r))) \geq 1$ .

Seeking a contradiction. Assume that  $\dim(\ker(\mu(r))) > 1$ . Let  $\Gamma : X \times X \rightarrow \mathbb{R}_{>0}$  be the Parry cocycle, and let  $f_\alpha(x) = r^{|x|} \Gamma(\alpha, \alpha \cdot x)$  for  $\alpha \in X$  and  $x \in \mathcal{M}_\alpha$ . Fix  $\alpha_0 \in X$  and define the positive vector  $u = (u_\alpha)_{\alpha \in X}$  by  $u_\alpha = \Gamma(\alpha_0, \alpha)$ . By Proposition 7.4.2,  $u \in \ker(\mu(r))$ .

Let  $v \in \ker(\mu(r)) \setminus \{0\}$ , non proportional to  $u$ . Choose arbitrary  $\epsilon > 0$  such that  $w = u + \epsilon v > 0$ . Let  $\Delta : X \times X \rightarrow \mathbb{R}_{>0}$  be the cocycle defined by:

$$\forall (\alpha, \beta) \in X \times X \quad \Delta(\alpha, \beta) = \frac{w_\beta}{w_\alpha}.$$

Then  $(\Gamma(\alpha_0, \alpha))_{\alpha \in X}$  is proportional to  $w$ , and thus is not proportional to  $u$ . So, one has:  $\Delta \neq \Gamma$ .

Let  $h_\alpha$  be the Möbius transform of  $f_\alpha$ , and let  $h'_\alpha$  be the Möbius transform of  $f'_\alpha$ . The formula for  $h'_\alpha$  can be computed as follows, for  $\alpha \in X$  and for  $c \in \mathcal{C}_\alpha$ :

$$\begin{aligned} h'_\alpha(c) &= \sum_{c' \in \mathcal{C} : c \leq c'} (-1)^{|c'| - |c|} r^{|c'|} \Delta(\alpha, \alpha \cdot c') \\ &= \frac{1}{w_\alpha} \cdot \left( u_\alpha \cdot h_\alpha(c) + \epsilon \cdot \sum_{c' \in \mathcal{C} : c \leq c'} (-1)^{|c'| - |c|} r^{|c'|} v_{\alpha \cdot c'} \right). \end{aligned} \quad (7.14)$$

In particular,

$$h'_\alpha(\varepsilon) = \frac{1}{w_\alpha} \cdot \left( u_\alpha \cdot h_\alpha(\varepsilon) + \varepsilon \cdot (\mu(r) \cdot v)_\alpha \right). \quad (7.15)$$

Since  $h_\alpha(\varepsilon) = 0$  and since  $v \in \ker(\mu(r))$ , we have  $h'_\alpha(\varepsilon) = 0$ , which is sufficient for the relation  $h'_\alpha(c) = f''_\alpha(c)g'_\alpha(c)$  as in (7.7).

Fix an arbitrary positive node  $(\alpha, c)$ , then  $h_\alpha(c) > 0$  by Lemma 7.3.1. Moreover, for arbitrary small  $\varepsilon > 0$ , we obtain that  $h'_\alpha(c) > 0$  from (7.14). Following the idea as in the proof of Theorem 7.4.1, we focus on the related terminal strongly connected component of  $(\alpha, c)$ . Both  $h_\alpha$  and  $h'_\alpha$  could give a positive eigenvector of the adjacency matrix of  $\text{ADSC}^+$ . However, the adjacency matrix of the terminal component is irreducible, and thus there exists a unique Perron eigenvector defined on the nodes of this component. Hence,  $h_\alpha(c)$  is proportional to  $h'_\alpha(c)$  for all  $(\alpha, c)$  in this component.

Consider two uniform measure  $\nu$  and  $\nu'$  with respect to  $f_\alpha$  and  $f'_\alpha$ . We adopt the same technique as in the proof of Theorem 7.4.1. After analyzing the transition matrix of the MCSC starting from  $(\alpha, c)$  of this terminal component, we obtain that  $f_\alpha = f'_\alpha$  for all  $\alpha \in X$ . This implies that  $\Gamma = \Delta$ , which contradicts the result derived from the assumption.  $\square$

## 7.5 Applications of spectral property

Within our probabilistic framework, we are able to prove some interesting properties by applying the spectral property. According to Theorem 6.2.2, a concurrent system which has the spectral property is equivalent to its irreducibility. Therefore, all results in this section are required an irreducible system.

### 7.5.1 Positive nodes and null nodes of DSC from a probabilistic point of view

In Section 5.3, the definition of positive nodes and of null nodes is depended on the existence of  $(\alpha, c)$ -protection, for  $\alpha \in X$  and for  $c \in \mathcal{M}_\alpha$ . The aim of the following theorem is to give an alternative, probabilistic characterization of positive and of null nodes of DSC.

**Theorem 7.5.1.** *Let  $(\mathcal{M}, X, \perp)$  be an irreducible concurrent system. Assume that  $(f_\alpha)_{\alpha \in X}$  is the fibred valuation induced by the uniform measure  $(\nu_\alpha)_{\alpha \in X}$ . Let  $h_\alpha$  be the Möbius transform of  $f_\alpha$ . Let  $(\alpha, c) \in \text{DSC}$ . Then the following statements are equivalent:*

1.  $(\alpha, c)$  is a positive node
2.  $h_\alpha(c) > 0$ .

Furthermore, for every  $\alpha \in X$ , the MCSC under  $\nu_\alpha$  only goes through the positive nodes of the DSC.

*Proof.* 1.  $\implies$  2. follows from Lemma 7.3.1. For the other direction, we prove by contrapositive. Suppose that  $(\alpha, c)$  is a null node, we want to show that  $h_\alpha(c) = 0$ .

Applying Proposition 5.3.3, point 3 for infinite traces on  $\partial\mathcal{M}$ , we can deduce that

$$\{\xi \in \partial\mathcal{M}_\alpha \mid C_1(\xi) = c\} \subseteq \bigcup_{a \in \Sigma} \partial\mathcal{M}_\alpha^{\setminus a} \quad (7.16)$$

where  $\mathcal{M}^{\setminus a} = \langle \Sigma \setminus \{a\} \rangle$ . Then the probability of (7.16) on  $\nu_\alpha$  has the relation:

$$\nu_\alpha([c]) \leq \sum_{a \in \Sigma} \nu_\alpha(\partial\mathcal{M}_\alpha^{\setminus a}).$$

Since  $\nu_\alpha([c]) = h_\alpha(c)$ , it is enough to show that  $\nu_\alpha(\partial\mathcal{M}_\alpha^{\setminus a}) = 0$ , for every  $a \in \Sigma$ .

We consider the following relation on  $\partial\mathcal{M}$ :

$$\forall n \geq 0 \quad \partial\mathcal{M}_\alpha^{\setminus a} \subseteq \bigcup_{x \in \mathcal{M}_\alpha^{\setminus a}(n)} (\uparrow x)$$

where the right hand side constraints only  $n$  first letters in  $\Sigma \setminus \{a\}$ . So, we have:

$$\nu_\alpha(\partial\mathcal{M}_\alpha^{\setminus a}) \leq \sum_{x \in \mathcal{M}_\alpha^{\setminus a}(n)} \nu(\alpha(\uparrow x)) \leq Kr^n \#(\mathcal{M}_\alpha^{\setminus a}(n))$$

where  $K$  is a bound of the Parry cocycle.

