Exploring the landscape of heterotic string compactifications

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Explorant le paysage des compactifications de la corde hétérotique

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Titre: Explorant le paysage des compactifications de la corde hétérotique
Mots clés: Corde Hétérotique, Compactifications, Symétrie de gauge, Vide de supercordes, Dualités

Résumé: L'objectif principal de cette thèse est d'étudier l'espace des modules d'un large ensemble de compactifications de la théorie des cordes hétérotiques et, en particulier, de trouver et classer la liste des groupes de jauge qui sont réalisés dans de telles théories. Nous commençons par analyser le cas des compactifications sur un cercle, en développant une technique pour calculer et représenter les régions dans l'espace des modules où il y a des groupes augmentés de symétrie.

A l'aide des techniques des réseaux, nous énonçons des critères généraux pour établir si un groupe de jauge est réalisé ou non dans les compactifications sur T^d, créant une série d'algorithmes pour explorer complètement ces espaces de modules. Pour $d = 2$, on trouve que les groupes de jauge respectifs coïncident avec toutes les fibres singulières possibles des surfaces extrêmes K3, corroborant la dualité avec Théorie F sur une surface K3. Nous construisons également une méthode pour transformer les modules sous T-dualité et construisons la carte qui relie les modules des cordes hétérotiques $E_8 \times E_8$ et $SO(32)$ sur un tore.

Nous analysons également les compactifications de la corde hétérotique sur les orbifolds asymétriques $T^d/\mathbb{Z}_2$ qui réalisent la dite corde CHL. Ceci est particulièrement intéressant car les cas $d = 2$ et $d = 3$ sont duaux respectivement de la théorie F et de la théorie M sur un K3 à singularité figée, qui ne sont pas bien compris. Nous étudions en détail ces théories et, avec quelques modifications à nos algorithmes, explorons et trouvons tous les groupes de symétrie, vérifiant qu'ils satisfont une condition de centre sans anomalie découverte très récemment.

Enfin, nous obtienons la liste complète des groupes de jauge qui sont réalisés dans la corde hétérotique en 7d et 6d, y compris les compactifications toroïdales ordinaires, le CHL et quatre autres composants réalisés via des triplets d'holonomie non triviaux. Nous dérivons une carte qui relie les groupes de jauge sur le compactifications toroïdales aux autres composants. En 7d, il coïncide avec le mécanisme de gel des singularités en théorie M sur K3; tandis qu'en 6d on montre que les gels possibles pour chaque groupe de jauge sont déterminés par sa topologie.

Title: Exploring the landscape of heterotic string compactifications
Keywords: Heterotic String, Compactifications, Gauge Symmetry, Superstring Vacua, Dualities

Abstract: The main goal of this thesis is to study the moduli space of a broad set of compactifications of heterotic string theory and, in particular, to find and classify the list of gauge groups that are realized in such theories. We start by analyzing the case of circle compactifications, developing a technique to compute and represent the regions of enhancement on the moduli space.

Using lattice embedding techniques, we state general criteria to establish whether a gauge group is realized or not on compactifications on T^d, creating a series of algorithms to completely explore these moduli spaces. For $d = 2$, we find that the respective gauge groups coincide with all possible singular fibers of extremal K3 surfaces, corroborating the duality with F-theory on a K3 surface. We also construct a method to transform the moduli under T-duality and build the map that relates the moduli of the $E_8 \times E_8$ and $SO(32)$ heterotic strings on a torus.

We also analyse compactifications of the heterotic string on $T^d/\mathbb{Z}_2$ asymmetric orbifolds which realize the so-called CHL string. This is of interest because the $d = 2$ and $d = 3$ cases are dual respectively to F-theory and M-theory on a K3 with a frozen singularity, which are not well understood. We study in detail these theories and, with some modifications to our algorithms, explore and find all the symmetry enhancements, verifying that they satisfy a condition for anomaly-free one-form center brought to light very recently.

Finally, we obtain the complete list of gauge groups that are realized in the heterotic string in 7d and 6d, including the ordinary toroidal compactifications, the CHL and four other components realized via non-trivial holonomy triples. We derive a map that relates the gauge groups on the toroidal compactifications to the other components. In 7d, it coincides with the singularity freezing mechanism in M-theory on K3; while in 6d we show that the possible freezings for each gauge group are determined by its topology.

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Título: Explorando el paisaje de las compactificaciones de la cuerda heterótica

Palabras clave: Cuerda Heterótica, Compactificaciónes, Simetría de gauge, Vacío de supercuerdas, Dualidades

Resumen: El objetivo principal de la presente tesis es estudiar el espacio de módulos de un amplio conjunto de compactificaciones de la teoría de cuerdas heterótica y, en particular, encontrar y clasificar la lista de grupos de calibre que se realizan en dichas teorías. Comenzamos analizando el caso de las compactificaciones circulares, desarrollando una técnica para calcular y representar las regiones de aumento en el espacio de módulos.

Usando técnicas de encajes de látices, enunciamos criterios generales para establecer si un grupo de calibre se realiza o no en T^d, creando una serie de algoritmos para explorar completamente estos espacios de módulos. Para $d = 2$, encontramos que los respectivos grupos de calibre coinciden con todas las posibles fibras singulares de las superficies extremas K3, lo que corrobora la dualidad con la teoría F en una superficie K3. También construimos un método para transformar los módulos bajo T-dualidad y construimos el mapa que relaciona los módulos de las teorías heteróticas $E_8 \times E_8$ y $SO(32)$ en un toro.

También analizamos las compactificaciones de la cuerda heterótica en orbifolds asimétricos $T^d/\mathbb{Z}_2$ que realizan la llamada cuerda CHL. Esto es de interés porque los casos $d = 2$ y $d = 3$ son duales respectivamente a la teoría F y la teoría M en un K3 con una singularidad congelada, que no están bien entendidos. Estudiamos en detalle estas teorías y, con algunas modificaciones a nuestros algoritmos, exploramos y encontramos todos los aumentos de simetría, verificando que satisfacen una condición de centro sin anomalías descubierta muy recientemente.

Finalmente, obtenemos la lista completa de grupos de gauge que se realizan en la cuerda heterótica en 7d y 6d, incluyendo las compactificaciones toroidales ordinarias, la CHL y otras cuatro componentes realizadas mediante triples de holonomía no triviales. Derivamos un mapa que relaciona los grupos de calibre en las compactificaciones toroidales con las otras componentes. En 7d, coincide con el mecanismo de congelamiento de singularidades en la teoría M en K3; mientras que en 6d mostramos que los posibles congelamientos para cada grupo de calibre están determinados por su topología.

To Sabrina,
for her immeasurable support.

Publications

This thesis is based on the following publications [1–5]:

- Fraiman, B., Graña, M., Nuñez, C.A. **A new twist on heterotic string compactifications.** J. High Energ. Phys. 2018, 78 (2018).

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- Font, A., Fraiman, B., Graña, M., Nuñez, C.A., Parra De Freitas, H. **Exploring the landscape of heterotic strings on T^d.** J. High Energ. Phys. 2020, 194 (2020).

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- Font, A., Fraiman, B., Graña, M., Nuñez, C.A., Parra De Freitas, H. **Exploring the landscape of CHL strings on T^d.** J. High Energ. Phys. 2021, 95 (2021).

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- Fraiman, B., Parra De Freitas, H. **Symmetry Enhancements in 7d Heterotic Strings.** J. High Energ. Phys. 2021, 2 (2021).

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- Fraiman, B., Parra De Freitas, H. **Freezing of Gauge Symmetries in the Heterotic String on T^4.** J. High Energ. Phys. 2022, 7 (2022).

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Contents

Chapter 1

Introduction

1.1 Motivation

String Theory is the most developed and promising framework to study quantum gravity. Many efforts have been made to try to reproduce the Standard Model as a low energy effective theory of some string model. The first string proposed, the bosonic string, only included bosons in its spectrum and predicted a tachyon, which signaled an inconsistency in the theory. This motivated the construction of the type I and II superstring theories, which have a tachyon-free supersymmetric spectrum and are formulated in 10 space-time dimensions (this number is fixed by consistency). Their low energy limit is supergravity in 10 dimensions.

Two important aspects of the Standard Model are that it is formulated in 4 dimensions and that it contains some particles with non-Abelian gauge symmetries. A way of reducing the number of dimensions from 10 to 4 is through the mechanism of compactification, that is, to take six of those dimensions to have finite length (e.g. by making them periodic), generalizing Kaluza-Klein theory. The 4-dimensional universe is then a low energy approximation of a more fundamental theory. When one compactifies string theory in special ways, the particles acquire non-Abelian gauge symmetries, making these theories plausible candidates for describing our universe. While in Kaluza-Klein theory the momentum in the fifth dimension is quantized, in string theory compactifications we have a similar situation, with the momentum associated to the string states living on a lattice.

The so called heterotic string combines the bosonic and type II string theories [6]. For consistency the low energy spectrum has two possible gauge group symmetries: $\frac{Spin(32)}{\mathbb{Z}_2}$ or $E_8 \times E_8$, defining two different string theories. These heterotic strings also have super-symmetry and live in a 10-dimensional space-time, so that they can also be compactified on different geometries [7]. This feature was considered very promising, since the gauge groups of both theories contain $SO(10)$ and $SU(5)$, while E_8 also contains E_6, which are the symmetries of the principal Grand Unified Theories. In compactifications to lower dimensions, it is quite simple to break the original gauge symmetries to their subgroups. Depending on the geometry of the compact space, it is also possible to get an enhancement of the original symmetry, creating a rich landscape of theories with very diverse gauge groups. The low energy limit of these theories is just supergravity coupled to super Yang-Mills with the corresponding gauge group.

A simple question one could ask is: what are the possible gauge groups that can be obtained from heterotic string theory in less than 10 dimensions? Can we classify them? This question is relevant from a phenomenological point of view (building realistic examples of our universe);

but it could be connected to a deeper puzzle: what are the possible theories that can be consistently coupled to gravity?

Even for relatively simple geometries, little was known about the possible gauge groups that can appear [1], let alone the special values for the parameters of the compactification where these enhancements occur. This motivated us to study in detail the structure of the space of these parameters (the moduli space) and to obtain a complete classification of the possible gauge groups for some types of geometries.

Since the strings are extended objects, they can be wrapped along compact dimensions. This originates an equivalence between different string configurations after compactification, which is known as T-duality [8].

It was shown that all the different superstrings are related by this and other dualities, leading to the idea that they are distinct limits of a higher dimensional theory coined M-theory [9]. A strong motivation to study toroidal compactifications of heterotic strings is the dualities they have with other theories: heterotic on T^4 is dual to type IIA on K3 [10], heterotic on T^3 is dual to M-theory on K3, while heterotic on T^2 is dual to F-theory on an elliptically fibered K3. This web of dualities provides a framework for exploring different aspects of string theory.

1.2 Results

The principal goal of this thesis is to answer the question of what gauge symmetries are allowed in heterotic string compactifications on geometries that preserve all of the supersymmetry. Although this is a very concrete and relatively simple question, an answer was only known for the two simplest cases: circle compactifications through the Generalized Dynkin Diagram (GDD) method [11–13] and, although indirectly, the 2-torus through duality with F-theory from the classification of ADE types of singular fibers in elliptic K3 surfaces [14, 15]. As we will explain in detail, for compact spaces of larger dimensions (which correspond to low-energy effective theories with less dimensions) and for non-geometrical spaces, a great deal of subtleties appears, turning the aforementioned goal into a highly no-trivial one. A related problem that we will address is the study of the corresponding moduli spaces: their structure, regions of symmetry enhancements and their relation to T-duality. We will take different but complementary approaches.

In [1] we studied compactifications on the circle from the point of view of the 17-dimensional moduli space defined by the parameters. Solving the mass and level matching equations as a function of the moduli, we developed a method for charting the special regions where there is symmetry enhancement to some non-Abelian group of rank 17, finding that the maximal ones (i.e. to groups without $U(1)$ factors) appear on the intersection points of curves where there are non-maximal enhancements. This point of view allowed us to develop tools that are ideal to gain intuition and understand this type of compactifications but, even though they predict a rich variety of gauge symmetries, it was not possible to get all of them. Changing the focus to the fact that the momenta of the spectrum of states are quantized in the even

[1] Compactifications on a circle being an exception.

self-dual Lorentzian Narain lattice $\Gamma_{1,17}$, we were able to obtain the complete classification of enhancements, together with the regions on moduli space where they arise, using the GDD method.

In [2] we treated the case of toroidal compactifications of arbitrary dimensions, where the models realized also have 16 supercharges and the gauge groups are of ADE type (i.e. $SU(n)$, $SO(2n)$, E_n) and have rank $16+d$. We found how the points in the moduli space are related by T-duality and defined the map relating the moduli spaces of the compactified $\frac{\text{Spin}(32)}{\mathbb{Z}_2}$ and $E_8 \times E_8$ theories. For T^d with $d > 1$, the momentum lattice $\Gamma_{d,16+d}$ is still even and self-dual but not Lorentzian, making it impossible to construct a GDD. A certain gauge group appears in the moduli space if its associated root lattice can be embedded in the corresponding momentum lattice (satisfying certain conditions). This way of looking at the problem allowed us to state some conditions for these groups to appear by using some theorems by Nikulin [16]. We carried out an exhaustive classification of the allowed gauge groups in $d = 2$, finding that it matches exactly the one obtained on the dual F-theory on elliptically fibered K3 surfaces [15], but also giving the moduli defining the compactification that realizes each case. To perform this classification we made an exploration algorithm which consists in moving from singular points of maximal enhancement in moduli space to others via manipulations of their associated root lattices in a controlled way. This algorithm was improved in subsequent works, allowing us to explore T^d compactifications with $d > 2$ and to compute the precise global data of the gauge groups.

In [3] we performed a detailed study of compactifications on special $T^d/\mathbb{Z}_2$ orbifolds[2] (realizing the CHL string [17, 18]), which preserve the supersymmetry and present enhancements of reduced rank $d+8$. One peculiarity is that the gauge groups are not always of ADE type, with groups $Sp(n)$, $SO(2n+1)$ or F_4 appearing in special regions. Another characteristic is that the momentum lattice is not self-dual for $d > 1$ [19]. The latter implies that the criteria stated before for lattices embedding does not apply anymore. This required many changes in the methods developed in the previous work. With a generalization of our exploration algorithm we found the list of gauge symmetries for $d = 2$, 3 and 4 ($d = 1$ enhancements can be easily found using the associated GDD, which turns out to be the Dynkin diagram of E_{10}). We computed the precise form of their respective topologies by adapting our methods using results of [20], and checked that they satisfy a condition for anomaly-free one-form center symmetries given in [21].

For $d = 1$ and 2 the only known possible ranks for heterotic string compactifications preserving 16 supercharges are $d+16$ and $d+8$, namely the Narain and CHL components[3]. In contrast, for $d \geq 3$ there are more possibilities: non-trivial holonomy triples of [23] produce vacua with reduced ranks $d+4$, $d+2$ and d. We turned to these constructions in [4], adapting and applying the machinery already developed in the previous works to the case of $d = 3$ and obtaining the full list of maximal enhancements. These new moduli spaces have less maximal enhancement points due to the reduced rank, but in some cases they present a richer variety

[2] It should be noted that this is just notation, as the $\mathbb{Z}_2$ acts on the gauge bundle.

[3] In 8d there are also the $\mathbb{Z}_2$-triple and the "no vector structure" constructions, which are dual to the CHL [22, 23].

of gauge groups (e.g. gauge group G_2 in the $\mathbb{Z}_3$-triple). Heterotic string on the T^3 is dual to M-theory on a K3 surface. The gauge groups with reduced rank are realized in the latter when there are partially frozen singularities on the K3 [23–25]. Exploiting lattices embedding techniques, we found the explicit freezing rules that relate the gauge groups on the Narain component with those appearing on the reduced rank ones. We found a perfect match of these rules from the heterotic string side with the known results on M-theory. A generalization of this analysis to $d = 4$ was done in [5], where we again obtained the complete classification of groups for all the $\mathbb{Z}_n$-triples via our exploration algorithm.[4] We discovered that the freezing mechanism here involves the topology of the gauge groups, acting on those and only those that are non-simply-connected. Strikingly, the freezing rules constructed coincide with the map relating the moduli space components of flat G-bundles on T^2 with G non-simply-connected.

We only include in the present thesis the lists of gauge groups for the $d = 1$, 2 and 3 cases. For $d = 4$ we refer to [27], where we list the maximal gauge groups for all the 16 supercharges preserving compactifications with 6 or more space-time dimensions. In addition to the Narain, CHL and triples components, there are many more listed there. In general they do not correspond to heterotic string constructions. They were studied in [26], which is left out of this thesis.

In Table 1.1 we list the main information concerning each of the compactifications studied. It is notable that the number of groups that appear on T^d or $T^d/\mathbb{Z}_2$ increases exponentially with d. Also, as expected, more types of groups are permitted when reducing the rank. Partial explorations for $d \geq 5$ exhibit the same exponential behavior, but the corresponding analysis is not included here. The rules relating the gauge groups in toroidal compactifications to the reduced rank theories found is summarized in Table 1.2.

[4] For $d = 4$ (as for $d = 1$ and 2) we have a recent confirmation that our lists are exhaustive from the results in our very recent work [26], where all these gauge groups were reobtained from the Niemeier lattices.

G	d	r	Algebras	Groups	Information	Types
$\emptyset$	0	16	2	2		DE
S^1	1	17	44	44		ADE
$S^1/\mathbb{Z}_2$	1	9	9	9	CHL	ADE
T^2	2	18	325	341	Dual to F-theory on a K3	ADE
$T^2/\mathbb{Z}_2$	2	10	61	61	CHL/$\mathbb{Z}_2$-triple	ACDE
T^3	3	19	1035	1232	Dual to M-theory on a K3	ADE
$T^3/\mathbb{Z}_2$	3	11	407	429	CHL/$\mathbb{Z}_2$-triple	ABCDEF
$T^3/\mathbb{Z}_3$	3	7	50	52	$\mathbb{Z}_3$-triple	ADEG
$T^3/\mathbb{Z}_4$	3	5	16	18	$\mathbb{Z}_4$-triple	AD
$T^3/\mathbb{Z}_5$	3	3	3	3	$\mathbb{Z}_5$-triple	A
$T^3/\mathbb{Z}_6$	3	3	3	3	$\mathbb{Z}_6$-triple	A
T^4	4	20	2252	3396	Dual to IIA on a K3	ADE
$T^4/\mathbb{Z}_2$	4	12	1988	2540	CHL/$\mathbb{Z}_2$-triple	ABCDEF
$T^4/\mathbb{Z}_3$	4	8	154	202	$\mathbb{Z}_3$-triple	ADEG
$T^4/\mathbb{Z}_4$	4	6	101	127	$\mathbb{Z}_4$-triple	ACDE
$T^4/\mathbb{Z}_5$	4	4	11	14	$\mathbb{Z}_5$-triple	AD
$T^4/\mathbb{Z}_6$	4	4	36	40	$\mathbb{Z}_6$-triple	ACG

Table 1.1: Geometry of the compactification space G, number of compactified dimensions d, rank r, number of maximal algebras and groups found and their types.

d	Algebra	Order	Transforms to
1	E_8	2	$\emptyset$
2	D_{n+8}	2	C_n
3	D_{n+4}	2	C_n
3	E_{n+4}	2	F_n
3	E_{n+6}	3	G_n
3	E_{n+7}	4	C_n
3	E_8	5, 6	$\emptyset$
4	A_{qn-1}	$q = 2, 3, 4, 5, 6$	A_{n-1}
4	D_{n+2}	2 (v)	C_n
4	D_{2n}	2 (s)	B_n
4	E_7	2	F_4
4	E_6	3	G_2
4	D_{2n+3}	4 (s)	C_n

Table 1.2: Freezing rules for the simple factors in the gauge groups for $d \leq 4$. We use the conventions $C_1 \equiv A_1$, $F_2 \equiv A_2$, $F_3 \equiv B_3$, $G_1 \equiv A_1$,

1.3 Structure of the thesis

This thesis is divided in six chapters: The first one is a short introduction to the thesis. Chapter 2 is an introduction to heterotic string and its toroidal compactification, with a detailed analysis

of the structure of the moduli space an its enhancements of symmetry for the circle case based on [1].

In Chapter 3, which is based on [2], we resume with the toroidal compactifications introduced earlier, this time analyzing them from the point of view of lattice embeddings. We introduce an exploration algorithm which we use to classify and give the list of symmetry enhancements of T^2 compactifications.

In Chapter 4 we analyze in detail the CHL string and its compactifications, constructing them from the heterotic string. We explore the moduli space and give the list of the rank reduced gauge symmetry enhancements for nine and eight space-time dimensions. To this end we improve and generalize the algorithm introduced in the previous chapter. This part is based on [3].

In Chapter 5, based on [4,5], we study different orbifold constructions in 7 and 6 space-time dimensions, obtaining the list of gauge groups that are realized not only on the Narain and CHL components but also four other components realized via non-trivial holonomy triples. Finally, we construct the general freezing rules relating the different components.

In Chapter 6 we summarize the conclusions of this thesis, while at the end we include five appendices: A, B and C with complements to the second, third and fifth chapters; and D with the lists of groups of maximal enhancement for the studied theories. Finally, Appendix E provides a summary in French.

Chapter 2

Heterotic string compactifications

A generic point in the moduli space of a compactified theory corresponds to one whose spectrum has Abelian gauge symmetry. To arrive at non-Abelian symmetries we have to choose special regions where there is an enhancement of symmetry. After a brief introduction to heterotic strings and general toroidal compactifications in Sections 2.1 and 2.2, we restrict to one-dimensional compactifications for the rest of this chapter, in order to attack the problem of exploring these special regions explicitly. These compactifications are described by the radius R of the circle and the 16 values of the background field A called Wilson line. This will be the subject of Section 2.3.

There are different configurations of these parameters giving equivalent theories, we say they are related by the T-duality group. Examining the action of this group, we can see that all points in moduli space where there is maximal symmetry enhancement[1], namely to groups that do not have $U(1)$ factors, are fixed points of T-duality. In the simplest cases, such as those listed below, the enhanced symmetry arises at the self-dual radius given by $R_{sd}^2 = 1 - \frac{1}{2}|A|^2$. We explore the action of T-duality and its fixed points in Section 2.3.1. One can have other points of symmetry enhancement, which are fixed points of duality symmetries that involve shifts of Wilson lines on top of the exchange of momentum and winding.

To explore the whole moduli space of circle compactifications, we split the discussion into the situations in which the Wilson line preserves the $E_8 \times E_8$ or $SO(32)$ gauge symmetry, and those where it breaks it. In Section 2.3.2 we explain the former case. The circle direction can give a further symmetry enhancement to $E_8 \times E_8 \times SU(2)$ at radius $R = 1$, and either to $SO(32) \times SU(2)$ at $R = 1$ or to $SO(34)$ at $R = \frac{1}{\sqrt{2}}$. When the Wilson line breaks the $E_8 \times E_8$ or $SO(32)$ gauge symmetry, the pattern of gauge symmetries is very interesting. Not only is it possible to restore the original $E_8 \times E_8$ or $SO(32)$ gauge symmetry for specific values of R and A, but also larger groups of rank 17 can be obtained. In Section 2.3.3 we explicitly work out enhancements of the HO theory to $SO(34)$ at $R^2 = \frac{1}{2}$; $SU(18)$ at $R^2 = \frac{1}{4}$; $E_{p+1} \times SO(32-2p)$ at $R^2 = 1 - \frac{p}{8}$; $E_{p+1} \times SU(16-p)$ at $R^2 = 1 - \frac{8}{16-p}$, and in the HE to $SO(34)$ at $R^2 = \frac{1}{18}$; $SU(18)$ at $R^2 = \frac{1}{9}$; $SO(18) \times E_8$ at $R^2 = \frac{1}{2}$; $SU(2) \times E_8 \times E_8$ at $R^2 = \frac{1}{4}$. We depict slices of the moduli space for different values of R and Wilson lines in several figures, which clarify the analysis and neatly exhibit the curves and points with special properties. The technique used is presented in Section 2.3.4

Finally, switching to a more lattice-oriented approach that will accompany us for the reminder of the thesis, we find in Section 2.4 all the possible maximal enhancement groups, and

[1] "Maximal" stands here for an enhanced semisimple and simply-laced symmetry group.

the point in the fundamental region of moduli space where they arise, using the Generalized Dynkin Diagram of the lattice $\Gamma_{1,17}$ [11, 13].

We have included three appendices. Appendix A.1 collects some known facts about lattices that are used throughout this chapter. Details of the procedures leading to construct the curves of enhancement and more slices of the moduli space are contained in Appendices A.2 and A.3 , respectively.

2.1 Heterotic String

Both in the bosonic and type II strings, the equation of motion for the fields implies that we can separate them in a left and right-moving part. This separation is translated to the commutation of the operators associated to the left-moving and right-moving oscillations. This decoupling between left and right degrees of freedoms allows us to define a new type of consistent theory, the heterotic string, combining the left-moving fields of the bosonic string and the right-moving fields of one of the superstrings. As for all closed strings, the states are just tensorial products of left and right-moving states subject to the level matching constraint.

For consistency, the heterotic string is defined in 10 space-time dimensions. While the right supersymmetric side is just the same as for the type II strings, the left side needs $26 - 10 = 16$ extra degrees of freedom as a consequence of this. There are two standard solutions: adding 16 left-moving bosons or adding 32 left-moving fermions. We will use the former, called the bosonic formulation, as it is the best suited for studying toroidal compactifications. The world-sheet left-moving degrees of freedom are the 10 bosonic fields $X_L^\mu(\tau+\sigma)$ with $\mu = 0, \ldots, 9$ the space-time directions plus 16 internal chiral bosons Y^I with $I = 1, \ldots, 16$. On the right side we have the usual 10 bosonic fields $X_R^\mu(\tau-\sigma)$ and their superpartners $\psi_R^\mu(\tau-\sigma)$. The presence of these fermionic fields guarantees, in combination with a GSO projection, that the spectrum of states is supersymmetric.

The fields X correspond to space-time coordinates while Y correspond to internal degrees of freedom. To achieve a consistent string theory, the Y^I have to satisfy periodic boundary conditions. Since they are chiral bosons (constrained by $\partial_+ Y^I = 0$), the only way of doing this besides taking their associated momentum $p^I = 0$ is by making them periodic. Just as in usual string compactifications (or Kaluza-Klein theory), this periodicity induces a quantization of their momentum, forcing it to live on some compact space Υ_{16}. Moreover, for consistency this space has to be a product of 16 circles of equal radius $R = \sqrt{\alpha'}$ (from now on we are taking $\alpha' = 1$).

On a well defined string theory, the 1-loop partition function presents modular invariance. By computing and demanding it to be invariant under the transformation $\tau \to \tau + 1$, we find that $\frac{1}{2}|p^I|^2 \in \mathbb{Z}$ (Υ_{16} is an even lattice); and by demanding invariance under $\tau \to \frac{-1}{\tau}$ and using the Poisson summation formula, we get $\Upsilon_{16}^* = \Upsilon_{16}$, with Υ_{16}^* the dual lattice. In conclusion the heterotic momentum lies on a very special type of lattice, an Euclidean, even and self-dual one.[2]

[2] As we will see in Section 2.2, in toroidal compactifications of heterotic strings, the momentum will also live on an even and self-dual (but not Euclidean) lattice.

Even self-dual definite lattices exist only for rank multiple of 8. There are two possibilities for rank 16: the lattice of weights of the group $\mathrm{Spin}(32)/\mathbb{Z}_2$ and the lattice of weights of $\mathrm{E}_8 \times \mathrm{E}_8$, denoted respectively by Γ_{16} and $\Gamma_8 \oplus \Gamma_8$ with Γ_N being defined as the set of N-dimensional vectors with integer components together with the set of vectors with half-integer components whose sum is an even number (see Appendix A.1 for details on these lattices). We have then two different supersymmetric heterotic string theories, the so called $\mathrm{Spin}(32)/\mathbb{Z}_2$ heterotic string or HO, with heterotic momentum $p^I \in \Gamma_{16}$ and the $\mathrm{E}_8 \times \mathrm{E}_8$ heterotic string or HE with $p^I \in \Gamma_8 \times \Gamma_8$:

$$\Upsilon_{16} = \begin{cases} \Gamma_8 \oplus \Gamma_8\,, & \text{for (HE)} \\ \Gamma_{16}\,, & \text{for (HO)} \end{cases} . \tag{2.1.1}$$

2.1.1 Spectrum

We now describe the spectrum of the HE and HO theories in the light-cone gauge. Since they are closed strings, the possible states are made by taking the tensorial product between one state from the left sector and one from the right, subject to the level matching condition.

The mass equation is $m_L^2 = N_L + \frac{1}{2}|p_L|^2 - 1$, with N_L the number operator for the left-moving oscillators α_{-n}^{μ} and α_{-n}^{A} and p_L the heterotic momentum.

For $N_L = p_L = 0$ we have the tachyonic vacuum state of bosonic string $|0\rangle$.

On the massless level we have:

- $(N_L = 1,\, p_L = 0)$: $\alpha_{-1}^{\mu}\,|0\rangle$ space-time vector with $\mu = 2, ..., 9$.
- $(N_L = 1,\, p_L = 0)$: $\alpha_{-1}^{I}\,|0\rangle$ 16 Abelian gauge bosons ($U(1)^{16}$ Cartan subalgebra of $E_8 \times E_8$ or $SO(32)$).
- $(N_L = 0, |p_L|^2 = 2)$: $|p_L\rangle = e^{ip^I Y^I}\,|0\rangle$, with p_L a root of $E_8 \times E_8$ or $SO(32)$ (480 non-Abelian gauge bosons).

We have an infinite tower of massive states for bigger values of N or $|p_L|^2$.

The right sector is identical to the other superstrings, the mass equation is

$$m_R^2 = \begin{cases} N_R & \text{in } R \\ N_R - \frac{1}{2} & \text{in } NS \end{cases} ,$$

with N_R the number operator for the right-moving oscillators, $\bar{\psi}_{-n}^{\mu}$ and $\bar{\alpha}_{-n}^{\mu}$. with the zero-point energy being different in the Ramond (R) and Neveu-Schwarz (NS) sectors. We apply the same GSO projection as in the type II strings, requiring for an odd (even) number of fermions on the NS (R) sector; therefore, eliminating the tachyon $|0\rangle_{NS}$ from the spectrum. The massless states are:

- $(N_R = \frac{1}{2})$: $\bar{\psi}_{-\frac{1}{2}}^{\mu}\,|0\rangle_{NS}$ space-time vector with $\mu = 2, ..., 9$.
- $(N_R = 0)$: $|s\rangle_R$ space-time spinor, superpartner of the vector.

We again have an infinite number of massive states with bigger N_R on both the NS and R sectors.

The level matching condition is just $m_L^2 = m_R^2$:

$$N + \frac{1}{2}|p_L|^2 - 1 = \begin{cases} N_R & \text{in } R \\ N_R - \frac{1}{2} & \text{in } NS \end{cases} \tag{2.1.2}$$

The states of heterotic strings will be the product between one state on the left and one on the right that satisfy this equation. The only negative value of the LHS is -1, but the RHS is always bigger than $-\frac{1}{2}$, expelling from the spectrum the tachyon inherited from the bosonic string. The GSO projection on the right sector guarantees that each heterotic string state has a superpartner, making the spectrum $\mathcal{N} = 1$ supersymmetric in 10 dimensions.

The massless states are

- Supergravity sector:
  - $(N_L = 1, p_L = 0, N_R = \frac{1}{2})$: $\alpha_{-1}^{\mu}\bar{\psi}_{-\frac{1}{2}}^{\nu}|0\rangle_{NS} \longrightarrow$ graviton, Kalb-Ramond two-form and dilaton.
  - $(N_L = 1, p_L = 0, N_R = 0)$: $\alpha_{-1}^{\mu}|a\rangle_R \longrightarrow$ gravitino and dilatino.
- Gauge bosons of $SO(32)$ or $E_8 \times E_8$:
  - $(N_L = 1, p_L = 0, N_R = \frac{1}{2})$: $\alpha_{-1}^{I}\bar{\psi}_{-\frac{1}{2}}^{\mu}|0\rangle_{NS} \longrightarrow$ 16 space-time vectors associated to the Cartan subalgebra.
  - $(N_L = 0, |p_L|^2 = 2, N_R = \frac{1}{2})$: $e^{i\vec{p}^I Y^I}\bar{\psi}_{-\frac{1}{2}}^{\mu}|0\rangle_{NS} \longrightarrow$ 480 space-time vectors associated to the roots.
  - $(N_L + \frac{1}{2}|p_L|^2 = 1, N_R = 0)$: $\alpha_{-1}^{I}|a\rangle_R$, $e^{ip^I Y^I}|a\rangle_R \longrightarrow$ the gaugini.

The low energy limit is then 10-dimensional $\mathcal{N} = 1$ supergravity coupled to super Yang-Mills theory with gauge groups $\mathrm{Spin}(32)/\mathbb{Z}_2$ or $\mathrm{E}_8 \times \mathrm{E}_8$. Since both groups have dimension 496, both strings have the same number of massless states. In fact, it is easy to prove that at each massive level the number of states is the same for both theories.

2.2 Toroidal compactifications

As shown in the previous section, the states of the heterotic string live on a 10-dimensional space-time: to make contact with our universe it is necessary to compactify 6 of these dimensions. If the characteristic length of the compact geometry is small enough, the low-energy effective theory will be 4-dimensional. The simplest compactifications of d dimensions occur for d-torus. As we will see, a rich and interesting pattern of gauge symmetries and some subtleties appear already for the smallest values of d. In this section we will describe the compactification on a torus of arbitrary dimension, focusing on the case $d = 1$ for the remainder of this chapter. For a more complete review see [28].

The bosonic part of the action for the non-linear sigma model of heterotic string is given by $S = S_b + S_h$, with

$$S_b = \frac{1}{4\pi}\int d\tau d\sigma\Big(\delta^{\alpha\beta}G_{\mu\nu}\partial_\alpha X^\mu\partial_\beta X^\nu - i\epsilon^{\alpha\beta}B_{\mu\nu}\partial_\alpha X^\mu\partial_\beta X^\nu\Big) \tag{2.2.1}$$

where we have the usual background fields of the gravitational sector: $G_{\mu\nu}$ and $B_{\mu\nu}$ (we are taking $\Phi = 0$) and

$$S_h = \frac{1}{8\pi}\int d\tau d\sigma\Big(\delta^{\alpha\beta}\partial_\alpha Y^I\partial_\beta Y^I - 2i\epsilon^{\alpha\beta}A^I_\mu\partial_\alpha X^\mu\partial_\beta Y^I\Big) \tag{2.2.2}$$

where we have an additional background field A^I_μ, with a space-time index $\mu = 0,\ldots,9$ and a heterotic chiral index $I = 1,\ldots,16$. $\alpha = 1,2$ are the world-sheet coordinates σ and τ. For both actions we gauge fixed the world-sheet metric to $\delta^{\alpha\beta}$ and defined $\epsilon^{01} = 1$.

Now we compactify on a T^d, denoting the compact directions as $i,j = 1,\ldots,d$. We turn on constant background metric G_{ij}, antisymmetric two-form B_{ij} and $U(1)^{16}$ gauge field A^I_i. For simplicity we take the background dilaton to be zero. The d fields A^I_i have null field-strength and are called Wilson lines, we will have one of them for each toroidal direction.

The set of vectors e_i define a basis in the compactification lattice Λ^d such that the internal part of the target space is the d-dimensional torus $T^d = \mathbb{R}^d/\pi\Lambda^d$, they satisfy $G = e^t e$ $\Big(\Rightarrow G_{ij} = e^a{}_i\delta_{ab}e^b{}_j\Big)$. The vectors $\hat{e}_a$ constitute the canonical basis for the dual lattice Λ^{d*}, i.e. $\hat{e}_a{}^i e^a{}_j = \delta^i{}_j$, and thus they obey $\hat{e}^t\hat{e} = G^{-1}$ $\Big(\Rightarrow \hat{e}_a{}^i\delta^{ab}\hat{e}_b{}^j = G^{ij}\Big)$.

The contribution from the internal sector to the world-sheet action (we consider only the bosonic sector here) is

$$\begin{aligned} S &= \frac{1}{4\pi}\int_M d\tau d\sigma\left(\delta^{\alpha\beta}G_{ij} - i\epsilon^{\alpha\beta}B_{ij}\right)\partial_\alpha Y^i\partial_\beta Y^n \\ &\quad + \frac{1}{8\pi}\int_M d\tau d\sigma\left(\delta^{\alpha\beta}\partial_\alpha Y^I\partial_\beta Y^I - 2i\epsilon^{\alpha\beta}A^I_i\partial_\alpha Y^i\partial_\beta Y^I\right), \end{aligned} \tag{2.2.3}$$

(recall we are always taking $\alpha' = 1$), Y^I are chiral bosons and the currents ∂Y^I form a maximal commuting set of the $SO(32)$ or $E_8 \times E_8$ current algebra. The internal string coordinate fields satisfy

$$Y^i(\tau,\sigma+2\pi) \simeq Y^i(\tau,\sigma) + 2\pi w^i\ , \tag{2.2.4}$$

where $w^i \in \mathbb{Z}$ are the winding numbers. It is convenient to define holomorphic $Y^i_L(z)$ and antiholomorphic $Y^i_R(\bar{z})$ fields as

$$Y^i(z,\bar{z}) = \tfrac{1}{\sqrt{2}}\left[Y^i_L(z) + Y^i_R(\bar{z})\right]\ ,\quad z = \exp(\tau+i\sigma)\ ,\ \bar{z} = \exp(\tau - i\sigma)\ , \tag{2.2.5}$$

with Laurent expansion

$$Y^i_L(z) = y^i_L - ip^i_L\, lnz + \cdots\ , \qquad Y^I_L(z) = y^I_L - ip^I\, lnz + \cdots\ , \tag{2.2.6}$$

$$Y^i_R(\bar{z}) = y^i_R - ip^i_R\, ln\bar{z} + \cdots\ , \tag{2.2.7}$$

the dots standing for the oscillators contribution. Then the periodicity condition is

$$Y^i(\tau,\sigma+2\pi)-Y^i(\tau,\sigma)=2\pi\left(\tfrac{1}{\sqrt{2}}\right)(p_L^i-p_R^i)=2\pi w^i\ . \tag{2.2.8}$$

The canonical momentum has components[3]

$$\begin{aligned}\Pi_i &= i\frac{\delta S}{\delta\partial_\tau Y^i}=\frac{1}{2\pi}\left[iG_{ij}\partial_\tau Y^n+B_{ij}\partial_\sigma Y^j+\frac{1}{2}A_i^I\partial_\sigma Y^I\right]\ ,\\ &= \frac{1}{2\pi\sqrt{2}}\left[G_{ij}(p_L^j+p_R^j)+B_{ij}(p_L^j-p_R^j)\right]+\frac{1}{4\pi}A_i^Ip^I\\ \Pi^I &= i\frac{\delta S}{\delta\partial_\tau Y^I}=\frac{1}{4\pi}\left(i\partial_\tau Y^I-A_i^I\partial_\sigma Y^i\right)=\frac{1}{2\pi}\left[p^I-\tfrac{1}{\sqrt{2}}A_i^I(p_L^i-p_R^i)\right]\ .\end{aligned}$$

The chirality constraint on Y^I and the condition of vanishing Dirac brackets between momentum components require the redefinitions $\Pi_I\to\tilde\Pi_I=2\Pi_I$ and $\Pi_i\to\tilde\Pi_i=\Pi_i+\frac{1}{2}A_i^I\tilde\Pi_I$. Integrating over σ, we get the center of mass momenta

$$\pi_i = \int d\sigma\tilde\Pi_i=2\pi\left(\Pi_i+\frac{1}{2}A_i^I\tilde\Pi_I\right)=n_i\in\mathbb{Z}\,, \tag{2.2.9a}$$

$$\pi^I = \int d\sigma\tilde\Pi^I=p^I-A_i^Iw^i\,, \tag{2.2.9b}$$

where we used univaluedness of the wave function in the first line. $\pi_i\equiv n_i$ are integer vectors, while modular invariance requires $\pi^I\in\Gamma_{16}$ or $\Gamma_8\oplus\Gamma_8$, corresponding to the HO or HE theory, respectively.

From these equations we get

$$p_{Ra} = \frac{1}{\sqrt{2}}\hat e_a{}^i\left[n_i-(G_{ij}+B_{ij})w^j-\pi^IA_i^I-\frac{1}{2}A_j^IA_i^Iw^j\right]\ , \tag{2.2.10a}$$

$$p_{La} = \frac{1}{\sqrt{2}}\hat e_a{}^i\left[n_i+(G_{ij}-B_{ij})w^j-\pi^IA_i^I-\frac{1}{2}A_j^IA_i^Iw^j\right]\,, \tag{2.2.10b}$$

$$p^I = \pi^I+w^iA_i^I\,. \tag{2.2.10c}$$

The momentum $\mathbf{p}=(\mathbf{p_R},\mathbf{p_L})$, with $\mathbf{p_R}=p_{Ra}$, $\mathbf{p_L}=(p_{La},p^I)$, transforms as a vector under $O(d,d+16;\mathbb{R})$. It expands the $2d+16$-dimensional momentum lattice $\Gamma_{d,d+16}\subset\mathbb{R}^{2d+16}$, satisfying

$$\mathbf{p}\cdot\mathbf{p}' = p_{La}\delta^{ab}p'_{Lb}+p^Ip'^I-p_{Ra}\delta^{ab}p'_{Rb}=n_i\omega'^i+n'_i\omega^i+\pi^I\pi'^I\in\mathbb{Z}\,, \tag{2.2.11}$$

and

$$\mathbf{p}\cdot\mathbf{p}=\mathbf{p_L}^2-\mathbf{p_R}^2=2w^in_i+\pi^I\pi^I\in2\mathbb{Z}\,, \tag{2.2.12}$$

because π^I is on an even lattice, and therefore $\mathbf{p}$ forms an even $(d,d+16)$ Lorentzian lattice for any values of the background fields called the Narain lattice. In addition, self-duality $\Gamma_{d,d+16}=\Gamma^*_{d,d+16}$ follows from modular invariance [7, 29]. Note that $\mathbf{p_L},\mathbf{p_R}$ depend on $2d+16$

[3]The unusual i factors are due to the use of Euclidean world-sheet metric.

integer parameters n_i, w^i and π^I, and on the background fields G, B and A.

The space of inequivalent lattices and inequivalent backgrounds reduces to

$$\frac{O(d,d+16;\mathbb{R})}{O(d+16;\mathbb{R})\times O(d;\mathbb{R})\times O(d,d+16;\mathbb{Z})}\,, \tag{2.2.13}$$

where $O(d,d+16;\mathbb{Z})$ is the T-duality group (we give more details about it in the next section).

The mass of the states and the level matching condition are respectively given by

$$m^2 = \mathbf{p_L}^2 + \mathbf{p_R}^2 + 2\left(N_L + N_R - \begin{cases} 1 & \text{R sector} \\ \frac{3}{2} & \text{NS sector} \end{cases}\right), \tag{2.2.14a}$$

$$0 = \mathbf{p_L}^2 - \mathbf{p_R}^2 + 2\left(N_L - N_R - \begin{cases} 1 & \text{R sector} \\ \frac{1}{2} & \text{NS sector} \end{cases}\right). \tag{2.2.14b}$$

the difference between $\mathbf{p_L}^2$ and $\mathbf{p_R}^2$ is given by $2w^i n_i + \pi^I\pi^I$, but the sum depends on the background fields:

$$\begin{aligned}\mathbf{p_L}^2 + \mathbf{p_R}^2 = {} & G^{ij}n_i n_j + \left(G_{ij} + G^{kl}C_{jl}C_{ik} + A_i^I A_j^I\right)\omega^i\omega^j + \\ & \left(G^{kl}A_r^I A_l^J + \delta^{IJ}\right)\pi^I\pi^J - 2G^{ik}C_{kj}n_i w^j - 2G^{ik}A_r^I n_i \pi^I + 2\left(G^{ik}C_{ij}A_k^I + A_j^I\right)w^j\pi^I\end{aligned} \tag{2.2.15}$$

where $C_{ij} \equiv B_{ij} + \frac{1}{2}A_i^I A_j^I$.

Because the Narain lattice is even and self-dual, we have modular invariance guaranteed for all toroidal compactifications. It can be shown that, unlike the Euclidean case, these lattices are unique except for a pseudo-orthogonal transformation $O(d,d+16)$. Orthogonal transformations $O(d)\times O(d+16)$ that do not mix $\mathbf{p_R}$ and $\mathbf{p_L}$ produce equivalent lattices, then the moduli space of inequivalent compactifications would be given by $\frac{O(d,d+16)}{O(d)\times O(d+16)}$[4], with the number of moduli equal to

$$\frac{d^2}{2} + 16d = \frac{d(d+1)}{2} + \frac{d(d-1)}{2} + 16d \tag{2.2.16}$$

showing that the moduli space is parameterized by the components of G_{ij}, B_{ij} and the d Wilson lines A_i^I.

2.2.1 Massless spectrum

In compactifications on T^d, the spectrum depends on the background fields. When $(N_L, N_R) = (1, \frac{1}{2})$ there are the same number of massless states at any point in moduli space (we refer to this as sector 1). When $(N_L, N_R) = (0, \frac{1}{2})$, we see from (2.2.10) that there are no massless states for generic values of the metric, B-field and Wilson lines A_i^I, while for certain values of these fields the momenta can lie in the weight lattice of a rank $2d+16$ group $G_L \times G_R$ (we refer to this as sector 2). In this case, there is a subgroup with $|(\mathbf{p_R}, \mathbf{p_L})|^2 = 2$ which can give rise to massless states. Subtracting (2.2.14a) and (2.2.14b) we see that massless states have $\mathbf{p_R} = \mathbf{0}$,

[4]Taking into account the equivalence of points on moduli space due to T-duality, the coset manifold characterizing the distinct backgrounds is $\frac{O(d,d+16)}{O(d)\times O(d+16)\times O(d,d+16;\mathbb{Z})}$.

and thus (unlike in the bosonic string theory), the non-Abelian gauge symmetry comes from the left sector only. The group $G_L \times U(1)_R^d$ in which the massless states transform defines the gauge group of the theory, with G_L a simply-laced group of rank $16+d$ and dimension N, that depends on the point in moduli space (which is spanned by G_{ij}, B_{ij}, A_j^I). Specifically, the $10-d$ dimensional massless bosonic spectrum are given by ($\mu,\nu = 0,\ldots,9-d$; $i,j = 1,\ldots,d$; $I = 1,\ldots,16$):

1. $N_L = 1, N_R = \frac{1}{2}$, $\mathbf{p_L} = \mathbf{p_R} = 0$:

   - Common gravitational sector: $G_{\mu\nu}, B_{\mu\nu}, D$

   $$\alpha_{-1}^{\mu}\bar{\psi}_{-\frac{1}{2}}^{\nu}\left|0,k\right\rangle_{NS} \tag{2.2.17}$$

   - d KK left Abelian gauge vectors: $G_{i\mu} + B_{i\mu} \equiv a_{i\mu}$ and 16 Cartan generators of $SO(32)$ or $E_8 \times E_8$: a_μ^I

   $$\alpha_{-1}^{\hat{I}}\bar{\psi}_{-\frac{1}{2}}^{\mu}\left|0,k\right\rangle_{NS} \tag{2.2.18}$$

     where the index $\hat{I} = (I,i)$ includes both the chiral "heterotic" directions and the compact toroidal ones, labeling the Cartan sector of the gauge group G_L.

   - d KK right Abelian gauge vectors: $G_{i\mu} - B_{i\mu} \equiv \bar{a}_{i\mu}$

   $$\alpha_{-1}^{\mu}\bar{\chi}_{-\frac{1}{2}}^{i}\left|0,k\right\rangle_{NS} \tag{2.2.19}$$

   - $d(d+16)$ scalars: G_{ij}, B_{ij}, a_i^I

   $$\alpha_{-1}^{\hat{I}}\bar{\chi}_{-\frac{1}{2}}^{i}\left|0,k\right\rangle_{NS} \tag{2.2.20}$$

2. $N_L = 0$, $N_R = \frac{1}{2}$, $\boldsymbol{p_L^2} = 2$, $\boldsymbol{p_R} = 0$:

   - $(N-d-16)$ root vectors: a_μ^α

   $$\bar{\psi}_{-\frac{1}{2}}^{\mu}\left|0,k,\pi_\alpha\right\rangle_{NS}\ , \tag{2.2.21}$$

     where α are the roots of G_L (or equivalently the left momenta).

   - $(N-d-16)\times d$ scalars: $a_{\alpha i}$

   $$\bar{\chi}_{-\frac{1}{2}}^{i}\left|0,k,\pi_\alpha\right\rangle_{NS} \tag{2.2.22}$$

The massive states are obtained increasing the oscillation numbers N_L and N_R or choosing $|(p_R,p_L)|^2 \geq 4$.

Due to the uniqueness of Lorentzian self-dual lattices [11] both heterotic theories on T^d can be connected continuously [7, 30], i.e. they belong to the same moduli space. The possible enhanced non-Abelian gauge symmetry groups are those with root lattices admitting an embedding into $\Gamma_{d,d+16}$. Although some theorems on lattice embeddings are known [16], it is a

non-trivial problem to determine which groups admit an embedding[5], as we will study in the next chapter. Here we present a general discussion.

Using that $\mathbf{p_R} = 0$, we get from (2.2.10) that the massless states have left-moving momentum

$$\mathbf{p_L} = \left(\sqrt{2}\,\hat{e}_{ai}w^i, \pi^I + w^i A_i^I\right) , \tag{2.2.23}$$

while their momentum number on the torus is given by

$$n_i = (G_{ij} + B_{ij})\, w^j + \pi^I A_i^I + \frac{1}{2} A_j^I A_i^I w^j . \tag{2.2.24}$$

Note that quantization of momentum number on the torus is a further condition to be imposed on top of $\mathbf{p_L}^2 = 2$.

In the absence of Wilson lines $A_i^I = 0$, the d torus directions decouple from the 16 chiral "heterotic directions" Y^I; $p^I = \pi^I$ is a vector of the weight lattice of $SO(32)$ or $E_8 \times E_8$ and then $|p^I|^2 \in 2\mathbb{N}$. The only possible massless states then have either momenta $\mathbf{p_L} = (0, \pi^I)$ with $|\pi|^2 = 2$, or $\mathbf{p_L} = (\sqrt{2}\, e_{aj} w^j, 0)$ with $w^i G_{ij} w^j = 1$ (and additionally $n_i w^i = 1$). The former are the root vectors of $SO(32)$ or $E_8 \times E_8$, while the latter have solutions only for certain values of the metric and B-field on the torus and lead to the same groups as in the (left sector of) bosonic string theory, namely all simply-laced groups H of rank d. The total gauge group is then $SO(32) \times H \times U(1)_R^d$ or $E_8 \times E_8 \times H \times U(1)_R^d$. For $d = 1$, i.e. a circle compactification, H is $SU(2)$ at $G_{11} = R^2 = 1$, and $U(1)$ for any other value of the radius. For compactifications on T^2, the possible groups of maximal enhancement (see footnote 1) are $SO(32) \times SU(2)_L^2 \times U(1)_R^2$ (for a diagonal metric with both circles at the self-dual radius and no B-field) or $SO(32) \times SU(3)_L \times U(1)_R^2$ (equivalently $SO(32) \to E_8 \times E_8$). See [32] for details.

Turning on Wilson lines, the pattern of gauge symmetries is more complicated, and also richer. In the sector with zero winding numbers, $w^i = 0$, we have $p^I = \pi^I$ as before, but now requiring a quantized momentum number imposes $\pi^I A_i^I \in \mathbb{Z}$ (see (2.2.24)) which, for a generic Wilson line breaks all the gauge symmetry leaving only $\pi^I = 0$, which corresponds to the $U(1)^{16}$ Cartan subgroup. The opposite situation corresponds to $A_i^I \in \Gamma_g^*$ [6]. For HE, since $\Gamma_g^* = \Gamma_8 \oplus \Gamma_8$, A_i^I is innocuous because the momentum (2.2.10) remains the same if one substitutes $n_i \to n_i + \pi^I A_i^I - \frac{1}{2} A_j^I A_i^I w^j$ and $\pi^I \to \pi^I - w^i A_i^I$; and thus the pattern of gauge symmetries is as for no Wilson line. In the HO theory, the same conclusions hold if $A \in \Gamma_{16}$, but one has the more interesting possibility $A \in \Gamma_v$ or $A \in \Gamma_c$, where the $SO(32)$ symmetry is not broken, and the 16 chiral heterotic directions can be combined with the torus ones, giving larger groups which are not products. When analyzing the sectors with non-zero winding numbers we can have an incredible variety of enhancements for $A_i^I \notin \Gamma_g^*$, as we will show in the next section for the circle case and in the next chapter for T^2.

Let us discuss the different groups that can arise in points of moduli space where the enhancement is maximal. In that case, the matrices that embed the internal sector of the heterotic theory on T^d into a $16 + d$-dimensional bosonic theory are related to the Cartan

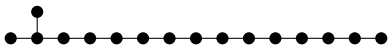
[5] A preliminary attempt can be found in [31].

[6] We denote Γ_g^* the dual of the root lattice, and one has $\Gamma_g^* = \Gamma_8 \times \Gamma_8$ for HE and $\Gamma_g^* = \Gamma_w = \Gamma_{16} + \Gamma_v + \Gamma_c$ for HO (see Appendix A.1 for more details).

matrix $\mathcal{C}$ by [28]

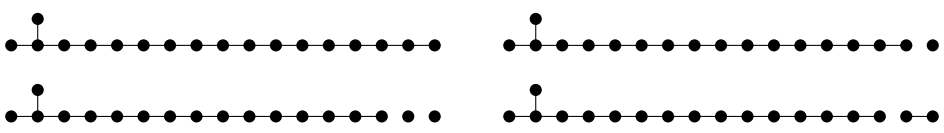
$$\begin{pmatrix} (G+\frac{1}{2}A^IA^I)_{ij} & \frac{1}{2}A_i{}^I \\ \frac{1}{2}A^I{}_j & G_{IJ} \end{pmatrix} = \frac{1}{2}\mathcal{C}_{\hat{I}\hat{J}} \ , \qquad \begin{pmatrix} B_{ij} & \frac{1}{2}A_i{}^I \\ -\frac{1}{2}A^I{}_j & B_{IJ} \end{pmatrix} = \begin{cases} \frac{1}{2}\mathcal{C}_{\hat{I}\hat{J}} & \text{for } \hat{I}<\hat{J} \\ -\frac{1}{2}\mathcal{C}_{\hat{I}\hat{J}} & \text{for } \hat{I}>\hat{J} \\ 0 & \text{for } \hat{I}=\hat{J} \end{cases} \tag{2.2.25}$$

One can then view the possible maximal enhancements from Dynkin diagrams. Let us first consider Wilson lines that do not break the original gauge group, i.e. $A \in \Gamma_g^*$. We start with the HO theory. The Dynkin diagram of $SO(32)$ is



The Dynkin diagrams of the gauge symmetry groups arising at points of maximal enhancement in the compactification of the HO theory on T^d have d extra nodes, with or without lines in between. Since the resulting groups have to be in the ADE class (they are all simply laced), one cannot add nodes with lines on the left side. Therefore, the nodes should be added on the right side, linked or not to the last node, and additionally add lines linking them to each other, or not. For one dimensional compactifications ($d=1$), the only possibilities are

corresponding respectively to $SO(32)\times SU(2)$ and $SO(34)$. Since a line in the Dynkin diagram means that the new simple root is not orthogonal to the former one, then the Cartan matrix for this situation should have an off-diagonal term in the row corresponding to the new node and the column of the previous node, which according to (2.2.25) means that there is a non-zero Wilson line. Thus, no Wilson line (or a line in Γ_{16}, which is equivalent to no Wilson line) gives the enhancement group $SO(32)\times SU(2)$ and, as explained above, this enhancement works as in the bosonic theory, at $R=1$. The enhancement symmetry group $SO(34)$ is obtained with a Wilson line in the vector or negative-chirality spinor conjugacy classes, and will be presented in detail in Section 2.3.3.1. For compactifications on T^d, the d extra nodes give as largest enhancement symmetry group $SO(32+2d)$, and this happens when Wilson lines in all directions are turned on. For less symmetric Wilson lines one gets smaller groups, and it is easy to see from the Dynkin diagrams what are all the possible groups. Here we draw all the possibilities for $d=2$ only



corresponding respectively to $SO(36)$, $SO(34)\times SU(2)$, $SO(32)\times SU(2)^2$ and $SO(32)\times SU(3)$.

For the HE theory, the situation is less rich in the cases in which the dimension of the resulting group is larger than that of $E_8\times E_8$. As we explained above, since $\Gamma_g^*=\Gamma_8\oplus\Gamma_8$, a

Wilson line that preserves the $E_8 \times E_8$ symmetry should be in the lattice, and thus equivalent to no Wilson line. This can also be seen from the Dynkin diagram of $E_8 \times E_8$

where we see immediately that the extra nodes cannot be linked to any of the E_8's, as any extra line would get us away from ADE. Then the possible enhancements are groups which are products of the form $E_8 \times E_8 \times H$, where H is any semisimple group of rank d, and each H arises at the same point in moduli space as in the compactifications of the bosonic theory on T^d [32]. However, maximal enhancement can still be obtained by breaking one of the E_8 to $SO(16)$, and then the richness of the $SO(32)$ case is recovered (e.g. enhancement to $SO(18) \times E_8$).

Points of enhancement are fixed points of some $O(d, d+16; \mathbb{Z})$ symmetry. Enhancement groups that are not semisimple, i.e. that contain $U(1)$ factors, arise at curves, surfaces or hypersurfaces in moduli space. On the contrary, maximal enhancement occurs at isolated points in moduli space. This is developed in Section 2.3.1 for some compactifications on a circle, to which we now turn.

2.3 Compactifications on a circle

In this section we will study in great detail the case of circle compactifications. We start by specializing the previous chapter observations to $d = 1$.

The momentum components (2.2.10) are[7]

$$\begin{aligned} p_R &= \frac{1}{\sqrt{2}R}\left[n - R^2 w - \pi \cdot A - \frac{1}{2}|A|^2 w\right], \\ p_L &= \frac{1}{\sqrt{2}R}\left[n + R^2 w - \pi \cdot A - \frac{1}{2}|A|^2 w\right], \\ p^I &= \pi^I + w A^I, \end{aligned} \tag{2.3.1}$$

where $|A|^2 = A^I A^I = A\kappa A^t$.[8] The massless states, which satisfy $\mathbf{p_R} = 0$, have left-moving momenta

$$\mathbf{p_L} = (\sqrt{2}Rw, \pi^I + wA^I) = (\sqrt{2}Rw, p^I), \tag{2.3.2}$$

and momentum number on the circle

$$n = \left(R^2 + \frac{1}{2}|A|^2\right) w + \pi \cdot A. \tag{2.3.3}$$

The condition $|\mathbf{p_L}|^2 = 2$ can be written in the following form, that we shall use

$$|\pi + wA|^2 = 2(1 - w^2 R^2). \tag{2.3.4}$$

[7] From now on, suppressed indices in p are orthonormal indices, i.e. $p_R \equiv p_{Ra}, p_L \equiv p_{La}$.

[8] We are abusing notation, as $|A|^2 = A\kappa A$ is not a scalar under reparameterizations of the circle coordinate, i.e. our definition is $|A|^2 = A_i^I A_i^I$ where i here is just the circle coordinate. The scalar quantity is $\mathbb{A}^2 = |A|^2/R^2$ (see (2.3.6) below).

In the sector $\mathbf{p_L} = 0$ one has $n = w = \pi^I = 0$, and the massless spectrum corresponds to the common gravitational sector and 18 Abelian gauge bosons: 16 from the Cartan sector of $E_8 \times E_8$ or $SO(32)$ and 2 KK vectors, forming the $U(1)^{18}$ gauge group.

The condition $\mathbf{p_L}^2 = 2$ can be achieved in two possible ways:

1) $\mathbf{p_L} = (0, p^I)$, with $|p^I|^2 = 2$,

2) $\mathbf{p_L} = (\pm s, p^I)$, with $0 < s \leq \sqrt{2}$, $s^2 + |p^I|^2 = 2$.

From (2.3.2) we see that sector 1 has $w = 0$ and then (2.3.1) implies $p^I = \pi^I$. The condition on the norm says that these are the roots of $SO(32)$ or $E_8 \times E_8$. But as explained in the previous section, one has to impose further that $n \in \mathbb{Z}$ and thus from (2.3.3), $\pi \cdot A \in \mathbb{Z}$. We divide the discussion into two cases, one in which this condition does not break the $SO(32)$ or $E_8 \times E_8$ symmetry, and the second one in which it does. This distinction is useful to understand the enhancement process but, as we will see, is somewhat artificial: all enhancement groups, including those with $SO(32)$ or $E_8 \times E_8$ as subgroups, can be achieved with Wilson lines that are not in the dual lattice by appropriately choosing the radius.

2.3.1 T-duality

In this section we discuss the action of T-duality in the heterotic string compactified on a circle. By T-duality we mean the action of certain type of transformations in $O(1, 17, \mathbb{Z})$ that relate a given heterotic theory with 16-dimensional lattice Γ, compactified on a circle of radius R and Wilson line A, to another heterotic theory with lattice Γ', compactified on a circle of radius R' and Wilson line A'.

The "generalized metric" of the circle, given by a 18×18 scalar matrix, is

$$\mathcal{M} = \begin{pmatrix} R^2(1 + \frac{1}{2}\mathbb{A}^2)^2 & -\frac{1}{2}\mathbb{A}^2 & (1 + \frac{1}{2}\mathbb{A}^2)A \\ -\frac{1}{2}\mathbb{A}^2 & \frac{1}{R^2} & -\frac{1}{R^2}\,A \\ (1 + \frac{1}{2}\mathbb{A}^2)\,A^t & -\frac{1}{R^2}\,A^t & \mathrm{I} + \frac{1}{R^2}A^tA \end{pmatrix} , \tag{2.3.5}$$

where we have defined the scalar

$$\mathbb{A}^2 \equiv \frac{|A|^2}{R^2} . \tag{2.3.6}$$

This is a symmetric element of $O(1, 17)$, accounting for the degrees of freedom of the $\frac{O(1,17)}{O(17)}$ coset.

As in [33], $\mathbf{p} = (p_R; p_L, p^I)$ can be expanded as

$$\mathbf{p} = w\overline{k} + nk + \pi \cdot l , \tag{2.3.7}$$

with basis

$$k = \frac{1}{\sqrt{2}}\left(\frac{1}{R}; \frac{1}{R}, 0\right), \quad \overline{k} = \frac{1}{\sqrt{2}}\left(-R - \frac{A^2}{2R}; R - \frac{A^2}{2R}, \sqrt{2}A^I\right), \quad l^I = \left(-\frac{A^I}{\sqrt{2}R}; -\frac{A^I}{\sqrt{2}R}, u^I\right) . \tag{2.3.8}$$

Here u^I is a Cartesian 16-dimensional basis vector. The inner product is taken with the Lorentzian metric $(-; +, \ldots, +)$. Thus $k \cdot k = \overline{k} \cdot \overline{k} = 0$, $k \cdot \overline{k} = 1$, $l^I \cdot l^J = \delta^{IJ}$, $k \cdot l^I = \overline{k} \cdot l^I = 0$. We

define the charge vector as the $O(1,17)$ vector

$$|Z\rangle = |w, n; \pi^I\rangle \,. \tag{2.3.9}$$

The change of basis to **p** is easily read from (2.3.7). In the rest of the thesis we will work extensively on this basis.

The mass formula (2.2.14a) and level matching condition (2.2.14b) read

$$m^2 = 2\left(N_L + N_R - \begin{cases} 1 & \text{R sector} \\ \frac{3}{2} & \text{NS sector} \end{cases}\right) + \langle Z|\mathcal{M}|Z\rangle \ , \tag{2.3.10}$$

$$0 = 2\left(N_L - N_R - \begin{cases} 1 & \text{R sector} \\ \frac{1}{2} & \text{NS sector} \end{cases}\right) + \langle Z|Z\rangle \ , \tag{2.3.11}$$

respectively, with the inner product defined by the $O(1,17)$ invariant metric

$$\eta = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & \delta_{IJ} \end{pmatrix} \tag{2.3.12}$$

and giving

$$\langle Z'|Z\rangle = w'n + n'w + \pi' \cdot \pi \,. \tag{2.3.13}$$

Note that these equations are invariant under the T-duality group $O(1,17;\mathbb{Z})$ acting as

$$|Z\rangle \to O\eta|Z\rangle \ , \quad \mathcal{M} \to O\mathcal{M}O^t \ , \quad O \in O(1,17,\mathbb{Z}) \ . \tag{2.3.14}$$

The group $O(1,17;\mathbb{Z})$ is generated by:

- Λ-parameter shifts associated to the addition of a vector Λ^I to the Wilson line

$$O_\Lambda = \begin{pmatrix} 1 & -\frac{1}{2}|\Lambda|^2 & \Lambda \\ 0 & 1 & 0 \\ 0 & -\Lambda^t & 1_{16\times 16} \end{pmatrix} \ , \quad \Lambda \in \Upsilon_{16} \,, \tag{2.3.15}$$

- Factorized duality, which is the generalization of the $R \to 1/R$ circle duality, of the form

$$O_D = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1_{16\times 16} \end{pmatrix} \ , \tag{2.3.16}$$

The transformation of the charges under the action of O_Λ is

$$|w', n'; \pi'^I\rangle = \left|w, n - \tfrac{1}{2}|\Lambda|^2 + \Lambda \cdot \pi; \pi^I - \Lambda^I w\right\rangle \ . \tag{2.3.17}$$

The duality generated by O_D is the usual T-duality transformation exchanging momentum

and winding numbers

$$|w', n'; \pi'^I\rangle = |n, w; \pi^I\rangle \ . \tag{2.3.18}$$

Since π stays untouched, this duality is possible if $\Gamma' = \Gamma$. Its action on the background fields can be worked out from the generalized metric (2.3.5).The action of O_D transforms it into

$$\mathcal{M}' = O_D \mathcal{M} O_D^t = \mathcal{M}^{-1} = \begin{pmatrix} \frac{1}{R^2} & -\frac{1}{2}\mathbb{A}^2 & -\frac{1}{R^2}\,A \\ -\frac{1}{2}\mathbb{A}^2 & R^2(1+\frac{1}{2}\mathbb{A}^2)^2 & (1+\frac{1}{2}\mathbb{A}^2)A \\ -\frac{1}{R^2}\,A^t & (1+\frac{1}{2}\mathbb{A}^2)\,A^t & \mathrm{I} + \frac{1}{R^2}A^t A \end{pmatrix} , \tag{2.3.19}$$

and thus we get

$$A' = -\frac{A}{R^2(1+\frac{1}{2}\mathbb{A}^2)} \ , \qquad R' = \frac{1}{R\,(1+\frac{1}{2}\mathbb{A}^2)} \quad (\Rightarrow \ \frac{A'}{R'} = -\frac{A}{R}\,)$$

in agreement with the heterotic Buscher rules for scalars [34]. We get that a background has $R' = R$ for

$$R_{\rm sd}^2 = 1 - \frac{1}{2}|A|^2 \quad (\Rightarrow \ R' = R\,,\ A' = -A\,) \tag{2.3.20}$$

Additionally, if $2A \in \Gamma'$, then $A' = -A \sim A$, and therefore the background is fully self-dual, satisfying $\mathcal{M} = \mathcal{M}^{-1}$ up to discrete transformations.

All the examples of enhancement that we will discuss in Section 2.3.3 except for 2.3.3.8 satisfy the self-duality condition (2.3.20). By performing a Λ-shift to the Wilson line of 2.3.3.8 we can bring it to the equivalent one $A = \left((-\frac{3}{4})_2, (\frac{1}{4})_6, 0_8\right)$, which satisfies (2.3.20).

For Wilson lines with only one non-zero component, we have that the fixed "points" of this symmetry correspond actually to lines of non-maximal enhancement symmetry where the Wilson lines are functions of the radius ($A = A(R_{\rm sd})$), and are such that $A \sim A_{\rm sd}$, with $|A_{\rm sd}|^2 = 2(1 - R_{\rm sd}^2)$.

2.3.2 Enhancement of $SO(32)$ or $E_8 \times E_8$ symmetry

If we want the condition $\pi \cdot A \in \mathbb{Z}$ not to select a subset of the possible π^I in the root lattice, or in other words not to break the $SO(32)$ or $E_8 \times E_8$ gauge symmetry, we have to impose

$$A \in \Gamma_g^* \,, \tag{2.3.21}$$

with

$$\Gamma_g^* = \Gamma_8 \times \Gamma_8 \ \text{ for } E_8 \times E_8 \qquad \text{or} \qquad \Gamma_g^* = \Gamma_w = \Gamma_{16} + \Gamma_v + \Gamma_c \ \text{ for } SO(32)\,.$$

We restrict to this case now, and leave the discussion of the possible symmetry breakings to the next section.

Sector 2 contributes states only at radii $R^2 = s^2/(2w^2)$. The momentum number of these

states given in (2.3.3) becomes

$$n = \frac{1}{2}\left(\frac{s^2}{w} + |A|^2 w\right) + \pi \cdot A = \frac{1}{w}\left(1 - \frac{|\pi|^2}{2}\right) \in \mathbb{Z}\,, \tag{2.3.22}$$

where in the last equality we have used (2.3.2) and $|\mathbf{p_L}|^2 = 2$.

If $A \in \Gamma$,[9] the condition $|p^I|^2 < 2$ can only be satisfied for $p^I = \pi^I + wA^I = 0$. Then we have $s^2 = 2$ and the quantization condition is: $\frac{1}{w} + \frac{1}{2}|A|^2 w \in \mathbb{Z}$. One has $\frac{1}{2}|A|^2 \in \mathbb{Z}$, and thus the only way to satisfy it is with $w = \pm 1$ and $\pi = \mp A$ which gives two extra states at $R = 1$, with momentum number $n = \pm(1 - \frac{1}{2}|A|^2)$.

The condition $0 \neq |p^I|^2 < 2$ is only possible if A is not in the root lattice. And as it is required to be in the weight lattice, this possibility arises in the HO theory only, for $A \in \Gamma_v$ or $A \in \Gamma_c$. For $A \in \Gamma_v$, $\pi \cdot A \in \mathbb{Z}$ for $\pi \in \Gamma_g$ and $\frac{1}{2}|A|^2 = \frac{1}{2} \pmod 1$, so the only option is $s = 1$, giving extra states with $w = \pm 1$ at $R = 1/\sqrt{2}$. These states enhance $SO(32) \times U(1)$ to $SO(34)$. We present an explicit example of this case in Section 2.3.3.1. For $A \in \Gamma_c$, $\pi \cdot A \in \mathbb{Z}$ for $\pi \in \Gamma_g$ but now $\frac{1}{2}|A|^2 \in \mathbb{Z}$ and thus we cannot satisfy the quantization condition (2.3.22) this way. However $\pi \cdot A = \frac{1}{2} \pmod 1$ for $\pi \in \Gamma_s$ and thus we recover that for these Wilson lines there is an enhancement to $SO(34)$ at $R = 1/\sqrt{2}$ as well, by states with $w = \pm 1$. Note that $A \in \Gamma_c$ is equivalent by a Λ-shift with $\Lambda \in \Gamma_s$ to $A \in \Gamma_v$. As we can see from (2.3.17), by this shift the winding number remains invariant, while $\pi \in \Gamma_s$ gets shifted to $\pi' \in \Gamma_g$.

We conclude that in circle compactifications with Wilson lines that do not break the original $SO(32)$ or $E_8 \times E_8$ groups the pattern of gauge symmetry enhancement is (we give here only the groups on the left-moving side):

- $E_8 \times E_8 \times U(1) \to E_8 \times E_8 \times SU(2)$ at $R = 1$ if $A \in \Gamma_8 \oplus \Gamma_8$
- $SO(32) \times U(1) \to SO(32) \times SU(2)$ at $R = 1$ if $A \in \Gamma_{16}$, or
- $SO(32) \times U(1) \to SO(34)$ at $R = \frac{1}{\sqrt{2}}$ if $A \in \Gamma_v$ or $A \in \Gamma_c$

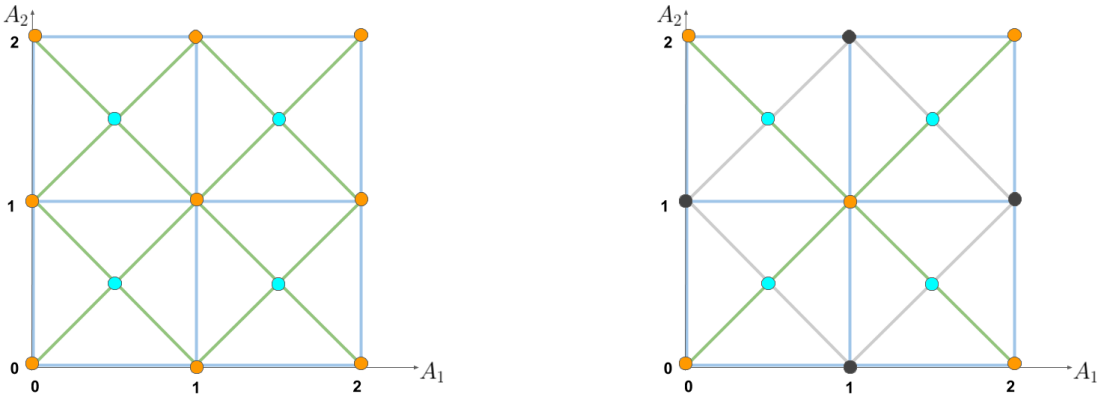
In the following figures we show slices of the moduli space. To exhibit the increase in the number of possible enhancement groups as the radius decreases and more winding numbers contribute, as well as the symmetries in the Wilson lines, we present figures 2.1, 2.2, 2.3, 2.4 and 2.5 corresponding to compactification on a circle of generic radius $R^2 > 1$ and at $R^2 = 1$, $R^2 = \frac{3}{4}$, $R^2 = \frac{1}{2}$ and $R^2 = \frac{1}{4}$, respectively.[10] The circles in figures 2.3, 2.4 and 2.5 reflect the dependence on $|A|^2$ and invariance under rotations. Two dimensional slices given by one parameter in the Wilson line and the radial direction are shown in figures 2.6 and 2.7. More figures of slices of moduli space are given in Appendix A.3.

The first item above corresponds to the red points in figures 2.2b and 2.6b, while the second and third ones correspond, respectively to the red and green points in figures 2.2a, 2.4a and 2.6a. Note that there are also red points in figure 2.5, but as we will see, these arise in a

[9] By Γ we mean Γ_{16} or $\Gamma_8 \oplus \Gamma_8$, according to which heterotic theory one is looking at.

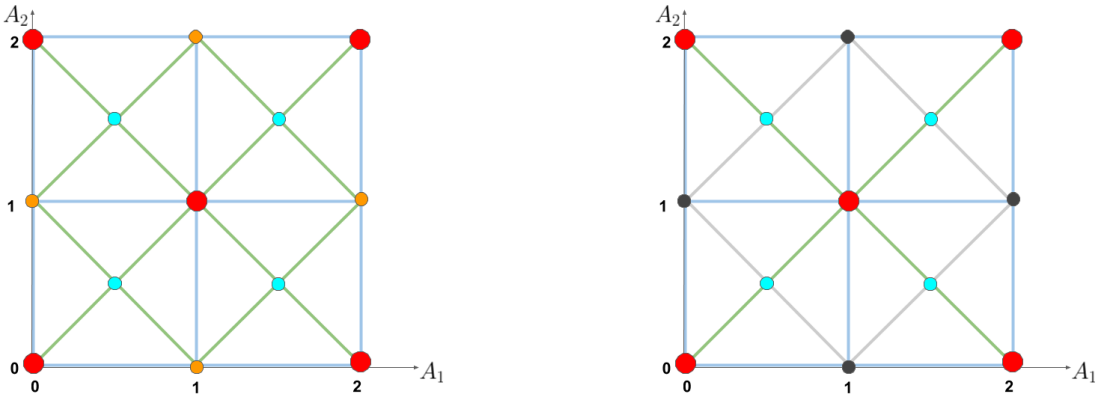
[10] For the HE theory, the Wilson lines chosen do not break the second E_8 factor and therefore we display the unbroken gauge group corresponding to the circle and first E_8 directions. Figure 2.1b can be found in [35].

different way as above, by a combination of breaking and enhancement. In the next section we will show how the enhancement at some of the other special points in the figures arise.



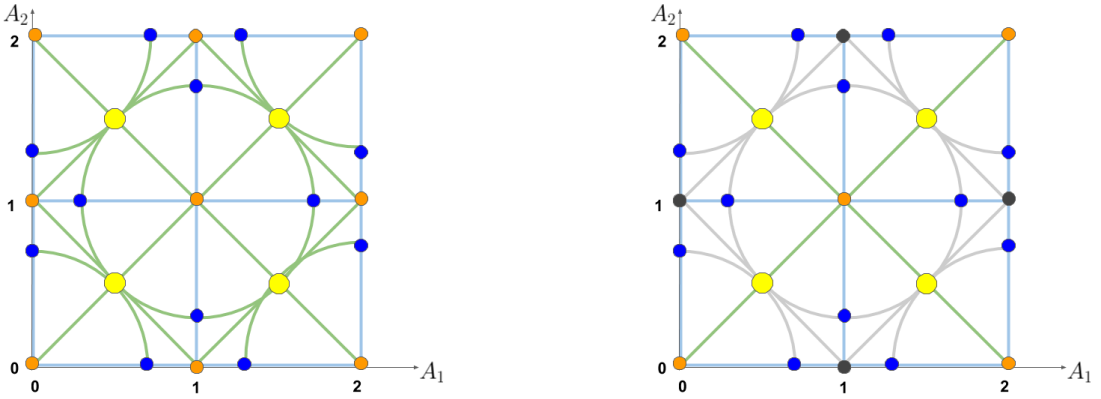
(a) HO theory **(b)** HE theory

Figure 2.1: Enhancement groups on the left sector of the heterotic theory on the slice of moduli space defined by $A^{3,\dots,16} = 0$, $R = R_0$ with a generic $R_0 > 1$



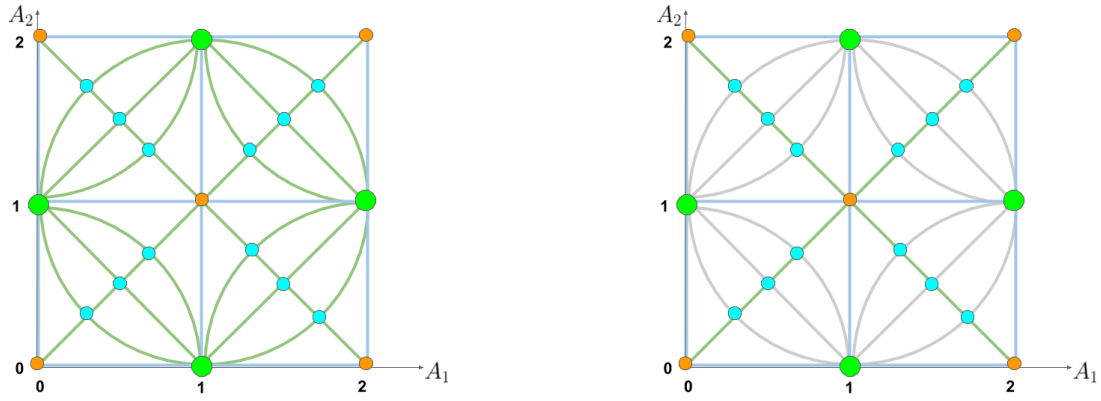
(a) HO **(b)** HE theory

Figure 2.2: Enhancement groups on the left sector of the heterotic theory on the slice of moduli space defined by $A^{3,\dots,16} = 0$, $R = 1$.



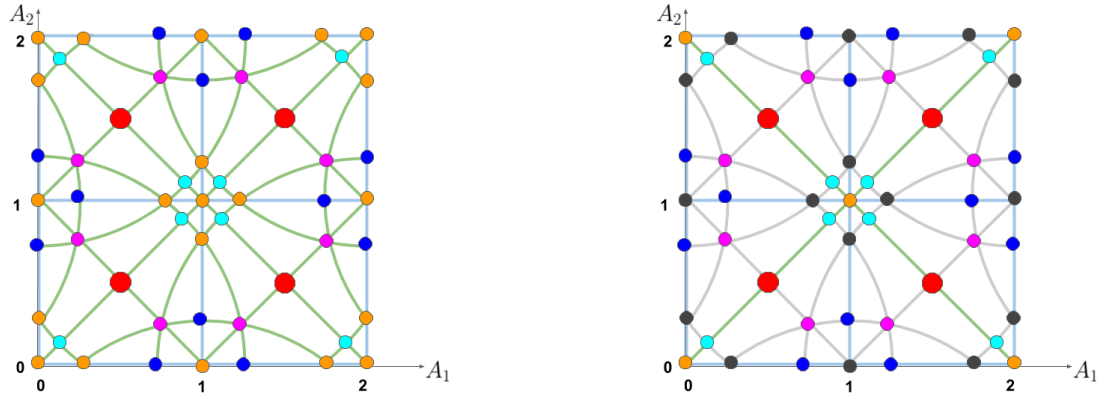
(a) HO theory **(b)** HE theory

Figure 2.3: Enhancement groups on the left sector of the heterotic theory on the slice of moduli space defined by $A^{3,\dots,16} = 0$, $R^2 = 3/4$.



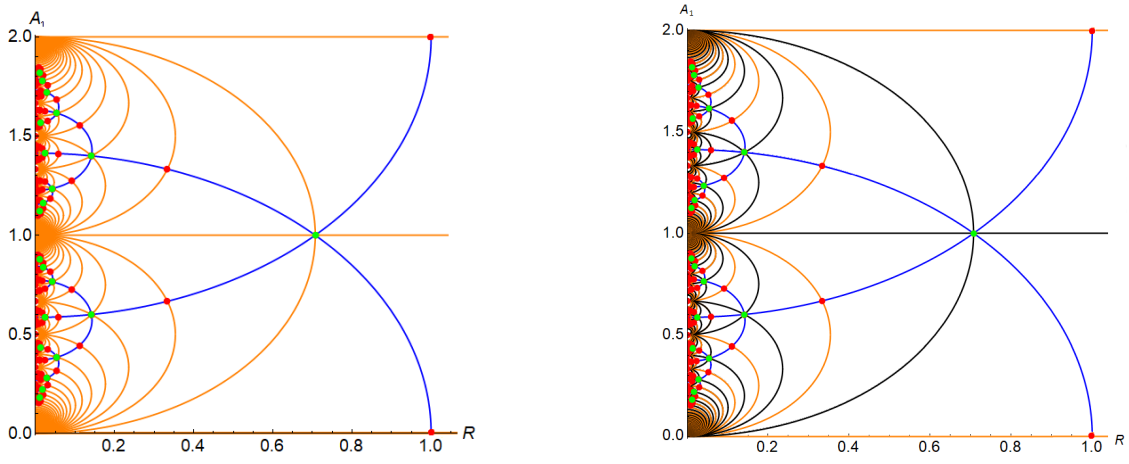
(a) HO theory **(b)** HE theory

Figure 2.4: Enhancement groups on the left sector of the heterotic theory on the slice of moduli space defined by $A^{3,\ldots,16} = 0$, $R^2 = 1/2$.



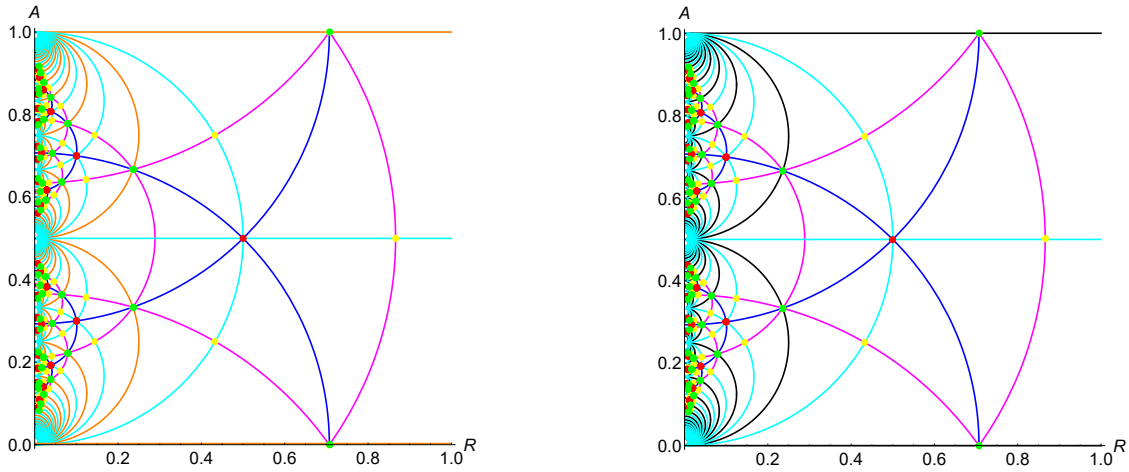
(a) HO theory **(b)** HE theory

Figure 2.5: Enhancement groups on the left sector of the heterotic theory on the slice of moduli space defined by $A^{3,\ldots,16} = 0$, $R^2 = 1/4$.



(a) HO theory **(b)** HE theory

Figure 2.6: Enhancement groups on the left sector of the heterotic theory on the slice of moduli space defined by $A^{2,\ldots,16} = 0$.



(a) HO theory **(b)** HE theory

Figure 2.7: Enhancement groups on the left sector of the heterotic theory on the slice of moduli space defined by A, R, with Wilson line $A^I = (A, 0_7, A+1, 0_7)$.

HO theory:

(2.1a to 2.5a) $\rightarrow$
- $U(1)^3 \times SO(28)$
- $U(1)^2 \times SU(2) \times SO(28)$
- $U(1)^2 \times SO(30)$

(2.6a) $\rightarrow$ $U(1)^2 \times SO(30)$

(2.7a) $\rightarrow$
- $U(1)^2 \times SU(2) \times SO(28)$
- $U(1) \times SU(2)^2 \times SO(28)$
- $U(1) \times SU(3) \times SO(28)$
- $U(1) \times SU(2) \times SO(30)$
- $U(1) \times SO(32)$
- $SU(2) \times SU(3) \times SO(28)$
- $SU(2) \times SO(32)$
- $SO(34)$

HE theory:

(2.1b to 2.5b) $\rightarrow$
- $U(1)^3 \times SO(12) \times E_8$
- $U(1)^2 \times SU(2) \times SU(12) \times E_8$
- $U(1)^2 \times SO(14) \times E_8$
- $U(1)^2 \times E_7 \times E_8$

(2.6b) $\rightarrow$ $U(1)^2 \times SO(14) \times E_8$

(2.7b) $\rightarrow$
- $U(1)^2 \times SU(2) \times SO(12) \times E_8$
- $U(1) \times SU(2) \times E_7 \times E_8$
- $U(1) \times SU(3) \times SO(12) \times E_8$
- $U(1) \times SU(2) \times SO(14) \times E_8$
- $U(1) \times SO(16) \times E_8$
- $U(1) \times E_8 \times E_8$
- $SU(3) \times E_7 \times E_8$
- $SU(2) \times E_8 \times E_8$
- $SO(18) \times E_8$

2.3.3 Enhancement-breaking of gauge symmetry

Whenever the Wilson line is not in the dual root lattice, part or all of the original $SO(32)$ or $E_8 \times E_8$ symmetry is broken. However, this does not imply that no symmetry enhancement from the circle direction is possible. The pattern of gauge symmetries can still be rich. We denote these cases enhancement-breaking of gauge symmetry. This nomenclature can be confusing however: for specific values of R and A, there is the possibility that the symmetry enhancement is so large that it restores the original $SO(32)$ or $E_8 \times E_8$ symmetry, or even leads to a larger group of rank 17. This means that we can have a maximal enhancement even if the Wilson line is not in the dual root lattice, either to the groups listed at the end of the previous section, or to any other simply-laced, semisimple group of rank 17, such as for example $SO(18) \times E_8$.

The massless states for an arbitrary Wilson line are the following:

Sector 1 has $w = 0$ (and thus $p^I = \pi^I$) and consists of the roots of $SO(32)$ or $E_8 \times E_8$ satisfying $\pi \cdot A \in \mathbb{Z}$, which form a subgroup $H \subset SO(32)$ or $H \subset E_8 \times E_8$. We give examples of Wilson lines preserving $U(1) \times SU(16) \subset SO(32)$, $SO(2p) \times SO(32-2p) \subset SO(32)$, $U(1) \times SU(p) \times SO(32-2p) \subset SO(32)$, $U(1) \times SU(9) \times SO(16) \subset E_8 \times E_8$, $U(1)^2 \times SU(8) \times SU(8) \subset E_8 \times E_8$, $SO(16) \times E_8 \subset E_8 \times E_8$, $SU(2) \times E_7 \times E_8 \subset E_8 \times E_8$ in the following sections.

Sector 2 contains states only at radii $R^2 = s^2/(2w^2)$. Quantization of momentum gives the condition (2.3.22).If there are states in this sector, there is an enhancement of $H \times U(1)$ to $H \times SU(2)$ (where the $SU(2)$ can be on the circle direction or along some direction mixing the circle with the heterotic directions) or to a group that is not a product, like for example enhancement of $SO(16) \times U(1)$ to $SO(18)$, as we will show in detail.

On figures 2.6 to 2.11 and A.1 to A.16 sector 1 is represented by the horizontal lines and sector 2 by the curves.

Now we show explicitly how the groups mentioned in sector 1 get enhanced respectively to $SO(34)$ at $R^2 = \frac{1}{2}$; $SU(18)$ at $R^2 = \frac{1}{4}$; $E_{p+1} \times SO(32-2p)$ at $R^2 = 1 - \frac{p}{8}$; $E_{p+1} \times SU(16-p)$ at $R^2 = 1 - \frac{8}{16-p}$ in the HO theory, and $SO(34)$ at $R^2 = \frac{1}{18}$; $SU(18)$ at $R^2 = \frac{1}{9}$; $SO(18) \times E_8$ at $R^2 = \frac{1}{2}$; $SU(2) \times E_8 \times E_8$ at $R^2 = \frac{1}{4}$ in the HE.

- Explicit examples for the HO theory

Here we present some examples of symmetry enhancement-breaking. The roots of $SO(32)$ are given by

$$SO(32) \ : \quad (\underline{\pm 1, \pm 1, 0^{14}}) \,, \tag{2.3.23}$$

where underline means all possible permutations of the entries.

2.3.3.1 $U(1) \times SO(32) \to SO(34)$

Consider the HO theory compactified on a circle of radius $R = 1/\sqrt{2}$ with a Wilson line $A = (1, 0, \ldots, 0) \in \Gamma_v$. The states with $p_R = 0$ have left-moving momenta

$$\mathbf{p_L} = (w,\, \pi^I + \delta_1^A w \,) \,, \tag{2.3.24}$$

where the first entry corresponds to the circle direction. In sector 1, with $w = 0$, all the momenta satisfy $|\pi^I|^2 = 2$ and $\pi \cdot A \in \mathbb{Z}$. The last condition holds for any $\pi^I \in \Gamma_g$, and thus in this sector one has all the root vectors of $SO(32)$ given in (2.3.23). In sector 2 we have $s = 1$ and $w = \pm 1$. Here we get massless states coming from three different sectors of the $SO(32)$ weight lattice, namely

2.a) $|\pi|^2 = 2$, with $\pi^1 = \pm 1$

$$\mathbf{p_L} = (\pm 1, 0, \underline{\pm 1, 0, 0, \ldots, 0}) \tag{2.3.25}$$

(where the signs are not correlated). These are 60 states with $n = 0$.

2.b) $|\pi|^2 = 0$,

$$\mathbf{p_L} = (\pm 1, \pm 1, 0, \ldots, 0)\,. \tag{2.3.26}$$

These are 2 states, which have $n = w$.

2.c) $|\pi|^2 = 4$, with $\pi^1 = \pm 2$

$$\mathbf{p_L} = (\mp 1, \pm 1, 0, \ldots, 0)\,. \tag{2.3.27}$$

Another 2 states with $n = -w$.

We thus get 64 extra states, which together with the Cartan direction of the circle, enhance the $SO(32)$ to $SO(34)$. This point in moduli space is illustrated in green in figures 2.4a, 2.6a and 2.7a. In figure 2.4a the other green points differ from this by a Λ-shift, while the other green points in figures 2.6a and 2.7a, that appear at a different radii, will be explained in Section 2.3.4.

2.3.3.2 $U(1)^2 \times SU(16) \to SU(18)$

We now take the Wilson line $A = \left(\left(\frac{1}{4}\right)_{15}, -\frac{3}{4}\right)$. In sector 1 ($w = 0$) we have the roots of $SO(32)$ that obey:

$$\tfrac{1}{4}\sum_{I=1}^{16} \pi^I - \pi^{16} \in \mathbb{Z}\,. \tag{2.3.28}$$

Since the sum cannot be a multiple of 4, it has to vanish. Then we have the roots with two non-zero entries of opposite signs, that is $SU(16)$. For a generic R this is the gauge group, but if $R^2 = \frac{1}{4}$ we get enhancement to the maximal group $SU(18)$. In this case, the mass formula (2.3.4) gives

$$\sum_{I=1}^{15}(\pi^I + \tfrac{w}{4})^2 + (\pi_{16} - w + \tfrac{w}{4})^2 = \sum_{I=1}^{16}(\hat{\pi}^I + \tfrac{w}{4})^2 = 2 - \tfrac{w^2}{2}$$

where we defined $\hat{\pi} = (\pi_1, \pi_2, \ldots, \pi_{15}, \pi_{16} - w)$. If w is even then $\hat{\pi}$ is in (0) or (s), but if it is odd then $\hat{\pi}$ is in (v) or (c). We also have the quantization condition:

$$\frac{\frac{1}{2}|\pi|^2 - 1}{w} = \frac{\frac{1}{2}|\hat{\pi}|^2 + \frac{1}{2}w^2 + w\hat{\pi}_{16} - 1}{w} = \frac{\frac{1}{2}|\hat{\pi}|^2 - 1}{w} + \tfrac{w}{2} + \hat{\pi}_{16} \in \mathbb{Z}\,. \tag{2.3.29}$$

For $w = 1$, $-\sum_{I=1}^{16}\hat{\pi}^I = 2|\hat{\pi}|^2 - 1$, and the solutions are $\hat{\pi} = -\left(\underline{1, 0_{15}}\right)$ on (v) and $\hat{\pi} = -\left(\underline{(\frac{1}{2})_{15}, -\frac{1}{2}}\right)$ on (c).

For $w = 2$, $\sum_{I=1}^{16}(\hat{\pi}_I + \frac{1}{2})^2 = 0$, with unique solution $\hat{\pi} = -\left(\left(\frac{1}{2}\right)_{16}\right)$.

They all obey the quantization condition, and add up to 66 additional states. Together with the 240 roots of $SU(16)$, they complete the 306 roots of $SU(18)$.

2.3.3.3 $U(1) \times SO(2p) \times SO(32-2p) \to E_{p+1} \times SO(32-2p)$

Now we take a Wilson line $A = \left(\left(\frac{1}{2}\right)_p, 0_{16-p}\right)$, $2 \le p \le 8$, in the HO theory[11].

[11] Note that $p > 8$ is equivalent, by a shift $\Lambda = -\left(\left(\frac{1}{2}\right)_{16}\right)$, to $p' = 16 - p < 8$.

The massless states that survive in sector 1 ($w = 0$) are those with momentum π^I satisfying

$$\frac{1}{2}\sum_{I=1}^{p}\pi^I \in \mathbb{Z}\,. \tag{2.3.30}$$

Then the surviving states have momenta

$$\begin{aligned}\mathbf{p_L} &= (0, \underline{\pm 1, \pm 1, 0_{p-2}}, 0_{16-p}) \longrightarrow SO(2p)\,,\\ \mathbf{p_L} &= (0, 0_p, \underline{\pm 1, \pm 1, 0_{14-p}}) \quad \longrightarrow SO(32-2p)\,.\end{aligned} \tag{2.3.31}$$

For generic radius there are no states with non-zero winding, and then we get $SO(2p) \times SO(32-2p)$. These points are illustrated for $p = 2$ by the cyan dots in figures 2.1a, 2.2a, 2.4a and 2.5a; for $p = 7$, on the horizontal cyan line in figure 2.7a and for other values of p, at half-integer values of the horizontal lines of the figures in Appendix A.3.

At special values of R some states with non-vanishing winding are massless. For example, when $R^2 = 1 - \frac{p}{8}$ for $p < 8$, the $U(1) \times SO(2p)$ is enhanced to E_{p+1}. In this case, the mass formula (2.3.4) is

$$\sum_{I=1}^{p}(\pi_I + \tfrac{w}{2})^2 + \sum_{I=p+1}^{16}\pi_I^2 = 2 - 2w^2(1-p/8) \leq \frac{p}{4}\,,$$

and then if $p < 8$ the LHS must be smaller than 2. If the π_I are half-integer, then the LHS is always bigger than 2. Consequently, π_I can only take integer values and we need $\sum_{I=p+1}^{16}\pi_I^2 = \beta = 0,\, 1$.

For $w = 1$ the solution must be of the form $\left(\underline{(-1)_k, 0_{p-k}}, \underline{\pm\beta, 0_{15-p}}\right)$ and the equation is solved for every p if $\beta = 0$. Then we get $\left(\underline{(-1)_k, 0_{p-k}}, 0_{16-p}\right)$.

There is an additional constraint because $|\pi|^2$ must be even, and then k must be even. The number of states is equal to the way of choosing the value of the first p components. Choosing the first $p-1$ components, the last one is fixed by the constraint. There are $2 \times 2^{p-1} = 2^p$ states with $|w| = 1$.

For $w = 2$ we get $\sum_{I=1}^{p}(\pi_I + 1)^2 = p - 6 - \beta$, which is only possible for $p = 6,\, 7$. The RHS can only take the values 0 or 1. In the first case, all the π_I must be equal to -1. Then we get the solutions $\left((-1)_7, \underline{\pm 1, 0_8}\right)$ for $p = 7$ and $((-1)_6, 0_{10})$ for $p = 6$. The second case is only possible for $p = 7$ and $\beta = 0$. One of the π_I can take the value 0 (or -2) and the rest must take the value -1: $\left(\underline{-1 \pm 1, (-1)_6}, 0_9\right)$ for $p = 7$. In total we have 2 states with $|w| = 2$ for $p = 6$ and $2 \times (18 + 14) = 64$ for $p = 7$.

For $w \geq 3$ the equation cannot be satisfied. Then for $p < 6$ we get 2^p states (all with

$|w| = 1$), while for $p = 6$ and 7 we get 2 and 64 extra states respectively with $|w| = 2$.

$$\begin{aligned} U(1) \times SO(4) &\to SU(2) \times SU(3) \equiv E_3 \text{ (4 extra states)} \\ U(1) \times SO(6) &\to SU(5) \equiv E_4 \text{ (8 extra states)} \\ U(1) \times SO(8) &\to SO(10) \equiv E_5 \text{ (16 extra states)} \\ U(1) \times SO(10) &\to E_6 \text{ (32 extra states)} \\ U(1) \times SO(12) &\to E_7 \text{ (66 extra states)} \\ U(1) \times SO(14) &\to E_8 \text{ (192 extra states)} \end{aligned} \tag{2.3.32}$$

Recalling that $E_2 = U(1) \times SU(2)$, this is also valid for $p = 1$, where we get the enhancement at $R^2 = \frac{7}{8}$:

$$U(1)^2 \equiv U(1) \times SO(2) \to U(1) \times SU(2) \equiv E_2 \text{ (2 extra states)}. \tag{2.3.33}$$

The enhancement group $U(1) \times SU(2) \times SO(30)$, as any non-maximal enhancement, does not arise at an isolated point, but at a line, displayed in blue in figure 2.6a.

Applying the statement to $p = 8$, appears an enhancement from $U(1) \times SO(16)$ to E_9 at $R = 0$. Since E_9 has infinite dimension, we would need infinite massless states with infinitely many different winding numbers. It is obvious that at $R = 0$ winding states do not cost any energy, and thus one can have all the windings. The mass equation is:

$$\sum_{I=1}^{8} (\pi_I + \tfrac{w}{2})^2 = 2 - \beta\,. \tag{2.3.34}$$

We see that for this value of p the RHS is independent of the winding number. If $w = 1$ then $\pi = \left(\underline{(-1)_k, 0_{8-k}}, 0_8\right)$ is a solution (if k is even). For any other odd value of w we have the solution: $\pi = -\left(\underline{\left(\frac{w+1}{2}\right)_k, \left(\frac{w-1}{2}\right)_{8-k}}, 0_8\right)$. These, together with the states with even w, give infinite massless states.

We can see all these enhancements at the intersections of the lines at $A = 1/2$ in figures A.1 to A.16 that occur at $R^2 = 1 - \frac{p}{8}$.

2.3.3.4 $U(1)^2 \times SO(2p) \times SU(16-p) \to SU(2) \times E_{p+1} \times SU(16-p)$

Consider the Wilson line $A = \left(\left(\frac{4}{16-p}\right)_p, 0_p\right)$, with $0 \le p \le 7$.

The massless states that survive in sector 1 ($w = 0$) are those with momentum π^I satisfying $\frac{4}{16-p} \sum_{I=1}^{16-p} \pi^I \in \mathbb{Z}$. Then the surviving states have momenta

$$\begin{aligned} \mathbf{p_L} &= (0, 0_{16-p}, \underline{\pm 1, \pm 1, 0_{p+2}}) \longrightarrow SO(2p) \\ \mathbf{p_L} &= (0, \underline{1, -1, 0_{14-p}}, 0_p) \quad \longrightarrow SU(16-p) \end{aligned} \tag{2.3.35}$$

For generic radii there cannot be states with non-zero winding, and then the symmetry group is $SO(2p) \times SU(16-p)$. This is illustrated in the white spaces of the figures in Appendix

A.3.

There are special values of R where some states with non-vanishing winding are massless. For example, when $R^2 = 1 - \frac{8}{16-p}$, the $U(1)^2 \times SO(2p)$ is enhanced to $SU(2) \times E_{p+1}$. To see this, consider the mass formula (2.3.4)

$$\sum_{I=1}^{q}(\pi_I + \tfrac{4w}{q})^2 + \sum_{I=q+1}^{16} \pi_I^2 = 2 - 2w^2(1 - 8/q) \text{ where } q \equiv 16 - p \ .$$

For $w \neq 0$, the RHS is smaller than or equal to $16/q$ and then the LHS must be smaller than 2. If the π_I are integer, then we need $\sum_{I=q+1}^{16} \pi_I^2 = \beta = 0, 1$ and it follows that

$$\sum_{I=1}^{q}(\pi_I + \tfrac{4w}{q})^2 = 2 - 2w^2(1 - 8/q) - \beta \, .$$

For $w = 1$, $\sum_{I=1}^{q}(\pi_I + \frac{4}{q})^2 = 16/q - \beta \leq \frac{16}{9}$. If one of the π_I is different from 0 or -1 then the LHS is larger than $16/q$. So the solution must be of the form $\left(\underline{(-1)_k, 0_{16-p-k}}, \underline{\pm\beta, 0_{p-1}}\right)$ and then $k = \beta = 0$. There are only two states (considering also $w = -1$) with momentum (0_{16}).

For $w = 2$ we get $\sum_{I=1}^{q}(\pi_I + 8/q)^2 = -6 + 64/q - \beta$ which is only possible for $q = 9$, 10 ($p = 7, 6$). If $p = 6$ then we need $\beta = 0$, the RHS is $\frac{2}{5}$ and we only have the solution $((-1)_{10}, 0_6)$. If $p = 7$ then, for $\beta = 0$ and $\beta = 1$ the RHS takes the values $\frac{10}{9}$ and $\frac{1}{9}$. The equation for $\beta = 0$ is impossible to satisfy, and then we get $\left((-1)_9, \underline{\pm 1, 0_6}\right)$. In total we have 2 states with $|w| = 2$ for $p = 6$ and $2 \times (14) = 28$ for $p = 7$.

For $w \geq 3$ we get $\sum_{I=1}^{q}(\pi_I + 12/q)^2 = 144/q - 16 - \beta \leq 0$. Then for $q > 10$ ($p < 6$) there are 2 states (both with $|w| = 1$), while for $p = 6$ and 7 there are 2 and 28 extra $|w| = 2$ states respectively.

If the π_I are half-integer, then the last p values have to be $\pm\frac{1}{2}$:

$$\sum_{I=1}^{q}(\pi_I + \tfrac{4w}{q})^2 = \tfrac{q}{4} - 2 - 2w^2(1 - 8/q) \tag{2.3.36}$$

For $w = 1$, $\sum_{I=1}^{q}(\pi_I + \frac{4}{q})^2 = \frac{(q-8)^2}{4q} \leq 1$ and the π_I can only take the values $\pm\frac{1}{2}$. The solutions are of the form $\left(\underline{\left(\frac{1}{2}\right)_k, \left(-\frac{1}{2}\right)_{16-p-k}}, \underline{\left(\pm\frac{1}{2}\right)_p}\right)$, and the equation implies $k = 0$. Then, for $|w| = 1$, we get the $2 \times 2^{p+1+\delta_{p,0}}$ solutions $\left(\left(-\frac{1}{2}\right)_{16-p}, \underline{\left(\pm\frac{1}{2}\right)_p}\right)$.

For $w = 2$ we obtain $\sum_{I=1}^{q}(\pi_I + \frac{8}{q})^2 = \frac{(q-8)(q-32)}{4q} \leq 0$, and then there are no states with $|w| > 1$.

In total, for $p < 6$ we get $2 + 2^{p+\delta_{p,0}}$ states (all of them with $|w| = 1$), while for $p = 6$ and 7

we get 2 and 28 extra states respectively with $|w| = 2$.

$$U(1)^2 \to SO(4) \equiv SU(2) \times E_1 \text{ (4 extra states)}$$
$$U(1)^3 \equiv U(1)^2 \times SO(2) \to SU(2) \times SU(2) \times U(1) \equiv SU(2) \times E_2 \text{ (4 extra states)}$$
$$U(1)^2 \times SO(4) \to SU(2) \times SU(2) \times SU(3) \equiv SU(2) \times E_3 \text{ (6 extra states)}$$
$$U(1)^2 \times SO(6) \to SU(2) \times SU(5) \equiv SU(2) \times E_4 \text{ (10 extra states)}$$
$$U(1)^2 \times SO(8) \to SU(2) \times SO(10) \equiv SU(2) \times E_5 \text{ (18 extra states)}$$
$$U(1)^2 \times SO(10) \to SU(2) \times E_6 \text{ (34 extra states)}$$
$$U(1)^2 \times SO(12) \to SU(2) \times E_7 \text{ (68 extra states)}$$
$$U(1)^2 \times SO(14) \to SU(2) \times E_8 \text{ (158 extra states)}$$

At $p = 8$ we seem to get an enhancement from $U(1)^2 \times SO(16)$ to $SU(2) \times E_9$ at $R = 0$.

All of these enhancements can be seen on the intersections of the red and purple curves of figures A.9 to A.16 that occur at $R^2 = 1 - \frac{8}{q}$.

- Explicit examples for the HE theory

The roots of $E_8 \times E_8$ are

$$E_8 \times E_8 \ : \quad (\underline{\pm 1, \pm 1, 0^6}, 0^8)\,,\ (0^8, \underline{\pm 1, \pm 1, 0^6})\ , \qquad (2.3.37)$$
$$((\pm\frac{1}{2})^8, 0^8)\,,\ (0^8, (\pm\frac{1}{2})^8)\ , \text{with even number of } +\ \text{signs}$$

2.3.3.5 $U(1)^2 \times SU(9) \times SO(16) \to SO(34)$

Consider the HE theory compactified with Wilson line $A = \left(\left(\frac{1}{6}\right)_7, \frac{5}{6}, 1, 0_7\right)$. In sector 1 ($w = 0$) we have the roots of $E_8 \times E_8$ that obey:

$$\tfrac{1}{6}\sum_{I=1}^{7} \pi^I + \tfrac{5}{6}\pi^8 + \pi^9 \in \mathbb{Z} \qquad (2.3.38)$$

This breaks into two conditions, one for each E_8:

$$\tfrac{1}{6}\sum_{I=1}^{7} \pi^I + \tfrac{5}{6}\pi^8 \in \mathbb{Z} \qquad \text{and} \qquad \pi^9 \in \mathbb{Z}\,. \qquad (2.3.39)$$

For the first condition we have (0) and (s) roots. The (0) roots are vectors of the form $\left(\underline{\pm 1, \pm 1, 0_6}\right)$. The condition implies that if $\pi_8 = 0$ then we need opposite signs for the two non-zero entries. If $\pi^8 = \pm 1$ then the other non-zero entry must have the same sign. We get $\left(\underline{1, -1, 0_5}, 0\right)$ and $\pm\left(\underline{1, 0_6}, 1\right)$. These are $42 + 14 = 56$ roots.

