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## Around stability for functional inequalities

Jordan Serres

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# THÈSE

En vue de l'obtention du

## DOCTORAT DE L'UNIVERSITÉ DE TOULOUSE

Délivré par : *l'Université Toulouse 3 Paul Sabatier (UT3 Paul Sabatier)*

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Présentée et soutenue le *11/10/2022* par :

**Jordan Serres**

**Around stability for functional inequalities**

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### JURY

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« Because of the clumsiness of the technique used at this point, it is likely that slightly better results could be obtained by using the basis identity, Lemma 2.1. [...] However, I believe that, in the long run, better results will be obtained by direct methods. » C. Stein, 1986.

« Un coup de dés jamais n'abolira le hasard. » S. Mallarmé, 1897.



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# Chapter 1

## Introduction (FR)

Au cours des 3 dernières années, j'ai principalement étudié les inégalités de Poincaré et leur stabilité, la méthode de Stein, les conditions de courbure-dimension, les problèmes isopérimétriques, ainsi que les formulations fonctionnelles de l'isopérimétrie telles que les inégalités de Bobkov. Ces sujets et les résultats que j'ai obtenus sont présentés de manière générale dans le premier chapitre de cette thèse. Les autres chapitres contiennent ces résultats et leur preuve en détail. Un plan plus détaillé de cette thèse se trouve à la fin du premier chapitre.

### 1.1 Inégalités de Poincaré

La notion d'inégalité de Poincaré remonte aux travaux de H. Poincaré à la fin du 19ème siècle [177, 178]. Une inégalité de Poincaré est un type d'inégalité fonctionnelle qui majore la norme  $L^2$  d'une fonction par la norme  $L^2$  de ses dérivées multipliée par une constante qui dépend d'une certaine manière de la géométrie de l'espace où la fonction est définie. Les inégalités de Poincaré les plus classiques qui nous intéressent sont les suivantes. Soit  $\Omega \subset \mathbb{R}^d$  un sous-ensemble ouvert borné et convexe de  $\mathbb{R}^d$ . Alors il existe une certaine constante  $C > 0$  telle que

$$(1.1) \quad \forall u \in C^1(\Omega), \quad \text{Var}_\Omega(u) \leq C \int_\Omega \|\nabla u(x)\|^2 dx$$

où la norme  $\|\cdot\|$  désigne la norme euclidienne dans  $\mathbb{R}^d$  et la variance est définie par

$$\text{Var}_\Omega(u) := \int_\Omega \left| u - \frac{1}{|\Omega|} \int_\Omega u(y) dy \right|^2 dx.$$

Le terme de droite est homogène à une énergie, donc les inégalités de Poincaré contrôlent la variance de l'état  $u$  par son énergie. Le plus grand ensemble de fonctions où ce type d'inégalité peut être étendu de façon non triviale est appelé espace de Sobolev. La notion d'espace de Sobolev peut être abordée d'un point de vue technique, en utilisant des notions telles que les dérivées faibles ou les traces. Cependant, dans notre contexte, même si ce n'est pas tout à fait rigoureux, l'idée à garder en tête est que le premier espace de Sobolev  $H^1(\Omega)$  est le plus grand espace de fonctions où l'inégalité de Poincaré (1.1) est non triviale.

Esquissons la preuve de (1.1). Nous désignons par

$$\rho_{x,y} : s \mapsto sy + (1-s)x, \quad s \in [0, 1],$$

la paramétrisation du segment entre  $x$  et  $y$ . Par la convexité et le caractère borné de  $\Omega$ , l'inégalité de Cauchy-Schwarz et l'inégalité de Jensen, nous avons

$$\begin{aligned}
\int_{\Omega} \left| u - \frac{1}{|\Omega|} \int_{\Omega} u(y) dy \right|^2 dx &= \frac{1}{|\Omega|^2} \int_{\Omega} \left| \int_{\Omega} (u(x) - u(y)) dy \right|^2 dx \\
&= \frac{1}{|\Omega|^2} \int_{\Omega} \left| \int_{\Omega} \int_0^1 \langle \nabla u(\rho_{x,y}(s)), \rho'_{x,y}(s) \rangle ds dy \right|^2 dx \\
&\leq \frac{1}{|\Omega|^2} \int_{\Omega} \left| \int_{\Omega} \int_0^1 \|\nabla u(\rho_{x,y}(s))\| \|y - x\| ds dy \right|^2 dx \\
&\leq \frac{1}{|\Omega|^2} \int_{\Omega} \int_{\Omega} \int_0^1 \|\nabla u(\rho_{x,y}(s))\|^2 \|y - x\|^2 ds dy dx \\
&\leq \frac{\text{diam}(\Omega)^2}{|\Omega|^2} \int_{\Omega} \int_{\Omega} \int_0^1 \|\nabla u(\rho_{x,y}(s))\|^2 ds dy dx.
\end{aligned}$$

En utilisant le théorème de Fubini,

$$\begin{aligned}
&\frac{\text{diam}(\Omega)^2}{|\Omega|^2} \int_{\Omega} \int_{\Omega} \int_0^1 \|\nabla u(\rho_{x,y}(s))\|^2 ds dy dx \\
&= \frac{\text{diam}(\Omega)^2}{|\Omega|^2} \left( \int_{\Omega} \int_0^{\frac{1}{2}} \int_{\Omega} \|\nabla u(\rho_{x,y}(s))\|^2 dx ds dy + \int_{\Omega} \int_{\frac{1}{2}}^1 \int_{\Omega} \|\nabla u(\rho_{x,y}(s))\|^2 dy ds dx \right).
\end{aligned}$$

D'une part, en utilisant le changement de variable  $R_{s,y} : \Omega \rightarrow \Omega$  donné par  $R_{s,y}(x) = sy + (1-s)x$ , puisque  $\det(\nabla R_{s,y}) = (1-s)^d$ , pour  $d \geq 2$ , il vient

$$\begin{aligned}
\int_{\Omega} \int_0^{\frac{1}{2}} \int_{\Omega} \|\nabla u(\rho_{x,y}(s))\|^2 dx ds dy &= \int_{\Omega} \int_0^{\frac{1}{2}} \frac{1}{(1-s)^d} \int_{\Omega} \|\nabla u(\rho_{x,y}(s))\|^2 (1-s)^d dx ds dy \\
&= \int_{\Omega} \int_0^{\frac{1}{2}} \frac{1}{(1-s)^d} \int_{\Omega} \|\nabla u(z)\|^2 dz ds dy \\
&\leq \frac{2^{d-1}}{d-1} |\Omega| \int_{\Omega} \|\nabla u(z)\|^2 dz.
\end{aligned}$$

D'autre part, en utilisant le changement de variable  $R_{s,x} : \Omega \rightarrow \Omega$  donné par  $R_{s,x}(y) = sy + (1-s)x$ , puisque  $\det(\nabla R_{s,x}) = s^d$ , nous obtenons

$$\begin{aligned}
\int_{\Omega} \int_{\frac{1}{2}}^1 \int_{\Omega} \|\nabla u(\rho_{x,y}(s))\|^2 dy ds dx &= \int_{\Omega} \int_{\frac{1}{2}}^1 \frac{1}{s^d} \int_{\Omega} \|\nabla u(\rho_{x,y}(s))\|^2 s^d dy ds dx \\
&= \int_{\Omega} \int_{\frac{1}{2}}^1 \frac{1}{s^d} \int_{\Omega} \|\nabla u(z)\|^2 dz ds dx \\
&\leq \frac{2^{d-1}}{d-1} |\Omega| \int_{\Omega} \|\nabla u(z)\|^2 dz.
\end{aligned}$$

D'où finalement

$$\frac{\text{diam}(\Omega)^2}{|\Omega|^2} \int_{\Omega} \int_{\Omega} \int_0^1 \|\nabla u(\rho_{x,y}(s))\|^2 ds dy dx \leq \frac{2^d \text{diam}(\Omega)^2}{(d-1)|\Omega|} \int_{\Omega} \|\nabla u(z)\|^2 dz,$$

ce qui conclut la preuve.

Donc la constante  $C = \frac{2^d \text{diam}(\Omega)^2}{(d-1)|\Omega|}$  satisfait (2.1). Soulignons le fait que cette constante est finie car  $\Omega$  est borné. De plus, la preuve fait un usage crucial de la convexité. Il est possible de la généraliser aux ensembles ouverts bornés avec une frontière Lipschitz (nous renvoyons au livre complet [152] de V. Maz'ya). Elle a également été étendue à des ensembles encore

moins réguliers tels que les ensembles sous-analytiques (voir [212]). Nous nous intéressons à la constante optimale de l'inégalité de Poincaré, c'est-à-dire à la plus petite constante  $C$  satisfaisant (1.1). Nous désignons cette constante optimale par  $C_P(\Omega)$ , et nous l'appelons généralement "la" constante de Poincaré :

$$(1.2) \quad C_P(\Omega) := \sup_{u \in H^1(\Omega) \setminus \{0\}} \frac{\text{Var}_\Omega(u)}{\int_\Omega \|\nabla u(x)\|^2 dx}.$$

La constante  $C_P(\Omega)$  dépend fortement de la géométrie du domaine  $\Omega$ . Par exemple, si  $\Omega = \mathbb{R}^d$  ou si  $\Omega$  est borné mais a une frontière trop singulière, alors il n'existe pas de telle inégalité de Poincaré. Nous sommes naturellement amenés à étudier l'extension de la notion d'inégalités de Poincaré pour les mesures de probabilité. Soit  $\mu$  une mesure de probabilité sur un certain  $\mathbb{R}^d$  (la même chose fonctionne encore dans un contexte plus large, comme nous le verrons plus tard). Nous disons que  $\mu$  satisfait une inégalité de Poincaré s'il existe une constante  $C > 0$  telle que

$$(1.3) \quad \forall u \in H^1(\mu), \quad \text{Var}_\mu(u) \leq C \int \|\nabla u(x)\|^2 d\mu(x),$$

où, de manière similaire à la présentation ci-dessus des inégalités de Poincaré,  $\text{Var}_\mu$  désigne la variance par rapport à  $\mu$ ,

$$\text{Var}_\mu(u) := \int \left| u(x) - \int u(y) d\mu(y) \right|^2 d\mu(x),$$

et  $H^1(\mu)$  doit être considéré comme l'ensemble de toutes les fonctions  $u$  telles que les quantités impliquées aient un sens. Par exemple, une façon courante de le décrire est de le présenter comme étant l'ensemble de toutes les fonctions  $u \in L^2(\mu)$  telles que leur gradient au sens des distributions (qui est toujours défini) soit aussi dans  $L^2(\mu)$ . De même, nous considérerons la constante optimale de Poincaré que l'on définit comme la plus petite constante  $C > 0$  satisfaisant (1.3) et nous la désignerons par  $C_P(\mu)$ .

D'après les calculs ci-dessus, un exemple standard de mesures de probabilité satisfaisant une inégalité de Poincaré sont les mesures uniformes sur un domaine convexe borné ouvert de  $\mathbb{R}^d$ . En fait, de nombreuses mesures de probabilité classiques la satisfont. Citons par exemple la loi normale, toutes les lois gamma sur  $\mathbb{R}_+$ , y compris la loi exponentielle et la loi du chi-2 et toutes les lois bêta sur  $[0, 1]$ .

Une propriété importante de l'inégalité de Poincaré est la propriété de tensorisation. Si l'on prend  $\mu_1, \dots, \mu_n$ , des mesures de probabilité satisfaisant toutes l'inégalité de Poincaré avec des constantes optimales respectives  $C_P(\mu_i)$ , alors leur mesure produit  $\mu_1 \otimes \dots \otimes \mu_n$  satisfait une inégalité de Poincaré avec une constante optimale  $\max_i C_P(\mu_i)$ . Cela découle facilement de la formule de tensorisation de la variance (voir par exemple [7, Proposition 1.4.1])

$$\text{Var}_{\mu_1 \otimes \dots \otimes \mu_n}(f) \leq \sum_{i=1}^n \mathbb{E}_{\mu_1 \otimes \dots \otimes \mu_n} [\text{Var}_{\mu_i}(f)].$$

La propriété de tensorisation suggère que pour une mesure de probabilité multidimensionnelle, la constante de Poincaré ne dépend en fait que d'une seule direction dans l'espace. Pour une mesure produit, cette direction est bien une coordonnée classique, mais ce n'est pas le cas pour une mesure non produit. Cette heuristique nous a guidé dans les résultats présentés dans les chapitres 3 et 5, où la coordonnée spectrale joue le rôle de la direction préférée. Il faut alors trouver quelle est cette direction pour comprendre le comportement de l'inégalité de Poincaré. Il est assez clair qu'en tant que fonction réelle, cette direction devrait être une fonction saturant l'inégalité, c'est-à-dire réalisant l'égalité dans l'inégalité de Poincaré. Nous verrons dans la section 1.1.2 que ces fonctions sont des fonctions propres d'un certain opérateur.

Dans les trois sections suivantes, nous présenterons certaines des principales conséquences des inégalités de Poincaré de notre point de vue combiné d'Analyse, de Géométrie et de Probabilité : le phénomène de concentration de mesure, l'existence d'un trou spectral, et une borne sur le profil isopérimétrique.

### 1.1.1 Concentration exponentielle de la mesure

L'inégalité de Poincaré implique un phénomène de concentration exponentielle de la mesure. Cette propriété quantifie à quel point la masse d'une mesure  $\mu$  est située autour de sa moyenne. L'étude de ce phénomène remonte à P. Lévy, M. Gromov et V. Milman [141, 106, 107, 105]. Nous renvoyons le lecteur au manuel [137] de M. Ledoux.

**Théorème.** (e.g. [137, Théorème 3.1]) *Si la distribution de probabilité  $\mu$  satisfait l'inégalité de Poincaré (1.3) avec une constante optimale  $C_P(\mu)$ , alors pour toutes les fonctions Lipschitz  $f$  et pour tous les  $t \geq 0$ ,*

$$(1.4) \quad \mu(|f - \mu(f)| \geq t) \leq 8 \exp\left(-t \sqrt{\frac{2}{C_P(\mu)}}\right).$$

Esquissons une façon dont ce résultat de concentration peut être prouvé. Désignons par  $C = C_P(\mu)$  la constante de Poincaré optimale, et appliquons l'inégalité de Poincaré (1.3) à la fonction  $u = \exp\left(\frac{1}{2}sf\right)$ . On obtient

$$\int e^{sf} d\mu - \left(\int e^{\frac{1}{2}sf} d\mu\right)^2 = \text{Var}_\mu(u) \leq C \int \left\| \nabla \exp\left(\frac{1}{2}sf\right) \right\|^2 d\mu \leq C \frac{s^2}{4} \int e^{sf} d\mu,$$

où dans la dernière inégalité nous avons utilisé que  $\|\nabla f\| \leq 1$  parce que  $f$  est supposée être 1-Lipschitz. Il s'ensuit que

$$(1.5) \quad \int e^{sf} d\mu \leq \left(1 - \frac{Cs^2}{4}\right)^{-1} \left(\int \exp\left(\frac{1}{2}sf\right) d\mu\right)^2.$$

On itère l'inégalité (1.5) avec  $\frac{s}{2}, \frac{s}{4}, \dots, \frac{s}{2^n}, \dots$ . Par une étude classique de convergence de série, on voit qu'il est possible de prendre la limite en  $n \rightarrow +\infty$ , et l'on obtient alors

$$(1.6) \quad \forall s \in \left[0, \sqrt{\frac{2}{C}}\right], \quad \int \exp(s(f - \mu(f))) d\mu \leq 2 \left(1 - \frac{Cs^2}{4}\right)^{-1}.$$

Enfin, nous avons, en utilisant l'inégalité de Markov et l'inégalité (1.6), pour tout  $s \in \left[0, \sqrt{\frac{2}{C}}\right]$ ,

$$\begin{aligned} \mu(|f - \mu(f)| \geq t) &= \mu\left(\exp(s|f - \mu(f)|) \geq e^{st}\right) \\ &\leq \frac{1}{e^{st}} \int \exp(s|f - \mu(f)|) d\mu \\ &\leq 2 \left(1 - \frac{Cs^2}{4}\right)^{-1} e^{-st} \end{aligned}$$

d'où nous pouvons alors conclure en prenant  $s = \sqrt{\frac{2}{C}}$ .

En particulier, cette propriété de concentration exponentielle nous renseigne sur les lois de probabilité ne satisfaisant pas à une inégalité de Poincaré. Par exemple, les mesures de probabilité à queue lourde ne satisfont pas l'inégalité de Poincaré. Un exemple en est la loi de Cauchy sur  $\mathbb{R}$  donnée par la densité  $\frac{1}{\pi} \frac{1}{1+x^2}$ . Signalons enfin que ce résultat de concentration exponentielle montre que, en un certain sens, les mesures à queue exponentielle (telles que la mesure exponentielle sur  $\mathbb{R}_+$  ou encore la mesure logistique) sont des points extrêmes parmi l'ensemble des mesures satisfaisant l'inégalité de Poincaré.

### 1.1.2 Trou spectral

Écrivons

$$(1.7) \quad d\mu(x) = e^{-V(x)} dx,$$

avec  $-V$  désignant le logarithme de la densité. Pour l'instant, nous pouvons considérer  $dx$  comme étant la mesure de Lebesgue sur  $\mathbb{R}^d$ , mais la forme (1.7) a toujours un sens dans le contexte plus général des variétés riemanniennes à poids ( $dx$  désignerait alors la mesure volume), et aussi dans le cadre RCD. Nous discutons de cette question un peu plus bas.

D'un point de vue probabiliste, nous savons que l'opérateur de Laplace est lié à la mesure de Lebesgue de la manière suivante. Soit  $\Omega$  un domaine borné de  $\mathbb{R}^d$ . Le caractère borné permet de définir une probabilité uniforme sur  $\Omega$  en y restreignant la mesure de Lebesgue. Alors toutes les fonctions  $C^1(\Omega)$  satisfaisant la condition de bord de Neumann  $\nabla f \cdot n = 0$  sur  $\partial\Omega$ , ont un Laplacien de moyenne nulle par rapport à la mesure de Lebesgue. En effet, par le théorème de la divergence (utilisé dans la deuxième égalité),

$$\int_{\Omega} \Delta f \, dx = \int_{\Omega} \operatorname{div} \nabla f \, dx = \int_{\partial\Omega} \nabla f \cdot n \, d\sigma = 0,$$

avec  $n$  la normale extérieure.

De même, puisque  $\mu$  a une densité par rapport à la mesure de Lebesgue, il existe un opérateur elliptique jouant le même rôle pour  $\mu$  que le Laplacien pour  $dx$ . Cet opérateur elliptique est le Laplacien auquel on ajoute une dérive égale au gradient de la log-densité de  $\mu$  :

$$(1.8) \quad L = \Delta - \nabla V \cdot \nabla$$

En effet, en utilisant à nouveau le théorème de la divergence, (à la quatrième égalité),

$$\begin{aligned} \int_{\Omega} Lf \, d\mu &= \int_{\Omega} (\Delta f - \nabla V \cdot \nabla f) e^{-V} \, dx \\ &= \int_{\Omega} (e^{-V} \operatorname{div} \nabla f + \nabla e^{-V} \cdot \nabla f) \, dx \\ &= \int_{\Omega} \operatorname{div} (e^{-V} \nabla f) \, dx \\ &= \int_{\partial\Omega} (e^{-V} \nabla f) \cdot n \, d\sigma \\ &= \int_{\partial\Omega} \nabla f \cdot n \, d\mu|_{\partial\Omega}. \end{aligned}$$

Lorsque  $\Omega = \mathbb{R}^d$ , le terme de bord n'apparaît pas, par conséquent cette propriété est vraie pour  $f$  dès que  $Lf$  est  $\mu$ -intégrable, c'est-à-dire pour les fonctions qui ne croissent pas trop vite à l'infini. Il se trouve qu'une propriété plus forte est également vérifiée. En effet, cet opérateur  $L$  est symétrique sur  $L^2(\mu)$ , c'est-à-dire que si l'on désigne par  $\mathcal{D} \subset L^2(\mu)$  le domaine de  $L$ ,

$$\forall f, g \in \mathcal{D}, \quad \int f Lg \, d\mu = \int g Lf \, d\mu.$$

Cette propriété est souvent appelée réversibilité de  $\mu$  par rapport à  $L$ . Cette terminologie provient de la théorie des processus de Markov. Une dernière propriété qui mérite d'être mentionnée dans cette section introductive est la formule suivante :

$$(1.9) \quad \int f Lf \, d\mu = - \int \|\nabla f\|^2 \, d\mu.$$

Cette égalité sera souvent désignée comme étant une formule d'intégration par parties dans la suite. En particulier, elle implique que l'opérateur  $-L$  est semi-défini positif dans  $L^2(\mu)$ . Donc toutes ses valeurs propres seront positives ou nulles.



En plus d'avoir une forme particulièrement agréable, l'opérateur elliptique  $L$  peut aussi bien être étudié afin de déduire des propriétés sur la mesure  $\mu$ , qu'inversement l'on peut étudier la mesure  $\mu$  afin d'en déduire des propriétés sur cet opérateur  $L$ . Pour en revenir aux inégalités de Poincaré, nous avons maintenant suffisamment de matière pour comprendre le lien entre l'inégalité de Poincaré pour  $\mu$  et la répartition des valeurs propres de  $L$ . Nous savons déjà que toutes les valeurs propres de  $-L$  sont positives ou nulles, mais sous une inégalité de Poincaré (1.3) ceci est renforcé : les valeurs propres sont dans  $\{0\} \cup [\frac{1}{C_P(\mu)}, +\infty)$ . De plus,  $\frac{1}{C_P(\mu)}$  est le plus petit nombre satisfaisant cette propriété, nous pouvons donc dire que  $\frac{1}{C_P(\mu)}$  est le trou spectral de l'opérateur  $-L$ . Soulignons que dans la suite nous parlons souvent de trou spectral de  $L$  ou de  $\mu$  par abus de langage. La raison pour laquelle l'inégalité de Poincaré implique un trou dans le spectre est simple. Soit  $\lambda$  une valeur propre de  $-L$  et  $f$  une fonction propre associée que nous choisissons centrée. En évaluant l'inégalité de Poincaré (1.3) en  $f$  et en utilisant la formule d'intégration par parties (1.9), nous obtenons

$$\int f^2 d\mu = \text{Var}_\mu(f) \leq C_P(\mu) \int \|\nabla f\|^2 d\mu = -C_P(\mu) \int f L f d\mu = C_P(\mu) \lambda \int f^2 d\mu,$$

d'où nous déduisons effectivement que si  $\lambda \neq 0$  alors

$$\lambda \geq \frac{1}{C_P(\mu)}.$$

On peut se demander quelles sont les conditions de bord requises lorsque justement il y a un bord. La réponse consiste en la description de l'ensemble  $H^1$  sur lequel se vérifie l'inégalité de Poincaré. Pour satisfaire l'inégalité, les fonctions doivent avoir leurs dérivées normales égales à zéro à la frontière. Nous avons déjà vu ces conditions en utilisant le théorème de la divergence dans les calculs ci-dessus. Par conséquent, les conditions aux bord correspondantes pour l'opérateur  $L$  sont des conditions de Neumann. Nous verrons plus tard que ces conditions sont nécessaires pour que le processus sous-jacent soit markovien. D'un point de vue physique, ceci correspond à la réflexion orthogonale des ondes de vibration à la frontière.

La signification spectrale de l'inégalité de Poincaré peut être explicitée par le théorème min-max pour les opérateurs auto-adjoints. En effet, après avoir défini l'opérateur auto-adjoint  $L$  dans  $L^2(\mu)$ , le théorème min-max nous dit que les minimisations sur des espaces appropriés du quotient de Rayleigh

$$\frac{\int \|\nabla f\|^2 d\mu}{\int f^2 d\mu}$$

donnent le spectre discret de  $-L$ . En gardant ce théorème min-max à l'esprit, il apparaît clairement que par définition (1.2), la constante de Poincaré optimale coïncide avec l'inverse de la première valeur propre après zéro de  $-L$ , chaque fois que son spectre discret n'est pas réduit à zéro. Lorsque c'est le cas, toutes les fonctions propres associées à cette première valeur propre ont une énergie minimale et réalisent donc le cas d'égalité dans l'inégalité de Poincaré. Dans toute cette thèse, nous utiliserons cette approche variationnelle des valeurs propres afin d'étudier leur stabilité. De plus, nous utiliserons cette caractérisation des fonctions propres comme égalisateurs de l'inégalité de Poincaré afin de définir les espaces propres de manière plus générale dans le chapitre 4. Soulignons que par "la première valeur propre", nous entendons "la première valeur propre après zéro", juste pour alléger certaines formulations. L'interprétation spectrale des inégalités de Poincaré est très importante, d'ailleurs les inégalités de Poincaré sont souvent appelées inégalités de trou spectral. Lorsque  $\mu$  est la mesure volume d'une variété compacte,  $L$  est alors l'opérateur de Laplace-Beltrami. L'opérateur de Laplace-Beltrami sur une variété compacte a un spectre discret et le problème de la compréhension du comportement de sa première valeur propre a été étudié depuis très longtemps en géométrie spectrale, et remonte au moins pour les domaines dans  $\mathbb{R}^d$  à J. Rayleigh en 1896 (voir [200]). Citons également la célèbre question "Peut-on entendre la forme d'un tambour ?" posée en 1966 par M. Kac dans [121]. D'un point de vue physique, une peau de tambour ne peut vibrer que pour un

ensemble discret de fréquences. Modélisées par un domaine dans le plan, ces fréquences de la peau de tambour correspondent exactement aux valeurs propres de l'opérateur de Laplace restreint à ce domaine avec des conditions de bord de Dirichlet, c'est-à-dire que les fonctions doivent s'annuler à la limite du domaine. Les conditions de Dirichlet correspondent au fait que la peau de tambour est fixée au tambour et ne peut donc pas bouger lorsqu'elle vibre. Ces fréquences de vibration sont évidemment déterminées par la "forme" (c'est-à-dire la géométrie) du domaine. Inversement, M. Kac s'est demandé si la forme est déterminée par l'ensemble des fréquences de vibration. En langage rigoureux, cela revient à demander si les domaines isospectraux (c'est-à-dire les domaines ayant le même spectre pour le Laplacien de Dirichlet) sont isométriques ? Si l'on ne se restreint pas à la dimension 2, la question a été rapidement résolue par J. Milnor [164] qui a construit deux tores plats non isométriques de dimension 16 partageant le même spectre. En dimension 2, le problème a finalement été résolu négativement par C. Gordon, D. Webb et S. Wolpert en 1992 dans [101, 102]. Il existe différentes formes dans le plan qui partagent le même spectre. La figure la plus célèbre est la figure 1.1. Pour

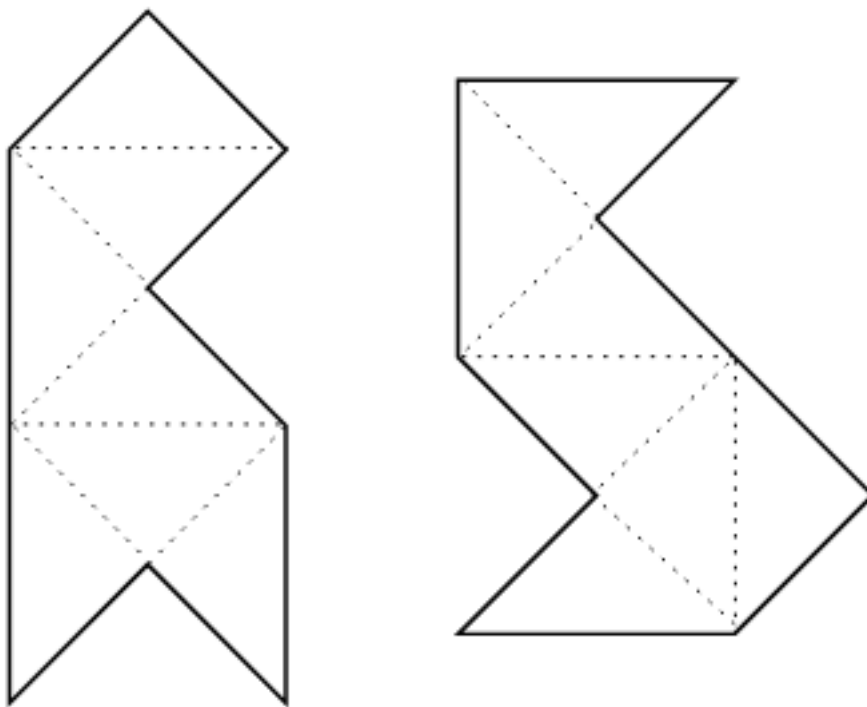


Figure 1.1: Deux domaines du plan isospectraux mais non isométriques

plus de détails sur l'histoire de ce problème, nous renvoyons le lecteur à l'étude [99].

Un autre résultat classique en géométrie concernant la première valeur propre de l'opérateur de Laplace Beltrami est le théorème suivant, de A. Lichnerowicz [142].

**Théorème.** (Lichnerowicz [142]) *Si  $(M, g)$  est une variété riemannienne sans bord de dimension  $d$  ayant une courbure de Ricci minorée par  $d-1$ , alors la première valeur propre non nulle de son opérateur de Laplace-Beltrami est supérieure ou égale à  $d$ .*

Nous présenterons cette notion d'espaces à courbure positive dans la section 1.3.

### 1.1.3 L'inégalité isopérimétrique de Cheeger

Discutons brièvement des inégalités isopérimétriques de Cheeger. Nous traiterons des inégalités isopérimétriques de manière plus détaillée dans la deuxième partie de cette thèse.

On dit qu'une mesure de probabilité  $\mu$  satisfait l'inégalité isopérimétrique de Cheeger s'il existe une certaine constante  $h > 0$  telle que pour tout sous-ensemble mesurable  $A$ ,

$$\text{Per}_\mu(A) \geq h \min(\mu(A), 1 - \mu(A)),$$

où

$$\text{Per}_\mu(A) := \liminf_{\varepsilon \rightarrow 0_+} \frac{\mu(A_\varepsilon \setminus A)}{\varepsilon}$$

est le périmètre pondéré par  $\mu$ , et  $A_\varepsilon := \{x \mid d(x, A) < \varepsilon\}$ . La constante de Cheeger optimale est notée  $h_\mu$  et est donnée par

$$h_\mu := \inf_A \frac{\text{Per}_\mu(A)}{\min(\mu(A), 1 - \mu(A))}.$$

La constante de Cheeger  $h_\mu$  donne des informations importantes sur la géométrie de l'espace muni de la distribution de probabilité  $\mu$ . Pour un aperçu de cette notion, nous renvoyons le lecteur à [140].

L'inégalité de Poincaré et l'inégalité de Cheeger sont profondément liées. L'une des principales connexions a été établie en 1970 par J. Cheeger lui-même dans [64].

**Théorème.** ([64, Théorème 1])

$$\frac{1}{C_P(\mu)} \geq \frac{h_\mu^2}{4}$$

Nous avons vu à la section précédente 1.1.2 que la quantité  $\frac{1}{C_P(\mu)}$  représente le trou spectral de la mesure  $\mu$ . Dans le cas où ce trou spectral est atteint, c'est-à-dire lorsque  $\frac{1}{C_P(\mu)}$  est effectivement la première valeur propre non nulle  $\lambda_1(\mu)$ , cela donne  $\lambda_1(\mu) \geq \frac{h_\mu^2}{4}$ . Il s'agit du résultat original énoncé par Cheeger dans [64] lorsque  $\mu$  est la mesure de probabilité uniforme sur une variété riemannienne compacte. Alors que la preuve de Cheeger est plutôt de teneur géométrique, M. Ledoux en a donné une plus analytique dans [135] en utilisant la caractérisation des inégalités isopérimétriques de Cheeger en termes d'inégalités de Poincaré  $L^1$ .

### 1.1.4 Un critère classique

Nous avons vu trois des principales conséquences de l'inégalité de Poincaré. Parmi celles-ci, le phénomène de concentration de la mesure nous a donné des exemples de mesures de probabilité qui ne satisfont pas l'inégalité de Poincaré. Un problème naturel est alors de dériver des conditions suffisantes pour qu'une mesure  $\mu$  satisfasse une inégalité de Poincaré. Nous présentons ici l'un des critères les plus utiles, le critère de Bakry-Émery, également appelé condition de courbure-dimension, sur lequel nous reviendrons dans la section 1.3 de cette introduction. Ce critère a été introduit pour la première fois par D. Bakry et M. Émery dans l'article fondateur [12]. Soit  $\mu$  une mesure de probabilité ayant une densité  $e^{-V(x)} dx$  et soit  $L$  l'opérateur

$$L = \Delta - \nabla V \cdot \nabla$$

comme vu à la section 1.1.2 de cette introduction. Si l'inégalité suivante entre 2-tenseurs

$$(1.10) \quad \text{Hess } V + \text{Ric} \geq Kg$$

est ponctuellement satisfaite au sens des formes quadratiques, où Ric désigne le tenseur de Ricci,  $g$  désigne le tenseur métrique, et  $K$  est une constante strictement positive, alors la mesure  $\mu = e^{-V} dx$  satisfait une inégalité de Poincaré avec la constante  $0 < C_P(\mu) \leq \frac{1}{K}$ . Soulignons que dans  $\mathbb{R}^d$ , ce critère se réduit à

$$\text{Hess } V \geq KI_d$$

où  $I_d$  désigne la matrice identité. Par conséquent, dans  $\mathbb{R}^d$ , la condition de Bakry-Émery nous dit que pour un potentiel uniformément convexe  $V$ , la mesure  $e^{-V} dx$  satisfait une inégalité de Poincaré. L'exemple typique est la loi normale. En effet,

$$d\gamma(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{\|x\|^2}{2}} dx$$

satisfait

$$\text{Hess } V = I_d.$$

Par conséquent, la condition de Bakry-Émery nous indique que la loi normale satisfait une inégalité de Poincaré avec la constante  $C_P(\gamma) \leq 1$ . De plus, en testant l'inégalité pour les fonctions affines, nous pouvons voir que 1 est en fait la constante optimale de la loi Normale, et donc dans ce cas le critère de Bakry-Émery donne la meilleure constante.

Le critère de Bakry-Emery peut être déduit de l'inégalité de Brascamp-Lieb. L'inégalité de Brascamp-Lieb [42] dans  $\mathbb{R}^d$  dit que si le potentiel  $V$  est strictement convexe (c'est-à-dire que  $\text{Hess } V > 0$ ), alors la distribution  $\mu = e^{-V(x)} dx$  satisfait à

$$(1.11) \quad \forall f \in H^1(\mu), \quad \text{Var}_\mu(f) \leq \int_{\mathbb{R}^d} (\text{Hess } V)^{-1} (\nabla f, \nabla f) d\mu.$$

De plus, les fonctions  $f$  atteignant l'égalité sont toutes des fonctions de la forme  $\langle \nabla V, c \rangle$  avec une constante  $c \in \mathbb{R}$ . Il est alors immédiat de déduire l'inégalité de Poincaré à partir de l'inégalité (1.11) lorsque le potentiel est uniformément convexe (i.e.  $\text{Hess } V \geq K I_d$  pour un certain  $K > 0$ ).

Mentionnons que l'inégalité de Brascamp-Lieb a été étendue au cadre riemannien pondéré par A. Kolesnikov et E. Milman dans [130]. Cependant, dans ce cadre, les fonctions qui atteignent l'égalité sont inconnues.

Terminons cette section en citant les travaux de T. Bodineau et R. Bauerschmidt [24] qui ont généralisé le critère de Bakry-Émery au cas des mesures non log-concaves en utilisant une méthode de renormalisation. Leur critère s'applique en particulier aux modèles issus de la physique statistique.

### 1.1.5 Inégalités de Poincaré améliorées

Dans cette section, nous décrivons une façon d'améliorer les inégalités de Poincaré en exigeant une condition d'orthogonalité sur les fonctions test. Nous ne faisons qu'esquisser ici la philosophie des inégalités de Poincaré améliorées, les définitions formelles peuvent être trouvées dans le chapitre 4. Soit  $\mu$  une mesure de probabilité sur  $\mathbb{R}^d$  satisfaisant l'inégalité de Poincaré (1.1), et soit  $Sp_1(\mu)$  l'ensemble de toutes les fonctions  $f_1$  atteignant l'égalité dans (1.1). Nous avons alors que pour toutes les fonctions  $f \in H^1(\mu)$  orthogonales à  $Sp_1(\mu)$  dans  $L^2(\mu)$ ,

$$(1.12) \quad \int f^2 d\mu \leq C_P^2(\mu) \int |\nabla f|^2 d\mu,$$

avec  $0 < C_P^2(\mu) < C_P(\mu)$ . L'inégalité (1.12) est l'inégalité de Poincaré améliorée d'ordre 2. On peut alors itérer cette procédure, et définir  $Sp_{k+1}(\mu)$  comme l'ensemble de toutes les fonctions  $f_{k+1} \in H^1(\mu) \cap (Sp_1(\mu) \oplus \dots \oplus Sp_k(\mu))^\perp$  atteignant l'égalité dans l'inégalité de Poincaré améliorée d'ordre  $k$ . On obtient alors que pour toutes les fonctions  $f \in H^1(\mu) \cap (Sp_1(\mu) \oplus \dots \oplus Sp_{k+1}(\mu))^\perp$ ,

$$(1.13) \quad \int f^2 d\mu \leq C_P^{k+2}(\mu) \int |\nabla f|^2 d\mu,$$

avec  $0 < C_P^{k+2}(\mu) < C_P^{k+1}(\mu) < \dots < C_P(\mu)$ . L'interprétation spectrale de l'inégalité de Poincaré nous permet de voir que lorsque  $\mu$  est la mesure réversible d'un générateur de diffusion  $L$ , alors les espaces  $Sp_k(\mu)$  sont les espaces propres de l'opérateur  $L$  avec les valeurs propres associées  $\frac{-1}{C_P^k(\mu)}$ . Par conséquent, les inégalités de Poincaré améliorées sont un moyen de coder le spectre entier de l'opérateur  $L$  par des inégalités fonctionnelles.

## 1.2 La méthode de Stein

La méthode de Stein est un ensemble de techniques largement développées à partir de l'article fondateur [197] de C. Stein publié en 1972. Nous renvoyons le lecteur aux résumés [63, 185, 66]. Le but de ces techniques est de quantifier la distance entre deux mesures de probabilité. Elle a d'abord été introduite pour la loi normale unidimensionnelle, permettant de saisir des taux de convergence effectifs dans le théorème de la limite centrale, largement connus sous le nom de bornes de Berry-Esséen (voir par exemple [94, 69, 68, 170]). Décrivons brièvement les principales idées de la méthode.

L'observation clef est que la loi normale  $\gamma$  sur  $\mathbb{R}$  est la seule à satisfaire la formule d'intégration par parties suivante

$$(1.14) \quad \forall f \in C^1(\mathbb{R}), \quad \int f'(x) d\gamma(x) = \int x f(x) d\gamma(x).$$

Par conséquent, on peut avoir l'intuition que si une mesure de probabilité  $\nu$  satisfait presque cette formule d'intégration par parties (1.14), alors *dans un certain sens*, elle devrait être proche de la loi normale. Pour rendre l'intuition rigoureuse, on doit donc essayer de lier le terme d'erreur mesurant à quel point une loi de probabilité  $\nu$  satisfait la formule (1.14) avec une notion plus usuelle quantifiant la différence (ou la distance) entre deux mesures de probabilité.

D'une part, le meilleur terme que nous puissions trouver pour caractériser l'erreur dans l'intégration par parties pour  $\nu$  est évidemment

$$(1.15) \quad \sup_{f \in \mathcal{F}} \left| \int_{\mathbb{R}} f'(x) - x f(x) d\nu \right|$$

où le choix de l'ensemble fonctionnel  $\mathcal{F}$  reste à déterminer. Cet ensemble ne doit pas être choisi trop petit afin de fournir des informations pertinentes sur la formule d'intégration par parties, mais il ne doit pas non plus être trop grand afin d'être utilisable en pratique.

D'autre part, deux lois  $\nu$  et  $\beta$  sont égales si pour toute fonction test  $g$ , elles donnent la même valeur  $\nu(g) = \beta(g)$ . Une façon naturelle de mesurer la différence entre deux mesures est donc d'évaluer leurs différences sur un ensemble de fonctions tests :

$$(1.16) \quad \sup_{g \in \mathcal{G}} |\nu(g) - \beta(g)| = \sup_{g \in \mathcal{G}} \left| \int g d\nu - \int g d\beta \right|.$$

Cette quantité est très courante dans la littérature, et satisfait des propriétés très utiles dès lors que l'ensemble fonctionnel  $\mathcal{G}$  est bien choisi. Par exemple,

- pour  $\mathcal{G} = \{\mathbf{1}_{(-\infty, t)} \mid t \in \mathbb{R}\}$ , elle correspond à la distance de Kolmogorov : c'est la norme  $L^\infty$  entre les fonctions de répartition,
- pour  $\mathcal{G} = \{\mathbf{1}_A \mid A \subset \mathbb{R} \text{ mesurable}\}$ , c'est la distance de variation totale,
- pour  $\mathcal{G} = \{g \in C^1(\mathbb{R}) \mid \|g'\|_\infty \leq 1\}$ , c'est la distance de Wasserstein-1, qui est liée au transport optimal par la formule de Kantorovich-Rubinstein (voir [213, Chapitre 5]),
- pour  $\mathcal{G} = \{g \in C^k(\mathbb{R}) \mid \|g^{(k)}\|_\infty \leq 1\}$ , elle correspond à la  $k$ -ième distance de Zolotarev qui contrôle la même topologie que la distance de Wasserstein- $k$  (voir [26]) qui est elle-même équivalente à la convergence faible et à la convergence des  $k$ -ième premiers moments.

L'intuition serait rendue rigoureuse si nous montrions que la quantité (1.15) contrôle certaines des quantités classiques (1.16). Pour atteindre ce but, une stratégie consiste à résoudre l'équation différentielle ordinaire suivante pour  $g \in \mathcal{G}$ ,

$$(1.17) \quad f'(x) - x f(x) = g(x) - \int g d\gamma$$

et trouver des solutions  $f$  bornées et ayant une certaine régularité. L'ensemble fonctionnel  $\mathcal{F}$  sera alors choisi comme l'ensemble des fonctions satisfaisant ces propriétés de régularité. C'est cette idée qui a été introduite pour la première fois par C. Stein dans [185].

Dans le cas de la distribution normale et de la distance Wasserstein-1, ce problème est résolu par le célèbre lemme de Stein.

**Lemme 1.** (*Lemme de Stein*) Si  $g : \mathbb{R} \rightarrow \mathbb{R}$  est absolument continue et centrée pour la loi normale  $\gamma$ , alors on peut trouver un solution  $f$  de (1.17) vérifiant

$$(1.18) \quad \|f\|_\infty \leq 2\|g'\|_\infty, \quad \|f'\|_\infty \leq \sqrt{\frac{2}{\pi}}\|g'\|_\infty, \quad \text{and} \quad \|f''\|_\infty \leq 2\|g'\|_\infty.$$

Désignons par  $\mathcal{F}_s$  l'ensemble de toutes les fonctions de  $C^2(\mathbb{R})$  satisfaisant à

$$\|f\|_\infty \leq 2, \quad \|f'\|_\infty \leq \sqrt{\frac{2}{\pi}}, \quad \text{and} \quad \|f''\|_\infty \leq 2.$$

Toutes les étapes du raisonnement se combinent donc et donnent

$$(1.19) \quad W_1(\nu, \gamma) = \sup_{\|g'\| \leq 1} \left| \int \left( g - \int g d\gamma \right) d\nu \right| \leq \sup_{f \in \mathcal{F}_s} \left| \int f'(x) - xf(x) d\nu \right|.$$

Il s'agit d'une expression quantitative du fait que si une mesure de probabilité  $\nu$  satisfait presque à la formule d'intégration par parties (1.14), alors elle est proche de la loi normale en distance Wasserstein-1. On peut également faire de même pour les distances classiques ci-dessus.

Disons un mot sur la preuve du lemme de Stein. L'équation (1.17) est une équation différentielle ordinaire du premier ordre. Nous savons que l'ensemble des solutions est un espace affine de dimension un, et nous voulons simplement trouver une fonction dans cet espace qui satisfait les conditions de régularité du lemme de Stein. Comme enseigné aux étudiants de premier cycle, la solution générale est donnée par

$$e^{\frac{x^2}{2}} \int_x^{+\infty} \left( g(x) - \int g d\gamma \right) e^{-\frac{x^2}{2}} dx + K e^{\frac{x^2}{2}}, \quad K \in \mathbb{R}.$$

Il est clair que pour  $K \neq 0$  la solution est non bornée. Par conséquent, prouver le lemme de Stein revient à prouver que

$$(1.20) \quad f_g(x) := e^{\frac{x^2}{2}} \int_x^{+\infty} \left( g(x) - \int g d\gamma \right) e^{-\frac{x^2}{2}} dx$$

est la seule solution de (1.17) satisfaisant les conditions (1.18). Puisque  $f_g$  est explicite, on peut effectuer des calculs explicites afin de démontrer (1.18). De tels calculs sont par exemple présentés en détail par L. Chen, L. Goldstein et Q. Shao dans [66, Appendix p.37]. Comme on peut le deviner à partir de la formule (1.20), ces calculs sont basés sur le contrôle de la fonction de répartition de la loi normale. Dans la section 3.6 du chapitre 3, nous généraliserons cette approche à une classe plus large de mesures de probabilité.

Il s'avère que la solution donnée par (1.20) possède une interprétation probabiliste. Cette interprétation est la pierre angulaire de l'approche de Barbour de la méthode de Stein. Afin de l'expliquer, nous allons faire un détour par la théorie des processus de Markov.

Soit  $(X_t)_{t \geq 0}$  un processus aléatoire vivant sur un certain espace fixé. Pour simplifier, nous pouvons considérer que cet espace est  $\mathbb{R}$ . Tout espace étant caractérisé par l'ensemble des façons possibles de le transformer, une façon d'étudier le processus aléatoire est d'étudier les espérances des fonctions réelles le long des trajectoires de ce processus. Soit étudier

$$P_t f(x) := \mathbb{E}[f(X_t) | X_0 = x].$$

Lorsque le futur du processus  $(X_t)_t$  ne dépend que de son présent (et non de son passé), on dit que  $(X_t)_t$  est markovien. Dans un langage mathématique rigoureux, cela correspond aux équations de Chapman-Kolmogorov :

$$(1.21) \quad \forall t, s \geq 0, \quad P_{t+s} = P_t \circ P_s.$$

Les équations (1.21) disent que les opérateurs  $(P_t)_t$  ont la propriété de semigroupe. Un exemple simple d'une famille d'opérateurs satisfaisant cette propriété est  $(e^{tL})_{t \geq 0}$  lorsque  $L$  est un opérateur tel que toutes les exponentielles précédentes soient bien définies. Un fait important est que cet exemple est générique : la théorie de Hille-Yosida stipule que sous certaines conditions naturelles sur le semigroupe, il existe un opérateur  $L$  défini sur un espace fonctionnel suffisamment grand  $\mathcal{D}$ , et tel que, au moins formellement, (voir [217, Chapitre IX])

$$P_t = e^{tL}.$$

L'opérateur  $L$  est appelé le générateur infinitésimal (ou simplement générateur) du processus, et est donné pour toute fonction  $f$  dans son domaine  $\mathcal{D}$ , par

$$Lf = \lim_{s \rightarrow 0} \frac{P_s f - f}{s}.$$

L'approche par générateur des processus de Markov constitue un point de vue analytique de la théorie des processus aléatoires. Passons maintenant au point de vue probabiliste de cette théorie. Un problème classique dans la théorie des processus aléatoires est la question de l'existence (et de l'unicité) de la loi d'équilibre, aussi appelée mesure invariante. Une loi  $\mu$  est dite invariante par rapport à un processus aléatoire  $(X_t)_t$  si le tirage de  $X_0$  avec une distribution  $\mu$  a pour résultat que tous les  $X_t$  suivent la loi  $\mu$ . Ceci est équivalent à

$$\forall f \in \mathcal{D}, \quad \int Lf d\mu = 0.$$

Présentons un moyen de situer la loi normale dans cette théorie générale. Il existe un processus de Markov simple  $(X_t)_{t \geq 0}$  sur  $\mathbb{R}$  dont la loi d'équilibre est la gaussienne  $\gamma$ . Ce processus de Markov est largement connu sous le nom de processus d'Ornstein-Uhlenbeck en référence à l'article de L. Ornstein et G. Uhlenbeck [210]. Ce processus est la solution de l'équation différentielle stochastique suivante

$$dX_t = -X_t dt + dB_t,$$

où  $B_t$  désigne le mouvement brownien, et son générateur infinitésimal est

$$(1.22) \quad Lf(x) = f''(x) - x f'(x).$$

La similitude avec l'équation de Stein (1.17) est claire : le générateur d'Ornstein-Uhlenbeck est le même que le côté gauche de l'équation de Stein appliqué à  $f'$  au lieu de  $f$ . Par conséquent, les solutions de l'équation de Stein (1.17) sont les dérivées premières des solutions de l'équation suivante

$$(1.23) \quad Lf = g - \int g d\gamma.$$

L'équation (1.23) est connue comme étant l'équation de Poisson du processus de Markov généré par  $L$  et de mesure invariante  $\gamma$ . La terminologie vient du fait que, dans le cas du mouvement brownien (qui est l'un des processus de Markov les plus naturels), le générateur est le laplacien, de sorte que l'équation (1.23) est alors l'équation de Poisson classique. Pour de telles équations de Poisson dans un contexte markovien, nous savons que (au moins formellement),

$$(1.24) \quad f(x) = - \int_0^{+\infty} \left( P_t g(x) - \int g d\gamma \right) dt$$

est solution de (1.23). En effet, formellement, nous avons

$$\begin{aligned}
-Lf &= -L \int_0^{+\infty} \left( P_t g(x) - \int g d\gamma \right) dt \\
&= - \int_0^{+\infty} L P_t g(x) dt \\
&= - \int_0^{+\infty} \frac{d}{ds} \Big|_{s=t} P_t g(x) dt \\
&= - [P_t g]_0^{+\infty} \\
&= P_0 g - P_\infty g \\
&= g - \int g d\gamma.
\end{aligned}$$

Nous avons maintenant abordé suffisamment de notions en théorie de Markov pour pouvoir présenter l'interprétation probabiliste de la solution (1.20) de l'équation de Stein. Cette solution (1.20) est la dérivée de la solution (1.24) de l'équation de Poisson associée au processus d'Ornstein-Uhlenbeck. Cette remarque peut sembler mineure, mais elle a en fait des conséquences plus profondes. En effet, la formule (1.24) peut être écrite pour tout semigroupe de Markov  $(P_t)_t$  ayant un générateur  $L$  et une mesure invariante  $\mu$ , et on a toujours, au moins formellement, que

$$f_g(x) = - \int_0^{+\infty} \left( P_t g(x) - \int g d\mu \right) dt$$

est solution de l'équation de Poisson

$$Lf = g - \int g d\mu,$$

et de plus,

$$(1.25) \quad \forall f \in \mathcal{D}, \quad \int Lf d\mu = 0.$$

L'équation (1.25) peut être vue comme une formule d'intégration par parties. En effet, lorsque  $L$  est l'opérateur d'Ornstein-Uhlenbeck, (1.25) se réduit à la formule classique d'intégration par parties (1.14) qui était le point de départ de la méthode de Stein. Ces considérations permettent d'étendre la méthode de Stein à des mesures de probabilité autres que la loi normale. Si  $\mu$  est une mesure de probabilité, nous trouverons un processus de Markov pour lequel  $\mu$  est la loi d'équilibre. Alors le générateur infinitésimal  $L$  de ce processus de Markov donnera une formule d'intégration par parties qui sera un bon candidat pour caractériser la mesure  $\mu$ . Afin de montrer que l'erreur dans la formule d'intégration par parties (1.25) contrôle une certaine distance usuelle entre les mesures de probabilité, nous considérerons la solution (1.24) de l'équation de Poisson. Tout le travail restant consistera alors effectivement à trouver un tel générateur  $L$  et à prouver des estimations de régularité pour la solution (1.24). Si l'on y parvient, l'inégalité suivante du type de l'inéquation (1.19) sera prouvée pour la loi  $\mu$  :

$$(1.26) \quad W_1(\nu, \mu) \leq \sup_{f \in \mathcal{G}} \left| \int Lf(x) d\nu \right|.$$

Cette stratégie d'utilisation de la méthode de Stein sera utilisée tout au long de la première partie de cette thèse. L'approche par générateur de la méthode de Stein a été initialement introduite par A. Barbour en 1990 dans [17]. Grâce à celle-ci, A. Barbour a pu utiliser la méthode de Stein pour la mesure de Wiener au lieu de la loi normale, et par conséquent, il a obtenu une borne quantitative dans le théorème de Donsker. Rappelons que le théorème de Donsker constitue l'extension fonctionnelle du théorème central limite. Le résultat de Barbour correspond donc à une extension des bornes de Berry-Esseen données par la méthode de Stein



pour la loi normale. Concluons cette section en présentant une façon d'obtenir par la méthode de Stein un théorème central limite quantitatif.

Soit  $X_1, \dots, X_n$  des variables aléatoires réelles indépendantes et identiquement distribuées normalisées de telle sorte que  $\mathbb{E}[X_1] = 0$  et  $\mathbb{E}[X_1^2] = 1$ . Supposons que  $X_1$  admette un troisième et un quatrième moment finis. Soit  $S_N := \frac{1}{\sqrt{N}} \sum_{i=1}^N X_i$ . D'après le théorème central limite, nous savons que  $S_N$  converge faiblement vers la loi normale  $\gamma$ . Nous donnerons un taux de convergence quantitatif en bornant la distance de Wasserstein  $W_1(\gamma, \mu_N)$  entre  $\gamma$  et la distribution  $\mu_N$  de la variable aléatoire réelle  $S_N$ . Soit  $f : \mathbb{R} \rightarrow \mathbb{R}$  une fonction telle que

$$\|f\|_\infty \leq 2, \quad \|f'\|_\infty \leq \sqrt{\frac{2}{\pi}}, \quad \text{and} \quad \|f''\|_\infty \leq 2.$$

En notant  $W_i := \frac{1}{\sqrt{N}} \sum_{j \neq i} X_j = S_N - \frac{1}{\sqrt{N}} X_i$ , nous pouvons alors calculer

$$\mathbb{E}[S_N f(S_N)] = \mathbb{E} \left[ \frac{1}{\sqrt{N}} \sum_i X_i (f(S_N) - f(W_i) - (S_N - W_i) f'(S_N)) + \frac{1}{\sqrt{N}} \sum_i X_i (S_N - W_i) f'(S_N) \right].$$

Ainsi

$$\begin{aligned} |\mathbb{E}[f'(S_N) - S_N f(S_N)]| &\leq \left| \mathbb{E} \left[ \frac{1}{\sqrt{N}} \sum_i X_i \left( f(S_N) - f(W_i) - \frac{X_i}{\sqrt{N}} f'(S_N) \right) \right] \right| \\ &\quad + \left| \mathbb{E} \left[ f'(S_N) \left( 1 - \frac{1}{\sqrt{N}} \sum_i \frac{1}{\sqrt{N}} X_i^2 \right) \right] \right| \end{aligned}$$

Maintenant, d'une part en utilisant le développement de Taylor de la fonction  $f$ ,

$$\left| \mathbb{E} \left[ \frac{1}{\sqrt{N}} \sum_i X_i \left( f(S_N) - f(W_i) - \frac{X_i}{\sqrt{N}} f'(S_N) \right) \right] \right| \leq \frac{\|f''\|_\infty}{2\sqrt{N}} \sum_i \mathbb{E} \left| \frac{X_i^3}{N} \right| = \frac{\|f''\|_\infty}{2\sqrt{N}} \mathbb{E} |X_1^3|,$$

et d'autre part,

$$\begin{aligned} \left| \mathbb{E} \left[ f'(S_N) \left( 1 - \frac{1}{\sqrt{N}} \sum_i \frac{1}{\sqrt{N}} X_i^2 \right) \right] \right| &\leq \frac{\|f'\|_\infty}{N} \mathbb{E} \left| \sum_i (1 - X_i^2) \right| \\ &\leq \frac{\|f'\|_\infty}{N} \sqrt{\text{Var} \left( \sum_i X_i^2 \right)} \\ &\leq \frac{\|f'\|_\infty}{N} \sqrt{N \mathbb{E} |X_1^4|}. \end{aligned}$$

Finalement, la formule (1.19) et les calculs précédents donnent

$$W_1(\gamma, \mu_N) \leq \frac{1}{\sqrt{N}} \left( \mathbb{E} |X_1^3| + \sqrt{\frac{2}{\pi}} \mathbb{E} |X_1^4| \right),$$

qui est effectivement une borne quantitative dans la convergence donnée par le théorème central limite.

### 1.3 La condition de Courbure Dimension

Cette section est consacrée à la présentation des espaces de courbure-dimension riemanniens (RCD en anglais). Cette notion s'inscrit dans le contexte plus large de la théorie de la Courbure Dimension. Nous renvoyons le lecteur aux exposés complets de J. Lott et C. Villani [145, 214] et de K-T. Sturm [201, 202]. Nous ne discuterons ici que du cadre technique et de la vision de ces concepts qui nous ont guidés dans cette thèse.

De nombreux domaines des mathématiques utilisent la notion très générale d'espace. La géométrie traite des espaces dotés d'une métrique. Depuis l'Antiquité, on sait que la notion de métrique donne lieu à une notion de volume. Cela ressort clairement des formules classiques des volumes des formes géométriques de base. La théorie des probabilités traite des espaces avec une notion de certitude. Depuis la théorie moderne des probabilités de Kolmogorov, qui est basée sur la théorie de la mesure de Lebesgue, nous savons que ces deux notions, volume et certitude, sont équivalentes. En effet, le degré de certitude d'un événement peut être vu comme le volume occupé par cet événement dans l'espace de tous les événements possibles.

Plus formellement, en géométrie, une variété riemannienne est un espace doté d'un tenseur métrique, qui est une forme quadratique lisse sur les espaces tangents. A partir de ce tenseur métrique, on peut définir la distance géodésique. Une variété riemannienne est donc un exemple particulier d'espace métrique. En outre, on peut également définir la forme volume à partir du tenseur métrique. Une variété riemannienne est donc aussi un exemple particulier d'espace mesuré, la généralisation des espaces de probabilité où le volume total peut être infini comme dans l'espace euclidien. Une variété riemannienne est donc un exemple de ce que l'on appelle communément un espace métrique mesuré. Dans ce cas, les deux notions de distance et de volume sont liées car elles proviennent toutes deux du tenseur métrique. Une question naturelle est alors de comprendre ce lien. La seule explication selon laquelle ce lien provient de formules et de définitions mathématiques formelles n'est pas satisfaisante, il doit y avoir une raison plus conceptuelle derrière. Pour comprendre cette raison, revenons à la théorie des processus de Markov.

Comme nous l'avons vu brièvement dans la section 1.2, deux objets mathématiques importants sont liés à un processus de Markov : son générateur  $L$  (qui est un opérateur), et sa loi d'équilibre  $\mu$ . A partir de l'opérateur  $L$ , on peut définir les opérateurs bilinéaires successifs  $\Gamma$  (voir [13, Chapitre 1]), le premier

$$(1.27) \quad \Gamma(f, g) := \frac{1}{2} [L(fg) - fLg - gLf],$$

et le second

$$(1.28) \quad \Gamma_2(f, g) := \frac{1}{2} [L\Gamma(f, g) - \Gamma(f, Lg) - \Gamma(Lf, g)].$$

Pour alléger les notations, nous écrirons  $\Gamma(f)$  au lieu de  $\Gamma(f, f)$ , et de même  $\Gamma_2(f)$  au lieu de  $\Gamma_2(f, f)$ . Nous parlerons de l'opérateur  $\Gamma_2$  un peu plus tard. Occupons-nous tout d'abord du premier opérateur. Ce premier opérateur  $\Gamma$  est appelé l'opérateur carré du champ et quantifie dans quelle mesure l'opérateur  $L$  n'est pas une dérivation, c'est-à-dire ne vérifie pas la formule de Leibniz. De plus, dans le cas de l'opérateur  $L = \Delta - \nabla V \cdot \nabla$  de la forme (1.8), on peut calculer que l'opérateur carré du champ est le produit scalaire du gradient des fonctions :

$$(1.29) \quad \Gamma(f, g) = \nabla f \cdot \nabla g.$$

Le produit scalaire est le tenseur de Riemann, donc l'opérateur carré du champ est aussi la métrique riemannienne (restreinte aux champs vectoriels de gradients). Dans le cas général,  $\Gamma$  est en fait la co-métrique, c'est-à-dire, dans une carte locale, l'inverse du tenseur métrique. Dans le cas d'un générateur  $L$  de la forme (1.8), le tenseur métrique est la matrice identité donc la métrique et la co-métrique coïncident. L'opérateur carré du champ nous permet de définir la distance de la manière suivante :

$$\forall x, y, \quad d(x, y) := \sup_{\Gamma(f) \leq 1} |f(x) - f(y)|.$$

Lorsque  $\Gamma$  a la forme (1.29), on reconnaît la distance de Wasserstein-1 définie dans (1.16) entre la mesure de Dirac en  $x$  et la mesure de Dirac en  $y$ , qui coïncide avec la distance riemannienne

entre les points  $x$  et  $y$ . Nous avons maintenant suffisamment de matériel pour pouvoir répondre à la question du lien entre les notions de distance et de volume d'une variété riemannienne.

Sur toutes les variétés riemanniennes, il existe une marche aléatoire particulière qui correspond à la fois à la métrique et au volume. Cette marche aléatoire est appelée mouvement brownien. Le mouvement brownien est la formalisation mathématique du phénomène physique observé pour la première fois par le botaniste R. Brown en 1827, qui a décrit le mouvement d'une particule de pollen immergée dans l'eau. La trajectoire de la particule de pollen est modifiée chaque fois qu'elle rencontre des molécules d'eau, ce qui donne lieu à un processus de Markov. Dans le cas du mouvement brownien, le générateur est l'opérateur de Laplace-Beltrami, et la loi d'équilibre est la mesure de volume riemannienne. L'opérateur carré du champ du mouvement brownien encode donc la métrique, comme vu précédemment. Ainsi, le lien entre les notions de distance et de volume d'une variété riemannienne devient clair : la métrique est codée par l'opérateur carré du champ, et le volume est la mesure invariante, tous deux du mouvement brownien. Leur lien est donc le mouvement brownien : la métrique et le volume dérivent du processus de Markov brownien. Dès lors, nous n'étudions pas un espace avec une métrique (géométrie) ou un espace avec une mesure (théorie des probabilités), mais un espace sur lequel existe un processus de Markov. Bien sûr, pour que l'étude soit intéressante, nous devons faire des hypothèses sur ce processus de Markov. Ceci est similaire au contexte géométrique, où la métrique doit satisfaire certaines conditions, par exemple être hyperbolique, ou au contexte probabiliste où la distribution est supposée être log-concave par exemple. Les espaces de courbure-dimension riemanniens (appelés espaces RCD dans la suite) sont des espaces équipés d'un processus de Markov satisfaisant deux conditions : la propriété de diffusion et la condition de courbure-dimension. Nous présentons ici le cadre lisse des espaces RCD, c'est-à-dire lorsque nous supposons qu'ils ont une structure différentielle. Le cadre général où nous ne supposons pas avoir de structure différentielle est présenté dans le chapitre 5. Cependant, la philosophie reste la même dans les deux cas.

On dit qu'un processus de Markov est une diffusion lorsque son générateur satisfait pour tout  $\phi \in C^2(\mathbb{R}, \mathbb{R})$ ,

$$(1.30) \quad L(\phi \circ f) = \phi'(f)Lf + \phi''(f)\Gamma(f, f).$$

D'un point de vue analytique, cette propriété de diffusion est la règle de la chaîne des opérateurs différentiels du second ordre dont le Laplacien constitue le prototype. D'un point de vue probabiliste, la propriété de diffusion du processus  $(X_t)_{t \geq 0}$  avec générateur  $L$  implique la continuité de tous les processus  $(f(X_t))_{t \geq 0}$  lorsque  $f$  est dans le domaine du générateur  $L$  (voir [13, Section 1.11.1]). Dans la première partie de cette thèse, la propriété de diffusion nous permettra d'obtenir des processus de Markov unidimensionnels en prenant le pushforward du processus étudié par des fonctions propres. Rappelons que l'opérateur de Laplace-Beltrami est toujours une diffusion, de sorte que le mouvement brownien sur une variété riemannienne est toujours une diffusion de Markov.