Let  $r$  be the characteristic root of the concurrent system  $(\mathcal{M}, X, \perp)$ . According to Theorem 6.2.2, the irreducible concurrent system has the spectral property and thus  $r^{\setminus a} > r$ . Hence,

$$\nu_\alpha(\partial\mathcal{M}_\alpha^{\setminus a}) \sim K \frac{(r^{\setminus a})^n}{r^n} \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

which completes the proof.  $\square$

Retelling the contraposition of Theorem 7.5.1, we obtain the following corollary.

**Corollary 7.5.2.** *Let  $(\mathcal{M}, X, \perp)$  be an irreducible concurrent system. Assume that  $(f_\alpha)_{\alpha \in X}$  is the fibred valuation induced by the uniform measure  $(\nu_\alpha)_{\alpha \in X}$ . Let  $h_\alpha$  be the Möbius transform of  $f_\alpha$ . Let  $(\alpha, c) \in \text{DSC}$ . Then the following statements are equivalent:*

1.  $(\alpha, c)$  is a null node
2.  $h_\alpha(c) = 0$ .

## 7.5.2 Property of tip-top action

In our theory for concurrent systems, the characteristic root  $r$  is crucial, which requires us to calculate the spectral radius of  $\text{ADSC}^+$  (Ref. Cor. 6.3.1). As the system becomes larger in size, the complexity increases rapidly. Thanks for the structure of a tip-top action, we have the following lemma, which leads us to prove that the characteristic root of a concurrent system of a tip-top action could simply determined by the root of the Möbius polynomial of its associated trace monoid, instead of the Möbius matrix of the system.

**Lemma 7.5.3.** *Let  $(\mathcal{M}, X, \perp)$  be an irreducible concurrent system of a tip-top action. For every state  $\alpha \in X$ , the uniform valuation  $f_\alpha$  satisfies:*

$$f_\alpha(\alpha) = 1.$$

*Proof.* For  $\alpha \in X$ , let  $\xi \in \partial\mathcal{M}_\alpha$  such that  $\xi \not\uparrow \alpha$ . That is to say that certain letter of  $\alpha$  isn't executed for the first clique of  $\xi$ . Then there exists  $a \in \alpha$  such that  $\xi \not\uparrow a$ . By induction, the letter  $a$  is parallel to every clique of  $\xi$  and thus

$$\forall i \geq 1 \quad a \notin C_i.$$

We can deduce that

$$\partial\mathcal{M}_\alpha \setminus (\uparrow \alpha) \subseteq \bigcup_{a \in \Sigma} \partial\mathcal{M}_\alpha \setminus a \tag{7.17}$$

Using again the spectral property as the proof of Theorem 7.5.1, we know that

$$\forall a \in \Sigma \quad \nu_\alpha(\partial\mathcal{M}_\alpha \setminus a) = 0.$$

Therefore, the probability of (7.17) is

$$1 - \nu_\alpha(\uparrow \alpha) \leq 0$$

which implies that  $f_\alpha(\alpha) = \nu_\alpha(\uparrow \alpha) = 1$ . □

**Theorem 7.5.4.** *Let  $(\mathcal{M}, X, \perp)$  be an irreducible concurrent system of a tip-top action, and let  $p$  be the root of smallest modulus of the Möbius polynomial  $\mu_{\mathcal{M}}(z)$ . Then the characteristic root  $r$  of the system is given by  $r = \sqrt{p}$ .*

*Proof.* By Lemma 7.5.3, we have:

$$\forall \alpha \in X \quad f_\alpha(\alpha) = r^{|\alpha|} \Gamma(\alpha, \varepsilon) = 1,$$

which implies that  $\Gamma(\varepsilon, \alpha) = \Gamma(\alpha, \varepsilon)^{-1} = r^{|\alpha|}$ .

Then by (7.5) one has:

$$\begin{aligned} 0 = h_\varepsilon(\varepsilon) &= \sum_{\alpha \in \mathcal{C}_\varepsilon} (-1)^{|\alpha|} f_\varepsilon(\alpha) \\ &= \sum_{\alpha \in \mathcal{C}} (-1)^{|\alpha|} r^{2|\alpha|} \\ &= \mu_{\mathcal{M}}(r^2). \end{aligned}$$

which implies that  $r^2$  is a root of  $\mu_{\mathcal{M}}$ .

Since  $h_\varepsilon(\varepsilon) = h(\varepsilon) = 0$  and  $h_\varepsilon(c) = h(c) \geq 0$  for all  $c \in X$ , the Möbius conditions hold. According to [2, Theorem 1.6], the only candidate is the root of smallest modulus. This completes the proof. □

We refer to two concrete models in Section 4.3.3 to verify the theorem proved above.

## 7.6 ADSC and the spectral radii of its components

We refer to some known results related to the spectral radius for reducible matrices, and obtain a more clear picture how the Markov chain works on ADSC.

**Proposition 7.6.1.** *Let  $(\mathcal{M}, X, \perp)$  be an irreducible concurrent system of characteristic root  $r$ . Then, among the strongly connected components of  $\text{ADSC}^+$ , those of spectral radius  $r^{-1}$  are exactly the terminal components.*

*Proof.* Let  $B$  be the adjacency matrix of  $\text{ADSC}^+$ . According to Corollary 6.3.1, the spectral radius of  $B$  is  $r^{-1}$ . By the existence of the uniform measure and Lemma 7.3.2, there exists a positive right eigenvector of  $B$  corresponding to  $r^{-1}$ . Therefore, according to Proposition 1.3.1, point 1, we obtain the desired result.  $\square$

In other words, the basic components of  $\text{ADSC}^+$  are exactly the terminal components.

**Corollary 7.6.2.** *Let  $(\mathcal{M}, X, \perp)$  be an irreducible and non trivial concurrent system. Let  $(\alpha, c, i)$  be a null node of the ADSC. Then the irreducible component of  $(\alpha, c, i)$  can be reached from at least one basic components of the ADSC.*

Following the results of Remark 6.3.2, Proposition 7.6.1 and Corollary 7.6.2, we show a schematic diagram of the condensation of ADSC in Figure 7.1. Each node represent the irreducible component in ADSC, where the purple one with double lines are represented as the basic components. The Markov chain under the uniform measure defined on infinite traces does not cross the dashed line in this figure, which means only stay on the components of positive nodes.

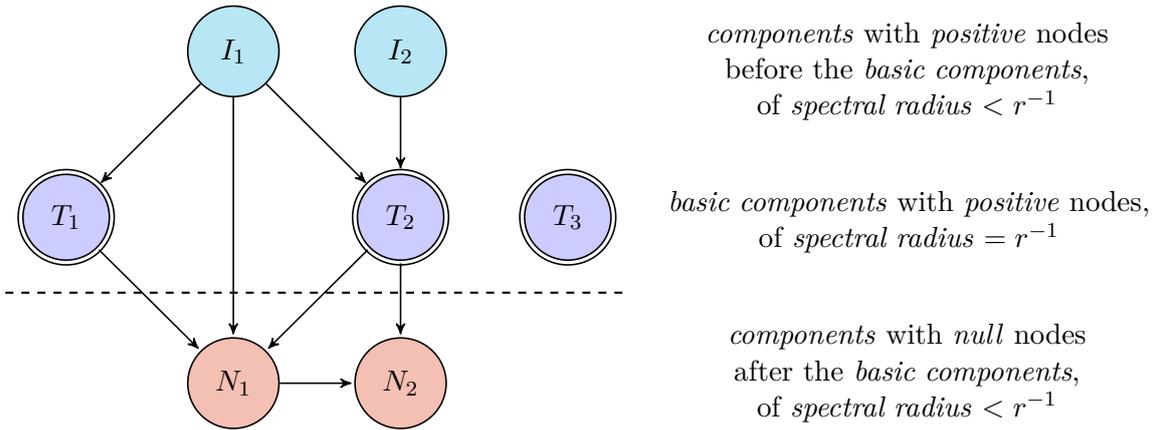


Figure 7.1: Schematic condensation of ADSC.