The (s) roots are vectors of the form $\left(\underline{\left(\pm\frac{1}{2}\right)_8}\right)$ with an even number of minus signs. The condition is $\sum_{I=1}^{7} \pi^I + 5\pi^8 = 0$ mod 6. The absolute value of the LHS can only be 0 or 6. In the first case one of the first 5 components must have a different sign than the rest, and in the

second case all the 8 components must have the same sign and we get $\pm\left(\underline{\left(\frac{1}{2}\right)_6, -\frac{1}{2}, -\frac{1}{2}}\right)$ and $\pm\left(\left(\frac{1}{2}\right)_8\right)$. These are $14+2=16$ roots.

In total we have the $56+16=72$ roots of $SU(9)$.

The second condition leaves only the integer roots, and then we have $SO(16)$.

For an arbitrary value of R there cannot be states with non-zero winding, and then the gauge group is $SU(9)\times SO(16)$.

Now we show that when $R^2=\frac{1}{18}$ there is enhancement of the gauge symmetry to $SO(34)$. The mass formula (2.3.4) is:

$$\sum_{I=1}^{7}(\pi_I+\tfrac{w}{6})^2+(\pi_8+\tfrac{5w}{6})^2+(\pi_9+w)^2+\sum_{I=10}^{16}\pi_I^2=2-\tfrac{w^2}{9}<2\,. \tag{2.3.40}$$

Then $\sum_{I=10}^{16}\pi_I^2$ can only take the values 0, 1 or $\frac{7}{4}$. In the last case, we also have that $(\pi_9+w)^2\geq\frac{1}{4}$, which means that there are no spinorial roots in the last 8 components. The only possibilities are: $(-w,0_7)$ and $(-w,0_7)\pm\left(\underline{1,0_7}\right)$. The first (second) case requires w to be even (odd). Defining $\hat{\pi}=(\pi_1,\pi_2,\ldots,\pi_7,-\pi_8-w)$, we have:

$$\sum_{I=1}^{8}(\hat{\pi}_I+\tfrac{w}{6})^2=2-\tfrac{w^2}{9}-\tfrac{1-(-1)^w}{2} \tag{2.3.41}$$

but now the condition for the integer vectors is $\sum_{I=1}^{8}\hat{\pi}_I$ odd (even) when w is odd (even); and for the half-integer vectors we have the (s) conditions if w is odd and the (c) conditions if w is even.

The quantization condition is

$$\frac{\frac{1}{2}|\pi|^2-1}{w}\in\mathbb{Z}\rightarrow\begin{cases}|\pi|^2=0 \bmod 2 & \text{for } |w|=1\\ |\pi|^2=2 \bmod 4 & \text{for } |w|=2\\ |\pi|^2=2 \bmod 6 & \text{for } |w|=3\end{cases} \tag{2.3.42}$$

If $w=1$, $-\sum_{I=1}^{8}\hat{\pi}_I=3|\hat{\pi}_I|^2-2$. The minimum value for $|\hat{\pi}_I|^2$ is 1, and in that case we have $\hat{\pi}=-\left(\underline{1,0_7}\right)$.

$|\hat{\pi}_I|^2=2$ can only be achieved for the (s) conjugacy class, and then $\hat{\pi}=-\left(\left(\frac{1}{2}\right)_8\right)$.

$|\hat{\pi}|^2=3$ is for the (v) conjugacy class, $-\sum_{I=1}^{8}\hat{\pi}_I=7$, but this cannot be achieved. The same happens for greater values of $|\hat{\pi}|^2$.

If $w=2$, $-\sum_{I=1}^{8}\hat{\pi}_I=\frac{3}{2}|\hat{\pi}_I|^2-1$. Then $|\hat{\pi}|^2$ has to be even. The minimum value is 0, which could be achieved only on (0), and the equation cannot be solved. $|\hat{\pi}|^2=2$ can only be achieved for (0) and we get $\left(\underline{(-1)_2,0_6}\right)$. $|\hat{\pi}|^2=4$ has the solution $\hat{\pi}=-\left(\underline{\frac{3}{2},(\frac{1}{2})_7}\right)$. And for $|\hat{\pi}|^2=6$ the equation cannot be satisfied.

If $w=3$, $\sum_{I=1}^{8}(\hat{\pi}_I+\frac{1}{2})^2=0$, and the only solution is $\hat{\pi}=\left(\left(-\frac{1}{2}\right)_8\right)$. That is $\pi=\left((-\frac{1}{2})_7,-\frac{5}{2},-3,0_7\right)+\left(0_8,\underline{\pm1,0_7}\right)$. This has $|\pi|^2=12$, 18 or 24, which do not obey the quantization condition.

If $w=4$, $-\sum_{I=1}^{8}\hat{\pi}_I=\frac{3}{4}|\hat{\pi}|^2+\frac{1}{6}$. But this equation cannot be solved for integer $|\hat{\pi}|^2$.

Defining 8 more components for a 16 dimensional $\hat{\pi}$ such that $\hat{\pi}_9 = \pi_9 - w$ and the rest equal to the last 7 components of π, one can write the additional states $\hat{\pi}$ for $R^2 = \frac{1}{18}$ as $\left(\underline{\pm 1, 0_7}, \underline{\pm 1, 0_7}\right)$ and $\left(\pm\left(\frac{1}{2}\right)_8, \underline{\pm 1, 0_7}\right)$ for $|w| = 1$ and $\pm\left(\underline{(1)_2, 0_6}, 0_8\right)$ and as $\pm\left(\left(\underline{\frac{3}{2}, (\frac{1}{2})_7}\right), 0_8\right)$ for $|w| = 2$. The former are $256 + 32 = 288$ states and the latter $56 + 16 = 72$ states. In total these 360 additional states added to the 184 roots of $SU(9) \times SO(16)$ give the 544 roots of $SO(34)$.

In figure 2.9 we show this maximal enhancement on the intersection between one red, two yellow and one green curves. The integer states with $|w| = 1$ and $|w| = 2$ give the red curve, the half-integer states with $|w| = 1$ give the green curve and the ones with $|w| = 2$ are represented by the yellow curve. The additional states without winding are those in the yellow line.

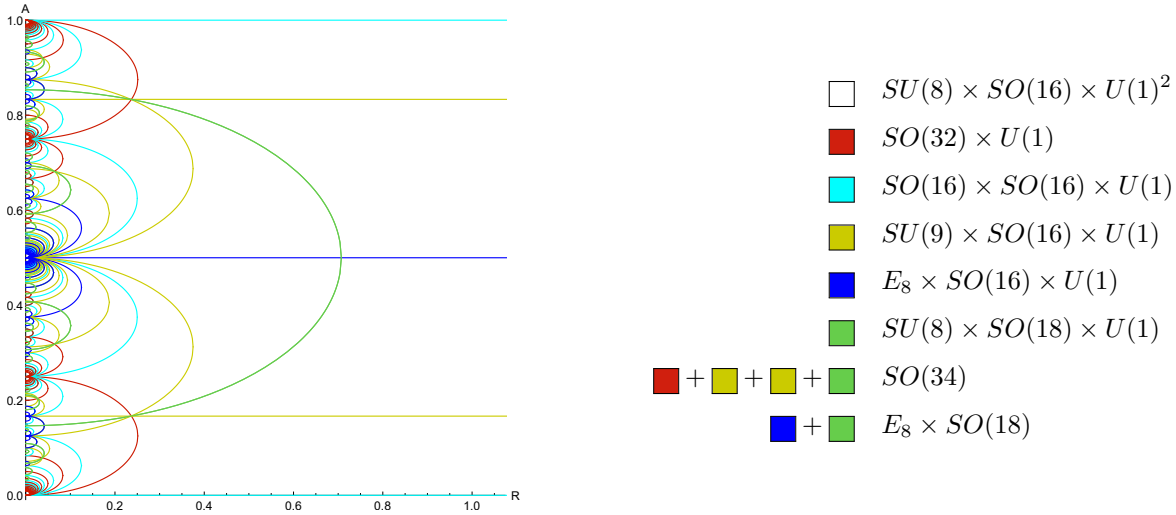


Figure 2.9: HE with Wilson line $A^I = ((A)_7, 1 - A, 1, 0_7)$

2.3.3.6 $U(1)^3 \times SU(8) \times SU(8) \to SU(18)$

Consider the Wilson line $A = \left(\left(\frac{1}{6}\right)_7, \frac{5}{6}, \left(\frac{1}{6}\right)_7, \frac{5}{6}\right)$ in the HE theory.

In sector 1 ($w = 0$) we have the first condition of (2.3.39) for each of the E_8, then we get the 144 roots of $SU(9) \times SU(9)$. For an arbitrary value of R this is the gauge group.

For $R^2 = \frac{1}{9}$ there is enhancement of the gauge symmetry to $SU(18)$. To see this, take the mass formula (2.3.4)

$$\sum_{I=1}^{7}(\pi_I + \tfrac{w}{6})^2 + (\pi_8 + \tfrac{5w}{6})^2 + \sum_{I=9}^{15}(\pi_I + \tfrac{w}{6})^2 + (\pi_{16} + \tfrac{5w}{6})^2 = 2 - \tfrac{2w^2}{9} < 2\,. \tag{2.3.43}$$

Defining $\hat{\pi} = (\pi_1, \pi_2, \ldots, \pi_7, -\pi_8 - w, \pi_9, \pi_{10}, \ldots, \pi_{15}, -\pi_{16} - w)$ we have:

$$\sum_{I=1}^{16}(\hat{\pi}_I + \tfrac{w}{6})^2 = 2 - \tfrac{2w^2}{9}\,, \tag{2.3.44}$$

but now $\hat{\pi}$ has to be on the conjugacy classes (ss), (vv), (sv) or (vs) if w is odd and on (cc), (00), $(0c)$, $(c0)$ if w is even.

We also have to obey the quantization condition $\frac{\frac{1}{2}|\pi|^2 - 1}{w} \in \mathbb{Z}$.

If $w=1$, $-\sum_{I=1}^{16}\hat{\pi}_I = 3|\hat{\pi}|^2-4$ and $\hat{\pi}$ is on (vv), (ss), (vs) or (sv). The minimum value for $|\hat{\pi}_I|^2$ is 2, and in that case $(\hat{\pi},\hat{\pi}') = -\left(\underline{1,0_7},\underline{1,0_7}\right)$.

$|\hat{\pi}|^2=3$ can only be achieved for the (vs) and (sv) conjugacy classes, and $\hat{\pi} = -\left(\underline{1,0_7},(\frac{1}{2})_8\right)$, $-\left((\frac{1}{2})_8,\underline{1,0_7}\right)$. $|\hat{\pi}|^2=4$ is for the (ss) and (vv) conjugacy classes, and $\hat{\pi} = -\left((\frac{1}{2})_{16}\right)$. $|\hat{\pi}|^2=5$ is for the (sv) and (vs) conjugacy classes, $-\sum_{I=1}^{16}\hat{\pi}_I = 11$ which cannot be achieved. The same happens for greater values of $|\hat{\pi}|^2$.

If $w=2$, $-\sum_{I=1}^{16}\hat{\pi}_I = \frac{3}{2}|\hat{\pi}|^2+1$ implies (00), (cc), $(c0)$ or $(0c)$. The minimum value for $|\hat{\pi}_I|^2$ is 0, but then the equation cannot be solved.

$|\hat{\pi}|^2=2$ can only be achieved for (00), $(0c)$ or $(c0)$, but there is no solution.

$|\hat{\pi}|^2=4$ implies $-\sum_{I=1}^{16}\hat{\pi}_I = 7$ and this cannot be achieved. The same happens for greater values of $|\hat{\pi}|^2$.

If $w=3$, $\sum_{I=1}^{16}(\hat{\pi}_I+\frac{1}{2})^2=0$ has only a solution belonging to (ss), namely $\hat{\pi} = -\left(\left(\frac{1}{2}\right)_{16}\right)$.

It can be shown that all of these states obey the quantization condition. Then, the additional states for $R^2=\frac{1}{18}$ are $\pm\left(\underline{1,0_7},\underline{1,0_7}\right)$, $\pm\left(\underline{1,0_7},(\frac{1}{2})_8\right)$ and $\pm\left((\frac{1}{2})_8,\underline{1,0_7}\right)$ for $|w|=1$ and $\hat{\pi} = \left(\left(-\frac{1}{2}\right)_{16}\right)$ for $|w|=3$ $\left(\underline{\pm1,0_7},\underline{\pm1,0_7}\right)$ and $\left(\pm\left(\frac{1}{2}\right)_8,\underline{\pm1,0_7}\right)$ for $|w|=1$ and $\pm\left(\underline{(1)_2,0_6},0_8\right)$ and $\pm\left(\left(\frac{1}{2}\right)_{16}\right)$ for $|w|=2$. The former are $128+32=160$ states and the latter 2 states. In total these are 162 additional states, which added to the 144 roots of $SU(9)\times SU(9)$ give the 306 roots of $SU(18)$.

In figure 2.10 we show this maximal enhancement on the intersection between one red, two yellow and one green curves. The integer states with $|w|=1$ are represented by the red curve, the half-integer states with $|w|=1$ give the yellow curve, the states with $|w|=3$ are represented by the green curve and the additional states with $w=0$ give the yellow horizontal line.

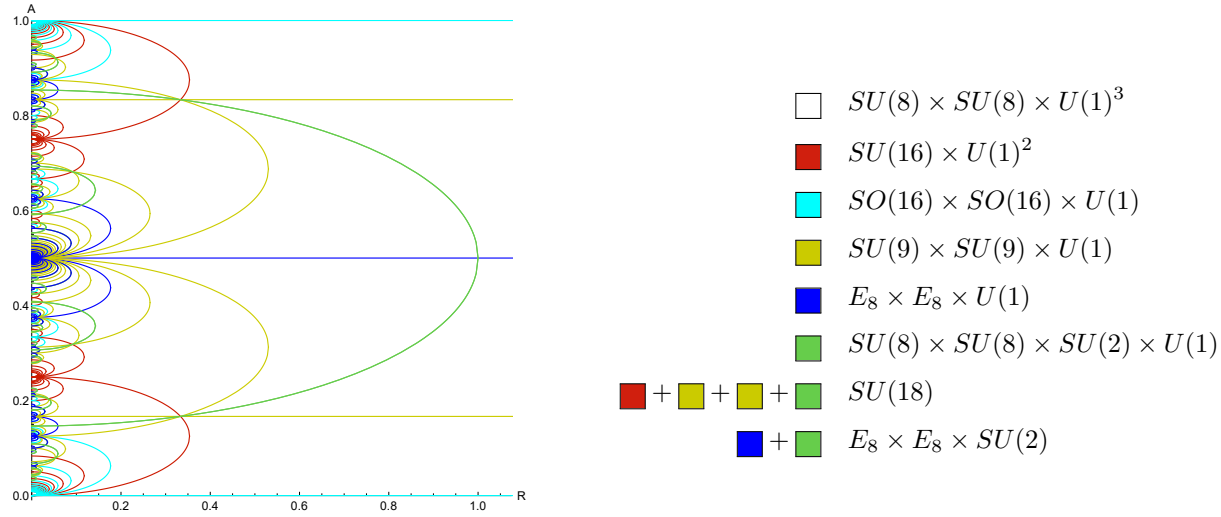


Figure 2.10: HE with Wilson line $A^I=((A)_7,1-A,(A)_7,1-A)$

2.3.3.7 $U(1)\times SO(16)\times E_8 \to SO(18)\times E_8$

Consider the HE string compactified on a circle of radius $R=\frac{1}{\sqrt{2}}$, with Wilson line $A=(1,0^7,0^8)$, which is of the form $(v0)$ according to the notation of Appendix A.1 (see (A.1.10) in particular). This Wilson line leaves the second E_8 unbroken, while from the first E_8, the

surviving states in sector 1 are the ones with integer entries, i.e. those in the first line of (2.3.37). The group H from sector 1 is then $SO(16) \times E_8$ and the corresponding points in moduli space are illustrated by the grey dots in figure 2.1b.

In sector 2 we have states with $w = \pm 1$ such that $s = 1$, $|\mathbf{p}^I|^2 = 1$. The surviving states have the following momenta

$$\begin{aligned}
\mathbf{p_L} &= (0, \underline{\pm 1, \pm 1, 0_6}),\ w = 0,\ |\pi|^2 = 2 \quad 112 \text{ roots} \\
\mathbf{p_L} &= (\pm 1, 0, \underline{\pm 1, 0_6}),\ w = \pm 1, |\pi|^2 = 2, \quad 28 \text{ roots} \\
\mathbf{p_L} &= (\pm 1, \pm 1, 0_7), \quad w = \pm 1, \pi = 0, \quad 2 \text{ roots} \\
\mathbf{p_L} &= (\pm 1, \mp 1, 0_7), \quad w = \pm 1, |\pi|^2 = 4, \quad 2 \text{ roots}\,,
\end{aligned}$$

where the first entry corresponds to the circle and the subsequent ones to the 8 directions along the Cartan of the first E_8 factor. The first line contains the states of sector 1. These are the 144 roots of SO(18). This point in moduli space, together with its equivalent ones, are illustrated by the green dots in figure 2.4b, 2.6b and 2.7b.

2.3.3.8 $U(1) \times SU(2) \times E_7 \times E_8 \to SU(2) \times E_8 \times E_8$

This is an interesting example of enhancement-breaking in the HE theory, where first the E_8 is broken to $SU(2) \times E_7$ by the Wilson line $A = \left(\left(\frac{1}{4}\right)_8, 0_8\right)$ and then enhanced by the circle direction to $SU(2) \times E_8$.

The Wilson line leaves the second E_8 unbroken, while the surviving roots from the first E_8 have 9-momenta

$$\begin{aligned}
\mathbf{p_L} &= \pm(0, \underline{1, -1, 0_6}) \\
\mathbf{p_L} &= \pm\left(0, \left(\tfrac{1}{2}\right)_8\right) \\
\mathbf{p_L} &= \left(0, \underline{\left(\tfrac{1}{2}\right)_4, -\left(\tfrac{1}{2}\right)_4}\right)
\end{aligned} \tag{2.3.45}$$

This, gives 128 roots, which together with the 8 Cartan directions, gives an unbroken gauge group $H = SU(2) \times E_7 \subset E_8$.

Additionally at $R = \frac{1}{2}$ there are 114 states in sector 2: two with $w = \pm 2$ and 112 with $w = \pm 1$ and momentum

$$\begin{aligned}
\mathbf{p_L} &= \left(\pm\tfrac{\sqrt{2}}{2}, \underline{\mp\left(\tfrac{3}{4}\right)_2, \pm\left(\tfrac{1}{4}\right)_6}\right) \\
\mathbf{p_L} &= \left(\pm\tfrac{\sqrt{2}}{2}, \underline{\pm\left(\tfrac{3}{4}\right)_2, \mp\left(\tfrac{1}{4}\right)_6}\right)
\end{aligned} \tag{2.3.46}$$

These states give a total of 114 extra states that add up to the previous 136 states, plus the circle direction, adding up to the 251 states of $SU(2) \times E_8$. So at $R = \frac{1}{2}$ we get enhancement to $SU(2) \times E_8 \times E_8$, which works very differently than the enhancement occurring at $R = 1$, mentioned in Section 2.3.2.

In figure 2.11 we present these maximal enhancements for the HE theory, and we also show a maximal enhancement to $SU(3) \times E_7 \times E_8$. The additional states with $w = 0$ are represented by the cyan line and the states with $|w| = 1$ together with the ones with $|w| = 2$ are represented

by the orange curve.

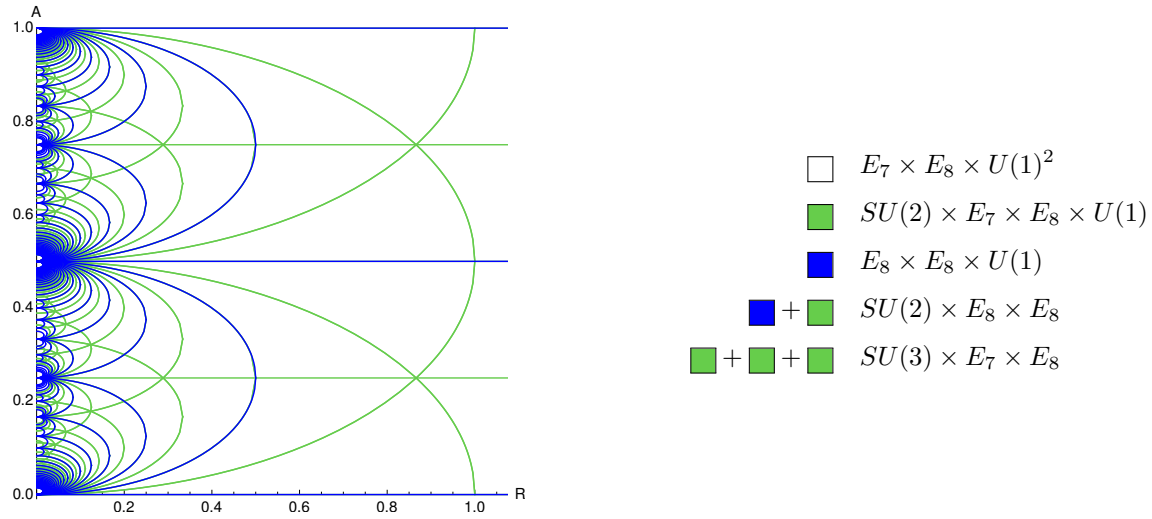


Figure 2.11: HE with Wilson line $A^I = ((A)_8, 0_8)$

2.3.4 Exploring a slice of moduli space

In this section we present a detailed analysis of the slice of moduli space for compactifications of the heterotic theory on a circle at any radius and Wilson line given by

$$A = (A_1, 0_{15}) \ . \tag{2.3.47}$$

The results of this section are displayed in figure 2.6. Here we present the main ingredients of the calculations, and leave further details to Appendix A.2.

For this type of Wilson line, the states with $w = 0$ (sector 1) that survive, are those satisfying

$$\pi_1 A_1 \in \mathbb{Z} \, . \tag{2.3.48}$$

This preserves all the roots only if $A_1 \in \mathbb{Z}$ for the Γ_{16} case, or $A_1 \in 2\mathbb{Z}$ for the $\Gamma_8 \oplus \Gamma_8$ case. These correspond to the horizontal orange lines in figure 2.6, where at any generic radius, the gauge symmetry is $U(1) \times SO(32)$, or $U(1) \times E_8 \times E_8$. If A_1 is an odd number, then the $SO(32)$ symmetry is unbroken, but the $E_8 \times E_8$ is broken to $SO(16) \times E_8$, which is depicted with a black line at $A_1 = 1$ in figure 2.6b.

If $A_1 \notin \mathbb{Z}$, then we have just the roots with $\pi_1 = 0$. That is, the 420 roots of $SO(30)$ or the 324 roots of $SO(14) \times E_8$. This corresponds to the white regions in figure 2.6.

Now, depending on the value of R, we can have additional states in sector 2, i.e. states with non-zero winding[12] which momenta satisfy $|\mathbf{p_L}|^2 = 2$ and have a quantized momentum number

[12]From now on we take $w > 0$, keeping in mind that for every massless state with w there is also a massless state with $-w$.

on the circle. Then, according to (2.3.2) and (2.3.22), they should obey

$$\begin{aligned} |\pi + wA|^2 &= 2(1 - w^2R^2)\,, \\ \frac{1}{w}\left(1 - \tfrac{1}{2}|\pi|^2\right) &\in \mathbb{Z}\ . \end{aligned} \tag{2.3.49}$$

The first equation implies $R^{-1} \geq w$, and the simplest solution is

$$\pi = \left(\pm\sqrt{2(1 - w^2R^2)} - wA_1, 0_{15}\right)\ .$$

But π is in an even lattice, which implies $\pi_1 = -2q$, $q \in \mathbb{Z}$. The quantization condition for n yields

$$\tfrac{2q^2-1}{w} \in \mathbb{Z}\,,$$

so we have only the winding numbers that are divisors of the numbers that can be written as $2q^2 - 1$, for some integer q. In terms of q, the Wilson lines are of the form

$$A_1 = \frac{2q \pm \sqrt{2 - 2w^2R^2}}{w} \equiv a_{w,q}(R)\ , \quad \{w, q, \tfrac{2q^2-1}{w}\} \in \mathbb{Z}\,. \tag{2.3.50}$$

If the radius also satisfies $R < \frac{1}{\sqrt{2}w} < \frac{1}{w}$, we have additional solutions where some of the other components of π are non-zero, such that

$$\begin{aligned} \pi + wA &= \left(\pm\sqrt{1 - 2w^2R^2}, \underline{\pm 1, 0_{14}}\right) \text{ for } \Gamma_{16}\,, \\ \pi + wA &= \left(\pm\sqrt{1 - 2w^2R^2}, \underline{\pm 1, 0_6}, 0_8\right) \text{ for } \Gamma_8 \oplus \Gamma_8\ . \end{aligned}$$

The quantization conditions are the same as before, but now the Wilson lines have the following behavior as a function of the radius

$$A_1 = \frac{2q + 1 \pm \sqrt{1 - 2w^2R^2}}{w} \equiv b_{w,q}(R)\ , \quad \{w, q, \tfrac{2q^2-1}{w}\} \in \mathbb{Z}\,. \tag{2.3.51}$$

If additionally $R < (2\sqrt{2}w)^{-1}$ we have yet other possible solutions, but only for the HE theory, where

$$\pi + wA = \left(\pm\tfrac{1}{2}\sqrt{1 - 8w^2R^2}, \underline{(\pm\tfrac{1}{2})_7}, 0_8\right) \text{ for } \Gamma_8 \oplus \Gamma_8.$$

The lines and quantization conditions are:

$$A_1 = \frac{q + \frac{1}{2} \pm \sqrt{\frac{1}{4} - 2w^2R^2}}{w} \equiv c_{w,q}(R)\ , \quad \{w, q, \tfrac{q(q+1)}{2w}\} \in \mathbb{Z}\,, \tag{2.3.52}$$

where we used $(\pi_1)^2 = |\pi|^2 - \frac{7}{4}$ and $\pi_1 = -\left(q + \frac{1}{2}\right)$.

For a given q and w, whenever the Wilson line is of the form $a_{w,q}$ in (2.3.50), we get 2 massless states (one for $w > 0$ and another one for $w < 0$). If there are no more states, then we have enhancement to $U(1) \times SU(2) \times SO(30)$ and $U(1) \times SU(2) \times SO(14) \times E_8$. These correspond to the blue lines in figure 2.6, where for example in figure 2.6a, the long blue line going from $(R, A_1) = (0, \sqrt{2})$ to $(1, 0)$ corresponds to $a_{1,0} = \sqrt{2(1 - R^2)}$, while its mirror one

along the axis $A_1 = 1$ is $a_{1,1} = 2 - a_{1,0}$.

For Wilson lines of the form $b_{w,q}$ in (2.3.51), we get 60 extra states for the Γ_{16}, and 28 for $\Gamma_8 \oplus \Gamma_8$. The former promote the enhancement to $U(1) \times SO(32)$, while the latter to $U(1) \times SO(16) \times E_8$, and they correspond respectively to the orange lines in figure 2.6a and the black lines in figure 2.6b. The largest curved orange line in the former and black line in the latter going from $(0,0)$ to $(0,2)$ corresponds to $b_{0,1} = 1 \pm \sqrt{1-2R^2}$, where the plus sign is for the upper half of the curve, and the minus sign for the lower half.

Finally, Wilson lines of the form $c_{w,q}$ in (2.3.52) give in the HE theory, $2 \times 2^6 = 128$ states (the sign of one of the seven $(\pm\frac{1}{2})$ is determined by the sign of the other 6 and the sign chosen for the Wilson line). Note that $c_{w,q}(R) = b_{2w,q}(R)$. It is not hard to show that a Wilson line that can be written as $c_{w,q}(R)$ can always be written as $b_{2w,q}(R)$, but the function b can also have an odd w. Wilson lines b that can also be written as c bring then a total of $28+128 = 156$ states, which corresponds to the enhancement to $U(1) \times E_8 \times E_8$ in the orange lines of figure 2.6b.

There are only two kinds of intersections between lines, and the points of intersection correspond to points of maximal enhancement (see Appendix A.2 for details):

- between a blue curve $a(R)$ with w_1 and an orange curve $b(R)$ with w_2, where the enhancement group is $SU(2) \times SO(32)$ ($SU(2) \times E_8 \times E_8$) in the HO (HE) theory. These are the red dots of figure 2.6, and arise at

$$(R, A_1) = \left(\frac{1}{\sqrt{w_1^2 + 2w_2^2}}, \frac{2}{w_1}\left(q \pm w_2 R\right) \right) = \left(\frac{1}{C}, \frac{2k}{C} \right) ,$$

  for some integer k, with $C = 1, 3, 9, 11, ...$ are all the integers whose prime divisors are 1 or 3 (mod 8).

- between two blue $a(R)$ with w_1 and w_2 and two orange (black) curves $b(R)$ with w_3 and w_4, where the enhancement group is $SO(34)$ ($SO(18) \times E_8$) for the $SO(32)$ ($E_8 \times E_8$) theory. These are the green dots of figure 2.6, and arise at[13]

$$(R, A_1) = \left(\frac{1}{\sqrt{w_1^2 + w_2^2}}, \frac{2}{w_1}\left(q \pm \tfrac{1}{\sqrt{2}} w_2 R\right) \right) = \left(\frac{1}{\sqrt{2}C}, \frac{k}{C} \right) ,$$

  for some integer k, with $C = 1, 5, 13, 17, ...$ are all the integers whose prime divisors are Pythagorean primes.

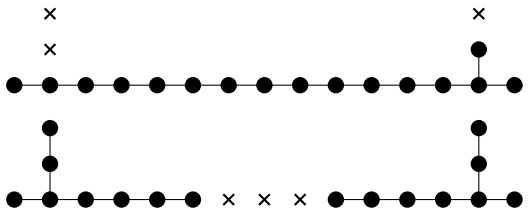
In Appendix A.2 we give the details of the calculations and also prove that these are the only possible intersections for this type of Wilson lines. In Appendix A.3 we present other slices of moduli space given by the radius and Wilson lines determined by a single parameter A.

[13] We get additionally $R = \frac{1}{\sqrt{w_1^2+w_2^2}} = \frac{1}{\sqrt{2}\sqrt{w_3^2+w_4^2}}$.

2.4 Generalized Dynkin Diagram of $\Gamma_{1,17}$

As was explained previously, the momentum of the states associated to heterotic strings compactified on a torus lies in an even self-dual lattice. In the case of the circle this is $\Gamma_{1,17}$. It is quite simple to obtain all the gauge group and their respective moduli for circle compactifications of both heterotic strings by means of the Generalized Dynkin Diagram (GDD) of this lattice. We refer to [11] for an introduction to root systems and associated GDDs of Lorentzian $\Gamma_{1,8m+1}$ lattices. The special case of $\Gamma_{1,17}$ is discussed in detail in [33] and [13], precisely in connection to circle compactifications of the heterotic string. It was originally considered by Vinberg [36]. The reflective part of its group of automorphisms, which is actually the duality group $\mathrm{O}(1,17,\mathbb{Z})$ [13], can be encoded in the GDD as we review shortly.

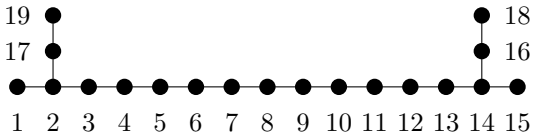
The equivalence of the two heterotic strings on S^1 is determined by the uniqueness of the $\Gamma_{1,17}$ root lattice. The Generalized Dynkin Diagram (GDD) of $\Gamma_{1,17}$ is obtained by adding roots associated with the crosses in the following extension of the $SO(32)$ and $E_8 \times E_8$ Dynkin diagrams respectively



(2.4.1)

(2.4.2)

The 17-dimensional moduli space of inequivalent compactifications can be chosen to be delimited by 19 boundaries, each of them associated with one of the nodes of the GDD



(2.4.3)

A possible fundamental region for the moduli space is determined by the points satisfying all of the inequalities in Table 2.12

Node	Fund region for Γ_{16}	Fund region $\Gamma_8 \oplus \Gamma_8$
$1 \leq i \leq 6$	$A_i \leq A_{i+1}$	$A_i \leq A_{i+1}$
7	$A_7 \leq A_8$	$A_7 \leq A_8 + 1$
8	$A_8 \leq A_9$	$\sum_{i=1}^{16} A_i^2 \geq 2 - 2R^2$
9	$A_9 \leq A_{10}$	$A_9 \leq A_{10} + 1$
$10 \leq i \leq 15$	$A_i \leq A_{i+1}$	$A_i \leq A_{i+1}$
16	$A_{16} \leq 1 - A_{15}$	$A_{16} \leq -A_{15}$
17	$-A_2 \leq A_1$	$-A_2 \leq A_1$
18	$\sum_{i=1}^{16} A_i^2 \geq 2 - 2R^2$	$\sum_{i=9}^{16} A_i \geq 0$
19	$\sum_{i=1}^{16} (A_i - \frac{1}{2})^2 \geq 2 - 2R^2$	$\sum_{i=1}^{8} A_i \leq 0$

Figure 2.12: Fundamental region for HE and HO in $d = 1$.

This defines a 17-dimensional surface resembling a chimney [13]. In Γ_{16}, the first 17 nodes define walls parallel to the R direction and the last two nodes define hyperspheres which delimit the bottom of the chimney. In $\Gamma_8 \oplus \Gamma_8$, there are 18 walls and only one hypersphere at the bottom defined by the 8th node. At the borders of the fundamental region, where some equalities are saturated, the gauge symmetry is enhanced. The enhanced gauge group is obtained by removing all the nodes of the Generalized Dynkin Diagram except those with saturated inequality. Hence, the maximally enhanced symmetries saturate all but 2 of the inequalities[14]. It can be shown that all the possible combinations of saturated inequalities produce Dynkin diagrams of the ADE classification.

Some sections of the bottom of the chimney are represented below in figures A.1 to A.16 by the red curves that intersect the horizontal axis and the purple curves that intersect the $A = \frac{1}{2}$ line. These are the sections of the hypersphere associated respectively to the nodes 18 and 19 in the Γ_{16} case. The absence of purple curves in the first eight figures is related to the fact that for Wilson lines with more than 7 zeros there are no spinorial roots which makes the inequality of the 19th node impossible to saturate.

All the possible enhancement groups in S^1 compactifications can be obtained from the GDDs [11–13] . Here we list all the possible maximal enhancements for the Γ_{16} and $\Gamma_8 \oplus \Gamma_8$ theories, together with the point in the fundamental region that gives that enhancement ($p, q \in \mathbb{Z}$, $1 \leq p, q \leq 8$)

Wilson line	R^{-2}	Gauge group
$\left(0_{8-p}, \left(\frac{q}{2(p+q)}\right)_{p+q}, \left(\frac{1}{2}\right)_{8-q}\right)$	$8\left(\frac{1}{p}+\frac{1}{q}\right)$	$E_{9-p} \times E_{9-q} \times SU(p+q)$
$\left(-\frac{q}{2(6+q)}, \left(\frac{q}{2(6+q)}\right)_{7+q}, \left(\frac{1}{2}\right)_{8-q}\right)$	$2 - \frac{2}{q+9} + \frac{8}{q}$	$SU(9+q) \times E_{9-q}$
$\left(-\frac{1}{4}, \left(\frac{1}{4}\right)_{14}, -\frac{1}{4}\right) + (0_{15}, 1)$	4	$SU(18)$
$\left(0_{8+q}, (\frac{1}{2})_{8-q}\right)$	$\frac{8}{q}$	$SO(16+2q) \times E_{9-q}$
$(0_{15}, 1)$	2	$SO(34)$

Figure 2.13: Maximal enhancements for the HO theory.

Wilson line	R^2	Gauge group
$\left(0_{8-p}, \left(\frac{1}{p}\right)_p, \left(-\frac{1}{q}\right)_q, 0_{8-q}\right) + (0_7, -1, 1, 0_7)$	$\frac{1}{2}\left(\frac{1}{p}+\frac{1}{q}\right)$	$E_{9-p} \times E_{9-q} \times SU(p+q)$
$\left(-\frac{1}{6}, \left(\frac{1}{6}\right)_7, \left(-\frac{1}{q}\right)_q, 0_{8-q}\right) + (0_7, -1, 1, 0_7)$	$\frac{1}{2}\left(\frac{1}{9}+\frac{1}{q}\right)$	$SU(9+q) \times E_{9-q}$
$\left(-\frac{1}{6}, \left(\frac{1}{6}\right)_7, \left(-\frac{1}{6}\right)_7, \frac{1}{6}\right) + (0_7, -1, 1, 0_7)$	$\frac{1}{9}$	$SU(18)$
$\left(0_8, \left(-\frac{1}{q}\right)_q, 0_{8-q}\right) + (0_7, -1, 1, 0_7)$	$\frac{1}{2q}$	$SO(16+2q) \times E_{9-q}$
$\left(0_8, \left(-\frac{1}{6}\right)_7, \frac{1}{6}\right) + (0_7, -1, 1, 0_7)$	$\frac{1}{18}$	$SO(34)$

Figure 2.14: Maximal enhancements for the HE theory.

When p and/or q equal 7 one gets $E_2 = SU(2) \times U(1)$ and the enhancement is not maximal.

[14] Actually, if the group has one or two E_2, 3 or 4 nodes have to be removed instead of 2.

Now we will explain the construction of the GDD in more detail for the HE case. The HO is analogous an can be read for example in [2].

2.4.1 Embedding of $\Gamma_8 \oplus \Gamma_8$

We will describe the embedding of the HE lattice $\Gamma_8 \oplus \Gamma_8$ in $\Gamma_{1,17}$.

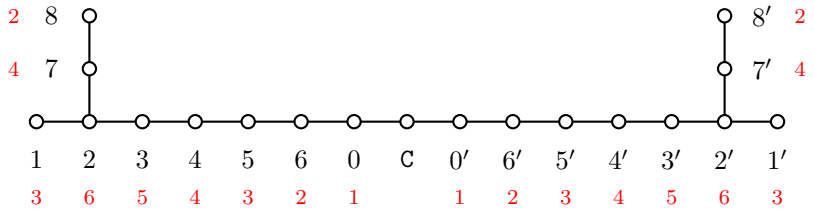


Figure 2.15: Generalized Dynkin Diagram for the $\Gamma_{1,17}$ lattice, with labels showing the embedding of the extended Dynkin diagrams of $E_8 + E_8'$. The Kac marks are shown in red.

The GDD is shown in Figure 2.15. It is composed by the extended Dynkin diagrams of E_8 and E_8' joined by a central node. The nodes can be specified in terms of the charge vectors (2.3.9)

$$\begin{aligned} &\varphi_i = |0,0;\alpha_i,0^8\rangle, \qquad \varphi_{i'} = |0,0;0^8,\alpha_i'\rangle, \qquad i = 1,...,8\,, \\ &\varphi_0 = |0,-1;\alpha_0,0^8\rangle, \qquad \varphi_{\mathfrak{c}} = |1,1;0^8,0^8\rangle, \qquad \varphi_{0'} = |0,-1;0^8,\alpha_0'\rangle\,. \end{aligned} \tag{2.4.4}$$

where α_i and α_i' are the simple roots of E_8 and E_8', given together with their fundamental weights in Table 2.1[15]. We have also written down the lowest root $\alpha_0 = -\sum_{k=1}^{8} \kappa_k \alpha_k$, and similarly for α_0'. The κ_i and κ_i' are the Kac marks, they are listed on the table and also shown in red in the Figure 2.15. By definition $\kappa_0 = \kappa_0' = 1$ and sometimes we will set $w_0 = 0$, $w_0' = 0$.

i	κ_i	α_i	w_i
1	3	(1,-1,0,0,0,0,0,0)	-(-$\frac{1}{2}$,$\frac{1}{2}$,$\frac{1}{2}$,$\frac{1}{2}$,$\frac{1}{2}$,$\frac{1}{2}$,$\frac{1}{2}$,-$\frac{5}{2}$)
2	6	(0,1,-1,0,0,0,0,0)	-(0,0,1,1,1,1,1,-5)
3	5	(0,0,1,-1,0,0,0,0)	-(0,0,0,1,1,1,1,-4)
4	4	(0,0,0,1,-1,0,0,0)	-(0,0,0,0,1,1,1,-3)
5	3	(0,0,0,0,1,-1,0,0)	-(0,0,0,0,0,1,1,-2)
6	2	(0,0,0,0,0,1,-1,0)	(0,0,0,0,0,0,-1,1)
7	4	-(1,1,0,0,0,0,0,0)	-($\frac{1}{2}$,$\frac{1}{2}$,$\frac{1}{2}$,$\frac{1}{2}$,$\frac{1}{2}$,$\frac{1}{2}$,$\frac{1}{2}$,-$\frac{7}{2}$)
8	2	($\frac{1}{2}$,$\frac{1}{2}$,$\frac{1}{2}$,$\frac{1}{2}$,$\frac{1}{2}$,$\frac{1}{2}$,$\frac{1}{2}$,$\frac{1}{2}$)	(0,0,0,0,0,0,0,2)
0	1	(0,0,0,0,0,0,1,-1)	(0,0,0,0,0,0,0,0)

Table 2.1: Simple roots α_i, Kac marks κ_i and fundamental weights w_i of E_8.

[15] On our convention, α_i' and w_i' are given by changing the sign and reflecting the components of α_i and w_i.

In [13] (see also [36]), the generators of the duality group $\mathrm{O}(1,17,\mathbb{Z})$ were identified with Weyl reflections in the lattice. To be more concrete, let us consider the transformations of the charge vector $|Z\rangle$ about the simple roots of $\Gamma_{1,17}$ in (2.4.4), denoted collectively $|\varphi\rangle$. Since $\langle\varphi|\varphi\rangle = 2$, the Weyl transformation is

$$|Z'\rangle = |Z\rangle - \langle\varphi|Z\rangle|\varphi\rangle\,. \tag{2.4.5}$$

Once $|Z'\rangle$ is found, the action on the moduli is deduced by imposing that $p_R = 0$ transforms into $p'_R = 0$, i.e. $n - Ew - \pi\cdot A = 0$ goes into $n' - E'w' - \pi'\cdot A' = 0$, where we have defined the moduli space parameter $E = R^2 + \frac{1}{2}A^2$ as an alternative to R. This is a shortcut to requiring invariance of the spectrum. For example, writing only the transformed quantities, from the nodes 1, 0, and C we obtain

$$\varphi_1 : \pi'^1 = \pi^2, \pi'^2 = \pi^1 \;\Rightarrow\; A'^1 = A^2, A'^2 = A^1, \tag{2.4.6a}$$

$$\varphi_0 : n' = n - w + \pi^7 - \pi^8, \pi'^7 = \pi^8 + w, \pi'^8 = \pi^7 - w \;\Rightarrow\; A'^7 = A^8 - 1, A'^8 = A^7 + 1,$$
$$E' = E + A^7 - A^8 + 1, \tag{2.4.6b}$$

$$\varphi_{\mathtt{C}} : w' = -n, n' = -w \;\Rightarrow\; E' = \frac{1}{E}, A' = \frac{A}{E}. \tag{2.4.6c}$$

Clearly, (2.4.6a) is a permutation of the first two components of the Wilson line. In general, the reflections about nodes φ_i, or φ'_i, $i = 1,\ldots,8$, induce transformations of the Wilson line A^I which are just elements of the Weyl group of E_8, or E'_8. In (2.4.6b) we recognize a translation of A^I by $\alpha_0 \times 0$, which belongs to $\Gamma_8 \oplus \Gamma_8$, combined with a permutation of A^7 and A^8. Finally, (2.4.6c) is the generalization of the T-duality $R \to 1/R$ when $A \neq 0$, already given in Section 2.3.1.

The fundamental region described in Table 2.12 can be rewritten in terms of the simple roots of $\Gamma_{1,17}$ in Table 2.1. On the HE case, the inequality for the nodes $1 \leq i \leq 8$ on figure 2.15 can be expressed as $A\cdot(\alpha_i \times 0) \geq 0$; for the 0 node it is $A\cdot(\alpha_0 \times 0) \geq -1$ and for the C node $E \geq 1$. For the primed nodes it is the same as before but changing $(\alpha_k \times 0)$ with $(0 \times \alpha'_k)$.

As explained above, the prescription to obtain a non-Abelian gauge group G_r is to delete $19 - r$ nodes of the GDD such that the remaining ones give the Dynkin diagram of the desired semisimple Lie Algebra. The total gauge group is $\mathrm{G}_r \times \mathrm{U}(1)^{17-r}$. The Wilson line and the radius are determined by saturating the inequalities corresponding to the r undeleted nodes. In this manner one can obtain all the allowed groups and the corresponding moduli. For example, for maximal enhancement, all but 2 of the inequalities are saturated. The allowed groups of maximal rank are precisely found by deleting one node in the E_8 side and one node in the E'_8 side, while the central node C corresponding to $E = 1$ cannot be erased. We note that the i-th node forming the first E_8 will have its inequality saturated only when the Wilson line A is orthogonal to α_i, while the presence of the node 0 implies $A\cdot\alpha_0 = -1$. From this it is easy to see that the Wilson lines that give maximal enhancements are:

$$A = \frac{w_k}{\kappa_k} \times \frac{w'_m}{\kappa'_m}\,, \tag{2.4.7}$$

with the fundamental weights of E_8 given in Table 2.1. As a check, see that the LHS of the inequalities is

$$A \cdot (\alpha_i \times 0) = \frac{w_k}{\kappa_k} \cdot \alpha_i = \frac{1}{\kappa_i}\delta_{ik} \ ,$$

$$A \cdot (\alpha_0 \times 0) = \frac{w_k}{\kappa_k} \cdot \left(-\sum_{j=1}^{8} \kappa_j \alpha_j \right) = \begin{cases} -1 & \text{for} \quad 1 \leq k \leq 8 \\ 0 & \text{for} \quad k = 0 \end{cases} \ ,$$

confirming that this Wilson line kills the nodes k and m' while preserving the rest.

If the Wilson line A and the radius R are supplied, the resulting group can be determined by checking which boundary conditions are saturated and keeping only the associated nodes in the GDD. To this end we might need to first bring the given A and R to the fundamental region by transformations including shifts and Weyl reflections of A in $\Gamma_8 \oplus \Gamma_8$, and the T-duality (2.4.6c).

From the GDD we can also determine the automorphisms of the lattice corresponding to any enhanced gauge group. They are just generated by Weyl reflections (2.4.5) associated to the surviving nodes. The fixed points of each reflection determine a 16-dimensional hyperplane in moduli space where the inequality associated to the given node is saturated. The intersection of r of these hyperplanes gives the $(17-r)$-dimensional subspace of moduli space where the given rank r gauge group is realized (maximal enhancements are realized at a point). This subspace is invariant under the subgroup of $O(1, 17, \mathbb{Z})$ generated by the r Weyl reflections associated to the surviving nodes.

There are 44 allowed groups with maximal rank $r = 17$, as can be seen by removing pairs of nodes from the GDD. They were given in a condensed form in Tables 2.13 and 2.14 and are shown with more details such as the global structure in Table D.1[16]. On the other hand, there are 1093 forbidden groups with $r = 17$, which clearly cannot be obtained from the GDD. One interesting case is the $r = 16$ enhancement $SO(16) \times SO(16) \times U(1)$. From the GDD it is evident that there is only one way of selecting the nodes and it is impossible to add a node without getting some non-ADE diagram. This group has an expected trivial enhancement to $SO(16) \times SO(16) \times SU(2)$, but this is one of the forbidden groups mentioned. It turns out that the necessary self-dual radius (eq. (2.3.20)) would be equal to 0, which is out of the moduli space. Another fact is that this is the only non-maximal group that cannot be enhanced. This is reflected on figure A.8, where the cyan curves do not have any intersection inside the moduli space[17]. In concordance to the previous observation, from this figure one can see that for $R = 0$ there are indeed intersections. To finish, we mention that the property of this moduli space of having one non-maximal non-enhanceable point is also shared by the moduli spaces for T^2 and T^3 compactifications of heterotic strings [37]. We will revisit this in the following chapters.

[16] Our method of removing nodes from the GDD only accounts for the algebra associated to the group, on next chapters we will describe different methods for computing the global part of it.

[17] All the other curves of enhancement displayed in this thesis have at least one intersection.

Chapter 3

Toroidal compactifications as lattice embeddings

In the previous chapter, we introduced T^d compactifications of heterotic strings and focused on the simplest case, $d = 1$, finding the 44 allowed gauge groups with maximal rank 17 and the corresponding moduli. The main objective of this chapter is to generalize this result to any value of d. To be able to obtain robust results it will be necessary to change the approach followed in most of the previous chapter (moving around in moduli space) and focus on the lattices that quantize the momenta. As it is well known, modular invariance requires that the momenta of the world-sheet fields take values on the even self-dual lattice $\Gamma_{d,d+16}$ [7]. As a consequence, the allowed gauge groups are such that their even positive definite root lattice can be embedded there. Thus, they can in principle be found using lattice embedding techniques, in particular the machinery developed by Nikulin [16], as advocated in [38]. For instance, Theorem 1.12.4 in [16] implies that any ADE group of rank less or equal than $(d + 8)$ can be embedded in $\Gamma_{d,d+16}$, and is thus realized in compactifications of the heterotic theory on T^d.

For $d = 2$ all allowed gauge groups are known from the work of Shimada and Zhang who classified all possible ADE types of singular fibers in elliptic K3 surfaces [14, 15]. As we will explain, the classification provides all possible heterotic gauge groups because the lattice embedding conditions are identical in the K3 and heterotic frameworks. This is consistent with duality between heterotic on T^2 and F-theory on K3.

Another problem is to obtain the resulting gauge group for specific moduli. On the last chapter we addressed this by organizing the left-moving component of the momenta into roots of an ADE group (see also [31]). However, since this method is cumbersome, it is desirable to develop a more powerful approach which could also be applied to the question of finding all possible groups. When $d = 1$ both problems can be solved using the Generalized Dynkin Diagram (GDD) associated to the Narain lattice $\Gamma_{1,17}$.

The generalization of the powerful GDD algorithm to higher dimensional compactifications clashes with the fact that, unlike $\mathrm{O}(1, 17; \mathbb{Z})$, the T-duality group $\mathrm{O}(d, d + 16; \mathbb{Z})$ is no longer generated by simple reflections. In the absence of a Dynkin diagram to describe $\Gamma_{d,d+16}$, what we can do to explore the landscape of heterotic strings on T^d, for generic dimension $d > 1$, is to develop alternative methods.

To begin we will revisit Nikulin's criteria, and apply them to compactifications of the heterotic string on T^d. The study of embeddings in $\Gamma_{d,d+16}$ will enable us to characterize the allowed gauge groups in terms of lattice data consisting of the pair (L, T), where L is the root

lattice of the group, and T is the dual lattice of the right-moving momenta. Conversely, (L, T) can be determined from the moduli that originate the group.

We also present other method to examine the toroidal landscape. We focus mainly on maximal enhancement in T^2 compactifications of the HE theory, but the algorithm works in higher dimensions. In particular, we will obtain all semisimple groups with maximal rank $d + 16$, occurring in $d = 1, 2$. Moreover, the moduli in the HO theory can be deduced from those of the HE theory by making use of an $\mathrm{O}(d, d + 16)$ transformation that generalizes the map constructed in [39] for $d = 1$.

The idea of this exploration algorithm is to start from a point of maximal enhancement, i.e. a rank $d + 16$ group with no U(1) factors, move along lines in moduli space where there is a breaking to a group with one U(1) factor, and then find all maximal enhancements that can be reached from the neighborhood of the initial point. We have fully exploited this technique in $d = 2$, finding all enhancements reported in [14]. We have also done explorations for $d = 3$ and 4, in connection to the work explained in Chapter 5.

This chapter is organized as follows. In Section 3.1, we briefly review the basics of heterotic compactification on T^d and present a simple method to find the transformation of the background fields under the action of $\mathrm{O}(d, d+16)$. We also review the map relating the charge vectors and moduli of the HE and HO theories on the circle, and formulate it for generic d. In Section 3.2 we state criteria, based on lattice embedding techniques, that can be used to detect whether a group is allowed or not. We additionally explain how to translate between heterotic moduli and lattice data. The notation and essential concepts about lattices that supplement this section are contained in Appendices B.1 and B.2. Compactifications on T^2 are the subject of Section 3.3. In Section 3.3.1, we present a computational algorithm to obtain the moduli underlying semisimple groups of maximal rank, exploring the neighborhood of points of maximal enhancement. In Section 3.3.2 we discuss several features of the models appearing in $d = 2$. Tables containing all the groups of maximal enhancement in one and two dimensions, and the points in moduli space where they arise, are presented in Appendix D.1.

3.1 Toroidal compactification of the heterotic string

Now we return to the toroidal compactifications introduced in Section 2.2. On the following we will derive the transformation rules for the moduli and the map that relates the HE and HO theories. It will be convenient to define the tensor E_{ij} given by

$$E_{ij} = G_{ij} + \frac{1}{2} A_i \cdot A_j + B_{ij} \,, \tag{3.1.1}$$

where $A_i \cdot A_j = A_i^I A_j^I$. The $d(d+16)$ parameters of the compactification are the $d \times d$ matrix E_{ij} and the d vectors A_i^I. The momenta 2.2.10 can be expressed as[1]

$$p_R = \frac{1}{\sqrt{2}} \left[n_i - E_{ij} w^j - \pi \cdot A_i \right] \hat{e}^{*i} \,, \tag{3.1.2a}$$

$$p_L = \frac{1}{\sqrt{2}} \left[n_i + (2G_{ij} - E_{ij})\, w^j - \pi \cdot A_i \right] \hat{e}^{*i} = \sqrt{2}\, w^i e_i + p_R \,, \tag{3.1.2b}$$

$$p^I = \pi^I + A_i^I w^i \,. \tag{3.1.2c}$$

Recall the mass and level matching equations invariant under $O(d, d+16; \mathbb{Z})$ (2.2.14a) and (2.2.14b). In the NS sector the lowest lying states have $N_R = \dfrac{1}{2}$ and their supersymmetric partners in R have $N_R = 0$. These states can be massless only if

$$\mathbf{p_R} = 0, \quad \mathbf{p_L}^2 + 2(N_L - 1) = 0 \,. \tag{3.1.3}$$

where the left and right momenta were given in 2.2.10.

Moreover, from (2.2.12) it follows that

$$\mathbf{p_L}^2 = 2 w^i n_i + \pi \cdot \pi \,. \tag{3.1.4}$$

For generic values of the moduli the only solution is $w^i = 0$, $n_i = 0$, $\pi^I = 0$, implying $\mathbf{p_L} = 0$, and $N_L = 1$ in (3.1.3). It gives rise to the gravity multiplet plus gauge multiplets of $\mathrm{U}(1)^{d+16}$. On the other hand, for special values of the moduli there can exist solutions with $N_L = 0$, and $\mathbf{p_L}^2 = 2$. The set of $\mathbf{p_L}$ then gives the roots of a Lie group G_r of rank $r \leq d+16$. In this case there will be gauge multiplets of a group $\mathrm{G}_r \times \mathrm{U}(1)^{d+16-r}$. The non-Abelian piece G_r is in turn a product of ADE factors of total rank r. Our main task for the next sections is to study which groups can occur and to determine the underlying moduli.

We will mostly work with the HE theory. The results for the HO can be deduced from the map discussed in Section 3.1.2.

3.1.1 Duality transformations of the moduli

In this section we present a simple way of finding the action of $\mathrm{O}(d, d+16)$ transformations on the background fields (G_{ij}, B_{ij}, A_i^I).

We first start by the transformation of the $2d+16$ charge vectors, defined as

$$|Z\rangle = |w^i, n_i; \pi^I\rangle \,. \tag{3.1.5}$$

which are just generalizations to T^d of the one introduced in Section 2.4.1 for the circle. The

[1] We clarify that the momenta can be written as a function of just E and A since G_{ij} can be written as $\frac{1}{2}(E_{ij} + E_{ji} - A_i \cdot A_j)$.

inner product between charge vectors is computed using the $\mathrm{O}(d, d+16)$ invariant metric

$$\eta = \begin{pmatrix} 0 & 1_{d\times d} & 0 \\ 1_{d\times d} & 0 & 0 \\ 0 & 0 & \delta_{IJ} \end{pmatrix} . \tag{3.1.6}$$

and is given by

$$\langle Z'|Z\rangle = w'^i n_i + n'_i w^i + \pi'^I \pi^I \ . \tag{3.1.7}$$

Given the generators $O \in \mathrm{O}(d, d+16; \mathbb{Z})$ presented in [1][2], the transformation of $|\tilde{Z}\rangle \equiv \eta|Z\rangle$ is simply [3]

$$|\tilde{Z}\rangle \ \to O\,|\tilde{Z}\rangle\,, \tag{3.1.8}$$

The transformation of the moduli can be obtained from the transformation of the generalized metric for the torus. It is generally simpler though to find the transformation of the moduli using the vielbein $\mathcal{E}$ for the generalized metric. This vielbein can be built using that the left and right-moving momenta (3.1.2) are

$$\mathbf{p} = \mathcal{E}|\tilde{Z}\rangle \ . \tag{3.1.9}$$

Under $\mathrm{O}(d, d+16)$, the vielbein transforms as

$$\mathcal{E} \to \mathcal{E}\,\eta O^T \eta \ . \tag{3.1.10}$$

From this transformation law it follows that the first d rows of $\eta\mathcal{E}$, which we write as

$$|\tilde{\mathcal{E}}_a\rangle \equiv \frac{1}{\sqrt{2}}\hat{e}_a^{i*}|E_{ik}, -\delta_i{}^j; A_i{}^I\rangle, \qquad a = 1, ..., d, \tag{3.1.11}$$

are $\mathrm{O}(d, d+16)$ vectors. Taking the transpose of (3.1.10) we find

$$|\tilde{\mathcal{E}}_a\rangle \to O\,|\tilde{\mathcal{E}}_a\rangle\,. \tag{3.1.12}$$

These vectors also form a negative definite orthonormal set:

$$\langle\tilde{\mathcal{E}}_a|\tilde{\mathcal{E}}_b\rangle = \frac{1}{2}\hat{e}_a^{i*}\hat{e}_b^{j*}(-2E_{ij} + A_i \cdot A_j) = \frac{1}{2}\hat{e}_a^{i*}\hat{e}_b^{j*}(-2G_{ij}) = -\delta_{ab}. \tag{3.1.13}$$

To get the transformation laws for the moduli under an $\mathrm{O}(d, d+16; \mathbb{R})$ element we simply construct the vectors $|\tilde{\mathcal{E}}_a\rangle$, transform them to $|\tilde{\mathcal{E}}'_a\rangle = O\,|\tilde{\mathcal{E}}_a\rangle$, and extract the transformed moduli E'_{ij}, A'_i. In practice, however, this procedure can be simplified as follows. Construct the $d \times (2d+16)$ matrix

$$\mathcal{A} \equiv \begin{pmatrix} E_{ij} & -\delta_i{}^j & A_i{}^I \end{pmatrix}, \tag{3.1.14}$$

with rows labeled $\mathcal{A}_i$. These differ from the vectors $|\tilde{\mathcal{E}}_a\rangle$ in that the factor $(1/\sqrt{2})\hat{e}_a^{*i}$ is missing (cf. eq. (3.1.11)). We may however interpret this as taking $\hat{e}_a^{*i} = \sqrt{2}\delta_a^i$, so that the rows $\mathcal{A}_i$ can

[2] This specific content was not included on this thesis.

[3] For instance, when $B_{ij} \to B_{ij} + \Theta_{ij}$, with $\Theta_{ij} = -\Theta_{ji} \in \mathbb{Z}$, $|Z\rangle \to |w^i, n_i + \Theta_{ij}w^j; \pi^I\rangle$.

also be transformed as $\mathrm{O}(d,d+16)$ vectors, $\mathcal{A}_i \to \mathcal{A}'_i = O\mathcal{A}_i$. From the new matrix $\mathcal{A}'$ one then extracts the moduli with the formula

$$\begin{pmatrix} E'_{ij} & -\delta_i{}^j & A_i^{I'} \end{pmatrix} = -\begin{pmatrix} \mathcal{A}'_{1,d+1} & \cdots & \mathcal{A}'_{1,2d} \\ \vdots & \ddots & \vdots \\ \mathcal{A}'_{d,d+1} & \cdots & \mathcal{A}'_{d,2d} \end{pmatrix}^{-1} \mathcal{A}' \,, \tag{3.1.15}$$

where on the right hand side we multiply by minus the inverse of the $d \times d$ middle block of $\mathcal{A}'$, which is the vielbein for the transformed metric e'_{ai}.

We now proceed to illustrate this method with a pair of examples where we restrict to the T-duality group $\mathrm{O}(d,d+16,\mathbb{Z})$. Consider first the case $d=2$, and apply the transformation given by the matrix

$$O_{\Lambda_1} = \begin{pmatrix} 1 & 0 & -\frac{1}{2}\Lambda_1^2 & 0 & \Lambda \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -\Lambda_1^t & 0 & 1_{16\times 16} \end{pmatrix}, \qquad \Lambda_1 \in \Upsilon_{16} \,, \tag{3.1.16}$$

which shifts A_1 by Λ_1. After transforming the rows of $\mathcal{A}$ with O_{Λ_1}, we obtain

$$\mathcal{A}' = \begin{pmatrix} E_{11} + \frac{1}{2}\Lambda^2 + \Lambda_1 \cdot A_1 & E_{12} & -1 & 0 & A_1 + \Lambda_1 \\ E_{21} + \Lambda_1 \cdot A_2 & E_{21} & 0 & -1 & A_2 \end{pmatrix}. \tag{3.1.17}$$

Since the second 2×2 block of $\mathcal{A}$ remains invariant, minus its inverse, which appears in (3.1.15), is the identity. The transformed E_{ij} and A_i can then be read off from eq. (3.1.17). In terms of the background fields G_{ij}, B_{12}, A_i, we see that

$$O_{\Lambda_1}: \qquad G'_{ij} = G_{ij}, \qquad B'_{12} = B_{12} - \frac{1}{2}\Lambda_1 \cdot A_2, \qquad A'_1 = A_1 + \Lambda, \qquad A'_2 = A_2. \tag{3.1.18}$$

This result highlights the fact that, generically, a shift of one Wilson line A_i by a vector $\Lambda_i \in \Upsilon_{16}$ must be accompanied by a b-field shift $B'_{ij} = B_{ij} - \frac{1}{2}\Lambda_i \cdot A_j$. The components of the charge vector $|Z\rangle$ transform as

$$O_{\Lambda_i}: \quad \pi^I \to \pi^I - \Lambda_i^I w^i, \quad n_i \to n_i - \frac{1}{2}\Lambda_i^2 w^i + \pi \cdot \Lambda_i, \quad n_j \to n_j \ (j \neq i) \,, \quad w^i \to w^i \,. \tag{3.1.19}$$

Now let us use this method to obtain the factorized duality O_{D_1}, which exchanges $n_1 \leftrightarrow w^1$ in generic dimension d. The action of O_{D_1} on the matrix $\mathcal{A}$ exchanges the first and the $(d+1)$th columns, and so

$$\begin{pmatrix} E'_{ij} & -\delta_i{}^j & A_i^{I'} \end{pmatrix} = \begin{pmatrix} -E_{11} & \delta^i{}_1 \\ \vdots & \vdots \\ -E_{d1} & \delta_d{}^i \end{pmatrix}^{-1} \begin{pmatrix} -\delta_1^1 & E_{1i} & E_{11} & -\delta^i{}_1 & A_1{}^I \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -\delta_d^1 & E_{di} & E_{d1} & -\delta_d{}^i & A_d{}^I \end{pmatrix}, \qquad i = 2, ..., d. \tag{3.1.20}$$

After performing this matrix operation, we obtain the transformation rules

$$E' = \frac{1}{E_{11}} \begin{pmatrix} 1 & -E_{1j} \\ -E_{i1} & E_{11}E_{ij} - E_{i1}E_{1j} \end{pmatrix}, \quad A'_i = \frac{1}{E_{11}} \begin{pmatrix} -A_1 \\ E_{11}A_i - E_{i1}A_1 \end{pmatrix}, \quad i,j = 2,...,d\,. \tag{3.1.21}$$

This result generalizes to a factorized duality in an arbitrary direction θ,

$$\begin{aligned} O_{D_\theta}: \quad E'_{\theta\theta} &= \frac{1}{E_{\theta\theta}}, \quad E'_{\theta j} = -\frac{E_{\theta j}}{E_{\theta\theta}}, \quad E'_{i\theta} = -\frac{E_{i\theta}}{E_{\theta\theta}}, \quad E'_{ij} = \frac{E_{\theta\theta}E_{ij} - E_{i\theta}E_{\theta j}}{E_{\theta\theta}}, \\ A'_\theta &= -\frac{A_\theta}{E_{\theta\theta}}, \qquad A'_i = \frac{E_{\theta\theta}A_i - E_{i\theta}A_\theta}{E_{\theta\theta}}, \qquad\qquad i,j = 1,...,d \neq \theta \end{aligned} \tag{3.1.22}$$

in agreement with the heterotic Buscher rules found originally in [40] and discussed also in [34].

3.1.2 The HE $\leftrightarrow$ HO map

Due to the uniqueness of the Narain lattices, the HO and HE theories compactified on T^d share the same moduli space. For the circle, an explicit map relating the charge lattices of both theories was given in [33] and the precise relation between the moduli was worked out in [39].

The $\mathrm{O}(1,17)$ transformation relating a basis of vectors of the $\Gamma_8 \oplus \Gamma_8$ embedding into $\Gamma_{1,17}$ to another one of the Γ_{16} embedding is given by [33]

$$\Theta_{\mathrm{E}\to\mathrm{O}} = O_{\Lambda_\mathrm{O}} O_\Omega O_{P_1} O_{D_1} O_{-\Lambda_\mathrm{E}}\,, \tag{3.1.23}$$

where O_{Λ_E}, O_{Λ_O} are shifts of the Wilson line by

$$\Lambda_\mathrm{E} = (0^7, 1, -1, 0^7)\,, \qquad \Lambda_\mathrm{O} = \left(\tfrac{1}{2}^8, 0^8\right)\,, \tag{3.1.24}$$

O_{D_1} is a T-duality in the circle direction, O_{P_1} an inversion and O_Ω a rescaling. Their action on the charge vectors and moduli is given by

$$\begin{aligned} O_\Lambda: &\quad |w,n;\pi\rangle \to |w, n+\pi\cdot\Lambda - \tfrac{1}{2}w\Lambda^2; \pi - w\Lambda\rangle\,, && (R,A) \to (R, A+\Lambda), \\ O_{D_1}: &\quad |w,n;\pi\rangle \to |n,w;\pi\rangle\,, && (R,A) \to \left(\frac{R}{R^2+\frac{1}{2}A^2}, -\frac{A}{R^2+\frac{1}{2}A^2}\right), \\ O_{P_1}: &\quad |w,n;\pi\rangle \to |-w,-n;\pi\rangle\,, && (R,A) \to (R,-A), \\ O_\Omega: &\quad |w,n;\pi\rangle \to |2w, \tfrac{1}{2}n;\pi\rangle\,, && (R,A) \to (\tfrac{1}{2}R, \tfrac{1}{2}A). \end{aligned} \tag{3.1.25}$$

Hence the total transformation (3.1.23) gives

$$\begin{aligned} \Theta_{\mathrm{E}\to\mathrm{O}}: &\quad w \to 2w - 2n + 2\pi\cdot\Lambda_\mathrm{E}, \quad n \to -2w + 2n + \pi\cdot(\Lambda_\mathrm{O} - 2\Lambda_\mathrm{E}), \\ &\quad \pi \to w(\Lambda_\mathrm{E} - 2\Lambda_\mathrm{O}) + 2n\Lambda_\mathrm{O} + \pi - 2\Lambda_\mathrm{O}(\Lambda_\mathrm{E}\cdot\pi), \\ &\quad R \to \frac{R}{2R^2 + (A-\Lambda_\mathrm{E})^2}, \quad A \to \frac{A-\Lambda_\mathrm{E}}{2R^2+(A-\Lambda_\mathrm{E})^2} + \Lambda_\mathrm{O}, \end{aligned} \tag{3.1.26}$$

corresponding to the $O(1,17,\mathbb{R})$ matrix

$$\Theta_{\mathrm{E}\to\mathrm{O}} = \begin{pmatrix} 2 & -2 & \Lambda_{\mathrm{O}} - 2\Lambda_{\mathrm{E}} \\ -2 & 2 & 2\Lambda_{\mathrm{E}} \\ 2\Lambda_{\mathrm{O}}^t & \Lambda_{\mathrm{E}}^t - 2\Lambda_{\mathrm{O}}^t & \mathbb{1}_{16} - 2\Lambda_{\mathrm{O}} \otimes \Lambda_{\mathrm{E}} \end{pmatrix}, \tag{3.1.27}$$

where $\otimes$ is an outer product.

Labeling $E_{\mathrm{E}} = R_{\mathrm{E}}^2 + \frac{1}{2}A_{\mathrm{E}}^2$ and the Wilson line A_{E} in the HE theory, the transformation (3.1.26) gives the HO moduli as [39]

$$(E_{\mathrm{O}}, A_{\mathrm{O}}) = \left(1 + \frac{A_{\mathrm{E}} \cdot \Lambda_{\mathrm{O}}}{2(E_{\mathrm{E}} + 1 - A_{\mathrm{E}} \cdot \Lambda_{\mathrm{E}})} \ , \ \frac{A_{\mathrm{E}} - \Lambda_{\mathrm{E}}}{2(E_{\mathrm{E}} + 1 - A_{\mathrm{E}} \cdot \Lambda_{\mathrm{E}})} + \Lambda_{\mathrm{O}}\right). \tag{3.1.28}$$

The map from HO to HE is simply obtained by exchanging $(E_{\mathrm{O}}, A_{\mathrm{O},}, \Lambda_{\mathrm{O}}) \leftrightarrow (E_{\mathrm{E}}, A_{\mathrm{E}}, \Lambda_{\mathrm{E}})$.

To extend (3.1.27) from the circle to T^d, it is sufficient to consider a decomposition of the Narain lattice of the form

$$\Gamma_{d,d+16} = \Gamma_{1,1} \oplus \cdots \oplus \Gamma_{1,1} \oplus \Gamma_8 \oplus \Gamma_8, \tag{3.1.29}$$

where the number of $\Gamma_{1,1}$ lattices is d.[4] We use $\Theta_{\mathrm{E}\to\mathrm{O}}$ to transform

$$\Theta_{\mathrm{E}\to\mathrm{O}}: \quad \Gamma_{1,1} \oplus \Gamma_8 \oplus \Gamma_8 \ \to \ \Gamma_{1,1} \oplus \Gamma_{16}, \tag{3.1.30}$$

choosing $\Gamma_{1,1}$ to be in the direction given by the torus lattice vector e_1, without loss of generality. This brings the Narain lattice into the form

$$\Gamma_{d,d+16} = \Gamma_{1,1} \oplus \cdots \oplus \Gamma_{1,1} \oplus \Gamma_{16}. \tag{3.1.31}$$

It follows that the desired extension is

$$\Theta_{\mathrm{E}\to\mathrm{O}}^{(d)} \to \mathbb{1}_{(2d-2)\times(2d-2)} \oplus \Theta_{\mathrm{E}\to\mathrm{O}} = \begin{pmatrix} \mathbb{1}_{(2d-2)\times(2d-2)} & 0 \\ 0 & \Theta_{\mathrm{E}\to\mathrm{O}} \end{pmatrix}, \tag{3.1.32}$$

which holds provided the ordering $|Z\rangle = |w^2, n_2, ..., w^d, n_d, w^1, n_1; \pi\rangle$ is used. In practice one may wish to keep the order in (3.1.5) and rearrange the entries of $\Theta_{\mathrm{E}\to\mathrm{O}}^{(d)}$ instead, which is reasonable for low values of d.

To get the transformation rules for the moduli, we proceed constructively using the factorized form of $\Theta_{\mathrm{E}\to\mathrm{O}}$ in (3.1.23), and generalizing each intermediate transformation. Each of the generalized transformation rules can be obtained by the method detailed in Section 3.1.1, which is valid not only for T-dualities but for generic $\mathrm{O}(d, d+16)$ transformations such as $O_{\Lambda_{\mathrm{O}}}$ (in HE) and O_{D_1}.

Let us first take a detailed look at the map $\Theta_{\mathrm{E}\to\mathrm{O}}$ for $d = 2$. The generalization to arbitrary d is straightforward. Preserving the usual ordering of the components of $|Z\rangle$, namely

[4] $\Gamma_{1,1}$ is the hyperbolic lattice with Gram matrix $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$.

$|w^1, w^2, n_1, n_2; \pi\rangle$, we write

$$\Theta^{(2)}_{\mathrm{E}\to\mathrm{O}} = \begin{pmatrix} 2 & 0 & -2 & 0 & \Lambda_{\mathrm{O}} - 2\Lambda_{\mathrm{E}} \\ 0 & 1 & 0 & 0 & 0 \\ -2 & 0 & 2 & 0 & 2\Lambda_{\mathrm{E}} \\ 0 & 0 & 0 & 1 & 0 \\ 2\Lambda^t_{\mathrm{O}} & 0 & 2\Lambda^t_{\mathrm{E}} - \Lambda^t_{\mathrm{O}} & 0 & \mathbb{1}_{16\times 16} - 2\Lambda_{\mathrm{O}} \otimes \Lambda_{\mathrm{E}} \end{pmatrix}, \tag{3.1.33}$$

The transformation rules for the quantum numbers are exactly the same as in the $d = 1$ case for w^1,n_1 and π, while w^2 and n_2 are invariant, as expected.

To work out the map, we proceed by applying the transformations in the RHS of (3.1.23) in succession. The Wilson line shift in direction 1 acts as

$$O_\Lambda : \quad E \to \begin{pmatrix} E_{11} - \Lambda \cdot A_1 + 1 & E_{12} \\ E_{21} - \Lambda \cdot A_2 & E_{22} \end{pmatrix}, \quad A_1 \to A_1 - \Lambda\,, \quad A_2 \to A_2. \tag{3.1.34}$$

Note that E_{12} is invariant since the b-field is also shifted (see the footnote 3). The factorized duality acts as

$$O_{D_1} : \quad E \to \frac{1}{E_{11}} \begin{pmatrix} 1 & -E_{12} \\ E_{21} & \det E \end{pmatrix}, \quad A_1 \to -\frac{A_1}{E_{11}}\,, \quad A_2 \to A_2 - \frac{E_{21}}{E_{11}} A_1\,, \tag{3.1.35}$$

and finally O_{P_1} and O_Ω produce the transformations

$$O_{P_1} \quad : \qquad E \to \begin{pmatrix} E_{11} & -E_{12} \\ -E_{21} & E_{22} \end{pmatrix}, \quad A_1 \to -A_1\,, \quad A_2 \to A_2\,, \tag{3.1.36}$$

$$O_\Omega \quad : \qquad E \to \begin{pmatrix} \frac{1}{4}E_{11} & \frac{1}{2}E_{12} \\ \frac{1}{2}E_{21} & E_{22} \end{pmatrix}, \quad A_1 \to \frac{1}{2} A_1\,, \quad A_2 \to A_2\,. \tag{3.1.37}$$

Putting all together, we get

$$\begin{aligned} \begin{pmatrix} E_{11} & E_{12} & A_1 \\ E_{21} & E_{22} & A_2 \end{pmatrix} \to & \begin{pmatrix} 1 & 0 & \Lambda_{\mathrm{O}} \\ \Lambda_{\mathrm{O}} \cdot A_2 & E_{22} & A_2 \end{pmatrix} \\ & + \frac{1}{E_{11} - \Lambda_{\mathrm{E}} \cdot A_1 + 1} \begin{pmatrix} \frac{1}{2} \\ \Lambda_{\mathrm{E}} \cdot A_2 - E_{21} \end{pmatrix} \begin{pmatrix} \Lambda_{\mathrm{O}} \cdot A_1 & E_{12} & A_1 - \Lambda_{\mathrm{E}} \end{pmatrix}. \end{aligned} \tag{3.1.38}$$

The map for generic d can be worked out in a similar fashion. The final result reads

$$\begin{pmatrix} E_{11} & E_{12} & \cdots & E_{1d} & A_1 \\ E_{21} & E_{22} & \cdots & E_{2d} & A_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ E_{d1} & E_{d2} & \cdots & E_{dd} & A_d \end{pmatrix} \to \begin{pmatrix} 1 & 0 & \cdots & 0 & \Lambda_{\rm O} \\ \Lambda_{\rm O}\cdot A_2 & E_{22} & \cdots & E_{2d} & A_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \Lambda_{\rm O}\cdot A_d & E_{d2} & \cdots & E_{dd} & A_d \end{pmatrix}$$
$$+ \frac{1}{E_{11}-\Lambda_{\rm E}\cdot A_1+1} \begin{pmatrix} \frac{1}{2} \\ \Lambda_{\rm E}\cdot A_2 - E_{21} \\ \vdots \\ \Lambda_{\rm E}\cdot A_d - E_{d1} \end{pmatrix} \begin{pmatrix} \Lambda_{\rm O}\cdot A_1 & E_{12} & \cdots & E_{1d} & A_1-\Lambda_{\rm E} \end{pmatrix}. \quad (3.1.39)$$

In the forthcoming sections we will apply the HE-HO map in compactifications to $d=1$ and 2 and give some examples for other values of d.

3.2 Embedding in Narain lattices

In this section we discuss how to determine which gauge groups $\mathrm{G}_r \times \mathrm{U}(1)^{d+16-r}$ occur in the compactification of perturbative heterotic strings on T^d. We are mostly interested in heterotic compactification on T^2, which is dual to F-theory compactifications on elliptic K3 surfaces [41]. Not surprisingly, for $d=2$ the problem of finding all allowed G_r happens to be related to the classification of possible singular fibers of ADE type in elliptic K3 surfaces. The explicit solution has been obtained in the K3 framework in [14, 15], using Nikulin's formalism. The results are expected to hold in the heterotic context too. The reason is that in the K3 context, the condition on the allowed G_r is that its even positive definite root lattice can be embedded in $\Gamma_{2,18}$ which is precisely the Narain lattice.

According to Theorem 1.12.4 in [16], any G_r of type ADE with $r \leq 10$ is allowed for $d=2$, as indeed found in [15]. For larger r more complicated conditions have to be verified as we will explain shortly. This program has been carried out in [15]. It turns out that for $r=11,12$, also all ADE G_r can be embedded in $\Gamma_{2,18}$. For $r=13$, only $13\mathrm{A}_1$ and $11\mathrm{A}_1+\mathrm{A}_2$ are precluded. Henceforth G_r will be denoted by the chain of ADE factors of its algebra. For $r=14$, except $8\mathrm{A}_1+\mathrm{E}_6$, all other forbidden groups, e.g. $14\mathrm{A}_1$, were predicted to be prohibited because singular fibers with such G_r could not fit in a K3 where the vanishing degree of the discriminant must be 24. For $r \geq 15$ there are many more forbidden groups. In particular, there are 1599 ADE groups of rank 18 [15] but according to the analysis of [14, 15], only 325 are expected to be realized in compactifications of the heterotic string on T^2. A natural question is why some groups are forbidden. To answer it, we will present some tools that can be applied to decide when a group is allowed or not. Our purpose is to illustrate the main ideas, not to do a systematic search as in [14, 15] for $d=2$.

We will mostly focus on the case of maximal enhancement, i.e. G_r with $r=d+16$. In 3.2.1, we will first discuss three criteria that can be applied for generic d. We then specialize to $d=1,2$, and in less detail to $d=8$. The criteria for groups with $r<16+d$ are presented

in Appendix B.2.1. The connection of the criteria to heterotic compactifications is addressed in Section 3.2.2 for generic d and in Section 3.2.3 for $d = 1$. We refer to [42–44] for short expositions of the main results of Nikulin's [16] relevant for our analysis, see also [45–48]. On Appendix B.1 we introduce the notation and some basic concepts that will be used in the following.

3.2.1 Embeddings of groups with maximal rank $r = d + 16$

The problem is to embed a lattice L of signature $(0, d + 16)$ in the even unimodular Narain lattice $\Gamma_{d,d+16}$. In the heterotic context L is the root lattice of a group of maximal rank arising upon compactification on T^d. Nikulin [16] provides powerful results that serve to determine whether or not such embedding exists. In particular, adapting respectively Corollary 1.12.3 and Theorem 1.12.4(c) of [16] to the case at hand leads to the criteria

Criterion 1
If $\ell(A_L) < d$ then L has a primitive embedding in $\Gamma_{d,d+16}$.

Criterion 2
L has a primitive embedding in $\Gamma_{d,d+16}$ if and only if there exists a lattice T of signature $(0, d)$ such that (A_T, q_T) is isomorphic to (A_L, q_L).

Here A_L and q_L are respectively the discriminant group and the quadratic discriminant form of L, whereas $\ell(A_L)$ is the minimal number of generators of A_L, and analogously for T (see Appendix B.1 for details). Since $\ell(A_T) \leq d$, groups with $\ell(A_L) = d$ could pass criterion 2 which actually requires $d(L) = d(T)$. We will shortly explain how the lattice T can be determined when $d = 1, 2$. There could exist more than one T, as found for some groups in [14]. Notice that in our conventions $(0, d)$ means positive signature.

Now, criteria 1 and 2 cannot be the whole story. We know groups with $\ell(A_L) > d$ that can be realized in heterotic compactifications on T^d. For example, when $d = 2$, heterotic moduli that give $L = 3\mathrm{E}_6$ are known. Hence, there should be an embedding of this L in $\Gamma_{2,18}$ even though $\ell(A_L) = 3$. We also know examples with $d = 1$. In particular, $L = \mathrm{D}_{16} + \mathrm{A}_1$ with $\ell(A_L) = 3$, would be forbidden by criterion 2 but must admit an embedding in $\Gamma_{1,17}$ because it certainly arises in the heterotic string on S^1. For $d = 1$, the 44 groups with maximal rank in Table 2.13 have $\ell(A_L) \leq 3$. Only the groups with $\ell(A_L) = 1$, e.g. $L = 2\mathrm{E}_8 + \mathrm{A}_1$, could possibly be allowed by criterion 2. The problem is that criteria 1 and 2 refer to *primitive* embeddings and this need not be the case. From the arguments in [14, 15] it transpires that this condition can be relaxed by demanding that L has an overlattice M which can be embedded primitively in the Narain lattice. For instance, we know that D_{16} has an overlattice given by the even unimodular HO lattice Γ_{16} with trivial discriminant group. Therefore, $L = \mathrm{D}_{16} + \mathrm{A}_1$ has an overlattice $M = \Gamma_{16} + \mathrm{A}_1$ with $A_M = \mathbb{Z}_2$ and $\ell(A_M) = 1$. The overlattice M could then pass criterion 2 with an even 1 dimensional lattice T equal to the A_1 lattice.

The above arguments lead to a third criterion obtained adapting Theorem 7.1 [15]. It reads

Criterion 3
L has an embedding in $\Gamma_{d,d+16}$ if and only if L has an overlattice M with the following properties:

(i) *there exists an even lattice T of signature $(0,d)$ such that (A_T, q_T) is isomorphic to (A_M, q_M),*

(ii) *the sublattice M_{root} of M coincides with L.*

Since L is an overlattice of itself, criterion 2 is a subcase of criterion 3. As explained in Appendix B.1, for an overlattice M to exist, there must be an isotropic subgroup H_L of A_L such that $M/L \cong H_L$ and $|H_L|^2 = d(L)/d(M)$. When criterion 3 is satisfied, $d(M) = d(T)$. We then obtain the useful relation

$$d(L) = d(T)|H_L|^2 \,. \tag{3.2.1}$$

We will refer to T as the complementary lattice in the following.

In the K3 framework, in which $d = 2$, H_L corresponds to the torsion part of the Mordell-Weil group, called MW in [14]. It can be checked that all pairs (L, T) in Table 2 of [14], reproduced in our Table D.2, satisfy the relation (3.2.1). We remark that there could exist more than one M, as found for some groups in [14].

In the work of Shimada and Zhang [14], the focus is on the classification of all possible ADE types of singular fibers of *extremal* elliptic K3 surfaces. Such a surface, called X, is characterized by having Picard number, $\rho(X)$, equal to 20, and finite Mordell-Weil group [42]. In this case the Néron-Severi lattice, NS_X, and the transcendental lattice, T_X, have signatures $(1, 19)$ and $(2, 0)$ respectively[5]. The lattice W_X has signature $(0, 18)$ and contains the sublattice $L(\Sigma)$ of rank 18, where Σ is the formal sum of the ADE types of singular fibers (determined by the Kodaira classification). It follows that $L(\Sigma)$ must admit an embedding in $\Gamma_{2,18}$. Now, in the heterotic compactification on T^2, the semisimple ADE groups of maximal rank 18 that can occur are such that their root lattice can be embedded in the Narain lattice $\Gamma_{2,18}$. Thus, the results of [14] for all possible $L(\Sigma)$ translate into all possible maximal enhancements in the heterotic compactification on T^2. Notice that the complementary lattice of criteria 2 and 3 above is related to the transcendental lattice by a change of sign of the Gram metric, i.e. $T = T_X\langle -1\rangle$. In Section 3.2.2 we will discuss to greater extent the connection to heterotic compactifications.

We illustrate below the application of criteria 1,2,3 to the cases $d = 1, 2$. We will also comment briefly on $d = 8$. In practice we first try criterion 1. If L passes it, then it is allowed. If not, we continue with criterion 2. If L satisfies it, we are done, otherwise we apply criterion 3. If L also fails criterion 3 we conclude that L is not allowed. A consistency check is that if L passes criterion 1 it must also fulfill criterion 3. Let us mention that the steps taken by Shimada and Zhang to compile their list, cf. section 3 in [14], indicate that they run a computer program based on the more general criterion 3.

[5]By definition, $NS_X = H^{1,1}(X,\mathbb{R}) \cap H^2(X,\mathbb{Z})$ and has signature $(1, \rho(X) - 1)$. The transcendental lattice is the orthogonal complement of NS_X in $H^2(X,\mathbb{Z})$ and has signature $(2, 20 - \rho(X))$. With the intersection form of X, the second cohomology group $H^2(X,\mathbb{Z})$ is isometric to $\Gamma_{3,19}$. The Néron-Severi lattice can be decomposed as $NS_X = \Gamma_{1,1} \oplus W_X$, where $\Gamma_{1,1}$ is generated by the zero section and the generic fiber. The lattice W_X is the orthogonal complement of $\Gamma_{1,1}$ in NS_X and has signature $(0, \rho(X) - 2)$. Thus, $\Gamma_{1,1} \oplus W_X \oplus T_X \subset \Gamma_{3,19}$.

3.2.1.1 $d = 1$

As a warm up we will study the $d = 1$ case which is simple yet instructive. Moreover, all allowed groups of maximal enhancement appearing in heterotic compactification on S^1 were already listed in Section 2.4. Thus, there are many examples to illustrate the application of the lattice embedding techniques.

When $d = 1$ the easy criterion 1 gives no information. When $\ell(A_L) = 1$ we then apply criterion 2. In Table 3.1 we give some examples of allowed groups. It is easy to propose the corresponding T because it must be $d(T) = d(L)$ and the (0,1) even lattices are of type $\mathrm{A}_1\langle m\rangle$, defined to be the A_1 lattice rescaled so that its basis vector has norm $u_1^2 = 2m$. One still has to check that the discriminant forms do match, more precisely that there is an isomorphism $(A_L, q_L) \cong (A_T, q_T)$. For example, for $L = \mathrm{D}_{17}$, A_L is generated by the spinor class with $s^2 = \frac{17}{4} = \frac{1}{4} \bmod 2$, so q_L takes values $\frac{j^2}{4} \bmod 2$, $j = 0, \ldots 3$. This matches the q_T of $\mathrm{A}_1\langle 2\rangle$ which takes the same values because $(u_1^*)^2 = \frac{1}{4}$. It is more challenging to check $L = \mathrm{E}_7 + \mathrm{A}_{10}$. For the proposed T, A_T is generated by u_1^* with $(u_1^*)^2 = \frac{1}{22}$, whereas A_L is generated by $w_{\mathbf{56}} \times w_1$ with $w_{\mathbf{56}}^2 = \frac{3}{2}$ and $w_1^2 = \frac{10}{11}$. To see that q_L and q_T match it suffices to verify that $\left(\frac{3}{2} + \frac{10j^2}{11} = \frac{1}{22} + 2k\right)$ is satisfied by integers j and k, e.g. $j = 4$, $k = 8$.

L	A_L	T
$2\mathrm{E}_8 + \mathrm{A}_1$	$\mathbb{Z}_2$	A_1
D_{17}	$\mathbb{Z}_4$	$\mathrm{A}_1\langle 2\rangle$
$\mathrm{E}_8 + \mathrm{D}_9$	$\mathbb{Z}_4$	$\mathrm{A}_1\langle 2\rangle$
$\mathrm{E}_7 + \mathrm{A}_{10}$	$\mathbb{Z}_2 \times \mathbb{Z}_{11} \cong \mathbb{Z}_{22}$	$\mathrm{A}_1\langle 11\rangle$

Table 3.1: Examples of allowed L with $\ell(A_L) = 1$, when $d = 1$.

The allowed groups with maximal enhancement of the form $L = \mathrm{E}_8 + \mathrm{E}_{9-p} + A_p$, $p = 1, \ldots, 9$, $p \neq 7$, all have $\ell(A_L) = 1$. Only for $p = 8$ there is an isotropic subgroup (actually for the A_8 component) but the M_{root} of the associated M is larger than L. Hence, all these groups should be allowed by criterion 2. We find that the corresponding T is $\mathrm{A}_1\langle\frac{p(p+1)}{2}\rangle$, $p = 1, \ldots, 6$, and $\mathrm{A}_1\langle\frac{(10-p)(p+1)}{2}\rangle$, $p = 8, 9$.

It is straightforward but cumbersome to check exhaustively which of the known groups with maximal enhancement and $\ell(A_L) = 1$ satisfy criterion 2, and if not apply criterion 3. In many cases, e.g. $L = \mathrm{E}_7 + \mathrm{E}_6 + \mathrm{A}_4$, $A_L = \mathbb{Z}_{30}$, one can quickly see that an overlattice cannot exist because there is no isotropic subgroup. Since this L is known to appear, criterion 2 should allow it, and indeed $T = \mathrm{A}_1\langle 15\rangle$ fulfills the conditions.

A neat example with $\ell(A_L) = 1$ is $L = \mathrm{A}_{17}$, $A_L = \mathbb{Z}_{18}$. The candidate T would be $\mathrm{A}_1\langle 9\rangle$ but the discriminant forms do not match because there are no integers j and k such that $\left(\frac{17j^2}{18} = \frac{1}{18} + 2k\right)$ is satisfied. Fortunately, A_{17} has an overlattice M associated to the isotropic subgroup $H_L = \mathbb{Z}_3$, generated by w_6 with $w_6^2 = 4 = 0 \bmod 2$. From (3.2.1) we see that $d(T) = 2$ so it must be $T = \mathrm{A}_1$. Since $d(M) = d(T)$ also $A_M = \mathbb{Z}_2$. It remains to check that the discriminant forms of A_M and A_T coincide. To this end we need to determine the orthogonal complement $H_L^\perp$ of H_L in A_L and restrict q_L to $H_L^\perp/H_L$. We then look for weights orthogonal to the generator w_6, i.e. weights such that $w_i \cdot w_6 = 0 \bmod 1$. Besides w_6 and w_{12} which belong

to H_L, w_3, w_9 and w_{15} are orthogonal. Now, $w_i^2 = \frac{1}{2} \bmod 2$, for $i = 3, 9, 15$. This confirms that $A_M = \mathbb{Z}_2$, with the discriminant form q_M taking values 0 and $\frac{1}{2}$. These are the same values taken by q_T. Finally, the root sublattice of M is equal to L because $w_6^2 = 4$.

We can also study known allowed groups with $\ell(A_L) \geq 2$ where criterion 3 must be applied. An example is the group with $L = \mathrm{E}_6 + \mathrm{A}_{11}$, $A_L = \mathbb{Z}_3 \times \mathbb{Z}_{12}$. There exists an overlattice with $H_L = \mathbb{Z}_3$ and it can be shown that criterion 3 is satisfied with $T = \mathrm{A}_1\langle 2\rangle$. For a second example take $L = \mathrm{A}_1 + \mathrm{A}_2 + \mathrm{A}_{14}$, $A_L = \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_{15} \cong \mathbb{Z}_6 \times \mathbb{Z}_{15}$. The piece $\tilde{L} = \mathrm{A}_2 + \mathrm{A}_{14}$ has an overlattice $\tilde{M}$ with $d(\tilde{M}) = 5$ so necessarily $A_{\tilde{M}} = \mathbb{Z}_5$. Thus, L has an overlattice $M = \mathrm{A}_1 + \tilde{M}$, $A_M = \mathbb{Z}_2 \times \mathbb{Z}_5 \cong \mathbb{Z}_{10}$ and a candidate T is $\mathrm{A}_1\langle 5\rangle$. With $\ell(A_L) = 3$ we already discussed how $L = \mathrm{D}_{16} + \mathrm{A}_1$ passes the test. In Table D.1 we give full results.

So far we have discussed groups with maximal enhancement which are known to occur. It is reassuring that they are allowed by the lattice embedding criteria but our main motivation was to understand why some groups are forbidden. Let us then finally offer a couple of examples of forbidden groups. Take $L = \mathrm{A}_6 + \mathrm{D}_{11}$, $A_L = \mathbb{Z}_{28}$. A candidate T is $A_1\langle 14\rangle$, but $q_T \ncong q_L$. An overlattice cannot exist because there is no isotropic subgroup of A_L. Thus, this L fails criteria 2 and 3. A less trivial example is $L = 2\mathrm{D}_8 + \mathrm{A}_1$, $A_L = \mathbb{Z}_2^5$. In Appendix B.1 we explained that D_8 admits E_8 as an overlattice. For L this leads to a full overlattice given by $M = 2\mathrm{E}_8 + \mathrm{A}_1$. Now $A_M = \mathbb{Z}_2$ and an adequate T would be A_1. However, condition (ii) in criterion 3 is not satisfied. As remarked in Appendix B.1, the root sublattice of $2\mathrm{E}_8$ is not equal to $2\mathrm{D}_8$. Actually, L admits also an overlattice $M' = \mathrm{E}_8 + \mathrm{D}_8 + \mathrm{A}_1$ with $A_{M'} = \mathbb{Z}_2^3$ and $\ell(A_{M'}) = 3$ so there can be no associated T. It would be interesting to study more examples of forbidden groups.

3.2.1.2 $d = 2$

When $d = 2$, criterion 1 implies that lattices with $\ell(A_L) = 1$ give allowed groups. In Table 3.2 we present a few examples of this type.

L	A_L	T
A_{18}	$\mathbb{Z}_{19}$	[2, 1, 10]
$\mathrm{A}_4 + \mathrm{E}_6 + \mathrm{E}_8$	$\mathbb{Z}_5 \times \mathbb{Z}_3 \cong \mathbb{Z}_{15}$	[2, 1, 8]
$\mathrm{A}_2 + \mathrm{A}_{16}$	$\mathbb{Z}_3 \times \mathbb{Z}_{17} \cong \mathbb{Z}_{51}$	[6, 3, 10]
$\mathrm{A}_8 + \mathrm{A}_{10}$	$\mathbb{Z}_9 \times \mathbb{Z}_{11} \cong \mathbb{Z}_{99}$	[10, 1, 10]
$\mathrm{A}_6 + \mathrm{A}_{12}$	$\mathbb{Z}_7 \times \mathbb{Z}_{13} \cong \mathbb{Z}_{91}$	[2, 1, 46]
$\mathrm{E}_6 + \mathrm{A}_{12}$	$\mathbb{Z}_3 \times \mathbb{Z}_{13} \cong \mathbb{Z}_{39}$	[4, 1, 10]

Table 3.2: Examples of allowed L with $\ell(A_L) = 1$, when $d = 2$. T is denoted by its Gram matrix $[u_1^2, u_1 \cdot u_2, u_2^2]$.

Before considering examples with $\ell(A_L) = 2$ let us describe how to find the lattice T. To begin, $d(T)$ is known because it must be equal to $d(L)$ or $d(M)$. Next, the even 2 dimensional lattices of determinant less than 50 are listed in Table 15.1 of [49], and for larger $d(T)$ they can be found using the `SageMath` module on binary quadratic forms [50]. Given T, the pair (A_T, q_T) can be deduced as explained in Appendix B.1. We then check if $(A_T, q_T) \cong (A_L, q_L)$.

Criterion 2 must also hold when $\ell(A_L) = 1$ since in this case the existence of a primitive

embedding is guaranteed by criterion 1. In Table 3.2 we have shown the corresponding matrices T. For example, with $d(T) = 19$ there is only the lattice $[2, 1, 10]$. It can be checked that $A_T \cong \mathbb{Z}_{19}$ and that the values of q_T are such that indeed (A_T, q_T) is isomorphic to (A_L, q_L) for $L = A_{18}$. For $L = \mathrm{A}_4 + \mathrm{E}_6 + \mathrm{E}_8$ we need a T with $d(T) = 15$. In this case there are two possible lattices, $[2, 1, 8]$ and $[4, 1, 4]$, both with $A_T = \mathbb{Z}_{15}$. It can be checked that only the discriminant form of the first does match q_L.

The allowed L's are given in Table 2 in [14]. It is a simple task to find A_L and $\ell(A_L)$. Groups accepted by criterion 2 have $\ell(A_L) = 2$ and $MW = [0]$. In our language trivial MW means trivial H_L, i.e. trivial overlattice $M = L$. There are many examples of this type. In Table 3.3 we show a few. To find T we proceed as explained before, looking first for even lattices of determinant $d(T) = d(L)$ and $A_T = A_L$. There might be more than one, the correct ones must have $(A_T, q_T) \cong (A_L, q_L)$. In Table 3.3 we have displayed in red candidates for T that are discarded because q_T is incongruent with q_L. The incorrect T's are more or less obvious. Checking the isomorphism for the correct ones is more laborious. For instance, for $L = \mathrm{E}_6 + \mathrm{D}_{12}$, the distinct values that can appear in q_L are in the set $\{0, \frac{1}{3}, 1, \frac{4}{3}\}$. Both T's have $A_T = \mathbb{Z}_2 \times \mathbb{Z}_6$, but the values of q_L can only be matched to the values in the T with $Q^{-1} = [\frac{1}{3}, -\frac{1}{6}, \frac{1}{3}]$.

L	A_L	T
$2\mathrm{D}_9$	$\mathbb{Z}_4 \times \mathbb{Z}_4$	[4, 0, 4]
$\mathrm{A}_4 + 2\mathrm{E}_7$	$\mathbb{Z}_5 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \cong \mathbb{Z}_{10} \times \mathbb{Z}_2$	[4, 2, 6] [2,0,10]
$\mathrm{E}_6 + \mathrm{D}_{12}$	$\mathbb{Z}_3 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \cong \mathbb{Z}_6 \times \mathbb{Z}_2$	[4, 2, 4] [2,0,6]
$\mathrm{A}_1 + \mathrm{A}_{17}$	$\mathbb{Z}_2 \times \mathbb{Z}_{18}$	[4, 2, 10] [2,0,18]

Table 3.3: Examples of allowed L with $\ell(A_L) = 2$, when $d = 2$. The candidates for T with $d(T) = d(L)$, but with $(A_T, q_T) \ncong (A_L, q_L)$, are displayed in red.

The example $L = \mathrm{A}_1 + \mathrm{A}_{17}$ is interesting because it also admits an overlattice. Indeed, in section 3.2.1.1 we saw that $\tilde{L} = \mathrm{A}_{17}$ has an overlattice $\tilde{M}$ with $\tilde{M}/\tilde{L} \cong \mathbb{Z}_3$, $A_{\tilde{M}} = \mathbb{Z}_2$ and $q_{\tilde{M}} = \{0, \frac{1}{2}\}$. Thus, the full L has an overlattice $M = \mathrm{A}_1 + \tilde{M}$ with $A_M = \mathbb{Z}_2 \times \mathbb{Z}_2$ and $M/L \cong \mathbb{Z}_3$. Now criterion 3 can be fulfilled with $T = [2, 0, 2]$. This agrees with results of [14] for this L.

When $\ell(A_L) \geq 3$ we can check that the allowed groups pass criterion 3 with the data given in Table 2 of [14]. One example is $L = 3\mathrm{A}_6$, $A_L = \mathbb{Z}_7^3$. There is an isotropic subgroup $H_L = \mathbb{Z}_7$ generated by $\mu = w_1(1) \times w_2(2) \times w_4(3)$, where $w_i(a)$ denotes weights of the a^{th} A_6 factor. Notice that $\mu^2 = 4 = 0 \bmod 2$. From (3.2.1), $d(M) = 7$ so necessarily $A_M = \mathbb{Z}_7$. Following the procedure to determine q_M shows that it matches the q_T of $T = [2, 1, 4]$ which is the unique even 2-dimensional lattice with $d(T) = 7$.

Finally we come to forbidden groups. Let us discuss the examples in Table 3.4. In all three there are no suitable lattices T. The possible candidates, shown in red, are discarded because their q_T does not match q_L. We conclude that these groups do not satisfy criterion 2 and continue to check criterion 3. In example 1 we know that D_8 has an overlattice E_8 so the full L has an overlattice $M = 2\mathrm{E}_8 + \mathrm{A}_2$, $M/L \cong \mathbb{Z}_2$ so $d(M) = 3$, consistent with $A_M = \mathbb{Z}_3$.

Now q_M matches the q_T of $T = [2,1,2]$ but still criterion 3 fails because $M_{\rm root} \neq L$. In example 2, there is an isotropic subgroup $H_L = \mathbb{Z}_2$ generated by $\mu = v \times w_2$, where v is the vector weight of D_{15} and w_2 is the weight of the **10** of A_3. Since $v^2 = 1$ and $w_2^2 = 1$, $\mu^2 = 2$. From (3.2.1), $d(M) = \frac{16}{2^2} = 4$. The only possible T with $d(T) = 4$ is $[2,0,2]$ and it could be that q_T matches q_M. However, M has elements $y + n\mu$, $y \in L$, $n = 0, 1$ and since $\mu^2 = 2$, $M_{\rm root} \neq L$. Hence, example 2 does not pass criterion 3. Concerning example 3, it flops criterion 3 because there is no isotropic subgroup of A_L. To see this, first observe that (3.2.1) implies that only $|H_L| = 7$ would be consistent with $d(M)$ being an integer. Thus, H_L would have to be $\mathbb{Z}_7$ and its generator would have to be a product of weights of the A_6's, say $\mu = w_i(1) \times w_j(2)$. However it is not possible to obtain $\mu^2 = 0 \bmod 2$.

#	L	A_L	T
1	$\mathrm{E}_8 + \mathrm{D}_8 + \mathrm{A}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \cong \mathbb{Z}_2 \times \mathbb{Z}_6$	[2, 0, 6] [4,2,4]
2	$\mathrm{D}_{15} + \mathrm{A}_3$	$\mathbb{Z}_4 \times \mathbb{Z}_4$	[4,0,4]
3	$2\mathrm{A}_6 + \mathrm{E}_6$	$\mathbb{Z}_7 \times \mathbb{Z}_7 \times \mathbb{Z}_3 \cong \mathbb{Z}_7 \times \mathbb{Z}_{21}$	[14,7,14]

Table 3.4: Examples of forbidden L when $d = 2$.

In summary, we have provided several examples where it was relatively simple to apply by hand the criteria that serve to determine whether a group of maximal rank is allowed or not. Clearly, to make a full search, or even to check more complicated examples, would require computer aid.

In Table D.2 we give the subgroups H_L and the lattice T for all the allowed L's found in the K3 framework [14]. They correspond to all maximal enhancements arising in heterotic compactifications on T^2.

3.2.1.3 $d = 8$

The case $d = 8$ is peculiar because there exists an even unimodular lattice of signature $(0,8)$, namely E_8. To see how this enters the analysis, consider $L = 3\mathrm{E}_8$ which has trivial A_L. Since $\ell(A_L) = 0$, this L easily passes criterion 1. Now, since criterion 2 must also be fulfilled there has to be an even lattice of signature $(0,8)$ and trivial A_T. This requires $d(T) = 1$ so $T = \mathrm{E}_8$. This indicates that in the heterotic on T^8 it is possible to obtain the group $3\mathrm{E}_8$. Indeed, it can be found in the HE by setting all the Wilson lines to zero and taking the internal torus with metric $G_{ij} = \frac{1}{2}\tilde{G}_{ij}$, where $\tilde{G}_{ij}$ is the Cartan matrix of E_8. The antisymmetric field must be chosen as

$$B_{ij} = \begin{cases} \frac{1}{2}\tilde{G}_{ij}, & i < j, \\ -\frac{1}{2}\tilde{G}_{ij}, & i > j, \\ 0, & i = j. \end{cases} \quad . \tag{3.2.2}$$

This is an example of the general type discussed in [11,33] in which $p_L - p_R$ belongs to the root lattice of an ADE group of rank d.

A second interesting example is $L = 24\mathrm{A}_1$, $A_L = \mathbb{Z}_2^{24}$. Since $\ell(A_L) = 24$, L fails criterion 1 and criterion 2 as well because $\ell(T) \leq 8$. To apply criterion 3 we recall that this L admits an

even unimodular overlattice given by one of the Niemeier lattices, say N_ψ, with $N_\psi/L \cong \mathbb{Z}_2^{12}$ (see chapter 16 in [49]). It is also known that the root lattice of N_ψ and L coincide. Thus, L fulfills criterion 3 with $M = N_\psi$ and $T = \mathrm{E}_8$. By the same token $L = 12A_2$ is also allowed by criterion 3. Niemeier lattices in heterotic compactifications on T^8 have appeared in [51].

3.2.2 Connection to heterotic compactifications

We have seen that the groups of maximal rank that can be embedded in $\Gamma_{d,d+16}$ are characterized by an ADE lattice L of rank $d+16$, the isotropic subgroup $H_L \subset A_L$, the associated overlattice M and the complementary even lattice T of rank d, satisfying $(A_T, q_T) \cong (A_M, q_M)$. The isotropic subgroup H_L is the torsion part of the embedding, in the sense that $M/L \cong H_L$. For an embedding to exist, it must be that $d(M) = d(T) = d(L)/|H_L|^2$. In the heterotic framework L is the root lattice of some gauge group with maximal enhancement. We now want to identify T, which we call the complementary lattice.

There is a natural candidate for an even lattice of rank d, namely the sublattice of $\Gamma_{d,d+16}$, denoted K, obtained by setting $\mathbf{p_L} = 0$. This is

$$K = \left\{ (\mathbf{p_R}; \mathbf{p_L}) \in \Gamma_{d,d+16} \;||\; \mathbf{p_L} = 0 \right\}. \tag{3.2.3}$$

Let us next examine the consequences of setting $\mathbf{p_L} = 0$. First, from (3.1.2c) we find that $p^I = 0$ implies

$$\pi^I = -w^i A_i^I \,. \tag{3.2.4}$$

Second, imposing $p_L = 0$ leads to

$$n_i = -w^j E_{ji} \,, \tag{3.2.5}$$

after substituting (3.2.4) in (3.1.2b). From $p_L = 0$ it further follows that

$$p_R = -\sqrt{2} w^i e_i \,. \tag{3.2.6}$$

Thus, p_R lies in a lattice of rank d as long as all the windings w^i are allowed to be different from zero. Since π is a vector in the gauge lattice Υ_{16}, the condition (3.2.4) can only be fulfilled with $w^i \neq 0$ if the Wilson lines A_i are quantized, in the sense that they are given by a vector in Υ_{16}, divided by a positive integer. We define the order of the Wilson line A_i as the smallest positive integer N_i such that

$$N_i A_i \in \Upsilon_{16} \quad (\text{no sum in } i) \,. \tag{3.2.7}$$

If $A_i = 0$, its order is 1. All A_i must be quantized so that (3.2.4) does not force some windings w^i to be identically zero. The quantization condition in (3.2.5) is also very restrictive. It clearly demands the E_{ij} to be rational numbers. Taking into account quantization of the Wilson lines then requires the T^d metric components $G_{ij} = e_i \cdot e_j$ to be rational numbers, which is consistent with p_R^2 being even. From now on we assume that K has rank d.

The constraints on the A_i and E_{ij} are compatible with having a gauge group of maximal

enhancement, which is the case under study. In fact, recall that to this end there must exist solutions to $\mathbf{p_R} = 0$ and $\mathbf{p_L}^2 = 2$. The former implies the condition $n_i = E_{ij}w^j + \pi \cdot A_i \in \mathbb{Z}$ (eq. (2.2.24)), which can be achieved with quantized A_i and rational E_{ij}.

The even lattice $K \subset \Gamma_{d,d+16}$ has signature $(d, 0)$ by construction. Applying Nikulin's Theorem 1.12.4 in [16], we learn that K admits a primitive embedding in $\Gamma_{d,d+16}$. It follows that the orthogonal complement of K in $\Gamma_{d,d+16}$ also admits a primitive embedding in $\Gamma_{d,d+16}$. This orthogonal complement is just the sublattice of $\Gamma_{d,d+16}$ defined by $\mathbf{p_R} = 0$ which we denote M, i.e.

$$M = \{(\mathbf{p_R}; \mathbf{p_L}) \in \Gamma_{d,d+16} \;||\; \mathbf{p_R} = 0\}\,. \tag{3.2.8}$$

The name M is appropriate because it is indeed the overlattice of criteria 3 with $M_{\text{root}} = L$. The reason is that M_{root} is the sublattice of M generated by vectors with $\mathbf{p_L}^2 = 2$ and it has rank $(d + 16)$ by the assumption of maximal enhancement.