On dit d'un processus de Markov qu'il satisfait la condition de courbure-dimension  $CD(K, N)$  lorsqu'il satisfait à la formule de Bochner suivante

$$(1.31) \quad \Gamma_2(f, f) \geq K \Gamma(f, f) + \frac{1}{N} (Lf)^2.$$

Dans le cas où le processus de Markov est le mouvement brownien sur la sphère unitaire  $\mathbb{S}^N$ , le générateur est l'opérateur de Laplace-Beltrami  $\Delta$ , l'opérateur carré du champ est  $\Gamma(f) = |\nabla f|^2$  et l'opérateur  $\Gamma_2$  est

$$\Gamma_2(f) = \text{Ric}(\nabla f, \nabla f) + \|\text{Hess}(f)\|_{HS}^2,$$

où  $\|\cdot\|_{HS}$  désigne la norme de Hilbert-Schmidt des opérateurs. Par l'inégalité de Cauchy-Schwarz, on a toujours

$$\Delta f = \text{Tr Hess}(f) = \langle I_d, \text{Hess}(f) \rangle_{HS} \leq \sqrt{N} \|\text{Hess}(f)\|_{HS},$$

donc dans ce cas, la condition  $CD(K, N)$  est équivalente à

$$(1.32) \quad \text{Ric}(\nabla f, \nabla f) \geq K|\nabla f|^2.$$

Par conséquent, lorsque le processus de Markov est un mouvement brownien sur une variété riemannienne, les espaces  $RCD(K, N)$  sont des variétés riemanniennes de dimension au plus égale à  $N$  et dont la courbure de Ricci est minorée par  $K$ . Ainsi, l'idée principale derrière la formule de Bochner (1.31) est de donner une borne supérieure sur la dimension de l'espace, et une borne inférieure (uniforme) sur la courbure. Les espaces RCD sont des généralisations des variétés riemanniennes avec une minoration de la courbure de Ricci, et coïncident exactement avec ceux-ci lorsque la diffusion de Markov ambiante est le mouvement brownien. L'inégalité de Bochner (1.31) avec des paramètres  $K = N - 1$  est une égalité lorsque l'espace est la sphère unitaire  $\mathbb{S}^N$  équipée de son mouvement brownien. En d'autres termes, l'espace modèle des espaces  $RCD(N - 1, N)$  est la sphère unitaire  $\mathbb{S}^N$  équipée de son mouvement brownien. Les espaces  $RCD(K, N)$  sont donc des espaces que l'on peut raisonnablement comparer à des sphères. Nous avons déjà vu cette idée de comparer un espace avec une sphère à la fin de la section 1.1.2 lorsque nous avons présenté le théorème de Lichnerowicz. Rappelons-le.

**Théorème.** (*Lichnerowicz [142]*) *Si  $(M, g)$  est une variété riemannienne de dimension  $N$  et de courbure de Ricci minorée par  $N - 1$ , alors la première valeur propre de son opérateur de Laplace-Beltrami est plus grande ou égale que  $N$ .*

Nous avons vu ici qu'en terme d'espaces  $RCD$ , une variété riemannienne de dimension  $N$  avec une courbure de Ricci minorée par  $N - 1$  est un espace  $RCD(N - 1, N)$ . Nous avons également vu que l'espace modèle parmi les espaces  $RCD(N - 1, N)$  est la sphère unitaire  $\mathbb{S}^N$ . De plus, la valeur propre  $N$  du théorème de Lichnerowicz correspond à la première valeur propre non nulle de l'opérateur de Laplace-Beltrami de la sphère unité. Le théorème de Lichnerowicz établit donc la comparaison de la première valeur propre des espaces  $RCD(N - 1, N)$  avec celle de leur espace modèle. Mentionnons que de ce résultat on peut facilement obtenir le même résultat pour les espaces  $RCD(K, N)$  par une normalisation correcte. En effet, l'espace modèle de  $RCD(K, N)$  est la sphère de rayon  $\frac{N-1}{K}$  et sa première valeur propre non nulle est  $\frac{NK}{N-1}$ .

De nombreux autres résultats de comparaison existent. Par exemple, on peut comparer le diamètre d'un espace  $RCD(K, N)$  avec celui de la sphère  $\mathbb{S}^N$ , ou encore comparer leurs profils isopérimétriques (théorème de Levy-Gromov voir [104]). L'estimation de Lichnerowicz a été étendue aux espaces RCD non lisses par M. Erbar, K. Kuwada et K-T. Sturm dans [83, Theorem 4.22]. Prouvons-le dans le cadre lisse avec la formule de Bochner. Soit  $L$  le générateur du processus de diffusion de Markov d'un espace  $RCD(N - 1, N)$ . Soit  $\lambda_1(L)$  la première valeur propre non nulle de l'opérateur semi-défini positif  $-L$ , et  $f$  une fonction propre associée. En évaluant l'inégalité de Bochner (1.31) avec la fonction propre  $f$ , et en l'intégrant contre la distribution d'équilibre  $\mu$ , on obtient

$$\int \Gamma_2(f) d\mu \geq (N - 1) \int \Gamma(f) d\mu + \frac{\lambda_1(L)^2}{N} \int f^2 d\mu.$$

Maintenant par la définition de l'opérateur  $\Gamma_2$  et la formule d'intégration par parties (1.9), on obtient

$$\begin{aligned} \lambda_1(L)^2 \int f^2 d\mu &= \int (Lf)^2 d\mu = \int \Gamma_2(f) d\mu \geq (N - 1) \int \Gamma(f) d\mu + \frac{\lambda_1(L)^2}{N} \int f^2 d\mu \\ &= \lambda_1(L)(N - 1) \int f^2 d\mu + \frac{\lambda_1(L)^2}{N} \int f^2 d\mu, \end{aligned}$$

ce qui entraîne effectivement

$$\lambda_1(L) \geq N.$$

Nous savons que les sphères atteignent l'égalité dans l'estimation de trou spectral de Lichnerowicz. Une question naturelle se pose alors sur la rigidité dans ce cas d'égalité : les sphères

sont-elles les seuls espaces atteignant l'égalité parmi toutes les variétés riemanniennes de dimension  $N$  avec une courbure de Ricci uniformément minorée par  $K$  ? M. Obata a répondu par l'affirmative à cette question dans [172].

**Théorème.** (*Théorème de rigidité d'Obata [172]*) Soit  $(M, g)$  une variété riemannienne de dimension  $N$  avec une courbure de Ricci minorée par  $N - 1$ . Si la première valeur propre de l'opérateur de Laplace-Beltrami est égale à  $N$ , alors  $(M, g)$  est isométrique à la sphère unitaire  $\mathbb{S}^N$ .

Comme pour l'estimation de Lichnerowicz, le théorème de rigidité d'Obata a été étendu aux espaces RCD généraux. La rigidité est directement étendue dans le cadre lisse : les sphères restent les seuls espaces à atteindre l'égalité. Cependant, dans le cadre non lisse, il existe des espaces autres que les sphères qui atteignent l'égalité : ce sont les suspensions sphériques. Tous les cas d'égalité ont été décrits par Christian Ketterer dans [124, 123]. Nous ne nous étendrons pas davantage sur les suspensions sphériques car nous ne les utilisons pas dans cette thèse. En effet, dans le chapitre 5, après avoir effectué un pushforward par une fonction réelle, nous sommes réduits à comparer des espaces RCD unidimensionnels. Or, il s'avère que, à une reparamétrisation près, il n'existe qu'un seul espace modèle de dimension un. Cet espace modèle est le segment muni de la distribution Beta, et peut être vu comme la distribution d'une coordonnée sur la sphère  $\mathbb{S}^N$  (voir la section 5.2.3 du chapitre 5).

Notons que la question de la rigidité présente quelques similitudes avec la question "d'entendre" la forme d'un tambour évoquée dans la section 1.1.2. En effet, dans les deux cas, on a une hypothèse sur le spectre de deux opérateurs et on veut en déduire une information sur la géométrie. Nous avons vu dans la section 1.1.2 que la réponse était négative, alors qu'ici sous la condition de courbure-dimension, la réponse est positive. La principale différence est que dans la présente section, nous avons considérablement réduit les degrés de liberté du problème en fixant certains espaces modèles et en restreignant la comparaison aux seuls espaces  $RCD(K, N)$ . Nous verrons dans la section suivante que dans l'étude de la stabilité des inégalités fonctionnelles, il est crucial de choisir un espace modèle approprié d'une part, mais aussi une classe d'espaces appropriés auquel le modèle peut être comparé d'autre part.

## 1.4 Stabilité des résultats de comparaison

Lorsqu'il existe un résultat de comparaison tel que le théorème de Lichnerowicz, la première question naturelle est de savoir ce qui se passe lorsque l'égalité est atteinte. Nous avons un résultat de rigidité lorsque l'égalité n'est atteinte que pour un espace particulier, comme dans le théorème de rigidité d'Obata. Cet espace particulier constitue alors l'espace modèle. Dans le cas du théorème de Lichnerowicz, l'espace modèle est la sphère (à laquelle il faut ajouter les suspensions sphériques dans le cas non lisse). La question suivante est alors de savoir si le résultat de la comparaison est *stable* ou non. Si l'égalité n'est pas atteinte, mais *presque* atteinte, l'espace ressemble-t-il à l'espace modèle ? Dans quel sens y ressemble-t-il ? En particulier, nous avons besoin d'une notion appropriée à chaque contexte pour comparer deux espaces.

La question de la stabilité peut être résumée comme suit. Nous commençons par un résultat de comparaison de la forme

$$\forall F \in \mathcal{F}, \quad \Omega(F) \geq \Omega(M),$$

où  $M \in \mathcal{F}$  est l'espace modèle, et le seul élément de  $\mathcal{F}$  atteignant l'égalité, et  $\Omega(F)$  est une quantité d'intérêt dépendant de la structure de  $F$ . Un *résultat de stabilité* est alors un résultat de la forme suivante :

$$\text{si } F \in \mathcal{F} \text{ est tel que } \Omega(F) \approx \Omega(M), \text{ alors } F \approx M,$$

où le second symbole  $\approx$  doit être compris en un sens qui dépend du type d'objets  $F \in \mathcal{F}$  considérés. Un *résultat de stabilité quantitatif* est alors un résultat de la forme suivante :

$$(1.33) \text{ si } F \in \mathcal{F} \text{ est tel que pour un assez petit } \varepsilon > 0, \Omega(F) = \Omega(M) + \varepsilon, \text{ alors } d(F, M) \leq \omega(\varepsilon),$$

où  $d$  est une distance sur l'espace  $\mathcal{F}$  et  $\omega$  est un module de continuité, c'est-à-dire une fonction continue positive de  $\mathbb{R}_+$  à valeurs dans  $\mathbb{R}_+$ , et ne s'annulant qu'en zéro. Notons que la plupart du temps, on ne contrôle pas directement la distance  $d(F, M)$  entre  $F$  et  $M$ , mais la distance entre  $F'$  et  $M'$  où  $F'$  et  $M'$  sont des espaces plus simples que  $F$  et  $M$ . En particulier dans cette thèse,  $F$  et  $M$  seront des lois de probabilité, et  $F'$ ,  $M'$  sera le pushforward de  $F$  et  $M$  par une certaine fonction.

Le choix de l'espace  $\mathcal{F}$  est crucial. En effet, s'il est trop grand, alors la propriété de rigidité pourrait ne pas avoir lieu, et s'il est trop petit, l'information donnée par de tels résultats serait triviale. Dans la première partie de cette thèse, nous étudions et dérivons des résultats de stabilité pour la constante de Poincaré. Ainsi, l'ensemble  $\mathcal{F}$  sera un ensemble de mesures de probabilité, et le résultat de comparaison avec lequel nous commencerons sera de la forme

$$\forall \nu \in \mathcal{F}, \quad C_P(\nu) \geq C_P(\mu),$$

où  $\mu$  constituera la mesure modèle. Nous étudierons la stabilité de tels résultats de comparaison dans deux contextes : une condition de normalisation, et une condition de courbure-dimension.

### 1.4.1 Stabilité sous une hypothèse de normalisation

Traitons d'abord le cas des conditions de normalisation, c'est-à-dire lorsque l'ensemble des mesures  $\mathcal{F}$  est l'ensemble des mesures de probabilité dont certains moments ont une valeur fixée à l'avance. La mesure modèle sera la distribution d'équilibre déterminée par un certain processus de diffusion de Markov. Cette approche remonte à l'article de L. Chen de 1987 [67], où le résultat suivant est prouvé.

**Théorème.** (L. Chen [67, Corollaire 2.1]) *Si  $\nu$  est une mesure de probabilité sur  $\mathbb{R}$  de moyenne nulle et de variance égale à 1, c'est-à-dire*

$$\int x d\nu = 0, \text{ et } \int x^2 d\nu = 1,$$

*et si elle satisfait à l'inégalité de Poincaré*

$$\forall u \in C^1(\mathbb{R}) \cap L^2(\nu), \quad \text{Var}_\nu(u) \leq C_P(\nu) \int u'(x)^2 d\nu(x),$$

*de constante optimale  $C_P(\nu)$ , alors*

$$C_P(\nu) \geq 1.$$

*De plus, sous ces hypothèses sur  $\nu$ , si  $C_P(\nu) = 1$ , alors  $\nu = \gamma$  est égale à la loi normale sur  $\mathbb{R}$ .*

Nous avons donc un résultat de comparaison sur l'ensemble  $\mathcal{F}$  des mesures de probabilité de moyenne zéro et de variance un, la mesure modèle est la gaussienne, et de plus l'inégalité est rigide. Tous les ingrédients sont réunis pour étudier la stabilité de ce résultat. Et en effet, le résultat de comparaison de Chen est stable : en 1989 dans [211], S. Utev a quantifié la différence entre les constantes de Poincaré avec la distance de variation totale.

**Théorème.** (S. Utev [211]) *Sous les mêmes hypothèses sur  $\nu$  que dans le théorème de comparaison de Chen ci-dessus, nous avons :*

$$C_P(\nu) \geq 1 + \frac{1}{9} d_{TV}(\nu, \gamma)^2,$$

*avec  $d_{TV}$  la distance de variation totale (voir section 2.2).*

En 2019 dans le papier [75], T. Courtade, M. Fathi et A. Pananjady ont montré que toutes les mesures de probabilité de l'ensemble  $\mathcal{F}$  des mesures de probabilité de moyenne zéro et de variance un satisfont une formule d'intégration par parties approchée dont le terme d'erreur est contrôlé par la différence  $C_P(\nu) - 1$  entre les constantes de Poincaré. En revenant à la méthode de Stein, cela signifie que le côté droit de l'inégalité (1.19) est contrôlé par un terme dépendant explicitement de  $C_P(\nu) - 1$ . Par conséquent, ils ont montré le résultat de stabilité suivant.

**Théorème.** (*T. Courtade, M. Fathi, A. Pananjady [75]*) *Sous les mêmes hypothèses sur  $\nu$  que dans le théorème de comparaison de Chen ci-dessus, nous avons :*

$$C_P(\nu) \geq 1 + W_1(\nu, \gamma)^2,$$

avec  $W_1$  la distance Wasserstein-1.

En fait, ils obtiennent un résultat de stabilité multidimensionnel par rapport à la distance de Wasserstein-2 avec la constante multiplicative  $\frac{1}{d}$  devant la distance. Le théorème énoncé peut s'en déduire puisque  $W_2 \geq W_1$ .

Notre étude a commencé avec l'idée de généraliser ce résultat en remplaçant la mesure modèle  $\gamma$  par la loi d'équilibre  $\mu$  d'un processus de Markov général. Les lois d'équilibre des processus de Markov sont des candidats idéaux pour jouer le rôle de mesures modèles. En effet, dans la preuve des résultats de stabilité d'Utev et de Courtade-Fathi-Pananjady, la méthode de Stein est utilisée. Cependant, afin de généraliser ces preuves, nous avons besoin de mesures de probabilité pour lesquelles nous sommes capables d'implémenter cette méthode. Comme expliqué dans la section 1.2, l'approche de Barbour suggère comment implémenter la méthode de Stein dans le cas de la mesure invariante d'un processus de Markov. La deuxième chose à faire, pour au moins généraliser le résultat de Courtade-Fathi-Pananjady à d'autres mesures invariantes de processus de Markov, est de trouver quelles conditions de normalisation doivent être requises dans l'ensemble  $\mathcal{F}$  des mesures test. Pour les trouver, un résultat de comparaison tel que le théorème de Chen doit être satisfait. À première vue, on pourrait penser que ce qui est requis est que  $\nu$  ait les mêmes premier et deuxième moments que  $\mu$ , puisque c'est ce qui est requis dans le théorème de Chen lorsque  $\mu$  est la gaussienne. Cependant ces conditions ne sont pas les bonnes. En effet, un contre-exemple est le suivant. Soit  $d\mu(x) = \frac{1}{Z}e^{-x^4-x^2}dx$  avec  $Z = \int_{\mathbb{R}} e^{-x^4-x^2}dx$  tel que  $\mu$  soit une mesure de probabilité. Alors comme on le verra au chapitre 3 section 3.6.3,  $\mu$  est la mesure invariante d'un processus de Markov satisfaisant à une inégalité de Poincaré de constante optimale égale à un. Cependant, si on prend  $\nu := \mathcal{N}(0, \sigma^2)$  la loi normale de moyenne zéro et de variance  $\sigma^2 := \frac{1}{Z} \int x^2 d\mu(x)$ , alors  $\mu$  et  $\nu$  ont les mêmes premier et second moments, mais la constante de Poincaré de  $\nu$  est  $\sigma^2$  qui est strictement inférieure à 1 (une estimation numérique donne  $\sigma^2 \approx 0.2$ ). Par conséquent, la constante de Poincaré  $C_P(\mu) = 1$  ne peut pas être un minimum sur l'ensemble des mesures ayant les mêmes premier et second moments que  $\mu$ .

À ce stade, nous devons rappeler la propriété de tensorisation de l'inégalité de Poincaré, qui implique que la constante optimale ne dépend que d'une seule direction, et cette direction est la coordonnée spectrale donnée par la première fonction propre, puisque celle-ci réalise l'égalité dans l'inégalité de Poincaré (voir section 1.1). Ainsi, dans le cas gaussien, lorsque nous avons exigé que les deux premiers moments soient respectivement zéro et un, ce n'est pas parce que ce sont des moments, mais parce que la première fonction propre est l'identité  $f_1(x) = x$ , donc le premier moment est sa moyenne, et le second moment est sa variance. Par conséquent, nous sommes amenés à prendre ce qui suit comme normalisation :

$$\int f_1 d\nu = \int f_1 d\mu, \text{ et } \int f_1^2 d\nu = \int f_1^2 d\mu,$$

où  $f_1$  est la première fonction propre de la mesure modèle  $\mu$ . Cependant, ces deux conditions ne sont toujours pas suffisantes pour dériver un résultat de comparaison. En effet, nous devons également exiger que

$$\int \Gamma(f_1) d\nu = \int \Gamma(f_1) d\mu,$$

où  $\Gamma$  est l'opérateur carré du champ du processus de Markov. (Nous verrons au chapitre 3 que cette condition d'égalité peut être relaxée en inégalité " $\leq$ "). Cette troisième condition est vide dans le cas gaussien car la dérivée de l'identité est toujours égale à un, donc son opérateur carré du champ est aussi égal à 1. Sous ces trois conditions de normalisation, on obtient finalement le résultat de comparaison suivant, qui sera le point de départ d'un résultat de stabilité.

**Théorème.** (Voir l'introduction du chapitre 3) Si  $\nu$  est normalisée comme ci-dessus, alors  $C_P(\nu) \geq C_P(\mu)$ .

Par conséquent, restreinte à cet ensemble de mesures normalisées, qui est un ensemble de codimension 3 dans l'ensemble de dimension infini constitué de toutes les mesures de probabilité, la constante de Poincaré possède un minimum global qui est atteint pour la mesure modèle  $\mu$ .

### 1.4.2 Stabilité sous la condition RCD

Passons à la condition de la courbure-dimension riemannienne. L'ensemble  $\mathcal{F}$  sera l'ensemble de toutes les mesures des espaces  $RCD(N-1, N)$ . La mesure modèle  $\mu$  sera la loi d'équilibre du mouvement brownien sur la sphère unitaire  $\mathbb{S}^N$ , c'est-à-dire la mesure de Lebesgue normalisée ( $\mu(\mathbb{S}^N) = 1$ ) sur  $\mathbb{S}^N$ . Le résultat de comparaison dont nous partirons sera le théorème de Lichnerowicz (voir la section 1.3 de l'introduction). Dans ce cas, puisque  $C_P(\nu) = \frac{1}{\lambda_1(\nu)}$  (voir section 1.1.2 de l'Introduction) il sera de la forme

$$\forall \nu \in \mathcal{F}, \quad C_P(\nu) \leq C_P(\mu),$$

où la constante de Poincaré de l'espace modèle est un maximum et non un minimum. Nous aurons alors tous les ingrédients pour étudier la stabilité. Nous avons vu dans la présentation des espaces RCD de la section 1.3 que le problème de rigidité est résolu par le théorème d'Obata. Puisque le cas d'égalité n'est pas seulement satisfait par un espace modèle mais par la sphère et les suspensions sphériques, nous ne pouvons pas nous attendre à un résultat de stabilité au sens fort défini par (1.33). Ceci a été prouvé en 1990 par M. Anderson dans [6]. Il existe néanmoins le résultat de stabilité suivant prouvé par J. Bertrand dans [27].

**Théorème.** ([27, Theorem 5]) Si la première valeur propre non nulle  $\lambda_1$  d'un espace  $RCD(N-1, N)$  est proche de  $N$ , alors cet espace RCD contient une partie qui est proche de la sphère  $\mathbb{S}^{k-1}$  en distance de Gromov-Hausdorff, où  $k \leq N+1$  est la dimension de l'espace propre associé à la valeur propre  $\lambda_1$ .

### 1.4.3 Utilisation de la méthode de Stein pour obtenir des résultats de stabilité

Le point de départ est le même que l'idée principale de la preuve du résultat de Courtade-Fathi-Pananjady. Premièrement, la mesure modèle satisfait une formule d'intégration par parties, disons pour tout  $g \in H^1(\mu)$ ,

$$\int Lg \, d\mu = 0.$$

Deuxièmement, toutes les mesures test  $\nu$  (normalisées ou RCD) satisfont la même formule d'intégration par parties, à l'exception d'un terme d'erreur, qui est contrôlé par la différence entre la constante de Poincaré de  $\nu$  et la constante de Poincaré de  $\mu$  :

$$(1.34) \quad \left| \int Lg \, d\nu \right|^2 \leq |C_P(\mu) - C_P(\nu)| \int \Gamma(g) \, d\nu.$$

Ainsi, si la méthode de Stein peut être appliquée à  $\mu$ , alors nous savons que le terme d'erreur de la formule d'intégration par parties approchée contrôle la distance de Wasserstein-1 entre  $\mu$  et  $\nu$ . Il s'ensuivrait donc un résultat de stabilité de la forme

$$W_1(\mu, \nu)^2 \leq K |C_P(\mu) - C_P(\nu)|$$

où  $K$  serait une constante. Cependant, nous savons qu'un tel résultat ne peut être vrai en toute généralité. En effet, en dimension supérieure à un, nous avons vu que la constante de Poincaré d'une mesure produit ne dépend que d'une seule direction. Il est donc impossible que



la différence entre les constantes de Poincaré contrôle la distance entre  $\nu$  et  $\mu$  dans toutes les directions. Par exemple, dans  $\mathbb{R}^2$ , les mesures normales

$$\mathcal{N}(0, I_2) \quad \text{et} \quad \mathcal{N}\left(0, \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}\right)$$

ont la même constante de Poincaré égale à 1. Elles ne sont pas égales dans toutes les directions, mais elles sont égales dans la direction de cette constante de Poincaré 1 qui est leur première coordonnée. Ainsi, nous n'attendons pas un résultat de stabilité entre  $\mu$  et  $\nu$  dans toutes les directions, mais seulement dans la direction maximale pour la constante de Poincaré de  $\mu$ . Cette direction est la première fonction propre de  $\mu$ , et comparer les distributions dans cette direction revient à comparer leur pushforward par  $f_1$ . Nous nous attendons donc à un résultat de stabilité de la forme

$$W_1\left(f_1^\#(\mu), f_1^\#(\nu)\right)^2 \leq K |C_P(\mu) - C_P(\nu)|$$

où  $f_1^\#$  représente le pushforward par  $f_1$  et  $K$  est une constante. Dans le cas de la loi normale unidimensionnelle, puisque  $f_1$  est l'identité, le pushforward ne change pas  $\nu$ , et c'est pourquoi la nécessité du pushforward n'était pas si claire à première vue.

Par conséquent, nous n'appliquerons pas la méthode de Stein à  $\mu$ , mais à la mesure de probabilité unidimensionnelle  $f_1^\# \mu$ . Nous savons que  $\nu$  satisfait à une formule d'intégration par parties approchée, et nous devons dériver une telle formule approximative pour  $f_1^\# \mu$ . Par définition du pushforward, la chose naturelle à faire est d'évaluer (1.34) avec  $g$  de la forme  $g = \phi \circ f_1$  où  $\phi : \mathbb{R} \rightarrow \mathbb{R}$ . Nous avons donc affaire à des quantités de la forme  $L(\phi \circ f_1)$  et  $\Gamma(\phi \circ f_1)$ , et nous voudrions qu'elles soient aussi  $f_1$ -mesurables, c'est-à-dire de la forme  $\Phi(f_1)$ . Nous avons vu dans la section 1.3 que sous une propriété de diffusion du générateur  $L$ , la première quantité satisfait (voir formule (1.30))

$$L(\phi \circ f_1) = \phi'(f_1)Lf_1 + \phi''(f_1)\Gamma(f_1).$$

De plus, cela implique pour cette dernière que

$$\Gamma(\phi \circ f_1) = \phi'(f_1)^2 \Gamma(f_1).$$

Par conséquent, ces quantités seront  $f_1$ -mesurables en fonction de la  $f_1$ -mesurabilité de l'opérateur carré du champ de la première fonction propre. Cela revient à poser la question de l'existence d'une fonction  $h : \mathbb{R} \rightarrow \mathbb{R}_+$  telle que  $\Gamma(f_1) = h \circ f_1$ . Nous utilisons deux approches distinctes pour ce problème, selon que nous considérons le cas des conditions de normalisation ou le cas des conditions RCD. Dans le cas des conditions de normalisation, nous supposons directement l'existence de la fonction de factorisation  $h$ . Nous justifions cette hypothèse en dimension 1 par le lemme de monotonie 13 du chapitre 3. Sous la condition RCD, nous montrons que la quantité  $\Gamma(f_1) + f_1^2$  est proche d'une constante (voir le lemme 64), et donc  $\Gamma(f_1)$  est presque  $f_1$ -mesurable puisqu'elle se comporte comme  $f_1^2$ .

#### 1.4.4 Utilisation de la technique de localisation

La localisation est une méthode issue de l'analyse convexe qui permet de réduire un problème multidimensionnel à une étude unidimensionnelle. Elle a été développée par Gromov-Milman [107] et R. Kannan, L. Lovasz, M. Simonovits [146, 122]. R. Eldan [82], Y. T. Lee et S. Vempala [138] ont développé sur cette base la localisation stochastique, conduisant au meilleur résultat dans le sens d'une preuve de la conjecture KLS par Klartag-Lehec [127] (voir aussi [70] par Y. Chen). La conjecture KLS affirme que la constante de Poincaré d'une mesure isotrope log-concave dans  $\mathbb{R}^d$  ne dépend pas de la dimension.

Un lien entre les problèmes de transport optimal et la technique de localisation a été découvert par B. Klartag dans [126]. Le théorème de localisation a ainsi été établi dans le

cadre des variétés riemanniennes lisses. Suite à ces travaux, F. Cavalletti et A. Mondino ont prouvé dans [60, 61, 59] un théorème de localisation pour les espaces métriques mesurés vérifiant la condition  $CD(K, N)$ , qui est une condition plus générale que la condition  $RCD$  (voir [58]). Le principe est le suivant. Prenons une fonction intégrable  $f : X \rightarrow \mathbb{R}$  sur un espace  $RCD(K, N)$   $(X, d, \mu)$  avec  $K \in \mathbb{R}$  et  $N \geq 1$ . Supposons que  $f$  soit centrée par rapport à  $\mu$ , et qu'il existe un certain  $x_0 \in X$  tel que  $x \mapsto |f(x)|d(x, x_0) \in L^1(\mu)$ . Alors il existe une désintégration  $(\mu_q)_{q \in Q}$  telle que

i ) il y a une mesure de probabilité  $m$  sur  $Q$  telle que

$$\mu = \int_Q \mu_q dm,$$

ii ) et pour  $m$ -presque tout  $q \in Q$ , le support  $X_q$  de la mesure  $\mu_q$  est une géodésique, et l'espace  $(X_q, d|_{X_q}, \mu_q)$  est  $RCD(K, N)$ .

Les espaces unidimensionnels  $X_q$  sont souvent appelés aiguilles, et cette technique est donc appelée décomposition en aiguilles. Entre autres, la décomposition en aiguilles a été utilisée pour dériver des inégalités isopérimétriques rigides dans les espaces  $CD$  [60], des estimations quantitatives pour les inégalités isopérimétriques dans les variétés riemanniennes à poids satisfaisant la condition  $CD(1, \infty)$  [149, 150], et le théorème de rigidité d'Obata pour les espaces  $CD$  [62]. Le dernier résultat est le plus proche de notre théorème 60 du chapitre 5. Énonçons-le brièvement.

**Théorème.** [62, Theorem 1.3] *Soit  $(M, d, \mu)$  un espace  $CD(N-1, N)$  sans ramifications, avec un trou spectral  $\lambda_1$  et la fonction propre normalisée associée  $u$ , il existe un point  $x_0 \in M$  tel que*

$$\|u - \sqrt{N+1} \cos(d(\cdot, x_0))\|_2 \leq C(N)(\lambda_1 - N)^{1/(8N+4)}; \quad \pi - \text{diam}(M) \leq C(N)(\lambda_1 - N)^{1/N}.$$

Nous savons que les espaces modèles sont les sphères unitaires. En dimension un, il s'agit donc d'un intervalle, et la première fonction propre est le cosinus. Ce résultat donne donc une stabilité de la constante de Poincaré en termes de déviation de la première fonction propre par rapport au modèle de dimension un. Dans cette thèse, au lieu de la technique de localisation, nous prenons le pushforward par une fonction réelle pour réduire le problème à un problème unidimensionnel.

## 1.5 Inégalités isopérimétriques

Le problème isopérimétrique a été posé pour la première fois il y a fort longtemps. La question était de savoir quelle surface maximale peut être délimitée par une corde de longueur prescrite. Les Grecs anciens savaient déjà que la meilleure forme pour délimiter l'aire maximale est le cercle. Ils savaient que l'aire  $\mathcal{A}$  et le périmètre  $\mathcal{P}$  d'une forme plane satisfont à la condition

$$\mathcal{P}^2 \geq 4\pi\mathcal{A},$$

avec égalité pour les disques. Le problème a également été posé pour les objets de dimension 3: quelle forme doit avoir une surface de volume prescrit pour posséder une aire la plus petite possible? Des progrès significatifs vers une preuve ont été faits plus tard avec l'utilisation du calcul différentiel. En mécanique, le problème isopérimétrique a été lié au principe de moindre action. En effet, par exemple, une bulle de savon doit minimiser sa surface afin de minimiser la tension qu'elle subit. Au 19ème siècle, J. Steiner [199] a introduit une méthode qui a permis à K. Weierstrass et H. Minkowski de prouver le théorème isopérimétrique de manière rigoureuse. Cette méthode, maintenant connue sous le nom de symétrisation de Steiner, est

basée sur le fait que si un ensemble isopérimétrique est divisé en deux par une ligne, alors les deux moitiés donnent un nouvel ensemble isopérimétrique en complétant chacune d'elles symétriquement autour de la ligne. Introduisons maintenant le cadre général des problèmes isopérimétriques. Nous renvoyons à la présentation [182] de A. Ros, et au livre [47] de Y. Burago et V. Zalgaller pour une exposition complète. La première notion dont nous avons besoin pour énoncer rigoureusement le problème isopérimétrique est la notion de volume. Nous avons donc besoin d'un espace mesuré. La seconde est la notion d'aire, c'est-à-dire une façon de mesurer le volume des ensembles de codimension 1. Il existe plusieurs façons non triviales d'étendre la mesure donnée d'un espace mesuré aux bords des ensembles mesurables. La plus courante est le contenu de Minkowski, qui utilise une notion de distance. Nous avons donc besoin d'un espace métrique mesuré. Soit  $(X, d, m)$  un tel espace, et  $A \subset X$  un ensemble mesurable, on définit son  $r$ -élargissement par

$$A_r := \{x \mid d(x, A) < r\}.$$

Alors le contenu de Minkowski de  $A$  est défini par

$$(1.35) \quad m^+(A) := \liminf_{r \rightarrow 0} \frac{m(A_r \setminus A)}{r},$$

et constitue un bon moyen de définir rigoureusement la mesure de la surface  $\partial A$ . Soulignons que le contenu de Minkowski est la dérivée prise en  $r = 0$  de la fonction  $r \mapsto m(A_r \setminus A)$ . Informellement, cette quantité doit être comprise comme

$$m^+(A) = \int |\nabla \mathbf{1}_A| \, dm.$$

En effet, en prenant

$$f_r(x) = \left(1 - \frac{d(x, A)}{r}\right)_+,$$

avec  $h_+ := \max(0, h)$ , on a  $f_r \xrightarrow{r \rightarrow 0} \mathbf{1}_A$  et  $|\nabla f_r| = \frac{1}{r} \mathbf{1}_{A_r \setminus A}$ . Donc, si la liminf de la définition est en fait une limite, ce qui est le cas lorsque la frontière de  $A$  est suffisamment régulière, alors

$$m^+(A) = \lim_{r \rightarrow 0} \frac{m(A_r \setminus A)}{r} = \lim_{r \rightarrow 0} \int |\nabla f_r| \, dm = \int |\nabla \mathbf{1}_A| \, dm,$$

et la dernière interversion entre limite, intégrale et gradient est justifiée lorsque  $A$  a une frontière régulière. La deuxième façon principale de définir la notion de périmètre est du point de vue de la topologie faible. On se réfère à [2], [85] et [218, Chapitre 5]. Comme la précédente, l'idée de cette approche est de considérer que puisque le volume d'un ensemble  $A$  est donné par l'intégrale de sa fonction indicatrice  $\mathbf{1}_A$  par rapport à la mesure  $m$ , alors le périmètre de  $A$  doit être donné par l'intégrale de cette fonction indicatrice  $\mathbf{1}_A$  par rapport à la dérivée de  $m$  au sens des distributions. Autrement dit, et après intégration par parties, le périmètre de  $A$  doit être l'intégrale de la dérivée de sa fonction indicatrice  $\mathbf{1}_A$  par rapport à  $m$ . Dans cette approche, cette idée est rendue rigoureuse au moyen de la topologie faible. La dérivée de  $\mu$  appliquée à la fonction indicatrice est comprise comme la variation totale de la fonction indicatrice  $\mathbf{1}_A$ . Par exemple, si  $m$  est la mesure de Lebesgue sur  $\mathbb{R}^d$ , alors

$$m^+(A) := \sup \left\{ \int_A \operatorname{div} g \, dx \mid g \in \mathcal{C}_c^\infty(\mathbb{R}^d), \|g\|_\infty \leq 1 \right\}.$$

Ces deux notions de périmètre coïncident lorsque  $A$  a une frontière  $\partial A$  suffisamment lisse [80, Theorem 5]. Une fois ces notions de volume et de périmètre définies, le problème isopérimétrique prend tout son sens. Il s'agit de minimiser le périmètre parmi des ensembles dont le volume est prescrit. On s'intéresse à

$$\inf \{ \operatorname{Per}(A) \mid m(A) = v \}.$$

Trois questions naturelles se posent :

1. L'infimum est-il atteint?
2. Quelle est la valeur de l'infimum?
3. Si l'infimum est atteint, par quelles ensembles l'est-il?

### 1.5.1 L'inégalité isopérimétrique euclidienne

Dans le cas où  $X$  est l'espace euclidien  $\mathbb{R}^d$  et  $m = \lambda$  est la mesure de Lebesgue, les réponses sont oui l'infimum est atteint, il est atteint pour les boules euclidiennes  $B_d$ , et sa valeur est  $d\lambda(B_d)$ . Ceci conduit à l'inégalité isopérimétrique euclidienne : pour tout  $A \subset \mathbb{R}^d$  mesurable, on a

$$(1.36) \quad \lambda^+(A) \geq d\lambda(B_d)^{\frac{1}{d}} \lambda(A)^{1-\frac{1}{d}}.$$

Nous nous référons au résumé [174] de R. Osserman pour ce résultat classique. Il existe de nombreuses preuves différentes de l'inégalité isopérimétrique euclidienne. L'une des plus classiques est celle basée sur la symétrisation de Steiner. Au cœur des méthodes de symétrisation se trouve l'idée [10] que si nous avons un moyen de construire à partir de tout ensemble  $E$  un ensemble  $E^*$  avec le même volume que  $E$  mais avec un périmètre plus petit, alors en répétant ce processus encore et encore, il montrera que les ensembles isopérimétriques possèdent les mêmes symétries que le processus itéré. En particulier, la symétrisation de Steiner est basée sur la construction de  $E^*$  à partir des symétries par rapport aux hyperplans. L'heuristique à la fin de l'itération est alors que les boules sont les seuls ensembles symétriques par rapport à tous les hyperplans passant par leur barycentre. Nous nous référons au cours [117] donné par H. Hopf en 1940. Les cas d'égalité dans la symétrisation de Steiner ont été étudiés dans le cours [73].

### L'inégalité de Brunn-Minkowski

Une preuve très courte de l'inégalité isopérimétrique euclidienne peut être donnée en utilisant l'inégalité de Brunn-Minkowski. Voir [91, Chapitre 3]. L'inégalité de Brunn-Minkowski stipule que pour tout  $A, B \subset \mathbb{R}^d$ ,

$$|A + B|^{\frac{1}{d}} \geq |A|^{\frac{1}{d}} + |B|^{\frac{1}{d}}$$

où  $|\cdot|$  est le volume de Lebesgue, et la somme d'ensembles est définie par

$$A + B := \{a + b \mid a \in A, b \in B\}.$$

Avec cette notion de somme d'ensembles, on voit que le  $r$ -élargissement d'un ensemble  $A$  satisfait à

$$A_r = A + rB_d$$

où  $B_d$  est la boule unitaire de  $\mathbb{R}^d$ . Alors l'inégalité de Brunn-Minkowski donne

$$|A_r| \geq \left(|A|^{\frac{1}{d}} + |rB_d|^{\frac{1}{d}}\right)^d \geq |A| + d|A|^{\frac{d-1}{d}} |B_d|^{\frac{1}{d}} r$$

où l'inégalité classique  $\forall x \geq 0, (1+x)^d \geq x^d + dx^{d-1}$  est utilisé pour la seconde inéquation. On obtient alors

$$\frac{|A_r \setminus A|}{r} = \frac{|A_r| - |A|}{r} \geq d|A|^{\frac{d-1}{d}} |B_d|^{\frac{1}{d}},$$

qui donne l'inégalité isopérimétrique euclidienne (1.36) en prenant la liminf. L'inégalité de Brunn-Minkowski a été découverte et prouvée à la fin du XIXe siècle. Pour plus de détails sur son histoire, voir le livre [190]. Pour une exposition très claire de l'inégalité de Brunn-Minkowski, nous renvoyons le lecteur au livre [95] de R. Gardner. En particulier, soulignons

l'inégalité de Prekopa-Leindler comme la formulation fonctionnelle de l'inégalité de Brunn-Minkowski : pour toutes les fonctions  $f, g, h : \mathbb{R}^d \rightarrow [0, +\infty)$  satisfaisant à  $\forall x, y \in \mathbb{R}^d, \forall \lambda \in [0, 1], h((1 - \lambda)x + \lambda y) \geq f(x)^{1-\lambda}g(y)^\lambda$ , l'inégalité suivante à lieu

$$\int h(x) dx \geq \left( \int f(x) dx \right)^{1-\lambda} \left( \int g(x) dx \right)^\lambda.$$

L'inégalité de Prekopa-Leindler trouve son origine dans l'article [112] de 1953, où R. Henstock et A. Macbeath ont prouvé l'inégalité de Brunn-Minkowski à partir d'une formulation fonctionnelle.

## Transport optimal

Une autre preuve très importante utilise le transport optimal. La théorie du transport optimal est très ancienne, puisqu'elle remonte à la fin du 18ème siècle avec sa formulation mathématique par G. Monge. Ce problème n'a cessé de s'enrichir au fil du temps, donnant lieu aujourd'hui à de nombreux outils puissants aux applications très variées, comme la géométrie (avec par exemple la notion synthétique de courbure-dimension), les statistiques (voir [175] par exemple), ou l'apprentissage automatique (voir [54]). Nous présentons ici brièvement les principaux concepts du transport optimal, et nous renvoyons au cours d'introduction [103] de N. Gozlan, P-M. Samson et P-A. Zitt, et au livre complet [213] de C. Villani. Étant donné deux mesures de probabilité  $\mu$  et  $\nu$  sur  $\mathbb{R}^d$ , le but est de comprendre comment transporter la masse de  $\mu$  sur la masse de  $\nu$  de la manière la plus efficace, c'est-à-dire en minimisant une certaine quantité vue comme un coût par les économistes comme L. Kantorovich, vue comme une énergie par les physiciens comme A-M. Ampère. Plus formellement, l'objectif est d'étudier

$$(1.37) \quad \inf_{\pi} \int c(x, y) d\pi(x, y),$$

où le supremum s'étend sur toutes les mesures de probabilité  $\pi$  sur  $\mathbb{R}^d \times \mathbb{R}^d$  ayant  $\mu$  pour première marginale, et  $\nu$  pour seconde marginale, c'est-à-dire telle que

$$\forall A \subset \mathbb{R}^d, \quad \pi(A \times \mathbb{R}^d) = \mu(A) \quad \text{and} \quad \pi(\mathbb{R}^d \times A) = \nu(A).$$

La fonction  $c$  est traditionnellement appelée fonction de coût, et dans ce qui suit nous ne traitons que de la plus étudiée, qui est le coût quadratique euclidien  $c(x, y) = \frac{1}{2}|x - y|^2$ . Néanmoins, il convient de mentionner le nombre croissant de travaux sur les fonctions de coût générales, on peut citer par exemple la thèse de M. Prodhomme [180]. Par le théorème très général de Prokhorov, l'infimum dans (1.37) est atteint sous des hypothèses naturelles sur la fonction de coût  $c$ , qui sont satisfaites pour le coût quadratique euclidien. Un couplage  $\pi$  atteignant le minimum est appelé couplage optimal. Lorsque le couplage optimal prend la forme  $\pi = (Id, T)^\#(\mu)$ , où  $(\cdot)^\#$  dénote le pushforward, le couplage est dit déterministe et est souvent identifié à l'application  $T$  qui satisfait par conséquent  $T^\#\mu = \nu$ . Un résultat majeur est le théorème de Brenier [45, 46].

**Théorème.** (*Théorème de Brenier*) Soit  $c(x, y) = |x - y|^2$  le coût quadratique euclidien, et  $\mu$  une mesure de probabilité absolument continue par rapport à la mesure de Lebesgue sur  $\mathbb{R}^d$ . Si l'infimum (2.37) est fini, alors il existe une application  $T : \mathbb{R}^d \rightarrow \mathbb{R}^d$  telle que  $\pi = (Id, T)^\#(\mu)$  soit un couplage optimal. De plus,  $T = \nabla\phi$  est le gradient d'une certaine fonction convexe  $\phi : \mathbb{R}^d \rightarrow \mathbb{R}$ . Cette fonction optimale  $T$  est appelée la l'application de Brenier, et  $\phi$  est appelé son potentiel.

Non seulement le théorème de Brenier donne un couplage déterministe, mais il donne également une structure convexe sous-jacente. Supposons encore que la mesure de probabilité

$\nu$  ait une densité  $\rho_2$  par rapport à la mesure de Lebesgue, et désignons par  $\rho_1$  la densité de  $\mu$ . Le fait que  $T^\#(\mu) = \nu$  signifie que pour toute fonction mesurable bornée  $f : \mathbb{R}^d \rightarrow \mathbb{R}$ ,

$$\int f(x)\rho_2(x)dx = \int f(T(x))\rho_1(x)dx.$$

Si nous appliquons ensuite formellement la formule de changement de variable au côté gauche de l'égalité, nous obtenons

$$\int f(T(x))\rho_2(T(x)) \det(\nabla T(x)) dx = \int f(T(x))\rho_1(x)dx,$$

ce qui implique que presque sûrement par rapport à la mesure de Lebesgue,

$$\rho_1(x) = \rho_2(T(x)) \det(\nabla T(x)).$$

En réécrivant cette équation pour le potentiel convexe  $\phi$  tel que  $T = \nabla\phi$ , nous en déduisons l'équation différentielle partielle de Monge-Ampère

$$(1.38) \quad \rho_1(x) = \rho_2(\nabla\phi) \det(\nabla^2\phi(x)),$$

où  $\nabla^2$  désigne le hessien. La justification rigoureuse de l'utilisation de la formule de changement de variable nécessite des outils très profonds et fastidieux d'analyse convexe et de théorie de la mesure. Ceci est dû au fait que le théorème de Brenier ne donne pas une régularité forte pour le gradient  $\nabla\phi$ . Le lien avec l'équation de Monge-Ampère est très important car ce type d'EDP était déjà largement utilisé en mécanique des fluides. L. Caffarelli a prouvé le théorème suivant en étudiant la régularité de l'équation de Monge-Ampère (1.38) pour l'application de Brenier.

**Théorème.** (*Théorème de contraction de Caffarelli*) [52, 53] Soit  $\mu$  la mesure gaussienne sur  $\mathbb{R}^d$ , et soit  $\nu$  une loi de probabilité avec une densité de Lebesgue donnée par  $d\nu(x) = e^{-V(x)}dx$  pour un potentiel uniformément convexe  $\text{Hess } V \geq \rho I_d$ , avec  $\rho > 0$ , et  $I_d$  la matrice d'identité. Alors l'application de Brenier transportant la gaussienne  $\mu$  sur la mesure uniformément log-concave  $\nu$  est 1-Lipschitz :

$$\|\nabla T\| \leq 1.$$

Alors que la preuve originale de Caffarelli est basée sur une étude précise de l'équation de Monge-Ampère, une nouvelle preuve a été établie par régularisation entropique dans [90] et simplifiée dans [72].

Revenons à l'inégalité isopérimétrique. En 1957, dans l'article [129], H. Knothe a prouvé l'inégalité isopérimétrique euclidienne (1.36) en utilisant un couplage entre la mesure uniforme sur un ensemble  $A$  et la mesure uniforme sur la boule  $B_d$ . À l'origine, le couplage utilisé était le couplage de Knothe-Rosenblatt. Ce couplage a l'avantage d'être défini de manière constructive. Cependant, à la place de l'application de Brenier, l'application de Knothe-Rosenblatt n'a pas de structure convexe sous-jacente. Par conséquent, M. Gromov a utilisé l'application de Brenier comme couplage déterministe entre la distribution uniforme sur un ensemble  $A$  et la boule  $B_d$  dans [163, Appendice] afin de simplifier la preuve. Néanmoins, la philosophie de la preuve avec transport optimal reste la même. Nous allons maintenant l'esquisser. Soit  $A \subset \mathbb{R}^d$  et soit  $B$  une boule euclidienne de dimension  $d$  de rayon  $r$  centrée en 0 telle que  $B$  ait le même volume que  $A$  :  $|A| = |B|$ . Soit  $T = \nabla\phi$  l'application de Brenier entre la mesure uniforme  $\frac{1}{|A|}\mathbf{1}_A$  sur  $A$  et la mesure uniforme  $\frac{1}{|B|}\mathbf{1}_B$  sur  $B$ . L'équation de Monge-Ampère donne que, presque sûrement,

$$\frac{|B|}{|A|} = \det \nabla T \mathbf{1}_A.$$

En intégrant cette relation sur  $A$ , on trouve

$$(1.39) \quad d|B|^{\frac{1}{d}}|A|^{1-\frac{1}{d}} = \int_A d \cdot (\det \nabla T)^{\frac{1}{d}} dx.$$

Mais puisque  $\nabla T$  est le Hessien du potentiel  $\phi$  qui est convexe, en notant  $\lambda_i \geq 0$  les valeurs propres de  $\nabla T$ , et en utilisant l'inégalité arithmético-géométrique, on a

$$(\det \nabla T)^{\frac{1}{d}} = \left( \prod \lambda_i \right)^{\frac{1}{d}} \leq \frac{1}{d} \sum \lambda_i = \frac{1}{d} \operatorname{div} T.$$

Ensuite,

$$\int_A d \cdot (\det \nabla T)^{\frac{1}{d}} dx \leq \int_A \operatorname{div} T dx = \int_{\partial A} T \cdot \nu_A, d\sigma \leq r \operatorname{Per}(A),$$

où l'égalité utilise le théorème de Stokes, et la dernière inégalité vient du fait que  $T \in B$ , donc  $|T| \leq r$ . Enfin, avec (1.39) et puisque  $|B| = r^d |B_d|$  où  $B_d$  est la boule euclidienne unitaire, on retrouve l'égalité (1.36).  $\square$

Cette preuve basée sur le transport optimal a été affinée par A. Figalli, F. Maggi et A. Pratelli dans [93] pour obtenir un résultat de stabilité de l'inégalité isopérimétrique. Le théorème de Brenier et la preuve de Gromov de l'inégalité isopérimétrique sont au cœur de leur approche. Les résultats de stabilité consistent en une étude quantitative du déficit

$$\lambda^+(A) - d\lambda(B_d)^{\frac{1}{d}} \lambda(A)^{1-\frac{1}{d}}$$

dans l'inégalité isopérimétrique. Le but est de quantifier l'optimalité de l'inégalité isopérimétrique lorsque  $A$  est en un certain sens proche de la boule. La notion de proximité utilisée par A. Figalli, F. Maggi et A. Pratelli est l'asymétrie de Fraenkel :

$$F(A) := \inf \left\{ \frac{|A \Delta B(x, r)|}{|A|} \mid x \in \mathbb{R}^d, r > 0, \text{ tel que } |B(x, r)| = |A| \right\},$$

où  $\Delta$  désigne la différence symétrique des ensembles  $A \Delta B := (A \setminus B) \cup (B \setminus A)$ , et  $B(x, r)$  désigne la boule euclidienne de rayon  $r$  centrée en  $x$ . Leur principal résultat est le suivant.

**Théorème.** [93, Thm 1.1] *Il existe une constante finie  $C_d$  dépendant de la dimension, telle que*

$$\operatorname{Per}(A) \geq d\lambda(B_d)^{\frac{1}{d}} \lambda(A)^{1-\frac{1}{d}} \left( 1 + \left( \frac{F(A)}{C_d} \right)^2 \right).$$

*De plus l'exposant 2 est optimal.*

Pour plus d'informations sur la question de la stabilité dans les inégalités isopérimétriques, nous renvoyons le lecteur à la présentation [147] de F. Maggi.

## Calibration

Mentionnons une dernière approche pour prouver l'inégalité isopérimétrique euclidienne en dimension 2. Cette approche est basée sur le concept de calibration (voir [109]). Esquissons l'idée principale. Soit  $\beta$  une  $(d-1)$ -forme différentielle sur  $\mathbb{R}^d$  telle que  $|\beta| \leq 1$ . Si  $N \subset \mathbb{R}^d$  est un ensemble ouvert et connexe tel que  $\beta$  soit la forme volume sur  $N$ , alors  $\operatorname{Per}(N)$  est minimal parmi tous les ensembles  $N'$  de même  $\beta$ -volume :  $\int_N d\beta = \int_{N'} d\beta$ . En effet, puisque  $\beta$  est la forme volume sur  $N$ , on a

$$\operatorname{Per}(N) = \int_{\partial N} \beta = \int_N d\beta = \int_{N'} d\beta = \int_{\partial N'} \beta \leq \operatorname{Per}(N'),$$

où nous avons utilisé le théorème de Stokes dans les deuxième et quatrième égalités, l'hypothèse sur  $N'$  dans la troisième égalité, et l'hypothèse  $|\beta| \leq 1$  dans la dernière inégalité. La construction d'une telle forme différentielle sur  $\mathbb{R}^2$  a été faite par F. Helein dans [110, 111]. L'avantage de cette méthode est qu'elle donne une preuve très courte essentiellement basée sur le théorème de Stokes, l'inconvénient est que la construction d'une telle forme différentielle  $\beta$  se fait à la main au cas par cas.

### Approche variationnelle

Par définition, les ensembles isopérimétriques sont des minima globaux de la fonctionnelle périmètre restreinte aux ensembles de volume fixé. De ce point de vue, il y a deux conditions nécessaires pour qu'un ensemble soit isopérimétrique. Il doit être un point critique de la fonctionnelle périmètre et, de plus, le hessien du périmètre doit être défini positif. Comme pour les fonctions, la condition définie positive assure que l'ensemble est un minimum local du périmètre. L'étude des points critiques de la fonctionnelle de périmètre conduit à la notion d'ensembles stationnaires. L'étude des points pour lesquels le hessien de la fonctionnelle de périmètre est positif conduit à la notion d'ensembles stables. Ces notions de points critiques et de hessien défini positif sont formellement mises en œuvre en termes de première et deuxième variation. Les articles pionniers de cette approche sont les articles de 1982 et 1983 [168, 167] de M. Hiroshi, et l'article de 1984 [16] de L. Barbosa et M. Do Carmo. Décrivons brièvement les principaux concepts.

Soit  $A \subset \mathbb{R}^d$  un ensemble ouvert lisse et  $\partial A = \Sigma$  sa frontière. Soit  $N$  le vecteur normal unitaire dirigé vers l'intérieur de  $\Sigma$ . Soit  $(\phi_t : \mathbb{R}^d \rightarrow \mathbb{R}^d)_t$  une variation à un paramètre de champ de vecteurs infinitésimal  $X = \left. \frac{d\phi_t}{dt} \right|_{t=0}$ . Soit  $u = \langle X, N \rangle$  la composante normale de  $X$ . Définissons  $A_t := \phi_t(A)$ ,  $\Sigma_t := \phi_t(\Sigma)$ , et prenons  $V(t) = \text{Vol}(A_t)$  la variation de volume, et  $P(t) = \text{Per}(A_t)$  la variation de périmètre. En utilisant des formules standard de géométrie différentielle (voir par exemple [195]), on peut calculer la première variation du volume et du périmètre :

$$(1.40) \quad V'(0) = - \int_{\Sigma} u \, da, \quad \text{and} \quad P'(0) = - \int_{\Sigma} (d-1)H u \, da,$$

où  $da$  est la mesure de surface sur  $\Sigma$  et  $H$  est la courbure moyenne euclidienne de  $\Sigma$  par rapport à  $N$ , ( $H$  est la trace de la différentielle  $dN$  de  $N$ ). On peut également calculer la seconde variation du volume et du périmètre de  $A$  :

$$(1.41) \quad (P - HV)''(0) = Q(u, u) := \int_{\Sigma} (|\nabla_{\Sigma} u|^2 - |\sigma|^2 u^2) \, da,$$

où  $\nabla_{\Sigma} u$  est le gradient de  $u$  calculé dans la géométrie de  $\Sigma$ , et  $|\sigma|^2$  est la somme des carrés des courbures principales de  $\Sigma$ . Nous pouvons maintenant introduire les définitions suivantes.

**Definition 1.** *Nous disons qu'une variation  $(\phi_t)_t$  préserve le volume si  $V(t) = \text{constante}$ , On dit que l'ensemble  $A$  est stationnaire si  $P'(0) = 0$  pour toute variation  $(\phi_t)_t$  préservant le volume, et nous disons que  $A$  est stable lorsque  $P''(0) \geq 0$  pour toute variation préservant le volume  $(\phi_t)$ .*

Le calcul des formules de première et deuxième variations (1.40) et (1.41) nous permet alors d'énoncer la caractérisation suivante des ensembles stationnaires et stables.

**Proposition 2.** *L'ensemble  $A$  est stationnaire si et seulement si sa frontière  $\Sigma$  a une courbure moyenne constante  $H_0$ .*

*L'ensemble  $A$  est stable si et seulement si pour toute fonction lisse  $u$  à support compact dans  $\Sigma$  telle que  $\int_{\Sigma} u \, da = 0$ , on a  $Q(u, u) \geq 0$ .*

La caractérisation de la stationnarité au moyen d'une courbure moyenne constante permet de relier directement ce problème à la théorie des surfaces minimales qui sont les surfaces à courbure moyenne nulle. La théorie des surfaces minimales a été fortement développée depuis la création du calcul des variations par L. Euler et J-L. Lagrange au 18ème siècle (voir [154] pour un aperçu des avancées modernes).

La caractérisation de la stabilité donne un lien avec les inégalités de Poincaré. En effet, avec la formule (1.41), on peut voir que la condition  $Q(u, u) \geq 0$  est en fait une inégalité de Poincaré pondérée sur  $\Sigma$ . Par conséquent, la connaissance de la constante de Poincaré optimale sur  $\Sigma$  fournira des informations sur les ensembles stables (voir chapitre 6 Proposition 87).



## 1.5.2 Inégalité isopérimétriques dans les espaces de probabilité

Le problème isopérimétrique classique concerne la mesure de Lebesgue sur  $\mathbb{R}^d$ . Cependant, nous avons vu que la notion de périmètre a un sens pour toute mesure  $\mu$ . En particulier, le problème isopérimétrique est très intéressant lorsque  $\mu$  est une mesure de probabilité. A propos du problème de l'existence des ensembles isopérimétriques dans ce contexte pondéré, nous renvoyons à l'article [166] de F. Morgan et A. Pratelli. Mentionnons dans ce domaine la preuve récente de la conjecture de la double bulle par M. Hutchings, F. Morgan, M. Ritoré et A. Ros en 2002 [119] et la preuve des conjectures de la double bulle et de la multi-bulle dans l'espace gaussien par E. Milman et J. Neeman en 2018 [162].

### Le profil isopérimétrique

Le profil isopérimétrique  $I_\mu$  d'une mesure de probabilité  $\mu$  est une fonction qui encode beaucoup d'informations sur le problème isopérimétrique associé. Il est défini par

$$I_\mu : [0, 1] \rightarrow \mathbb{R}_+,$$

$$I_\mu(p) := \inf \{ \text{Per}_\mu(A) \mid \mu(A) = p \}.$$

Précisons que le profil isopérimétrique dépend de la mesure  $\mu$ , mais aussi de la distance utilisée dans la définition du périmètre, bien que cette distance n'apparaisse pas dans la notation  $I_\mu$ . Dans la suite, nous nous intéresserons principalement aux distances euclidiennes et uniformes. En dimension un, le profil isopérimétrique ne dépend pas de la distance, comme on peut facilement le voir dans la définition du contenu de Minkowski (1.35). Le profil isopérimétrique est symétrique autour de  $\frac{1}{2}$  et s'annule en 0 et en 1. Dans [32], S. Bobkov a montré que si  $\mu$  est une mesure log-concave paire sur  $\mathbb{R}$ , alors son profil isopérimétrique est donné par

$$I_\mu = \phi \circ \Phi^{-1}$$

où  $\phi$  est la densité de Lebesgue de  $\mu$ , et  $\Phi$  est sa fonction de répartition. Rappelons qu'une mesure sur  $\mathbb{R}^d$  est dite log-concave lorsqu'elle possède une densité de Lebesgue de la forme  $e^{-V}$  pour un potentiel convexe  $V$ . Cette formule signifie que les demi-espaces  $(-\infty, t)$  sont des ensembles isopérimétriques. Par exemple, on peut donner les profils isopérimétriques de deux lois importantes : lorsque  $\mu = \gamma$  est la gaussienne, alors  $I_\gamma = \phi \circ \Phi^{-1}$  avec  $\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$ , et lorsque  $\mu$  est la loi exponentielle sur  $\mathbb{R}_+$ , alors  $I_\mu(p) = \min(p, 1-p)$ . Les premières preuves du cas gaussien ont été données indépendamment en 1974 par C. Borell [41] et V. Sudakov et B. Tsirel'son [203]. S. Bobkov a également montré que sur la droite réelle, la log-concavité de la mesure est équivalente à la concavité de son profil isopérimétrique (voir [32, Proposition A.1]). Mentionnons la thèse de doctorat de V. Bayle [25] sur la généralisation de cette propriété de concavité du profil isopérimétrique lorsque  $\mu$  est la mesure de volume sur les variétés riemanniennes. Dans la suite, nous serons souvent intéressés par le profil isopérimétrique des mesures produits. Mentionnons la comparaison immédiate

$$(1.42) \quad I_{\mu^{\otimes n}} \leq I_{\mu^{\otimes (n-1)}}$$

qui résulte de l'évaluation de la fonction périmètre sur des ensembles de la forme  $A \times \mathbb{R}^{d-1}$ . Une quantité importante dans ce domaine d'étude est le profil isopérimétrique de dimension infinie, défini par

$$I_{\mu^\infty} := \inf_n I_{\mu^{\otimes n}}.$$

Évidemment, à partir de la définition et de la propriété de monotonie (1.42), le profil isopérimétrique de dimension infinie est majoré de la manière suivante :

$$I_{\mu^\infty} \leq I_\mu.$$

Une question intéressante est de savoir quand est-ce que l'inégalité inverse se produit, c'est-à-dire de comprendre quand est-ce qu'il existe une constante  $c \in (0, 1)$  telle que

$$(1.43) \quad I_{\mu^\infty} \geq c I_\mu.$$

L'inégalité isopérimétrique de dimension infinie (1.43) signifie que pour tout  $n$ , il existe un demi-espace de la forme  $A \times \mathbb{R}^{n-1}$  qui est isopérimétrique à la constante multiplicative  $c^{-1}$  près. Une mesure de probabilité  $\mu$  satisfaisant (1.43) a un comportement intermédiaire entre l'exponentielle et la gaussienne. On se réfère à l'article [21], de F. Barthe, P. Cattiaux et C. Roberto. Inversement, les mêmes auteurs ont prouvé ce qui suit :

**Théorème.** [20, théorème 46] *Pour tout  $\alpha \in [1, 2]$ , soit  $d\mu_\alpha(t) = \frac{1}{Z_\alpha} e^{-|t|^\alpha} dt$ , avec  $Z_\alpha$  une constante de normalisation. Alors il existe une constante universelle  $K > 0$  telle que pour tout  $\alpha \in [1, 2]$ ,*

$$I_{\mu_\alpha^\infty} \geq K I_{\mu_\alpha}.$$

### Approche variationnelle

Dans la section 1.5.1, nous avons présenté l'approche variationnelle du problème isopérimétrique dans le cas euclidien. Nous traitons ici le cas pondéré, c'est-à-dire lorsque la mesure a une densité par rapport à la mesure de Lebesgue. La philosophie est la même que dans le cas de la mesure de Lebesgue, cependant les calculs de la première et de la deuxième variation changent légèrement en raison de la présence de la fonction de poids. Les ensembles stationnaires sont alors caractérisés par leur courbure moyenne généralisée constante au lieu de la courbure moyenne classique, et les ensembles stables sont encore caractérisés par certaines inégalités de Poincaré pondérées (avec un poids différent du cas euclidien). Nous faisons référence à l'article [184] de C. Rosales, A. Canete, V. Bayle et F. Morgan.

**Proposition 3.** [184, proposition 3.2 et lemme 3.8] *Soit  $\mu$  une mesure sur  $\mathbb{R}^d$  avec une densité de Lebesgue  $f(x) = e^{\psi(x)}$ . On a alors:*

- *l'ensemble  $A$  est stationnaire si et seulement si son bord  $\Sigma$  a une courbure moyenne généralisée constante*

$$(d-1)H - \langle \nabla\psi, N \rangle = \text{constante},$$

*où  $H$  est la courbure moyenne de  $\Sigma$  par rapport à son vecteur normal unitaire  $N$  pointant vers l'intérieur,*

- *l'ensemble  $A$  est stable si et seulement si, pour toute fonction  $u \in \mathcal{C}_c^\infty(\Sigma)$  à support compact dans  $\Sigma$ , telle que  $\int_\Sigma u f da = 0$ , on a*

$$\int_\Sigma (|\nabla_\Sigma u|^2 + (\nabla^2\psi(N, N) - |\sigma|^2) u^2) da \geq 0,$$

*où  $\nabla_\Sigma u$  est le gradient de  $u$  dans  $\Sigma$ , et  $|\sigma|^2$  est la somme des carrés des courbures principales de  $\Sigma$ , et  $\nabla^2$  désigne le Hessien.*

Dans l'article [183] publié en 2021, C. Rosales étend ces résultats au cas d'une variété riemannienne à poids possédant un bord. Le seul ajout dans la caractérisation des ensembles stationnaires est l'exigence que le bord  $\Sigma$  de l'ensemble coupe orthogonalement le bord de la variété (voir [183, Lemme 2.3]). Notons que cette condition est très naturelle, par exemple lorsqu'on pense à une bulle de savon sur une surface. Nous utiliserons le résultat de Rosales dans la section 6.4 du chapitre 6.