**Example 7.6.3** (Example 5.1.3 continued). The node  $(M_0, c)$  in ADSC (Fig. 5.1) is a null node which never go through under the uniform measure  $\nu$ . With the aid of *spectral property*, the null nodes can be detected by comparing the spectral radii between the

strongly connected components. This helps us reduce the state space of the Markov chains and improve the efficiency and the accuracy of the random generation (more details in next chapter).

## Chapter 8

# Realization of random generation for concurrent systems

The probabilistic model of executions of concurrent systems constructed in Chapter 7 highly depends on the combinatorics of trace monoids. For practical application, two approaches are presented in this chapter. One is via the MCSC with some efficient algorithms for the initial distribution and for the transition matrix. Another approach is via the Parry measure on ADSC. At the end of this chapter, we finish with some examples.

### 8.1 Computation of the parameters of the uniform measure for concurrent systems

In this entire section, we fix an *irreducible* concurrent system  $(\mathcal{M}, X, \perp)$  with  $\mathcal{M} = \mathcal{M}(\Sigma, I)$ . Let  $\mu(z) = (\mu_{\alpha, \beta}(z))_{(\alpha, \beta) \in X \times X}$  denote the Möbius matrix of the system  $(\mathcal{M}, X, \perp)$ . We introduce the following notations.

As a particular case, we define :  $\mathcal{M}^{\setminus c} = \langle \Sigma \setminus \{c\} \rangle$ ,  $\mathcal{M}^{\parallel c} = \langle \Sigma_{\parallel c} \rangle$  with  $\Sigma_{\parallel c} = \{b \in \Sigma \mid c = a_1 \cdots a_p \text{ with } a_i \in \Sigma, (a_i, b) \in I\}$  for every clique  $c \in \mathcal{C}$ . And we put:

$$\mathcal{C}_{\alpha, \beta}^{\setminus c} = \mathcal{M}_{\alpha, \beta}^{\setminus c} \cap \mathcal{C} \quad \text{and} \quad \mathcal{C}_{\alpha, \beta}^{\parallel c} = \mathcal{M}_{\alpha, \beta}^{\parallel c} \cap \mathcal{C}.$$

#### 8.1.1 Computation of the Möbius matrix

Let  $\mathcal{M}' = \langle H \rangle$  for some subset  $H \subseteq \Sigma$ . The restriction to  $\mathcal{M}'$  of the action  $(X \cup \{\perp\}) \times \mathcal{M} \rightarrow X \cup \{\perp\}$  is still an action  $(X \cup \{\perp\}) \times \mathcal{M}' \rightarrow X \cup \{\perp\}$  defining a concurrent system  $(\mathcal{M}', X, \perp)$ . In particular, we denote by  $\mu^{\setminus c}(z) = (\mu_{\alpha, \beta}^{\setminus c}(z))_{(\alpha, \beta) \in X \times X}$  and  $\mu^{\parallel c}(z) = (\mu_{\alpha, \beta}^{\parallel c}(z))_{(\alpha, \beta) \in X \times X}$  the Möbius matrices corresponding to the induced concurrent systems  $(\mathcal{M}^{\setminus c}, X, \perp)$  and  $(\mathcal{M}^{\parallel c}, X, \perp)$  for every clique  $c \in \mathcal{C}$ .

To compute the Möbius matrix effectively, we use the following formulae.

**Lemma 8.1.1.** *Let  $(\mathcal{M}, X, \perp)$  be an irreducible concurrent system with Möbius matrix  $\mu(z)$ . Let  $a \in \Sigma$  and let  $\mu^{\setminus a}(z)$  and  $\mu^{\parallel a}(z)$  be the Möbius matrices introduced above. Then:*

$$\forall (\alpha, \beta) \in X \times X \quad \begin{cases} \forall a \in \Sigma_\alpha & \mu_{\alpha, \beta}(z) = \mu_{\alpha, \beta}^{\setminus a}(z) - z \cdot \mu_{\alpha, a, \beta}^{\parallel a}(z) \\ \forall a \notin \Sigma_\alpha & \mu_{\alpha, \beta}(z) = \mu_{\alpha, \beta}^{\setminus a}(z) \end{cases} \quad (8.1)$$

*Proof.* For  $a \notin \Sigma_\alpha$ , it is obvious that the formula holds.

For  $a \in \Sigma_\alpha$ , we compute  $\mu_{\alpha, \beta}(z)$  according to (4.1):

$$\begin{aligned} \mu_{\alpha, \beta}(z) &= \sum_{c \in \mathcal{C}_{\alpha, \beta}} (-1)^{|c|} z^{|c|} \\ &= \sum_{c \in \mathcal{C}_{\alpha, \beta} : a \notin c} (-1)^{|c|} z^{|c|} + \sum_{c \in \mathcal{C}_{\alpha, \beta} : a \in c} (-1)^{|c|} z^{|c|} \\ &= \mu_{\alpha, \beta}^{\setminus a}(z) + \sum_{c_1 \in \mathcal{C}_{\alpha, a, \beta} : c_1 \parallel a} (-1)^{|a \cdot c_1|} z^{|a \cdot c_1|} \\ &= \mu_{\alpha, \beta}^{\setminus a}(z) - z \cdot \mu_{\alpha, a, \beta}^{\parallel a}(z). \end{aligned} \quad (*)$$

(\*) The set  $\{c \in \mathcal{C}_{\alpha, \beta}, a \in c\}$  bijects to the set  $\{a \cdot c_1 \mid c_1 \in \mathcal{C}_{\alpha, a, \beta}^{\parallel a}\}$ .  $\square$

Based on the result of Lemma 8.1.1, the following recursive algorithm computes the Möbius matrix of a concurrent system. The algorithm computes recursively the Möbius matrices of the successive concurrent system  $(\Sigma_0, X, \perp), \dots, (\Sigma_k, X, \perp)$  with  $\Sigma = \{a_1, \dots, a_k\}$ ,  $\Sigma_0 = \emptyset$  and  $\Sigma_i = \{a_1, \dots, a_i\}$  for  $i \in \{1, \dots, k\}$ .

To compute more effectively, one can adopt the standard technique — “memoization” for recording the results of the Möbius matrix on a subset of  $\Sigma$ . We omit this part in Algorithm 4 in order to focus on the main idea of the algorithm.

### 8.1.2 Computation of the Möbius transform

Let  $\nu = (\nu_\alpha)_{\alpha \in X}$  be the uniform measure of an *irreducible* concurrent system  $S$ . Let  $f = (f_\alpha)_{\alpha \in X}$  be the fibred valuation induced by  $\nu$ . It satisfies:

$$\forall \alpha \in X \quad \begin{cases} \forall x \in \mathcal{M}_\alpha & f_\alpha(x) = r^{|x|} \Gamma(\alpha, \alpha \cdot x) \\ \forall x \notin \mathcal{M}_\alpha & f_\alpha(x) = 0 \end{cases}$$

where  $r$  is the characteristic root of the system and  $\Gamma$  is its *Parry cocycle*. The *Möbius transform* of  $f_\alpha$  is the function  $h_\alpha : \mathcal{C} \rightarrow \mathbb{R}$  defined by:

$$\forall c \in \mathcal{C}_\alpha \quad h_\alpha(c) = \sum_{c' \in \mathcal{C}_\alpha : c \leq c'} (-1)^{|c'| - |c|} f_\alpha(c').$$

One way to effectively compute the Möbius transform  $h_\alpha$  is to relate it with the Möbius matrix of the system.