So far we have argued that M of signature $(0, d + 16)$ is the orthogonal complement in $\Gamma_{d,d+16}$ of K of signature $(d, 0)$, and that K as well as M are primitively embedded in $\Gamma_{d,d+16}$. In fact, $\Gamma_{d,d+16}$ is an overlattice of $M \oplus K$. We can then apply Lemma 2.4 in [14] to conclude that there is an isomorphism $(A_M, q_M) \cong (A_K, -q_K)$. A proof of this lemma is presented in Appendix B.2.2. Finally, by Nikulin's Proposition 1.12.1 [16] there exists T of signature $(0, d)$ satisfying $(A_M, q_M) \cong (A_T, q_T)$. It is obtained by changing the sign of the Gram matrix of K, i.e.

$$T = K\langle -1\rangle\ . \tag{3.2.9}$$

Summarizing, the two rationality conditions $N_i A_i \in \Upsilon_{16}$ and $E_{ij} \in \mathbb{Q}$, guarantee the existence of the even $(0, d)$ lattice T, which in turn implies the existence of the even $(0, d + 16)$ lattice M with $(A_M, q_M) \cong (A_T, q_T)$. Thus, the rationality conditions are necessary to have maximal enhancement to a group of rank $d + 16$. However, these conditions are not sufficient to ensure that the sub-lattice M_{root} has rank $d + 16$. The additional constraint in criterion 3 is precisely that the gauge lattice L of rank $d + 16$ coincides with M_{root}.

3.2.2.1 Moduli from lattice data

If we have the set of charge vectors $|Z\rangle$ that generate certain L, it is quite easy to obtain the associated moduli A_i and E_{ij}. To see this, we rewrite the condition for massless states $\mathbf{p_R} = 0$ in terms of $|Z\rangle$ and the moduli vectors $|\tilde{\mathcal{A}}_i\rangle = |-\delta_i^j, E_{ik}; A_i^I\rangle$[6]

$$\langle \tilde{\mathcal{A}}_i | Z\rangle = 0,\ i = 1, \ldots, d \tag{3.2.10}$$

We have $d \times (d+16)$ unknowns and d equations for each of the $(d+16)$ independent vectors $|Z\rangle$, which makes it possible to get the moduli solving this system.

For simple Wilson lines there is a basis that is quite useful. For the HE theory, we write them as:

$$A_i = \left(\sum_{k=1}^{8} \gamma_k w_k\right) \times \left(\sum_{k=1}^{8} \gamma'_k w_k\right) \tag{3.2.11}$$

[6] $\mathbf{p_L}^2 = 2$ translates to $\langle Z|Z\rangle = 2$ and is satisfied trivially.

for some rational coefficients γ_k and γ'_k with w the fundamental weights of E_8 given in Table 2.1.

If $|Z\rangle$ corresponds to one of the first eight simple roots of the original $\mathrm{E}_8 \times \mathrm{E}_8$ symmetry, $|Z\rangle = |0,0,0,0;\alpha_j,0^8\rangle$, and eq. (3.2.10) is:

$$\left(\sum_{k=1}^{8} \gamma_k w_k\right) \cdot \alpha_j = \gamma_j = 0 \tag{3.2.12}$$

If we want most of the original symmetry to be preserved, the Wilson lines should be orthogonal to most of the roots. That is, most of the γ_k and γ'_k on (3.2.11) are zero. It is quite useful to write the Wilson lines on this basis, as one can see from the subscripts of the w_k that appear on the A_i, which of the original simple roots are killed. We use this basis to express the results in this chapter.[7]

3.2.2.2 Lattice data from moduli

Once we know the data (L,T) of the allowed groups G_r we still have to determine specific moduli A_i and E_{ij} that give rise to them. Conversely, given A_i and E_{ij}, in principle L is obtained from the solutions of $\mathbf{p_R} = 0$, $\mathbf{p_L}^2 = 2$, which correspond to the roots of G_r. On the other hand, T can be derived directly from the moduli as explained below.

The elements of T are of the form (3.2.6). Besides, the moduli must comply with the conditions (3.2.4) and (3.2.5). To make more concrete statements, consider first the case in which the E_{ij} are integers so that (3.2.6) is satisfied by any w^i. Then, a class of allowed values for the w^i are multiples of the Wilson lines orders, namely $w^i = \ell_i N_i$ (no sum over i), with $\ell_i \in \mathbb{Z}$. If we assume that this class exhausts all possibilities, T will be generated by a basis

$$u_1 = \sqrt{2}N_1 e_1,\ u_2 = \sqrt{2}N_2 e_2, \ldots,\ u_d = \sqrt{2}N_d e_d\,, \tag{3.2.13}$$

where we dropped an irrelevant sign. The Gram matrix of T will then be given by

$$Q_{ij} = u_i \cdot u_j = 2N_i N_j G_{ij} = N_i N_j (E_{ij} + E_{ji} - A_i \cdot A_j)\,. \tag{3.2.14}$$

Since this is valid for E_{ij} integers and $N_i A_i \in \Upsilon_{16}$, we see that the Q_{ij} are integers and the diagonal components are even, as required for an even lattice.

In some cases there might be more admissible values of the winding numbers w^i. In general, the allowed values are sets of integers $(M_1, M_2, \ldots, M_d)$ that satisfy

$$M_1 A_1 + M_2 A_2 + \cdots + M_d A_d \in \Upsilon_{16}\,, \tag{3.2.15a}$$

$$M_1 E_{1i} + M_2 E_{2i} + \cdots + M_d E_{di} \in \mathbb{Z}\,,\ i = 1,\ldots,d\,. \tag{3.2.15b}$$

In this situation a way to proceed is to obtain d solutions $(M_1^{(k)}, \ldots, M_d^{(k)})$, $k = 1,\ldots,d$,

[7] The original reason for using this basis in [2] was as a byproduct of the shift algorithm and some related techniques.

linearly independent (with Euclidean metric), such that the vectors

$$u_k = \sum_{\ell=1}^{d} \sqrt{2} M_\ell^{(k)} e_\ell \tag{3.2.16}$$

generate a lattice with the least volume. For instance, the vectors in (3.2.13) are recovered when $E_{ij} \in \mathbb{Z}$ and the only solutions of (3.2.15a) are $M_\ell^{(k)} = N_\ell \delta_{\ell k}$ (no sum over ℓ). In the general case we have to impose the condition of least volume. To be more precise, define the matrix C with elements $C_{k\ell} = M_\ell^{(k)}$, i.e. the rows of C are the solutions of (3.2.15). The Gram matrix of T then reads

$$Q_{k\ell} = u_k \cdot u_\ell = 2(C\, G\, C^t)_{k\ell}\,, \tag{3.2.17}$$

where we used $G_{ij} = e_i \cdot e_j$. Therefore, $\det Q = 2^d (\det C)^2 \det G$. Since the determinant of the torus metric is fixed by the choice of moduli A_i and E_{ij}, to obtain the least lattice volume it suffices to choose C with least determinant. Hadamard's inequality then instructs us to choose d independent solutions $(M_1^{(k)}, \ldots, M_d^{(k)})$ of (3.2.15) with the least norm. To check that $Q_{k\ell}$ are integers and the diagonal elements are even, we write $G_{ij} = \frac{1}{2}(E_{ij} + E_{ji} - A_i \cdot A_j)$, and take into account that the $M_i^{(k)}$ verify (3.2.15). Finally, Q is unique up to the action of $\mathrm{GL}(d, \mathbb{Z})$. For $d = 2$ we can use the procedure described in section 3, Chapter 15, of [49] to bring Q to the standard reduced form used in [14].

In the next sections we will discuss systematic methods to determine moduli associated to groups of maximal enhancement when $d = 1$ and $d = 2$. We will then exemplify further how T computed from the moduli matches the T from the lattice embedding data. Meanwhile it is instructive to illustrate the main points in cases with generic d.

For a simple example, consider moduli $A_i = 0$, $G_{ij} = \frac{1}{2}\tilde{G}_{ij}$, where $\tilde{G}_{ij}$ is the Cartan matrix of an ADE group $\tilde{\mathrm{G}}_d$ of rank d, and B_{ij} is given in (3.2.2). The E_{ij} moduli are found to be

$$E_{ij} = \begin{cases} \frac{1}{2}\tilde{G}_{ij}, & i = j, \\ \tilde{G}_{ij}, & i < j, \\ 0, & i > j \end{cases} . \tag{3.2.18}$$

Therefore, the E_{ij} are either 1, -1 or 0. In this setup the gauge group of the heterotic string on T^d is $2\mathrm{E}_8 + \tilde{\mathrm{G}}_d$ in the HE or $\mathrm{D}_{16} + \tilde{\mathrm{G}}_d$ in the HO. This example is of the general type in which all Wilson lines are set to zero and $p_L - p_R \in \tilde{\Gamma}_d$, where $\tilde{\Gamma}_d$ is the root lattice of $\tilde{\mathrm{G}}_d$ [11, 33]. From the lattice formalism we find that $T = \tilde{\Gamma}_d$. From the moduli we obtain the same result for T because the basis is given in (3.2.13) with $e_i = \frac{1}{\sqrt{2}}\tilde{e}_i$ and $N_i = 1$.

A second example in the HO on T^d has moduli [11, 33]

$$e_i^a = \frac{1}{\sqrt{2}}\delta_i^a\,, \quad B_{ij} = 0\,, \quad A_i^I = \delta_i^I \ \text{ with } i \le d \tag{3.2.19}$$

It can be shown that the resulting group is D_{d+16}. All Wilson lines have order $N_i = 2$. Besides, $E_{ij} = \delta_{ij}$ so that the condition (3.2.15b) does not constrain the M_i. For $d = 1$ we can just take $M_1 = N_1 = 2$ so that $u_1 = 2$ and $T = \mathrm{A}_1\langle 2\rangle$ as we found with the lattice formalism in Section 3.2.1.1. For $d \ge 2$ there are solutions to (3.2.15a) other than $M_i^{(j)} = 2\delta_{ij}$. For instance,

$A_1 \pm A_2 \in \Gamma_{16}$. The $M_i^{(j)}$ can be chosen so that the u_i are the roots of D_d. Thus, $T = \mathrm{D}_d$.

Another important question in the heterotic context is the meaning of the quadratic discriminant form q_T. The answer is that the values that p_R^2 can take are precisely given by q_T mod 2. This follows because p_R generically lies in the dual lattice T^*. When T has basis (3.2.13), it is easy to see from (3.1.2a) that p_R indeed takes values in a lattice generated by $u^{*i} = \frac{1}{\sqrt{2}N_i}\hat{e}^{*i}$, with Gram matrix the inverse of Q in (3.2.14). When there are additional solutions to (3.2.15), so that the basis for T is given by (3.2.16), p_R lies in a lattice spanned by

$$u^{*i} = \frac{1}{\sqrt{2}} \sum_{k=1}^{d} C^{ki} \hat{e}^{*k} \,, \tag{3.2.20}$$

where $C^{ki} = C_{ki}^{-1}$ and as before $C_{k\ell} = M_\ell^{(k)}$. Thus, $u^{*i} \cdot u^{*j} = Q^{ij} = Q_{ij}^{-1}$, with Q the Gram matrix in (3.2.17). The fact that q_T gives the values of p_R^2 is useful to determine the spectrum of massive states.

3.2.3 Circle compactifications

In this section, we consider again compactifications of the heterotic string on the circle, where the moduli are the radius R and the 16-dimensional Wilson line A^I. The problems of finding all possible gauge groups $\mathrm{G}_r \times \mathrm{U}(1)^{17-r}$ and the corresponding moduli (R, A^I), was solved in Section 2.4 by means of the Generalized Dynkin Diagram (GDD) associated to $\Gamma_{1,17}$. Here we will discuss the connection with the lattice embedding formalism.

As mentioned previously, there are 44 different groups of maximal rank that are realized in heterotic compactification on S^1. We collect them in Table D.1 in Appendix D.1, where they are denoted by its root lattice L. The Table includes the moduli $(R_{\mathrm{E}}, A_{\mathrm{E}})$ and $(R_{\mathrm{O}}, A_{\mathrm{O}})$ in the HE and HO theories respectively. For both the moduli lie in the fundamental regions defined in Table 2.12. As explained previously, they can be obtained by just saturating the inequalities associated with the nodes we are keeping. In the case of the HE it is given by (2.4.7), with an analogous expression for the HO. Also, the latter can be derived from the map (3.1.28) too. In all cases $E_{\mathrm{E}} = E_{\mathrm{O}} = 1$. By looking at the GDD we deduce that all ADE G_r of $r \leq 9$ are allowed, which is consistent with Theorem 1.12.4 in the Nikulin formalism [16]. The diagram also shows that for $r = 10$ all ADE G_r can appear and that for $r = 11$ only $11\mathrm{A}_1$ is forbidden.

For each maximal group in Table D.1 we also give its discriminant group $A_L = L^*/L$, its appropriate isotropic subgroup H_L, and its complementary lattice T. For the lattice T, the notation $\mathrm{A}_1\langle m\rangle$ is simplified to $\langle m\rangle$. Besides, $d(T) = 2m$. It is easy to check that in all cases $d(L) = d(T)|H_L|^2$ holds. For all groups we have verified the isomorphism $(A_M, q_M) \cong (A_T, q_T)$, which is less trivial when $H_L \neq \mathbb{1}$. Some examples were worked out in Section 3.2.1.1.

It is a compelling exercise to deduce the lattice T from the moduli as explained in Section 3.2.2. For $d = 1$ there is only one Wilson line and the simple result (3.2.13) is valid. Thus, T is generated by

$$u = \sqrt{2}NR \,, \tag{3.2.21}$$

where N is the order of A and we used $e_1 = R$. The Gram matrix is then $Q = 2N^2R^2 = d(T)$.

On the lattice side, $T = \mathrm{A}_1\langle m\rangle$ with $d(T) = 2m$. Therefore, it must be that

$$2N^2R^2 = 2N^2(1 - \tfrac{1}{2}A^2) = 2m\,, \tag{3.2.22}$$

where we used that $E = 1$ in all cases of maximal enhancement. It is straightforward to confirm this relation using the data for m and A in Table D.1. In the HE case the Wilson line A_{E} is given in (2.4.7) and the order is

$$N_{\mathrm{E}} = \frac{\kappa_k\,\kappa'_m}{\gcd(\kappa_k, \kappa'_m)}\,. \tag{3.2.23}$$

Another interesting question is the relation of generic p_R to the complementary lattice T. In Section 3.2.2 we argued that in general p_R takes values in T^*. When $d = 1$ the proof is rather simple. Since $E = 1$, (2.3.1) reduces to

$$p_R = \frac{1}{\sqrt{2}R}(n - w - \pi\cdot A)\,. \tag{3.2.24}$$

We now use that A has order N to set $\pi\cdot A = \tilde{l}/N$, $\tilde{l}\in\mathbb{Z}$. Inserting in p_R above gives $p_R = \frac{l}{\sqrt{2}NR}$, with l integer. Hence, p_R lies in a lattice generated by u^*, with u the generator of T in (3.2.21). We conclude that p_R lies on T^* and the allowed values of p_R^2 are $q_T \bmod 2$.

3.3 Compactifications on T^2

In heterotic compactification on T^2 there are 36 real moduli, namely $\{G_{11}, G_{12}, G_{22}, B_{12}\}$, plus two 16-dimensional Wilson lines $\{A_1^I, A_2^I\}$. The $\Gamma_{2,18}$ lattice vectors $(p_R; p_L, p^I)$, which depend on these moduli, are given in (3.1.2). For the purpose of studying enhancement of symmetries it is actually more appropriate to use as moduli the components E_{ij}, cf. (3.1.1), together with the A_i^I. Indeed, as we have seen in Section 3.2.2, enhancement requires the E_{ij} to be rational numbers and the A_i to be quantized in the sense of eq. (3.2.7). In Section 3.3.1 we treat the problem of determining all gauge groups $\mathrm{G}_r\times\mathrm{U}(1)^{18-r}$ that can appear, and the corresponding moduli.

The extension of the systematic procedure discussed in Section 2.4 to compactifications on T^2 would require the construction of a Generalized Dynkin Diagram for $\Gamma_{2,18}$. However, it has been argued that the even, self-dual lattices of signature (p, q) with both $p, q > 1$ (that is, with a signature with more that one negative sign), do not possess a system of simple roots and cannot be described in terms of generators and relations similar to Kac-Moody or Borcherds algebras [52]. Nevertheless, although the addition of a new Kac-Moody simple root introduces multiple links and loops in the structure of the quadruple extension of simple Lie algebras, it was shown in [53] that the "simple-links" structure can be preserved if the extra root is a Borcherds (imaginary) simple root. In any case, a GDD for $\Gamma_{2,18}$ is not known and it is not even clear whether it exists. Hence, we will proceed in a constructive way.

In Section 3.2 we explained that all allowed groups $\mathrm{G}_r\times\mathrm{U}(1)^{d+16-r}$ in heterotic compactification on T^d can be obtained by lattice embedding techniques. For T^2 the full results are known from the work of Shimada and Zhang who classified all possible ADE types of singular

fibers in elliptic K3 surfaces [14, 15]. The classification translates into all possible heterotic gauge groups because the lattice embedding conditions are the same in the K3 and heterotic contexts.

Knowing all allowed groups it remains to compute the corresponding moduli. We will focus in the HE since the moduli in the HO can be derived from the map elaborated in Section 3.1.2. We will mostly consider the case of maximal enhancement, i.e. $r = 18$. As argued in Section 3.2.2, this can occur only if the E_{ij} are rational numbers and the A_i are quantized.

In Section 3.3.1 we will develop a procedure in order to obtain all groups of maximal rank. The results are summarized in Section 3.3.2.

3.3.1 Exploring the moduli space (Neighborhood algorithm)

To tackle the problem of getting all the maximal enhancements on T^2 compactifications, we pursued different methods: construction of Extended Dynkin diagrams, the Fixed Wilson line algorithm and the Neighborhood algorithm. Even though each one of them gives different insights, we will only include in this thesis the latter, which is the one that succeeded in this task. The former methods can be consulted in [2]. We will apply the algorithm in the HE theory. The moduli for the $\mathrm{Spin}(32)/\mathbb{Z}_2$ theory will then be determined using the map described in Section 3.1.2.

The main idea is to find new maximal enhancements that are *close* to those already found. More precisely, we start at a point of maximal enhancement where the group G_{18}, and its 18 simple roots, are known. Then we move along surfaces in moduli space where the symmetry is broken to $\mathrm{G}_{17} \times \mathrm{U}(1)$. On each of these 18 surfaces G_{17} will have 17 of the 18 original simple roots. For each surface we collect the candidate extra simple root that would give back an ADE group of rank 18. For each candidate we compute the moduli, A_i and E_{ij}, by imposing that the 18 simple roots correspond to states that satisfy the massless conditions[8]

$$n_1 = E_{11}w^1 + E_{12}w^2 + \pi \cdot A_1, \quad n_2 = E_{21}w^1 + E_{22}w^2 + \pi \cdot A_2\,, \tag{3.3.1a}$$

$$\pi^2 + 2w^1 n_1 + 2w^2 n_2 = 2\,. \tag{3.3.1b}$$

We then check that the torus metric G_{ij} is well defined and finally read the gauge group from the simple roots.[9] We also compute the Gram matrix Q corresponding to the moduli, as explained in Section 3.2.2.2, giving us the complementary lattice T. We end with a list of points of maximal enhancement that are on the neighborhood of the original point, i.e. they are connected through a 17-dimensional enhancement surface. The algorithm can be repeated to explore regions of the moduli space that are far away from the starting point.

We illustrate the algorithm with an example defined by the starting point $A_1 = A_2 = 0$,

[8] Which in practice is done by imposing eq. (3.2.10).

[9] We developed a routine that takes a base of simple roots and detects if its Dynkin diagram is of ADE type and, in that case, it identifies the group.

$E_{ij} = \delta_{ij}$, where the gauge group is $2A_1 + 2E_8$. The charge vectors of the 18 simple roots are

$$\begin{aligned} \varphi_j &= |0,0,0,0;\alpha_j,0^8\rangle\,, \quad \varphi'_j = |0,0,0,0;0^8,\alpha'_j\rangle\,, \quad j=1,\ldots,8, \\ \varphi_{\mathtt{C}_1} &= |1,0,1,0;0^8,0^8\rangle\,, \quad \varphi_{\mathtt{C}_2} = |0,1,0,1;0^8,0^8\rangle\,. \end{aligned} \tag{3.3.2}$$

They form the DD of Figure 3.1.

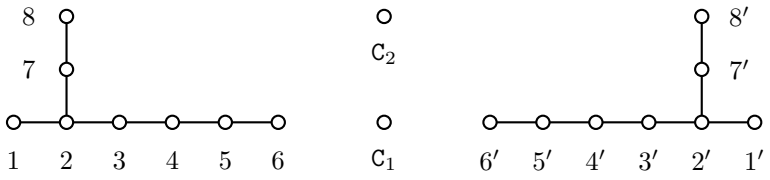


Figure 3.1: Dynkin diagram of $2A_1 + 2E_8$

Now we want to move along directions that preserve 17 of the 18 simple roots by deleting one node. Since the DD is symmetric under the interchange of the node $[j]$ with the node $[j']$, it suffices to remove one of the nodes $[j]$. We are then effectively breaking $E_8 + 2A_1$ by erasing one node. The nodes $\mathtt{C}_1$ and $\mathtt{C}_2$ are also interchangeable. We choose to always keep $\mathtt{C}_2$. There are thus only 9 inequivalent breakings, obtained by deleting either $\mathtt{C}_1$ or one of the 8 nodes of E_8. Altogether, the 17 surviving simple roots are the 18 original ones in (3.3.2), except for the one corresponding to the removed node. Afterwards we add a new node which clearly cannot be connected to any of the 8 nodes $[j']$ associated to the second E_8, since the resulting diagram has to be of type ADE. Hence, only algebras of the form $G_{10} + E_8$ can arise. For convenience we ignore the second E_8 unless otherwise stated.

To further elaborate on the algorithm we analyze first the case in which the node $\mathtt{C}_1$ is removed. The effect is simply to break $E_8 + 2A_1$ to $E_8 + A_1$. We then add one node, called $\mathtt{N}$, to its Dynkin diagram. The 2 possibilities for the connections of the new node are displayed in Figure 3.2. Generically, the charge vector corresponding to $\mathtt{N}$ is

$$\varphi_{\mathtt{N}} = |w^1, w^2, n_1, n_2; \pi^1, \ldots, \pi^8, 0^8\rangle\ . \tag{3.3.3}$$

The last 8 components of π are zero just because the new node is always disconnected from the second E_8. The way that $\mathtt{N}$ is linked in each of the possible Dynkin diagrams gives 9 conditions for the 12 unknowns w^i, n_i, plus the eight non-zero components of π. We use these conditions to determine all except 3 of the unknowns. It is convenient, and always possible, to leave w^1 and w^2 undetermined.

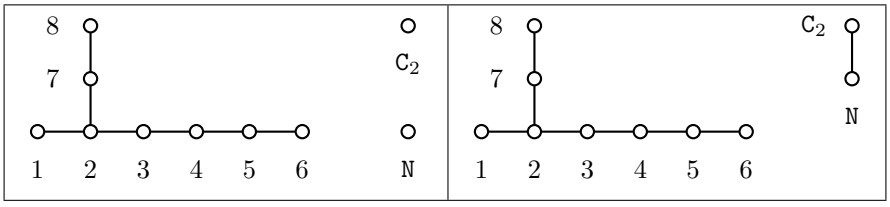


Figure 3.2: Dynkin diagrams corresponding to the possible ways of adding a node $\mathtt{N}$ to the diagram of $E_8 + A_1$.

We just consider all possible values for the 3 unknowns, with a fixed bound for the maximum

of their absolute values. For computational reasons, this truncation is necessary to avoid infinitely many possibilities[10]. Concretely, we introduce two parameters λ_1 and λ_2, which define the truncation, and consider only states with

$$|w^i| \leq \lambda_1, \quad |n_i| \leq \lambda_2, \quad |\pi^I| \leq \lambda_2 \,. \tag{3.3.4}$$

For this example it is enough to use $\lambda_1 = 1$ and $\lambda_2 = 2$. Afterwards, we filter all the candidates by imposing that $\varphi_{\mathtt{N}}$ has norm squared 2 and $\pi \in \Upsilon_{16}$. In some cases it might occur that, regardless of the values of λ_1 and λ_2, there are actually no solutions with $w^i, n_i \in \mathbb{Z}$ and $\pi \in \Upsilon_{16}$.

The case of $\mathrm{E}_8 + 2\mathrm{A}_1$, on the left in Figure 3.2, is rather trivial because we are just restoring the deleted node $\mathtt{C}_1$. The algorithm will find charge vectors $\varphi_{\mathtt{N}}$ which are not necessarily equal to $\varphi_{\mathtt{C}_1}$, but at the end of the day all of them should be equivalent to it. When we compute the moduli we obviously get $E_{ij} = \delta_{ij}$, $A_1 = A_2 = 0$, or some T-dual point. We just restored the simple root that we removed, thus returning to the original point in the moduli space. In general, this possibility will occur in all the breakings.

In the less trivial case $\mathrm{E}_8 + \mathrm{A}_2$, on the right of Figure 3.2, $\mathtt{N}$ is linked to $\mathtt{C}_2$. Imposing $\langle \varphi_{\mathtt{C}_2} | \varphi_{\mathtt{N}} \rangle = -1$, implies $n_2 = -1 - w^2$. Considering all the possible values for the 3 unknowns w^1, w^2 and n_1, with the bounds in (3.3.4), and filtering by requiring $\langle \varphi_{\mathtt{N}} | \varphi_{\mathtt{N}} \rangle = 2$, gives the list

$$|1,0,1,-1;0^8,0^8\rangle, \quad |-1,-1,-1,0;0^8,0^8\rangle, \quad |1,-1,1,0;0^8,0^8\rangle, \quad |-1,0,-1,-1;0^8,0^8\rangle \,. \tag{3.3.5}$$

We next deduce the moduli by demanding that the charge vectors of the full set of 18 simple roots satisfy the quantization conditions (3.3.1a). This is a well posed problem because in general there are 36 moduli to determine and the 18 simple roots give two equations each. In this case we readily find $A_1 = 0$ and $A_2 = 0$. From $\varphi_{\mathtt{C}_2}$ we obtain $E_{12} = 0$ and $E_{22} = 1$, whereas from $\varphi_{\mathtt{N}}$, $n_1 = E_{11} w^1$ and $-2w^2 - 1 = E_{21} w_1$. The 4 elements in the list (3.3.5) solve these equations with $E_{11} = 1$, and E_{21} equal to 1 or -1. It is easy to see that the corresponding G_{ij} is well defined and that these points are T-dual to each other.

The algorithm proceeds in the same fashion for all the 9 possible breakings of $\mathrm{E}_8 + 2\mathrm{A}_1$. For a more fruitful example, let us consider the breaking to $\mathrm{A}_7 + 2\mathrm{A}_1$, obtained by removing the node φ_1. Appending a new node $\mathtt{N}$ leads to various possible enhancements. For instance, $\mathtt{N}$ can connect only to φ_8 to form $\mathrm{A}_8 + 2\mathrm{A}_1$. With $\lambda_1 = 1$ and $\lambda_2 = 2$ in the bounds (3.3.4), we find that the charge vectors of φ_N can be one of

$$|-1,0,1,0;-w_8,0^8\rangle \,, \quad |0,-1,0,1;-w_8,0^8\rangle \,, \quad |1,0,-1,0;-w_8,0^8\rangle \,, \quad |0,1,0,-1;-w_8,0^8\rangle \,. \tag{3.3.6}$$

The moduli are determined as explained before. Taking into account all nodes except $\mathtt{N}$, we arrive at

$$E_{ij} = \delta_{ij}, \quad A_1 = \gamma_1 w_1 \times 0, \quad A_2 = \gamma_2 w_1 \times 0 \,, \tag{3.3.7}$$

where (γ_1, γ_2) are some free parameters. The above moduli determine a slice of moduli space with group $\mathrm{SU}(8) \times \mathrm{SU}(2)^2 \times \mathrm{E}_8 \times \mathrm{U}(1)$. Finally imposing the quantization conditions (3.3.1a) to the possible charge vectors for $\varphi_{\mathtt{N}}$, cf. (3.3.6), fixes $(\gamma_1, \gamma_2) = \pm(\underline{\tfrac{2}{5}, 0})$, where underlining means permutations. With

[10] This limitation was overcome in subsequent more general versions of the algorithm made for the exploration of the CHL string, which is the subject of the following chapter (see Section 4.3).

these values we reach the rank 18 group with algebra $\mathrm{A}_8 + 2\mathrm{A}_1 + \mathrm{E}_8$.

There is a feature of the algorithm than can be explained considering again the enhancement to $\mathrm{A}_8 + 2\mathrm{A}_1$, but now with A_8 formed by connecting $\varphi_{\mathtt{N}}$ to φ_6. The algorithm finds the charge vector $|0,0,0,0;-w_6,0^8\rangle$ for $\varphi_{\mathtt{N}}$. The moduli are again of the form (3.3.7), but now the quantization conditions from $\varphi_{\mathtt{N}}$ imply $(\gamma_1,\gamma_2) = (0,0)$. Thus, the predicted moduli are $A_1 = A_2 = 0$, $E = \delta_{ij}$, and we know that this point has trivial enhancement to $2\mathrm{E}_8 + 2\mathrm{A}_1$. On the other hand, the Dynkin diagram that results adding $\mathtt{N}$ indicates enhancement to $\mathrm{A}_8 + 2\mathrm{A}_1 + \mathrm{E}_8$. The problem here is that the $\varphi_{\mathtt{N}}$, which has zero winding and momenta, corresponds to a root of E_8. In fact, $-w_6 = \alpha_0$ is the lowest root. Since the quantization conditions are linear equations, if we replace one of the original simple roots of $2\mathrm{E}_8 + 2\mathrm{A}_1$ with any other root, the moduli that solve the system will be the same, but the other root is no longer simple. Our prescription to solve it is to classify all the enhancements, originating from the same starting point, by the resulting moduli. If there is more than one enhancement for the same moduli we just pick the one with higher dimensional group[11]. In this case, we choose $2\mathrm{E}_8 + 2\mathrm{A}_1$ over $\mathrm{E}_8 + \mathrm{A}_8 + 2\mathrm{A}_1$.

In Table 3.5 we collect the maximal enhancements in the neighborhood of the original point $A_1 = A_2 = 0$, $E = \delta_{ij}$, which has $\mathrm{G}_{18} = 2\mathrm{E}_8 + 2\mathrm{A}_1$. The node shown in the first column is removed from the set in (3.3.2) at the start. The effect is to break G_{18} to $\mathrm{G}_9 \times \mathrm{E}_8 \times \mathrm{U}(1)$, with G_9 given in the second column. Appending a new node then leads to $\mathrm{G}_{10} \times \mathrm{E}_8$, with the various possibilities for G_{10} listed in the third column. To arrive at this list we have only kept the groups of higher dimension as explained before, and we have used $\lambda_1 = 1$ and $\lambda_2 = 2$ in the bounds in (3.3.4).

deleted node	G_9	G_{10}
$\mathtt{C}_1$	$\mathrm{E}_8 + \mathrm{A}_1$	$\mathrm{E}_8 + 2\mathrm{A}_1,\ \mathrm{E}_8 + \mathrm{A}_2$
1	$\mathrm{A}_7 + 2\mathrm{A}_1$	$\mathrm{A}_9 + \mathrm{A}_1,\ \mathrm{A}_8 + 2\mathrm{A}_1,\ \mathrm{D}_{10}$
2	$\mathrm{A}_4 + \mathrm{A}_2 + 3\mathrm{A}_1$	$\mathrm{D}_7 + \mathrm{A}_2 + \mathrm{A}_1,\ \mathrm{D}_5 + \mathrm{A}_4 + \mathrm{A}_1,\ \mathrm{A}_6 + \mathrm{A}_2 + 2\mathrm{A}_1,\ 2\mathrm{A}_4 + 2\mathrm{A}_1$
3	$\mathrm{A}_4 + \mathrm{A}_3 + 2\mathrm{A}_1$	$\mathrm{D}_6 + \mathrm{A}_4, \mathrm{A}_8 + 2\mathrm{A}_1, \mathrm{A}_6 + \mathrm{A}_3 + \mathrm{A}_1, \mathrm{E}_6 + \mathrm{A}_3 + \mathrm{A}_1, \mathrm{A}_5 + \mathrm{A}_4 + \mathrm{A}_1$
4	$\mathrm{D}_5 + \mathrm{A}_2 + 2\mathrm{A}_1$	$2\mathrm{D}_5,\ \mathrm{D}_7 + \mathrm{A}_2 + \mathrm{A}_1,\ \mathrm{E}_7 + \mathrm{A}_2 + \mathrm{A}_1,\ \mathrm{D}_5 + \mathrm{A}_4 + \mathrm{A}_1$
5	$\mathrm{E}_6 + 3\mathrm{A}_1$	$\mathrm{E}_6 + \mathrm{D}_4,\ \mathrm{E}_6 + \mathrm{A}_3 + \mathrm{A}_1$
6	$\mathrm{E}_7 + 2\mathrm{A}_1$	$\mathrm{E}_7 + \mathrm{A}_3, \mathrm{E}_7 + \mathrm{A}_2 + \mathrm{A}_1$
7	$\mathrm{A}_6 + 3\mathrm{A}_1$	$\mathrm{D}_9 + \mathrm{A}_1,\ \mathrm{A}_8 + 2\mathrm{A}_1,\ \mathrm{A}_6 + \mathrm{A}_3 + \mathrm{A}_1,\ \mathrm{A}_6 + \mathrm{A}_2 + 2\mathrm{A}_1$
8	$\mathrm{D}_7 + 2\mathrm{A}_1$	$\mathrm{D}_9 + \mathrm{A}_1,\ \mathrm{D}_7 + \mathrm{A}_2 + \mathrm{A}_1$

Table 3.5: Maximal enhancements $\mathrm{G}_{10} + \mathrm{E}_8$ in the neighborhood of $A_1 = 0$, $A_2 = 0$, $E_{ij} = \delta_{ij}$, found setting $\lambda_1 = 1$ and $\lambda_2 = 2$ in the bounds of (3.3.4).

The Neighborhood algorithm can be iterated and can ramify from a different point of maximal rank. In particular, in this way we can find the maximal enhancements $\mathrm{A}_3 + \mathrm{A}_6 + \mathrm{A}_9$ and $3\mathrm{A}_6$, which, as argued in [2], cannot be deduced using the algorithm with fixed Wilson lines. To this end we will set the bounds (3.3.4) as before. We will see that this is enough to obtain the missing groups, although a priori there was no guarantee for it. We now start at a point with group $\mathrm{G}_{18} = \mathrm{A}_6 + \mathrm{A}_3 + \mathrm{A}_1 + \mathrm{E}_8$, which in turn was found by the algorithm initiating from the point $E_{ij} = \delta_{ij}$, $A_1 = 0$, $A_2 = 0$ (cf. Table 3.5). Concretely, G_{18} arises after deleting the node φ_3 in (3.3.2) and then appending

[11] This issue can be solved in a more systematic and elegant way with the *saturation* method described in Section 4.3.1 (see footnote 10).

the extra node $\mathtt{N}$ with charge vector $\varphi_{\mathtt{N}} = |0,-1,-1,1;w_3-w_1,0^8\rangle$. The corresponding moduli are $A_1 = -\frac{1}{8}w_3 \times 0$, $A_2 = \frac{1}{4}w_3 \times 0$, $E_{ij} = \delta_{ij}$. We can now readily apply the algorithm to G_{18} whose Dynkin diagram is shown in Figure 3.3.a. All the enhancement points on the neighborhood of this point can be computed. However, to reach the desired maximal enhancements, the nodes $\mathtt{C}_1$ and $\mathtt{C}_2$ will be maintained during the whole process. Therefore, E_{ij} will remain equal to the identity as we move through the neighborhood. To proceed we remove the node $1'$, thereby breaking G_{18} to $\mathrm{G}_{17} \times \mathrm{U}(1)$, with $\mathrm{G}_{17} = \mathrm{A}_1 + \mathrm{A}_3 + \mathrm{A}_6 + \mathrm{A}_7$, as shown in Figure 3.3.b. The neighboring point is on the surface characterized by $A_1 = -\frac{1}{8}w_3 \times \gamma_1 w_1'$, $A_2 = \frac{1}{4}w_3 \times \gamma_2 w_1'$. The algorithm then searches for new nodes that can be consistently added. It finds $\mathtt{N}'$ with charge vector $|-1,-1,1,0;0^8,-w_8'\rangle$, which leads to $\mathrm{A}_3 + \mathrm{A}_6 + \mathrm{A}_9$, as seen in Figure 3.3.c. The point is $(\gamma_1,\gamma_2) = (-\frac{2}{5},-\frac{1}{5})$. Luckily, from this point we can attain $3\mathrm{A}_6$ in a couple of steps. With the algorithm it is easy to see what is needed. As displayed in Figure 3.3.d, the node $8'$ is next removed to break the symmetry to $2\mathrm{A}_6 + \mathrm{A}_3 + \mathrm{A}_2$, plus U(1). The surface is given by $A_1 = -\frac{1}{8}w_3 \times (-\frac{1}{2}w_8' + \mu_1(4w_1' - 5w_8'))$, $A_2 = \frac{1}{4}w_3 \times (-\frac{1}{4}w_8' + \mu_2(4w_1' - 5w_8'))$. The algorithm then discovers the extra node $\mathtt{S}$, with charge vector $|-1,0,1,-1;-w_6,w_8'-w_1'\rangle$, which has enhancement to $3\mathrm{A}_6$, as indicated in Figure 3.3.e. The point is $(\mu_1,\mu_2) = (-\frac{1}{8},0)$.

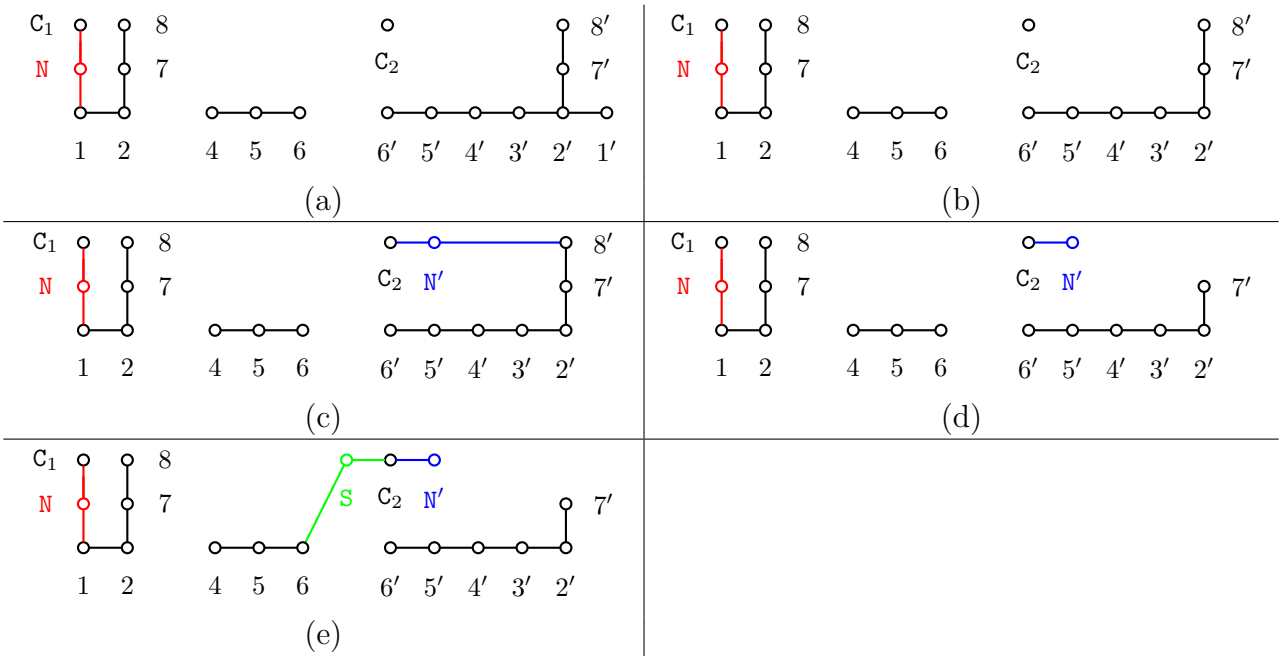


Figure 3.3: Dynkin diagrams for the steps leading to the enhancements $\mathrm{A}_3 + \mathrm{A}_6 + \mathrm{A}_9$ (c) and $3\mathrm{A}_6$ (e), starting from a point with $\mathrm{A}_6 + \mathrm{A}_3 + \mathrm{A}_1 + \mathrm{E}_8$ (a). Intermediate stages where the symmetry is broken are shown in (b) and (d).

In conclusion, we have arrived at $\mathrm{A}_3 + \mathrm{A}_6 + \mathrm{A}_9$ and $3\mathrm{A}_6$. The former has Wilson lines $A_1 = -(\frac{1}{8}w_3 \times \frac{2}{5}w_1')$, $A_2 = \frac{1}{4}w_3 \times (-\frac{1}{5}w_1')$, and complimentary lattice T with Gram matrix $Q = [2,0,140]$. For the latter $A_1 = -\frac{1}{8}w_3 \times (-\frac{1}{2}w_1' + \frac{1}{8}w_8')$, $A_2 = \frac{1}{4}w_3 \times (-\frac{1}{4}w_8')$, and $Q = [2,1,4]$. For both, $E = \delta_{ij}$.

For $d = 1$ and 2, we know that this algorithm is exhaustive. By removing arbitrary roots from the maximal enhancements we can obtain the non-maximal ones. As mentioned previously, there is only one case for $d = 1$ and 2 that is missing, namely $2\mathrm{D}_8$ and $4\mathrm{D}_4$, respectively. To arrive at these enhancements we can just use this algorithm but instead of starting with a maximal enhancement, we start with any non-maximal enhancement of type $\mathrm{G}_{16} \times \mathrm{U}(1)^d$.

3.3.2 All maximal rank groups for $d = 2$

From the results in [14] we infer that there are 359 distinct maximally enhanced heterotic models on T^2, some of which share the same gauge group [12]. The number of distinct maximal rank gauge groups found is 325 (if we do not take into account the global structure). Using the extended diagram formalism developed in [2], we are able to obtain the moduli for 331 of these models. The more powerful computational method described in 3.3.1 allow us to obtain the moduli for the remaining 28 models, as well as alternative moduli for the other 331.

In Table D.2, displayed in Appendix D.1, we show a representative for each of the 359 models in the $\mathrm{E}_8 \times \mathrm{E}_8'$ heterotic theory. The data for each point consists of the root lattice L, which gives the gauge group, the isotropic subgroup H_L, the complementary lattice T, and the moduli E_{ij}, A_1, A_2. The lattice T is conveyed by its Gram matrix, computed from the moduli as described in Section 3.2.2.2. Once T is known we can determine the order of H_L using the relation (3.2.1). We can then check that the appropriate isotropic subgroup of A_L exists as in the examples worked out in Section 3.2.1.2. In this way we can confirm the results of [14] for the H_L corresponding to each pair (L, T).

In contrast to the $d = 1$ case, we do not have an explicit form of the fundamental domain of the moduli space, which would give us a clear criterion for choosing the moduli. Instead, we have selected those that have the simplest form. In some cases we present two different sets of moduli, one in which the Wilson lines are simple but the E_{ij} have non-integer entries, and another where the opposite happens. As expected from the general arguments of Section 3.2.2, in all cases the E_{ij} are rational numbers and the Wilson lines are quantized in the sense of eq. (3.2.1). Moreover, it can be shown that for every pair (L, T), it is always possible to find Wilson lines such that $E_{ij} = \delta_{ij}$. Examples of this result are # 15 or # 19 in Table D.2. The torus metric and the b-field can be easily derived from the moduli E_{ij} and A_i substituting in $G_{ij} = \frac{1}{2}(E_{ij} + E_{ji} - A_i \cdot A_j)$ and $B_{ij} = \frac{1}{2}(E_{ij} - E_{ji})$.

For each model in Table D.2, the moduli in the HO theory can be obtained by using the map described in Section 3.1.2. We have explicitly verified that the Gram matrices of the lattices L and T are preserved under this map, which is to be expected from an orthogonal transformation. Some examples of these transformed HO models are given in Table D.3.

[12] During this work we lacked a method for computing the overlattice of L. Later, we developed an algorithm which was used to verify all of the H_L reported here. This algorithm will be explained in Section 4.3.3.1. By computing the precise global structure it is possible to distinguish two points both with $L = \mathrm{A}_1 + \mathrm{A}_3 + \mathrm{A}_5 + \mathrm{A}_9$, $H = \mathbb{Z}_2$ and $T = [10, 0, 12]$ but inequivalent generators for H: one is generated by the element $k = (1, 2, 0, 5)$ and the other by $k = (0, 0, 3, 5)$, in the notation of Section 4.3.3. This results in 360 distinct models, with two of them being identified as the same (#61) in Table D.2 of this thesis and Table 2 of [14].

Chapter 4

Chaudhuri-Hockney-Lykken string

With the motivation to exhaustively explore the landscape of heterotic compactifications with maximal supersymmetry now we extend the analysis of the previous chapters to compactifications of the HE string on $T^d/\mathbb{Z}_2$ asymmetric orbifolds which realize the so-called CHL string [17, 18] (in $10-d$ dimensions with $d \geq 1$). This $\mathbb{Z}_2$ acts by exchanging the two E_8 components of the momentum lattice, together with a shift by half a period along one of the compact directions. One of the effects of this (freely acting) $\mathbb{Z}_2$ modding is to remove eight of the U(1) gauge bosons from the spectrum, thereby reducing the rank of the gauge group by eight. The moduli space of the CHL string in $10-d$ dimensions is locally $\frac{SO(d,d+8)}{SO(d)\times SO(d+8)}$ and world-sheet current algebras can be realized at level 2 or 1.

The momenta of the physical states of the 9-dimensional CHL string belong to the Lorentzian even self-dual lattice $\Gamma_{1,9}$ [19]. At generic points of the moduli space the (left) gauge symmetry is Abelian, namely $U(1)^9$. In the absence of Wilson lines and for generic values of the radius, some vector states of the untwisted sector become massless and enhance the gauge group to $E_8 \times U(1)$. At the self-dual orbifold radius $R = \sqrt{2}$ (taking $\alpha' = 1$), two twisted states become massless and a further enhancement to $E_8 \times SU(2)$ takes place. Eight other non-Abelian ADE groups of maximal rank 9 can be found at other special points of the moduli space. All of these groups have world-sheet current algebras realized at level 2. In Section 4.1 we list these groups, which can be easily obtained by deleting nodes from the Generalized Dynkin Diagram for $\Gamma_{1,9}$ (which is the same as the Dynkin diagram of the group E_{10}), in an analogous way as we did in Section 2.4 for the S^1 compactification.

In less than nine dimensions the lattice of momenta of the physical states (the so-called Mikhailov lattice) is even but not self-dual [19]. This can be understood by noting that there is an asymmetry between the possible winding states along the orbifolded direction and those along the remaining ones, obstructing an automorphism that would make the lattice self-dual. On the other hand, this asymmetry enriches the pattern of gauge symmetries with respect to those found from the Narain lattice, and we find in addition gauge groups of BCF type.

As in the case of T^d compactifications, there does not seem to exist a Generalized Dynkin Diagram (GDD) from which one can extract all possible enhancements for $d > 1$. Although many different GDDs can be constructed, it is uncertain whether they can produce the whole set of enhanced gauge groups. Hence, we adapt the exploration algorithm that was introduced in Section 3.3.1 for the Narain lattices to the Mikhailov lattices. We find that, for $d = 2$, the algorithm generates a list of 61 groups of maximal enhancement. In this case, the CHL string is a realization of the anomaly free theories with 16 supercharges and rank 10 gauge groups [54]. We find that the ADE groups arise at level 2 while C groups appear at level 1 (A_1 also appears at level 1, but $A_1 = C_1$). Taking into account that the exploration algorithm produces all possible maximal enhancements in T^2 compactifications strongly indicates that these results exhaust all the possibilities.

Roughly half of the enhanced gauge groups in the CHL string in eight dimensions are multiply

connected, while the rest are simply connected. Importantly, they are seen to satisfy the condition derived in [21] for anomaly free 8d $\mathcal{N}=1$ supergravities, a result that can be proven in general also for T^2 compactifications [20].

We note that our results for the 8d CHL string extend under T-duality to compactifications of the HO theory string on tori without vector structure [22], as discovered in [55] and further discussed in [23]. These theories are also dual to F-theory on K3 with a frozen singularity [22,56], hence it should be possible to reproduce our results in that context. Similarly, the $d=1$ case is dual to M-theory on the Möbius strip [57,58]. We do not dwell on these dualities here, focusing our attention only on the $(\mathrm{E}_8\times\mathrm{E}_8)$ heterotic side.

The chapter is organized as follows. In Section 4.1 we review the construction of the CHL string in nine dimensions as an $S^1/\mathbb{Z}_2$ orbifold of the HE string. We then find all the maximal enhancements from the Generalized Dynkin Diagram and list them in Table D.4. The T-duality map among the states of the theory is also checked. The more general setting of the CHL string in $D=10-d$ dimensions (with $D\leq 9$) is considered in Section 4.2, where the theory is realized as an orbifold of heterotic compactifications on T^d. In Section 4.3 we explain the methods used in the algorithm that searches for maximal enhancement points and illustrate them with an explicit example. We then present the maximal enhancements generated by this procedure in the eight dimensional theory and collect the final results in Table D.5 with the corresponding explanation in Section 4.3.4.

4.1 The nine-dimensional CHL String

In this section we review the construction of the CHL string in nine dimensions [17] as an $S^1/\mathbb{Z}_2$ orbifold of the HE string [18] and fix our conventions. We recall the massless spectrum and study the possible gauge symmetries from the point of view of lattice embeddings. We will see that, as in the case of the heterotic string on S^1, this problem is well under control.

4.1.1 Constructing the theory from the heterotic string

In Chapter 2, we considered the HE string with the coordinate x^9 compactified on a circle of radius R. We explained that varying R and turning on the Wilson line A on the compact direction we sweep through the Narain moduli space

$$\mathcal{M}_{\text{Narain}}=\frac{\mathrm{O}(1,17,\mathbb{R})}{\mathrm{O}(17,\mathbb{R})}\Big/\mathrm{O}(1,17,\mathbb{Z}), \tag{4.1.1}$$

with the discrete T-duality group $\mathrm{O}(1,17,\mathbb{Z})$ determining its global structure. These compactifications yield theories with gauge group of rank 17 (ignoring the graviphoton). However, the class of nine-dimensional theories with 16 unbroken supercharges also contains reduced rank theories, with gauge groups of rank 9 and 1. Those of rank 9 are realized in the CHL string, and have moduli space

$$\mathcal{M}_{\text{CHL}}=\frac{\mathrm{O}(1,9,\mathbb{R})}{\mathrm{O}(9,\mathbb{R})}\Big/\mathrm{O}(1,9,\mathbb{Z}), \tag{4.1.2}$$

as will be made clear at the end of this section.

For our purposes, it is convenient to construct the CHL string as an orbifold of the $\mathrm{E}_8\times\mathrm{E}_8'$ heterotic string following [18]. The orbifold symmetry $g=\mathrm{RT}$ consists of the outer automorphism R of the

$\mathrm{E}_8 \times \mathrm{E}_8'$ lattice accompanied by a half turn T around the compactification circle, namely

$$\mathrm{R}: \quad \Gamma_{\mathrm{E}_8} \oplus \Gamma_{\mathrm{E}_8'} \to \Gamma_{\mathrm{E}_8'} \oplus \Gamma_{\mathrm{E}_8}, \qquad \mathrm{T}: \quad x^9 \to x^9 + \pi R. \tag{4.1.3}$$

Since $x^9 \sim x^9 + 2\pi R$ in the parent theory, $g^2 = 1$ and g defines a freely-acting $\mathbb{Z}_2$ orbifold.

To find the spectrum of this theory, we start by recalling the components of the internal momentum of the heterotic string in nine dimensions eq. (2.3.1):

$$p_R = \frac{1}{\sqrt{2}R}\left[n - (R^2 + \tfrac{1}{2}A^2)w - \Pi \cdot A\right], \tag{4.1.4a}$$

$$p_L = \frac{1}{\sqrt{2}R}\left[n + (R^2 - \tfrac{1}{2}A^2)w - \Pi \cdot A\right], \tag{4.1.4b}$$

$$p^{\hat{I}} = \Pi^{\hat{I}} + A^{\hat{I}} w\,, \tag{4.1.4c}$$

where $\hat{I} = 1, ..., 16$, $n \in \mathbb{Z}$ is the momentum number on the circle, $w \in \mathbb{Z}$ is the winding number and $\Pi \in \Gamma_8 \oplus \Gamma_8$, with $\Gamma_8 \equiv \Gamma_{\mathrm{E}_8}$. The momenta form the unique even self-dual Lorentzian lattice $\Gamma_{1,17}$ (up to $\mathrm{SO}(1,17)$ boosts given by the moduli), with vectors labeled by the quantum numbers $w, n, \Pi^{\hat{I}}$. We use the convention $\alpha' = 1$.

On the $S^1/\mathbb{Z}_2$ orbifold, the Wilson lines are restricted to take the form

$$A = (a, a)\ , \quad a \in \mathbb{R}^8. \tag{4.1.5}$$

Similarly, it is convenient to decompose the heterotic momenta as

$$\Pi = (\pi, \pi'), \qquad \pi, \pi' \in \Gamma_8, \tag{4.1.6}$$

and to define the symmetric and antisymmetric combinations

$$p_+^I = \frac{1}{\sqrt{2}}(p^I + p^{I+8}), \qquad p_-^I = \frac{1}{\sqrt{2}}(p^I - p^{I+8}), \quad I = 1, ..., 8 \tag{4.1.7}$$

Defining moreover the symmetric combination

$$\rho = \pi + \pi' \quad \in \Gamma_8\ , \tag{4.1.8}$$

the components (4.1.4) can be written as

$$p_R = \frac{1}{\sqrt{2}R}\left[n - R^2 w - a^2 w - \rho \cdot a\right], \tag{4.1.9a}$$

$$p_L = \frac{1}{\sqrt{2}R}\left[n + R^2 w - a^2 w - \rho \cdot a\right] = p_R + \sqrt{2}Rw\,, \tag{4.1.9b}$$

$$p_+ = \frac{1}{\sqrt{2}}\left(\rho + 2aw\right)\ , \tag{4.1.9c}$$

$$p_- = \frac{1}{\sqrt{2}}(\pi - \pi')\,, \tag{4.1.9d}$$

and the total internal momentum vector is $P = (\mathbf{p_R}; \mathbf{p_L}) \equiv (p_R; p_L, p_+, p_-)$.

The orbifold action on the momenta can be written as

$$g\ |p_R; p_L, p_+, p_-\rangle = e^{2i\pi v\cdot P}|p_R; p_L, p_+, -p_-\rangle\,, \tag{4.1.10}$$

where the inner product is defined with respect to the metric $\mathrm{diag}(-1,+1,\ldots,+1)$. The shift vector v is constrained by the condition that g has order two. Choosing $v_- = 0$ implies that $2v$ belongs to the Narain lattice $\Gamma_{1,17}$. Besides, the condition that the shift corresponds to the geometric translation of x^9 by half a period amounts to $e^{2i\pi v\cdot P} = e^{i\pi n}$ and leads to

$$v = \frac{1}{2\sqrt{2}}\left(-R - \frac{a^2}{R}; R - \frac{a^2}{R}, 2a,\ 0\right). \tag{4.1.11}$$

Notice that $2v$ equals the Narain lattice vector obtained by substituting $w = 1, n = 0$, and $\pi = \pi' = 0$ in the formulae (4.1.9). The lattice vectors can be conveniently traded for states $|w, n, \pi, \pi'\rangle$, which depend on the quantum numbers and transform as

$$g\ |w, n, \pi, \pi'\rangle = e^{i\pi n}|w, n, \pi', \pi\rangle, \tag{4.1.12}$$

for all values of the moduli.

The action of g on the left-moving bosons living on $\Gamma_{\mathrm{E}_8} \oplus \Gamma_{\mathrm{E}_8'}$, denoted Y^I and $Y'^I = Y^{I+8}$, $I = 1,\ldots,8$, is the exchange $Y^I \leftrightarrow Y'^I$, or $Y^I_\pm \to \pm Y^I_\pm$ where

$$Y^I_\pm = \frac{1}{\sqrt{2}}(Y^I \pm Y'^I). \tag{4.1.13}$$

The action on the space-time coordinates is just the translation in x^9. The corresponding oscillators then transform as

$$g(\alpha^I) = \alpha^{I+8}\,, \quad g(\alpha^{I+8}) = \alpha^I\ , \quad g(\alpha^\mu) = \alpha^\mu, \tag{4.1.14}$$

where $\mu = 2, ..., 9$ refers to the space-time transverse coordinates. Notice also that $g(\alpha^I_\pm) = \pm\alpha^I_\pm$ for the $Y^I_\pm$ oscillators.

In the untwisted sector, the spectrum consists of states of the parent theory invariant under the orbifold action. The invariant states are superpositions of the form

$$|\varphi\rangle_{\text{untwisted}} = \frac{1}{\sqrt{2}}\Big(\alpha\,|w, n, \pi, \pi'\rangle + (-1)^n g(\alpha)\,|w, n, \pi', \pi\rangle\Big)\,, \tag{4.1.15}$$

where α denotes any possible combination of oscillators and $g(\alpha)$ its image under g, given by (4.1.14).

In the twisted sector, the internal chiral bosons Y^I and Y'^I satisfy the boundary conditions

$$Y^I(\sigma + 2\pi) = Y'^I(\sigma) + Q^I, \qquad Y'^I(\sigma + 2\pi) = Y^I(\sigma) + Q'^I, \tag{4.1.16}$$

where Q, Q' are arbitrary (fixed) vectors in Γ_8 which specify the precise way of exchanging $\mathrm{E}_8 \leftrightarrow \mathrm{E}_8'$ [59]. The $Y^I_\pm$ then obey

$$Y^I_\pm(\sigma + 2\pi) = \pm Y^I_\pm(\sigma) + \frac{1}{\sqrt{2}}(Q^I \pm Q'^I)\,, \tag{4.1.17}$$

and have oscillator expansions

$$\begin{aligned} Y_+^I(\tau+\sigma) &= \tfrac{1}{2}y_{+,0}^I + \tfrac{1}{2\pi}p_+^I(\tau+\sigma) + \frac{i}{\sqrt{2}}\sum_{n\neq 0}\frac{\alpha_{+,n}^I}{n}e^{-in(\tau+\sigma)}\,, \\ Y_-^I(\tau+\sigma) &= \tfrac{1}{2}y_{-,0}^I + \frac{i}{\sqrt{2}}\sum_{s\,\in\,\mathbb{Z}+\frac{1}{2}}\frac{\alpha_{-,s}^I}{s}e^{-is(\tau+\sigma)}\,, \end{aligned} \tag{4.1.18}$$

where $p_+^I \equiv \frac{1}{\sqrt{2}}(Q^I + Q'^I)$ and $y_{-,0}^I \equiv \frac{1}{\sqrt{2}}(Q^I - Q'^I)$. The boson corresponding to the compact x^9 dimension satisfies

$$X^9(\sigma+2\pi) = X^9(\sigma) + \pi R + 2\pi R\tilde{w} \equiv X^9(\sigma) + 2\pi R w, \tag{4.1.19}$$

with $\tilde{w} \in \mathbb{Z}$, and hence $w \in \mathbb{Z} + \frac{1}{2}$.

The twisted states have three distinctive features: they have half-integer winding w, the occupation numbers of their oscillators can be half-integer or integer valued, and they do not have antisymmetric momentum p_-^I. We write them as

$$|\varphi\rangle_{\text{twisted}} = |w, n, \rho\rangle\ , \tag{4.1.20}$$

up to the action of oscillators. Note that upon quantisation the symmetric momentum takes the form $p_+ = \frac{1}{\sqrt{2}}(\rho + 2aw)$, with $\rho = Q + Q' \in \Gamma_8$, coinciding with the untwisted symmetric momentum in (4.1.7).

In the NS sector for the right movers (which gives the space-time bosons), the mass and level matching conditions are

$$M^2 = \mathbf{p_L}^2 + \mathbf{p_R}^2 + 2(N_L + N_R) + 2\mathfrak{a} - 1\ , \tag{4.1.21}$$

$$0 = \mathbf{p_L}^2 - \mathbf{p_R}^2 + 2(N_L - N_R) + 2\mathfrak{a} + 1\ , \tag{4.1.22}$$

where the zero point energy $\mathfrak{a}$ is -1 in the untwisted sector, as usual, and $-\frac{1}{2}$ in the twisted sector, since the left-moving side part receives contributions from 16 periodic bosons $\{Y_+^I, X^\mu\}$ (with μ labeling the 8 transverse directions) and 8 anti-periodic bosons $\{Y_-^I\}$. Concretely,

$$\mathfrak{a}_{\text{twisted}} = 16 \times \mathfrak{a}_{\text{periodic}} + 8 \times \mathfrak{a}_{\text{anti-periodic}} = -16 \times \frac{1}{24} + 8 \times \frac{1}{48} = -\frac{1}{2}\ . \tag{4.1.23}$$

It is convenient to define the modified 'oscillator number'

$$N_L' = N_L + \delta, \qquad \delta = \begin{cases} \frac{1}{2}\,p_-^2 & \text{Untwisted} \\ \frac{1}{2} & \text{Twisted} \end{cases}\ , \tag{4.1.24}$$

where p_-^2 is an integer (cf. (4.1.9)), and the nine-dimensional momentum

$$P_L = (p_L, p_+), \tag{4.1.25}$$

which allows to rewrite the formulas (4.1.21) and (4.1.22) in an $\mathrm{O}(1,9)$ covariant form as

$$M^2 = P_L^2 + p_R^2 + 2(N'_L + N_R) - 3 \tag{4.1.26a}$$

$$= \frac{1}{2} Z^T \mathcal{H} Z + 2(N'_L + N_R) - 3 \tag{4.1.26b}$$

$$0 = P_L^2 - p_R^2 + 2(N'_L - N_R) - 1 \tag{4.1.27a}$$

$$= \frac{1}{2} Z^T \eta Z + 2(N'_L - N_R) - 1 \ . \tag{4.1.27b}$$

Here we have defined the charge vector

$$Z \equiv |\ell, n; \rho\rangle \ , \tag{4.1.28}$$

with

$$\ell \equiv 2w, \tag{4.1.29}$$

and ρ is defined in (4.1.8). Note that ℓ is always an integer, and is odd (even) for twisted (untwisted) states. $\mathcal{H}$ is the so-called 'generalized metric' (cf. (2.3.5))

$$\mathcal{H} = \frac{1}{R^2} \begin{pmatrix} E^2/2 & -a^2 & Ea \\ -a^2 & 2 & -2a \\ Ea^T & -2a^T & R^2 + 2a^T a \end{pmatrix} , \tag{4.1.30}$$

where a is taken to be a row vector and the lower right R^2 term is implicitly multiplied by $\mathbb{1}_8$ so that $\mathcal{H}$ is a 10×10 matrix, and

$$E \equiv R^2 + a^2 \, . \tag{4.1.31}$$

Finally, η is the $\mathrm{O}(1,9)$ metric

$$\eta = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & \mathbb{1}_8 \end{pmatrix} . \tag{4.1.32}$$

The important result

$$Z^2 \equiv Z^T \eta Z = 2\ell n + \rho^2 \in 2\mathbb{Z} \tag{4.1.33}$$

implies that the charge vectors Z span the even self-dual Lorentzian lattice $\Gamma_{1,9} \simeq \Gamma_{1,1} \oplus \Gamma_8$, since $\ell, n \in \mathbb{Z}$ and $\rho \in \Gamma_8$. The correspondence between the states of the theory and the elements of $\Gamma_{1,9}$ was derived in [19].

It can now be seen that the local structure of the moduli space (4.1.2) is $\mathrm{O}(1,9,\mathbb{R})/\mathrm{O}(9,\mathbb{R})$ due to the reduction of the Wilson line from 16 to 8 components and the invariance of eqs. (4.1.26a) and (4.1.27a) under $\mathrm{O}(9,\mathbb{R})$ rotations of P_L. Furthermore, the automorphism group $\mathrm{O}(1,9,\mathbb{Z})$ of $\Gamma_{1,9}$ corresponds to the T-duality group of the theory, giving the global structure for $\mathcal{M}_{\mathrm{CHL}}$. The similarities between $\mathcal{M}_{\mathrm{CHL}}$ and $\mathcal{M}_{\mathrm{Narain}}$ (cf. eq. (4.1.1)) allow to carry out an analysis of the nine-dimensional CHL string mirroring the one performed for S^1 compactifications in [13] and mentioned in Section 2.4, namely constructing the fundamental region of the moduli space whose codimension $r \leq 9$ boundaries give enhanced semisimple gauge groups of rank r. This ensures that we are able to easily find all possible gauge group enhancements in the theory, as we explain shortly.

4.1.2 Massless vectors

From equations (4.1.26a) and (4.1.27a) we see that the NS sector contains massless states with $N_R = \frac{1}{2}$, $p_R = 0$ and

$$P_L^2 = 2(1 - N'_L) \qquad \Rightarrow \qquad N'_L = 0, 1, \frac{1}{2}. \tag{4.1.34}$$

Of these, untwisted states can have $N_L = 0, 1$ and twisted states $N_L = 0$ (cf. eq. (4.1.24)). For $N_L = 1$, besides the universal gravitational sector, the massless spectrum contains the left Abelian KK gauge vector

$$\alpha^9_{-1}\bar{\psi}^\mu_{-\frac{1}{2}}|0\rangle_{\rm NS}\ , \tag{4.1.35}$$

with $\bar{\psi}^\mu_{-\frac{1}{2}}$ the coefficient of the Laurent expansion of the right-moving fermions, $\mu = 2, ..., 8$, and the 8 symmetric combinations of the Cartan sector of the heterotic theory that survive the R projection

$$\frac{1}{\sqrt{2}}(\alpha^I_{-1} + \alpha^{I+8}_{-1})\ \bar{\psi}^\mu_{-\frac{1}{2}}|0\rangle_{\rm NS}\,, \tag{4.1.36}$$

implying that the gauge group of the theory has rank 9.

For $N_L = 0$, the set of massless states depends on the point in moduli space. The $p_R = 0$ condition reads (cf. eq. (4.1.9a))

$$\tfrac{1}{2}E\,\ell - n + a\cdot\rho = 0\,, \tag{4.1.37}$$

with E defined in (4.1.31), while the level matching condition (4.1.27b) becomes a constraint on the norm of $\Gamma_{1,9}$ vectors:

$$Z^2 = 2\ell n + \rho^2 = 4(1 - N'_L) = 4 \text{ or } 2. \tag{4.1.38}$$

The states with $Z^2 = 4$ correspond to $N'_L = 0$, and from the definition of N'_L given in (4.1.24) we see that this is only possible in the untwisted sector, with $\pi = \pi'$. From eq. (4.1.15) we see that such states could only exist with n even. However, substitution in (4.1.38) gives $2\ell n + \rho^2 = 4q + 4\pi^2 = 4$, with q even, or $\pi^2 = 1 - q =$ odd, which is inconsistent since $\pi \in \Gamma_8$. In compactifications to lower dimensions, such massless states do appear, and correspond to roots of gauge algebras at level 1, being long roots for non-ADE algebras (see Section 4.2). On the other hand, states with $Z^2 = 2$ are allowed in this case, and correspond to roots of ADE algebras at level 2. They come both from the twisted and untwisted sectors (the latter with $\pi = 0$ or $\pi' = 0$). We summarize this in Table 4.1.

	twisted	untwisted
Z^2	2	2
ℓ	odd	even
n	integer	integer
ρ	Γ_8	Γ_8

Table 4.1: Quantum numbers of the massless states in the twisted and untwisted sector in nine dimensions. The states must satisfy (4.1.37) to be massless.

At a generic point in the moduli space there are no massless states (twisted or untwisted) other than (4.1.35)-(4.1.36), since condition (4.1.37) can only be satisfied generically for $Z = 0$, and therefore generically the gauge group is $\mathrm{U}(1)^9$. Enhanced gauge symmetry appears at special points in the moduli space, as we will show.

Let us look at the simple situation where $a = 0$. The massless condition (4.1.37) is trivially satisfied

for states with $\ell = n = 0$, and the level matching condition (4.1.38) with $\rho^2 = 2$, hence we get the massless untwisted sates with charge vectors

$$Z = |0, 0; \rho\rangle\,, \qquad \rho^2 = 2. \tag{4.1.39}$$

These are just the 240 roots of the E_8 arising from the symmetric combination of the two E_8's in the parent theory. In the twisted sector, since ℓ is odd, eq. (4.1.37) is not satisfied for generic values of the compactification radius, since $R = \sqrt{E}$ in this case. The surviving gauge group for $a = 0$ and generic R is then $E_8 \times U(1)$. Interestingly enough, taking $R = 1$ when $a = 0$ does not lead to additional states that enhance the U(1) to SU(2), as occurs in the S^1 compactification. For this enhancement to occur we must actually take $R = \sqrt{2}$, i.e. $E = 2$, so that equations (4.1.37) and (4.1.38) are solved by

$$Z = \pm\,|1, 1; 0\rangle\,, \tag{4.1.40}$$

corresponding to two *twisted* states with winding number $m = \pm\frac{1}{2}$.

It can be shown that in this example the world-sheet realization of the $E_8 \times SU(2)$ space-time gauge symmetry is provided by a Kac-Moody algebra at level $k = 2$. It is interesting to compare the radius that gives this enhancement in the orbifold theory with the self-dual radius R_k in the standard S^1 compactification where the enhancement occurs at $R_k = 1$ and the gauge group is realized at level 1. They are related as $R = \sqrt{k} R_k$. For generic Wilson lines this enhancement occurs at

$$E_k = k^{-1} E \; = 1. \tag{4.1.41}$$

In the following section we show that this is a generic feature: while maximal enhancement in the heterotic string on S^1 occurs at $E = 1$ and the Kac-Moody algebra is realized at $k = 1$, in the nine-dimensional orbifold theory they occur at $E = 2$ and $k = 2$, i.e. both enhancements occur at $E_k = 1$. This is actually expected from T-duality. We will shortly explain that in the orbifold theory the self-dual point is $E = 2$.

4.1.3 Maximal enhancements from the Generalized Dynkin Diagram

As we have commented in Section 4.1.1, the structure of the moduli space of the nine-dimensional CHL string, $\mathcal{M}_{\rm CHL}$, is similar to that of S^1 compactifications of the heterotic string, $\mathcal{M}_{\rm Narain}$ (studied in Chapter 2). In particular, its global structure is given by $O(1, 9, \mathbb{Z})$, the group of automorphisms of a Lorentzian even self-dual lattice. This group is reflexive, meaning that it can be generated by a finite set of Weyl reflections on the moduli space cover $O(1, 9, \mathbb{R})/O(9, \mathbb{R})$, each of which fixes an hyperplane at the boundary of the fundamental domain. Each one of these reflections is uniquely associated to a short root quantum state that becomes massless on its fixed hyperplane, such that all possible enhanced semisimple gauge groups of rank r may be found at their r-fold intersections (for details see [13]).

The upshot is that given the set of 10 roots corresponding to the boundaries of $\mathcal{M}_{\rm CHL}$, we may simply impose that some of them satisfy the massless condition (4.1.37) (condition (4.1.38) is satisfied by construction), so that they become the simple roots of some simply laced gauge algebra. This can be done neatly by introducing the Generalized Dynkin Diagram (GDD) [11] for the lattice $\Gamma_{1,9}$ shown in Figure 4.1, which is the over-extended Dynkin Diagram for E_8, usually denoted E_{10}. The roots 1

through 8 are the simple roots of E_8, and we take them to have the following embedding in $\Gamma_{1,9}$

$$Z_i = |0, 0; \alpha_i\rangle \ , \quad i = 1, ..., 8, \tag{4.1.42}$$

where the α_i are listed in Table 2.1. The root 0 corresponds to the lowest root of E_8 with the additional property that it has $n = -1$, i.e.

$$Z_0 = |0, -1; \alpha_0\rangle \, . \tag{4.1.43}$$

Finally, the root $\mathtt{C}$ lies in the hyperbolic sublattice $\Gamma_{1,1}$ and reads

$$Z_{\mathtt{C}} = |1, 1; 0\rangle \, . \tag{4.1.44}$$

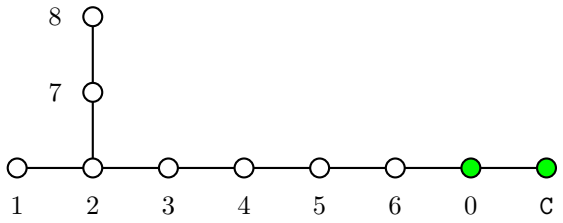


Figure 4.1: Generalized Dynkin Diagram for the lattice $\Gamma_{1,9}$. The coloring of the nodes 0 and $\mathtt{C}$ reflects the fact the the associated states have non-zero momentum and/or winding, as opposed to the white nodes.

Analogously to the case of the torus of Section 2.4, maximally enhanced (rank 9) non-Abelian gauge groups are then found by deleting one node in the GDD such that the remaining nodes form the Dynkin diagram of an ADE algebra. Imposing the condition (4.1.37) on the roots associated to the remaining nodes gives rise to 9 constraints on the moduli and defines a singular point (E, a) at the boundary of the fundamental domain with maximally enhanced gauge group. More generally, deleting s nodes defines a subvariety of dimension $s - 1$ with generic semisimple gauge group of rank $10 - s$, given by the remaining Dynkin diagram.

Note that for maximal enhancements the node $\mathtt{C}$ cannot be broken, since the remaining diagram corresponds to the infinite dimensional algebra E_9. This means that all maximal enhancements must contain this node, and from eq. (4.1.37) this implies that $E = 2$. The massless condition then reduces to

$$a \cdot \rho = \ell - n. \tag{4.1.45}$$

Deletion of the ith node, $i = 0, ..., 8$, corresponds to the Wilson line

$$a = \frac{1}{\kappa_i} w_i, \tag{4.1.46}$$

with no sum over i, where w_i and κ_i are respectively the fundamental weight and Kac mark listed in Table 2.1, similarly to 2.4.7. It is easy to show that this prescription exactly solves eq. (4.1.45) for the remaining roots $Z_{j\neq i}$, while violating the one for Z_i since $w_i \cdot \alpha_i / \kappa_i \notin \mathbb{Z}$, $i \neq 0$. In fact, these values for the Wilson line correspond to those for a shift vector breaking E_8 to a maximal regular subgroup [60].

The maximal enhancements are listed in Table D.4, where in all the cases the world-sheet Kac-Moody algebra is realized at level 2. Moreover, note that the relation (4.1.41) is satisfied in all cases, since $E = 2$.

4.1.4 T-duality

The T-duality group of the nine-dimensional CHL string is $\mathrm{O}(1,9,\mathbb{Z})$, the automorphism group of $\Gamma_{1,9}$. Of particular interest is the Weyl reflection, say T, generated by the root $Z_{\mathtt{C}}$, whose action on the moduli and the quantum numbers ℓ, n, ρ is

$$T: \qquad E \leftrightarrow \frac{4}{E}, \qquad a \leftrightarrow \frac{2a}{E}, \qquad \ell \leftrightarrow n, \qquad \rho \leftrightarrow -\rho \tag{4.1.47}$$

while N'_L is invariant. Note that this transformation is not inherited from the T-duality group of the parent theory on S^1, although it is analogous to the transformation $E \to 1/E$ found there. In fact, in the $S^1/\mathbb{Z}_2$ orbifold some states in the untwisted sector are transformed under T to states in the twisted sector. Twisted states with ℓ odd and n even are mapped to untwisted states with ℓ even and n odd (cf. Table 4.1), and vice versa. This mixing of the two sectors under T-duality was originally noted in [19].

It can be shown that the partition function of the $S^1/\mathbb{Z}_2$ orbifold is invariant under T. One can also see explicitly how the mixing of untwisted and twisted states occurs at the level of the Hilbert space by taking into account the difference in the ground states and internal oscillators of the two sectors. As a simple example consider the twisted state with $\ell = 1$, $n = 0$, $\rho = r$, with r a root of E_8, and no left oscillators. Since T-duality preserves the norms of the momenta p_R^2 and P_L^2, it should also preserve the value of N'_L to leave the mass (4.1.26a) unaffected. In this case, $N'_L = \frac{1}{2}$, and so the transformed untwisted state must have $p_-^2 = 1$ (cf. eq. (4.1.24)). It is not hard to see that it should take the form

$$\frac{1}{\sqrt{2}}\left(|0,1;r,0\rangle - |0,1;0,r\rangle\right), \tag{4.1.48}$$

where the notation is that of eq. (4.1.15).

The mapping is more complicated when oscillators are involved. Consider for instance the set of twisted states with charge vector $Z = |1,0;0\rangle$ and $N'_L = 2$, i.e. $N_L = \frac{3}{2}$. The allowed combinations of oscillators along the eight directions I that can act on Z are

$$\alpha^I_{-,-\frac{1}{2}}\alpha^J_{-,-\frac{1}{2}}\alpha^K_{-,-\frac{1}{2}}, \quad \alpha^I_{+,-1}\alpha^J_{-,-\frac{1}{2}}, \quad \alpha^I_{-,-\frac{3}{2}}, \quad I,J,K = 1,\ldots,8, \tag{4.1.49}$$

giving $120 + 64 + 8 = 192$ states. Their T-dual untwisted states, labeled by $|\ell, n; \pi, \pi'\rangle$, must have $\ell = 0$, $n = 1$, $\pi' = -\pi$ since $\rho = 0$, and they must also add up to 192 states. For the first 120 twisted states the T-duality is

$$\alpha^I_{-,-\frac{1}{2}}\alpha^J_{-,-\frac{1}{2}}\alpha^K_{-,-\frac{1}{2}}, |1,0;0,0\rangle \quad \leftrightarrow \quad \frac{1}{\sqrt{2}}\left(|0,1;r,-r\rangle - |0,1;-r,r\rangle\right), \tag{4.1.50}$$

where r is any of the 120 positive roots of E_8 (the other 120 give the same states up to an overall irrelevant sign). We see that $p_-^2 = 2r^2 = 4$, hence $N'_L = 2$ as required.

For the remaining states the mapping reads

$$\begin{aligned} \alpha^I_{+,-1}\alpha^J_{-,-\frac{1}{2}}|1,0;0,0\rangle \quad &\leftrightarrow \quad \alpha^I_{+,-1}\alpha^J_{-,-1}|0,1;0,0\rangle, \\ \alpha^I_{-,-\frac{3}{2}}, |1,0;0,0\rangle \quad &\leftrightarrow \quad \alpha^I_{-,-2}|0,1;0,0\rangle. \end{aligned} \tag{4.1.51}$$

Here we have used that in the untwisted sector the α^I_- oscillators have integer occupation number and under the orbifold action pick up a minus sign so that the full states are invariant.

4.2 The CHL string in D dimensions

We now consider the more general setting of the CHL string in D external dimensions, with $D \leq 9$. It is realized as an orbifold of heterotic compactifications on T^d (with $d = 10 - D$), where the orbifold symmetry is again $g = \mathrm{RT}$ (cf. eq. (4.1.3)), with T a half-turn around one of the cycles of T^d. We will choose this cycle to be along x^9, while the others remain unaffected.

4.2.1 Extending the nine-dimensional construction

The moduli of the $\mathrm{E}_8 \times \mathrm{E}_8$ heterotic string on T^d are the torus metric G_{ij}, the antisymmetric tensor B_{ij} and the Wilson lines A_i, where $i, j = 1, ..., d$. Again, the Wilson lines have to be invariant under the R symmetry, which implies that they are of the form $A_i = (a_i, a_i)$. Generalizing (4.1.31), we define the moduli

$$E_{ij} = G_{ij} + B_{ij} + a_i \cdot a_j, \tag{4.2.1}$$

and the quantum numbers

$$\ell^i \equiv 2w^i, \qquad n_i, \qquad \rho^I \equiv \pi^I + \pi'^I, \tag{4.2.2}$$

where w^i and n_i are the winding and momentum numbers along the ith direction and π^I, π'^I are the same as in (4.1.6). The momenta (4.1.9) are then generalized to

$$p_R = \frac{1}{\sqrt{2}} \left(n_i - \tfrac{1}{2} E_{ij} \ell^j - a_i \cdot \rho \right) \hat{e}^i, \tag{4.2.3a}$$

$$p_L = \frac{1}{\sqrt{2}} \left(n_i + (G_{ij} - \tfrac{1}{2} E_{ij}) \ell^j - a_i \cdot \rho \right) \hat{e}^i = p_R + \frac{1}{\sqrt{2}} \ell^i e_i \,, \tag{4.2.3b}$$

$$p_+ = \frac{1}{\sqrt{2}} \left(\rho + \ell^i a_i \right) \,, \tag{4.2.3c}$$

$$p_- = \frac{1}{\sqrt{2}} (\pi - \pi') \,, \tag{4.2.3d}$$

where e_i is the vielbein for the torus metric, i.e. $e_i \cdot e_j = G_{ij}$, and $\hat{e}^i$ its inverse.

The construction of the spectrum in Section 4.1 carries over with some differences. Basically, the $i = 1$ direction behaves as in the nine-dimensional case, while the other directions $i \geq 2$ behave as in the usual T^d compactification. In particular, the charge vectors

$$Z \equiv |\ell^1, ..., \ell^d, n_1, ..., n_d; \rho\rangle \tag{4.2.4}$$

have ℓ^1 odd (even) for twisted (untwisted) states, but $\ell^2, ..., \ell^d$ are always even, while in general, $n_1, ..., n_d \in \mathbb{Z}$ and $\rho \in \Gamma_8$.

The Lorentzian metric (4.1.32) generalizes to

$$\eta = \begin{pmatrix} 0 & \mathbb{1}_d & 0 \\ \mathbb{1}_d & 0 & 0 \\ 0 & 0 & \mathbb{1}_8 \end{pmatrix} \tag{4.2.5}$$

and, together with the allowed values for the quantum numbers, already suggests that the vectors Z span the lattice

$$\Gamma_{(d)} \simeq \Gamma_{d-1,d-1}(2) \oplus \Gamma_{1,9}. \tag{4.2.6}$$

The (2) at the right of $\Gamma_{d-1,d-1} \simeq \bigoplus^{d-1} \Gamma_{1,1}$ means that the norm squared of its vectors is scaled by a factor of 2, in this case due to $\ell^2, ..., \ell^d$ always being even. This is in agreement with [19], where these lattices were initially introduced. We therefore refer to $\Gamma_{(d)}$ in this context as the Mikhailov lattice. This is the analog of the Narain lattice $\Gamma_{d,d+16}$, but with the important difference that it is not self-dual (except for the $d = 1$ case reviewed in Section 4.1).

The left-moving sector of the theory now includes d Abelian KK gauge vectors like (4.1.35), so that the gauge group is of rank $8 + d$. A generic point in the moduli space has gauge group $\mathrm{U}(1)^{d+8}$, but at special points this group is enhanced. The novel feature for $d > 1$ compactifications is that states with $Z^2 = 4$ can become massless and certain enhanced gauge groups are not simply laced, as we now show.

The zero mass and level matching conditions generalizing (4.1.37) and (4.1.38) are

$$\frac{1}{2} E_{ij} \ell^j - n_i + a_i \cdot \rho = 0, \qquad i = 1, ..., d\,, \tag{4.2.7}$$

$$Z^2 = 2\ell^i n_i + \rho^2 = 4 \text{ or } 2\,. \tag{4.2.8}$$

Let us take for the moment $d = 2$. An untwisted state with $Z^2 = 4$ has n_1 even and $\rho^I = 2\pi^I$. Substituting in (4.2.8) gives $2\ell^1 n_1 + 2\ell^2 n_2 + \rho^2 = 2\ell^2 n_2 + 4q + 4\pi^2 = 4$, with q even, but in contrast to the situation in $d = 1$, it can be solved by an appropriate choice of ℓ^2 and n_2. Indeed, the product $\ell^2 n_2$ can be any even number, say $2p$ with $p \in \mathbb{Z}$. Then (4.2.8) reduces to $\pi^2 = 1 - q - p$, which admits solutions in $\Gamma_{(2)}$ if p is odd. As before, we have that $Z^2 = 4(1 - N'_L)$, which implies $N'_L = 0$ and, from (4.1.24), $\pi = \pi'$, then $\rho \in \Gamma_8(2)$. These states give rise to C_n gauge algebras at level 1, where they play the role of long roots when $n \geq 2$ ($\mathrm{C}_1 = \mathrm{A}_1$). For $d \geq 3$ there are more possibilities such as B_n and F_4 algebras (see [61] for an introduction on this topic). In Table 4.2 we record the values of the quantum numbers that massless states can have for $d \geq 2$, together with the squared length Z^2 of the charge vector.

	twisted	untwisted	
Z^2	2	2	4
ℓ^1	odd	even	even
n_1	integer	integer	even
ℓ^i	even	even	even
n_i	integer	integer	integer
ρ	Γ_8	Γ_8	$\Gamma_8(2)$

Table 4.2: Quantum numbers of the massless states in the twisted and untwisted sector. The index $i > 1$ corresponds to further compactifications of the nine-dimensional theory. States with $Z^2 = 4$ can only be massless in $D < 9$ dimensions. The states must satisfy (4.2.7) to be massless.

4.3 Exploring the moduli space

As in T^d compactifications of the heterotic string, there does not seem to exist a GDD for $d > 1$ from which one can extract all possible enhancements. One obstruction to obtaining such a GDD is that the group of automorphisms for the Mikhailov lattice, similarly to the Narain lattice, is not generated by simple reflections when $d > 1$. A workaround was found in the previous chapter: we developed an algorithm for T^d compactifications which, starting from a point p_0 of the moduli space

corresponding to a (semisimple) gauge group of maximal rank $r_{max} = d + 16$, gives a set of new points of maximal enhancement. Heuristically, it searches for maximal enhancement points which are connected to p_0 through some variety with generic gauge group of rank $r_{max} - 1$. In the case of S^1 and T^2 compactifications, this algorithm was proven to be exhaustive by comparing with previous results [13, 14].

We have modified this algorithm in order to apply it to the CHL compactifications. This is required by the technicalities of working with Mikhailov lattices as opposed to Narain lattices, specially for compactifications to spacetime dimensions lower than nine, where non-ADE root lattices appear.

In the following section we explain the methods used in our algorithm and illustrate them with an explicit example. We explain the final results in Section 4.3.4 and present the maximal enhancements generated by iterating this procedure in Table D.5.

4.3.1 Exploration algorithm

The purpose of our algorithm is to take as input some point p_0 of maximal enhancement and return a list of other such points p_k related to p_0 in some specific, controllable way. To this end, it is best to specify p_0 not by its moduli, but by its root lattice L_0 via some generating matrix (in general, by generating matrix we mean a matrix whose rows are a basis for some lattice) of simple roots embedded in the Mikhailov lattice. Both sets of data are equivalent as one can recover one from the other using equations (4.2.7) and (4.2.8). However, the lattice L_0 is more amenable to discrete operations, which we now describe.

Consider the $(10-d)$-dimensional ($d \geq 1$) CHL string at a point p_0 in moduli space specified by a set of $d+8$ simple roots with quantum numbers ℓ^i, n_i and ρ. Substituting each one of them in (4.2.7) gives d real constraints on the $d \times (d+8)$ moduli. It follows that deletion of some simple root r_0 defines a d-dimensional subvariety in moduli space which contains p_0. Generically, this subvariety contains many more maximal enhancement points p_k, each one corresponding to a distinct simple root r_k replacing r_0, $r_k \neq r_0$. It is in this sense that the p_k are neighbors of p_0. To generate such a root r_k we solve a system of equations stating that r_k must have inner product $0, -1$ or -2 with all other roots, its squared length must be 2 or 4 and it must be embedded in the Mikhailov lattice $\Gamma_{(d)}$ in accordance with Table 4.2.

In order to make sure that the root lattice obtained by replacing $r_0 \to r_k$ corresponds to the gauge group G_k at p_k, we have to take care of an ambiguity in the relation between the moduli of p_k and the root lattice $L_k \equiv L$ of G_k. Even though the embedding of L in $\Gamma_{(d)}$ specifies the moduli via the constraints mentioned above, it is also true that any sublattice $L' \subseteq L$ with $\text{rank}(L') = \text{rank}(L)$ will give the same moduli. When we replace $r_0 \to r_k$ there is therefore the possibility that the lattice obtained will not be L but some L'. This ambiguity is eliminated if we implement a procedure, which we explain below, that takes L' and returns L by adding the missing roots. This adding of roots will be referred to as a *saturation* of L' to L.

To saturate L' we recall that all of its even overlattices are contained in the dual lattice L'^*, so that in particular $L' \subseteq L \subseteq L'^*$. It suffices then to compute the vectors dual to L', select those which correspond to roots embedded in $\Gamma_{(d)}$ and add them to L'. In practice this is done by iterating an algorithm which replaces one root vector in the generating matrix for L' such that $\det L'$ gets smaller (indicating that L' has been extended) and is still embedded in $\Gamma_{(d)}$. When all attempts to do this leave the determinant of the lattice invariant, L' has been saturated to the true root lattice L at p_k.

4.3.2 Example

To illustrate this procedure we first consider an exploration of the neighborhood of the point in moduli space corresponding to eight dimensional CHL with gauge algebra $(\mathrm{A}_1+\mathrm{A}_3+\mathrm{D}_6)_2$, with the subscript indicating that the world-sheet Kac-Moody algebra is realized at level 2, given by the moduli

$$E_{ij}=\begin{pmatrix}2&0\\0&1\end{pmatrix},\qquad a_1=(0^7,1),\qquad a_2=(0^3,-\tfrac{1}{2}^4,\tfrac{1}{2}). \tag{4.3.1}$$

The root lattice L_0 is generated by the rows $(\ell^1,\ell^2,n_1,n_2;\rho)$ of the 10×12 matrix

$$\mathcal{G}_0=\begin{bmatrix}
1&2&-1&-1&0&0&0&1&1&1&1&-2\\
0&0&0&0&1&-1&0&0&0&0&0&0\\
0&0&0&0&0&1&-1&0&0&0&0&0\\
0&0&0&0&-1&-1&0&0&0&0&0&0\\
1&0&1&0&0&0&0&0&0&0&0&0\\
-1&0&0&0&0&0&0&0&0&0&1&1\\
0&0&0&0&0&0&0&0&0&1&-1&0\\
0&0&0&0&0&0&0&0&1&-1&0&0\\
0&0&0&1&0&0&0&-1&-1&0&0&0\\
1&0&-1&-1&0&0&0&0&0&0&0&-2
\end{bmatrix}, \tag{4.3.2}$$

from which the gauge algebra is read by computing its Gram matrix $\mathcal{G}_0\eta\mathcal{G}_0^T$, with η given in (4.2.5). Note also that $\mathcal{G}_0$ is not a square matrix due to the fact that it gives an embedding of a rank 10 lattice into the rank 12 lattice $\Gamma_{(2)}$. We have chosen this particular vacuum because, as we explain below, it neighbors another vacuum with globally non-trivial gauge group.

Starting from $\mathcal{G}_0$, one of the paths that our algorithm will follow is to remove, for example, the 6th row. This breaks $(\mathrm{D}_6)_2\to(2\mathrm{A}_1+\mathrm{A}_3)_2$ and eliminates two real constraints on the moduli (cf. eq. (4.2.7)), which taking into account the remaining $20-2=18$ constraints read

$$E_{ij}=\begin{pmatrix}2&x\\0&y\end{pmatrix},\qquad a_1=(0_3,x,(-x)_3,1),\qquad a_2=(0_3,y-\tfrac{3}{2},(\tfrac{1}{2}-y)_3,\tfrac{1}{2}), \tag{4.3.3}$$

with the subindex 3 meaning that the quantity is repeated 3 times. In other words, the moduli are now constrained to a plane (x,y) with generic gauge algebra $(3\mathrm{A}_1+2\mathrm{A}_3)_2$. Our algorithm will now generate a new simple root α by picking out a solution to the set of equations

$$\begin{cases}\mathcal{G}_{0,mn}\alpha_n=k_m, & k_m\in\{0,-1,-2\},\quad m\neq 6\\ \alpha^2=N, & N\in\{2,4\},\end{cases} \tag{4.3.4}$$

where $\alpha=(\ell^i,n_i;\rho)$ is constrained to lie in $\Gamma_{(2)}$, meaning that $\ell^1,n_1,n_2\in\mathbb{Z}$, $\ell^2\in 2\mathbb{Z}$ and $\rho\in\Gamma_8$. One possible solution with $N=4$ is

$$\alpha=\begin{bmatrix}0&2&-2&-3&0&0&0&0&2&2&2&-2\end{bmatrix}. \tag{4.3.5}$$

The new matrix $\mathcal{G}_1$ resulting from this exchange of roots (α is now in the 6th row) is seen to generate the root lattice L_1 corresponding to the gauge algebra $(2\mathrm{A}_1+2\mathrm{A}_3)_2+(\mathrm{C}_2)_1$ and the moduli are fixed to

$$E_{ij}=\begin{pmatrix}2&0\\0&\frac{5}{4}\end{pmatrix},\qquad a_1=(0_7,1),\qquad a_2=(0_3,-\tfrac{1}{4},(-\tfrac{3}{4})_3,\tfrac{1}{2}). \tag{4.3.6}$$

To check that L_1 contains all the solutions to equations (4.2.7) and (4.2.8), our algorithm calculates the generating matrix $\mathcal{G}_1^*$ for the dual lattice L_1^*:

$$\mathcal{G}_1^* = \begin{bmatrix} \frac{1}{2} & 1 & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -1 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & -\frac{3}{2} & 0 & 0 & 0 & 0 & 1 & 1 & 1 & -1 \\ \frac{1}{2} & 1 & -\frac{1}{2} & -\frac{3}{2} & 0 & 0 & 0 & 0 & 1 & 1 & 1 & -1 \\ 0 & 0 & 0 & \frac{1}{4} & 0 & 0 & 0 & -\frac{1}{4} & \frac{1}{4} & \frac{1}{4} & -\frac{3}{4} & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & \frac{3}{4} & 0 & 0 & 0 & 0 & -\frac{3}{4} & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & 0 \\ \frac{1}{2} & 0 & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}. \tag{4.3.7}$$

It then constructs generic integer linear combinations of the rows corresponding to roots lying in $\Gamma_{(2)}$ and adds them to L_1 by replacing one of the rows of $\mathcal{G}_1$. This is done in an exhaustive way, but in this particular case no such replacement decreases the determinant of $\mathcal{G}_1$, hence L_1 is saturated. This means that the gauge algebra at this point in moduli space is indeed $(2\mathrm{A}_1 + 2\mathrm{A}_3)_2 + (\mathrm{C}_2)_1$.

4.3.3 Matter states and global data

There are two other sets of data of importance that can be obtained by our methods, namely the matter states in the lowest massive level associated to fundamental representations of the gauge group G, and the global structure of G, i.e. the fundamental group $\pi_1(G)$. Both of these problems involve finding overlattices of root lattices which are primitively embedded in the momentum lattice $\Gamma_{(d)}$ or its dual $\Gamma^*_{(d)}$, as we now explain.

4.3.3.1 Computing the overlattice

By primitively embedded overlattice we mean the intersection of the real span of the root lattice, $L \otimes \mathbb{R}$, and the momentum lattice $\Gamma_{(d)}$ in the ambient space $\mathbb{R}^{d+8,d}$. In terms of the momenta $p_{L,R}$ this means all vectors which satisfy the constraint $p_R = 0$ but p_L is unconstrained. Generally such an overlattice M corresponds to an extension of L by a set of fundamental weights $\{\mu, \mu', ...\}$, and the quotient M/L can be put in correspondence with a subgroup K of the center of the universal cover $\tilde{G}$ of G, denoted $Z(\tilde{G})$ (cf. Table 4.3). It follows that the overlattice data can be encoded in the generators $\{k, k', ...\}$ of K.

$\tilde{G}$	$Z(\tilde{G})$
$\mathrm{SU}(n+1)$	$\mathbb{Z}_{n+1}$
$\mathrm{Spin}(2n+1)$, $\mathrm{Sp}(2n)$, E_7	$\mathbb{Z}_2$
E_6	$\mathbb{Z}_3$
$\mathrm{Spin}(4n+2)$	$\mathbb{Z}_4$
$\mathrm{Spin}(4n)$	$\mathbb{Z}_2 \times \mathbb{Z}_2$
E_8, F_4, G_2	$\mathbb{1}$

Table 4.3: Center $Z(\tilde{G})$ of compact connected simple groups $\tilde{G}$.

Computing the weight vectors μ_i can be done by a slight generalization of the saturation algorithm described at the end of Section 4.3.1. Indeed, what it basically does is a computation of an overlattice of L which is also a root lattice. By relaxing this last constraint, the same algorithm can be used to compute M. Returning to the example of Section 4.3.2, we apply this algorithm and find that L can

be extended to an overlattice M in $\Gamma_{(2)}$ by adding the weight vector

$$\mu = |2, 2, -1, -2; 0, 0, -1, 0, 2, 1, 1, -3\rangle \ . \tag{4.3.8}$$

In other words, the vector μ satisfies $p_R = 0$ (cf. eq. (4.2.7)) with the moduli given in (4.3.6), but is not in L. Determining the precise $K \subset Z(\tilde{G})$ now amounts to determining the element in $Z(\tilde{G})$ to which μ corresponds. To do this we recall that

$$Z(\tilde{G}) = \Lambda_{\text{weight}}/\Lambda_{\text{root}} \tag{4.3.9}$$

where Λ_{weight} is the weight lattice, which in particular contains M, and $\Lambda_{\text{root}} = L$. The weight μ together with all its L-translations constitutes an equivalence class $[\mu] \in Z(\tilde{G})$.

In general, for $\tilde{G}$ a semisimple group with s simple factors, $Z(\tilde{G})$ is a product of $s+t$ cyclic groups,

$$Z(\tilde{G}) = \mathbb{Z}_{p_1} \times \cdots \times \mathbb{Z}_{p_{s+t}} \, , \tag{4.3.10}$$

where t is the number of D_{2n} factors since they contribute each a $\mathbb{Z}_2 \times \mathbb{Z}_2$ group (see Table 4.3). Any element of $Z(\tilde{G})$ can therefore be written as a tuple

$$k = (k_1, ..., k_{s+t}) \, , \tag{4.3.11}$$

where $k_i \sim k_i + p_i$, and the ordering of the k_i's is appropriately specified in each case. In our example, we have

$$\tilde{G} = \mathrm{SU}(2)^2 \times \mathrm{SU}(4)^2 \times \mathrm{Sp}(2), \qquad Z(\tilde{G}) = \mathbb{Z}_2^2 \times \mathbb{Z}_4^2 \times \mathbb{Z}_2 \, , \tag{4.3.12}$$

and each central element is of the form

$$k = (k_1, k_2, k_3, k_4, k_5) \mod (2, 2, 4, 4, 2) \, . \tag{4.3.13}$$

To determine which equivalence class k contains the weight vector μ, we first note that each possible k can be put in correspondence with a combination of fundamental weights of $\tilde{G}$. If for example one looks at the fundamental weights w_i of $\mathrm{SU}(n)$, one finds that $[w_i] = i \in \mathbb{Z}_n$ (up to the outer automorphism of $\mathrm{SU}(n)$ which maps $i \to -i \mod n$). For $\mathrm{Sp}(2)$, the only non-trivial element of the center contains the weight corresponding to the short simple root (or equivalently the spinor class in $\mathrm{Spin}(5) = \mathrm{Sp}(2)$). Using these facts one finds that the μ given in (4.3.8) is contained in

$$k = (1, 1, 2, 2, 1) \, . \tag{4.3.14}$$

To verify this, one can compute the fundamental weights (labeled by i) $w_{j,i}$ of each simple factor (labeled by j) and check that the vector

$$w_{1,1} + w_{2,1} + w_{3,2} + w_{4,2} + w_{5,1} \tag{4.3.15}$$

can be translated by roots in L to the given μ. Keep in mind that these calculations are performed with respect to the particular embedding of L and M in $\Gamma_{(2)}$.

Having determined the explicit form of $k = [\mu] \in Z(\tilde{G})$, we immediately find that $K = \mathbb{Z}_2$, since $2k = (2, 2, 4, 4, 2) = (0, 0, 0, 0, 0)$, i.e. k is an order 2 element. Moreover, it is uniquely in correspondence with the fundamental representation $(\mathbf{2}, \mathbf{2}, \mathbf{6}, \mathbf{6}, \mathbf{4})$ of $\tilde{G}$. Indeed, one can explicitly find all the states

which form this representation with mass $M^2 = 4$. It suffices to construct such a state from the weight vector (4.3.8) and act on it with the Weyl group of the enhanced gauge group, which is a subset of the subgroup of T-dualities that leave the moduli invariant. In this way all the states forming the corresponding representation of $\tilde{G}$ are obtained.

4.3.3.2 Computing the fundamental group

As explained in [20] (see also [19]) the fundamental group of G can be computed as the quotient $M^\vee/L^\vee$, where $L^\vee$ and $M^\vee$ are respectively the coroot lattice and the cocharacter lattice of G. For every G, $L^\vee$ is embedded in the dual Mikhailov lattice $\Gamma^*_{(d)}(2)$, where the (2) means that it is also rescaled by a factor of $\sqrt{2}$ to make it even, and $M^\vee$ corresponds to its overlattice primitively embedded in $\Gamma^*_{(d)}(2)$. In practice this means that to compute the fundamental groups we need to find embeddings of the lattices $L^\vee$ in the dual Mikhailov lattice and then apply the procedure explained before to get the respective $M^\vee$.

Even though the exploration algorithm was designed to find points of maximal symmetry enhancement in moduli space, it can be considered on its own as an algorithm for finding embeddings of lattices into other lattices. For this reason it can be used also to compute all possible root lattices in $\Gamma^*_{(d)}(2)$. This is due to the fact that the data that we manipulate through this algorithm corresponds to the lattice vectors themselves and not the moduli or the momenta. A point that has to be made clear however is that the condition for a vector in the lattice to be a root is that it is of norm 2, or that it is of norm 4 and furthermore has even inner product with all other vectors in the lattice. This is the statement which generalizes the conditions for massless states shown in Table 4.2 to any basis for the momentum lattice that we choose. It applies both to $\Gamma_{(d)}$ and $\Gamma^*_{(d)}(2)$.

In eight dimensions, for example, we have

$$\Gamma_{(2)} = \Gamma_{1,1}(2) \oplus \Gamma_{1,1} \oplus \mathrm{E}_8 \quad \Rightarrow \quad \Gamma^*_{(2)}(2) = \Gamma_{1,1} \oplus \Gamma_{1,1}(2) \oplus \mathrm{E}_8(2)\,. \tag{4.3.16}$$

We can take as a starting point for the exploration the root lattice of, say, B_{10}, which can be constructed by hand and is expected to embed into $\Gamma^*_{(2)}(2)$ since it is the coroot lattice of C_{10} which embeds into $\Gamma_{(2)}$. After a few steps, the algorithm produces a list of root lattices which correspond exactly to the coroot lattices of the gauge algebras found by exploring the original lattice $\Gamma_{(2)}$. In particular, we find the root lattice

$$L = 2\mathrm{A}_1(2) \oplus 2\mathrm{A}_3(2) \oplus \mathrm{B}_2, \tag{4.3.17}$$

which corresponds to the coroot lattice $L^\vee$ of the model used in the examples of Sections 4.3.2 and 4.3.3.1. One may apply exactly the same procedure of the last section to compute its overlattice and the subgroup of $Z(\tilde{G}^\vee)$ to which it corresponds, where $\tilde{G}^\vee$ is the simply connected gauge group with root lattice in (4.3.17). Since this subgroup coincides with $M^\vee/L^\vee$, its generators k_i give precisely the fundamental group $\pi_1(G) \subset Z(\tilde{G}) \simeq Z(\tilde{G}^\vee)$, which we refer to as H, i.e. $G = \tilde{G}/H$. In this case, we find two generators

$$k = (0,1,0,2,1)\,, \qquad k' = (1,0,2,0,1) \tag{4.3.18}$$

of order 2, so that $H = \mathbb{Z}_2 \times \mathbb{Z}_2$, and the gauge group is

$$G = \frac{\mathrm{SU}(2)^2 \times \mathrm{SU}(4)^2 \times \mathrm{Spin}(5)}{\mathbb{Z}_2 \times \mathbb{Z}_2}\,. \tag{4.3.19}$$

This result is in agreement with that of [20].

4.3.3.3 Anomaly for center symmetries

It has been shown in [21] that in order for an 8d $\mathcal{N}=1$ supergravity theory with global gauge group $G=\tilde{G}/H$ to be consistent, the following condition must be satisfied:

$$\sum_{i=1}^{s}\alpha_{\tilde{G}_i}m_i k_i^2=0 \mod 1\,, \tag{4.3.20}$$

where $\tilde{G}_i$ are the s simple factors in $\tilde{G}$, $\alpha_{\tilde{G}_i}$ are the conformal dimensions of the Kac-Moody representations which generate the center [62], m_i are free parameters in the supergravity theory and $k=(k_1,...,k_s)$ is the generator of $H\in Z(\tilde{G})$. This condition ensures that the H center symmetry is free of anomalies. In the string theory whose low energy limit corresponds to this supergravity theory, m_i are the levels of the world-sheet current algebra of $\tilde{G}_i$. It can be shown in general that (4.3.20) is satisfied by construction for all $G=\tilde{G}/H$ obtained from the heterotic string on T^2 and the 8d CHL string [20]. Here we give a brief alternative proof for this fact in the T^2 case, and comment briefly on the CHL case.

The fact that the gauge groups that arise from the Narain lattice $\Gamma_{2,18}$ satisfy (4.3.20) by construction is relatively easy to see. For this we recall that the conformal dimension $\alpha_{\tilde{G}_i}$ can be written as

$$\alpha_{\tilde{G}_i}=\frac{w_i^2}{\alpha_\ell^2}\,, \tag{4.3.21}$$

where w_i is the fundamental weight that generates the center of the group $\tilde{G}_i$ and α_ℓ is the highest root, which is a long root. In this case, all possible gauge groups are of ADE type, so that $\alpha_\ell^2=2$, and have $m_i=1$. We can therefore rewrite (4.3.20) as

$$\sum_{i=1}^{s}(w_i k_i)^2=0 \mod 2\,, \tag{4.3.22}$$

which is the statement that the weight vector $\sum_{i=1}^{s}w_i k_i$ is even. For ADE groups, the root and coroot lattices are the same, and since the Narain lattice is also self-dual, the global structure is given by the overlattice M which embeds primitively into $\Gamma_{2,18}$ and is given by precisely this weight vector (cf. Sections 4.3.3.1 and 4.3.3.2). It is of course possible that there is more than one weight vectors involved, in which case the situation is analogous. Since the Narain lattice is even, all overlattices M must also be even and so the condition (4.3.20) is satisfied by construction.

For the CHL string the situation is more subtle since the Mikhailov lattice is not self dual and there are symplectic groups. One can understand why groups occurring in this case should satisfy (4.3.20) by noting that all of them can be constructed from groups arising from the Narain lattice by a suitable projection [20], and so they must also preserve condition (4.3.20). It is straightforward to verify that this is the case given the H generators displayed in Table D.6.

4.3.3.4 Globally non-trivial groups of lower rank

So far we have discussed maximally enhanced gauge groups. For non-Abelian groups of lower rank there are of course many more possibilities. In particular, the list of all possible gauge groups arising in T^2 compactifications of the heterotic string is 3279, of which only 325 are of maximal rank[1]; this

[1] In this count we are ignoring the global part of the groups.

was determined by Shimada in [15] from the point of view of elliptic K3 surfaces, and was verified for heterotic string on T^2 as explained in Chapter 3.

An important fact that was noticed in [15] is that all possible gauge groups of rank lower than 18 (the maximal rank in T^2 compactifications) which are simply-connected can be obtained from those of rank 18 which are also simply-connected by deleting nodes in the corresponding Dynkin diagram (e.g. $\mathrm{A}_{m+n+1} \to \mathrm{A}_m + \mathrm{A}_n$). For groups with non-trivial fundamental group H, this is not necessarily true. For example, the gauge group $\mathrm{Spin}(8)^4/(\mathbb{Z}_2 \times \mathbb{Z}_2)$ cannot be enhanced to a higher rank group, so that, conversely, it cannot be found by deleting a node as just described. We note that Shimada has given a set of rules for obtaining such gauge groups (see theorems 2.4-2.7 of [15]), but they do not correspond to arbitrary node deletion and are rather involved.

Here we will not attempt to repeat this analysis for the CHL string, but instead ask the following question: what gauge groups with non-trivial H can be obtained by breaking maximally enhanced groups via node deletion? Given that all maximal enhancements in 9d have trivial H (cf. Table D.4), we will restrict ourselves to the 8d theory. In this case, there are 29 such groups, 24 with $H = \mathbb{Z}_2$ and 5 with $H = \mathbb{Z}_2 \times \mathbb{Z}_2$ (cf. Table D.5).We record them with their corresponding k's in Table D.6.