## Périmètre anisotropique

Dans cette section, nous abordons brièvement les problèmes isopérimétriques anisotropes. Ces problèmes consistent en un problème isopérimétrique où le périmètre ne provient pas d'une distance mais d'une semi-norme  $H$  correspondant à une fonction non négative, convexe et positivement homogène de degré un sur  $\mathbb{R}^d$ . Le périmètre associé est alors défini par

$$\text{Per}_H(A) := \int_{\partial A} H(N(x)) ds(x),$$

où  $N(x)$  est la normale unitaire pointant vers l'extérieur de  $\partial A$  en  $x$ . Lorsque  $H$  est la norme euclidienne, on retrouve le périmètre euclidien standard. Les problèmes isopérimétriques anisotropes consistent à minimiser le périmètre  $\text{Per}_H$  sous une contrainte de volume euclidien fixe. La forme de Wulff joue alors le rôle de la boule euclidienne, et est définie par

$$W_H := \{x \in \mathbb{R}^d \mid x \cdot v < H(v), \forall v \in \mathbb{S}^{d-1}\}.$$

Dans le cas où  $H$  est la norme euclidienne, la forme de Wulff est la boule euclidienne. Le problème de l'isopérimétrie anisotrope est résolu par le célèbre théorème de Wulff. G. Wulff l'a énoncé sans preuve dans [216] en 1901, et J. Taylor l'a prouvé dans [207, 208] en 1974-1975.

**Théorème.** (*Inégalité de Wulff [216, 207, 208]*) Pour tout  $A \subset \mathbb{R}^d$  de volume fini non trivial, on a

$$\frac{\text{Per}_H(A)}{|A|^{\frac{d-1}{d}}} \geq \frac{\text{Per}_H(W)}{|W|^{\frac{d-1}{d}}}.$$

De plus, l'égalité est atteinte si, et seulement si  $A = aW + b$  pour un certain  $a > 0$  et  $b \in \mathbb{R}^d$ .

Dans ce contexte anisotrope, X. Cabré, X. Ros-Oton et J. Serra ont donné une nouvelle preuve de l'inégalité de Wulff avec une méthode basée sur l'estimation Alexandroff-Backelman-Pucci. Cette estimation est une borne uniforme sur la solution du problème de Dirichlet associé aux opérateurs uniformément elliptiques du second ordre (voir [49] pour les détails sur cette estimation). Cette méthode avait déjà été utilisée par X. Cabré dans [48] pour retrouver l'inégalité isopérimétrique euclidienne classique. Dans le cas anisotrope, cette méthode a permis aux trois auteurs de généraliser l'inégalité de Wulff aux cônes convexes ouverts de  $\mathbb{R}^d$ . Signalons que A. Figalli, F. Maggi et A. Pratelli ont obtenu un résultat de stabilité quantitative pour l'inégalité de Wulff sur  $\mathbb{R}^d$  dans [93]. Leur preuve est basée sur le transport optimal, comme mentionné dans la section sur le transport optimal dans 1.5.1.

## 1.6 Formulations fonctionnelles de l'isopérimétrie

Les inégalités isopérimétriques peuvent être encodées par des inégalités fonctionnelles. Les plus courantes sont les inégalités de Poincaré, les inégalités de Sobolev logarithmique et les inégalités de type Bobkov.

### 1.6.1 Inégalités de Poincaré

Nous avons vu dans la section 1.5.2 que le profil isopérimétrique de la loi exponentielle est  $I(t) = \min(t, 1-t)$ . Les inégalités isopérimétriques associées pour une mesure de probabilité  $\mu$  sont de la forme

$$(1.44) \quad \text{Per}_\mu(A) \geq h_\mu \min(\mu(A), 1 - \mu(A)),$$

où  $h_\mu$  est la plus petite constante telle que (1.44) soit vérifiée. Cette constante  $h_\mu$  est la constante de Cheeger, déjà mentionnée à la section 1.1.3 de cette introduction. Nous avons vu que la constante de Cheeger, et donc les inégalités isopérimétriques de Cheeger, sont liées

aux inégalités de Poincaré, dans le sens où les inégalités de Poincaré impliquent les inégalités isopérimétriques de Cheeger. Par conséquent, l'inégalité de Poincaré est une formulation fonctionnelle de l'inégalité de Cheeger. Plus précisément, introduisons les inégalités de Poincaré  $(p, q)$ . Soit  $p, q \in [1, +\infty]$ . Nous disons que la mesure  $\mu$  satisfait à une inégalité Poincaré- $(p, q)$  s'il existe une constante  $C_{p,q} > 0$  telle que pour toutes fonction suffisamment régulière  $f$ , l'on ait

$$(1.45) \quad \|f - M_\mu(f)\|_{L^p(\mu)} \leq C_{p,q}(\mu) \|\nabla f\|_{L^q(\mu)},$$

où  $M_\mu(f)$  désigne une médiane de  $f$ . Notons que l'inégalité de Poincaré classique correspond dans cette terminologie à l'inégalité Poincaré- $(2, 2)$ . Avec cette définition, l'inégalité isopérimétrique de Cheeger est équivalente à l'inégalité Poincaré- $(1, 1)$ . Ce résultat est chronologiquement dû à H. Federer et W. Fleming [92], V. Maz'ja [151], S. Bobkov et C. Houdré [36].

**Théorème.** [36, 92, 151]

*L'inégalité isopérimétrique de Cheeger (1.44) a lieu si et seulement si l'inégalité de Poincaré- $(1, 1)$  a également lieu. De plus, dans ce cas, les constantes optimales des deux inégalités satisfont à la relation suivante :  $h_\mu = \frac{1}{C_{1,1}(\mu)}$ .*

Esquissons brièvement la preuve contenue dans le papier [36]. De la même manière que pour l'inégalité de Poincaré classique (voir la section 1.1), l'inégalité isopérimétrique de Cheeger découle de l'inégalité Poincaré- $(1, 1)$  en évaluant cette dernière avec les fonctions caractéristiques des ensembles mesurables. Cette idée est rendue rigoureuse par l'approximation des indicatrices avec des fonctions lipschitziennes. L'énoncé inverse est basé sur la formule de la co-aire. Pour simplifier, supposons que  $M_\mu(f) = 0$ . Alors

$$\int |\nabla f| d\mu = \int_{\mathbb{R}} \mu^+(f > t) dt \geq h_\mu \left( \int_{-\infty}^0 (1 - \mu(f > t)) dt + \int_0^\infty \mu(f > t) dt \right) = h_\mu \int |f| d\mu,$$

où les première et dernière égalités proviennent de la formule de la co-aire, et l'inéquation est l'inégalité isopérimétrique de Cheeger (1.44).  $\square$

Contrairement à l'inégalité classique de Poincaré- $(2, 2)$ , l'inégalité de Poincaré- $(1, 1)$  n'a pas d'interprétation spectrale. Cependant, sa constante optimale est liée à la constante de Cheeger de la même manière que la constante de Poincaré  $C_P(\mu)$  est liée à la première valeur propre du générateur associé. De plus, le lien entre ces différentes inégalités est plus profond : lorsque  $\mu$  est une distribution log-concave, E. Milman a prouvé que toutes les inégalités de Poincaré  $(p, q)$  sont équivalentes [159].

**Théorème.** (E. Milman, 2009, [159, Théorème 2.4])

*Lorsque  $\mu$  est log-concave, alors pour tout  $1 \leq p \leq q \leq \infty$  et tout  $1 \leq p' \leq q' \leq \infty$ , on a*

$$C_{p'q'}(\mu) \leq K p' C_{p,q}(\mu),$$

où  $K > 0$  est une constante universelle.

En rapport avec la constante de Cheeger et les mesures log-concaves, mentionnons la célèbre conjecture de Kannan-Lovasz-Simonovits.

**Conjecture 4.** (R. Kannan, L. Lovasz, and M. Simonovits, 1995, [122])

*Il existe une constante universelle  $C > 0$ , telle que pour tout  $d \geq 1$  et pour toute mesure log-concave  $\mu$  sur  $\mathbb{R}^d$ ,*

$$h_\mu \geq C \inf_H \frac{\mu^+(\partial H)}{\min(\mu(H), 1 - \mu(H))},$$

où l'infimum s'étend sur tous les demi-espaces de  $\mathbb{R}^d$ .

Nous renvoyons le lecteur au résumé [139] pour ses multiples connexions avec d'autres conjectures dans l'étude de la géométrie des corps convexes. Les techniques de localisation sont au cœur des tentatives de preuve de la conjecture KLS. L'objectif des techniques de localisation est de réduire un problème multidimensionnel à un problème unidimensionnel en désintégrant une mesure le long d'"aiguilles" unidimensionnelles. Les aiguilles peuvent être obtenues en itérant des bisections d'hyperplans, comme dans [122], ou en suivant les trajectoires d'un processus stochastique (localisation stochastique d'Eldan, voir [138]), ou encore en suivant les lignes de transport de masse d'un problème de transport optimal (localisation de Klartag-Cavalletti-Mondino, voir [61, 126]). Les deux premiers schémas de localisation relèvent principalement de l'analyse convexe, tandis que le troisième relève de la géométrie des espaces de courbure-dimension. Le meilleur résultat vers une preuve de la conjecture KLS est le théorème suivant de B. Klartag et J. Lehec en 2022 contenu dans [127].

**Théorème.** [127, Théorème 1.1] *Il existe une constante universelle  $c' > 0$  telle que pour tous les  $d \geq 2$  et pour toutes les mesures isotropes log-concaves  $\mu$  dans  $\mathbb{R}^d$ ,*

$$h_\mu \geq c' (\log d)^{-5}.$$

La preuve est basée sur une implémentation minutieuse et technique de la localisation stochastique d'Eldan avec l'analyse de l'évolution des mesures log-concaves le long du flot de la chaleur.

## 1.6.2 Inégalités de Bobkov

Nous avons vu dans la section précédente que les inégalités de Poincaré constituent la forme fonctionnelle des inégalités isopérimétriques de Cheeger, qui sont elles-mêmes liées au profil isopérimétrique de la loi exponentielle. Nous avons vu que le profil isopérimétrique de la loi normale est donné par  $I_\gamma = \phi \circ \Phi^{-1}$ , avec  $\phi$  la densité de la loi gaussienne et  $\Phi$  sa fonction de répartition. On dit qu'une distribution de probabilité satisfait à une inégalité isopérimétrique gaussienne lorsque, pour tout ensemble mesurable  $A$ , l'inégalité suivante a lieu pour une certaine constante  $c \geq 0$  :

$$(1.46) \quad \text{Per}_\mu(A) \geq c I_\gamma(\mu(A)).$$

La formulation fonctionnelle la plus reconnue de l'inégalité isopérimétrique gaussienne est l'inégalité de Bobkov introduite en 1997 dans [33] par S. Bobkov. Pour toute fonction localement Lipschitzienne  $f : \mathbb{R}^d \rightarrow [0, 1]$ ,

$$(1.47) \quad I_\gamma \left( \int_{\mathbb{R}^d} f d\mu \right) \leq \int_{\mathbb{R}^d} \sqrt{I_\gamma(f)^2 + |\nabla f|^2} d\mu.$$

Le fait que la loi normale  $\mu = \gamma$  satisfasse (1.47) admet de nombreuses preuves différentes. La preuve originale dans [33] consiste à d'abord montrer que l'inégalité de Bobkov satisfait la propriété de tensorisation, comme les inégalités de Poincaré (voir la fin de la section 1.1). En d'autres termes, si  $\mu_1$  et  $\mu_2$  satisfont (1.47), alors la mesure produit  $\mu_1 \otimes \mu_2$  satisfait également (1.47). Deuxièmement, la preuve consiste à montrer l'inégalité de Bobkov lorsque  $\mu$  est une mesure de Bernoulli. Dans ce cas, l'inégalité (1.47) se réduit à une inégalité à deux points. Ensuite, par le théorème de la limite centrale et la propriété de tensorisation, S. Bobkov a pu en déduire l'inégalité (1.47) pour la loi normale.

En 1996, dans [15], D. Bakry et M. Ledoux ont prouvé ce fait par une méthode de semi-groupes. Ils ont utilisé le semigroupe d'Ornstein-Uhlenbeck  $(P_t)_{t \geq 0}$  pour lequel la loi normale est la loi d'équilibre. Au cœur de leur preuve se trouve le fait que pour toutes les fonctions lisses  $f$  prenant leurs valeurs dans  $[0, 1]$ , la quantité

$$\int \sqrt{I_\gamma(P_t(f)) + |\nabla P_t(f)|^2} d\gamma$$

est décroissante en  $t$ . Nous renvoyons à [136] pour une preuve courte utilisant cette méthode.

Mentionnons la preuve par martingale de F. Barthe et B. Maurey [23]. Leur approche consiste à prouver l'inégalité de Bobkov en deux points en utilisant le théorème d'arrêt de Doob pour une martingale bien choisie. L'inégalité (1.47) découle alors de la propriété de tensorisation et du théorème de la limite centrale, comme dans la preuve originale de S. Bobkov.

L'inégalité de Bobkov est équivalente à l'inégalité isopérimétrique gaussienne dans le sens suivant.

**Théorème.** [23] Une mesure de probabilité  $\mu$  satisfait à  $I_\mu \geq cI_\gamma$  pour une certaine constante  $c \geq 0$ , si et seulement si, pour toute fonction Lipschitzienne  $f : \mathbb{R}^d \rightarrow [0, 1]$ ,

$$(1.48) \quad I_\gamma \left( \int_{\mathbb{R}^d} f \, d\mu \right) \leq \int_{\mathbb{R}^d} \sqrt{I_\gamma(f)^2 + \frac{1}{c^2} |\nabla f|^2} \, d\mu.$$

Pour prouver le sens "si", on procède de façon habituelle en évaluant (1.48) par des approximations de fonctions indicatrices. Pour prouver la réciproque, nous nous ramenons au cas  $c = 1$ , et la méthode est basée sur la technique de réarrangement, qui, dans ce contexte, donne que les ensembles de la forme  $R = \{(x, t) \mid \Phi(t) < f(x)\}$ , ont un périmètre égal à

$$\text{Per}_{\mu \otimes \gamma}(R) = \int_{\mathbb{R}^d} \sqrt{I_\gamma(f)^2 + |\nabla f|^2} \, d\mu.$$

La méthode de réarrangement est la généralisation pour les fonctions de la méthode de symétrisation des ensembles. Nous nous référons à [2]. Étant donné que

$$(\mu \otimes \gamma)(R) = \int \int_{-\infty}^{\Phi^{-1}(f(x))} d\gamma \, dx = \int f(x) \, d\mu(x),$$

il s'ensuit que

$$\text{Per}_{\mu \otimes \gamma}(R) \geq I_{\mu \otimes \gamma} \left( \int f(x) \, d\mu(x) \right).$$

Mais nous savons que  $I_\mu \geq I_\gamma$  implique  $I_{\mu \otimes \gamma} \geq I_{\gamma \otimes \gamma}$  (cf. [18, Théorème 8]), nous pouvons donc conclure grâce à la propriété de tensorisation du profil isopérimétrique de la loi gaussienne:  $I_{\gamma \otimes \gamma} = I_\gamma$ .  $\square$

L'inégalité isopérimétrique  $I_\mu \geq cI_\gamma$  est rigide : en effet, si  $\mu$  est une mesure de probabilité telle que  $I_\mu = cI_\gamma$ , alors  $\mu = \mathcal{N}(0, c^2 I_d)$  est égale à la gaussienne [38]. La stabilité n'a pas encore été étudiée. Une question intéressante est de savoir quelles fonctions  $f$  réalisent l'égalité dans (1.48). La réponse a été donnée par E. Carlen et J. Kerce dans [55]. L'égalité est obtenue pour les fonctions indicatrices des demi-espaces et pour les fonctions de la forme  $\Phi(v \cdot x + b)$  où  $\Phi$  désigne la fonction de répartition de la loi normale,  $v$  est un vecteur dans  $\mathbb{R}^d$ ,  $b$  est un nombre réel, et  $\cdot$  désigne le produit scalaire sur  $\mathbb{R}^d$ .

Une extension naturelle de (1.48) consiste à remplacer la loi normale par une autre mesure de référence  $\nu$ . Ceci a pour conséquence de remplacer le profil isopérimétrique gaussien  $I_\gamma$  par le profil isopérimétrique  $I_\nu$  de  $\nu$ . Nous disons qu'une mesure de probabilité  $\mu$  satisfait à une inégalité de type Bobkov par rapport à une mesure de probabilité de référence  $\nu$  lorsqu'il existe une certaine constante  $\alpha > 0$  telle que

$$(1.49) \quad I_\nu \left( \int f \, d\mu \right) \leq \int \sqrt{I_\nu(f)^2 + |\alpha \nabla f|^2} \, d\mu.$$

S. Bobkov a prouvé dans [35] que (1.49) est satisfaite par les mesures  $\mu$  ayant une constante de Cheeger positive lorsque la mesure de référence est la mesure logistique  $\nu$ . Rappelons que la mesure logistique est la distribution de probabilité sur  $\mathbb{R}$  ayant une densité de Lebesgue  $\frac{e^x}{(1+e^x)^2}$  et un profil isopérimétrique  $I_\nu(p) = p(1-p)$ . Une question naturelle est alors de savoir si (1.49)

équivalent à une certaine inégalité isopérimétrique de la forme  $I_\mu \geq \alpha I_\nu$  comme dans le cas où  $\nu = \gamma$  est la loi normale. La réponse est négative. La seule étape qui ne fonctionne pas lorsque nous remplaçons  $\gamma$  par un  $\nu$  général dans la preuve du théorème 1.6.2 est l'égalité  $I_{\nu \otimes \nu} = I_\nu$ . En effet, la loi normale est la seule mesure satisfaisant à cette propriété (cf. [38]). Cependant, B. Huou a montré la caractérisation suivante de l'inégalité de type Bobkov (1.49).

**Théorème.** [118, Section 5] *La mesure  $\mu$  satisfait (1.49) si, et seulement si,*

$$(1.50) \quad I_{\mu \otimes \nu_\alpha} = \frac{1}{\alpha} I_\nu,$$

où  $\nu_\alpha$  désigne la dilatation  $\alpha$  de  $\nu$ .

La dilatation  $\alpha$  de  $\nu$  est définie comme la loi de la variable aléatoire  $\alpha X$  lorsque  $X$  est tiré avec la loi  $\nu$ . Il est facile de voir avec le théorème 1.6.2 que l'équation (1.50) est équivalente à l'inégalité  $I_\mu \geq \frac{1}{\alpha} I_\gamma$  lorsque  $\nu = \gamma$ . Par conséquent, le théorème 1.6.2 généralise bien le théorème 1.6.2.

### 1.6.3 Inégalités de Sobolev logarithmiques

Concluons cette introduction aux formulations fonctionnelles de l'isopérimétrie en mentionnant brièvement les inégalités de Sobolev logarithmique (LSI). Nous renvoyons à [7] pour une exposition complète. Nous disons qu'une mesure de probabilité  $\mu$  dans  $\mathbb{R}^d$  satisfait une inégalité de Sobolev logarithmique lorsqu'il existe une constante  $C_{LSI} > 0$  telle que pour toutes les fonctions  $f \in \mathcal{C}^2(\mathbb{R}^d)$ , l'on ait

$$(1.51) \quad \text{Ent}_\mu(f^2) \leq C_{LSI}(\mu) \int |\nabla f|^2 d\mu.$$

L'entropie est définie pour les fonctions à valeurs positives par

$$\text{Ent}_\mu(f) := \int f \log(f) d\mu - \int f d\mu \log \int f d\mu.$$

Les inégalités de Sobolev logarithmiques ont été introduites par L. Gross en 1975 dans [108] afin de caractériser l'hypercontractivité d'un semigroupe de Markov à l'aide d'inégalités fonctionnelles. Plus tard, F. Otto et C. Villani ont montré des liens avec les inégalités de transport de Talagrand [206]. Les inégalités de Sobolev logarithmiques satisfont la propriété de tensorisation. De plus, elles impliquent les inégalités de Poincaré, comme on peut le montrer directement en prenant  $1 + \varepsilon f$  au lieu de  $f$  dans (1.51) et en faisant tendre  $\varepsilon$  vers zéro. Comme les inégalités de Poincaré, les inégalités de Sobolev logarithmique sont liées aux propriétés spectrales des générateurs de Markov. En particulier, elles impliquent le caractère discret du spectre ainsi que l'intégrabilité des fonctions propres (voir [13, Section 5.3]).

L'inégalité de Bobkov (1.48) implique l'inégalité de Sobolev logarithmique, reliant ainsi les inégalités LSI aux problèmes isopérimétriques. Ceci peut être vu directement à partir de l'inégalité (1.48) en prenant  $f = \varepsilon g^2$ , en utilisant l'équivalence

$$I_\gamma(x) \underset{x \rightarrow 0}{\sim} x \sqrt{2 \log \frac{1}{x}},$$

et en faisant tendre  $\varepsilon$  vers zéro.

## 1.7 Plan de la thèse

Dans cette section, nous présentons brièvement la structure de cette thèse.

Dans le chapitre 3, nous étudions la stabilité de la constante de Poincaré de la loi d'équilibre  $\mu$  d'un processus de diffusion de Markov, dans un espace de mesures satisfaisant à certaines conditions de normalisation. Des résultats de stabilité par rapport à la distance de variation totale et à la distance de Wasserstein-1 sont établis. La même stratégie de preuve est utilisée dans les deux résultats : nous montrons d'abord que sous ces conditions de normalisation, les mesures test  $\nu$  satisfont à une formule d'intégration par parties approchée dont le terme d'erreur est contrôlé par la différence entre les constantes de Poincaré. Ensuite, nous prenons le pushforward de  $\mu$  par la première fonction propre  $f_1$ . Pour ce faire, nous supposons que l'opérateur carré du champ  $\Gamma(f_1)$  se factorise en  $\Gamma(f_1) = h \circ f_1$  pour une fonction réelle  $h$ . Cette hypothèse est justifiée par le lemme 13. Nous mettons ensuite en œuvre la méthode de Stein et trouvons des conditions suffisantes sur  $h$  (c'est-à-dire sur  $\Gamma(f_1)$ ) pour avoir l'équivalent du lemme de Stein pour  $\mu$  (au lieu de la loi normale  $\gamma$  dans le résultat classique). Pour obtenir la stabilité en distance de variation totale, nous demandons à  $h$  d'être minorée de façon uniforme:  $h \geq \kappa > 0$ . Cette hypothèse est vérifiée par les mesures uniformément log-concaves, qui fournissent ainsi un exemple d'application de notre premier résultat (voir la section 3.6.3). Pour obtenir une stabilité quantifiée par la distance de Wasserstein-1, nous avons besoin de certaines conditions de croissance sur  $h$  à la frontière de son domaine de définition (cf. Proposition 36). Ces conditions sont introduites pour des raisons techniques, mais sont vérifiées dans tous les principaux exemples, à savoir les lois gamma sur  $\mathbb{R}_+$  et les lois beta (voir les sections 3.6.2 et 3.6.4).

Dans le chapitre 4, nous étudions la stabilité des constantes de Poincaré améliorées d'une loi d'équilibre  $\mu$  d'un processus de diffusion de Markov, par rapport à un espace de mesures satisfaisant à des conditions de normalisation comme dans le chapitre 3. Comme la constante de Poincaré correspond à l'inverse de la première valeur propre non nulle, la  $k$ -ième constante de Poincaré correspond à l'inverse de la  $k$ -ième valeur propre non nulle, comptée sans multiplicité. Nous prouvons un résultat de stabilité par rapport à la distance de Wasserstein-1, dans le cas où  $\mu$  est une mesure de probabilité sur  $\mathbb{R}$ . La stratégie de preuve est principalement la même que pour les résultats de stabilité du chapitre 3. D'abord, nous montrons que sous les conditions de normalisation, les mesures test  $\nu$  satisfont la formule d'intégration par parties approchée, dont le terme d'erreur est contrôlé par la différence entre les  $k$ -ièmes valeurs propres ainsi que par le défaut d'orthogonalité de la  $k$ -ième fonction propre. Ensuite, nous voulons prendre le pushforward par une fonction propre  $f_k$  associée à la  $k$ -ième valeur propre non nulle. Cette étape est plus compliquée que dans le cas de la première valeur propre, car il n'existe pas de lemme de monotonie pour  $f_k$  comme le lemme 13 dans le cas de  $f_1$ . Pour résoudre ce problème, nous restreignons le pushforward à l'extérieur des points critiques de la fonction, c'est-à-dire sur chaque intervalle où l'opérateur carré du champ  $\Gamma(f_k)$  ne s'annule pas. Ensuite, nous utilisons la méthode de Stein sur ces intervalles, et adaptons la proposition 36 pour dériver l'équivalent du lemme de Stein. Afin d'obtenir un résultat de stabilité, nous devons demander certaines conditions de croissance sur  $\Gamma(f_k)$  au voisinage de ses zéros. Ces conditions sont simplement techniques et nécessaires pour le calcul, mais sont vérifiées dans le cas des lois normales sur  $\mathbb{R}$ , des lois gamma sur  $\mathbb{R}_+$ , et des lois beta sur  $[-1, 1]$  (voir les sections 4.5, 4.6 et 4.7).

Dans le chapitre 5, nous étudions la stabilité de la constante de Poincaré des espaces modèles  $RCD(K, N)$ . Des résultats de stabilité par rapport à la distance de Wasserstein-1 sont obtenus pour le cas de la dimension positive et finie  $N > 1$ , pour le cas de dimension infinie  $N = +\infty$ , et pour le cas de dimension effective négative  $N \leq -1$ . Dans chaque cas, la mesure modèle est

- dans le cas de la dimension finie  $N > 1$ , la mesure modèle est la loi beta  $\mu = \beta\left(\frac{N}{2}, \frac{N}{2}\right)$ ,
- pour une dimension infinie  $N = +\infty$ , la mesure modèle est la gaussienne  $\mu = \gamma$ ,
- pour une dimension effective négative  $N \leq -1$ , la mesure modèle est la loi de Cauchy généralisée sur  $\mathbb{R}$ .



Dans tous les cas, la stratégie de preuve reste essentiellement la même. Tout d'abord, nous montrons que toutes les mesures test  $\nu$  satisfaisant à la condition RCD correspondante satisfont une formule d'intégration par parties approchée. Ceci est une conséquence de l'identité de Bochner, et d'un lemme technique ( Cf. Proposition 66 ou Lemme 69) qui diffère selon la valeur de  $N$ . Si  $N > 1$  est fini, alors la Proposition 66 est plus précise que le Lemme 69 grâce à l'ultracontractivité (5.12). La différence qui en résulte est que dans le cas de  $N > 1$  fini, la borne obtenue est d'ordre  $\varepsilon$ , alors que dans les autres cas elle est d'ordre  $\varepsilon \log(\frac{1}{\varepsilon})$ , où  $\varepsilon$  est la différence entre les premières valeurs propres de  $\mu$  et  $\nu$ . Deuxièmement, nous utilisons la méthode de Stein. Dans le cas des lois bêta, cela a déjà été fait dans [81, 100], il nous suffit donc de l'adapter dans ce contexte. De même, dans le cas de la loi normale, nous disposons déjà du lemme de Stein. Cependant, dans le cas de la loi de Cauchy généralisée, il n'y avait pas de lemme de Stein disponible dans la littérature, nous l'avons donc développé dans le Théorème 77.

Dans le chapitre 6, nous étudions la constante de Bobkov optimale en utilisant le résultat de B. Huou (voir le théorème 82 donnant une équivalence entre l'inégalité de Bobkov et un problème isopérimétrique pour les mesures produits en dimension 2). En utilisant cette interprétation et une approche variationnelle, nous dérivons une nouvelle borne pour la constante de Bobkov  $L^2$  de la mesure logistique sur  $\mathbb{R}$  (voir la section 6.3). De plus, nous effectuons dans ce cadre une étude numérique des ensembles stationnaires pour la mesure produit de l'exponentielle sur  $\mathbb{R}_+$  dilatée (voir Section 6.4). Cela nous donne une idée de la forme des courbes isopérimétriques. Les tracés et les approximations numériques sont réalisés à l'aide du logiciel libre Octave (équivalent libre de Matlab). Enfin, nous étudions le lien entre le problème isopérimétrique uniforme pour les mesures produit de dimension 2 donné par le théorème de Huou ( problème équivalent à l'inégalité de Bobkov  $L^1$ ) et les inégalités isopérimétriques uniformes de dimension infinie. En particulier, nous montrons que le problème du produit est plus fort que l'inégalité isopérimétrique de dimension infinie (voir la section 6.5.1). Ces résultats impliquant des constantes optimales sont illustrés en utilisant l'exponentielle sur  $\mathbb{R}_+$  dans la section 6.5.3.

Dans le chapitre 7, nous présentons quelques perspectives et travaux en cours pouvant découler des chapitres précédents. Dans la section 7.1, nous présentons un résultat de monotonie de la constante de Poincaré le long du flot de renormalisation de Polchinski. Nous obtenons ce résultat en généralisant un théorème de B. Klartag et E. Putterman avec les outils du critère multi-échelle de Bakry-Emery introduit par T. Bodineau et R. Bauerchmidt. Dans la section 7.2, nous présentons deux façons d'obtenir un lemme de Stein généralisé pour les mesures satisfaisant à une condition de courbure-dimension positive, en utilisant une méthode de couplage. Dans la section 7.3, nous présentons une méthode pour généraliser les résultats du chapitre 3 dans le cas d'une première valeur propre avec une multiplicité supérieure à 1. Nous dérivons une formule d'intégration par parties multidimensionnelles approchée, et l'utilisons pour obtenir un résultat de stabilité pour le cas d'une distribution strictement log-concave sur  $\mathbb{R}^d$  sous une hypothèse de courbure supplémentaire. Enfin, dans la section 7.4, nous discutons d'une manière d'implémenter la méthode de Stein dans un cadre isopérimétrique. En particulier, nous montrons qu'il est possible de retrouver l'inégalité isopérimétrique de Cheeger de la loi gaussienne en utilisant cette stratégie.

# Chapter 2

## Introduction (EN)

During the past 3 years, I mainly studied Poincaré inequalities and their stability, Stein's method, curvature-dimension conditions, isoperimetric problems, and also functional formulations of isoperimetry such as Bobkov's inequalities. These topics and the results I obtained are presented in a general way in the first chapter of this thesis. The remaining chapters contain these results and their proof in detail. A more detailed outline of this thesis can be found at the end of the first chapter.

### 2.1 Poincaré inequalities

The notion of Poincaré inequality dates back to the work of H. Poincaré in the late 19th century [177, 178]. A Poincaré inequality is a type of functional inequality bounding the  $L^2$  norm of a function by the  $L^2$  norm of its derivatives and a constant that depends in some way on the geometry of the space where the function is defined. The more classical Poincaré inequalities that we are interested in are the following. Let  $\Omega \subset \mathbb{R}^d$  be an open bounded and convex subset of  $\mathbb{R}^d$ . Then there exists some constant  $C > 0$  such that

$$(2.1) \quad \forall u \in C^1(\Omega), \quad \text{Var}_\Omega(u) \leq C \int_\Omega \|\nabla u(x)\|^2 dx$$

where the norm  $\|\cdot\|$  denotes the Euclidean norm in  $\mathbb{R}^d$  and the variance is defined by

$$\text{Var}_\Omega(u) := \int_\Omega \left| u - \frac{1}{|\Omega|} \int_\Omega u(y) dy \right|^2 dx.$$

The right hand side term is homogeneous to an energy, so Poincaré inequalities control the variance of state  $u$  by its energy. The largest set of functions where this type of inequality can be extended is called Sobolev space. The notion of Sobolev spaces can be approached from a technical point of view, using notions such as weak derivatives or traces. However, in our context, even if it is not entirely rigorous, the idea to always keep in mind is that the first Sobolev space  $H^1(\Omega)$  is the largest space of functions where the Poincaré inequality (2.1) is non trivial.

Let us sketch the proof of (2.1). We denote by

$$\rho_{x,y} : s \mapsto sy + (1-s)x, \quad s \in [0, 1],$$

the parametrization of the segment between  $x$  and  $y$ . By the convexity and boundedness of  $\Omega$ , the Cauchy-Schwarz inequality and Jensen's inequality, we have

$$\begin{aligned}
\int_{\Omega} \left| u - \frac{1}{|\Omega|} \int_{\Omega} u(y) dy \right|^2 dx &= \frac{1}{|\Omega|^2} \int_{\Omega} \left| \int_{\Omega} (u(x) - u(y)) dy \right|^2 dx \\
&= \frac{1}{|\Omega|^2} \int_{\Omega} \left| \int_{\Omega} \int_0^1 \langle \nabla u(\rho_{x,y}(s)), \rho'_{x,y}(s) \rangle ds dy \right|^2 dx \\
&\leq \frac{1}{|\Omega|^2} \int_{\Omega} \left| \int_{\Omega} \int_0^1 \|\nabla u(\rho_{x,y}(s))\| \|y - x\| ds dy \right|^2 dx \\
&\leq \frac{1}{|\Omega|^2} \int_{\Omega} \int_{\Omega} \int_0^1 \|\nabla u(\rho_{x,y}(s))\|^2 \|y - x\|^2 ds dy dx \\
&\leq \frac{\text{diam}(\Omega)^2}{|\Omega|^2} \int_{\Omega} \int_{\Omega} \int_0^1 \|\nabla u(\rho_{x,y}(s))\|^2 ds dy dx.
\end{aligned}$$

Using Fubini's theorem,

$$\begin{aligned}
&\frac{\text{diam}(\Omega)^2}{|\Omega|^2} \int_{\Omega} \int_{\Omega} \int_0^1 \|\nabla u(\rho_{x,y}(s))\|^2 ds dy dx \\
&= \frac{\text{diam}(\Omega)^2}{|\Omega|^2} \left( \int_{\Omega} \int_0^{\frac{1}{2}} \int_{\Omega} \|\nabla u(\rho_{x,y}(s))\|^2 dx ds dy + \int_{\Omega} \int_{\frac{1}{2}}^1 \int_{\Omega} \|\nabla u(\rho_{x,y}(s))\|^2 dy ds dx \right).
\end{aligned}$$

On the one hand, using the change of variable  $R_{s,y} : \Omega \rightarrow \Omega$  given by  $R_{s,y}(x) = sy + (1-s)x$ , since  $\det(\nabla R_{s,y}) = (1-s)^d$ , for  $d \geq 2$ , we get

$$\begin{aligned}
\int_{\Omega} \int_0^{\frac{1}{2}} \int_{\Omega} \|\nabla u(\rho_{x,y}(s))\|^2 dx ds dy &= \int_{\Omega} \int_0^{\frac{1}{2}} \frac{1}{(1-s)^d} \int_{\Omega} \|\nabla u(\rho_{x,y}(s))\|^2 (1-s)^d dx ds dy \\
&= \int_{\Omega} \int_0^{\frac{1}{2}} \frac{1}{(1-s)^d} \int_{\Omega} \|\nabla u(z)\|^2 dz ds dy \\
&\leq \frac{2^{d-1}}{d-1} |\Omega| \int_{\Omega} \|\nabla u(z)\|^2 dz.
\end{aligned}$$

On the other hand, using the change of variable  $R_{s,x} : \Omega \rightarrow \Omega$  given by  $R_{s,x}(y) = sy + (1-s)x$ , since  $\det(\nabla R_{s,x}) = s^d$ , we get

$$\begin{aligned}
\int_{\Omega} \int_{\frac{1}{2}}^1 \int_{\Omega} \|\nabla u(\rho_{x,y}(s))\|^2 dy ds dx &= \int_{\Omega} \int_{\frac{1}{2}}^1 \frac{1}{s^d} \int_{\Omega} \|\nabla u(\rho_{x,y}(s))\|^2 s^d dy ds dx \\
&= \int_{\Omega} \int_{\frac{1}{2}}^1 \frac{1}{s^d} \int_{\Omega} \|\nabla u(z)\|^2 dz ds dx \\
&\leq \frac{2^{d-1}}{d-1} |\Omega| \int_{\Omega} \|\nabla u(z)\|^2 dz.
\end{aligned}$$

So finally

$$\frac{\text{diam}(\Omega)^2}{|\Omega|^2} \int_{\Omega} \int_{\Omega} \int_0^1 \|\nabla u(\rho_{x,y}(s))\|^2 ds dy dx \leq \frac{2^d \text{diam}(\Omega)^2}{(d-1)|\Omega|} \int_{\Omega} \|\nabla u(z)\|^2 dz,$$

which concludes the proof.

Hence the constant  $C = \frac{2^d \text{diam}(\Omega)^2}{(d-1)|\Omega|}$  satisfies (2.1). Let us underline that it is finite because  $\Omega$  is bounded. Moreover, the proof makes crucial use of convexity. It is possible to generalize it to bounded open sets with Lipschitz boundary (we refer to the complete book [152] by V. Maz'ya). It has also been extended to sets with even less regularity such as subanalytical sets

(see [212]). We are interested in the sharp constant of the Poincaré inequality, that is in the lower constant  $C$  satisfying (2.1). We denote by  $C_P(\Omega)$  this sharp constant, and we usually refer to it as "the" Poincaré constant:

$$(2.2) \quad C_P(\Omega) := \sup_{u \in H^1(\Omega) \setminus \{0\}} \frac{\text{Var}_\Omega(u)}{\int_\Omega \|\nabla u(x)\|^2 dx}.$$

The constant  $C_P(\Omega)$  crucially depends on the geometry of the domain  $\Omega$ . For instance, if  $\Omega = \mathbb{R}^d$  or if  $\Omega$  is bounded but has a too singular boundary, then there is no such Poincaré inequality.

We are naturally lead to study the extension of the notion of Poincaré inequalities for probability distributions. Let  $\mu$  be a probability distribution on some  $\mathbb{R}^d$  (the same still works a broader context, as we will see latter). We say that  $\mu$  satisfies a Poincaré inequality if there exists some constant  $C > 0$  such that

$$(2.3) \quad \forall u \in H^1(\mu), \quad \text{Var}_\mu(u) \leq C \int \|\nabla u(x)\|^2 d\mu(x),$$

where similarly to the above presentation of Poincaré inequalities,  $\text{Var}_\mu$  denotes the variance with respect to  $\mu$ ,

$$\text{Var}_\mu(u) := \int \left| u(x) - \int u(y) d\mu(y) \right|^2 d\mu(x),$$

and  $H^1(\mu)$  has to be thought as the set of all functions  $u$  such that the quantities involved make sense. For instance, a common way to write it is as the set of all functions  $u \in L^2(\mu)$  such that their weak gradient (which is always defined) is also in  $L^2(\mu)$ . Similarly again, we will consider the sharp Poincaré constant defined as the smallest constant  $C > 0$  satisfying (2.3) and we will denote it by  $C_P(\mu)$ .

From the above calculations, a standard example of probability distributions satisfying a Poincaré inequality are uniform distributions on an open bounded convex domain of  $\mathbb{R}^d$ . In fact, many classical probability distributions satisfy it. Let us cite for instance the Normal distribution, all gamma distributions in  $\mathbb{R}_+$ , including the exponential distribution and the chi-2 distribution) and all beta distributions on  $[0, 1]$ .

An important property of the Poincaré inequality is the tensorization property. If one takes  $\mu_1, \dots, \mu_n$ , probability distributions all satisfying the Poincaré inequality with sharp constant  $C_P(\mu_i)$  respectively, then their product distribution  $\mu_1 \otimes \dots \otimes \mu_n$  satisfies a Poincaré inequality with sharp constant  $\max_i C_P(\mu_i)$ . That easily follows from the tensorization formula of the variance (see e.g. [7, Proposition 1.4.1]).

$$\text{Var}_{\mu_1 \otimes \dots \otimes \mu_n}(f) \leq \sum_{i=1}^n \mathbb{E}_{\mu_1 \otimes \dots \otimes \mu_n} [\text{Var}_{\mu_i}(f)]$$

The tensorization property suggests that for a multidimensional probability distribution, the sharp Poincaré constant only depends on one dimensional direction in the space. For a product measure, this direction actually is a classical coordinate, but this is not the case for a nonproduct measure. However, this heuristic has guided us in the results presented in the chapters 3 and 5, where spectral coordinate plays the role of the preferred direction. One has then to find which direction is it in order to understand the sharpness of the Poincaré inequality. It is rather clear that as a real function, that direction should be a function saturating the inequality, i.e. achieving the equality in the Poincaré inequality. We will see in Section 2.1.2 that these functions are eigenfunctions of a certain operator.

In the following three sections, we will present some of the main consequences of Poincaré inequalities from our combined Analysis, Geometry and Probability point of view: the concentration of measure phenomenon, the existence of a spectral gap, and a bound on the isoperimetric profile.

### 2.1.1 Exponential concentration of measure phenomenon

The Poincaré inequality implies an exponential concentration of measure phenomenon. This property quantifies the extent to which the mass of a measure  $\mu$  is located around its mean. The study of this phenomenon goes back to P. Lévy, M. Gromov and V. Milman [141, 106, 107, 105]. We refer the reader to the textbook [137] by M. Ledoux.

**Theorem.** (e.g. [137, Theorem 3.1]) *If the probability distribution  $\mu$  satisfies the Poincaré inequality (2.3) with sharp constant  $C_P(\mu)$ , then for all 1-Lipschitz functions  $f$  and for all  $t \geq 0$ ,*

$$(2.4) \quad \mu(|f - \mu(f)| \geq t) \leq 8 \exp\left(-t \sqrt{\frac{2}{C_P(\mu)}}\right).$$

Let us sketch out how this concentration result can be proved. Let us denote  $C = C_P(\mu)$  the sharp Poincaré constant, and apply the Poincaré inequality (2.3) to the function  $u = \exp\left(\frac{1}{2}sf\right)$ . We get

$$\int e^{sf} d\mu - \left(\int e^{\frac{1}{2}sf} d\mu\right)^2 = \text{Var}_\mu(u) \leq C \int \left\| \nabla \exp\left(\frac{1}{2}sf\right) \right\|^2 d\mu \leq C \frac{s^2}{4} \int e^{sf} d\mu,$$

where in the last inequality we have used that  $\|\nabla f\| \leq 1$  because  $f$  is assumed to be 1-Lipschitz. Hence it follows

$$(2.5) \quad \int e^{sf} d\mu \leq \left(1 - \frac{Cs^2}{4}\right)^{-1} \left(\int \exp\left(\frac{1}{2}sf\right) d\mu\right)^2.$$

We iterate Inequality (2.5) with  $\frac{s}{2}, \frac{s}{4}, \dots, \frac{s}{2^n}, \dots$ . By a standard study of the convergence of the series, we see that it is possible to take the limit as  $n \rightarrow +\infty$  and we get

$$(2.6) \quad \forall s \in \left[0, \sqrt{\frac{2}{C}}\right], \quad \int \exp(s(f - \mu(f))) d\mu \leq 2 \left(1 - \frac{Cs^2}{4}\right)^{-1}.$$

Finally we have, using the Markov inequality and Inequality (2.6), for all  $s \in \left[0, \sqrt{\frac{2}{C}}\right]$ ,

$$\begin{aligned} \mu(|f - \mu(f)| \geq t) &= \mu\left(\exp(s|f - \mu(f)|) \geq e^{st}\right) \\ &\leq \frac{1}{e^{st}} \int \exp(s|f - \mu(f)|) d\mu \\ &\leq 2 \left(1 - \frac{Cs^2}{4}\right)^{-1} e^{-st} \end{aligned}$$

and we conclude by taking  $s = \sqrt{\frac{2}{C}}$ .

In particular, this exponential concentration property tells us about probability distributions that do not satisfy a Poincaré inequality. For instance, heavy tailed probability measures do not satisfy the Poincaré inequality. An example is Cauchy distribution on  $\mathbb{R}$  given by the density  $\frac{1}{\pi} \frac{1}{1+x^2}$ . Let us finally point out that this exponential concentration phenomenon result shows that in some sense, exponential tailed distributions (such as the exponential measure on  $\mathbb{R}_+$  or the logistic measure) are extremal points among the set of distributions satisfying the Poincaré inequality.

### 2.1.2 Spectral gap

Let us write

$$(2.7) \quad d\mu(x) = e^{-V(x)} dx,$$

where  $-V$  is the logarithm of the density. At the moment we can think of  $dx$  as the Lebesgue measure on  $\mathbb{R}^d$ , but the form (2.7) still makes sense in the broader context of weighted Riemannian manifold ( $dx$  denotes then the volume measure) or the RCD setting. We discuss this matter below.

From a probabilistic point of view, we know that the Laplace operator is linked to the Lebesgue measure in the following manner. Let  $\Omega$  be a bounded domain of  $\mathbb{R}^d$ . Boundedness allows to define a uniform probability via Lebesgue measure restricted to  $\Omega$ . Then all  $C^1(\Omega)$  functions satisfying the Neumann boundary condition  $\nabla f \cdot n = 0$  on  $\partial\Omega$ , have zero mean Laplacian with respect to the Lebesgue measure. Indeed, by the divergence theorem (used in the second equality),

$$\int_{\Omega} \Delta f \, dx = \int_{\Omega} \operatorname{div} \nabla f \, dx = \int_{\partial\Omega} \nabla f \cdot n \, d\sigma = 0,$$

with  $n$  the outward-pointing normal.

Similarly, since  $\mu$  has a density with respect to the Lebesgue measure, there exists an elliptic operator playing the same role for  $\mu$  as the Laplacian does for  $dx$ . This elliptic operator is the Laplacian drifted by the gradient of the log-density of  $\mu$ :

$$(2.8) \quad L = \Delta - \nabla V \cdot \nabla$$

Indeed, using again the divergence theorem, (at the fourth equality),

$$\begin{aligned} \int_{\Omega} Lf \, d\mu &= \int_{\Omega} (\Delta f - \nabla V \cdot \nabla f) e^{-V} \, dx \\ &= \int_{\Omega} (e^{-V} \operatorname{div} \nabla f + \nabla e^{-V} \cdot \nabla f) \, dx \\ &= \int_{\Omega} \operatorname{div} (e^{-V} \nabla f) \, dx \\ &= \int_{\partial\Omega} (e^{-V} \nabla f) \cdot n \, d\sigma \\ &= \int_{\partial\Omega} \nabla f \cdot n \, d\mu|_{\partial\Omega}. \end{aligned}$$

When  $\Omega = \mathbb{R}^d$ , the boundary term does not appear, consequently this property is true for  $f$  as soon as  $Lf$  is  $\mu$ -integrable, i.e. for functions that do not grow too fast at infinity.

Moreover, a stronger property holds. Actually, this operator  $L$  is symmetric on  $L^2(\mu)$ , that is, denoting by  $\mathcal{D} \subset L^2(\mu)$  the domain of  $L$ ,

$$\forall f, g \in \mathcal{D}, \quad \int f Lg \, d\mu = \int g Lf \, d\mu.$$

This property is often referred to as reversibility of  $\mu$  with respect to  $L$ . This terminology comes from the theory of Markov processes. A last property that is worth mentioning in this introduction is the following formula:

$$(2.9) \quad \int f Lf \, d\mu = - \int \|\nabla f\|^2 \, d\mu.$$

This equality will be often referred to as an integration by parts formula in the sequel. In particular it implies that the operator  $-L$  is positive semi-definite in  $L^2(\mu)$ . So all its eigenvalues are non negative.

In addition to having a particularly nice form, the elliptic operator  $L$  can as well be studied in order to deduce properties of the measure  $\mu$ , or conversely one can study the measure  $\mu$  in order to deduce properties on this operator  $L$ . Going back to Poincaré inequalities, we have now enough material to understand the link between the Poincaré inequality for  $\mu$  and the repartition of eigenvalues of  $L$ . We already know that all eigenvalues of  $-L$  are non negative,

but under a Poincaré inequality (2.3) this is reinforced: its eigenvalues are in  $\{0\} \cup [\frac{1}{C_P(\mu)}, +\infty)$ . Moreover,  $\frac{1}{C_P(\mu)}$  is the lowest number satisfying this property, we can therefore say that  $\frac{1}{C_P(\mu)}$  is the spectral gap of the operator  $-L$ . Let us underline that in the sequel we often talk about the spectral gap of  $L$  or of  $\mu$  by misuse of language. The reason why the Poincaré inequality implies a gap in the spectrum is simple. Let  $\lambda$  be an eigenvalue of  $-L$  and  $f$  an associated eigenfunction which we choose to be centered. Evaluating the Poincaré inequality (2.3) with  $f$  and using the integration by parts formula (2.9), we get

$$\int f^2 d\mu = \text{Var}_\mu(f) \leq C_P(\mu) \int \|\nabla f\|^2 d\mu = -C_P(\mu) \int f L f d\mu = C_P(\mu) \lambda \int f^2 d\mu,$$

from which we effectively deduce that if  $\lambda \neq 0$  then

$$\lambda \geq \frac{1}{C_P(\mu)}.$$

One may ask what boundary conditions are required when we are in the presence of a non-trivial boundary. The answer consists of the description of the set  $H^1$  on which the Poincaré inequality holds. In order to satisfy the inequality, functions must have their normal derivatives equal to zero at the boundary. We have already seen these conditions when using the divergence theorem in the above calculations. Hence the corresponding boundary condition for the operator  $L$  are Neumann conditions. We will see later that these conditions are required for the underlying process to be Markovian. From a physical point of view, this corresponds for a vibration wave to be orthogonally reflected at the boundary.

The spectral meaning of the Poincaré inequality can be explained by classical min-max theorem for self-adjoint operators. Actually, after having defined the self-adjoint operator  $L$  in  $L^2(\mu)$ , the min-max theorem says us that minimizations over appropriate spaces of the Rayleigh ratio

$$\frac{\int \|\nabla f\|^2 d\mu}{\int f^2 d\mu}$$

give the discrete spectrum of  $-L$ . Keeping that min-max theorem in mind, it appears clearly that by definition (2.2), the sharp Poincaré constant coincides with the first eigenvalue after zero of  $-L$ , whenever its discrete spectrum is not reduced to zero. When this is the case, all eigenfunctions associated to this first eigenvalue have minimal energy and hence achieve the equality case in the Poincaré inequality. In all this thesis, we will use this min max variational approach of eigenvalues in order to study their stability. Moreover, we will use this characterization of eigenfunctions to be equalizers of the Poincaré inequality in order to define eigenspaces in the broader way in Chapter 4. Let us underline that by "the first eigenvalue" we mean "the first eigenvalue after zero" just to shorten some formulations. The spectral interpretation of Poincaré inequalities is very important, by the way Poincaré inequalities are often called spectral gap inequalities. When  $\mu$  is the volume measure of a compact manifold,  $L$  is then the Laplace-Beltrami operator. The Laplace-Beltrami operator on a compact manifold has a discrete spectrum and the problem of understanding the behavior of its first eigenvalue has been investigated from a very long time in spectral geometry, and dates back at least for domains in  $\mathbb{R}^d$  to J. Rayleigh in 1896 (see. [200]). For instance, let us mention the famous question "Can One Hear the Shape of a Drum?" asked in 1966 by M. Kac in [121]. From a physical point of view, a drumhead can only vibrate for a discrete set of frequencies. Modeled by a domain in the plane, these frequencies of the drumhead corresponds exactly to eigenvalues of the Laplace operator restricted to this domain with Dirichlet boundary conditions, i.e. functions are required to vanish at the boundary of the domain. The Dirichlet boundary condition corresponds to the fact that the drumhead is fixed to the drum and hence can not move when it vibrates. These vibration frequencies are obviously determined by the "shape" (i.e. the geometry) of the domain. Conversely, M. Kac asked whether the shape is determined by the set of vibration frequencies. In rigorous language, this amounts to asking whether isospectral

domains (i.e. domains having the same spectrum for the Dirichlet Laplacian) are isometric? If one does not restrict to dimension 2, the question was quickly solved by J. Milnor [164] who constructed two non isometric flat tori of dimension 16 sharing the same spectrum. In dimension 2, the problem was finally answered negatively by C. Gordon, D. Webb and S. Wolpert in 1992 in [101, 102]. There are different shapes in the plane which share the same spectrum. The most famous figure is Figure 2.1. For more details on the history of this problem, we refer the reader

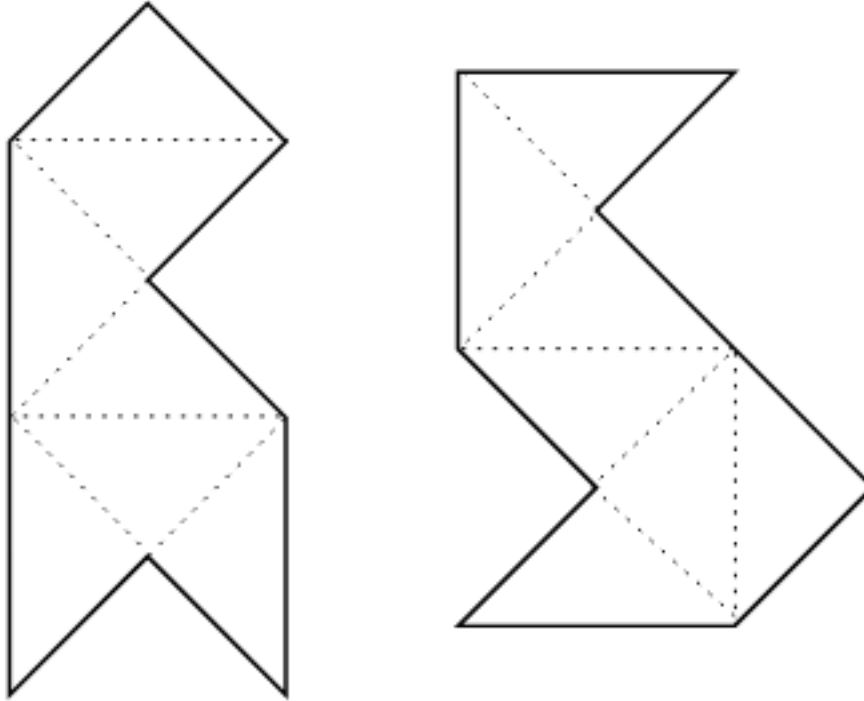


Figure 2.1: Two isospectral but non isometric domains in the plane.

to the survey [99].

An other classical result in Geometry about the first eigenvalue of Laplace Beltrami operator is the following theorem, by A. Lichnerowicz [142].

**Theorem.** (Lichnerowicz [142]) *If  $(M, g)$  is a  $d$  dimensionnal Riemannian manifold without boundary and with Ricci curvature bounded by below by  $d - 1$ , then the first eigenvalue of its Laplace Beltrami operator is greater than or equal to  $d$ .*

We will present this notion of curved spaces in Section 2.3.

### 2.1.3 Cheeger's isoperimetric inequality

Let us briefly discuss Cheeger's isoperimetric inequalities. We will deal with isoperimetric inequalities in more details in the second part of this thesis.

A probability measure  $\mu$  is said to satisfy the Cheeger isoperimetric inequality if there is some constant  $h > 0$  such that for all measurable subsets  $A$ ,

$$\text{Per}_\mu(A) \geq h \min(\mu(A), 1 - \mu(A)),$$

where

$$\text{Per}_\mu(A) := \liminf_{\varepsilon \rightarrow 0_+} \frac{\mu(A_\varepsilon \setminus A)}{\varepsilon}$$



is the  $\mu$ -weighted perimeter, and  $A_\varepsilon := \{x \mid d(x, A) < \varepsilon\}$ . The sharp constant in the Cheeger inequality is denoted by  $h_\mu$  and given by

$$h_\mu := \inf_A \frac{\text{Per}_\mu(A)}{\min(\mu(A), 1 - \mu(A))}.$$

The Cheeger constant  $h_\mu$  gives important information about the geometry of the space endowed with the probability distribution  $\mu$ . For an overview of this notion, we refer the reader to [140].

The Poincaré inequality and the Cheeger inequality are deeply connected. One of the main connections was established in 1970 by J. Cheeger himself in [64].

**Theorem.** ([64, Theorem 1])

$$\frac{1}{C_P(\mu)} \geq \frac{h_\mu^2}{4}$$

We have seen in the previous Section 2.1.2 that the quantity  $\frac{1}{C_P(\mu)}$  is the spectral gap of the measure  $\mu$ . In case where this spectral gap is attained, that is when  $\frac{1}{C_P(\mu)}$  is effectively the first non zero eigenvalue  $\lambda_1(\mu)$ , this gives  $\lambda_1(\mu) \geq \frac{h_\mu^2}{4}$ . This is the original result stated by Cheeger in [64] when  $\mu$  is the uniform probability distribution on a compact Riemannian manifold. While Cheeger's proof has a geometrical flavour, M. Ledoux has given an analytical one in [135] using the characterization of Cheeger isoperimetric inequalities in terms of  $L^1$ -Poincaré inequalities.

## 2.1.4 A classical criterion

We have seen three of the main consequences of a Poincaré inequality. Among these, the phenomenon of concentration of the measure has given us examples of probability distributions that do not satisfy the Poincaré inequality. A natural problem is then to derive sufficient conditions for a measure  $\mu$  to satisfy a Poincaré inequality. We present here one of the most useful criteria, the Bakry-Emery criterion, also called the Curvature-Dimension condition, on which we will elaborate on in Section 2.3 of this introduction. This criterion was first introduced by D. Bakry and M. Emery in the seminal paper [12]. Let  $\mu$  be a probability distribution with density  $e^{-V(x)}dx$  and  $L$  be the operator

$$L = \Delta - \nabla V \cdot \nabla$$

as seen in Section 2.1.2 of this introduction. If the following inequality between 2-tensors

$$(2.10) \quad \text{Hess } V + \text{Ric} \geq Kg$$

is satisfied pointwise in the sense of quadratic forms, where Ric denotes the Ricci tensor,  $g$  denotes the metric tensor, and  $K$  is a positive constant, then the measure  $\mu = e^{-V}dx$  satisfies a Poincaré inequality with constant  $0 < C_P(\mu) \leq \frac{1}{K}$ . Let us underline that in  $\mathbb{R}^d$ , this criterion reduces to

$$\text{Hess } V \geq KI_d$$

where  $I_d$  denotes the identity matrix. Hence in  $\mathbb{R}^d$  the Bakry-Emery condition tells us that for a uniformly convex potential  $V$ , the measure  $e^{-V}dx$  satisfies a Poincaré inequality. The typical example is the Normal distribution. Indeed,

$$d\gamma(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{\|x\|^2}{2}} dx$$

satisfies

$$\text{Hess } V = I_d.$$

Hence the Bakry-Emery condition tells us that the Normal distribution satisfies a Poincaré inequality with constant  $C_P(\gamma) \leq 1$ . Moreover, by testing the inequality for affine functions, we can see that 1 is actually the sharp Poincaré constant of the Normal distribution, and hence in this case the Bakry-Emery criterion gives the best constant.

The Bakry-Emery criterion can be deduced from the Brascamp-Lieb inequality. The Brascamp-Lieb inequality [42] in  $\mathbb{R}^d$  says that if the potential  $V$  is strictly convex (i.e.  $\text{Hess } V > 0$ ), then the distribution  $\mu = e^{-V(x)} dx$  satisfies

$$(2.11) \quad \forall f \in H^1(\mu), \quad \text{Var}_\mu(f) \leq \int_{\mathbb{R}^d} (\text{Hess } V)^{-1} (\nabla f, \nabla f) d\mu.$$

Moreover, the functions  $f$  reaching equality are all functions of the form  $\langle \nabla V, c \rangle$  where  $c \in \mathbb{R}$ . It is then immediate to derive the Poincaré inequality from Inequality (2.11) when the potential is uniformly convex (i.e.  $\text{Hess } V \geq K I_d$  for some  $K > 0$ ).

Let us mention that the Brascamp-Lieb inequality has been extended to the weighted Riemannian setting by A. Kolesnikov and E. Milman in [130]. However in this framework, the functions that reach equality are unknown.

Let us conclude this section by citing the work of T. Bodineau and R. Bauerschmidt [24] who generalized the Bakry-Emery criterion to the case of non-log-concave measures by using a renormalization method. Their criterion applies in particular to models from statistical physics.

### 2.1.5 Improved Poincaré inequalities

In this section we describe a way to improve Poincaré inequalities by requiring an orthogonality condition on the test functions. We only sketch here the philosophy of the improved Poincaré inequalities, the formal definitions can be found in the chapter 4. Let  $\mu$  be a probability distribution on  $\mathbb{R}^d$  satisfying the Poincaré inequality (2.1), and let  $Sp_1(\mu)$  be the set of all functions  $f_1$  reaching equality in (2.1). We then have that for all functions  $f \in H^1(\mu)$  orthogonal to  $Sp_1(\mu)$  in  $L^2(\mu)$ ,

$$(2.12) \quad \int f^2 d\mu \leq C_P^2(\mu) \int |\nabla f|^2 d\mu,$$

with  $0 < C_P^2(\mu) < C_P(\mu)$ . The inequality (2.12) is the improved Poincaré inequality of order 2. We can then iterate this procedure, and define  $Sp_{k+1}(\mu)$  as the set of all functions  $f_{k+1} \in H^1(\mu) \cap (Sp_1(\mu) \oplus \dots \oplus Sp_k(\mu))^\perp$  reaching equality in the improved Poincaré inequality of order  $k$ . We then obtain that for all functions  $f \in H^1(\mu) \cap (Sp_1(\mu) \oplus \dots \oplus Sp_{k+1}(\mu))^\perp$ , it holds

$$(2.13) \quad \int f^2 d\mu \leq C_P^{k+2}(\mu) \int |\nabla f|^2 d\mu,$$

with  $0 < C_P^{k+2}(\mu) < C_P^{k+1}(\mu) < \dots < C_P(\mu)$ . The spectral interpretation of the Poincaré inequality allows us to see that when  $\mu$  is the reversible distribution of a diffusion generator  $L$ , then the spaces  $Sp_k(\mu)$  are eigenspaces of the operator  $L$  with associated eigenvalues  $\frac{-1}{C_P^k(\mu)}$ . Therefore, the improved Poincaré inequalities are a way to encode the whole spectrum of the operator  $L$  by functional inequalities.

## 2.2 Stein's method

Stein's method is a set of techniques extensively developed starting with the seminal paper [197] of C. Stein in 1972. We refer the reader to the surveys [63, 185, 66]. The aim of these techniques is to bound the distance between two probability measures. First, it was introduced for the one dimensional Normal distribution, allowing to grasp effective rates of convergence in the Central

Limit Theorem, widely known as sharp Berry-Esséen bounds (see e.g. [94, 69, 68, 170]). Let us briefly sketch the main ideas of the method.

The key observation is that the normal distribution  $\gamma$  on  $\mathbb{R}$  is the only one satisfying the following integration by parts formula

$$(2.14) \quad \forall f \in C^1(\mathbb{R}), \quad \int f'(x) d\gamma(x) = \int x f(x) d\gamma(x).$$

Consequently, one can develop the intuition that if a probability distribution  $\nu$  almost satisfies this integration by parts formula (2.14), then *in some sense*, it should be close to the normal distribution. To make the intuition rigorous, one would therefore try to link the error term measuring the extent to which a probability distribution  $\nu$  is to satisfy Formula (2.14) with some classical notion measuring the difference (or the distance) between two probability distributions.

On the one hand, the best term we can find to characterizes the error in the integration by parts for  $\nu$  is obviously

$$(2.15) \quad \sup_{f \in \mathcal{F}} \left| \int_{\mathbb{R}} f'(x) - x f(x) d\nu \right|$$

where the choice of the functional set  $\mathcal{F}$  remains to be determined. This set should not be chosen too small in order to provide relevant information on the integration by parts formula, but it should also not be too large in order to be usable.