---

**Algorithm 4** Computes the Möbius matrix  $\mu$  of a concurrent system  $(\mathcal{M}, X, \perp)$ 


---

**Require:** Concurrent system  $(\mathcal{M}, X, \perp)$  where  $\mathcal{M} = \mathcal{M}(\Sigma, I)$ 

```

1:  $\Delta \leftarrow \Sigma$  ▷ initialization
2: if  $\Delta = \emptyset$  then
3:    $\mu \leftarrow \text{Id}$  ▷ initialization of  $\mu$  with the identity matrix
4: else
5:   Choose  $a \in \Sigma$ 
6:    $\Delta \leftarrow \Sigma \setminus \{a\}$ 
7:    $\Lambda \leftarrow \Sigma_{\parallel a}$ 
8:    $\theta \leftarrow$  Call Algorithm 4 on  $\Delta$ 
9:    $\pi \leftarrow$  Call Algorithm 4 on  $\Lambda$ 
10:  for all  $(\alpha, \beta) \in X \times X$  do
11:    if  $a \in \mathcal{M}_\alpha$  then ▷ Lemma 8.1.1
12:       $\mu_{\alpha, \beta} = \theta_{\alpha, \beta} - z \cdot \pi_{\alpha \cdot a, \beta}$ 
13:    else
14:       $\mu_{\alpha, \beta} = \theta_{\alpha, \beta}$ 
15:    end if
16:  end for
17: end if
18: return  $\mu$ 

```

---

**Lemma 8.1.2.** *Let  $(\mathcal{M}, X, \perp)$  be a concurrent system. Let  $f = (f_\alpha)_{\alpha \in X}$  be the fibred valuation induced by the uniform measure. The Möbius transform  $h_\alpha$  of  $f_\alpha$  satisfies the following formula:*

$$\forall \alpha \in X \quad \begin{cases} \forall c \in \mathcal{C}_\alpha & h_\alpha(c) = r^{|c|} \cdot \sum_{\beta \in X} \mu_{\alpha \cdot c, \beta}^{\parallel c}(r) \cdot \Gamma(\alpha, \beta) \\ \forall c \notin \mathcal{C}_\alpha & h_\alpha(c) = 0 \end{cases}$$

where  $r$  is the characteristic root of the system and  $\Gamma$  is its Parry cocycle.

*Proof.* For  $\alpha \in X$  and  $c \in \mathcal{C}_\alpha$ , we compute  $h_\alpha$  from the definition:

$$\begin{aligned} h_\alpha(c) &= \sum_{c' \in \mathcal{C}_\alpha : c \leq c'} (-1)^{|c'| - |c|} f_\alpha(c') \\ &= \sum_{c_1 \in \mathcal{C}_{\alpha \cdot c}^{\parallel c}} (-1)^{|c_1|} \cdot f_\alpha(c) \cdot f_{\alpha \cdot c}(c_1) \\ &= f_\alpha(c) \cdot \sum_{c_1 \in \mathcal{C}_{\alpha \cdot c}^{\parallel c}} (-1)^{|c_1|} r^{|c_1|} \cdot \Gamma(\alpha \cdot c, \alpha \cdot c \cdot c_1) \end{aligned}$$

Collect according to the arriving state,

$$\begin{aligned}
h_\alpha(c) &= f_\alpha(c) \cdot \sum_{\beta \in X} \left( \sum_{c_1 \in \mathcal{C}_{\alpha \cdot c, \beta}^{\parallel c}} (-1)^{|c_1| r^{|c_1|}} \right) \cdot \Gamma(\alpha \cdot c, \beta) \\
&= \left( r^{|c|} \cdot \Gamma(\alpha, \alpha \cdot c) \right) \cdot \sum_{\beta \in X} \mu_{\alpha \cdot c, \beta}^{\parallel c}(r) \cdot \Gamma(\alpha \cdot c, \beta) \\
&= r^{|c|} \cdot \sum_{\beta \in X} \mu_{\alpha \cdot c, \beta}^{\parallel c}(r) \cdot \Gamma(\alpha, \beta)
\end{aligned}$$

□

We perform the computation of  $h_\alpha(c)$  in Algorithm 5. In this algorithm, the computation of the Parry cocycle  $\Gamma$  is according to the vector in Proposition 7.4.2 instead of the theoretical definition (7.3).

---

**Algorithm 5** Computes the Möbius transform  $h_\alpha(c)$  of  $f_\alpha$  for all  $c \in \mathcal{C}_\alpha$

---

**Require:** Concurrent system  $S = (\mathcal{M}, X, \perp)$  where  $\mathcal{M} = \mathcal{M}(\Sigma, I)$ , the state  $\alpha$  and the clique  $c$

- 1:  $\mu \leftarrow$  Call Algorithm 4 on  $\Sigma$  ▷ the Möbius matrix of S
  - 2:  $r \leftarrow$  The smallest positive root of  $\det \mu$  ▷ the characteristic root of S
  - 3: Choose  $u \in \ker(\mu(r))$  ▷ Proposition 7.4.2
  - 4: **for all**  $(\alpha, \beta) \in X \times X$  **do**
  - 5:    $\Gamma(\alpha, \beta) \leftarrow \frac{u(\beta)}{u(\alpha)}$  ▷ Parry cocycle
  - 6: **end for**
  - 7:  $\Lambda \leftarrow \Sigma_{\parallel c}$
  - 8:  $\theta \leftarrow$  Call Algorithm 4 on  $\Lambda$  ▷ the Möbius matrix of  $(\mathcal{M}^{\parallel c}, X, \perp)$
  - 9:  $\text{sum} \leftarrow 0$
  - 10: **for**  $\beta \in X$  **do**
  - 11:    $\text{sum} = \text{sum} + \theta_{\alpha \cdot c, \beta}(r) \cdot \Gamma(\alpha, \beta)$  ▷ the part of summation in Lemma 8.1.2
  - 12: **end for**
  - 13: **return**  $r^{|c|} \cdot \text{sum}$
- 

### 8.1.3 Computation of the parameters of Markov chain of states-and-cliques (MCSC)

Let  $(\mathcal{M}, X, \perp)$  be an *irreducible* concurrent system and let  $(\nu_\alpha)_{\alpha \in X}$  be the associated uniform measure at infinity. Let  $\xi = (c_i)_{i \geq 1} \in \partial \mathcal{M}$  be an infinite trace. Fix an initial state  $\alpha \in X$  and define recursively  $\alpha_0 = \alpha$  and  $\alpha_{i+1} = \alpha_i \cdot c_{i+1}$  for all  $i \geq 0$ . Put  $Y_i(\xi) = (\alpha_i, c_{i+1})$  for all  $i \geq 0$ . According to Theorem 7.2.3, the process  $(Y_i)_{i \geq 0}$  is a discrete homogeneous Markov chain with values under  $\nu_\alpha$ . The transition matrix  $M$  of

the Markov chain  $(Y_i)_{i \geq 0}$  is:

$$M_{(\alpha,c),(\beta,d)} = \mathbf{1}_{\beta=\alpha \cdot c} \cdot \mathbf{1}_{c \rightarrow d} \cdot \frac{h_\beta(d)}{g_\alpha(c)}.$$

The following algorithm computes the transition matrix of the MCSC.