It is easier to find the answer to our question by brute force. Just delete one of the simple roots in the embedding of the rank 10 root lattice L into the Mikhailov lattice $\Gamma_{(2)}$ and check if the resulting rank 9 lattice $L' \subset L$ still has a non-trivial weight overlattice $W' \subset W$. This will give rank 9 semisimple gauge groups with $H = \mathbb{Z}_2$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$ (as there are no other possibilities). Repeating the same procedure gives groups of rank 8 with the same H, and so on. There is only one non-simply-connected gauge group of rank 4, namely $SU(2)^4/\mathbb{Z}_2$, and there are none for rank ≤ 3. On the other hand, all of the 29 rank 10 groups can be broken to the rank 4 one. Analogously, $SU(2)^7/(\mathbb{Z}_2 \times \mathbb{Z}_2)$ is the only gauge group of rank 7 with $H = \mathbb{Z}_2 \times \mathbb{Z}_2$. There are no groups with that H for rank ≤ 6 and all of the five rank 10 groups with that fundamental group can be broken to the rank 7 one. In Figure 4.2 we present a graph which encodes the breaking patterns that preserve the $\mathbb{Z}_2 \times \mathbb{Z}_2$.

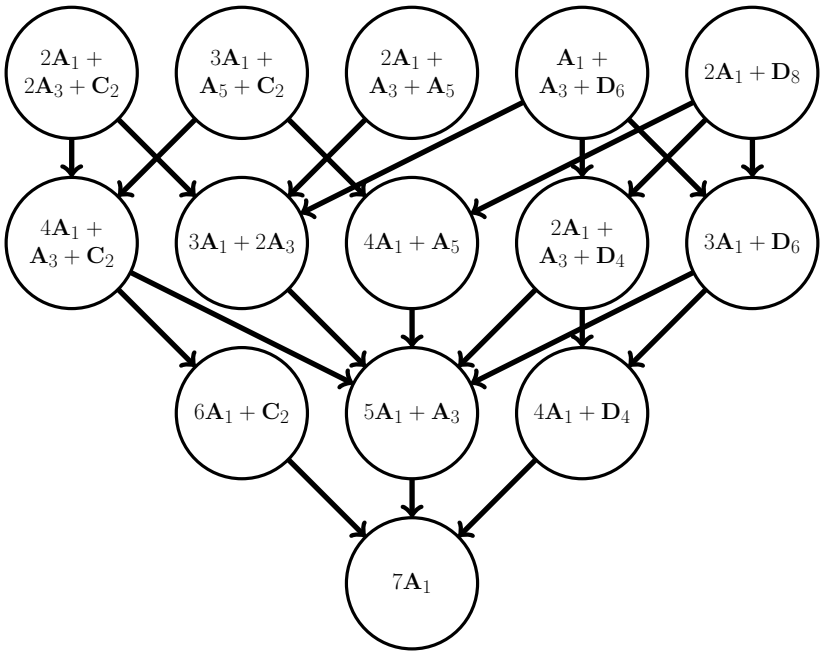


Figure 4.2: Scheme of how deleting nodes in the Dynkin diagrams of maximally enhanced groups with $H = \mathbb{Z}_2 \times \mathbb{Z}_2$ leads to gauge groups with lower rank and also with $H = \mathbb{Z}_2 \times \mathbb{Z}_2$.

4.3.4 Results

We collect in Table D.5 the 61 maximally enhanced groups $G = \tilde{G}/H$ that are realized in the eight-dimensional CHL string, and give the point in moduli space where they arise. ADE groups are realized at level 2 of the Kac-Moody algebra, while C groups arise at level 1. Our results for the algebras are in complete agreement with those obtained in [63] from F-theory, which appeared while [3] was being written.

There are 32 simply connected groups. The rest are of the form $\tilde{G}/H$ with $H = \mathbb{Z}_2$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$. The fundamental group H is generated in each case by the elements $k \in Z(\tilde{G})$ shown in Table D.6. Our results are in perfect agreement with those in [20].

Most of the groups shown lie in the subspace of moduli space given by $E_{ij} = \mathrm{diag}(2, 1)$, and it can actually be shown that the remaining ones can be mapped to this subspace by applying T-dualities. This is analogous to the situation in the heterotic string on T^2 with $E_{ij} = \mathrm{diag}(1, 1)$ encountered on Chapter 3. By performing the necessary T-dualities to realize the enhancement groups at such E_{ij}, however, the Wilson lines get much more complicated, and difficult to handle.

Chapter 5

Orbifolds and singularity freezing

By means of the asymmetric orbifold construction of [64], theories with 16 supercharges but with gauge symmetry with reduced rank can be obtained. In particular, one finds the so called CHL string component [17, 18] in 9d, studied in the previous chapter, for which the momentum lattice was constructed by Mikhailov in [19]. In 8d the story is the same, but in 7d one finds four extra components (six including the Narain and CHL component). These were constructed together with their momentum lattices in [23].

In this chapter we extend this work to the six aforementioned components in 7d and 6d. To this end it is necessary to state precisely how the enhanced symmetry groups can be obtained from the momentum lattices, which we do by a natural generalization of the case for the CHL string . We see that the lattice alone is not sufficient to determine the allowed gauge groups, but rather one must impose a constraint in the embeddings characterized by an integer, which comes from the string theory but is ad hoc from the point of view of the lattice (see Proposition 3). Implementing this constraint in our algorithm we obtain a list of maximally enhanced gauge algebras for each component.

On the other hand it is well known that the heterotic string on T^3 is dual to M-theory on K3. Gauge groups with reduced rank are realized in the later when there are so-called partially frozen singularities on the K3 [23–25]. It is then natural to ask how this mechanism of partial freezing appears in the heterotic string. We study this problem by exploiting relations between the reduced rank momentum lattices and the Narain lattice and find a match with the known results in the M-theory side. General freezing rules involving the topology of the gauge groups are obtained, generalizing the results of [20] for the 8d CHL string.

It turns out that for $d \leq 4$ all the gauge groups of the reduced rank components can be obtained from those of the Narain component by means of a suitable map. At the level of the algebras this has been known for a long time, for $d = 2$, in the dual frame of F-theory on K3, where reduced rank algebras are obtained by partially "freezing" the singular fibers [22,56]. This situation extends to $d = 3$ using M-theory on K3 [23–25]. In the heterotic string these results can be reproduced by using lattice embedding techniques, and in fact one can also see how the full gauge group is "frozen". For $d = 2$ this extension was done in [20], here we generalize these results for $d = 3$, by exploiting relations between the reduced rank momentum lattices and the Narain lattice, finding a match with the known results in the M-theory side. As will be shown later, for $d \leq 3$ the map can be naively applied at the algebras of the enhancements found on the toroidal compactification, giving the algebras associated to the reduced rank components. In contrast, in $d = 4$ in order to apply it one must know the fundamental group of the gauge group explicitly. This is due to the fact that the lattice which corresponds to the frozen singularity in the heterotic frame is a root lattice for $d = 1, 2, 3$ but the weight lattice of a non-simply-connected group for $d = 4$. Most remarkably, however, is the fact that the maps seem to be exactly those which relate the "topologically non-trivial" components of the moduli space of flat

connections of a non-simply-connected gauge group over T^2 (and not T^4) to the "topologically trivial" component [55, 65], although to our awareness an explanation for this is lacking.

As we did before for 9d and 8d compactifications, we have carried out an exploration of the possible maximally enhanced gauge groups realized in the six components of the moduli space of heterotic strings with 16 supercharges for 7d and 6d, with the results for 7d being listed in Appendix D.2 and for 6d in the corresponding tables at [27][1]. These results where in turn used to check the freezing maps we derived.

This chapter is organized as follows. In Section 5.1 we review the construction of rank reduced heterotic theories in nine to seven dimensions, emphasizing the role of outer automorphisms of the gauge lattice in the framework of asymmetric orbifolds. Then in Section 5.2 we state the criteria for gauge groups being realized in the relevant theories in terms of lattice embeddings. In Section 5.3 we explain how the freezing maps for $d = 1$ to 4 are constructed for the CHL case, extending them to the holonomy triples of [23] in 7d and 6d in Section 5.4. Finally, the results obtained with the exploration algorithm are presented and discussed in Section 5.5.

5.1 Construction of rank reduced theories

In this section we review how rank reduced theories with 16 supercharges are constructed from the heterotic string in nine to seven dimensions. The idea is to get an intuitive understanding of these constructions through the manipulation of Dynkin Diagrams, illustrating the asymmetric orbifold construction with an outer automorphism. This complements the more general (and abstract) treatment in [23]. We go through the CHL string, the $\mathrm{Spin}(32)/\mathbb{Z}_2$ heterotic theory compactification without vector structure and the $\mathbb{Z}_m$-triples.

5.1.1 CHL string

As we detailed in the previous chapter, the CHL string in 9d can be realized as the HE string compactified on an orbifold of a circle involving the outer automorphism θ which exchanges both E_8's and a half-period shift a along the circle [18]. The resulting target space has an holonomy θ along the compact direction which breaks the gauge group $\mathrm{E}_8 \times \mathrm{E}_8$ to its diagonal E_8. The shift a obstructs the recovery of the broken E_8 in the twisted sector and so it ensures that the rank of the total gauge group is reduced.

Since θ is an outer automorphism of a gauge group, its implementation as an orbifold symmetry naturally leads to a picture of Dynkin Diagram folding. In the case of the CHL string, one "folds one E_8 into the other", and finds that the gauge group of the resulting theory is E_8 (with an extra U(1) for arbitrary radius). Turning on a Wilson line does not change this picture since it must break both E_8's in the same way, and one then just folds one of the broken groups into the other.

Even though the length of a root is not by itself a meaningful concept, it is helpful to think that the nodes that get superposed in folding a diagram correspond to shortened roots. The reason is that this maps naturally to an increase in the level of the associated gauge algebra by a factor equal to the order of the automorphism θ. In this case, the $\mathrm{E}_8 \times \mathrm{E}_8$ at level 1 becomes an E_8 at level 2. On the other hand, connected diagrams containing invariant nodes correspond to algebras at level 1.

[1] For 6d its exhaustivity was proven in a recent work [26].

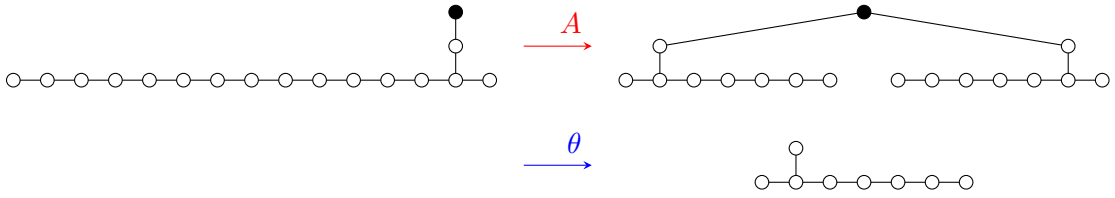
The main idea here is that using the symmetry (a, θ) one constructs a vacuum of the heterotic string with an holonomy that in particular projects out Cartan generator states. Such an holonomy can not be implemented in the theory by merely turning on Wilson lines, as outer automorphisms are not connected to the identity element in the gauge group. However, the set of holonomies that can be obtained by orbifolding the target manifold is larger and includes those of this type. Together with the diagram folding picture, this story generalizes to the other constructions reviewed below.

5.1.2 Compactification without vector structure

There is a theory dual to the 8d CHL string which is obtained from the $\mathrm{Spin}(32)/\mathbb{Z}_2$ heterotic string by compactifying it on a T^2 without vector structure [22]. The basic idea is that the spectrum of the 10d theory does not contain vector representations of $\mathrm{Spin}(32)$, and so one should consider topologies of the gauge bundle which do not admit such representations. An obstruction of this type is measured by a mod two cohomology class $\tilde{w}_2$, analogous to the second Stieffel-Whitney class w_2 which obstructs spin structure.

This compactification is characterized by the fact that the two holonomies g_1, g_2 on the torus commute as elements of $\mathrm{Spin}(32)/\mathbb{Z}_2$, but do *not* commute when lifted to elements of the double cover $\mathrm{Spin}(32)$. In other words, the commutator of these holonomies is lifted to a non-trivial element in $\mathrm{Spin}(32)$ which is identified with the identity upon quotienting by one of the spinor classes in its center. The lifting $\mathrm{Spin}(32)/\mathbb{Z}_2 \to \mathrm{Spin}(32)$ is therefore obstructed and no vector representations are allowed.

Two such holonomies can not be put simultaneously on a maximal torus of the gauge group. Similarly to the CHL string, one of them has to be realized by orbifolding the theory. The difference in this case is that the 10d gauge group $\mathrm{Spin}(32)/\mathbb{Z}_2$ does not have any outer automorphism. One can however turn on a Wilson line along one of the compact directions such that from the point of view of the remaining dimensions the gauge group is actually broken to one which does in fact have an outer automorphism. Concretely, we turn on a Wilson line $A = (\frac{1}{2}^8, 0^8)$ which breaks $\mathrm{Spin}(32)/\mathbb{Z}_2 \to \mathrm{Spin}(16)^2/\mathbb{Z}_2$. This can be represented diagrammatically as



(5.1.1)

where the white nodes are simple roots and the black nodes represent the fundamental weight which generates the $\mathbb{Z}_2$ in each case. We see that the RHS corresponds to a group with outer automorphism θ. Orbifolding the theory by this symmetry and a half period shift along the second compact direction we obtain a theory with gauge group $\mathrm{Spin}(16) \times \mathrm{U}(1)^2$ (for arbitrary values of the torus metric and B-field). We note that the fundamental weight gets projected out by the orbifold symmetry, but the gauge group is $\mathrm{Spin}(16)/\mathbb{Z}_2$, as can be seen by breaking group #49 in Table D.6.

The commutator of the holonomies chosen is the exponential of

$$A - \theta(A) = (\tfrac{1}{2}^8, 0^8) - (0^8, \tfrac{1}{2}^8) = (\tfrac{1}{2}^8, -\tfrac{1}{2}^8), \tag{5.1.2}$$

which does not yield the identity in $\mathrm{Spin}(32)$ but rather the element which gets identified with it in

$\mathrm{Spin}(32)/\mathbb{Z}_2$. This corresponds to the discussion above. More generally one can deform this Wilson line by adding vectors symmetric in the first and last eight components, i.e. those of the form (δ, δ), as to respect condition (5.1.2). One can also turn on another Wilson line A' in the second compact direction such that $\theta(A') = A'$, since the product of two holonomies on the same direction should commute. Together with deformations of the metric and the B-field we reach other points in the moduli space, which is equivalent to that of the 8d CHL string, where the equivalence is given by T-duality [23].

5.1.3 Holonomy triples in 7d

The basic idea behind the construction just described can be applied to the heterotic string on a circle and further compactifying two dimensions on a torus. This comes from the fact that there are various 9d gauge groups analogous to the 10d $\mathrm{Spin}(32)/\mathbb{Z}_2$. It is enough to consider the following five:

$$\frac{(\mathrm{E}_7 \times \mathrm{SU}(2))^2}{\mathbb{Z}_2}, \quad \frac{(\mathrm{E}_6 \times \mathrm{SU}(3))^2}{\mathbb{Z}_3}, \quad \frac{(\mathrm{Spin}(10) \times \mathrm{SU}(4))^2}{\mathbb{Z}_4}, \quad \frac{\mathrm{SU}(5)^4}{\mathbb{Z}_5}, \quad \frac{(\mathrm{SU}(2) \times \mathrm{SU}(3) \times \mathrm{SU}(6))^2}{\mathbb{Z}_6}\,. \tag{5.1.3}$$

These correspond to breakings of $\mathrm{E}_8 \times \mathrm{E}_8$ by a Wilson line A, so that it is most natural to work in the framework of the HE string. Natural choices for these Wilson lines are, respectively,

$$A = \begin{cases} (0^6, -\frac{1}{2}, \frac{1}{2}) \times (-\frac{1}{2}, \frac{1}{2}, 0^6) & (\mathbb{Z}_2) \\ (0^5, -\frac{1}{3}^2, \frac{2}{3}) \times (-\frac{2}{3}, \frac{1}{3}^2, 0^5) & (\mathbb{Z}_3) \\ (0^4, -\frac{1}{4}^3, \frac{3}{4}) \times (-\frac{3}{4}, \frac{1}{4}^3, 0^4) & (\mathbb{Z}_4) \\ (0^3, -\frac{1}{5}^4, \frac{4}{5}) \times (-\frac{4}{5}, \frac{1}{5}^4, 0^3) & (\mathbb{Z}_5) \\ (0^2, -\frac{1}{6}^5, \frac{5}{6}) \times (-\frac{5}{6}, \frac{1}{6}^5, 0^2) & (\mathbb{Z}_6) \end{cases}. \tag{5.1.4}$$

The $\mathbb{Z}_m$'s correspond not only to the fundamental group of each broken gauge group but also to the cyclic group generated by the outer automorphism θ to be implemented. The name '$\mathbb{Z}_m$-triple' refers to this group together with the three holonomies consisting of (5.1.4) and the pair analogous to the one discussed in the previous section, which we now discuss.

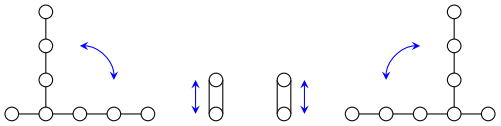
5.1.3.1 $\mathbb{Z}_2$-triple

Consider first the $\mathbb{Z}_2$-triple. From the point of view of the T^2 on which the 9d theory is compactified, the gauge group is $(\mathrm{E}_7 \times \mathrm{SU}(2))^2/\mathbb{Z}_2$, which indeed has an order two outer automorphism, exchanging the $\mathrm{E}_7 \times \mathrm{SU}(2)$ factors. However, using this symmetry to orbifold the theory just gives us the CHL string, as discussed in Section 5.1.1. Consider instead turning on a Wilson line A' on one of the T^2 directions (x^1), of the form

$$A' = (0^5, -\tfrac{1}{2}, \tfrac{1}{2}, 0) \times (0, -\tfrac{1}{2}, \tfrac{1}{2}, 0^5)\,. \tag{5.1.5}$$

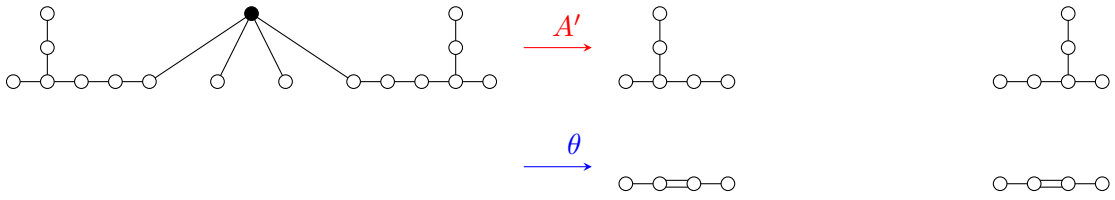
It has the effect of further breaking the gauge group to $(\mathrm{E}_6 \times \mathrm{U}(1)^2)^2$. From the point of view of the other T^2 direction (x^2), the gauge group has then an order 2 outer automorphism corresponding to the symmetry of each E_6 diagram. To get a consistent theory (meaning that the partition function is modular invariant), however, we have to take into account how the orbifold symmetry acts on the 16 internal directions and not only the 12 corresponding to the E_6's. Fortunately, it is not hard to find such a consistent automorphism. One just has to take the one corresponding to the symmetry of the

affine diagram of the original gauge algebra $2\mathrm{E}_7 + 2\mathrm{A}_1$:



(5.1.6)

It can then be shown that, together with an order 2 shift in x^2, one obtains a consistent theory with an holonomy that breaks 8 Cartan generators, and the gauge group is $\mathrm{F}_4 \times \mathrm{F}_4$ at level 1 times $\mathrm{U}(1)^3$, for arbitrary metric and B-field. The former is due to the automorphism having an associated projector $P_\theta = 1 + \theta$ of rank 8. The later comes from the fact that each E_6 folds into an F_4, where two nodes are left invariant (cf. discussion in Section 5.1.1). As in the previous construction, we can represent this breaking diagrammatically:



(5.1.7)

Let us now consider the commutator of the holonomies along the T^2. We find that

$$\theta(A') - A' = (0^5, 1, -1, 0) \times (0, 1, -1, 0^5)\,, \tag{5.1.8}$$

which is just the fundamental weight represented as a black node in the above diagram. Its exponential is a non-trivial element of $(\mathrm{E}_7 \times \mathrm{SU}(2)^2)$ which gets identified with the identity in the quotient $(\mathrm{E}_7 \times \mathrm{SU}(2)^2)/\mathbb{Z}_2$, mirroring the situation in the compactification without vector structure as expected. One may also deform the Wilson lines along all directions by adding vectors invariant under θ. This restriction reduces the degrees of freedom of the theory with respect to the Narain moduli space in the appropriate way.

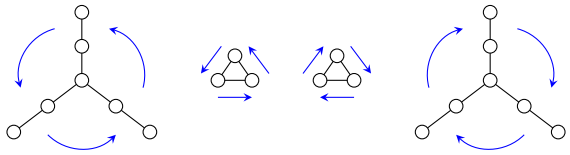
Finally we note that here we have obtained a particular gauge group, $\mathrm{F}_4 \times \mathrm{F}_4 \times \mathrm{U}(1)^3$, out of the many possibilities that exist in the moduli space of the theory. The general construction carried out in [23] leads to a momentum lattice analogous to the Narain lattice, with which we may systematically explore this moduli space (as we discuss in next section). In this case, the momentum lattice is just the Mikhailov lattice in 7d and the theory is equivalent to the 7d CHL string. We emphasize that the $\mathbb{Z}_2$-triple does not involve the exchange of the E_8's (or subgroups thereof), and so strictly speaking it does not correspond to the CHL string. Indeed, one can construct the CHL string but not the $\mathbb{Z}_2$-triple in 9d. When they exist, they are equivalent by T-duality.

5.1.3.2 $\mathbb{Z}_3$-triple

Starting in the $\mathbb{Z}_3$-triple we find genuinely new rank-reduced moduli space components with respect to the 8d case. Here the gauge group from the point of view of the T^2 is $(\mathrm{E}_6 \times \mathrm{SU}(3))^2/\mathbb{Z}_3$. We turn on a Wilson line along x^1 of the form

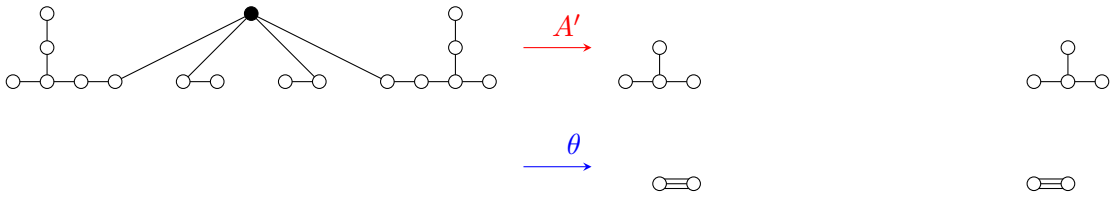
$$A' = (0^4, -\tfrac{1}{3}, \tfrac{2}{3}, \tfrac{1}{3}, 0) \times (0, -\tfrac{1}{3}, -\tfrac{2}{3}, \tfrac{1}{3}, 0^4). \tag{5.1.9}$$

This breaks the gauge group to $(\mathrm{SO}(8)\times \mathrm{U}(1)^4)^2$. To get the rank 3 automorphism we again consider the symmetry of the affine diagram of the original group:



(5.1.10)

This descends to the triality of each SO(8) and folds them into $\mathrm{G}_2\times\mathrm{G}_2$ at level 1. The projector $P_\theta = 1+\theta+\theta^2$ is of rank 4, eliminating 12 Cartan generators, and so the resulting gauge group is $\mathrm{G}_2\times\mathrm{G}_2\times\mathrm{U}(1)^3$ for arbitrary metric and B-field. Again, the orbifold includes an order 3 shift in x^2. The corresponding breaking diagram is



(5.1.11)

The commutator of A' and θ is given by

$$\theta(A') - A' = (0^4, 1, -1, 0^2)\times(0^2, 1, -1, 0^4)\,, \tag{5.1.12}$$

corresponding to the weight represented by the black node in the diagram above, and the story is the same as before for the $\mathbb{Z}_2$-triple. In this case one can deform the three Wilson lines with four degrees of freedom each, which is the rank of the projector P_θ. Together with the nine degrees of freedom coming from the metric and B-field, the dimension of the moduli space is 21, and its local geometry is given by the coset

$$\mathrm{SO}(7,3,\mathbb{R})/(\mathrm{SO}(7,\mathbb{R})\times\mathrm{SO}(3,\mathbb{R})). \tag{5.1.13}$$

In [23] it was proposed that the global structure is given by the automorphism group of the momentum lattice of the theory, which was determined to be

$$\Lambda_3 = \Gamma_{3,3}\oplus \mathrm{A}_2\oplus\mathrm{A}_2, \tag{5.1.14}$$

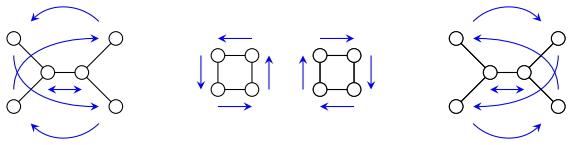
extending the results for the first two components of the moduli space where the Narain and the Mikhailov lattice respectively play this role.

5.1.3.3 $\mathbb{Z}_4$-triple

For the $\mathbb{Z}_4$-triple we start with the 9d gauge group $(\mathrm{Spin}(10)\times\mathrm{SU}(4))^2/\mathbb{Z}_4$ and turn on the Wilson line

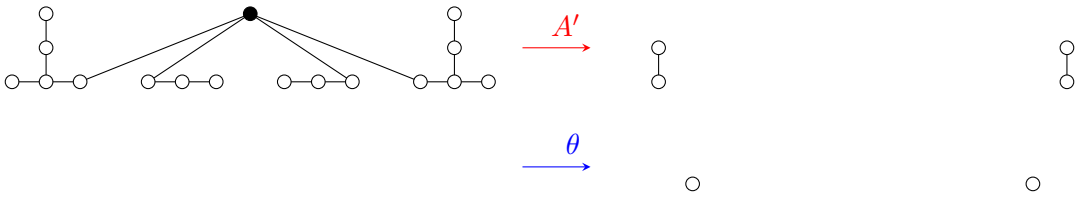
$$A' = \frac{1}{8}(1,-1,-1,-3,3,1,-1,5)\times(-5,1,-1,-3,3,1,1,-1), \tag{5.1.15}$$

which breaks it to $\mathrm{SU}(3)^2 \times \mathrm{U}(1)^{12}$. The affine diagram of the original group has an order 4 symmetry:



(5.1.16)

The surviving SU(3)'s under the action of A' correspond to the innermost nodes of the affine Spin(10)'s, and they get identified under θ into $\mathrm{SU}(2) \times \mathrm{SU}(2)$ at level 1. The rank of the projector $P_\theta = 1 + \theta + \theta^2 + \theta^3$ is 2, and so 14 Cartan generators are eliminated. There is again an order 4 shift in x^2 in the orbifold symmetry, and we get the gauge group $\mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{U}(1)^3$ for generic metric and B-field. The breaking diagram is



(5.1.17)

We remark that the roots obtained after the folding have norm 8, this being the reason that the SU(2)'s are at level 1. This can be understood by noting that the affine diagram for D_5 gets folded into a pair of linked nodes with norms 2 and 8, respectively. Four nodes collapse into one corresponding to a root with norm smaller by a factor of 4, while two linked nodes fold into one with invariant length. Upon scaling, the shorter root that gets broken is of norm 2, while the remaining has norm 8.

We find that

$$\theta(A') - A' = (0^3, 1, 0^3, -1) \times (1, 0^3, -1, 0^3)\,, \tag{5.1.18}$$

which is the weight in the LHS of the diagram above modulo a translation in the A_3 sublattices. The moduli space is of dimension 15, locally of the form

$$\mathrm{SO}(5,3,\mathbb{R})/(\mathrm{SO}(5,\mathbb{R}) \times \mathrm{SO}(3,\mathbb{R})), \tag{5.1.19}$$

and the momentum lattice is

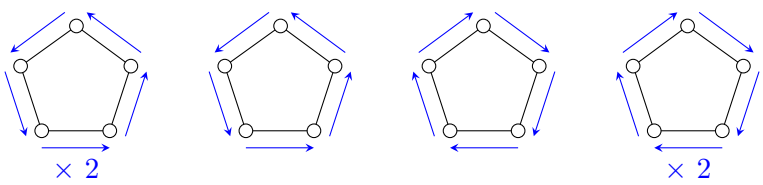
$$\Lambda_4 = \Gamma_{3,3} \oplus \mathrm{A}_1 \oplus \mathrm{A}_1. \tag{5.1.20}$$

5.1.3.4 $\mathbb{Z}_5$ and $\mathbb{Z}_6$-triples

For the $\mathbb{Z}_5$-triple we use Wilson line

$$A' = \frac{1}{5}(0, -1, -2, 3, 2, 1, 0, -1) \times (1, 0, -1, -2, -3, 2, 1, 0)\,, \tag{5.1.21}$$

which breaks $\mathrm{SU}(5)^4/\mathbb{Z}_5$ to $\mathrm{U}(1)^{16}$. The automorphism θ corresponds to the symmetry



(5.1.22)

and has projector $P_\theta = 0$. The rank of the gauge group is reduced by a factor of 16 and only the Cartans coming from the T^3 compactification are present. We have that

$$\theta(A') - A' = (0^2, 1, -1, 0^4) \times (0^4, 1, -1, 0^2)\,, \tag{5.1.23}$$

which is the weight associated to the $\mathbb{Z}_5$ quotient. The moduli space has dimension 9 and is locally of the form

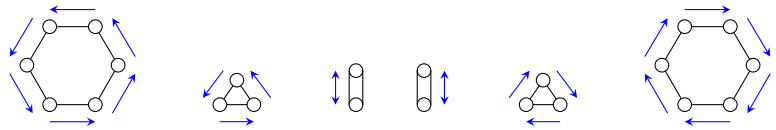
$$\mathrm{SO}(3,3,\mathbb{R})/(\mathrm{SO}(3,\mathbb{R}) \times \mathrm{SO}(3,\mathbb{R})), \tag{5.1.24}$$

and the momentum lattice is just $\Gamma_{3,3}$.

The story for the $\mathbb{Z}_6$-triple is basically the same, the only differences being that the Wilson line used is

$$A' = \frac{1}{12}(1, -5, 7, 5, 3, 1, -1, -3) \times (3, 1, -1, -3, -5, -7, 5, -1), \tag{5.1.25}$$

the automorphism θ corresponds to the symmetry of the affine $(\mathrm{SU}(2) \times \mathrm{SU}(3) \times \mathrm{SU}(6))^2$ diagram,



(5.1.26)

and

$$\theta(A') - A' = (0, 1, -1, 0^5) \times (0^5, 1, -1, 0^1)\,. \tag{5.1.27}$$

As in the previous case there are no Wilson line degrees of freedom, and the local and global data for the moduli space are the same. One should note however that the groups which are realized at level 5 in the $\mathbb{Z}_5$-triple are realized in this case at level 6. Indeed, this information is not contained implicitly in the momentum lattice.

5.2 Momentum lattices

Here we explain the basic machinery of how gauge symmetry groups can be obtained from the momentum lattices corresponding to certain heterotic string compactifications with 16 supercharges. These include the Narain lattice for T^n compactifications and the Mikhailov lattice for the CHL string (which we will revisit as they were already described in Sections 3.2.1 and 4.3.3) and the four extra momentum lattices for components with further rank reduction obtained in [23].

5.2.1 The Narain construction

It was shown in [7] that the perturbative spectrum of the heterotic string on T^d can be put in correspondence with an even self-dual Lorentzian lattice $\Gamma_{d,d+16}$ of signature $(-^d, +^{d+16})$. This lattice is spanned by vectors $(p_R; p_L, P)$, where P is the left gauge lattice momentum and $p_{L,R}$ are the right and left internal space momenta.

The only massless states in the spectrum have $p_R = 0$, and those which realize the adjoint representation of the gauge algebra $\mathfrak{g}$ also have $P^2 + p_L^2 = 2$. They correspond therefore to a set of length $\sqrt{2}$ vectors in $\Gamma_{d,d+16}$ spanning a positive definite sublattice L, which is just the root lattice of $\mathfrak{g}$. The question of what gauge algebras can be realized in the theory is then equivalent to the question of what root lattices L can be embedded in the Narain lattice. Note that this embedding

has to be such that the intersection of the real span of L with $\Gamma_{d,d+16}$ does not contain a larger root lattice L', since this would leave out extra states that do form part of the massless spectrum. We can be more precise about the relation between gauge symmetries and lattice embeddings and in the way gain more information. As discussed in Section 3.2.2, relaxing the condition $P^2 + p_L^2 = 2$ while keeping $p_R = 0$ defines an overlattice $M \supseteq L$ corresponding to the weight lattice of the global gauge group G. In this case, M is such that the intersection of its real span with $\Gamma_{d,d+16}$ is M itself, i.e. it is *primitively* embedded in $\Gamma_{d,d+16}$. The full statement regarding the possibility of some gauge group G being realized in the heterotic string on T^d is as follows:

Proposition 1. *Let $G = \tilde{G}/H$ be some semisimple group of rank $r \leq 16 + d$, where $\tilde{G}$ and H are respectively the universal cover and the fundamental group. $G \times U(1)^{16+d-r}$ is realized in the heterotic string on T^d as a gauge symmetry group if and only if its weight lattice M admits a primitive embedding in the Narain lattice $\Gamma_{d,d+16}$ such that the vectors in M of length $\sqrt{2}$ are roots.*

At the end of the day, the classification of the possible gauge groups that can be obtained in the heterotic string on T^d turns out to be a (conceptually) simple problem of lattice embeddings. As explained previously, using the exploration algorithm described in 4.3.1, we have collected a set of points of maximal enhancement characterized by their root lattices L, i.e. their gauge algebras $\mathfrak{g}$. For each point we compute the weight lattice M and from it the generators of the fundamental group H, using the methods described in 4.3.3. The results are presented in Section 5.5.

5.2.2 The CHL string and Mikhailov lattice

The analog of the Narain lattice for this theory was constructed by Mikhailov in [19] and can be written as

$$\Gamma_{(d)} = \Gamma_{d-1,d-1}(2) \oplus \Gamma_{1,1} \oplus \mathrm{E}_8 \,, \tag{5.2.1}$$

where the (2) indicates that $\Gamma_{d-1,d-1}$ is scaled by a factor of $\sqrt{2}$. Depending on the dimension d, this lattice may be rewritten in different ways using lattice isomorphisms. For $d = 3$, we have

$$\Gamma_{2,2}(2) \oplus \Gamma_{1,1} \oplus \mathrm{E}_8 \quad \simeq \quad \Gamma_{3,3} \oplus \mathrm{D}_4 \oplus \mathrm{D}_4 \quad \simeq \quad \Gamma_{3,3} \oplus \mathrm{F}_4 \oplus \mathrm{F}_4 \,. \tag{5.2.2}$$

Here we have used the root lattice isomorphism $\mathrm{D}_4 \simeq \mathrm{F}_4$ (the corresponding root *systems* are of course not isomorphic) to reflect the fact that the 'canonical' point in the theory has gauge algebra $2\mathrm{F}_4$ and not $2\mathrm{D}_4$, as shown in Section 5.1.3.1.

The relation between lattice embeddings and realizability of gauge groups in the CHL string is more complicated than for the usual heterotic string on tori. In the latter, the roots of the gauge algebra correspond to the length $\sqrt{2}$ vectors in some positive definite lattice Λ primitively embedded into $\Gamma_{d,d+16}$. In the CHL string the mass formulas are such that it is also possible for some but not all vectors of length 2 to give roots. In order for such a vector v to correspond to a root, it must satisfy the condition that its inner product with all other vectors in the whole Mikhailov lattice is even [19]. In this case we say that v is a level 2 vector (not to be confused with the level of the Kac-Moody algebra for the gauge group). More generally, a vector v in a lattice Λ is said to be at embedding level ℓ if the product of v with every vector in Λ is divisible by ℓ.

On the other hand, the statement that the global structure of the gauge group is given by the

primitively weight overlattice M does not generalize to the case where the momentum lattice is not self-dual and the gauge algebras are not of ADE type. The problem of obtaining this global data was studied in detail in [20]. It was shown in particular that the fundamental group $\pi_1(G)$ of the gauge group G is given by the quotient of the cocharacter lattice $M^\vee$ and the coroot lattice $L^\vee$ where the later is embedded in the dual momentum lattice $\Gamma^*_{(d)}$ and the former is the corresponding overlattice which is primitively embedded in $\Gamma^*_{(d)}$.

As explained in Section 4.3.3.2, one strategy to obtain all the possible gauge groups in the theory is to apply the exploration algorithm described in 4.3.1 to the dual lattice $\Gamma^*_{(d)}$ (which usually has to be rescaled to be made even) and compute the lattices L and M the same way as for the Narain lattice, but dualizing the algebra $\mathfrak{g} \to \mathfrak{g}^\vee$ at the end. It can be shown that the embedding level condition for vectors to be roots is the same as for the original lattice $\Gamma_{(d)}$. This was the method employed on Chapter 4 to obtain the list of gauge groups for the CHL string in 8d.

Having dealt with this subtlety, a statement generalizing proposition 1 for the usual heterotic string to the CHL string on T^d can be made as follows:

Proposition 2. *Let $G = \tilde{G}/H$ be some semisimple group of rank $r \leq d+8$, where $\tilde{G}$ and H are respectively the universal cover and the fundamental group. $G \times U(1)^{d+8-r}$ is realized in the CHL string on T^d as a gauge symmetry group if and only if the weight lattice $M^\vee$ of the dual group $G^\vee$ admits a primitive embedding in the dual Mikhailov lattice $\Gamma^*_{(d)}(2)$ such that the vectors in $M^\vee$ of length $\sqrt{2\ell}$ at embedding level $\ell = 1, 2$ in $\Gamma^*_{(d)}(2)$ belong to $L^\vee$.*

We see that the embedding level ℓ plays an important role in the theory, allowing to treat the problem of finding the possible gauge groups without reference to the string theory itself, as in the case of the original heterotic string.

Finally let us recall that the simple factors in G have associated Kac-Moody algebras at level $\mathfrak{m} = 1, 2$ where $4/\mathfrak{m}$ is the squared length of the corresponding longest root. For $d = 2$ there are only ADE groups at level 2 and symplectic groups at level 1 (including $\mathrm{Sp}(1) = \mathrm{SU}(2)$). For $d \geq 3$ there are more interesting possibilities including B_3 and F_4 at level 1.

5.2.3 Momentum lattices from Triples

Let us now turn to the $\mathbb{Z}_m$-triples reviewed in Section 5.1.3. The respective momentum lattices in 7d are given in Table 5.1, where we also show the rank reduction of the respective gauge groups. Here again we have chosen to write the lattices in terms of the canonical point groups using the lattice isomorphisms $\mathrm{D}_4 \simeq \mathrm{F}_4$ and $\mathrm{A}_2 \simeq \mathrm{G}_2$. We also record the frozen singularity for each lattice Λ_m, which in this context corresponds to the orthogonal complement of the embedding $\Lambda_m \hookrightarrow \Gamma_{3,19}$. This point is discussed in more detail in the next section.

It is natural to ask whether we can extend propositions 1 and 2 to these lattices. An obvious ansatz is the following:

m	Λ_m	Ω	r_-
1	$\Gamma_{3,3} \oplus \mathrm{E}_8 \oplus \mathrm{E}_8$	$\emptyset$	0
2	$\Gamma_{3,3} \oplus \mathrm{F}_4 \oplus \mathrm{F}_4$	$\mathrm{D}_4 \oplus \mathrm{D}_4$	8
3	$\Gamma_{3,3} \oplus \mathrm{G}_2 \oplus \mathrm{G}_2$	$\mathrm{E}_6 \oplus \mathrm{E}_6$	12
4	$\Gamma_{3,3} \oplus \mathrm{A}_1 \oplus \mathrm{A}_1$	$\mathrm{E}_7 \oplus \mathrm{E}_7$	14
5	$\Gamma_{3,3}$	$\mathrm{E}_8 \oplus \mathrm{E}_8$	16
6	$\Gamma_{3,3}$	$\mathrm{E}_8 \oplus \mathrm{E}_8$	16

Table 5.1: Momentum lattices Λ_m for the moduli spaces of 7d heterotic $\mathbb{Z}_m$-triples. The gauge group rank for $m = 1$ is 19, which is just the Narain component. The case $m = 2$ is dual to but not the same as the CHL component [23]. Ω is the frozen singularity corresponding to the orthogonal complement of $\Lambda_m \hookrightarrow \Gamma_{3,19}$.

Proposition 3. *Let $G = \tilde{G}/H$ be some semisimple group of rank $r \leq r_m$, where $\tilde{G}$ and H are respectively the universal cover and the fundamental group, and $r_m = 19, 11, 7, 5, 3, 3$ respectively for $m = 1, ..., 6$. $G \times U(1)^{19-r_m}$ is realized in the $\mathbb{Z}_m$-triple as a gauge symmetry group if and only if the weight lattice $M^\vee$ of the dual group $G^\vee$ admits a primitive embedding in the dual momentum lattice $\Lambda_m^*(m)$ such that the vectors in $M^\vee$ of length $\sqrt{2\ell}$ at embedding level $\ell = 1, m$ in $\Lambda_m^*(m)$ belong to $L^\vee$. Simple factors are realized at level $\mathfrak{m} = 2m/\alpha_{long}^2$, where α_{long} is a long root in $L \hookrightarrow \Lambda_m$.*

The key ingredient is that the vectors of length $\sqrt{2m}$ at embedding level m correspond to massless states and give e.g. long roots for non-ADE gauge groups. This can in fact be explicitly proved in the particular construction used in [23] to obtain the momentum lattices. This roughly corresponds to the fact that in this construction there is a rescaling by a factor of $\sqrt{m}$ involved, such that the product of long roots, coming from invariant states in the parent theory of the orbifold, with all other vectors is scaled by a factor of m. We will however confirm this for the general case by showing in Section 5.4.1 that assuming this ansatz one can reproduce the mechanism of singularity freezing in the dual M-theory on K3 from the heterotic side.

An extension of the exploration algorithm used for the CHL string to these lattices is straightforward and produces the results presented in Section 5.5.2. In Sections 5.3 and 5.4 we will see that these can be reproduced by applying an appropriate projection map to the Narain component.

It was already noted by Mikhailov in [19] that the momentum lattice for the CHL string is primitively embedded in the Narain lattice such that its orthogonal complement corresponds to the frozen singularity on the dual F/M-theories on K3 (for $d = 2, 3$, respectively). This observation was extended in [23] to the $\mathbb{Z}_m$-triples in 7d. On sections 5.3 and 5.4 we make use of it together with Proposition 3 to determine precisely how the ADE singularities are partially frozen in 7d (usually to give non-ADE algebras) and recover the known "freezing rules" on the K3 side. We will also derive analogous rules in 6d. As we will see in Section 5.4.2, Proposition 3 has to be generalized for 6d by permitting the embedding level to be any divisor of m. This makes a difference only in the non-prime cases $m = 4$ and $m = 6$.

5.3 Mapping gauge groups from Narain to CHL

In this section we explain the general method for determining the map which connects the Narain component with the CHL component and explicitly derive it for $d = 1, 2, 3, 4$. The case $d = 2$ was

first obtained in [20] and the case $d = 3$ in [4]. Extensions to other rank reduced components are considered in Section 5.4.

5.3.1 Setup and basic facts

In order to determine the map which applies to the gauge groups of the Narain component of the moduli space to give those of the rank reduced components we have to relate the way in which these are obtained in each case from the corresponding momentum lattices. We will illustrate this procedure using the CHL string, and so the focus is on the Narain lattice Γ_N and the Mikhailov lattice Γ_M, which can be written as

$$\begin{aligned} \Gamma_N &\simeq \Gamma_{d,d} \oplus \mathrm{E}_8 \oplus \mathrm{E}_8 \,, \\ \Gamma_M &\simeq \Gamma_{d-1,d-1}(2) \oplus \Gamma_{1,1} \oplus \mathrm{E}_8 \,. \end{aligned} \tag{5.3.1}$$

Here $\Gamma_{d,d} \simeq \bigoplus_{i=1}^{d} \Gamma_{1,1}$ is the unique even self-dual lattice with signature $(+^d, -^d)$. The symbol (2) denotes a rescaling of the lattice by $\sqrt{2}$, hence a rescaling of the Gram matrix by 2. The lattice E_8 is just the lattice generated by the roots of the algebra $\mathfrak{e}_8$, but for the latter, as well as for the groups, we will use the symbol E_8 when there is no risk of ambiguity. The same applies for any other root lattice of A-to-G type. We convene in taking the momentum lattices to have signature with mostly pluses, unless stated otherwise.

For the Narain component of the moduli space one obtains all the possible gauge algebras by finding embeddings of root lattices L into Γ_N such that the intersection of $L \otimes \mathbb{R}$ with Γ_N is an overlattice $M \hookleftarrow L$ whose maximal root sublattice is L itself. Here we mean by overlattice any lattice of the same rank containing the lattice in question. Intersections of real slices such as $L \otimes \mathbb{R}$ with Γ_N give lattices which are said to be primitively embedded, in this case in Γ_N, hence the embedding $M \hookrightarrow \Gamma_N$ is primitive but $L \hookrightarrow \Gamma_N$ is not unless $M = L$. By roots we mean vectors $v \in \Gamma_N$ with norm $v \cdot v = 2$, since these are the ones associated to root states in the adjoint representation of the gauge algebra.

This discussion extends to the CHL component of the moduli space, with the only difference being that roots are not only vectors with norm 2 but also vectors v with norm 4 satisfying the condition $v \cdot u = 0 \mod 2$ for all vectors $u \in \Gamma_M$ [19]. This last condition is equivalent to the statement that the coroot $v^\vee = \frac{1}{2} v$ is in the dual lattice Γ_M^*, which is the language used in [20]. Note that $v^\vee \cdot v^\vee = 1$, hence this condition cannot be satisfied by any vector in the Narain lattice which is even and self-dual. The same applies to Γ_M when $d = 1$. For $d \geq 2$, however, Γ_M is not self dual and Γ_M^* indeed contains vectors with norm 1. The appearance of non-simply-laced algebras seems therefore to be intimately connected with the non-self-duality of the momentum lattice for the moduli space component in question.

These facts allow to obtain the possible gauge algebras $\mathfrak{g}$ in these moduli space components, but we are also interested in the full gauge groups G. For this we need to compute the fundamental group $\pi_1(G)$, which we denote by H. If $\tilde{G}$ is the universal cover of G, then $G = \tilde{G}/H$. In the Narain component it suffices to compute the lattice quotient M/L, which gives a finite Abelian group isomorphic to H due to the self-duality of Γ_N, as discussed in the previous chapters. For example, if $M = L$, then G is simply-connected. For the CHL component one must do a more precise analysis [20], but the upshot is that H is given by the quotient $M^\vee/L^\vee$, where $L^\vee$ is the coroot lattice of $\mathfrak{g}$ embedded in the dual lattice Γ_M^*, and $M^\vee$ its overlattice which embeds primitively into Γ_M^*. Clearly, this is a

generalization of the computation for Γ_N. In both cases H is a subgroup of the center $Z(G)$, specified by a set of elements $k_i \in Z(G)$.

5.3.2 Construction of the map in $d = 1, 2, 3$

To relate the Narain and the CHL constructions just outlined we require some additional facts. For $d = 1, 2, 3, 4$, Γ_M can be written respectively as [19]

$$\Gamma_M \simeq \Gamma_{d,d} \oplus \Omega\,, \qquad \Omega = \begin{cases} \mathrm{E}_8 & d = 1 \\ \mathrm{D}_8 & d = 2 \\ \mathrm{D}_4 \oplus \mathrm{D}_4 & d = 3 \\ \mathrm{D}_8^*(2) & d = 4 \end{cases}, \tag{5.3.2}$$

to which we restrict our attention in the following. In each case there is an embedding

$$\Gamma_M \oplus \Omega \hookrightarrow \Gamma_N\,, \tag{5.3.3}$$

where $\Gamma_M \hookrightarrow \Gamma_N$ and $\Omega \hookrightarrow \Gamma_N$ are primitive. Furthermore, the primitive embedding of Ω into Γ_N is unique (up to automorphisms of Γ_N), so that by constructing any such embedding one may take its orthogonal complement which by necessity is just Γ_M. As we will review, Ω can be interpreted as the K3 frozen singularity (or singularities) in the dual geometric frame both for 8d and 7d, and so we will refer to it as the *frozen sublattice* in the heterotic string context. We also use the terms mapping (from Narain to reduced rank) and freezing interchangeably.

Consider now a lattice[2] M' primitively embedded into Γ_M, with root sublattice L'. It follows from (5.3.3) that there is an embedding

$$M' \oplus \Omega \hookrightarrow \Gamma_N \tag{5.3.4}$$

with M' (but not necessarily $M' \oplus \Omega$) primitively embedded into Γ_N. The intersection $(M' \oplus \Omega) \otimes \mathbb{R} \cap \Gamma_N$ gives a lattice M primitively embedded into Γ_N, with root sublattice L. This gives a priori a map φ from a gauge algebra $\mathfrak{g}_{\mathrm{CHL}}$ in CHL moduli space to another $\mathfrak{g}_{\mathrm{Narain}}$ in Narain moduli space, but since we are dealing with the full embedding data for each lattice, we can also obtain the fundamental group of the gauge group and promote this map to one at the level of groups,

$$\varphi: \quad G_{\mathrm{CHL}} \mapsto G_{\mathrm{Narain}}\,. \tag{5.3.5}$$

Consider conversely a lattice M primitively embedded into Γ_N, with root sublattice L, such that Ω is in turn primitively embedded into M (note that primitivity in this case is guaranteed by the fact that $\Omega \hookrightarrow \Gamma_N$ is primitive). It follows that M has a sublattice of the form $M' \oplus \Omega$, where both M' and Ω are primitively embedded into M. Since the orthogonal complement of Ω in Γ_N is just Γ_M, it follows that M' is primitively embedded into Γ_M, and defines a gauge group G_{CHL}. This gives a map

$$\varphi^{-1}: \quad G_{\mathrm{Narain}} \mapsto G_{\mathrm{CHL}}\,. \tag{5.3.6}$$

We note however that the embedding $\Omega \hookrightarrow M$ is not necessarily unique so that this map is generically

[2]Here we prime the lattice M in the Mikhailov lattice since we will later focus on the map *to* and not *from* the CHL component.

one-to-many. As we will see, the form of this map has markedly different qualitative features depending on the value of d. In the following we study explicitly the cases $d = 1, 2, 3, 4$.

5.3.2.1 $d = 1$

For $d = 1$, we have that $\Gamma_M \simeq \Gamma_{1,1} \oplus \mathrm{E}_8$ and $\Omega = \mathrm{E}_8$ are even self-dual. Therefore, eq. (5.3.3) can be replaced by a stronger statement (cf. eq. (5.3.1)),

$$\Gamma_N \simeq \Gamma_M \oplus \mathrm{E}_8\,, \qquad d = 1\,. \tag{5.3.7}$$

In this case, the lattice M' that we consider is a root lattice L', since in nine dimensions all gauge groups are simply-connected (as shown in Chapter 4). Therefore we have an embedding $L' \oplus \mathrm{E}_8 \hookrightarrow \Gamma_N$. This embedding *is* primitive, since $L' \hookrightarrow \Gamma_M$ is primitive and E_8 is unimodular, so there does not exist an even overlattice of $L' \oplus \mathrm{E}_8$ in Γ_N. Moreover, $L' \oplus \mathrm{E}_8$ is a root lattice corresponding to a simply-connected gauge group in Narain moduli space. We see therefore that to every gauge group G_{CHL} in the CHL component we can associate another group G_{Narain} in the Narain component by some map

$$\varphi: \quad G_{\mathrm{CHL}} \mapsto G_{\mathrm{Narain}} = G_{\mathrm{CHL}} \times \mathrm{E}_8\,, \qquad d = 1. \tag{5.3.8}$$

Conversely, consider some root lattice of the form $L' \oplus \mathrm{E}_8$ in Γ_N. Similarly to the CHL component, all of the associated groups are simply-connected. Since the primitive embedding of E_8 into Γ_N is unique, it follows that L' is primitively embedded into $\mathrm{E}_8^{\perp} \simeq \Gamma_M$. This means that any gauge group of the form $G \times \mathrm{E}_8$ in the Narain component necessarily has $G = G_{\mathrm{CHL}}$ some group in the CHL component. At the end of the day, the result is that by taking all gauge groups in the Narain component which contain an E_8 factor and deleting it one obtains all of the gauge groups in the CHL component. If there are two E_8 factors, they are equivalent by an automorphism of Γ_N, so that there is no ambiguity in deleting one or the other.

This same result can be obtained in a more concrete way by considering the GDDs for the lattices Γ_N and Γ_M, shown in Figure 5.1. Gauge algebras in the Narain moduli space can be obtained by deleting two or more nodes of the diagram such that the result is the Dynkin diagram for an ADE root lattice. The same applies to the CHL component, but the minimum number of nodes one can delete is one. As we can see, deleting the node $0'$ in the GDD for Γ_N gives the GDD for Γ_M accompanied by an E_8 Dynkin diagram, from which it follows that the gauge algebras that can be obtained in each moduli space component are related as deduced above. As commented, all of the relevant groups are simply-connected.

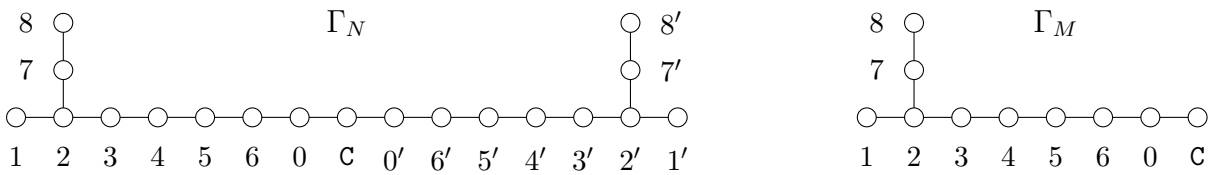


Figure 5.1: Generalized Dynkin Diagrams for the Narain lattice $\Gamma_N \simeq \Gamma_{1,17}$ and the Mikhailov lattice $\Gamma_M \simeq \Gamma_{9,1}$ in nine dimension.

5.3.2.2 $d = 2$

The map from the Narain to the CHL components of the moduli space for $d = 2$ was obtained at the level of the full gauge groups in [20] using more group-theoretical language, and proven explicitly by projecting the cocharacter lattice, which determines the topology, from Γ_N to Γ_M. Here we briefly explain how it can be obtained in the framework of this chapter.

In eight dimensions we have $\Gamma_M \simeq \Gamma_{2,2} \oplus \mathrm{D}_8$ and $\Omega = \mathrm{D}_8$. We will consider a lattice M primitively embedded into Γ_N, which is the overlattice of a root lattice L, containing in turn a primitively embedded D_8 lattice. This condition restricts L to be of the form

$$L \simeq \mathrm{D}_{8+n} \oplus N\,, \tag{5.3.9}$$

where n is some non-negative integer and N is some other ADE lattice. The orthogonal complement of D_8 in L is of the form $\mathrm{D}_n \oplus N$, and has an overlattice M' primitively embedded into Γ_M.

The question is if $\mathrm{D}_n \oplus N$ is the maximal root sublattice of M', according to the definition of roots in the CHL moduli space. This can indeed be verified for all points of symmetry enhancement. The subtlety here is that as lattices, D_n and C_n are equivalent. The actual contribution to the gauge algebra depends on which vectors correspond to massless states, and we find that in this case it is actually $\mathfrak{sp}_n$ and not $\mathfrak{so}_{2n}$. We therefore write $L = \mathrm{C}_n \oplus N$. We have then a simple rule for mapping gauge algebras from the Narain component to the CHL component of the moduli space. Just take any gauge algebra with a D_{8+n} factor and replace it with C_n. Since it is possible to have gauge algebras with terms $\mathrm{D}_{8+n} \oplus \mathrm{D}_{8+m}$, with $m \neq n$, this map is generically one-to-many. We recover the freezing rule for F-theory on K3 in the reverse. Indeed, applying these rules to all the possible gauge algebras in the Narain component gives those in the CHL component (Table D.5).

To promote this map to one at the level of groups we compute the fundamental group of the gauge group associated to the embeddings $L \hookrightarrow \Gamma_N$ and $L \hookrightarrow \Gamma_M$ using the lattice methods outlined above, and then see how they are related. We explain how this works by considering a gauge group, obtained from the Narain lattice, of the form

$$G = \tilde{G}/H = G_1 \times \cdots \times G_s \times \mathrm{Spin}(2n+16)/H\,, \tag{5.3.10}$$

where H is generated by an element $k = (k_1, ..., k_s, \hat{k})$ of the center $Z(\tilde{G})$. The corresponding group in the CHL string will be of the form

$$G' = G_1 \times \cdots \times G_s \times \mathrm{Sp}(n)/H'\,, \tag{5.3.11}$$

with H' generated by the element $k' = (k_1, ..., k_s, \hat{k}')$ of the center $Z(\tilde{G}')$. As can be expected, only the contribution of the partially frozen factor will change. Indeed the center of $\mathrm{Spin}(2n+16)$ and that of $\mathrm{Sp}(n)$ are different. For n odd, we have $\hat{k} \in \mathbb{Z}_4$ and $\hat{k}' \in \mathbb{Z}_2$, and the projection reads

$$\hat{k} \to \hat{k}' = \hat{k} \mod 2 \qquad (\{0,1,2,3\} \to \{0,1,0,1\})\,, \quad n = \text{odd}\,. \tag{5.3.12}$$

For n even, we have $\hat{k} \equiv (\hat{k}^{(1)}, \hat{k}^{(2)}) \in \mathbb{Z}_2 \times \mathbb{Z}_2$ and again $\hat{k}' \in \mathbb{Z}_2$, and the projection reads

$$\hat{k} \to \hat{k}' = \hat{k}^{(1)} + \hat{k}^{(2)} \mod 2 \qquad (\{0,\mathrm{s},\mathrm{c},\mathrm{v}\} \to \{0,1,1,0\})\,, \quad n = \text{even}\,, \tag{5.3.13}$$

where $\{0,\mathrm{s},\mathrm{c},\mathrm{v}\} \equiv \{(0,0),(1,0),(0,1),(1,1)\}$.

As a simple example, consider the gauge group $\frac{\mathrm{Spin}(32)}{\mathbb{Z}_2} \times \mathrm{SU}(2)^2$, whose fundamental group is generated by only one element $k = (\mathrm{s}, 0, 0)$. Using the rules above, the associated gauge group in the CHL component is $\frac{\mathrm{Sp}(8)}{\mathbb{Z}_2} \times \mathrm{SU}(2)^2$ with $k = (1, 0, 0)$. If we had the gauge group $\mathrm{SO}(32) \times \mathrm{SU}(2)^2$ with $k = (\mathrm{v}, 0, 0)$, it would map to the simply-connected $\mathrm{Sp}(8) \times \mathrm{SU}(2)^2$. However, $\mathrm{SO}(2n)$ factors are not present in the theory so that this last example does not arise. It's interesting to note that $\mathrm{SO}(2n)$ would map to the same gauge group as $\mathrm{Spin}(2n)$, making the mapping generically many-to-many and not one-to-many.

Note also that the fundamental group of any two groups connected by this mapping are isomorphic. This is in accordance with the fact that the topology of the gauge groups in the dual frame of F-theory on elliptically fibered K3 is given by the torsional part of the Mordell-Weil group [66,67] which can be obtained from the Weierstrass model for the fibration (see e.g. [68]), as the mechanism of singularity freezing does not alter the Weierstrass model itself [56].

5.3.2.3 $d = 3$

In seven dimensions we have $\Omega = \mathrm{D}_4 \oplus \mathrm{D}_4$. Each D_4 factor can be contained in algebras of D_{4+n} type, in which case the analysis for $d = 2$ goes through, including the way in which the contribution of these factors to the fundamental group transform. The difference now is that we have two such factors transforming simultaneously, e.g. $\mathrm{D}_{n+4} \oplus \mathrm{D}_{m+4} \to \mathrm{C}_n \oplus \mathrm{C}_m$. This is not the only possibility, however.

It is also possible for D_4 to be primitively embedded into E_6, E_7 and E_8. Taking the orthogonal complement of D_4 in each case we obtain the lattices $\mathrm{A}_2(2)$, $3\mathrm{A}_1 \simeq \mathrm{B}_3$ and $\mathrm{D}_4 \simeq \mathrm{F}_4$, respectively. Similarly to the previous case, we can look at the points of symmetry enhancement in the CHL component and determine that the contributions to the algebra are respectively $\mathfrak{su}_3$, $\mathfrak{so}_7$ and $\mathfrak{f}_4$, hence the use of these lattice isomorphisms. With respect to the gauge group's topology, we have that $Z(\mathrm{SU}(3)) \simeq Z(\mathrm{E}_6) \simeq \mathbb{Z}_3$, $Z(\mathrm{Spin}(7)) \simeq Z(\mathrm{E}_7) \simeq \mathbb{Z}_2$ and $Z(\mathrm{F}_4) \simeq Z(\mathrm{E}_8) \simeq \{0\}$, and that the contributions of these factors to the $\{k_i\}$ remain invariant. This means that as for $d = 2$, the fundamental group of two gauge groups related by the mapping are isomorphic. As in the previous case, this coincides at the algebra level with results on the dual geometrical frame's mechanism of singularity freezing [23–25], in this case M-theory on K3 with two D_4 frozen singularities. We are not aware of how the fundamental group of the gauge group is encoded in the M-theory compactification, but it should in any case be invariant under singularity freezing.

5.3.3 Algebra projection

In the previous constructions we have seen that the root system of the CHL gauge algebra corresponds to a subset of the orthogonal complement lattice of Ω in the root lattice L'. This algebra is determined precisely by checking each case algorithmically and the result is seen to correspond to a simple general rule. Now we give a procedure whose result predicts this algebra directly, mapping the simple roots of $\mathfrak{g}_{\mathrm{Narain}}$ to those of $\mathfrak{g}_{\mathrm{CHL}}$. This procedure gives the correct results for $d = 1, 2, 3, 4$. We will illustrate it case by case starting with $d = 2$, which exhibits the non-trivial features that generalize to larger d.

5.3.3.1 $d = 2$

We start by considering a primitive embedding of $\Omega = \mathrm{D}_8$ into $\Gamma_N \simeq \Gamma_{2,2} \oplus \Gamma_{16}$, where Γ_{16} is the weight lattice of $\frac{\mathrm{Spin}(32)}{\mathbb{Z}_2}$. This description makes calculations easier because D_8 embeds primitively

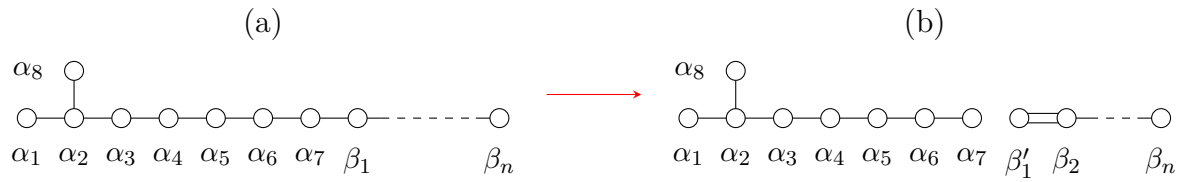


Figure 5.2: (a) Primitive embedding of D_8 in $\Gamma_{2,18}$ with simple roots α_i extended to D_{8+n} by β_j (see eqs. (5.3.14) and (5.3.15)). (b) Projection of the roots β_j to the orthogonal complement of D_8 gives a C_n lattice and associated $\mathfrak{sp}_n$ algebra in the CHL component.

into Γ_{16} but not into $E_8 \oplus E_8$. A particularly simple embedding is

$$\begin{aligned} \alpha_i &= |0,0,0,0;0^{i-1},1,-1,0^{15-i}\rangle \ , \ i=1,...,7\,, \\ \alpha_8 &= |0,0,0,0;-1,-1,0^{14}\rangle \ , \end{aligned} \tag{5.3.14}$$

where the first four entries correspond to the $\Gamma_{2,2}$ part and the other 16 to Γ_{16}. Suppose the associated gauge algebra is enhanced to D_{8+n} by adding n simple $\beta_1,...,\beta_n$ roots forming an A_n chain, with $\beta_1 \cdot \alpha_7 = -1$. For example, take

$$\beta_i = |0,0,0,0;0^{i+6},1,-1,0^{8-i}\rangle \ , \quad i=1,...,n\leq 8. \tag{5.3.15}$$

We will take the projection of the roots β_i along the space orthogonal to D_8. The roots $\beta_2,...,\beta_8$ are obviously invariant under this projection, but β_1 gets projected as

$$\beta_1 \to |0,0,0,0;0^8,-1,0^7\rangle\,. \tag{5.3.16}$$

However, this projection is not in Γ_N, and so we multiply it by 2 to get a simple root $\beta'_1 = |0,0,0,0;0^8,-2,0^7\rangle$. We see then that the simple roots of the A_n chain get projected into the simple roots of C_n. This construction is represented in Figure 5.2, and applies to any other primitive embedding of D_8 since it is unique up to automorphisms of Γ_N.

5.3.3.2 $d = 3$

For $d = 3$ we have $\Omega = D_4 \oplus D_4$, which has an easily describable primitive embedding into $E_8 \oplus E_8$, so we use the basis $\Gamma_N \simeq \Gamma_{3,3} \oplus E_8 \oplus E_8$. This embedding reads

$$\begin{aligned} \alpha_1 &= |0,0,0,0;1,-1,0^6;0^8\rangle \ , & \alpha_2 &= |0,0,0,0;0,1,-1,0^5;0^8\rangle \ , \\ \alpha_3 &= |0,0,0,0;0^2,1,-1,0^4;0^8\rangle \ , & \alpha_4 &= |0,0,0,0;-1,-1,0^6;0^8\rangle \ , \\ \alpha'_1 &= |0,0,0,0;0^8;1,-1,0^6\rangle \ , & \alpha'_2 &= |0,0,0,0;0^8;0,1,-1,0^5\rangle \ , \\ \alpha'_3 &= |0,0,0,0;0^8;0^2,1,-1,0^4\rangle \ , & \alpha'_4 &= |0,0,0,0;0^8;-1,-1,0^6\rangle \ . \end{aligned} \tag{5.3.17}$$

As in the previous case, we can extend each D_4 to D_{4+n} with an A_n chain, which gets projected to the orthogonal complement of Ω as a C_n. However, D_4 can also be extended to E_8 passing through D_5, E_6 and E_7. This D_5 coincides with that of the generic extension D_{4+n} with $n = 1$, and so it gives rise to an $A_1(2)$ lattice with simple root, say,

$$\beta'_1 = |0,0,0,0;0,0,0,0,-2,0^3;0^8\rangle \ , \tag{5.3.18}$$

which arises from projecting $|0,0,0,0;0,0,0,1,-1,0^3;0^8\rangle$. Extending D_5 to E_6 can be done by adding the root $|0,0,0,0;\frac{1}{2}^8,0^8\rangle$. Its projection multiplied by 2 is

$$\beta_2' = |0,0,0,0;0^4,1^4;0^8\rangle \ , \tag{5.3.19}$$

and so we see that β_1' and β_2' give rise to an $A_2(2)$ lattice, as expected. We can further add the roots $|0,0,0,0;0^4,1,-1,0,0\rangle$ and $|0,0,0,0;0^5,1,-1,0\rangle$, extending E_6 to E_7 and then E_8. Since these roots are orthogonal to Ω, they are invariant under the projection and we see that they extend $A_2(2)$ to B_3 and then F_4 as predicted.

5.3.3.3 $d = 4$

Here we have $\Omega = D_8^*(2)$. This lattice has a root sublattice $L_\Omega = 8A_1$ and can be in fact interpreted as the weight lattice of $\frac{SU(2)^8}{\mathbb{Z}_2}$ with $\mathbb{Z}_2$ diagonal, i.e. $k = (1,...,1)$. A suitable primitive embedding of this lattice into $\Gamma_N \simeq \Gamma_{4,4} \oplus E_8 \oplus E_8$ has simple roots

$$\begin{aligned}
\alpha_1 &= |0,0,0,0;1,-1,0^6;0^8\rangle \ , & \alpha_2 &= |0,0,0,0;0,0,1,-1,0^4;0^8\rangle \ , \\
\alpha_3 &= |0,0,0,0;0^4,1,-1,0^2;0^8\rangle \ , & \alpha_4 &= |0,0,0,0;0^6,1,-1;0^8\rangle \ , \\
\alpha_5 &= |0,0,0,0;0^8;1,-1,0^6\rangle \ , & \alpha_6 &= |0,0,0,0;0^8;0,0,1,-1,0^4\rangle \ , \\
\alpha_7 &= |0,0,0,0;0^8;0^4,1,-1,0^2\rangle \ , & \alpha_8 &= |0,0,0,0;0^8;0^6,1,-1\rangle \ .
\end{aligned} \tag{5.3.20}$$

The weight vector extending this root lattice to Ω is just

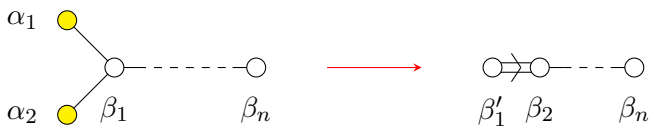
$$w = \frac{1}{2}\sum_{i=1}^{8}\alpha_i = |0,0,0,0;1,0,1,0,1,0,1,0;1,0,1,0,1,0,1,0\rangle \ . \tag{5.3.21}$$

Requiring orthogonality with the roots is enough to get orthogonality with Ω, so we will not worry about w. However we note that there exists also a primitive embedding of $8A_1$ into Γ_N, which should not be confused with Ω.

The first thing to note is that the lattice $L_\Omega = 8A_1$ can be naively extended in many different ways but not all of them are allowed extensions of Ω itself. For example, no A_1 factor can be individually extended to A_2 with a root orthogonal to the other A_1 factors. Any attempt to do this is easily seen to fail. The next logical step is to attach a root to two A_1 factors at the same time, e.g. with $|0,0,0,0;0,1,-1,0^5;0^8\rangle$, in this case giving an A_3. This vector gets projected to

$$\beta = |0,0,0,0;1,1,-1,-1,0^4;0^8\rangle \ , \tag{5.3.22}$$

and so we have that A_3 freezes to[3] $A_1(2)$. This is equivalent to $D_3 \to C_1$, and forms part of the more general rule $D_{2+n} \to C_n$, or $\mathfrak{so}_{2n+4} \to \mathfrak{sp}_n$, in analogy with those we have for $d = 2, 3$. This is depicted as



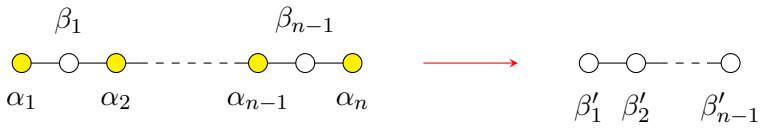
(5.3.23)

[3]It is more precise to say that $A_3 \oplus 6A_1$ freezes to $A_1(2)$, but we are now focusing on the behavior under projection of sublattices corresponding to simple algebras and not the whole lattice containing Ω.

The next possibility is to attach $n-1$ roots to n A_1 factors in pairs such that one gets an A_{2n-1} chain. The case $\mathrm{A}_3 \to \mathrm{A}_1(2)$ above can be generalized e.g. to $\mathrm{A}_5 \to \mathrm{A}_2(2)$ with roots

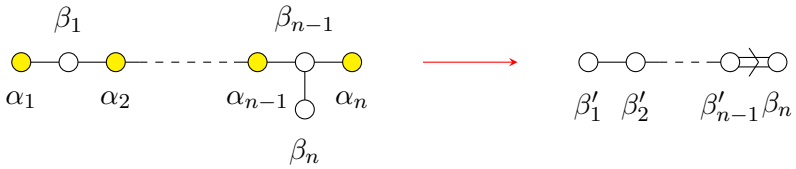
$$\beta_1 = |0,0,0,0;1,1,-1,-1,0^4;0^8\rangle \ , \qquad \beta_2 = |0,0,0,0;0^2,1,1,-1,-1,0^2;0^8\rangle \ , \tag{5.3.24}$$

and more generally we find the rule $\mathrm{A}_{2n-1} \to \mathrm{A}_{n-1}(2)$, or $\mathfrak{su}_{2n} \to \mathfrak{su}_n$, depicted as



(5.3.25)

From this rule we can actually get another by simply attaching a root β_n to β_{n-1}, namely $\mathrm{D}_{2n} \to \mathrm{B}_n$, or $\mathfrak{so}_{4n} \to \mathfrak{so}_{2n+1}$,



(5.3.26)

Finally, we can take the particular case $n = 4$ and attach a root to β_3 to get the rule $\mathrm{E}_7 \to \mathrm{F}_4$, or $\mathfrak{e}_7 \to \mathfrak{f}_4$,

(5.3.27)

In summary we have found the following freezing rules at the level of the algebras:

$$\begin{aligned} \mathfrak{so}_{2n+4} &\to \mathfrak{sp}_n \\ \mathfrak{su}_{2n} &\to \mathfrak{su}_n \\ \mathfrak{so}_{4n} &\to \mathfrak{so}_{2n+1} \\ \mathfrak{e}_7 &\to \mathfrak{f}_4 \end{aligned} \tag{5.3.28}$$

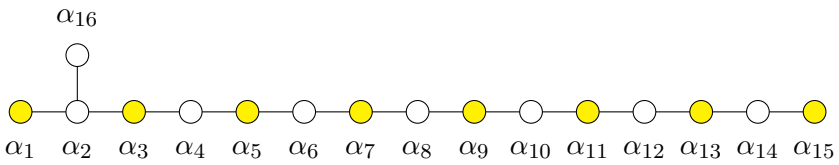
where both the LHS and RHS algebras are at level 1 (the algebras unaffected by the freezing become level 2). These rules cannot be applied arbitrarily, however. In order for the LHS algebras to be reduced to those in the RHS, their roots must be connected with those of Ω as specified in each case above. Any root of Ω left by itself is simply projected out, $\mathfrak{su}_2 \to \emptyset$.

5.3.4 Applying the map in $d = 4$

Having seen the possible ways in which subalgebras of a gauge algebra in the Narain component in six dimensions can be transformed when mapping to the CHL component, we now treat the problem of when these rules are applicable for a given gauge group G. In the cases $d = 1, 2, 3$ this problem is trivial because the root lattices associated to Ω are uniquely embedded, so one always knows for any

gauge group if its weight lattice contains Ω by a simple reading of the algebra. For $d = 4$, however, the relevant root lattice is $8\mathrm{A}_1$, which may or may not be associated to Ω. It is necessary therefore to check explicitly, for each $8\mathrm{A}_1$ sublattice, if it corresponds to Ω or not.

As a simple example let us consider the gauge group $\frac{\mathrm{Spin}(32)}{\mathbb{Z}_2}$, ignoring the extra four U(1) factors for now. It turns out that the weight lattice of this group contains Ω as a sublattice, whose $8\mathrm{A}_1$ sublattice correspond to the yellow nodes in the diagram



(5.3.29)

This can be shown explicitly by deleting the white nodes and checking that the weight vector of the $\frac{\mathrm{SU}(2)^8}{\mathbb{Z}_2}$ weight lattice is in the Narain lattice (cf. eq. (5.3.21)). At the level of the algebras, then, we have that $\mathfrak{so}_{32}$ goes to $\mathfrak{so}_{17}$. This is to be contrasted with the gauge group Spin(32), which is simply-connected and therefore does not contain Ω in its weight lattice (which is a root lattice in this case). From this we learn that the topology of the group dictates what are the allowed freezings. Furthermore, we can explicitly compute the fundamental group of the gauge groups using the methods described in Section 4.3.3, which extend to any d, and find that $\frac{\mathrm{Spin}(32)}{\mathbb{Z}_2}$ gets mapped to Spin(17). In other words, the element $k' = (1, 0)$ which generates $\pi_1(\frac{\mathrm{Spin}(32)}{\mathbb{Z}_2})$ gets mapped to $k = 0$ in $\pi_1(\mathrm{Spin}(17))$.

In general, the gauge group to be mapped has more than one non-trivial element in its fundamental group, which makes things more complicated. Consider for example

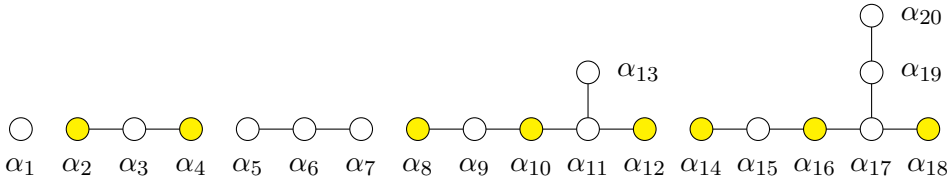
$$G = \frac{\mathrm{SU}(2) \times \mathrm{SU}(4) \times \mathrm{SU}(4) \times \mathrm{Spin}(12) \times \mathrm{E}_7}{\mathbb{Z}_2 \times \mathbb{Z}_2}\,, \tag{5.3.30}$$

where the $\mathbb{Z}_2 \times \mathbb{Z}_2$ consists of

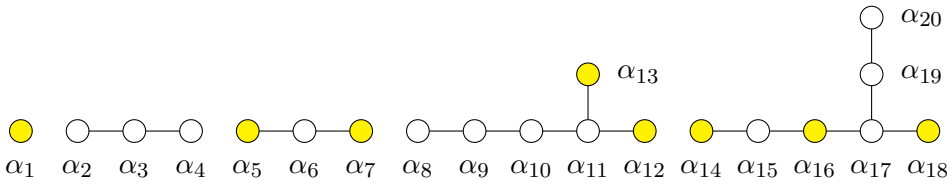
$$k_1 = (0, 2, 0, (1, 0), 1)\,, \qquad k_2 = (1, 0, 2, (1, 1), 1)\,, \qquad k_3 = (1, 2, 2, (0, 1), 0)\,. \tag{5.3.31}$$

Any pair of these elements, which generate $\pi_1(G)$, corresponds to two vectors which extend the root lattice L of G to its weight lattice M. They are inequivalent under translations in L. One can then delete nodes in the Dynkin diagram of L such that the reduced root lattice still has a non-trivial weight overlattice which might correspond to Ω, at which point any other reduction will not contain weight vectors. In this special case, all such reductions lead to inequivalent embeddings of Ω in M,

represented by the yellow nodes in the diagrams



(5.3.32)



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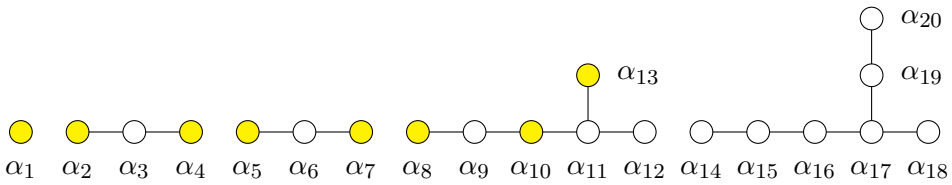


(5.3.34)

To each of these embeddings corresponds a different way of mapping G to the CHL component. Using the rules in (5.3.28) and computing the fundamental group in each case we get, respectively,

$$G \rightarrow \frac{\mathrm{SU}(2)\times\mathrm{SU}(2)\times\mathrm{SU}(4)\times\mathrm{Spin}(7)\times\mathrm{F}_4}{\mathbb{Z}_2}\,, \quad k=(1,0,2,1,0)\,, \tag{5.3.35}$$

$$G \rightarrow \frac{\mathrm{SU}(4)\times\mathrm{SU}(2)\times\mathrm{Sp}(4)\times\mathrm{F}_4}{\mathbb{Z}_2}\,, \quad k=(2,0,1,0)\,, \tag{5.3.36}$$

$$G \rightarrow \frac{\mathrm{SU}(2)\times\mathrm{SU}(2)\times\mathrm{Spin}(7)\times\mathrm{E}_7}{\mathbb{Z}_2}\,, \quad k=(0,0,1,1)\,. \tag{5.3.37}$$

The first thing to note is that in the resulting gauge group the fundamental group always reduces by a factor of $\mathbb{Z}_2$ (as already happened in the $\frac{\mathrm{Spin}(32)}{\mathbb{Z}_2}\rightarrow\mathrm{Spin}(17)$ case above). This can be understood by noting that one is taking the orthogonal complement of Ω, which contains weight vectors. These are also weight vectors in M, equivalent under translations in $8\mathrm{A}_1$, so they can be related to one of the elements in $\pi_1(G)$. For any such weight vector w, we have that $2w\in 8\mathrm{A}_1$ and so the associated $k\in\pi_1(G)$ generates a $\mathbb{Z}_2$. This is precisely the factor which is eliminated in mapping G, corresponding respectively to k_1, k_2 and k_3 above.

Now we need to know how the remaining k's get transformed in each case. What we find is that it suffices to mod every k by the one that is eliminated, call it k_Ω,

$$k\rightarrow k \mod k_\Omega \tag{5.3.38}$$

and then project it into the center of the resulting gauge group. In the case of a $\mathrm{Spin}(4n)$ factor, we project the modded k contribution to $1\in\pi_1(\mathrm{Spin}(2n+1))=\mathbb{Z}_2$ if it is not $(0,0)$. Of course, we also have that $k_\Omega\rightarrow 0$ so that this rule applies equally well to all the k's of $\pi_1(G)$.

We see then that the only information we require to know how to map a group G to the CHL component is the embedding of the roots of Ω into the root lattice L of G and its associated $k_\Omega\in\pi_1(G)$. In fact, however, these two pieces of data are the same. One can take any $k\in\pi_1(G)$ of order 2 in $Z(G)$ and check if it corresponds to Ω in the following way. For each simple factor in G, if the corresponding

entry in k is non-zero, its Dynkin diagram should be labeled according to one of the diagrams of Section 5.3.3.3. The only simple factor which contains more than one order 2 element is D_{2n}, in which case $k_{D_{2n}} = (1,1)$ corresponds to the diagram (5.3.23) and $k_{D_{2n}} = (0,1), (1,0)$ correspond to (5.3.26). Coloring the nodes appropriately lead to those shown in the example above, as one can easily check. If there are in total eight yellow nodes, this labeling will correspond to an embedding of Ω into M. With this we can apply the mapping rules to the algebra and to the fundamental group of G.

We verified all of these statements by applying the procedures outlined above to a reasonably exhaustive list of gauge groups in the Narain component, and checking the results against a list for the CHL string. In the next section we look at other rank reduced components, where the results are similarly verified against lists of symmetry enhancements. These lists can be obtained with the same method as those of the 7d case and can be accessed at [27]. We provide various examples in Appendix C.

5.4 Other rank reduced components

In this section we extend the freezing procedure explained above to other rank reduced components in the moduli space of heterotic strings which appear in seven dimensions and below. These correspond to the holonomy triples constructed in [23] and their torus compactifications.

5.4.1 Freezing map in 7d

As explained in Section 5.1, in seven dimensions there are six connected components in the moduli space of supersymmetric heterotic strings. They can be obtained as asymmetric orbifolds of the T^3 compactifications at points in the moduli space where the Narain lattice exhibits appropriate symmetries. These orbifolds are of order 2 to 6, and they correspond to non-trivial holonomy triples in the target space, hence they are called $\mathbb{Z}_m$-triples with $m = 2, ..., 6$.

For each $\mathbb{Z}_m$-triple, the momentum lattice can be obtained as the orthogonal complement of some other lattice Ω. This data was shown in Table 5.1. The $\mathbb{Z}_2$-triple is equivalent to the CHL string treated in Section 5.3.

For the $\mathbb{Z}_3$-triple, we have $\Omega = \mathrm{E}_6 \oplus \mathrm{E}_6$, which can only be embedded into $\mathrm{E}_p \oplus \mathrm{E}_q$ with $p, q = 6, 7, 8$. For each E_p factor, we have the algebra mapping

$$\mathfrak{e}_6 \to \emptyset\,, \qquad \mathfrak{e}_7 \to \mathfrak{su}_2\,, \qquad \mathfrak{e}_8 \to \mathfrak{g}_2\,, \tag{5.4.1}$$

while the corresponding contribution to any element k of the fundamental group is preserved. As with the $m = 2$ component, we have that the gauge groups related by the mapping have isomorphic fundamental groups.

For the $\mathbb{Z}_4$-triple, we have $\Omega = \mathrm{E}_7 \oplus \mathrm{E}_7$, which embeds only into $\mathrm{E}_p \oplus \mathrm{E}_q$ with $p, q = 7, 8$. For each E_p factor we have the algebra mapping

$$\mathfrak{e}_7 \to \emptyset\,, \qquad \mathfrak{e}_8 \to \mathfrak{su}_2\,. \tag{5.4.2}$$

The $\mathbb{Z}_5$ and $\mathbb{Z}_6$-triples both have $\Omega = \mathrm{E}_8 \oplus \mathrm{E}_8$ and so the only mapping allowed is $\mathfrak{e}_8 \to \emptyset$. All the possible gauge groups involved in this mapping are simply-connected so here again they have isomorphic fundamental groups, namely trivial ones.

The full set of rules is summarized in Table 5.2

Algebra	m	Transforms to	Contribution to Ω
D_{n+4}	2	C_n	D_4
E_{n+4}	2	F_n	D_4
E_{n+6}	3	G_n	E_6
E_{n+7}	4	C_n	E_7
E_8	5, 6	$\emptyset$	E_8

Table 5.2: Freezing rules for the simple factors in the gauge groups for 7d. Transformed algebras appear at level 1 on the rank reduced theory.

where we are using the conventions:

$$C_1 \equiv A_1\,,\ F_2 \equiv A_2\,,\ F_3 \equiv B_3\,,\ G_1 \equiv A_1\,, \tag{5.4.3}$$

The map for the group explained in Section 5.3.2.2 can be directly generalized to all the different components in the moduli space of 7d theories treated here. Similarly, only the contributions to the fundamental group coming from the partially frozen factors change. In the 7d CHL string the rules for going from D_{n+4} to C_n are equivalent to those for going from D_{n+8} to C_n described above. For example, we find that $(\mathrm{Spin}(24)/\mathbb{Z}_2) \times \mathrm{Spin}(14)$ maps to $(\mathrm{Sp}(8)/\mathbb{Z}_2) \times \mathrm{Sp}(3)$. For the freezing $E_{4+n} \to F_n$ (cf. Table 5.2), the center of the gauge group is unaltered and so is the corresponding contribution to the fundamental group, i.e. $\hat{k} \to \hat{k}' = \hat{k}$. This is also true for the freezing $E_{6+n} \to G_n$ in the $m = 3$ case.

For $m = 5, 6$, the rule $E_8 \to \emptyset$ has no effect on the fundamental group other than shortening $(k_1, ..., k_s, \hat{k})$ to $(k_1, ..., k_s)$. With these generalized freezing rules, one can project the enhancements in the Narain component of the moduli space to the other five components treated in this chapter to reproduce the results found with our exploration algorithm.

Like we encountered for the CHL string, these rules agree perfectly with the freezing mechanism in M-theory on K3 [23, 25]. When applied to the enhancements found in the Narain moduli space one reproduces the results, at the level of the algebras, obtained with the exploration algorithm applied to the remaining momentum lattices, as expected.

5.4.2 Extension of the freezing map in 6d

Let us now consider the compactifications of the 7d $\mathbb{Z}_m$-triples to 6d with $m = 3, 4, 5, 6$. Not surprisingly, the mappings that we find here generalize naturally those of the $m = 2$ case.

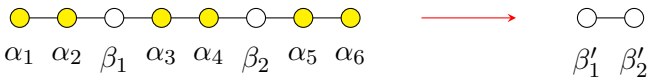
5.4.2.1 6d $\mathbb{Z}_3$-triple

For $m = 3$, the momentum lattice is

$$\Gamma_{3,3} \oplus \Gamma_{1,1}(3) \oplus A_2 \oplus A_2\,, \tag{5.4.4}$$

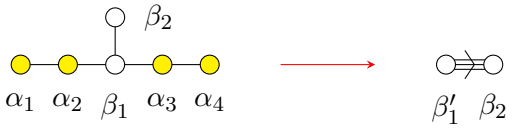
which can be shown to be the orthogonal complement of a lattice Ω in $\Gamma_{4,20}$ isomorphic to the weight lattice of $\frac{\mathrm{SU}(3)^6}{\mathbb{Z}_3}$, with $\mathbb{Z}_3$ diagonal. There are two types of root lattices which can be obtained by attaching nodes to the Dynkin diagram of this $\mathrm{SU}(3)^6$. First, we have those of the type A_{3n-1},

obtained by adding roots between each pair of A_2's consecutively. These map to A_n. For example, we have that $A_8 \to A_2$,



(5.4.5)

The other possibility is to map E_6 to G_2,

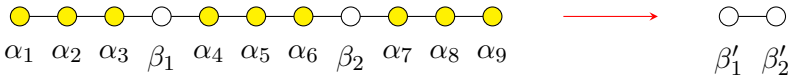


(5.4.6)

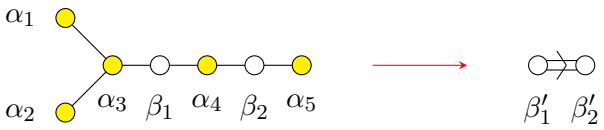
A gauge group G in the Narain component can be mapped to this moduli space if $\pi_1(G)$ contains an order 3 element k_Ω such that its entries label exactly 12 nodes in the associated Dynkin diagram, in a manner completely analogous to the case for the CHL string (see Section 5.3.4). The procedure for mapping all the elements of $\pi_1(G)$ is the same. For example, the gauge group $\frac{E_6^3}{\mathbb{Z}_3}$ with π_1 generator $k = (1,1,1)$ maps to G_2^3, and $\frac{SU(3)^2\times SU(6)^2\times Spin(10)}{\mathbb{Z}_6}$ with π_1 generator $k = (1,1,1,1,2)$ maps to $\frac{SU(2)^2\times Spin(10)}{\mathbb{Z}_2}$ with $k = (1,1,2)$. Similarly to the CHL string, the unaltered simple factors correspond to level 3 algebras and the altered to level 1 ones, so that e.g. the latter has algebra $(\mathfrak{su}_2 \oplus \mathfrak{su}_2)_1 \oplus (\mathfrak{spin}_{10})_3$.

5.4.2.2 6d $\mathbb{Z}_4$-triple

For $m = 4$, the momentum lattice is $\Gamma_{3,3} \oplus \Gamma_{1,1}(4) \oplus A_1 \oplus A_1$, whose associated Ω is the weight lattice of $\frac{SU(2)^2\times SU(4)^4}{\mathbb{Z}_4}$ with $\mathbb{Z}_4$ generated by $k = (1,1,1,1,1,1)$. The roots of this lattice can be extended in particular to A_{4n-1} and D_{2n+3}, the latter with $n = 1, 2$. The algebras are respectively mapped to $\mathfrak{su}_n$ and $\mathfrak{sp}_n$. For example, we have



(5.4.7)



(5.4.8)

In the latter case we see that the two A_1's of Ω are used up, so that one cannot extend to D_9 and beyond. The resulting gauge groups have current algebras at level 1, except for the case of A_3 which only involves two frozen A_1's and produces an A_1 at level 2. Unaffected factors become level 4.

The element k_Ω associated to this mapping is of order 4. In particular this means that $2k_\Omega$ is an order 2 element, which turns out to be associated to the freezing to the CHL component of the moduli space. This is reflected in the fact that the frozen sublattice of this moduli space component contains the one for the CHL component. Indeed, the $2A_1$ part of L_Ω can be extended to D_n and frozen to C_{n-2}, as for the CHL freezing rule. This will be the case if k_Ω has an order 2 contribution to a D_n factor.

For example, the group $\frac{SU(2)^3\times SU(4)\times SU(8)^2}{\mathbb{Z}_8}$ with π_1 generator $k = (1,1,1,1,1,1)$ maps to $\frac{SU(2)^6}{\mathbb{Z}_2}$

with π_1 generator $k = (1,1,1,1,1,1)$ and algebra $(\mathfrak{su}_2 \oplus \mathfrak{su}_2)_1 \oplus (\mathfrak{su}_2)_2 \oplus (\mathfrak{su}_2 \oplus \mathfrak{su}_2 \oplus \mathfrak{su}_2)_4$, showcasing the possible ways in which current algebra levels can mix; here the mapping is associated to the order four element $2k \simeq (0,0,0,2,2,2)$. Another interesting example is given by the group $\mathrm{SU}(3) \times \frac{\mathrm{SU}(12) \times \mathrm{Spin}(14)}{\mathbb{Z}_4}$ with π_1 generator $k = (0,3,3)$, which maps to $\mathrm{SU}(3)^2 \times \mathrm{Sp}(2)$, with algebra $(\mathfrak{sp}_2 \oplus \mathfrak{su}_3)_1 \oplus (\mathfrak{su}_3)_4$; this involves the freezing rule for D_7, producing a simply-connected gauge group.

5.4.2.3 6d $\mathbb{Z}_{5,6}$-triples

For $m = 5$ the momentum lattice is $\Gamma_{3,3} \oplus \Gamma_{1,1}(5)$, whose associated Ω is the weight lattice of $\mathrm{SU}(5)^4/\mathbb{Z}_5$ with $\mathbb{Z}_5$ generated by $k = (1,1,1,1)$. The only extension allowed here is A_{5n-1}, which maps to A_{n-1}, generalizing the similar freezings in the previous cases.

For $n = 6$, we have momentum lattice $\Gamma_{3,3} \oplus \Gamma_{1,1}(6)$, whose Ω is the weight lattice of $\frac{\mathrm{SU}(2)^2 \times \mathrm{SU}(3)^2 \times \mathrm{SU}(6)^2}{\mathbb{Z}_6}$ with $\mathbb{Z}_6$ generated by $(1,1,1,1,1,1)$. Again, the only allowed freezing here will be from A_{6n-1} to A_{n-1}, associated to an order 6 element in $\pi_1(G)$. However, this Ω includes the frozen sublattices of $m = 2$ and $m = 3$. Similarly to the $m = 4$ case including $m = 2$ freezing rules, here we also have the $m = 2$ and $m = 3$ rules which can be realized by two A_1 factors and two A_2 factors, respectively.

5.4.3 Freezing rules in 6d

In 6d, the connected components of moduli space of the heterotic string studied in this chapter have momentum lattices and corresponding orthogonal complements in $\Gamma_{4,20}$ (frozen sublattices) as shown in Table 5.3. Here we have given Ω in terms of its root sublattice L_Ω and the fundamental group of the gauge group associated to Ω[4]. The gauge symmetry groups that can be realized in the $n = 2, ..., 6$ components can be obtained by applying a set of "freezing rules" to those of the $n = 1$ one.

n	Momentum Lattice Γ	Frozen root lattice L_Ω	$\pi_1(G_\Omega)$
1	$\Gamma_{4,20}$	$\emptyset$	
2	$\Gamma_{3,3} \oplus \Gamma_{1,1}(2) \oplus \mathrm{D}_4 \oplus \mathrm{D}_4$	$8\mathrm{A}_1$	$\mathbb{Z}_2$
3	$\Gamma_{3,3} \oplus \Gamma_{1,1}(3) \oplus \mathrm{A}_2 \oplus \mathrm{A}_2$	$6\mathrm{A}_2$	$\mathbb{Z}_3$
4	$\Gamma_{3,3} \oplus \Gamma_{1,1}(4) \oplus \mathrm{A}_1 \oplus \mathrm{A}_1$	$2\mathrm{A}_1 \oplus 4\mathrm{A}_3$	$\mathbb{Z}_4$
5	$\Gamma_{3,3} \oplus \Gamma_{1,1}(5)$	$4\mathrm{A}_4$	$\mathbb{Z}_5$
6	$\Gamma_{3,3} \oplus \Gamma_{1,1}(6)$	$2\mathrm{A}_1 \oplus 2\mathrm{A}_2 \oplus 2\mathrm{A}_5$	$\mathbb{Z}_6$

Table 5.3: Momentum lattices and corresponding orthogonal complements in $\Gamma_{4,20}$, given in terms of their root sublattices and fundamental group of the associated gauge group.

For 7d it is guaranteed that the rules in 5.2 can be applied as long as the contributions to Ω result in the ones listed in Table 5.1. In contrast, there is a particularity in 6d: to check if one of these freezings can be done with a certain G, one looks for order n elements k_Ω in $\pi_1(G)$ such that they define an embedding of L_Ω into the root lattice L of G. If this is the case, one applies the rules shown in Table 5.4 according to this embedding, and obtains the fundamental group of the resulting gauge group G' by modding the elements of $\pi_1(G)$ by k_Ω and projecting them onto the center of G'.

[4]The L_Ω's correspond to the singularities of $\mathrm{K3} \times S^1$ orbifolds of order n in the dual M-theory [23].

Algebra	k_Ω	Order of k_Ω	Transforms to	Contribution to Ω
A_{qn-1}	n	$q = 2, 3, 4, 5, 6$	A_{n-1}	nA_{q-1}
D_{n+2}	v	2	C_n	$2A_1$
D_{2n}	s	2	B_n	nA_1
E_7	1	2	F_4	$3A_1$
E_6	1	3	G_2	$2A_2$
D_{2n+3}	$1 \simeq s$	4	C_n	$nA_1 + A_3$

Table 5.4: Freezing rules in 6d for the simple factors in the gauge groups according to the element k_Ω of the fundamental group associated to the freezing. For all the cases, the longest roots are of length twice the order of k_Ω. v and s denote the vector and spinor classes of the orthogonal groups.

These transformations also appear in a seemingly unrelated problem, namely in the relation between the moduli space components of flat bundles over T^2 with non-simply-connected structure group G [55] when G is simply-laced. The transformed group is simply-connected and describes the so-called topologically non-trivial components of the moduli space for a certain G. In this sense, what we find in the moduli space of 6d heterotic strings is a generalization to semisimple lie groups with many factors and more complicated fundamental groups[5].

5.5 Classification of gauge groups

The full tables with maximal enhancements and their global data for the Narain components and the $\mathbb{Z}_m$-triples are given in Appendix D.2 for 7d and in the corresponding tables in [27] for 6d. Here we give tables with the counting of the different gauge symmetries which are realized in each component in 7d.

5.5.1 Narain Component

As explained before, obtaining the gauge groups for the Narain component is done with a straightforward extension of the original exploration algorithm developed in Chapter 3. However, here we have also computed the full global data for each group, giving the explicit generators for the fundamental groups using the methods in chapter 4 based on [20]. All the maximally enhanced groups in this component are listed in Table D.7, with the generators of their fundamental groups in Table D.12.

We have for example the gauge group (#421)

$$\frac{\mathrm{SU}(8) \times \mathrm{SU}(8) \times \mathrm{Spin}(10)}{\mathbb{Z}_8}, \tag{5.5.1}$$

where the fundamental group $\mathbb{Z}_8$ is generated by the element $(1, 3, 1)$ of the center $\mathbb{Z}_8 \times \mathbb{Z}_8 \times \mathbb{Z}_4$ of the universal cover $\mathrm{SU}(8) \times \mathrm{SU}(8) \times \mathrm{Spin}(10)$.

For each generator we give a sequence of numbers representing the contribution from the center of each simple factor. In the example just given, the generator is 131. Note that the ordering of the sequence corresponds to the ordering of the listed ADE type. To properly read the sequence one must write expressions of the form $A_3^2 D_4^3$ as $(A_3, A_3, D_4, D_4, D_4)$, assigning each number in the sequence to each subsequent ADE factor. For D_{2n} factors there are four order two elements in the center denoted

[5] We are not aware of a treatment of this general problem in the literature.

v, c, s and 1, corresponding to the vector class, spinor classes and the identity, respectively. Note that in some cases the fundamental group has more than one generator.

For 7d, we list the total number of distinct gauge algebras and distinct gauge groups for different ranks of the semisimple part in Table 5.5. These have been obtained by deleting nodes in the Dynkin Diagrams of the maximally enhanced groups, and we assume that this gives all the possibilities, as discussed in Section 4.3.

Rank	1	$\mathbb{Z}_2$	${\mathbb{Z}_2}^2$	$\mathbb{Z}_3$	$\mathbb{Z}_4$	${\mathbb{Z}_2}^3$	${\mathbb{Z}_2}^4$	${\mathbb{Z}_2}^5$	$\mathbb{Z}_5$	$\mathbb{Z}_6$	${\mathbb{Z}_3}^2$	$\mathbb{Z}_2\mathbb{Z}_4$	$\mathbb{Z}_7$	$\mathbb{Z}_2\mathbb{Z}_6$	${\mathbb{Z}_4}^2$	$\mathbb{Z}_8$	Algebras	Groups
19	652	381	68	51	37	5	1		6	16	3	2	1	2	1	2	1035	1232
18	852	492	89	52	35	9	1		4	10	3	6	1	1	1	1	1180	1557
17	827	442	73	39	23	8	1		2	4	2	3					1024	1424
16	694	334	47	25	12	4	1	1	1	1	1	1					794	1122
15	528	217	24	12	4	2	1										567	788
14	389	128	11	6	1	1											403	536
13	272	66	3	2													276	343
12	192	33	1	1													193	227
11	128	14															128	142
10	88	6															88	94
9	57	2															57	59
8	39	1															39	40
7	24																24	24
6	16																16	16
5	9																9	9
4	6																6	6
3	3																3	3
2	2																2	2
1	1																1	1
All	4779	2116	316	188	112	29	5	1	13	31	9	12	2	3	2	3	5845	7625

Table 5.5: Number of algebras and groups of each rank with a certain fundamental group for the heterotic string on T^3. The gauge group with $\pi_1 = \mathbb{Z}_2^5$ (cf. eq. (5.5.2)) does not admit further enhancements.

We note that there are many cases in which two gauge groups have isomorphic fundamental groups with inequivalent inclusions in the center of the universal covering (meaning that they are not related by outer automorphisms of the group, as is the case e.g. for $\mathrm{SO}(2n)$ versus $\mathrm{Spin}(2n)/\mathbb{Z}_2$ for $n \neq 4$). These were not distinguished in Table D.2, where we recorded 339 groups (the ordering goes only up to 325 because we label them only with the algebra). The inequivalence is taken into account on the T^2 table in [27] resulting in 341 different groups.

It is natural to assume that all points of maximal enhancement in moduli space can be reached with our exploration algorithm, and it is in fact true for the cases $d = 1, 2$. Non-maximal enhancements can be obtained from the maximal ones by simply removing an arbitrary number of roots. Remarkably, for $d = 1, 2$ there are respectively only two gauge groups which can not be obtained in this way, namely $\mathrm{Spin}(16)^2/\mathbb{Z}_2$ for $d = 1$ and $\mathrm{Spin}(8)^4/\mathbb{Z}_2^2$ for $d = 2$. In $d = 3$ there is also such gauge group,

$$G = \frac{\mathrm{SU}(2)^{16}}{\mathbb{Z}_2^5}\,, \tag{5.5.2}$$

where the fundamental group is given by

$$\begin{array}{c} 0000000011111111 \\ 0000111100001111 \\ 0011001100110011 \\ 0101010101010101 \\ 1001011001101001 \end{array} \tag{5.5.3}$$

The pattern for all d is $\frac{\mathrm{Spin}(2^{5-d})^{2^d}}{\mathbb{Z}_2^{(2^d-d)}}$. At the level of gauge algebras this was already noted in [37]. To obtain them we just use the exploration algorithm starting from an adequate non-maximal gauge group. For 6d this pattern predicts an Abelian symmetry and is irrelevant for our study. Nonetheless, there is a peculiarity in the T^4 moduli space: one maximal enhancement, $\frac{\mathrm{SU}(2)^{20}}{\mathbb{Z}_2^8}$, cannot be obtained by just removing and adding a node to other maximal enhancements. To obtain it with our methods it is necessary to first find $\frac{\mathrm{SU}(2)^{16}}{\mathbb{Z}_2^5}$ by exploring the neighborhood of some non-maximal enhancement we already have (or alternatively do a trivial compactification of (5.5.2) to 6d). We get all the missing groups by continuously adding nodes, obtaining sequentially:

$$\frac{\mathrm{SU}(2)^{16}}{\mathbb{Z}_2^5} \to \frac{\mathrm{SU}(2)^{16}}{\mathbb{Z}_2^5} \times \mathrm{SU}(2) \to \frac{\mathrm{SU}(2)^{18}}{\mathbb{Z}_2^6} \to \frac{\mathrm{SU}(2)^{19}}{\mathbb{Z}_2^7} \to \frac{\mathrm{SU}(2)^{20}}{\mathbb{Z}_2^8} \tag{5.5.4}$$

5.5.2 Triples

The results for the components of the moduli space with rank reduction are obtained by an extension of the exploration algorithm taking into account Proposition 3 for 7d and its generalization to 6d where we accept as levels all the divisors of m for each $\mathbb{Z}_m$-triple. For 7d, the gauge groups are recorded in tables D.8 to D.11 in Appendix D.2.1, while the generators for the fundamental groups are recorded in Tables D.13 and D.14 in Appendix D.2.2. For 6d the gauge groups are recorded in the corresponding tables of [27]. In the case of the $\mathbb{Z}_5$ and $\mathbb{Z}_6$-triples in 7d, we see that all the gauge groups are simply connected and so no global data is required to specify them. As explained in Section 5.4, all the gauge groups for the non-trivial $\mathbb{Z}_m$ triples can be obtained from those of the Narain component using a projection map generalizing the one obtained in [20] for the 8d CHL string (see Tables 5.2 and 5.4). In Appendix C we give some examples of these freezings in 6d. For 7d, the total number of distinct gauge algebras and distinct gauge groups are listed in Table 5.6.

$\mathbb{Z}_2$ triple

Rank	1	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_3$	$\mathbb{Z}_4$	$\mathbb{Z}_2^3$	$\mathbb{Z}_2^4$	Algebras	Groups
11	224	143	44	7	3	7	1	407	429
10	307	192	51	5	3	8	1	473	567
9	284	161	37	2	2	4	1	372	491
8	214	101	18	1	1	2		244	337
7	137	45	5					143	187
6	84	17	1					85	102
5	46	4						46	50
4	26	1						26	27
3	12							12	12
2	6							6	6
1	2							2	2
All	1342	664	156	15	9	21	3	1816	2210

$\mathbb{Z}_3$ triple

Rank	1	$\mathbb{Z}_2$	$\mathbb{Z}_3$	Algebras	Groups
7	41	6	5	50	52
6	37	5	4	41	46
5	24	2	2	24	28
4	15	1	1	15	17
3	8			8	8
2	5			5	5
1	2			2	2
All	132	14	12	145	158

$\mathbb{Z}_4$ triple

Rank	1	$\mathbb{Z}_2$	Algebras	Groups
5	13	5	16	18
4	10	4	11	14
3	6	2	6	8
2	4	1	4	5
1	2		2	2
All	35	12	39	47

$\mathbb{Z}_5$ and $\mathbb{Z}_6$ triples

Rank	1	Algebras	Groups
3	3	3	3
2	2	2	2
1	1	1	1
All	6	6	6

Table 5.6: Number of algebras and groups of each rank with a certain fundamental group for the heterotic $\mathbb{Z}_2$, $\mathbb{Z}_3$, $\mathbb{Z}_4$, $\mathbb{Z}_5$ and $\mathbb{Z}_6$ triples.

Chapter 6

Conclusions

The main objective of this thesis was to obtain and classify all the gauge groups appearing as symmetries of the massless spectrum of the half-maximal supersymmetric compactifications of heterotic string. We succeeded in this task for theories with 9, 8, 7 and 6 space-time dimensions, with the results collected in [27] and also on Appendix D for 7 or more dimensions[1]. With this goal in mind we studied diverse aspects of the compactified theories, developing techniques and arriving at results serving not only for answering elementary questions (e.g. what are the relations between these theories?) but also for treating many unrelated problems. In the following we summarize these results and give possible directions for future work.

In Chapter 2, after a brief introduction to heterotic string theory, we presented its toroidal compactification and analyzed some generalities. The main goal of this chapter was to understand in an explicit manner the structure of the moduli spaces, gaining intuition on symmetry enhancements. To achieve this we quickly restricted to the circle case. The 17-dimensional moduli space of S^1 compactifications, involving the radius of the circle and the 16 components of the Wilson line along the Cartan directions of the $SO(32)$ or $E_8 \times E_8$ gauge group, was studied in detail.

We presented the action of the standard T-duality exchanging momentum and winding number, and studied its fixed points, which are at $R^2 = 1 - \frac{1}{2}|A|^2$. At these points, the dual background has the same radius and opposite Wilson line, $A' = -A$. If $2A$ is in the root lattice, then $A' = -A \sim A$ and the full background is self-dual. For Wilson lines with only one non-zero component, as those explored in Section 2.3.4, the fixed "points" of the T-duality symmetry are not really points, but in this two-dimensional subspace of moduli space they correspond to lines of non-maximal enhancement symmetry, where the Wilson line is a function of the radius ($A = A(R_{\rm sd})$), and is such that $A \sim A_{\rm sd}$, with $|A_{\rm sd}|^2 = 2(1 - R_{\rm sd}^2)$.