On the other hand, two distributions  $\nu$  and  $\beta$  are equals if for all test functions  $g$ , they give the same value  $\nu(g) = \beta(g)$ . So a natural way to measure the difference between two distributions is to evaluate their differences on test functions:

$$(2.16) \quad \sup_{g \in \mathcal{G}} |\nu(g) - \beta(g)| = \sup_{g \in \mathcal{G}} \left| \int g d\nu - \int g d\beta \right|.$$

This quantity is very common in the litterature, and satisfies very useful properties as soon as the functional set  $\mathcal{G}$  is well chosen. For instance,

- for  $\mathcal{G} = \{\mathbf{1}_{(-\infty, t)} \mid t \in \mathbb{R}\}$ , it corresponds to the Kolmogorov distance: it is the  $L^\infty$  norm between cumulative distribution functions,
- for  $\mathcal{G} = \{\mathbf{1}_A \mid A \subset \mathbb{R} \text{ measurable}\}$ , it is the Total Variation distance,
- for  $\mathcal{G} = \{g \in C^1(\mathbb{R}) \mid \|g'\|_\infty \leq 1\}$ , it is the  $L^1$ -Wasserstein distance, which is related to optimal transport by the Kantorovich-Rubinstein formula (see [213, Chapter 5]),
- for  $\mathcal{G} = \{g \in C^k(\mathbb{R}) \mid \|g^{(k)}\|_\infty \leq 1\}$ , it corresponds to the  $k$ -th Zolotarev distance which controls the same topology as the  $L^k$ -Wasserstein distance (see [26]) which is itself equivalent to weak convergence and convergence of the  $k$ -th first moments.

The intuition would be made rigorous if we showed that the quantity (2.15) controls some of the classical quantities (2.16). To achieve this goal, one strategy is to solve for  $g \in \mathcal{G}$  the ordinary differential equation

$$(2.17) \quad f'(x) - x f(x) = g(x) - \int g d\gamma$$

and find solutions  $f$  with some regularity and boundedness. The functional set  $\mathcal{F}$  will then be chosen as the set of functions satisfying these regularities. This is this idea which was introduced for the first time by C. Stein in [185].

In case of the normal distribution and the  $L^1$ -Wasserstein distance, this problem is solved by the famous Stein's lemma.

**Lemma 5.** (Stein's lemma) *If  $g : \mathbb{R} \rightarrow \mathbb{R}$  is absolutely continuous and centered for the normal distribution  $\gamma$ , then one can find a solution  $f$  of (2.17) satisfying*

$$(2.18) \quad \|f\|_\infty \leq 2\|g'\|_\infty, \quad \|f'\|_\infty \leq \sqrt{\frac{2}{\pi}}\|g'\|_\infty, \quad \text{and} \quad \|f''\|_\infty \leq 2\|g'\|_\infty.$$

Let us denote by  $\mathcal{F}_s$  the set of all  $C^2(\mathbb{R})$  functions satisfying

$$\|f\|_\infty \leq 2, \quad \|f'\|_\infty \leq \sqrt{\frac{2}{\pi}}, \quad \text{and} \quad \|f''\|_\infty \leq 2.$$

All steps of the reasoning therefore combine and give

$$(2.19) \quad W_1(\nu, \gamma) = \sup_{\|g'\| \leq 1} \left| \int \left( g - \int g d\gamma \right) d\nu \right| \leq \sup_{f \in \mathcal{F}_s} \left| \int f'(x) - xf(x) d\nu \right|.$$

This is a quantitative expression of the fact that if a probability distribution  $\nu$  almost satisfies integration by parts formula (2.14), then it is close to the normal distribution in 1-Wasserstein distance. The same can be done for the above classical distances.

Let us say a word about the proof of Stein's lemma. Equation (2.17) is a first order Ordinary Differential Equation. We know that the set of solutions is an affine space of dimension one, and we just want to find one function in this space which satisfies the boundedness conditions of Stein's lemma. As taught to undergraduate students, the general solution is

$$e^{\frac{x^2}{2}} \int_x^{+\infty} \left( g(x) - \int g d\gamma \right) e^{-\frac{x^2}{2}} dx + Ke^{\frac{x^2}{2}}, \quad K \in \mathbb{R}.$$

It is clear that for  $K \neq 0$  the solution is unbounded. Hence proving Stein's lemma is proving that

$$(2.20) \quad f_g(x) := e^{\frac{x^2}{2}} \int_x^{+\infty} \left( g(x) - \int g d\gamma \right) e^{-\frac{x^2}{2}} dx$$

is the only solution of (2.17) satisfying boundedness conditions (2.18). Since  $f_g$  is explicit, one can conduct explicit calculations in order to show (2.18). Such calculations are for instance presented in details by L. Chen, L. Goldstein and Q. Shao in [66, Appendix p.37]. As can be guessed from Formula (2.20), these calculations are based on control of the cumulative distribution function of the normal distribution. In Section 3.6, we shall generalize this approach to a broader class of probability distributions.

It turns out that the solution given by (2.20) has a probabilistic interpretation. This interpretation is a cornerstone in Stein's method and is now known as the Barbour generator approach. In order to explain this, let us take a detour into Markov process theory.

Assume that  $(X_t)_{t \geq 0}$  is a random process living on some fixed space. To simplify, we can consider that this space is  $\mathbb{R}$ . All spaces being characterized by the set of possible ways to transform them, a way to study the random process is to study expectations of real functions randomized by this process. That is

$$P_t f(x) := \mathbb{E}[f(X_t) | X_0 = x].$$

When the future of the process  $(X_t)_t$  only depends on its present (and not on its past), we say that  $(X_t)_t$  is Markov. In a rigorous mathematical language, this corresponds to the Chapman-Kolmogorov equations:

$$(2.21) \quad \forall t, s \geq 0, \quad P_{t+s} = P_t \circ P_s.$$

Equation (2.21) says that operators  $(P_t)_t$  have the semigroup property. A simple example of a family of operators satisfying it is  $(e^{tL})_{t \geq 0}$  when  $L$  is an operator such that all the previous

exponential make sense. An important fact is that this example is generic: the Hille-Yosida theory states that under some natural conditions on the semigroup, there exists an operator  $L$  defined on a large enough functional space  $\mathcal{D}$ , and such that, at least formally, (see [217, Chapter IX])

$$P_t = e^{tL}.$$

The operator  $L$  is called the infinitesimal generator (or just generator) of the process, and is given for all function  $f$  in its domain  $\mathcal{D}$ , by

$$Lf = \lim_{s \rightarrow 0} \frac{P_s f - f}{s}.$$

The Markov process generator approach is the analytical viewpoint of random process theory. Let us move on the probabilistic point of view in this theory. A classical problem in the theory of random processes is the question of the existence (and uniqueness) of equilibrium distribution, also called invariant measure. A distribution  $\mu$  is said to be invariant with respect to a random process  $(X_t)_t$  if the drawing of  $X_0$  with a distribution  $\mu$  results in all  $X_t$  following the distribution  $\mu$ . This is equivalent to

$$\forall f \in \mathcal{D}, \quad \int Lf d\mu = 0.$$

Let us present a way to link the Normal distribution to this general theory. There exists a simple Markov process  $(X_t)_{t \geq 0}$  on  $\mathbb{R}$  which equilibrium distribution is the normal distribution  $\gamma$ . This Markov process is widely known as the Ornstein-Uhlenbeck process from the 1930 paper of L. Ornstein and G. Uhlenbeck [210]. This process is the solution of the following Stochastic Differential Equation

$$dX_t = -X_t dt + dB_t,$$

where  $B_t$  denotes the Brownian motion, and its infinitesimal generator is

$$(2.22) \quad Lf(x) = f''(x) - x f'(x).$$

The similarity with Stein's equation (2.17) is clear: the Ornstein-Uhlenbeck generator is the same as the left hand side of Stein's equation applied to  $f'$  instead of  $f$ . Hence solutions of the Stein equation (2.17) are first derivatives of solutions of the following equation

$$(2.23) \quad Lf = g - \int g d\gamma.$$

Equation (2.23) is known as the Poisson equation of the Markov process generated by  $L$  and with invariant measure  $\gamma$ . The terminology comes from the fact that in case of the Brownian motion (which is one of the most natural Markov process), the generator is the Laplacian, so equation (2.23) is the classical Poisson equation. For such Poisson equations in a Markovian context, we know that (at least formally),

$$(2.24) \quad f(x) = - \int_0^{+\infty} \left( P_t g(x) - \int g d\gamma \right) dt$$

is a solution of (2.23). Indeed, formally we have

$$\begin{aligned} -Lf &= -L \int_0^{+\infty} \left( P_t g(x) - \int g d\gamma \right) dt \\ &= - \int_0^{+\infty} L P_t g(x) dt \\ &= - \int_0^{+\infty} \frac{d}{ds} \Big|_{s=t} P_t g(x) dt \\ &= - [P_t g]_0^{+\infty} \\ &= P_0 g - P_\infty g \\ &= g - \int g d\gamma. \end{aligned}$$

We have now enough concept in Markov theory to present the probabilistic interpretation of the solution (2.20) of Stein's equation. This solution (2.20) is the derivative of the solution (2.24) of the Poisson equation associated with the Ornstein-Uhlenbeck process. This remark may seem minor, however it is very deep. Actually, Formula (2.24) can be written for any Markov semigroup  $(P_t)_t$  with generator  $L$  and invariant measure  $\mu$ , and we always have, at least formally, that

$$f_g(x) = - \int_0^{+\infty} \left( P_t g(x) - \int g d\mu \right) dt$$

is a solution of the Poisson equation

$$Lf = g - \int g d\mu,$$

and moreover,

$$(2.25) \quad \forall f \in \mathcal{D}, \quad \int Lf d\mu = 0.$$

Equation (2.25) can be seen as an integration by parts formula. Indeed, when  $L$  is the Ornstein-Uhlenbeck operator, (2.25) is reduced to the classical integration by parts formula (2.14) which was the starting point of Stein's method. These considerations make it possible to extend Stein's method to probability distributions other than the Normal. If  $\mu$  is a probability distribution, we will find a Markov process for which  $\mu$  is the equilibrium distribution. Then the infinitesimal generator  $L$  of this Markov process will offer an integration by parts formula which will be a good candidate to characterize the distribution  $\mu$ . In order to show that the error in the integration by parts formula (2.25) controls some classical distance between probability distributions, we will consider the solution (2.24) of the Poisson equation. All the remaining work will be to effectively find such a generator  $L$  and prove regularity estimates for the solution (2.24). If one succeeds in doing this, the following inequality of the type of Inequation (2.19) will be proven for the distribution  $\mu$ :

$$(2.26) \quad W_1(\nu, \mu) \leq \sup_{f \in \mathcal{G}} \left| \int Lf(x) d\nu \right|.$$

This strategy for implementing Stein's method will be used throughout the first part of this thesis. This principle of implementing Stein's method with a generator approach was originally introduced by A. Barbour in 1990 in [17]. Thanks to this generator approach, A. Barbour was able to use Stein's method for the Wiener measure instead of the normal distribution, and as a result he obtained a quantitative limit in Donsker's theorem. Recall that the Donsker theorem is the functional extension to the central limit theorem. So Barbour's result corresponds to an extension of Berry-Esseen bounds given by Stein's method for the Normal distribution. Let us conclude this section by presenting a classical proof by Stein's method of a quantitative bounds in the central limit theorem.

Let  $X_1, \dots, X_n$  be independant and identically distributed real random variables normalized such that  $\mathbb{E}[X_1] = 0$  and  $\mathbb{E}[X_1^2] = 1$ . Let us assume that  $X_1$  has finite third and fourth moments. Let  $S_N := \frac{1}{\sqrt{N}} \sum_{i=1}^N X_i$ . According to the central limit theorem, we know that  $S_N$  converges weakly towards the normal distribution  $\gamma$ . We will give a quantitative convergence rate by bounding the 1-Wasserstein distance  $W_1(\gamma, \mu_N)$  between  $\gamma$  and the distribution  $\mu_N$  of the real random variable  $S_N$ . Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a function such that

$$\|f\|_\infty \leq 2, \quad \|f'\|_\infty \leq \sqrt{\frac{2}{\pi}}, \quad \text{and} \quad \|f''\|_\infty \leq 2.$$

Denoting by  $W_i := \frac{1}{\sqrt{N}} \sum_{j \neq i} X_j = S_N - \frac{1}{\sqrt{N}} X_i$ , we compute

$$\mathbb{E}[S_N f(S_N)] = \mathbb{E} \left[ \frac{1}{\sqrt{N}} \sum_i X_i (f(S_N) - f(W_i) - (S_N - W_i) f'(S_N)) + \frac{1}{\sqrt{N}} \sum_i X_i (S_N - W_i) f'(S_N) \right].$$

So

$$\begin{aligned} |\mathbb{E}[f'(S_N) - S_N f(S_N)]| &\leq \left| \mathbb{E} \left[ \frac{1}{\sqrt{N}} \sum_i X_i \left( f(S_N) - f(W_i) - \frac{X_i}{\sqrt{N}} f'(S_N) \right) \right] \right| \\ &\quad + \left| \mathbb{E} \left[ f'(S_N) \left( 1 - \frac{1}{\sqrt{N}} \sum_i \frac{1}{\sqrt{N}} X_i^2 \right) \right] \right| \end{aligned}$$

Now, on the one hand, using the Taylor expansion of  $f$ ,

$$\left| \mathbb{E} \left[ \frac{1}{\sqrt{N}} \sum_i X_i \left( f(S_N) - f(W_i) - \frac{X_i}{\sqrt{N}} f'(S_N) \right) \right] \right| \leq \frac{\|f''\|_\infty}{2\sqrt{N}} \sum_i \mathbb{E} \left| \frac{X_i^3}{N} \right| = \frac{\|f''\|_\infty}{2\sqrt{N}} \mathbb{E} |X_1^3|,$$

and on the other hand,

$$\begin{aligned} \left| \mathbb{E} \left[ f'(S_N) \left( 1 - \frac{1}{\sqrt{N}} \sum_i \frac{1}{\sqrt{N}} X_i^2 \right) \right] \right| &\leq \frac{\|f'\|_\infty}{N} \mathbb{E} \left| \sum_i (1 - X_i^2) \right| \\ &\leq \frac{\|f'\|_\infty}{N} \sqrt{\text{Var} \left( \sum_i X_i^2 \right)} \\ &\leq \frac{\|f'\|_\infty}{N} \sqrt{N \mathbb{E} |X_1^4|}. \end{aligned}$$

Finally, Formula (2.19) and the above computations give

$$W_1(\gamma, \mu_N) \leq \frac{1}{\sqrt{N}} \left( \mathbb{E} |X_1^3| + \sqrt{\frac{2}{\pi}} \mathbb{E} |X_1^4| \right),$$

which is effectively a quantitative bound for the convergence given by the central limit theorem.

## 2.3 Curvature Dimension condition

This section is devoted to the presentation of Riemannian Curvature Dimension (RCD) spaces. This notion is part of the broader context of Curvature Dimension theory. We refer the reader to the complete expositions of J. Lott and C. Villani [145, 214] and K-T. Sturm [201, 202]. We will only discuss here the technical framework and the vision of these concepts that have guided us in this thesis.

Many areas of mathematics use the very general notion of space. Geometry deals with spaces with a metric. Since ancient times, we know that the notion of metrics gives rise to a notion of volume. This is clear from the classical formulas for volumes of basic geometric forms. Probability theory deals with spaces with a notion of certainty. Since the modern Kolmogorov probability theory, which is based on Lebesgue's measure theory, we know that these two notions of volume and certainty are equivalent. In fact, the degree of certainty of an event can be seen as the volume occupied by this event in the space of all possible events.

More formally, in geometry, a Riemannian manifold is a space equipped with a metric tensor, which is a smooth quadratic form on every tangent space. From this metric tensor, one can define the geodesic distance. A Riemannian manifold is thus a particular example of a metric space. In addition, one can also define the volume form from the metric tensor. A Riemannian manifold is therefore also a particular example of a measured space, the generalization of probability spaces where the total volume can be infinite as in Euclidean space. A Riemannian manifold is therefore an example of what is commonly called a metric measured space. In this case, the two notions of distance and volume are related because they both come from the metric tensor. A natural question then is to understand this link. The only explanation that this link comes from formal mathematical formulae and definitions is not satisfactory, there

must be a more conceptual reason behind it. To understand this reason, let us return to the theory of Markov processes.

As we briefly seen in Section 2.2, two important mathematical objects are linked to a Markov process: its generator  $L$  (which is an operator), and its equilibrium distribution  $\mu$ . From the operator  $L$ , one can define the successive  $\Gamma$  bilinear operators (see [13, Chapter 1]), the first

$$(2.27) \quad \Gamma(f, g) := \frac{1}{2} [L(fg) - fLg - gLf],$$

and the second

$$(2.28) \quad \Gamma_2(f, g) := \frac{1}{2} [L\Gamma(f, g) - \Gamma(f, Lg) - \Gamma(Lf, g)].$$

To lighten the notation, we will write  $\Gamma(f)$  instead of  $\Gamma(f, f)$ , and similarly  $\Gamma_2(f)$  instead of  $\Gamma_2(f, f)$ . We will talk about the operator  $\Gamma_2$  a little bit latter. Let us deal with the first one. The first operator  $\Gamma$  is called the carré du champ operator and quantifies to what extent the operator  $L$  is not a derivation. Moreover, in the case of the operator  $L = \Delta - \nabla V \cdot \nabla$  of the form (2.8), one can compute that the carré du champ operator is the scalar product of gradients of functions:

$$(2.29) \quad \Gamma(f, g) = \nabla f \cdot \nabla g.$$

The scalar product is the Riemann tensor, so the carré du champ operator is also the Riemannian metric (restricted to gradients vector fields). In the general case,  $\Gamma$  is in fact the co-metric, that is in a local chart, the inverse of the metric tensor. In the case of  $L$  of the form (2.8), the metric tensor is the identity matrix so the metric and the co-metric coincide. The carré du champ operator allow us to define the distance in the following way:

$$\forall x, y, \quad d(x, y) := \sup_{\Gamma(f) \leq 1} |f(x) - f(y)|.$$

When  $\Gamma$  has the form (2.29), we recognize the Wasserstein-1 distance defined in (2.16) between the Dirac measure at  $x$  and the Dirac measure at  $y$ , which coincide with the Riemannian distance between points  $x$  and  $y$ . We now have enough material to answer the question of the link between the notions of distance and volume of a Riemannian manifold.

On all Riemannian manifolds, there is a particular random walk that fits both the metric and the volume. This random walk is called Brownian motion. Brownian motion is the mathematical formalisation of the physical phenomenon first observed by the botanist R. Brown in 1827, who described the motion of a pollen particle immersed in water. The trajectory of the pollen particle is modified each time it encounters water molecules, giving rise to a Markov process. In the case of Brownian motion, the generator is the Laplace-Beltrami operator, and the equilibrium distribution is the Riemannian volume measure. The carré du champ operator of Brownian motion thus encodes the metric, as seen earlier. Thus the link between the notions of distance and volume of a Riemannian manifold becomes clear: the metric is encoded by the carré du champ operator, and the volume is the invariant measure, both of Brownian motion. Their link is therefore Brownian motion: the metric and the volume derive from the Brownian Markov process. From this point on, we are not studying a space with a metric (Geometry) or a space with a measure (Probability theory), but a space on which a Markov process exists. Of course, for the study to be interesting, we will have to make assumptions about this Markov process. This is similar to the geometric context, where the metric is required to satisfy certain conditions, for example to be hyperbolic, or to the probabilistic context where the distribution is assumed to be log-concave for example. Riemannian curvature dimension spaces (called RCD spaces in the following) are spaces equipped with a Markov process satisfying two conditions: the diffusion property and the curvature dimension condition. We present here the smooth framework of RCD spaces, i.e. when we assume that they have a differential structure.



The general framework where we do not assume to have a differential structure is presented in chapter 5. However, the philosophy remains the same in both cases.

A Markov process is said to be a diffusion when its generator satisfies for all  $\phi \in C^2(\mathbb{R}, \mathbb{R})$ ,

$$(2.30) \quad L(\phi \circ f) = \phi'(f)Lf + \phi''(f)\Gamma(f, f).$$

From an analytical point of view, this diffusion property is the chain rule of second order differential operators whose prototype is the Laplacian. From a probabilistic point of view, the diffusion property of the process  $(X_t)_{t \geq 0}$  with generator  $L$  implies the continuity of all processes  $(f(X_t))_{t \geq 0}$  when  $f$  is in the domain of the generator  $L$  (see [13, Section 1.11.1]). In the first part of this thesis, the diffusion property will allow us to obtain one-dimensional Markov processes by taking the pushforward of the process under study by eigenfunctions. Recall that the Laplace Beltrami operator is always a diffusion, so that the Brownian motion on a Riemannian manifold is always a Markov diffusion.

A Markov process is said to satisfy the Curvature Dimension Condition  $CD(K, N)$  when it satisfies the following Bochner's formula

$$(2.31) \quad \Gamma_2(f, f) \geq K\Gamma(f, f) + \frac{1}{N}(Lf)^2.$$

In the case where the Markov process is the Brownian motion on the unit sphere  $\mathbb{S}^N$ , the generator is the Laplace-Beltrami operator  $\Delta$ , the carré du champ operator is  $\Gamma(f) = |\nabla f|^2$  and the  $\Gamma_2$  operator is

$$\Gamma_2(f) = \text{Ric}(\nabla f, \nabla f) + \|\text{Hess}(f)\|_{HS}^2,$$

where  $\|\cdot\|_{HS}$  denotes the Hilbert-Schmidt norm of operators. By the Cauchy-Schwarz inequality, one always has

$$\Delta f = \text{Tr Hess}(f) = \langle I_d, \text{Hess}(f) \rangle_{HS} \leq \sqrt{N} \|\text{Hess}(f)\|_{HS},$$

so in that case, the  $CD(K, N)$  condition is equivalent to

$$(2.32) \quad \text{Ric}(\nabla f, \nabla f) \geq K|\nabla f|^2.$$

Hence, when the Markov process is Brownian motion on a Riemannian manifold,  $RCD(K, N)$  spaces are Riemannian manifolds of dimension at most  $N$  and with Ricci curvature bounded by below by  $K$ . So the main idea behind Bochner formula (2.31) is to ask for an upper bound on the dimension of the space, and for a (uniform) lower bound on the curvature.  $RCD$  spaces are generalizations of Riemannian manifolds with lower Ricci bound, and coincide exactly with them when the ambient Markov diffusion is the Brownian motion. The Bochner inequality (2.31) with parameters  $K = N - 1$  is an equality when the space is the unit sphere  $\mathbb{S}^N$  equipped with its Brownian motion. In other words, the model space of  $RCD(N - 1, N)$  spaces is the unit sphere  $\mathbb{S}^N$  equipped with its Brownian motion. Hence  $RCD(K, N)$  are spaces which can reasonably be compared with spheres. We have already seen this idea to compare a space with a sphere at the end of Section 2.1.2 when we have presented the Lichnerowicz theorem. Let us recall it.

**Theorem.** (Lichnerowicz [142]) *If  $(M, g)$  is a  $N$  dimensionnal Riemannian manifold with Ricci curvature bounded by below by  $N - 1$ , then the first eigenvalue of its Laplace Beltrami operator is greater or equal to  $N$ .*

We have seen here that in term of  $RCD$  spaces, a  $N$  dimensionnal Riemannian manifold with Ricci curvature bounded by below by  $N - 1$  is a  $RCD(N - 1, N)$  space. We have also seen that the model space among  $RCD(N - 1, N)$  spaces is the unit sphere  $\mathbb{S}^N$ . Moreover, the eigenvalue  $N$  in the Lichnerowicz theorem corresponds to the first non zero eigenvalue of the Laplace-Beltrami operator on the unit sphere. Therefore the Lichnerowicz theorem establishes

the comparison for the first eigenvalue of  $RCD(N-1, N)$  spaces and their model space. Let us mention that from this result one can easily obtain the same result for  $RCD(K, N)$  spaces by a correct normalization. Indeed, the model space of  $RCD(K, N)$  is the sphere of radius  $\frac{N-1}{K}$  and its first non zero eigenvalue is  $\frac{NK}{N-1}$ .

Many other comparative results exist. For example, one can compare the diameter of a  $RCD(K, N)$  space with the one of the sphere  $\mathbb{S}^N$ , or also compare their isoperimetric profiles (Levy-Gromov theorem see [104]). The Lichnerowicz estimate was extended to RCD spaces by M. Erbar, K. Kuwada and K-T. Sturm in [83, Theorem 4.22]. Let us prove it in the smooth setting with the Bochner formula. Let  $L$  be the generator of the Markov diffusion process of a  $RCD(N-1, N)$  space. Let  $\lambda_1(L)$  be the first eigenvalue after zero of the positive semi-definite operator  $-L$ , and  $f$  be an associated eigenfunction. By evaluating the Bochner inequality (2.31) with the eigenfunction  $f$ , and integrating it against the equilibrium distribution  $\mu$ , one gets

$$\int \Gamma_2(f) d\mu \geq (N-1) \int \Gamma(f) d\mu + \frac{\lambda_1(L)^2}{N} \int f^2 d\mu.$$

Now by the definition of the  $\Gamma_2$  operator and the integration by parts formula (2.9), one obtains

$$\begin{aligned} \lambda_1(L)^2 \int f^2 d\mu &= \int (Lf)^2 d\mu = \int \Gamma_2(f) d\mu \geq (N-1) \int \Gamma(f) d\mu + \frac{\lambda_1(L)^2}{N} \int f^2 d\mu \\ &= \lambda_1(L)(N-1) \int f^2 d\mu + \frac{\lambda_1(L)^2}{N} \int f^2 d\mu, \end{aligned}$$

which actually implies

$$\lambda_1(L) \geq N.$$

We know that spheres reach equality in the Lichnerowicz spectral gap estimate. A natural question then arises about the rigidity of this equality case: are spheres the only spaces reaching equality among all  $N$  dimensional Riemannian manifolds with Ricci curvature uniformly bounded by below by  $K$ ? This question was answered in the affirmative by M. Obata in [172].

**Theorem.** (*Obata's rigidity theorem [172]*) *Let  $(M, g)$  be a  $N$  dimensionnal Riemannian manifold with Ricci curvature bounded by below by  $N-1$ . If the first non zero eigenvalue of its Laplace-Beltrami operator is  $N$ , then  $(M, g)$  is isometric to the unit sphere  $\mathbb{S}^N$ .*

As for the Lichnerowicz estimate, Obata's rigidity theorem has been extended to RCD spaces. The rigidity is directly extended in the smooth framework: spheres remain the only spaces to reach equality. However, in the non-smooth framework, there are spaces other than spheres that achieve equality: these are the spherical suspensions. All cases of equality have been described by Christian Ketterer in [124, 123]. We will not elaborate further on spherical suspensions because we do not use them in this thesis. Indeed, in the chapter 5, after having performed a pushforward by a real function, we are reduced to comparing one-dimensional RCD spaces. However, it turns out that, up to a reparametrization, there is only one model space of dimension one. This model space is the Beta distribution, and can be seen as the distribution of a coordinate on the sphere  $\mathbb{S}^N$  (see Section 5.2.3 of Chapter 5).

Note that the question of rigidity has some similarities with the question of hearing the shape of a drum mentioned in Section 2.1.2. Indeed, in both cases, we have an assumption about the spectrum of two operators and we want to deduce strong information about the geometry. We saw in Section 2.1.2 that the answer was negative, whereas here under the curvature dimension condition the answer is positive. The main difference is that in the present section we have considerably reduced the degrees of freedom of the problem by fixing some model spaces and restricting the comparison to  $RCD(K, N)$  spaces only. We will see in the next section that in the study of the stability of functional inequalities, it is crucial to choose an appropriate model space on the one hand, but also an appropriate space to which the model can be compared on the other hand.

## 2.4 Stability of comparison results

When there is some comparison result such as the Lichnerowicz theorem, the first natural question is what happens when equality is reached? We have a rigidity result when equality is reached only for a particular space, as in the Obata rigidity theorem. This particular space will therefore constitute the model space. In the case of the Lichnerowicz theorem, the model space is the sphere (to which one must add the spherical suspensions in the non-smooth frame). The next question is then whether the comparison result is *stable* or not. If equality is not achieved, but *almost* achieved, does the space resemble the model space? In what sense does it resemble it? In particular, we need a notion appropriate to each context to compare two spaces.

The question of stability can be summarized as follows. We begin with a comparison result of the form

$$\forall F \in \mathcal{F}, \quad \Omega(F) \geq \Omega(M),$$

where  $M \in \mathcal{F}$  is the model space and the only element of  $\mathcal{F}$  reaching equality, and  $\Omega(F)$  is some quantity of interest depending on the structure of  $F$ . A *stability result* is then a result of the following form:

$$\text{if } F \in \mathcal{F} \text{ is such that } \Omega(F) \approx \Omega(M), \text{ then } F \approx M,$$

where the second symbol  $\approx$  must be understood in a sense that depends on the type of objects  $F \in \mathcal{F}$  considered. In turn, a *quantitative stability result* is a result of the following form:

$$(2.33) \text{ if } F \in \mathcal{F} \text{ is such that for a small enough } \varepsilon > 0, \Omega(F) = \Omega(M) + \varepsilon, \text{ then } d(F, M) \leq \omega(\varepsilon),$$

where  $d$  is a distance on the space  $\mathcal{F}$  and  $\omega$  is a modulus of continuity, that is a positive continuous function from  $\mathbb{R}_+$  to  $\mathbb{R}_+$ , and vanishing only at zero. Note that most of the time, we do not directly control the distance  $d(F, M)$  between  $F$  and  $M$ , but the distance between  $F'$  and  $M'$  where  $F'$  and  $M'$  are simpler spaces than  $F$  and  $M$ . In particular in this thesis,  $F$  and  $M$  will be probability distributions, and  $F', M'$  will be the pushforward of  $F$  and  $M$  by a certain function.

The choice of the  $\mathcal{F}$  space is crucial. Indeed, if it is too large, then the rigidity might not be satisfied, and if it is too small, the information given by such results would be trivial. In the first part of this thesis, we study and derive stability results for the Poincaré constant. Thus, the set  $\mathcal{F}$  will be a set of probability distributions, and the comparison result we start with will be of the form

$$\forall \nu \in \mathcal{F}, \quad C_P(\nu) \geq C_P(\mu),$$

where  $\mu$  will constitute the model measure. We will study the stability of such comparison results in two contexts: a normalization condition, and a curvature-dimension condition.

### 2.4.1 Stability under normalization conditions

Let us first deal with the case of normalization conditions, i.e. when the set of  $\mathcal{F}$  measures is the set of probability measures with some of their moments having a value fixed in advance. The model measure will be the equilibrium distribution determined by some Markov diffusion process. This approach goes back to L. Chen's 1987 paper [67], where the following was shown.

**Theorem.** (L. Chen [67, Corollary 2.1]) *If  $\nu$  is a probability distribution on  $\mathbb{R}$  with mean zero and variance one, that is*

$$\int x d\nu = 0, \text{ and } \int x^2 d\nu = 1,$$

*and if it satisfies the Poincaré inequality*

$$\forall u \in C^1(\mathbb{R}) \cap L^2(\nu), \quad \text{Var}_\nu(u) \leq C_P(\nu) \int u'(x)^2 d\nu(x),$$

with sharp constant  $C_P(\nu)$ , then

$$C_P(\nu) \geq 1.$$

Moreover, under these assumptions on  $\nu$ , if  $C_P(\nu) = 1$ , then  $\nu = \gamma$  is the normal distribution on  $\mathbb{R}$ .

We thus have a comparison result on the set  $\mathcal{F}$  of probability distributions of mean zero and variance one, the model distribution is the Gaussian, and moreover the inequality is rigid. All the ingredients are present to study the stability of this result. Actually, the Chen comparison result is stable: in 1989 in [211], S. Utev quantified the difference between Poincaré constants with the total variation distance.

**Theorem.** (*S. Utev [211]*) *Under the same assumptions about  $\nu$  as in the Chen comparison theorem above, we have:*

$$C_P(\nu) \geq 1 + \frac{1}{9}d_{TV}(\nu, \gamma)^2,$$

where  $d_{TV}$  is the total variation distance (see Section 2.2).

In 2019 in [75], T. Courtade, M. Fathi and A. Pananjady showed that all probability distributions in the set  $\mathcal{F}$  of probability distributions of mean zero and variance one satisfy an approximate integration by parts formula whose error term is controlled by the difference  $C_P(\nu) - 1$  between Poincaré sharp constants. Returning to Stein's method, this means that the right hand side of Inequality (2.19) is bounded by a term explicitly depending on  $C_P(\nu) - 1$ . Hence they showed the following stability result.

**Theorem.** (*T. Courtade, M. Fathi, A. Pananjady [75]*) *Under the same assumptions about  $\nu$  as in the Chen comparison theorem above, one has:*

$$C_P(\nu) \geq 1 + W_1(\nu, \gamma)^2,$$

where  $W_1$  is the 1-Wasserstein distance.

In fact, they get a multidimensional stability result with respect to the 2-Wasserstein distance with the constant  $\frac{1}{d}$  before the distance. The stated theorem can be deduced from this since  $W_2 \geq W_1$ .

Our study started with the idea of generalizing this result by replacing the model measure  $\gamma$  by the equilibrium distribution  $\mu$  of a Markov process. Equilibrium distributions of Markov processes are ideal candidates to play the role of model measures. Indeed, in the proof of the Utev and Courtade-Fathi-Pananjady stability results, the method used is Stein's method. However, in order to generalise those proofs, we need probability distributions for which we are able to implement Stein's method. As explained in Section 2.2, Barbour's approach suggests how to implement Stein's method in the case of the invariant measure of a Markov process. The second thing to do, to at least generalize the Courtade-Fathi-Pananjady result to other invariant measures of Markov processes, is to find out what normalization conditions must be required in the set  $\mathcal{F}$  of test measures. To find them, a comparison result such as Chen's theorem must be satisfied. At first sight, one might think that what is required is that  $\nu$  has the same first and second moment as  $\mu$ , since this is what is required in Chen's theorem when  $\mu$  is the Gaussian. However these conditions are not the right ones. Indeed, a counter-example is the following. Let  $d\mu(x) = \frac{1}{Z}e^{-x^4-x^2}dx$  with  $Z = \int_{\mathbb{R}} e^{-x^4-x^2}dx$  such that  $\mu$  is a probability distribution. Then as seen in the chapter 3 Section 3.6.3,  $\mu$  is the invariant measure of a Markov process with sharp Poincaré constant equals to one. However, if we take  $\nu := \mathcal{N}(0, \sigma^2)$  the normal distribution of mean zero and variance  $\sigma^2 := \frac{1}{Z} \int x^2 d\mu(x)$ , then  $\mu$  and  $\nu$  have the same first and second moments, but the Poincaré constant of  $\nu$  is  $\sigma^2$  which is strictly less than 1 (a numerical estimate gives  $\sigma^2 \approx 0.2$ ). Therefore, the Poincaré constant  $C_P(\mu) = 1$  cannot be a minimum on the set of measures having the same first and second moments as  $\mu$ .

At this point, we should recall the tensorization property of the Poincaré inequality, which implies that the sharp constant depends on only one direction, and this direction is the spectral coordinate given by the first eigenfunction, since this realizes the equality in the Poincaré inequality (see section 2.1). Thus, in the Gaussian case, when we required the first two moments to be respectively zero and one, it is not because they are moments, but because the first eigenfunction is the identity  $f_1(x) = x$ , so the first moment is its mean, and the second moment is its variance. Therefore, we are led to take the following as normalization:

$$\int f_1 d\nu = \int f_1 d\mu, \text{ and } \int f_1^2 d\nu = \int f_1^2 d\mu,$$

where  $f_1$  is the first eigenfunction of the model measure  $\mu$ . However, these two conditions are still not sufficient to derive a comparison result. Indeed, we must also require that

$$\int \Gamma(f_1) d\nu = \int \Gamma(f_1) d\mu,$$

where  $\Gamma$  is the operator carré du champ of the Markov process. (We will see in Chapter 3 that this equality condition can be relaxed to the inequality " $\leq$ "). This third condition is empty in the Gaussian case because the derivative of the identity is always equal to one, so its square field operator is also equal to 1. Under these three normalization conditions, we finally obtain the following comparison result, which will be the starting point of a stability result.

**Theorem.** (See the introduction of Chapter 3) *If  $\nu$  is normalized as above, then  $C_P(\nu) \geq C_P(\mu)$ .*

Therefore, restricted to this set of normalized measures, which is a 3 codimensional set in the set of all probability measures, the Poincaré constant has a global minimum reached for the model measure  $\mu$ .

## 2.4.2 Stability under the RCD condition

Let us move onto the Riemannian Curvature Dimension condition. The set  $\mathcal{F}$  will be the set of all measures of the  $RCD(N-1, N)$  spaces. The model measure  $\mu$  will be the equilibrium distribution of the Brownian motion on the unit Sphere  $\mathbb{S}^N$ , i.e. the normalized ( $\mu(\mathbb{S}^N) = 1$ ) Lebesgue measure on  $\mathbb{S}^N$ . The comparison result from which we start will be the Lichnerowicz theorem (see Section 2.3 of the Introduction). In this case, since  $C_P(\nu) = \frac{1}{\lambda_1(\nu)}$  (see Section 2.1.2 of the Introduction) it will be of the form

$$\forall \nu \in \mathcal{F}, \quad C_P(\nu) \leq C_P(\mu),$$

where the Poincaré constant of the model space is a maximum and not a minimum. We then have all the ingredients to study stability. We saw in the presentation of RCD spaces in Section 2.3 that the rigidity problem is answered by the Obata theorem. Since the equality case is not only satisfied by a model space but by the sphere and spherical suspension, we can not expect a stability result in the strong sense defined by (2.33). This was proved in 1990 by M. Anderson in [6]. There is nevertheless the following stability result proved by J. Bertrand in [27].

**Theorem.** ([27, Theorem 5]) *If the first non zero eigenvalue  $\lambda_1$  of a  $RCD(N-1, N)$  space is close to  $N$ , then this RCD space contains a piece that is close to the sphere  $\mathbb{S}^{k-1}$  in Gromov-Hausdorff distance, where  $k \leq N+1$  is the dimension of the eigenspace associated to the eigenvalue  $\lambda_1$ .*

### 2.4.3 Using Stein's method to derive stability results

The starting link between Stein's method and the stability of the Poincaré constant is the same as the main idea of the proof of the Courtade-Fathi-Pananjady result. First, the model measure satisfies an integration by parts formula, say for any  $g \in H^1(\mu)$ ,

$$\int Lg d\mu = 0.$$

Second, all test measures (normalized or RCD) satisfy the same integration-by-parts formula, except for one error term, which is controlled by the difference between its Poincaré constant and the Poincaré constant of  $\mu$  :

$$(2.34) \quad \left| \int Lg d\nu \right|^2 \leq |C_P(\mu) - C_P(\nu)| \int \Gamma(g) d\nu.$$

So if Stein's method can be applied to  $\mu$ , then we know that the error term of the approximate integration by parts formula controls the Wasserstein-1 distance between  $\mu$  and  $\nu$ . We would thus have a stability result of the form

$$W_1(\mu, \nu)^2 \leq K |C_P(\mu) - C_P(\nu)|$$

where  $K$  would be a constant. However, we know that such a result cannot be true in all generality. Indeed, in dimension greater than one, we have seen that the Poincaré constant of a product measure depends on only one direction. It is therefore impossible for the difference between the Poincaré constants to control the distance between  $\nu$  and  $\mu$  in all directions. For example, in  $\mathbb{R}^2$ , the normal distributions

$$\mathcal{N}(0, I_2) \quad \text{and} \quad \mathcal{N}\left(0, \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}\right)$$

have the same Poincaré constant equal to 1. They are not equal in all directions, but they are equal in the direction of this Poincaré constant 1 which is their first coordinate. Thus, we do not expect a stability result between  $\mu$  and  $\nu$  in all directions, but only in the direction of the Poincaré constant of  $\mu$ . This direction is the first eigenfunction of  $\mu$ , and comparing the distributions in this direction is like comparing their pushforward by  $f_1$ . We therefore expect a stability result of the form

$$W_1(f_1^\#(\mu), f_1^\#(\nu))^2 \leq K |C_P(\mu) - C_P(\nu)|$$

where  $f_1^\#$  denotes the pushforward by  $f_1$  and  $K$  is a constant. In case of the one dimensional Normal distribution, since  $f_1$  is the identity, the pushforward does not change  $\nu$ , and this is why the necessity of the pushforward was not so clear at first sight.

Therefore, we will not apply the Stein method to  $\mu$ , but to the one-dimensional probability distribution  $f_1^\# \mu$ . We know that  $\nu$  satisfies an approximate integration by parts formula, and we need to derive such an approximate formula for  $f_1^\# \mu$ . By definition of pushforward, the natural thing to do is to evaluate (2.34) with  $g$  of the form  $g = \phi \circ f_1$  where  $\phi : \mathbb{R} \rightarrow \mathbb{R}$ . We are therefore dealing with quantities of the form  $L(\phi \circ f_1)$  and  $\Gamma(\phi \circ f_1)$ , and we would like these to be  $f_1$ -measurable too, that is of the form  $\Phi(f_1)$ . We saw in Section 2.3 that under a diffusion property of the generator  $L$ , the first quantity satisfies (see Formula (2.30))

$$L(\phi \circ f_1) = \phi'(f_1)Lf_1 + \phi''(f_1)\Gamma(f_1).$$

Furthermore, this implies for the latter that

$$\Gamma(\phi \circ f_1) = \phi'(f_1)^2 \Gamma(f_1).$$

Therefore, these quantities will be  $f_1$ -measurable as a function of the  $f_1$ -measurability of the carré du champ operator of the first eigenfunction. This amounts to asking the question of the existence of a function  $h : \mathbb{R} \rightarrow \mathbb{R}_+$  such that  $\Gamma(f_1) = h \circ f_1$ . We use two distinct approaches to this problem, depending on whether we consider the case of normalization conditions or the case of RCD conditions. Under the normalization conditions, we directly assume the existence of the factorization function  $h$ . We justify this assumption in dimension 1 by the monotonicity lemma 13 from chapter 3. Under the RCD condition, we show that the quantity  $\Gamma(f_1) + f_1^2$  is almost constant (see Lemma 64), and hence  $\Gamma(f_1)$  is almost  $f_1$ -measurable since it behaves like  $f_1^2$ .

#### 2.4.4 The use of the localization technique

Localization is a method from convex analysis that allows a multidimensional problem to be reduced to a one-dimensional study. It was developed by Gromov-Milman [107] and R. Kannan, L. Lovasz, M. Simonovits [146, 122]. R. Eldan [82], Y. T. Lee and S. Vempala [138] developed on this basis the stochastic localization, leading to the best result in the direction of a proof of the KLS conjecture by Klartag-Lehec [127] (see also [70] by Y. Chen). The KLS conjecture states that the Poincaré constant of a log-concave isotropic distribution in  $\mathbb{R}^d$  does not depend on the dimension.

A link between optimal transportation problems and the localization technique was discovered by B. Klartag in [126]. The localization theorem was thus established in the framework of smooth Riemannian manifolds. Following this work, F. Cavalletti and A. Mondino proved in [60, 61, 59] a localization theorem for metric measure spaces verifying the  $CD(K, N)$  condition, which is a weaker condition than the RCD one (see [58]). The principle is the following. Let us take an integrable function  $f : X \rightarrow \mathbb{R}$  on a  $RCD(K, N)$  space  $(X, d, \mu)$  with  $K \in \mathbb{R}$  and  $N \geq 1$ . Assume that  $f$  is centered with respect to  $\mu$ , and that there is some  $x_0 \in X$  such that  $x \mapsto |f(x)|d(x, x_0) \in L^1(\mu)$ . Then there is desintegration  $(\mu_q)_{q \in Q}$  such that

- i ) there is a probability distribution  $m$  over  $Q$  such that

$$\mu = \int_Q \mu_q dm,$$

- ii ) and for  $m$  almost every  $q \in Q$ , the support  $X_q$  of the distribution  $\mu_q$  is a geodesic, and the space  $(X_q, d|_{X_q}, \mu_q)$  is  $RCD(K, N)$ .

One-dimensional spaces  $X_q$  are often called needles, and this technique is therefore called needle decomposition. Among these, the needle decomposition has been used to derive rigid isoperimetric inequalities in  $CD$  spaces [60], quantitative estimates for isoperimetric inequalities in weighted Riemannian manifolds satisfying the condition  $CD(1, \infty)$  [149, 150], and Obata's rigidity theorem for  $CD$  spaces [62]. The last result is the closest to our Theorem 60 in Chapter 5. Let us state it briefly.

**Théorème 6.** [62, Theorem 1.3] *On an essentially non-branching  $CD(N-1, N)$  space  $(M, d, \mu)$  with spectral gap  $\lambda_1$  and associated normalized eigenfunction  $u$  there exists a point  $x_0 \in M$  such that*

$$\|u - \sqrt{N+1} \cos(d(\cdot, x_0))\|_2 \leq C(N)(\lambda_1 - N)^{1/(8N+4)}; \quad \pi - \text{diam}(M) \leq C(N)(\lambda_1 - N)^{1/N}.$$

We know that the model spaces are the unit spheres. In dimension one, this is an interval and the first eigenfunction is the cosine. This result therefore gives a stability of the sharp Poincaré constant in terms of the deviation of the first eigenfunction from the one-dimensional model. In this thesis, instead of the localization technique, we take the pushforward by a real function to reduce the problem to a one-dimensional problem.

## 2.5 Isoperimetric inequalities

The isoperimetric problem was first posed a long time ago. The question was what maximum area can be bounded by a string of prescribed length. The ancient Greeks already knew that the best shape to bound the maximum area is the circle. They knew that the area  $\mathcal{A}$  and the perimeter  $\mathcal{P}$  of a plane shape satisfy the following conditions

$$\mathcal{P}^2 \geq 4\pi\mathcal{A},$$

with equality for disks. The problem has also been posed for objects of dimension 3: what shape must have a minimum area for a prescribed volume? Significant progress towards a proof was made later with the use of differential calculus. In Mechanics, the isoperimetric problem was linked to the principle of least action. Indeed, for example, a soap bubble must minimise its surface area in order to minimise the tension it undergoes. In the 19th century, J. Steiner [199] introduced a method that allowed K. Weierstrass and H. Minkowski to prove the isoperimetric theorem in a rigorous manner. This method, now known as Steiner's symmetrization, is based on the fact that if an isoperimetric set is divided in two by a line, then the two halves give a new isoperimetric set by completing each of them symmetrically about the line. Let us now introduce the general framework of isoperimetric problems. We refer to the presentation [182] by A. Ros, and to the book [47] by Y. Burago and V. Zalgaller for a complete exposition.

The first notion we need to rigorously state the isoperimetric problem is the notion of volume. So we need a measured space. The second is the notion of area, i.e. a way of measuring the volume of sets of codimension 1. There are several non-trivial ways of extending the given measure of a measured space to the boundary of measurable sets. The most common is the Minkowski content, which uses a notion of distance. We therefore need a metric measured space. Let  $(X, d, m)$  be such a space, and  $A \subset X$  be a measurable set, we define its  $r$ -enlargement by

$$A_r := \{x \mid d(x, A) < r\}.$$

Then the Minkowski content of  $A$  is defined by

$$(2.35) \quad m^+(A) := \liminf_{r \rightarrow 0} \frac{m(A_r \setminus A)}{r},$$

and is a good way to rigorously define the measure of the surface  $\partial A$ . Let us stress that the Minkowski content is the derivative taken in  $r = 0$  of the function  $r \mapsto m(A_r \setminus A)$ . Informally, this quantity should be understood as

$$m^+(A) = \int |\nabla \mathbf{1}_A| \, dm.$$

Indeed, taking

$$f_r(x) = \left(1 - \frac{d(x, A)}{r}\right)_+,$$

with  $h_+ := \max(0, h)$ , we have  $f_r \xrightarrow{r \rightarrow 0} \mathbf{1}_A$  and  $|\nabla f_r| = \frac{1}{r} \mathbf{1}_{A_r \setminus A}$ . So if the liminf in the definition is in fact a limit, which is the case when the boundary of  $A$  is sufficiently regular, then

$$m^+(A) = \lim_{r \rightarrow 0} \frac{m(A_r \setminus A)}{r} = \lim_{r \rightarrow 0} \int |\nabla f_r| \, dm = \int |\nabla \mathbf{1}_A| \, dm,$$

and the last reversal between limit, integral and gradient is justified when  $A$  has a regular boundary.

The second main way to define the notion of perimeter is from the point of view of weak topology. We refer to [2], [85] and [218, Chapter 5]. Like the previous one, the idea of this approach is to consider that since the volume of a set  $A$  is given by the integral of its indicator



function  $\mathbf{1}_A$  with respect to the measure  $m$ , then the perimeter of  $A$  must be given by the integral of this indicator function  $\mathbf{1}_A$  with respect to the derivative of  $m$  in the sense of distributions. In other words, and after integration by parts, the perimeter of  $A$  must be the integral of the derivative of its indicator function  $\mathbf{1}_A$  with respect to  $m$ . In this approach, this idea is made rigorous by means of weak topology. The derivative of  $\mu$  applied to the indicator function is understood as the total variation of the indicator function  $\mathbf{1}_A$ . For example, if  $m$  is the Lebesgue measure on  $\mathbb{R}^d$ , then

$$m^+(A) := \sup \left\{ \int_A \operatorname{div} g \, dx \mid g \in \mathcal{C}_c^\infty(\mathbb{R}^d), \|g\|_\infty \leq 1 \right\}.$$

These two notions of perimeter coincide when  $A$  has sufficiently smooth boundary  $\partial A$  [80, Theorem 5]. Once these notions of volume and perimeter are defined, the isoperimetric problem takes on its full meaning. It is a question of minimising the perimeter among sets whose volume is prescribed. We are interested in

$$\inf \{ \operatorname{Per}(A) \mid m(A) = v \}.$$

Three natural questions arise:

1. Is the infimum reached?
2. What is the value of the infimum?
3. If the infimum is reached, which sets achieve the minimum?

### 2.5.1 The Euclidean isoperimetric inequality

In case where  $X$  is the Euclidean space  $\mathbb{R}^d$  and  $m = \lambda$  is the Lebesgue measure, the answers are yes the infimum is reached, it is reached for the Euclidean balls  $B_d$ , and its value is  $d\lambda(B_d)$ . This leads to the Euclidean isoperimetric inequality: for any measurable  $A \subset \mathbb{R}^d$ , one has

$$(2.36) \quad \lambda^+(A) \geq d\lambda(B_d)^{\frac{1}{d}} \lambda(A)^{1-\frac{1}{d}}.$$

We refer to the survey [174] by R. Osserman for this classic result. There are many different proofs of the Euclidean isoperimetric inequality. One of the most classical is the one based on Steiner symmetrization. At the heart of symmetrization methods is the idea [10] that if we have a way of constructing from any set  $E$  a set  $E^*$  with the same volume as  $E$  but with a smaller perimeter, then by repeating this process over and over again, it will show that isoperimetric sets possess the same symmetries as the iterated process. In particular, Steiner's symmetrization is based on the construction of  $E^*$  from the symmetries with respect to the hyperplanes. The heuristic at the end of the iteration is then that the balls are the only sets symmetric with respect to all hyperplanes passing through their barycentre. We refer to the course [117] given by H. Hopf in 1940. The cases of equality in the Steiner symmetrization were studied in the course [73].

#### The Brunn-Minkowski inequality

A very short proof of the Euclidean isoperimetric inequality can be given using the Brunn-Minkowski inequality. See [91, Chapter 3]. The Brunn-Minkowski inequality states that for any  $A, B \subset \mathbb{R}^d$ ,

$$|A + B|^{\frac{1}{d}} \geq |A|^{\frac{1}{d}} + |B|^{\frac{1}{d}}$$

where  $|\cdot|$  is the Lebesgue volume, and the set sum is defined by

$$A + B := \{a + b \mid a \in A, b \in B\}.$$

With this notion of sum of sets, we see that the  $r$ -enlargement of a set  $A$  satisfies

$$A_r = A + rB_d$$

where  $B_d$  is the unit ball of  $\mathbb{R}^d$ . Then the Brunn-Minkowski inequality gives

$$|A_r| \geq \left( |A|^{\frac{1}{d}} + |rB_d|^{\frac{1}{d}} \right)^d \geq |A| + d|A|^{\frac{d-1}{d}} |B_d|^{\frac{1}{d}} r$$

where the classic inequality  $\forall x \geq 0, (1+x)^d \geq x^d + dx^{d-1}$  is used to recover the second inequation. Then one gets

$$\frac{|A_r \setminus A|}{r} = \frac{|A_r| - |A|}{r} \geq d|A|^{\frac{d-1}{d}} |B_d|^{\frac{1}{d}},$$

which gives the Euclidean isoperimetric inequality (2.36) by taking the liminf. The Brunn-Minkowski inequality was first discovered and proved in the late 19th century. For more details on its history, see the book [190]. For a very clear exposition of the Brunn-Minkowski inequality, we refer the reader to the book [95] by R. Gardner. In particular, let us highlight the Prekopa-Leindler inequality as the functional formulation of the Brunn-Minkowski inequality: for all functions  $f, g, h : \mathbb{R}^d \rightarrow [0, +\infty)$  satisfying  $\forall x, y \in \mathbb{R}^d, \forall \lambda \in [0, 1], h((1-\lambda)x + \lambda y) \geq f(x)^{1-\lambda} g(y)^\lambda$ , it holds

$$\int h(x) dx \geq \left( \int f(x) dx \right)^{1-\lambda} \left( \int g(x) dx \right)^\lambda.$$

The Prekopa-Leindler inequality has its origin in the 1953 paper [112], where R. Henstock and A. Macbeath proved the Brunn-Minkowski inequality from a functional formulation.

### Optimal transport

Another very important proof uses optimal transport. The theory of optimal transport is very old, dating back to the end of the 18th century with its mathematical formulation by G. Monge. This problem has never ceased to be enriched over time, giving rise today to many powerful tools with a wide range of applications, such as geometry (with for example the synthetic notion of curvature dimension), statistics (see [175] for example), or machine learning (see [54]). We briefly present here the main concepts of optimal transport, and we refer to the introductory course [103] by N. Gozlan, P-M. Samson and P-A. Zitt, and to the complete book [213] by C. Villani. Given two probability distributions  $\mu$  and  $\nu$  on  $\mathbb{R}^d$ , the goal is to understand how to transport the mass of  $\mu$  on the mass of  $\nu$  in the most efficient way, i.e. by minimising a certain quantity seen as a cost by economists like L. Kantorovich, seen as an energy by physicists such as A-M. Ampère. More formally, the objective is to study

$$(2.37) \quad \inf_{\pi} \int c(x, y) d\pi(x, y),$$

where the supremum runs over all probability distributions  $\pi$  on  $\mathbb{R}^d \times \mathbb{R}^d$  with first marginal  $\mu$  and second marginal  $\nu$ , i.e. such that

$$\forall A \subset \mathbb{R}^d, \quad \pi(A \times \mathbb{R}^d) = \mu(A) \quad \text{and} \quad \pi(\mathbb{R}^d \times A) = \nu(A).$$

The function  $c$  is traditionally called a cost function, and in the following we only deal with the most studied one, which is the Euclidean quadratic cost  $c(x, y) = \frac{1}{2}|x - y|^2$ . Nevertheless, it is worth mentioning the growing number of works on general cost functions, we can mention for example the thesis of M. Prodhomme [180]. By the very general theorem of Prokhorov, the infimum in (2.37) is reached under natural assumptions on the cost function  $c$ , which are satisfied for the Euclidean quadratic cost. A coupling  $\pi$  reaching the minimum is called optimal coupling. When the optimal coupling takes the form  $\pi = (Id, T)^\#(\mu)$ , where  $(, )^\#$  denotes the pushforward, the coupling is said to be deterministic and is often identified with the map  $T$  which consequently satisfies  $T^\# \mu = \nu$ . A major result is the Brenier theorem [45, 46].

**Theorem.** (*Brenier's theorem*) Let  $c(x, y) = |x - y|^2$  be the Euclidean quadratic cost, and  $\mu$  be an absolutely continuous probability distribution with respect to the Lebesgue measure on  $\mathbb{R}^d$ . If the infimum (2.37) is finite, then it exists a map  $T : \mathbb{R}^d \rightarrow \mathbb{R}^d$  such that  $\pi = (\text{Id}, T)^\#(\mu)$  is an optimal coupling. Moreover,  $T = \nabla\phi$  is the gradient of a certain convex function  $\phi : \mathbb{R}^d \rightarrow \mathbb{R}$ . This optimal map  $T$  is called the Brenier map, and  $\phi$  is called its potential.

Not only does Brenier's theorem give a deterministic coupling, but it also gives an underlying convex structure to this map. Suppose further that the probability distribution  $\nu$  has a density  $\rho_2$  with respect to the Lebesgue measure, and let us denote by  $\rho_1$  the density of  $\mu$ . The fact that  $T^\#(\mu) = \nu$  means that for any bounded measurable function  $f : \mathbb{R}^d \rightarrow \mathbb{R}$ ,

$$\int f(x)\rho_2(x)dx = \int f(T(x))\rho_1(x)dx.$$

If we then formally apply the change of variable formula to the left-hand side of the equality, we obtain

$$\int f(T(x))\rho_2(T(x)) \det(\nabla T(x)) dx = \int f(T(x))\rho_1(x)dx,$$

which implies that almost surely with respect to the Lebesgue measure,

$$\rho_1(x) = \rho_2(T(x)) \det(\nabla T(x)).$$

Rewriting this equation for the convex potential  $\phi$  such that  $T = \nabla\phi$ , we derive the Monge-Ampère partial differential equation

$$(2.38) \quad \rho_1(x) = \rho_2(\nabla\phi) \det(\nabla^2\phi(x)),$$

where  $\nabla^2$  denotes the Hessian. The rigorous justification of the use of the change of variable formula requires very deep and tedious tools of convex analysis and measure theory. This is because Brenier's theorem does not give a strong regularity for the gradient  $\nabla\phi$ . The link with the Monge-Ampère equation is very important because this type of PDE was already widely used in fluid mechanics.

L. Caffarelli proved the following theorem by studying the regularity of the Monge-Ampère equation (2.38) for the Brenier map.

**Theorem.** (*Caffarelli's contraction theorem*) [52, 53] Let  $\mu$  be the Gaussian distribution on  $\mathbb{R}^d$ , and let  $\nu$  be a probability distribution with Lebesgue density given by  $d\nu(x) = e^{-V(x)}dx$  for some uniformly convex potential  $\text{Hess } V \geq \rho I_d$ , with  $\rho > 0$ , and  $I_d$  the identity matrix. Then the Brenier map carrying the Gaussian  $\mu$  to the uniformly log-concave measure  $\nu$  is 1-Lipschitz:

$$\|\nabla T\| \leq 1.$$

While Caffarelli's original proof is based on a precise study of the Monge-Ampère equation, a new proof has been established by entropic regularization in [90] and simplified [72].

Let us return to the isoperimetric inequality. In 1957, in the paper [129], H. Knothe proved the Euclidean isoperimetric inequality (2.36) by using a coupling between the uniform measure on a set  $A$  and the uniform measure on the ball  $B_d$ . Originally, the coupling used was the Knothe-Rosenblatt coupling. This coupling has the advantage of being constructively defined. However, instead of the Brenier map, the Knothe-Rosenblatt map does not have an underlying convex structure. Therefore, M. Gromov used the Brenier map as a deterministic coupling between the uniform distribution on a set  $A$  and the ball  $B_d$  in [163, Appendix] in order to simplify the proof. Nevertheless, the philosophy of the proof with optimal transport remains the same. We will now sketch it. Let  $A \subset \mathbb{R}^d$  and let  $B$  be a  $d$ -dimensional Euclidean ball of radius  $r$  centered at 0 such that  $B$  has the same volume as  $A$ :  $|A| = |B|$ . Let  $T = \nabla\phi$  be the

Brenier map between the uniform distribution  $\frac{1}{|A|}\mathbf{1}_A$  on  $A$  and the uniform distribution  $\frac{1}{|B|}\mathbf{1}_B$  on  $B$ . The Monge-Ampère equation gives that, almost surely,

$$\frac{|B|}{|A|} = \det \nabla T \mathbf{1}_A.$$

By integrating this relation over  $A$ , one gets

$$(2.39) \quad d|B|^{\frac{1}{d}}|A|^{1-\frac{1}{d}} = \int_A d \cdot (\det \nabla T)^{\frac{1}{d}} dx.$$

But since  $\nabla T$  is the Hessian of the potential  $\phi$  which is convex, denoting  $\lambda_i \geq 0$  the eigenvalues of  $\nabla T$ , and using the inequality of the arithmetic and geometric means, we have

$$(\det \nabla T)^{\frac{1}{d}} = \left( \prod \lambda_i \right)^{\frac{1}{d}} \leq \frac{1}{d} \sum \lambda_i = \frac{1}{d} \operatorname{div} T.$$

Then,

$$\int_A d \cdot (\det \nabla T)^{\frac{1}{d}} dx \leq \int_A \operatorname{div} T dx = \int_{\partial A} T \cdot \nu_A d\sigma \leq r \operatorname{Per}(A),$$

where the equality uses Stokes' theorem, and the last inequality comes from the fact that  $T \in B$ , so  $|T| \leq r$ . Finally, with (2.39) and since  $|B| = r^d |B_d|$  where  $B_d$  is the unit Euclidean ball, we recover equality (2.36).  $\square$

This proof based on optimal transport was refined by A. Figalli, F. Maggi and A. Pratelli in [93] to obtain a stability result in the isoperimetric inequality. The Brenier theorem and Gromov's proof of the isoperimetric inequality are at the heart of their approach. The stability results consist of quantitative studies of the deficit

$$\lambda^+(A) - d\lambda(B_d)^{\frac{1}{d}}\lambda(A)^{1-\frac{1}{d}}$$

in the isoperimetric inequality. The goal is to quantify the sharpness of the isoperimetric inequality when  $A$  is in some sense close to the ball. The notion of proximity used by A. Figalli, F. Maggi and A. Pratelli is the Fraenkel asymmetry:

$$F(A) := \inf \left\{ \frac{|A \Delta B(x, r)|}{|A|} \mid x \in \mathbb{R}^d, r > 0, \text{ such that } |B(x, r)| = |A| \right\},$$

where  $\Delta$  denotes the symmetric difference of sets  $A \Delta B := (A \setminus B) \cup (B \setminus A)$ , and  $B(x, r)$  denotes the Euclidean ball of radius  $r$  centered at  $x$ . Their main result is the following.

**Theorem.** [93, Thm 1.1] *It exists a finite constant  $C_d$  depending on the dimension such that*

$$\operatorname{Per}(A) \geq d\lambda(B_d)^{\frac{1}{d}}\lambda(A)^{1-\frac{1}{d}} \left( 1 + \left( \frac{F(A)}{C_d} \right)^2 \right).$$

*Moreover, the exponent 2 is sharp.*

For more information on the issue of stability in isoperimetric inequalities, we refer the reader to the [147] by F. Maggi.

## Calibration

Let us mention a last approach to prove the Euclidean isoperimetric inequality in dimension 2. This approach is based on the concept of calibration (see [109]). Let us sketch the main idea. Let  $\beta$  be a  $(d-1)$ -differential form on  $\mathbb{R}^d$  such that  $|\beta| \leq 1$ . If  $N \subset \mathbb{R}^d$  is an open and

connected set such that  $\beta$  is the volume form on  $N$ , then  $\text{Per}(N)$  is minimal among all sets  $N'$  of the same  $\beta$ -volume:  $\int_N d\beta = \int_{N'} d\beta$ . Indeed, since  $\beta$  is the volume form on  $N$ , one has

$$\text{Per}(N) = \int_{\partial N} \beta = \int_N d\beta = \int_{N'} d\beta = \int_{\partial N'} \beta \leq \text{Per}(N'),$$

where we used Stokes' theorem in the second and fourth equality, the assumption on  $N'$  in the third equality, and the assumption  $|\beta| \leq 1$  in the last inequality. The construction of such a differential form on  $\mathbb{R}^2$  was done by F. Helein in [110, 111]. The advantage of this method is that it gives a very short proof essentially based on Stokes' theorem, the disadvantage is that the construction of such a differential form  $\beta$  is done by hand on a case by case basis.

## Variational approach

By definition, isoperimetric sets are global minima of the Perimeter functional restricted to sets of fixed volume. From this point of view, there are two necessary conditions for a set to be isoperimetric. It must be a critical point of the perimeter functional and, in addition, the Hessian of the perimeter must be non-negative. As with functions, the positive-definite condition ensures that the set is a local minimum of the perimeter. The study of the critical points of the perimeter functional leads to the notion of stationary sets. The study of the points for which the Hessian of the perimeter functional is positive leads to the notion of stable sets. These notions of critical points and positive definite Hessian are formally implemented in terms of first and second variation. The pionner papers in this approach are the 1982 and 1983 articles [168, 167] by M. Hiroshi, and the 1984 article [16] by L. Barbosa and M. Do Carmo. Let us briefly describe the main concepts.