---

**Algorithm 6** Computes the parameters of the MCSC for the nodes in  $\text{DSC}^+$

---

**Require:** Concurrent system  $(\mathcal{M}, X, \perp)$  where  $\mathcal{M} = \mathcal{M}(\Sigma, I)$

```

1:  $M \leftarrow 0$  ▷ Initialization of the transition matrix
2: for all  $(\alpha, c), (\beta, d) \in X \times \mathcal{C}$  do
3:   if  $c \rightarrow d$  and  $\alpha \cdot c = \beta$  then ▷ the normal form relation  $\rightarrow$ 
4:      $M_{(\alpha,c),(\beta,d)} \leftarrow$  Call Algorithm 5 on  $\beta$  and  $d$ 
5:   end if
6: end for
7:  $l \leftarrow |X \times \mathcal{C}|$ 
8:  $u \leftarrow (1, \dots, 1)_{1 \times l}$ 
9:  $n \leftarrow M \cdot u$  ▷ the parameters of normalization
10: for all  $(\alpha, c), (\beta, d) \in X \times \mathcal{C}$  do
11:    $M_{(\alpha,c),(\beta,d)} = \frac{M_{(\alpha,c),(\beta,d)}}{n_{(\alpha,c)}}$  ▷ normalization
12: end for
13: return  $M$ 

```

---

Depending on the chosen initial state of the concurrent system, the initial law of the MCSC is  $\delta_\alpha \otimes h_\alpha$ . The computation of  $h_\alpha$  can be found in Section 8.1.2. So, we get all parameters we need for random generation via MCSC.

## 8.2 Parry measure on concurrent systems

The approach in previous section relies on the calculation of the determinant of the Möbius matrix to get the characteristic root of the concurrent system. For large models, this requires the cumbersome calculation.

With the aid of spectral property (See in Chapter 6) and the discussion in Section 7.6), we know that the characteristic root of the system equals to the spectral radius of one of the terminal components in  $\text{ADSC}^+$ . By comparing the spectral radii between the components of the  $\text{ADSC}$ , we could focus on  $\text{ADSC}^+$  and calculate the transition matrix via the Parry measure on concurrent systems defined as follows.

**Definition 8.2.1** (Parry measure on concurrent systems). *Let  $(\mathcal{M}, X, \perp)$  be an irreducible concurrent system. Define the Parry measure on the concurrent system as the Parry measure on  $\text{ADSC}$ , given by:*

$$\forall (\alpha, c, i), (\beta, d, j) \in \text{ADSC} \quad P_{(\alpha,c,i),(\beta,d,j)} = A_{(\alpha,c,i),(\beta,d,j)} \cdot \frac{u_{(\beta,d,j)}}{\rho \cdot u_{(\alpha,c,i)}}$$

where  $A$  is the adjacency matrix of ADSC with spectral radius  $\rho$ , and  $u = (u_{(\alpha,c,i)})_{(\alpha,c,i) \in \text{ADSC}}$  is a positive right eigenvector respect to  $\rho$ . The initial distribution is represented as a vector  $v = (v_{(\alpha,c,i)})$  given by

$$v_{(\alpha,c,i)} = \frac{u_{(\alpha,c,i)} \cdot \delta_{(\alpha,c,i)}}{\sum_{(\beta,d,j)} u_{(\beta,d,j)} \cdot \delta_{(\beta,d,j)}}$$

where  $\delta$  is the initial vector with  $\delta_{(\alpha,c,i)} = 1$  if  $i = 1$  and  $\delta_{(\alpha,c,i)} = 0$  otherwise.

**Proposition 8.2.2.** *Let  $(\mathcal{M}, X, \perp)$  be an irreducible concurrent system with characteristic root  $r$ . Define the Parry measure on the concurrent system as in Definition 8.2.1. Then its related transition matrix restricted to  $\text{ADSC}^+$  coincides the transition matrix of MCSC.*

*Proof.* Let  $A$  be the adjacency matrix of the  $\text{ADSC}^+$ . Fix an arbitrary state  $\alpha_0 \in X$ . According to Corollary 6.3.1 and Lemma 7.3.2, the spectral radius of  $A$  is  $r^{-1}$  associated with a positive eigenvector given by:

$$\forall (\alpha, c, i) \in \text{ADSC}^+ \quad u_{(\alpha,c,i)} = \frac{1}{r^{i-1}} \cdot \Gamma(\alpha_0, \alpha) \cdot h_\alpha(c).$$

Let  $P$  be the the transition matrix of the Parry measure restricted on  $\text{ADSC}^+$  and let  $M$  be the the transition matrix of MCSC as in (7.8). Then, for  $(\alpha, c, i), (\beta, d, j) \in \text{ADSC}^+$ :

$$\begin{aligned} P_{(\alpha,c,i),(\beta,d,j)} &= A_{(\alpha,c,i),(\beta,d,j)} \cdot \frac{u_{(\beta,d,j)}}{\rho \cdot u_{(\alpha,c,i)}} \\ &= A_{(\alpha,c,i),(\beta,d,j)} \cdot r^{i-j+1} \cdot \frac{\Gamma(\alpha_0, \beta) h_\beta(d)}{\Gamma(\alpha_0, \alpha) h_\alpha(c)}. \end{aligned}$$

If  $\alpha \cdot c = \beta$ ,  $c \rightarrow d$ ,  $i = |c|$  and  $j = 1$ ,

$$\begin{aligned} P_{(\alpha,c,|c|),(\beta,d,1)} &= 1 \cdot r^{|c|} \cdot \Gamma(\alpha, \beta) \cdot \frac{h_\beta(d)}{h_\alpha(c)} \\ &= f_\alpha(c) \cdot \frac{h_\beta(d)}{h_\alpha(c)} && \text{using (7.7)} \\ &= M_{(\alpha,c),(\beta,d)}. \end{aligned}$$

If  $(\alpha, c) = (\beta, d)$  and  $j = i + 1$ ,

$$P_{(\alpha,c,i),(\alpha,c,i+1)} = 1$$

which represents the transition between the extended states of a multiple clique.

Otherwise,  $P_{(\alpha,c,i),(\beta,d,j)} = 0$ .

□

## 8.3 Examples of random generation for concurrent systems

### 8.3.1 Safe Petri nets

Let  $(\mathcal{M}, X, \perp)$  be the irreducible concurrent system depicted in Figure 5.1. The characteristic root  $r \approx 0.618$  is the positive root of smallest modulus of  $z^2 + z - 1 = 0$ .

#### 8.3.1.1 Direct method via the MCSC

In this example, we could calculate all parameters of MCSC in symbolic computation. According to Proposition 7.4.2, the Parry cocycle  $\Gamma$  is in the kernel of the Möbius matrix evaluated at  $r$ , which is:

$$\Gamma = (\Gamma(\alpha, \beta))_{\alpha, \beta \in X} = \begin{matrix} M_0 \\ M_1 \end{matrix} \begin{pmatrix} 1 & \frac{1}{r} \\ r & 1 \end{pmatrix}.$$

The following table presents the values of all fibred valuations  $f_\alpha$  and its Möbius transform  $h_\alpha$ .

clique	$a$	$b$	$c$	$ac$
$f_{M_0}$	$r \cdot \frac{1}{r} = 1$	0	$r \approx 0.618$	$r^2 \cdot \frac{1}{r} \approx 0.618$
$h_{M_0}$	$1 - r \approx 0.382$	0	$r - r = 0$	$r \approx 0.618$
$f_{M_1}$	0	$r^2 \approx 0.382$	$r \approx 0.618$	0
$h_{M_1}$	0	$r^2 \approx 0.382$	$r \approx 0.618$	0

We see that  $(M_0, c)$  is a node in DSC with  $h_{M_0}(c) = 0$ . Then, by Corollary 7.5.2, we deduce that  $(M_0, c)$  is a null node where the MCSC doesn't go through.