The discussion of the explicit enhancement process was split into compactifications with $\pi \cdot A \in \mathbb{Z}$ and $\pi \cdot A \notin \mathbb{Z}$. Although all the enhancements can be obtained with Wilson lines that are not on any lattice by appropriately choosing R, the distinction is useful to understand the enhancement process. When the Wilson line has zero vacuum expectation value, or equivalently when the vacuum expectation value is on the root lattice Γ_g, the gauge group of the uncompactified theory is unbroken at generic radius, and the total gauge group on the external space is $U(1)_R \times (U(1) \times SO(32))_L$ or $U(1)_R \times (U(1) \times E_8 \times E_8)_L$. At $R = 1$, there are additional states with momentum and winding that become massless and enhance the $U(1)_L$ to $SU(2)_L$. For other values of Wilson lines and generic R, the gauge symmetry is determined by the subset of heterotic momenta π that have integer inner product with the Wilson line. In the HO theory, one has the interesting possibility of a Wilson line that has integer inner product with all π, i.e. a Wilson line in the dual root lattice, but which is not in the lattice, namely $A \in \Gamma_v$ or $A \in \Gamma_c$. These two possibilities lead to an unbroken $SO(32)$ gauge

[1] We left out of this thesis the tables for 6d because they are too long.

symmetry at any radius, while at $R^2 = \frac{1}{2}$ there are extra massless states with non-zero momentum and/or winding number on the circle, giving a total 17-component left-moving momentum with mixed circle and chiral heterotic directions which enhance the gauge symmetry to $SO(34)$.

We developed a method for computing and drawing two dimensional slices of the 17-dimensional moduli space which neatly exhibit the distribution of the enhanced groups. The family of functions corresponding to each of the curves and the heterotic momentum of the additional massless states can be obtained from this analysis. While non-maximal enhancement occurs at lines, maximal enhancement occurs at isolated points. More interesting figures arise at smaller radii, and the smaller the radius, the richer the pattern of enhanced gauge symmetries, as there are more winding numbers that lead to massless states[2]. Moreover, we were able to univocally relate the intersections of the curves in the figures with the enhanced groups obtained from the Generalized Dynkin Diagram (GDD). An interesting output of the construction is that, in order to obtain groups that contain $SO(32)$ from the HE theory or groups that contain $E_8 \times E_8$ from the HO theory it is necessary to choose a slice where, for a generic point, the group is $SO(16) \times SO(16)$ or a subgroup of it.

From the GDD for the Narain lattice $\Gamma_{1,17}$, we found all the possible maximal enhancements for heterotic string on a circle. In particular, we showed that the same enhancements can be achieved in both heterotic theories (e.g. $SO(34)$ enhancement from the HE string) and explained how to obtain them.

The emphasis in this thesis has been to study gauge symmetry enhancement in compactifications of perturbative heterotic string theory for the characterization of the string theory moduli space. One interesting direction that could be studied is the inclusion of non-perturbative effects, where the physics of symmetry enhancement plays an important part. In particular, winding heterotic $E_8 \times E_8$ states are related to the dynamics of D-particles in the presence of $D8$-branes and orientifold planes in type I' superstring theory, and have been crucial in the understanding of subtle aspects of the Type I/heterotic duality [13, 35, 70, 71]. We hope that the methods developed here are useful to analyze these questions further.

In Chapter 3 we aimed at generalizing the results of Chapter 2 to compactifications to lower dimensions. To this end, we focused on the lattices that quantize the momenta. At special points in moduli space, the $(d + 16)$ U(1) symmetries can get enhanced, and we stated lattice embedding criteria to determine whether a given gauge group is realized or not in a toroidal compactification. The use of these criteria was explained in several examples.

We also introduced an algorithm to systematically explore the moduli space and applied it to obtain all the semisimple groups of maximal rank for $d = 1$ and $d = 2$, as well as the values of the corresponding background fields. Specifying the moduli is important for various reasons. First of all, the vertex operators and the full 1-loop modular invariant partition function of the theory explicitly depend on the momenta (3.1.2) [7, 30]. Besides, the moduli could be relevant in the study of dualities with other constructions and in phenomenological applications (combining with additional orbifold actions).

Actually, our results include not only the groups with maximal enhancement, but also groups $\mathrm{G}_r \times \mathrm{U}(1)^{d+16-r}$, with $r \leq (16 + d)$. For $d = 1$ all possible G_r can be deduced from the GDD as mentioned previously, and for $d = 2$ they are listed in [15]. A natural question is whether different G_r could arise in other non-chiral string constructions with 16 supercharges. For $d = 2$, our results contain the groups with maximal enhancement found in the covariant lattice formulation [72]. It would

[2]These figures could be useful when interpreting $R \to 0$ as a decompactification limit, where a tower of massless states appears [69].

be interesting to know if some other CFT construction could give for instance 8-dimensional theories with 16 supercharges and an ADE gauge group of rank 18 that is forbidden in the heterotic on T^2 (such as $\mathrm{E}_8 \times \mathrm{SO}(14) \times \mathrm{SU}(4)$). It would also be helpful to understand if a theory with a forbidden group could suffer from global anomalies as discussed in [73].

Our results show that all maximal enhancements in the heterotic compactification on T^2 coincide with all possible singular fibers of extremal K3 surfaces classified in [14]. This gives relevant information for the study of extremal K3 surfaces. Some realizations of these surfaces have been studied in detail, see [42, 74–77] and references therein. In the early days some examples were found by analyzing F-theory on orbifold limits of K3 [78]. Other examples have been obtained more recently by considering enhancements at special points in the moduli space of K3 surfaces with Picard number less than 20 [79–81].

In Chapter 4 we have studied heterotic string compactifications that realize the CHL branch of superstring vacua with 16 supercharges in $(10-d)$ dimensions, $d \geq 1$. Such vacua, characterized by left-moving gauge group of rank $d+8$, were first obtained in the context of type I strings [82] and later derived in heterotic strings both in the fermionic [17] and bosonic formalism [18]. We have followed the latter approach, based on compactification of the HE string on an asymmetric orbifold $T^d/\mathbb{Z}_2$, which enables a description at any point of the moduli space. In particular, we have focused on the question of which non-Abelian groups of maximal rank can appear. We have given a complete answer in $d = 1, 2$ in the form of a list of allowed groups and the corresponding moduli. We know that these lists are exhaustive, as they can also be obtained from the exhaustive lists of gauge groups for the S^1 and T^2 compactifications, respectively, by using the rules developed in Chapter 5.

Our analysis relies on the Mikhailov lattice $\Gamma_{(d)}$ underlying the $T^d/\mathbb{Z}_2$ asymmetric orbifold. In analogy with the Narain lattice $\Gamma_{d,d+16}$ associated to heterotic compactification on T^d, the momenta of all states in the orbifold spectrum lie in $\Gamma_{(d)}$ and symmetries of the spectrum correspond to automorphisms of the lattice [19]. For our purposes an essential fact is that the root lattice of the resulting non-Abelian groups must admit an embedding in $\Gamma_{(d)}$, which is even but not self-dual for $d > 1$. This last property leads to both simply-laced and non-simply-laced groups realized at Kac-Moody levels 1 or 2. The embedding condition gives a systematic prescription to determine the groups that can arise or not. Moreover, studying embeddings of the coroot and cocharacter lattices in the dual Mikhailov lattice allows to determine the global structure of the gauge group [20]. In this way we have proven that for $d = 1$ the groups are simply-laced and simply-connected whereas for $d = 2$ there are also symplectic and doubly-connected groups.

Our results for the global groups exactly match those obtained in [20], where they were shown to satisfy the condition for anomaly-free one-form center symmetries [21]. It would be interesting to check if these results are also consistent with constraints imposed by triviality of cobordism classes [83]. A partial check was carried out in [20].

As mentioned above, the automorphisms of the Mikhailov lattice are T-dualities of the theory. As such they restrict the moduli space, and fixed points of discrete transformations are expected to display gauge symmetry enhancement. Indeed, we have shown that this is the case in $d = 1$. A striking feature of the T-duality in $d = 1$ is that it mixes untwisted and twisted states.

The 8-dimensional CHL string is known to have a dual F-theory description in terms of compactification on a K3 surface with frozen singularities [22, 56]. The gauge groups arising in F-theory on such K3 surfaces were worked out very recently in [63], and agree perfectly with the heterotic groups of maximal enhancement given in Table D.5, giving yet more support to the exhaustiveness of our algorithm.

It is known that for $d \geq 3$, superstring vacua with 16 supercharges exhibit a broader pattern of rank reduction. In particular, in $d = 3$ there are components with rank 7, 5 and 3 for which an underlying momentum lattice analogous to $\Gamma_{(d)}$ has been constructed [23]. In Chapter 5 we extended our analysis to these theories. This was done by finding embeddings of weight lattices into the momentum lattices constructed in [23], taking into account an extra constraint on the role of the lattice vectors as stated in Proposition 3, stating the precise criteria for gauge groups being realized in the relevant theories.

Generalizing our previous algorithms we produced the list of maximally enhanced gauge groups that are realized in the heterotic string in 7d and 6d, encompassing the usual Narain component, and five other components with rank reduction realized via non-trivial holonomy triples. For 6d we verified exhaustiveness of these lists in [26] by computing the possible embeddings of rank 20 Lie algebra lattices on the Niemeier lattices. For 7d, since the explorations are simpler than for 6d, we conjecture that they are also exhaustive.

Exploiting the relations between the lattices corresponding to different compactifications, we found the general freezing rules for all these components, generalizing the results of [20] for the 8d CHL string. We described this map as a projection of the root sublattice of the gauge group in the toroidal compactification to the orthogonal complement of the singularity associated to each component. For 7d this map does not alter the fundamental group, and we found an explicit match with the mechanism of singularity freezing in M-theory on K3. For 6d we found a novel feature, since the freezing map explicitly involves the topology of the gauge groups, in particular acting only on non-simply-connected ones. This relation is equivalent to that of connected components of the moduli space of flat G-bundles over T^2 with G non-simply-connected.

The moduli space components that we have studied are not all. In [23] it was shown that there is at least another component in 6d: a $\mathbb{Z}_2 \times \mathbb{Z}_2$-quadruple, and an exhaustive list of the components of the moduli space of heterotic strings in 6d with maximal supersymmetry is not known. However, the map we have obtained is defined in terms of the fundamental group elements of the gauge groups and seems to naturally extend to many other cases that may correspond to other moduli space components, some of which require an M-theory description. This extension was the subject of [26].

On the other hand, the relation between these freezing rules and the problem of non-simply-connected flat G-bundles over T^2 is not clear, as in the heterotic string we are considering bundles over T^4. It may be better understood, perhaps, in a dual frame such as F-theory on K3 $\times T^2$ where one can more naturally isolate tori such as the fibers of the K3. As the former problem is rather involved, it is tantalizing to think that it may play a role in constraining the possible theories with 16 supercharges that can be coupled to gravity (see e.g. [84] for recent results in this direction).

Note that our exploration algorithm can be implemented in arbitrary dimension. A classification for the gauge groups appearing in less than 6 space-time dimensions was left for future work. An important open question is if it is possible to find a set of freezing rules for 5 or less dimensions analogous to the ones appearing in higher dimensions.

Finally, we note that our results serve to test Swampland conjectures [85], which are also easier to study in high-dimensional theories with a large amount of supersymmetry (see e.g. [21, 63, 83]). More generally, it would be very interesting if the methods developed here could be generalized to theories with less supersymmetry such as [86].

Appendix A

Supplements to Chapter 2

A.1 Lie algebras and lattices

Modular invariance of the one-loop partition function of the heterotic string implies that the 16-dimensional internal momenta must take values in an even self-dual Euclidean lattice, $\Gamma = \Gamma^*$, of dimension 16. There are only two of these: $\Gamma_8 \oplus \Gamma_8$, where Γ_8 is the root lattice of E_8, and Γ_{16}, which is the root lattice of $SO(32)$ in addition to the (s) or (c) conjugacy class

$$\begin{aligned} \Gamma_8 \oplus \Gamma_8 &= \Gamma_g \qquad \text{for } \mathrm{E}_8 \times \mathrm{E}_8 \\ \Gamma_{16} &= \Gamma_g + \Gamma_s \quad \text{for SO(32)} \end{aligned} \tag{A.1.1}$$

In this Appendix we summarize some basic notions on these lattices.

Given a Lie algebra g of rank n, taking arbitrary integer linear combinations of root vectors, one generates an n-dimensional Euclidean lattice Γ_g, called the root lattice. E.g., for the rank n orthogonal groups $SO(2n)$, the n component simple root vectors are

$$(\pm 1, \pm 1, 0, \dots) \;\text{ all other entries zero,} \tag{A.1.2}$$

and all permutations of these. For E_8, the eight component vectors

$$\begin{gathered} (\pm 1, \pm 1, 0, 0, 0, 0, 0, 0) \;+\; \text{permutations} \\ \left(\pm\tfrac{1}{2}, \pm\tfrac{1}{2}, \pm\tfrac{1}{2}, \pm\tfrac{1}{2}, \pm\tfrac{1}{2}, \pm\tfrac{1}{2}, \pm\tfrac{1}{2}, \pm\tfrac{1}{2}\right) \;\text{ even number of “}-\text{” signs} \end{gathered} \tag{A.1.3}$$

contain the 240 roots, i.e. the 112 root vectors of $SO(16)$ and 128 additional vectors.

Any Lie group G has infinitely many irreducible representations which are characterized by their weight vectors. Irreducible representations fall into different conjugacy classes, and Γ_g can be thought of as the (0) conjugacy class. Two different representations are said to be in the same conjugacy class if the difference between their weight vectors is a vector of the root lattice.

While E_8 has only one conjugacy class, namely (0), the $SO(2n)$ algebras have four inequivalent conjugacy classes:

- The (0) conjugacy class, i.e. the root lattice, contains vectors of the form

$$(n_1, \dots, n_n)\,, \quad n_i \in \mathbb{Z}\,, \quad \sum_{i=1}^{n} n_i = 0 \bmod 2\,. \tag{A.1.4}$$

- The vector conjugacy class, denoted by (v), contains vectors of the form

$$(n_1, \ldots, n_n)\,, \quad n_i \in \mathbb{Z}\,, \quad \sum_{i=1}^{n} n_i = 1 \bmod 2\,. \tag{A.1.5}$$

- The spinor conjugacy class, denoted by (s), contains vectors of the form

$$(n_1 + \frac{1}{2}, \ldots, n_n + \frac{1}{2})\,, \quad n_i \in \mathbb{Z}\,, \quad \sum_{i=1}^{n} n_i = 0 \bmod 2\,. \tag{A.1.6}$$

- The (c) conjugacy class contains vectors of the form

$$(n_1 + \frac{1}{2}, \ldots, n_n + \frac{1}{2})\,, \quad n_i \in \mathbb{Z}\,, \quad \sum_{i=1}^{n} n_i = 1 \bmod 2\,. \tag{A.1.7}$$

The weight lattice Γ_w is formed by all weights of all conjugacy classes including the root lattice itself. Clearly $\Gamma_g \subset \Gamma_w$, and for a simply-laced Lie algebra, which roots have squared modulus 2, it can be shown that $\Gamma_g = \Gamma_w^*$. Therefore, the weight lattice of E_8 contains the weights of the form

$$\Gamma_w^8 : \begin{cases} (n_1, \ldots, n_8) \\ \left(n_1 + \frac{1}{2}, \ldots, n_8 + \frac{1}{2}\right)\,, \end{cases} \qquad \sum_{i=1}^{8} n_i = \text{even integer} \tag{A.1.8}$$

with $n_i \in \mathbb{Z}$, is identical to its root lattice, which implies that it is even self-dual. It is also identical to the $SO(16)$ lattice with the (0) and (s) conjugacy classes

A necessary condition for a self-dual lattice is that it be unimodular. The $SO(2n)$ Lie algebra lattices are unimodular if they contain two conjugacy classes. The weight lattice of $Spin(32)/\mathbb{Z}_2$ is identical to the $SO(32)$ lattice with the (0) and (s) conjugacy classes. It is even self-dual and it's vectors are:

$$\Gamma_w^{16} : \begin{cases} (n_1, \ldots, n_{16}) \\ \left(n_1 + \frac{1}{2}, \ldots, n_{16} + \frac{1}{2}\right) \end{cases} \qquad \sum_{i=1}^{16} n_i = \text{even integer} \tag{A.1.9}$$

Both the root lattice of $E_8 \times E_8$ and the weight lattice of $Spin(32)/\mathbb{Z}_2$ contain 480 vectors of $(\text{length})^2 = 2$ which are the roots of $E_8 \times E_8$ and $SO(32)$, respectively.

It is convenient to write the conjugacy classes of $SO(32)$ in terms of conjugacy classes of representations of $SO(16) \times SO(16)$. We denote by (xy) a vector with the first eight components in the conjugacy class (x) of $SO(16)$ and the last eight in the class (y). x and y can be 0, s, v or c. We then have 16 conjugacy classes (xy). The $SO(32)$ conjugacy classes correspond to the following $SO(16) \times SO(16)$ pairs

$$\begin{aligned} (0) &= (00), (vv) \\ (s) &= (ss), (cc) \\ (c) &= (sc), (cs) \\ (v) &= (0v), (v0) \end{aligned} \tag{A.1.10}$$

We have then

$$\begin{aligned} \Gamma_{16} = \Gamma_0^{16} + \Gamma_s^{16} &= (00), (vv), (ss), (cc) \\ \Gamma_8 \oplus \Gamma_8 \equiv \Gamma_{8+8} = (\Gamma_0^8 + \Gamma_s^8) \oplus (\Gamma_0^8 + \Gamma_s^8) &= (00), (ss), (0s), (s0) \end{aligned} \tag{A.1.11}$$

The dual to the root lattice of $SO(32)$ is

$$(\Gamma_0^{16})^* = \Gamma_g^* = (00), (vv), (ss), (cc), (0v), (v0), (sc), (cs). \tag{A.1.12}$$

We also use the following properties of the lattices

$$\begin{aligned}
\Gamma_{8+8}\backslash\Gamma_{16} &= (0s), (s0) \\
\Gamma_{16}\backslash\Gamma_{8+8} &= (vv), (cc) \\
\Gamma_{16}\cap\Gamma_{8+8} &= (00), (ss) \\
(\Gamma_{16}\cap\Gamma_{8+8})^* &= (00), (ss), (vv), (cc), (vc), (cv), (0s), (s0) \\
(\Gamma_{16}\cap\Gamma_{8+8})^*\backslash(\Gamma_{16}\cup\Gamma_{8+8}) &= (vc), (cv)\ .
\end{aligned} \tag{A.1.13}$$

A.2 Maximal enhancement points for $A = (A_1, 0_{15})$

In this Appendix we show how to obtain the maximal enhancement points for the particular case of Wilson lines with only one non-zero component, treated in Section 2.3.4. We also prove that the only possible maximal enhancements for Wilson lines with only one non-zero entry are to $SU(2)\times SO(32)$, $SO(34)$, $SU(2)\times E_8\times E_8$ and $SO(18)\times E_8$.

The maximal enhancement points are those where two or more curves intersect. There are three types of intersections: $a_{w_1,q_1}(R) = a_{w_2,q_2}(R), b_{w_1,q_1}(R) = b_{w_2,q_2}(R)$ and $a_{w_1,q_1}(R) = b_{w_2,q_2}(R)$, that we treat separately. In the case of $\Gamma_8\times\Gamma_8$, the curves b can in principle have a curve c on top of them.

A.2.1 $\mathbf{a_{w_1,q_1}(R) = a_{w_2,q_2}(R)}$

$$\begin{aligned}
a_{w_1,q_1}(R) &= \tfrac{2q_1\pm_1\sqrt{2-2w_1^2R^2}}{w_1}, \quad \tfrac{2q_1^2-1}{w_1}\in\mathbb{Z} \\
a_{w_2,q_2}(R) &= \tfrac{2q_2\pm_2\sqrt{2-2w_2^2R^2}}{w_2}, \quad \tfrac{2q_1^2-1}{w_2}\in\mathbb{Z}
\end{aligned} \tag{A.2.1}$$

imply

$$\mp_1 w_2\sqrt{2-2w_1^2R^2}\pm_2 w_1\sqrt{2-2w_2^2R^2} = 2q_1w_2 - 2q_2w_1 \equiv C' = 2C\in 2\mathbb{Z}\,. \tag{A.2.2}$$

The case $C=0$ is trivial, so we must assume $C\neq 0$, which leads to

$$R^2 = \frac{2}{C'^2} - \frac{(2w_1^2+2w_2^2-C'^2)^2}{8w_1^2w_2^2C'^2}\,. \tag{A.2.3}$$

Defining $N = \frac{(1-2q_1^2)}{w_1}w_2 + \frac{(1-2q_2^2)}{w_2}w_1 + 4q_1q_2\in\mathbb{Z}$, we can rewrite (A.2.3) as

$$N^2 = 4 - 2C'^2R^2\,. \tag{A.2.4}$$

Since $(1-2q_i^2)$ and w_i are odd, N is even. Also, since C' and R are non-zero we get $N^2<4$, which implies $N=0$, then $R^2 = \frac{2}{C'^2}$. Then the radius where a curve a with winding w_1 intersects another curve a with winding w_2 is

$$R^{-2} = w_1^2 + w_2^2\,. \tag{A.2.5}$$

The constraint

$$|q_1 w_2 - q_2 w_1| = \sqrt{\tfrac{w_1^2+w_2^2}{2}} \implies \tfrac{w_1^2+w_2^2}{2} \text{ must be a perfect square .}$$

If $w_1 = w_2 = w$, then $q_1 = q_2 \pm 1$. The winding must be a divisor of both $2q_1^2 - 1$ and $2q_2^2 - 1$, but these numbers are coprime $\forall q_1$. Then the only possible value of w is 1. In conclusion, the only curves a with the same winding number that intersect are $a_{1,q}(R)$ and $a_{1,q\pm1}(R)$. And the intersection is on $R = \frac{1}{\sqrt{2}}$.

A.2.2 $\mathbf{b_{w_1,q_1}(R) = b_{w_2,q_2}(R)}$

$$\begin{aligned} b_{w_1,q_1}(R) &= \tfrac{2q_1+1\pm_1\sqrt{1-2w_1^2R^2}}{w_1}, \quad \tfrac{2q_1(q_1+1)}{w_1} \in \mathbb{Z} \\ b_{w_2,q_2}(R) &= \tfrac{2q_2+1\pm_2\sqrt{1-2w_2^2R^2}}{w_2}, \quad \tfrac{2q_2(q_2+1)}{w_2} \in \mathbb{Z} \end{aligned} \tag{A.2.6}$$

In this case,

$$\mp_1 w_2\sqrt{1-2w_1^2R^2} \pm_2 w_1\sqrt{1-2w_2^2R^2} = (2q_1+1)w_2 - (2q_2+1)w_1 \equiv C \in \mathbb{Z}\,. \tag{A.2.7}$$

If $C = 0$, then $w_1 = w_2$ and $q_1 = q_2$. If $C \neq 0$

$$R^2 = \tfrac{1}{2C^2} - \tfrac{(w_1^2+w_2^2-C^2)^2}{8w_1^2w_2^2C^2}\,. \tag{A.2.8}$$

Defining $N = \frac{2q_1(q_1+1)}{w_1}w_2 + \frac{2q_2(q_2+1)}{w_2}w_1 - (2q_1+1)(2q_2+1) \in \mathbb{Z}$, we get

$$N^2 = 1 - 2R^2C^2 < 1 \implies N = 0\,, \tag{A.2.9}$$

and then $R^2 = \frac{1}{2C^2}$. Replacing in (A.2.8), $C^2 = w_1^2 + w_2^2$, and then the radius where curve b with winding w_1 intersects curve b with winding w_2 is $R^{-2} = 2(w_1^2 + w_2^2)$.

The constraint

$$|(2q_1+1)w_2 - (2q_2+1)w_1| = \sqrt{w_1^2+w_2^2} \implies w_1^2+w_2^2 \text{ is a perfect square}\,. \tag{A.2.10}$$

If $w_1 = w_2 = w$ then $|2(q_1+q_2)| = \sqrt{2}w$. The LHS. is integer and the RHS is irrational, then there is no winding such that $b_{w,q_1}(R) = b_{w,q_2}(R)$.

A.2.3 $\mathbf{a_{w_1,q_1}(R) = b_{w_2,q_2}(R)}$

$$a_{w_1,q_1}(R) = \tfrac{2q_1\pm_1\sqrt{2-2w_1^2R^2}}{w_1}, \quad \tfrac{2q_1^2-1}{w_1} \in \mathbb{Z} \tag{A.2.11}$$

$$b_{w_2,q_2}(R) = \tfrac{2q_2+1\pm_2\sqrt{1-2w_2^2R^2}}{w_2}, \quad \tfrac{2q_2(q_2+1)}{w_2} \in \mathbb{Z} \tag{A.2.12}$$

$$\mp_1 w_2\sqrt{2-2w_1^2R^2} \pm_2 w_1\sqrt{1-2w_2^2R^2} = 2q_1w_2 - (2q_2+1)w_1 = C \in \mathbb{Z}\,. \tag{A.2.13}$$

Since w_1 is always odd, then C is also odd (in particular it is non-zero). Then

$$R^2 = \frac{1}{C^2} - \frac{(w_1^2+2w_2^2-C^2)^2}{8w_1^2w_2^2C^2} \quad \text{and} \quad N^2 = 2 - 2C^2R^2\,, \tag{A.2.14}$$

where $N = \frac{(1-2q_1^2)}{w_1}w_2 - \frac{2q_2(q_2+1)}{w_2}w_1 + q_1(2q_2+1) \in \mathbb{Z}$, and then $N = 0$ or 1, which give $R^2 = \frac{1}{C^2}$ or $R^2 = \frac{1}{2C^2}$. From (A.2.14) we obtain $C^2 = w_1^2 + 2w_2^2$ or $C^2 = (w_1 - w_2)^2 + w_2^2$. Then the radii where a curve a with w_1 intersects another curve b with w_2 are:

$$R^{-2} = w_1^2 + 2w_2^2 \quad \text{or} \quad R^{-2} = 2((w_1 - w_2)^2 + w_2^2) \tag{A.2.15}$$

For each case we have one of these constraints:

$$|2q_1w_2 - (2q_2+1)w_1| = \sqrt{w_1^2 + 2w_2^2} \quad \text{or } |2q_1w_2 - (2q_2+1)w_1| = \sqrt{(w_1 - w_2)^2 + w_2^2}$$

and then $w_1^2 + 2w_2^2$ or $(w_1 - w_2)^2 + w_2^2$ must be a perfect square. If $w_1 = w_2 = w$ we get the constraints:

$$|2q_1 - (2q_2+1)| = \sqrt{3} \quad \text{or} \quad |2q_1 - (2q_2+1)| = 1 \tag{A.2.16}$$

leaving only the second case, with $q_2 = q_1$ or $q_1 - 1$. The quantization conditions imply that w must be a divisor of both $2q_1^2 - 1$ and $2q_1(q_1 \pm 1)$. But it can be shown that these numbers are coprime, and then $w = 1$. The only curves with the same windings that intersect are $a_{1,q}(R)$ and $b_{1,q}(R)$ or $b_{1,q-1}(R)$. The intersections are at $R = \frac{1}{\sqrt{2}}$.

Summarizing, we have:

$$\begin{aligned}
a_{w_1,q_1} = a_{w_2,q_2} &\iff R^{-2} = w_1^2 + w_2^2 = C^2 \\
b_{w_1,q_1} = b_{w_2,q_2} &\iff R^{-2} = 2(w_1^2 + w_2^2) = 2C^2 \\
a_{w_1,q_1} = b_{w_2,q_2} &\iff R^{-2} = \begin{cases} w_1^2 + 2w_2^2 = C^2 \\ 2((w_1 - w_2)^2 + w_2^2) = 2C^2 \end{cases}
\end{aligned} \tag{A.2.17}$$

The winding numbers on b can in principle be any positive integer and those on a can only be the divisors of some number of the form $2q^2 - 1$, $q \in \mathbb{Z}$.

A.2.4 Enhancements to $SO(34)$ or $SO(18) \times E_8$

Here we prove that $a_{w_1,q_1}(R) = a_{w_2,q_2}(R)$ implies that there exist integers w_3, q_3, w_4 and q_4 such that $a_{w_1,q_1}(R) = b_{w_3,q_3}(R) = b_{w_4,q_4}(R)$.

We start with $R^{-2} = w_1^2 + w_2^2$. If $w_1 > w_2$, there are integers w_3 and w_4 such that $w_1 = w_3 + w_4$ and $w_2 = w_3 - w_4$, because w_1 and w_2 are odd numbers. Then

$$R^{-2} = w_1^2 + (2w_3 - w_1)^2 = 2(w_1^2 - 2w_3w_1 + w_3^2 + w_3^2) = 2((w_1 - w_3)^2 + w_3^2) \tag{A.2.18}$$

Since $R^{-2} = 2((w_1 - w_4)^2 + w_4^2)$ as well, there exist integers w_3, w_4, q_3 and q_4 such that $a_{w_1,q_1}(R) = b_{w_3,q_3}(R) = b_{w_4,q_4}(R)$. Note that we can always find q_3 and q_4 because the functions b admit any value of w.

Replacing $w_3 = \frac{1}{2}(w_1 + w_2)$ and $w_4 = \frac{1}{2}(w_1 - w_2)$ we get

$$a_{w_1,q_1}(R) = a_{w_2,q_2}(R) \implies a_{w_1,q_1}(R) = a_{w_2,q_2}(R) = b_{(w_1+w_2)/2,q_3}(R) = b_{(w_1-w_2)/2,q_4}(R)$$

Note that we can also write the radius as $2(w_3^2 + w_4^2)$. We want to satisfy

$$(\sqrt{2}R)^{-1} = |2q_1w_3 - (2q_3+1)w_1| = |2q_1w_4 - (2q_4+1)w_1| = |(2q_3+1)w_4 - (2q_4+1)w_3|\,,$$

and we have that

$$\begin{aligned}(\sqrt{2}R)^{-1} &= |q_1w_2 - q_2w_1| = |2q_1w_3 - (q_1+q_2)w_1| = |2q_1w_4 - (q_1-q_2)w_1| \\ &= |(q_1+q_2)w_4 - (q_1-q_2)w_3|\,.\end{aligned}$$

Then we need to identify $q_1+q_2 = 2q_3+1\,, q_1-q_2 = 2q_4+1$.

We still have to prove that $2q_3(q_3+1)$ and $2q_4(q_4+1)$ are divisible by w_3 and w_4, respectively, which amounts to proving that

$$\begin{aligned}&w_i \text{ is a divisor of } 2q_i^2-1 \text{ and } |q_1w_2 - q_2w_1| = \sqrt{\tfrac{w_1^2+w_2^2}{2}} \\ &\qquad\qquad \Longrightarrow\ w_1 \pm w_2 \text{ is a divisor of } (q_1\pm q_2)^2 - 1\end{aligned} \tag{A.2.19}$$

We checked that this is satisfied for the first 300 values of q_i.

Then we have that

$$a_{w_1,q_1}(R) \;=\; a_{w_2,q_2}(R) \implies b_{(w_1+w_2)/2,(q_1+q_2-1)/2}(R) = b_{(w_1-w_2)/2,(q_1-q_2-1)/2}(R)\,.$$

To prove that $b_{w_3,q_3}(R) = b_{w_4,q_4}(R)$ implies that there exists integers w_1, q_1, w_2 and q_2 such that $b_{w_3,q_3}(R) = a_{w_1,q_1}(R) = a_{w_2,q_2}(R)$, we start with $R^{-2} = 2(w_3^2+w_4^2)$. Define integers w_1 and w_2 such that $w_3 = \frac{1}{2}(w_1+w_2)$ and $w_4 = \frac{1}{2}(w_1-w_2)$ (we assume $w_3 > w_4$),

$$R^{-2} = 2((w_1-w_3)^2+w_3^2) \quad \text{and} \quad R^{-2} = 2((w_2-w_3)^2+w_3^2)\,.$$

But we still need to satisfy the constraint that w_1 and w_2 are divisors of $2q_1^2-1$ and $2q_2^2-1$ for two integers q_1 and q_2. With the identifications $q_1+q_2 = 2q_3+1\,, q_1-q_2 = 2q_4+1$, we get the correct radius

$$R^{-1} = \sqrt{2}|(2q_3+1)w_4 - (2q_4+1)w_3| = \sqrt{2}|2q_1w_3 - (2q_3+1)w_1|\,,$$

$$b_{w_3,q_3}(R) = b_{w_4,q_4}(R) \implies b_{w_3,q_3}(R) = b_{w_4,q_4}(R) = a_{w_3+w_4,q_3+q_4+1}(R) = a_{w_3-w_4,q_3-q_4}(R)\,.$$

We still have to prove that $2q_1^2-1$ and $2q_2^2-1$ are divisible by w_1 and w_2, respectively. This is the same as proving that

$$\begin{aligned}&q_i \text{ is a divisor of } 2q_i(q_i+1) \text{ and } |(2q_3+1)w_4 - (2q_4+1)w_3| = \sqrt{w_3^2+w_4^2} \\ &\qquad\qquad \Longrightarrow\ w_3 \pm w_4 \text{ is a divisor of } 2\left[(q_3+1/2)\pm(q_4+1/2)\right]^2 - 1\,,\end{aligned} \tag{A.2.20}$$

which we checked is satisfied.

In conclusion, we have that, for $R^{-2} = w_1^2+w_2^2$, $a_{w_1,q_1}(R) = a_{w_2,q_2}(R) \iff$

$$\begin{aligned}a_{w_1,q_1}(R) = a_{w_2,q_2}(R) &= b_{(w_1+w_2)/2,(q_1+q_2-1)/2}(R) = b_{(w_1-w_2)/2,(q_1-q_2-1)/2}(R) \\ &\iff b_{(w_1+w_2)/2,(q_1+q_2-1)/2}(R) = b_{(w_1-w_2)/2,(q_1-q_2-1)/2}(R)\,.\end{aligned}$$

The Wilson lines that give this enhancement can be written in four different ways

$$\frac{2q_1}{w_1} \pm_1 \sqrt{2}R\frac{w_2}{w_1} = \frac{2q_2}{w_2} \pm_2 \sqrt{2}R\frac{w_1}{w_2} = \frac{2q_3+1}{w_3} \pm_3 \sqrt{2}R\frac{w_4}{w_3} = \frac{2q_4+1}{w_4} \pm_4 \sqrt{2}R\frac{w_3}{w_4}$$

Using that $w_3 = \frac{w_1+w_2}{2}$, $w_4 = \frac{w_1-w_2}{2}$, $q_3 = \frac{q_1+q_2-1}{2}$ and $q_4 = \frac{q_1-q_2-1}{2}$, after a few steps, we get

$\mp_4 = \pm_3 = \pm_2 = \mp_1$ and then the Wilson lines are

$$\begin{aligned} A_1 = \frac{2q_1}{w_1} \pm \frac{w_2}{w_1}\sqrt{2}R, \quad A_1 = \frac{2q_2}{w_2} \mp \frac{w_1}{w_2}\sqrt{2}R, \\ A_1 = \frac{2q_3+1}{w_3} \mp \frac{w_4}{w_3}\sqrt{2}R, \quad A_1 = \frac{2q_4+1}{w_4} \pm \frac{w_3}{w_4}\sqrt{2}R \end{aligned} \tag{A.2.21}$$

From here,

$$(\sqrt{2}R)^{-1} = \mp(q_1 w_2 - q_2 w_1) \in \mathbb{Z} \tag{A.2.22}$$

and then, after a few steps, we can prove that

$$\frac{1}{\sqrt{2}R}, \quad \frac{\mathbb{A}}{\sqrt{2}}, \quad \frac{R}{\sqrt{2}}\left(\tfrac{1}{2}\mathbb{A}^2 + 1\right) \quad \in \mathbb{Z}\,, \tag{A.2.23}$$

Defining integers $m = (\sqrt{2}R)^{-1}$ and $n = \mathbb{A}/\sqrt{2}$, all this type of enhancement points are given by

$$(R, A_1) = \left(\frac{1}{m\sqrt{2}}, \frac{n}{m}\right) \quad \text{such that } \frac{n^2+1}{2m} \in \mathbb{Z} \tag{A.2.24}$$

and then

$$R^{-2} = 2, 50, 338, 578, 1250, 1682, 2738, 3362, 5618, 7442, 8450, 10658, \ldots \tag{A.2.25}$$

These are all of the form $2C^2$ with C an integer with prime divisors congruent to 1 mod 4. That is: $1, 5, 13, 17, 25, 29, 37, 41, 53, 61, 65, 73, 85, 89, 97, 101, 109, \ldots$. Except for the 1, these numbers are all Pythagorean primes or multiples of them.

We want to see if the b lines considered here can be interposed with a c line. q_3 and q_4 are suitable for curves b with w_3 and w_4. For curves c to coincide with them, we need w_i even and $\frac{q_i(q_i+1)}{w_i} \in \mathbb{Z}$. If one of the two curves b has also a curve c then we have an intersection between an a and a c curve. Analyzing all the possibilities, it can be shown that there are no c curves that intersect with more than one other curve.

A.2.5 Enhancements to $SU(2) \times SO(32)$ or $SU(2) \times E_8 \times E_8$

The equality $a_{w_1,q_1}(R) = b_{w_2,q_2}(R)$ arises for two type of radius

$$R^{-2} = w_1^2 + 2w_2^2 \quad \text{or} \quad R^{-2} = 2((w_1 - w_2)^2 + w_2^2)\,. \tag{A.2.26}$$

The second type gives $R^{-2} = w_1^2 + w_3^2$ if $w_2 = \frac{w_3+w_1}{2}$, which implies that there is an intersection with another curve a of winding w_3. Then, we restrict to the first type, where R^{-2} is odd for odd w_1^2. Thus the even R^{-2} found in the previous section cannot have additional curves a or b on the intersection.

For $R^{-2} = w_1^2 + 2w_2^2$, the constraints are

$$|2q_1 w_2 - (2q_2+1)w_1| = \sqrt{w_1^2 + 2w_2^2}\,, \quad \frac{2q_1^2-1}{w_1} \in \mathbb{Z}\,, \quad \frac{2q_2(q_2+1)}{w_2} \in \mathbb{Z} \tag{A.2.27}$$

The Wilson line can be written as

$$A_1 = \frac{2q_1 \pm_1 2Rw_2}{w_1} \quad \text{or} \quad A_1 = \frac{2q_2+1 \pm_2 Rw_1}{w_2}\,, \tag{A.2.28}$$

and equating them leads to $\pm_2 = \mp_1$ and

$$R^{-1} = \mp \left(2q_1 w_2 - (2q_2 + 1)w_1\right) , \tag{A.2.29}$$

implying that R^{-1} is an odd number. After some algebra, we get

$$\frac{1}{R}, \quad \mathbb{A}, \quad R\left(\tfrac{1}{2}\mathbb{A}^2 + 1\right) \quad \in \mathbb{Z} , \tag{A.2.30}$$

and then all this type of enhancement points satisfy

$$(R, A_1) = \left(\frac{1}{m}, \frac{2n}{m}\right) \tag{A.2.31}$$

for integer $m = R^{-1}$ and $n = \frac{R^{-1}A_1}{2}$, such that

$$\frac{2n^2 + 1}{m} \in \mathbb{Z}. \tag{A.2.32}$$

We obtain

$$R^{-1} = 3, 9, 11, 17, 19, 27, 33, 41, 43, 51, 57, 59, \ldots \tag{A.2.33}$$

all integer numbers with prime divisors congruent to 1 or 3 (mod 8).

It is not hard to prove that all the curves b that intersect just one curve a are superimposed by a curve c (in the $\Gamma_8 \oplus \Gamma_8$ case).

A.3 Other slices of moduli space

Here we analyze two-dimensional slices of moduli space given by the radius and one parameter in the Wilson lines. First we consider the HO theory compactified with Wilson lines of the form $A^I = ((A)_p, 0_{16-p})$. We then show how the Generalized Dynkin Diagrams give us the points of enhancement located in the fundamental region (in the conventions of Section 2.4).

A.3.1 Slices for the HO theory

The results are summarized in the following figures, after which we present the calculations leading to them.

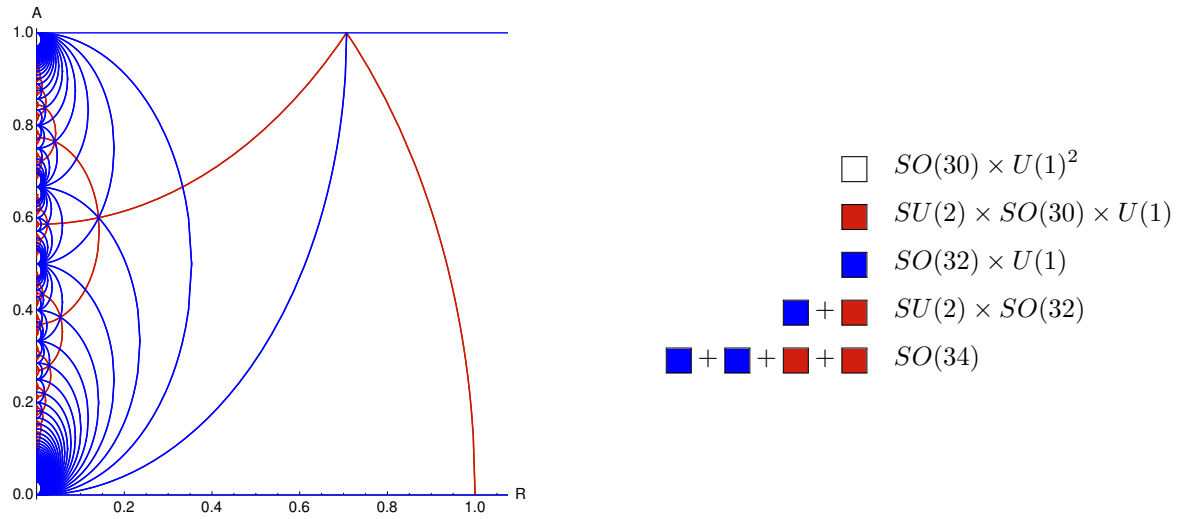


Figure A.1: HO with Wilson line $A^I = (A, 0_{15})$

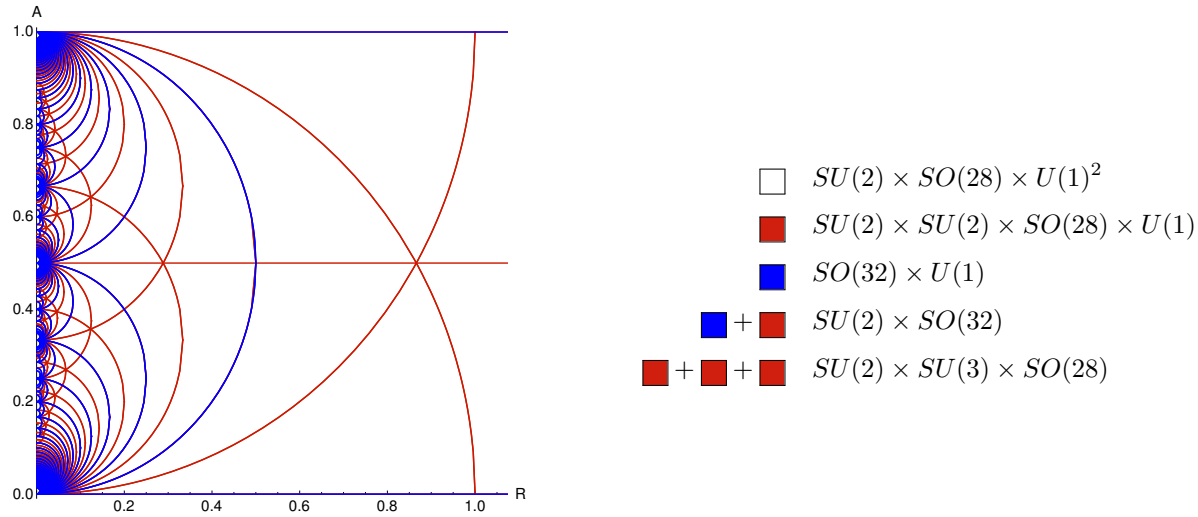


Figure A.2: HO with Wilson line $A^I = ((A)_2, 0_{14})$

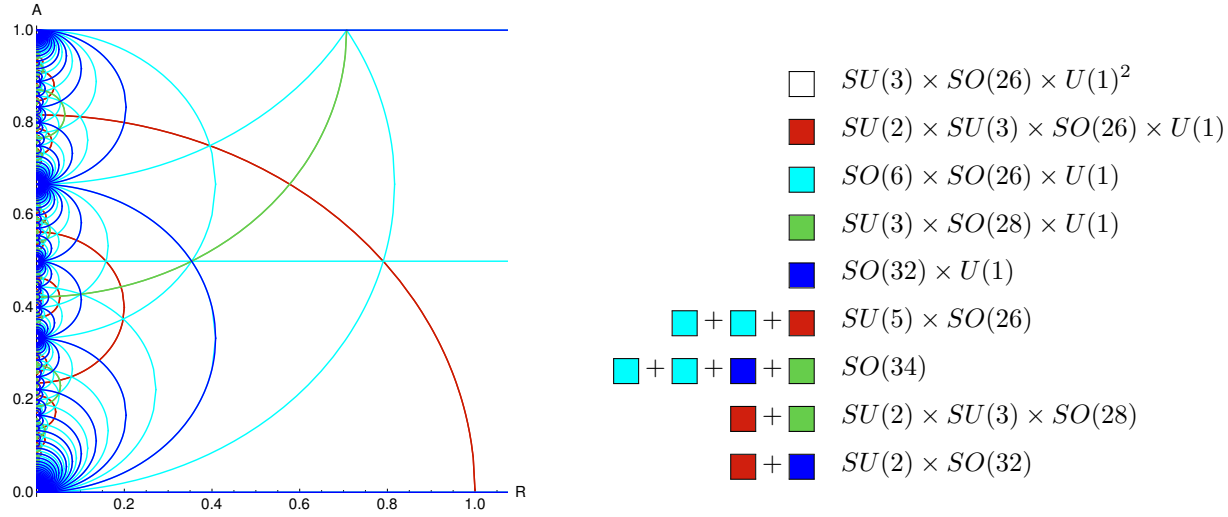


Figure A.3: HO with Wilson line $A^I = ((A)_3, 0_{13})$

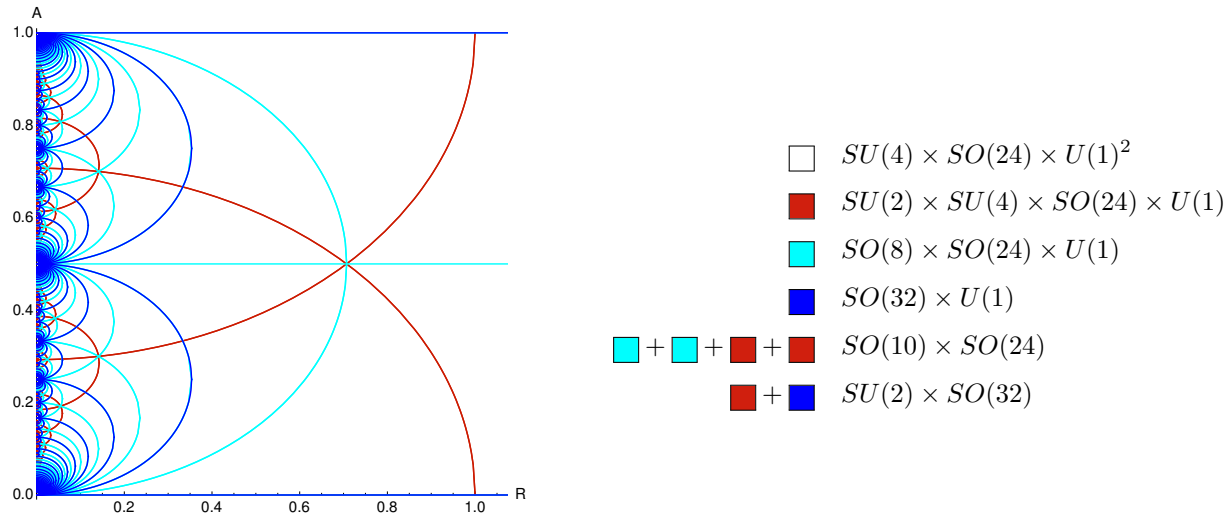


Figure A.4: HO with Wilson line $A^I = ((A)_4, 0_{12})$

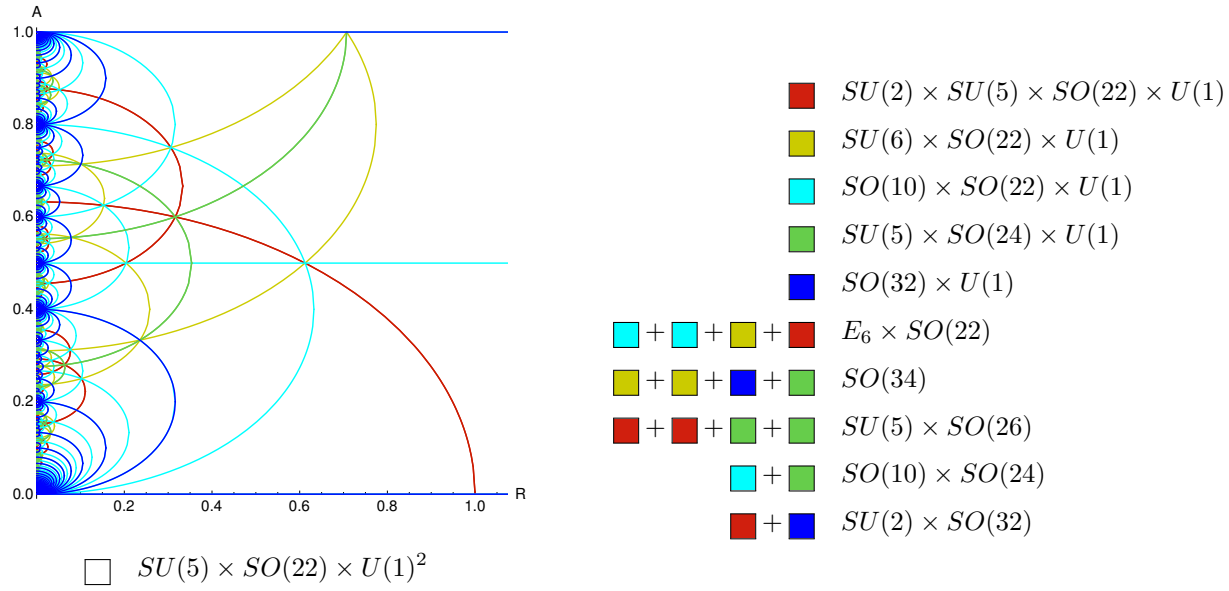


Figure A.5: HO with Wilson line $A^I = ((A)_5, 0_{11})$

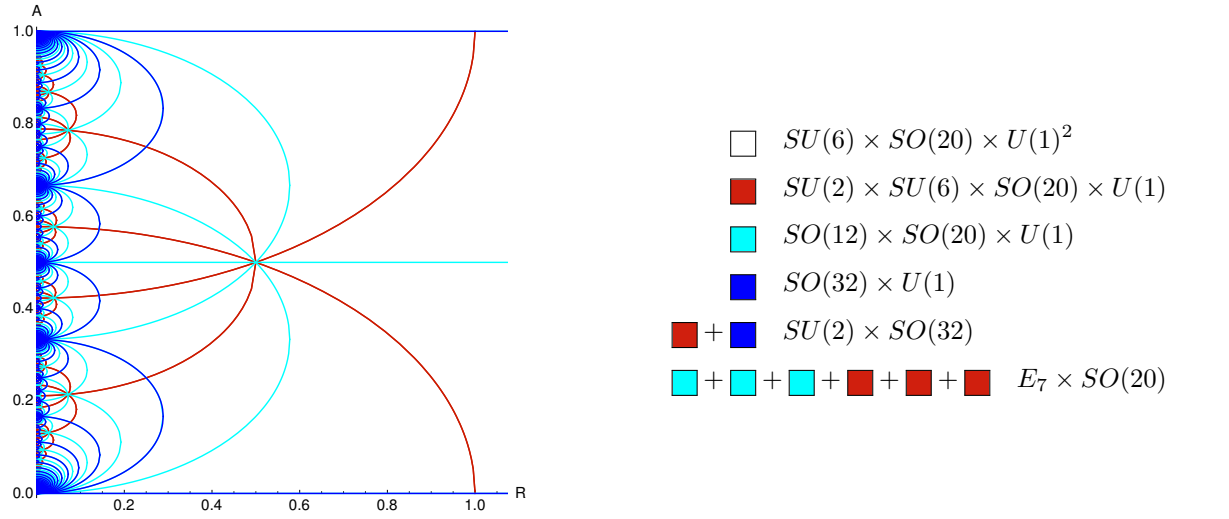


Figure A.6: HO with Wilson line $A^I = ((A)_6, 0_{10})$

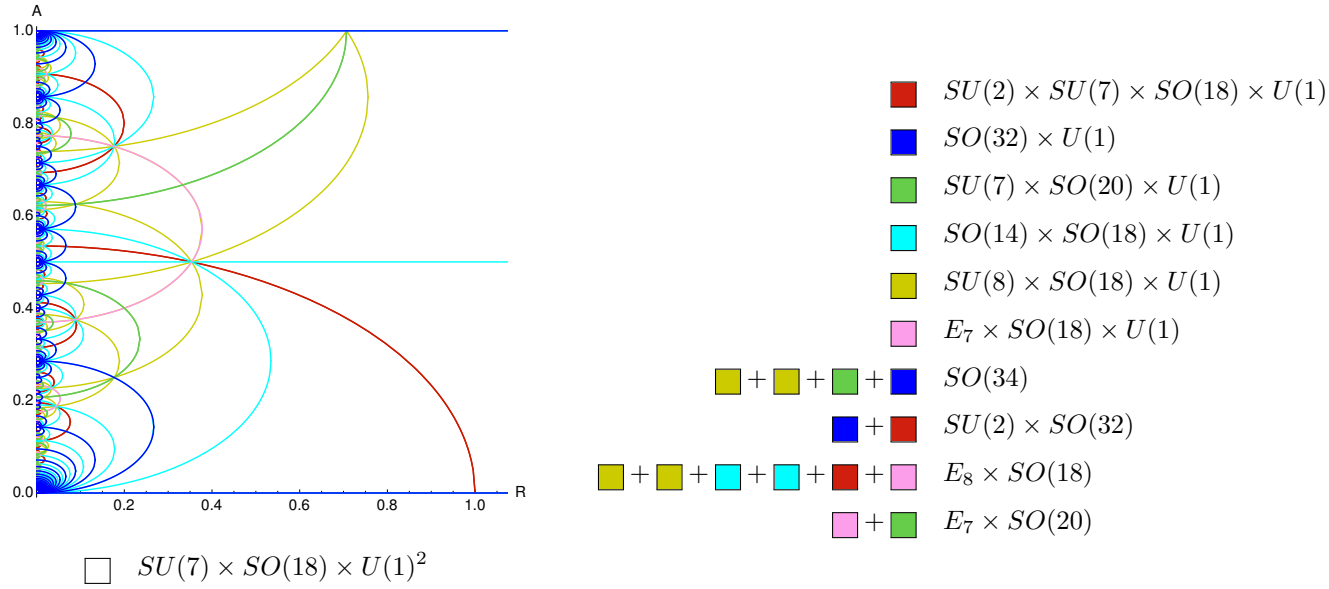


Figure A.7: HO with Wilson line $A^I = ((A)_7, 0_9)$

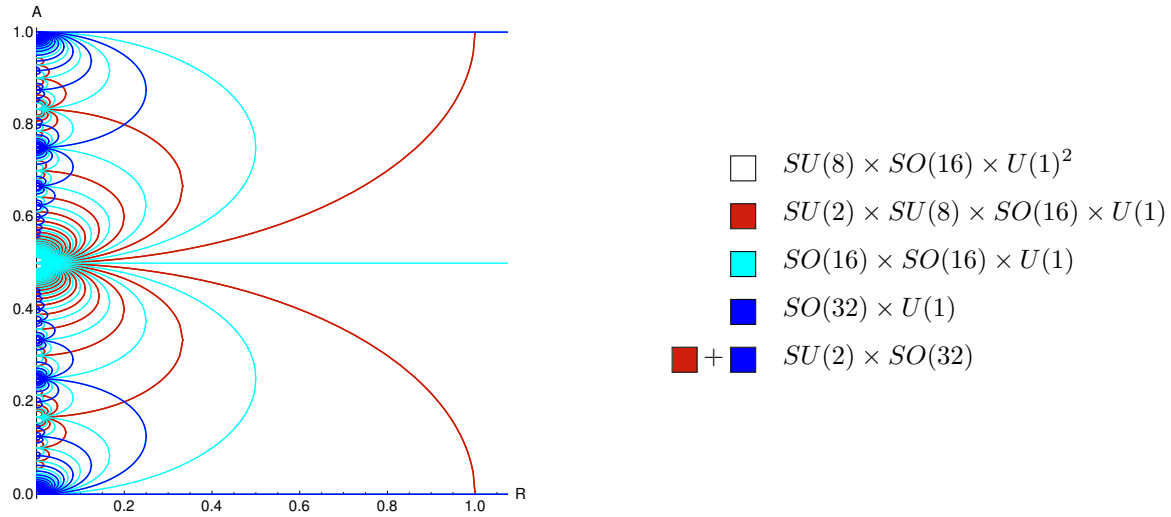


Figure A.8: HO with Wilson line $A^I = ((A)_8, 0_8)$

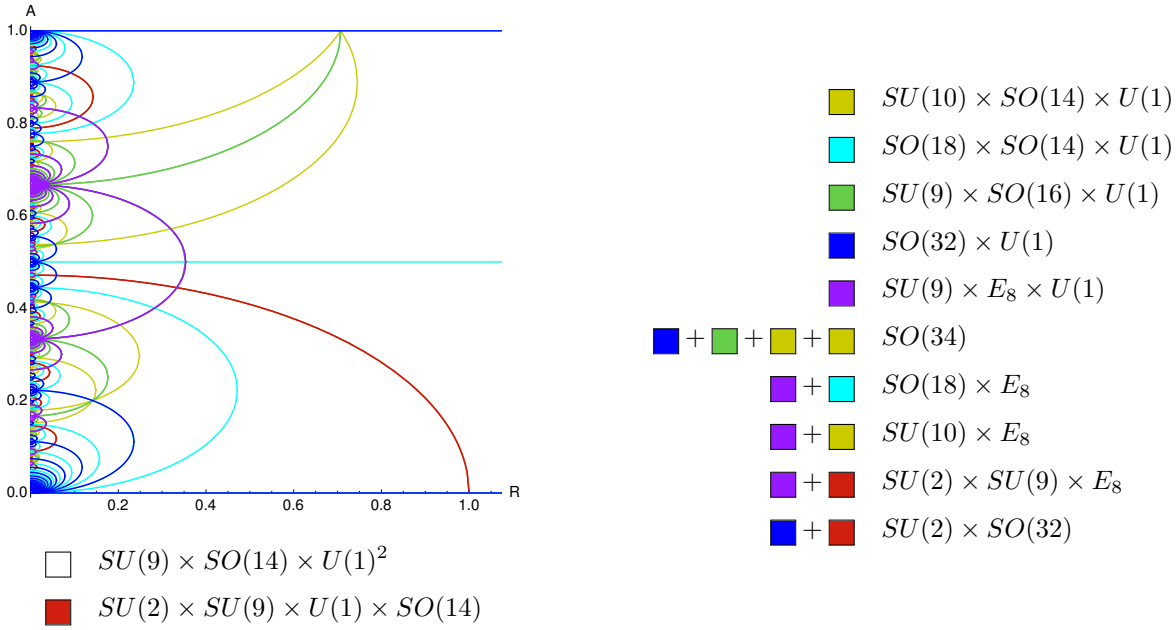


Figure A.9: HO with Wilson line $A^I = ((A)_9, 0_7)$

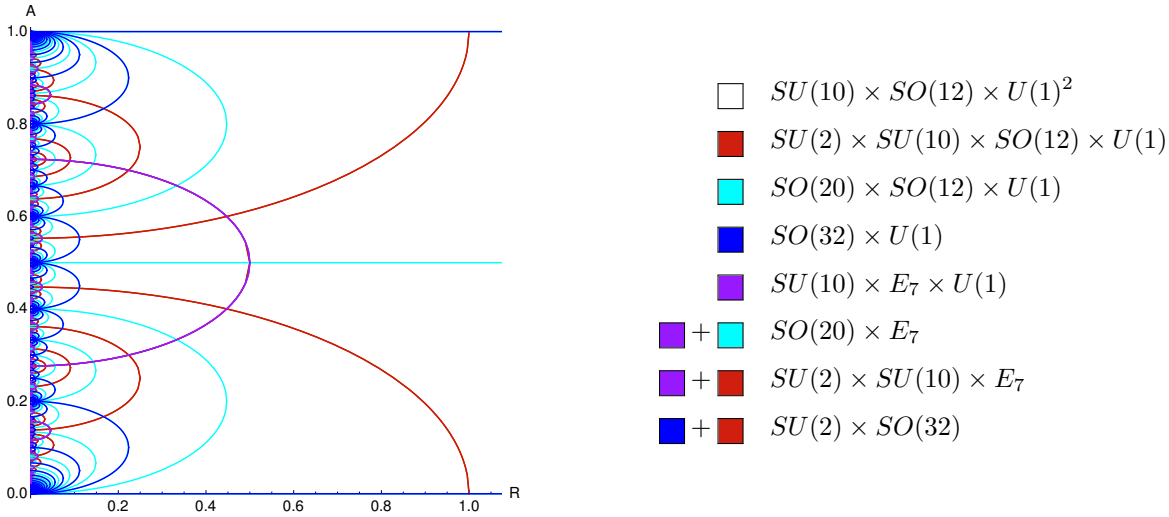


Figure A.10: HO with Wilson line $A^I = ((A)_{10}, 0_6)$

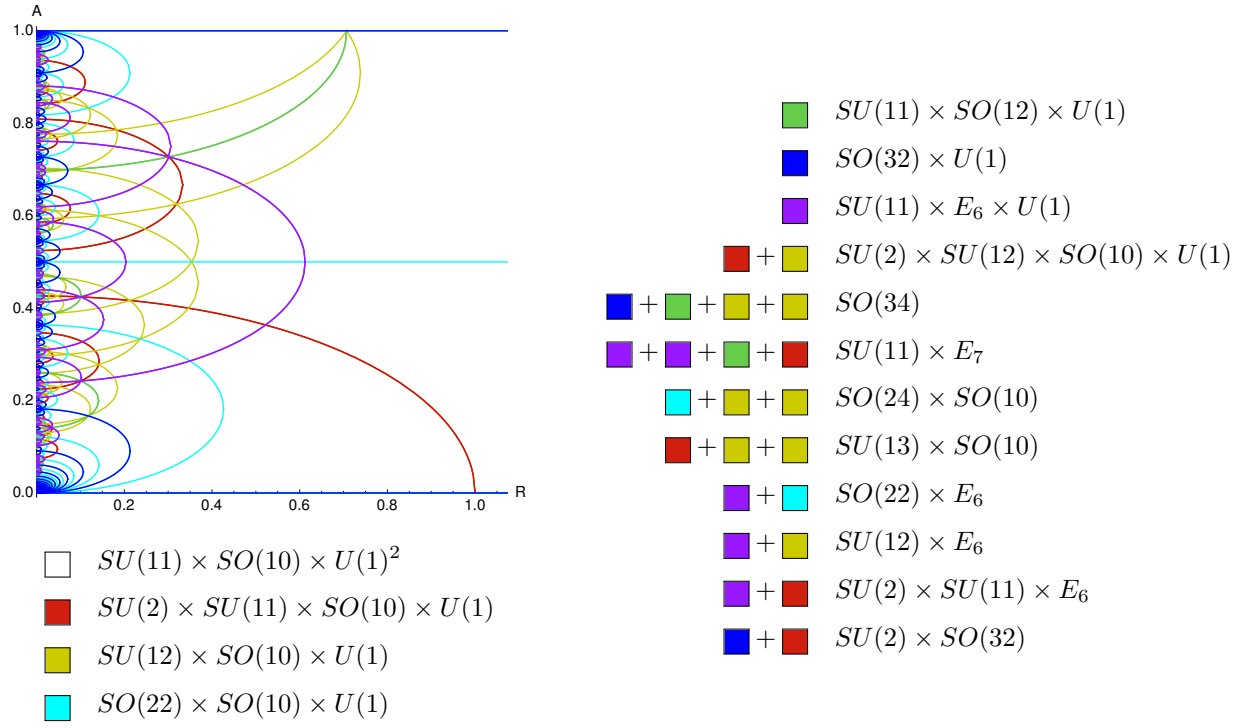


Figure A.11: HO with Wilson line $A^I = ((A)_{11}, 0_5)$

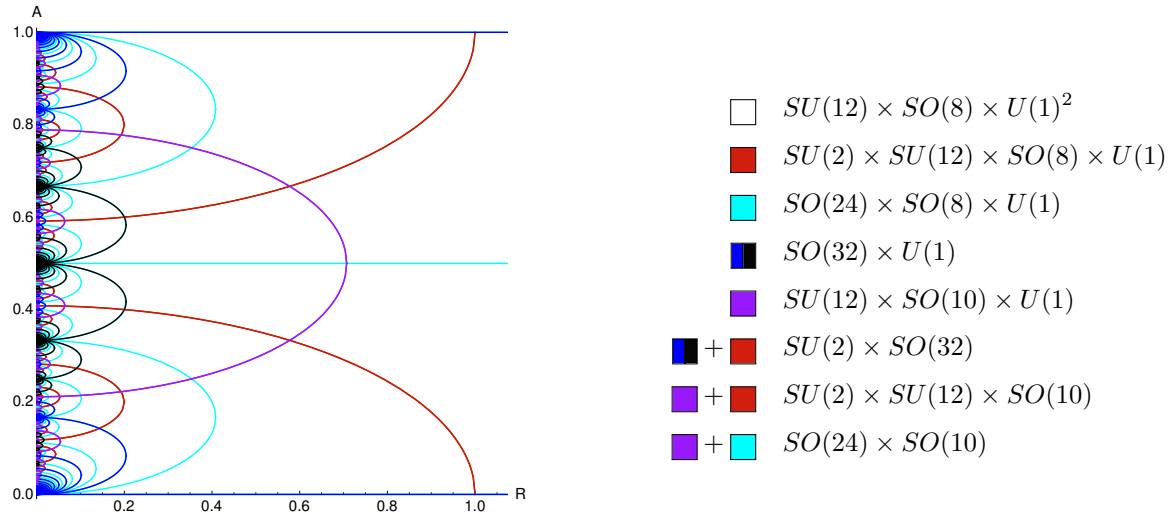
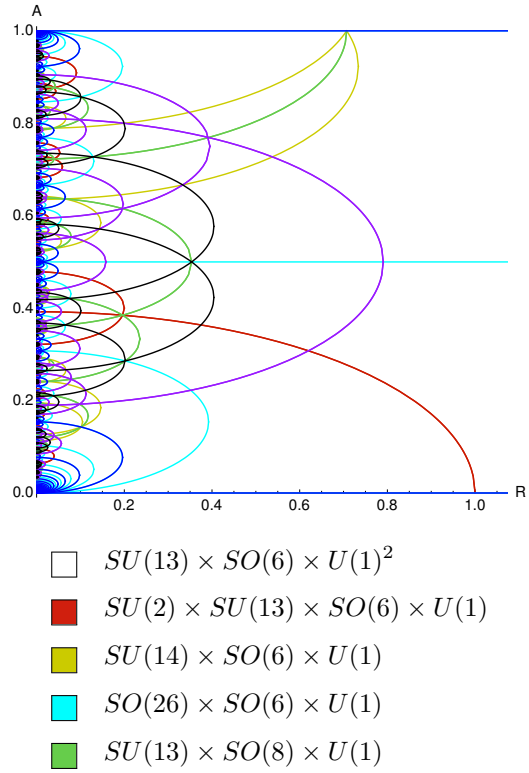


Figure A.12: HO with Wilson line $A^I = ((A)_{12}, 0_4)$



$SU(13) \times SO(6) \times U(1)^2$

$SU(2) \times SU(13) \times SO(6) \times U(1)$

$SU(14) \times SO(6) \times U(1)$

$SO(26) \times SO(6) \times U(1)$

$SU(13) \times SO(8) \times U(1)$

$SO(32) \times U(1)$

$SU(13) \times SU(5) \times U(1)$

$SU(17) \times U(1)$

$SO(34)$

$SU(13) \times SO(10)$

$SO(28) \times SO(6)$

$SU(2) \times SU(13) \times SU(5)$

$SU(13) \times SO(10)$

$SU(2) \times SO(32)$

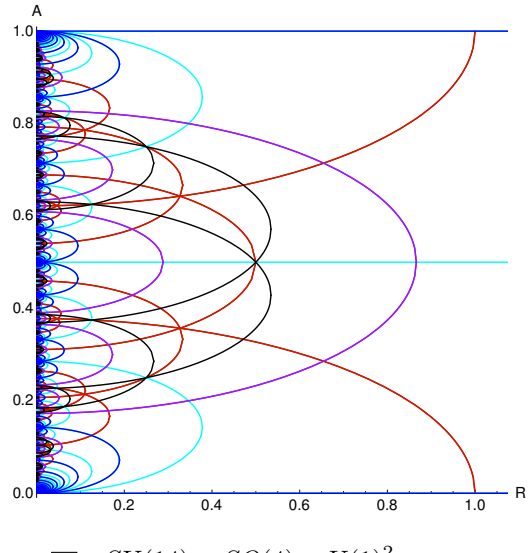
$SU(2) \times SU(2) \times SU(17)$

$SO(26) \times SU(5)$

$SU(18)$

$SU(14) \times SU(5)$

Figure A.13: HO with Wilson line $A^I = ((A)_{13}, 0_3)$



$SU(14) \times SO(4) \times U(1)^2$

$SU(2) \times SU(14) \times SO(4) \times U(1)$

$SO(28) \times SO(4) \times U(1)$

$SO(32) \times U(1)$

$SU(14) \times SU(3) \times SU(2) \times U(1)$

$SU(16) \times SU(2) \times U(1)$

$SU(2) \times SO(32)$

$SU(3) \times SU(14) \times SO(4)$

$SO(28) \times SU(3) \times SU(2)$

$SU(2) \times SU(14) \times SU(3) \times SO(4)$

$SU(2) \times SO(32)$

$SU(2) \times SU(16) \times SO(4)$

Figure A.14: HO with Wilson line $A^I = ((A)_{14}, 0_2)$

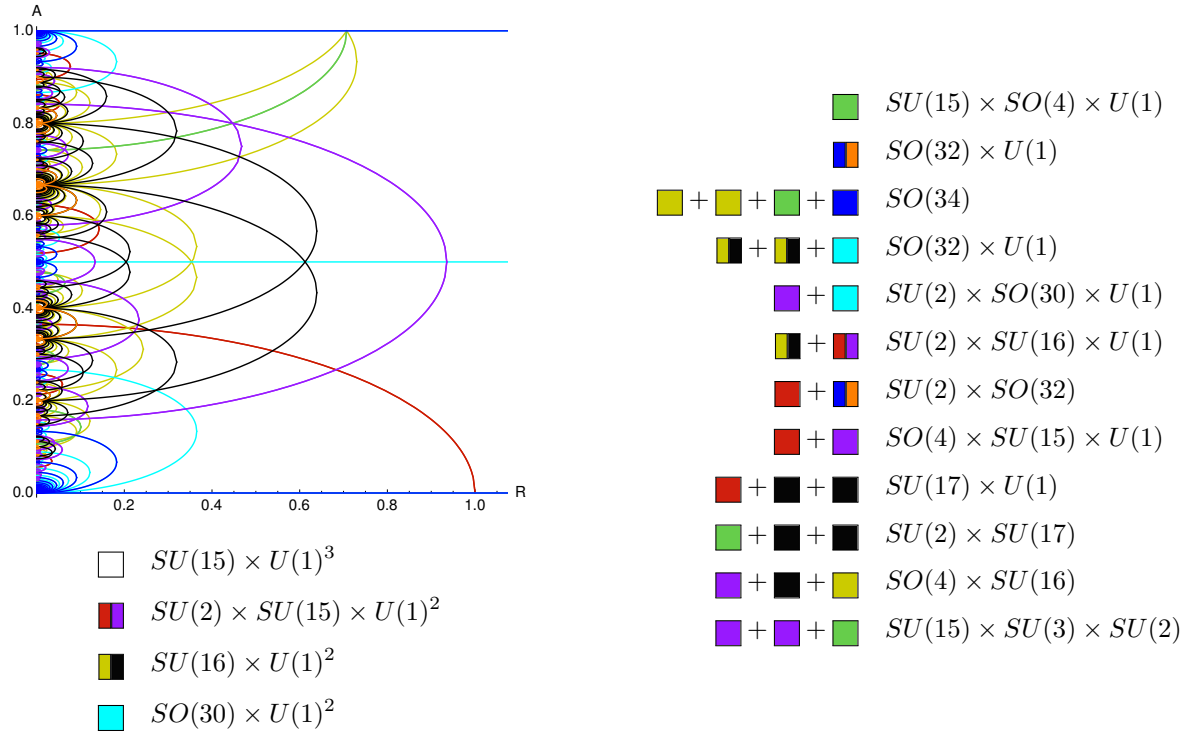


Figure A.15: HO with Wilson line $A^I = ((A)_{15}, 0)$

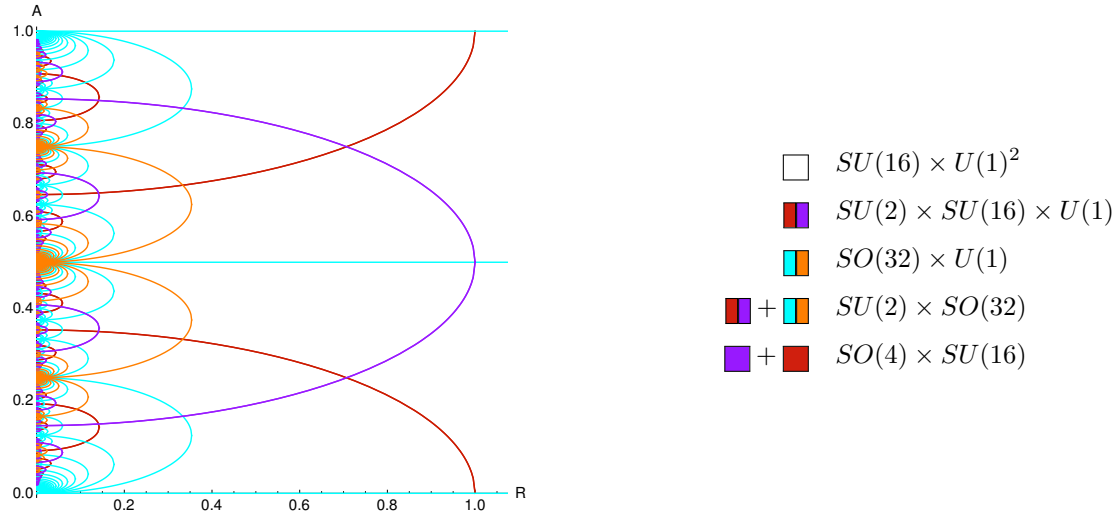


Figure A.16: HO with Wilson line $A^I = ((A)_{16})$

For Wilson lines of type $A^I = \left((A)_p, 0_{16-p}\right)$ there are families of curves of enhancement parameterized by three integer numbers α, β and δ. Inside each family there are different curves corresponding to different winding numbers and different integer values for q. If R is sufficiently small then w can be arbitrarily large.

$$\begin{aligned} A_{w,\alpha,\beta,\delta}(R) &= \frac{pq + \alpha - \frac{p\delta}{2} \pm \sqrt{\left(\alpha - \frac{p\delta}{2}\right)^2 - p\left(|\alpha| - \delta\alpha + \beta + 4\delta - 2 + 2w^2R^2\right)}}{pw} \\ &= \frac{pq + \mu \pm \sqrt{\mu^2 - p\left(\lambda + 2w^2R^2\right)}}{pw}, \end{aligned} \tag{A.3.1}$$

where we defined:

$$\mu \;=\; \alpha - \frac{p\delta}{2} \qquad \text{and} \qquad \lambda \;=\; |\alpha| - \delta\alpha + \beta + 4\delta - 2\,.$$

The massless states associated with each family of curves are

$$\pi = \Big(\underline{(q \pm (1 - \tfrac{3}{2}\delta))_\alpha, (q \pm \tfrac{1}{2}\delta)_{p-\alpha}}, \underline{\beta \pm \tfrac{1}{2}\delta, (\pm\tfrac{1}{2}\delta)_{15-p}} \Big)\,. \tag{A.3.2}$$

The possible values of the parameters are listed in the following table, with the color we use to identify them on the figures and the corresponding gauge group.

Colour	δ	β	$\lvert\alpha\rvert$	Gauge group
				$A_{p-1} \times D_{16-p}$
	0	0	0	$A_1 \times A_{p-1} \times D_{16-p}$
	0	0	1	$A_p \times D_{16-p}$
	0	0	2	$D_p \times D_{16-p}$
	0	0	3	$E_p \times D_{16-p}$
	0	1	0	$A_{p-1} \times D_{17-p}$
	0	1	1	D_{16}
	1	0	0	$A_{p-1} \times E_{17-p}$
	1	0	1	D_{16} for $p = 12$, A_{16} for $p = 13$, $A_{15} \times A_1$ for $p = 14$, $A_{15} \times D_1$ for $p = 15$
	1	0	2	D_{16}

The number of states for each of these curves is given by

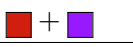
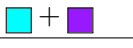
$$2\binom{p}{|\alpha|}(32 - 2p)^\beta 2^{(15-p+\delta_{p,16})\delta}\,. \tag{A.3.3}$$

The allowed values for q and w are the ones that satisfy the quantization condition

$$\frac{pq^2 + 2\mu q + \lambda}{2w} \in \mathbb{Z}\,. \tag{A.3.4}$$

For arbitrary A, we get the $3p^2 - 63p + 480$ roots of $U(1)^2 \times SU(p) \times SO(32-2p)$. If A is half-integer we get the $4p^2 - 64p + 480$ roots of $U(1) \times SO(2p) \times SO(32 - 2p)$, so we can think of them as part of the family with $(\delta, \beta, \alpha) = (0, 0, 2)$ and $w = 0$ which give $p^2 - p$ additional states. For $p = 2$ $(0, 0, 2)$ is equivalent to $(0, 0, 0)$.

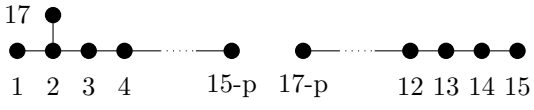
If A_1 is integer we get the 480 roots of $SO(32) \times U(1)$, so we can think of them as part of the family with $(\delta, \beta, \alpha) = (0, 1, 1)$ and $w = 0$ superimposed with another one of the family with $(\delta, \beta, \alpha) = (0, 0, 2)$ and $w = 0$, which give $63p - 3p^2 = (64p - 4p^2) + (p^2 - p)$ additional states. For $p = 16$ we only have the $(0, 0, 2)$. We can classify some of the enhancements by the colors of the curves that intersect, we list them on the table below:

Colours	Gauge group
	$A_1 \times D_{16}$
	$A_1 \times A_{p-1} \times E_{17-p}$
	$E_{p+1} \times D_{16-p}$
	$A_p \times E_{17-p}$
	$A_2 \times A_{p-1} \times D_{16-p}$

A.3.2 Relation to Generalized Dynkin Diagrams

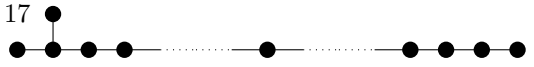
Here we show how some of the previous enhancement curves and points can be obtained from the Generalized Dynkin Diagram in (2.4.3).

For Wilson lines of the form $(0_{16-p}, (A)_p)$ and at any radius, then the inequality $-A_2 \leq A_1$, as well as all the $A_i \leq A_{i+1}$ inequalities are saturated except for $A_{16-p} = A_{17-p}$. This means that the gauge group is given by the generalized diagram with all the nodes except for 16, 18, 19 and $16-p$. Then the diagram that gives the enhancement symmetry is:

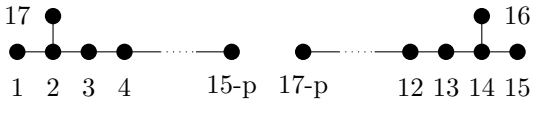


(A.3.5)

which corresponds to the $A_{p-1} \times D_{16-p}(\times U(1)^2) = SU(p) \times SO(32-2p)(\times U(1)^2)$ at a generic value of A and R. Choosing particular values for them, we can saturate one or more inequalities associated to the missing nodes. To obtain the horizontal lines we have to pick an arbitrary R, which discards the nodes 18 and 19. To get the nodes $16-p$ or 16 we have only one possibility: $A = 0$ for the former, and $A = \frac{1}{2}$ for the latter. We get, respectively:

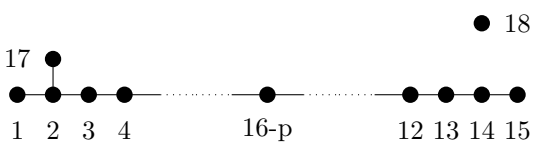


(A.3.6)



(A.3.7)

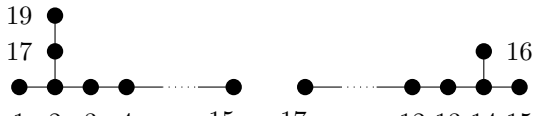
and hence the gauge groups are $D_{16} = SO(32)$ $(\times U(1))$ and $D_p \times D_{16-p} = SO(2p) \times SO(32-2p)$ $(\times U(1))$ (blue and cyan lines). Finally, choosing a specific value of R, the inequality associated to the 18th or 19th node (not both at the same time) can be saturated. This gives maximal enhancements. In the D_{16} case, the only possibility is to add the 18th node, which gives $A_1 \times D_{16}$ (intersection between a blue and a red curve):



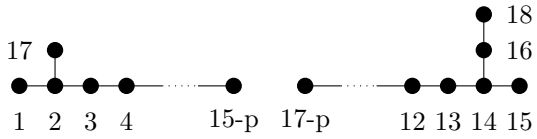
(A.3.8)

In the $D_p \times D_{16-p}$ case, one can add the 18th or the 19th node, depending on which part of the

diagram has less than 8 nodes



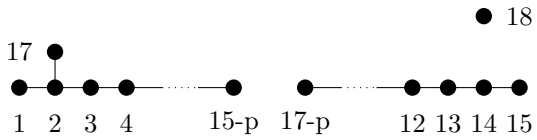
(A.3.9)



(A.3.10)

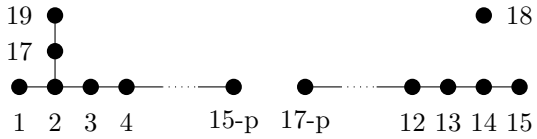
This accounts for $D_p \times E_{17-p}$ (intersection between a cyan and other curves) and $E_{p+1} \times D_{16-p}$ (intersection between a cyan and a purple curve).

For $R(A)$ (with arbitrary A) saturating the inequality associated to the 18th node, we obtain $A_1 \times A_{p-1} \times D_{16-p}$ (red curves):



(A.3.11)

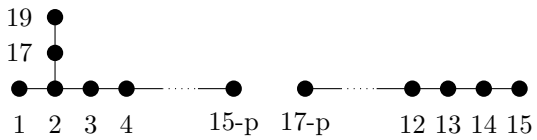
And in particular for $A = \frac{4}{p}$, we have:



(A.3.12)

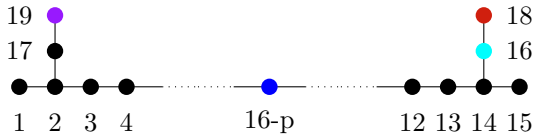
which gives the gauge group $A_1 \times E_{17-p} \times A_{p-1}$, considered in Section 2.3.3.4 and seen in the figures at the intersections between the red and purple curves.

On the other hand, choosing $R(A)$ so that it saturates the inequality associated to the 19th node, we obtain $E_{17-p} \times A_{p-1}$ (purple curves):



(A.3.13)

Then we can color the dots on the Generalized Dynkin Diagram depending on which curves saturate their inequality:

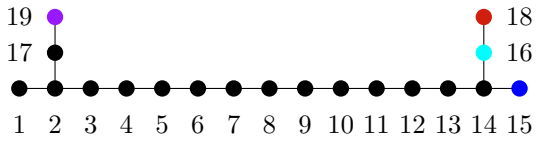


(A.3.14)

The enhancements corresponding to each curve are obtained by removing all the colored nodes except the node with that color. The intersections of curves give the group associated to the diagram obtained

by keeping the nodes with the colors of the involved curves.

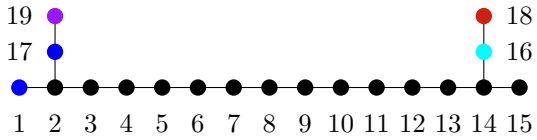
Something odd happens for $p = 1$



(A.3.15)

For generic A and R, this is D_{15}. For $A = 1$ (cyan dot) we get D_{16} and if we also take $R^2 = \frac{1}{2}$ (red dot) we get D_{17}. If, on the other hand, we take $A = 0$ (blue dot) then we get D_{16} and if we also select $R^2 = 1$ (red dot) we get $A_1 \times D_{16}$. If we only take the appropriate R to have the red dot, then we get $A_1 \times D_{15}$. To compare with figure A.1 we have to take into account that the cyan solutions are not well defined for $p < 2$, and then we see them as blue curves.

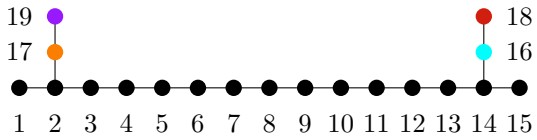
For $p = 15$, the equation for the seventeenth node no longer holds, and then we have:



(A.3.16)

For generic A and R this is A_{14}. Selecting a specific R, we can turn on the red and/or the purple nodes to get $A_1 \times A_{14}$ or $D_2 \times A_{14}$. Selecting $A = \frac{1}{2}$ (cyan dot) we obtain D_{15} and for $A = 0$ both blue dots are turned on and we get D_{16}. Only choosing $R = 1$ (red dot) we get $A_1 \times D_{16}$.

For $p = 16$ we have a very different situation:



(A.3.17)

For generic A and R this is A_{15}. Selecting a specific R, the red and/or the purple nodes are turned on and we get $A_1 \times A_{15}$ or $D_2 \times A_{15}$. Selecting $A = \frac{1}{2}$ (cyan dot) we obtain D_{16} and for $A = 0$ (orange dot) we get D_{16}. Only choosing $R = 1$ (red or purple dot) we get $A_1 \times D_{16}$.

The enhancements of the curves that correspond to the other colors cannot be obtained with this construction. On one hand we see from the figures that the Wilson lines that give these curves are not in the fundamental region in the conventions of Section 2.4. On the other hand, if this region is the fundamental region, it should contain all the possible enhancement groups, and as such all the curves with the different colors. However, it is easy to see that using this method, the Wilson lines in the fundamental region that give the missing enhancement groups are not of the form chosen, with p equal components and the other zero. For example, to obtain the enhancement $A_p \times D_{16-p}$ corresponding to the yellow curves, we would need to replace the 15th node with the 16th one (and then add the 18th one), which requires $A_{16} = 1 - A_{15}$ which is not within the ansatz chosen for the Wilson lines.

Appendix B

Supplements to Chapter 3

B.1 Notation and basics concerning lattices

L, even positive definite lattice of rank r

Typically L will be the sum of ADE root lattices. There is a basis formed by roots α_i with $\alpha_i^2 = 2$. The Gram matrix of L has elements $\alpha_i \cdot \alpha_j$. It is equal to the Cartan matrix when L is the root lattice of an ADE group.

$d(L)$, discriminant of L

It is defined to be the determinant of the Gram matrix of L. By assumption $d(L) \neq 0$.

L^*, dual lattice

Lattice generated by the weights w_i defined by $w_i \cdot \alpha_j = \delta_{ij}$. Clearly $L \subset L^*$.

A_L, discriminant group

It is defined as $A_L = L^*/L$, also named D_L or G_L in the literature.

It can be shown that A_L is a finite Abelian group of order $d(L)$.

Since E_8 is unimodular, its discriminant group is trivial. For $L = \mathrm{A}_n, \mathrm{D}_{2m+1}, \mathrm{D}_{2m}, \mathrm{E}_6, \mathrm{E}_7$, $A_L \cong \mathbb{Z}_{n+1}, \mathbb{Z}_4, \mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_2$.

$\ell(A_L)$, minimal number of generators of A_L

For example, for $L = 2\mathrm{E}_6 + \mathrm{A}_6$, $\ell(A_L) = 2$, because $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_7 \sim \mathbb{Z}_3 \times \mathbb{Z}_{21}$. Notice that $\ell(A_L) \leq r$.

q_L, discriminant quadratic form

It is a map $q_L : A_L \to \mathbb{Q}/2\mathbb{Z},\ x + L \mapsto x^2 \bmod 2$.

For example for $L = \mathrm{A}_n$, $A_L = \mathbb{Z}_{n+1}$ is generated by the class of the fundamental weight $[w_1]$. Thus $q_L([w_1]) = w_1^2 = \frac{n}{n+1}$, whereas $q_L([w_j]) = w_j^2 = \frac{j(n+1-j)}{n+1} = \frac{j^2 n}{n+1}$, with equalities mod 2.

For $L = \mathrm{D}_{2m+1}$, $A_L = \mathbb{Z}_4$ is generated by the spinor class $[s]$ with $q_L([s]) = \frac{2m+1}{4}$.

For D_{2m}, $A_L = \mathbb{Z}_2 \times \mathbb{Z}_2$. One $\mathbb{Z}_2$ is generated by the spinor class $[s]$ with $q_L([s]) = \frac{m}{2}$, and the other $\mathbb{Z}_2$ by the vector class $[v]$ with $q_L([v]) = 1$.

For E_6, $A_L = \mathbb{Z}_3$ is generated by the fundamental weights of $[\mathbf{27}]$ with $q_L([\mathbf{27}]) = \frac{4}{3}$.

For E_7, $A_L = \mathbb{Z}_2$ is generated by the fundamental weights of $[\mathbf{56}]$ with $q_L([\mathbf{56}]) = \frac{3}{2}$.

T, even positive definite lattice of rank d

It is characterized by the Gram matrix $(Q)_{ij} = u_i \cdot u_j$, where u_i are the basis vectors.

A generic even 1 dimensional lattice, denoted $\mathrm{A}_1\langle m \rangle$, is a multiple by m of the A_1 lattice. It is generated by a vector u_1 with $u_1^2 = 2m$ and has discriminant group $\mathbb{Z}_{2m}$, in turn generated by $(u_1^*)^2 = \frac{1}{2m}$.

We will mostly consider $d = 2$ and as in [14], represent Q as $[u_1^2, u_1 \cdot u_2, u_2^2]$. For classification of even 2-dimensional lattices see chapter 15 in [49], and section 2 in [14] for a short account. Q can

be brought to Smith normal form $\mathrm{diag}(s_1, s_2)$, with positive integer entries. Then $A_T \cong \mathbb{Z}_{s_1} \times \mathbb{Z}_{s_2}$. Notice that if s_1 and s_2 are coprimes then $A_T \cong \mathbb{Z}_{s_1 s_2}$. We will also need to compute the discriminant form q_T. From Q^{-1} we can read off $u_i^* \cdot u_j^*$, where u_1^*, u_2^* are the basis vectors of the dual lattice T^*. Besides, Q^{-1} gives the e_i^* in terms of e_i. With this data we can then find the generators of A_T and derive q_T. For example, for T with $Q = [2, 1, 4]$, $A_T \cong \mathbb{Z}_7$ and $Q^{-1} = [\frac{4}{7}, -\frac{1}{7}, \frac{2}{7}]$. The generator of A_T can be taken to be u_2^* which satisfies $7u_2^* = -u_1 + 2u_2 \in T$, and has the lowest norm. Then q_T takes values $\frac{2j^2}{7} \bmod 2$, $j = 0, \ldots, 6$.

H_L, isotropic subgroup of A_L

$H_L \subset A_L$ is isotropic if $q_L\big|_H = 0$.

For instance, for $L = \mathrm{A}_8$, with $A_L = \mathbb{Z}_9$, the subgroup $H_L = \mathbb{Z}_3$ generated by $w_3 \sim 3w_1$ is isotropic because $q_L([w_3]) = \frac{18}{9} = 2 = 0 \bmod 2$.

Another example is $L = \mathrm{D}_8$, with $A_L = \mathbb{Z}_2 \times \mathbb{Z}_2$. Now there is an isotropic $H_L = \mathbb{Z}_2$ generated by the spinor class with $s^2 = \frac{8}{4} = 2 = 0 \bmod 2$.

An important example is $L = \mathrm{D}_{16}$ which has an isotropic group $H_L = \mathbb{Z}_2$ generated by the spinor weight with $s^2 = \frac{16}{4} = 4 = 0 \bmod 2$.

Orthogonal complement

Given a sublattice S of Γ, $S \subset \Gamma$, the orthogonal complement of S in Γ is defined to be the set $S^\perp = \{x \in \Gamma \,|\, x.y = 0 \ \forall y \in S\}$.

M, overlattice of L

If $L \subset M$ and the index $[M : L]$ is finite then M is an overlattice of L. This means that M and L have the same rank. In fact, $[M : L]^2 = d(L)/d(M)$. The index is also denoted by $|M/L|$.

The important Proposition 1.4.1 of Nikulin states that the set of even overlattices of L corresponds bijectively with the set of isotropic subgroups of A_L [16]. The overlattice corresponding to H_L can be constructed as $M_H = \{x \in L^* | [x \bmod L] \in H_L\}$. (see e.g. proposition α in [44]). This means that the elements of M_H are weights that can be written as roots plus generators in H_L. Besides, the discriminant form q_{M_H} is given by the discriminant form q_L restricted to $H_L^\perp/H_L$. Orthogonality is defined with respect to the bilinear quadratic form b_L [44]. In practice, $y \in H_L^\perp$ if $y \in A_L$ and $y \cdot x =$ integer for all $x \in H_L$. To avoid cluttering we will drop the subscript in M_H when H_L has been specified.

As an example, take $L = \mathrm{A}_8$ and $H_L = \mathbb{Z}_3$ so that $M/L \cong \mathbb{Z}_3$ and $d(M) = \frac{9}{3^2} = 1$. Then M has elements $x = y + nw_3$, with $y \in L$ and $n = 0, 1, 2$. It can be shown that this M is isomorphic to E_8, which is the unique rank 8 even unimodular lattice.

For $L = \mathrm{D}_8$ the overlattice associated to $H_L = \mathbb{Z}_2$ has elements $x = y + ns$, with $y \in L$ and $n = 0, 1$. This is nothing but E_8, as expected since the overlattice has $d(M) = \frac{4}{2^2} = 1$.

For $L = \mathrm{D}_{16}$ the overlattice corresponding to $H_L = \mathbb{Z}_2$ is the even unimodular lattice Γ_{16} with elements $x = y + ns$, with $y \in L$ and $n = 0, 1$. Unimodularity follows from $M/L \cong \mathbb{Z}_2$ implying $d(M) = \frac{4}{2^2} = 1$. Γ_{16} is the HO lattice.

M_{root}, root sublattice of M

It is the sublattice of M generated by roots, i.e. by vectors of norm 2.

For example, for the overlattice of $L = \mathrm{D}_{16}$, $M_{\mathrm{root}} = L$. For $L = \mathrm{D}_8$ this is not the case because the overlattice E_8 has many more roots. This reflects the fact that for D_8 the additional element s in the overlattice has $s^2 = 2$.

Primitive embedding

A lattice S is primitively embedded in another lattice Γ if $S \subset \Gamma$ and Γ/S is torsion-free.

For example, $\mathrm{A}_8 \subset \mathrm{E}_8$ but the embedding is not primitive because $\mathrm{E}_8/\mathrm{A}_8 \cong \mathbb{Z}_3$ as explained above. An example of primitive embedding is $\mathrm{A}_3 \subset \mathrm{E}_8$. Since A_3 has rank 3 and E_8 is even unimodular, this follows from Theorem 1.12.4 of Nikulin [16] quoted below. It can then be shown that $\mathrm{D}_5 \subset \mathrm{E}_8$ is primitive because D_5 is the orthogonal complement of A_3 in E_8, and also that E_8 is an overlattice of $\mathrm{D}_5 + \mathrm{A}_3$.

Nikulin's Theorem 1.12.4 [16]

Every even lattice of signature $(t_{(-)}, t_{(+)})$ admits a primitive embedding in an even unimodular lattice of signature $(l_{(-)}, l_{(+)})$, with $l_{(+)} - l_{(-)} \equiv 0 \bmod 8$, if

$$t_{(+)} \leq l_{(+)}, \qquad t_{(-)} \leq l_{(-)}, \qquad t_{(+)} + t_{(-)} \leq \frac{1}{2}(l_{(+)} + l_{(-)})\,. \tag{B.1.1}$$

In particular, if $r \leq (8+d)$ then L of signature $(0,r)$ admits a primitive embedding in $\Gamma_{d,d+16}$.

B.2 Complements to Section 3.2

In this appendix we present some additional material for the discussion of the lattice embedding formalism.

B.2.1 Embeddings of groups with rank $r < d + 16$

The problem is now to embed L of signature $(0,r)$, $r < d+16$, in the even unimodular Narain lattice $\Gamma_{d,d+16}$. In this case there are also three criteria that read

Criterion 1, from Corollary 1.12.3 [16]
If $\ell(A_L) < 16 + 2d - r$ then L has an embedding in $\Gamma_{d,d+16}$.

Criterion 2, from Theorem 1.12.2(c) [16]
L has a primitive embedding in $\Gamma_{d,d+16}$ if and only if there exists a lattice T of signature $(d, d+16-r)$ such that (A_T, q_T) is isomorphic to (A_L, q_L).

Criterion 3, from Theorem 7.1 [15]
L has an embedding in $\Gamma_{d,d+16}$ if and only if L has an overlattice M with the following properties:

(i) *there exists an even lattice T of signature $(d, d+16-r)$ such that (A_T, q_T) is isomorphic to (A_M, q_M),*

(ii) *the sublattice M_{root} of M coincides with L.*

Recall that Theorem 1.12.4 [16] further implies that when $r \leq (8+d)$ there is always a primitive embedding. The above criteria clearly reduce to those in Section 3.2.1 setting $r = d+16$. The lattice T now has indefinite signature so the application would be more complicated.

B.2.2 More on the complementary lattice T of signature $(0, d)$

In Section 3.2.2 we have argued that $T = K\langle -1\rangle$. To complete the proof that $(A_M, q_M) \cong (A_K, -q_K)$ we can use the following theorem of [49]: Let L_1 and L_2 be two sublattices of a unimodular lattice L_3 such that[1]

$$L_1 \oplus L_2 \subset L_3, \qquad L_1 = (L_1 \otimes \mathbb{R}) \cap L_3, \qquad L_2 = (L_2 \otimes \mathbb{R}) \cap L_3.$$

Then the discriminant groups L_1^*/L_1 and L_2^*/L_2 are isomorphic. The isomorphism is given by $y_1 + L_1 \to y_2 + L_2$, where $y_1 \in L_1^*/L_1$ and $y_2 \in L_2^*/L_2$, whenever $y = y_1 + y_2$ generates an isotropic subgroup of $L_1 \oplus L_2$.

To apply this theorem to our problem we take $L_1 = M$, $L_2 = K$, and $L_3 = \Gamma_{d,d+16}$, with K and M given in (3.2.3) and (3.2.8). We have $M \otimes \mathbb{R} = \mathbb{R}^{0,d+16}$ and $K \otimes \mathbb{R} = \mathbb{R}^{d,0}$. Moreover, $\mathbb{R}^{0,d+16} \cap \Gamma_{d,d+16} = M$ and $\mathbb{R}^{d,0} \cap \Gamma_{d,d+16} = K$. It follows that M and K have isomorphic discriminant groups. It remains to show that they have isomorphic discriminant forms. The Narain lattice $\Gamma_{d,d+16}$ is generated by the lattice sum $M \oplus K$ together with some isotropic vectors (glue vectors in the language of [49]). These vectors are generically of the form $y = y_1 + y_2$, where y_1 and y_2 are non-trivial vectors in the discriminant groups of M and K, respectively, and are connected by the discriminant group isomorphism. Since y must be even, we have $y^2 = 0 \mod 2$. Therefore, $y_1^2 + y_2^2 = 0 \mod 2$, because M and K are orthogonal. We thus find $y_1^2 = -y_2^2 \mod 2$. This shows that $q_M \cong -q_K$, and so T as defined is the complementary lattice of M.

[1] $L \otimes \mathbb{R}$ means the set of all points obtained by real linear combinations of the basis vectors of L.

Appendix C

Supplements to Chapter 5

Examples of freezings

Here we give some examples of freezings of gauge groups in 6d heterotic strings. For simplicity we use the A-to-G notation for gauge *groups*. Whenever the length of an A factor is not 2, superscript indicates half its length.

a) $2\mathrm{A}_1 + \mathrm{A}_{11} + \mathrm{D}_7$ with $H = \mathbb{Z}_4$ generated by $k = (0, 0, 3, 3)$, with center $(2, 2, 12, 4)$ can be frozen to

k_Ω	Singularity	L	H	k	Center
(0,0,6,2)	$8\mathrm{A}_1/\mathbb{Z}_2$	$2\mathrm{A}_1 + \mathrm{C}_5 + \mathrm{A}_5^2$	$\mathbb{Z}_2$	(0,0,1,3)	(2,2,2,6)
(0,0,9,1)	$(2\mathrm{A}_1 + 4\mathrm{A}_3)/\mathbb{Z}_4$	$2\mathrm{A}_1 + \mathrm{C}_2^2 + \mathrm{A}_2^4$	1		(2,2,2,3)

b) $\mathrm{A}_3 + \mathrm{A}_{11} + \mathrm{E}_6$ with $H = \mathbb{Z}_6$ generated by $k = (2, 2, 2)$, with center $(4, 12, 3)$ can be frozen to

k_Ω	Singularity	L	H	k	Center
(2,6,0)	$8\mathrm{A}_1/\mathbb{Z}_2$	$\mathrm{E}_6 + \mathrm{A}_1^2 + \mathrm{A}_5^2$	$\mathbb{Z}_3$	(1,0,2)	(3,2,6)
(0,4,1)	$6\mathrm{A}_2/\mathbb{Z}_3$	$\mathrm{A}_3 + \mathrm{G}_2 + \mathrm{A}_3^3$	$\mathbb{Z}_2$	(2,0,2)	(4,1,4)
(2,2,2)	$(2\mathrm{A}_1 + 2\mathrm{A}_2 + 2\mathrm{A}_5)/\mathbb{Z}_6$	$\mathrm{G}_2 + \mathrm{A}_1^2 + \mathrm{A}_1^6$	1		(1,2,2)

c) $3\mathrm{A}_1 + 2\mathrm{A}_4 + \mathrm{A}_9$ with $H = \mathbb{Z}_{10}$ generated by $k = (1, 1, 1, 4, 4, 1)$, with center $(2, 2, 2, 5, 5, 10)$ can be frozen to

k_Ω	Singularity	L	H	k	Center
(1,1,1,0,0,5)	$8\mathrm{A}_1/\mathbb{Z}_2$	$2\mathrm{A}_4 + \mathrm{A}_4^2$	$\mathbb{Z}_5$	(4,4,1)	(5,5,5)
(0,0,0,3,3,2)	$4\mathrm{A}_4/\mathbb{Z}_5$	$3\mathrm{A}_1 + \mathrm{A}_1^5$	$\mathbb{Z}_2$	(1,1,1,1)	(2,2,2,2)

d) $2\mathrm{A}_2 + 2\mathrm{A}_5 + \mathrm{E}_6$ with $H = \mathbb{Z}_3^2$ generated by $k_1 = (0, 0, 2, 4, 1)$ and $k_2 = (1, 2, 0, 2, 1)$, with center $(3, 3, 6, 6, 3)$ can be frozen to

k_Ω	Singularity	L	H	k	Center
(0,0,2,4,1)	$6\mathrm{A}_2/\mathbb{Z}_3$	$2\mathrm{A}_2 + \mathrm{G}_2 + 2\mathrm{A}_1^3$	$\mathbb{Z}_3$	(1,1,0,0,0)	(3,3,1,2,2)
(2,1,2,2,0)	$6\mathrm{A}_2/\mathbb{Z}_3$	$\mathrm{E}_6 + 2\mathrm{A}_1^3$	$\mathbb{Z}_3$	(1,0,0)	(3,2,2)
(1,2,0,2,1)	$6\mathrm{A}_2/\mathbb{Z}_3$	$\mathrm{A}_5 + \mathrm{G}_2 + \mathrm{A}_1^3$	$\mathbb{Z}_3$	(2,0,0)	(6,1,2)

e) $3A_1 + D_{10} + E_7$ with $H = \mathbb{Z}_2^2$ generated by $k_1 = (0,0,0,1,0,1)$ and $k_2 = (1,1,1,0,1,0)$, with center $(2,2,2,(2,2),2)$ can be frozen to

k_Ω	Singularity	L	H	k	Center
(1,1,1,1,1,1)	$8A_1/\mathbb{Z}_2$	$C_8 + F_4$	$\mathbb{Z}_2$	(1,0)	(2,1)
(1,1,1,0,1,0)	$8A_1/\mathbb{Z}_2$	$B_5 + E_7$	$\mathbb{Z}_2$	(1,1)	(2,2)
(0,0,0,1,0,1)	$8A_1/\mathbb{Z}_2$	$3A_1 + B_5 + F_4$	$\mathbb{Z}_2$	(1,1,1,1,0)	(2,2,2,2,1)

f) $2A_1 + 2A_2 + A_3 + A_{11}$ with $H = \mathbb{Z}_{12}$ generated by $k = (1,1,2,2,1,1)$, with center $(2,2,3,3,4,12)$ can be frozen to

k_Ω	Singularity	L	H	k	Center
(0,0,0,0,2,6)	$8A_1/\mathbb{Z}_2$	$2A_1 + 2A_2 + A_1^2 + A_5^2$	$\mathbb{Z}_6$	(1,1,2,2,1,1)	(2,2,3,3,2,6)
(0,0,2,2,0,4)	$6A_2/\mathbb{Z}_3$	$2A_1 + A_3 + A_3^3$	$\mathbb{Z}_4$	(1,1,1,1)	(2,2,4,4)
(1,1,0,0,3,3)	$(2A_1 + 4A_3)/\mathbb{Z}_4$	$2A_2 + A_2^4$	$\mathbb{Z}_3$	(2,2,1)	(3,3,3)
(0,0,1,1,2,2)	$(2A_1 + 2A_2 + 2A_5)/\mathbb{Z}_6$	$2A_1 + A_1^2 + A_1^6$	$\mathbb{Z}_2$	(1,1,1,1)	(2,2,2,2)

g) $A_1 + 3A_5 + D_4$ with $H = \mathbb{Z}_2 \times \mathbb{Z}_6$ generated by $k_1 = (0,0,3,3,(1,1))$ and $k_2 = (0,1,1,2,(0,1)))$, with center $(2,6,6,6,(2,2))$ can be frozen to

k_Ω	Singularity	L	H	k	Center
(0,0,3,3,(1,1))	$8A_1/\mathbb{Z}_2$	$A_1 + A_5 + C_2 + 2A_2^2$	$\mathbb{Z}_6$	(0,5,1,1,2)	(2,6,2,3,3)
(0,2,2,4,(0,0))	$6A_2/\mathbb{Z}_3$	$A_1 + D_4 + 3A_1^3$	$\mathbb{Z}_2^2$	$(0,(1,0),0,1,1)$ $(0,(0,1),1,0,1)$	(2,(2,2),2,2,2)
(0,1,1,2,(0,1))	$(2A_1 + 2A_2 + 2A_5)/\mathbb{Z}_6$	$A_1 + C_2 + A_1^3$	$\mathbb{Z}_2$	(0,1,1)	(2,2,2)

h) $4A_1 + A_2 + 2A_7$ with $H = \mathbb{Z}_2 \times \mathbb{Z}_4$ generated by $k_1 = (0,0,1,1,0,2,2)$ and $k_2 = (1,1,1,1,0,0,4)$, with center $(2,2,2,2,3,8,8)$ can be frozen to

k_Ω	Singularity	L	H	k	Center
(1,1,1,1,0,0,4)	$8A_1/\mathbb{Z}_2$	$A_2 + A_7 + A_3^2$	$\mathbb{Z}_4$	(0,2,2)	(3,8,4)
(0,0,0,0,0,4,4)	$8A_1/\mathbb{Z}_2$	$4A_1 + A_2 + 2A_3^2$	$\mathbb{Z}_2^2$	$(0,1,0,1,0,2,2)$ $(1,0,1,0,0,2,2)$	(2,2,2,2,3,4,4)
(0,0,1,1,0,2,2)	$(2A_1 + 4A_3)/\mathbb{Z}_4$	$2A_1 + A_2 + 2A_1^4$	$\mathbb{Z}_2$	(1,1,0,0,0)	(2,2,3,2,2)

i) $4A_1 + 2A_3 + 2D_5$ with $H = \mathbb{Z}_2 \times \mathbb{Z}_4$ generated by $k_1 = (0,0,0,0,1,3,1,3)$ and $k_2 = (1,1,1,1,0,0,2,2)$, with center $(2,2,2,2,4,4,4,4)$ can be frozen to

k_Ω	Singularity	L	H	k	Center
(1,1,1,1,2,2,0,0)	$8A_1/\mathbb{Z}_2$	$2D_5 + 2A_1^2$	$\mathbb{Z}_4$	(1,3,1,1)	(4,4,2,2)
(1,1,1,1,0,0,2,2)	$8A_1/\mathbb{Z}_2$	$2A_3 + 2C_3$	$\mathbb{Z}_4$	(1,3,1,1)	(4,4,2,2)
(0,0,0,0,2,2,2,2)	$8A_1/\mathbb{Z}_2$	$4A_1 + 2C_3 + 2A_1^2$	$\mathbb{Z}_2^2$	$(1,1,1,1,0,0,0,0)$ $(0,0,0,0,1,1,1,1)$	(2,2,2,2,2,2,2,2)
(0,0,0,0,1,3,1,3)	$(2A_1 + 4A_3)/\mathbb{Z}_4$	$4A_1 + 2A_1^4$	$\mathbb{Z}_2$	(1,1,1,1,0,0)	(2,2,2,2,2,2)

j) $5\mathrm{A}_1 + \mathrm{D}_4 + \mathrm{D}_5 + \mathrm{D}_6$ with $H = \mathbb{Z}_2^3$ generated by $k_1 = (0,0,0,0,1,(0,1),2,(0,1))$, $k_2 = (0,1,1,1,0,(0,0),2,(1,0)))$ and $k_3 = (1,0,0,1,0,(1,1),2,(1,1)))$, with center $(2,2,2,2,2,(2,2),4,(2,2))$ can be frozen to

k_Ω	Singularity	L	H	k	Center
(0,0,0,0,1,(0,1),2,(0,1))	$8\mathrm{A}_1/\mathbb{Z}_2$	$4\mathrm{A}_1 + \mathrm{B}_3 + \mathrm{C}_2 + \mathrm{C}_3$	$\mathbb{Z}_2^2$	$(0,1,1,1,0,1,0)$ $(1,0,0,1,1,1,0)$	(2,2,2,2,2,2)
(0,1,1,1,0,(0,0),2,(1,0))	$8\mathrm{A}_1/\mathbb{Z}_2$	$2\mathrm{A}_1 + \mathrm{B}_3 + \mathrm{C}_3 + \mathrm{D}_4$	$\mathbb{Z}_2^2$	$(0,1,1,0,(0,1))$ $(1,0,1,0,(1,1))$	(2,2,2,2,(2,2))
(0,1,1,1,1,(0,1),0,(1,1))	$8\mathrm{A}_1/\mathbb{Z}_2$	$\mathrm{A}_1 + \mathrm{C}_2 + \mathrm{C}_4 + \mathrm{D}_5$	$\mathbb{Z}_2^2$	$(0,0,1,2)$ $(1,1,0,2)$	(2,2,2,4)
(1,0,0,1,0,(1,1),2,(1,1))	$8\mathrm{A}_1/\mathbb{Z}_2$	$3\mathrm{A}_1 + \mathrm{C}_2 + \mathrm{C}_3 + \mathrm{C}_4$	$\mathbb{Z}_2^2$	$(0,1,1,0,0,1)$ $(1,0,0,1,0,1)$	(2,2,2,2,2,2)
(1,0,0,1,1,(1,0),0,(1,0))	$8\mathrm{A}_1/\mathbb{Z}_2$	$2\mathrm{A}_1 + \mathrm{B}_3 + \mathrm{C}_2 + \mathrm{D}_5$	$\mathbb{Z}_2^2$	$(0,0,1,1,2)$ $(1,1,0,0,2)$	(2,2,2,2,4)
(1,1,1,0,0,(1,1),0,(0,1))	$8\mathrm{A}_1/\mathbb{Z}_2$	$2\mathrm{A}_1 + \mathrm{B}_3 + \mathrm{C}_2 + \mathrm{D}_5$	$\mathbb{Z}_2^2$	$(1,0,1,0,2)$ $(0,1,0,1,2)$	(2,2,2,2,4)
(1,1,1,0,1,(1,0),2,(0,0))	$8\mathrm{A}_1/\mathbb{Z}_2$	$\mathrm{A}_1 + \mathrm{C}_2 + \mathrm{C}_3 + \mathrm{D}_6$	$\mathbb{Z}_2^2$	$(0,1,0,(1,0))$ $(1,0,0,(0,1))$	(2,2,2,(2,2))

Appendix D

Tables of gauge groups

D.1 Groups of maximal enhancement in 9d and 8d

In this appendix we present the Tables containing all the groups of maximal enhancement in one and two dimensions. The list of groups realized in S^1 compactifications of the heterotic string is displayed in Table D.1. The groups realized in T^2 compactifications of the $\mathrm{E}_8 \times \mathrm{E}_8'$ heterotic string are shown in Table D.2. To simplify notation we dropped the primes in the E_8' weights. In Table D.3 we give the realization of some of these groups in the $\mathrm{Spin}(32)/\mathbb{Z}_2$ theory. Tables D.4, D.5 and D.6 contain the groups for the CHL string in 9d and 8d.