Let  $A \subset \mathbb{R}^d$  be a smooth open set and  $\partial A = \Sigma$  its boundary. Let  $N$  be the inward-pointing unit normal vector to  $\Sigma$ . Let  $(\phi_t : \mathbb{R}^d \rightarrow \mathbb{R}^d)_t$  be a one parameter variation with associated infinitesimal vector field  $X = \left. \frac{d\phi_t}{dt} \right|_{t=0}$ . Let  $u = \langle X, N \rangle$  be the normal component of  $X$ . Set  $A_t := \phi_t(A)$ ,  $\Sigma_t := \phi_t(\Sigma)$ , and let  $V(t) = \text{Vol}(A_t)$  be the volume variation, and  $P(t) = \text{Per}(A_t)$  be the perimeter variation. Using standard formulae of differential geometry (see for example [195]), one can calculate the first variation of volume and perimeter:

$$(2.40) \quad V'(0) = - \int_{\Sigma} u \, da, \quad \text{and} \quad P'(0) = - \int_{\Sigma} (d-1)H u \, da,$$

where  $da$  is the surface measure on  $\Sigma$  and  $H$  is the Euclidean mean curvature of  $\Sigma$  with respect to  $N$ , ( $H$  is the trace of the differential  $dN$  of  $N$ ). One can also compute the second variation of the volume and perimeter of  $A$ :

$$(2.41) \quad (P - HV)''(0) = Q(u, u) := \int_{\Sigma} (|\nabla_{\Sigma} u|^2 - |\sigma|^2 u^2) \, da,$$

where  $\nabla_{\Sigma} u$  is the gradient of  $u$  computed in the geometry of  $\Sigma$ , and  $|\sigma|^2$  is the squared sum of the principal curvatures of  $\Sigma$ . Now, we can introduce definitions.

**Definition 7.** *We say that a variation  $(\phi_t)_t$  preserves volume if  $V(t) = \text{constant}$ . We say that the set  $A$  is stationary if  $P'(0) = 0$  for all volume preserving  $(\phi_t)$ , and we say that  $A$  is stable when  $P''(0) \geq 0$  for all volume preserving variation  $(\phi_t)$ .*

The calculation of first and second variations formulas (2.40) and (2.41) allows us then to state the following characterization of the stationary and stable sets.

**Proposition 8.** *The set  $A$  is stationary if and only if its boundary  $\Sigma$  has constant mean curvature  $H_0$ .*

*The set  $A$  is stable if and only if for any smooth function  $u$  with compact support in  $\Sigma$  such that  $\int_{\Sigma} u \, da = 0$ , we have  $Q(u, u) \geq 0$ .*

The characterisation of the stationarity by means of a constant mean curvature allows this problem to be linked directly to the theory of minimal surfaces which are surfaces with zero mean curvature. The theory of minimal surfaces has been strongly developed since the creation of calculus of variations by L. Euler and J-L. Lagrange in the 18th century (see [154] for an overview on modern advances).

The characterization of the stability gives a link with Poincaré inequalities. Indeed, with Formula (2.41), we can see that the condition  $Q(u, u) \geq 0$  is in fact a weighted Poincaré inequality on  $\Sigma$ . Therefore, the knowledge of the sharp Poincaré constant on  $\Sigma$  will provide information on the stable sets (see Chapter 6 Proposition 87).

## 2.5.2 Isoperimetric inequalities in probability spaces

The classic isoperimetric problem concerns the Lebesgue measure on  $\mathbb{R}^d$ . However, we have seen that the notion of perimeter has a meaning for any measure  $\mu$ . In particular, the isoperimetric problem is very interesting when  $\mu$  is a probability measure. About the problem of existence of isoperimetric sets in this weighted context, we refer to the paper [166] by F. Morgan and A. Pratelli. Let us mention in this field the recent proof of the double bubble conjecture by M. Hutchings, F. Morgan, M. Ritoré and A. Ros in 2002 [119] and the proof of the double bubble and multi-bubble conjectures in Gaussian space by E. Milman and J. Neeman in 2018 [162].

### The isoperimetric profile

The isoperimetric profile  $I_\mu$  of a probability distribution  $\mu$  is a function that encodes much information about the associated isoperimetric problem. It is defined by

$$I_\mu : [0, 1] \rightarrow \mathbb{R}_+,$$

$$I_\mu(p) := \inf\{\text{Per}_\mu(A) \mid \mu(A) = p\}.$$

Let us specify that the isoperimetric profile depends on the measure  $\mu$ , but also on the distance used in the definition of the perimeter, although this distance does not appear in the notation  $I_\mu$ . In the following, we will mainly be interested in Euclidean and uniform distances. In dimension one, the isoperimetric profile does not depend on the distance, as can be easily seen in the definition of the Minkowski content (2.35). The isoperimetric profile is symmetric around  $\frac{1}{2}$  and vanishes at 0 and 1. In [32], S. Bobkov showed that if  $\mu$  is an even log-concave distribution on  $\mathbb{R}$ , then its isoperimetric profile is given by

$$I_\mu = \phi \circ \Phi^{-1},$$

where  $\phi$  is the Lebesgue density of  $\mu$ , and  $\Phi$  is its cumulative distribution function. Recall that a measure on  $\mathbb{R}^d$  is said to be log-concave when it has a Lebesgue density of the form  $e^{-V}$  for a convex potential  $V$ . This formula means that half-spaces  $(-\infty, t)$  are isoperimetric sets. For example, we can give the isoperimetric profiles of two central distributions: when  $\mu = \gamma$  is the Gaussian, then  $I_\gamma = \phi \circ \Phi^{-1}$  with  $\phi(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$ , and when  $\mu$  is the one-sided exponential distribution, then  $I_\mu(p) = \min(p, 1-p)$ . The first proofs of the Gaussian case were given independently in 1974 by C. Borell [41] and V. Sudakov and B. Tsirel'son [203]. S. Bobkov also showed that on the real line, the log-concavity of the measure is equivalent to the concavity of its isoperimetric profile (see [32, Proposition A.1]). Let us mention the PhD thesis of V. Bayle [25] on the generalization of this concavity property of the isoperimetric profile when  $\mu$  is the volume measure on Riemannian manifold. In the sequel, we will often be interested in the isoperimetric profile of product measures. Let us mention the following immediate comparison

$$(2.42) \quad I_{\mu^{\otimes n}} \leq I_{\mu^{\otimes(n-1)}},$$

which results from the evaluation of the perimeter function at sets of the form  $A \times \mathbb{R}^{d-1}$ . An important quantity in this field of study is the infinite dimensional isoperimetric profile, defined by

$$I_{\mu^\infty} := \inf_n I_{\mu^{\otimes n}}.$$

Obviously, from the definition and the monotonicity property (2.42), the infinite dimensional isoperimetric profile is bounded by above in the following manner:

$$I_{\mu^\infty} \leq I_\mu.$$

An interesting question is to know when the reverse inequality holds, i.e. to understand when there is a certain constant  $1 > c > 0$  such that

$$(2.43) \quad I_{\mu^\infty} \geq c I_\mu.$$

The infinite dimensional isoperimetric inequality (2.43) means that for all  $n$ , there is some half-space of the form  $A \times \mathbb{R}^{n-1}$  which is isoperimetric up to the multiplicative constant  $c^{-1}$ . A probability measure  $\mu$  satisfying (2.43) has an intermediate behaviour between the exponential and the Gaussian. We refer to the article [21], by F. Barthe, P. Cattiaux and C. Roberto. Conversely, the same authors have proved the following:

**Theorem.** [20, Theorem 46] For all  $\alpha \in [1, 2]$ , let  $d\mu_\alpha(t) = \frac{1}{Z_\alpha} e^{-|t|^\alpha} dt$ , where  $Z_\alpha$  is a normalization constant. Then there exists a universal constant  $K > 0$  such that for all  $\alpha \in [1, 2]$ ,

$$I_{\mu_\alpha^\infty} \geq K I_{\mu_\alpha}.$$

## Variational approach

In Section 2.5.1, we presented the variational approach to the isoperimetric problem in the Euclidean case. Here we deal with the weighted case, i.e. when the measure has a density with respect to the Lebesgue measure. The philosophy of the approach is the same as in the case of the Lebesgue measure, however the computations of the first and second variation change slightly due to the presence of the weight function. Stationary sets are then characterised by their constant generalised mean curvature instead of the classical mean curvature, and stable sets are still characterised by some weighted Poincaré inequalities (with a different weight than in the Euclidean case). We refer to the article [184] by C. Rosales, A. Canete, V. Bayle and F. Morgan.

**Proposition 9.** [184, Proposition 3.2 and Lemma 3.8] Let  $\mu$  be a measure on  $\mathbb{R}^d$  with Lebesgue density  $f(x) = e^{\psi(x)}$ . The following holds.

- The set  $A$  is stationary if and only if its boundary  $\Sigma$  has constant generalized mean curvature

$$(d-1)H - \langle \nabla \psi, N \rangle = \text{constant},$$

where  $H$  is the mean curvature of  $\Sigma$  with respect to its inward unit normal vector  $N$ .

- The set  $A$  is stable if and only if for all smooth  $u \in \mathcal{C}_c^\infty$  with compact support in  $\Sigma$ , such that  $\int_\Sigma u f da = 0$ , we have

$$\int_\Sigma \left( |\nabla_\Sigma u|^2 + \left( \nabla^2 \psi(N, N) - |\sigma|^2 \right) u^2 \right) da \geq 0,$$

where  $\nabla_\Sigma u$  is the gradient of  $u$  relative to  $\Sigma$ , and  $|\sigma|^2$  is the squared sum of the principal curvatures of  $\Sigma$ , and  $\nabla^2$  denotes the Hessian.

In the paper [183] published in 2021, C. Rosales extends these results in the case of a weighted Riemannian manifold with boundary. The only addition in the characterization of stationary sets is the requirement that the boundary  $\Sigma$  of the set orthogonally meets the boundary of the manifold (see [183, Lemma 2.3]). Note that this requirement is very natural, for example when thinking of a soap bubble on a surface. We will use Rosales' result in the section 6.4.

### Anisotropic perimeters

In this section, we briefly mention anisotropic isoperimetric problems. These problems consist of isoperimetric problem where the perimeter does not come from a distance but from a semi-norm  $H$  which is a non-negative, convex and positively homogeneous function of degree one on  $\mathbb{R}^d$ . The associated perimeter is then defined as

$$\text{Per}_H(A) := \int_{\partial A} H(N(x)) ds(x),$$

where  $N(x)$  is the outward-pointing unit normal at  $x$  of  $\partial A$ . When  $H$  is the Euclidean norm, we find the standard Euclidean perimeter. Anisotropic isoperimetric problems consist in minimising the perimeter  $\text{Per}_H$  under a fixed Euclidean volume constraint. The Wulff form then plays the role of the Euclidean ball, and is defined by

$$W_H := \{x \in \mathbb{R}^d \mid x \cdot v < H(v), \forall v \in \mathbb{S}^{d-1}\}.$$

In the case where  $H$  is the Euclidean norm, the Wulff form is the Euclidean ball. The problem of anisotropic isoperimetry is solved by the famous Wulff theorem. G. Wulff stated it without proof in [216] in 1901, and J. Taylor proved it in [207, 208] in 1974-1975.

**Theorem.** (*Wulff's inequality [216, 207, 208]*) For all  $A \subset \mathbb{R}^d$  with finite and non zero volum, it holds

$$\frac{\text{Per}_H(A)}{|A|^{\frac{d-1}{d}}} \geq \frac{\text{Per}_H(W)}{|W|^{\frac{d-1}{d}}}.$$

Moreover, equality holds if and only if  $A = aW + b$  for some  $a > 0$  and  $b \in \mathbb{R}^d$ .

In this anisotropic context, X. Cabré, X. Ros-Oton and J. Serra have given a new proof of the Wulff inequality with a method based on the Alexandroff-Backelman-Pucci estimate. This estimate is a uniform bound on the solution of the Dirichlet problem associated with uniformly elliptic operators of the second order (see [49] for details on this estimate). This method has already been used by X. Cabré in [48] to derive the classical Euclidean isoperimetric inequality. In the anisotropic case, this method allowed the three authors to generalize Wulff's inequality to open convex cones of  $\mathbb{R}^d$ . Let us mention that A. Figalli, F. Maggi and A. Pratelli have obtained a quantitative stability result for Wulff's inequality on  $\mathbb{R}^d$  in [93]. Their proof is based on optimal transport, as mentioned in the section on optimal transport in 2.5.1.

## 2.6 Functional formulations of isoperimetry

Isoperimetric inequalities can be encoded by functional inequalities. The most common are Poincaré inequalities, logarithmic Sobolev inequalities and Bobkov type inequalities.

### 2.6.1 Poincaré inequalities

We saw in Section 2.5.2 that the isoperimetric profile of the exponential distribution is  $I(t) = \min(t, 1 - t)$ . The associated isoperimetric inequalities for a probability measure  $\mu$  are of the form

$$(2.44) \quad \text{Per}_\mu(A) \geq h_\mu \min(\mu(A), 1 - \mu(A)),$$

where  $h_\mu$  is the smaller constant such that (2.44) holds. This constant  $h_\mu$  is the Cheeger constant, already mentioned in Section 2.1.3 of this introduction. We have seen that the Cheeger constant, and thus isoperimetric Cheeger inequalities, are related to Poincaré inequalities, in the sense that Poincaré inequalities imply isoperimetric Cheeger inequalities. Therefore, the Poincaré inequality is a functional formulation of the Cheeger inequality. More precisely, let



us introduce the  $(p, q)$ -Poincaré inequalities. Let  $p, q \in [1, +\infty]$ . We say that the measure  $\mu$  satisfies a  $(p, q)$ -Poincaré inequality if there exists a constant  $C_{p,q} > 0$  such that for all sufficiently smooth functions  $f$ ,

$$(2.45) \quad \|f - M_\mu(f)\|_{L^p(\mu)} \leq C_{p,q}(\mu) \|\nabla f\|_{L^q(\mu)},$$

where  $M_\mu(f)$  is a median of  $f$ . Note that the classic Poincaré inequality corresponds in this terminology to the  $(2, 2)$ -Poincaré inequality. With this definition, the Cheeger isoperimetric inequality is equivalent to the  $(1, 1)$ -Poincaré inequality. This result is chronologically due to H. Federer and W. Fleming [92], V. Maz'ja [151], S. Bobkov and C. Houdré [36].

**Theorem.** [36, 92, 151]

The Cheeger isoperimetric inequality (2.44) holds if and only if the  $(1, 1)$ -Poincaré inequality (2.45) holds. Moreover, in this case, the sharp constants of the two inequalities satisfy the following relationship:  $h_\mu = \frac{1}{C_{1,1}(\mu)}$ .

Let us briefly sketch the proof of [36]. In the same way as for the classic Poincaré inequality (see Section 2.1), the Cheeger isoperimetric inequality is recovered from the  $(1, 1)$ -Poincaré inequality by evaluating the latter with characteristic functions of Borell sets. The rigorous formulation is then based on the approximation of the indicators by Lipschitz functions. The converse statement is based on the co-area formula. For simplicity, assume that  $M_\mu(f) = 0$ . Then

$$\int |\nabla f| d\mu = \int_{\mathbb{R}} \mu^+(f > t) dt \geq h_\mu \left( \int_{-\infty}^0 (1 - \mu(f > t)) dt + \int_0^{\infty} \mu(f > t) dt \right) = h_\mu \int |f| d\mu,$$

where the first and last equalities come from the co-area formula, and the inequality is the Cheeger isoperimetric inequality (2.44).  $\square$

Unlike the classical  $(2, 2)$ -Poincaré inequality, the  $(1, 1)$ -Poincaré inequality has no spectral interpretation. However, its sharp constant is related to the Cheeger constant in the same way as the Poincaré constant  $C_P(\mu)$  is related to the first eigenvalue of the associated generator. Moreover, the link between these different inequalities is deeper: when  $\mu$  is a log-concave distribution, E. Milman proved that all Poincaré inequalities  $(p, q)$  are equivalent [159].

**Theorem.** (E. Milman, 2009, [159, Theorem 2.4])

If  $\mu$  is log-concave, then for any  $1 \leq p \leq q \leq \infty$  and  $1 \leq p' \leq q' \leq \infty$ ,

$$C_{p'q'}(\mu) \leq K p' C_{p,q}(\mu),$$

where  $K > 0$  a universal constant.

In connection with the Cheeger constant and log-concave measures, let us mention the famous Kannan-Lovasz-Simonovits conjecture.

**Conjecture 10.** (R. Kannan, L. Lovasz, and M. Simonovits, 1995, [122])

There exists a universal constant  $C > 0$ , such that for all  $d \geq 1$  and for all log-concave measures  $\mu$  in  $\mathbb{R}^d$ ,

$$h_\mu \geq C \inf_H \frac{\mu^+(\partial H)}{\min(\mu(H), 1 - \mu(H))},$$

where the infimum runs over all half-spaces of  $\mathbb{R}^d$ .

We refer the reader to the survey [139] for its multiple connections with other conjectures in the study of the geometry of convex bodies. Localization techniques are at the heart of attempts to prove the KLS conjecture. The aim of localisation techniques is to reduce a

multidimensional problem to a one-dimensional one by disintegrating a measure along one-dimensional "needles". The needles can be obtained by iterating bisections of hyperplanes, as in [122], or by following the trajectories of a stochastic process (Eldan stochastic localization, see [138]), or by following the mass transport lines of an optimal transport problem (Klartag-Cavalletti-Mondino localization, see [61, 126]). The first two localisation schemes are mainly part of convex analysis, while the third one is part of the geometry of curvature dimension spaces. The best result towards the proof of the KLS conjecture is the following theorem by B. Klartag and J. Lehec in 2022 in [127].

**Theorem.** [127, Theorem 1.1] *There exists a universal constant  $c' > 0$  such that for all  $d \geq 2$  and for all isotropic log-concave measures  $\mu$  in  $\mathbb{R}^d$ ,*

$$h_\mu \geq c' (\log d)^{-5}.$$

The proof is based on a carefull and technical implementation of Eldan's stochastic localization with the analysis of the evolution of log-concave measurements along the heat flow.

### 2.6.2 Bobkov's inequalities

We saw in the previous section that Poincaré inequalities are the functional form of Cheeger isoperimetric inequalities, which are themselves related to the isoperimetric profile of the exponential distribution. We have seen that the isoperimetric profile of the normal distribution is given by  $I_\gamma = \phi \circ \Phi^{-1}$ , with  $\phi$  the density of the Gaussian distribution and  $\Phi$  its cumulative distribution function. We say that a probability distribution satisfies a Gaussian isoperimetric inequality when for any measurable set  $A$ , it holds for a certain constant  $c \geq 0$ :

$$(2.46) \quad \text{Per}_\mu(A) \geq c I_\gamma(\mu(A)).$$

The most efficient functional formulation of the Gaussian isoperimetric inequality is the Bobkov inequality introduced in 1997 in [33] by S. Bobkov.

$$(2.47) \quad \text{For all } f : \mathbb{R}^d \rightarrow [0, 1], \text{ locally Lipschitz, } I_\gamma \left( \int_{\mathbb{R}^d} f d\mu \right) \leq \int_{\mathbb{R}^d} \sqrt{I_\gamma(f)^2 + |\nabla f|^2} d\mu.$$

The fact that the normal distribution  $\mu = \gamma$  satisfies (2.47) admits many different proofs. The original proof in [33] consists in proving first that the Bobkov inequality satisfies the tensorization property, like the Poincaré inequalities (see end of Section 2.1). In other words, if  $\mu_1$  and  $\mu_2$  satisfy (2.47), then the product measure  $\mu_1 \otimes \mu_2$  also satisfies (2.47). Secondly, the proof consists in showing (2.47) when  $\mu$  is a Bernoulli measure. In this case, Inequality (2.47) reduces to a two-points inequality. Then, by the Central Limit Theorem and the tensorization property, S. Bobkov derived Inequality (2.47) for the normal distribution.

In 1996, in [15], D. Bakry and M. Ledoux proved this fact by the semigroup method. They used the Ornstein-Uhlenbeck semigroup  $(P_t)_{t \geq 0}$  for which the normal distribution is the equilibrium distribution. At the heart of their proof is the fact that for all smooth functions  $f$  taking their values in  $[0, 1]$ , the quantity

$$\int \sqrt{I_\gamma(P_t(f)) + |\nabla P_t(f)|^2} d\gamma$$

is non increasing in  $t$ . We refer to [136] for a short proof using this method.

Let us mention the martingale proof by F. Barthe and B. Maurey [23]. Their approach consists in proving the two-points Bobkov inequality by use of Doob's optional sampling theorem for a well-choosen martingale. Inequality (2.47) then follows from the tensorization property and the Central Limit Theorem, as in the original proof of S. Bobkov.

The Bobkov inequality is equivalent to the Gaussian isoperimetric inequality in the following sense.

**Theorem.** [23] A probability distribution  $\mu$  satisfies  $I_\mu \geq cI_\gamma$  for some constant  $c \geq 0$ , if, and only if, for all Lipschitz functions  $f : \mathbb{R}^d \rightarrow [0, 1]$ ,

$$(2.48) \quad I_\gamma \left( \int_{\mathbb{R}^d} f d\mu \right) \leq \int_{\mathbb{R}^d} \sqrt{I_\gamma(f)^2 + \frac{1}{c^2} |\nabla f|^2} d\mu.$$

To prove the "if" direction, we proceed as usual by evaluating (2.48) in approximations of indicator functions. To prove the converse statement, we return to the case  $c = 1$ , and the method is based on the rearrangement technique in this context, which gives that if  $R = \{(x, t) \mid \Phi(t) < f(x)\}$ , then

$$\text{Per}_{\mu \otimes \gamma}(R) = \int_{\mathbb{R}^d} \sqrt{I_\gamma(f)^2 + |\nabla f|^2} d\mu.$$

The rearrangement method is the generalization for functions of the set symmetrization method. We refer to [2]. Given that

$$\mu \otimes \gamma(R) = \int \int_{-\infty}^{\Phi^{-1}(f(x))} d\gamma dx = \int f(x) d\mu(x),$$

it follows that

$$\text{Per}_{\mu \otimes \gamma}(R) \geq I_{\mu \otimes \gamma} \left( \int f(x) d\mu(x) \right).$$

But we know that  $I_\mu \geq I_\gamma$  implies  $I_{\mu \otimes \gamma} \geq I_{\gamma \otimes \gamma}$  (cf. [18, Theorem 8]), so we can conclude thanks to the infinite dimensional property of the isoperimetric profile of the Gaussian distribution:  $I_{\gamma \otimes \gamma} = I_\gamma$ .  $\square$

The isoperimetric inequality  $I_\mu \geq cI_\gamma$  is rigid: if  $\mu$  is a probability distribution such that  $I_\mu = cI_\gamma$ , then  $\mu = \mathcal{N}(0, c^2 I_d)$  is Gaussian [38]. The stability has not yet been studied. An interesting question is which functions  $f$  achieve equality in (2.48). The answer was given by E. Carlen and J. Kerse in [55]. Equality is achieved for indicator functions of half-spaces and for functions of the form  $\Phi(v \cdot x + b)$  where  $\Phi$  denotes the cumulative distribution function of the normal distribution,  $v$  is a vector in  $\mathbb{R}^d$ ,  $b$  is a real number and  $\cdot$  denotes the scalar product on  $\mathbb{R}^d$ .

A natural extension of (2.48) consists in replacing the normal distribution by another reference measure  $\nu$ . This has the consequence of replacing the Gaussian isoperimetric profile  $I_\gamma$  by the isoperimetric profile  $I_\nu$  of  $\nu$ . We say that a probability measure  $\mu$  satisfies a Bobkov type inequality with respect to a reference probability measure  $\nu$  when there exists a certain constant  $\alpha > 0$  such that

$$(2.49) \quad I_\nu \left( \int f d\mu \right) \leq \int \sqrt{I_\nu(f)^2 + |\alpha \nabla f|^2} d\mu.$$

S. Bobkov proved in [35] that (2.49) is satisfied by measures  $\mu$  having a positive Cheeger constant when the reference measure is the logistic measure  $\nu$ . Recall that the logistic measure is the probability distribution on  $\mathbb{R}$  with a Lebesgue density  $\frac{e^x}{(1+e^x)^2}$  and an isoperimetric profile  $I_\nu(p) = p(1-p)$ .

A natural question then is whether (2.49) is equivalent to some isoperimetric inequality of the form  $I_\mu \geq \alpha I_\nu$  as in the case where  $\nu = \gamma$  is the normal distribution. The answer is negative. The only step that does not work when we replace  $\gamma$  by a general  $\nu$  in the proof of Theorem 2.6.2 is the equality  $I_{\nu \otimes \nu} = I_\nu$ . Indeed, the normal distribution is the only measure satisfying this property (cf. [38]). However, B. Huou has shown the following characterization of the Bobkov type inequality (2.49).

**Theorem.** [118, Section 5] The measure  $\mu$  satisfies (2.49) if and only if

$$(2.50) \quad I_{\mu \otimes \nu_\alpha} = \frac{1}{\alpha} I_\nu,$$

where  $\nu_\alpha$  denotes the  $\alpha$  dilatation of  $\nu$ .

The  $\alpha$  dilatation of  $\nu$  is defined as the distribution of the random variable  $\alpha X$  when  $X$  is drawn with distribution  $\nu$ . It is easy to see with Theorem 2.6.2 that the equation (2.50) is equivalent to the inequality  $I_\mu \geq \frac{1}{\alpha} I_\gamma$  when  $\nu = \gamma$ . Consequently, Theorem 2.6.2 generalizes Theorem 2.6.2.

### 2.6.3 Logarithmic Sobolev inequality

Let us conclude this introduction to functional formulations of isoperimetry by briefly mentioning Logarithmic Sobolev inequalities (LSI). We refer to [13] for a full exposition. We say that a probability distribution  $\mu$  in  $\mathbb{R}^d$  satisfies a Logarithmic Sobolev inequality when there exists a constant  $C_{LSI} > 0$  such that for all functions  $f \in \mathcal{C}^2(\mathbb{R}^d)$ ,

$$(2.51) \quad \text{Ent}_\mu(f^2) \leq C_{LSI}(\mu) \int |\nabla f|^2 d\mu.$$

The entropy is defined for non-negative functions by

$$\text{Ent}_\mu(f) := \int f \log(f) d\mu - \int f d\mu \log \int f d\mu.$$

Logarithmic Sobolev inequalities were introduced by L. Gross in 1975 in [108] in order to characterize hypercontractivity of a Markov semigroup using functional inequalities. Later, F. Otto and C. Villani showed connections with Talagrand transportation inequalities [206]. Logarithmic Sobolev inequalities satisfy the tensorization property. Moreover, they imply Poincaré inequalities, as can be shown directly by taking  $1 + \varepsilon f$  instead of  $f$  in (2.51) and letting  $\varepsilon$  go to zero. Like the Poincaré inequalities, the Logarithmic Sobolev inequalities are related to the spectral properties of Markov generators. In particular, they imply discreteness of the spectrum and integrability of eigenfunctions (see [13, Section 5.3]).

The Bobkov inequality (2.48) implies the Logarithmic Sobolev inequality, thus linking LSI inequalities to isoperimetric problems. This can be seen directly from the inequality (2.48) by taking  $f = \varepsilon g^2$ , using the equivalence

$$I_\gamma(x) \underset{x \rightarrow 0}{\sim} x \sqrt{2 \log \frac{1}{x}},$$

and letting  $\varepsilon$  go to zero.

## 2.7 Outline of the thesis

In this section, we briefly present the structure of this thesis.

In the chapter 3, we study the stability of the Poincaré constant of the equilibrium distribution  $\mu$  of a Markov diffusion process, in a space of measures satisfying certain normalisation conditions. Stability results with respect to the total variation distance and the 1-Wasserstein distance are established. The same proof strategy is used in both results: we first show that under these normalisation conditions, the test distributions  $\nu$  satisfy an approximate integration by parts formula whose error term is controlled by the difference of the Poincaré constants. Next, we take the pushforward of  $\mu$  by the first eigenfunction  $f_1$ . To do this, we assume that the carré du champ operator  $\Gamma(f_1)$  factorizes as  $\Gamma(f_1) = h \circ f_1$  for some real function  $h$ . This assumption is justified by the Lemma 13. We then implement Stein's method and find sufficient conditions on  $h$  (i.e. on  $\Gamma(f_1)$ ) to have the equivalent of Stein's lemma for  $\mu$  (instead of the normal distribution  $\gamma$  in the classical result). To derive stability in total variation distance, we ask  $h$  to be uniformly bounded by below:  $h \geq \kappa > 0$ . This assumption is verified by uniformly strictly log-concave measures, which provide an example of application of our first result (see Section 3.6.3).

To obtain a stability quantified by the 1-Wasserstein distance, we require certain growth conditions on  $h$  at the boundary of its domain of definition (cf. Proposition 36). These conditions are introduced for technical purposes, but are verified in all our main examples of Gamma distributions on  $\mathbb{R}_+$  and Beta distributions (see sections 3.6.2 and 3.6.4).

In Chapter 4, we study the stability of the improved Poincaré constant of an equilibrium distribution  $\mu$  of a Markov diffusion process, among a space of measures satisfying normalization conditions as in Chapter 3. As the Poincaré constant corresponds to the inverse of the first non-zero eigenvalue, the  $k$ -th sharp Poincaré constant corresponds to the inverse of the  $k$ -th non-zero eigenvalue, counted without multiplicity. We derive a stability result with respect to the 1-Wasserstein distance, in the case where  $\mu$  is a probability distribution on  $\mathbb{R}$ . The strategy of proof is mainly the same as for the stability results of the chapter 3. First, we show that under the normalization conditions, the test measures  $\nu$  satisfy the approximate integration by parts formula, whose default term is controlled by the difference between the  $k$ -th eigenvalues. Next, we want to take the pushforward by an eigenfunction  $f_k$  associated with the  $k$ -th non-zero eigenvalue. This step is more complicated than in the case of the first eigenvalue, because there is no monotonicity lemma for  $f_k$  like the 13 lemma in the case of  $f_1$ . To solve this problem, we restrict the pushforward to outside the critical points of the function, i.e. on each interval where the carré du champ operator  $\Gamma(f_k)$  does not vanish. Then we implement Stein's method on these intervals, and adapt the proposition 36 to derive the equivalent of Stein's lemma. In order to obtain a stability result, we have to ask for some growth conditions on  $\Gamma(f_k)$  in the neighbourhood of its zeros. These conditions are simply technical and necessary for the calculation, but are verified in the case of the normal distributions on  $\mathbb{R}$ , Gamma distributions on  $\mathbb{R}_+$ , and Beta distributions on  $[-1, 1]$  (see sections 4.5, 4.6 and 4.7).

In Chapter 5, we study the stability of the sharp Poincaré constant of  $RCD(K, N)$  model spaces. Stability results with respect to 1-Wasserstein distance are derived for the positive and finite dimension case  $N > 1$ , for the infinite dimension case  $N = +\infty$ , and for the the negative effective dimension case  $N \leq -1$ . In each case, the model distribution is

- for finite  $N > 1$ , the model measure is the beta distribution  $\mu = \beta\left(\frac{N}{2}, \frac{N}{2}\right)$ ,
- for infinite  $N = +\infty$ , the model measure is the Gaussian  $\mu = \gamma$ ,
- for negative  $N \leq -1$ , the model measures is the generalized Cauchy distribution on  $\mathbb{R}$ .

In all cases, the strategy of proof is essentially the same. First, we show that all test measures  $\nu$  satisfying the correspond RCD condition satisfy an approximate integration by parts formula. This is a consequence of the Bochner identity, and of a technical lemma ( Cf. Proposition 66 or Lemma 69) which differs according to  $N$ . If  $N > 1$  is finite, then Proposition 66 is simpler than Lemma 69 cases because of the ultracontractive bounds (5.12). The resulting difference is that in the case of finite  $N > 1$ , the quantitative bound obtained is of order  $\varepsilon$ , while in the other cases it is of order  $\varepsilon \log\left(\frac{1}{\varepsilon}\right)$ , where  $\varepsilon$  is the difference between first eigenvalues of  $\mu$  and  $\nu$ . Second, we implement Stein's method. In the case of the Beta distributions, this has already been done in [81, 100], so we just need to adapt it in this context. Similarly, in the case of the Normal distribution, we already have Stein's lemma. However, in the case of the generalized Cauchy distribution, there was no Stein lemma available in the litterature, so we developed it in Theorem 77.

In Chapter 6, we study the sharp Bobkov constant using Huou's result (see Theorem 82 giving an equivalence between Bobkov's inequality and an isoperimetric problem for 2-dimensional product measures). Using this interpretation and a classical variational approach, we derive a new bound for the  $L^2$ -Bobkov constant of the logistic measure on  $\mathbb{R}$  (see Section 6.3). In addition, we carry out in this framework a numerical study of stationary sets for the product

measure of the dilated one-sided exponential (see Section 6.4). This gives us an idea of the shape of the isoperimetric curves. The plots and numerical approximations are carried out using the free software Octave (free equivalent of Matlab). Finally, we study the link between the uniform isoperimetric problem for 2-dimensional product measures given by Huou's theorem (equivalent to the  $L^1$ -Bobkov inequality) and infinite dimensional uniform isoperimetric inequalities. In particular, we show that the product problem is stronger than the infinite-dimensional isoperimetric inequality (see Section 6.5.1). These results involving optimal constants are illustrated using the one-sided exponential in Section 6.5.3.

In Chapter 7, we present some perspectives and works in progress that may arise from the previous chapters. In Section 7.1, we present a monotonicity result of the Poincaré constant along the Polchinski renormalization flow. We obtain this result by generalizing a theorem of B. Klartag and E. Putterman with the Bakry-Emery multi-scale criterion tools introduced by T. Bodineau and R. Bauerchmidt. In Section 7.2, we present two ways to obtain a generalized Stein lemma for distributions satisfying a positive curvature-dimension condition, using a coupling method. In Section 7.3, we discuss a method to generalize the results of Chapter 3 in the case of a first eigenvalue with multiplicity greater than 1. We derive an approximate multidimensional integration by parts, and use it to obtain a stability result for the case of a strictly log-concave distribution on  $\mathbb{R}^d$  under an additional curvature assumption. Finally, in Section 7.4, we discuss a way to implement Stein's method in isoperimetric settings. In particular, we show that it is possible to recover the Cheeger isoperimetric inequality for the Gaussian distribution using this strategy.



# Chapter 3

## Stability of the Poincaré constant

This chapter is a slightly modified version of the paper [191].

### 3.1 Introduction

A probability measure  $\mu$  on  $\mathbb{R}^d$  is said to satisfy a Poincaré inequality when there exists a positive finite constant  $C$  such that for all functions  $f$  in the Sobolev space  $H^1(\mu)$ ,

$$(3.1) \quad \text{Var}_\mu(f) \leq C \int |\nabla f|^2 d\mu.$$

We denote by  $C_P(\mu)$  the smallest constant for which the above inequality holds. Poincaré inequalities have many applications (see for instance the survey [7]). For instance they can be seen as embeddings of weighted Sobolev spaces in  $L^2$ , or as quantifying concentration of measure phenomenon (see e.g. [209]). The sharp Poincaré constant governs ergodicity of the underlying dynamic in the  $L^2$  sense, and the convergence rate of some algorithms used for numerical simulations (see e.g. [122, 139]). When  $\mu$  is reversible for a Markov process, a Poincaré inequality has a spectral interpretation: the infinitesimal generator  $L$  of the Markov process is symmetric on  $L^2(\mu)$  and the quantity  $\lambda_\mu := \frac{1}{C_P(\mu)}$  is then the spectral gap of the positive symmetric operator  $-L$  (see [13, section 4.2.1]).

Stability results for Poincaré constant began to appear in the late 80's. Chen [67, Corollary 2.1] showed that all isotropic probability measures on  $\mathbb{R}^d$  have sharp Poincaré constant greater than 1. He proved furthermore that the standard Gaussian is the only one attaining 1. Then Utev [211] refined this result in dimension one, quantifying the difference between Poincaré constants in term of total variation distance:

$$C_P(\nu) \geq 1 + \frac{1}{9} d_{TV}(\nu, \gamma)^2$$

where  $\nu$  is a normalized probability measure on  $\mathbb{R}$ ,  $\gamma$  is the standard Gaussian and  $d_{TV}$  is the total variation distance.

More recently, Courtade, Fathi and Pananjady [75], extended it to the multidimensional case with the Wasserstein-2 distance:

$$(3.2) \quad C_P(\nu) \geq 1 + \frac{W_2(\nu, \gamma)^2}{d}$$

where  $\nu$  is a centered probability measure on  $\mathbb{R}^d$ , normalized such that  $\int |x|^2 d\nu = d$ ,  $\gamma$  denotes the Gaussian  $\mathcal{N}(0, I_d)$  and  $W_2$  is the 2-Wasserstein distance (see [213, chapter 6]). Our goal is to get such stability results in a more abstract framework, say for a general reference probability measure  $\mu$  on a manifold instead of the Gaussian on  $\mathbb{R}^d$ . What we call stability results for Poincaré constant with respect to some distance  $d$ , are inequations of the form

$$(3.3) \quad d(\mu, \nu) \leq \phi(C_P(\mu), C_P(\nu))$$



where  $d$  is a distance on the space of probability measures,  $\phi : \{(x, y) \in \mathbb{R}^2 \mid y \geq x > 0\} \rightarrow \mathbb{R}_+$  is a continuous function such that  $\forall x > 0, \phi(x, x) = 0$ , and  $C_P(\mu), C_P(\nu)$  are respectively sharp Poincaré constants for  $\mu$  a reference measure, and  $\nu$  satisfying some constraints.

From now on, let us consider  $L$  an infinitesimal reversible Markov diffusion generator on a Riemannian manifold  $M$  with domain  $\mathcal{D}$ , see [13, Sect. 1.4] for further details. Let  $\Gamma(f, g) := \frac{1}{2}(L(fg) - fLg - gLf)$  be the carré du champ operator. To lighten the notation, we set  $\Gamma(f) := \Gamma(f, f)$ . The diffusion property means that  $\forall \phi \in C^2(\mathbb{R}), \Gamma(\phi(f)) = \phi'(f)^2\Gamma(f)$  and also  $L(\phi(f)) = \phi''(f)\Gamma(f) + \phi'(f)L(f)$ , see [13, Section 1.11]. Let us assume  $\mu$  to be the only reversible probability measure on  $M$  for the Markovian process generated by  $L$ . This existence and uniqueness assumption is verified for instance if the process is irreducible and strongly Feller (see [125, Chapter 4]).

Assume that  $\mu$  satisfies the following Poincaré inequality:

$$(3.4) \quad \forall f \in H^1(\mu), \text{Var}_\mu(f) \leq C_P(\mu) \int \Gamma(f) d\mu$$

where

$$(3.5) \quad H^1(\mu) := \{f \in L^2(\mu) \mid \int f d\mu = 0 \text{ and } \int \Gamma(f) d\mu < \infty\}$$

and  $C_P(\mu) = \sup_{f \in H^1(\mu)} \frac{\text{Var}_\mu(f)}{\int \Gamma(f) d\mu} < \infty$  is the sharp Poincaré constant.

**Assumptions on  $(L, \mu)$ .**

1. *There exists an eigenfunction  $f_1$  attaining the spectral gap:  $-Lf_1 = \lambda_\mu f_1$ . We will always choose it to be normalized with respect to  $\mu$ , i.e. such that  $\int f_1 d\mu = 0$  and  $\int f_1^2 d\mu = 1$ .*
2.  *$\Gamma(f_1)$  can be written as  $h \circ f_1$  for some smooth function  $h : I \rightarrow \mathbb{R}_+$  with  $I := f_1(M)$ .*
3. *The function  $h$  does not vanish on the interior of  $I$ .*

Assumption 1 is used to push forward the diffusion from  $M$  to  $I \subset \mathbb{R}$  through  $f_1$  (see Section 3.2). This assumption is necessary. Indeed for  $M = \mathbb{R}$  and  $Lf(x) = f''(x) - \text{sgn}(x)f'(x)$  where  $\text{sgn}(x)$  denotes the sign of  $x$ , the spectral gap is not attained (see [13, Section 4.4.1]). Assumption 2 guarantees the push forward process to be Markovian (see Section 3.2). This assumption would obviously be verified as soon as  $f_1$  is injective. In dimension 1, it is always the case (see Lemma 13). Assumption 3 guarantees the diffusion on  $I$  to be irreducible (see Section 3.2.1).

Let  $\nu$  be another measure on  $M$  satisfying the same Poincaré inequality with sharp constant  $C_P(\nu)$ :

$$\forall f \in H^1(\nu), \text{Var}_\nu(f) \leq C_P(\nu) \int \Gamma(f) d\nu.$$

where  $H^1(\nu)$  is defined similarly as above. We also ask  $\nu$  to be  $f_1$ -normalized, that is  $f_1 \in H^1(\nu)$  and

$$(3.6) \quad \int f_1 d\nu = 0, \int f_1^2 d\nu = 1, \text{ and } \int \Gamma(f_1) d\nu \leq \frac{1}{C_P(\mu)}.$$

The class of measures satisfying the three equations  $\int f_1 d\nu = 0$ ,  $\int f_1^2 d\nu = 1$ , and  $\int \Gamma(f_1) d\nu = \frac{1}{C_P(\mu)}$  being a space of codimension 3 in the space of all probability measures on  $M$ , one sees that (3.6) is satisfied on a half-space of codimension 3 in the infinite dimension space of probability measures.

Applying Poincaré inequality for  $\nu$  to  $f_1$ , one immediately gets  $C_P(\nu) \geq C_P(\mu)$ . We refine this minorization by proving the following stability theorem:

**Theorem 11.** *Under Assumptions 1, 2 and 3,*

$$(3.7) \quad W_1(\mu^*, \nu^*) \leq C_h \left( \sqrt{\delta} + \sqrt{C_P(\nu) \delta} \right),$$

where  $\mu^*$  (resp.  $\nu^*$ ) is the pushforward of  $\mu$  (resp.  $\nu$ ) by  $f_1$ ,  $W_1$  is the 1-Wasserstein distance (see Definition 25),  $\delta := \frac{C_P(\nu) - C_P(\mu)}{C_P(\nu) C_P(\mu)}$ , and  $C_h$  is a constant defined in Proposition 29 whose finiteness only depends on the behavior of  $h$  at the boundary of  $I$ .

The method followed in this article is to derive some approximate integration by part formula for  $\nu$  (see Section 3.3) and then to use Stein's method (see Section 3.4). The problem is then reduced to bound the carré du champ operator of the solution (see Section 3.4.1).

A particular case, treated by Courtade and Fathi [75, 76], is when  $L = \Delta - x \cdot \nabla$  is the Ornstein-Uhlenbeck process on  $\mathbb{R}^d$ . The invariant measure is the standard Gaussian measure  $\gamma$  and the  $d$  coordinate projections  $x_1, \dots, x_d$  are orthogonal eigenfunctions satisfying Assumption 1, 2 and 3. Choosing any of these projections gives the one dimensional Gaussian  $\mathcal{N}(0, 1)$  for the pushforward measure. In this case, the approach described above works using only classical results of Stein's method for the Gaussian (see [185, Section 2]). Moreover, if we use the same approach with  $f_1 = (x_1, \dots, x_d)$  a vector valued function, it works thanks to technical bounds obtained in [171, Lemma 3.3] and it gives similar results as (4.2). Taking vector valued eigenfunction when the dimension of the eigenspace is greater than 1 appears to be the natural extension of this method. Technical issues come here from the fact that Stein's method involves control of the Hessian of solutions to a family of PDE.

In Section 3.6.2, we obtain stability result for the Laguerre process as Corollary of Theorem 11. As far as we know, stability results had not yet been investigated in this case.

One can ask for another distance than the 1-Wasserstein in the stability inequality (3.7). The main difference lies in the set of target function chosen using Stein's method. We will see in Section 3.5 that replacing the condition  $C_h < \infty$  by  $h > \kappa > 0$ , where  $\kappa$  is a constant, gives a stability result in total variation distance (and hence in Kolmogorov distance). We illustrate this result in Section 3.6.3 with uniformly log-concave measures.

A natural question arising is then to compare stability results involving different distances. We will say that a  $d_1$  stability result

$$(3.8) \quad d_1(\mu, \nu) \leq \phi_1(C_P(\mu), C_P(\nu))$$

is stronger than a  $d_2$  one

$$(3.9) \quad d_2(\mu, \nu) \leq \phi_2(C_P(\mu), C_P(\nu))$$

if (3.8) implies the existence of  $\phi_2$  in (3.9). Comparison between stability results in Kolmogorov distance obtained from Theorems 23 and 30 shall be discussed in Section 3.7.

## 3.2 The quotient process

In the one dimensional Ornstein-Uhlenbeck process, the first non zero eigenfunction is injective. So one may expect the important properties of  $L$  to be preserved when mapping it onto  $I$  through such eigenfunction. In a broader context, according to [13, page 60], Assumption 2 implies that all of the Markovian structure on the manifold is mapped onto a Markovian structure on  $I := f_1(M) \subset \mathbb{R}$ . Assuming that  $M$  is connected,  $I$  is an interval since  $f_1$  is

continuous. We define  $a := \inf I$  and  $b := \sup I$ . Moreover, the diffusion property allows to write that

$$\forall \phi \in C^2(\mathbb{R}), \quad L(\phi(f_1)) = \phi''(f_1)\Gamma f_1 + \phi'(f_1)Lf_1 = \phi''(f_1)h(f_1) - \phi'(f_1)\frac{1}{C_P(\mu)}f_1.$$

Hence, the induced Markov process has generator

$$(3.10) \quad L^*(\phi)(x) := h(x)\phi''(x) - \frac{x}{C_P(\mu)}\phi'(x)$$

and reversible measure  $\mu^* := f_1^\# \mu$ . This one dimensional Markov process will be called the quotient process.

If  $h$  is constant, then it is equal to  $\frac{1}{C_P(\mu)}$ . Indeed, one gets that  $\Gamma(f_1) = h$ , but we know that  $\int \Gamma(f_1)d\mu = \frac{1}{C_P(\mu)}$ . In this case,  $L^*$  is reduced to the Ornstein-Uhlenbeck generator with reversible measure  $\mathcal{N}(0, \frac{1}{C_P(\mu)})$ . Therefore,  $\mu^* = \mathcal{N}(0, \frac{1}{C_P(\mu)})$ . Similarly,  $\nu^* := f_1^\# \nu$  will satisfy Poincaré inequality with sharp constant  $C_P(\nu^*) \leq \frac{C_P(\nu)}{C_P(\mu)}$ . Indeed, using Poincaré inequality for  $\nu$ ,

$$\text{var}_{\nu^*}(\phi) = \text{var}_\nu(\phi \circ f_1) \leq C_P(\nu) \int \Gamma(\phi \circ f_1)d\nu = C_P(\nu) \int \phi'(f_1)^2 h(f_1)d\nu = \frac{C_P(\nu)}{C_P(\mu)} \int \phi'^2 d\nu^*.$$

Of course the carré du champ operator used in Poincaré inequalities on  $\mathbb{R}$  is the square of the first derivative. At that point, we can apply known stability results (see [75]) and obtain:

$$\frac{C_P(\nu)}{C_P(\mu)} \geq C_P(\nu^*) \geq C_P(\mu) + W_2(\nu^*, \mu^*)^2.$$

In the sequel, the goal will be to prove such stability inequalities in a broader context, that is without assuming  $h$  to be constant. Let us now compute the carré du champ of the quotient process.

**Proposition 12.** *The carré du champ operator associated to  $(L^*, \mu^*)$  is*

$$(\Gamma^*\psi)(t) = h(t)\psi'(t)^2.$$

Moreover, with this operator,  $\mu^*$  satisfies a Poincaré inequality with sharp constant  $C_P(\mu)$  and the inequality becomes an equality for  $\psi = Id$ .

**Proof** It is simply a computation using the basic definition of  $\Gamma f := \frac{1}{2}[L(f^2) - 2fLf]$ . The Poincaré inequality for  $\mu^*$  follows from the Poincaré inequality for  $\mu$ :

$$\text{var}_{\mu^*}(\psi) = \text{var}_\mu(\psi \circ f_1) \leq C_P(\mu) \int \Gamma(\psi \circ f_1)d\mu = C_P(\mu) \int \psi'(f_1)^2 h(f_1)d\mu = C_P(\mu) \int \Gamma^*\psi d\mu^*.$$

Now taking  $\psi = Id$ , the inequality becomes an equality because of the definition of  $f_1$ , showing that  $C_P(\mu)$  is sharp.  $\square$

Replacing  $\mu^*$  by  $\nu^*$  in the proof above, we get that  $\nu^*$  satisfies a Poincaré inequality with constant  $C_P(\nu)$ . However in this case,  $C_P(\nu)$  may not be the sharp constant for  $\nu^*$ .

### 3.2.1 The non vanishing assumption

Let us consider the case where  $Lf := \alpha f'' + \beta f'$  is a diffusion operator on an interval  $M = J := (c, d) \subset \mathbb{R}$ , with  $\alpha$  and  $\beta$  two continuous functions on  $J$  such that  $\alpha > 0$  and

$$\rho(x) := \frac{1}{\alpha(x)} \exp\left(\int_c^x \frac{\beta(t)}{\alpha(t)} dt\right)$$

is well defined on  $J$ . So  $d\mu := \rho(x)dx$  is reversible for  $L$ . Assume the operator  $L$  satisfies both a Poincaré inequality with sharp constant  $C_P$  and Assumption 1. Then the following holds.

**Lemma 13.** *The first eigenfunction  $f_1$  is strictly monotone.*

**Proof** We extend the proof in e.g. [186, Section 2] which treats the case  $\alpha \equiv 1$ . First, recall that  $f_1$  is a minimizer of the Rayleigh ratio  $\frac{\int_J \Gamma(f_1) d\mu}{\text{var}_\mu(f_1)}$ , and introduce  $g(x) := \int_c^x |f_0'(t)| dt$ . Then  $g'(x) = |f_0'(x)|$  hence  $\Gamma(g) = \Gamma(f_1)$ . So  $g$  has same energy than the eigenfunction  $f_1$ :

$$\int_J \Gamma(g) d\mu = \int_J \Gamma(f_1) d\mu$$

On the other hand,  $g$  has greater variance than  $f_1$ . Indeed,

$$\text{Var}_\mu(g) = \frac{1}{2} \int_J \int_J \left( \int_x^y |f_1'(t)| dt \right)^2 d\mu(x) d\mu(y) \geq \frac{1}{2} \int_J \int_J \left( \int_x^y f_1'(t) dt \right)^2 d\mu(x) d\mu(y) = \text{Var}_\mu(f_1)$$

But  $f_1$  being a minimizer of the Rayleigh ratio, one can infer, since  $f_1 \in C^1(J)$ , that  $f_0'$  has same sign on the interval  $J$ , hence  $f_1$  is monotone. Assume now  $f_1' \geq 0$  and let us show that  $f_1' > 0$ . The eigenfunction  $f_1$  being centered and continuous, it is negative in a neighborhood of  $c$ , then it is zero, and then positive up to  $d$ . Using classical ODE tools for the equation  $\alpha f_0'' + \beta f_0' = \frac{-1}{C_P} f_1$ , one gets the formulas:

$$(3.11) \quad f_1'(x) = \frac{1}{C_P \alpha(x) \rho(x)} \int_x^d f_1(t) d\mu(t)$$

$$(3.12) \quad = \frac{-1}{C_P \alpha(x) \rho(x)} \int_c^x f_1(t) d\mu(t)$$

Hence if  $f_1'(x) = 0$  with  $x \in (c, d)$ , then either  $f_1(x) < 0$  but Formula 3.12 implies  $f_1'(x) > 0$ , or  $f_1(x) > 0$  but Formula 3.11 implies  $f_1'(x) > 0$ , or  $f_1(x) = 0$  but both formulas give then again  $f_1'(x) > 0$ .

Hence  $f_0'$  is positive on  $J$ , justifying the claim that  $f_1$  is strictly monotone.  $\square$

In this case the monotonicity implies that  $h$  cannot vanish in the interior of  $I$ . This motivates Assumption 3 already mentioned in the introduction. Furthermore, since  $a$  (respectively  $b$ ) is the global minimum (resp. maximum) of  $f_1$ , it can be reformulated as:

**Assumption 3.** *All  $x_0 \in M$  such that  $\Gamma(f_1)(x_0) = 0$  are global extrema of  $f_1$ .*

If  $\Gamma = |\nabla|^2$  for the metric of  $M$ , then the assumption can be reformulated: all critical points of  $f_1$  are global extrema. The following is immediate and justify why we often write  $I$  instead of its interior  $\overset{\circ}{I}$  in the sequel.

**Proposition 14.** *The eigenfunction  $f_1$  satisfies Assumption 3 if, and only if,  $h$  does not vanish on  $\overset{\circ}{I}$ .*

**Proof** Assume that Assumption 3 holds, and let  $t \in I$  such that  $h(t) = 0$ . Then there exists  $x \in M$  such that  $t = f_1(x)$  and so  $\Gamma(f_1)(x) = 0$ . Hence  $x$  is a global extremum of  $f_1$ . Hence  $t = a \notin \overset{\circ}{I}$  or  $t = b \notin \overset{\circ}{I}$ . Conversely let  $x \in M$  such that  $\Gamma(f_1)(x) = 0$ . Then  $f_1(x)$  is a zero of  $h$ . Hence by assumption it is on the boundary of  $I$ , so it is a global extremum of  $f_1$ .  $\square$

### 3.2.2 Density of $\mu^*$

First, we point out that  $\int f_1 d\mu \in I$ . Indeed  $f_1$  is continuous, so  $I = f_1(M)$  is connected in  $\mathbb{R}$ , hence convex. Therefore its expectation belongs to  $I$ . Recall that  $\lambda_\mu := \frac{1}{C_P(\mu)}$ . We now define the function  $v : I \rightarrow \mathbb{R}_+$  by

$$(3.13) \quad v(t) := \exp \left( -\lambda_\mu \int_0^t \frac{u du}{h(u)} \right)$$

Taking 0 as reference point in the integral is justified by the above observation since  $f_1$  was chosen to be centered. Moreover Assumption 3 guarantees that dividing by  $h$  makes sense.

**Proposition 15.** *The measure*

$$\rho(x)dx := \frac{1}{h(x)} \exp\left(-\lambda_\mu \int_0^x \frac{u du}{h(u)}\right) \mathbf{1}_I dx$$

is invariant for the quotient process, where  $\mathbf{1}$  denotes an indicator.

**Proof** It is enough to show that

$$\int_I L^*(f) \frac{1}{h(x)} \exp\left(-\lambda_\mu \int_0^x \frac{u du}{h(u)}\right) dx = 0$$

for all functions  $f$  compactly supported in  $I$ . Let  $f$  be such a function. Recall that  $L^*f = hf'' - \frac{x}{C_P(\mu)}f'$ . We first compute by an integration by parts:

$$\int_I f''(x)v(x)dx = \underbrace{[f'(x)v(x)]_a^b}_{=0} + \lambda_\mu \int_I \frac{xf'(x)}{h(x)}v(x)dx,$$

and this allows us to conclude.  $\square$

We know that the quotient process admits  $f_1^\# \mu$  as invariant probability measure. Since we assumed  $\mu$  to be the only invariant probability measure for  $L$ , Proposition 15 gives that the measure  $\rho(x)dx$  is finite and then  $f_1^\# \mu = \frac{1}{Z}\rho(x)dx$  where  $Z$  is a normalization constant. Let us conclude this section with a control of the tail of  $\mu^*$ . We denote the cumulative distribution of  $\mu^*$  by

$$q(t) := \mu^*([-\infty, t]) = \int_{-\infty}^t \frac{\mathbf{1}_I}{Zh(y)} \exp\left(-\lambda_\mu \int_0^y \frac{u du}{h(u)}\right) dy.$$

Recall that  $a := \inf I$ ,  $b := \sup I$  and let us denote  $I_- := I \cap \mathbb{R}_-$  and  $I_+ := I \cap \mathbb{R}_+$ .

**Lemma 16.** *One can bound the tail  $q$  of  $\mu^*$  as follows:*

- $\forall t \in I_-$ ,

$$q(t) \leq \min\left(q(0), \frac{-C_P(\mu)}{Zt}\right) v(t).$$

- $\forall t \in I_+$ ,

$$1 - q(t) \leq \min\left((1 - q(0)), \frac{C_P(\mu)}{Zt}\right) v(t).$$

**Proof** We only process on  $I_-$ . The proof is similar on  $I_+$ .

1. Let us define  $f : I_- \rightarrow \mathbb{R}$ , by  $f(t) = q(t) - q(0)v(t)$ . We want to show that  $f \leq 0$ . Compute

$$f'(t) = \frac{1}{Zh(t)} \exp\left(-\lambda_\mu \int_a^t \frac{u du}{h(u)}\right) + \lambda_\mu q(0) \frac{t}{h(t)} v(t) = \frac{v(t)}{h(t)} \left(\frac{1}{Z} + \lambda_\mu q(0)t\right).$$

If  $-\frac{C_P(\mu)}{q(0)Z} < a$ , then  $f' \geq 0$  so  $f \leq f(0) = 0$ .

Else,  $f$  decreases on  $(a, -\frac{C_P(\mu)}{q(0)Z})$  and increases on  $(-\frac{C_P(\mu)}{q(0)Z}, 0)$  but  $\lim_{t \rightarrow a} f(t) \leq 0$  and  $f(0) = 0$ , so we have the first claim.

2. Let us define  $f : I_- \rightarrow \mathbb{R}$ , by  $f(t) = q(t) + \frac{C_P(\mu)}{Zt}v(t)$ . We want to show that  $f \leq 0$ . Compute:

$$f'(t) = \frac{1}{Zh(t)} \exp\left(-\lambda_\mu \int_0^t \frac{u du}{h(u)}\right) - \frac{C_P(\mu)}{Zt^2}v(t) - \lambda_\mu \frac{tC_P(\mu)}{Zh(t)t}v(t) = -\frac{v(t)C_P(\mu)}{Zt^2} < 0.$$

Hence  $f \leq \lim_{t \rightarrow a} f(t) \leq 0$ .

$\square$

### 3.3 Exact and approximate integration by parts formulas

Let us recall a classical result in  $\Gamma$ -calculus which we will often use in the sequel. For all  $f, g \in H^1(\mu)$ , the following integration by parts formula holds (see formula (3.5) for the definition of  $H^1(\mu)$ ).

$$(3.14) \quad \int \Gamma(f, g) d\mu = - \int f Lg d\mu$$

Indeed, since  $\mu$  is the invariant measure of the process generated by  $L$ ,  $\int L\phi d\mu = 0$  for all functions  $\phi \in \mathcal{D}$ . Hence, integrating the definition of the carré du champ operator  $\Gamma(f, g) := \frac{1}{2}(L(fg) - fLg - gLf)$  and using the reversibility of  $\mu$ , one gets the result. Let us now state an integration by parts formula for the quotient process.

**Proposition 17.** *For all  $C^1(I)$  functions  $\psi : I \rightarrow \mathbb{R}$  such that  $\psi \circ f_1 \in H^1(\mu)$ , it holds that:*

$$(3.15) \quad \int_{\mathbb{R}} x\psi(x) d\mu^*(x) = C_P(\mu) \int_{\mathbb{R}} h(x)\psi'(x) d\mu^*(x)$$

**Proof** As a consequence of the integration by part formula (3.14) for the initial diffusion process, one gets

$$\int \Gamma(f_1, g) d\mu = - \int g Lf_1 d\mu = \frac{1}{C_P(\mu)} \int g f_1 d\mu.$$

We then use this equality with  $g := \phi \circ f_1$  and use the diffusion property of  $\Gamma$  and the definition of  $h$ .  $\square$

Let us state now an extension of the approximate integration by part formula which extend a Lemma from Courtade and Fathi [76, Lemma 2.3]. Let us recall that

$$H^1(\nu) := \{f \in L^2(\nu) \cap \mathcal{D} \mid \int f d\nu = 0 \text{ and } \int \Gamma(f) d\nu < \infty\}.$$

**Theorem 18.** *The following inequality holds for all  $g \in H^1(\nu)$ :*

$$(3.16) \quad \left| \int f_1 g d\nu - C_P(\nu) \int \Gamma(f_1, g) d\nu \right| \leq (C_P(\nu) - C_P(\mu))^{\frac{1}{2}} \left( \frac{C_P(\nu)}{C_P(\mu)} \right)^{\frac{1}{2}} \left( \int \Gamma(g) d\nu \right)^{\frac{1}{2}}$$

**Proof** Let  $t \in \mathbb{R}$  and  $g \in H^1(\nu)$ . Let us apply the Poincaré inequality for  $\nu$  to  $\alpha := f_1 + tg$ . Computing that

$$\text{Var}_{\nu}(\alpha) = \int (f_1 + tg)^2 d\nu = \underbrace{\int f_1^2 d\nu}_{=1} + 2t \int f_1 g d\nu + t^2 \int g^2 d\nu,$$

and that

$$\int \Gamma(\alpha) d\nu = \underbrace{\int \Gamma(f_1) d\nu}_{\leq C_P(\mu)^{-1}} + 2t \int \Gamma(f_1, g) d\nu + t^2 \int \Gamma(g) d\nu,$$

we get that

$$1 + 2t \int f_1 g d\nu + t^2 \int g^2 d\nu \leq C_P(\nu) \left( C_P(\mu)^{-1} + 2t \int \Gamma(f_1, g) d\nu + t^2 \int \Gamma(g) d\nu \right).$$

Now, writting  $C_P(\nu) = C_P(\nu) - C_P(\mu) + C_P(\mu) =: \Delta C_P + C_P(\mu)$ , we obtain :

$$2t \int f_1 g d\nu + t^2 \int g^2 d\nu \leq \frac{\Delta C_P}{C_P(\mu)} + 2C_P(\nu)t \int \Gamma(f_1, g) d\nu + t^2 C_P(\nu) \int \Gamma(g) d\nu.$$

Hence, for all  $t \in \mathbb{R}$ ,

$$\frac{\Delta C_P}{C_P(\mu)} + 2t \left( C_P(\nu) \int \Gamma(f_1, g) d\nu - \int f_1 g d\nu \right) + t^2 C_P(\nu) \int \Gamma(g) d\nu \geq 0.$$

This polynomial of degree 2 takes only non-negative values, hence its discriminant is non-positive:

$$b^2 - 4ac = 4 \left( C_P(\nu) \int \Gamma(f_1, g) d\nu - \int f_1 g d\nu \right)^2 - 4C_P(\nu) \int \Gamma(g) d\nu \frac{\Delta C_P}{C_P(\mu)} \leq 0,$$

which yields the result.  $\square$

Using Theorem 18 with  $g := \psi \circ f_1$  and dividing by  $C_P(\nu)$ , we can state now the approximate integration by part formula for the quotient process.

**Corollary 19.** *For all  $C^1(I)$  functions  $\psi : I \rightarrow \mathbb{R}$  such that  $\psi \circ f_1 \in H^1(\mu)$ ,*

$$\left| \int \left( h(x)\psi'(x) - \frac{x}{C_P(\nu)}\psi(x) \right) d\nu^* \right| \leq \left( \frac{C_P(\nu) - C_P(\mu)}{C_P(\nu)C_P(\mu)} \right)^{\frac{1}{2}} \left( \int h(x)\psi'(x)^2 d\nu^* \right)^{\frac{1}{2}}.$$

The original point of view of Courtade and Fathi is the following [76, Section 1.3]. The Poincaré constant is considered as the minimizer of the energy  $\int \Gamma(f) d\mu$  for  $f$  with variance 1 in a large enough functional set. One can then write the Euler-Lagrange equation. The heuristic is that if another measure almost satisfies this equation, its Poincaré constant would not be so far from the one of  $\mu$ . This is made rigorous with Stein's method. Here the integration by part formula (3.15) plays the role of the Euler-Lagrange equation.