The difference between  $f_{M_0}(c)$  and  $h_{M_0}(c)$  can be interpreted as follows. The value of  $f_{M_0}(c) \approx 0.618$  represents the probability of an infinite execution  $\xi$  starting from the state  $M_0$  such that  $c \leq C_1(\xi)$ . On the other hand, the value of  $h_{M_0}(c)$  represents the probability of an infinite execution  $\xi$  starting from the state  $M_0$  such that  $C_1(\xi) = c$ . Since letter  $c$  and letter  $a$  are concurrent in this example, we wouldn't have the chance to execute  $b$ . Once we fix the first clique to be  $c$ , the only possibility is to execute  $\xi = (cc\dots)$ , which is an event of probability 0.

The transition matrix of the MCSC on  $\text{DSC}^+$  in numerical way is given by:

$$M = \begin{matrix} (M_0, a) \\ (M_0, ac) \\ (M_1, b) \\ (M_1, c) \end{matrix} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0.382 & 0.618 \\ 0.382 & 0.618 & 0 & 0 \\ 0 & 0 & 0.382 & 0.618 \end{pmatrix}. \quad (8.2)$$

### 8.3.1.2 Alternative method via the Parry measure

The adjacency matrix of ADSC of the system is present in Example 6.3.3. The Perron eigenvector restricted to  $\text{ADSC}^+$  is:

$$u = \begin{matrix} (M_0, a) \\ (M_0, ac_1) \\ (M_0, ac_2) \\ (M_1, b) \\ (M_1, c) \end{matrix} \begin{pmatrix} r \\ 1 \\ 1+r \\ 1 \\ 1+r \end{pmatrix}.$$

Then the transition matrix via Parry measure is:

$$P = \begin{matrix} (M_0, a) \\ (M_0, ac) \\ (M_1, b) \\ (M_1, c) \end{matrix} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{r}{1+r} & r \\ r^2 & r & 0 & 0 \\ 0 & 0 & \frac{r}{1+r} & r \end{pmatrix},$$

which coincides with (8.2). Notice that

$$P_{(M_0, ac), (M_1, b)} = r \cdot \frac{u_{(M_0, ac_2)}}{u_{(M_1, b)}} \quad \text{and} \quad P_{(M_1, b), (M_0, ac)} = r \cdot \frac{u_{(M_1, b)}}{u_{(M_0, ac_1)}}.$$

The probabilities occur at the moment when the state changes, instead of between the extended states of a multiple clique.

## 8.3.2 The dining philosopher model

In Section 4.3.3.2, we present the dining philosopher model which is an irreducible concurrent system of the tip-top action. Thanks to Theorem 7.5.4, the characteristic root of the system is the square root of the smallest modulus of the Möbius polynomial of the associated monoid  $\mathcal{M}$ , given by:

$$r = \sqrt{\frac{1}{2} - \frac{\sqrt{5}}{10}} \approx 0.526.$$

By the discussion in Section 7.6, we could calculate the spectral radius of each component and compare with  $r^{-1} \approx 1.902$ . The condensation of DSC is depicted in Figure 5.3. For the ADSC, it keeps the same number of the strongly connected components of size 30, 20, 4,4,4,4 and 4. The one with 30 elements has the spectral radius close to 1.902, which determines the structure. The rest components have the spectral radius close to 1.414. Hence, we could remove the nodes after the components of size 30 and reduce the state space.

# Conclusion and future work

**Conclusion** In this thesis, we cover the uniform measure defined on free monoids, on trace monoids and on concurrent systems. All this uniform measures have two approaches – directly from a Markov chain or via the Parry measure.

The theory of trace monoids has the strong combinatorial foundation around the Möbius polynomial. The irreducibility of trace monoids implies the strong connectivity of the ADC. Hence, a dominant eigenvalue exists and determines the growth rate of trace monoids. This completes the probabilistic theory and proves that the finite distributions on trace monoids converge weakly to the uniform measure defined on infinite traces.

The structure of concurrent systems keeps similar combinatorial properties to trace monoids, but is more complicated. The difficulty to extend to concurrent systems is that the Perron-Frobenius theorem is not applicable. Therefore, we need the results on nonnegative reducible matrices. To resolve this problem, we found the spectral property of irreducible concurrent system, which allows us to distinguish the main components determined by the characteristic root of the system. The transition matrix can be obtained either from the MCSC with the algorithmic tool or from the Parry measure with the spectral radius of the dominant components.

In the following paragraphs, we present three possible directions to improve our theory.

**Limit of finite uniform distributions** Let  $(\mathcal{M}, X, \perp)$  be a non trivial irreducible concurrent system. For each state  $\alpha \in X$  and for each integer  $n \geq 0$ , let  $\nu_\alpha(n)$  denote the uniform distribution on  $\mathcal{M}_\alpha(n)$ . To generate larger executions of fixed-length for concurrent systems, one piece is missing to complete the theory. The question is to clarify whether the finite distributions  $(\nu_\alpha(n))_{n \geq 0}$  converge weakly to the uniform measure  $\nu_\alpha$  on  $\partial\mathcal{M}_\alpha$  as defined in Chapter 7.

Let us recall the case of trace monoids. Due to the problem of commutativity, we are obligated to define the uniform measure on infinite traces. When considering the finite distributions on trace monoids, we first view each finite distribution as a probability measure on the compact space  $\overline{\mathcal{M}}$ . The uniqueness of the uniform measure guarantees a dominant eigenvalue of the adjacency matrix of ADC, and determines the asymptotic approximations of  $\mathcal{M}(n)$ . This implies the weak convergence of the finite distributions on trace monoids (See in Section 3.5.2).

Back to concurrent systems, the finite executions in  $\mathcal{M}_\alpha(n)$  correspond to the paths

of length  $n$  in the ADSC with some constraints on the initial node and the final node, as the bijection mentioned in Proposition 5.2.1. Therefore, the finite distribution  $\nu_\alpha(n)$  on concurrent systems is equivalent to the finite distribution on a subset of the paths of length  $n$  in the ADSC, say  $\tilde{\nu}_\alpha(n)$ .

With the aid of the spectral property for irreducible concurrent systems, we could retrieve the dominant eigenvalue of ADSC. However, this doesn't allow us to claim that the growth of the subset of the paths of length  $n$  is determined by the spectral radius of the ADSC. Since the ADSC is not strongly connected in general, we need more work together with the literature on reducible nonnegative matrices.

Rephrasing the original question, it suffices to show that the finite distributions  $(\tilde{\nu}_\alpha(n))_{n \geq 0}$  converges weakly as  $n \rightarrow \infty$  toward the uniform measure on infinite paths in ADSC, corresponding to a Markov chain on ADSC. According to Proposition 7.6.1, the digraph has the property that its basic components coincides the terminal components, restricted to the positive nodes by this Markov chain.

The study of  $(\tilde{\nu}_\alpha(n))_{n \geq 0}$  inspires us to study the convergence of finite uniform distributions on general finite graphs. We present in next paragraph an initial study to define a uniform measure on general finite graphs.

**Generalized Parry measure on a finite graph** Let  $\Sigma$  be a finite alphabet. Consider a digraph  $G = (N, E)$  whose edges are labeled by letters of  $\Sigma$ . Define an action of the free monoid  $\Sigma^*$  over the finite set  $N$  as a mapping  $\phi : N \times \Sigma^* \rightarrow N$ , denoted by  $\phi(\alpha, x) = \alpha \cdot x$ .