#	L	A_L	H_L	T	R^2_{E}	A_{E}	R^2_{O}	A_{O}
1	$2\mathrm{E}_8 + \mathrm{A}_1$	$\mathbb{Z}_2$	$\mathbb{1}$	$\langle 1\rangle$	1	0×0	$\frac{1}{16}$	$\frac{1}{4}(\mathrm{w}_7 + \mathrm{w}_9)$
2	$\mathrm{E}_8 + \mathrm{E}_7 + \mathrm{A}_2$	$\mathbb{Z}_6$	$\mathbb{1}$	$\langle 3\rangle$	$\frac{3}{4}$	$\frac{1}{2}w_6 \times 0$	$\frac{1}{12}$	$\frac{1}{6}(\mathrm{w}_6 + 2\mathrm{w}_9)$
3	$\mathrm{E}_8 + \mathrm{E}_6 + \mathrm{A}_3$	$\mathbb{Z}_{12}$	$\mathbb{1}$	$\langle 6\rangle$	$\frac{2}{3}$	$\frac{1}{3}w_5 \times 0$	$\frac{3}{32}$	$\frac{1}{8}(\mathrm{w}_5 + 3\mathrm{w}_9)$
4	$\mathrm{E}_8 + \mathrm{D}_9$	$\mathbb{Z}_4$	$\mathbb{1}$	$\langle 2\rangle$	$\frac{1}{2}$	$0 \times \frac{1}{2}w_8$	$\frac{1}{8}$	$\frac{1}{2}\mathrm{w}_7$
5	$\mathrm{E}_8 + \mathrm{D}_5 + \mathrm{A}_4$	$\mathbb{Z}_{20}$	$\mathbb{1}$	$\langle 10\rangle$	$\frac{5}{8}$	$\frac{1}{4}w_4 \times 0$	$\frac{1}{10}$	$\frac{1}{10}(\mathrm{w}_4 + 4\mathrm{w}_9)$
6	$\mathrm{E}_8 + \mathrm{A}_9$	$\mathbb{Z}_{10}$	$\mathbb{1}$	$\langle 5\rangle$	$\frac{5}{9}$	$0 \times \frac{1}{3}w_1$	$\frac{5}{49}$	$\frac{1}{7}(3\mathrm{w}_7 + \mathrm{w}_{15})$
7	$\mathrm{E}_8 + \mathrm{A}_8 + \mathrm{A}_1$	$\mathbb{Z}_{18}$	$\mathbb{1}$	$\langle 9\rangle$	$\frac{9}{16}$	$\frac{1}{4}w_7 \times 0$	$\frac{1}{9}$	$\frac{4}{9}\mathrm{w}_9$
8	$\mathrm{E}_8 + \mathrm{A}_6 + \mathrm{A}_2 + \mathrm{A}_1$	$\mathbb{Z}_{42}$	$\mathbb{1}$	$\langle 21\rangle$	$\frac{7}{12}$	$\frac{1}{6}w_2 \times 0$	$\frac{3}{28}$	$\frac{1}{14}(\mathrm{w}_2 + 6\mathrm{w}_9)$
9	$\mathrm{E}_8 + \mathrm{A}_5 + \mathrm{A}_4$	$\mathbb{Z}_{30}$	$\mathbb{1}$	$\langle 15\rangle$	$\frac{3}{5}$	$\frac{1}{5}w_3 \times 0$	$\frac{5}{48}$	$\frac{1}{12}(\mathrm{w}_3 + 5\mathrm{w}_9)$
10	$\mathrm{E}_7 + \mathrm{E}_7 + \mathrm{A}_3$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4$	$\mathbb{Z}_2$	$\langle 2\rangle$	$\frac{1}{2}$	$\frac{1}{2}w_6 \times \frac{1}{2}w_6$	$\frac{1}{8}$	$\frac{1}{4}(\mathrm{w}_6 + \mathrm{w}_{10})$
11	$\mathrm{E}_7 + \mathrm{E}_6 + \mathrm{A}_4$	$\mathbb{Z}_{30}$	$\mathbb{1}$	$\langle 15\rangle$	$\frac{5}{12}$	$\frac{1}{3}w_5 \times \frac{1}{2}w_6$	$\frac{3}{20}$	$\frac{1}{10}(2\mathrm{w}_5 + 3\mathrm{w}_{10})$
12	$\mathrm{E}_7 + \mathrm{D}_{10}$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2$	$\langle 1\rangle$	$\frac{1}{4}$	$\frac{1}{2}w_6 \times \frac{1}{2}w_8'$	$\frac{1}{4}$	$\frac{1}{2}\mathrm{w}_6$
13	$\mathrm{E}_7 + \mathrm{D}_5 + \mathrm{A}_5$	$\mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_6$	$\mathbb{Z}_2$	$\langle 6\rangle$	$\frac{3}{8}$	$\frac{1}{4}w_4 \times \frac{1}{2}w_6$	$\frac{1}{6}$	$\frac{1}{6}(\mathrm{w}_4 + 2\mathrm{w}_{10})$
14	$\mathrm{E}_7 + \mathrm{A}_{10}$	$\mathbb{Z}_{22}$	$\mathbb{1}$	$\langle 11\rangle$	$\frac{11}{36}$	$\frac{1}{2}w_6 \times \frac{1}{3}w_1$	$\frac{11}{64}$	$\frac{1}{8}(3\mathrm{w}_6 + 2\mathrm{w}_{15})$
15	$\mathrm{E}_7 + \mathrm{A}_9 + \mathrm{A}_1$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{10}$	$\mathbb{Z}_2$	$\langle 5\rangle$	$\frac{5}{16}$	$\frac{1}{4}w_7 \times \frac{1}{2}w_6$	$\frac{1}{5}$	$\frac{2}{5}\mathrm{w}_{10}$
16	$\mathrm{E}_7 + \mathrm{A}_7 + \mathrm{A}_2 + \mathrm{A}_1$	$\mathbb{Z}_2 \times \mathbb{Z}_6 \times \mathbb{Z}_8$	$\mathbb{Z}_2$	$\langle 12\rangle$	$\frac{1}{3}$	$\frac{1}{6}w_2 \times \frac{1}{2}w_6$	$\frac{3}{16}$	$\frac{1}{8}(\mathrm{w}_2 + 3\mathrm{w}_{10})$
17	$\mathrm{E}_7 + \mathrm{A}_6 + \mathrm{A}_4$	$\mathbb{Z}_{70}$	$\mathbb{1}$	$\langle 35\rangle$	$\frac{7}{20}$	$\frac{1}{5}w_3 \times \frac{1}{2}w_6$	$\frac{5}{28}$	$\frac{1}{14}(2\mathrm{w}_3 + 5\mathrm{w}_{10})$
18	$\mathrm{E}_6 + \mathrm{E}_6 + \mathrm{A}_5$	$\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_6$	$\mathbb{Z}_3$	$\langle 3\rangle$	$\frac{1}{3}$	$\frac{1}{3}w_5 \times \frac{1}{3}w_5$	$\frac{3}{16}$	$\frac{1}{4}(\mathrm{w}_5 + \mathrm{w}_{11})$
19	$\mathrm{E}_6 + \mathrm{D}_{11}$	$\mathbb{Z}_{12}$	$\mathbb{1}$	$\langle 6\rangle$	$\frac{1}{6}$	$\frac{1}{3}w_5 \times \frac{1}{2}w_8$	$\frac{3}{8}$	$\frac{1}{2}\mathrm{w}_5$
20	$\mathrm{E}_6 + \mathrm{D}_5 + \mathrm{A}_6$	$\mathbb{Z}_{84}$	$\mathbb{1}$	$\langle 42\rangle$	$\frac{7}{24}$	$\frac{1}{4}w_4 \times \frac{1}{3}w_5$	$\frac{3}{14}$	$\frac{1}{14}(3\mathrm{w}_4 + 4\mathrm{w}_{11})$
21	$\mathrm{E}_6 + \mathrm{A}_{11}$	$\mathbb{Z}_3 \times \mathbb{Z}_{12}$	$\mathbb{Z}_3$	$\langle 2\rangle$	$\frac{2}{9}$	$\frac{1}{3}w_1 \times \frac{1}{3}w_5$	$\frac{2}{9}$	$\frac{1}{3}(\mathrm{w}_1 + \mathrm{w}_{11})$
22	$\mathrm{E}_6 + \mathrm{A}_{10} + \mathrm{A}_1$	$\mathbb{Z}_{66}$	$\mathbb{1}$	$\langle 33\rangle$	$\frac{11}{48}$	$\frac{1}{4}w_7 \times \frac{1}{3}w_5$	$\frac{3}{11}$	$\frac{4}{11}\mathrm{w}_{11}$

23	$\mathrm{E}_6+\mathrm{A}_8+\mathrm{A}_2+\mathrm{A}_1$	$\mathbb{Z}_3\times\mathbb{Z}_6\times\mathbb{Z}_9$	$\mathbb{Z}_3$	$\langle 9\rangle$	$\frac{1}{4}$	$\frac{1}{6}w_2\times\frac{1}{3}w_5$	$\frac{1}{4}$	$\frac{1}{6}(\mathrm{w}_2+2\mathrm{w}_{11})$
24	$\mathrm{E}_6+\mathrm{A}_7+\mathrm{A}_4$	$\mathbb{Z}_{120}$	$\mathbb{1}$	$\langle 60\rangle$	$\frac{4}{15}$	$\frac{1}{5}w_3\times\frac{1}{3}w_5$	$\frac{15}{64}$	$\frac{1}{16}(3\mathrm{w}_3+5\mathrm{w}_{11})$
25	D_{17}	$\mathbb{Z}_4$	$\mathbb{1}$	$\langle 2\rangle$	$\frac{1}{18}$	$\frac{1}{3}w_1\times\frac{1}{2}w_8$	$\frac{1}{2}$	w_1
26	$\mathrm{D}_{16}+\mathrm{A}_1$	$\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2$	$\mathbb{Z}_2$	$\langle 1\rangle$	$\frac{1}{16}$	$\frac{1}{4}w_7\times\frac{1}{2}w_8$	1	0
27	$\mathrm{D}_{14}+\mathrm{A}_2+\mathrm{A}_1$	$\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_6$	$\mathbb{Z}_2$	$\langle 3\rangle$	$\frac{1}{12}$	$\frac{1}{6}w_2\times\frac{1}{2}w_8$	$\frac{3}{4}$	$\frac{1}{2}\mathrm{w}_2$
28	$\mathrm{D}_{13}+\mathrm{A}_4$	$\mathbb{Z}_{20}$	$\mathbb{1}$	$\langle 10\rangle$	$\frac{1}{10}$	$\frac{1}{5}w_3\times\frac{1}{2}w_8$	$\frac{5}{8}$	$\frac{1}{2}\mathrm{w}_3$
29	$\mathrm{D}_{12}+\mathrm{D}_5$	$\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_4$	$\mathbb{Z}_2$	$\langle 2\rangle$	$\frac{1}{8}$	$\frac{1}{4}w_4\times\frac{1}{2}w_8$	$\frac{1}{2}$	$\frac{1}{2}\mathrm{w}_4$
30	$\mathrm{D}_5+\mathrm{A}_{12}$	$\mathbb{Z}_{52}$	$\mathbb{1}$	$\langle 26\rangle$	$\frac{13}{72}$	$\frac{1}{4}w_4\times\frac{1}{3}w_1$	$\frac{13}{50}$	$\frac{1}{10}(3\mathrm{w}_4+4\mathrm{w}_{15})$
31	$\mathrm{D}_5+\mathrm{A}_{11}+\mathrm{A}_1$	$\mathbb{Z}_4\times\mathbb{Z}_{12}\times\mathbb{Z}_2$	$\mathbb{Z}_4$	$\langle 3\rangle$	$\frac{3}{16}$	$\frac{1}{4}w_7\times\frac{1}{4}w_4$	$\frac{1}{3}$	$\frac{1}{3}\mathrm{w}_{12}$
32	$\mathrm{D}_5+\mathrm{A}_9+\mathrm{A}_2+\mathrm{A}_1$	$\mathbb{Z}_4\times\mathbb{Z}_{10}\times\mathbb{Z}_6$	$\mathbb{Z}_2$	$\langle 30\rangle$	$\frac{5}{24}$	$\frac{1}{6}w_2\times\frac{1}{4}w_4$	$\frac{3}{10}$	$\frac{1}{10}(2\mathrm{w}_2+3\mathrm{w}_{12})$
33	$\mathrm{D}_5+\mathrm{A}_8+\mathrm{A}_4$	$\mathbb{Z}_{180}$	$\mathbb{1}$	$\langle 90\rangle$	$\frac{9}{40}$	$\frac{1}{5}w_3\times\frac{1}{4}w_4$	$\frac{5}{18}$	$\frac{1}{18}(4\mathrm{w}_3+5\mathrm{w}_{12})$
34	$\mathrm{D}_5+\mathrm{D}_5+\mathrm{A}_7$	$\mathbb{Z}_4\times\mathbb{Z}_4\times\mathbb{Z}_8$	$\mathbb{Z}_4$	$\langle 4\rangle$	$\frac{1}{4}$	$\frac{1}{4}w_4\times\frac{1}{4}w_4$	$\frac{1}{4}$	$\frac{1}{4}(\mathrm{w}_4+\mathrm{w}_{12})$
35	A_{17}	$\mathbb{Z}_{18}$	$\mathbb{Z}_3$	$\langle 1\rangle$	$\frac{1}{9}$	$\frac{1}{3}w_1\times\frac{1}{3}w_1$	$\frac{1}{4}$	$\frac{1}{2}(\mathrm{w}_1+\mathrm{w}_{15})$
36	$\mathrm{A}_{16}+\mathrm{A}_1$	$\mathbb{Z}_{34}$	$\mathbb{1}$	$\langle 17\rangle$	$\frac{17}{144}$	$\frac{1}{4}w_7\times\frac{1}{3}w_1$	$\frac{17}{49}$	$\frac{4}{7}\mathrm{w}_{15}$
37	$\mathrm{A}_{15}+\mathrm{A}_1+\mathrm{A}_1$	$\mathbb{Z}_{16}\times\mathbb{Z}_2\times\mathbb{Z}_2$	$\mathbb{Z}_4$	$\langle 2\rangle$	$\frac{1}{8}$	$\frac{1}{4}w_7\times\frac{1}{4}w_7$	$\frac{1}{2}$	$\frac{1}{2}\mathrm{w}_{16}$
38	$\mathrm{A}_{14}+\mathrm{A}_2+\mathrm{A}_1$	$\mathbb{Z}_{15}\times\mathbb{Z}_6$	$\mathbb{Z}_3$	$\langle 5\rangle$	$\frac{5}{36}$	$\frac{1}{3}w_1\times\frac{1}{6}w_2$	$\frac{5}{16}$	$\frac{1}{4}(2\mathrm{w}_1+\mathrm{w}_{14})$
39	$\mathrm{A}_{13}+\mathrm{A}_4$	$\mathbb{Z}_{70}$	$\mathbb{1}$	$\langle 35\rangle$	$\frac{7}{45}$	$\frac{1}{5}w_3\times\frac{1}{3}w_1$	$\frac{35}{121}$	$\frac{1}{11}(3\mathrm{w}_3+5\mathrm{w}_{15})$
40	$\mathrm{A}_{13}+\mathrm{A}_2+2\mathrm{A}_1$	$\mathbb{Z}_{14}\times\mathbb{Z}_6\times\mathbb{Z}_2$	$\mathbb{Z}_2$	$\langle 21\rangle$	$\frac{7}{48}$	$\frac{1}{4}w_7\times\frac{1}{6}w_2$	$\frac{3}{7}$	$\frac{2}{7}\mathrm{w}_{14}$
41	$\mathrm{A}_{12}+\mathrm{A}_4+\mathrm{A}_1$	$\mathbb{Z}_{130}$	$\mathbb{1}$	$\langle 65\rangle$	$\frac{13}{80}$	$\frac{1}{5}w_3\times\frac{1}{4}w_7$	$\frac{5}{13}$	$\frac{1}{13}(4\mathrm{w}_3+5\mathrm{w}_{16})$
42	$\mathrm{A}_{11}+2\mathrm{A}_2+2\mathrm{A}_1$	$\mathbb{Z}_{12}\times\mathbb{Z}_6\times\mathbb{Z}_6$	$\mathbb{Z}_6$	$\langle 6\rangle$	$\frac{1}{6}$	$\frac{1}{6}w_2\times\frac{1}{6}w_2$	$\frac{3}{8}$	$\frac{1}{4}(\mathrm{w}_2+\mathrm{w}_{14})$
43	$\mathrm{A}_{10}+\mathrm{A}_4+\mathrm{A}_2+\mathrm{A}_1$	$\mathbb{Z}_{330}$	$\mathbb{1}$	$\langle 165\rangle$	$\frac{11}{60}$	$\frac{1}{5}w_3\times\frac{1}{6}w_2$	$\frac{15}{44}$	$\frac{1}{22}(6\mathrm{w}_3+5\mathrm{w}_{14})$
44	$\mathrm{A}_9+2\mathrm{A}_4$	$\mathbb{Z}_{10}\times\mathbb{Z}_5\times\mathbb{Z}_5$	$\mathbb{Z}_5$	$\langle 5\rangle$	$\frac{1}{5}$	$\frac{1}{5}w_3\times\frac{1}{5}w_3$	$\frac{5}{16}$	$\frac{1}{4}(\mathrm{w}_3+\mathrm{w}_{13})$

Table D.1: Data for allowed groups of maximal rank, $d=1$. $(R_\mathrm{E},A_\mathrm{E})$ and $(R_\mathrm{O},A_\mathrm{O})$ are the radius and Wilson line in the $\mathrm{E}_8\times\mathrm{E}_8'$ and $\mathrm{Spin}(32)/\mathbb{Z}_2$ heterotic theory.

#	L	H_L	T	E_{11}	E_{21}	E_{22}	E_{12}	A_1	A_2
1	$6\mathrm{A}_3$	$\mathbb{Z}_4\times\mathbb{Z}_4$	$[4,0,4]$	1	0	1	$\frac{1}{2}$	$\frac{w_6}{2}\times\frac{w_6}{2}$	$\left(\frac{w_2}{4}-\frac{3w_6}{4}\right)\times\left(\frac{w_2}{4}-\frac{3w_6}{4}\right)$
2	$2\mathrm{A}_1+4\mathrm{A}_4$	$\mathbb{Z}_5$	$[10,0,10]$	1	$\frac{1}{5}$	1	$-\frac{1}{5}$	$\frac{w_3}{5}\times\frac{w_3}{5}$	$\left(w_1-\frac{3w_3}{5}\right)\times\left(w_1-\frac{3w_3}{5}\right)$
3	$2\mathrm{A}_2+2\mathrm{A}_3+2\mathrm{A}_4$	$\mathbb{1}$	$[60,0,60]$	1	0	$\frac{11}{12}$	$\frac{5}{12}$	$\frac{w_5}{3}\times\frac{w_6}{2}$	$\left(\frac{w_2}{3}-\frac{2w_5}{3}\right)\times\left(\frac{w_2}{4}-\frac{3w_6}{4}\right)$
4	$3\mathrm{A}_1+3\mathrm{A}_5$	$\mathbb{Z}_2\times\mathbb{Z}_6$	$[2,0,6]$	1	0	1	0	$\frac{w_6}{2}\times\frac{w_8}{2}$	$\left(\frac{3w_6}{4}-\frac{w_2}{4}\right)\times\left(\frac{w_2}{2}-\frac{3w_4}{4}\right)$
5	$4\mathrm{A}_2+2\mathrm{A}_5$	$\mathbb{Z}_3\times\mathbb{Z}_3$	$[6,0,6]$	1	0	1	0	$\frac{w_5}{3}\times\frac{w_5}{3}$	$\left(\frac{w_2}{3}-\frac{2w_5}{3}\right)\times\left(\frac{w_2}{3}-\frac{2w_5}{3}\right)$
6	$\mathrm{A}_3+3\mathrm{A}_5$	$\mathbb{Z}_2\times\mathbb{Z}_3$	$[4,0,6]$	1	0	1	0	$\frac{w_5}{3}\times\frac{w_5}{3}$	$\left(\frac{w_1}{2}-\frac{w_5}{2}\right)\times\left(\frac{w_1}{2}-\frac{w_5}{2}\right)$
7	$2\mathrm{A}_1+2\mathrm{A}_3+2\mathrm{A}_5$	$\mathbb{Z}_2\times\mathbb{Z}_2$	$[12,0,12]$	1	0	1	0	$\frac{w_4}{4}\times\frac{w_6}{2}$	$\left(\frac{w_7}{2}-\frac{w_4}{2}\right)\times\left(\frac{w_2}{4}-\frac{3w_6}{4}\right)$
8	$\mathrm{A}_1+2\mathrm{A}_2+\mathrm{A}_3+2\mathrm{A}_5$	$\mathbb{Z}_2\times\mathbb{Z}_3$	$[6,0,12]$	1	0	1	0	$\frac{w_5}{3}\times\frac{w_2}{6}$	$\left(\frac{w_2}{3}-\frac{2w_5}{3}\right)\times\left(\frac{2w_4}{3}-\frac{w_2}{3}\right)$
9	$2\mathrm{A}_4+2\mathrm{A}_5$	$\mathbb{1}$	$[30,0,30]$	1	0	1	0	$\frac{w_3}{5}\times 0$	$0\times\frac{w_3}{5}$
10	$2\mathrm{A}_2+\mathrm{A}_4+2\mathrm{A}_5$	$\mathbb{Z}_3$	$[6,0,30]$	1	0	1	0	$\frac{w_5}{3}\times\frac{w_5}{3}$	$\left(\frac{w_1}{2}-\frac{w_5}{2}\right)\times\left(\frac{w_2}{3}-\frac{2w_5}{3}\right)$
11	$\mathrm{A}_1+\mathrm{A}_3+\mathrm{A}_4+2\mathrm{A}_5$	$\mathbb{Z}_2$	$[12,0,30]$	1	0	1	0	$\frac{w_4}{4}\times\frac{w_6}{2}$	$\left(\frac{w_7}{2}-\frac{w_4}{2}\right)\times\left(\frac{w_7}{3}-\frac{2w_6}{3}\right)$
12	$\mathrm{A}_1+\mathrm{A}_2+2\mathrm{A}_3+\mathrm{A}_4+\mathrm{A}_5$	$\mathbb{Z}_2$	$[24,12,36]$	1	0	1	0	$\frac{w_6}{2}\times\frac{w_2}{6}$	$\left(\frac{w_7}{3}-\frac{2w_6}{3}\right)\times\left(\frac{3w_3}{4}-\frac{w_2}{2}\right)$
13	$3\mathrm{A}_6$	$\mathbb{Z}_7$	$[2,1,4]$	1	0	1	0	$\frac{w_6}{2}\times\frac{w_8}{2}$	$\left(\frac{2w_6}{3}-\frac{w_7}{3}\right)\times\left(\frac{2w_8}{3}-\frac{w_1}{3}\right)$
14	$2\mathrm{A}_1+2\mathrm{A}_2+2\mathrm{A}_6$	$\mathbb{1}$	$[42,0,42]$	1	0	1	0	$\frac{w_2}{6}\times 0$	$0\times\frac{w_2}{6}$
15	$2\mathrm{A}_3+2\mathrm{A}_6$	$\mathbb{1}$	$[28,0,28]$	1	0	$\frac{19}{20}$	$\frac{7}{20}$	$\frac{w_3}{5}\times\frac{w_6}{2}$	$\left(w_1-\frac{3w_3}{5}\right)\times\left(\frac{w_2}{4}-\frac{3w_6}{4}\right)$
				1	0	1	0	$\frac{2w_5}{7}\times\left(\frac{4w_1}{7}+\frac{w_5}{7}-\frac{2w_7}{7}\right)$	$\left(\frac{w_1}{2}-\frac{5w_5}{8}\right)\times\left(\frac{w_1}{4}+\frac{3w_7}{8}-\frac{3w_5}{4}\right)$
16	$\mathrm{A}_2+\mathrm{A}_4+2\mathrm{A}_6$	$\mathbb{1}$	$[28,7,28]$	1	0	1	0	$\frac{w_3}{5}\times\frac{w_6}{2}$	$0\times\left(\frac{w_3}{3}-\frac{5w_6}{6}\right)$
17	$2\mathrm{A}_1+\mathrm{A}_2+2\mathrm{A}_4+\mathrm{A}_6$	$\mathbb{1}$	$[50,20,50]$	1	$\frac{1}{5}$	$\frac{13}{15}$	$-\frac{4}{15}$	$\frac{w_2}{6}\times\frac{w_3}{5}$	$\left(w_7-\frac{2w_2}{3}\right)\times\left(w_1-\frac{3w_3}{5}\right)$
				1	0	1	0	$\left(\frac{w_3}{3}+\frac{w_6}{2}-\frac{5w_7}{12}\right)\times\frac{w_7}{4}$	$\left(\frac{2w_3}{5}-w_6\right)\times\left(\frac{2w_2}{5}-\frac{w_7}{2}\right)$

18	$A_1+A_3+2A_4+A_6$	$\mathbb{1}$	$[20,0,70]$	1	0	1	0	$\frac{w_3}{5}\times\frac{w_6}{2}$	$0\times\left(\frac{w_2}{4}-\frac{3w_6}{4}\right)$
			$[10,0,140]$	1	0	1	0	$\frac{w_6}{2}\times\frac{w_7}{4}$	$\left(\frac{w_7}{3}-\frac{2w_6}{3}\right)\times\left(\frac{8w_2}{15}-\frac{2w_7}{3}\right)$
19	$A_2+2A_3+A_4+A_6$	$\mathbb{1}$	$[24,12,76]$	1	0	$\frac{21}{20}$	$\frac{7}{20}$	$\frac{w_3}{5}\times\frac{w_6}{2}$	$\left(w_8-\frac{2w_3}{5}\right)\times\left(\frac{w_2}{4}-\frac{3w_6}{4}\right)$
				1	0	1	0	$\frac{w_6}{2}\times\frac{w_2}{6}$	$\left(\frac{w_1}{5}+\frac{w_7}{5}-\frac{7w_6}{10}\right)\times\left(\frac{3w_3}{4}-\frac{w_2}{2}\right)$
20	$A_1+2A_2+A_3+A_4+A_6$	$\mathbb{1}$	$[30,0,84]$	1	0	1	0	$\frac{w_4}{4}\times\frac{w_5}{3}$	$\left(\frac{w_7}{2}-\frac{w_4}{2}\right)\times\left(\frac{w_2}{3}-\frac{2w_5}{3}\right)$
21	$2A_1+2A_5+A_6$	$\mathbb{Z}_2$	$[12,6,24]$	1	0	1	0	$\frac{w_4}{4}\times\frac{w_5}{3}$	$\left(\frac{w_2}{2}-\frac{3w_4}{4}\right)\times\left(\frac{w_1}{2}-\frac{w_5}{2}\right)$
22	$A_1+2A_3+A_5+A_6$	$\mathbb{Z}_2$	$[4,0,84]$	1	0	1	0	$\frac{w_4}{4}\times\frac{w_5}{3}$	$\left(\frac{w_7}{2}-\frac{w_4}{2}\right)\times\left(\frac{w_1}{2}-\frac{w_5}{2}\right)$
23	$A_1+A_2+A_4+A_5+A_6$	$\mathbb{1}$	$[18,6,72]$	1	0	1	$-\frac{5}{12}$	$\frac{w_5}{3}\times\frac{w_6}{2}$	$\left(\frac{w_7}{2}-\frac{2w_5}{3}\right)\times\left(\frac{w_7}{3}-\frac{2w_6}{3}\right)$
				1	0	1	0	$\left(\frac{w_1}{3}+\frac{w_4}{3}-\frac{2w_7}{3}\right)\times\frac{w_2}{6}$	$\left(\frac{10w_1}{21}+\frac{w_7}{21}-\frac{4w_4}{21}\right)\times\left(w_8-\frac{w_2}{3}\right)$
			$[30,0,42]$	1	0	1	0	$\frac{w_2}{6}\times 0$	$0\times\frac{w_3}{5}$
24	$A_3+A_4+A_5+A_6$	$\mathbb{1}$	$[12,0,70]$	1	0	1	0	$\frac{w_3}{5}\times\frac{w_6}{2}$	$0\times\left(\frac{w_7}{3}-\frac{2w_6}{3}\right)$
25	$4A_1+2A_7$	$\mathbb{Z}_2\times\mathbb{Z}_4$	$[4,0,4]$	1	0	1	0	$\frac{w_4}{4}\times\frac{w_4}{4}$	$\left(\frac{w_2}{2}-\frac{3w_4}{4}\right)\times\left(\frac{w_2}{2}-\frac{3w_4}{4}\right)$
26	$2A_2+2A_7$	$\mathbb{Z}_2$	$[12,0,12]$	1	0	1	0	$\frac{w_6}{2}\times\left(\frac{w_3}{3}-\frac{5w_6}{6}\right)$	$\left(\frac{w_3}{3}-\frac{5w_6}{6}\right)\times\frac{w_6}{2}$
		$\mathbb{1}$	$[24,0,24]$	1	0	1	0	$\frac{w_3}{5}\times\frac{w_5}{3}$	$\left(w_1-\frac{3w_3}{5}\right)\times\left(\frac{w_2}{3}-\frac{2w_5}{3}\right)$
27	$A_1+A_3+2A_7$	$\mathbb{Z}_8$	$[2,0,4]$	1	0	1	$\frac{1}{2}$	$\frac{w_6}{2}\times\frac{w_6}{2}$	$\left(\frac{w_1}{2}-\frac{3w_6}{4}\right)\times\left(\frac{w_2}{4}-\frac{3w_6}{4}\right)$
				1	0	1	0	$\frac{w_6}{2}\times\left(\frac{w_2}{2}-\frac{w_7}{2}\right)$	$\left(\frac{w_7}{3}-\frac{2w_6}{3}\right)\times\left(-\frac{2w_2}{3}+\frac{2w_3}{3}+\frac{w_7}{3}\right)$
28	$2A_1+3A_3+A_7$	$\mathbb{Z}_2\times\mathbb{Z}_4$	$[4,0,8]$	1	0	1	0	$\frac{w_4}{4}\times\frac{w_4}{4}$	$\left(\frac{w_4}{2}-\frac{w_7}{2}\right)\times\left(\frac{w_4}{2}-\frac{w_7}{2}\right)$
29	$A_2+3A_3+A_7$	$\mathbb{Z}_4$	$[4,0,24]$	1	0	1	0	$\frac{w_1}{6}\times\frac{w_5}{3}$	$\frac{w_1}{3}\times\left(\frac{w_2}{3}-\frac{2w_5}{3}\right)$
30	$2A_2+A_3+A_4+A_7$	$\mathbb{1}$	$[12,0,120]$	1	0	1	0	$\frac{w_3}{5}\times\frac{w_5}{3}$	$0\times\left(\frac{w_2}{3}-\frac{2w_5}{3}\right)$
31	$2A_1+A_2+A_3+A_4+A_7$	$\mathbb{Z}_2$	$[20,0,24]$	1	0	1	0	$\frac{w_2}{6}\times\frac{w_6}{2}$	$0\times\left(\frac{w_2}{4}-\frac{3w_6}{4}\right)$
32	$A_1+2A_5+A_7$	$\mathbb{Z}_2$	$[6,0,24]$	1	0	1	0	$\frac{w_2}{6}\times\frac{w_6}{2}$	$\left(w_7-\frac{2w_2}{3}\right)\times\left(\frac{w_7}{3}-\frac{2w_6}{3}\right)$
33	$3A_1+A_3+A_5+A_7$	$\mathbb{Z}_2\times\mathbb{Z}_2$	$[8,0,12]$	1	0	1	0	$\frac{w_4}{4}\times\frac{w_4}{4}$	$\left(\frac{w_2}{2}-\frac{3w_4}{4}\right)\times\left(\frac{w_4}{2}-\frac{w_7}{2}\right)$
34	$A_1+A_2+A_3+A_5+A_7$	$\mathbb{Z}_2$	$[12,0,24]$	1	0	1	0	$\frac{w_2}{6}\times\frac{w_6}{2}$	$0\times\left(\frac{w_7}{3}-\frac{2w_6}{3}\right)$
35	$2A_1+A_4+A_5+A_7$	$\mathbb{Z}_2$	$[2,0,120]$	1	0	$\frac{23}{24}$	$\frac{5}{12}$	$\frac{w_5}{3}\times\frac{w_6}{2}$	$\left(\frac{w_3}{2}-\frac{5w_5}{6}\right)\times\left(\frac{w_2}{4}-\frac{3w_6}{4}\right)$
				1	0	1	0	$\frac{w_6}{2}\times\left(\frac{w_2}{2}-\frac{w_7}{2}\right)$	$\left(\frac{2w_3}{5}-w_6\right)\times\left(-\frac{w_2}{2}+\frac{w_3}{2}+\frac{w_7}{4}\right)$
36	$A_2+A_4+A_5+A_7$	$\mathbb{1}$	$[6,0,120]$	1	0	1	0	$\frac{w_3}{5}\times\frac{w_5}{3}$	$0\times\left(\frac{w_1}{2}-\frac{w_5}{2}\right)$
			$[24,0,30]$	1	0	1	0	$\frac{w_6}{2}\times\left(\frac{w_3}{3}-\frac{5w_6}{6}\right)$	$\left(\frac{w_7}{3}-\frac{2w_6}{3}\right)\times\frac{w_6}{2}$
37	$A_1+2A_2+A_6+A_7$	$\mathbb{1}$	$[24,0,42]$	1	0	1	0	$\frac{w_2}{6}\times\frac{w_6}{2}$	$0\times\left(\frac{w_3}{3}-\frac{5w_6}{6}\right)$
38	$2A_1+A_3+A_6+A_7$	$\mathbb{Z}_2$	$[12,4,20]$	1	0	1	0	$\frac{w_4}{4}\times\frac{w_5}{3}$	$\left(\frac{w_4}{2}-\frac{w_7}{2}\right)\times\left(\frac{2w_5}{3}-\frac{w_7}{2}\right)$
39	$A_2+A_3+A_6+A_7$	$\mathbb{1}$	$[4,0,168]$	1	0	1	0	$\frac{3w_1}{16}\times\frac{w_6}{2}$	$\frac{w_1}{3}\times\left(\frac{w_7}{3}-\frac{2w_6}{3}\right)$
40	$A_1+A_4+A_6+A_7$	$\mathbb{1}$	$[18,4,32]$	1	0	1	0	$\frac{w_3}{5}\times\frac{w_5}{3}$	$0\times\left(\frac{w_3}{2}-\frac{5w_5}{6}\right)$
			$[2,0,280]$	1	0	$\frac{37}{40}$	$-\frac{7}{20}$	$\frac{w_3}{5}\times\left(\frac{w_7}{2}-\frac{w_4}{2}\right)$	$\left(w_1-\frac{3w_3}{5}\right)\times\frac{w_4}{4}$
				1	0	1	0	$\frac{w_6}{2}\times\left(\frac{w_2}{2}-\frac{w_7}{2}\right)$	$\left(\frac{w_3}{3}-\frac{5w_6}{6}\right)\times\left(-\frac{7w_2}{12}+\frac{7w_3}{12}+\frac{7w_7}{24}\right)$
41	$A_5+A_6+A_7$	$\mathbb{1}$	$[16,4,22]$	1	0	1	0	$\frac{w_3}{5}\times\frac{w_5}{3}$	$\left(w_1-\frac{3w_3}{5}\right)\times\left(\frac{w_1}{2}-\frac{w_5}{2}\right)$
42	$2A_1+2A_8$	$\mathbb{1}$	$[18,0,18]$	1	0	1	0	$\frac{w_7}{4}\times 0$	$0\times\frac{w_7}{4}$
		$\mathbb{Z}_3$	$[4,2,10]$	1	0	1	0	$\frac{w_3}{5}\times\frac{w_4}{4}$	$\left(w_1-\frac{3w_3}{5}\right)\times\left(\frac{w_2}{2}-\frac{3w_4}{4}\right)$
43	$A_1+3A_2+A_3+A_8$	$\mathbb{Z}_3$	$[12,0,18]$	1	0	1	0	$\frac{w_2}{6}\times\frac{w_5}{3}$	$0\times\left(\frac{w_2}{3}-\frac{2w_5}{3}\right)$
44	$2A_1+2A_4+A_8$	$\mathbb{1}$	$[20,10,50]$	1	0	1	0	$\frac{w_3}{5}\times\frac{w_4}{4}$	$0\times\left(\frac{w_2}{2}-\frac{3w_4}{4}\right)$
45	$3A_2+A_4+A_8$	$\mathbb{Z}_3$	$[12,3,12]$	1	0	1	0	$\frac{w_5}{3}\times\left(\frac{w_3}{3}-\frac{5w_6}{6}\right)$	$\left(\frac{w_2}{3}-\frac{2w_5}{3}\right)\times\frac{w_6}{2}$
46	$A_1+A_2+A_3+A_4+A_8$	$\mathbb{1}$	$[6,0,180]$	1	0	1	0	$\frac{w_3}{5}\times\frac{w_4}{4}$	$0\times\left(\frac{w_4}{2}-\frac{w_7}{2}\right)$
47	$A_1+2A_2+A_5+A_8$	$\mathbb{Z}_3$	$[6,0,18]$	1	0	1	0	$\frac{w_2}{6}\times\frac{w_5}{3}$	$0\times\left(\frac{w_1}{2}-\frac{w_5}{2}\right)$
48	$A_2+A_3+A_5+A_8$	$\mathbb{Z}_3$	$[4,0,18]$	1	0	1	0	$\frac{w_5}{3}\times\left(\frac{w_3}{3}-\frac{5w_6}{6}\right)$	$\left(\frac{w_1}{2}-\frac{w_5}{2}\right)\times\frac{w_6}{2}$
49	$A_1+A_4+A_5+A_8$	$\mathbb{1}$	$[18,0,30]$	1	0	1	0	$0\times\frac{w_7}{4}$	$\frac{w_3}{5}\times 0$
50	$2A_1+A_2+A_6+A_8$	$\mathbb{1}$	$[18,0,42]$	1	0	1	0	$0\times\frac{w_7}{4}$	$\frac{w_2}{6}\times 0$
51	$A_1+A_3+A_6+A_8$	$\mathbb{1}$	$[10,4,52]$	1	0	1	0	$\frac{w_3}{5}\times\frac{w_4}{4}$	$\left(w_1-\frac{3w_3}{5}\right)\times\left(\frac{w_4}{2}-\frac{w_7}{2}\right)$
52	$A_4+A_6+A_8$	$\mathbb{1}$	$[18,9,22]$	1	0	1	0	$\frac{w_3}{5}\times\frac{w_6}{2}$	$0\times\left(\frac{w_1}{2}-\frac{3w_6}{4}\right)$
53	$A_1+A_2+A_7+A_8$	$\mathbb{1}$	$[18,0,24]$	1	0	1	0	$\frac{w_2}{6}\times\frac{w_6}{2}$	$0\times\left(\frac{w_1}{2}-\frac{3w_6}{4}\right)$
54	$2A_9$	$\mathbb{Z}_5$	$[2,0,2]$	1	0	1	0	$\frac{w_3}{5}\times\frac{w_3}{5}$	$\left(w_1-\frac{3w_3}{5}\right)\times\left(w_1-\frac{3w_3}{5}\right)$
		$\mathbb{1}$	$[10,0,10]$	1	0	1	0	$\frac{w_1}{3}\times 0$	$0\times\frac{w_1}{3}$
55	$A_1+A_2+2A_3+A_9$	$\mathbb{Z}_2$	$[4,0,60]$	1	0	1	0	$\frac{w_4}{4}\times\left(\frac{w_3}{3}-\frac{5w_6}{6}\right)$	$\left(\frac{w_7}{2}-\frac{w_4}{2}\right)\times\frac{w_6}{2}$
56	$2A_1+2A_2+A_3+A_9$	$\mathbb{Z}_2$	$[6,0,60]$	1	0	1	0	$\frac{w_2}{6}\times\frac{w_4}{4}$	$0\times\left(\frac{w_4}{2}-\frac{w_7}{2}\right)$
57	$A_1+2A_4+A_9$	$\mathbb{Z}_5$	$[2,0,10]$	1	0	1	0	$\frac{w_3}{5}\times\frac{w_3}{5}$	0×0
58	$3A_1+A_2+A_4+A_9$	$\mathbb{Z}_2$	$[20,10,20]$	1	0	1	0	$\frac{w_2}{6}\times\frac{w_4}{4}$	$0\times\left(\frac{w_2}{2}-\frac{3w_4}{4}\right)$
59	$2A_1+A_3+A_4+A_9$	$\mathbb{Z}_2$	$[10,0,20]$	1	0	1	0	$\frac{w_6}{2}\times\frac{w_7}{4}$	$\left(\frac{w_2}{4}-\frac{3w_6}{4}\right)\times 0$
60	$2A_1+A_2+A_5+A_9$	$\mathbb{Z}_2$	$[12,6,18]$	1	0	1	0	$\frac{w_4}{4}\times\left(\frac{w_3}{3}-\frac{5w_6}{6}\right)$	$\left(\frac{w_2}{2}-\frac{3w_4}{4}\right)\times\frac{w_6}{2}$
61	$A_1+A_3+A_5+A_9$	$\mathbb{Z}_2$	$[10,0,12]$	1	0	1	0	$\frac{w_6}{2}\times\frac{w_7}{4}$	$\left(\frac{w_7}{3}-\frac{2w_6}{3}\right)\times 0$
62	$A_4+A_5+A_9$	$\mathbb{1}$	$[10,0,30]$	1	0	1	0	$\frac{w_1}{3}\times 0$	$0\times\frac{w_3}{5}$
		$\mathbb{Z}_2$	$[10,5,10]$	1	0	1	0	$\frac{w_6}{2}\times\left(\frac{w_1}{2}-\frac{3w_6}{4}\right)$	$\left(\frac{w_7}{3}-\frac{2w_6}{3}\right)\times\frac{w_6}{2}$
63	$3A_1+A_6+A_9$	$\mathbb{Z}_2$	$[4,2,36]$	1	0	1	0	$\frac{w_2}{6}\times\frac{w_4}{4}$	$\left(w_7-\frac{2w_2}{3}\right)\times\left(\frac{w_2}{2}-\frac{3w_4}{4}\right)$
64	$A_1+A_2+A_6+A_9$	$\mathbb{1}$	$[10,0,42]$	1	0	1	0	$\frac{w_1}{3}\times 0$	$0\times\frac{w_2}{6}$

65	$A_3 + A_6 + A_9$	$\mathbb{1}$	$[2, 0, 140]$	1	0	1	0	$\frac{w_6}{2} \times \frac{w_8}{2}$	$\left(\frac{w_7}{3} - \frac{2w_6}{3}\right) \times \left(\frac{4w_8}{5} - \frac{8w_1}{15}\right)$
66	$A_2 + A_7 + A_9$	$\mathbb{1}$	$[10, 0, 24]$	1	0	1	0	$\left(\frac{w_1}{2} - \frac{3w_6}{4}\right) \times \frac{w_6}{2}$	$\frac{w_6}{2} \times \left(\frac{w_3}{3} - \frac{5w_6}{6}\right)$
67	$A_1 + A_8 + A_9$	$\mathbb{1}$	$[10, 0, 18]$	1	0	1	0	$\frac{w_1}{3} \times 0$	$0 \times \frac{w_7}{4}$
68	$A_{10} + A_2 + 2A_3$	$\mathbb{1}$	$[24, 12, 28]$	1	0	1	0	$\left(\frac{w_3}{8} - \frac{3w_1}{8}\right) \times \frac{w_6}{2}$	$\left(\frac{2w_1}{3} - \frac{2w_3}{9}\right) \times \left(\frac{w_7}{3} - \frac{2w_6}{3}\right)$
69	$A_1 + A_{10} + 2A_2 + A_3$	$\mathbb{1}$	$[12, 0, 66]$	1	0	1	0	$\frac{w_5}{3} \times \frac{w_7}{4}$	$\left(\frac{w_2}{3} - \frac{2w_5}{3}\right) \times 0$
70	$A_{10} + 2A_4$	$\mathbb{1}$	$[10, 5, 30]$	1	0	1	0	$\left(\frac{2w_4}{5} - \frac{w_1}{5}\right) \times \frac{w_3}{5}$	$\left(\frac{3w_4}{5} - \frac{4w_1}{5}\right) \times 0$
71	$A_{10} + 2A_2 + A_4$	$\mathbb{1}$	$[6, 3, 84]$	1	0	1	0	$\frac{w_3}{5} \times \left(\frac{w_3}{3} - \frac{5w_6}{6}\right)$	$0 \times \frac{w_6}{2}$
			$[24, 9, 24]$	1	0	1	0	$\frac{w_5}{3} \times \left(\frac{w_1}{2} - \frac{3w_6}{4}\right)$	$\left(\frac{w_2}{3} - \frac{2w_5}{3}\right) \times \frac{w_6}{2}$
72	$2A_1 + A_{10} + A_2 + A_4$	$\mathbb{1}$	$[2, 0, 330]$	1	0	1	0	$\frac{w_2}{6} \times \frac{w_3}{5}$	0×0
73	$A_1 + A_{10} + A_3 + A_4$	$\mathbb{1}$	$[12, 4, 38]$	1	0	1	0	$\frac{w_2}{6} \times \frac{w_3}{5}$	$\left(w_7 - \frac{2w_2}{3}\right) \times 0$
			$[20, 0, 22]$	1	0	1	0	$\frac{w_1}{3} \times \frac{w_6}{2}$	$0 \times \left(\frac{w_2}{4} - \frac{3w_6}{4}\right)$
74	$A_1 + A_{10} + A_2 + A_5$	$\mathbb{1}$	$[6, 0, 66]$	1	0	1	0	$\frac{w_5}{3} \times \frac{w_7}{4}$	$\left(\frac{w_1}{2} - \frac{w_5}{2}\right) \times 0$
			$[18, 6, 24]$	1	0	1	0	$\frac{w_2}{6} \times \frac{w_3}{5}$	$0 \times \left(w_1 - \frac{3w_3}{5}\right)$
75	$A_{10} + A_3 + A_5$	$\mathbb{1}$	$[4, 0, 66]$	1	0	1	0	$\frac{w_5}{3} \times \left(\frac{w_1}{2} - \frac{3w_6}{4}\right)$	$\left(\frac{w_1}{2} - \frac{w_5}{2}\right) \times \frac{w_6}{2}$
			$[12, 0, 22]$	1	0	1	0	$\frac{w_1}{3} \times \frac{w_6}{2}$	$0 \times \left(\frac{w_7}{3} - \frac{2w_6}{3}\right)$
76	$2A_1 + A_{10} + A_6$	$\mathbb{1}$	$[12, 2, 26]$	1	0	1	0	$\frac{w_5}{3} \times \frac{w_7}{4}$	$\left(\frac{w_3}{2} - \frac{5w_5}{6}\right) \times 0$
77	$A_{10} + A_2 + A_6$	$\mathbb{1}$	$[4, 1, 58]$	1	0	1	0	$\frac{w_3}{5} \times \left(\frac{w_3}{3} - \frac{5w_6}{6}\right)$	$\left(w_1 - \frac{3w_3}{5}\right) \times \frac{w_6}{2}$
			$[16, 5, 16]$	1	0	1	0	$\frac{w_1}{3} \times \frac{w_6}{2}$	$0 \times \left(\frac{w_3}{3} - \frac{5w_6}{6}\right)$
78	$A_1 + A_{10} + A_7$	$\mathbb{1}$	$[2, 0, 88]$	1	0	1	0	$\frac{w_2}{6} \times \frac{w_3}{5}$	$\left(w_7 - \frac{2w_2}{3}\right) \times \left(w_1 - \frac{3w_3}{5}\right)$
			$[10, 2, 18]$	1	0	1	0	$\frac{w_2}{6} \times \frac{w_3}{5}$	$\left(w_7 - \frac{2w_2}{3}\right) \times \left(w_8 - \frac{2w_3}{5}\right)$
79	$A_{10} + A_8$	$\mathbb{1}$	$[10, 1, 10]$	1	0	1	0	$\frac{w_1}{3} \times \frac{w_6}{2}$	$0 \times \left(\frac{w_1}{2} - \frac{3w_6}{4}\right)$
80	$A_1 + A_{11} + 3A_2$	$\mathbb{Z}_3$	$[6, 0, 12]$	1	0	1	0	$\frac{w_2}{6} \times \left(\frac{w_3}{3} - \frac{5w_6}{6}\right)$	$0 \times \frac{w_6}{2}$
81	$3A_1 + A_{11} + 2A_2$	$\mathbb{Z}_2 \times \mathbb{Z}_3$	$[2, 0, 12]$	1	0	1	0	$\frac{w_2}{6} \times \frac{w_2}{6}$	0×0
82	$A_1 + A_{11} + 2A_3$	$\mathbb{Z}_4$	$[4, 0, 6]$	1	0	1	0	$\frac{w_4}{4} \times \left(\frac{w_1}{2} - \frac{3w_6}{4}\right)$	$\left(\frac{w_7}{2} - \frac{w_4}{2}\right) \times \frac{w_6}{2}$
83	$A_{11} + 2A_2 + A_3$	$\mathbb{Z}_3$	$[4, 0, 12]$	1	0	1	0	$\frac{w_1}{3} \times \frac{w_5}{3}$	$0 \times \left(\frac{w_2}{3} - \frac{2w_5}{3}\right)$
		$\mathbb{Z}_2 \times \mathbb{Z}_3$	$[4, 2, 4]$	1	0	1	0	$\left(\frac{w_3}{3} - \frac{5w_6}{6}\right) \times \left(\frac{w_3}{3} - \frac{5w_6}{6}\right)$	$\frac{w_6}{2} \times \frac{w_6}{2}$
84	$2A_1 + A_{11} + A_2 + A_3$	$\mathbb{Z}_4$	$[6, 0, 6]$	1	0	1	0	$\frac{w_4}{4} \times \frac{w_7}{4}$	$\left(\frac{w_4}{2} - \frac{w_7}{2}\right) \times 0$
		$\mathbb{Z}_2$	$[12, 0, 12]$	1	0	1	0	$\frac{w_2}{6} \times \frac{w_2}{6}$	$0 \times \left(w_7 - \frac{2w_2}{3}\right)$
85	$3A_1 + A_{11} + A_4$	$\mathbb{Z}_2$	$[6, 0, 20]$	1	0	1	0	$\frac{w_4}{4} \times \frac{w_7}{4}$	$\left(\frac{w_2}{2} - \frac{3w_4}{4}\right) \times 0$
86	$A_1 + A_{11} + A_2 + A_4$	$\mathbb{1}$	$[12, 0, 30]$	1	0	1	0	$\left(\frac{2w_4}{5} - \frac{w_1}{5}\right) \times \frac{w_2}{6}$	$\left(\frac{3w_4}{5} - \frac{4w_1}{5}\right) \times 0$
87	$2A_1 + A_{11} + A_5$	$\mathbb{Z}_2 \times \mathbb{Z}_3$	$[2, 0, 4]$	1	0	1	0	$\frac{w_2}{6} \times \frac{w_2}{6}$	$\left(w_7 - \frac{2w_2}{3}\right) \times \left(w_7 - \frac{2w_2}{3}\right)$
		$\mathbb{Z}_2$	$[6, 0, 12]$	1	0	1	0	$\frac{w_2}{6} \times \frac{w_2}{6}$	$\left(w_7 - \frac{2w_2}{3}\right) \times \left(w_8 - \frac{w_2}{3}\right)$
88	$A_{11} + A_2 + A_5$	$\mathbb{Z}_3$	$[4, 0, 6]$	1	0	1	0	$\frac{w_1}{3} \times \frac{w_5}{3}$	$0 \times \left(\frac{w_1}{2} - \frac{w_5}{2}\right)$
89	$A_1 + A_{11} + A_6$	$\mathbb{1}$	$[4, 0, 42]$	1	0	1	0	$\frac{w_1}{3} \times \frac{w_5}{3}$	$0 \times \left(\frac{w_3}{2} - \frac{5w_5}{6}\right)$
90	$2A_1 + A_{12} + 2A_2$	$\mathbb{1}$	$[12, 6, 42]$	1	0	1	0	$\frac{w_2}{6} \times \left(\frac{2w_3}{3} - \frac{5w_7}{6}\right)$	$0 \times \left(\frac{w_3}{3} - \frac{w_7}{6}\right)$
91	$A_1 + A_{12} + A_2 + A_3$	$\mathbb{1}$	$[6, 0, 52]$	1	0	1	0	$\frac{w_1}{3} \times \frac{w_4}{4}$	$0 \times \left(\frac{w_4}{2} - \frac{w_7}{2}\right)$
92	$2A_1 + A_{12} + A_4$	$\mathbb{1}$	$[2, 0, 130]$	1	0	1	0	$\frac{w_3}{5} \times \frac{w_7}{4}$	0×0
			$[18, 8, 18]$	1	0	1	0	$\frac{w_1}{3} \times \frac{w_4}{4}$	$0 \times \left(\frac{w_2}{2} - \frac{3w_4}{4}\right)$
93	$A_{12} + A_2 + A_4$	$\mathbb{1}$	$[6, 3, 34]$	1	0	1	0	$\frac{w_3}{5} \times \left(\frac{w_1}{2} - \frac{3w_6}{4}\right)$	$0 \times \frac{w_6}{2}$
94	$A_1 + A_{12} + A_5$	$\mathbb{1}$	$[10, 2, 16]$	1	0	1	0	$\frac{w_3}{5} \times \frac{w_7}{4}$	$\left(w_1 - \frac{3w_3}{5}\right) \times 0$
95	$A_{12} + A_6$	$\mathbb{1}$	$[2, 1, 46]$	1	0	1	0	$\frac{w_3}{5} \times \left(\frac{w_1}{2} - \frac{3w_6}{4}\right)$	$\left(w_1 - \frac{3w_3}{5}\right) \times \frac{w_6}{2}$
96	$A_1 + A_{13} + 2A_2$	$\mathbb{1}$	$[6, 0, 42]$	1	0	1	0	$\left(\frac{w_3}{3} - \frac{5w_6}{6}\right) \times \frac{w_7}{4}$	$\frac{w_6}{2} \times 0$
		$\mathbb{Z}_2$	$[6, 3, 12]$	1	0	1	0	$\frac{w_2}{6} \times \left(\frac{w_1}{2} - \frac{3w_6}{4}\right)$	$0 \times \frac{w_6}{2}$
97	$3A_1 + A_{13} + A_2$	$\mathbb{Z}_2$	$[2, 0, 42]$	1	0	1	0	$\frac{w_2}{6} \times \frac{w_7}{4}$	0×0
98	$2A_1 + A_{13} + A_3$	$\mathbb{Z}_2$	$[6, 2, 10]$	1	0	1	0	$\frac{w_2}{6} \times \frac{w_7}{4}$	$\left(w_7 - \frac{2w_2}{3}\right) \times 0$
99	$A_{13} + A_2 + A_3$	$\mathbb{1}$	$[4, 0, 42]$	1	0	1	0	$\left(\frac{w_1}{2} - \frac{3w_6}{4}\right) \times \left(\frac{w_3}{3} - \frac{5w_6}{6}\right)$	$\frac{w_6}{2} \times \frac{w_6}{2}$
100	$A_1 + A_{13} + A_4$	$\mathbb{1}$	$[2, 0, 70]$	1	0	1	0	$\frac{w_1}{3} \times \frac{w_3}{5}$	0×0
			$[8, 2, 18]$	1	0	1	0	$\left(\frac{2w_4}{5} - \frac{w_1}{5}\right) \times \frac{w_7}{4}$	$\left(\frac{3w_4}{5} - \frac{4w_1}{5}\right) \times 0$
		$\mathbb{Z}_2$	$[2, 1, 18]$	1	0	1	0	$\frac{w_2}{6} \times \left(\frac{w_1}{2} - \frac{3w_6}{4}\right)$	$\left(w_7 - \frac{2w_2}{3}\right) \times \frac{w_6}{2}$
101	$A_{13} + A_5$	$\mathbb{1}$	$[4, 2, 22]$	1	0	1	0	$\frac{w_1}{3} \times \frac{w_3}{5}$	$0 \times \left(w_1 - \frac{3w_3}{5}\right)$
102	$A_{14} + 2A_2$	$\mathbb{Z}_3$	$[4, 1, 4]$	1	0	1	0	$\frac{w_1}{3} \times \left(\frac{w_3}{3} - \frac{5w_6}{6}\right)$	$0 \times \frac{w_6}{2}$
103	$2A_1 + A_{14} + A_2$	$\mathbb{Z}_3$	$[2, 0, 10]$	1	0	1	0	$\frac{w_1}{3} \times \frac{w_2}{6}$	0×0
		$\mathbb{1}$	$[12, 6, 18]$	1	0	1	0	$\left(\frac{2w_3}{3} - \frac{5w_7}{6}\right) \times \frac{w_7}{4}$	$\left(\frac{w_3}{3} - \frac{w_7}{6}\right) \times 0$
104	$A_1 + A_{14} + A_3$	$\mathbb{1}$	$[10, 0, 12]$	1	0	1	0	$\frac{w_1}{3} \times \frac{w_2}{6}$	$0 \times \left(w_7 - \frac{2w_2}{3}\right)$
105	$A_{14} + A_4$	$\mathbb{1}$	$[10, 5, 10]$	1	0	1	0	$\frac{w_1}{3} \times \left(\frac{2w_4}{5} - \frac{w_1}{5}\right)$	$0 \times \left(\frac{3w_4}{5} - \frac{4w_1}{5}\right)$
106	$3A_1 + A_{15}$	$\mathbb{Z}_4$	$[2, 0, 4]$	1	0	1	0	$\frac{w_7}{4} \times \frac{w_7}{4}$	0×0
107	$A_1 + A_{15} + A_2$	$\mathbb{Z}_2$	$[4, 0, 6]$	1	0	1	0	$\left(\frac{w_1}{2} - \frac{3w_6}{4}\right) \times \frac{w_7}{4}$	$\frac{w_6}{2} \times 0$
		$\mathbb{1}$	$[10, 2, 10]$	1	0	1	0	$\frac{w_1}{3} \times \left(\frac{2w_3}{3} - \frac{5w_7}{6}\right)$	$0 \times \left(\frac{w_3}{3} - \frac{w_7}{6}\right)$
108	$A_{15} + A_3$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$[2, 0, 2]$	1	0	1	0	$\left(\frac{w_1}{2} - \frac{3w_6}{4}\right) \times \left(\frac{w_1}{2} - \frac{3w_6}{4}\right)$	$\frac{w_6}{2} \times \frac{w_6}{2}$
109	$2A_1 + A_{16}$	$\mathbb{1}$	$[4, 2, 18]$	1	0	1	-1	$\frac{w_7}{4} \times 0$	$0 \times \frac{w_7}{4}$
			$[2, 0, 34]$	1	0	1	0	$\frac{w_1}{3} \times \frac{w_7}{4}$	0×0

110	$A_{16} + A_2$	$\mathbb{1}$	$[6, 3, 10]$	1	0	1	0	$\frac{w_1}{3} \times \left(\frac{w_1}{2} - \frac{3w_6}{4}\right)$	$0 \times \frac{w_6}{2}$
111	$A_1 + A_{17}$	$\mathbb{1}$	$[4, 2, 10]$	1	0	1	-1	$\frac{w_1}{3} \times 0$	$0 \times \frac{w_7}{4}$
		$\mathbb{Z}_3$	$[2, 0, 2]$	1	0	1	0	$\frac{w_1}{3} \times \frac{w_1}{3}$	0×0
112	A_{18}	$\mathbb{1}$	$[2, 1, 10]$	1	0	1	-1	$\frac{w_1}{3} \times 0$	$0 \times \frac{w_1}{3}$
113	$2A_4 + 2D_5$	$\mathbb{1}$	$[20, 0, 20]$	1	0	1	0	$\frac{w_4}{4} \times 0$	$0 \times \frac{w_4}{4}$
114	$A_3 + 2A_5 + D_5$	$\mathbb{Z}_2$	$[12, 0, 12]$	1	0	1	0	$\frac{w_4}{4} \times \frac{w_6}{2}$	$0 \times \left(\frac{w_7}{3} - \frac{2w_6}{3}\right)$
115	$2A_4 + A_5 + D_5$	$\mathbb{1}$	$[20, 0, 30]$	1	0	1	0	$\frac{w_3}{5} \times 0$	$0 \times \frac{w_4}{4}$
116	$A_1 + A_3 + A_4 + A_5 + D_5$	$\mathbb{Z}_2$	$[12, 0, 20]$	1	0	1	0	$\frac{w_4}{4} \times \frac{w_6}{2}$	$0 \times \left(\frac{w_2}{4} - \frac{3w_6}{4}\right)$
117	$A_1 + 2A_6 + D_5$	$\mathbb{1}$	$[14, 0, 28]$	1	0	1	0	$\frac{w_4}{4} \times \frac{w_5}{3}$	$0 \times \left(\frac{w_3}{2} - \frac{5w_5}{6}\right)$
118	$2A_2 + A_3 + A_6 + D_5$	$\mathbb{1}$	$[12, 0, 84]$	1	0	1	0	$\frac{w_4}{4} \times \frac{w_5}{3}$	$0 \times \left(\frac{w_2}{3} - \frac{2w_5}{3}\right)$
119	$A_1 + A_2 + A_4 + A_6 + D_5$	$\mathbb{1}$	$[20, 0, 42]$	1	0	1	0	$\frac{w_2}{6} \times 0$	$0 \times \frac{w_4}{4}$
120	$A_2 + A_5 + A_6 + D_5$	$\mathbb{1}$	$[6, 0, 84]$	1	0	1	0	$\frac{w_4}{4} \times \frac{w_5}{3}$	$0 \times \left(\frac{w_1}{2} - \frac{w_5}{2}\right)$
			$[12, 0, 42]$	1	0	1	0	$\frac{w_4}{4} \times \frac{w_6}{2}$	$0 \times \left(\frac{w_3}{3} - \frac{5w_6}{6}\right)$
121	$A_1 + A_7 + 2D_5$	$\mathbb{Z}_4$	$[2, 0, 8]$	1	0	1	0	$\frac{w_4}{4} \times \frac{w_4}{4}$	0×0
122	$A_1 + A_2 + A_3 + A_7 + D_5$	$\mathbb{Z}_4$	$[6, 0, 8]$	1	0	1	0	$\frac{w_4}{4} \times \frac{w_4}{4}$	$0 \times \left(\frac{w_4}{2} - \frac{w_7}{2}\right)$
123	$2A_1 + A_4 + A_7 + D_5$	$\mathbb{Z}_2$	$[8, 0, 20]$	1	0	1	0	$\frac{w_4}{4} \times \frac{w_4}{4}$	$\left(\frac{w_2}{2} - \frac{3w_4}{4}\right) \times 0$
124	$A_8 + 2D_5$	$\mathbb{1}$	$[8, 4, 20]$	1	0	1	0	$\frac{w_3}{5} \times \frac{w_4}{4}$	$\left(w_2 - \frac{6w_3}{5}\right) \times 0$
125	$A_1 + A_4 + A_8 + D_5$	$\mathbb{1}$	$[2, 0, 180]$	1	0	1	0	$\frac{w_3}{5} \times \frac{w_4}{4}$	0×0
			$[18, 0, 20]$	1	0	1	0	$0 \times \frac{w_7}{4}$	$\frac{w_4}{4} \times 0$
126	$A_5 + A_8 + D_5$	$\mathbb{1}$	$[12, 0, 18]$	1	0	1	0	$\frac{w_3}{5} \times \frac{w_4}{4}$	$\left(w_1 - \frac{3w_3}{5}\right) \times 0$
127	$2A_2 + A_9 + D_5$	$\mathbb{1}$	$[6, 0, 60]$	1	0	1	0	$\frac{w_4}{4} \times \left(\frac{w_3}{3} - \frac{5w_6}{6}\right)$	$0 \times \frac{w_6}{2}$
128	$2A_1 + A_2 + A_9 + D_5$	$\mathbb{Z}_2$	$[2, 0, 60]$	1	0	1	0	$\frac{w_2}{6} \times \frac{w_4}{4}$	0×0
129	$A_1 + A_3 + A_9 + D_5$	$\mathbb{Z}_2$	$[8, 4, 12]$	1	0	1	0	$\frac{w_2}{6} \times \frac{w_4}{4}$	$\left(w_7 - \frac{2w_2}{3}\right) \times 0$
130	$A_4 + A_9 + D_5$	$\mathbb{1}$	$[10, 0, 20]$	1	0	1	0	$\frac{w_1}{3} \times 0$	$0 \times \frac{w_4}{4}$
131	$A_1 + A_{10} + A_2 + D_5$	$\mathbb{1}$	$[14, 4, 20]$	1	0	1	0	$\frac{w_2}{6} \times \frac{w_3}{5}$	$0 \times \left(w_2 - \frac{6w_3}{5}\right)$
132	$2A_1 + A_{11} + D_5$	$\mathbb{Z}_4$	$[2, 0, 6]$	1	0	1	0	$\frac{w_4}{4} \times \frac{w_7}{4}$	0×0
133	$A_{11} + A_2 + D_5$	$\mathbb{Z}_2$	$[6, 0, 6]$	1	0	1	0	$\frac{w_4}{4} \times \left(\frac{w_1}{2} - \frac{3w_6}{4}\right)$	$0 \times \frac{w_6}{2}$
134	$A_1 + A_{12} + D_5$	$\mathbb{1}$	$[2, 0, 52]$	1	0	1	0	$\frac{w_1}{3} \times \frac{w_4}{4}$	0×0
			$[6, 2, 18]$	1	0	1	0	$\frac{w_3}{5} \times \frac{w_7}{4}$	$\left(w_2 - \frac{6w_3}{5}\right) \times 0$
135	$A_{13} + D_5$	$\mathbb{1}$	$[6, 2, 10]$	1	0	1	0	$\frac{w_1}{3} \times \frac{w_3}{5}$	$0 \times \left(w_2 - \frac{6w_3}{5}\right)$
136	$3D_6$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$[2, 0, 2]$	1	0	1	0	$\frac{w_6}{2} \times \frac{w_6}{2}$	$\left(\frac{w_8}{2} - \frac{w_6}{2}\right) \times \left(\frac{w_8}{2} - w_6\right)$
137	$2A_3 + 2D_6$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$[4, 0, 4]$	1	0	1	0	$\frac{w_6}{2} \times \frac{w_6}{2}$	$\left(\frac{w_6}{2} - \frac{w_8}{2}\right) \times \left(\frac{w_6}{2} - \frac{w_8}{2}\right)$
138	$2A_2 + 2A_4 + D_6$	$\mathbb{1}$	$[30, 0, 30]$	1	0	1	0	$\frac{w_5}{3} \times \frac{w_6}{2}$	$\left(\frac{w_2}{3} - \frac{2w_5}{3}\right) \times \left(\frac{w_6}{2} - \frac{w_8}{2}\right)$
139	$2A_1 + 2A_5 + D_6$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$[6, 0, 6]$	1	0	1	0	$\frac{w_4}{4} \times \frac{w_6}{2}$	$\left(\frac{w_2}{2} - \frac{3w_4}{4}\right) \times \left(\frac{w_6}{2} - \frac{w_8}{2}\right)$
140	$A_1 + 2A_3 + A_5 + D_6$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$[4, 0, 12]$	1	0	1	0	$\frac{w_4}{4} \times \frac{w_6}{2}$	$\left(\frac{w_4}{2} - \frac{w_7}{2}\right) \times \left(\frac{w_6}{2} - \frac{w_8}{2}\right)$
141	$A_3 + A_4 + A_5 + D_6$	$\mathbb{Z}_2$	$[4, 0, 30]$	1	0	1	0	$\frac{w_5}{3} \times \frac{w_6}{2}$	$\left(\frac{w_1}{2} - \frac{w_5}{2}\right) \times \left(\frac{w_6}{2} - \frac{w_8}{2}\right)$
142	$2A_6 + D_6$	$\mathbb{1}$	$[14, 0, 14]$	1	0	1	0	$\frac{w_3}{5} \times \frac{w_6}{2}$	$\left(w_1 - \frac{3w_3}{5}\right) \times \left(\frac{w_6}{2} - \frac{w_8}{2}\right)$
143	$A_2 + A_4 + A_6 + D_6$	$\mathbb{1}$	$[6, 0, 70]$	1	0	1	0	$\frac{w_3}{5} \times \frac{w_6}{2}$	$0 \times \left(\frac{w_6}{2} - \frac{w_8}{2}\right)$
144	$A_1 + 2A_2 + A_7 + D_6$	$\mathbb{Z}_2$	$[6, 0, 24]$	1	0	1	0	$\frac{w_2}{6} \times \frac{w_6}{2}$	$0 \times \left(\frac{w_6}{2} - \frac{w_8}{2}\right)$
145	$A_2 + A_3 + A_7 + D_6$	$\mathbb{Z}_2$	$[4, 0, 24]$	1	0	1	0	$\frac{w_6}{2} \times \left(\frac{w_3}{3} - \frac{5w_6}{6}\right)$	$\left(\frac{w_8}{2} - \frac{w_6}{2}\right) \times \frac{w_6}{2}$
146	$A_1 + A_4 + A_7 + D_6$	$\mathbb{Z}_2$	$[6, 2, 14]$	1	0	1	0	$\frac{w_2}{6} \times \frac{w_6}{2}$	$\left(w_7 - \frac{2w_2}{3}\right) \times \left(\frac{w_6}{2} - \frac{w_8}{2}\right)$
147	$A_4 + A_8 + D_6$	$\mathbb{1}$	$[4, 2, 46]$	1	0	1	0	$\frac{w_3}{5} \times \frac{w_4}{4}$	$0 \times \left(w_8 - \frac{w_4}{2}\right)$
148	$A_1 + A_2 + A_9 + D_6$	$\mathbb{Z}_2$	$[4, 2, 16]$	1	0	1	0	$\frac{w_2}{6} \times \frac{w_4}{4}$	$0 \times \left(w_8 - \frac{w_4}{2}\right)$
			$[6, 0, 10]$	1	0	1	0	$\frac{w_6}{2} \times \frac{w_7}{4}$	$\left(\frac{w_8}{2} - \frac{w_6}{2}\right) \times 0$
149	$A_3 + A_9 + D_6$	$\mathbb{Z}_2$	$[4, 0, 10]$	1	0	1	0	$\frac{w_6}{2} \times \left(\frac{w_1}{2} - \frac{3w_6}{4}\right)$	$\left(\frac{w_8}{2} - \frac{w_6}{2}\right) \times \frac{w_6}{2}$
150	$A_{10} + A_2 + D_6$	$\mathbb{1}$	$[6, 0, 22]$	1	0	1	0	$\frac{w_1}{3} \times \frac{w_6}{2}$	$0 \times \left(\frac{w_6}{2} - \frac{w_8}{2}\right)$
151	$A_1 + A_{11} + D_6$	$\mathbb{Z}_2$	$[4, 0, 6]$	1	0	1	0	$\frac{w_4}{4} \times \frac{w_7}{4}$	$\left(w_8 - \frac{w_4}{2}\right) \times 0$
152	$A_{12} + D_6$	$\mathbb{1}$	$[4, 2, 14]$	1	0	1	0	$\frac{w_1}{3} \times \frac{w_4}{4}$	$0 \times \left(w_8 - \frac{w_4}{2}\right)$
153	$A_2 + A_5 + D_5 + D_6$	$\mathbb{Z}_2$	$[6, 0, 12]$	1	0	1	0	$\frac{w_4}{4} \times \frac{w_6}{2}$	$0 \times \left(\frac{w_6}{2} - \frac{w_8}{2}\right)$
154	$A_7 + D_5 + D_6$	$\mathbb{Z}_2$	$[4, 0, 8]$	1	0	1	0	$\frac{w_4}{4} \times \frac{w_4}{4}$	$0 \times \left(w_8 - \frac{w_4}{2}\right)$
155	$2A_2 + 2D_7$	$\mathbb{1}$	$[12, 0, 12]$	1	0	1	$\frac{1}{2}$	$\frac{w_5}{3} \times \left(\frac{w_4}{2} - w_8\right)$	$\left(\frac{w_2}{3} - \frac{2w_5}{3}\right) \times \left(\frac{w_4}{2} - \frac{w_8}{2}\right)$ †
				1	0	1	0	$\left(\frac{w_1}{4} + \frac{w_2}{4} - \frac{w_7}{2}\right) \times \frac{w_8}{2}$	$\left(\frac{w_1}{8} + \frac{5w_7}{4} - \frac{7w_2}{8}\right) \times \frac{3w_8}{8}$
156	$A_2 + 3A_3 + D_7$	$\mathbb{Z}_4$	$[8, 4, 8]$	1	0	1	0	$\left(\frac{w_6}{3} - \frac{w_1}{3}\right) \times \frac{w_5}{3}$	$\left(\frac{w_1}{3} - \frac{w_6}{3}\right) \times \left(\frac{w_2}{3} - \frac{2w_5}{3}\right)$
157	$A_1 + A_2 + 2A_4 + D_7$	$\mathbb{1}$	$[10, 0, 60]$	1	0	1	0	$\left(\frac{w_1}{4} + \frac{w_4}{4} - \frac{w_7}{2}\right) \times \frac{w_8}{2}$	$\left(\frac{3w_1}{4} + \frac{w_4}{12} - \frac{w_7}{2}\right) \times \frac{w_8}{4}$
158	$A_2 + A_3 + A_6 + D_7$	$\mathbb{1}$	$[8, 4, 44]$	1	0	1	0	$\left(\frac{w_1}{2} - \frac{w_7}{2}\right) \times \frac{3w_8}{8}$	$\left(\frac{2w_1}{3} - \frac{w_7}{3}\right) \times \frac{w_8}{3}$
159	$A_1 + A_4 + A_6 + D_7$	$\mathbb{1}$	$[4, 0, 70]$	1	0	1	0	$\frac{w_3}{5} \times \frac{w_6}{2}$	$0 \times \left(\frac{w_4}{2} - w_6\right)$
160	$A_5 + A_6 + D_7$	$\mathbb{1}$	$[2, 0, 84]$	1	0	1	0	$\left(\frac{w_5}{2} - \frac{w_1}{2}\right) \times \left(w_8 - \frac{w_4}{2}\right)$	$\frac{w_5}{3} \times \frac{w_4}{4}$
161	$2A_1 + A_2 + A_7 + D_7$	$\mathbb{Z}_2$	$[4, 0, 24]$	1	0	1	0	$\frac{w_2}{6} \times \frac{w_6}{2}$	$0 \times \left(\frac{w_4}{2} - w_6\right)$
162	$A_1 + A_3 + A_7 + D_7$	$\mathbb{Z}_4$	$[2, 0, 8]$	1	0	1	0	$\frac{w_4}{4} \times \frac{w_4}{4}$	$\left(\frac{w_7}{2} - \frac{w_4}{2}\right) \times \left(w_8 - \frac{w_4}{2}\right)$
163	$2A_1 + A_9 + D_7$	$\mathbb{Z}_2$	$[4, 0, 10]$	1	0	1	0	$\frac{w_6}{2} \times \frac{w_7}{4}$	$\left(\frac{w_4}{2} - w_6\right) \times 0$
164	$A_2 + A_9 + D_7$	$\mathbb{1}$	$[2, 0, 60]$	1	0	1	0	$\frac{w_4}{4} \times \left(\frac{w_3}{3} - \frac{5w_6}{6}\right)$	$\left(w_8 - \frac{w_4}{2}\right) \times \frac{w_6}{2}$

165	$A_1 + A_{10} + D_7$	$\mathbb{1}$	$[4, 0, 22]$	1	0	1	0	$\frac{w_1}{3} \times \frac{w_6}{2}$	$0 \times \left(\frac{w_4}{2} - w_6\right)$
166	$A_{11} + D_7$	$\mathbb{Z}_4$	$[2, 1, 2]$	1	0	1	0	$\frac{w_4}{4} \times \left(\frac{w_1}{2} - \frac{3w_6}{4}\right)$	$\left(w_8 - \frac{w_4}{2}\right) \times \frac{w_6}{2}$
167	$A_1 + A_5 + D_5 + D_7$	$\mathbb{Z}_2$	$[4, 0, 12]$	1	0	1	0	$\frac{w_4}{4} \times \frac{w_6}{2}$	$0 \times \left(\frac{w_4}{2} - w_6\right)$
168	$A_5 + D_6 + D_7$	$\mathbb{Z}_2$	$[2, 0, 12]$	1	0	1	0	$\left(w_8 - \frac{w_4}{2}\right) \times \frac{w_6}{2}$	$\frac{w_4}{4} \times \left(\frac{w_6}{2} - \frac{w_8}{2}\right)$
169	$2A_1 + 2D_8$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$[2, 0, 2]$	1	0	1	0	$\frac{w_6}{2} \times \left(\frac{w_4}{2} - w_6\right)$	$\left(\frac{w_4}{2} - w_6\right) \times \frac{w_6}{2}$
170	$2A_2 + 2A_3 + D_8$	$\mathbb{Z}_2$	$[12, 0, 12]$	1	0	1	0	$\frac{w_6}{2} \times \frac{w_2}{6}$	$\left(\frac{w_8}{2} - \frac{w_6}{2}\right) \times \left(\frac{3w_3}{4} - \frac{w_2}{2}\right)$
171	$2A_5 + D_8$	$\mathbb{Z}_2$	$[6, 0, 6]$	1	0	1	0	$\frac{w_5}{3} \times \left(w_8 - \frac{w_4}{2}\right)$	$\left(\frac{w_5}{2} - \frac{w_1}{2}\right) \times \frac{w_4}{4}$
172	$2A_1 + A_3 + A_5 + D_8$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$[2, 0, 12]$	1	0	1	0	$\frac{w_6}{2} \times \left(\frac{w_4}{2} - w_6\right)$	$\left(\frac{w_2}{4} - \frac{3w_6}{4}\right) \times \frac{w_6}{2}$
173	$A_1 + A_4 + A_5 + D_8$	$\mathbb{Z}_2$	$[2, 0, 30]$	1	0	1	0	$\frac{w_6}{2} \times \left(\frac{w_4}{2} - w_6\right)$	$\left(\frac{w_7}{3} - \frac{2w_6}{3}\right) \times \frac{w_6}{2}$
174	$2A_2 + A_6 + D_8$	$\mathbb{1}$	$[12, 6, 24]$	1	0	1	0	$\left(w_8 - \frac{w_4}{2}\right) \times \frac{w_5}{3}$	$\frac{w_4}{4} \times \left(\frac{w_2}{3} - \frac{2w_5}{3}\right)$
175	$A_1 + A_2 + A_7 + D_8$	$\mathbb{Z}_2$	$[2, 0, 24]$	1	0	1	0	$\frac{w_6}{2} \times \left(\frac{w_4}{2} - w_6\right)$	$\left(\frac{w_3}{3} - \frac{5w_6}{6}\right) \times \frac{w_6}{2}$
176	$A_1 + A_9 + D_8$	$\mathbb{Z}_2$	$[2, 0, 10]$	1	0	1	0	$\frac{w_6}{2} \times \left(\frac{w_4}{2} - w_6\right)$	$\left(\frac{w_1}{2} - \frac{3w_6}{4}\right) \times \frac{w_6}{2}$
177	$2D_5 + D_8$	$\mathbb{Z}_2$	$[4, 0, 4]$	1	0	1	0	$\frac{w_4}{4} \times \frac{w_4}{4}$	$0 \times \left(w_5 - \frac{w_4}{2}\right)$
178	$A_1 + A_3 + D_6 + D_8$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$[2, 0, 4]$	1	0	1	0	$\frac{w_6}{2} \times \left(\frac{w_4}{2} - w_6\right)$	$\left(\frac{w_8}{2} - \frac{w_6}{2}\right) \times \frac{w_6}{2}$
179	$2D_9$	$\mathbb{1}$	$[4, 0, 4]$	1	0	1	0	$\frac{w_8}{2} \times 0$	$0 \times \frac{w_8}{2}$
180	$A_1 + 2A_2 + A_4 + D_9$	$\mathbb{1}$	$[12, 0, 30]$	1	0	1	0	$\frac{w_5}{3} \times \left(\frac{w_4}{2} - w_6\right)$	$\left(\frac{w_2}{3} - \frac{2w_5}{3}\right) \times \frac{w_6}{2}$
181	$A_1 + A_3 + A_5 + D_9$	$\mathbb{Z}_2$	$[4, 0, 12]$	1	0	1	0	$\frac{w_4}{4} \times \left(w_8 - \frac{w_4}{2}\right)$	$\left(\frac{w_7}{2} - \frac{w_4}{2}\right) \times \frac{w_4}{4}$
182	$A_4 + A_5 + D_9$	$\mathbb{1}$	$[4, 0, 30]$	1	0	1	0	$0 \times \frac{w_8}{2}$	$\frac{w_3}{5} \times 0$
183	$A_1 + A_2 + A_6 + D_9$	$\mathbb{1}$	$[4, 0, 42]$	1	0	1	0	$0 \times \frac{w_8}{2}$	$\frac{w_2}{6} \times 0$
184	$2A_1 + A_7 + D_9$	$\mathbb{Z}_2$	$[4, 0, 8]$	1	0	1	0	$\frac{w_4}{4} \times \left(w_8 - \frac{w_4}{2}\right)$	$\left(\frac{w_2}{2} - \frac{3w_4}{4}\right) \times \frac{w_4}{4}$
185	$A_1 + A_8 + D_9$	$\mathbb{1}$	$[4, 0, 18]$	1	0	1	0	$0 \times \frac{w_8}{2}$	$\frac{w_7}{4} \times 0$
186	$A_9 + D_9$	$\mathbb{1}$	$[4, 0, 10]$	1	0	1	0	$0 \times \frac{w_8}{2}$	$\frac{w_1}{3} \times 0$
187	$A_4 + D_5 + D_9$	$\mathbb{1}$	$[4, 0, 20]$	1	0	1	0	$0 \times \frac{w_8}{2}$	$\frac{w_4}{4} \times 0$
188	$2A_1 + 2A_3 + D_{10}$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$[4, 0, 4]$	1	0	1	0	$\frac{w_4}{4} \times \left(\frac{w_4}{2} - w_6\right)$	$\left(\frac{w_7}{2} - \frac{w_4}{2}\right) \times \frac{w_6}{2}$
189	$2A_4 + D_{10}$	$\mathbb{1}$	$[10, 0, 10]$	1	0	1	0	$\frac{w_3}{5} \times \left(w_8 - \frac{w_4}{2}\right)$	$0 \times \frac{w_4}{4}$
190	$A_1 + A_3 + A_4 + D_{10}$	$\mathbb{Z}_2$	$[2, 0, 20]$	1	0	1	0	$\frac{w_6}{2} \times \frac{w_8}{2}$	$\left(\frac{w_2}{4} - \frac{3w_6}{4}\right) \times 0$
191	$3A_1 + A_5 + D_{10}$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$[4, 2, 4]$	1	0	1	0	$\frac{w_4}{4} \times \left(\frac{w_4}{2} - w_6\right)$	$\left(\frac{w_2}{2} - \frac{3w_4}{4}\right) \times \frac{w_6}{2}$
192	$A_3 + A_5 + D_{10}$	$\mathbb{Z}_2$	$[2, 0, 12]$	1	0	1	0	$\frac{w_6}{2} \times \frac{w_8}{2}$	$\left(\frac{w_7}{3} - \frac{2w_6}{3}\right) \times 0$
193	$A_2 + A_6 + D_{10}$	$\mathbb{1}$	$[2, 0, 42]$	1	0	1	0	$\frac{w_6}{2} \times \frac{w_8}{2}$	$\left(\frac{w_3}{3} - \frac{5w_6}{6}\right) \times 0$
194	$A_8 + D_{10}$	$\mathbb{1}$	$[2, 0, 18]$	1	0	1	0	$\frac{w_6}{2} \times \frac{w_8}{2}$	$\left(\frac{w_1}{2} - \frac{3w_6}{4}\right) \times 0$
195	$A_1 + A_2 + D_{10} + D_5$	$\mathbb{Z}_2$	$[4, 0, 6]$	1	0	1	0	$\frac{w_4}{4} \times \left(\frac{w_4}{2} - w_6\right)$	$0 \times \frac{w_6}{2}$
196	$A_2 + D_{10} + D_6$	$\mathbb{Z}_2$	$[2, 0, 6]$	1	0	1	0	$\frac{w_6}{2} \times \frac{w_8}{2}$	$\left(\frac{w_6}{2} - \frac{w_8}{2}\right) \times 0$
197	$A_1 + D_{10} + D_7$	$\mathbb{Z}_2$	$[2, 0, 4]$	1	0	1	0	$\frac{w_6}{2} \times \frac{w_8}{2}$	$\left(\frac{w_4}{2} - w_6\right) \times 0$
198	$2A_2 + A_3 + D_{11}$	$\mathbb{1}$	$[12, 0, 12]$	1	0	1	0	$\frac{w_5}{3} \times \frac{w_8}{2}$	$\left(\frac{w_2}{3} - \frac{2w_5}{3}\right) \times 0$
199	$A_1 + A_2 + A_4 + D_{11}$	$\mathbb{1}$	$[6, 0, 20]$	1	0	1	0	$\frac{w_2}{6} \times \left(w_8 - \frac{w_4}{2}\right)$	$0 \times \frac{w_4}{4}$
200	$A_2 + A_5 + D_{11}$	$\mathbb{1}$	$[6, 0, 12]$	1	0	1	0	$\frac{w_5}{3} \times \frac{w_8}{2}$	$\left(\frac{w_1}{2} - \frac{w_5}{2}\right) \times 0$
201	$A_1 + A_6 + D_{11}$	$\mathbb{1}$	$[6, 2, 10]$	1	0	1	0	$\frac{w_5}{3} \times \frac{w_8}{2}$	$\left(\frac{w_3}{2} - \frac{5w_5}{6}\right) \times 0$
202	$2A_1 + 2A_2 + D_{12}$	$\mathbb{Z}_2$	$[6, 0, 6]$	1	0	1	0	$\frac{w_2}{6} \times \left(\frac{w_4}{2} - w_6\right)$	$0 \times \frac{w_6}{2}$
203	$A_1 + A_2 + A_3 + D_{12}$	$\mathbb{Z}_2$	$[4, 0, 6]$	1	0	1	0	$\frac{w_4}{4} \times \frac{w_8}{2}$	$\left(\frac{w_4}{2} - \frac{w_7}{2}\right) \times 0$
204	$2A_1 + A_4 + D_{12}$	$\mathbb{Z}_2$	$[4, 2, 6]$	1	0	1	0	$\frac{w_4}{4} \times \frac{w_8}{2}$	$\left(\frac{w_2}{2} - \frac{3w_4}{4}\right) \times 0$
205	$A_1 + D_{12} + D_5$	$\mathbb{Z}_2$	$[2, 0, 4]$	1	0	1	0	$\frac{w_4}{4} \times \frac{w_8}{2}$	0×0
206	$D_{12} + D_6$	$\mathbb{Z}_2$	$[2, 0, 2]$	1	0	1	0	$\frac{w_4}{4} \times \frac{w_8}{2}$	$\left(w_8 - \frac{w_4}{2}\right) \times 0$
207	$A_1 + A_4 + D_{13}$	$\mathbb{1}$	$[2, 0, 20]$	1	0	1	0	$\frac{w_3}{5} \times \frac{w_8}{2}$	0×0
208	$A_5 + D_{13}$	$\mathbb{1}$	$[2, 0, 12]$	1	0	1	0	$\frac{w_3}{5} \times \frac{w_8}{2}$	$\left(w_1 - \frac{3w_3}{5}\right) \times 0$
209	$D_{13} + D_5$	$\mathbb{1}$	$[4, 0, 4]$	1	0	1	0	$\frac{w_3}{5} \times \frac{w_8}{2}$	$\left(w_2 - \frac{6w_3}{5}\right) \times 0$
210	$2A_2 + D_{14}$	$\mathbb{1}$	$[6, 0, 6]$	1	0	1	0	$\left(\frac{w_3}{3} - \frac{5w_6}{6}\right) \times \frac{w_8}{2}$	$\frac{w_6}{2} \times 0$
211	$2A_1 + A_2 + D_{14}$	$\mathbb{Z}_2$	$[2, 0, 6]$	1	0	1	0	$\frac{w_2}{6} \times \frac{w_8}{2}$	0×0
212	$A_1 + A_3 + D_{14}$	$\mathbb{Z}_2$	$[2, 0, 4]$	1	0	1	0	$\frac{w_2}{6} \times \frac{w_8}{2}$	$\left(w_7 - \frac{2w_2}{3}\right) \times 0$
213	$A_4 + D_{14}$	$\mathbb{1}$	$[4, 2, 6]$	1	0	1	0	$\frac{w_1}{3} \times \left(w_8 - \frac{w_4}{2}\right)$	$0 \times \frac{w_4}{4}$
214	$A_1 + A_2 + D_{15}$	$\mathbb{1}$	$[4, 0, 6]$	1	0	1	0	$\frac{w_1}{3} \times \left(\frac{w_4}{2} - w_6\right)$	$0 \times \frac{w_6}{2}$
215	$2A_1 + D_{16}$	$\mathbb{Z}_2$	$[2, 0, 2]$	1	0	1	0	$\frac{w_7}{4} \times \frac{w_8}{2}$	0×0
216	$A_2 + D_{16}$	$\mathbb{Z}_2$	$[2, 1, 2]$	1	0	1	0	$\left(\frac{w_1}{2} - \frac{3w_6}{4}\right) \times \frac{w_8}{2}$	$\frac{w_6}{2} \times 0$
217	$A_1 + D_{17}$	$\mathbb{1}$	$[2, 0, 4]$	1	0	1	0	$\frac{w_1}{3} \times \frac{w_8}{2}$	0×0
218	D_{18}	$\mathbb{1}$	$[2, 0, 2]$	1	0	1	-1	$\frac{w_1}{3} \times 0$	$0 \times \frac{w_8}{2}$
219	$3E_6$	$\mathbb{Z}_3$	$[2, 1, 2]$	1	0	1	-1	$\frac{w_5}{3} \times \frac{w_5}{3}$	0×0
220	$2A_3 + 2E_6$	$\mathbb{1}$	$[12, 0, 12]$	1	0	1	0	$\frac{w_5}{3} \times 0$	$0 \times \frac{w_5}{3}$
221	$A_1 + A_3 + 2A_4 + E_6$	$\mathbb{1}$	$[20, 0, 30]$	1	0	1	0	$\frac{w_5}{3} \times \frac{w_6}{2}$	$0 \times \left(\frac{w_2}{4} - \frac{3w_6}{4}\right)$
222	$A_1 + A_5 + 2E_6$	$\mathbb{Z}_3$	$[2, 0, 6]$	1	0	1	0	$\frac{w_5}{3} \times \frac{w_5}{3}$	0×0
223	$A_2 + 2A_5 + E_6$	$\mathbb{Z}_3$	$[6, 0, 6]$	1	0	1	0	$\frac{w_5}{3} \times \frac{w_5}{3}$	$\left(\frac{w_1}{2} - \frac{w_5}{2}\right) \times 0$
224	$2A_2 + A_3 + A_5 + E_6$	$\mathbb{Z}_3$	$[6, 0, 12]$	1	0	1	0	$\frac{w_5}{3} \times \frac{w_5}{3}$	$\left(\frac{w_2}{3} - \frac{2w_5}{3}\right) \times 0$
225	$A_3 + A_4 + A_5 + E_6$	$\mathbb{1}$	$[12, 0, 30]$	1	0	1	0	$\frac{w_3}{5} \times 0$	$0 \times \frac{w_5}{3}$
226	$A_6 + 2E_6$	$\mathbb{1}$	$[6, 3, 12]$	1	0	1	0	$\frac{w_4}{4} \times \frac{w_5}{3}$	$\left(w_3 - \frac{5w_4}{4}\right) \times 0$

227	$A_1 + A_2 + A_3 + A_6 + E_6$	$\mathbb{1}$	$[6, 0, 84]$	1	0	1	0	$\frac{w_4}{4} \times \frac{w_5}{3}$	$\left(\frac{w_7}{2} - \frac{w_4}{2}\right) \times 0$
			$[12, 0, 42]$	1	0	1	0	$\frac{w_2}{6} \times 0$	$0 \times \frac{w_5}{3}$
228	$2A_1 + A_4 + A_6 + E_6$	$\mathbb{1}$	$[20, 10, 26]$	1	0	1	0	$\frac{w_4}{4} \times \frac{w_5}{3}$	$\left(\frac{w_2}{2} - \frac{3w_4}{4}\right) \times 0$
229	$A_2 + A_4 + A_6 + E_6$	$\mathbb{1}$	$[18, 3, 18]$	1	0	1	0	$\frac{w_5}{3} \times \frac{w_6}{2}$	$0 \times \left(\frac{w_3}{3} - \frac{5w_6}{6}\right)$
230	$A_1 + A_5 + A_6 + E_6$	$\mathbb{1}$	$[6, 0, 42]$	1	0	1	0	$\frac{w_5}{3} \times \frac{w_5}{3}$	$0 \times \left(\frac{2w_5}{3} - \frac{w_7}{2}\right)$
231	$A_1 + A_4 + A_7 + E_6$	$\mathbb{1}$	$[2, 0, 120]$	1	0	1	0	$\frac{w_3}{5} \times \frac{w_5}{3}$	0×0
232	$A_5 + A_7 + E_6$	$\mathbb{1}$	$[6, 0, 24]$	1	0	1	0	$\frac{w_3}{5} \times \frac{w_5}{3}$	$\left(w_1 - \frac{3w_3}{5}\right) \times 0$
233	$2A_2 + A_8 + E_6$	$\mathbb{Z}_3$	$[6, 3, 6]$	1	0	1	0	$\frac{w_5}{3} \times \left(\frac{w_3}{3} - \frac{5w_6}{6}\right)$	$0 \times \frac{w_6}{2}$
234	$2A_1 + A_2 + A_8 + E_6$	$\mathbb{Z}_3$	$[2, 0, 18]$	1	0	1	0	$\frac{w_2}{6} \times \frac{w_5}{3}$	0×0
235	$A_1 + A_3 + A_8 + E_6$	$\mathbb{1}$	$[12, 0, 18]$	1	0	1	0	$0 \times \frac{w_7}{4}$	$\frac{w_5}{3} \times 0$
236	$A_4 + A_8 + E_6$	$\mathbb{1}$	$[12, 3, 12]$	1	0	1	0	$\frac{w_5}{3} \times \frac{w_6}{2}$	$0 \times \left(\frac{w_1}{2} - \frac{3w_6}{4}\right)$
237	$A_1 + A_2 + A_9 + E_6$	$\mathbb{1}$	$[12, 6, 18]$	1	0	1	0	$\frac{w_2}{6} \times \frac{w_4}{4}$	$0 \times \left(w_3 - \frac{5w_4}{4}\right)$
238	$A_3 + A_9 + E_6$	$\mathbb{1}$	$[10, 0, 12]$	1	0	1	0	$\frac{w_1}{3} \times 0$	$0 \times \frac{w_5}{3}$
239	$2A_1 + A_{10} + E_6$	$\mathbb{1}$	$[2, 0, 66]$	1	0	1	0	$\frac{w_5}{3} \times \frac{w_7}{4}$	0×0
240	$A_{10} + A_2 + E_6$	$\mathbb{1}$	$[6, 3, 18]$	1	0	1	0	$\frac{w_5}{3} \times \left(\frac{w_1}{2} - \frac{3w_6}{4}\right)$	$0 \times \frac{w_6}{2}$
241	$A_1 + A_{11} + E_6$	$\mathbb{Z}_3$	$[2, 0, 4]$	1	0	1	0	$\frac{w_1}{3} \times \frac{w_5}{3}$	0×0
		$\mathbb{1}$	$[6, 0, 12]$	1	0	1	0	$\frac{w_4}{4} \times \frac{w_7}{4}$	$\left(w_3 - \frac{5w_4}{4}\right) \times 0$
242	$A_{12} + E_6$	$\mathbb{1}$	$[4, 1, 10]$	1	0	1	0	$\frac{w_1}{3} \times \frac{w_4}{4}$	$0 \times \left(w_3 - \frac{5w_4}{4}\right)$
243	$A_3 + A_4 + D_5 + E_6$	$\mathbb{1}$	$[12, 0, 20]$	1	0	1	0	$\frac{w_4}{4} \times 0$	$0 \times \frac{w_5}{3}$
244	$A_1 + A_6 + D_5 + E_6$	$\mathbb{1}$	$[2, 0, 84]$	1	0	1	0	$\frac{w_4}{4} \times \frac{w_5}{3}$	0×0
245	$A_7 + D_5 + E_6$	$\mathbb{1}$	$[8, 0, 12]$	1	0	1	0	$\frac{w_4}{4} \times \frac{w_4}{4}$	$\left(w_3 - \frac{5w_4}{4}\right) \times 0$
246	$D_6 + 2E_6$	$\mathbb{1}$	$[6, 0, 6]$	1	0	1	-1	$\frac{w_5}{3} \times \frac{w_6}{2}$	$0 \times \left(\frac{w_6}{2} - \frac{w_8}{2}\right)$
247	$A_2 + A_4 + D_6 + E_6$	$\mathbb{1}$	$[6, 0, 30]$	1	0	1	0	$\frac{w_5}{3} \times \frac{w_6}{2}$	$0 \times \left(\frac{w_6}{2} - \frac{w_8}{2}\right)$
248	$A_6 + D_6 + E_6$	$\mathbb{1}$	$[4, 2, 22]$	1	0	1	0	$\frac{w_4}{4} \times \frac{w_5}{3}$	$\left(w_8 - \frac{w_4}{2}\right) \times 0$
249	$A_1 + A_4 + D_7 + E_6$	$\mathbb{1}$	$[4, 0, 30]$	1	0	1	0	$\frac{w_5}{3} \times \frac{w_6}{2}$	$0 \times \left(\frac{w_4}{2} - w_6\right)$
250	$D_5 + D_7 + E_6$	$\mathbb{1}$	$[4, 0, 12]$	1	0	1	0	$\frac{w_4}{4} \times \frac{w_8}{2}$	$0 \times \frac{w_8}{4}$
251	$A_4 + D_8 + E_6$	$\mathbb{1}$	$[8, 2, 8]$	1	0	1	0	$\left(w_8 - \frac{w_4}{2}\right) \times \frac{w_5}{3}$	$\frac{w_4}{4} \times 0$
252	$A_1 + A_2 + D_9 + E_6$	$\mathbb{1}$	$[6, 0, 12]$	1	0	1	0	$\frac{w_5}{3} \times \left(\frac{w_4}{2} - w_6\right)$	$0 \times \frac{w_6}{2}$
253	$A_3 + D_9 + E_6$	$\mathbb{1}$	$[4, 0, 12]$	1	0	1	0	$0 \times \frac{w_8}{2}$	$\frac{w_5}{3} \times 0$
254	$A_1 + D_{11} + E_6$	$\mathbb{1}$	$[2, 0, 12]$	1	0	1	0	$\frac{w_5}{3} \times \frac{w_8}{2}$	0×0
255	$D_{12} + E_6$	$\mathbb{1}$	$[4, 2, 4]$	1	0	1	0	$\frac{w_4}{4} \times \frac{w_8}{2}$	$\left(w_3 - \frac{5w_4}{4}\right) \times 0$
256	$2A_2 + 2E_7$	$\mathbb{1}$	$[6, 0, 6]$	1	0	1	0	$\frac{w_6}{2} \times 0$	$0 \times \frac{w_6}{2}$
257	$A_1 + A_3 + 2E_7$	$\mathbb{Z}_2$	$[2, 0, 4]$	1	0	1	0	$\frac{w_6}{2} \times \frac{w_6}{2}$	0×0
258	$A_4 + 2E_7$	$\mathbb{1}$	$[4, 2, 6]$	1	0	1	0	$\left(w_4 - \frac{4w_5}{3}\right) \times 0$	$\frac{w_5}{3} \times \frac{w_6}{2}$
259	$A_1 + 2A_3 + A_4 + E_7$	$\mathbb{Z}_2$	$[4, 0, 20]$	1	0	1	0	$\frac{w_6}{2} \times \frac{w_6}{2}$	$\left(\frac{w_2}{4} - \frac{3w_6}{4}\right) \times 0$
260	$2A_2 + A_3 + A_4 + E_7$	$\mathbb{1}$	$[12, 0, 30]$	1	0	1	0	$\frac{w_5}{3} \times \frac{w_6}{2}$	$\left(\frac{w_2}{3} - \frac{2w_5}{3}\right) \times 0$
261	$2A_3 + A_5 + E_7$	$\mathbb{Z}_2$	$[4, 0, 12]$	1	0	1	0	$\frac{w_6}{2} \times \frac{w_6}{2}$	$\left(\frac{w_7}{3} - \frac{2w_6}{3}\right) \times 0$
262	$A_1 + A_2 + A_3 + A_5 + E_7$	$\mathbb{Z}_2$	$[6, 0, 12]$	1	0	1	0	$\frac{w_4}{4} \times \frac{w_6}{2}$	$\left(\frac{w_7}{2} - \frac{w_4}{2}\right) \times 0$
263	$2A_1 + A_4 + A_5 + E_7$	$\mathbb{Z}_2$	$[8, 2, 8]$	1	0	1	0	$\frac{w_4}{4} \times \frac{w_6}{2}$	$\left(\frac{w_2}{2} - \frac{3w_4}{4}\right) \times 0$
264	$A_2 + A_4 + A_5 + E_7$	$\mathbb{1}$	$[6, 0, 30]$	1	0	1	0	$\frac{w_3}{5} \times 0$	$0 \times \frac{w_6}{2}$
265	$A_1 + 2A_2 + A_6 + E_7$	$\mathbb{1}$	$[6, 0, 42]$	1	0	1	0	$\frac{w_2}{6} \times 0$	$0 \times \frac{w_6}{2}$
266	$A_2 + A_3 + A_6 + E_7$	$\mathbb{1}$	$[4, 0, 42]$	1	0	1	0	$\frac{w_6}{2} \times \frac{w_6}{2}$	$\left(\frac{w_3}{3} - \frac{5w_6}{6}\right) \times 0$
267	$A_1 + A_4 + A_6 + E_7$	$\mathbb{1}$	$[2, 0, 70]$	1	0	1	0	$\frac{w_3}{5} \times \frac{w_6}{2}$	0×0
			$[8, 2, 18]$	1	0	1	0	$\frac{w_5}{3} \times \frac{w_6}{2}$	$\left(\frac{w_3}{2} - \frac{5w_5}{6}\right) \times 0$
268	$A_5 + A_6 + E_7$	$\mathbb{1}$	$[4, 2, 22]$	1	0	1	0	$\frac{w_3}{5} \times \frac{w_6}{2}$	$\left(w_1 - \frac{3w_3}{5}\right) \times 0$
269	$2A_2 + A_7 + E_7$	$\mathbb{1}$	$[6, 0, 24]$	1	0	1	0	$\left(\frac{w_3}{3} - \frac{5w_6}{6}\right) \times \frac{w_6}{2}$	$\frac{w_6}{2} \times 0$
270	$2A_1 + A_2 + A_7 + E_7$	$\mathbb{Z}_2$	$[2, 0, 24]$	1	0	1	0	$\frac{w_2}{6} \times \frac{w_6}{2}$	0×0
271	$A_1 + A_3 + A_7 + E_7$	$\mathbb{Z}_2$	$[4, 0, 8]$	1	0	1	0	$\frac{w_2}{6} \times \frac{w_6}{2}$	$\left(w_7 - \frac{2w_2}{3}\right) \times 0$
272	$A_4 + A_7 + E_7$	$\mathbb{1}$	$[6, 2, 14]$	1	0	1	0	$\frac{w_3}{5} \times \frac{w_5}{3}$	$0 \times \left(w_4 - \frac{4w_5}{3}\right)$
273	$A_1 + A_2 + A_8 + E_7$	$\mathbb{1}$	$[6, 0, 18]$	1	0	1	0	$0 \times \frac{w_7}{4}$	$\frac{w_6}{2} \times 0$
274	$A_3 + A_8 + E_7$	$\mathbb{1}$	$[4, 0, 18]$	1	0	1	0	$\frac{w_6}{2} \times \frac{w_6}{2}$	$\left(\frac{w_1}{2} - \frac{3w_6}{4}\right) \times 0$
275	$2A_1 + A_9 + E_7$	$\mathbb{Z}_2$	$[2, 0, 10]$	1	0	1	0	$\frac{w_6}{2} \times \frac{w_7}{4}$	0×0
276	$A_2 + A_9 + E_7$	$\mathbb{Z}_2$	$[4, 1, 4]$	1	0	1	0	$\left(\frac{w_1}{2} - \frac{3w_6}{4}\right) \times \frac{w_6}{2}$	$\frac{w_6}{2} \times 0$
		$\mathbb{1}$	$[6, 0, 10]$	1	0	1	0	$\frac{w_1}{3} \times 0$	$0 \times \frac{w_6}{2}$
277	$A_1 + A_{10} + E_7$	$\mathbb{1}$	$[2, 0, 22]$	1	0	1	0	$\frac{w_1}{3} \times \frac{w_6}{2}$	0×0
			$[6, 2, 8]$	1	0	1	0	$\frac{w_5}{3} \times \frac{w_7}{4}$	$\left(w_4 - \frac{4w_5}{3}\right) \times 0$
278	$A_{11} + E_7$	$\mathbb{1}$	$[4, 0, 6]$	1	0	1	0	$\frac{w_1}{3} \times \frac{w_5}{3}$	$0 \times \left(w_4 - \frac{4w_5}{3}\right)$
279	$D_4 + 2E_7$	$\mathbb{Z}_2$	$[2, 0, 2]$	1	0	1	-1	$\frac{w_6}{2} \times \frac{w_6}{2}$	0×0
280	$A_2 + A_4 + D_5 + E_7$	$\mathbb{1}$	$[6, 0, 20]$	1	0	1	0	$\frac{w_4}{4} \times 0$	$0 \times \frac{w_6}{2}$
281	$A_1 + A_5 + D_5 + E_7$	$\mathbb{Z}_2$	$[2, 0, 12]$	1	0	1	0	$\frac{w_4}{4} \times \frac{w_6}{2}$	0×0
282	$A_6 + D_5 + E_7$	$\mathbb{1}$	$[6, 2, 10]$	1	0	1	0	$\frac{w_3}{5} \times \frac{w_6}{2}$	$\left(w_2 - \frac{6w_3}{5}\right) \times 0$
283	$A_2 + A_3 + D_6 + E_7$	$\mathbb{Z}_2$	$[4, 0, 6]$	1	0	1	0	$\frac{w_6}{2} \times \frac{w_6}{2}$	$0 \times \left(\frac{w_6}{2} - \frac{w_8}{2}\right)$
284	$A_5 + D_6 + E_7$	$\mathbb{Z}_2$	$[4, 2, 4]$	1	0	1	0	$\frac{w_4}{4} \times \frac{w_6}{2}$	$\left(w_8 - \frac{w_4}{2}\right) \times 0$
285	$D_5 + D_6 + E_7$	$\mathbb{Z}_2$	$[2, 0, 4]$	1	0	1	-1	$\frac{w_6}{2} \times \frac{w_4}{4}$	0×0
286	$A_1 + A_3 + D_7 + E_7$	$\mathbb{Z}_2$	$[4, 0, 4]$	1	0	1	0	$\frac{w_6}{2} \times \frac{w_6}{2}$	$\left(\frac{w_4}{2} - w_6\right) \times 0$

287	$A_4 + D_7 + E_7$	$\mathbb{1}$	$[2, 0, 20]$	1	0	1	-1	$\frac{w_6}{2} \times \frac{w_3}{5}$	0×0
288	$A_1 + A_2 + D_8 + E_7$	$\mathbb{Z}_2$	$[2, 0, 6]$	1	0	1	-1	$\frac{w_6}{2} \times \frac{w_2}{6}$	0×0
289	$A_2 + D_9 + E_7$	$\mathbb{1}$	$[4, 0, 6]$	1	0	1	0	$0 \times \frac{w_8}{2}$	$\frac{w_6}{2} \times 0$
290	$A_1 + D_{10} + E_7$	$\mathbb{Z}_2$	$[2, 0, 2]$	1	0	1	0	$\frac{w_6}{2} \times \frac{w_8}{2}$	0×0
291	$D_{11} + E_7$	$\mathbb{1}$	$[2, 0, 4]$	1	0	1	-1	$\frac{w_6}{2} \times \frac{w_1}{3}$	0×0
292	$A_2 + A_3 + E_6 + E_7$	$\mathbb{1}$	$[6, 0, 12]$	1	0	1	0	$\frac{w_5}{3} \times 0$	$0 \times \frac{w_6}{2}$
293	$A_1 + A_4 + E_6 + E_7$	$\mathbb{1}$	$[2, 0, 30]$	1	0	1	0	$\frac{w_5}{3} \times \frac{w_6}{2}$	0×0
294	$A_5 + E_6 + E_7$	$\mathbb{1}$	$[6, 0, 6]$	1	0	1	0	$\frac{w_5}{3} \times \frac{w_5}{3}$	$\left(w_4 - \frac{4w_5}{3}\right) \times 0$
295	$D_5 + E_6 + E_7$	$\mathbb{1}$	$[2, 0, 12]$	1	0	1	-1	$\frac{w_6}{2} \times \frac{w_5}{3}$	0×0
296	$2A_1 + 2E_8$	$\mathbb{1}$	$[2, 0, 2]$	1	0	1	0	0×0	0×0
297	$A_2 + 2E_8$	$\mathbb{1}$	$[2, 1, 2]$	1	0	1	-1	0×0	0×0
298	$2A_2 + 2A_3 + E_8$	$\mathbb{1}$	$[12, 0, 12]$	1	0	1	0	$\frac{w_5}{3} \times 0$	$\left(\frac{w_2}{3} - \frac{2w_5}{3}\right) \times 0$
299	$2A_1 + 2A_4 + E_8$	$\mathbb{1}$	$[10, 0, 10]$	1	0	1	0	$\frac{w_4}{4} \times 0$	$\left(\frac{w_2}{2} - \frac{3w_4}{4}\right) \times 0$
300	$A_1 + A_2 + A_3 + A_4 + E_8$	$\mathbb{1}$	$[6, 0, 20]$	1	0	1	0	$\frac{w_4}{4} \times 0$	$\left(\frac{w_4}{2} - \frac{w_7}{2}\right) \times 0$
301	$2A_5 + E_8$	$\mathbb{1}$	$[6, 0, 6]$	1	0	1	0	$\frac{w_3}{5} \times 0$	$\left(w_1 - \frac{3w_3}{5}\right) \times 0$
302	$A_2 + A_3 + A_5 + E_8$	$\mathbb{1}$	$[6, 0, 12]$	1	0	1	0	$\frac{w_5}{3} \times 0$	$\left(\frac{w_1}{2} - \frac{w_5}{2}\right) \times 0$
303	$A_1 + A_4 + A_5 + E_8$	$\mathbb{1}$	$[2, 0, 30]$	1	0	1	0	$\frac{w_3}{5} \times 0$	0×0
304	$2A_2 + A_6 + E_8$	$\mathbb{1}$	$[6, 3, 12]$	1	0	1	0	$\left(\frac{w_3}{3} - \frac{5w_6}{6}\right) \times 0$	$\frac{w_6}{2} \times 0$
305	$2A_1 + A_2 + A_6 + E_8$	$\mathbb{1}$	$[2, 0, 42]$	1	0	1	0	$\frac{w_2}{6} \times 0$	0×0
306	$A_1 + A_3 + A_6 + E_8$	$\mathbb{1}$	$[6, 2, 10]$	1	0	1	0	$\frac{w_2}{6} \times 0$	$\left(w_7 - \frac{2w_2}{3}\right) \times 0$
307	$A_4 + A_6 + E_8$	$\mathbb{1}$	$[2, 1, 18]$	1	0	1	-1	$0 \times \frac{w_3}{5}$	0×0
308	$A_1 + A_2 + A_7 + E_8$	$\mathbb{1}$	$[2, 0, 24]$	1	0	1	-1	$0 \times \frac{w_2}{6}$	0×0
309	$2A_1 + A_8 + E_8$	$\mathbb{1}$	$[2, 0, 18]$	1	0	1	0	$\frac{w_7}{4} \times 0$	0×0
310	$A_2 + A_8 + E_8$	$\mathbb{1}$	$[6, 3, 6]$	1	0	1	0	$\left(\frac{w_1}{2} - \frac{3w_6}{4}\right) \times 0$	$\frac{w_6}{2} \times 0$
311	$A_1 + A_9 + E_8$	$\mathbb{1}$	$[2, 0, 10]$	1	0	1	0	$\frac{w_1}{3} \times 0$	0×0
312	$A_{10} + E_8$	$\mathbb{1}$	$[2, 1, 6]$	1	0	1	-1	$0 \times \frac{w_1}{3}$	0×0
313	$2D_5 + E_8$	$\mathbb{1}$	$[4, 0, 4]$	1	0	1	0	$\frac{w_4}{4} \times \frac{w_4}{4}$	$0 \times -\frac{w_4}{4}$
314	$A_1 + A_4 + D_5 + E_8$	$\mathbb{1}$	$[2, 0, 20]$	1	0	1	0	$\frac{w_4}{4} \times 0$	0×0
315	$A_5 + D_5 + E_8$	$\mathbb{1}$	$[2, 0, 12]$	1	0	1	-1	$0 \times \frac{w_4}{4}$	0×0
316	$2A_2 + D_6 + E_8$	$\mathbb{1}$	$[6, 0, 6]$	1	0	1	0	$\frac{w_6}{2} \times 0$	$\left(\frac{w_6}{2} - \frac{w_8}{2}\right) \times 0$
317	$A_4 + D_6 + E_8$	$\mathbb{1}$	$[4, 2, 6]$	1	0	1	0	$\left(w_8 - \frac{w_4}{2}\right) \times 0$	$\frac{w_4}{4} \times 0$
318	$A_1 + A_2 + D_7 + E_8$	$\mathbb{1}$	$[4, 0, 6]$	1	0	1	0	$\left(\frac{w_4}{2} - w_6\right) \times 0$	$\frac{w_6}{2} \times 0$
319	$A_1 + D_9 + E_8$	$\mathbb{1}$	$[2, 0, 4]$	1	0	1	0	$\frac{w_8}{2} \times 0$	0×0
320	$D_{10} + E_8$	$\mathbb{1}$	$[2, 0, 2]$	1	0	1	-1	$0 \times \frac{w_8}{2}$	0×0
321	$A_1 + A_3 + E_6 + E_8$	$\mathbb{1}$	$[2, 0, 12]$	1	0	1	0	$\frac{w_5}{3} \times 0$	0×0
322	$A_4 + E_6 + E_8$	$\mathbb{1}$	$[2, 1, 8]$	1	0	1	-1	$0 \times \frac{w_5}{3}$	0×0
323	$D_4 + E_6 + E_8$	$\mathbb{1}$	$[4, 2, 4]$	1	0	1	0	$\left(\frac{w_4}{2} - \frac{w_8}{2}\right) \times 0$	$\left(\frac{w_4}{2} - w_8\right) \times 0$
324	$A_1 + A_2 + E_7 + E_8$	$\mathbb{1}$	$[2, 0, 6]$	1	0	1	0	$\frac{w_6}{2} \times 0$	0×0
325	$A_3 + E_7 + E_8$	$\mathbb{1}$	$[2, 0, 4]$	1	0	1	-1	$\frac{w_6}{2} \times 0$	0×0

Table D.2: Data for all allowed groups of maximal rank, for the $E_8 \times E_8'$ heterotic on T^2.