### 3.4 Implementing Stein's method

The original idea of Stein's method is to control some distance between two probability measures by bounding the solution of some equation called Stein equation. For more details, see the survey [185]. We start with the approximated integration by parts formula given in Corollary 19 above. The goal is to get stability inequalities of the form

$$C_P(\nu) \geq C_P(\mu) + \alpha d(\mu^*, \nu^*)^2,$$

where  $\alpha > 0$  is a multiplicative constant and  $d$  is a metric on the space of probability measures defined by

$$d(\mu, \nu) := \sup_{f \in \mathcal{F}} \left| \int f d\mu - \int f d\nu \right|$$

for some set of functions  $\mathcal{F}$ . Metrics of this form include 1-Wasserstein, Kolmogorov and total variation distances. Denoting  $\mu^*(f) := \int f d\mu^*$ , this goal would be achieved if one could solve the equation

$$S_\nu(\psi)(x) := h(x)\psi'(x) - \frac{x}{C_P(\nu)}\psi(x) = f(x) - \mu^*(f), \quad x \in I,$$

for all  $f \in \mathcal{F}$ , and bound the term  $\int h(x)\psi'(x)^2 d\nu^*$  independently of  $f \in \mathcal{F}$ . The actual Stein equation is

$$S_\mu(\psi)(x) := h(x)\psi'(x) - \frac{x}{C_P(\mu)}\psi(x) = f(x) - \mu^*(f), \quad x \in \overset{\circ}{I}.$$

Since  $S_\mu(\psi') = L^*(\psi)$ , for any solution  $\phi$  of the Poisson equation  $L\phi = f - \mu^*(f)$ ,  $\phi'$  solves the Stein equation. Hence one can study the Stein equation via the probabilistic analysis of the Poisson equation. Observing that

$$S_\mu\psi = S_\nu\psi + \left( \frac{1}{C_P(\nu)} - \frac{1}{C_P(\mu)} \right) x\psi,$$

Corollary 19 gives

$$\left| \int S_\mu(\psi) d\nu^* \right| \leq \left( \frac{C_P(\nu) - C_P(\mu)}{C_P(\nu) C_P(\mu)} \right)^{\frac{1}{2}} \left( \int h(x) \psi'(x)^2 d\nu^* \right)^{\frac{1}{2}} + \left( \frac{C_P(\nu) - C_P(\mu)}{C_P(\nu) C_P(\mu)} \right) \left| \int x \psi d\nu^* \right|.$$

Now, using that  $\nu^*$  is centered, the Cauchy-Schwarz inequality and the Poincaré inequality for  $\nu^*$ , we obtain, denoting  $\text{Var}_{\nu^*}(\psi) := \int (\psi - \int \psi d\nu^*)^2 d\nu^*$ ,

$$\begin{aligned} \left| \int x \psi d\nu^* \right| &= \left| \int x (\psi - \int \psi d\nu^*) d\nu^* \right| \\ &\leq (\text{Var}_{\nu^*}(\psi))^{\frac{1}{2}} \left( \int x^2 d\nu^* \right)^{\frac{1}{2}} \\ &= (\text{Var}_{\nu^*}(\psi))^{\frac{1}{2}} \left( \int f_1^2 d\nu \right)^{\frac{1}{2}} \\ &= (\text{Var}_{\nu^*}(\psi))^{\frac{1}{2}} \\ &\leq \sqrt{C_P(\nu)} \left( \int \Gamma^*(\psi) d\nu^* \right)^{\frac{1}{2}}. \end{aligned}$$

Setting,

$$\delta := \frac{C_P(\nu) - C_P(\mu)}{C_P(\nu) C_P(\mu)}$$

we have proved that

$$(3.17) \quad \left| \int S_\mu(\psi) d\nu^* \right| \leq \left( \sqrt{\delta} + \sqrt{C_P(\nu) \delta} \right) \left( \int \Gamma^*(\psi) d\nu^* \right)^{\frac{1}{2}}.$$

So our goal would be achieved if for all  $f \in \mathcal{F}$ , we can solve the Stein equation  $S_\mu(\psi) = f - \mu^*(f)$ , and get a bound for  $\int \Gamma^*(\psi) d\nu^* = \int_a^b h(x) \psi'(x)^2 d\nu^*$  independently of  $f$ . Note that it would be enough to get a bound for  $\|\sqrt{h}\psi'\|_\infty$ , or for  $\|\psi'\|_\infty$  since  $\int h(x) \psi'(x)^2 d\nu^* \leq \|\psi'\|_\infty^2 \int h d\nu^* = \|\psi'\|_\infty^2 \int \Gamma(f_1) d\nu \leq \frac{\|\psi'\|_\infty^2}{C_P(\mu)}$ , where  $\|\cdot\|_\infty$  denotes the supremum norm.

### 3.4.1 Stein solution

Let  $f \in L^1(\mu^*)$ . We consider the Stein equation

$$(3.18) \quad h(x) \psi'(x) - \frac{x}{C_P(\mu)} \psi(x) = f(x) - \mu^*(f), \quad x \in I$$

In the sequel, we will call  $f$  the target function. This equation is a first order linear ODE, which can be explicitly solved.

**Lemma 20.** *The function*

$$(3.19) \quad \psi(x) := \exp \left( \int_0^x \frac{u du}{C_P(\mu) h(u)} \right) \int_a^x (f(y) - \mu^*(f)) \frac{1}{h(y)} \exp \left( - \int_0^y \frac{u du}{C_P(\mu) h(u)} \right) dy$$

is a solution of (5.22). Moreover, it also can be written as

$$(3.20) \quad \psi(x) = - \exp \left( \int_0^x \frac{u du}{C_P(\mu) h(u)} \right) \int_x^b (f(y) - \mu^*(f)) \frac{1}{h(y)} \exp \left( - \int_0^y \frac{u du}{C_P(\mu) h(u)} \right) dy$$

**Proof** The solution (3.19) is obtained by classical tools for first order linear differential equations. The second formula is obtained from Proposition 15.  $\square$

Let us emphasize that this approach is only possible in dimension 1.



### 3.4.2 Boundedness of solutions for bounded target functions

Under the current framework one can give the following bounds for the solution of the Stein equation.

**Proposition 21.** *Let  $f$  be a bounded measurable function on  $I$ , and let  $\psi$  be the solution of (5.22) given by Lemma 20. Then*

$$\|\psi\|_\infty \leq Z \max(q(0), 1 - q(0)) \|f - \mu^*(f)\|_\infty \leq Z \|f - \mu^*(f)\|_\infty,$$

with  $Z$  the normalization constant (see Proposition 15) and  $q$  the cumulative distribution of  $\mu^*$ . Moreover,

$$\|x\psi(x)\|_\infty \leq C_P(\mu) \|f - \mu^*(f)\|_\infty.$$

**Proof** Let us recall that  $v(t) = \exp\left(-\lambda_\mu \int_0^t \frac{u du}{h(u)}\right)$ . If  $x \leq 0$  we use the writing (3.19) of  $\psi$ , and get:

$$|\psi(x)| \leq \|f - \mu^*(f)\|_\infty v(x)^{-1} Z q(x).$$

and by Lemma 79:

$$\sup_{x \in I_-} |\psi(x)| \leq Z q(0) \|f - \mu^*(f)\|_\infty.$$

If  $x > 0$ , we use the writing (3.20) of  $\psi$  in Lemma 20 together with Lemma 79:

$$\sup_{x \in I_+} |\psi(x)| \leq Z (1 - q(0)) \|f - \mu^*(f)\|_\infty.$$

To prove the second inequation, we split off again the case  $x > 0$  and  $x < 0$  using in each case the appropriate representation of  $\psi$  in Lemma 20 and we conclude with Lemma 79. If  $x \in I_-$ , then  $|\psi(x)| \leq v(x)^{-1} \|f - \mu^*(f)\|_\infty Z q(x)$  but  $q(x) \leq \frac{C_P(\mu)}{Z|x|} v(x)$  so  $|x\psi(x)| \leq C_P(\mu) \|f - \mu^*(f)\|_\infty$ . If  $x \in I_+$ , then  $|\psi(x)| \leq v(x)^{-1} \|f - \mu^*(f)\|_\infty Z (1 - q(x))$  but  $(1 - q(x)) \leq \frac{C_P(\mu)}{Zx} v(x)$  so  $|x\psi(x)| \leq C_P(\mu) \|f - \mu^*(f)\|_\infty$ .  $\square$

## 3.5 Stability in total variation under a uniform ellipticity condition

Let us recall the definition of the total variation distance.

**Definition 22.** *The total variation distance between two measures  $\alpha, \beta$  on  $I$  is*

$$d_{TV}(\alpha, \beta) = \sup_{A \subset I} \left| \int \mathbf{1}_A d\alpha - \int \mathbf{1}_A d\beta \right|,$$

where the supremum is running over all measurable subsets of  $I$ .

We assume the following uniform ellipticity condition on  $h$ :

$$(3.21) \quad \kappa := \inf_{x \in \mathbb{R}} h(x) > 0$$

Under this ellipticity condition, one can get better estimates on the solutions of Stein equation. The vocabulary of ellipticity comes from PDE theory, see [84, Chapter 6]. This condition extends the Gaussian case discussed at the beginning of Section 3.2. We can now state the main result of this section.

**Theorem 23.** *Let  $(L, \mu)$  be a Markov diffusion on a Riemannian manifold  $M$  satisfying a Poincaré inequality with sharp constant  $C_P(\mu)$ . Suppose that Assumptions 1, 2 and 3 are verified. If  $\kappa := \inf_{x \in \mathbb{R}} h(x) > 0$ , then for all measures  $\nu$  on  $M$  normalized as in (3.6) and satisfying a Poincaré inequality with sharp constant  $C_P(\nu)$ ,*

$$(3.22) \quad d_{TV}(\mu^*, \nu^*) \leq \frac{4}{\sqrt{\kappa}} \left( \sqrt{\delta} + \sqrt{C_P(\nu) \delta} \right),$$

where  $\mu^*$  (resp.  $\nu^*$ ) is the pushforward of  $\mu$  (resp.  $\nu$ ) by the first eigenfunction  $f_1$ , and  $\delta := \frac{C_P(\nu) - C_P(\mu)}{C_P(\nu) C_P(\mu)}$ .

In order to prove Theorem 23, we need the following bounds on the solution of the Stein equation.

**Proposition 24.** *Let  $f$  be a bounded function on  $\mathbb{R}$ , and let  $\psi$  be the solution of the Stein equation given by Lemma 20. Then*

$$\|\psi'\|_\infty \leq \frac{2}{\kappa} \|f - \mu^*(f)\|_\infty,$$

and

$$\|h\psi'^2\|_\infty \leq \frac{4}{\kappa} \|f - \mu^*(f)\|_\infty^2.$$

**Proof** By direct calculation:

$$\psi'(x) = \frac{1}{h(x)} \left( f - \mu^*(f) + \frac{x}{C_P(\mu)} \psi(x) \right).$$

The lower bound on  $h$  and Proposition 21 yield the first claim. Similarly,

$$h(x)\psi'(x)^2 = \frac{1}{h(x)} \left( f - \mu^*(f) + \frac{x}{C_P} \psi(x) \right)^2.$$

So from the elementary inequality  $(a + b)^2 \leq 2a^2 + 2b^2$ , and the same reasoning as above, we get the result.  $\square$

**Proof of Theorem 23** Let  $f : I \rightarrow \mathbb{R}$  be an indicator function. We solve the Stein equation with Lemma 20 and denote  $\psi$  the chosen solution. Then (3.17) and Proposition 24 give:

$$\left| \int S_\mu(\psi) d\nu^* \right| \leq \left( \sqrt{\delta} + \sqrt{C_P(\nu) \delta} \right) \left( \int \Gamma^*(\psi) d\nu^* \right)^{\frac{1}{2}} \leq \left( \sqrt{\delta} + \sqrt{C_P(\nu) \delta} \right) \frac{2}{\sqrt{\kappa}} \|f - \mu^*(f)\|_\infty.$$

But:

$$\left| \int S_\mu(\psi) d\nu^* \right| = \left| \int (f - \mu^*(f)) d\nu^* \right| = |\nu^*(f) - \mu^*(f)|.$$

Hence for any indicator function  $f$ ,

$$|\nu^*(f) - \mu^*(f)| \leq \frac{4}{\sqrt{\kappa}} \left( \sqrt{\delta} + \sqrt{C_P(\nu) \delta} \right).$$

Taking the supremum over all such  $f$  concludes the proof.  $\square$

Let us point out that (3.22) is of the form (3.3) with  $\phi(x, y) = \frac{4}{\sqrt{\kappa}} \left( \sqrt{\frac{y-x}{xy}} + \frac{y-x}{x\sqrt{y}} \right)$ .

When specialized to the Gaussian case, Theorem 23 gives that if  $\nu$  is a normalized probability measure on  $\mathbb{R}$  and  $C_P(\nu) - 1 \leq 1$ , then:

$$C_P(\nu) \geq 1 + \frac{1}{16(1 + \sqrt{2})} d_{TV}(\gamma, \nu)^2.$$

Our constant  $16(1 + \sqrt{2})$  is worse than the constant 9 from [211].

### 3.6 Stability in $W_1$ distance

Let us recall the Kantorovitch-Rubinstein formula, which we will take as the definition of the  $W_1$  distance.

**Definition 25.** (see [213, Chapter 5]) *The 1-Wasserstein distance between two measures  $\alpha, \beta$  on  $I$  is*

$$W_1(\alpha, \beta) = \sup_{\substack{f: I \rightarrow \mathbb{R} \\ 1\text{-Lipschitz}}} \left| \int f d\alpha - \int f d\beta \right|.$$

We can now state the main theorem of this article.

**Theorem 26.** *Let  $(L, \mu)$  be a Markov diffusion on a Riemannian manifold  $M$  satisfying a Poincaré inequality with sharp constant  $C_P(\mu)$ . Suppose that Assumptions 1, 2 and 3 are verified. Let*

$$C_h := \sup_{x \in I} \left[ \sqrt{\Gamma^*(a_1)(x)} \int_a^x q(t) dt + \sqrt{\Gamma^*(a_2)(x)} \int_x^b (1 - q(t)) dt \right],$$

where  $a_1$  and  $a_2$  are defined in (3.27) and (3.28). Then for all measures  $\nu$  on  $M$  normalized as in (3.6) and satisfying a Poincaré inequality with sharp constant  $C_P(\nu)$ , it holds:

$$W_1(\mu^*, \nu^*) \leq C_h \left( \sqrt{\delta} + \sqrt{C_P(\nu)} \delta \right) \leq C_h \left( \frac{\sqrt{C_P(\nu) - C_P(\mu)}}{C_P(\mu)} + \frac{C_P(\nu) - C_P(\mu)}{C_P(\mu)^{3/2}} \right),$$

where  $\mu^*$  (resp.  $\nu^*$ ) is the pushforward of  $\mu$  (resp.  $\nu$ ) by  $f_1$ ,  $W_1$  is the 1-Wasserstein distance (see Definition 25), and  $\delta := \frac{C_P(\nu) - C_P(\mu)}{C_P(\nu) C_P(\mu)}$ .

**Proof** The proof is the same as in Theorem 23 but we replace the indicator functions by 1-Lipschitz functions, and we use Proposition 29 instead of Proposition 24.  $\square$

All the sequel of this section is devoted to prove Proposition 29. In [66, Appendix p. 37], Chen, Goldstein and Shao bound the derivatives of the solution of the Stein equation in the case where  $\mu$  is the standard Gaussian measure. Here, we extend their approach to a broader class of measures. Let  $f : I \rightarrow \mathbb{R}$  be absolutely continuous, and let  $\psi$  be the solution of (5.22) given by Lemma 20. We are looking for inequalities of the form

$$\|h\psi'^2\|_\infty \leq C \|f'\|_\infty,$$

where  $C > 0$  is a constant which does not depend on  $f$ . Recall that we denote by  $q(t) = \int_a^t d\mu^*$  the cumulative distribution function of the measure  $\mu^*$ , and  $v(t) = \exp\left(-\lambda_\mu \int_0^x \frac{u du}{h(u)}\right)$  (see (3.13)).

**Proposition 27.** *Let  $f : I \rightarrow \mathbb{R}$  be in  $C^1(I) \cap L^1(\mu^*)$ . The associated solution (3.20) of Lemma 20 can be written as:*

$$(3.23) \quad \psi(x) = -Z \frac{1 - q(x)}{v(x)} \int_{t=a}^x f'(t) q(t) dt - Z \frac{q(x)}{v(x)} \int_{t=x}^b f'(t) (1 - q(t)) dt$$

**Proof** First, write

$$\begin{aligned} f(x) - \mu^*(f) &= \int_a^x (f(x) - f(y)) d\mu^*(y) - \int_b^x (f(x) - f(y)) d\mu^*(y) \\ &= \int_a^x \int_y^x f'(t) dt d\mu^*(y) - \int_b^x \int_y^x f'(t) dt d\mu^*(y) \\ &= \int_a^x f'(t) \int_a^t d\mu^*(y) dt - \int_x^b f'(t) \int_t^b d\mu^*(y) dt \end{aligned}$$

$$(3.24) \quad = \int_a^x f'(t)q(t) dt - \int_x^b f'(t)(1-q(t)) dt.$$

It follows that

$$\begin{aligned} \psi(x) &= \frac{1}{v(x)} \int_a^x (f(y) - \mu^*(f)) Z d\mu^*(y) \\ &= \frac{1}{v(x)} \int_a^x \left( \int_a^y f'(t)q(t) dt - \int_y^b f'(t)(1-q(t)) dt \right) Z d\mu^*(y). \end{aligned}$$

Now, on the one hand:

$$\int_a^x \int_a^y f'(t)q(t) dt d\mu^*(y) = \int_a^x \int_t^x f'(t)q(t) d\mu^*(y) dt = \int_a^x f'(t)q(t) (q(x) - q(t)) dt,$$

on the other hand:

$$\begin{aligned} \int_a^x \int_y^b f'(t)(1-q(t)) dt d\mu^*(y) &= \int_x^b \int_a^x f'(t)(1-q(t)) d\mu^*(y) dt + \int_a^x \int_a^t f'(t)(1-q(t)) d\mu^*(y) dt \\ &= \int_x^b f'(t)(1-q(t))q(x) dt + \int_a^x f'(t)(1-q(t))q(t) dt. \end{aligned}$$

From which one gets (3.23). □

By definition  $\psi$  satisfies

$$h(x)\psi'(x) = \left( f(x) - \mu^*(f) + \frac{x}{C_P} \psi(x) \right).$$

Hence, using Proposition 27 and Formula (3.24), we are now able to give a new formula for the derivative of the solution.

**Corollary 28.** *We have:*

$$\begin{aligned} h(x)\psi'(x) &= \left( 1 - Z(1-q(x)) \frac{x}{C_P(\mu)} \exp \left( \int_0^x \frac{u du}{C_P(\mu)h(u)} \right) \right) \int_a^x f'(t)q(t) dt \\ &\quad - \left( 1 + Zq(x) \frac{x}{C_P(\mu)} \exp \left( \int_0^x \frac{u du}{C_P(\mu)h(u)} \right) \right) \int_x^b f'(t)(1-q(t)) dt. \end{aligned}$$

In order to simplify the expression above, one can see that

$$(3.25) \quad \left| 1 - Z(1-q(x)) \frac{x}{C_P(\mu)} \exp \left( \int_0^x \frac{u du}{C_P(\mu)h(u)} \right) \right| \frac{1}{\sqrt{h(x)}} = \sqrt{\Gamma^*(a_1)(x)}$$

and

$$(3.26) \quad \left| 1 + Zq(x) \frac{x}{C_P(\mu)} \exp \left( \int_0^x \frac{u du}{C_P(\mu)h(u)} \right) \right| \frac{1}{\sqrt{h(x)}} = \sqrt{\Gamma^*(a_2)(x)}$$

where

$$(3.27) \quad a_1(x) := Z(1-q(x)) \exp \left( \int_0^x \frac{u du}{C_P(\mu)h(u)} \right)$$

$$(3.28) \quad a_2(x) := Zq(x) \exp \left( \int_0^x \frac{u du}{C_P(\mu)h(u)} \right)$$

The following is then immediate.

**Proposition 29.** *Let  $f : I \rightarrow \mathbb{R}$  be in  $C^1(I) \cap L^1(\mu^*)$ , and let  $\psi$  the associated solution (3.20). The following bound holds:*

$$\|\sqrt{h}\psi'\|_\infty \leq C_h \|f'\|_\infty,$$

where

$$(3.29) \quad C_h := \sup_{x \in I} \left[ \sqrt{\Gamma^*(a_1)(x)} \int_a^x q(t) dt + \sqrt{\Gamma^*(a_2)(x)} \int_x^b (1-q(t)) dt \right]$$

### 3.6.1 Finiteness of $C_h$

If  $C_h = \infty$ , the result of Theorem 26 degenerates. That is why in this section, we give explicit conditions on  $h$  that ensure the finiteness of  $C_h$ .

**Theorem 30.** *Let  $(L, \mu)$  be a Markov diffusion on a Riemannian manifold  $M$  satisfying a Poincaré inequality with sharp constant  $C_P(\mu)$ . Suppose that Assumptions 1, 2 and 3 are verified. Recall that we denote  $a = \inf f_1(M) < 0$  and  $b = \sup f_1(M) > 0$ . Assume that one of these two conditions is verified at  $a$ :*

- either  $a = -\infty$  and  $c_1|t|^{2\alpha-2} \leq h(t) \leq c_2|t|^\alpha$  for  $t \rightarrow -\infty$  with  $\alpha \leq 2$  and  $c_1, c_2 > 0$ ,
- or  $a > -\infty$  and  $c_1(t-a)^2 \leq h(t) \leq c_2(t-a)$  for  $t \rightarrow a^+$  with  $c_1, c_2 > 0$ ,

and one of these two conditions is satisfied at  $b$ :

- either  $b = +\infty$  and  $c_1t^{2\alpha-2} \leq h(t) \leq c_2t^\alpha$  for  $t \rightarrow +\infty$  with  $\alpha \leq 2$  and  $c_1, c_2 > 0$ ,
- or  $b < +\infty$  and  $c_1(b-t)^2 \leq h(t) \leq c_2(b-t)$  for  $t \rightarrow b^-$  with  $c_1, c_2 > 0$ .

Then for all measures  $\nu$  on  $M$  normalized as in (3.6) and satisfying a Poincaré inequality with sharp constant  $C_P(\nu)$ , it holds:

$$W_1(\mu^*, \nu^*) \leq C_h \left( \sqrt{\delta} + \sqrt{C_P(\nu) \delta} \right),$$

where  $C_h > 0$  is a constant,  $\mu^*$  (resp.  $\nu^*$ ) is the pushforward of  $\mu$  (resp.  $\nu$ ) by  $f_1$ ,  $W_1$  is the 1-Wasserstein distance (see Definition 25), and  $\delta := \frac{C_P(\nu) - C_P(\mu)}{C_P(\nu)C_P(\mu)}$ .

Before proving Theorem 30, let us formulate an immediate corollary.

**Corollary 31.** *Let  $\mu$  and  $\nu$  such as in Theorem 30. If moreover  $C_P(\nu) = C_P(\mu)$ , then the pushforward by  $f_1$  gives the same measure for  $\mu$  and  $\nu$ :*

$$f_1^\# \nu = f_1^\# \mu.$$

The converse of Corollary 31 is false. This is due to the fact that the Poincaré constant depends only on one direction. We already discussed that fact when presenting the tensorization property of Poincaré inequalities in Section 2.4.3 of the introduction chapter. A counter example is given by choosing in  $\mathbb{R}^2$ , the normal distributions

$$\mu = \mathcal{N}(0, I_2) \quad \text{and} \quad \nu = \mathcal{N}\left(0, \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}\right),$$

and  $f_1(x, y) = x$ . Then  $f_1^\# \mu = f_1^\# \nu = \gamma = \mathcal{N}(0, 1)$ , but  $C_P(\nu) = 2 \neq C_P(\mu) = 1$ .

In order to prove Theorem 30, we will use the following results.

**Theorem 32.** *Assume that:*

1. *There exists a non-negative function  $g_1$ , defined on a neighborhood of  $a$ , satisfying*

$$(3.30) \quad \frac{x(g_1(x) - 1)}{\sqrt{h(x)}} = O(1), \quad x \rightarrow a^+$$

and

$$(3.31) \quad q(x) \geq -\frac{C_P(\mu)}{Z} v(x) \frac{g_1(x)}{x}, \quad x \rightarrow a^+$$

2. There exists a non-negative function  $g_2$ , defined on a neighborhood of  $b$ , satisfying

$$\frac{x(g_2(x) - 1)}{\sqrt{h(x)}} = O(1), \quad x \rightarrow b^-$$

and

$$(3.32) \quad 1 - q(x) \geq \frac{C_P(\mu)}{Z} v(x) \frac{g_2(x)}{x}, \quad x \rightarrow b^-$$

Then the constant  $C_h$  defined in Theorem 26 is finite.

**Proof** We only study boundedness of  $\sqrt{\Gamma^*(a_1)(x)} \int_{-\infty}^x q(t) dt$  at  $a$  and  $b$  and conclude by continuity. The case of  $\sqrt{\Gamma^*(a_2)(x)} \int_x^{+\infty} (1 - q(t)) dt$  is similar. We only deal with the case of  $a$ , the one of  $b$  is similar. Since  $v'(x) = -\lambda_\mu h(x)v(x)$  with an integration by parts, we get

$$\int_a^x q(t) dt = [tq(t)]_a^x - \int_a^x \frac{1}{Z} t \frac{v(t)}{h(t)} dt \leq xq(x) + \frac{C_P(\mu)}{Z} v(x).$$

The last inequality comes from Lemma 79 which gives  $\lim_{t \rightarrow a} (-tq(t) - \frac{C_P(\mu)}{Z} v(t)) \leq 0$ . Moreover, it is obvious that  $1 - Z\lambda_\mu(1 - q(x))x \exp\left(\lambda_\mu \int_0^x \frac{u du}{h(u)}\right) \geq 0$ , for  $x < 0$ . Hence for  $x < 0$ , using (3.25),

$$\begin{aligned} \sqrt{\Gamma^*(a_1)(x)} \int_a^x q(t) dt &= \frac{1}{\sqrt{h(x)}} \left( 1 - Z\lambda_\mu(1 - q(x))x \exp\left(\lambda_\mu \int_0^x \frac{u du}{h(u)}\right) \right) \int_a^x q(t) dt \\ &\leq \frac{1}{\sqrt{h(x)}} \left( xq(x) + \frac{C_P(\mu)}{Z} v(x) \right) - Z\lambda_\mu \frac{1 - q(x)}{\sqrt{h(x)}} \frac{x^2}{v(x)} q(x) - \frac{1 - q(x)}{\sqrt{h(x)}} x. \end{aligned}$$

On the one hand, using first the assumption on  $g_1$ , and next Lemma 79, one gets that for  $x \rightarrow a^+$ ,

$$\frac{1}{\sqrt{h(x)}} \left( xq(x) + \frac{C_P(\mu)}{Z} v(x) \right) \leq -\frac{C_P(\mu)}{Z\sqrt{h(x)}} v(x)g_1(x) + \frac{C_P(\mu)}{Z\sqrt{h(x)}} v(x) \leq \frac{(g_1(x) - 1)xq(x)}{\sqrt{h(x)}} \xrightarrow{t \rightarrow a} 0.$$

On the other hand, the existence of  $g_1$  gives

$$-Z\lambda_\mu \frac{1 - q(x)}{\sqrt{h(x)}} \frac{x^2}{v(x)} q(x) - \frac{1 - q(x)}{\sqrt{h(x)}} x \leq x \frac{g_1(x)}{\sqrt{h(x)}} (1 - q(x)) - \frac{1 - q(x)}{\sqrt{h(x)}} x = \frac{x(g_1(x) - 1)}{\sqrt{h(x)}} (1 - q(x)),$$

which is bounded by assumption.

The boundedness at  $b$  is obtained in the same way. By Lemma 79, one gets that  $1 - Z\lambda_\mu(1 - q(x))x \exp\left(\lambda_\mu \int_0^x \frac{u du}{h(u)}\right) \geq 0$ , for  $x > 0$ . By the assumption on  $g_2$ , one gets

$$1 - Z\lambda_\mu(1 - q(x))x \exp\left(\lambda_\mu \int_0^x \frac{u du}{h(u)}\right) \leq 1 - g_2(x).$$

Hence using the same integration by parts as before,

$$\sqrt{\Gamma^*(a_1)(x)} \int_a^x q(t) dt \leq \frac{1}{\sqrt{h(x)}} \left( xq(x) + \frac{C_P(\mu)}{Z} v(x) \right) (1 - g_2(x)).$$

Now  $\frac{1 - g_2(x)}{\sqrt{h(x)}} v(x) = \frac{x(1 - g_2(x))}{\sqrt{h(x)}} \frac{v(x)}{x}$  goes to zero when  $x$  goes to  $b$  since (3.32) gives  $\frac{v(x)}{x} \xrightarrow{x \rightarrow b} 0$ . This concludes the proof.  $\square$

The standard Gaussian satisfies the requirements of Theorem 32 with  $g_1(x) = g_2(x) = \frac{x^2}{1+x^2}$ . Let us now give sufficient conditions on  $h$  to ensure that such  $g_1, g_2$  functions exist. We prove the case of  $g_1$  at  $a$ , the case of  $g_2$  at  $b$  is similar.

**Lemma 33.** *Let  $g_1$  be a bounded non-negative  $C^1$  function defined on a neighborhood of  $a$  such that for  $t \rightarrow a^+$ ,  $g_1(t) > tg_1'(t)$  and*

$$(3.33) \quad h(t) \leq \lambda_\mu \frac{t^2(1 - g_1(t))}{g_1(t) - tg_1'(t)}, \quad t \rightarrow a^+$$

*If  $\lim_{t \rightarrow a} v(t) = 0$ , then*

$$q(t) \geq -\frac{C_P}{Z} v(t) \frac{g_1(t)}{t}, \quad t \rightarrow a^+.$$

**Proof** Let  $f(t) := q(t) + \frac{C_P}{Z} \frac{g_1(t)}{t} v(t)$ . Compute

$$f'(t) = \frac{v(t)}{Z} \left( \frac{C_P g_1'(t)}{t} - \frac{C_P g_1(t)}{t^2} + \frac{1}{h(t)} - \frac{g_1(t)}{h(t)^2} \right)$$

The assumption implies then that  $f' \geq 0$ . Hence  $f(t) \geq \lim_{x \rightarrow a} f(x) = 0$ .  $\square$

**Lemma 34.** *Let  $g_2$  be a bounded non-negative  $C^1$  function defined on a neighborhood of  $b$  such that for  $t \rightarrow b^-$ ,  $g_2(t) > tg_2'(t)$  and*

$$(3.34) \quad h(t) \leq \lambda_\mu \frac{t^2(1 - g_2(t))}{g_2(t) - tg_2'(t)}, \quad t \rightarrow b^-$$

*If  $\lim_{t \rightarrow b} v(t) = 0$ , then*

$$1 - q(t) \geq \frac{C_P(\mu)}{Z} v(t) \frac{g_2(t)}{t}, \quad t \rightarrow b^-.$$

Under a growth control on  $h$ , the condition  $v(t) \xrightarrow{x \rightarrow a} 0$  can easily be verified. One needs however to distinguish the case where  $a$  is finite from the one where it is infinite. For  $a = -\infty$ : if  $h(t) \leq c|t|^\alpha$  for  $t \rightarrow -\infty$  with  $c > 0$  and  $\alpha \leq 2$ , then for  $x \rightarrow -\infty$ ,

$$-\lambda_\mu \int_0^x \frac{u}{h(u)} du \leq -\frac{\lambda_\mu}{c} \int_x^0 \frac{1}{|u|^{\alpha-1}} du \xrightarrow{x \rightarrow -\infty} -\infty,$$

and hence  $v(t) \xrightarrow{x \rightarrow -\infty} 0$ . For  $a > -\infty$ : if  $h(t) \leq c(t-a)^\alpha$  for  $t \sim a$  with  $c > 0$  and  $\alpha \geq 1$ , then for  $x \in ]a, 0[$ :

$$-\lambda_\mu \int_0^x \frac{u}{h(u)} du \leq \lambda_\mu \int_x^0 \frac{u}{(u-a)^\alpha} du = \lambda_\mu \int_x^0 \frac{du}{(u-a)^{\alpha-1}} + a\lambda_\mu \int_x^0 \frac{du}{(u-a)^\alpha} \xrightarrow{x \rightarrow a} -\infty,$$

and hence  $v(t) \xrightarrow{x \rightarrow a} 0$ . The same results are valid at  $b$ , only replacing  $(t-a)^\alpha$  by  $(b-t)^\alpha$  when  $b < +\infty$ .

**Proposition 35.** *If  $a = -\infty$  and  $c_1|t|^\beta \leq h(t) \leq c_2|t|^\alpha$  for  $t \rightarrow -\infty$  with  $c_1, c_2 > 0$ ,  $\alpha \leq 2$ , and  $\beta \in [2\alpha - 2, \alpha]$ , then  $q$  satisfies (3.31) with  $g_1(t) = 1 - \frac{c_2}{\lambda_\mu} |t|^{\alpha-2}$ . Moreover, (3.30) is also satisfied.*

*If  $a > -\infty$  and  $c_1(t-a)^\beta h(t) \leq c_2(t-a)^\alpha$  for  $t \rightarrow a^+$  with  $c_1, c_2 > 0$ ,  $\alpha \geq 1$  and  $\beta \leq 2\alpha$ , then  $q$  satisfies (3.31) with  $g_1(t) = 1 - \frac{4c_2}{\lambda_\mu a^2} (t-a)^\alpha$ .*

**Proof** Let us compute, if  $a = -\infty$  and  $t < 0$ :

$$\lambda_\mu \frac{t^2(1 - g_1(t))}{g_1(t) - tg_1'(t)} = \frac{c_2 t^2 |t|^{\alpha-2}}{1 - \frac{c_2}{\lambda_\mu} (3 - \alpha) |t|^{\alpha-2}} \geq c_2 |t|^\alpha,$$

for  $t \sim -\infty$ . If  $a > -\infty$  and  $t \in ]a, \frac{a}{2}[$ ,

$$\lambda_\mu \frac{t^2(1 - g_1(t))}{g_1(t) - tg_1'(t)} = \frac{4c_2}{a^2} \frac{t^2(t - a)^\alpha}{1 + \frac{4c_2}{\lambda_\mu a^2}(t - a)^{\alpha-1}[\alpha t - (t - a)]} \geq c_2(t - a)^\alpha,$$

since  $\alpha t - (t - a) < 0$  and  $t^2 \geq \frac{a^2}{4}$ .

Hence Lemma 33 and the above remark give us that  $q$  satisfies (3.31) in both case. In regards to (3.30), in the first case, one has

$$\left| \frac{x(1 - g_1(x))}{\sqrt{h(x)}} \right| = \frac{c_2}{\lambda_\mu \sqrt{h(x)}} |x|^{\alpha-1} \leq \frac{c_2}{\lambda_\mu \sqrt{c_1}} |x|^{\alpha-1-\frac{\beta}{2}}$$

which is bounded at  $-\infty$  because  $\alpha - 1 - \frac{\beta}{2} \leq \alpha - 1 - \frac{2\alpha-2}{2} = 0$ . Similarly in the second case,

$$\left| \frac{x(1 - g_1(x))}{\sqrt{h(x)}} \right| = \frac{4c_2 |x|}{\lambda_\mu a^2} \cdot \frac{|x - a|^\alpha}{\sqrt{h(x)}} \leq \frac{4c_2}{\lambda_\mu |a|} \frac{1}{\sqrt{c_1}} |x - a|^{\alpha-\frac{\beta}{2}}$$

which is bounded at  $a > -\infty$  because  $\beta \leq 2\alpha$ . □

The same result is valid at  $b$ , using Lemma 34. We summarize it in the following Proposition:

**Proposition 36.** *Assume that one of these two conditions is verified at  $a$ :*

- either  $a = -\infty$  and  $c_1 |t|^{2\alpha-2} \leq h(t) \leq c_2 |t|^\alpha$  for  $t \rightarrow -\infty$  with  $\alpha \leq 2$  and  $c_1, c_2 > 0$ ,
- or  $a > -\infty$  and  $c_1 (t - a)^2 \leq h(t) \leq c_2 (t - a)$  for  $t \rightarrow a^+$  with  $c_1, c_2 > 0$ ,

and one of these two conditions is satisfied at  $b$ :

- either  $b = +\infty$  and  $c_1 t^{2\alpha-2} \leq h(t) \leq c_2 t^\alpha$  for  $t \rightarrow +\infty$  with  $\alpha \leq 2$  and  $c_1, c_2 > 0$ ,
- or  $b < +\infty$  and  $c_1 (b - t)^2 \leq h(t) \leq c_2 (b - t)$  for  $t \rightarrow b^-$  with  $c_1, c_2 > 0$ .

Then the constant  $C_h$  defined in Theorem 26 is finite.

Let us underline that sufficient conditions for  $h$  to satisfy the requirements of Proposition 36 are the following. One of these two conditions if verified at  $a$ :

- either  $a = -\infty$  and there is some  $\alpha \leq 2$  such that  $h(t) \sim |t|^\alpha$  at  $t \rightarrow -\infty$ ,
- or  $a > -\infty$  and there is some  $\alpha \in [0, 2]$  such that  $h(t) \sim (t - a)^\alpha$  at  $t \rightarrow a^+$ ,

and one of these two are verified at  $b$ :

- either  $b = +\infty$  and there is some  $\alpha \leq 2$  such that  $h(t) \sim t^\alpha$  at  $t \rightarrow +\infty$ ,
- or  $b < +\infty$  and there is some  $\alpha \in [0, 2]$  such that  $h(t) \sim (b - t)^\alpha$  at  $t \rightarrow b^-$ .

### 3.6.2 Application: stability of the $\Gamma(s, \theta)$ distributions, $s > 0, \theta > 0$

Let us consider the  $\Gamma(s, \theta)$  distribution  $d\mu(x) := \frac{x^{s-1} e^{-\frac{x}{\theta}}}{\Gamma(s)\theta^s} \mathbf{1}_{\mathbb{R}_+}(x) dx$ , where  $\Gamma$  denotes the Euler's Gamma function. This probability measure is the reversible measure of the Laguerre process with following generator on  $[0, \infty)$ :

$$Lf(x) = xf''(x) + \left(s - \frac{1}{\theta}x\right)f'(x).$$

We have  $\Gamma(f) = x(f')^2$ ,  $C_P(\mu) = \theta$  and the first Laguerre polynomial  $f_1(x) = \frac{s-\frac{x}{\theta}}{\sqrt{s}}$  is a normalized eigenfunction attaining the spectral gap. Hence  $I = ]-\infty, \sqrt{s}]$ , so we get  $h(x) =$



$\theta^{-1}(s - \sqrt{s}x)$ . So Assumptions 1, 2 and 3 are satisfied. Moreover, one can see that we are also under the conditions of Proposition 36, with at the infimum,  $a = -\infty$  and  $c_1 \leq h(t) \leq c_2|t|$  for  $c_1 = \frac{s}{\theta}$ ,  $c_2 = 2\frac{\sqrt{s}}{\theta}$  (and  $\alpha = 1$ ), and at the supremum,  $b = \sqrt{s} < +\infty$ ,  $c_1(b-t) \leq h(t) \leq c_2(b-t)$  for  $c_1 = c_2 = \frac{\sqrt{s}}{\theta}$ . In this case the three conditions (3.6) are reduced to only two:

$$(3.35) \quad \int x d\nu = s\theta \quad \text{and} \quad \int x^2 d\nu = s(s+1)\theta^2$$

meaning that we only ask  $\nu$  to have the same first and second moments as  $\mu$ . So we can apply Theorem 30 for the gamma distributions  $\Gamma(s, \theta)$ , with  $s, \theta > 0$ : for all measures  $\nu$  on  $\mathbb{R}_+$  normalized as in (3.35) and satisfying the following Poincaré inequality with sharp constant  $C_P(\nu)$ :

$$(3.36) \quad \text{Var}_\nu(f) \leq C_P(\nu) \int_0^{+\infty} x f'(x)^2 d\nu(x),$$

it holds for a finite constant  $C_h$ :

$$(3.37) \quad W_1(\mu^*, \nu^*) \leq C_h \left( \sqrt{\delta} + \sqrt{C_P(\nu)} \delta \right),$$

where  $W_1$  is the 1-Wasserstein distance (see Definition 25),  $\delta := \frac{C_P(\nu) - \theta}{C_P(\nu)}$ ,  $\nu^* := f_0^\# \nu$ , and

$$d\mu^*(x) = \left( \frac{\sqrt{s}}{e} \right)^s \frac{1}{\Gamma(s)} (\sqrt{s} - x)^{s-1} e^{\sqrt{s}x} \mathbf{1}_{]-\infty, \sqrt{s}[}(x) dx.$$

### 3.6.3 Application : stability for Brascamp-Lieb inequalities

Let  $M = \mathbb{R}$  and  $d\mu = e^{-\phi} dx$  be a probability measure with  $\phi : \mathbb{R} \rightarrow \mathbb{R}$  strictly convex and  $C^2$ . Then  $\mu$  satisfies the following inequality introduced by Brascamp and Lieb in 1976 in [42] :

$$\text{Var}_\mu(f) \leq \int \frac{f'^2}{\phi''} d\mu$$

for all functions  $f$  in the weighted Sobolev space  $H_{\phi, \mu}^{1,2} = \{g \in L^2(\mu) \mid \frac{g'}{\sqrt{\phi''}} \in L^2(\mu)\}$ . Then

$$Lf = \frac{1}{\phi''} f'' - \left( \frac{\phi'}{\phi''} + \frac{\phi^{(3)}}{\phi''^2} \right) f'$$

is a Markov diffusion generator with reversible measure  $\mu$  satisfying a Poincaré inequality with sharp constant 1 and carré du champ operator  $\Gamma(f) = \frac{f'^2}{\phi''}$ . Moreover

$$L(\phi') = -\phi',$$

hence the spectral gap is attained with  $f_1 = \phi'$  which is centered. One easily sees that  $\phi'$  is a bijection, so  $I = \mathbb{R}$  and Assumption 2 is satisfied with  $h = \phi'' \circ \phi'^{-1}$ . The push forward measure  $\mu^*$  is then the moment measure of  $\phi$  (see [74, 187]) and has density

$$d\mu^* = \frac{\exp(-\phi(\phi'^{-1}(t)))}{\phi''(\phi'^{-1}(t))} dt.$$

In this setting, we can get two stability results with respect to two different distances. Firstly, Theorem 23 says that if  $\phi'' \geq \kappa > 0$  (meaning that  $\phi$  is uniformly convex) then  $\mu^*$  satisfies a stability result in total variation distance. It is a direct generalization of the Gaussian case [211].

**Corollary 37.** *Let  $M = \mathbb{R}$  and  $d\mu = e^{-\phi} dx$  be a  $C^2$  uniformly log-concave probability measure, i.e.  $\forall x \in \mathbb{R}, \phi''(x) \geq \kappa > 0$ . Let  $\nu$  be a probability measure on  $\mathbb{R}$  verifying the following Poincaré inequality: for all functions  $f$  in the weighted Sobolev space  $H_{\phi, \nu}^{1,2} = \{g \in L^2(\nu) \mid \frac{g'}{\sqrt{\phi''}} \in L^2(\nu)\}$ ,*

$$\text{Var}_\nu(f) \leq C_P(\nu) \int \frac{f'^2}{\phi''} d\nu,$$

and such that  $\int \phi' d\nu = 0$ ,  $\int \phi'^2 d\nu = \int \phi'^2 d\mu$  and  $\int \phi'' d\nu \leq \int \phi'' d\mu$ . So  $C_P(\nu) \geq 1$ , and moreover  $\delta := \frac{C_P(\nu)-1}{C_P(\nu)}$  satisfies

$$d_{TV}(\mu^*, \nu^*) \leq \frac{4}{\sqrt{\kappa}} \left( \sqrt{\delta} + \sqrt{C_P(\nu) \delta} \right),$$

where  $\mu^* = \phi'_{\#} \mu$  and  $\nu^* = \phi'_{\#} \nu$ .

Secondly, Theorem 30 yields a stability in 1-Wasserstein distance under a growth control of the second derivative of the potential  $\phi$  at infinity.

**Corollary 38.** *Let  $M = \mathbb{R}$  and  $d\mu = e^{-\phi} dx$  be a  $C^2$  strictly log-concave probability measure. Let  $\nu$  be a probability measure on  $\mathbb{R}$  verifying the following Poincaré inequality: for all functions  $f$  in the weighted Sobolev space  $H_{\phi}^{1,2} = \{g \in L^2(\nu) \mid \frac{g'}{\sqrt{\phi''}} \in L^2(\nu)\}$ ,*

$$\text{Var}_\nu(f) \leq C_P(\nu) \int \frac{f'^2}{\phi''} d\nu,$$

and such that  $\int \phi' d\nu = 0$ ,  $\int \phi'^2 d\nu = \int \phi'^2 d\mu$  and  $\int \phi'' d\nu \leq \int \phi'' d\mu$ . So  $C_P(\nu) \geq 1$ , and if for some  $\alpha \leq 2$ ,

$$c_1 |t|^{2\alpha-2} \leq \phi''(t) \leq c_2 |t|^\alpha, \quad t \rightarrow \pm\infty,$$

then  $\delta := \frac{C_P(\nu)-1}{C_P(\nu)}$  satisfies

$$W_1(\mu^*, \nu^*) \leq C_h \left( \sqrt{\delta} + \sqrt{C_P(\nu) \delta} \right),$$

where  $\mu^* = \phi'_{\#} \mu$  and  $\nu^* = \phi'_{\#} \nu$  and  $C_h$  is a finite constant defined by (3.29).

### 3.6.4 Application : stability of the Poincaré constant on the sphere $\mathbb{S}^d$

Let  $M = \mathbb{S}^d$ ,  $d \geq 1$  and  $\mu = \frac{\Gamma(\frac{d}{2})}{2\pi^{\frac{d}{2}}} \text{vol}_{\mathbb{S}^d}$  the Riemannian volume measure on the sphere, normalized to be a probability measure, where  $\Gamma$  denotes Euler's Gamma function. Then  $\mu$  is the reversible probability measure of the spherical Laplacian generating the Brownian motion on the sphere, which can be written as the restriction to the sphere of

$$Lf(x) = \sum_{i,j=1}^{d+1} (\delta_{ij} - x_i x_j) \partial_{ij}^2 f - d \sum_{i=1}^{d+1} x_i \partial_i f,$$

if  $f$  is the restriction to the sphere of a smooth function on a neighborhood of the sphere in  $\mathbb{R}^{d+1}$  (see [13, Section 2.2.1] for more details), and where  $\delta_{ij}$  denotes the Kronecker delta.

We have  $\Gamma(f) = \sum_{i,j=1}^{d+1} (\delta_{ij} - x_i x_j) \partial_i f \partial_j f$ ,  $C_P(\mu) = \frac{1}{d}$  and all coordinate functions  $x \mapsto x_j$  restricted to the sphere are centered eigenfunctions attaining the spectral gap. Let us choose  $f_1(x) = \sqrt{d+1} x_1$  which has variance one. Hence  $I = [-\sqrt{d+1}, \sqrt{d+1}]$  and  $\Gamma(f_1) = (d+1)(1-x_1^2)$ , so we get  $h(t) = (d+1) - t^2$ . So Assumptions 1, 2 and 3 are satisfied. Furthermore, at the infimum  $a = -\sqrt{d+1}$ , one sees  $\sqrt{d+1}(\sqrt{d+1} + t) \leq h(t) \leq 2\sqrt{d+1}(\sqrt{d+1} + t)$ , and

at the supremum  $b = \sqrt{d+1}$ , one similarly sees  $\sqrt{d+1}(\sqrt{d+1}-t) \leq h(t) \leq 2\sqrt{d+1}(\sqrt{d+1}-t)$ . Hence the conditions of Theorem 30 are satisfied. In this case the three conditions (3.6) are reduced to only two:

$$(3.38) \quad \int_{\mathbb{S}^d} x_1 d\nu = 0 \quad \text{and} \quad \int_{\mathbb{S}^d} x_1^2 d\nu = \frac{1}{d+1}.$$

So Theorem 30 applies to the canonical probability distribution on  $\mathbb{S}^d$ .

**Corollary 39.** *Let  $M = \mathbb{S}^d$ ,  $d \geq 1$  and  $\mu = \frac{\Gamma(\frac{d}{2})}{2\pi^{\frac{d}{2}}}\text{vol}_{\mathbb{S}^d}$ . Let  $\nu$  be a probability measure on  $\mathbb{S}^d$  normalized as in (3.38) and satisfying the following Poincaré inequality with sharp constant  $C_P(\nu)$ : for all functions  $f$  which is the restriction to the sphere of a  $C^2$  function on  $\mathbb{R}^{d+1}$ ,*

$$(3.39) \quad \text{Var}_\nu(f) \leq C_P(\nu) \int_{\mathbb{S}^d} \sum_{i,j=1}^{d+1} (\delta_{ij} - x_i x_j) \partial_i f \partial_j f d\nu(x).$$

Then it holds for a finite constant  $C_h$ ,

$$(3.40) \quad W_1(\mu^*, \nu^*) \leq C_h \left( \sqrt{\delta} + \sqrt{C_P(\nu) \delta} \right),$$

where  $W_1$  is the 1-Wasserstein distance (see Definition 25),  $\delta := \frac{C_P(\nu)-d}{C_P(\nu)}$ ,  $\nu^* := x_1^\# \nu$ , and

$$d\mu^*(t) = \frac{1}{Z} \left( d+1-t^2 \right)^{\frac{d}{2}-1} \mathbf{1}_{[-\sqrt{d+1}, \sqrt{d+1}]} dx,$$

with  $Z$  a normalization constant.

### 3.7 Comparison

A natural question is to compare the scope of application of Theorem 23 and Theorem 30. Indeed, Theorem 23 require  $h$  to be uniformly bounded by below, while Theorem 30 only require a growth control on  $h$  at the boundary of  $I$ . In this section we get stability results involving the Kolmogorov distance under the assumptions of Theorems 23 and 30. Afterwards, we will see that the order of the bound obtained from Theorem 23 is stronger. Let us begin recalling:

**Proposition 40.** [185, Prop 1.2] *If  $\mu^*$  has bounded density  $\rho$  with respect to the Lebesgue measure on  $\mathbb{R}$ , then*

$$d_K(\nu^*, \mu^*) \leq 2\sqrt{C W_1(\nu^*, \mu^*)}$$

where  $\rho \leq C$ ,  $W_1$  is the 1-Wasserstein distance, and  $d_K(\nu^*, \mu^*) = \sup_{t \in \mathbb{R}} \left| \int_{-\infty}^t d\nu^* - \int_{-\infty}^t d\mu^* \right|$  is the Kolmogorov distance.

Combining Proposition 40 above with Theorem 30 gives the following bound on the Kolmogorov distance.

**Corollary 41.** *Let  $(L, \mu)$  and  $h$  satisfy the requirements of Theorem 30. If moreover  $\frac{v(x)}{h(x)}$  is bounded by a constant  $C$ , then it holds for  $\delta := \frac{C_P(\nu)-C_P(\mu)}{C_P(\nu)C_P(\mu)}$ ,*

$$d_K(\mu^*, \nu^*) \leq 2\sqrt{C C_h} \sqrt{\sqrt{\delta} + \sqrt{C_P(\nu) \delta}},$$

for all measures  $\nu$  normalized as in (3.6) and satisfying a Poincaré inequality with sharp constant  $C_P(\nu)$ .

On the other hand, it is well-known that the total variation distance controls the Kolmogorov one:  $d_K(\nu^*, \mu^*) \leq d_{TV}(\nu^*, \mu^*)$ . Hence, Theorem 23 implies the following.

**Corollary 42.** *Let  $(L, \mu)$  and  $h \geq \kappa > 0$  satisfy the requirements of Theorem 23. Then it holds for  $\delta := \frac{C_P(\nu) - C_P(\mu)}{C_P(\nu)C_P(\mu)}$ ,*

$$d_K(\mu^*, \nu^*) \leq \frac{4}{\sqrt{\kappa}} \left( \sqrt{\delta} + \sqrt{C_P(\nu) \delta} \right),$$

for all measures  $\nu$  normalized as in (3.6) and satisfying a Poincaré inequality with sharp constant  $C_P(\nu)$ .

In case where  $(L, \mu)$  satisfies both assumptions of Corollary 41 and Corollary 42, one can see that the bound of Corollary 42 is better because it is of order  $\frac{1}{2}$  against  $\frac{1}{4}$  for the bound of Corollary 41. However, in the example of the gamma distributions, while Corollary 42 does not apply, although non-optimal, Corollary 41 gives us the following stability result in Kolmogorov distance.

**Corollary 43.** *For all  $\theta, s > 0$  and for all measures  $\nu$  on  $\mathbb{R}_+$  normalized as in (3.35) and satisfying the Poincaré inequality (3.36) with sharp constant  $C_P(\nu)$ , it holds:*

$$d_K(\mu^*, \nu^*) \leq 2\sqrt{C C_h} \sqrt{\sqrt{\delta} + \sqrt{C_P(\nu) \delta}},$$

where  $\delta := \frac{C_P(\nu) - \theta}{C_P(\nu)}$ ,  $C = (\sqrt{s})^{s+\sqrt{s}} \frac{e^{1-s}}{\Gamma(s)}$ ,  $\nu^* := \left(\sqrt{s} - \frac{x}{\theta\sqrt{s}}\right)^\# \nu$ , and

$$d\mu^*(x) = \left(\frac{\sqrt{s}}{e}\right)^s \frac{1}{\Gamma(s)} (\sqrt{s} - x)^{s-1} e^{\sqrt{s}x} \mathbf{1}_{]-\infty, \sqrt{s}[}(x) dx.$$



# Chapter 4

## Stability of higher order eigenvalues

This chapter is a slightly modified version of the paper [192].

### 4.1 Introduction

A classical question in Spectral Geometry is to identify properties of a manifold from the knowledge of eigenvalues of a canonical differential operator. The most extensively studied case is when the differential operator is the Laplace-Beltrami operator of a Riemannian manifold. This problem has been formulated by the famous "Can one hear the shape of a drum?" by M.Kac [121]. We refer the reader to [101, 102, 164] and to the survey [99]. The Hille-Yosida theory gives that under certain natural conditions, a differential operator generates a contractive semigroup (see [217]). In particular, in the case of the Laplace-Beltrami operator, it is the heat semigroup. In this range of ideas, Kato's formula implies comparison of semigroups and hence comparison of eigenvalues (see [30, 113, 196]). A large part of the literature is also devoted to estimates of the growth of eigenvalues of Schrodinger operators. These include the works of M.Bordoni [40], A.Laptev [134] and E.Lieb and W.Thirring [143, 144].

There are many classical comparison results involving only the first or second eigenvalues of operators. Let us cite among them the celebrated Faber-Krahn inequality: balls uniquely minimize the first Dirichlet eigenvalue of the Laplacian in  $\mathbb{R}^d$  among sets with given volume [86, 132], and the Hong-Krahn-Szego inequality: disjoint pair of equal balls uniquely minimize the second Dirichlet eigenvalue among sets with given volume [87, 116, 179].

In terms of functional inequalities, the first eigenvalue is encoded by the Poincaré constant. A probability measure  $\mu$  on  $\mathbb{R}^d$  is said to satisfy a Poincaré inequality when for all functions  $f$  in the Sobolev space  $H^1(\mu)$ ,

$$(4.1) \quad \text{Var}_\mu(f) \leq C_P(\mu) \int |\nabla f|^2 d\mu,$$

where  $C_P(\mu)$  denotes the smallest constant for which the above inequality holds. Poincaré inequalities have many applications (see for instance the survey [7]). When  $\mu$  is reversible for a Markov process, the infinitesimal generator  $L$  of the Markov process is symmetric on  $L^2(\mu)$  and the quantity  $\lambda_1(\mu) := \frac{1}{C_P(\mu)}$  is then the spectral gap of the positive symmetric operator  $-L$  (see [13, section 4.2.1]).

Stability results for Poincaré constant began to appear in the late 80's. Chen [67, Corollary 2.1] showed that all isotropic probability measures on  $\mathbb{R}^d$  have sharp Poincaré constant greater than 1. He proved furthermore that the standard Gaussian is the only one attaining 1. Then Utev [211] refined this result in dimension one, quantifying the difference between Poincaré constants in term of total variation distance:

$$C_P(\nu) \geq 1 + \frac{1}{9} d_{TV}(\nu, \gamma)^2$$

where  $\nu$  is a normalized probability measure on  $\mathbb{R}$ ,  $\gamma$  is the standard Gaussian and  $d_{TV}$  is the total variation distance. More recently, Courtade, Fathi and Pananjady [75], extended it to the multidimensional case with the Wasserstein-2 distance:

$$(4.2) \quad C_P(\nu) \geq 1 + \frac{W_2(\nu, \gamma)^2}{d}$$

where  $\nu$  is a centered probability measure on  $\mathbb{R}^d$ , normalized such that  $\int |x|^2 d\nu = d$ ,  $\gamma$  denotes the Gaussian  $\mathcal{N}(0, I_d)$  and  $W_2$  is the 2-Wasserstein distance (see [213, chapter 6]). This result has been extended in a more abstract setting, for a general reference probability measure  $\mu$  on a manifold instead of the Gaussian on  $\mathbb{R}^d$ .

**Theorem 44.** [191, Theorem 16] *Let  $L$  be a Markov reversible generator with respect to a probability measure  $\mu$ , carré du champ operator  $\Gamma$ , and with spectral gap  $C_P(\mu)^{-1}$  and associated eigenfunction  $f_1$ . If any other measure  $\nu$  satisfies the normalization conditions*

$$\int f_1 d\nu = 0, \quad \int f_1^2 d\nu = 1, \quad \int \Gamma(f_1) d\nu \leq \frac{1}{C_P(\mu)},$$

and the Poincaré inequality

$$\forall f, \quad \text{Var}_\nu(f) \leq C_P(\nu) \int \Gamma(f) d\nu,$$

then  $C_P(\nu) \geq C_P(\mu)$  and moreover the closeness between  $C_P(\nu)$  and  $C_P(\mu)$  bounds the 1-Wasserstein distance between the laws of the pushforwards of  $\mu$  and  $\nu$  by  $f_0$ :

$$(4.3) \quad W_1(f_0^\#(\mu), f_0^\#(\nu)) \leq \text{Const} \left( \frac{1}{C_P(\mu)} \sqrt{C_P(\nu) - C_P(\mu)} + \frac{\sqrt{C_P(\nu)}}{C_P(\mu)^2} (C_P(\nu) - C_P(\mu)) \right).$$

The constant is finite when the generalized gradient  $\Gamma(f_1)$  of the eigenfunction satisfies some growth conditions (see Proposition 29).

In this paper we will consider a diffusion process  $L$  on an interval with reversible probability measure  $\mu$  and carré du champ operator  $\Gamma$ . The reason why we obtain results only in dimension one will appear clear in Section 4.4. However, the entire framework and results outlined up to section 4.4 remain valid in higher dimensions. We will derive a stability result for higher order eigenvalues of the generator  $L$ . In [191], we used the following min-max theorem as definition of the first eigenvalue:

$$(4.4) \quad \lambda_1(\nu) = \inf_{f \in H^1(\nu)} \frac{\int \Gamma(f) d\nu}{\int f^2 d\nu}.$$

There are other min-max theorems for higher order eigenvalues, that require to change the functional space over which the infimum in Formula (4.4) runs. This can be seen as improving the Poincaré constant by decreasing the domain of the inequality. Despite the changes, we shall see that the main ingredients used for the stability of the first eigenvalue can still be used to establish stability results for higher order eigenvalues.

Let  $k \in \mathbb{N}$ ,  $k \geq 1$  and  $0 < \lambda_1(\mu) < \lambda_2(\mu) < \dots < \lambda_k(\mu) < \dots$  be the sequence of eigenvalues of  $-L$ , counted without multiplicity, and let  $f_k$  be a normalized eigenfunction associated with  $\lambda_k(\mu)$ , i.e.

$$-L f_k = \lambda_k f_k, \quad \int f_k d\mu = 0, \quad \text{and} \quad \int f_k^2 d\mu = 1.$$

We set  $I_k := \text{Im}(f_k)$ ,  $a_k := \inf I_k$  and  $b_k := \sup I_k$ . Let  $\nu$  be another probability measure on  $M$ , normalized so that

$$\int f_k d\nu = 0, \quad \int f_k^2 d\nu = 1, \quad \int \Gamma(f_k) d\nu \leq \lambda_k(\mu),$$

and satisfying the following improved Poincaré inequality

$$\int f^2 d\nu \leq \frac{1}{\lambda_k(\nu)} \int \Gamma(f) d\nu \quad \forall f \in H^1(\nu) \cap (Sp_1(\nu) \oplus \dots \oplus Sp_{k-1}(\nu))^\perp,$$

where  $Sp_i(\nu)$  denotes the  $i$ -th eigenspace of  $\nu$ ,  $\lambda_k(\nu)$  denotes the  $k$ -th eigenvalue of  $\nu$ , and the orthogonal complement is to be understood in the  $L^2(\nu)$  sense. We will show (see Lemma 47) that under these normalization conditions,  $\nu$  satisfies

$$\lambda_k(\nu) \leq \lambda_k(\mu) + \sum_{i=1}^{k-1} (\lambda_k(\nu) - \lambda_i(\nu)) d(f_k, Sp_i(\nu)^\perp)^2,$$

where  $d(f_k, Sp_i(\nu)^\perp)^2$  denotes the squared distance between  $f_k$  and  $Sp_i(\nu)^\perp$ . We refine this by proving the following stability result for the  $k$ -th eigenvalue.

**Theorem 45.** *For all one dimensional probability measures  $\nu$  normalized as in (4.5), satisfying the improved Poincaré inequalities (4.7), and the technical Assumption 1, it holds for some finite constant  $C > 0$ :*

$$\sum_j \nu_j(I_k^j) W_1(\nu_j^*, \mu_j^*) \leq C \left[ \sqrt{|\lambda_k(\mu) - \lambda_k(\nu)|} + \frac{|\lambda_k(\mu) - \lambda_k(\nu)|}{\sqrt{\lambda_1(\nu)}} + \sum_{i=1}^{k-1} C_i d(f_k, Sp_i(\nu)^\perp) \right]$$

where  $(I_k^j)_j$  are the images by  $f_k$  of the connected components of the complementary of its critical points,  $\nu_j^*$  (resp.  $\mu_j^*$ ) is the pushforward of  $\nu$  (resp.  $\mu$ ) restricted to  $I_k^j$ , constants  $C_i$  are given by

$$C_i = \sqrt{\lambda_k(\nu) - \lambda_i(\nu)} + \frac{\lambda_k(\nu) - \lambda_i(\nu)}{\sqrt{\lambda_i(\nu)}},$$

and  $d(f_k, Sp_i(\nu)^\perp)$  is defined in Remark 1. The value  $C = \sum_j C_{h_j}^2$  suffices, with  $C_{h_j}$  given in Proposition 29.

The technical Assumption 1 (see Section 4.4.2) ensures the finiteness of the constant  $C$  (see Proposition 54), and asks that the carré du champ operator  $\Gamma(f_k)$  of the eigenfunction satisfies a certain polynomial growth condition. The proof of Theorem 45 is based on an approximate integration by parts formula satisfied by  $\nu$  with respect to the  $k$ -th eigenfunction  $f_k$  (see Corollary 50) and the use of Stein's method to the pushforward of  $\mu$  by  $f_k$  (see Section 4.4.2). Let us mention that the exact integration by parts satisfied by the  $k$ -th Hermite polynomial in case of the Normal distribution, was used in [88] to define the notion of higher order Stein's kernels in the context of Gaussian approximation.

Our result applies in particular to the normal distribution (see Section 4.5), Gamma distributions on  $\mathbb{R}_+$  (see Section 4.6), and Beta distributions on  $[-1, 1]$  (see Section 4.7). Applying this to the second Hermite polynomial, we obtain the following Chi-2 approximation result. For all measure  $\nu$  on  $\mathbb{R}$  normalized as

$$\int x^2 d\nu = 1, \quad \text{and} \quad \int x^4 d\nu = 3,$$

it holds for some finite positive constant  $C > 0$ :

$$W_1\left(\frac{1}{\sqrt{2}}(\chi_2 - 1), \nu^*\right) \leq C \left[ \sqrt{|2 - \lambda_2(\nu)|} + \frac{|2 - \lambda_2(\nu)|}{\sqrt{\lambda_1(\nu)}} + C_\nu d\left(\frac{1}{\sqrt{2}}(x^2 - 1), Sp_1(\nu)^\perp\right) \right]$$

where  $\chi_2$  is the  $\chi_2$ -distribution on  $\mathbb{R}_+$ ,  $\nu^*$  is the pushforward of  $\nu$  by the second Hermite polynomial  $\frac{1}{\sqrt{2}}(x^2 - 1)$ , the constant  $C_\nu$  is given by  $C_\nu = \sqrt{\lambda_2(\nu) - \lambda_1(\nu)} + \frac{\lambda_2(\nu) - \lambda_1(\nu)}{\sqrt{\lambda_1(\nu)}}$ , and



$d\left(\frac{1}{\sqrt{2}}(x^2 - 1), Sp_1(\nu)^\perp\right)$  quantifies the orthogonality error between  $\frac{1}{\sqrt{2}}(x^2 - 1)$  and the first eigenspace of  $\nu$  (see Section 4.2.1).

Let us say a few words about stability results from a geometric setting. The Lichnerowicz theorem asserts that among all Riemannian manifolds with Ricci curvature bounded by below by  $N - 1$ , unit spheres of dimension  $N$  uniquely minimizes the first eigenvalue of the Laplace-Beltrami operator [142]. The Bakry-Emery criterion [12, 11] extends this result to Gaussian spaces: if  $\mu = e^{-V} dx$  is a probability distribution which is more log-concave than the Gaussian (i.e.  $\text{Hess } V \geq I_d$ ) then its Poincaré constant is smaller than 1 which is that of the Gaussian.