Imitating the uniform measure defined on the executions of concurrent systems, we define a family of uniform measures on the words read from the paths in the digraph  $G$ . The graph is not necessarily strongly connected. Therefore, it is not clear how to choose an appropriate Perron eigenvector in the sense of Parry measure.

**Definition** (Generalized Parry measure). *Let  $G = (N, E)$  be a digraph. A generalized Parry measure on  $G$  is defined as a family of probability measure  $\nu = (\nu_\alpha)_{\alpha \in N}$  such that, for  $q > 0$  and for some cocycle  $\theta$ :*

$$\forall \alpha \in N \quad \forall x \in \Sigma^* \quad \nu_\alpha(C[x]) = q^{|x|} \cdot \theta(\alpha, \alpha \cdot x)$$

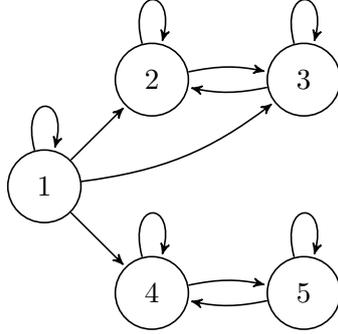
where  $C[x]$  is the elementary cylinder of base  $x$  on  $G$ .

**Observation.** *Let  $G = (N, E)$  be a digraph with spectral radius  $\rho$ . For  $\alpha \in N$ , let  $G_\alpha = (N_\alpha, E_\alpha)$  be the accessible subgraph of  $G$  from  $\alpha$ , assume that its terminal components coincide its basic components. Then, for each Perron eigenvector of  $G$ , there is a generalized Parry measure  $\nu_\alpha$  as defined in previous definition, given by:*

$$q = \rho^{-1} \quad \text{and} \quad \forall \alpha, \beta \in N \quad \theta(\alpha, \beta) = \frac{r_\beta}{r_\alpha},$$

where  $(r_\alpha)_{\alpha \in N}$  is any right Perron eigenvector of  $G$ .

For instance, a general finite graph is given in the following figure.



Its adjacency matrix is:

$$A = \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix} \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}.$$

The spectral radius of  $A$  is 2. There are two terminal components which are basic. Hence, we have to decide which Perron eigenvector we choose for generalized Parry measure.

Once we get a well-defined Parry measure, can we claim that the finite distributions on paths converges weakly to this generalized Parry measure?

**Detecting null nodes** Since the MCSC goes through only positive nodes, the detection of the null nodes in application could effectively reduce the state space and simplify the computation.

In Section 5.3, we introduced a theoretical definition for null nodes, which is not practical for implementation. An alternative way is to apply Corollary 7.5.2 by calculating the probability  $h_\alpha(c)$  for  $(\alpha, c) \in \text{ADSC}$ . The node  $(\alpha, c)$  is null if and only if  $h_\alpha(c) = 0$ . This approach requires an exact computation for all parameters, which is doable with the aid of symbolic computation software. However, the exhaustion of computational resources leads soon as the size of the systems grow.

Instead, we calculate the parameters numerically in order to deal with larger systems. This improves the efficiency, but we lose the information of the probability identical to zero. In other words, we couldn't distinguish a probability between relatively small and asymptotic to zero via numerical computation.

The last approach is to apply Corollary 7.6.2 by comparing the spectral radius of different components (refer to the schematic diagram of ADSC in Section 7.6). We first determine the spectral radius  $r^{-1}$  of ADSC, then compute numerically the spectral radius of all components and compare with  $r^{-1}$ . When the system has only one basic component, it can be determined by computing numerically, and thus the null nodes can be found. Otherwise, the same computation errors take arise.

In our study, we have observed that most systems have only one basic component, that is to say that its spectral radius equals the spectral radius of the digraph. Therefore, it is useful to develop the sufficient conditions on th models for this property to hold.

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# Appendix

## Implementation of safe Petri nets

To test the model of our concurrent systems and generate its executions, part of our experiments focus on 1-safe Petri nets. We adapt the Python library – SNAKES [36] to fit better with our model. We keep the data structure in `data.py` and rewrite `net.py` for safe Petri nets. The original script of Petri nets allows to add the conditions for transitions. We simplify this part and replace by the simple firing rule. Since the nets are 1-bounded, a place can be added at most one token.

The following shows the main objects that we use and emphasizes the new functions or classes which are not in the original library.

```
1 from snakes.data import *
2 class Place:
3     [...]
4 class Transition:
5     [...]
6 class Marking:
7     [...]
8 class PetriNet:
9     def enabled_clique(self, clique):
10        # check the concurrency of transitions
11        for trans in clique:
12            if not trans.enabled():
13                return False
14        return True
15    def dependent(self, trans1, trans2):
16        t1 = trans1._input|trans1._output
17        t2 = trans2._input|trans2._output
18        if t1&t2 == set():
19            return False
20        else:
21            return True
22    def dependent_set(self, net): #dependency relation on transitions
23        l = set()
24        for trans1 in net.transition():
25            for trans2 in net.transition():
26                if self.dependent(trans1, trans2) and trans1!= trans2:
27                    l.add((trans1, trans2))
28                    ll = [tuple(element) for element in set(frozenset(i) for
29                                                                i in l)]
29        return ll
30    [...]
31 class StateGraph: # build the marking graph of the nets
32    [...]
```

## Implementation of trace monoids

Given a trace monoid with its alphabet and its dependence relation. We record it as the adjacency dictionary of its Coxeter graph. The key point is to find all possible cliques and able to build the traces in Cartier-Foata normal form. Then we could construct the edges in the ADC and in the ADSC. The main functions for trace monoids are presented as follows.

```

1  from copy import deepcopy
2  class DictionnaireAdjacence(object):
3      # an adjacency dictionary for an undirected graph
4      [...]
5  class Monoid(object):
6      def __init__(self, alphabet = [], dependences = []):
7          # a trace monoid is recorded as the dictionary of its Coxeter graph
8          [...]
9      def dependencies(self, e):
10         # find the nodes connected with e in Coxeter graph
11         [...]
12     def layerDependencies(self, s):
13         # find the dependencies with the current clique
14         res = set()
15         for e in s:
16             for a in self.dependencies(e):
17                 res.add(a)
18         return res
19     def findAllPossibleLayers(self, alphabet): # all possible cliques
20         return self.findAllPossibleLayersAux([], alphabet)
21     def findAllPossibleLayersAux(self, prec, possible):
22         res = set()
23         for e in possible:
24             layer = set()
25             for el in prec:
26                 layer.add(el)
27                 layer.add(e)
28             nextPossible = set()
29             for old in possible:
30                 if not e in self.dependencies(old):
31                     nextPossible.add(old)
32                     res.add(frozenset(layer))
33             for l in self.findAllPossibleLayersAux(layer, nextPossible):
34                 res.add(l)
35         return res
36     def clique_dict(self):
37         # a numbering dictionary for all possible cliques
38         [...]
39     def dictPossibleCliques(self, alphabet):
40         # a dictionary storing (clique, next possible cliques)
41         d = dict()
42         for clique in self.findAllPossibleLayers(alphabet):
43             nextClique = list()
44             possible = self.layerDependencies(clique)
45             possibleClique = self.findAllPossibleLayersAux([], possible)
46             nextClique = [[y for y in x] for x in possibleClique]
47             clique = tuple(clique)
48             d[clique]=nextClique
49         return d

```

## Implementation of the Möbius matrix with SageMath

With previous data structures, we are able to define our concurrent system. First, we calculate the necessary parameters for the system.

```

1 net = PetriNet('mynet')
2 trans_set = set(net.transition()) # alphabet
3 dependency = net.dependent_set() # dependence relation
4 monoid = Monoid(trans_set, dependency) # related monoid
5 stategraph = StateGraph(net) # state space
6 stategraph.build()
7 dim = len(stategraph._done) # size of the state space

```

The calculation of the Möbius matrix is crucial, which allows us to obtain the characteristic root and the parameters of the transition matrix via the MCSC.