#	L	E_{11}	E_{21}	E_{22}	E_{12}	A_1	A_2
1	$6A_3$	1	0	$\frac{3}{2}$	$\frac{1}{4}$	$\frac{1}{4}(w_6 + w_{10})$	$\frac{1}{4}(w_2 - w_6 - w_{10} + w_{14})$
2	$2A_1 + 4A_4$	1	0	1	$-\frac{1}{4}$	$\frac{1}{4}(w_3 + w_{13})$	$w_1 + w_{15} + \frac{1}{5}(w_8 - 3w_3 - 3w_{13})$
13	$3A_6$	1	0	1	0	$\frac{1}{2}w_6$	$\frac{1}{3}(2w_6 - w_9 - w_{15})$
21	$2A_1 + 2A_5 + A_6$	1	0	1	0	$\frac{1}{14}(3w_4 + 4w_{11})$	$\frac{1}{28}(14w_2 - 15w_4 - 6w_{11} + 14w_{15})$
65	$A_3 + A_6 + A_9$	1	0	1	0	$\frac{1}{2}w_6$	$\frac{1}{30}(3w_6 - 16w_{15})$
177	$2D_5 + D_8$	1	0	1	0	$\frac{1}{2}(w_4 + w_{12})$	$\frac{1}{4}(w_4 - w_{12}) - w_9 - w_{11}$
196	$A_2 + D_{10} + D_6$	1	$-\frac{1}{2}$	1	0	$\frac{1}{2}w_6$	0
219	$3E_6$	1	0	1	$-\frac{3}{4}$	$\frac{1}{4}(w_{11} + w_5)$	0
297	$A_2 + 2E_8$	1	0	1	$-\frac{1}{4}$	$\frac{1}{4}(w_7 + w_9)$	0
319	$A_1 + D_9 + E_8$	$\frac{5}{4}$	0	1	0	$\frac{1}{2}w_9$	0

Table D.3: Data for some groups of maximal rank, for the HO theory on T^2.

i	Gauge group root lattice	E	$-a$
1	A_9	2	$(-\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},-\frac{5}{6})$
2	$\mathrm{A}_1+\mathrm{A}_2+\mathrm{A}_6$	2	$(0,0,\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},-\frac{5}{6})$
3	$\mathrm{A}_4+\mathrm{A}_5$	2	$(0,0,0,\frac{1}{5},\frac{1}{5},\frac{1}{5},\frac{1}{5},-\frac{4}{5})$
4	$\mathrm{D}_5+\mathrm{A}_4$	2	$(0,0,0,0,\frac{1}{4},\frac{1}{4},\frac{1}{4},-\frac{3}{4})$
5	$\mathrm{E}_6+\mathrm{A}_3$	2	$(0,0,0,0,0,\frac{1}{3},\frac{1}{3},-\frac{2}{3})$
6	$\mathrm{E}_7+\mathrm{A}_2$	2	$(0,0,0,0,0,0,\frac{1}{2},-\frac{1}{2})$
7	$\mathrm{A}_1+\mathrm{A}_8$	2	$(\frac{1}{8},\frac{1}{8},\frac{1}{8},\frac{1}{8},\frac{1}{8},\frac{1}{8},\frac{1}{8},-\frac{7}{8})$
8	D_9	2	$(0,0,0,0,0,0,0,-1)$
0	$\mathrm{E}_8+\mathrm{A}_1$	2	$(0,0,0,0,0,0,0,0)$

Table D.4: Maximal enhancements in the nine-dimensional CHL string, obtained by deleting the ith node in the GDD shown in Figure 4.1. All groups arise at level 2. The Wilson line is always of the form $a = w_i/\kappa_i$ (cf. Table 2.1).

#	L	H	E_{11}	E_{21}	E_{22}	E_{12}	a_1	a_2
1	$2\mathrm{A}_2+2\mathrm{A}_3$	$\mathbb{Z}_2$	2	0	3	1	$\frac{w_3}{4}$	$\frac{w_3}{4}-\frac{w_2}{3}$
2	$2\mathrm{A}_5$	$\mathbb{Z}_2$	2	0	1	-1	$\frac{w_6}{2}$	$\frac{w_7}{3}-\frac{2w_6}{3}$
3	$2\mathrm{A}_1+\mathrm{A}_3+\mathrm{A}_5$	$\mathbb{Z}_2^2$	2	0	2	2	0	$\frac{w_7}{3}-\frac{w_6}{6}$
4	$\mathrm{A}_1+\mathrm{A}_4+\mathrm{A}_5$	$\mathbb{Z}_2$	2	0	2	2	0	$\frac{2w_7}{5}-\frac{w_5}{5}$
5	$2\mathrm{A}_2+\mathrm{A}_6$	$\mathbb{1}$	2	0	3	$\frac{7}{2}$	$\frac{w_2}{6}$	0
6	$\mathrm{A}_1+\mathrm{A}_2+\mathrm{A}_7$	$\mathbb{Z}_2$	2	0	$\frac{3}{2}$	2	0	$\frac{w_2}{6}$
7	$\mathrm{A}_1+\mathrm{A}_9$	$\mathbb{Z}_2$	2	0	3	-2	0	$\frac{w_3}{3}$
8	$\mathrm{A}_1+2\mathrm{A}_2+\mathrm{A}_4+\mathrm{C}_1$	$\mathbb{1}$	2	0	1	0	$\frac{w_2}{6}$	$\frac{w_2}{3}-\frac{2w_5}{3}$
9	$\mathrm{A}_1+\mathrm{A}_3+\mathrm{A}_5+\mathrm{C}_1$	$\mathbb{Z}_2$	2	0	1	0	$\frac{w_8}{2}$	$\frac{5w_8}{6}-\frac{w_3}{3}$
10	$\mathrm{A}_4+\mathrm{A}_5+\mathrm{C}_1$	$\mathbb{1}$	2	0	1	0	$\frac{w_3}{5}$	0
11	$\mathrm{A}_1+\mathrm{A}_2+\mathrm{A}_6+\mathrm{C}_1$	$\mathbb{1}$	2	0	1	0	$\frac{w_2}{6}$	0
12	$2\mathrm{A}_1+\mathrm{A}_7+\mathrm{C}_1$	$\mathbb{Z}_2$	2	0	1	0	$\frac{w_8}{2}$	$\frac{w_2}{3}-\frac{2w_8}{3}$
13	$\mathrm{A}_1+\mathrm{A}_8+\mathrm{C}_1$	$\mathbb{1}$	2	0	1	0	$\frac{w_7}{4}$	0
14	$\mathrm{A}_9+\mathrm{C}_1$	$\mathbb{1}$	2	0	1	0	$\frac{w_1}{3}$	0
15	$2\mathrm{A}_1+2\mathrm{A}_3+\mathrm{C}_2$	$\mathbb{Z}_2^2$	2	0	1	0	$\frac{w_8}{2}$	$\frac{w_3}{4}-\frac{w_8}{4}$
16	$\mathrm{A}_1+\mathrm{A}_3+\mathrm{A}_4+\mathrm{C}_2$	$\mathbb{Z}_2$	2	0	1	0	$\frac{w_2}{5}$	$\frac{w_2}{2}-w_1$
17	$2\mathrm{A}_4+\mathrm{C}_2$	$\mathbb{1}$	2	0	1	0	$w_4-\frac{2w_3}{3}$	$\frac{w_3}{5}$
18	$3\mathrm{A}_1+\mathrm{A}_5+\mathrm{C}_2$	$\mathbb{Z}_2^2$	2	0	1	0	$\frac{w_8}{2}$	$\frac{w_2}{4}-\frac{3w_8}{8}$
19	$\mathrm{A}_3+\mathrm{A}_5+\mathrm{C}_2$	$\mathbb{Z}_2$	2	0	1	2	0	$\frac{w_3}{8}$
20	$\mathrm{A}_2+\mathrm{A}_6+\mathrm{C}_2$	$\mathbb{1}$	2	0	1	2	0	$\frac{w_7}{7}$
21	$\mathrm{A}_8+\mathrm{C}_2$	$\mathbb{1}$	2	0	1	-2	0	$\frac{w_3}{7}$
22	$2\mathrm{A}_2+\mathrm{A}_3+\mathrm{C}_3$	$\mathbb{1}$	2	0	1	0	$\frac{w_3}{4}$	$\frac{w_3}{2}-\frac{w_2}{3}$
23	$\mathrm{A}_1+\mathrm{A}_2+\mathrm{A}_4+\mathrm{C}_3$	$\mathbb{1}$	2	0	1	0	$\frac{w_2}{10}$	$-\frac{w_2}{6}$
24	$\mathrm{A}_2+\mathrm{A}_5+\mathrm{C}_3$	$\mathbb{1}$	2	0	1	0	$\frac{w_6}{2}$	$\frac{w_7}{3}-\frac{2w_6}{3}$
25	$\mathrm{A}_1+\mathrm{A}_6+\mathrm{C}_3$	$\mathbb{1}$	2	0	1	0	$\frac{w_2}{6}$	$w_8-\frac{w_2}{3}$
26	$2\mathrm{A}_1+2\mathrm{A}_2+\mathrm{C}_4$	$\mathbb{Z}_2$	2	0	1	0	$\frac{w_4}{6}$	$\frac{w_2}{6}$
27	$\mathrm{A}_1+\mathrm{A}_2+\mathrm{A}_3+\mathrm{C}_4$	$\mathbb{Z}_2$	2	0	1	0	$\frac{w_7}{2}-\frac{w_4}{3}$	$\frac{w_4}{4}$
28	$2\mathrm{A}_1+\mathrm{A}_4+\mathrm{C}_4$	$\mathbb{Z}_2$	2	0	1	0	$\frac{w_2}{5}$	$\frac{w_2}{10}$
29	$\mathrm{A}_1+\mathrm{A}_4+\mathrm{C}_5$	$\mathbb{1}$	2	0	1	0	0	$\frac{w_3}{5}$
30	$\mathrm{A}_5+\mathrm{C}_5$	$\mathbb{1}$	2	0	1	0	$\frac{w_7}{3}-\frac{w_6}{6}$	$\frac{w_6}{2}$
31	$2\mathrm{A}_1+\mathrm{A}_2+\mathrm{C}_6$	$\mathbb{Z}_2$	2	0	1	0	0	$\frac{w_2}{6}$

#	L	H	E_{11}	E_{21}	E_{22}	E_{12}	a_1	a_2
32	$2\mathrm{A}_2+\mathrm{C}_6$	$\mathbb{1}$	2	0	1	0	$\frac{w_2}{2}-w_1$	$\frac{w_2}{6}$
33	$\mathrm{A}_1+\mathrm{A}_3+\mathrm{C}_6$	$\mathbb{Z}_2$	2	0	1	0	$\frac{w_3}{4}$	$\frac{w_3}{8}$
34	$\mathrm{A}_4+\mathrm{C}_6$	$\mathbb{1}$	2	0	1	0	$\frac{w_6}{2}$	$\frac{w_3}{5}$
35	$\mathrm{A}_1+\mathrm{A}_2+\mathrm{C}_7$	$\mathbb{1}$	2	0	1	-2	$\frac{w_2}{6}$	0
36	$2\mathrm{A}_1+\mathrm{C}_8$	$\mathbb{Z}_2$	2	0	1	0	0	$\frac{w_7}{4}$
37	$\mathrm{A}_2+\mathrm{C}_8$	$\mathbb{Z}_2$	2	-1	1	0	0	$\frac{w_2}{6}$
38	$\mathrm{A}_1+\mathrm{C}_9$	$\mathbb{1}$	2	0	1	0	0	$\frac{w_1}{3}$
39	C_{10}	$\mathbb{1}$	2	0	1	-2	$\frac{w_1}{3}$	0
40	$2\mathrm{D}_5$	$\mathbb{Z}_2$	2	0	1	-1	0	$\frac{w_4}{4}$
41	$\mathrm{A}_4+\mathrm{C}_1+\mathrm{D}_5$	$\mathbb{1}$	2	0	1	0	$\frac{w_4}{4}$	0
42	$\mathrm{A}_1+\mathrm{A}_2+\mathrm{C}_2+\mathrm{D}_5$	$\mathbb{Z}_2$	2	0	1	0	$\frac{w_4}{6}$	$\frac{w_4}{4}$
43	$\mathrm{A}_1+\mathrm{C}_4+\mathrm{D}_5$	$\mathbb{Z}_2$	2	0	1	0	0	$\frac{w_4}{4}$
44	$\mathrm{C}_5+\mathrm{D}_5$	$\mathbb{1}$	2	0	1	-2	$\frac{w_4}{4}$	0
45	$\mathrm{A}_1+\mathrm{A}_3+\mathrm{D}_6$	$\mathbb{Z}_2^2$	2	0	2	2	0	$\frac{w_3}{4}$
46	$\mathrm{A}_2+\mathrm{C}_2+\mathrm{D}_6$	$\mathbb{Z}_2$	2	0	1	2	0	$\frac{w_4}{6}$
47	$\mathrm{C}_4+\mathrm{D}_6$	$\mathbb{Z}_2$	2	0	1	0	$\frac{w_8}{2}$	$\frac{w_6}{2}$
48	$\mathrm{A}_1+\mathrm{C}_2+\mathrm{D}_7$	$\mathbb{Z}_2$	2	0	1	0	$\frac{w_8}{2}$	$\frac{w_8}{4}$
49	$2\mathrm{A}_1+\mathrm{D}_8$	$\mathbb{Z}_2^2$	2	0	1	-1	$\frac{w_4}{2}-w_6$	$\frac{w_6}{2}$
50	$\mathrm{C}_1+\mathrm{D}_9$	$\mathbb{1}$	2	0	1	0	$\frac{w_8}{2}$	0
51	$\mathrm{A}_4+\mathrm{E}_6$	$\mathbb{1}$	2	0	1	-1	0	$\frac{w_5}{3}$
52	$\mathrm{A}_1+\mathrm{A}_2+\mathrm{C}_1+\mathrm{E}_6$	$\mathbb{1}$	2	0	1	0	$\frac{w_5}{6}$	$\frac{w_5}{3}$
53	$\mathrm{A}_3+\mathrm{C}_1+\mathrm{E}_6$	$\mathbb{1}$	2	0	1	0	$\frac{w_5}{3}$	0
54	$\mathrm{A}_1+\mathrm{C}_3+\mathrm{E}_6$	$\mathbb{1}$	2	0	1	0	0	$\frac{w_5}{3}$
55	$\mathrm{C}_4+\mathrm{E}_6$	$\mathbb{1}$	2	0	1	-2	$\frac{w_5}{3}$	0
56	$\mathrm{A}_1+\mathrm{A}_2+\mathrm{E}_7$	$\mathbb{Z}_2$	2	0	1	-1	0	$\frac{w_6}{2}$
57	$\mathrm{A}_2+\mathrm{C}_1+\mathrm{E}_7$	$\mathbb{1}$	2	0	1	0	$\frac{w_6}{2}$	0
58	$\mathrm{A}_1+\mathrm{C}_2+\mathrm{E}_7$	$\mathbb{Z}_2$	2	0	1	0	0	$\frac{w_6}{2}$
59	$\mathrm{C}_3+\mathrm{E}_7$	$\mathbb{1}$	2	0	1	0	$\frac{w_6}{2}$	$\frac{w_6}{2}$
60	$\mathrm{A}_1+\mathrm{C}_1+\mathrm{E}_8$	$\mathbb{1}$	2	0	1	0	0	0
61	$\mathrm{C}_2+\mathrm{E}_8$	$\mathbb{1}$	2	0	1	-2	0	0

Table D.5: All groups of maximal enhancement in the 8-dimensional CHL string. The Wilson lines are given in terms of the fundamental weights of E_8, see Table 2.1. ADE groups arise at level 2 and C groups at level 1.

#	L	H	k
1	$2\mathrm{A}_2 + 2\mathrm{A}_3$	$\mathbb{Z}_2$	0 0 2 2
2	$2\mathrm{A}_5$	$\mathbb{Z}_2$	3 3
3	$2\mathrm{A}_1 + \mathrm{A}_3 + \mathrm{A}_5$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	0 1 0 3 1 0 2 3
4	$\mathrm{A}_1 + \mathrm{A}_4 + \mathrm{A}_5$	$\mathbb{Z}_2$	1 0 3
6	$\mathrm{A}_1 + \mathrm{A}_2 + \mathrm{A}_7$	$\mathbb{Z}_2$	0 0 4
7	$\mathrm{A}_1 + \mathrm{A}_9$	$\mathbb{Z}_2$	1 5
9	$\mathrm{A}_1 + \mathrm{A}_3 + \mathrm{A}_5 + \mathrm{C}_1$	$\mathbb{Z}_2$	1 2 3 0
12	$2\mathrm{A}_1 + \mathrm{A}_7 + \mathrm{C}_1$	$\mathbb{Z}_2$	1 1 4 0
15	$2\mathrm{A}_1 + 2\mathrm{A}_3 + \mathrm{C}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	0 1 0 2 1 1 0 2 0 1
16	$\mathrm{A}_1 + \mathrm{A}_3 + \mathrm{A}_4 + \mathrm{C}_2$	$\mathbb{Z}_2$	1 2 0 1
18	$3\mathrm{A}_1 + \mathrm{A}_5 + \mathrm{C}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	0 0 0 3 1 1 1 1 0 1
19	$\mathrm{A}_3 + \mathrm{A}_5 + \mathrm{C}_2$	$\mathbb{Z}_2$	0 3 1
26	$2\mathrm{A}_1 + 2\mathrm{A}_2 + \mathrm{C}_4$	$\mathbb{Z}_2$	1 1 0 0 1
27	$\mathrm{A}_1 + \mathrm{A}_2 + \mathrm{A}_3 + \mathrm{C}_4$	$\mathbb{Z}_2$	0 0 2 1
28	$2\mathrm{A}_1 + \mathrm{A}_4 + \mathrm{C}_4$	$\mathbb{Z}_2$	1 1 0 1
31	$2\mathrm{A}_1 + \mathrm{A}_2 + \mathrm{C}_6$	$\mathbb{Z}_2$	0 1 0 1
33	$\mathrm{A}_1 + \mathrm{A}_3 + \mathrm{C}_6$	$\mathbb{Z}_2$	1 0 1
36	$2\mathrm{A}_1 + \mathrm{C}_8$	$\mathbb{Z}_2$	0 0 1
37	$\mathrm{A}_2 + \mathrm{C}_8$	$\mathbb{Z}_2$	0 1
40	$2\mathrm{D}_5$	$\mathbb{Z}_2$	2 2
42	$\mathrm{A}_1 + \mathrm{A}_2 + \mathrm{C}_2 + \mathrm{D}_5$	$\mathbb{Z}_2$	1 0 1 2
43	$\mathrm{A}_1 + \mathrm{C}_4 + \mathrm{D}_5$	$\mathbb{Z}_2$	0 1 2
45	$\mathrm{A}_1 + \mathrm{A}_3 + \mathrm{D}_6$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	0 2 $(1,1)$ 1 0 $(0,1)$
46	$\mathrm{A}_2 + \mathrm{C}_2 + \mathrm{D}_6$	$\mathbb{Z}_2$	0 1 $(1,0)$
47	$\mathrm{C}_4 + \mathrm{D}_6$	$\mathbb{Z}_2$	1 $(1,1)$
48	$\mathrm{A}_1 + \mathrm{C}_2 + \mathrm{D}_7$	$\mathbb{Z}_2$	1 1 2
49	$2\mathrm{A}_1 + \mathrm{D}_8$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	0 0 $(0,1)$ 1 1 $(1,0)$
56	$\mathrm{A}_1 + \mathrm{A}_2 + \mathrm{E}_7$	$\mathbb{Z}_2$	1 0 1
58	$\mathrm{A}_1 + \mathrm{C}_2 + \mathrm{E}_7$	$\mathbb{Z}_2$	0 1 1

Table D.6: Maximal enhancement groups with non-trivial global structure for the 8-dimensional CHL string. The k's are the generators of H. All ADE groups arise at level 2 while C groups arise at level 1.

D.2 Maximal enhancements for 7d

In this appendix we record the maximally enhanced gauge groups realized in the 7d $\mathbb{Z}_m$-triples constructed from the heterotic string. The algebras and the fundamental groups are presented in Appendix D.2.1, while the generators of the fundamental group are presented in Appendix D.2.2. The way in which the data is encoded is explained in Section 5.5.

D.2.1 Maximally enhanced algebras and fundamental groups

1	$A_1A_3^6$	$\mathbb{Z}_4^2$	48	$A_2^4A_5A_6$	$\mathbb{Z}_3$	95	$A_1^2A_5^2A_7$	$\mathbb{Z}_2$	142	$A_1^2A_4A_5A_8$		189	$A_1^4A_6A_9$	$\mathbb{Z}_2$
2	$A_3^5A_4$	$\mathbb{Z}_4$	49	$A_1A_2^2A_3A_5A_6$		96	$A_1^2A_5^2A_7$	$\mathbb{Z}_2^2$	143	$A_2A_4A_5A_8$		190	$A_1^2A_2A_6A_9$	
3	$A_1A_2^2A_3^2A_4^2$		50	$A_1^2A_3^2A_5A_6$	$\mathbb{Z}_2$	97	$A_2A_5^2A_7$		144	$A_2A_4A_5A_8$	$\mathbb{Z}_3$	191	$A_2^2A_6A_9$	
4	$A_1^2A_3^3A_4^2$	$\mathbb{Z}_2$	51	$A_2A_3^2A_5A_6$		98	$A_1^2A_2^2A_6A_7$		145	$A_1A_5^2A_8$		192	$A_1A_3A_6A_9$	
5	$A_2A_3^3A_4^2$		52	$A_1^2A_2A_4A_5A_6$		99	$A_2^3A_6A_7$		146	$A_1^3A_2A_6A_8$		193	$A_1A_3A_6A_9$	$\mathbb{Z}_2$
6	$A_2^2A_3A_4^3$		53	$A_2^2A_4A_5A_6$		100	$A_1^3A_3A_6A_7$	$\mathbb{Z}_2$	147	$A_1A_2^2A_6A_8$		194	$A_4A_6A_9$	
7	$A_1^3A_4^4$	$\mathbb{Z}_5$	54	$A_1A_3A_4A_5A_6$		101	$A_1A_2A_3A_6A_7$		148	$A_1^2A_3A_6A_8$		195	$A_1^3A_7A_9$	$\mathbb{Z}_2$
8	$A_1A_2A_4^4$	$\mathbb{Z}_5$	55	$A_4^2A_5A_6$		102	$A_3^2A_6A_7$		149	$A_2A_3A_6A_8$		196	$A_1A_2A_7A_9$	
9	$A_3A_4^4$	$\mathbb{Z}_5$	56	$A_1^3A_5^2A_6$	$\mathbb{Z}_2$	103	$A_3^2A_6A_7$	$\mathbb{Z}_2$	150	$A_1A_4A_6A_8$		197	$A_3A_7A_9$	
10	$A_2^4A_3^2A_5$	$\mathbb{Z}_3$	57	$A_1A_2A_5^2A_6$		104	$A_1^2A_4A_6A_7$		151	$A_5A_6A_8$		198	$A_1^2A_8A_9$	
11	$A_1^2A_2A_3^2A_4A_5$	$\mathbb{Z}_2$	58	$A_3A_5^2A_6$		105	$A_2A_4A_6A_7$		152	$A_1^4A_7A_8$	$\mathbb{Z}_2$	199	$A_2A_8A_9$	
12	$A_2^2A_3^2A_4A_5$		59	$A_3A_5^2A_6$	$\mathbb{Z}_2$	106	$A_1A_5A_6A_7$		153	$A_1^2A_2A_7A_8$		200	$A_1A_9^2$	
13	$A_1A_3^3A_4A_5$	$\mathbb{Z}_2$	60	$A_1^3A_2^2A_6^2$		107	$A_1A_5A_6A_7$	$\mathbb{Z}_2$	154	$A_2^2A_7A_8$		201	$A_1A_9^2$	$\mathbb{Z}_5$
14	$A_1^3A_3A_4^2A_5$	$\mathbb{Z}_2$	61	$A_1A_2^3A_6^2$		108	$A_6^2A_7$		155	$A_1A_3A_7A_8$		202	$A_1^2A_2^2A_3A_{10}$	
15	$A_1A_2A_3A_4^2A_5$		62	$A_1^2A_2A_3A_6^2$		109	$A_1^5A_7^2$	$\mathbb{Z}_2\mathbb{Z}_4$	156	$A_4A_7A_8$		203	$A_2^3A_3A_{10}$	
16	$A_3^2A_4^2A_5$		63	$A_2^2A_3A_6^2$		110	$A_1^3A_2A_7^2$	$\mathbb{Z}_4$	157	$A_1^3A_8^2$		204	$A_1A_2A_3^2A_{10}$	
17	$A_1^2A_4^3A_5$		64	$A_1A_3^2A_6^2$		111	$A_1A_2^2A_7^2$		158	$A_1^3A_8^2$	$\mathbb{Z}_3$	205	$A_3^3A_{10}$	
18	$A_1A_2^4A_5^2$	$\mathbb{Z}_3^2$	65	$A_1^3A_4A_6^2$		112	$A_1A_2^2A_7^2$	$\mathbb{Z}_2$	159	$A_1A_2A_8^2$		206	$A_1^3A_2A_4A_{10}$	
19	$A_1^2A_2^2A_3A_5^2$	$\mathbb{Z}_6$	66	$A_1A_2A_4A_6^2$		113	$A_1^2A_3A_7^2$	$\mathbb{Z}_2$	160	$A_1A_2A_8^2$	$\mathbb{Z}_3$	207	$A_1A_2^2A_4A_{10}$	
20	$A_2^3A_3A_5^2$	$\mathbb{Z}_3$	67	$A_3A_4A_6^2$		114	$A_1^2A_3A_7^2$	$\mathbb{Z}_2^2$	161	$A_3A_8^2$		208	$A_1^2A_3A_4A_{10}$	
21	$A_2^3A_3A_5^2$	$\mathbb{Z}_6$	68	$A_1^2A_5A_6^2$		115	$A_1^2A_3A_7^2$	$\mathbb{Z}_2\mathbb{Z}_4$	162	$A_3A_8^2$	$\mathbb{Z}_3$	209	$A_2A_3A_4A_{10}$	
22	$A_1^3A_3^2A_5^2$	$\mathbb{Z}_2^2$	69	$A_2A_5A_6^2$		116	$A_1^2A_3A_7^2$	$\mathbb{Z}_8$	163	$A_1^3A_2^2A_3A_9$	$\mathbb{Z}_2$	210	$A_1A_4^2A_{10}$	
23	$A_1A_2A_3^2A_5^2$	$\mathbb{Z}_2$	70	$A_1A_6^3$	$\mathbb{Z}_7$	117	$A_2A_3A_7^2$		164	$A_1A_2^3A_3A_9$		211	$A_1^2A_2A_5A_{10}$	
24	$A_3^3A_5^2$	$\mathbb{Z}_2$	71	$A_1^2A_2^2A_3^2A_7$	$\mathbb{Z}_4$	118	$A_2A_3A_7^2$	$\mathbb{Z}_2$	165	$A_1A_2^3A_3A_9$	$\mathbb{Z}_2$	212	$A_2^2A_5A_{10}$	
25	$A_1A_2^2A_4A_5^2$	$\mathbb{Z}_3$	72	$A_1^3A_3^3A_7$	$\mathbb{Z}_2\mathbb{Z}_4$	119	$A_2A_3A_7^2$	$\mathbb{Z}_4$	166	$A_1^2A_2A_3^2A_9$	$\mathbb{Z}_2$	213	$A_1A_3A_5A_{10}$	
26	$A_1^2A_3A_4A_5^2$	$\mathbb{Z}_2$	73	$A_1A_2A_3^3A_7$	$\mathbb{Z}_4$	120	$A_1A_4A_7^2$		167	$A_2^2A_3^2A_9$		214	$A_4A_5A_{10}$	
27	$A_2A_3A_4A_5^2$		74	$A_3^4A_7$	$\mathbb{Z}_4$	121	$A_1A_4A_7^2$	$\mathbb{Z}_2$	168	$A_1A_3^3A_9$	$\mathbb{Z}_2$	215	$A_1^3A_6A_{10}$	
28	$A_2A_3A_4A_5^2$	$\mathbb{Z}_2$	75	$A_3^4A_7$	$\mathbb{Z}_2\mathbb{Z}_4$	122	$A_5A_7^2$		169	$A_1^4A_2A_4A_9$	$\mathbb{Z}_2$	216	$A_1A_2A_6A_{10}$	
29	$A_1A_4^2A_5^2$		76	$A_1^3A_2A_3A_4A_7$	$\mathbb{Z}_2$	123	$A_5A_7^2$	$\mathbb{Z}_2$	170	$A_1^2A_2^2A_4A_9$		217	$A_3A_6A_{10}$	
30	$A_1^4A_5^3$	$\mathbb{Z}_2\mathbb{Z}_6$	77	$A_1A_2^2A_3A_4A_7$		124	$A_1^2A_2^3A_3A_8$	$\mathbb{Z}_3$	171	$A_2^3A_4A_9$		218	$A_1^2A_7A_{10}$	
31	$A_1^2A_2A_5^3$	$\mathbb{Z}_6$	78	$A_1^2A_3^2A_4A_7$	$\mathbb{Z}_2$	125	$A_2^4A_3A_8$	$\mathbb{Z}_3$	172	$A_1^3A_3A_4A_9$	$\mathbb{Z}_2$	219	$A_2A_7A_{10}$	
32	$A_2^2A_5^3$	$\mathbb{Z}_3$	79	$A_1^2A_3^2A_4A_7$	$\mathbb{Z}_4$	126	$A_1A_2^2A_3^2A_8$		173	$A_1A_2A_3A_4A_9$		220	$A_1A_8A_{10}$	
33	$A_2^2A_5^3$	$\mathbb{Z}_3^2$	80	$A_2A_3^2A_4A_7$		127	$A_1^2A_3^3A_8$	$\mathbb{Z}_2$	174	$A_1A_2A_3A_4A_9$	$\mathbb{Z}_2$	221	A_9A_{10}	
34	$A_1A_3A_5^3$	$\mathbb{Z}_2$	81	$A_2A_3^2A_4A_7$	$\mathbb{Z}_2$	128	$A_2A_3^3A_8$		175	$A_3^2A_4A_9$		222	$A_1^4A_2^2A_{11}$	$\mathbb{Z}_6$
35	$A_1A_3A_5^3$	$\mathbb{Z}_6$	82	$A_1^4A_4^2A_7$	$\mathbb{Z}_2$	129	$A_1A_2^3A_4A_8$	$\mathbb{Z}_3$	176	$A_1^2A_4^2A_9$		223	$A_1^2A_2^3A_{11}$	$\mathbb{Z}_3$
36	$A_4A_5^3$		83	$A_1^2A_2A_4^2A_7$		130	$A_1^2A_2A_3A_4A_8$		177	$A_1^2A_4^2A_9$	$\mathbb{Z}_5$	224	$A_1^2A_2^3A_{11}$	$\mathbb{Z}_6$
37	$A_4A_5^3$	$\mathbb{Z}_3$	84	$A_2^2A_4^2A_7$		131	$A_2^2A_3A_4A_8$		178	$A_2A_4^2A_9$	$\mathbb{Z}_5$	225	$A_2^4A_{11}$	$\mathbb{Z}_3$
38	$A_1A_2^3A_3^2A_6$		85	$A_1A_3A_4^2A_7$		132	$A_1A_3^2A_4A_8$		179	$A_1^3A_2A_5A_9$	$\mathbb{Z}_2$	226	$A_1^3A_2A_3A_{11}$	$\mathbb{Z}_2$
39	$A_1^2A_2^2A_3A_4A_6$		86	$A_1^4A_3A_5A_7$	$\mathbb{Z}_2^2$	133	$A_1^3A_4^2A_8$		180	$A_1A_2^2A_5A_9$		227	$A_1^3A_2A_3A_{11}$	$\mathbb{Z}_4$
40	$A_2^3A_3A_4A_6$		87	$A_1^2A_2A_3A_5A_7$	$\mathbb{Z}_2$	134	$A_1A_2A_4^2A_8$		181	$A_1A_2^2A_5A_9$	$\mathbb{Z}_2$	228	$A_1A_2^2A_3A_{11}$	
41	$A_1A_2A_3^2A_4A_6$		88	$A_2^2A_3A_5A_7$		135	$A_3A_4^2A_8$		182	$A_1^2A_3A_5A_9$	$\mathbb{Z}_2$	229	$A_1A_2^2A_3A_{11}$	$\mathbb{Z}_2$
42	$A_3^3A_4A_6$		89	$A_1A_3^2A_5A_7$	$\mathbb{Z}_2$	136	$A_1^2A_2^2A_5A_8$	$\mathbb{Z}_3$	183	$A_2A_3A_5A_9$		230	$A_1A_2^2A_3A_{11}$	$\mathbb{Z}_3$
43	$A_1^3A_2A_4^2A_6$		90	$A_1A_3^2A_5A_7$	$\mathbb{Z}_2^2$	137	$A_2^3A_5A_8$	$\mathbb{Z}_3$	184	$A_2A_3A_5A_9$	$\mathbb{Z}_2$	231	$A_1A_2^2A_3A_{11}$	$\mathbb{Z}_6$
44	$A_1A_2^2A_4^2A_6$		91	$A_1^3A_4A_5A_7$	$\mathbb{Z}_2$	138	$A_1^3A_3A_5A_8$	$\mathbb{Z}_2$	185	$A_1A_4A_5A_9$		232	$A_1^2A_3^2A_{11}$	$\mathbb{Z}_2$
45	$A_1^2A_3A_4^2A_6$		92	$A_1A_2A_4A_5A_7$		139	$A_1A_2A_3A_5A_8$		186	$A_1A_4A_5A_9$	$\mathbb{Z}_2$	233	$A_1^2A_3^2A_{11}$	$\mathbb{Z}_4$
46	$A_2A_3A_4^2A_6$		93	$A_1A_2A_4A_5A_7$	$\mathbb{Z}_2$	140	$A_1A_2A_3A_5A_8$	$\mathbb{Z}_3$	187	$A_5^2A_9$		234	$A_2A_3^2A_{11}$	
47	$A_1A_4^3A_6$		94	$A_3A_4A_5A_7$		141	$A_3^2A_5A_8$		188	$A_5^2A_9$	$\mathbb{Z}_2$	235	$A_2A_3^2A_{11}$	$\mathbb{Z}_2$

236	$A_2A_3^2A_{11}$ $\mathbb{Z}_4$	283	A_6A_{13}	330	$A_2A_3A_4A_6D_4$	377	$A_2A_3^3D_4^2$ $\mathbb{Z}_2^2$	424	$A_1^2A_4A_8D_5$
237	$A_1^4A_4A_{11}$ $\mathbb{Z}_2$	284	$A_1^3A_2A_{14}$	331	$A_1A_4^2A_6D_4$	378	$A_1A_5^2D_4^2$ $\mathbb{Z}_2^2$	425	$A_2A_4A_8D_5$
238	$A_1^2A_2A_4A_{11}$	285	$A_1^3A_2A_{14}$ $\mathbb{Z}_3$	332	$A_1A_3A_5A_6D_4$ $\mathbb{Z}_2$	379	$A_2^2A_7D_4^2$ $\mathbb{Z}_2$	426	$A_1A_5A_8D_5$
239	$A_1^2A_2A_4A_{11}$ $\mathbb{Z}_2$	286	$A_1A_2^2A_{14}$	333	$A_4A_5A_6D_4$	380	$A_1^3D_4^4$ $\mathbb{Z}_2^4$	427	$A_6A_8D_5$
240	$A_2^2A_4A_{11}$	287	$A_1A_2^2A_{14}$ $\mathbb{Z}_3$	334	$A_1A_2A_6^2D_4$	381	$A_1A_2^2A_3^3D_5$ $\mathbb{Z}_4$	428	$A_1^3A_2A_9D_5$ $\mathbb{Z}_2$
241	$A_2^2A_4A_{11}$ $\mathbb{Z}_3$	288	$A_1^2A_3A_{14}$	335	$A_3A_6^2D_4$	382	$A_2A_3^4D_5$ $\mathbb{Z}_4$	429	$A_1A_2^2A_9D_5$
242	$A_1A_3A_4A_{11}$	289	$A_2A_3A_{14}$	336	$A_1^2A_2^3A_7D_4$ $\mathbb{Z}_2$	383	$A_2^2A_3^2A_4D_5$	430	$A_1A_2^2A_9D_5$ $\mathbb{Z}_2$
243	$A_1A_3A_4A_{11}$ $\mathbb{Z}_2$	290	$A_2A_3A_{14}$ $\mathbb{Z}_3$	337	$A_1A_2^2A_3A_7D_4$ $\mathbb{Z}_2$	384	$A_1A_3^3A_4D_5$ $\mathbb{Z}_4$	431	$A_1^2A_3A_9D_5$ $\mathbb{Z}_2$
244	$A_4^2A_{11}$	291	$A_1A_4A_{14}$	338	$A_1^2A_3^2A_7D_4$ $\mathbb{Z}_2^2$	385	$A_1^2A_2^2A_4^2D_5$	432	$A_2A_3A_9D_5$
245	$A_1^3A_5A_{11}$ $\mathbb{Z}_2$	292	A_5A_{14}	339	$A_1^2A_3^2A_7D_4$ $\mathbb{Z}_2\mathbb{Z}_4$	386	$A_1A_2A_3A_4^2D_5$	433	$A_1A_4A_9D_5$
246	$A_1^3A_5A_{11}$ $\mathbb{Z}_6$	293	$A_1^4A_{15}$ $\mathbb{Z}_4$	340	$A_2A_3^2A_7D_4$ $\mathbb{Z}_2$	387	$A_1^2A_4^3D_5$	434	$A_1A_4A_9D_5$ $\mathbb{Z}_2$
247	$A_1A_2A_5A_{11}$	294	$A_1^2A_2A_{15}$	341	$A_2A_3^2A_7D_4$ $\mathbb{Z}_4$	388	$A_1^3A_3^2A_5D_5$ $\mathbb{Z}_2^2$	435	$A_5A_9D_5$
248	$A_1A_2A_5A_{11}$ $\mathbb{Z}_3$	295	$A_1^2A_2A_{15}$ $\mathbb{Z}_2$	342	$A_1^2A_2A_4A_7D_4$ $\mathbb{Z}_2$	389	$A_1A_2A_3^2A_5D_5$ $\mathbb{Z}_2$	436	$A_5A_9D_5$ $\mathbb{Z}_2$
249	$A_3A_5A_{11}$	296	$A_1^2A_2A_{15}$ $\mathbb{Z}_4$	343	$A_2^2A_4A_7D_4$	390	$A_1^2A_3A_4A_5D_5$ $\mathbb{Z}_2$	437	$A_1^2A_2A_{10}D_5$
250	$A_3A_5A_{11}$ $\mathbb{Z}_2$	297	$A_2^2A_{15}$	344	$A_1A_3A_4A_7D_4$ $\mathbb{Z}_2$	391	$A_2A_3A_4A_5D_5$	438	$A_2^2A_{10}D_5$
251	$A_3A_5A_{11}$ $\mathbb{Z}_3$	298	$A_2^2A_{15}$ $\mathbb{Z}_2$	345	$A_1^3A_5A_7D_4$ $\mathbb{Z}_2^2$	392	$A_1A_4^2A_5D_5$	439	$A_1A_3A_{10}D_5$
252	$A_3A_5A_{11}$ $\mathbb{Z}_6$	299	$A_1A_3A_{15}$	346	$A_1A_2A_5A_7D_4$ $\mathbb{Z}_2$	393	$A_1^4A_5^2D_5$ $\mathbb{Z}_2^2$	440	$A_4A_{10}D_5$
253	$A_1^2A_6A_{11}$	300	$A_1A_3A_{15}$ $\mathbb{Z}_2$	347	$A_3A_5A_7D_4$ $\mathbb{Z}_2$	394	$A_2^2A_5^2D_5$ $\mathbb{Z}_3$	441	$A_1^3A_{11}D_5$ $\mathbb{Z}_2$
254	$A_1^2A_6A_{11}$ $\mathbb{Z}_2$	301	$A_1A_3A_{15}$ $\mathbb{Z}_4$	348	$A_1^2A_6A_7D_4$ $\mathbb{Z}_2$	395	$A_2^2A_5^2D_5$ $\mathbb{Z}_6$	442	$A_1^3A_{11}D_5$ $\mathbb{Z}_4$
255	$A_2A_6A_{11}$	302	A_4A_{15}	349	$A_2A_6A_7D_4$	396	$A_1A_3A_5^2D_5$ $\mathbb{Z}_2$	443	$A_1A_2A_{11}D_5$
256	$A_1A_7A_{11}$	303	A_4A_{15} $\mathbb{Z}_2$	350	$A_1A_7^2D_4$ $\mathbb{Z}_4$	397	$A_4A_5^2D_5$	444	$A_1A_2A_{11}D_5$ $\mathbb{Z}_2$
257	A_8A_{11}	304	$A_1^3A_{16}$	351	$A_1A_2^3A_8D_4$ $\mathbb{Z}_3$	398	$A_4A_5^2D_5$ $\mathbb{Z}_2$	445	$A_1A_2A_{11}D_5$ $\mathbb{Z}_4$
258	$A_1^3A_2^2A_{12}$	305	$A_1A_2A_{16}$	352	$A_1A_2A_4A_8D_4$	399	$A_1A_2^2A_3A_6D_5$	446	$A_3A_{11}D_5$ $\mathbb{Z}_2$
259	$A_1A_2^3A_{12}$	306	A_3A_{16}	353	$A_2A_5A_8D_4$ $\mathbb{Z}_3$	400	$A_2A_3^2A_6D_5$	447	$A_3A_{11}D_5$ $\mathbb{Z}_4$
260	$A_1^2A_2A_3A_{12}$	307	$A_1^2A_{17}$	354	$A_1A_6A_8D_4$	401	$A_1^2A_2A_4A_6D_5$	448	$A_1^2A_{12}D_5$
261	$A_2^2A_3A_{12}$	308	$A_1^2A_{17}$ $\mathbb{Z}_3$	355	$A_1^2A_2^2A_9D_4$ $\mathbb{Z}_2$	402	$A_2^2A_4A_6D_5$	449	$A_2A_{12}D_5$
262	$A_1A_3^2A_{12}$	309	A_2A_{17}	356	$A_1A_2A_3A_9D_4$ $\mathbb{Z}_2$	403	$A_1A_3A_4A_6D_5$	450	$A_1A_{13}D_5$
263	$A_1^3A_4A_{12}$	310	A_2A_{17} $\mathbb{Z}_3$	357	$A_1^2A_4A_9D_4$ $\mathbb{Z}_2$	404	$A_4^2A_6D_5$	451	$A_1A_{13}D_5$ $\mathbb{Z}_2$
264	$A_1A_2A_4A_{12}$	311	A_1A_{18}	358	$A_2A_4A_9D_4$	405	$A_1A_2A_5A_6D_5$	452	$A_{14}D_5$
265	$A_3A_4A_{12}$	312	A_{19}	359	$A_1A_5A_9D_4$ $\mathbb{Z}_2$	406	$A_3A_5A_6D_5$	453	$A_2^2A_3^2D_4D_5$ $\mathbb{Z}_2$
266	$A_1^2A_5A_{12}$	313	$A_3^5D_4$ $\mathbb{Z}_2\mathbb{Z}_4$	360	$A_6A_9D_4$	407	$A_1^2A_6^2D_5$	454	$A_2^3A_4D_4D_5$
267	$A_2A_5A_{12}$	314	$A_2A_3^3A_4D_4$ $\mathbb{Z}_2$	361	$A_1A_2^2A_{10}D_4$	408	$A_2A_6^2D_5$	455	$A_1A_2^2A_5D_4D_5$ $\mathbb{Z}_2$
268	$A_1A_6A_{12}$	315	$A_1A_2^3A_4^2D_4$	362	$A_2A_3A_{10}D_4$	409	$A_1^3A_2^2A_7D_5$ $\mathbb{Z}_2$	456	$A_1^2A_3A_5D_4D_5$ $\mathbb{Z}_2^2$
269	A_7A_{12}	316	$A_2^2A_3A_4^2D_4$	363	$A_1A_4A_{10}D_4$	410	$A_1^2A_2A_3A_7D_5$ $\mathbb{Z}_2$	457	$A_1A_4A_5D_4D_5$ $\mathbb{Z}_2$
270	$A_1^4A_2A_{13}$ $\mathbb{Z}_2$	317	$A_1A_2A_4^3D_4$	364	$A_5A_{10}D_4$	411	$A_1^2A_2A_3A_7D_5$ $\mathbb{Z}_4$	458	$A_5^2D_4D_5$ $\mathbb{Z}_2$
271	$A_1^2A_2^2A_{13}$	318	$A_1^2A_2A_3^2A_5D_4$ $\mathbb{Z}_2^2$	365	$A_1^2A_2A_{11}D_4$ $\mathbb{Z}_2$	412	$A_2^2A_3A_7D_5$	459	$A_2^2A_6D_4D_5$
272	$A_1^2A_2^2A_{13}$ $\mathbb{Z}_2$	319	$A_1A_3^3A_5D_4$ $\mathbb{Z}_2^2$	366	$A_2^2A_{11}D_4$ $\mathbb{Z}_3$	413	$A_2^2A_3A_7D_5$ $\mathbb{Z}_2$	460	$A_1A_2A_7D_4D_5$ $\mathbb{Z}_2$
273	$A_2^3A_{13}$	320	$A_1A_2A_3A_4A_5D_4$ $\mathbb{Z}_2$	367	$A_2^2A_{11}D_4$ $\mathbb{Z}_6$	414	$A_1A_3^2A_7D_5$ $\mathbb{Z}_4$	461	$A_1A_9D_4D_5$ $\mathbb{Z}_2$
274	$A_1^3A_3A_{13}$ $\mathbb{Z}_2$	321	$A_2A_4^2A_5D_4$	368	$A_1A_3A_{11}D_4$ $\mathbb{Z}_2$	415	$A_1^3A_4A_7D_5$ $\mathbb{Z}_2$	462	$A_1^2A_2^2A_3D_5^2$ $\mathbb{Z}_2$
275	$A_1A_2A_3A_{13}$	322	$A_1^3A_2A_5^2D_4$ $\mathbb{Z}_2^2$	369	$A_1A_3A_{11}D_4$ $\mathbb{Z}_4$	416	$A_1A_2A_4A_7D_5$	463	$A_1A_2A_3^2D_5^2$ $\mathbb{Z}_4$
276	$A_1A_2A_3A_{13}$ $\mathbb{Z}_2$	323	$A_1A_2^2A_5^2D_4$ $\mathbb{Z}_6$	370	$A_4A_{11}D_4$ $\mathbb{Z}_2$	417	$A_3A_4A_7D_5$ $\mathbb{Z}_2$	464	$A_1A_2^2A_4D_5^2$
277	$A_3^2A_{13}$	324	$A_1^2A_3A_5^2D_4$ $\mathbb{Z}_2^2$	371	$A_1A_2A_{12}D_4$	418	$A_1^2A_5A_7D_5$ $\mathbb{Z}_2$	465	$A_1A_4^2D_5^2$
278	$A_1^2A_4A_{13}$	325	$A_1A_4A_5^2D_4$ $\mathbb{Z}_2$	372	$A_1^2A_{13}D_4$ $\mathbb{Z}_2$	419	$A_2A_5A_7D_5$	466	$A_1^2A_2A_5D_5^2$ $\mathbb{Z}_2$
279	$A_1^2A_4A_{13}$ $\mathbb{Z}_2$	326	$A_5^3D_4$ $\mathbb{Z}_6$	373	$A_2A_{13}D_4$	420	$A_1A_6A_7D_5$	467	$A_1A_3A_5D_5^2$ $\mathbb{Z}_2$
280	$A_2A_4A_{13}$	327	$A_5^3D_4$ $\mathbb{Z}_2\mathbb{Z}_6$	374	$A_1A_{14}D_4$	421	$A_7^2D_5$ $\mathbb{Z}_8$	468	$A_4A_5D_5^2$
281	$A_1A_5A_{13}$	328	$A_3^3A_6D_4$ $\mathbb{Z}_2$	375	$A_{15}D_4$ $\mathbb{Z}_4$	422	$A_2^3A_8D_5$ $\mathbb{Z}_3$	469	$A_1A_2A_6D_5^2$
282	$A_1A_5A_{13}$ $\mathbb{Z}_2$	329	$A_1A_2^2A_4A_6D_4$	376	$A_1^2A_3^3D_4^2$ $\mathbb{Z}_2^3$	423	$A_1A_2A_3A_8D_5$	470	$A_1^2A_7D_5^2$ $\mathbb{Z}_2$

471 $A_1^2A_7D_5^2$	$\mathbb{Z}_4$	518 $A_5A_8D_6$		565 $A_1D_6^3$	$\mathbb{Z}_2^2$	612 $A_2^2A_4D_4D_7$		659 $A_2^2A_7D_8$	$\mathbb{Z}_2$
472 $A_2A_7D_5^2$	$\mathbb{Z}_4$	519 $A_1^2A_2A_9D_6$	$\mathbb{Z}_2$	566 $A_1A_2A_3^3D_7$	$\mathbb{Z}_4$	613 $A_1A_2A_5D_4D_7$	$\mathbb{Z}_2$	660 $A_1A_3A_7D_8$	$\mathbb{Z}_2$
473 $A_1A_8D_5^2$		520 $A_2^2A_9D_6$		567 $A_3^4D_7$	$\mathbb{Z}_4$	614 $A_2A_6D_4D_7$		661 $A_4A_7D_8$	$\mathbb{Z}_2$
474 $A_9D_5^2$		521 $A_2^2A_9D_6$	$\mathbb{Z}_2$	568 $A_1^2A_2^3A_4D_7$		615 $A_1A_7D_4D_7$	$\mathbb{Z}_2$	662 $A_1A_2A_8D_8$	
475 $A_1^2A_2D_5^3$	$\mathbb{Z}_2$	522 $A_1A_3A_9D_6$	$\mathbb{Z}_2$	569 $A_1A_2^2A_3A_4D_7$		616 $A_1A_2^3D_5D_7$		663 $A_1^2A_9D_8$	$\mathbb{Z}_2$
476 $A_4D_5^3$		523 $A_4A_9D_6$		570 $A_1^2A_3^2A_4D_7$	$\mathbb{Z}_2$	617 $A_2^2A_3D_5D_7$		664 $A_2A_9D_8$	
477 $D_4D_5^3$	$\mathbb{Z}_2$	524 $A_4A_9D_6$	$\mathbb{Z}_2$	571 $A_2A_3^2A_4D_7$		618 $A_1A_2A_4D_5D_7$		665 $A_1A_{10}D_8$	
478 $A_1A_2^3A_3^2D_6$	$\mathbb{Z}_2$	525 $A_1A_2A_{10}D_6$		572 $A_1^2A_2A_4^2D_7$		619 $A_1^2A_5D_5D_7$	$\mathbb{Z}_2$	666 $A_{11}D_8$	$\mathbb{Z}_2$
479 $A_2^3A_3A_4D_6$		526 $A_3A_{10}D_6$		573 $A_2^2A_4^2D_7$		620 $A_2A_5D_5D_7$		667 $A_1^2A_2A_3D_4D_8$	$\mathbb{Z}_2^2$
480 $A_1A_2A_3^2A_4D_6$	$\mathbb{Z}_2$	527 $A_1^2A_{11}D_6$	$\mathbb{Z}_2$	574 $A_1A_3A_4^2D_7$		621 $A_1A_6D_5D_7$		668 $A_2^2A_3D_4D_8$	$\mathbb{Z}_2$
481 $A_3^3A_4D_6$	$\mathbb{Z}_2$	528 $A_2A_{11}D_6$		575 $A_4^3D_7$		622 $A_7D_5D_7$	$\mathbb{Z}_2$	669 $A_1A_3^2D_4D_8$	$\mathbb{Z}_2^2$
482 $A_1A_2^2A_4^2D_6$		529 $A_2A_{11}D_6$	$\mathbb{Z}_2$	576 $A_1^2A_2A_3A_5D_7$	$\mathbb{Z}_2$	623 $A_1A_2A_3D_6D_7$	$\mathbb{Z}_2$	670 $A_1^2A_5D_4D_8$	$\mathbb{Z}_2^2$
483 $A_2A_3A_4^2D_6$		530 $A_1A_{12}D_6$		577 $A_1A_3^2A_5D_7$	$\mathbb{Z}_2$	624 $A_3^2D_6D_7$	$\mathbb{Z}_2$	671 $A_1^2A_2^2D_5D_8$	$\mathbb{Z}_2$
484 $A_2^4A_5D_6$	$\mathbb{Z}_3$	531 $A_{13}D_6$		578 $A_1^3A_4A_5D_7$	$\mathbb{Z}_2$	625 $A_2A_4D_6D_7$		672 $A_1^3A_3D_5D_8$	$\mathbb{Z}_2^2$
485 $A_1^3A_2A_3A_5D_6$	$\mathbb{Z}_2^2$	532 $A_1^3A_3^2D_4D_6$	$\mathbb{Z}_2^3$	579 $A_1A_2A_4A_5D_7$		626 $A_1A_5D_6D_7$	$\mathbb{Z}_2$	673 $A_1A_2A_3D_5D_8$	$\mathbb{Z}_2$
486 $A_1A_2^2A_3A_5D_6$	$\mathbb{Z}_2$	533 $A_1A_2A_3^2D_4D_6$	$\mathbb{Z}_2^2$	580 $A_3A_4A_5D_7$		627 $A_6D_6D_7$		674 $A_1^2A_4D_5D_8$	$\mathbb{Z}_2$
487 $A_1^2A_3^2A_5D_6$	$\mathbb{Z}_2^2$	534 $A_1A_3A_5D_4D_6$	$\mathbb{Z}_2^2$	581 $A_1^2A_5^2D_7$	$\mathbb{Z}_2$	628 $A_1D_5D_6D_7$	$\mathbb{Z}_2$	675 $A_1A_5D_5D_8$	$\mathbb{Z}_2$
488 $A_2A_3^2A_5D_6$	$\mathbb{Z}_2$	535 $A_4A_5D_4D_6$	$\mathbb{Z}_2$	582 $A_1^2A_2^2A_6D_7$		629 $D_6^2D_7$	$\mathbb{Z}_2$	676 $A_1^2D_4D_5D_8$	$\mathbb{Z}_2^2$
489 $A_1^2A_2A_4A_5D_6$	$\mathbb{Z}_2$	536 $A_2A_7D_4D_6$	$\mathbb{Z}_2$	583 $A_1A_2A_3A_6D_7$		630 $A_1A_2^2D_7^2$		677 $A_1D_5^2D_8$	$\mathbb{Z}_2$
490 $A_2^2A_4A_5D_6$		537 $A_9D_4D_6$	$\mathbb{Z}_2$	584 $A_3^2A_6D_7$		631 $A_1^2A_3D_7^2$	$\mathbb{Z}_2$	678 $A_1^3A_2D_6D_8$	$\mathbb{Z}_2^2$
491 $A_1A_3A_4A_5D_6$	$\mathbb{Z}_2$	538 $A_1^2A_3D_4^2D_6$	$\mathbb{Z}_2^3$	585 $A_1^2A_4A_6D_7$		632 $A_2A_3D_7^2$		679 $A_1A_2^2D_6D_8$	$\mathbb{Z}_2$
492 $A_4^2A_5D_6$		539 $A_1A_2^2A_3D_5D_6$	$\mathbb{Z}_2$	586 $A_2A_4A_6D_7$		633 $A_1A_4D_7^2$		680 $A_1^2A_3D_6D_8$	$\mathbb{Z}_2^2$
493 $A_1^3A_5^2D_6$	$\mathbb{Z}_2^2$	540 $A_1^2A_3^2D_5D_6$	$\mathbb{Z}_2^2$	587 $A_1A_5A_6D_7$		634 $A_5D_7^2$		681 $A_2A_3D_6D_8$	$\mathbb{Z}_2$
494 $A_1A_2A_5^2D_6$	$\mathbb{Z}_2$	541 $A_2^2A_4D_5D_6$		588 $A_6^2D_7$		635 $A_1^3A_2A_3^2D_8$	$\mathbb{Z}_2^2$	682 $A_1A_4D_6D_8$	$\mathbb{Z}_2$
495 $A_3A_5^2D_6$	$\mathbb{Z}_2$	542 $A_1A_3A_4D_5D_6$	$\mathbb{Z}_2$	589 $A_1^3A_2A_7D_7$	$\mathbb{Z}_2$	636 $A_1A_2^2A_3^2D_8$	$\mathbb{Z}_2$	683 $A_5D_6D_8$	$\mathbb{Z}_2$
496 $A_3A_5^2D_6$	$\mathbb{Z}_2^2$	543 $A_4^2D_5D_6$		590 $A_1A_2^2A_7D_7$		637 $A_1^2A_3^3D_8$	$\mathbb{Z}_2^2$	684 $A_1D_4D_6D_8$	$\mathbb{Z}_2^2$
497 $A_1A_2^3A_6D_6$		544 $A_1^3A_5D_5D_6$	$\mathbb{Z}_2^2$	591 $A_1^2A_3A_7D_7$	$\mathbb{Z}_2$	638 $A_2A_3^3D_8$	$\mathbb{Z}_2$	685 $D_5D_6D_8$	$\mathbb{Z}_2$
498 $A_2^2A_3A_6D_6$		545 $A_1A_2A_5D_5D_6$	$\mathbb{Z}_2$	592 $A_1^2A_3A_7D_7$	$\mathbb{Z}_4$	639 $A_1^2A_2A_3A_4D_8$	$\mathbb{Z}_2$	686 $A_1^2A_2D_7D_8$	$\mathbb{Z}_2$
499 $A_1A_3^2A_6D_6$	$\mathbb{Z}_2$	546 $A_3A_5D_5D_6$	$\mathbb{Z}_2$	593 $A_2A_3A_7D_7$		640 $A_2^2A_3A_4D_8$		687 $A_2^2D_7D_8$	
500 $A_1A_2A_4A_6D_6$		547 $A_2A_6D_5D_6$		594 $A_2A_3A_7D_7$	$\mathbb{Z}_2$	641 $A_1A_3^2A_4D_8$	$\mathbb{Z}_2$	688 $A_1A_3D_7D_8$	$\mathbb{Z}_2$
501 $A_3A_4A_6D_6$		548 $A_1A_7D_5D_6$	$\mathbb{Z}_2$	595 $A_2A_3A_7D_7$	$\mathbb{Z}_4$	642 $A_1A_2A_4^2D_8$		689 $A_1^3D_8^2$	$\mathbb{Z}_2^2$
502 $A_1^2A_5A_6D_6$	$\mathbb{Z}_2$	549 $A_8D_5D_6$		596 $A_1A_4A_7D_7$		643 $A_1^4A_2A_5D_8$	$\mathbb{Z}_2^2$	690 $A_1A_2D_8^2$	$\mathbb{Z}_2$
503 $A_2A_5A_6D_6$		550 $A_1A_3D_4D_5D_6$	$\mathbb{Z}_2^2$	597 $A_5A_7D_7$		644 $A_1^3A_3A_5D_8$	$\mathbb{Z}_2^2$	691 $A_3D_8^2$	$\mathbb{Z}_2^2$
504 $A_1A_6^2D_6$		551 $A_1A_2D_5^2D_6$	$\mathbb{Z}_2$	598 $A_1^2A_2A_8D_7$		645 $A_1A_2A_3A_5D_8$	$\mathbb{Z}_2$	692 $A_1A_2^3A_3D_9$	
505 $A_1^2A_2^2A_7D_6$	$\mathbb{Z}_2$	552 $A_1^4A_3D_6^2$	$\mathbb{Z}_2^3$	599 $A_1A_3A_8D_7$		646 $A_3^2A_5D_8$	$\mathbb{Z}_2$	693 $A_2^2A_3^2D_9$	
506 $A_2^3A_7D_6$		553 $A_1^2A_2A_3D_6^2$	$\mathbb{Z}_2^2$	600 $A_4A_8D_7$		647 $A_1^2A_4A_5D_8$	$\mathbb{Z}_2$	694 $A_1^2A_2^2A_4D_9$	
507 $A_1^3A_3A_7D_6$	$\mathbb{Z}_2^2$	554 $A_2^2A_3D_6^2$	$\mathbb{Z}_2$	601 $A_1^3A_9D_7$	$\mathbb{Z}_2$	648 $A_2A_4A_5D_8$		695 $A_2^3A_4D_9$	
508 $A_1A_2A_3A_7D_6$	$\mathbb{Z}_2$	555 $A_1A_3^2D_6^2$	$\mathbb{Z}_2^2$	602 $A_1A_2A_9D_7$		649 $A_1A_5^2D_8$	$\mathbb{Z}_2$	696 $A_1A_2A_3A_4D_9$	
509 $A_3^2A_7D_6$	$\mathbb{Z}_2$	556 $A_3A_4D_6^2$	$\mathbb{Z}_2$	603 $A_1A_2A_9D_7$	$\mathbb{Z}_2$	650 $A_1A_5^2D_8$	$\mathbb{Z}_2^2$	697 $A_1^2A_4^2D_9$	
510 $A_3^2A_7D_6$	$\mathbb{Z}_4$	557 $A_1^2A_5D_6^2$	$\mathbb{Z}_2^2$	604 $A_3A_9D_7$		651 $A_1A_2^2A_6D_8$		698 $A_2A_4^2D_9$	
511 $A_1^2A_4A_7D_6$	$\mathbb{Z}_2$	558 $A_2A_5D_6^2$	$\mathbb{Z}_2$	605 $A_1^2A_{10}D_7$		652 $A_1^2A_3A_6D_8$	$\mathbb{Z}_2$	699 $A_1A_2^2A_5D_9$	
512 $A_2A_4A_7D_6$		559 $A_7D_6^2$	$\mathbb{Z}_2$	606 $A_2A_{10}D_7$		653 $A_2A_3A_6D_8$		700 $A_1^2A_3A_5D_9$	$\mathbb{Z}_2$
513 $A_1A_5A_7D_6$	$\mathbb{Z}_2$	560 $A_1^3D_4D_6^2$	$\mathbb{Z}_2^3$	607 $A_1A_{11}D_7$		654 $A_1A_4A_6D_8$		701 $A_2A_3A_5D_9$	
514 $A_6A_7D_6$		561 $A_1A_2D_4D_6^2$	$\mathbb{Z}_2^2$	608 $A_1A_{11}D_7$	$\mathbb{Z}_2$	655 $A_5A_6D_8$		702 $A_1A_4A_5D_9$	
515 $A_1A_2^2A_8D_6$		562 $A_3D_4D_6^2$	$\mathbb{Z}_2^2$	609 $A_1A_{11}D_7$	$\mathbb{Z}_4$	656 $A_1^4A_7D_8$	$\mathbb{Z}_2^2$	703 $A_5^2D_9$	
516 $A_2A_3A_8D_6$		563 $A_1^2D_5D_6^2$	$\mathbb{Z}_2^2$	610 $A_{12}D_7$		657 $A_1^2A_2A_7D_8$	$\mathbb{Z}_2$	704 $A_5^2D_9$	$\mathbb{Z}_2$
517 $A_1A_4A_8D_6$		564 $A_2D_5D_6^2$	$\mathbb{Z}_2$	611 $A_2A_3^2D_4D_7$	$\mathbb{Z}_2$	658 $A_2^2A_7D_8$		705 $A_1^2A_2A_6D_9$	

706	$A_2^2A_6D_9$	753	$A_1A_4D_4D_{10}$ $\mathbb{Z}_2$	800	D_7D_{12} $\mathbb{Z}_2$	847	$A_3A_5^2E_6$	894	$A_1A_4^2D_4E_6$
707	$A_1A_3A_6D_9$	754	$A_5D_4D_{10}$ $\mathbb{Z}_2$	801	$A_1^2A_2^2D_{13}$	848	$A_3A_5^2E_6$ $\mathbb{Z}_2$	895	$A_2^2A_5D_4E_6$ $\mathbb{Z}_3$
708	$A_4A_6D_9$	755	$A_1^2A_2D_5D_{10}$ $\mathbb{Z}_2$	802	$A_1A_2A_3D_{13}$	849	$A_3A_5^2E_6$ $\mathbb{Z}_3$	896	$A_1A_3A_5D_4E_6$ $\mathbb{Z}_2$
709	$A_1^3A_7D_9$ $\mathbb{Z}_2$	756	$A_2^2D_5D_{10}$	803	$A_1^2A_4D_{13}$	850	$A_3A_5^2E_6$ $\mathbb{Z}_6$	897	$A_4A_5D_4E_6$
710	$A_1A_2A_7D_9$	757	$A_1A_3D_5D_{10}$ $\mathbb{Z}_2$	804	$A_2A_4D_{13}$	851	$A_1^2A_2A_3A_6E_6$	898	$A_1A_2A_6D_4E_6$
711	$A_3A_7D_9$ $\mathbb{Z}_2$	758	$A_4D_5D_{10}$	805	$A_1A_5D_{13}$	852	$A_2^2A_3A_6E_6$	899	$A_1^2A_7D_4E_6$ $\mathbb{Z}_2$
712	$A_1^2A_8D_9$	759	$A_1^3D_6D_{10}$ $\mathbb{Z}_2^2$	806	A_6D_{13}	853	$A_1A_3^2A_6E_6$	900	$A_1A_8D_4E_6$
713	$A_2A_8D_9$	760	$A_1A_2D_6D_{10}$ $\mathbb{Z}_2$	807	$A_1D_5D_{13}$	854	$A_1^3A_4A_6E_6$	901	$A_9D_4E_6$
714	$A_1A_9D_9$	761	$A_3D_6D_{10}$ $\mathbb{Z}_2$	808	D_6D_{13}	855	$A_1A_2A_4A_6E_6$	902	$A_1^2A_3^2D_5E_6$ $\mathbb{Z}_2$
715	$A_1A_9D_9$ $\mathbb{Z}_2$	762	$A_1^2D_7D_{10}$ $\mathbb{Z}_2$	809	$A_1^3A_2D_{14}$ $\mathbb{Z}_2$	856	$A_3A_4A_6E_6$	903	$A_1^2A_2A_4D_5E_6$
716	$A_{10}D_9$	763	$A_2D_7D_{10}$	810	$A_1A_2^2D_{14}$	857	$A_1^2A_5A_6E_6$	904	$A_1A_3A_4D_5E_6$
717	$A_1A_5D_4D_9$ $\mathbb{Z}_2$	764	$A_1D_8D_{10}$ $\mathbb{Z}_2$	811	$A_1A_2^2D_{14}$ $\mathbb{Z}_2$	858	$A_2A_5A_6E_6$	905	$A_4^2D_5E_6$
718	$A_1A_2^2D_5D_9$	765	D_9D_{10}	812	$A_1^2A_3D_{14}$ $\mathbb{Z}_2$	859	$A_1A_6^2E_6$	906	$A_1^3A_5D_5E_6$ $\mathbb{Z}_2$
719	$A_1A_4D_5D_9$	766	$A_1A_2^2A_3D_{11}$	813	$A_2A_3D_{14}$	860	$A_1^4A_2A_7E_6$ $\mathbb{Z}_2$	907	$A_3A_5D_5E_6$
720	$A_5D_5D_9$	767	$A_1^2A_3^2D_{11}$ $\mathbb{Z}_2$	814	$A_1A_4D_{14}$	861	$A_1^3A_3A_7E_6$ $\mathbb{Z}_2$	908	$A_1^2A_6D_5E_6$
721	$D_5^2D_9$	768	$A_2A_3^2D_{11}$	815	$A_1A_4D_{14}$ $\mathbb{Z}_2$	862	$A_1A_2A_3A_7E_6$	909	$A_2A_6D_5E_6$
722	$A_2^2D_6D_9$	769	$A_1^2A_2A_4D_{11}$	816	A_5D_{14}	863	$A_1^2A_4A_7E_6$	910	$A_1A_7D_5E_6$
723	$A_1A_3D_6D_9$ $\mathbb{Z}_2$	770	$A_2^2A_4D_{11}$	817	$A_1D_4D_{14}$ $\mathbb{Z}_2$	864	$A_2A_4A_7E_6$	911	$A_8D_5E_6$
724	$A_4D_6D_9$	771	$A_1A_3A_4D_{11}$	818	D_5D_{14}	865	$A_1A_5A_7E_6$	912	$A_4D_4D_5E_6$
725	$A_1A_2D_7D_9$	772	$A_4^2D_{11}$	819	$A_1^2A_2D_{15}$	866	$A_1A_5A_7E_6$ $\mathbb{Z}_2$	913	$A_1A_2D_5^2E_6$
726	$A_1^2D_8D_9$ $\mathbb{Z}_2$	773	$A_1^3A_5D_{11}$ $\mathbb{Z}_2$	820	$A_2^2D_{15}$	867	$A_6A_7E_6$	914	$A_3D_5^2E_6$
727	$A_1D_9^2$	774	$A_1A_2A_5D_{11}$	821	$A_1A_3D_{15}$	868	$A_1^3A_2A_8E_6$ $\mathbb{Z}_3$	915	$A_2^2A_3D_6E_6$
728	$A_1^3A_2^3D_{10}$ $\mathbb{Z}_2$	775	$A_3A_5D_{11}$	822	A_4D_{15}	869	$A_1A_2^2A_8E_6$ $\mathbb{Z}_3$	916	$A_1A_3^2D_6E_6$ $\mathbb{Z}_2$
729	$A_1^2A_2^2A_3D_{10}$ $\mathbb{Z}_2$	776	$A_1^2A_6D_{11}$	823	$A_1^3D_{16}$ $\mathbb{Z}_2$	870	$A_1^2A_3A_8E_6$	917	$A_1A_2A_4D_6E_6$
730	$A_1^3A_3^2D_{10}$ $\mathbb{Z}_2^2$	777	$A_2A_6D_{11}$	824	$A_1A_2D_{16}$	871	$A_2A_3A_8E_6$	918	$A_3A_4D_6E_6$
731	$A_1A_2A_3^2D_{10}$ $\mathbb{Z}_2$	778	$A_1A_7D_{11}$	825	$A_1A_2D_{16}$ $\mathbb{Z}_2$	872	$A_2A_3A_8E_6$ $\mathbb{Z}_3$	919	$A_1^2A_5D_6E_6$ $\mathbb{Z}_2$
732	$A_3^3D_{10}$ $\mathbb{Z}_2$	779	A_8D_{11}	826	A_3D_{16} $\mathbb{Z}_2$	873	$A_1A_4A_8E_6$	920	$A_2A_5D_6E_6$
733	$A_1^3A_2A_4D_{10}$ $\mathbb{Z}_2$	780	$A_2^2D_4D_{11}$	827	$A_1^2D_{17}$	874	$A_5A_8E_6$	921	$A_1A_6D_6E_6$
734	$A_1A_2^2A_4D_{10}$	781	$A_1A_2D_5D_{11}$	828	A_2D_{17}	875	$A_1^4A_9E_6$ $\mathbb{Z}_2$	922	$A_7D_6E_6$
735	$A_1^2A_3A_4D_{10}$ $\mathbb{Z}_2$	782	$A_2D_6D_{11}$	829	A_1D_{18}	876	$A_1^2A_2A_9E_6$	923	$A_2D_5D_6E_6$
736	$A_2A_3A_4D_{10}$	783	$A_1D_7D_{11}$	830	D_{19}	877	$A_2^2A_9E_6$	924	$A_1^2A_2^2D_7E_6$
737	$A_1A_4^2D_{10}$	784	$A_1^3A_2^2D_{12}$ $\mathbb{Z}_2$	831	$A_1^2A_2A_3^3E_6$ $\mathbb{Z}_2$	878	$A_1A_3A_9E_6$	925	$A_1A_2A_3D_7E_6$
738	$A_1^4A_5D_{10}$ $\mathbb{Z}_2^2$	785	$A_1A_2^3D_{12}$	832	$A_2^2A_3^3E_6$	879	$A_1A_3A_9E_6$ $\mathbb{Z}_2$	926	$A_1^2A_4D_7E_6$
739	$A_1^2A_2A_5D_{10}$ $\mathbb{Z}_2$	786	$A_1^2A_2A_3D_{12}$ $\mathbb{Z}_2$	833	$A_1A_2A_3^2A_4E_6$	880	$A_4A_9E_6$	927	$A_2A_4D_7E_6$
740	$A_1A_3A_5D_{10}$ $\mathbb{Z}_2$	787	$A_2^2A_3D_{12}$	834	$A_1^3A_2A_4^2E_6$	881	$A_1^3A_{10}E_6$	928	$A_1A_5D_7E_6$
741	$A_1A_3A_5D_{10}$ $\mathbb{Z}_2^2$	788	$A_2^2A_3D_{12}$ $\mathbb{Z}_2$	835	$A_1^2A_3A_4^2E_6$	882	$A_1A_2A_{10}E_6$	929	$A_6D_7E_6$
742	$A_4A_5D_{10}$	789	$A_1A_3^2D_{12}$ $\mathbb{Z}_2$	836	$A_2A_3A_4^2E_6$	883	$A_3A_{10}E_6$	930	$A_1D_5D_7E_6$
743	$A_4A_5D_{10}$ $\mathbb{Z}_2$	790	$A_1^3A_4D_{12}$ $\mathbb{Z}_2$	837	$A_1A_4^3E_6$	884	$A_1^2A_{11}E_6$	931	$D_6D_7E_6$
744	$A_1^3A_6D_{10}$ $\mathbb{Z}_2$	791	$A_1A_2A_4D_{12}$	838	$A_2^4A_5E_6$ $\mathbb{Z}_3^2$	885	$A_1^2A_{11}E_6$ $\mathbb{Z}_2$	932	$A_1^2A_3D_8E_6$ $\mathbb{Z}_2$
745	$A_1A_2A_6D_{10}$	792	$A_3A_4D_{12}$ $\mathbb{Z}_2$	839	$A_1A_2^2A_3A_5E_6$ $\mathbb{Z}_3$	886	$A_1^2A_{11}E_6$ $\mathbb{Z}_3$	933	$A_1A_4D_8E_6$
746	$A_3A_6D_{10}$	793	$A_1^2A_5D_{12}$ $\mathbb{Z}_2$	840	$A_1^2A_3^2A_5E_6$ $\mathbb{Z}_2$	887	$A_1^2A_{11}E_6$ $\mathbb{Z}_6$	934	$A_5D_8E_6$
747	$A_1^2A_7D_{10}$ $\mathbb{Z}_2$	794	$A_2A_5D_{12}$	841	$A_2A_3^2A_5E_6$	888	$A_2A_{11}E_6$	935	$D_5D_8E_6$
748	$A_2A_7D_{10}$	795	$A_1A_6D_{12}$	842	$A_2^2A_4A_5E_6$ $\mathbb{Z}_3$	889	$A_2A_{11}E_6$ $\mathbb{Z}_3$	936	$A_1^2A_2D_9E_6$
749	$A_1A_8D_{10}$	796	$A_1A_2D_4D_{12}$ $\mathbb{Z}_2$	843	$A_1A_3A_4A_5E_6$	890	$A_1A_{12}E_6$	937	$A_2^2D_9E_6$
750	A_9D_{10}	797	$A_1^2D_5D_{12}$ $\mathbb{Z}_2$	844	$A_4^2A_5E_6$	891	$A_{13}E_6$	938	$A_1A_3D_9E_6$
751	$A_1A_2^2D_4D_{10}$ $\mathbb{Z}_2$	798	$A_2D_5D_{12}$ $\mathbb{Z}_2$	845	$A_1^3A_5^2E_6$ $\mathbb{Z}_6$	892	$A_1A_2^4D_4E_6$ $\mathbb{Z}_3$	939	$A_4D_9E_6$
752	$A_1^2A_3D_4D_{10}$ $\mathbb{Z}_2^2$	799	$A_1D_6D_{12}$ $\mathbb{Z}_2$	846	$A_1A_2A_5^2E_6$ $\mathbb{Z}_3$	893	$A_1A_2^2A_4D_4E_6$	940	$D_4D_9E_6$

941	$A_1^3D_{10}E_6$	$\mathbb{Z}_2$	979	$A_3A_4A_5E_7$	$\mathbb{Z}_2$	1017	$A_1A_2A_5D_4E_7$	$\mathbb{Z}_2$	1055	$D_5D_7E_7$		1093	$A_1^2A_2A_3A_4E_8$		1131	$A_2A_4D_5E_8$			
942	$A_1A_2D_{10}E_6$		980	$A_1^2A_5^2E_7$	$\mathbb{Z}_2$	1018	$A_3A_5D_4E_7$	$\mathbb{Z}_2$	1056	$A_1^4D_8E_7$	$\mathbb{Z}_2^2$	1094	$A_2^2A_3A_4E_8$		1132	$A_1A_5D_5E_8$			
943	$A_3D_{10}E_6$		981	$A_2A_5^2E_7$		1019	$A_2A_6D_4E_7$		1057	$A_1^2A_2D_8E_7$	$\mathbb{Z}_2$	1095	$A_1A_3^2A_4E_8$		1133	$A_6D_5E_8$			
944	$A_1^2D_{11}E_6$		982	$A_1^2A_2^2A_6E_7$		1020	$A_1A_7D_4E_7$	$\mathbb{Z}_2$	1058	$A_2^2D_8E_7$		1096	$A_1^3A_4^2E_8$		1134	$A_1D_5^2E_8$			
945	$A_2D_{11}E_6$		983	$A_2^3A_6E_7$		1021	$A_8D_4E_7$		1059	$A_1A_3D_8E_7$	$\mathbb{Z}_2$	1097	$A_1A_2A_4^2E_8$		1135	$A_1A_2^2D_6E_8$			
946	$A_1D_{12}E_6$		984	$A_1A_2A_3A_6E_7$		1022	$A_1^3A_2^2D_5E_7$	$\mathbb{Z}_2$	1060	$A_4D_8E_7$		1098	$A_3A_4^2E_8$		1136	$A_2A_3D_6E_8$			
947	$D_{13}E_6$		985	$A_3^2A_6E_7$		1023	$A_1^2A_2A_3D_5E_7$	$\mathbb{Z}_2$	1061	$A_1A_2D_9E_7$		1099	$A_1^3A_3A_5E_8$	$\mathbb{Z}_2$	1137	$A_1A_4D_6E_8$			
948	$A_1^3A_2^2E_6^2$	$\mathbb{Z}_3$	986	$A_1^2A_4A_6E_7$		1024	$A_2^2A_3D_5E_7$		1062	$A_3D_9E_7$		1100	$A_1A_2A_3A_5E_8$		1138	$A_5D_6E_8$			
949	$A_2^2A_3E_6^2$	$\mathbb{Z}_3$	987	$A_2A_4A_6E_7$		1025	$A_1A_3^2D_5E_7$	$\mathbb{Z}_2$	1063	$A_1^2D_{10}E_7$	$\mathbb{Z}_2$	1101	$A_3^2A_5E_8$		1139	$D_5D_6E_8$			
950	$A_1A_3^2E_6^2$		988	$A_1A_5A_6E_7$		1026	$A_1A_2A_4D_5E_7$		1064	$A_2D_{10}E_7$		1102	$A_1^2A_4A_5E_8$		1140	$A_1^2A_2D_7E_8$			
951	$A_1^3A_4E_6^2$		989	$A_6^2E_7$		1027	$A_3A_4D_5E_7$		1065	$A_2D_{10}E_7$	$\mathbb{Z}_2$	1103	$A_2A_4A_5E_8$		1141	$A_2^2D_7E_8$			
952	$A_3A_4E_6^2$		990	$A_1^3A_2A_7E_7$	$\mathbb{Z}_2$	1028	$A_1^2A_5D_5E_7$	$\mathbb{Z}_2$	1066	$A_1D_{11}E_7$		1104	$A_1A_5^2E_8$		1142	$A_1A_3D_7E_8$			
953	$A_1^2A_5E_6^2$	$\mathbb{Z}_3$	991	$A_1A_2^2A_7E_7$		1029	$A_2A_5D_5E_7$		1067	$D_{12}E_7$		1105	$A_1^3A_2A_6E_8$		1143	$A_4D_7E_8$			
954	$A_2A_5E_6^2$	$\mathbb{Z}_3$	992	$A_1A_2^2A_7E_7$	$\mathbb{Z}_2$	1030	$A_2A_5D_5E_7$	$\mathbb{Z}_2$	1068	$A_1^3A_3E_6E_7$	$\mathbb{Z}_2$	1106	$A_1A_2^2A_6E_8$		1144	$A_1A_2D_8E_8$			
955	$A_1A_6E_6^2$		993	$A_1^2A_3A_7E_7$	$\mathbb{Z}_2$	1031	$A_1A_6D_5E_7$		1069	$A_1A_2A_3E_6E_7$		1107	$A_1^2A_3A_6E_8$		1145	$A_1^2D_9E_8$			
956	$A_7E_6^2$		994	$A_2A_3A_7E_7$		1032	$A_7D_5E_7$		1070	$A_3^2E_6E_7$		1108	$A_2A_3A_6E_8$		1146	$A_2D_9E_8$			
957	$A_1A_2D_4E_6^2$		995	$A_1A_4A_7E_7$		1033	$A_1A_2D_4D_5E_7$	$\mathbb{Z}_2$	1071	$A_1^2A_4E_6E_7$		1109	$A_1A_4A_6E_8$		1147	$A_1D_{10}E_8$			
958	$A_3D_4E_6^2$		996	$A_1A_4A_7E_7$	$\mathbb{Z}_2$	1034	$A_1^2D_5^2E_7$	$\mathbb{Z}_2$	1072	$A_2A_4E_6E_7$		1110	$A_5A_6E_8$		1148	$D_{11}E_8$			
959	$A_1^2D_5E_6^2$		997	$A_5A_7E_7$		1035	$A_2D_5^2E_7$		1073	$A_1A_5E_6E_7$		1111	$A_1^4A_7E_8$	$\mathbb{Z}_2$	1149	$A_1^3A_2E_6E_8$			
960	$A_1D_6E_6^2$		998	$A_1^2A_2A_8E_7$		1036	$A_1^2A_2^2D_6E_7$	$\mathbb{Z}_2$	1074	$A_6E_6E_7$		1112	$A_1^2A_2A_7E_8$		1150	$A_1^2A_3E_6E_8$			
961	$D_7E_6^2$		999	$A_2^2A_8E_7$		1037	$A_2^3D_6E_7$		1075	$A_2D_4E_6E_7$		1113	$A_2^2A_7E_8$		1151	$A_2A_3E_6E_8$			
962	$A_1E_6^3$	$\mathbb{Z}_3$	1000	$A_1A_3A_8E_7$		1038	$A_1^3A_3D_6E_7$	$\mathbb{Z}_2^2$	1076	$A_1D_5E_6E_7$		1114	$A_1A_3A_7E_8$		1152	$A_1A_4E_6E_8$			
963	$A_1^2A_2^2A_3^2E_7$	$\mathbb{Z}_2$	1001	$A_4A_8E_7$		1039	$A_1A_2A_3D_6E_7$	$\mathbb{Z}_2$	1077	$D_6E_6E_7$		1115	$A_4A_7E_8$		1153	$A_5E_6E_8$			
964	$A_2^3A_3^2E_7$		1002	$A_1^3A_9E_7$	$\mathbb{Z}_2$	1040	$A_3^2D_6E_7$	$\mathbb{Z}_2$	1078	$E_6^2E_7$		1116	$A_1^3A_8E_8$		1154	$A_1D_4E_6E_8$			
965	$A_1A_2^2A_3A_4E_7$		1003	$A_1A_2A_9E_7$		1041	$A_1^2A_4D_6E_7$	$\mathbb{Z}_2$	1079	$A_1^3A_2E_7^2$	$\mathbb{Z}_2$	1117	$A_1A_2A_8E_8$		1155	$D_5E_6E_8$			
966	$A_1^2A_3^2A_4E_7$	$\mathbb{Z}_2$	1004	$A_1A_2A_9E_7$	$\mathbb{Z}_2$	1042	$A_2A_4D_6E_7$		1080	$A_1A_2^2E_7^2$		1118	$A_3A_8E_8$		1156	$A_1^2A_2E_7E_8$			
967	$A_2A_3^2A_4E_7$		1005	$A_3A_9E_7$		1043	$A_1A_5D_6E_7$	$\mathbb{Z}_2$	1081	$A_1^2A_3E_7^2$	$\mathbb{Z}_2$	1119	$A_1^2A_9E_8$		1157	$A_2^2E_7E_8$			
968	$A_1^2A_2A_4^2E_7$		1006	$A_3A_9E_7$	$\mathbb{Z}_2$	1044	$A_6D_6E_7$		1082	$A_2A_3E_7^2$		1120	$A_2A_9E_8$		1158	$A_1A_3E_7E_8$			
969	$A_2^2A_4^2E_7$		1007	$A_1^2A_{10}E_7$		1045	$A_2D_4D_6E_7$	$\mathbb{Z}_2$	1083	$A_2A_3E_7^2$	$\mathbb{Z}_2$	1121	$A_1A_{10}E_8$		1159	$A_4E_7E_8$			
970	$A_1A_3A_4^2E_7$		1008	$A_2A_{10}E_7$		1046	$A_1D_5D_6E_7$	$\mathbb{Z}_2$	1084	$A_1A_4E_7^2$		1122	$A_{11}E_8$		1160	$D_4E_7E_8$			
971	$A_1^4A_3A_5E_7$	$\mathbb{Z}_2^2$	1009	$A_1A_{11}E_7$		1047	$D_6^2E_7$	$\mathbb{Z}_2$	1085	$A_5E_7^2$		1123	$A_1A_2^3D_4E_8$		1161	$A_1^3E_8^2$			
972	$A_1^2A_2A_3A_5E_7$	$\mathbb{Z}_2$	1010	$A_{12}E_7$		1048	$A_1A_2^2D_7E_7$		1086	$A_1D_4E_7^2$	$\mathbb{Z}_2$	1124	$A_2^2A_3D_4E_8$		1162	$A_1A_2E_8^2$			
973	$A_2^2A_3A_5E_7$		1011	$A_1A_2^2A_3D_4E_7$	$\mathbb{Z}_2$	1049	$A_1^2A_3D_7E_7$	$\mathbb{Z}_2$	1087	$D_5E_7^2$		1125	$A_1A_2A_4D_4E_8$		1163	$A_3E_8^2$			
974	$A_2^2A_3A_5E_7$	$\mathbb{Z}_2$	1012	$A_1^2A_3^2D_4E_7$	$\mathbb{Z}_2^2$	1050	$A_2A_3D_7E_7$		1088	$D_5E_7^2$	$\mathbb{Z}_2$	1126	$A_2A_5D_4E_8$						
975	$A_1A_3^2A_5E_7$	$\mathbb{Z}_2$	1013	$A_2^2A_4D_4E_7$		1051	$A_1A_4D_7E_7$		1089	$A_1A_2^2A_3^2E_8$		1127	$A_1A_6D_4E_8$						
976	$A_1^3A_4A_5E_7$	$\mathbb{Z}_2$	1014	$A_1A_3A_4D_4E_7$	$\mathbb{Z}_2$	1052	$A_5D_7E_7$		1090	$A_1^2A_3^3E_8$	$\mathbb{Z}_2$	1128	$A_1^2A_2^2D_5E_8$						
977	$A_1A_2A_4A_5E_7$		1015	$A_4^2D_4E_7$		1053	$A_5D_7E_7$	$\mathbb{Z}_2$	1091	$A_2A_3^3E_8$		1129	$A_1A_2A_3D_5E_8$						
978	$A_3A_4A_5E_7$		1016	$A_1^3A_5D_4E_7$	$\mathbb{Z}_2^2$	1054	$A_1D_4D_7E_7$	$\mathbb{Z}_2$	1092	$A_1^3A_2^2A_4E_8$		1130	$A_1^2A_4D_5E_8$						

Table D.7: Algebras of maximal rank for the heterotic string on T^3.