While the original proof of Bakry-Emery is based on the semigroup method, another powerful method is the contraction principle [161]: if  $\mu$  is the pushforward of  $\nu$  by a  $L$ -Lipshitz map, then  $C_P(\mu) \leq L C_P(\nu)$ . In particular, Caffarelli's contraction theorem [52] states that the optimal transport between the Gaussian and a more log-concave distribution given by the Brenier map is 1-Lipschitz, recovering the Bakry-Emery criterion. E.Milman [161] pointed out that the contraction principle does not only entail a comparison between the first eigenvalues, but a comparison between the entire spectra. In that area of spectral comparison by the contraction principle, let us cite the recent works of D.Mikulincer and Y.Shenfeld [158, 157].

The question of stability of spectral estimates has been addressed in various works. We refer the reader to the survey [43] by L.Brasco and G. De Philippis for a view of quantitative sharp inequalities for first (and second) eigenvalues of the Laplacian in  $\mathbb{R}^d$ . Let us mention in particular the quantitative form of Faber-Krahn inequality. L.Brasco, G. De Philippis and B.Velichkov proved in [44] that there exists a constant  $\sigma > 0$  depending on the dimension such that for all  $\Omega \subset \mathbb{R}^d$  of volum 1,

$$\lambda_1(\Omega) \geq \lambda_1(B) + \sigma \mathcal{A}(\Omega)^2$$

where  $\lambda_1(\Omega)$  denotes the first Dirichlet eigenvalue of the Laplacian on  $\Omega$ ,  $B$  denotes the unit ball in  $\mathbb{R}^d$ ,  $\mathcal{A}(\Omega)$  is the Fraenkel asymmetry of  $\Omega$ , and the exponent 2 is sharp.

The study of the stability of the spectral gap of a diffusion operator falls within this framework. Under the curvature-dimension condition, let us cite the work [28] of J.Bertrand and M.Fathi which treats the case of the positive curvature and the infinite dimension. In particular, they show that any  $RCD(1, \infty)$  space reaching almost the Bakry-Emery bound 1 for its spectral gap, admits approximately all integers in its spectrum. The stability is quantified in terms of a spectral comparison with the Gaussian, since the integers are eigenvalues of the Ornstein-Uhlenbeck generator for which the Gaussian is reversible. In case of the positive curvature and the finite dimension, the Lichnerowicz theorem has been extended in the following way.

**Theorem 46.** [89, Theorem 1.1] *Let  $(M, d, \mu)$  be an  $RCD(N - 1, N)$  space with  $N > 1$  and spectral gap  $\lambda_1 \leq N + \varepsilon$  for some  $\varepsilon > 0$ , with  $f$  an eigenfunction of the Laplacian, with eigenvalue  $\lambda_1$  and normalized so that  $\|\Gamma(f)\|_1 = N/(N + 1)$ . There is a constant  $C(N) > 0$  (independent of  $M$ ) such that the 1-Wasserstein distance between the pushforward of  $\mu$  by  $f$  and a symmetrized Beta distribution with parameters  $(N/2, N/2)$  is smaller than  $C(N)\varepsilon$ .*

Let us emphasize that the stability is quantified in terms of  $W_1$  distance between pushforward by the first eigenfunction, since the symmetrized Beta distribution with parameters  $(N/2, N/2)$  is the distribution of the pushforward by a first eigenfunction of the reversible law of the Laplacian on a sphere. Let us conclude this introduction by mentioning that under the normalisation approach used in this paper, the spectral gap of the model space is maximal, whereas under the curvature condition approach, it is minimal.

## 4.2 The space of normalized probability distributions $\nu$

In this section, we explicitly describe the space of normalized probability distributions on which our stability result holds. We consider a probability measure  $\nu$  such that

$$(4.5) \quad \int f_k d\nu = 0, \quad \int f_k^2 d\nu = 1, \quad \text{and} \quad \int \Gamma(f_k) d\nu \leq \lambda_k(\mu)$$

Let us underline that these normalization conditions correspond to

$$\int f_k d\nu = \int f_k d\mu, \quad \int f_k^2 d\nu = \int f_k^2 d\mu, \quad \int \Gamma(f_k) d\nu \leq \int \Gamma(f_k) d\mu.$$

But since  $f_k$  is an eigenfunction associated to the  $k$ -th eigenvalue of  $\mu$ , we have  $\int f_k d\mu = 0$  and  $\int \Gamma(f_k) d\mu = \lambda_k(\mu) \int f_k^2 d\mu$ , hence we normalize  $f_k$  by  $\int f_k^2 = 1$  in order to make the conditions more readable.

### 4.2.1 Eigenspaces of $\nu$

We define the eigenspaces of  $\nu$  in the following way: first

$$Sp_1(\nu) := \left\{ f \in H^1(\nu) \mid \forall g \in H^1(\nu), \int fg d\nu = \frac{1}{\lambda_1(\nu)} \int \Gamma(f, g) d\nu \right\},$$

where

$$H^1(\nu) := \left\{ f \in L^2(\nu) \mid \int f d\nu = 0, \int \Gamma(f) d\nu < \infty \right\},$$

and

$$(4.6) \quad \lambda_1(\nu) = \inf_{f \in H^1(\nu) \setminus \{0\}} \frac{\int \Gamma(f) d\nu}{\int f^2 d\nu}.$$

This definition corresponds to eigenspace in a weak sense. It is clearly a linear space and a subset of  $\{f \in H^1(\nu) \mid \int f^2 d\nu = \frac{1}{\lambda_1(\nu)} \int \Gamma(f) d\nu\}$ . Moreover, if  $\nu$  is reversible for some generator  $L_\nu$  with carré du champ operator  $\Gamma$ , then the converse set inclusion holds and  $Sp_1(\nu)$  is an eigenspace of  $L_\nu$  in the classical sense. We can then recursively define higher order eigenspaces in a similar way.

$$Sp_{k+1}(\nu) := \left\{ f \in H^1(\nu) \mid \forall g \in H^1(\nu) \cap (Sp_1(\nu) \oplus \dots \oplus Sp_k(\nu))^\perp, \int fg d\nu = \frac{1}{\lambda_{k+1}(\nu)} \int \Gamma(f, g) d\nu \right\} \\ \cap (Sp_1(\nu) \oplus \dots \oplus Sp_k(\nu))^\perp,$$

where the orthogonal complement is to be understood in the  $L^2(\nu)$  sense and

$$\lambda_{k+1}(\nu) := \inf_{\substack{f \in H^1(\nu) \cap (Sp_1(\nu) \oplus \dots \oplus Sp_k(\nu))^\perp \\ f \neq 0}} \frac{\int \Gamma(f) d\nu}{\int f^2 d\nu}.$$

Note that by construction, eigenspaces are pairwise orthogonal in  $L^2(\nu)$ , and eigenvalues are ordered:  $\lambda_1(\nu) \leq \lambda_2(\nu) \leq \dots \leq \lambda_k(\nu) \leq \dots$

Let us emphasize that the integration by parts formula

$$\int fg d\nu = \frac{1}{\lambda_k(\nu)} \int \Gamma(f, g) d\nu,$$

when  $f$  is an eigenfunction can be interpreted as an "isometry along  $f$  in  $Sp_k(\nu)$ " between the  $L^2(\nu)$ -norm and the  $H^1(\nu)$ -norm. This property is the keystone of Lemma 47 and Theorem 18.

### 4.2.2 Improved Poincaré inequalities

By definition of eigenvalues and associated eigenspaces, the probability measure  $\nu$  always satisfies the following improved Poincaré inequalities.

$$(4.7) \quad \int f^2 d\nu \leq \frac{1}{\lambda_k(\nu)} \int \Gamma(f) d\nu \quad \forall f \in H^1(\nu) \cap (Sp_1(\nu) \oplus \dots \oplus Sp_{k-1}(\nu))^\perp$$

where

$$\lambda_k(\nu) := \inf_{f \in H^1(\nu) \cap (Sp_1(\nu) \oplus \dots \oplus Sp_{k-1}(\nu))^\perp} \frac{\int \Gamma(f) d\nu}{\int f^2 d\nu} \geq \lambda_{k-1}(\nu).$$

Even if the eigenvalue  $\lambda_k(\nu)$  is trivial (i.e. is zero), the improved Poincaré inequality becomes itself trivial, but remains true.

### 4.2.3 Projection of the eigenfunction $f_k$ onto eigenspaces of $\nu$

The first idea used in the previous study on the spectral gap [191] was to evaluate the Poincaré inequality satisfied by  $\nu$  with the first eigenfunction of  $\mu$ . We want to do the same in the general case, however it is impossible to evaluate the improved Poincaré inequality (4.7) with  $f_k$  since we have no guarantee that  $f_k \in H^1(\nu) \cap (Sp_1(\nu) \oplus \dots \oplus Sp_{k-1}(\nu))^\perp$ . But this space is a linear subspace of  $L^2(\nu)$ , hence it seems natural to think that (4.7) should not be evaluated with  $f_k$ , but with the  $L^2$ -projection of  $f_k$  on  $(Sp_1(\nu) \oplus \dots \oplus Sp_{k-1}(\nu))^\perp$ .

Let  $p_k^\perp$  be the  $L^2(\nu)$  orthogonal projection of  $f_k$  onto  $(Sp_1(\nu) \oplus \dots \oplus Sp_{k-1}(\nu))^\perp$  and  $p_k$  the  $L^2(\nu)$  orthogonal projection of  $f_k$  onto  $Sp_1(\nu) \oplus \dots \oplus Sp_{k-1}(\nu)$ . Hence we have the following formulas that we will repeatedly use in the sequel:

$$(4.8) \quad f_k = p_k + p_k^\perp, \quad p_k \in Sp_1(\nu) \oplus \dots \oplus Sp_{k-1}(\nu), \quad p_k^\perp \in (Sp_1(\nu) \oplus \dots \oplus Sp_{k-1}(\nu))^\perp,$$

$$(4.9) \quad p_k = p_k^1 + \dots + p_k^{k-1}, \quad p_k^1 \in Sp_1(\nu), \dots, p_k^{k-1} \in Sp_{k-1}(\nu)$$

Let us point out that in the case of the spectral gap (i.e.  $k = 1$ ),  $p_1$  would correspond to the projection of  $f_1$  onto the kernel of  $L$ , which is the set of constant functions, and  $p_1^\perp$  would hence be the projection of  $f_1$  onto the set of centered functions. But since  $f_1$  is centered, we would have  $p_1 = f_1$  and so this coincides with the general case where we will use  $p_k^\perp$  to evaluate in (4.7).

### 4.2.4 Eigenvalue comparisons

In this section, we will show that any probability distribution  $\nu$  normalized as (4.5) has its  $k$ -th eigenvalue  $\lambda_k(\nu)$  controlled by  $\lambda_k(\mu)$ , some terms quantifying the distance between  $f_k$  and  $(Sp_1(\nu) \oplus \dots \oplus Sp_{k-1}(\nu))^\perp$ , and the gap between successive eigenvalues of  $\nu$ . This estimate holds without any additional assumption on  $\mu$ .

**Lemma 47.** *Let  $\nu$  be a probability distribution normalized as in (4.5). Then*

$$(4.10) \quad \lambda_k(\nu) \leq \lambda_k(\mu) + \sum_{i=1}^{k-1} (\lambda_k(\nu) - \lambda_i(\nu)) \int (p_k^i)^2 d\nu,$$

where  $p_k^i$  are the projections defined in (4.9).

*Proof.* The proof only consists in evaluating (4.7) with  $f = p_k^\perp$ , which actually belongs to the correct space. On the one hand, using (4.9)

$$\int (f_k - p_k)^2 d\nu = \int f_k^2 d\nu + \sum_{i=1}^{k-1} \int (p_k^i)^2 d\nu - 2 \int f_k p_k d\nu = 1 - \sum_{i=1}^{k-1} \int (p_k^i)^2 d\nu.$$

On the other hand, using that all  $p_k^i$  are eigenfunctions, Formula (4.8) and Formula (4.9),

$$\begin{aligned} \int \Gamma(f_k - p_k) d\nu &= \int \Gamma(f_k) d\nu + \sum_{i=1}^{k-1} \int \Gamma(p_k^i) d\nu - 2 \int \Gamma(f_k, p_k) d\nu \\ &= \lambda_k(\mu) - \sum_{i=1}^{k-1} \int \Gamma(p_k^i) d\nu \\ &= \lambda_k(\mu) - \sum_{i=1}^{k-1} \lambda_i(\nu) \int (p_k^i)^2 d\nu. \end{aligned}$$

We then apply (4.7) to  $p_k^\perp = f_k - p_k \in (Sp_1(\nu) \oplus \dots \oplus Sp_{k-1}(\nu))^\perp$  and get

$$\lambda_k(\nu) - \lambda_k(\nu) \sum_{i=1}^{k-1} \int (p_k^i)^2 d\nu \leq \lambda_k(\mu) - \sum_{i=1}^{k-1} \lambda_i(\nu) \int (p_k^i)^2 d\nu$$

which gives the result.  $\square$

Let us point out that equality holds in (4.10) if  $\nu = \mu$  because in that case,  $f_k$  belongs to  $(Sp_1(\nu) \oplus \dots \oplus Sp_{k-1}(\nu))^\perp$  and thus  $p_k = 0$ . The first natural question is then about rigidity of inequality (4.10). What can we say about  $\nu$  if the inequality (4.10) is in fact an equality? We have seen in case of the spectral gap ( $k = 1$ ) that this implies the pushforward measures  $f_1^\# \mu$  and  $f_1^\# \nu$  to be equal, but the measures  $\mu$  and  $\nu$  themselves can be different. We will see that for general  $k \geq 2$ , the equality case also implies some link between the pushforward  $f_k^\# \mu$  and  $f_k^\# \nu$ , which itself implies equality of the pushforward measures in case where  $\nu$  allocate the same weight as  $\mu$  on each non critical sets of  $f_k$  (see Section 4.4.1).

**Remark 1.** *The quantity  $\int (p_k^i)^2 d\nu$  is the square distance between  $f_k$  and  $Sp_i(\nu)^\perp$  (by definition of the projection), and quantifies therefore the orthogonality error between  $f_k$  and eigenspaces of lower orders of  $\nu$ . We denote it by  $d(f_k, Sp_i(\nu))^\perp$ . Therefore (4.10) becomes:*

$$(4.11) \quad \lambda_k(\nu) \leq \lambda_k(\mu) + \sum_{i=1}^{k-1} (\lambda_k(\nu) - \lambda_i(\nu)) d(f_k, Sp_i(\nu))^\perp{}^2$$

### 4.3 Approximate integration by parts formulas

In this section, we derive approximate integration by parts formulas, for the measure  $\nu$ , with an error term involving quantities appearing in the comparison (4.11) between the eigenvalues of  $\mu$  and  $\nu$ . This approximate integration by parts formula will be the keystone to use Stein's method in this context. We shall proceed as for the spectral gap, with a difference: we now use  $p_k^\perp$  instead of directly using  $f_k$  as minimizer in the improved Poincaré inequality. Hence, in a first step, we will derive an approximate integration by parts formula with  $p_k^\perp$  and only valid on  $(Sp_1(\nu) \oplus \dots \oplus Sp_{k-1}(\nu))^\perp$ . In a second step, we will replace  $p_k^\perp$  by  $f_k$ , and finally in a third step we will extend it to the whole space  $H^1(\nu)$ .

**Theorem 48.** *We have the following inequality for all  $g \in H^1(\nu) \cap (Sp_1(\nu) \oplus \dots \oplus Sp_{k-1}(\nu))^\perp$ ,*

$$(4.12) \quad \left| \int (\lambda_k(\nu) p_k^\perp g - \Gamma(p_k^\perp, g)) d\nu \right| \leq \left[ \lambda_k(\mu) - \lambda_k(\nu) + \sum_{i=1}^{k-1} (\lambda_k(\nu) - \lambda_i(\nu)) d(f_k, Sp_i(\nu))^\perp{}^2 \right]^{\frac{1}{2}} \sqrt{\int \Gamma(g) d\nu}$$

where  $p_k^\perp$  is defined in (4.8).

**Proof** Let  $t \in \mathbb{R}$  and  $g \in H^1(\nu) \cap (Sp_1(\nu) \oplus \dots \oplus Sp_{k-1}(\nu))^\perp$ . Let us apply (4.7) to  $\alpha := p_k^\perp + tg \in H^1(\nu) \cap (Sp_1(\nu) \oplus \dots \oplus Sp_{k-1}(\nu))^\perp$ . Computing

$$\begin{aligned} \text{Var}_\nu(\alpha) &= \int (p_k^\perp + tg)^2 d\nu = \int (p_k^\perp)^2 d\nu + 2t \int p_k^\perp g d\nu + t^2 \int g^2 d\nu \\ &= \int (f_k - p_k)^2 d\nu + 2t \int p_k^\perp g d\nu + t^2 \int g^2 d\nu \\ &= 1 - \int p_k^2 d\nu + 2t \int p_k^\perp g d\nu + t^2 \int g^2 d\nu \\ &= 1 - \sum_{i=1}^{k-1} \int (p_k^i)^2 d\nu + 2t \int p_k^\perp g d\nu + t^2 \int g^2 d\nu \\ &= 1 - \sum_{i=1}^{k-1} \frac{1}{\lambda_i(\nu)} \int \Gamma(p_k^i) d\nu + 2t \int p_k^\perp g d\nu + t^2 \int g^2 d\nu, \end{aligned}$$

and

$$\begin{aligned} \int \Gamma(\alpha) d\nu &= \int \Gamma(p_k^\perp) d\nu + 2t \int \Gamma(p_k^\perp, g) d\nu + t^2 \int \Gamma(g) d\nu \\ &= \int \Gamma(f_k - p_k) d\nu + 2t \int \Gamma(p_k^\perp, g) d\nu + t^2 \int \Gamma(g) d\nu \\ &= \lambda_k(\mu) + \int \Gamma(p_k) d\nu - 2 \int \Gamma(f_k, p_k) d\nu + 2t \int \Gamma(p_k^\perp, g) d\nu + t^2 \int \Gamma(g) d\nu \\ &= \lambda_k(\mu) - \int \Gamma(p_k) d\nu + 2t \int \Gamma(p_k^\perp, g) d\nu + t^2 \int \Gamma(g) d\nu \\ &= \lambda_k(\mu) - \sum_{i=1}^{k-1} \int \Gamma(p_k^i) d\nu + 2t \int \Gamma(p_k^\perp, g) d\nu + t^2 \int \Gamma(g) d\nu, \end{aligned}$$

where we have used at line 4 that, since all  $p_k^i$  are eigenfunctions,

$$\begin{aligned} \int \Gamma(f_k, p_k) d\nu &= \sum_i \int \Gamma(p_k^\perp, p_k^i) d\nu + \sum_{i,j} \int \Gamma(p_k^i, p_k^j) d\nu \\ &= \sum_i \lambda_i(\nu) \int p_k^\perp p_k^i d\nu + \sum_{i,j} \lambda_i(\nu) \int p_k^i p_k^j d\nu \\ &= \sum_i \lambda_i(\nu) \int (p_k^i)^2 d\nu \\ &= \sum_i \int \Gamma(p_k^i) d\nu \\ &= \int \Gamma(p_k) d\nu, \end{aligned}$$

we get that for all  $t \in \mathbb{R}$ , the degree two polynomial

$$\left( - \int \Gamma(g) d\nu \right) t^2 + 2 \left( \int (\lambda_k(\nu) p_k^\perp g - \Gamma(p_k^\perp, g)) d\nu \right) t + \left( \lambda_k(\nu) - \lambda_k(\mu) + \sum_{i=1}^{k-1} \left( 1 - \frac{\lambda_k(\nu)}{\lambda_i(\nu)} \right) \int \Gamma(p_k^i) d\nu \right)$$

is non positive. Hence its discriminant is non positive:

$$4 \left( \int (\lambda_k(\nu) p_k^\perp g - \Gamma(p_k^\perp, g)) d\nu \right)^2 + 4 \int \Gamma(g) d\nu \left( \lambda_k(\nu) - \lambda_k(\mu) + \sum_{i=1}^{k-1} \left( 1 - \frac{\lambda_k(\nu)}{\lambda_i(\nu)} \right) \int \Gamma(p_k^i) d\nu \right) \leq 0$$

which gives the result since  $\int \Gamma(p_k^i) d\nu = \lambda_i(\nu) \int (p_k^i)^2 d\nu = \lambda_i(\nu) d(f_k, Sp_i(\nu))^\perp)^2$ .  $\square$

Now it is easy to see that one can replace  $p_k^\perp$  by  $f_k$  without any additional cost.

**Corollary 49.** *We have the following inequality for all  $g \in H^1(\nu) \cap (Sp_1(\nu) \oplus \dots \oplus Sp_{k-1}(\nu))^\perp$ ,*  
(4.13)

$$\left| \int (\lambda_k(\nu) f_k g - \Gamma(f_k, g)) d\nu \right| \leq \left[ \lambda_k(\mu) - \lambda_k(\nu) + \sum_{i=1}^{k-1} (\lambda_k(\nu) - \lambda_i(\nu)) d(f_k, Sp_i(\nu)^\perp)^2 \right]^{\frac{1}{2}} \sqrt{\int \Gamma(g) d\nu}$$

**Proof** Use the fact that  $p_k^\perp = f_k - p_k$  in Theorem 48 and both  $\int p_k g d\nu = 0$  and  $\int \Gamma(p_k, g) d\nu = 0$  since  $p_k \in Sp_1(\nu) \oplus \dots \oplus Sp_{k-1}(\nu)$  and  $g \in H^1(\nu) \cap (Sp_1(\nu) \oplus \dots \oplus Sp_{k-1}(\nu))^\perp$ .  $\square$

Finally, one can extend the approximate integration by parts (4.13) on the whole  $H^1(\nu)$  and it only adds a term which is again controlled by the orthogonal error of the eigenfunction.

**Corollary 50.** *We have the following inequality for all  $g \in H^1(\nu)$ :*

$$(4.14) \quad \left| \int (\lambda_k(\nu) f_k g - \Gamma(f_k, g)) d\nu \right| \leq \left[ \sqrt{|\lambda_k(\mu) - \lambda_k(\nu)|} + \sum_{i=1}^{k-1} C_i d(f_k, Sp_i(\nu)^\perp) \right] \sqrt{\int \Gamma(g) d\nu}$$

where

$$(4.15) \quad C_i = \sqrt{|\lambda_k(\nu) - \lambda_i(\nu)|} + \frac{\lambda_k(\nu) - \lambda_i(\nu)}{\sqrt{\lambda_i(\nu)}}$$

**Remark 2.** *We cannot avoid the absolute value under the square root because we only know that (4.11) holds, which does not imply  $\lambda_k(\nu) \leq \lambda_k(\mu)$  except for  $k = 1$ .*

**Proof** Let  $g \in H^1(\nu)$ . Let  $g = g_p + g_\perp$  with  $g_p \in H^1(\nu) \cap (Sp_1(\nu) \oplus \dots \oplus Sp_{k-1}(\nu))$  and  $g_\perp \in H^1(\nu) \cap (Sp_1(\nu) \oplus \dots \oplus Sp_{k-1}(\nu))^\perp$ . We have:

$$\int (\lambda_k(\nu) f_k g - \Gamma(f_k, g)) d\nu = \int (\lambda_k(\nu) f_k g_p - \Gamma(f_k, g_p)) d\nu + \int (\lambda_k(\nu) f_k g_\perp - \Gamma(f_k, g_\perp)) d\nu$$

We apply (4.13) to the second term in the sum. For the first one, since  $g_p = \sum_{i=1}^{k-1} g_p^i \in Sp_1(\nu) \oplus \dots \oplus Sp_{k-1}(\nu)$ , we have

$$\begin{aligned} \int (\lambda_k(\nu) f_k g_p - \Gamma(f_k, g_p)) d\nu &= \sum_{i=1}^{k-1} (\lambda_k(\nu) - \lambda_i(\nu)) \int f_k g_p^i d\nu \\ &= \sum_{i=1}^{k-1} (\lambda_k(\nu) - \lambda_i(\nu)) \int p_k g_p^i d\nu \text{ using (4.8)} \\ &= \sum_{i=1}^{k-1} (\lambda_k(\nu) - \lambda_i(\nu)) \int p_k^i g_p^i d\nu \text{ using (4.9)} \\ &\leq \sum_{i=1}^{k-1} (\lambda_k(\nu) - \lambda_i(\nu)) \sqrt{\int (p_k^i)^2 d\nu} \sqrt{\int (g_p^i)^2 d\nu} \\ &= \sum_{i=1}^{k-1} (\lambda_k(\nu) - \lambda_i(\nu)) d(f_k, Sp_i(\nu)^\perp) \sqrt{\frac{1}{\lambda_i(\nu)} \int \Gamma(g_p^i) d\nu} \\ &\leq \sum_{i=1}^{k-1} \frac{\lambda_k(\nu) - \lambda_i(\nu)}{\sqrt{\lambda_i(\nu)}} d(f_k, Sp_i(\nu)^\perp) \sqrt{\int \Gamma(g) d\nu}. \end{aligned}$$

which allows to conclude.  $\square$

## 4.4 Stability results in dimension one

From now on, our results will only apply when  $L$  is a one-dimensional diffusion operator on a (possibly infinite) interval. We will prove the following stability result for the  $k$ -th eigenvalue of  $-L$ .

**Theorem 51.** *Let  $L$  be a diffusion generator on an interval  $M \subset \mathbb{R}$ , let  $0 < \lambda_1(\mu) < \lambda_2(\mu) < \dots < \lambda_k(\mu)$  be its  $k \geq 1$  first eigenvalues, counted without multiplicity, and let  $f_k$  be an eigenfunction associated with  $\lambda_k(\mu)$ , satisfying Assumption 1. Let  $\text{Crit}(f_k)$  be the set of all critical points of  $f_k$ . Then for all probability measures  $\nu$  on  $M$  normalized as in (4.5) and satisfying the improved Poincaré inequalities (4.7), it holds for some finite constant  $C > 0$ :*

$$\sum_j \nu_j(I_k^j) W_1(\nu_j^*, \mu_j^*) \leq C \left[ \sqrt{|\lambda_k(\mu) - \lambda_k(\nu)|} + \frac{|\lambda_k(\mu) - \lambda_k(\nu)|}{\sqrt{\lambda_1(\nu)}} + \sum_{i=1}^{k-1} C_i d(f_k, Sp_i(\nu)^\perp) \right]$$

where  $(I_k^j)_j$  are the images by  $f_k$  of the connected components of  $M \setminus \text{Crit}(f_k)$ ,  $\nu_j^*$  (resp.  $\mu_j^*$ ) is the pushforward of  $\nu$  (resp.  $\mu$ ) restricted to  $I_k^j$ , constants  $C_i$  are given by

$$C_i = \sqrt{\lambda_k(\nu) - \lambda_i(\nu)} + \frac{\lambda_k(\nu) - \lambda_i(\nu)}{\sqrt{\lambda_i(\nu)}},$$

and  $d(f_k, Sp_i(\nu)^\perp)$  is defined in Remark 1. The value  $C = \sum_j C_{h_j}^2$  suffices, with  $C_{h_j}$  given in Proposition 29.

Let us point out that Theorem 51 implies that if  $\lambda_k(\nu) = \lambda_k(\mu)$  and if  $f_k$  is orthogonal in  $L^2(\nu)$  to all lower order eigenspaces of  $\nu$ , then the conditional pushforward of  $\nu$  and  $\mu$  are all equal :  $\forall j, \nu_j^* = \mu_j^*$ . In this case, one can compute that for all bounded  $\phi : I_k \rightarrow \mathbb{R}$ ,

$$\begin{aligned} \int \phi(f_k) d\nu - \int \phi(f_k) d\mu &= \sum_j \left( \int_{J_j} \phi(f_k) d\nu - \int_{J_j} \phi(f_k) d\mu \right) \\ &= \sum_j \left( \int_{I_k^j} \nu(J_j) \phi d\nu_j^* - \int_{I_k^j} \mu(J_j) \phi d\mu_j^* \right) \\ &= \sum_j (\nu(J_j) - \mu(J_j)) \int_{I_k^j} \phi d\mu_j^*. \end{aligned}$$

We then deduce the following corollary.

**Corollary 52.** *Let  $\mu$  and  $\nu$  such as required in Theorem 51. If moreover*

- $\lambda_k(\nu) = \lambda_k(\mu)$ ,
- $\forall i \leq k, f_k \perp Sp_i(\nu)$  in  $L^2(\nu)$ , and
- $\forall j, \nu(J_j) = \mu(J_j)$ ,

then the pushforwards by  $f_k$  are the same, that is

$$f_k^\# \nu = f_k^\# \mu.$$

When  $k = 1$  the last two conditions are trivially satisfied, so we recover the result in [191].

### 4.4.1 The pushforward by $f_k$

In [191], after obtaining the approximate integration by parts formula, we pushforwarded it by the first eigenfunction  $f_1$ . The integration by parts formula then became a one dimensional ODE that we explicitly solved. But taking the pushforward was possible because we assumed that the carré du champ of the first eigenfunction  $\Gamma(f_1)$  could be factorize as  $\Gamma(f_1) = h \circ f_1$  for some non-negative function  $h : I_1 \rightarrow \mathbb{R}_+$ . We justified this assumption by the fact that in case of dimension one, where  $M \subset \mathbb{R}$  is an interval, all first eigenfunctions are known to be strictly monotone, and hence injective, so one can simply take  $h := \Gamma(f_1) \circ f_1^{-1}$ . In a multidimensional space  $M$ ,  $f_1$  cannot be injective. However, we have the classical example of eigenfunctions on Spheres which also satisfy this factorization assumption. Hence this assumption does not seem so odd.

But now considering higher order eigenfunctions, in dimension one we no longer have monotonicity, and no injectivity either, so the obvious choice  $h := \Gamma(f_k) \circ f_k^{-1}$  is no longer available. Let us nevertheless point out that in our classical examples (i.e. the Normal, Gamma and Beta distributions), the second eigenfunctions of the Normal and Beta distributions still satisfy this factorization assumption, despite being non-injective. This is due to the symmetry of the Gaussian and Beta distributions. There is no other such eigenfunctions of any order in these three examples which factorize. However, between critical points, the derivative of any eigenfunction is obviously either positive or negative, and hence we have a local injectivity between critical points, so a local factorization of the carré du champ. Our approach is then to locally take the pushforward of (4.14) on each connected component of  $M \setminus \mathcal{C}_k$  where we denote the set of critical points of  $f_k$  by  $\mathcal{C}_k$ .

Let

$$\mathcal{C}_k := \{x \in M \subset \mathbb{R} \mid \Gamma(f_k)(x) = 0\} = \text{Crit}(f_k)$$

be the set of critical points of  $f_k$  and assume this set to be finite. This is always the case in classical examples, where  $\mathcal{C}_k$  has  $k-1$  elements. Let then  $(J_j)_j$  be the connected components of  $M \setminus \mathcal{C}_k$ . Hence we have that for all index  $j$ ,  $J_j = (\inf J_j, \sup J_j)$  and  $\inf J_j, \sup J_j \in \mathcal{C}_k$ . Recall that in dimension one, the carré du champ operator takes the form  $\Gamma(f)(x) = a(x)f'(x)^2$  with a function  $a$  positive in the interior of  $M$ . So  $\mathcal{C}_k$  corresponds to the classical notion of critical points in the interior of  $M$ .

We now split off the integration by parts formula as follows:

$$\int_M \lambda_k(\nu) f_k g - \Gamma(f_k, g) d\nu = \sum_j \int_{J_j} \lambda_k(\nu) f_k g - \Gamma(f_k, g) d\nu$$

Taking  $g \in H^1(\nu)$  of the form  $g = \phi \circ f_k$  with  $\phi : I_k \rightarrow \mathbb{R}$ , the above expression becomes:

$$\sum_j \int_{J_j} \lambda_k(\nu) f_k \phi(f_k) - \Gamma(f_k) \phi'(f_k) d\nu$$

On each  $J_j$ ,  $f_k$  is injective since its derivative has constant sign by construction. Hence one can define  $h_j : I_k^j \rightarrow \mathbb{R}_+$  by

$$(4.16) \quad h_k^j(t) := \Gamma(f_k)((f_k|_{J_j})^{-1}(t))$$

where  $I_k^j := f_k(J_j)$ . So we get that on each  $J_j$ ,  $\Gamma(f_k) = h_j \circ f_k$ . Therefore it is possible to take the pushforward by  $f_k$  on each integral on  $J_j$ , and the integration by parts formula is transformed into

$$\sum_j \int_{I_k^j} \lambda_k(\nu) t \phi(t) - h_j(t) \phi'(t) d\nu_j$$

where  $\nu_j := (f_k|_{J_j})^\#(\nu)$  is the pushforward of  $\nu$  restricted to  $J_j$  by  $f_k$ . Note that  $\nu_j$  is not necessarily a probability distribution (it has a total mass  $\nu(J_j)$ ).



Using the reasoning above, Corollary 50 can be pushforwarded by  $f_k$  and turns into

$$(4.17) \quad \left| \sum_j \int_{I_k^j} \lambda_k(\nu) t \phi(t) - h_j(t) \phi'(t) d\nu_j(t) \right| \leq \left[ \sqrt{|\lambda_k(\mu) - \lambda_k(\nu)|} + \sum_{i=1}^{k-1} C_i d(f_k, Sp_i(\nu)^\perp) \right] \sqrt{\sum_j \int_{I_k^j} h_j \phi'^2 d\nu_j}$$

where  $h$  is defined in Formula (4.16) and  $\nu_j := (f_k)^\#_{|J_j}(\nu)$  is the pushforward of  $\nu$  restricted to  $J_j$  by  $f_k$ .

Let us emphasize an important feature: while the sets  $J_j$  are pairwise disjoint (by construction), their images  $I_k^j$  are not disjoint (because  $f_k$  is not necessarily injective). Hence the distributions  $\nu_j$  do not have disjoint supports in general.

#### 4.4.2 Implementing Stein's method

Inequality (4.17) will allow us to implement Stein's method. The difference with the usual method is that we will now implement Stein's method on each subinterval  $(I_k^j)_j$ , and not globally on  $I_k$ .

Since the Stein equation only depends on the target distribution  $\mu$ , the quantity  $\lambda_k(\nu)$  must not appear anymore on the left hand side of Inequation (4.17). That is why we begin with writing

$$\int_M \lambda_k(\mu) f_k g - \Gamma(g, f_k) d\nu = \int_M \lambda_k(\nu) f_k g - \Gamma(g, f_k) d\nu + (\lambda_k(\mu) - \lambda_k(\nu)) \int_M f_k g d\nu$$

But using the Cauchy-Schwarz inequality, the normalization condition (4.5) and the Poincaré inequality (4.7) with  $k = 1$ , we have

$$\left| \int_M f_k g d\nu \right| \leq \sqrt{\int_M f_k^2 d\nu} \sqrt{\int_M g^2 d\nu} \leq \frac{1}{\sqrt{\lambda_1(\nu)}} \sqrt{\int_M \Gamma(g) d\nu}$$

Hence we have

$$(4.18) \quad \left| \int \lambda_k(\mu) f_k g - \Gamma(g, f_k) d\nu \right| \leq \left[ \sqrt{|\lambda_k(\mu) - \lambda_k(\nu)|} + \frac{|\lambda_k(\mu) - \lambda_k(\nu)|}{\sqrt{\lambda_1(\nu)}} + \sum_{i=1}^{k-1} C_i d(f_k, Sp_i(\nu)^\perp) \right] \sqrt{\int \Gamma(g) d\nu}$$

Hence for each  $j$ ,  $\phi \mapsto \lambda_k(\mu) t \phi(t) - h_j(t) \phi'(t)$  is a good candidate to be a Stein operator on  $I_k^j$  for the probability distribution

$$\mu_j^* := \frac{1}{\mu(J_j)} (f_k)^\#_{|J_j}(\mu).$$

Let us point out that  $\mu_j^*$  corresponds to the pushforward by the eigenfunction  $f_k$  of the probability distribution  $\mu$  restricted to  $J_j$ . Our strategy is to implement Stein's method on each  $I_k^j$  and then use the approximate integration by parts formula (4.18) to deduce a more global result on  $I_k$ .

**Stein's method on  $I_k^j = (a_j, b_j)$**

On  $I_k^j$ , the probability measure  $\mu_j^*$  is invariant with respect to the diffusion process

$$L_j(\psi)(t) = h_j(t) \psi''(t) - \lambda_k(\mu) t \psi'(t), \quad \forall \psi \in \mathcal{C}^2(I_k^j).$$

As one can see with a classical integration by parts argument,  $\mu_j^*$  has therefore the following density with respect to the Lebesgue measure on  $I_k^j$ :

$$d\mu_j^*(t) = \frac{1}{Z_j h_j(t)} \exp\left(-\lambda_k(\mu) \int_{a_j}^t \frac{u}{h_j(u)} du\right) dt,$$

where  $Z_j$  is a normalization constant. Indeed, let  $\psi \in \mathcal{C}^2(I_k^j)$  be compactly supported. Then

$$\begin{aligned} & \int_{I_k^j} h_j(t) \psi''(t) \left( \frac{1}{h_j(t)} \exp\left(-\lambda_k(\mu) \int_{a_j}^t \frac{u}{h_j(u)} du\right) \right) dt \\ &= \left[ \psi'(t) \exp\left(-\lambda_k(\mu) \int_{a_j}^t \frac{u}{h_j(u)} du\right) \right]_{a_j}^{b_j} + \int_{I_k^j} \lambda_k(\mu) t \psi'(t) \left( \frac{1}{h_j(t)} \exp\left(-\lambda_k(\mu) \int_{a_j}^t \frac{u}{h_j(u)} du\right) \right) dt \\ &= \int_{I_k^j} \lambda_k(\mu) t \psi'(t) \left( \frac{1}{h_j(t)} \exp\left(-\lambda_k(\mu) \int_{a_j}^t \frac{u}{h_j(u)} du\right) \right) dt, \end{aligned}$$

We will then use  $h_j(t)\psi'(t) - \lambda_k(\mu) t \psi(t)$  as a Stein operator for  $\mu_j^*$  and the uniqueness of the invariant probability measure for the diffusion process with generator  $L(\psi)(t) = h_j(t)\psi''(t) - \lambda_k(\mu) t \psi'(t)$  allows us to conclude.

Let  $g_j : I_k^j \rightarrow \mathbb{R}$  be 1-Lipschitz, and  $\psi_j : I_k^j \rightarrow \mathbb{R}$  given by

$$(4.19) \quad \psi_j(t) := \exp\left(\lambda_k \int_{a_j}^t \frac{u}{h_j(u)} du\right) \int_{a_j}^t (g_j(y) - \mu_j^*(g_j)) \frac{1}{h_j(y)} \exp\left(-\lambda_k \int_{a_j}^y \frac{u}{h_j(u)} du\right) dy.$$

Then one can easily verify that  $\psi_j$  is solution of the Stein equation on  $I_k^j$ :

$$h_j \psi_j' - \lambda_k(\mu) t \psi_j = g_j - \int_{I_k^j} g_j d\mu_j^*.$$

The following estimate holds:

**Proposition 53.** [191, Proposition 18] *Let  $g_j : I_k^j \rightarrow \mathbb{R}$  be in  $C^1(I_k^j) \cap L^1(\mu_j^*)$ , and let  $\psi_j$  the associated solution (4.19). Then*

$$\|\sqrt{h_j} \psi_j'\|_\infty \leq C_{h_j} \|g_j'\|_\infty,$$

where

$$\begin{aligned} C_{h_j} := \sup_{t \in I_j} & \left[ \left| 1 - Z_j(1 - q_j(t)) \lambda_k(\mu) t \exp\left(\lambda_k(\mu) \int_{a_j}^t \frac{u}{h_j(u)} du\right) \right| \frac{1}{\sqrt{h_j(t)}} \int_{a_j}^t q_j(y) dy \right. \\ & \left. + \left| 1 + Z_j q_j(x) \lambda_k(\mu) t \exp\left(\lambda_k(\mu) \int_{a_j}^t \frac{u}{h_j(u)} du\right) \right| \frac{1}{\sqrt{h_j(t)}} \int_t^{b_j} (1 - q_j(y)) dy \right] \end{aligned}$$

and  $q_j$  is the cumulative distribution function of  $\mu_j^*$ .

The issue of finiteness of the constant  $C_{h_j}$  is addressed in the following variant of [191, Proposition 25].

**Proposition 54.** *Assume that one of the two following conditions is verified at  $a_j$ :*

- either  $a_j = -\infty$  and  $c_1 |t|^{2\alpha-2} \leq h_j(t) \leq c_2 |t|^\alpha$  for  $t \rightarrow -\infty$  with  $\alpha \leq 2$  and  $c_1, c_2 > 0$ ,
- or  $a_j > -\infty$  and  $c_1 (t - a_j)^2 \leq h_j(t) \leq c_2 (t - a_j)$  for  $t \rightarrow a_j^+$  with  $c_1, c_2 > 0$ ,

and one of these two conditions is satisfied at  $b_j$ :

- either  $b_j = +\infty$  and  $c_1 t^{2\alpha-2} \leq h_j(t) \leq c_2 t^\alpha$  for  $t \rightarrow +\infty$  with  $\alpha \leq 2$  and  $c_1, c_2 > 0$ ,
- or  $b_j < +\infty$  and  $c_1 (b_j - t)^2 \leq h_j(t) \leq c_2 (b_j - t)$  for  $t \rightarrow b^-$  with  $c_1, c_2 > 0$ .

Then the constant  $C_{h_j}$  defined in Proposition 53 is finite.

In order to ensure the finiteness of these constants  $C_{h_j}$ , we are lead to make the following assumption.

**Assumption 1.** All  $h_j$  satisfy the requirements of Proposition 54.

### Proof of Theorem 51

For any  $j$ , let  $\psi_j$  be the Stein solution given by (4.19) and define

$$\phi(x) := \sum_j \mathbf{1}_{J_j}(x) \psi_j(f_k(x)).$$

On the one hand,

$$\begin{aligned} \int_M \lambda_k(\mu) f_k \phi - \Gamma(\phi, f_k) d\nu &= \sum_j \int_{J_j} \lambda_k(\mu) f_k \psi_j(f_k) - \Gamma(f_k, \psi_j(f_k)) d\nu \\ &= \sum_j \int_{I_k^j} \lambda_k(\mu) t \psi_j(t) - h_j(t) \psi_j'(t) d\nu_j(t), \end{aligned}$$

and by construction,

$$\begin{aligned} \sum_j \int_{I_k^j} \lambda_k(\mu) t \psi_j(t) - h_j(t) \psi_j'(t) d\nu_j(t) &= \sum_j \int_{I_k^j} g_j(t) - \mu_j^*(g_j) d\nu_j(t) \\ &= \sum_j \nu_j(g_j) - \nu_j(I_k^j) \mu_j^*(g_j) \\ &= \sum_j \nu_j(I_k^j) \left( \nu_j^*(g_j) - \mu_j^*(g_j) \right), \end{aligned}$$

where

$$\nu_j^* := \frac{1}{\nu_j(I_k^j)} \nu_j$$

is now a probability distribution on  $I_k^j$ . This is the pushforward by the eigenfunction  $f_k$  of the probability distribution  $\nu$  restricted to  $J_j$ . Let us point out that  $\nu_j(I_k^j) = \nu(J_j)$ .

On the other hand,

$$\begin{aligned} \sum_j \int_{I_k^j} h_j(t) (\psi_j'(t))^2 d\nu_j(t) &= \sum_j \int_{J_j} h_j(f_k(x)) (\psi_j'(f_k(x)))^2 d\nu(x) \\ &= \sum_j \int_{J_j} \Gamma(\psi_j(f_k))(x) d\nu(x) \\ &= \int_M \Gamma(\phi)(x) d\nu(x), \end{aligned}$$

hence Proposition 53 gives

$$\int_M \Gamma(\phi)(x) d\nu(x) \leq \sum_j \nu_j(I_k^j) C_{h_j}^2 \leq \sum_j C_{h_j}^2.$$

So taking  $g = \phi$  in (4.18), one gets

$$\sup_{(g_j)_j} \left| \sum_j \nu_j(I_k^j) \left( \nu_j^*(g_j) - \mu_j^*(g_j) \right) \right| \leq C \left[ \sqrt{|\lambda_k(\mu) - \lambda_k(\nu)|} + \frac{|\lambda_k(\mu) - \lambda_k(\nu)|}{\sqrt{\lambda_1(\nu)}} + \sum_{i=1}^{k-1} C_i d(f_k, Sp_i(\nu)^\perp) \right]$$

where the supremum runs over all  $(\#\mathcal{C}_k)$ -uple of functions  $(g_j)_j$  with for all  $j$ ,  $g_j : I_k^j \rightarrow \mathbb{R}$  being 1-Lipschitz, and  $C := \sum_j C_{h_j}^2$ . Finally

$$\sup_{(g_j)_j} \left| \sum_j \nu_j^*(I_k^j) \left( \nu_j^*(g_j) - \mu_j^*(g_j) \right) \right| = \sum_j \nu_j(I_k^j) W_1(\nu_j^*, \mu_j^*).$$

Indeed, the inequality " $\leq$ " easily follows from the triangle inequality. To see the other direction " $\geq$ ", let  $\varepsilon > 0$  small enough and for all  $j$ , pick a 1-Lipschitz function  $g_j$  such that  $\mu_j^*(g_j) - \nu_j^*(g_j) \geq W_1(\nu_j^*, \mu_j^*) - \varepsilon$ . Then

$$\begin{aligned} \left| \sum_j \nu_j(I_k^j) \left( \mu_j^*(g_j) - \nu_j^*(g_j) \right) \right| &= \sum_j \nu_j(I_k^j) \left( \mu_j^*(g_j) - \nu_j^*(g_j) \right) \\ &\geq \sum_j \nu_j(I_k^j) \left( W_1(\nu_j^*, \mu_j^*) - \varepsilon \right) \\ &= \sum_j \nu_j(I_k^j) W_1(\nu_j^*, \mu_j^*) - \varepsilon. \end{aligned}$$

Letting  $\varepsilon$  go to zero concludes the proof.

## 4.5 Application to the Gaussian distribution

In this section, we consider the case of the one dimensional Ornstein-Uhlenbeck operator, where  $M = \mathbb{R}$ ,  $Lf = f'' - xf'$  and  $\mu = \gamma := \mathcal{N}(0, 1)$  is the equilibrium distribution. The carré du champ operator is  $\Gamma(f, g) = f'g'$ . The eigenvalues are all integers:  $\lambda_k(\gamma) = k$ , with multiplicity 1, and the associated normalized eigenfunctions are the Hermite polynomials  $H_k$ ,  $k \geq 1$  given by

$$H_k = \frac{1}{\sqrt{n!}} P_k$$

where  $P_0(x) = 1$ ,  $P_1(x) = x$  and

$$P_{n+1}(x) = xH_n(x) - nP_{n-1}(x).$$

Note that the polynomial  $H_0 := P_0 = 1$  (which is not centered) corresponds to the zero-th eigenvalue  $\lambda_0 = 0$ , so only the  $H_k$  with  $k \geq 1$  are relevant.

### 4.5.1 The second eigenvalue

The case of the second eigenvalue  $\lambda_2 = 2$ , and  $f_2(x) = \frac{1}{\sqrt{2}}(x^2 - 1)$ , is quite specific. Indeed, the pushforward measure  $f_2^\#(\gamma) = \frac{1}{\sqrt{2}}(\chi_2 - 1)$  corresponds to a translation of the Chi-2 distribution. Let us mention that Chi-2 approximation have been investigated through the tools of the Stein-Malliavin method in [8, 96].

Let us apply our result. One could directly apply Theorem 51: zero is the only critical point, so we can inverse  $f_2$  on the connected components  $\mathbb{R}_-$  and  $\mathbb{R}_+$  and deduce factorizations for  $\Gamma(f_2) = 2x^2$  on each of these connected components, allowing to implement Stein's method after taking the pushforward by  $f_2$ . One can then see that Assumption 1 is satisfied, and deduce the split stability estimate of Theorem 51.

However, in this case, though  $f_2$  is not injective,  $\Gamma(f_2)$  can anyway be globally factorized on  $\mathbb{R}$ . Actually,  $\Gamma(f_2)(x) = 2x^2 = 2\sqrt{2} \left( \frac{1}{\sqrt{2}}(x^2 - 1) + \frac{1}{\sqrt{2}} \right) = 2\sqrt{2} \left( f_2(x) + \frac{1}{\sqrt{2}} \right)$ . Hence  $h : [-\frac{1}{\sqrt{2}}, \infty) \rightarrow \mathbb{R}_+$  given by  $h(t) = 2\sqrt{2}(t + \frac{1}{\sqrt{2}})$  globally factorizes  $\Gamma(f_2)$ . So from the approximate integration by parts formula (4.14), instead of using the method of Section 4.4.1, we can proceed

as for the first eigenfunction and taking the global pushforward by  $f_2$  on all  $\mathbb{R}$ . The following Stein operator is obtained:

$$2t f(t) - 2\sqrt{2} \left( t + \frac{1}{\sqrt{2}} \right) f'(t) \quad \text{on} \quad \left( -\frac{1}{\sqrt{2}}, +\infty \right).$$

Let us underline that this generator of the  $f_2^\#(\gamma) = \frac{1}{\sqrt{2}}(\chi_2 - 1)$  distribution corresponds to the one used in [96]. One can see that this  $h$  satisfies the conditions required by Proposition 54. Indeed  $h$  is an affine function, so the vanishing rate condition at  $a = -\frac{1}{\sqrt{2}}$  is obviously satisfied, and  $\alpha = 1$  is a suitable choice for the growth condition at  $b = +\infty$ . Moreover, the normalization conditions (4.5) are here reduced to the following two normalizations on the moments of order 2 and 4:  $\nu$  is asked to have the same moments of order 2 and 4 than the standard normal distribution.

So the following is proven:

**Theorem 55.** *For all measure  $\nu$  on  $\mathbb{R}$  normalized as*

$$\int x^2 d\nu = 1, \quad \text{and} \quad \int x^4 d\nu = 3,$$

*and satisfying an improved Poincaré inequality with sharp constant  $\frac{1}{\lambda_2(\nu)}$ , it holds for some finite positive constant  $C > 0$ :*

$$W_1 \left( \frac{1}{\sqrt{2}}(\chi_2 - 1), \nu^* \right) \leq C \left[ \sqrt{|2 - \lambda_2(\nu)|} + \frac{|2 - \lambda_2(\nu)|}{\sqrt{\lambda_1(\nu)}} + C_\nu d \left( \frac{1}{\sqrt{2}}(x^2 - 1), Sp_1(\nu)^\perp \right) \right]$$

where  $\chi_2$  is the  $\chi_2$ -distribution on  $\mathbb{R}_+$ ,  $\nu^*$  is the pushforward of  $\nu$  by  $f_2 = \frac{1}{\sqrt{2}}(x^2 - 1)$ ,  $W_1$  is the 1-Wasserstein distance, the constant  $C_\nu$  is given by  $C_\nu = \sqrt{\lambda_2(\nu) - \lambda_1(\nu)} + \frac{\lambda_2(\nu) - \lambda_1(\nu)}{\sqrt{\lambda_1(\nu)}}$ , and  $d \left( \frac{1}{\sqrt{2}}(x^2 - 1), Sp_1(\nu)^\perp \right)$  quantifies the orthogonality error between  $\frac{1}{\sqrt{2}}(x^2 - 1)$  and the first eigenspace of  $\nu$ .

### 4.5.2 The $k$ -th eigenvalue, $k \geq 3$

As soon as  $k \geq 3$ , the global factorization  $\Gamma(f_k) = h \circ f_k$  does not hold anymore. We are therefore led to use the method explained in Section 4.4.1. Since

$$(4.20) \quad \forall n \geq 0, \quad H'_{n+1} = \sqrt{n+1} H_n,$$

one gets that the critical points are  $\mathcal{C}_k = \{H_{k-1} = 0\}$ . So the connected components of  $\mathbb{R} \setminus \mathcal{C}_k$  are the  $k$  nodal sets of  $H_{k-1}$ . The eigenfunction  $H_k$  is injective on each of the connected component,  $J_j$ , so  $\Gamma(H_k)$  factorizes as  $\Gamma(H_k) = h_j \circ H_k$ . Since  $\Gamma(H_k) = (H'_k)^2 = k H_{k-1}^2$  we get

$$h_j = k H_{k-1}^2 \circ (H_k)_{|J_j}^{-1}.$$

Let us show that these functions  $h_j$  satisfy Assumption 1. At an infinite boundary, since  $H_k(x) \sim \frac{x^k}{\sqrt{k!}}$ , we get  $h_j(t) \sim_C t^{2\frac{k-1}{k}}$ , where  $f \sim_C g$  means that  $\frac{f}{g}$  tends to a constant. Therefore  $\alpha = 2\frac{k-1}{k}$  is a suitable choice in Proposition 54. Let us now treat the case of a finite boundary. We start by showing the following fact for Hermite polynomials:

**Fact.** *For all  $n \geq 0$ , if  $x_0 \in \{\inf J_j, \sup J_j\}$  and  $y_0 := H_{n+1}(x_0)$ , then for  $y \in I_k^j$  close enough to  $y_0$ , one has*

$$\left| (H_{n+1})_{|J_j}^{-1}(y) - y_0 \right| \leq c\sqrt{y - y_0},$$

for some  $c > 0$ .

**Proof** Since  $\inf J_j$  and  $\sup J_j$  are critical points of  $H_{n+1}$ , this fact is equivalent to the fact that  $H_{n+1}$  is quadratic at the neighborhood of all of its critical points, i.e. there is some non zero  $c$  such that  $H_{n+1}(x) - H_{n+1}(x_0) = c(x - x_0)^2 + o(x - x_0)^2$  for all critical points  $x_0$ . To show this we are reduced to check that  $H_{n+1}''(x_0) \neq 0$ . But using again formula (4.20), this is true because two consecutive Hermite polynomials never have a common root.  $\square$

Then, since  $\Gamma(H_k) = k H_{k-1}^2$  and since the roots of Hermite polynomials have only multiplicity one, we deduce that at all finite boundaries of  $I_k^j$ , the function  $h_j$  vanishes at a linear rate. Finally the requirement of Proposition 54 is satisfied, so that Assumption 1 is satisfied. Therefore in this case the normalization conditions (4.5) take the form

$$\int H_k d\nu = 0, \quad \int H_k^2 d\nu = 1 \text{ and } \int H_{k-1}^2 d\nu = 1.$$

Unlike what happened for the first and the second eigenfunctions, these three conditions can not be reduced to two. For example, for  $k = 3$ , they are:

$$\int (x^3 - 3x) d\nu = 0, \quad \int (x^6 - 6x^4 + 9x^2) d\nu = 6 \text{ and } \int (x^4 - 2x^2) d\nu = 1.$$

The requirements of Theorem 51 being satisfied, we can then apply it and get the following stability result for higher order eigenvalues of the one dimensional normal distribution.

**Theorem 56.** *Let  $k \geq 3$  and  $H_k$  (resp.  $H_{k-1}$ ) be the  $k$ -th (resp.  $(k-1)$ -th) Hermite polynomial. Then for all probability measures  $\nu$  on  $M$  normalized as*

$$\int H_k d\nu = 0, \quad \int H_k^2 d\nu = 1 \text{ and } \int H_{k-1}^2 d\nu = 1,$$

and satisfying the improved Poincaré inequalities (4.7), it holds for some finite constant  $C > 0$ :

$$\sum_j \nu(J_j) W_1(\nu_j^*, \gamma_j^*) \leq C \left[ \sqrt{|k - \lambda_k(\nu)|} + \frac{|k - \lambda_k(\nu)|}{\sqrt{\lambda_1(\nu)}} + \sum_{i=1}^{k-1} C_i d(H_k, Sp_i(\nu)^\perp) \right]$$

where  $(J_j)_j$  are the connected components of the complementary of critical points of  $H_k$ ,  $\nu_j^*$  (resp.  $\gamma_j^*$ ) is the pushforward of  $\nu$  (resp.  $\gamma$ ) restricted to  $H_k(J_j)$ , constants  $C_i$  are given by

$$C_i = \sqrt{\lambda_k(\nu) - \lambda_i(\nu)} + \frac{\lambda_k(\nu) - \lambda_i(\nu)}{\sqrt{\lambda_i(\nu)}},$$

and  $d(H_k, Sp_i(\nu)^\perp)$  is defined in Remark 1 and quantifies the orthogonality error between  $H_k$  and eigenspaces of lower orders of  $\nu$ .

## 4.6 Application to Gamma distributions

In this section, we consider the case of the Laguerre operator, where  $M = \mathbb{R}_+$ ,  $Lf = xf'' + (s - \frac{x}{\theta})f'$  and  $\mu$  is the  $\Gamma(s, \theta)$  distribution on  $\mathbb{R}_+$  given by the density  $d\mu(x) = \frac{x^{s-1} e^{-\frac{x}{\theta}}}{\Gamma(s)\theta^s} \mathbf{1}_{\mathbb{R}_+}$ , where  $\Gamma$  denotes the Euler  $\Gamma$  function. The carré du champ operator is  $\Gamma(f, g) = x f' g'$ . The eigenvalues are  $\lambda_k(\mu) = \frac{k}{\theta}$ , with multiplicity 1, and the associated normalized eigenfunctions are the generalized normalized Laguerre polynomials  $L_{k,s}$  given by

$$(4.21) \quad L_{k,s}(x) = \sqrt{\frac{k! \Gamma(s)}{\Gamma(k+s)}} l_{k,s}\left(\frac{x}{\theta}\right)$$

where  $l_{0,s}(x) = 1$ ,  $l_{1,s}(x) = s - x$ , and

$$\forall n \geq 1, \quad l_{n+1,s}(x) = \left(2 + \frac{s-2-x}{n+1}\right) l_{n,s}(x) - \left(1 + \frac{s-2}{n+1}\right) l_{n-1,s}(x).$$

Moreover,

$$(4.22) \quad L'_{n,s}(x) = \frac{-1}{\theta} \sqrt{\frac{n}{s}} L_{n-1,s+1}(x).$$

So the critical points  $\mathcal{C}_k = \{x \in \mathbb{R}_+ \mid \Gamma(L_{k,s})(x) = 0\}$  of  $L_{k,s}$  are the zeros of  $L_{k-1,s+1}$  and 0. This means that there are  $k$  connected components  $J_j$  of  $\mathbb{R}_+ \setminus \mathcal{C}_k$ , which are the nodal sets of  $L_{k-1,s+1}$ .

The eigenfunction  $L_{k,s}$  is injective on each connected component  $J_j$ , so  $\Gamma(L_{k,s})$  factorizes as  $\Gamma(L_{k,s}) = h_j \circ L_{k,s}$ . Since  $\Gamma(L_{k,s})(x) = x (L'_{k,s}(x))^2 = \frac{kx}{s\theta^2} (L_{n-1,s+1}(x))^2$  we get

$$(4.23) \quad h_j(t) = \frac{k}{s\theta^2} (L_{k,s}|_{J_j})^{-1}(t) (L_{n-1,s+1} \circ (L_{k,s}|_{J_j})^{-1}(t))^2.$$

Let us show that these functions  $h_j$  satisfy Assumption 1. At  $+\infty$ , since  $L_{k,s}(x) \sim \frac{1}{\theta^k} \sqrt{\frac{n! \Gamma(s)}{\Gamma(n+s)}} x^k$ , we get

$$h_j(t) \sim_C t^{\frac{1}{k} + \frac{2(k-1)}{k}} = t^{2 - \frac{1}{k}}.$$

Therefore  $\alpha = 2 - \frac{1}{k}$  is a suitable choice for  $\alpha$  in Proposition 54. Let us treat now the case of a finite boundary. In the same way as for Hermite polynomials, we can see the following.

**Fact.** *Let  $n \geq 0$ , let  $x_0 \in \{\inf J_j, \sup J_j\}$  such that  $L'_{n,s}(x_0) = 0$ , and set  $y_0 := L_{n,s}(x_0)$ . Then for  $y \in I_k^j$  close enough to  $y_0$ , one has*

$$\left| (L_{n,s}|_{J_j})^{-1}(y) - y_0 \right| \leq c \sqrt{y - y_0},$$

for some  $c > 0$ .

**Proof** This fact is equivalent to the fact that  $L_{n,s}$  is quadratic at the neighborhood of all of its critical points, i.e. there is some non zero  $c$  such that  $L_{n,s}(x) - L_{n,s}(x_0) = c(x - x_0)^2 + o(x - x_0)^2$  for all critical points  $x_0$ . To show this we are reduced to check that  $L''_{n,s}(x_0) \neq 0$ . But using Formula (4.22), this would imply that  $x_0$  is a root of  $L_{n-1,s+1}$  with multiplicity at least two. However, all Laguerre polynomials only have roots with multiplicity one. So the fact is proven.  $\square$

Using the above fact and Formulas (4.22) and (4.23), we get that at every finite boundary of  $I_k^j$ , the function  $h_j$  vanishes at a linear rate. Hence Assumption 1 is satisfied.

Moreover in this case the normalization conditions (4.5) takes the form

$$\int L_{k,s} d\nu = 0, \quad \int L_{k,s}^2 d\nu = 1 \quad \text{and} \quad \int x (L_{k-1,s+1}(x))^2 d\nu = s\theta,$$

where  $\Gamma$  is the Euler Gamma function, and ! denotes the factorial. For  $k = 1$  these three conditions reduced to only two, but as soon as  $k \geq 2$  it is not the case anymore. For example, if  $k = 2$ , and  $s = \theta = 1$  (in that case  $\mu$  is the exponential distribution), then these three conditions are:

$$\int \frac{x^2}{2} - 2x d\mu = 1, \quad \int \frac{x^4}{4} - 2x^3 + 5x^2 - 4x d\mu = 0, \quad \text{and} \quad \int x^3 - 4x^2 + 4x d\mu = 2.$$

The requirements of Theorem 51 being satisfied, we can then apply it and get the following stability result for higher order eigenvalues of the Gamma distributions  $\Gamma(s, \theta)$  on  $\mathbb{R}_+$ .

**Theorem 57.** *Let  $k \geq 1$ ,  $s > 0$ ,  $\theta > 0$ , and  $L_{k,s}$  be the Laguerre polynomials defined in (4.21). Then for all probability measures  $\nu$  on  $\mathbb{R}_+$  normalized with*

$$\int L_{k,s} d\nu = 0, \quad \int L_{k,s}^2 d\nu = 1 \quad \text{and} \quad \int x (L_{k-1,s+1}(x))^2 d\nu = s\theta,$$

where  $\Gamma$  is the Euler Gamma function, and satisfying the improved Poincaré inequalities (4.7), it holds for some finite constant  $C > 0$ :

$$\sum_j \nu_j(I_k^j) W_1(\nu_j^*, \mu_j^*) \leq C \left[ \sqrt{\left| \frac{k}{\theta} - \lambda_k(\nu) \right|} + \frac{\left| \frac{k}{\theta} - \lambda_k(\nu) \right|}{\sqrt{\lambda_1(\nu)}} + \sum_{i=1}^{k-1} C_i d(L_{k,s}, Sp_i(\nu)^\perp) \right]$$

where  $(I_k^j)_j$  are the images by  $L_{k,s}$  of the connected components of the complementary of its critical points,  $\nu_j^*$  (resp.  $\mu_j^*$ ) is the pushforward of  $\nu$  (resp.  $\mu$ ) restricted to  $I_k^j$ , constants  $C_i$  are given by

$$C_i = \sqrt{\lambda_k(\nu) - \lambda_i(\nu)} + \frac{\lambda_k(\nu) - \lambda_i(\nu)}{\sqrt{\lambda_i(\nu)}},$$

and  $d(L_{k,s}, Sp_i(\nu)^\perp)$  is defined in Remark 1.