The following function for the Möbius matrix is based on the Algorithm 4 in Chapter 8. To compute more effectively, we record all visited Möbius matrices on the subset of the alphabet during the recursive process of (8.1). We use SageMath which is a software of computer algebra written in Python to handle the symbolic computation.

```

1 # recording the results of the Mobius matrix on a subset
2 record_mobius=dict()
3 x = var('x')
4 def mobius_matrix(trans_set):
5     global record_mobius, x
6     M = matrix(SR, dim, dim) # empty symbolic matrix
7     M1 = matrix(SR, dim, dim)
8     if frozenset(trans_set) in record_mobius:
9         return record_mobius[frozenset(trans_set)]
10    else:
11        if not trans_set:
12            M = matrix.identity(dim) # identity matrix
13            record_mobius[frozenset(trans_set)] = M
14        else:
15            trans_subset = trans_set.copy()
16            letter = trans_subset.pop()
17            dependent_set = monoid.graphe.dictionnaire[letter]
18            parallel_set = trans_set - dependent_set
19            if frozenset(trans_subset) in record_mobius:
20                M1 = record_mobius[frozenset(trans_subset)]
21            else:
22                M1 = mobius_matrix(trans_subset)
23            record_mobius[frozenset(trans_subset)] = M1
24            for i in range(dim):
25                for j in range(dim):
26                    marking_i = stategraph._marking[i]
27                    net.set_marking(marking_i) #assign marking to net
28                    if letter.enabled():
29                        net.set_marking(marking_i)
30                        letter.fire()
31                        i1 = stategraph._state[n.get_marking()]
32                        M[i, j] = M1[i, j] - x * mobius_matrix(parallel_set)[i1, j]
33                    else:
34                        M[i, j] = M1[i, j]
35            record_mobius[frozenset(trans_set)] = M
36    return record_mobius[frozenset(trans_set)]

```

## Implementation of DSC and its spectral radius with SageMath

In this thesis, the analysis of the DSC and the ADSC of different concurrent systems help us to understand the structure of the system. Moreover, the characteristic root of the system can be calculated by the inverse of the spectral radius of a terminal component of the ADSC<sup>+</sup>.

For simplicity, we present the function for the construction of the DSC. We start from the initial state and check all possible cliques. If one clique is enabled, add (state, clique) to the state space and add the new arriving state to the stack. Follow the same process until we go through all states in the stack. Then we get all edges in the DSC. Following the similar idea, we obtain the function for the edges of the ADSC.

```

1 net = PetriNet('mynet')
2 trans_set = set(net.transition()) # alphabet
3 dependency = net.dependent_set() # dependence relation
4 monoid = Monoid(trans_set, dependency) # related monoid
5 next_possible_cliques = monoid.dictPossibleCliques(trans_set)
6 def DSC_edge(net, possible_cliques):
7     res = list()
8     stack = list() # state
9     visited = list() # state, clique
10    stack.append(net.get_marking()) # add the initial state
11    while len(stack)!=0:
12        marking=stack.pop(0)
13        net.set_marking(marking) #assign marking to net
14        marking_label=net.marking()
15        trans_set = possible_cliques.keys()
16        for clique in trans_set:
17            net.set_marking(marking) # back to the original marking!
18            if (marking, clique) in visited:
19                continue
20            if enabled_clique(net, clique):
21                for trans in clique:
22                    trans.fire()
23                    marking2 = net.get_marking()
24                    stack.append(marking2)
25                    state=(marking, clique)
26                    visited.append(state)
27                    for clique2 in possible_cliques[clique]:
28                        if enabled_clique(net, clique2):
29                            state2 = (marking2, clique2)
30                            res.append((state, state2))
31    return res
32 def ADSC_edge(net, possible_cliques):
33    [...]
```

Once we have the edges of the ADSC in hand, we import the Python package Networkx [20], which includes the functions to deal with the strongly connectivity, strongly connected components and the adjacency matrix. Finally, the spectral radius can be computed by SageMath.

```

1 import networkx as nx
2 G=nx.DiGraph()
3 G.add_edges_from(ADSC_edge(net, next_possible_cliques))
4 Adj = matrix(nx.adjacency_matrix(G).todense())
5 spectral_radius = max([abs(i) for i in Adj.eigenvalues()])
```

## Examples of random generation

With previous tools, we are able to randomly generate the executions of the concurrent systems. We presents the outputs of two models here.

### The dining philosopher model

trajectory(trans\_matrix, h\_init, 100)

```
>> ((ε, d), (d, i), (ε, c), (c, h), (ε, d), (d, i), (ε, c), (c, h), (ε, b), (b, g),
      (ε, a), (a, f), (ε, e), (e, j), (ε, d), (d, i), (ε, c), (c, h), (ε, c), (c, h),
      (ε, c), (c, h), (ε, c), (c, h), (ε, bd), (bd, gi), (ε, be), (be, gj), (ε, e), (e, j),
      (ε, d), (d, i), (ε, e), (e, j), (ε, d), (d, i), (ε, ce), (aε, hj), (ε, d), (d, i),
      (ε, d), (d, i), (ε, c), (c, h), (ε, c), (c, h), (ε, d), (d, i), (ε, d), (d, i),
      (ε, ce), (aε, hj), (ε, c), (c, h), (ε, c), (c, h), (ε, c), (c, h), (ε, c), (c, h),
      (ε, d), (d, i), (ε, d), (d, i), (ε, e), (e, j), (ε, ad), (ad, fi), (ε, d), (d, i),
      (ε, c), (c, h), (ε, bd), (bd, gi), (ε, c), (c, h), (ε, bd), (bd, gi), (ε, e), (e, j),
      (ε, a), (a, f), (ε, a), (a, f), (ε, e), (e, j), (ε, d), (d, i), (ε, d), (d, i),
      (ε, d), (d, i), (ε, d), (d, i), (ε, d), (d, i), (ε, c), (c, h), (ε, c), (c, h))
```

### The mosaic model

trajectory(trans\_matrix, h\_init, 50)

```
>> ((11111, 02), (00001, 02), (11111, 03), (00100, 0), (11100, 1), (10000, 2), (10110, 2),
      (10000, 13), (11111, 03), (00100, 3), (00111, 2), (00001, 1), (01101, 1), (00001, 02),
      (11111, 0), (00111, 0), (11111, 0), (00111, 0), (11111, 14), (00010, 0), (11010, 0),
      (00010, 14), (11111, 13), (10000, 13), (11111, 3), (11100, 3), (11111, 3), (11100, 3),
      (11111, 4), (01110, 4), (11111, 3), (11100, 3), (11111, 24), (01000, 24), (11111, 1),
      (10011, 1), (11111, 1), (10011, 1), (11111, 2), (11001, 2), (11111, 1), (10011, 1),
      (11111, 02), (00001, 1), (01101, 1), (00001, 02), (11111, 1), (10011, 1), (11111, 2), (11001, 2))
```