1	$A_1^2A_3^3$	$\mathbb{Z}_2^3$	48	$A_3A_4C_2^2$	$\mathbb{Z}_2$	95	$A_1A_5C_5$	$\mathbb{Z}_2$	142	$A_4C_1^2D_5$		189	$C_1C_3E_7$		236	$A_1^2C_7F_2$							
2	$A_2A_3^3$	$\mathbb{Z}_2^2$	49	$A_1^2A_5C_2^2$	$\mathbb{Z}_2^2$	96	$A_1A_2^2C_1C_5$		143	$A_1A_3C_2D_5$	$\mathbb{Z}_2^2$	190	$A_1C_1^2E_8$		237	$A_2C_7F_2$							
3	$A_1A_5^2$	$\mathbb{Z}_2^2$	50	$A_2A_5C_2^2$	$\mathbb{Z}_2$	97	$A_1A_4C_1C_5$		144	$A_1A_2C_1C_2D_5$	$\mathbb{Z}_2$	191	$C_1C_2E_8$		238	$A_1C_8F_2$							
4	$A_2^2A_7$	$\mathbb{Z}_2$	51	$A_7C_2^2$	$\mathbb{Z}_2$	98	$A_5C_1C_5$		145	$A_1^2C_2^2D_5$	$\mathbb{Z}_2^2$	192	$A_1A_2^4F_2$	$\mathbb{Z}_3$	239	C_9F_2							
5	$A_2^2A_3^2C_1$	$\mathbb{Z}_2$	52	$A_2A_3^2C_3$	$\mathbb{Z}_2$	99	$A_2^2C_2C_5$		146	$A_2C_2^2D_5$	$\mathbb{Z}_2$	193	$A_1A_2^2A_4F_2$		240	$A_4C_1D_4F_2$							
6	$A_2^3A_4C_1$		53	$A_2^2A_4C_3$		100	$A_1A_3C_2C_5$	$\mathbb{Z}_2$	147	$A_1C_2C_3D_5$	$\mathbb{Z}_2$	194	$A_1A_4^2F_2$		241	$C_5D_4F_2$							
7	$A_1A_2^2A_5C_1$	$\mathbb{Z}_2$	54	$A_1A_2A_5C_3$	$\mathbb{Z}_2$	101	$A_4C_2C_5$		148	$A_1^2C_4D_5$	$\mathbb{Z}_2^2$	195	$A_2^2A_5F_2$	$\mathbb{Z}_3$	242	$A_4D_5F_2$							
8	$A_1^2A_3A_5C_1$	$\mathbb{Z}_2^2$	55	$A_2A_6C_3$		102	$A_1A_2C_3C_5$		149	$A_1C_1C_4D_5$	$\mathbb{Z}_2$	196	$A_1A_3A_5F_2$	$\mathbb{Z}_2$	243	$A_1A_2C_1D_5F_2$							
9	$A_1A_4A_5C_1$	$\mathbb{Z}_2$	56	$A_1A_7C_3$	$\mathbb{Z}_2$	103	$A_1^2C_4C_5$	$\mathbb{Z}_2$	150	$C_2C_4D_5$	$\mathbb{Z}_2$	197	$A_4A_5F_2$		244	$A_3C_1D_5F_2$							
10	$A_5^2C_1$	$\mathbb{Z}_2$	57	$A_1A_2^3C_1C_3$		104	$A_1C_5^2$		151	$C_1C_5D_5$		198	$A_1A_2A_6F_2$		245	$A_2C_2D_5F_2$							
11	$A_2^2A_6C_1$		58	$A_2^2A_3C_1C_3$		105	$A_1A_2^2C_6$	$\mathbb{Z}_2$	152	$C_1^2D_4D_5$	$\mathbb{Z}_2$	199	$A_1^2A_7F_2$	$\mathbb{Z}_2$	246	$A_1C_3D_5F_2$							
12	$A_1A_2A_7C_1$	$\mathbb{Z}_2$	59	$A_1A_2A_4C_1C_3$		106	$A_1^2A_3C_6$	$\mathbb{Z}_2^2$	153	$C_1D_5^2$	$\mathbb{Z}_2$	200	$A_1A_8F_2$		247	$C_4D_5F_2$							
13	$A_1A_9C_1$	$\mathbb{Z}_2$	60	$A_1^2A_5C_1C_3$	$\mathbb{Z}_2$	107	$A_1A_4C_6$	$\mathbb{Z}_2$	154	$A_1^2A_3D_6$	$\mathbb{Z}_2^3$	201	A_9F_2		248	$A_2C_1D_6F_2$							
14	$A_1^2A_2^2A_3C_1^2$	$\mathbb{Z}_2$	61	$A_2A_5C_1C_3$		108	A_5C_6	$\mathbb{Z}_2$	155	$A_1A_3C_1D_6$	$\mathbb{Z}_2^2$	202	$A_1^2A_3^2C_1F_2$	$\mathbb{Z}_2$	249	$C_3D_6F_2$							
15	$A_1A_2A_3^2C_1^2$	$\mathbb{Z}_4$	62	$A_1A_6C_1C_3$		109	$A_1^2A_2C_1C_6$	$\mathbb{Z}_2$	156	$A_1A_2C_1^2D_6$	$\mathbb{Z}_2$	203	$A_1^2A_2A_4C_1F_2$		250	$A_1C_1D_7F_2$							
16	$A_1A_2^2A_4C_1^2$		63	$A_7C_1C_3$	$\mathbb{Z}_2$	110	$A_2^2C_1C_6$		157	$A_1^3C_2D_6$	$\mathbb{Z}_2^3$	204	$A_1A_3A_4C_1F_2$		251	$C_2D_7F_2$							
17	$A_1A_4^2C_1^2$		64	$A_1A_2A_3C_2C_3$	$\mathbb{Z}_2$	111	$A_1A_3C_1C_6$	$\mathbb{Z}_2$	158	$A_1A_2C_2D_6$	$\mathbb{Z}_2^2$	205	$A_4^2C_1F_2$		252	$C_1D_8F_2$							
18	$A_1^2A_2A_5C_1^2$	$\mathbb{Z}_2$	65	$A_3^2C_2C_3$	$\mathbb{Z}_2$	112	$A_4C_1C_6$		159	$A_3C_2D_6$	$\mathbb{Z}_2^2$	206	$A_1^3A_5C_1F_2$	$\mathbb{Z}_2$	253	D_9F_2							
19	$A_1A_3A_5C_1^2$	$\mathbb{Z}_2$	66	$A_2A_4C_2C_3$		113	$A_1^3C_2C_6$	$\mathbb{Z}_2^2$	160	$A_1^2C_1C_2D_6$	$\mathbb{Z}_2^2$	207	$A_3A_5C_1F_2$		254	$A_1A_2E_6F_2$							
20	$A_4A_5C_1^2$		67	$A_1A_5C_2C_3$	$\mathbb{Z}_2$	114	$A_1A_2C_2C_6$	$\mathbb{Z}_2$	161	$A_2C_1C_2D_6$	$\mathbb{Z}_2$	208	$A_1^2A_6C_1F_2$		255	$A_3E_6F_2$							
21	$A_1A_2A_6C_1^2$		68	$A_6C_2C_3$		115	$A_3C_2C_6$	$\mathbb{Z}_2$	162	$A_1C_2^2D_6$	$\mathbb{Z}_2^2$	209	$A_2A_6C_1F_2$		256	$A_1^2C_1E_6F_2$							
22	$A_1^2A_7C_1^2$	$\mathbb{Z}_2$	69	$A_1A_2^2C_3^2$		116	$A_1^2C_3C_6$	$\mathbb{Z}_2$	163	$A_1C_1C_3D_6$	$\mathbb{Z}_2$	210	$A_1A_7C_1F_2$		257	$A_1C_2E_6F_2$							
23	$A_1^2A_7C_1^2$	$\mathbb{Z}_4$	70	$A_1^2A_3C_3^2$	$\mathbb{Z}_2$	117	$A_2C_3C_6$		164	$C_2C_3D_6$	$\mathbb{Z}_2$	211	$A_8C_1F_2$		258	$C_3E_6F_2$							
24	$A_2A_7C_1^2$	$\mathbb{Z}_4$	71	$A_2A_3C_3^2$		118	$A_1C_4C_6$	$\mathbb{Z}_2$	165	$A_1C_4D_6$	$\mathbb{Z}_2^2$	212	$A_2^2A_3C_2F_2$		259	$A_2E_7F_2$							
25	$A_1A_8C_1^2$		72	$A_1A_4C_3^2$		119	C_5C_6		166	$C_1C_4D_6$	$\mathbb{Z}_2$	213	$A_1A_3^2C_2F_2$	$\mathbb{Z}_2$	260	$A_1C_1E_7F_2$							
26	$A_9C_1^2$		73	$A_5C_3^2$		120	$A_2^2C_7$		167	$A_1C_1C_2D_7$	$\mathbb{Z}_2$	214	$A_1A_2A_4C_2F_2$		261	$C_2E_7F_2$							
27	$A_1^3A_3^2C_2$	$\mathbb{Z}_2^3$	74	$A_1^2A_2A_3C_4$	$\mathbb{Z}_2^2$	121	$A_1A_2C_1C_7$		168	$C_2^2D_7$	$\mathbb{Z}_2$	215	$A_3A_4C_2F_2$		262	$A_1E_8F_2$							
28	$A_1A_2A_3^2C_2$	$\mathbb{Z}_2^2$	75	$A_2^2A_3C_4$	$\mathbb{Z}_2$	122	$A_2C_2C_7$		169	$A_1^2C_1D_8$	$\mathbb{Z}_2^2$	216	$A_1^2A_5C_2F_2$	$\mathbb{Z}_2$	263	$C_1E_8F_2$							
29	$A_1A_3A_5C_2$	$\mathbb{Z}_2^2$	76	$A_1A_3^2C_4$	$\mathbb{Z}_2^2$	123	$A_1C_3C_7$		170	$A_1C_1^2D_8$	$\mathbb{Z}_2$	217	$A_2A_5C_2F_2$		264	$A_1^3A_2^2F_2^2$	$\mathbb{Z}_3$						
30	$A_4A_5C_2$	$\mathbb{Z}_2$	77	$A_1^2A_5C_4$	$\mathbb{Z}_2^2$	124	$A_1A_2C_8$	$\mathbb{Z}_2$	171	$A_1C_2D_8$	$\mathbb{Z}_2^2$	218	$A_1A_6C_2F_2$		265	$A_2^2A_3F_2^2$	$\mathbb{Z}_3$						
31	$A_2A_7C_2$	$\mathbb{Z}_2$	78	$A_1^2A_2^2C_1C_4$	$\mathbb{Z}_2$	125	$A_1^2C_1C_8$	$\mathbb{Z}_2$	172	$C_1C_2D_8$	$\mathbb{Z}_2$	219	$A_7C_2F_2$		266	$A_1A_3^2F_2^2$							
32	A_9C_2	$\mathbb{Z}_2$	79	$A_1^3A_3C_1C_4$	$\mathbb{Z}_2^2$	126	$A_2C_1C_8$	$\mathbb{Z}_2$	173	$C_1^2D_9$		220	$A_1^2A_2^2C_3F_2$		267	$A_1^3A_4F_2^2$							
33	$A_1A_2^2A_3C_1C_2$	$\mathbb{Z}_2$	80	$A_1A_2A_3C_1C_4$	$\mathbb{Z}_2$	127	$A_1C_2C_8$	$\mathbb{Z}_2$	174	$A_4C_1E_6$		221	$A_1A_2A_3C_3F_2$		268	$A_3A_4F_2^2$							
34	$A_1^2A_3^2C_1C_2$	$\mathbb{Z}_2^2$	81	$A_1^2A_4C_1C_4$	$\mathbb{Z}_2$	128	C_3C_8	$\mathbb{Z}_2$	175	$A_1A_2C_1^2E_6$		222	$A_1^2A_4C_3F_2$		269	$A_1^2A_5F_2^2$	$\mathbb{Z}_3$						
35	$A_2^2A_4C_1C_2$		82	$A_1A_5C_1C_4$	$\mathbb{Z}_2$	129	$A_1C_1C_9$		176	$A_3C_1^2E_6$		223	$A_2A_4C_3F_2$		270	$A_2A_5F_2^2$	$\mathbb{Z}_3$						
36	$A_1A_3A_4C_1C_2$	$\mathbb{Z}_2$	83	$A_1^3A_2C_2C_4$	$\mathbb{Z}_2^2$	130	C_2C_9		177	$A_2C_1C_2E_6$		224	$A_1A_5C_3F_2$		271	$A_1A_6F_2^2$							
37	$A_4^2C_1C_2$		84	$A_1A_2^2C_2C_4$	$\mathbb{Z}_2$	131	A_1C_{10}	$\mathbb{Z}_2$	178	$A_1C_1C_3E_6$		225	$A_6C_3F_2$		272	$A_7F_2^2$							
38	$A_1^3A_5C_1C_2$	$\mathbb{Z}_2^2$	85	$A_1^2A_3C_2C_4$	$\mathbb{Z}_2^2$	132	C_1C_{10}		179	$C_2C_3E_6$		226	$A_1^2A_3C_4F_2$	$\mathbb{Z}_2$	273	$A_1A_2D_4F_2^2$							
39	$A_1A_2A_5C_1C_2$	$\mathbb{Z}_2$	86	$A_2A_3C_2C_4$	$\mathbb{Z}_2$	133	$A_1^2A_3C_2D_4$	$\mathbb{Z}_2^3$	180	$C_1C_4E_6$		227	$A_1A_4C_4F_2$		274	$A_3D_4F_2^2$							
40	$A_3A_5C_1C_2$	$\mathbb{Z}_2$	87	$A_1A_4C_2C_4$	$\mathbb{Z}_2$	134	$A_1A_3C_1C_2D_4$	$\mathbb{Z}_2^2$	181	C_5E_6		228	$A_5C_4F_2$		275	$A_1^2D_5F_2^2$							
41	$A_2A_6C_1C_2$		88	$A_5C_2C_4$	$\mathbb{Z}_2$	135	$A_1^3C_2^2D_4$	$\mathbb{Z}_2^3$	182	$A_1A_2C_1E_7$	$\mathbb{Z}_2$	229	$A_1^2A_2C_5F_2$		276	$A_1D_6F_2^2$							
42	$A_1A_7C_1C_2$	$\mathbb{Z}_2$	89	$A_1^2A_2C_3C_4$	$\mathbb{Z}_2$	136	$A_1A_2C_2^2D_4$	$\mathbb{Z}_2^2$	183	$A_1^2C_1^2E_7$	$\mathbb{Z}_2$	230	$A_2^2C_5F_2$		277	$D_7F_2^2$							
43	$A_8C_1C_2$		90	$A_2^2C_3C_4$		137	$A_3C_2^2D_4$	$\mathbb{Z}_2^2$	184	$A_2C_1^2E_7$		231	$A_1A_3C_5F_2$		278	$A_1E_6F_2^2$	$\mathbb{Z}_3$						
44	$A_1^4A_3C_2^2$	$\mathbb{Z}_2^3$	91	$A_1A_3C_3C_4$	$\mathbb{Z}_2$	138	$A_1^2C_1C_4D_4$	$\mathbb{Z}_2^2$	185	$A_2C_2E_7$	$\mathbb{Z}_2$	232	$A_4C_5F_2$		279	$E_7F_2^2$							
45	$A_1^2A_2A_3C_2^2$	$\mathbb{Z}_2^2$	92	$A_1^3C_4^2$	$\mathbb{Z}_2^2$	139	$A_1C_2C_4D_4$	$\mathbb{Z}_2^2$	186	$A_1C_1C_2E_7$	$\mathbb{Z}_2$	233	$A_1^3C_6F_2$	$\mathbb{Z}_2$	280	$A_1A_2^2A_3F_3$	$\mathbb{Z}_2$						
46	$A_2^2A_3C_2^2$	$\mathbb{Z}_2$	93	$A_1A_2C_4^2$	$\mathbb{Z}_2$	140	$A_1^3D_4^2$	$\mathbb{Z}_2^4$	187	$C_2^2E_7$	$\mathbb{Z}_2$	234	$A_1A_2C_6F_2$		281	$A_1^2A_3^2F_3$	$\mathbb{Z}_2^2$						
47	$A_1A_3^2C_2^2$	$\mathbb{Z}_2^2$	94	$A_3C_4^2$	$\mathbb{Z}_2^2$	141	$A_1^2A_2C_1^2D_5$	$\mathbb{Z}_2$	188	$A_1C_3E_7$	$\mathbb{Z}_2$	235	$A_3C_6F_2$		282	$A_2^2A_4F_3$							

283	$A_1A_3A_4F_3$	$\mathbb{Z}_2$	307	$A_1^2A_4C_2F_3$	$\mathbb{Z}_2$	331	$A_1C_3D_4F_3$	$\mathbb{Z}_2$	355	$A_6F_2F_3$		379	$A_1A_5C_1F_4$	403	$A_1A_4F_2F_4$
284	$A_4^2F_3$		308	$A_2A_4C_2F_3$		332	$A_1A_2D_5F_3$	$\mathbb{Z}_2$	356	$A_2D_4F_2F_3$		380	$A_6C_1F_4$	404	$A_5F_2F_4$
285	$A_1^3A_5F_3$	$\mathbb{Z}_2^2$	309	$A_1A_5C_2F_3$	$\mathbb{Z}_2$	333	$A_1^2C_1D_5F_3$	$\mathbb{Z}_2$	357	$A_1D_5F_2F_3$		381	$A_1A_2^2C_2F_4$	405	$A_1D_4F_2F_4$
286	$A_1A_2A_5F_3$	$\mathbb{Z}_2$	310	$A_6C_2F_3$		334	$A_2C_1D_5F_3$		358	$D_6F_2F_3$		382	$A_2A_3C_2F_4$	406	$D_5F_2F_4$
287	$A_3A_5F_3$	$\mathbb{Z}_2$	311	$A_1A_2^2C_3F_3$		335	$A_1C_2D_5F_3$	$\mathbb{Z}_2$	359	$E_6F_2F_3$		383	$A_1A_4C_2F_4$	407	$A_1^2A_2F_3F_4$
288	$A_2A_6F_3$		312	$A_1^2A_3C_3F_3$	$\mathbb{Z}_2$	336	$C_3D_5F_3$		360	$A_1^3A_2F_3^2$	$\mathbb{Z}_2$	384	$A_5C_2F_4$	408	$A_2^2F_3F_4$
289	$A_1A_7F_3$	$\mathbb{Z}_2$	313	$A_2A_3C_3F_3$		337	$A_2D_6F_3$	$\mathbb{Z}_2$	361	$A_1A_2^2F_3^2$		385	$A_1^2A_2C_3F_4$	409	$A_1A_3F_3F_4$
290	A_8F_3		314	$A_1A_4C_3F_3$		338	$A_1C_1D_6F_3$	$\mathbb{Z}_2$	362	$A_1^2A_3F_3^2$	$\mathbb{Z}_2$	386	$A_2^2C_3F_4$	410	$A_4F_3F_4$
291	$A_1^3A_2^2C_1F_3$	$\mathbb{Z}_2$	315	$A_5C_3F_3$		339	$C_2D_6F_3$	$\mathbb{Z}_2$	363	$A_2A_3F_3^2$		387	$A_1A_3C_3F_4$	411	$D_4F_3F_4$
292	$A_1^2A_2A_3C_1F_3$	$\mathbb{Z}_2$	316	$A_5C_3F_3$	$\mathbb{Z}_2$	340	$A_1D_7F_3$	$\mathbb{Z}_2$	364	$A_2A_3F_3^2$	$\mathbb{Z}_2$	388	$A_4C_3F_4$	412	$A_1^3F_4^2$
293	$A_2^2A_3C_1F_3$		317	$A_1^4C_4F_3$	$\mathbb{Z}_2^2$	341	$C_1D_7F_3$		365	$A_1A_4F_3^2$		389	$A_1A_2C_4F_4$	413	$A_1A_2F_4^2$
294	$A_1A_3^2C_1F_3$	$\mathbb{Z}_2$	318	$A_1^2A_2C_4F_3$	$\mathbb{Z}_2$	342	$A_2E_6F_3$		366	$A_5F_3^2$		390	$A_1^2C_5F_4$	414	$A_3F_4^2$
295	$A_1A_2A_4C_1F_3$		319	$A_2^2C_4F_3$		343	$A_1C_1E_6F_3$		367	$A_1D_4F_3^2$	$\mathbb{Z}_2$	391	$A_2C_5F_4$		
296	$A_3A_4C_1F_3$		320	$A_1A_3C_4F_3$	$\mathbb{Z}_2$	344	$C_2E_6F_3$		368	$D_5F_3^2$		392	$A_1C_6F_4$		
297	$A_1^2A_5C_1F_3$	$\mathbb{Z}_2$	321	$A_4C_4F_3$		345	$A_1E_7F_3$	$\mathbb{Z}_2$	369	$D_5F_3^2$	$\mathbb{Z}_2$	393	C_7F_4		
298	$A_2A_5C_1F_3$		322	$A_1A_2C_5F_3$		346	$C_1E_7F_3$		370	$A_1A_2^3F_4$		394	$A_1C_1D_5F_4$		
299	$A_2A_5C_1F_3$	$\mathbb{Z}_2$	323	$A_3C_5F_3$		347	$C_1E_7F_3$	$\mathbb{Z}_2$	371	$A_2^2A_3F_4$		395	$C_2D_5F_4$		
300	$A_1A_6C_1F_3$		324	$A_1^2C_6F_3$	$\mathbb{Z}_2$	348	E_8F_3		372	$A_1A_2A_4F_4$		396	$C_1D_6F_4$		
301	$A_7C_1F_3$		325	$A_2C_6F_3$		349	$A_1^3A_3F_2F_3$	$\mathbb{Z}_2$	373	$A_2A_5F_4$		397	$A_1E_6F_4$		
302	$A_1^2A_2^2C_2F_3$	$\mathbb{Z}_2$	326	$A_2C_6F_3$	$\mathbb{Z}_2$	350	$A_1A_2A_3F_2F_3$		374	$A_1A_6F_4$		398	$C_1E_6F_4$		
303	$A_2^3C_2F_3$		327	$A_1C_7F_3$		351	$A_3^2F_2F_3$		375	$A_1^2A_2^2C_1F_4$		399	E_7F_4		
304	$A_1^3A_3C_2F_3$	$\mathbb{Z}_2^2$	328	C_8F_3		352	$A_1^2A_4F_2F_3$		376	$A_1A_2A_3C_1F_4$		400	$A_1^3A_2F_2F_4$		
305	$A_1A_2A_3C_2F_3$	$\mathbb{Z}_2$	329	$A_1A_2C_1D_4F_3$	$\mathbb{Z}_2$	353	$A_2A_4F_2F_3$		377	$A_1^2A_4C_1F_4$		401	$A_1^2A_3F_2F_4$		
306	$A_3^2C_2F_3$	$\mathbb{Z}_2$	330	$A_2C_2D_4F_3$	$\mathbb{Z}_2$	354	$A_1A_5F_2F_3$		378	$A_2A_4C_1F_4$		402	$A_2A_3F_2F_4$		

Table D.8: Algebras of maximal rank for the heterotic $\mathbb{Z}_2$ triple.

1	$A_1^3A_2^2\,\mathbb{Z}_3$	7	A_2A_5	$\mathbb{Z}_3$	13	A_1D_6		19	$A_3^2G_1$	25	$A_1D_5G_1$	31	$A_2A_3G_1^2$	37	$D_5G_1^2$ $\mathbb{Z}_2$	43	$A_1D_4G_2$	49	$D_4G_1G_2$
2	$A_2^2A_3\,\mathbb{Z}_3$	8	A_1A_6		14	D_7		20	$A_1^2A_4G_1$	26	D_6G_1	32	$A_2A_3G_1^2\,\mathbb{Z}_2$	38	$A_1^3A_2G_2$	44	D_5G_2	50	$A_1^3G_2^2$
3	$A_1A_3^2$	9	A_7		15	A_1E_6	$\mathbb{Z}_3$	21	$A_2A_4G_1$	27	E_6G_1	33	$A_1A_4G_1^2$	39	$A_1^2A_3G_2$	45	$A_1^2A_2G_1G_2$	51	$A_1A_2G_2^2$
4	$A_1^3A_4$	10	$A_1A_2D_4$		16	E_7		22	$A_1A_5G_1$	28	$A_1^3A_2G_1^2\,\mathbb{Z}_2$	34	$A_5G_1^2$	40	$A_2A_3G_2$	46	$A_2^2G_1G_2$	52	$A_3G_2^2$
5	A_3A_4	11	A_3D_4		17	$A_1^3A_3G_1$	$\mathbb{Z}_2$	23	A_6G_1	29	$A_1A_2^2G_1^2$	35	$A_1D_4G_1^2\,\mathbb{Z}_2$	41	$A_1A_4G_2$	47	$A_1A_3G_1G_2$		
6	$A_1^2A_5\,\mathbb{Z}_3$	12	$A_1^2D_5$		18	$A_1A_2A_3G_1$		24	$A_2D_4G_1$	30	$A_1^2A_3G_1^2\,\mathbb{Z}_2$	36	$D_5G_1^2$	42	A_5G_2	48	$A_4G_1G_2$		

Table D.9: Algebras of maximal rank for the heterotic $\mathbb{Z}_3$ triple.

1	$A_1^3A_2\,\mathbb{Z}_2$	3	$A_1^2A_3\,\mathbb{Z}_2$	5	$A_2A_3\,\mathbb{Z}_2$	7	A_5	9	D_5	11	$A_1^2A_2\mathbb{A}_1$	13	$A_1A_3\mathbb{A}_1$	15	$D_4\mathbb{A}_1$	17	$A_1A_2\mathbb{A}_1^2$
2	$A_1A_2^2$	4	A_2A_3	6	A_1A_4	8	$A_1D_4\,\mathbb{Z}_2$	10	$D_5\,\mathbb{Z}_2$	12	$A_2^2\mathbb{A}_1$	14	$A_4\mathbb{A}_1$	16	$A_1^3\mathbb{A}_1^2$	18	$A_3\mathbb{A}_1^2$

Table D.10: Algebras of maximal rank for the heterotic $\mathbb{Z}_4$ triple. $\mathbb{A}_1$ denotes to an A_1 at level 1.

1	A_1^3
2	A_1A_2
3	A_3

Table D.11: Algebras of maximal rank for the heterotic $\mathbb{Z}_5$ and $\mathbb{Z}_6$ triples.

D.2.2 Fundamental group generators

4	$A_1^2A_3^3A_4^2$	$\mathbb{Z}_2$	1122200	172	$A_1^3A_3A_4A_9$	$\mathbb{Z}_2$	001205	342	$A_1^2A_2A_4A_7D_4$	$\mathbb{Z}_2$	11004s	455	$A_1A_2^2A_5D_4D_5$	$\mathbb{Z}_2$	1003s2
11	$A_1^2A_2A_3^2A_4A_5$	$\mathbb{Z}_2$	0102203	172′	$A_1^3A_3A_4A_9$	$\mathbb{Z}_2$	111005	344	$A_1A_3A_4A_7D_4$	$\mathbb{Z}_2$	0204s	457	$A_1A_4A_5D_4D_5$	$\mathbb{Z}_2$	103s2
13	$A_1A_3^3A_4A_5$	$\mathbb{Z}_2$	102203	174	$A_1A_2A_3A_4A_9$	$\mathbb{Z}_2$	10205	346	$A_1A_2A_5A_7D_4$	$\mathbb{Z}_2$	10340	458	$A_5^2D_4D_5$	$\mathbb{Z}_2$	3302
14	$A_1^3A_3A_4^2A_5$	$\mathbb{Z}_2$	1112003	179	$A_1^3A_2A_5A_9$	$\mathbb{Z}_2$	000035	347	$A_3A_5A_7D_4$	$\mathbb{Z}_2$	204s	458′	$A_5^2D_4D_5$	$\mathbb{Z}_2$	33s0
23	$A_1A_2A_3^2A_5^2$	$\mathbb{Z}_2$	000233	179′	$A_1^3A_2A_5A_9$	$\mathbb{Z}_2$	111005	348	$A_1^2A_6A_7D_4$	$\mathbb{Z}_2$	1104s	460	$A_1A_2A_7D_4D_5$	$\mathbb{Z}_2$	004s2
23′	$A_1A_2A_3^2A_5^2$	$\mathbb{Z}_2$	102203	181	$A_1A_2^2A_5A_9$	$\mathbb{Z}_2$	00035	355	$A_1^2A_2^2A_9D_4$	$\mathbb{Z}_2$	01005s	461	$A_1A_9D_4D_5$	$\mathbb{Z}_2$	1502
24	$A_3^3A_5^2$	$\mathbb{Z}_2$	00233	182	$A_1^2A_3A_5A_9$	$\mathbb{Z}_2$	00035	356	$A_1A_2A_3A_9D_4$	$\mathbb{Z}_2$	1005s	461′	$A_1A_9D_4D_5$	$\mathbb{Z}_2$	15s0
26	$A_1^2A_3A_4A_5^2$	$\mathbb{Z}_2$	002033	182′	$A_1^2A_3A_5A_9$	$\mathbb{Z}_2$	01205	356′	$A_1A_2A_3A_9D_4$	$\mathbb{Z}_2$	10250	462	$A_1^2A_2^2A_3D_5^2$	$\mathbb{Z}_2$	1100222
26′	$A_1^2A_3A_4A_5^2$	$\mathbb{Z}_2$	110033	184	$A_2A_3A_5A_9$	$\mathbb{Z}_2$	0035	357	$A_1^2A_4A_9D_4$	$\mathbb{Z}_2$	0105s	466	$A_1^2A_2A_5D_5^2$	$\mathbb{Z}_2$	010322
28	$A_2A_3A_4A_5^2$	$\mathbb{Z}_2$	02033	186	$A_1A_4A_5A_9$	$\mathbb{Z}_2$	0035	359	$A_1A_5A_9D_4$	$\mathbb{Z}_2$	0350	467	$A_1A_3A_5D_5^2$	$\mathbb{Z}_2$	12302
34	$A_1A_3A_5^3$	$\mathbb{Z}_2$	02033	188	$A_5^2A_9$	$\mathbb{Z}_2$	035	359′	$A_1A_5A_9D_4$	$\mathbb{Z}_2$	105s	470	$A_1^2A_7D_5^2$	$\mathbb{Z}_2$	11402
50	$A_1^2A_3^2A_5A_6$	$\mathbb{Z}_2$	012230	189	$A_1^4A_6A_9$	$\mathbb{Z}_2$	011105	365	$A_1^2A_2A_{11}D_4$	$\mathbb{Z}_2$	0006s	475	$A_1^2A_2D_5^3$	$\mathbb{Z}_2$	110222
56	$A_1^3A_5^2A_6$	$\mathbb{Z}_2$	011330	193	$A_1A_3A_6A_9$	$\mathbb{Z}_2$	1205	365′	$A_1^2A_2A_{11}D_4$	$\mathbb{Z}_2$	11060	477	$D_4D_5^3$	$\mathbb{Z}_2$	s222
59	$A_3A_5^2A_6$	$\mathbb{Z}_2$	2330	195	$A_1^3A_7A_9$	$\mathbb{Z}_2$	11105	368	$A_1A_3A_{11}D_4$	$\mathbb{Z}_2$	006s	478	$A_1A_2^3A_3^2D_6$	$\mathbb{Z}_2$	100022s
76	$A_1^3A_2A_3A_4A_7$	$\mathbb{Z}_2$	0110204	226	$A_1^3A_2A_3A_{11}$	$\mathbb{Z}_2$	011006	370	$A_4A_{11}D_4$	$\mathbb{Z}_2$	06s	480	$A_1A_2A_3^2A_4D_6$	$\mathbb{Z}_2$	10220s
78	$A_1^2A_3^2A_4A_7$	$\mathbb{Z}_2$	110204	229	$A_1A_2^2A_3A_{11}$	$\mathbb{Z}_2$	00026	372	$A_1^2A_{13}D_4$	$\mathbb{Z}_2$	0170	481	$A_3^3A_4D_6$	$\mathbb{Z}_2$	2220v
81	$A_2A_3^2A_4A_7$	$\mathbb{Z}_2$	02204	232	$A_1^2A_3^2A_{11}$	$\mathbb{Z}_2$	11006	379	$A_2^2A_7D_4^2$	$\mathbb{Z}_2$	004ss	486	$A_1A_2^2A_3A_5D_6$	$\mathbb{Z}_2$	00023s
82	$A_1^4A_4^2A_7$	$\mathbb{Z}_2$	1111004	235	$A_2A_3^2A_{11}$	$\mathbb{Z}_2$	0026	389	$A_1A_2A_3^2A_5D_5$	$\mathbb{Z}_2$	102230	488	$A_2A_3^2A_5D_6$	$\mathbb{Z}_2$	0023s
87	$A_1^2A_2A_3A_5A_7$	$\mathbb{Z}_2$	010034	237	$A_1^4A_4A_{11}$	$\mathbb{Z}_2$	001106	390	$A_1^2A_3A_4A_5D_5$	$\mathbb{Z}_2$	012032	489	$A_1^2A_2A_4A_5D_6$	$\mathbb{Z}_2$	11003s
87′	$A_1^2A_2A_3A_5A_7$	$\mathbb{Z}_2$	110204	239	$A_1^2A_2A_4A_{11}$	$\mathbb{Z}_2$	11006	396	$A_1A_3A_5^2D_5$	$\mathbb{Z}_2$	00332	491	$A_1A_3A_4A_5D_6$	$\mathbb{Z}_2$	0203s
89	$A_1A_3^2A_5A_7$	$\mathbb{Z}_2$	10034	243	$A_1A_3A_4A_{11}$	$\mathbb{Z}_2$	0206	396′	$A_1A_3A_5^2D_5$	$\mathbb{Z}_2$	02330	491′	$A_1A_3A_4A_5D_6$	$\mathbb{Z}_2$	1203v
89	$A_1A_3^2A_5A_7$	$\mathbb{Z}_2$	12230	245	$A_1^3A_5A_{11}$	$\mathbb{Z}_2$	01106	396″	$A_1A_3A_5^2D_5$	$\mathbb{Z}_2$	12032	494	$A_1A_2A_5^2D_6$	$\mathbb{Z}_2$	0033v
91	$A_1^3A_4A_5A_7$	$\mathbb{Z}_2$	001034	250	$A_3A_5A_{11}$	$\mathbb{Z}_2$	206	398	$A_4A_5^2D_5$	$\mathbb{Z}_2$	0332	495	$A_3A_5^2D_6$	$\mathbb{Z}_2$	033v
93	$A_1A_2A_4A_5A_7$	$\mathbb{Z}_2$	10034	254	$A_1^2A_6A_{11}$	$\mathbb{Z}_2$	1106	409	$A_1^3A_2^2A_7D_5$	$\mathbb{Z}_2$	0110042	495′	$A_3A_5^2D_6$	$\mathbb{Z}_2$	203s
95	$A_1^2A_5^2A_7$	$\mathbb{Z}_2$	01034	270	$A_1^4A_2A_{13}$	$\mathbb{Z}_2$	000107	410	$A_1^2A_2A_3A_7D_5$	$\mathbb{Z}_2$	110240	495″	$A_3A_5^2D_6$	$\mathbb{Z}_2$	2330
95′	$A_1^2A_5^2A_7$	$\mathbb{Z}_2$	11330	272	$A_1^2A_2^2A_{13}$	$\mathbb{Z}_2$	01007	413	$A_2^2A_3A_7D_5$	$\mathbb{Z}_2$	00242	499	$A_1A_3^2A_6D_6$	$\mathbb{Z}_2$	1220s
100	$A_1^3A_3A_6A_7$	$\mathbb{Z}_2$	011204	274	$A_1^3A_3A_{13}$	$\mathbb{Z}_2$	00107	415	$A_1^3A_4A_7D_5$	$\mathbb{Z}_2$	011042	502	$A_1^2A_5A_6D_6$	$\mathbb{Z}_2$	1130s
103	$A_3^2A_6A_7$	$\mathbb{Z}_2$	2204	276	$A_1A_2A_3A_{13}$	$\mathbb{Z}_2$	1007	417	$A_3A_4A_7D_5$	$\mathbb{Z}_2$	2042	505	$A_1^2A_2^2A_7D_6$	$\mathbb{Z}_2$	01004s
107	$A_1A_5A_6A_7$	$\mathbb{Z}_2$	1304	279	$A_1^2A_4A_{13}$	$\mathbb{Z}_2$	0107	418	$A_1^2A_5A_7D_5$	$\mathbb{Z}_2$	01340	508	$A_1A_2A_3A_7D_6$	$\mathbb{Z}_2$	0024v
112	$A_1A_2^2A_7^2$	$\mathbb{Z}_2$	00044	282	$A_1A_5A_{13}$	$\mathbb{Z}_2$	107	418′	$A_1^2A_5A_7D_5$	$\mathbb{Z}_2$	11042	508′	$A_1A_2A_3A_7D_6$	$\mathbb{Z}_2$	1004s
113	$A_1^2A_3A_7^2$	$\mathbb{Z}_2$	11204	295	$A_1^2A_2A_{15}$	$\mathbb{Z}_2$	0008	428	$A_1^3A_2A_9D_5$	$\mathbb{Z}_2$	001052	509	$A_3^2A_7D_6$	$\mathbb{Z}_2$	024v
118	$A_2A_3A_7^2$	$\mathbb{Z}_2$	0044	298	$A_2^2A_{15}$	$\mathbb{Z}_2$	008	428′	$A_1^3A_2A_9D_5$	$\mathbb{Z}_2$	111050	511	$A_1^2A_4A_7D_6$	$\mathbb{Z}_2$	0104s
121	$A_1A_4A_7^2$	$\mathbb{Z}_2$	0044	300	$A_1A_3A_{15}$	$\mathbb{Z}_2$	008	430	$A_1A_2^2A_9D_5$	$\mathbb{Z}_2$	10052	511′	$A_1^2A_4A_7D_6$	$\mathbb{Z}_2$	1104v
123	$A_5A_7^2$	$\mathbb{Z}_2$	044	303	A_4A_{15}	$\mathbb{Z}_2$	08	431	$A_1^2A_3A_9D_5$	$\mathbb{Z}_2$	01052	513	$A_1A_5A_7D_6$	$\mathbb{Z}_2$	104s
127	$A_1^2A_3^3A_8$	$\mathbb{Z}_2$	112220	314	$A_2A_3^3A_4D_4$	$\mathbb{Z}_2$	02220s	431′	$A_1^2A_3A_9D_5$	$\mathbb{Z}_2$	01250	513′	$A_1A_5A_7D_6$	$\mathbb{Z}_2$	1340
138	$A_1^3A_3A_5A_8$	$\mathbb{Z}_2$	111230	320	$A_1A_2A_3A_4A_5D_4$	$\mathbb{Z}_2$	10203s	434	$A_1A_4A_9D_5$	$\mathbb{Z}_2$	1052	519	$A_1^2A_2A_9D_6$	$\mathbb{Z}_2$	0005s
152	$A_1^4A_7A_8$	$\mathbb{Z}_2$	111140	325	$A_1A_4A_5^2D_4$	$\mathbb{Z}_2$	0033s	436	$A_5A_9D_5$	$\mathbb{Z}_2$	350	519′	$A_1^2A_2A_9D_6$	$\mathbb{Z}_2$	0105v
163	$A_1^3A_2^2A_3A_9$	$\mathbb{Z}_2$	0010025	328	$A_3^3A_6D_4$	$\mathbb{Z}_2$	2220s	441	$A_1^3A_{11}D_5$	$\mathbb{Z}_2$	01160	521	$A_2^2A_9D_6$	$\mathbb{Z}_2$	005s
165	$A_1A_2^3A_3A_9$	$\mathbb{Z}_2$	100025	332	$A_1A_3A_5A_6D_4$	$\mathbb{Z}_2$	1230s	444	$A_1A_2A_{11}D_5$	$\mathbb{Z}_2$	0062	522	$A_1A_3A_9D_6$	$\mathbb{Z}_2$	005s
166	$A_1^2A_2A_3^2A_9$	$\mathbb{Z}_2$	010025	336	$A_1^2A_2^3A_7D_4$	$\mathbb{Z}_2$	110004s	446	$A_3A_{11}D_5$	$\mathbb{Z}_2$	062	522′	$A_1A_3A_9D_6$	$\mathbb{Z}_2$	105v
168	$A_1A_3^3A_9$	$\mathbb{Z}_2$	10025	337	$A_1A_2^2A_3A_7D_4$	$\mathbb{Z}_2$	00024s	451	$A_1A_{13}D_5$	$\mathbb{Z}_2$	170	522″	$A_1A_3A_9D_6$	$\mathbb{Z}_2$	1250
169	$A_1^4A_2A_4A_9$	$\mathbb{Z}_2$	0111005	340	$A_2A_3^2A_7D_4$	$\mathbb{Z}_2$	0024s	453	$A_2^2A_3^2D_4D_5$	$\mathbb{Z}_2$	0022s2	524	$A_4A_9D_6$	$\mathbb{Z}_2$	05s

527	$A_1^2A_{11}D_6$	$\mathbb{Z}_2$	006v	626	$A_1A_5D_6D_7$	$\mathbb{Z}_2$	13v2	711	$A_3A_7D_9$	$\mathbb{Z}_2$	242	798	$A_2D_5D_{12}$	$\mathbb{Z}_2$	02s
527′	$A_1^2A_{11}D_6$	$\mathbb{Z}_2$	1160	628	$A_1D_5D_6D_7$	$\mathbb{Z}_2$	12s2	715	$A_1A_9D_9$	$\mathbb{Z}_2$	152	799	$A_1D_6D_{12}$	$\mathbb{Z}_2$	0vs
529	$A_2A_{11}D_6$	$\mathbb{Z}_2$	06v	629	$D_6^2D_7$	$\mathbb{Z}_2$	ss2	717	$A_1A_5D_4D_9$	$\mathbb{Z}_2$	13s2	800	D_7D_{12}	$\mathbb{Z}_2$	2s
535	$A_4A_5D_4D_6$	$\mathbb{Z}_2$	03ss	631	$A_1^2A_3D_7^2$	$\mathbb{Z}_2$	11222	723	$A_1A_3D_6D_9$	$\mathbb{Z}_2$	12s2	809	$A_1^3A_2D_{14}$	$\mathbb{Z}_2$	0010s
536	$A_2A_7D_4D_6$	$\mathbb{Z}_2$	04sv	636	$A_1A_2^2A_3^2D_8$	$\mathbb{Z}_2$	00022s	726	$A_1^2D_8D_9$	$\mathbb{Z}_2$	11s2	811	$A_1A_2^2D_{14}$	$\mathbb{Z}_2$	100s
537	$A_9D_4D_6$	$\mathbb{Z}_2$	50s	638	$A_2A_3^3D_8$	$\mathbb{Z}_2$	0022s	728	$A_1^3A_2^3D_{10}$	$\mathbb{Z}_2$	111000s	812	$A_1^2A_3D_{14}$	$\mathbb{Z}_2$	010s
539	$A_1A_2^2A_3D_5D_6$	$\mathbb{Z}_2$	10022s	638′	$A_2A_3^3D_8$	$\mathbb{Z}_2$	0222v	729	$A_1^2A_2^2A_3D_{10}$	$\mathbb{Z}_2$	01002s	815	$A_1A_4D_{14}$	$\mathbb{Z}_2$	10s
542	$A_1A_3A_4D_5D_6$	$\mathbb{Z}_2$	1202s	639	$A_1^2A_2A_3A_4D_8$	$\mathbb{Z}_2$	11020s	731	$A_1A_2A_3^2D_{10}$	$\mathbb{Z}_2$	1002s	817	$A_1D_4D_{14}$	$\mathbb{Z}_2$	10s
545	$A_1A_2A_5D_5D_6$	$\mathbb{Z}_2$	0032s	641	$A_1A_3^2A_4D_8$	$\mathbb{Z}_2$	0220s	732	$A_3^3D_{10}$	$\mathbb{Z}_2$	222v	823	$A_1^3D_{16}$	$\mathbb{Z}_2$	000s
546	$A_3A_5D_5D_6$	$\mathbb{Z}_2$	032s	645	$A_1A_2A_3A_5D_8$	$\mathbb{Z}_2$	1003s	733	$A_1^3A_2A_4D_{10}$	$\mathbb{Z}_2$	11100s	825	$A_1A_2D_{16}$	$\mathbb{Z}_2$	00s
546′	$A_3A_5D_5D_6$	$\mathbb{Z}_2$	230s	645′	$A_1A_2A_3A_5D_8$	$\mathbb{Z}_2$	1023v	735	$A_1^2A_3A_4D_{10}$	$\mathbb{Z}_2$	0120s	826	A_3D_{16}	$\mathbb{Z}_2$	0s
548	$A_1A_7D_5D_6$	$\mathbb{Z}_2$	042v	646	$A_3^2A_5D_8$	$\mathbb{Z}_2$	220s	739	$A_1^2A_2A_5D_{10}$	$\mathbb{Z}_2$	0003s	831	$A_1^2A_2A_3^2E_6$	$\mathbb{Z}_2$	1102220
548′	$A_1A_7D_5D_6$	$\mathbb{Z}_2$	140s	647	$A_1^2A_4A_5D_8$	$\mathbb{Z}_2$	0103s	740	$A_1A_3A_5D_{10}$	$\mathbb{Z}_2$	003s	840	$A_1^2A_3^2A_5E_6$	$\mathbb{Z}_2$	012230
551	$A_1A_2D_5^2D_6$	$\mathbb{Z}_2$	1022s	649	$A_1A_5^2D_8$	$\mathbb{Z}_2$	033v	740′	$A_1A_3A_5D_{10}$	$\mathbb{Z}_2$	120s	848	$A_3A_5^2E_6$	$\mathbb{Z}_2$	2330
554	$A_2^2A_3D_6^2$	$\mathbb{Z}_2$	002ss	649′	$A_1A_5^2D_8$	$\mathbb{Z}_2$	103s	740″	$A_1A_3A_5D_{10}$	$\mathbb{Z}_2$	123v	860	$A_1^4A_2A_7E_6$	$\mathbb{Z}_2$	1111040
556	$A_3A_4D_6^2$	$\mathbb{Z}_2$	20ss	652	$A_1^2A_3A_6D_8$	$\mathbb{Z}_2$	1120s	743	$A_4A_5D_{10}$	$\mathbb{Z}_2$	03s	861	$A_1^3A_3A_7E_6$	$\mathbb{Z}_2$	011240
558	$A_2A_5D_6^2$	$\mathbb{Z}_2$	03sv	657	$A_1^2A_2A_7D_8$	$\mathbb{Z}_2$	0004s	744	$A_1^3A_6D_{10}$	$\mathbb{Z}_2$	1110s	866	$A_1A_5A_7E_6$	$\mathbb{Z}_2$	1340
559	$A_7D_6^2$	$\mathbb{Z}_2$	4vv	657′	$A_1^2A_2A_7D_8$	$\mathbb{Z}_2$	1104v	747	$A_1^2A_7D_{10}$	$\mathbb{Z}_2$	114v	875	$A_1^4A_9E_6$	$\mathbb{Z}_2$	011150
564	$A_2D_5D_6^2$	$\mathbb{Z}_2$	02ss	659	$A_2^2A_7D_8$	$\mathbb{Z}_2$	004s	751	$A_1A_2^2D_4D_{10}$	$\mathbb{Z}_2$	100ss	879	$A_1A_3A_9E_6$	$\mathbb{Z}_2$	1250
570	$A_1^2A_3^2A_4D_7$	$\mathbb{Z}_2$	112202	660	$A_1A_3A_7D_8$	$\mathbb{Z}_2$	004s	753	$A_1A_4D_4D_{10}$	$\mathbb{Z}_2$	10ss	885	$A_1^2A_{11}E_6$	$\mathbb{Z}_2$	1160
576	$A_1^2A_2A_3A_5D_7$	$\mathbb{Z}_2$	010232	660′	$A_1A_3A_7D_8$	$\mathbb{Z}_2$	024v	754	$A_5D_4D_{10}$	$\mathbb{Z}_2$	30s	896	$A_1A_3A_5D_4E_6$	$\mathbb{Z}_2$	123s0
577	$A_1A_3^2A_5D_7$	$\mathbb{Z}_2$	10232	661	$A_4A_7D_8$	$\mathbb{Z}_2$	04s	755	$A_1^2A_2D_5D_{10}$	$\mathbb{Z}_2$	0102s	899	$A_1^2A_7D_4E_6$	$\mathbb{Z}_2$	114s0
577′	$A_1A_3^2A_5D_7$	$\mathbb{Z}_2$	12230	663	$A_1^2A_9D_8$	$\mathbb{Z}_2$	015v	757	$A_1A_3D_5D_{10}$	$\mathbb{Z}_2$	102s	902	$A_1^2A_3^2D_5E_6$	$\mathbb{Z}_2$	112220
578	$A_1^3A_4A_5D_7$	$\mathbb{Z}_2$	111032	666	$A_{11}D_8$	$\mathbb{Z}_2$	6v	757′	$A_1A_3D_5D_{10}$	$\mathbb{Z}_2$	120s	906	$A_1^3A_5D_5E_6$	$\mathbb{Z}_2$	111320
581	$A_1^2A_5^2D_7$	$\mathbb{Z}_2$	00332	668	$A_2^2A_3D_4D_8$	$\mathbb{Z}_2$	002ss	760	$A_1A_2D_6D_{10}$	$\mathbb{Z}_2$	00ss	916	$A_1A_3^2D_6E_6$	$\mathbb{Z}_2$	122s0
581′	$A_1^2A_5^2D_7$	$\mathbb{Z}_2$	11330	671	$A_1^2A_2^2D_5D_8$	$\mathbb{Z}_2$	11002s	760′	$A_1A_2D_6D_{10}$	$\mathbb{Z}_2$	10vs	919	$A_1^2A_5D_6E_6$	$\mathbb{Z}_2$	113s0
589	$A_1^3A_2A_7D_7$	$\mathbb{Z}_2$	011042	673	$A_1A_2A_3D_5D_8$	$\mathbb{Z}_2$	0022s	761	$A_3D_6D_{10}$	$\mathbb{Z}_2$	0ss	932	$A_1^2A_3D_8E_6$	$\mathbb{Z}_2$	112s0
591	$A_1^2A_3A_7D_7$	$\mathbb{Z}_2$	11042	674	$A_1^2A_4D_5D_8$	$\mathbb{Z}_2$	1102s	762	$A_1^2D_7D_{10}$	$\mathbb{Z}_2$	012s	941	$A_1^3D_{10}E_6$	$\mathbb{Z}_2$	111s0
591′	$A_1^2A_3A_7D_7$	$\mathbb{Z}_2$	11240	675	$A_1A_5D_5D_8$	$\mathbb{Z}_2$	130s	764	$A_1D_8D_{10}$	$\mathbb{Z}_2$	1vs	963	$A_1^2A_2^2A_3^2E_7$	$\mathbb{Z}_2$	0100221
594	$A_2A_3A_7D_7$	$\mathbb{Z}_2$	0242	675′	$A_1A_5D_5D_8$	$\mathbb{Z}_2$	132v	767	$A_1^2A_3^2D_{11}$	$\mathbb{Z}_2$	11222	966	$A_1^2A_3^2A_4E_7$	$\mathbb{Z}_2$	012201
601	$A_1^3A_9D_7$	$\mathbb{Z}_2$	00152	677	$A_1D_5^2D_8$	$\mathbb{Z}_2$	022s	773	$A_1^3A_5D_{11}$	$\mathbb{Z}_2$	11132	972	$A_1^2A_2A_3A_5E_7$	$\mathbb{Z}_2$	000231
601′	$A_1^3A_9D_7$	$\mathbb{Z}_2$	11150	679	$A_1A_2^2D_6D_8$	$\mathbb{Z}_2$	100ss	784	$A_1^3A_2^2D_{12}$	$\mathbb{Z}_2$	01100s	974	$A_2^2A_3A_5E_7$	$\mathbb{Z}_2$	00231
603	$A_1A_2A_9D_7$	$\mathbb{Z}_2$	1052	681	$A_2A_3D_6D_8$	$\mathbb{Z}_2$	02vs	786	$A_1^2A_2A_3D_{12}$	$\mathbb{Z}_2$	0002s	975	$A_1A_3^2A_5E_7$	$\mathbb{Z}_2$	00231
608	$A_1A_{11}D_7$	$\mathbb{Z}_2$	062	682	$A_1A_4D_6D_8$	$\mathbb{Z}_2$	10ss	786′	$A_1^2A_2A_3D_{12}$	$\mathbb{Z}_2$	1100s	975′	$A_1A_3^2A_5E_7$	$\mathbb{Z}_2$	12201
611	$A_2A_3^2D_4D_7$	$\mathbb{Z}_2$	022s2	683	$A_5D_6D_8$	$\mathbb{Z}_2$	3sv	788	$A_2^2A_3D_{12}$	$\mathbb{Z}_2$	002s	975″	$A_1A_3^2A_5E_7$	$\mathbb{Z}_2$	12230
613	$A_1A_2A_5D_4D_7$	$\mathbb{Z}_2$	103s2	685	$D_5D_6D_8$	$\mathbb{Z}_2$	2vs	789	$A_1A_3^2D_{12}$	$\mathbb{Z}_2$	002s	976	$A_1^3A_4A_5E_7$	$\mathbb{Z}_2$	011031
615	$A_1A_7D_4D_7$	$\mathbb{Z}_2$	04s2	686	$A_1^2A_2D_7D_8$	$\mathbb{Z}_2$	1102s	790	$A_1^3A_4D_{12}$	$\mathbb{Z}_2$	0110s	979	$A_3A_4A_5E_7$	$\mathbb{Z}_2$	2031
619	$A_1^2A_5D_5D_7$	$\mathbb{Z}_2$	01322	688	$A_1A_3D_7D_8$	$\mathbb{Z}_2$	022s	792	$A_3A_4D_{12}$	$\mathbb{Z}_2$	20s	980	$A_1^2A_5^2E_7$	$\mathbb{Z}_2$	11031
622	$A_7D_5D_7$	$\mathbb{Z}_2$	422	690	$A_1A_2D_8^2$	$\mathbb{Z}_2$	00ss	793	$A_1^2A_5D_{12}$	$\mathbb{Z}_2$	110s	980′	$A_1^2A_5^2E_7$	$\mathbb{Z}_2$	11330
623	$A_1A_2A_3D_6D_7$	$\mathbb{Z}_2$	102s2	700	$A_1^2A_3A_5D_9$	$\mathbb{Z}_2$	01232	796	$A_1A_2D_4D_{12}$	$\mathbb{Z}_2$	00ss	990	$A_1^3A_2A_7E_7$	$\mathbb{Z}_2$	001041
624	$A_3^2D_6D_7$	$\mathbb{Z}_2$	22v2	704	$A_5^2D_9$	$\mathbb{Z}_2$	332	797	$A_1^2D_5D_{12}$	$\mathbb{Z}_2$	002s	992	$A_1A_2^2A_7E_7$	$\mathbb{Z}_2$	10041
626′	$A_1A_5D_6D_7$	$\mathbb{Z}_2$	03s2	709	$A_1^3A_7D_9$	$\mathbb{Z}_2$	01142	797′	$A_1^2D_5D_{12}$	$\mathbb{Z}_2$	110s	993	$A_1^2A_3A_7E_7$	$\mathbb{Z}_2$	01041

993′	$A_1^2A_3A_7E_7$	$\mathbb{Z}_2$	11240	1086	$A_1D_4E_7^2$	$\mathbb{Z}_2$	0s11	839	$A_1A_2^2A_3A_5E_6$	$\mathbb{Z}_3$	011021	442	$A_1^3A_{11}D_5$	$\mathbb{Z}_4$	00131
996	$A_1A_4A_7E_7$	$\mathbb{Z}_2$	1041	1088	$D_5E_7^2$	$\mathbb{Z}_2$	211	842	$A_2^2A_4A_5E_6$	$\mathbb{Z}_3$	11021	445	$A_1A_2A_{11}D_5$	$\mathbb{Z}_4$	1031
1002	$A_1^3A_9E_7$	$\mathbb{Z}_2$	00051	1090	$A_1^2A_3^3E_8$	$\mathbb{Z}_2$	112220	846	$A_1A_2A_5^2E_6$	$\mathbb{Z}_3$	00221	447	$A_3A_{11}D_5$	$\mathbb{Z}_4$	132
1002′	$A_1^3A_9E_7$	$\mathbb{Z}_2$	11150	1099	$A_1^3A_3A_5E_8$	$\mathbb{Z}_2$	111230	849	$A_3A_5^2E_6$	$\mathbb{Z}_3$	0221	463	$A_1A_2A_3^2D_5^2$	$\mathbb{Z}_4$	001111
1004	$A_1A_2A_9E_7$	$\mathbb{Z}_2$	0051	1111	$A_1^4A_7E_8$	$\mathbb{Z}_2$	111140	868	$A_1^3A_2A_8E_6$	$\mathbb{Z}_3$	000131	471	$A_1^2A_7D_5^2$	$\mathbb{Z}_4$	00211
1006	$A_3A_9E_7$	$\mathbb{Z}_2$	051	10	$A_2^4A_3^2A_5$	$\mathbb{Z}_3$	1111002	869	$A_1A_2^2A_8E_6$	$\mathbb{Z}_3$	00131	472	$A_2A_7D_5^2$	$\mathbb{Z}_4$	0211
1011	$A_1A_2^2A_3D_4E_7$	$\mathbb{Z}_2$	1002s1	20	$A_2^3A_3A_5^2$	$\mathbb{Z}_3$	011022	872	$A_2A_3A_8E_6$	$\mathbb{Z}_3$	1031	510	$A_3^2A_7D_6$	$\mathbb{Z}_4$	112v
1014	$A_1A_3A_4D_4E_7$	$\mathbb{Z}_2$	120s1	25	$A_1A_2^2A_4A_5^2$	$\mathbb{Z}_3$	011022	886	$A_1^2A_{11}E_6$	$\mathbb{Z}_3$	0041	566	$A_1A_2A_3^3D_7$	$\mathbb{Z}_4$	001111
1017	$A_1A_2A_5D_4E_7$	$\mathbb{Z}_2$	003s1	32	$A_2^2A_5^3$	$\mathbb{Z}_3$	00222	889	$A_2A_{11}E_6$	$\mathbb{Z}_3$	041	567	$A_3^4D_7$	$\mathbb{Z}_4$	01111
1018	$A_3A_5D_4E_7$	$\mathbb{Z}_2$	03s1	32′	$A_2^2A_5^3$	$\mathbb{Z}_3$	11022	892	$A_1A_2^4D_4E_6$	$\mathbb{Z}_3$	0111101	567′	$A_3^4D_7$	$\mathbb{Z}_4$	11112
1018′	$A_3A_5D_4E_7$	$\mathbb{Z}_2$	2301	37	$A_4A_5^3$	$\mathbb{Z}_3$	0222	895	$A_2^2A_5D_4E_6$	$\mathbb{Z}_3$	11201	592	$A_1^2A_3A_7D_7$	$\mathbb{Z}_4$	00121
1020	$A_1A_7D_4E_7$	$\mathbb{Z}_2$	1401	48	$A_2^4A_5A_6$	$\mathbb{Z}_3$	111120	948	$A_1^3A_2^2E_6^2$	$\mathbb{Z}_3$	0001111	595	$A_2A_3A_7D_7$	$\mathbb{Z}_4$	0121
1022	$A_1^3A_2^2D_5E_7$	$\mathbb{Z}_2$	1110021	124	$A_1^2A_2^3A_3A_8$	$\mathbb{Z}_3$	0011103	949	$A_2^2A_3E_6^2$	$\mathbb{Z}_3$	11011	609	$A_1A_{11}D_7$	$\mathbb{Z}_4$	031
1023	$A_1^2A_2A_3D_5E_7$	$\mathbb{Z}_2$	010221	125	$A_2^4A_3A_8$	$\mathbb{Z}_3$	011103	953	$A_1^2A_5E_6^2$	$\mathbb{Z}_3$	00211	7	$A_1^3A_4^4$	$\mathbb{Z}_5$	0001122
1025	$A_1A_3^2D_5E_7$	$\mathbb{Z}_2$	12201	129	$A_1A_2^3A_4A_8$	$\mathbb{Z}_3$	011103	954	$A_2A_5E_6^2$	$\mathbb{Z}_3$	0211	8	$A_1A_2A_4^4$	$\mathbb{Z}_5$	001122
1028	$A_1^2A_5D_5E_7$	$\mathbb{Z}_2$	00321	136	$A_1^2A_2^2A_5A_8$	$\mathbb{Z}_3$	000123	962	$A_1E_6^3$	$\mathbb{Z}_3$	0111	9	$A_3A_4^4$	$\mathbb{Z}_5$	01122
1028′	$A_1^2A_5D_5E_7$	$\mathbb{Z}_2$	11301	137	$A_2^3A_5A_8$	$\mathbb{Z}_3$	00123	2	$A_3^5A_4$	$\mathbb{Z}_4$	211110	177	$A_1^2A_4^2A_9$	$\mathbb{Z}_5$	00114
1030	$A_2A_5D_5E_7$	$\mathbb{Z}_2$	0321	137′	$A_2^3A_5A_8$	$\mathbb{Z}_3$	11103	71	$A_1^2A_2^2A_3^2A_7$	$\mathbb{Z}_4$	1100112	178	$A_2A_4^2A_9$	$\mathbb{Z}_5$	0114
1033	$A_1A_2D_4D_5E_7$	$\mathbb{Z}_2$	10s21	140	$A_1A_2A_3A_5A_8$	$\mathbb{Z}_3$	01023	73	$A_1A_2A_3^3A_7$	$\mathbb{Z}_4$	002112	201	$A_1A_9^2$	$\mathbb{Z}_5$	024
1034	$A_1^2D_5^2E_7$	$\mathbb{Z}_2$	01221	144	$A_2A_4A_5A_8$	$\mathbb{Z}_3$	1023	74	$A_3^4A_7$	$\mathbb{Z}_4$	02112	19	$A_1^2A_2^2A_3A_5^2$	$\mathbb{Z}_6$	0011255
1036	$A_1^2A_2^2D_6E_7$	$\mathbb{Z}_2$	1100s1	158	$A_1^3A_8^2$	$\mathbb{Z}_3$	00033	79	$A_1^2A_3^2A_4A_7$	$\mathbb{Z}_4$	111102	21	$A_2^3A_3A_5^2$	$\mathbb{Z}_6$	011255
1039	$A_1A_2A_3D_6E_7$	$\mathbb{Z}_2$	002s1	160	$A_1A_2A_8^2$	$\mathbb{Z}_3$	0033	110	$A_1^3A_2A_7^2$	$\mathbb{Z}_4$	011022	31	$A_1^2A_2A_5^3$	$\mathbb{Z}_6$	110255
1040	$A_3^2D_6E_7$	$\mathbb{Z}_2$	02s1	162	$A_3A_8^2$	$\mathbb{Z}_3$	033	119	$A_2A_3A_7^2$	$\mathbb{Z}_4$	0222	35	$A_1A_3A_5^3$	$\mathbb{Z}_6$	02255
1041	$A_1^2A_4D_6E_7$	$\mathbb{Z}_2$	110s1	223	$A_1^2A_2^3A_{11}$	$\mathbb{Z}_3$	000114	227	$A_1^3A_2A_3A_{11}$	$\mathbb{Z}_4$	011013	222	$A_1^4A_2^2A_{11}$	$\mathbb{Z}_6$	00111110
1043	$A_1A_5D_6E_7$	$\mathbb{Z}_2$	03v1	225	$A_2^4A_{11}$	$\mathbb{Z}_3$	00114	233	$A_1^2A_3^2A_{11}$	$\mathbb{Z}_4$	00213	224	$A_1^2A_2^3A_{11}$	$\mathbb{Z}_6$	1101110
1045	$A_2D_4D_6E_7$	$\mathbb{Z}_2$	0ss1	230	$A_1A_2^2A_3A_{11}$	$\mathbb{Z}_3$	01104	233′	$A_1^2A_3^2A_{11}$	$\mathbb{Z}_4$	11013	231	$A_1A_2^2A_3A_{11}$	$\mathbb{Z}_6$	011210
1046	$A_1D_5D_6E_7$	$\mathbb{Z}_2$	02s1	241	$A_2^2A_4A_{11}$	$\mathbb{Z}_3$	1104	236	$A_2A_3^2A_{11}$	$\mathbb{Z}_4$	0213	246	$A_1^3A_5A_{11}$	$\mathbb{Z}_6$	011210
1047	$D_6^2E_7$	$\mathbb{Z}_2$	sv1	248	$A_1A_2A_5A_{11}$	$\mathbb{Z}_3$	0024	293	$A_1^4A_{15}$	$\mathbb{Z}_4$	00114	252	$A_3A_5A_{11}$	$\mathbb{Z}_6$	2210
1049	$A_1^2A_3D_7E_7$	$\mathbb{Z}_2$	01221	251	$A_3A_5A_{11}$	$\mathbb{Z}_3$	024	296	$A_1^2A_2A_{15}$	$\mathbb{Z}_4$	1104	323	$A_1A_2^2A_5^2D_4$	$\mathbb{Z}_6$	01111s
1053	$A_5D_7E_7$	$\mathbb{Z}_2$	321	285	$A_1^3A_2A_{14}$	$\mathbb{Z}_3$	00015	301	$A_1A_3A_{15}$	$\mathbb{Z}_4$	024	326	$A_5^3D_4$	$\mathbb{Z}_6$	211s
1054	$A_1D_4D_7E_7$	$\mathbb{Z}_2$	1s21	287	$A_1A_2^2A_{14}$	$\mathbb{Z}_3$	0015	341	$A_2A_3^2A_7D_4$	$\mathbb{Z}_4$	0112s	367	$A_2^2A_{11}D_4$	$\mathbb{Z}_6$	112s
1057	$A_1^2A_2D_8E_7$	$\mathbb{Z}_2$	010s1	290	$A_2A_3A_{14}$	$\mathbb{Z}_3$	105	350	$A_1A_7^2D_4$	$\mathbb{Z}_4$	022s	395	$A_2^2A_5^2D_5$	$\mathbb{Z}_6$	11112
1059	$A_1A_3D_8E_7$	$\mathbb{Z}_2$	10s1	308	$A_1^2A_{17}$	$\mathbb{Z}_3$	006	369	$A_1A_3A_{11}D_4$	$\mathbb{Z}_4$	013s	845	$A_1^3A_5^2E_6$	$\mathbb{Z}_6$	011112
1059′	$A_1A_3D_8E_7$	$\mathbb{Z}_2$	12v1	310	A_2A_{17}	$\mathbb{Z}_3$	06	375	$A_{15}D_4$	$\mathbb{Z}_4$	4s	850	$A_3A_5^2E_6$	$\mathbb{Z}_6$	2112
1063	$A_1^2D_{10}E_7$	$\mathbb{Z}_2$	00s1	351	$A_1A_2^3A_8D_4$	$\mathbb{Z}_3$	011130	381	$A_1A_2^2A_3^3D_5$	$\mathbb{Z}_4$	1001111	887	$A_1^2A_{11}E_6$	$\mathbb{Z}_6$	1122
1065	$A_2D_{10}E_7$	$\mathbb{Z}_2$	0s1	353	$A_2A_5A_8D_4$	$\mathbb{Z}_3$	1230	382	$A_2A_3^4D_5$	$\mathbb{Z}_4$	011112	70	$A_1A_6^3$	$\mathbb{Z}_7$	0123
1068	$A_1^3A_3E_6E_7$	$\mathbb{Z}_2$	111201	366	$A_2^2A_{11}D_4$	$\mathbb{Z}_3$	1140	384	$A_1A_3^3A_4D_5$	$\mathbb{Z}_4$	111101	116	$A_1^2A_3A_7^2$	$\mathbb{Z}_8$	01115
1079	$A_1^3A_2E_7^2$	$\mathbb{Z}_2$	011011	394	$A_2^2A_5^2D_5$	$\mathbb{Z}_3$	11220	411	$A_1^2A_2A_3A_7D_5$	$\mathbb{Z}_4$	010121	421	$A_7^2D_5$	$\mathbb{Z}_8$	131
1081	$A_1^2A_3E_7^2$	$\mathbb{Z}_2$	00211	422	$A_2^3A_8D_5$	$\mathbb{Z}_3$	11130	414	$A_1A_3^2A_7D_5$	$\mathbb{Z}_4$	01122				
1083	$A_2A_3E_7^2$	$\mathbb{Z}_2$	0211	484	$A_2^4A_5D_6$	$\mathbb{Z}_3$	111120	414′	$A_1A_3^2A_7D_5$	$\mathbb{Z}_4$	10121				

22	$A_1^3A_3^2A_5^2$	$\mathbb{Z}_2$ 0012203 $\mathbb{Z}_2$ 0102230	493	$A_1^3A_5^2D_6$	$\mathbb{Z}_2$ 00033v $\mathbb{Z}_2$ 01103s	635	$A_1^3A_2A_3^2D_8$	$\mathbb{Z}_2$ 011002s $\mathbb{Z}_2$ 101020c	741	$A_1A_3A_5D_{10}$	$\mathbb{Z}_2$ 003s $\mathbb{Z}_2$ 120c
86	$A_1^4A_3A_5A_7$	$\mathbb{Z}_2$ 0001034 $\mathbb{Z}_2$ 0110204	493′	$A_1^3A_5^2D_6$	$\mathbb{Z}_2$ 01103s $\mathbb{Z}_2$ 10130s	637	$A_1^2A_3^3D_8$	$\mathbb{Z}_2$ 00022s $\mathbb{Z}_2$ 11200s	752	$A_1^2A_3D_4D_{10}$	$\mathbb{Z}_2$ 010ss $\mathbb{Z}_2$ 1020c
90	$A_1A_3^2A_5A_7$	$\mathbb{Z}_2$ 02204 $\mathbb{Z}_2$ 10034	496	$A_3A_5^2D_6$	$\mathbb{Z}_2$ 033v $\mathbb{Z}_2$ 203s	643	$A_1^4A_2A_5D_8$	$\mathbb{Z}_2$ 000103s $\mathbb{Z}_2$ 111003v	759	$A_1^3D_6D_{10}$	$\mathbb{Z}_2$ 000ss $\mathbb{Z}_2$ 1110c
96	$A_1^2A_5^2A_7$	$\mathbb{Z}_2$ 01034 $\mathbb{Z}_2$ 10304	507	$A_1^3A_3A_7D_6$	$\mathbb{Z}_2$ 00104s $\mathbb{Z}_2$ 110240	644	$A_1^3A_3A_5D_8$	$\mathbb{Z}_2$ 00103s $\mathbb{Z}_2$ 01023v	971	$A_1^4A_3A_5E_7$	$\mathbb{Z}_2$ 0011031 $\mathbb{Z}_2$ 1101201
114	$A_1^2A_3A_7^2$	$\mathbb{Z}_2$ 00044 $\mathbb{Z}_2$ 11204	533	$A_1A_2A_3^2D_4D_6$	$\mathbb{Z}_2$ 0022sv $\mathbb{Z}_2$ 1002cs	644′	$A_1^3A_3A_5D_8$	$\mathbb{Z}_2$ 00103s $\mathbb{Z}_2$ 11020s	1012	$A_1^2A_3^2D_4E_7$	$\mathbb{Z}_2$ 0102s1 $\mathbb{Z}_2$ 1020c1
318	$A_1^2A_2A_3^2A_5D_4$	$\mathbb{Z}_2$ 010023s $\mathbb{Z}_2$ 100203c	534	$A_1A_3A_5D_4D_6$	$\mathbb{Z}_2$ 003ss $\mathbb{Z}_2$ 120cs	650	$A_1A_5^2D_8$	$\mathbb{Z}_2$ 033v $\mathbb{Z}_2$ 103s	1016	$A_1^3A_5D_4E_7$	$\mathbb{Z}_2$ 0003s1 $\mathbb{Z}_2$ 1110c1
319	$A_1A_3^3A_5D_4$	$\mathbb{Z}_2$ 02220s $\mathbb{Z}_2$ 10023s	534′	$A_1A_3A_5D_4D_6$	$\mathbb{Z}_2$ 003ss $\mathbb{Z}_2$ 120sc	656	$A_1^4A_7D_8$	$\mathbb{Z}_2$ 00004s $\mathbb{Z}_2$ 11110s	1038	$A_1^3A_3D_6E_7$	$\mathbb{Z}_2$ 0012v1 $\mathbb{Z}_2$ 1100s1
322	$A_1^3A_2A_5^2D_4$	$\mathbb{Z}_2$ 000033s $\mathbb{Z}_2$ 111003c	534″	$A_1A_3A_5D_4D_6$	$\mathbb{Z}_2$ 0230s $\mathbb{Z}_2$ 103sv	667	$A_1^2A_2A_3D_4D_8$	$\mathbb{Z}_2$ 0002ss $\mathbb{Z}_2$ 1100cc	1056	$A_1^4D_8E_7$	$\mathbb{Z}_2$ 0001s1 $\mathbb{Z}_2$ 1110v1
324	$A_1^2A_3A_5^2D_4$	$\mathbb{Z}_2$ 00033s $\mathbb{Z}_2$ 01203c	540	$A_1^2A_3^2D_5D_6$	$\mathbb{Z}_2$ 01022s $\mathbb{Z}_2$ 10202c	669	$A_1A_3^2D_4D_8$	$\mathbb{Z}_2$ 002ss $\mathbb{Z}_2$ 020cc	313	$A_3^5D_4$	$\mathbb{Z}_2$ 20022s $\mathbb{Z}_4$ 01111s
324′	$A_1^2A_3A_5^2D_4$	$\mathbb{Z}_2$ 01203s $\mathbb{Z}_2$ 10230s	544	$A_1^3A_5D_5D_6$	$\mathbb{Z}_2$ 00032s $\mathbb{Z}_2$ 11102c	670	$A_1^2A_5D_4D_8$	$\mathbb{Z}_2$ 0130s $\mathbb{Z}_2$ 103sv	339	$A_1^2A_3^2A_7D_4$	$\mathbb{Z}_2$ 11004s $\mathbb{Z}_4$ 111160
338	$A_1^2A_3^2A_7D_4$	$\mathbb{Z}_2$ 00024s $\mathbb{Z}_2$ 112040	544′	$A_1^3A_5D_5D_6$	$\mathbb{Z}_2$ 00132v $\mathbb{Z}_2$ 11030s	672	$A_1^3A_3D_5D_8$	$\mathbb{Z}_2$ 01102s $\mathbb{Z}_2$ 10120c	30	$A_1^4A_5^3$	$\mathbb{Z}_2$ 0011033 $\mathbb{Z}_6$ 0101114
345	$A_1^3A_5A_7D_4$	$\mathbb{Z}_2$ 001340 $\mathbb{Z}_2$ 11004s	550	$A_1A_3D_4D_5D_6$	$\mathbb{Z}_2$ 02s2v $\mathbb{Z}_2$ 10c2s	676	$A_1^2D_4D_5D_8$	$\mathbb{Z}_2$ 00s2s $\mathbb{Z}_2$ 11c0c	327	$A_5^3D_4$	$\mathbb{Z}_2$ 033s $\mathbb{Z}_6$ 112c
377	$A_2A_3^3D_4^2$	$\mathbb{Z}_2$ 0022ss $\mathbb{Z}_2$ 0202cc	553	$A_1^2A_2A_3D_6^2$	$\mathbb{Z}_2$ 0102sv $\mathbb{Z}_2$ 1002vs	678	$A_1^3A_2D_6D_8$	$\mathbb{Z}_2$ 0010ss $\mathbb{Z}_2$ 1100vc	18	$A_1A_2^4A_5^2$	$\mathbb{Z}_3$ 0001122 $\mathbb{Z}_3$ 0110024
378	$A_1A_5^2D_4^2$	$\mathbb{Z}_2$ 0330s $\mathbb{Z}_2$ 103sc	555	$A_1A_3^2D_6^2$	$\mathbb{Z}_2$ 002ss $\mathbb{Z}_2$ 020cc	680	$A_1^2A_3D_6D_8$	$\mathbb{Z}_2$ 002vs $\mathbb{Z}_2$ 010sc	33	$A_2^2A_5^3$	$\mathbb{Z}_3$ 00222 $\mathbb{Z}_3$ 11024
388	$A_1^3A_3^2A_5D_5$	$\mathbb{Z}_2$ 0010232 $\mathbb{Z}_2$ 1102202	555′	$A_1A_3^2D_6^2$	$\mathbb{Z}_2$ 002ss $\mathbb{Z}_2$ 120sv	680′	$A_1^2A_3D_6D_8$	$\mathbb{Z}_2$ 010ss $\mathbb{Z}_2$ 102sv	838	$A_2^4A_5E_6$	$\mathbb{Z}_3$ 001121 $\mathbb{Z}_3$ 110022
393	$A_1^4A_5^2D_5$	$\mathbb{Z}_2$ 0011330 $\mathbb{Z}_2$ 1101032	557	$A_1^2A_5D_6^2$	$\mathbb{Z}_2$ 003sv $\mathbb{Z}_2$ 110ss	684	$A_1D_4D_6D_8$	$\mathbb{Z}_2$ 0svs $\mathbb{Z}_2$ 10sc	72	$A_1^3A_3^3A_7$	$\mathbb{Z}_4$ 0002112 $\mathbb{Z}_2$ 0112004
456	$A_1^2A_3A_5D_4D_5$	$\mathbb{Z}_2$ 0103s2 $\mathbb{Z}_2$ 1023c0	561	$A_1A_2D_4D_6^2$	$\mathbb{Z}_2$ 00sss $\mathbb{Z}_2$ 10ccv	689	$A_1^3D_8^2$	$\mathbb{Z}_2$ 000ss $\mathbb{Z}_2$ 011cv	75	$A_3^4A_7$	$\mathbb{Z}_4$ 02112 $\mathbb{Z}_2$ 22004
485	$A_1^3A_2A_3A_5D_6$	$\mathbb{Z}_2$ 001023v $\mathbb{Z}_2$ 110003s	562	$A_3D_4D_6^2$	$\mathbb{Z}_2$ 0sss $\mathbb{Z}_2$ 20cc	691	$A_3D_8^2$	$\mathbb{Z}_2$ 0ss $\mathbb{Z}_2$ 2cv	109	$A_1^5A_7^2$	$\mathbb{Z}_4$ 0001122 $\mathbb{Z}_2$ 0111104
487	$A_1^2A_3^2A_5D_6$	$\mathbb{Z}_2$ 00023s $\mathbb{Z}_2$ 01203v	563	$A_1^2D_5D_6^2$	$\mathbb{Z}_2$ 012sv $\mathbb{Z}_2$ 102vs	730	$A_1^3A_3^2D_{10}$	$\mathbb{Z}_2$ 00102s $\mathbb{Z}_2$ 01020c	115	$A_1^2A_3A_7^2$	$\mathbb{Z}_4$ 00222 $\mathbb{Z}_2$ 11204
487′	$A_1^2A_3^2A_5D_6$	$\mathbb{Z}_2$ 01220s $\mathbb{Z}_2$ 102230	565	$A_1D_6^3$	$\mathbb{Z}_2$ 0ssv $\mathbb{Z}_2$ 0cvs	738	$A_1^4A_5D_{10}$	$\mathbb{Z}_2$ 00003s $\mathbb{Z}_2$ 01110c	1	$A_1A_3^6$	$\mathbb{Z}_4$ 0011112 $\mathbb{Z}_4$ 0101231

376	$A_1^2A_3^3D_4^2$	$\mathbb{Z}_2$ 00022ss $\mathbb{Z}_2$ 00202cc $\mathbb{Z}_2$ 11002vv	538	$A_1^2A_3D_4^2D_6$	$\mathbb{Z}_2$ 002ssv $\mathbb{Z}_2$ 010scs $\mathbb{Z}_2$ 100csc	560	$A_1^3D_4D_6^2$	$\mathbb{Z}_2$ 000sss $\mathbb{Z}_2$ 001ccv $\mathbb{Z}_2$ 1100cc
532	$A_1^3A_3^2D_4D_6$	$\mathbb{Z}_2$ 00102ss $\mathbb{Z}_2$ 01020cs $\mathbb{Z}_2$ 100220c	552	$A_1^4A_3D_6^2$	$\mathbb{Z}_2$ 00002ss $\mathbb{Z}_2$ 00110cc $\mathbb{Z}_2$ 11010sv	380	$A_1^3D_4^4$	$\mathbb{Z}_2$ 000ssss $\mathbb{Z}_2$ 000cccc $\mathbb{Z}_2$ 0110scv $\mathbb{Z}_2$ 1010cvs

Table D.12: Groups of maximal rank with non-trivial fundamental group and their generators for the heterotic string on T^3.

4	$A_2^2A_7$	$\mathbb{Z}_2$	004	82	$A_1A_5C_1C_4$	$\mathbb{Z}_2$	1300	161	$A_2C_1C_2D_6$	$\mathbb{Z}_2$	001s	312	$A_1^2A_3C_3F_3$	$\mathbb{Z}_2$	01201
5	$A_2^2A_3^2C_1$	$\mathbb{Z}_2$	00220	82′	$A_1A_5C_1C_4$	$\mathbb{Z}_2$	1301	163	$A_1C_1C_3D_6$	$\mathbb{Z}_2$	100s	316	$A_5C_3F_3$	$\mathbb{Z}_2$	301
7	$A_1A_2^2A_5C_1$	$\mathbb{Z}_2$	10030	84	$A_1A_2^2C_2C_4$	$\mathbb{Z}_2$	10011	164	$C_2C_3D_6$	$\mathbb{Z}_2$	10s	318	$A_1^2A_2C_4F_3$	$\mathbb{Z}_2$	01011
9	$A_1A_4A_5C_1$	$\mathbb{Z}_2$	1030	86	$A_2A_3C_2C_4$	$\mathbb{Z}_2$	0201	166	$C_1C_4D_6$	$\mathbb{Z}_2$	01v	320	$A_1A_3C_4F_3$	$\mathbb{Z}_2$	1011
10	$A_5^2C_1$	$\mathbb{Z}_2$	330	87	$A_1A_4C_2C_4$	$\mathbb{Z}_2$	1011	167	$A_1C_1C_2D_7$	$\mathbb{Z}_2$	1012	320′	$A_1A_3C_4F_3$	$\mathbb{Z}_2$	1201
12	$A_1A_2A_7C_1$	$\mathbb{Z}_2$	0040	88	$A_5C_2C_4$	$\mathbb{Z}_2$	310	168	$C_2^2D_7$	$\mathbb{Z}_2$	112	324	$A_1^2C_6F_3$	$\mathbb{Z}_2$	0011
13	$A_1A_9C_1$	$\mathbb{Z}_2$	150	89	$A_1^2A_2C_3C_4$	$\mathbb{Z}_2$	11001	170	$A_1C_1^2D_8$	$\mathbb{Z}_2$	000s	326	$A_2C_6F_3$	$\mathbb{Z}_2$	011
14	$A_1^2A_2^2A_3C_1^2$	$\mathbb{Z}_2$	1100200	91	$A_1A_3C_3C_4$	$\mathbb{Z}_2$	0201	172	$C_1C_2D_8$	$\mathbb{Z}_2$	00s	329	$A_1A_2C_1D_4F_3$	$\mathbb{Z}_2$	100s1
18	$A_1^2A_2A_5C_1^2$	$\mathbb{Z}_2$	010300	93	$A_1A_2C_4^2$	$\mathbb{Z}_2$	0011	182	$A_1A_2C_1E_7$	$\mathbb{Z}_2$	1001	330	$A_2C_2D_4F_3$	$\mathbb{Z}_2$	01s1
19	$A_1A_3A_5C_1^2$	$\mathbb{Z}_2$	12300	95	$A_1A_5C_5$	$\mathbb{Z}_2$	130	183	$A_1^2C_1^2E_7$	$\mathbb{Z}_2$	01001	331	$A_1C_3D_4F_3$	$\mathbb{Z}_2$	10s1
22	$A_1^2A_7C_1^2$	$\mathbb{Z}_2$	11400	100	$A_1A_3C_2C_5$	$\mathbb{Z}_2$	1210	185	$A_2C_2E_7$	$\mathbb{Z}_2$	011	332	$A_1A_2D_5F_3$	$\mathbb{Z}_2$	1021
30	$A_4A_5C_2$	$\mathbb{Z}_2$	031	103	$A_1^2C_4C_5$	$\mathbb{Z}_2$	1110	186	$A_1C_1C_2E_7$	$\mathbb{Z}_2$	0011	333	$A_1^2C_1D_5F_3$	$\mathbb{Z}_2$	01021
31	$A_2A_7C_2$	$\mathbb{Z}_2$	040	105	$A_1A_2^2C_6$	$\mathbb{Z}_2$	1001	187	$C_2^2E_7$	$\mathbb{Z}_2$	011	335	$A_1C_2D_5F_3$	$\mathbb{Z}_2$	0121
32	A_9C_2	$\mathbb{Z}_2$	51	107	$A_1A_4C_6$	$\mathbb{Z}_2$	101	188	$A_1C_3E_7$	$\mathbb{Z}_2$	101	337	$A_2D_6F_3$	$\mathbb{Z}_2$	0s1
33	$A_1A_2^2A_3C_1C_2$	$\mathbb{Z}_2$	100201	108	A_5C_6	$\mathbb{Z}_2$	31	196	$A_1A_3A_5F_2$	$\mathbb{Z}_2$	1230	338	$A_1C_1D_6F_3$	$\mathbb{Z}_2$	00s1
36	$A_1A_3A_4C_1C_2$	$\mathbb{Z}_2$	12001	109	$A_1^2A_2C_1C_6$	$\mathbb{Z}_2$	01001	199	$A_1^2A_7F_2$	$\mathbb{Z}_2$	1140	339	$C_2D_6F_3$	$\mathbb{Z}_2$	0s1
39	$A_1A_2A_5C_1C_2$	$\mathbb{Z}_2$	00301	111	$A_1A_3C_1C_6$	$\mathbb{Z}_2$	1001	202	$A_1^2A_3^2C_1F_2$	$\mathbb{Z}_2$	112200	339′	$C_2D_6F_3$	$\mathbb{Z}_2$	1v1
40	$A_3A_5C_1C_2$	$\mathbb{Z}_2$	0301	111′	$A_1A_3C_1C_6$	$\mathbb{Z}_2$	1201	206	$A_1^3A_5C_1F_2$	$\mathbb{Z}_2$	111300	340	$A_1D_7F_3$	$\mathbb{Z}_2$	121
40′	$A_3A_5C_1C_2$	$\mathbb{Z}_2$	2301	114	$A_1A_2C_2C_6$	$\mathbb{Z}_2$	0011	213	$A_1A_3^2C_2F_2$	$\mathbb{Z}_2$	12210	345	$A_1E_7F_3$	$\mathbb{Z}_2$	011
42	$A_1A_7C_1C_2$	$\mathbb{Z}_2$	0400	114′	$A_1A_2C_2C_6$	$\mathbb{Z}_2$	1001	216	$A_1^2A_5C_2F_2$	$\mathbb{Z}_2$	11310	347	$C_1E_7F_3$	$\mathbb{Z}_2$	011
42′	$A_1A_7C_1C_2$	$\mathbb{Z}_2$	1401	115	$A_3C_2C_6$	$\mathbb{Z}_2$	011	226	$A_1^2A_3C_4F_2$	$\mathbb{Z}_2$	11210	349	$A_1^3A_3F_2F_3$	$\mathbb{Z}_2$	111201
46	$A_2^2A_3C_2^2$	$\mathbb{Z}_2$	00211	116	$A_1^2C_3C_6$	$\mathbb{Z}_2$	0101	233	$A_1^3C_6F_2$	$\mathbb{Z}_2$	11110	360	$A_1^3A_2F_3^2$	$\mathbb{Z}_2$	011011
48	$A_3A_4C_2^2$	$\mathbb{Z}_2$	2011	118	$A_1C_4C_6$	$\mathbb{Z}_2$	101	280	$A_1A_2^2A_3F_3$	$\mathbb{Z}_2$	10021	362	$A_1^2A_3F_3^2$	$\mathbb{Z}_2$	00211
50	$A_2A_5C_2^2$	$\mathbb{Z}_2$	0301	124	$A_1A_2C_8$	$\mathbb{Z}_2$	001	283	$A_1A_3A_4F_3$	$\mathbb{Z}_2$	1201	364	$A_2A_3F_3^2$	$\mathbb{Z}_2$	0211
51	$A_7C_2^2$	$\mathbb{Z}_2$	400	125	$A_1^2C_1C_8$	$\mathbb{Z}_2$	0001	286	$A_1A_2A_5F_3$	$\mathbb{Z}_2$	0031	367	$A_1D_4F_3^2$	$\mathbb{Z}_2$	0s11
52	$A_2A_3^2C_3$	$\mathbb{Z}_2$	0220	125′	$A_1^2C_1C_8$	$\mathbb{Z}_2$	1101	287	$A_3A_5F_3$	$\mathbb{Z}_2$	031	369	$D_5F_3^2$	$\mathbb{Z}_2$	211
54	$A_1A_2A_5C_3$	$\mathbb{Z}_2$	1030	126	$A_2C_1C_8$	$\mathbb{Z}_2$	001	287′	$A_3A_5F_3$	$\mathbb{Z}_2$	231	192	$A_1A_2^4F_2$	$\mathbb{Z}_3$	011111
56	$A_1A_7C_3$	$\mathbb{Z}_2$	040	127	$A_1C_2C_8$	$\mathbb{Z}_2$	001	289	$A_1A_7F_3$	$\mathbb{Z}_2$	141	195	$A_2^2A_5F_2$	$\mathbb{Z}_3$	1121
60	$A_1^2A_5C_1C_3$	$\mathbb{Z}_2$	01300	128	C_3C_8	$\mathbb{Z}_2$	01	291	$A_1^3A_2^2C_1F_3$	$\mathbb{Z}_2$	1110001	264	$A_1^3A_2^2F_2^2$	$\mathbb{Z}_3$	0001111
63	$A_7C_1C_3$	$\mathbb{Z}_2$	400	131	A_1C_{10}	$\mathbb{Z}_2$	11	292	$A_1^2A_2A_3C_1F_3$	$\mathbb{Z}_2$	010201	265	$A_2^2A_3F_2^2$	$\mathbb{Z}_3$	11011
64	$A_1A_2A_3C_2C_3$	$\mathbb{Z}_2$	10210	141	$A_1^2A_2C_1^2D_5$	$\mathbb{Z}_2$	110002	294	$A_1A_3^2C_1F_3$	$\mathbb{Z}_2$	12201	269	$A_1^2A_5F_2^2$	$\mathbb{Z}_3$	00211
65	$A_3^2C_2C_3$	$\mathbb{Z}_2$	2200	144	$A_1A_2C_1C_2D_5$	$\mathbb{Z}_2$	10012	297	$A_1^2A_5C_1F_3$	$\mathbb{Z}_2$	00301	270	$A_2A_5F_2^2$	$\mathbb{Z}_3$	0211
67	$A_1A_5C_2C_3$	$\mathbb{Z}_2$	0310	146	$A_2C_2^2D_5$	$\mathbb{Z}_2$	0112	297′	$A_1^2A_5C_1F_3$	$\mathbb{Z}_2$	11301	278	$A_1E_6F_2^2$	$\mathbb{Z}_3$	0111
67′	$A_1A_5C_2C_3$	$\mathbb{Z}_2$	1300	147	$A_1C_2C_3D_5$	$\mathbb{Z}_2$	1102	299	$A_2A_5C_1F_3$	$\mathbb{Z}_2$	0301	15	$A_1A_2A_3^2C_1^2$	$\mathbb{Z}_4$	001111
70	$A_1^2A_3C_3^2$	$\mathbb{Z}_2$	11200	149	$A_1C_1C_4D_5$	$\mathbb{Z}_2$	0012	302	$A_1^2A_2^2C_2F_3$	$\mathbb{Z}_2$	110011	23	$A_1^2A_7C_1^2$	$\mathbb{Z}_4$	00211
75	$A_2^2A_3C_4$	$\mathbb{Z}_2$	0021	150	$C_2C_4D_5$	$\mathbb{Z}_2$	012	305	$A_1A_2A_3C_2F_3$	$\mathbb{Z}_2$	00211	24	$A_2A_7C_1^2$	$\mathbb{Z}_4$	0211
78	$A_1^2A_2^2C_1C_4$	$\mathbb{Z}_2$	110001	152	$C_1^2D_4D_5$	$\mathbb{Z}_2$	00s2	306	$A_3^2C_2F_3$	$\mathbb{Z}_2$	0211				
80	$A_1A_2A_3C_1C_4$	$\mathbb{Z}_2$	00201	153	$C_1D_5^2$	$\mathbb{Z}_2$	022	307	$A_1^2A_4C_2F_3$	$\mathbb{Z}_2$	11011				
81	$A_1^2A_4C_1C_4$	$\mathbb{Z}_2$	11001	156	$A_1A_2C_1^2D_6$	$\mathbb{Z}_2$	1000s	309	$A_1A_5C_2F_3$	$\mathbb{Z}_2$	0301				

2	$A_2A_3^3$	$\mathbb{Z}_2$	0022	47′	$A_1A_3^2C_2^2$	$\mathbb{Z}_2$	00211	106	$A_1^2A_3C_6$	$\mathbb{Z}_2$	0101	158	$A_1A_2C_2D_6$	$\mathbb{Z}_2$	001s
		$\mathbb{Z}_2$	0202			$\mathbb{Z}_2$	12001			$\mathbb{Z}_2$	1021			$\mathbb{Z}_2$	100c
3	$A_1A_5^2$	$\mathbb{Z}_2$	033	49	$A_1^2A_5C_2^2$	$\mathbb{Z}_2$	00301	113	$A_1^3C_2C_6$	$\mathbb{Z}_2$	00011	159	$A_3C_2D_6$	$\mathbb{Z}_2$	01s
		$\mathbb{Z}_2$	103			$\mathbb{Z}_2$	11011			$\mathbb{Z}_2$	11110			$\mathbb{Z}_2$	20v
8	$A_1^2A_3A_5C_1$	$\mathbb{Z}_2$	01030	74	$A_1^2A_2A_3C_4$	$\mathbb{Z}_2$	00021	134	$A_1A_3C_1C_2D_4$	$\mathbb{Z}_2$	0200s	160	$A_1^2C_1C_2D_6$	$\mathbb{Z}_2$	0100s
		$\mathbb{Z}_2$	10230			$\mathbb{Z}_2$	11001			$\mathbb{Z}_2$	1001c			$\mathbb{Z}_2$	1001v
28	$A_1A_2A_3^2C_2$	$\mathbb{Z}_2$	00220	76	$A_1A_3^2C_4$	$\mathbb{Z}_2$	0021	136	$A_1A_2C_2^2D_4$	$\mathbb{Z}_2$	0011s	162	$A_1C_2^2D_6$	$\mathbb{Z}_2$	001s
		$\mathbb{Z}_2$	10021			$\mathbb{Z}_2$	0201			$\mathbb{Z}_2$	1001c			$\mathbb{Z}_2$	010c
29	$A_1A_3A_5C_2$	$\mathbb{Z}_2$	0031	77	$A_1^2A_5C_4$	$\mathbb{Z}_2$	0130	137	$A_3C_2^2D_4$	$\mathbb{Z}_2$	011s	165	$A_1C_4D_6$	$\mathbb{Z}_2$	01v
		$\mathbb{Z}_2$	1201			$\mathbb{Z}_2$	1031			$\mathbb{Z}_2$	200s			$\mathbb{Z}_2$	10s
29′	$A_1A_3A_5C_2$	$\mathbb{Z}_2$	0231	79	$A_1^3A_3C_1C_4$	$\mathbb{Z}_2$	011001	138	$A_1^2C_1C_4D_4$	$\mathbb{Z}_2$	0001s	169	$A_1^2C_1D_8$	$\mathbb{Z}_2$	000s
		$\mathbb{Z}_2$	1030			$\mathbb{Z}_2$	101200			$\mathbb{Z}_2$	1100c			$\mathbb{Z}_2$	110c
34	$A_1^2A_3^2C_1C_2$	$\mathbb{Z}_2$	010201	83	$A_1^3A_2C_2C_4$	$\mathbb{Z}_2$	001011	139	$A_1C_2C_4D_4$	$\mathbb{Z}_2$	001s	171	$A_1C_2D_8$	$\mathbb{Z}_2$	00s
		$\mathbb{Z}_2$	102001			$\mathbb{Z}_2$	110001			$\mathbb{Z}_2$	110s			$\mathbb{Z}_2$	11c
38	$A_1^3A_5C_1C_2$	$\mathbb{Z}_2$	000301	85	$A_1^2A_3C_2C_4$	$\mathbb{Z}_2$	00201	143	$A_1A_3C_2D_5$	$\mathbb{Z}_2$	0202	281	$A_1^2A_3^2F_3$	$\mathbb{Z}_2$	01021
		$\mathbb{Z}_2$	111001			$\mathbb{Z}_2$	01011			$\mathbb{Z}_2$	1012			$\mathbb{Z}_2$	10201
38′	$A_1^3A_5C_1C_2$	$\mathbb{Z}_2$	001300	85′	$A_1^2A_3C_2C_4$	$\mathbb{Z}_2$	01011	145	$A_1^2C_2^2D_5$	$\mathbb{Z}_2$	01012	285	$A_1^3A_5F_3$	$\mathbb{Z}_2$	00031
		$\mathbb{Z}_2$	110301			$\mathbb{Z}_2$	10210			$\mathbb{Z}_2$	10102			$\mathbb{Z}_2$	11101
45	$A_1^2A_2A_3C_2^2$	$\mathbb{Z}_2$	010201	92	$A_1^3C_4^2$	$\mathbb{Z}_2$	00011	148	$A_1^2C_4D_5$	$\mathbb{Z}_2$	0012	304	$A_1^3A_3C_2F_3$	$\mathbb{Z}_2$	001201
		$\mathbb{Z}_2$	100210			$\mathbb{Z}_2$	01101			$\mathbb{Z}_2$	1102			$\mathbb{Z}_2$	110011
47	$A_1A_3^2C_2^2$	$\mathbb{Z}_2$	00211	94	$A_3C_4^2$	$\mathbb{Z}_2$	011	155	$A_1A_3C_1D_6$	$\mathbb{Z}_2$	020v	317	$A_1^4C_4F_3$	$\mathbb{Z}_2$	000111
		$\mathbb{Z}_2$	02011			$\mathbb{Z}_2$	201			$\mathbb{Z}_2$	100s			$\mathbb{Z}_2$	111001

1	$A_1^2A_3^3$	$\mathbb{Z}_2$	00022	44	$A_1^4A_3C_2^2$	$\mathbb{Z}_2$	0000211	135	$A_1^3C_2^2D_4$	$\mathbb{Z}_2$	00011s	157	$A_3^3C_2D_6$	$\mathbb{Z}_2$	0001s
		$\mathbb{Z}_2$	00202			$\mathbb{Z}_2$	0011011			$\mathbb{Z}_2$	00101c			$\mathbb{Z}_2$	0010c
		$\mathbb{Z}_2$	11002			$\mathbb{Z}_2$	1101001			$\mathbb{Z}_2$	11000s			$\mathbb{Z}_2$	1100v
27	$A_1^3A_3^2C_2$	$\mathbb{Z}_2$	001021	133	$A_1^2A_3C_2D_4$	$\mathbb{Z}_2$	0020s	154	$A_1^2A_3D_6$	$\mathbb{Z}_2$	002v	140	$A_1^3D_4^2$	$\mathbb{Z}_2$	000ss
		$\mathbb{Z}_2$	010201			$\mathbb{Z}_2$	0101s			$\mathbb{Z}_2$	010s			$\mathbb{Z}_2$	000cc
		$\mathbb{Z}_2$	100221			$\mathbb{Z}_2$	1001c			$\mathbb{Z}_2$	100c			$\mathbb{Z}_2$	0110s
														$\mathbb{Z}_2$	1010c

Table D.13: Groups of maximal rank with non-trivial fundamental group and their generators for the heterotic $\mathbb{Z}_2$ triple.

17	$A_1^3A_3G_1$	$\mathbb{Z}_2$	11121	32	$A_2A_3G_1^2$	$\mathbb{Z}_2$	0211	1	$A_1^3A_2^2$	$\mathbb{Z}_3$	00011	7	A_2A_5	$\mathbb{Z}_3$	02
28	$A_1^3A_2G_1^2$	$\mathbb{Z}_2$	011011	35	$A_1D_4G_1^2$	$\mathbb{Z}_2$	0s11	2	$A_2^2A_3$	$\mathbb{Z}_3$	110	15	A_1E_6	$\mathbb{Z}_3$	01
30	$A_1^2A_3G_1^2$	$\mathbb{Z}_2$	00211	37	$D_5G_1^2$	$\mathbb{Z}_2$	211	6	$A_1^2A_5$	$\mathbb{Z}_3$	002				

Table D.14: Groups of maximal rank with non-trivial fundamental group and their generators for the heterotic $\mathbb{Z}_3$ triple.

1	$A_1^3A_2$	$\mathbb{Z}_2$	0110	5	A_2A_3	$\mathbb{Z}_2$	02	10	D_5	$\mathbb{Z}_2$	2	3	$A_1^2A_3$	$\mathbb{Z}_2$	002	8	A_1D_4	$\mathbb{Z}_2$	0s

Table D.15: Groups of maximal rank with non-trivial fundamental group and their generators for the heterotic $\mathbb{Z}_4$ triple.

Appendix E

Synthèse en Français

Motivation

La théorie des cordes est le cadre le plus développé et le plus prometteur pour étudier la gravité quantique. De nombreux efforts ont été déployés pour tenter de reproduire le modèle standard en tant que théorie effective à basse énergie d'un modèle de cordes. La première corde proposée, la corde bosonique, ne comprenait que des bosons dans son spectre et prédisait un tachyon, ce qui signalait une incohérence dans la théorie. Cela a motivé la construction des théories des supercordes de type I et II, qui ont un spectre supersymétrique sans tachyon et sont formulées en 10 dimensions spatio-temporelles (ce nombre est fixé par cohérence). Leur limite de basse énergie est la sueergravité dans 10 dimensions.

Deux aspects importants du modèle standard sont qu'il est formulé en 4 dimensions et qu'il contient certaines particules avec des symétries de jauge non abéliennes. Une façon de réduire le nombre de dimensions de 10 à 4 est de recourir au mécanisme de compactification, c'est-à-dire de faire en sorte que six de ces dimensions aient une longueur finie (par exemple en les rendant périodiques), en généralisant la théorie de Kaluza-Klein. Le modèle quadridimensionnel est alors une approximation à basse énergie d'une théorie plus fondamentale. Lorsqu'on compacte la théorie des cordes de manière particulière, les particules acquièrent des symétries de jauge non abéliennes, ce qui fait de ces théories des candidats plausibles pour décrire notre univers. Alors que dans la théorie de Kaluza-Klein, le momentum dans la cinquième dimension est quantifié, dans les compacités de la théorie des cordes, nous avons une situation similaire, avec le momentum associé aux états des cordes vivant sur un réseau.

Ce qu'on appelle la corde hétérotique combine les théories bosonique et des cordes de type II [6]. Par souci de cohérence, le spectre de basse énergie présente deux symétries de groupe de jauge possibles : $\frac{Spin(32)}{\mathbb{Z}_2}$ ou $E_8 \times E_8$, définissant deux théories de cordes différentes. Ces cordes hétérotiques possèdent également une super-symétrie et vivent dans un espace-temps à 10 dimensions, de sorte qu'elles peuvent également être compactées sur différentes géométries [7]. Cette caractéristique était considérée comme très prometteuse, puisque les groupes de jauge des deux théories contiennent $SO(10)$ et $SU(5)$, tandis que E_8 contient également E_6, qui sont les symétries des principales théories Grand Unified. Dans les compactages à des dimensions inférieures, il est assez simple de décomposer les symétries de jauge originales en leurs sous-groupes. Selon la géométrie de l'espace compact, il est également possible d'obtenir une amélioration de la symétrie originale, créant ainsi un riche paysage de théories avec des groupes de jauge très divers. La limite de basse énergie de ces théories est simplement la supergravité couplée au super Yang-Mills avec le groupe de jauge correspondant.

Une question simple que l'on pourrait poser est la suivante : quels sont les groupes de jauge

possibles qui peuvent être obtenus à partir de la théorie hétérotique des cordes en moins de 10 dimensions ? Pouvons-nous les classer ? Cette question est pertinente d'un point de vue phénoménologique (construction d'exemples réalistes de notre univers) ; mais elle pourrait être liée à une énigme plus profonde : quelles sont les théories possibles qui peuvent être couplées de manière cohérente à la gravité ?

Même pour des géométries relativement simples, on sait peu de choses sur les groupes de jauge possibles qui peuvent apparaître, et encore moins sur les valeurs spéciales des paramètres de la compactification où ces améliorations se produisent. Ceci nous a motivé à étudier en détail la structure de l'espace de ces paramètres (l'espace moduli) et à obtenir une classification complète des groupes de jauge possibles pour certains types de géométries.

Puisque les cordes sont des objets étendus, elles peuvent être enveloppées dans des dimensions compactes. Ceci donne lieu à une équivalence entre différentes configurations de cordes après compactification, connue sous le nom de T-dualité [8].

Il a été démontré que toutes les différentes supercordes sont liées par cette dualité et d'autres, ce qui conduit à l'idée qu'elles sont des limites distinctes d'une théorie de dimension supérieure appelée M-théorie [9]. Une forte motivation pour étudier les compactifications toroïdales des cordes hétérotiques est la dualité qu'elles ont avec d'autres théories : l'hétérotique sur T^4 est duale au type IIA sur K3 [10], l'hétérotique sur T^3 est duale à la M-théorie sur K3, tandis que l'hétérotique sur T^2 est duale à la F-théorie sur un K3 à fibres elliptiques. Ce réseau de dualités fournit un cadre pour l'exploration de différents aspects de la théorie des cordes.

Résultats

L'objectif principal de cette thèse est de répondre à la question de savoir quelles symétries de jauge sont autorisées dans les compactifications de cordes hétérotiques sur des géométries qui préservent toute la supersymétrie. Bien qu'il s'agisse d'une question très concrète et relativement simple, une réponse n'était connue que pour les deux cas les plus simples : les compactifications du cercle par la méthode du Diagramme de Dynkin Généralisé (GDD) [11–13] et, bien qu'indirectement, le 2-torus par la dualité avec la F-théorie à partir de la classification des types ADE des fibres singulières dans les surfaces K3 elliptiques [14,15]. Comme nous l'expliquerons en détail, pour les espaces compacts de plus grandes dimensions (qui correspondent aux théories effectives de basse énergie de moins de dimensions) et pour les espaces non géométriques, de nombreuses subtilités apparaissent, transformant l'objectif susmentionné en un objectif hautement non trivial. Un problème connexe que nous aborderons est l'étude des espaces de moduli correspondants : leur structure, les régions de renforcement de la symétrie et leur relation avec la T-dualité. Nous adopterons des approches différentes mais complémentaires.

Dans [1] nous avons étudié les compacités sur le cercle du point de vue de l'espace moduli à 17 dimensions défini par les paramètres. En résolvant les équations de correspondance de masse et de niveau en fonction des moduli, nous avons développé une méthode pour cartographier les régions spéciales où il y a un renforcement de symétrie vers un groupe non abélien de rang 17, en découvrant que les renforcements maximaux (c'est-à-dire vers des groupes sans facteurs $U(1)$) apparaissent sur les points d'intersection des courbes où il y a des renforcements non maximaux. Ce point de vue nous a permis de développer des outils qui sont idéaux pour avoir l'intuition et comprendre ce type de compactifications mais, même s'ils prédisent une riche variété de symétries de jauge, il n'a pas été possible de les obtenir toutes. En nous concentrant sur le fait que les moments du spectre d'états sont

quantifiés dans le réseau Lorentzien de Narain $\Gamma_{1,17}$, nous avons pu obtenir la classification complète des améliorations, ainsi que les régions de l'espace modulaire où elles apparaissent, en utilisant la méthode GDD.

En [2], nous avons traité le cas des compactifications toroïdales de dimensions arbitraires, où les modèles réalisés ont également 16 supercharges et les groupes de jauge sont de type ADE (c'est-à-dire $SU(n)$, $SO(2n)$, E_n) et ont un rang $16+d$. Nous avons trouvé comment les points de l'espace de moduli sont reliés par la dualité T et défini la carte reliant les espaces de moduli des théories compactives $\frac{\mathrm{Spin}(32)}{\mathbb{Z}_2}$ et $E_8 \times E_8$. Pour T^d avec $d > 1$, le treillis de momentum $\Gamma_{d,16+d}$ est toujours pair et auto-duel mais non Lorentzien, ce qui rend impossible la construction d'un GDD. Un certain groupe de jauge apparaît dans l'espace moduli si le réseau racine qui lui est associé peut être intégré dans le réseau de momentum correspondant (en satisfaisant certaines conditions). Cette façon de voir le problème nous a permis d'énoncer certaines conditions d'apparition de ces groupes en utilisant certains théorèmes de Nikulin [16]. Nous avons effectué une classification exhaustive des groupes de jauge autorisés en $d = 2$, trouvant qu'elle correspond exactement à celle obtenue sur la F-théorie duale sur les surfaces K3 à fibres elliptiques [15], mais donnant aussi les modules définissant la compactification qui réalise chaque cas. Pour effectuer cette classification, nous avons réalisé un algorithme d'exploration qui consiste à passer des points singuliers d'enrichissement maximal dans l'espace des moduli à d'autres via des manipulations de leurs treillis de racines associés de manière contrôlée. Cet algorithme a été amélioré dans des travaux ultérieurs, ce qui nous a permis d'explorer des compacités de T^d avec $d > 2$ et de calculer les données globales précises des groupes de jauge.

Dans [3], nous avons effectué une étude détaillée des compactifications sur des orbifolds spéciaux $T^d/\mathbb{Z}_2$ [1] (réalisant la corde CHL [17,18]), qui préservent la supersymétrie et présentent des enrichissements de rang réduit $d+8$. Une particularité est que les groupes de jauge ne sont pas toujours de type ADE, avec des groupes $Sp(n)$, $SO(2n+1)$ ou F_4 apparaissant dans des régions spéciales. Une autre caractéristique est que le treillis de momentum n'est pas auto-dual pour $d > 1$ [19]. Ce dernier point implique que les critères énoncés précédemment pour l'encastrement des treillis ne s'appliquent plus. Cela a nécessité de nombreux changements dans les méthodes développées dans le travail précédent. Avec une généralisation de notre algorithme d'exploration, nous avons trouvé la liste des symétries de jauge pour $d = 2$, 3 et 4 (les améliorations de $d = 1$ peuvent être facilement trouvées en utilisant le GDD associé, qui s'avère être le diagramme de Dynkin de E_{10}). Nous avons calculé la forme précise de leurs topologies respectives en adaptant nos méthodes à l'aide des résultats de [20], et vérifié qu'elles satisfont une condition pour les symétries centrales à une forme sans anomalie donnée dans [21].

Pour $d = 1$ et 2, les seuls rangs possibles connus pour les compacités de cordes hétérotiques préservant 16 supercharges sont $d + 16$ et $d + 8$, à savoir les composantes de Narain et CHL [2]. En revanche, pour $d \geq 3$, les possibilités sont plus nombreuses : les triples holonomiques non triviaux de [23] produisent des vacua de rangs réduits $d + 4$, $d + 2$ et d. Nous nous sommes tournés vers ces constructions dans [4], en adaptant et en appliquant la machinerie déjà développée dans les travaux précédents au cas de $d = 3$ et en obtenant la liste complète des améliorations maximales. Ces nouveaux espaces de modulation ont moins de points d'amélioration maximale en raison du rang réduit, mais dans certains cas ils présentent une variété plus riche de groupes de jauge (par exemple le groupe de jauge G_2 dans le triplé $\mathbb{Z}_3$). La corde hétérotique sur le T^3 est duale à la M-théorie sur une surface K3. Les groupes de jauge de rang réduit sont réalisés dans cette dernière lorsqu'il existe des singularités partiellement gelées sur la surface K3 [23–25]. En exploitant les techniques d'encastrement des treillis,

[1] Il convient de noter qu'il ne s'agit là que d'une notation, car le $\mathbb{Z}_2$ agit sur le faisceau de jauge.

[2] En 8d il y a aussi le $\mathbb{Z}_2$-triple et les constructions "sans structure vectorielle", qui sont duales à la CHL [22,23].

nous avons trouvé les règles de gel explicites qui relient les groupes de jauge sur la composante de Narain avec ceux apparaissant sur ceux de rang réduit. Nous avons trouvé une correspondance parfaite de ces règles du côté des cordes hétérotiques avec les résultats connus sur la M-théorie. Une généralisation de cette analyse à $d = 4$ a été faite dans [5], où nous avons à nouveau obtenu la classification complète des groupes pour tous les $\mathbb{Z}_n$-triples via notre algorithme d'exploration. [3] Nous avons découvert que le mécanisme de congélation fait intervenir la topologie des groupes de jauge, en agissant sur ceux et seulement ceux qui ne sont pas simplement connectés. Il est frappant de constater que les règles de congélation construites coïncident avec la carte reliant les composantes de l'espace modulaire des faisceaux plats de G sur T^2 avec G non-simplicitement-connecté.

Nous n'incluons dans la présente thèse que les listes de groupes de jauge pour les cas $d = 1$, 2 et 3. Pour $d = 4$, nous nous référons à [27], où nous listons les groupes de jauge maximaux pour l'ensemble des 16 supercharges préservant les compactages à 6 ou plus de dimensions spatio-temporelles. En plus des composantes Narain, CHL et triples, beaucoup d'autres sont listées ici. En général, elles ne correspondent pas à des constructions de cordes hétérotiques. Elles ont été étudiées dans [26], qui est laissé de côté dans cette thèse.

Dans le tableau 1.1, nous listons les principales informations concernant chacune des compacités étudiées. Il est notable que le nombre de groupes qui apparaissent sur T^d ou $T^d/\mathbb{Z}_2$ augmente exponentiellement avec d. De plus, comme prévu, plus de types de groupes sont autorisés lors de la réduction du rang. Des explorations partielles pour $d \geq 5$ présentent le même comportement exponentiel, mais l'analyse correspondante n'est pas incluse ici. Les règles reliant les groupes de jauge dans les compactifications toroïdales aux théories de rang réduit trouvées sont résumées dans le tableau 1.2.

G	d	r	Algèbres	Groupes	Information	Types
$\emptyset$	0	16	2	2		DE
S^1	1	17	44	44		ADE
$S^1/\mathbb{Z}_2$	1	9	9	9	CHL	ADE
T^2	2	18	325	341	Dual à la F-théorie sur un K3	ADE
$T^2/\mathbb{Z}_2$	2	10	61	61	CHL/$\mathbb{Z}_2$-triple	ACDE
T^3	3	19	1035	1232	Dual à la M-théorie sur un K3	ADE
$T^3/\mathbb{Z}_2$	3	11	407	429	CHL/$\mathbb{Z}_2$-triple	ABCDEF
$T^3/\mathbb{Z}_3$	3	7	50	52	$\mathbb{Z}_3$-triple	ADEG
$T^3/\mathbb{Z}_4$	3	5	16	18	$\mathbb{Z}_4$-triple	AD
$T^3/\mathbb{Z}_5$	3	3	3	3	$\mathbb{Z}_5$-triple	A
$T^3/\mathbb{Z}_6$	3	3	3	3	$\mathbb{Z}_6$-triple	A
T^4	4	20	2252	3396	Dual à la IIA sur un K3	ADE
$T^4/\mathbb{Z}_2$	4	12	1988	2540	CHL/$\mathbb{Z}_2$-triple	ABCDEF
$T^4/\mathbb{Z}_3$	4	8	154	202	$\mathbb{Z}_3$-triple	ADEG
$T^4/\mathbb{Z}_4$	4	6	101	127	$\mathbb{Z}_4$-triple	ACDE
$T^4/\mathbb{Z}_5$	4	4	11	14	$\mathbb{Z}_5$-triple	AD
$T^4/\mathbb{Z}_6$	4	4	36	40	$\mathbb{Z}_6$-triple	ACG

Table E.1: Géométrie de l'espace de compactification G, nombre de dimensions compactées d, rang r, nombre d'algèbres et de groupes maximaux trouvés et leurs types.

[3] Pour $d = 4$ (comme pour $d = 1$ et 2) nous avons une confirmation récente que nos listes sont exhaustives grâce aux résultats de notre travail très récent [26], où tous ces groupes de jauge ont été obtenus à nouveau à partir des treillis de Niemeier.

d	Algèbre	Ordre	Il se transforme en
1	E_8	2	$\emptyset$
2	D_{n+8}	2	C_n
3	D_{n+4}	2	C_n
3	E_{n+4}	2	F_n
3	E_{n+6}	3	G_n
3	E_{n+7}	4	C_n
3	E_8	5, 6	$\emptyset$
4	A_{qn-1}	$q = 2, 3, 4, 5, 6$	A_{n-1}
4	D_{n+2}	2 (v)	C_n
4	D_{2n}	2 (s)	B_n
4	E_7	2	F_4
4	E_6	3	G_2
4	D_{2n+3}	4 (s)	C_n

Table E.2: Règles de gel pour les facteurs simples dans les groupes de jauge pour $d \leq 4$. Nous utilisons les conventions $C_1 \equiv A_1$, $F_2 \equiv A_2$, $F_3 \equiv B_3$, $G_1 \equiv A_1$,

Structure de la thèse

Cette thèse est divisée en six chapitres : Le premier est une brève introduction à la thèse. Le chapitre 2 est une introduction à la corde hétérotique et à sa compactification toroïdale, avec une analyse détaillée de la structure de l'espace moduli et de ses améliorations de symétrie pour le cas du cercle basé sur [1].

Dans le chapitre 3, qui est basé sur [2], nous reprenons les compacités toroïdales introduites précédemment, en les analysant cette fois du point de vue des encastrements de treillis. Nous introduisons un algorithme d'exploration que nous utilisons pour classer et donner la liste des améliorations de symétrie des compacités T^2.

Dans le chapitre 4 nous analysons en détail la chaîne CHL et ses compactifications, en les construisant à partir de la chaîne hétérotique. Nous explorons l'espace moduli et donnons la liste des améliorations de symétrie de jauge de rang réduit pour neuf et huit dimensions espace-temps. À cette fin, nous améliorons et généralisons l'algorithme introduit dans le chapitre précédent. Cette partie est basée sur [3].

Dans le chapitre 5, basé sur [4, 5], nous étudions différentes constructions d'orbitaux en 7 et 6 dimensions spatio-temporelles, obtenant la liste des groupes de jauge qui sont réalisés non seulement sur les composantes Narain et CHL mais aussi sur quatre autres composantes réalisées via des triples holonomes non triviaux. Enfin, nous construisons les règles générales de congélation reliant les différentes composantes.

Dans le chapitre 6, nous résumons les conclusions de cette thèse, tandis qu'à la fin nous incluons quatre annexes : A, B et C avec les compléments aux deuxième, troisième et cinquième chapitres ; et D avec les listes des groupes d'enrichissement maximal pour les théories étudiées.

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