## 4.7 Application to $\beta\left(\frac{N}{2}, \frac{N}{2}\right)$ distributions

In this section, we consider the case of the Jacobi operator, where  $M = [-1, 1]$ ,

$$Lf(x) = (1 - x^2)f''(x) - Nx f'(x),$$

and  $\mu$  is the  $\beta\left(\frac{N}{2}, \frac{N}{2}\right)$  distribution on  $[-1, 1]$  given by the density

$$d\mu(x) = \frac{1}{Z} (1 - x^2)^{\frac{N}{2}-1} dx,$$

where

$$Z = 2^{2-2N} \pi \frac{\Gamma(d-1)}{\left(\frac{d-1}{2}\right) \left(\Gamma\left(\frac{d-1}{2}\right)\right)^2}$$

is the normalization constant and  $\Gamma$  denotes the Euler function. The carré du champ operator is given by  $\Gamma(f, g)(x) = (1 - x^2)^{\frac{N}{2}-1} f'(x)g'(x)$ . The eigenvalues are  $\lambda_k(\mu) = k(k + N - 1)$ , with multiplicity one, and the associated normalized eigenfunctions are the normalized Gegenbauer polynomials (see [204]) given by

$$(4.24) \quad G_{N,k}(x) = \binom{k + N - 2}{k}^{-1} \frac{2k + N - 1}{N - 1} P_{N,k}(x),$$

where  $P_{N,0}(x) = 1$ ,  $P_{N,1}(x) = (N - 1)x$ , and

$$P_{N,k}(x) = \frac{2x}{k} \left( k + \frac{N - 3}{2} \right) P_{N,k-1}(x) - \frac{1}{k} (k + N - 3) P_{N,k-2}(x).$$

The Jacobi operator corresponds to the Laplace-Beltrami operator on the sphere  $\mathbb{S}^N$  projected on one coordinate and normalized to stay in  $[-1, 1]$ . The Gegenbauer polynomials are particular case of Jacobi polynomials, when the two parameters of Jacobi polynomials are equals.



### 4.7.1 The second eigenvalue

Similarly to the case of the Normal distribution, the global factorization condition of the carré du champ is satisfied for  $k = 1$  and  $k = 2$ . Indeed, the second Gegenbauer polynomial is

$$G_{N,2}(x) = \frac{1}{2} \binom{N}{2}^{-1} (N+3) \left( (N+1)x^2 - 1 \right),$$

so we can compute

$$\Gamma(G_{N,2})(x) = K^2(1-x^2)x^2 = K^2 \left[ 1 - \frac{1}{N+1} \left( \frac{1}{K} G_{N,2}(x) + 1 \right) \right] \frac{1}{N+1} \left( \frac{1}{K} G_{N,2}(x) + 1 \right),$$

where  $K = \binom{N}{2}^{-1} (N+3)(N+1)$ . Hence  $\Gamma(G_{N,2})(x) = h(G_{N,2}(x))$ , with

$$h(t) := \frac{K}{N+1} \left( \frac{N}{N+1} - \frac{t}{K(N+1)} \right) \left( \frac{t}{K} + 1 \right).$$

As in the Gaussian case, this is due to the fact that the only critical point is zero, and the carré du champ is symmetric. This  $h$  satisfies the vanishing rate requirements (it vanishes at linear speed), so we have the following stability result.

**Theorem 58.** *For all measure  $\nu$  on  $\mathbb{R}$  satisfying*

$$\int x^2 d\nu = \frac{1}{N+1}, \quad \text{and} \quad \int x^4 d\nu = \frac{1}{N+1} \left[ 4 \binom{N}{2}^2 (N+3)^{-2} + 1 \right]$$

and an improved Poincaré inequality with sharp constant  $\frac{1}{\lambda_2(\nu)}$ , it holds for some finite positive constant  $C > 0$  that

$$\begin{aligned} & W_1 \left( L \left( (N+1)\beta_N^2 - 1 \right), \nu^* \right) \\ & \leq C \left[ \sqrt{|2(N+1) - \lambda_2(\nu)|} + \frac{|2(N+1) - \lambda_2(\nu)|}{\sqrt{\lambda_1(\nu)}} + C_\nu d \left( L \left( (N+1)\beta_N^2 - 1 \right), Sp_1(\nu)^\perp \right) \right] \end{aligned}$$

where  $\beta_N$  is the  $\beta \left( \frac{N}{2}, \frac{N}{2} \right)$ -distribution on  $[-1, 1]$ ,  $\nu^*$  is the pushforward of  $\nu$  by  $f_2 = L((N+1)x^2 - 1)$ , the constant  $L$  is given by  $L = \binom{N}{2}^{-1} (N+3)$ , the constant  $C_\nu$  is given by  $C_\nu = \sqrt{\lambda_2(\nu) - \lambda_1(\nu)} + \frac{\lambda_2(\nu) - \lambda_1(\nu)}{\sqrt{\lambda_1(\nu)}}$ , and  $d \left( L \left( (N+1)\beta_N^2 - 1 \right), Sp_1(\nu)^\perp \right)$  quantifies the orthogonality error between  $f_2$  and the first eigenspace of  $\nu$ .

### 4.7.2 The $k$ -th eigenvalue, $k \geq 3$

As soon as  $k \geq 3$ , the global factorization does not hold anymore. So we apply the general method presented in Section 4.4.1. We have

$$(4.25) \quad G'_{N,k}(x) = \binom{k+N-1}{k-1} \binom{k+N-2}{k}^{-1} (N+1) G_{N+2,k-1}(x).$$

So the critical points  $\mathcal{C}_k = \{x \in \mathbb{R}_+ \mid \Gamma(G_{N,k})(x) = 0\}$  of  $G_{N,k}$  are the zeros of  $G_{N+2,k-1}$  and  $-1$  and  $1$ . This means that there are  $k$  connected components  $J_j$  of  $[-1, 1] \setminus \mathcal{C}_k$  which are the nodal sets of  $G_{N+2,k-1}$ . The eigenfunctions  $G_{N,k}$  are injective on each of this connected component  $J_j$ , so  $\Gamma(G_{N,k})$  factorizes as  $\Gamma(G_{N,k}) = h_j \circ G_{N,k}$ . Since  $\Gamma(G_{N,k}) = C(1-x^2)(G_{N+2,k-1}(x))^2$  where

$$C = \binom{k+N-1}{k-1}^2 \binom{k+N-2}{k}^{-2} (N+1)^2,$$

we get

$$(4.26) \quad h_j(t) = C \left(1 - (G_{N,k|J_j}^{-1}(t))^2\right) \left(G_{N+2,k-1} \circ G_{N,k|J_j}^{-1}(t)\right)^2.$$

In order to Theorem 51 to apply, we have to verify that these functions  $h_j$  satisfy Assumption 1. Gegenbauer polynomials do not vanish at  $-1$  and  $1$ , so the rate at which  $h_j$  vanishes at boundaries  $G_{N,k}(-1)$  and  $G_{N,k}(1)$  is linear. We have then to treat the case of critical points in the interior of  $[-1, 1]$ . The reasoning is the same as for Laguerre polynomials: since Gegenbauer polynomials have only roots of multiplicity one, by a Taylor expansion we see that the functions  $h_j$  vanishes at boundaries of their domains of definition with linear rate.

Moreover, in this case, the normalization conditions (4.5) take the form

$$\int G_{N,k} d\mu = 0, \quad \int G_{N,k}^2 d\mu = 1,$$

$$\text{and} \quad \int (1-x^2) G_{N+2,k-1}^2 d\mu = \binom{k+N-1}{k-1}^{-1} \binom{k+N-2}{k} \frac{k(k+N-1)}{N+1}.$$

Finally, Theorem 51 can be applied, and one gets the following.

**Theorem 59.** *Let  $N > 1$ ,  $\mu$  be the  $\beta\left(\frac{N}{2}, \frac{N}{2}\right)$  distribution on  $[-1, 1]$ , and  $G_{N,k}$  be the Gegenbauer polynomials defined in (4.24). Then for all probability measures  $\nu$  on  $[-1, 1]$  normalized such that*

$$\int G_{N,k} d\mu = 0, \quad \int G_{N,k}^2 d\mu = 1,$$

$$\text{and} \quad \int (1-x^2) G_{N+2,k-1}^2 d\mu = \binom{k+N-1}{k-1}^{-1} \binom{k+N-2}{k} \frac{k(k+N-1)}{N+1},$$

and satisfying the improved Poincaré inequalities (4.7), it holds for some finite constant  $C_\beta > 0$ :

$$\sum_j \nu_j(I_k^j) W_1(\nu_j^*, \mu_j^*) \leq C_\beta \left[ \sqrt{|k(k+N-1) - \lambda_k(\nu)|} + \frac{|k(k+N-1) - \lambda_k(\nu)|}{\sqrt{\lambda_1(\nu)}} + \sum_{i=1}^{k-1} C_i d(L_{k,s}, Sp_i(\nu)^\perp) \right]$$

where  $(I_k^j)_j$  are the images by  $G_{N,k}$  of the connected components of the complementary of its critical points,  $\nu_j^*$  (resp.  $\mu_j^*$ ) is the pushforward of  $\nu$  (resp.  $\mu$ ) restricted to  $I_k^j$ , constants  $C_i$  are given by

$$C_i = \sqrt{\lambda_k(\nu) - \lambda_i(\nu)} + \frac{\lambda_k(\nu) - \lambda_i(\nu)}{\sqrt{\lambda_i(\nu)}},$$

and  $d(G_{N,k}, Sp_i(\nu)^\perp)$  is defined in Remark 1 and quantifies the orthogonality error between  $G_{N,k}$  and eigenspaces of lower orders of  $\nu$ .



# Chapter 5

## Stability estimates for the sharp spectral gap bound under a curvature-dimension condition

This chapter was written in collaboration with Ivan Gentil and Max Fathi, and is taken from the paper [89].

### 5.1 Introduction

Our goal in this work is to study the stability of sharp spectral gap bounds for Markov diffusion operators  $\mathcal{L}$  satisfying a curvature lower bound. Formally (and rigorously in the smooth setting), these operators satisfy both the Bochner inequality, (or curvature-dimension condition  $\text{CD}(K, N)$ ),

$$(5.1) \quad \frac{1}{2}\mathcal{L}\Gamma(f) - \Gamma(f, \mathcal{L}f) \geq K\Gamma(f) + \frac{1}{N}(\mathcal{L}f)^2$$

for all smooth functions, and the diffusion property: for any smooth bounded function  $\phi$

$$(5.2) \quad \mathcal{L}\phi(f) = \phi'(f)\mathcal{L}f + \phi''(f)\Gamma(f).$$

This definition has been proposed in the seminal paper [12] by the way of the operator  $\Gamma_2(f) := \frac{1}{2}\mathcal{L}\Gamma(f) - \Gamma(f, \mathcal{L}f)$ .

An alternative definition of the curvature-dimension condition (now known as the Lott-Sturm-Villani theory [145, 201, 202]) is to require geodesic convexity properties of the entropy in the space of probability measures. In the setting of smooth Riemannian manifolds endowed with a reference probability measure, the curvature-dimension condition is equivalent to a lower bound on the weighted Ricci curvature tensor. However, in the non-smooth setting, requiring the inequality (5.1) to hold pointwise is slightly too strong. We shall be interested in so-called RCD spaces, that satisfy the diffusion property, as well as a weak, integral form of the curvature-dimension condition [5, 83]. This definition is equivalent to the more classical definition via convexity properties of the entropy functional as in the Lott-Sturm-Villani theory when the space is infinitesimally Hilbertian.

For a smooth  $N$ -dimensional Riemannian manifold endowed with its Laplacian, if the Ricci curvature tensor is bounded from below by  $N - 1$  (which implies the  $\text{CD}(N - 1, N)$  condition), we have a lower bound on the spectral gap, or first positive eigenvalue of the Laplacian:

$$(5.3) \quad \lambda_1(-\Delta) \geq N.$$

This bound is sharp, since equality holds for the  $N$ -sphere of radius 1. Moreover, this bound is *rigid*: equality holds iff the manifold is isometric to the  $N$ -sphere [172]. This phenomenon is related to rigidity results for other sharp bounds, such as on the diameter or the volume.

The next question is whether the bound is stable: if a manifold with Ricci curvature bounded from below by  $N - 1$  has a spectral gap close to  $N$ , is it close in some sense to the  $N$ -sphere? The answer to this question is negative [6]. To get closeness in Gromov-Hausdorff distance, Aubry [9] showed that we must ask that the  $N$ -th eigenvalue is close to  $N$ , which is true for the sphere, since the multiplicity of  $N$  as an eigenvalue is  $N + 1$ , and that  $\lambda_{N-1}$  being close to one is not enough. This improved on an earlier result of Petersen [176] who showed that  $\lambda_{N+1} \approx N$  suffices. The reason for this phenomenon is that smooth manifolds are not quite the right setting for the problem: it is possible to extend the notion of Ricci curvature lower bounds to non-smooth weighted manifolds, and in that setting there are spaces other than the  $N$ -sphere for which equality holds in (5.3), namely spherical suspensions. We refer to [124] for their description.

When it is only  $\lambda_1$  that is close to one, Cheng [71] showed that the diameter is close to  $\pi$ , and Croke [77] proved the converse statement, still in the smooth, unweighted setting. Later, Bertrand [27] showed that  $\lambda_k \approx N$  for  $k \leq n$  implies that the manifold contains a piece that is close to  $\mathbb{S}^k$ .

We shall now discuss the literature in the non-smooth setting of RCD spaces. This curvature condition, that we shall present in details in Section 5.2, extends Ricci curvature bounds, and can be introduced using either the Bakry-Ledoux gradient estimates, a weak form of the Bochner inequality or convexity properties of the entropy along geodesics. This setting also makes sense when  $N$  is not an integer. The sharp spectral gap estimate for  $\text{RCD}(N - 1, N)$  spaces was proved in [83], and cases of equality were fully described in [124, 123]. More recently, Cavaletti and Mondino tackled rigidity and stability results for geometric comparison theorems in the non-smooth setting using the so-called needle decomposition, a technique pioneered by Klartag [126] to reduce such problems to one-dimensional statements. Most relevant to the present work is the quantitative obata theorem of [62], which states that on an essentially non-branching  $\text{CD}(N - 1, N)$  space  $(M, d, \mu)$  with spectral gap  $\lambda$  and associated normalized eigenfunction  $u$  there exists a point  $x_0 \in M$  such that

$$\|u - \sqrt{N + 1} \cos(d(\cdot, x_0))\|_2 \leq C(N)(\lambda_1 - N)^{1/(8N+4)}; \quad \pi - \text{diam}(M) \leq C(N)(\lambda_1 - N)^{1/N}.$$

A variant of the diameter estimate was obtained in [124] in the RCD setting, still with a dimension-dependent exponent. We shall include a comparison between the technical estimates on the eigenfunction used in [62] and those used here in Section 5.3.1. The main differences between Theorem 60 below and the results of [62], beyond the norms used being different, is that on the upside we get a dimension-independent exponent in our main estimate, but with the downside of requiring the RCD condition rather than the more general CD condition. Topological sphere theorems were also considered in the RCD setting, in [115].

Our main result is a sharp quantitative estimate on how far the distribution of the pushforward measure by an eigenfunction is of being a beta distribution. More precisely, we show that

**Theorem 60.** *Let  $(M, d, \mu)$  be an  $\text{RCD}(N - 1, N)$  space with  $N > 1$ , unit mass and spectral gap  $\lambda_1 \leq N + \varepsilon$  for some  $\varepsilon > 0$ . Let  $f$  be an eigenfunction of the Laplacian, with eigenvalue  $\lambda_1$ , and normalized so that  $\|\Gamma(f)\|_1 = N/(N + 1)$ . There is a constant  $C(N) > 0$  (independent of  $M$  and  $f$ ) such that the  $L^1$ -Wasserstein distance between the pushforward of  $\mu$  by  $f$  and a symmetrized Beta distribution with parameters  $(N/2, N/2)$  is smaller than  $C(N)\varepsilon$ .*

The order of magnitude  $\varepsilon$  in the bound is *sharp*, as can be checked by considering an  $N$ -sphere of radius  $1 - \varepsilon$ . The choice of the value of the normalization for  $\|\Gamma(f)\|_1$  is to match the value for coordinates on a unit sphere in dimension  $N$ , and without the pushforward would simply be close to a scaled Beta distribution.

Symmetrized Beta distributions (with parameters  $(N/2, N/2)$ ) have densities proportional to  $(1 - x^2)^{N/2-1}$  on  $[-1, 1]$ . They appear in this statement because they are precisely the distribution of coordinates on unit spheres, so our statement can be viewed as saying that the

space contains a piece that is close to a piece of the  $N$ -sphere (when the parameter  $N$  is an integer of course).

Our method follows an approach developed in [28] for proving stability of the sharp spectral gap estimate on  $\text{RCD}(1, \infty)$  spaces, where the model space is the Gauss space. It combines quantitative estimates on eigenfunctions with Stein's method for comparing probability distributions via approximate integration by parts formulas. We shall actually sharpen the quantitative bound of order  $\varepsilon^{1/2}$  of [28, Theorem 1.3] in the  $\text{RCD}(1, \infty)$  setting to  $\varepsilon \log(1/\varepsilon)$ , which is within a logarithmic factor of being sharp.

An important tool in our analysis is a new  $L^1$ -functional inequality (Proposition 66). It arises as a limit case of a family of  $L^p$ -functional inequalities introduced by Meyer [156] and more recently revisited in [155, 1] in the Gaussian setting for  $p > 1$ . As we shall see, the  $L^1$ -inequality fails in the Gaussian setting, but holds for  $\text{RCD}(N-1, N)$  spaces. Another new result of independent interest (Theorem 67) is a general criterion for proving that an eigenfunction has a distribution close to a Beta distribution, which is a variant of a result of E. Meckes [153] for the Gaussian setting.

As noted in [189, 173, 131, 14], the definition of the Bakry-Emery condition also makes sense for *negative* values of  $N$ . Sharp functional inequalities and model spaces in this setting were studied for example in [160]. Rigidity in the smooth setting was studied in [148]. Under this curvature condition, the manifold may have infinite volume, so we shall require finite volume as an extra condition. Then the one-dimensional model space is  $\mathbb{R}$  endowed with a non-constant metric and a generalized Cauchy distribution [97]. We shall derive a stability estimate similar to that of the  $\text{RCD}(1, \infty)$  case, under an additional integrability condition on the eigenfunction, as well as assuming smoothness. To do so, we derive a version of Stein's lemma for one-dimensional generalized Cauchy distributions, of independent interest. We shall also give a Cauchy counterpart to E. Meckes' theorem on Gaussianity of eigenfunctions we previously mentioned.

The sequel is as follows: in Section 5.2, we shall present background results on  $\text{RCD}$  spaces, and a description of the model spaces for both positive and negative values of the dimension parameter. Section 5.3 shall contain the proof of our main Theorem, while Sections 5.4 and 5.5 shall respectively contain the results on the infinite-dimensional case and the negative-dimensional case.

## 5.2 Preliminaries on the setting and curvature-dimension condition

We briefly explain in next sections what we call *smooth space* and *metric measure space*.

### 5.2.1 Some generalities on smooth setting

We briefly describe here the smooth setting, a complete description can be found in [13, Sec. 3.2].

The smooth setting that we consider in this paper, other possibilities are available in the literature, is a smooth weighted manifold. Let  $(M, g)$  be a  $(\mathcal{C}^\infty)$  connected and complete  $d$ -dimensional Riemannian manifold ( $d \geq 1$ ). Let  $W$  be a smooth function on  $M$  and let  $\mu = e^{-W} \text{Vol}$  be the reference measure where  $\text{Vol}$  is the Riemannian measure. We assume in this article that  $\mu$  is a probability measure and it is the case under the curvature-dimension condition used here.

The generator is described on smooth functions  $f \in \mathcal{C}_c^\infty(M)$  by

$$\mathcal{L}(f) = \Delta_g f - \Gamma(W, f),$$

where  $\Delta_g$  is the Laplace-Beltrami operator and  $\Gamma$  is the associated carré du champ operator. The Markov semigroup with generator  $\mathcal{L}$  is noted  $(P_t)_{t \geq 0}$ . For any smooth functions  $f, h$ , we

have

$$\Gamma(f, h) = \langle \nabla f, \nabla h \rangle_g = \nabla f \cdot \nabla h,$$

that is scalar product associated with the metric  $g$ .

The weighted manifold  $(M, g)$  associated with the measure  $\mu$  satisfies a  $\text{CD}(K, N)$  condition whenever the inequality (5.1) is satisfied for all smooth function  $f$ . Following for instance [13, Sec. C.6], if  $N \in \mathbb{R} \setminus [0, d]$  then the curvature-dimension condition  $\text{CD}(K, N)$  is equivalent to the following inequality on tensors,

$$(5.4) \quad \text{Ric}_g - \nabla \nabla^g W \geq Kg + \frac{1}{N-d} \nabla W \otimes \nabla W,$$

where  $\text{Ric}_g$  is the Ricci tensor of  $(M, g)$  and  $\nabla \nabla^g W$  is the Hessian of  $W$  with respect to the metric  $g$ .

### 5.2.2 Some generalities on metric measure spaces

We consider a complete, separable, metric measure space  $(M, d, \mu)$ , where  $\mu$  is a probability measure. We can define the Cheeger energy of an  $L^2$ -function  $f$  as

$$\text{Ch}(f) := \frac{1}{2} \inf_{(f_i)_{i \in \mathbb{N}}} \liminf_{i \rightarrow \infty} \int (\text{Lip } f_i)^2 d\mu$$

where the infimum runs over all sequences of locally-Lipschitz functions converging to  $f$  in  $L^2$ , and  $\text{Lip } f(x)$  is the local Lipschitz constant at  $x$ . If an  $L^2$ -function  $f$  has finite Cheeger energy, then there exists a minimal weak upper gradient, which we shall denote as  $\Gamma(f)^{1/2}$ , such that

$$\text{Ch}(f) = \frac{1}{2} \int \Gamma(f) d\mu.$$

The Sobolev space  $W^{1,2}(M, d, \mu)$  is the space of  $L^2$ -functions with finite Cheeger energy. We refer to [3, 5] for more about these notions. For smooth functions on a Riemannian manifold,  $\Gamma(f)$  coincides with  $|\nabla f|^2$ .

The space is said to be infinitesimally Hilbertian if the Cheeger energy is quadratic, that is

$$\text{Ch}(f + g) + \text{Ch}(f - g) = 2 \text{Ch}(f) + \text{Ch}(g)$$

for all  $f, g \in W^{1,2}$ . Smooth manifolds are of course infinitesimally Hilbertian spaces, but there are examples of spaces, such as Finsler spaces, that do not satisfy this condition.

If the space is infinitesimally Hilbertian, we can define the scalar product  $\Gamma(f, g) = \langle \nabla f, \nabla g \rangle \in L^1$  of two elements of  $W^{1,2}$  by polarization of  $\Gamma$ , as well as the Dirichlet form  $\mathcal{E}(f, g) = \int \Gamma(f, g) d\mu$ . The analogue of the Laplace operator for the space  $(M, d, \mu)$  is then the operator  $\mathcal{L} : D(\mathcal{L}) \rightarrow L^2$  such that

$$\mathcal{E}(f, g) = - \int g(\mathcal{L} f) d\mu.$$

The domain  $D(\mathcal{L})$  of the operator is dense in  $L^2$ . The associated Markov semigroup is also noted  $(P_t)_{t \geq 0}$ . We refer to [4] for background about this construction. We can then define RCD spaces as follows:

**Definition 61** (RCD( $K, N$ ) spaces). *A complete, separable, metric-measure space  $(M, d, \mu)$  is said to be an RCD( $K, N$ ) space with  $K \in \mathbb{R}$  and  $N \in \mathbb{R} \setminus [0, 1]$  if it is infinitesimally Hilbertian and if it satisfies the Bochner inequality in a weak form: for any  $f \in D(\mathcal{L})$  with  $\mathcal{L} f \in W^{1,2}(M, d, \mu)$  and  $g \in D(\mathcal{L}) \cap L^\infty(\mu)$  with  $g \geq 0$  and  $\mathcal{L} g \in L^\infty$  we have*

$$(5.5) \quad \frac{1}{2} \int \mathcal{L} g \Gamma(f) d\mu - \int g \Gamma(f, \mathcal{L} f) d\mu \geq K \int g \Gamma(f) d\mu + \frac{1}{N} \int g(\mathcal{L} f)^2 d\mu.$$

The main issue differentiating this weak formulation of the pointwise Bochner inequality (5.1) is that  $\mathcal{L}\Gamma(f)$  might not be well-defined. Examples of RCD spaces include classical smooth manifolds satisfying Ricci curvature bounds, but also their possible Gromov-Hausdorff limits, as well as certain stratified spaces that do not arise as limits of smooth manifolds [29]. When  $N > 1$ , this definition is equivalent to the Lott-Sturm-Villani definition of Ricci curvature lower bounds, up to the extra assumption of linearity of the heat flow.

The now classical bound on the spectral gap is the following, proved for example in [83, Theorem 4.22] in the RCD setting:

**Theorem 62** (Spectral gap of  $\mathcal{L}$ ). *Let  $(M, d, \mu)$  a  $RCD(\rho, N)$  metric measure space with  $\rho > 0$  and  $N \in \mathbb{R} \setminus [0, 1]$ , the first eigenvalue  $\lambda_1 > 0$  of  $-\mathcal{L}$  satisfies*

$$\lambda_1 \geq \frac{N\rho}{N-1}.$$

Moreover, existence of an eigenfunction associated with the spectral gap was proved in [98] under the  $RCD(\rho, \infty)$  condition with  $\rho > 0$ .

### 5.2.3 Model spaces in dimension 1

We describe here the two main model spaces in dimension 1 used in the paper. The idea is to briefly describe how they are defined.

Let consider  $\varphi$ , a smooth and positive function on an open interval  $I \subset \mathbb{R}$ . And let define the generator

$$\mathcal{L}f = \varphi f'' - (\beta - 1)\varphi' f',$$

for smooth function  $f$ . This generator has a reversible measure  $\varphi^{-\beta} dx$ , for any smooth and compactly supported functions on  $I$ ,

$$\int_I f \mathcal{L}g \varphi^{-\beta} dx = - \int_I \varphi f' g' \varphi^{-\beta} dx.$$

The carré du champ operator,  $\Gamma$  defined by

$$\Gamma(f) = \frac{1}{2} \mathcal{L}(f^2) - f \mathcal{L}f,$$

is given by  $\Gamma(f) = \varphi \times (f')^2$ . It says that we are working on the open interval  $I$  associated with the metric  $1/\varphi$ . From this new metric, the generator  $\mathcal{L}$  takes the form

$$\mathcal{L}f = \Delta_\varphi f - \Gamma(W, f)$$

where  $\Delta_\varphi f = \varphi f'' + \frac{\varphi'}{2} f'$  is the Laplace-Beltrami operator on the one dimensional manifold  $(I, 1/\varphi)$  and  $W = (\beta - \frac{1}{2}) \log \varphi$ .

Following [13, Sec. C.6], here in dimension 1, the operator  $\mathcal{L}$  satisfies the curvature-dimension condition  $CD(\rho, N)$  (with  $N \notin [0, 1]$ ) if and only if

$$(5.6) \quad \nabla \nabla^\varphi W - \frac{\rho}{\varphi} \geq \frac{(W')^2}{N-1},$$

where  $\nabla \nabla^\varphi W = W'' + W' \frac{\varphi'}{2\varphi}$  is the Hessian of  $W$  with respect to the metric  $1/\varphi$ . Equation (5.6) becomes,

$$(5.7) \quad \left( \beta - \frac{1}{2} \right) \left[ \varphi'' - \frac{\rho}{\beta - \frac{1}{2}} - \frac{\varphi'^2}{\varphi} \frac{N + 2(\beta - 1)}{2(N - 1)} \right] \geq 0.$$

The two main examples, used in the paper, are when there is equality in (5.7).



1. Let  $I = (-1, 1)$  and  $\varphi = 1 - x^2$ ,  $\rho = 1 - 2\beta$  and  $N = 2(1 - \beta)$ , ( $\beta < 1/2$ ). Then the generator

$$\mathcal{L}^+ f = (1 - x^2)f'' - Nx f'$$

is the so-called Jacobi operator and satisfies the curvature-dimension condition  $CD(N - 1, N)$  with  $N > 1$ . The carré du champ is  $\Gamma(f) = (1 - x^2)(f')^2$  and the reversible measure is a Beta distribution,

$$(5.8) \quad d\mu_N^+ = \frac{(1 - x^2)^{N/2-1}}{Z^+} 1_{[-1,1]} dx,$$

where  $Z^+$  is such that  $\mu_N^+$  is a probability measure. This measure is the so-called symmetrized Beta distribution with parameters  $(N/2, N/2)$ .

The first non-trivial eigenvalue for the operator  $-\mathcal{L}^+$  is  $N$ , associated with the eigenfunction  $f(x) = x$ . Moreover we have

$$\text{Var}_{\mu_N^+}(f) = \int x^2 d\mu_N^+ - \left( \int x d\mu_N^+ \right)^2 = \int x^2 d\mu_N^+ = \frac{1}{N+1}$$

while  $\int \Gamma(f) d\mu_N^+ = \frac{N}{N+1}$ .

When  $N \geq 2$  is a positive integer,  $\mu_N^+$  is the distribution of a coordinate on the  $N$ -dimensional unit sphere  $\mathbf{S}^N$ .

2. Let now  $I = \mathbb{R}$ ,  $\varphi = 1 + x^2$ ,  $\rho = 2\beta - 1$  and  $N = 2(1 - \beta)$ . We assume that  $\beta > 3/2$ , so that  $N < -1$ . For this model, we have

$$\mathcal{L}^- f = (1 + x^2)f'' + Nx f'$$

and  $\Gamma(f) = (1 + x^2)f'^2$ , and the reversible measure is a Cauchy type distribution (also called Student distribution),

$$(5.9) \quad d\mu_N^- = \frac{(1 + x^2)^{N/2-1}}{Z^-} dx,$$

where  $Z^-$  is such that  $\mu_N^-$  is a probability measure. This model satisfies the curvature-dimension condition  $CD(1 - N, N)$  with  $N < -1$ .

Again, the first non trivial eigenvalue for the operator  $\mathcal{L}^-$  is  $N$  associated with the eigenfunction is  $f(x) = x$ . We also have  $\text{Var}_{\mu_N^-}(f) = \frac{-1}{N+1}$  and  $\int \Gamma(f) d\mu_N^- = \frac{N}{N+1}$ .

The assumption  $N < -1$  is a necessary condition for the variance of  $f$  to exist.

**Remark 3.** *Of course, these models could be parametrized differently, without effect on the curvature or the spectral gap. In particular, our model for negative  $N$  is the same as the model space in [160], except that we parametrize it differently. It is important to notice that we chose to deal with these parametrizations of the two models (in dimension 1) so that the eigenfunction associated with the first eigenvalue is the identity function, and satisfies  $\int \Gamma(f) d\mu_N^\pm = \frac{N}{N+1}$ .*

#### 5.2.4 $L^1$ -Wasserstein distance

The  $W_1$  distance, also called the  $L^1$ -Wasserstein distance is the optimal transport distance in  $L^1$ . For any  $\mu, \nu$  probability measure on a metric space  $(M, d)$ ,

$$W_1(\mu, \nu) = \inf \int d(x, y) d\pi(x, y),$$

where the infimum is running over all probability measures  $\pi$  on  $M \times M$  which admit  $\mu$  and  $\nu$  as marginals. From the Kantorovich-Rubinstein's theorem (see [3] for instance) we have

$$W_1(\mu, \nu) = \sup_{\|g\|_{L^1} \leq 1} \left\{ \int g d\mu - \int g d\nu \right\},$$

and this is the definition used in this paper.

### 5.3 The positive and finite-dimensional case

Let  $N > 1$  be a real number and  $(M, d, \mu)$  be a  $\text{RCD}(N - 1, N)$  metric measure space as proposed in Definition 61. As is classical, we consider this case rather than more general  $\text{RCD}(\rho, N)$  spaces since we can normalize the value of  $\rho$  to  $N - 1$  by scaling the metric. From Theorem 62, its spectral gap satisfies  $\lambda_1 \geq N$ .

The next result propose an estimate depending only on  $(\lambda_1 - N)$ , of the  $L^1$ -Wasserstein distance between one direction of the measure  $\mu$  and the reference measure  $\mu_N^+$ .

**Theorem 63** (The positive dimensional case). *Let  $(M, d, \mu)$  be an  $\text{RCD}(N - 1, N)$  metric measure space with  $N > 1$  and generator  $\mathcal{L}$ . Let  $f$  be an eigenfunction of  $-\mathcal{L}$  with eigenvalue  $N + \varepsilon$ , for  $\varepsilon \in [0, 1)$ , satisfying  $\int \Gamma(f) d\mu = N/(N + 1)$ . Then*

$$W_1(\mu \circ f^{-1}, \mu_N^+) \leq C\varepsilon,$$

where  $C > 0$  is an explicit constant, depending only on  $N$ .

The proof of this result is postponed in Section 5.3.4. We first prove the two main ingredients:  $L^1$ -estimates on eigenfunctions, and a criterion for comparing the distribution of an eigenfunction to a Beta distribution.

#### 5.3.1 Estimates on the first eigenfunction

Our key technical estimate is the following:

**Lemma 64.** *Let assume that  $\mathcal{L}$  satisfies the  $\text{RCD}(N - 1, N)$  condition. Let  $f$  be an eigenfunction of  $-\mathcal{L}$  with eigenvalue  $N + \varepsilon$ , for  $\varepsilon \in [0, 1)$  and satisfying  $\int \Gamma(f) d\mu = N/(N + 1)$ . Then,*

$$\|\Gamma(f) + (1 + \varepsilon)f^2 - 1\|_1 \leq \varepsilon C,$$

for some constant  $C$  depending only on  $N$ . The value

$$C = 4 \left( 2 + \frac{N + 1}{N} \left( \frac{2}{N + 1} \log 2 + \log \left( 2 + \frac{2N}{(N - 1)^2} \right) \right) \right) + \frac{N - 1}{N(N + 1)}$$

suffices.

The requirement that  $\varepsilon < 1$  is for convenience, as it allows to simplify the writing of various bounds in the proof.

Estimates on  $h = \Gamma(f) + (1 + \varepsilon)f^2$  are at the core of all of the results on rigidity and stability of sharp functional inequalities in the  $\text{RCD}(N - 1, N)$  setting. For rigidity, [124, Thm. 3.7] proves the  $\varepsilon = 0$  case of our lemma. In the smooth unweighted setting [27] uses  $L^p$ -estimates on  $\text{Hess } f + f$  (which is related to the gradient of  $\Gamma(f) + f^2$ ), while [9] uses  $L^\infty$ -estimates on  $\Gamma(f) + f^2$ . In the non-smooth setting, [62] establishes  $L^2$ -estimates on  $\text{Hess } f + f$  along one-dimensional needles.

Unlike these previous work, we use a weaker norm to estimate  $h$ . One of the upsides is that it is easier to work with first order derivatives (instead of Hessians) in the non-smooth setting. But the main upside is that the quantitative bounds we derive are *stronger*. Indeed, one could use the self improvement in the Bakry-Emery-Bochner bound [188] to estimate Hessian-like quantities (this is the approach used in [28, 124] for example), but the quantitative bounds are of order  $\sqrt{\varepsilon}$  instead of  $\varepsilon$ . Our use of an  $L^1$ -norm leads instead to a *sharp* quantitative bound.

Our proof of Lemma 64 is based on the following result.

**Lemma 65.** *Let assume that  $\mathcal{L}$  satisfies the  $\text{RCD}(N - 1, N)$  condition with  $N > 1$ . Let  $f$  be an eigenfunction of  $-\mathcal{L}$ , with eigenvalue  $N + \varepsilon$  for some  $\varepsilon \in [0, 1)$ . Then for  $h = \Gamma(f) + (1 + \varepsilon)f^2$  we have*

$$\|\mathcal{L}P_s h\|_1 \leq 4N\varepsilon \|f^2\|_1$$

where  $P_s$  is the semigroup generated from  $\mathcal{L}$ . In particular, in the smooth setting,

$$\|\mathcal{L}h\|_1 \leq 4N\varepsilon\|f^2\|_1.$$

The use of the semigroup  $P_s$  in the non-smooth setting is to avoid giving a meaning to  $\mathcal{L}\Gamma(f)$ . To simplify the exposition, we shall first prove Lemma 65 in the smooth setting, (assuming the Bochner inequality (5.1)) and then in the general RCD setting.

*Proof of Lemma 65 in the smooth setting.* Applying the smooth  $CD(N-1, N)$  condition (5.1) to the eigenfunction  $f$  (recall that  $f$  is a smooth function since we are working in a smooth setting), we have

$$\frac{1}{2}\mathcal{L}\Gamma(f) + (1 + \varepsilon)\Gamma(f) - \frac{(N + \varepsilon)^2}{N}f^2 \geq 0.$$

Moreover, since  $2\Gamma(f) = \mathcal{L}(f^2) + 2(N + \varepsilon)f^2$ , we have

$$\mathcal{L}(\Gamma(f) + (1 + \varepsilon)f^2) + 2(\varepsilon(N - 1) + \varepsilon^2(1 - 1/N))f^2 \geq 0.$$

In particular,

$$(\mathcal{L}h)_- \leq 2\varepsilon(N - 1)\left(1 + \frac{\varepsilon}{N}\right)f^2 \leq 2\varepsilon N f^2,$$

using  $\varepsilon < 1$  (which is only used here to make notations less cluttered) and  $N > 1$ . Since  $\mathcal{L}h$  has zero average with respect to  $\mu$ , we get

$$\|\mathcal{L}h\|_1 = 2\|(\mathcal{L}h)_-\|_1 \leq 4\varepsilon N\|f^2\|_1.$$

Which concludes the proof.  $\square$

*Proof of Lemma 65 in the general RCD case.* As mentioned previously, to follow the above scheme in the general RCD setting, we encounter the problem of giving a meaning to  $\mathcal{L}\Gamma(f)$ . We shall exploit the particular structure of  $f$  as an eigenfunction to bypass this issue via a regularization procedure.

The weak Bochner inequality (5.5) applied to the eigenfunction  $f$  of  $-\mathcal{L}$  with eigenvalue  $N + \varepsilon$  takes the form:

$$\frac{1}{2} \int \mathcal{L}g\Gamma(f)d\mu + (N + \varepsilon) \int g\Gamma(f)d\mu \geq (N - 1) \int g\Gamma(f)d\mu + \frac{(N + \varepsilon)^2}{N} \int gf^2d\mu,$$

for any test function  $g$  such that  $g, \mathcal{L}g \in L^\infty(\mu)$ .

If we consider test functions of the form  $P_s g$  for  $s > 0$  and  $g, \mathcal{L}g \in L^\infty(\mu)$ , since  $\mathcal{L}$  and  $P_s$  commute and  $\mathcal{L}P_s\Gamma(f)$  is well defined we have

$$(5.10) \quad \frac{1}{2} \int g(\mathcal{L}P_s\Gamma(f))d\mu \geq -(\varepsilon + 1) \int gP_s\Gamma(f)d\mu + \frac{(N + \varepsilon)^2}{N} \int gP_s(f^2)d\mu.$$

We can now remove the restriction that  $\mathcal{L}g$  is  $L^\infty(\mu)$  by approximating a  $g \in L^\infty(\mu)$  with

$$S_\varepsilon g := \int_0^\infty P_{\varepsilon r} g \kappa(r) dr,$$

where  $\kappa$  is a smooth nonnegative function compactly supported in  $(0, \infty)$  with  $\int \kappa(r) dr = 1$ . One can check that

$$\mathcal{L}S_\varepsilon g = -\frac{1}{\varepsilon} \int_0^\infty P_{\varepsilon r} g \kappa'(r) dr$$

is indeed  $L^\infty(\mu)$ , and that  $S_\varepsilon g$  converges to  $g$  in  $L^2(\mu)$  when  $\varepsilon \rightarrow 0$ . Therefore (5.10) holds for  $g \in L^\infty(\mu)$ . But since  $P_s\Gamma(f)$  is a well-defined function, we deduce the pointwise inequality

$$\frac{1}{2}\mathcal{L}P_s\Gamma(f) \geq -(\varepsilon + 1)P_s\Gamma(f) + \frac{(\varepsilon + N)^2}{N}P_s(f^2), \quad s > 0.$$

Since  $\Gamma(f) = \frac{1}{2}\mathcal{L}f^2 + (\varepsilon + n)f^2$ , we get as in the smooth setting, for  $h = \Gamma(f) + (1 + \varepsilon)f^2$ ,

$$(\mathcal{L}P_s h)_- \leq 2\varepsilon N P_s(f^2).$$

Hence we can deduce that  $\|\mathcal{L}P_s h\|_1 \leq 4N\varepsilon\|f\|_2^2$ .  $\square$

Since the kernel of  $\mathcal{L}$  is the set of constants, we expect that  $h$  is concentrated around its average. A tool for proving this is provided by the following result.

**Proposition 66.** *Let assume that  $\mathcal{L}$  satisfies the condition  $RCD(N-1, N)$  with  $N > 1$ , then for any function  $g \in D(\mathcal{L})$  with  $\int g d\mu = 0$ , we have*

$$\|g\|_1 \leq \left(2 + \frac{N+1}{N} \left(\frac{2}{N+1} \log 2 + \log \left(2 + \frac{2N}{(N-1)^2}\right)\right)\right) \|\mathcal{L}g\|_1.$$

*Proof.* We have for any  $t > 0$ ,

$$\|g\|_1 \leq \|g - P_t g\|_1 + \|P_t g\|_1.$$

For any bounded test function  $u$ , we have

$$\int (g - P_t g) u d\mu = \int_0^t \int (-\mathcal{L} P_s g) u d\mu ds = \int_0^t \int (-\mathcal{L} g)(P_s u) d\mu ds \leq t \|u\|_\infty \|\mathcal{L}g\|_1.$$

Taking the supremum over all  $u$  with  $\|u\|_\infty \leq 1$ , we get

$$\|g - P_t g\|_1 \leq t \|\mathcal{L}g\|_1,$$

hence

$$(5.11) \quad \|g\|_1 \leq (t+1) \|\mathcal{L}g\|_1 + \|P_{t+1}g\|_1.$$

Since  $N > 1$ , the operator  $\mathcal{L}$  satisfies the weaker condition  $CD(N-1, N+1)$ . From this remark,  $\mathcal{L}$  satisfies a Sobolev inequality (cf. [120] for the smooth setting and extended to the RCD setting in [181]),

$$\|f\|_{2\frac{N+1}{N-1}}^2 \leq \|f\|_2^2 + B \int \Gamma(f) d\mu,$$

with  $B = \frac{4N}{(N+1)(N-1)^2}$  and for functions such that terms are well defined. And then, the ultracontractive bound

$$(5.12) \quad \|P_t g\|_\infty \leq C t^{-\frac{N+1}{2}} \|g\|_1, \quad t \leq 1$$

with  $C = \left(2 + \frac{2N}{(N-1)^2}\right)^{\frac{N+1}{2}}$ , see [13, Thm. 6.3.1 and Rmk. 6.3.2]. Using this bound for  $t = 1$  and the spectral gap, we have

$$\begin{aligned} \|P_{t+1}g\|_1 &\leq \|P_{t+1}g\|_2 \leq e^{-Nt} \|P_1g\|_2 \\ &\leq e^{-Nt} \left(2 + \frac{2N}{(N-1)^2}\right)^{\frac{N+1}{2}} \|g\|_1. \end{aligned}$$

Taking  $t = \frac{N+1}{2N} \left(\frac{2}{N+1} \log 2 + \log \left(2 + \frac{2N}{(N-1)^2}\right)\right)$  and using inequality (5.11), we get

$$\|g\|_1 \leq (t+1) \|\mathcal{L}g\|_1 + \frac{\|g\|_1}{2}.$$

Hence

$$\|g\|_1 \leq 2(t+1) \|\mathcal{L}g\|_1 = \left(2 + \frac{N+1}{N} \left(\frac{2}{N+1} \log 2 + \log \left(2 + \frac{2N}{(N-1)^2}\right)\right)\right) \|\mathcal{L}g\|_1,$$

which is the inequality desired.  $\square$

*Proof of Lemma 64.* Let apply Proposition 66 to

$$P_s \Gamma(f) + (1 + \varepsilon) P_s(f^2) - \frac{N}{N+1} \left(1 + \frac{1 + \varepsilon}{N + \varepsilon}\right)$$

for some  $s > 0$ , which has zero average since  $\int f^2 d\mu = (N + \varepsilon)^{-1} \int \Gamma(f) d\mu$ . We obtain,

$$\left\| P_s \Gamma(f) + (1 + \varepsilon) P_s(f^2) - \frac{N}{N+1} \left(1 + \frac{1 + \varepsilon}{N + \varepsilon}\right) \right\|_1 \leq C \|\mathcal{L}[P_s \Gamma(f) + (1 + \varepsilon) P_s(f^2)]\|_1,$$

where  $C$  is given by Proposition 66. And then, using Lemma 65,

$$\left\| P_s \Gamma(f) + (1 + \varepsilon) P_s(f^2) - \frac{N}{N+1} \left(1 + \frac{1 + \varepsilon}{N + \varepsilon}\right) \right\|_1 \leq 4CN\varepsilon \|f\|_2 \leq 4C\varepsilon.$$

Finally,

$$\left\| P_s \Gamma(f) + (1 + \varepsilon) P_s(f^2) - 1 \right\|_1 \leq \varepsilon \left(4C + \frac{N-1}{N(N+1)}\right),$$

from the triangle inequality. We let  $s$  go to zero to conclude the proof.  $\square$

### 5.3.2 Approximate Beta distribution for eigenfunctions

The last main ingredient of the proof is a result stating that (normalized) eigenfunctions with eigenvalue close to  $N$  and such that  $\Gamma(f) + f^2$  is close to a constant approximately follow a symmetrized Beta distribution. The result, of independent interest, is the following:

**Theorem 67.** *Let  $f$  be an eigenfunction of a diffusion operator  $-\mathcal{L}$  with eigenvalue  $\lambda$  and invariant probability measure  $\mu$ , and let  $\nu = \mu \circ f^{-1}$  be the pushforward of  $\mu$  by  $f$ . Then*

$$W_1(\nu, \text{Beta}(N/2, N/2)) \leq \left(\frac{N^2}{4} + \frac{5N}{4} + 2\right) \|\Gamma(f) + f^2 - 1\|_{L^1(\mu)} + \frac{|N - \lambda|}{N} \|f\|_{L^2(\mu)},$$

where  $\text{Beta}(N/2, N/2) = \mu_N^\dagger$  (defined in Section 5.2.3).

Note that this result is only interesting if  $\lambda$  is close to  $N$ , and if  $f$  has been normalized so that  $\int (\Gamma(f) + f^2) d\mu$  is close to one.

This result is a variant of a result of E. Meckes [153, Theorem 1], who proved that eigenfunctions whose gradient has small variance are close to normal. As in Meckes' work, the proof will mostly be an application of Stein's method.

#### Stein's method for Beta distributions

To prove Theorem 67, we shall rely on a variant of Stein's method for Beta distributions. Stein's method is a set of techniques, pioneered in [197, 198], for bounding distances between probability measures via such integration-by-parts formulas. We refer to [185, 63] for recent introductions and surveys of this field.

The following variant of Stein's lemma was proven in [81, 100], for non-symmetric Beta distributions on  $[0, 1]$  (and the symmetrized case is an immediate consequence). The values of the constants stated here are slightly worse than those stated in [81, Prop. 4.2], and used here for ease of writing.

**Theorem 68** ([81, 100]). *Let  $N > 1$ , the  $W_1$  distance between a probability measure  $\nu$  supported on  $[-1, 1]$  to a  $\text{Beta}(N/2, N/2)$  distribution can be estimated by*

$$(5.13) \quad W_1(\nu, \text{Beta}(N/2, N/2)) \leq \frac{1}{2} \sup \left\{ \int \left[ (1 - x^2)g'(x) + Nxg(x) \right] d\nu \right\},$$

where the supremum is running over all smooth function  $g$  on  $\mathbb{R}$  such that

$$\|g\|_\infty \leq \frac{2}{N} \quad \text{and} \quad \|g'\|_\infty \leq (2 + N).$$

### 5.3.3 Proof of Theorem 67

We want to apply Theorem 68 to the measure  $\nu = \mu \circ f^{-1}$ . We first derive an approximate integration by parts formula for the measure  $\nu$ . Let  $g$  be a smooth test function with  $\|g\|_\infty \leq 2/N$  and  $\|g'\|_\infty \leq (2+N)$ .

Using the diffusion property (5.2) and the definition of  $\nu$ , we have

$$\begin{aligned} \lambda \int xg(x)d\nu(x) &= \lambda \int fg \circ f d\mu \\ &= \int (-\mathcal{L}f)g \circ f d\mu = \int \Gamma(f, g \circ f)d\mu = \int g' \circ f \Gamma(f)d\mu. \end{aligned}$$

So

$$\begin{aligned} \int [\lambda xg(x) - (1-x^2)g'(x)]d\nu(x) &= \int g' \circ f \Gamma(f)d\mu - \int (1-x^2)g'(x)d\nu \\ &= \int [\Gamma(f) + f^2 - 1]g' \circ f d\mu \leq (2+N)\|\Gamma(f) + f^2 - 1\|_{L^1(\mu)}. \end{aligned}$$

where we use the definition of  $\nu = \mu \circ f^{-1}$ . Moreover

$$\left| \int xg(x)d\nu(x) \right| = \left| \int fg(f)d\mu \right| \leq \|g\|_\infty \|f\|_{L^2(\mu)} \leq \frac{2}{N} \|f\|_{L^2(\mu)}.$$

Therefore, we get

$$(5.14) \quad \left| \int [Nxg(x) - (1-x^2)g'(x)]d\nu(x) \right| \leq \left| \int [\lambda xg(x) - (1-x^2)g'(x)]d\nu(x) \right| + |N-\lambda| \left| \int xg(x)d\nu(x) \right| \leq (2+N)\|\Gamma(f) + f^2 - 1\|_{L^1(\mu)} + 2\frac{|N-\lambda|}{N} \|f\|_{L^2(\mu)}.$$

To apply Theorem 68, we still need to consider a measure supported on  $[-1, 1]$ . To do so, we introduce  $\tilde{\nu}$  the pushforward of  $\mu$  by  $\phi \circ f$ , with  $\phi$  a cutoff function, that is  $\phi(x) = x$  on  $[-1, 1]$ ,  $\phi(x) = 1$  on  $[1, +\infty[$ , and  $\phi(x) = -1$  on  $]-\infty, -1]$ . Note that

$$(5.15) \quad \int |f - \phi(f)|d\mu = \int (|f| - 1)\mathbf{1}_{|f| \geq 1}d\mu \leq \frac{1}{2} \int (f^2 - 1)\mathbf{1}_{|f| \geq 1}d\mu$$

$$(5.16) \quad = \frac{1}{2} \int (f^2 - 1)_+ d\mu \leq \frac{1}{2} \int (\Gamma(f) + f^2 - 1)_+ d\mu \leq \frac{1}{2} \|\Gamma(f) + f^2 - 1\|_{L^1(\mu)}.$$

Therefore by a coupling argument

$$(5.17) \quad W_1(\tilde{\nu}, \nu) \leq \int x d\nu(x) - \int x d\tilde{\nu}(x) \leq \int |f - \phi(f)|d\mu \leq \frac{1}{2} \|\Gamma(f) + f^2 - 1\|_1.$$

It is therefore enough to apply Theorem 68 to  $\tilde{\nu}$ . To do so, we shall show that it satisfies the same approximate integration by parts formula than  $\nu$ , up to an error of order  $\|\Gamma(f) + f^2 - 1\|_{L^1(\mu)}$ .

First we have,

$$\begin{aligned} \left| \int [fg(f) - \phi(f)g(\phi(f))]d\mu \right| &\leq \left| \int (f - \phi(f))g(f)d\mu \right| + \left| \int \phi(f)(g(f) - g(\phi(f)))d\mu \right| \\ &\leq \frac{2}{N} \|f - \phi(f)\|_{L^1(\mu)} + (2+N)\|f - \phi(f)\|_{L^1(\mu)} \leq \left(1 + \frac{1}{N} + \frac{N}{2}\right) \|\Gamma(f) + f^2 - 1\|_{L^1(\mu)}, \end{aligned}$$

since  $\|\phi(f)\|_\infty \leq 1$ . Secondly,

$$\begin{aligned} \left| \int [(1-f^2)g'(f) - (1-\phi(f)^2)g'(\phi(f))]d\mu \right| &\leq \left| \int (\phi(f)^2 - f^2)g'(f)d\mu \right| \\ &\quad + \left| \int (1-\phi(f)^2)(g'(f) - g'(\phi(f)))d\mu \right| \\ &\leq (2+N) \int (f^2 - 1)_+ d\mu + 0 \\ &\leq \left(1 + \frac{N}{2}\right) \|\Gamma(f) + f^2 - 1\|_{L^1(\mu)}. \end{aligned}$$

where we used again (5.15). Hence from the definition of  $\tilde{\nu}$ , we have

$$\begin{aligned} \left| \int (Nxg(x) - (1-x^2)g'(x))d\tilde{\nu}(x) \right| &= \left| \int (N\phi(f)g(\phi(f)) - (1-\phi(f)^2)g'(\phi(f)))d\mu \right| \leq \\ &N \left| \int [\phi(f)g(\phi(f)) - fg(f)]d\mu \right| + \left| \int [(1-f^2)g'(f) - (1-\phi(f)^2)g'(\phi(f))]d\mu \right| \\ &\quad + \left| \int [Nxg(x) - (1-x^2)g'(x)]d\nu(x) \right|. \end{aligned}$$

We can deduce from the previous estimates and (5.14),

$$\left| \int Nxg(x) - (1-x^2)g'(x)d\tilde{\nu} \right| \leq \left( \frac{N^2}{2} + \frac{5N}{2} + 3 \right) \|\Gamma(f) + f^2 - 1\|_{L^1(\mu)} + 2 \frac{|N-\lambda|}{N} \|f\|_{L^2(\mu)}.$$

At the end, from Theorem 68 to  $\tilde{\nu}$  and the estimate (5.17),

$$\begin{aligned} W_1(\nu, \text{Beta}(N/2, N/2)) &\leq W_1(\tilde{\nu}, \nu) + W_1(\tilde{\nu}, \text{Beta}(N/2, N/2)) \leq \\ &\left( \frac{N^2}{4} + \frac{5N}{4} + 2 \right) \|\Gamma(f) + f^2 - 1\|_{L^1(\mu)} + \frac{|N-\lambda|}{N} \|f\|_{L^2(\mu)}, \end{aligned}$$

which is the inequality desired.

### 5.3.4 Proof of Theorem 63

We can now straightforwardly combine Theorem 67 and Lemma 64 to conclude.

Note that under our normalization  $\int \Gamma(f)d\mu = N/(N+1)$ , we have

$$\|f^2\|_{L^1(\mu)} \leq \frac{1}{N+1}$$

and hence

$$\|\Gamma(f) + f^2 - 1\|_{L^1(\mu)} \leq \varepsilon \|f^2\|_{L^1(\mu)} + \|\Gamma(f) + (1+\varepsilon)f^2 - 1\|_{L^1(\mu)} \leq C\varepsilon.$$

for some explicit constant  $C > 0$  depending only on  $N$ .

## 5.4 Stability of the spectral gap for $\text{RCD}(1, \infty)$ spaces

In this section, we are working with  $(M, d, \mu)$  a  $\text{RCD}(1, \infty)$  metric measure space with generator  $\mathcal{L}$  and unit mass.

Arguing as in the finite-dimensional case, we can use the Bochner formula to get the following estimate on the gradient of a normalized eigenfunction  $f$ :

$$(5.18) \quad \|\mathcal{L}P_s\Gamma(f)\|_1 \leq C\varepsilon,$$

for all  $s > 0$ .

However, unlike  $\text{RCD}(N-1, N)$  spaces, in this situation we do not have ultracontractive estimates on the semigroup (inequality (5.12)) to prove an  $L^1$ -inequality as in Lemma 64. Indeed, as we shall see in Proposition 72 below, that inequality fails for the Gauss space. Therefore, we must rely on a weaker functional inequality:

**Lemma 69.** *Let assume that the spectral gap of  $-\mathcal{L}$  is greater than 1. Let  $g$  such that  $\int g d\mu = 0$  and  $g \in L^p(\mu)$  for some  $p > 1$ . Then*

$$\|g\|_1 \leq C \|\mathcal{L}g\|_1 \left( 1 + \log \left( \max \left( \frac{\|g\|_p}{\|\mathcal{L}g\|_1}, 1 \right) \right) \right),$$

for some constant  $C$  depending only on  $p$ .

Note that this lemma does not involve any assumption on curvature. We will actually be interested in applying it when  $\|g\|_p$  is bounded and  $\|\mathcal{L}g\|_1$  is small.

*Proof.* As before, using inequality (5.11), we obtain for  $t \geq 0$ ,

$$\|g\|_1 \leq t \|\mathcal{L}g\|_1 + \|P_t g\|_1 \leq t \|\mathcal{L}g\|_1 + \|P_t g\|_p.$$

According to [57, Thm 1.6], we have for some constant  $C_p$  (depending only  $p$ ),

$$\|P_t g\|_p \leq C_p \exp(-4(p-1)t/p^2) \|g\|_p.$$

Hence, we have

$$\|g\|_1 \leq t \|\mathcal{L}g\|_1 + C_p \exp(-4(p-1)t/p^2) \|g\|_p.$$

Assume for simplicity that  $C_p \geq 1$  and take

$$t = \frac{p^2}{4(p-1)} \log \left[ \max \left( \frac{\|g\|_p}{\|\mathcal{L}g\|_1}; 1 \right) C_p \right],$$

in order to get

$$C_p \exp(-4(p-1)t/p^2) \leq \frac{\|\mathcal{L}g\|_1}{\|g\|_p}.$$

we get

$$\|g\|_1 \leq \|\mathcal{L}g\|_1 \left[ 1 + \frac{p^2}{4(p-1)} \log \left( C_p \max \left( \frac{\|g\|_p}{\|\mathcal{L}g\|_1}, 1 \right) \right) \right],$$

which is the inequality expected.  $\square$

**Lemma 70.** *Let assume that  $\mathcal{L}$  satisfies the RCD(1,  $\infty$ ) condition. And let  $f$  be an eigenfunction of  $-\mathcal{L}$  with eigenvalue  $1 + \varepsilon$ , for  $\varepsilon \in [0, 1]$  and satisfying  $\int \Gamma(f) d\mu = 1$ . Then,*

$$(5.19) \quad \|P_s \Gamma(f) + \varepsilon P_s(f^2) - 1 - \varepsilon\|_1 \leq C \varepsilon \log(2/\varepsilon),$$

for some numerical  $C$ .

*Proof.* Following the same proof as Lemma 65 in the general RCD setting, we get

$$\|\mathcal{L}(P_s \Gamma(f) + \varepsilon P_s(f^2))\|_1 \leq 4\varepsilon \|f^2\|_1 \leq 4\varepsilon,$$

since  $\int f^2 d\mu = (1 + \varepsilon)^{-1} \leq 1$ .

If  $h = P_s(\Gamma(f) + \varepsilon f^2)$ , we have  $\int h d\mu = (1 + 2\varepsilon)(1 + \varepsilon)^{-1}$  and then

$$\left\| h - \frac{1 + 2\varepsilon}{1 + \varepsilon} \right\|_1 \leq C \|\mathcal{L}h\|_1 \left( 1 + \log \left( \max \left( \frac{\|h\|_p}{\|\mathcal{L}h\|_1}, 1 \right) \right) \right) \leq 4C\varepsilon \left( 1 + \log \left( \max \left( \frac{\|h\|_p}{4\varepsilon}, 1 \right) \right) \right).$$

Hence

$$\|h - 1\|_1 \leq \left\| h - \frac{1 + 2\varepsilon}{1 + \varepsilon} \right\|_1 + \frac{1 + 2\varepsilon}{1 + \varepsilon} - 1 \leq 4C\varepsilon \left( 1 + \log \left( \max \left( \frac{\|h\|_p}{4\varepsilon}, 1 \right) \right) \right) + \varepsilon.$$

From [28, Prop. 3.2] (and the RCD(1,  $\infty$ ) condition) we have,

$$\begin{aligned} \|h\|_p &\leq \|P_s \Gamma(f)\|_p + \|P_s f\|_{2p} \leq \|\Gamma(f)\|_p + \|f\|_{2p} \\ &\leq (8p - 4)^{(1+\varepsilon)/2} \|f^2\|_1 + (p - 1)^{(1+\varepsilon)/2} \|f^2\|_1 \leq D \end{aligned}$$

for some other constant  $D > 0$ , which is the last estimate to prove (5.19).  $\square$



Lemma 70 improves on a particular case of [28, Lem. 3.3], and is enough to improve [28, Thm. 1.3] into:

**Theorem 71.** *Let  $(M, d, \mu)$  a  $RCD(1, \infty)$  metric measure space with generator  $\mathcal{L}$ . Let  $f$  is an eigenfunction of the  $-\mathcal{L}$  with eigenvalue  $1 + \varepsilon$  ( $\varepsilon \in [0, 1]$ ) and satisfying  $\int \Gamma(f) d\mu = 1$ , then*

$$W_1(\mu \circ f^{-1}, \gamma) \leq C\varepsilon \log(2/\varepsilon),$$

for some numerical constant  $C > 0$  and  $\gamma$  is the standard Gaussian measure.

*Proof.* We follow the same line of arguments as in [28] (and the implementation of Stein's method is essentially the same as the argument in [153]). From Stein's lemma for the one-dimensional Gaussian distribution (see for example [28, Lem. 3.5]) we have,

$$W_1(\mu \circ f^{-1}, \gamma) \leq \sup \left\{ \int (g'(x) - g(x)x) d\mu \circ f^{-1}(x), \|g'\|_\infty \leq 4 \right\}.$$

Similarly we have  $(1 + \varepsilon) \int xg(x) d\mu \circ f^{-1}(x) = \int g' \circ f \Gamma(f) d\mu$ , then

$$\begin{aligned} \int (g'(x) - g(x)x) d\mu \circ f^{-1}(x) &= \frac{1}{1 + \varepsilon} \int [1 + \varepsilon - \Gamma(f) - \varepsilon f^2] g' \circ f d\mu \\ &\quad - \frac{\varepsilon}{1 + \varepsilon} \int f^2 g' \circ f d\mu. \end{aligned}$$

That is for  $g$  satisfying  $\|g'\|_\infty \leq 4$ ,

$$\int (g'(x) - g(x)x) d\mu \circ f^{-1}(x) \leq \frac{4}{1 + \varepsilon} \|\Gamma(f) + \varepsilon f^2 - 1 - \varepsilon\|_1 + 4\varepsilon.$$

From the estimate (5.19) we get

$$\int (g'(x) - g(x)x) d\mu \circ f^{-1}(x) \leq C\varepsilon \log(2/\varepsilon),$$

for some numerical constant  $C$ , which concludes the proof.  $\square$

We now show that indeed the  $L^1$ -estimate of Lemma 64 may fail in the  $RCD(1, \infty)$  setting.

**Proposition 72.** *The inequality*

$$\|f\|_1 \leq C \|\mathcal{L}f\|_1$$

for some constant  $C > 0$  and every centered functions  $f$  fails in the Gauss space  $(\mathbb{R}, |\cdot|, \gamma)$ .

The counterexample in the proof below is inspired by Naor and Schechtman's proof [169] that an analogous inequality on the hypercube  $\{0, 1\}^d$  cannot hold with a constant that is uniform in  $d$ . We thank Alexandros Eskenazis for pointing out this reference to us.

*Proof.* Let  $\mathcal{L}$  be the one-dimensional Ornstein-Uhlenbeck operator  $\mathcal{L}f = f'' - xf'$ . We shall exhibit a family of functions  $f_r$  such that the ratio

$$\|f_r\|_1 / \|\mathcal{L}f_r\|_1$$

is unbounded. Let  $f_r$  be the centered solution (in  $L^2(\gamma)$ ) to the Poisson equation

$$\mathcal{L}f_r = -\mathbf{1}_{(-\infty, -r]} + \mathbf{1}_{[r, +\infty)}.$$

Since the source term is antisymmetric, we are looking for an antisymmetric solution (which will then be centered). We can check that  $f'_r$  is given on  $\mathbb{R}_+$  by the formula

$$f'_r(x) = \begin{cases} -\sqrt{2\pi}(1 - \varphi(r))e^{x^2/2} & \text{if } 0 \leq x \leq r; \\ -\sqrt{2\pi}(1 - \varphi(x))e^{x^2/2} & \text{if } x > r, \end{cases}$$

and extended by symmetry to  $\mathbb{R}$ . Here  $\varphi$  stands for the Gaussian cumulative distribution function  $\varphi(x) = \int_{-\infty}^x (2\pi)^{-1/2} \exp(-t^2/2) dt$ . Then for  $x \in [0, r]$  we have

$$f_r(x) = -\sqrt{2\pi}(1 - \varphi(r)) \int_0^x e^{t^2/2} dt.$$

It is easy to check that for  $x \in [1, r]$  we have a lower bound of the form

$$f_r(x) \geq C(1 - \varphi(r)) \frac{e^{x^2/2}}{x}, \quad C > 0.$$

Therefore, for  $r > 1$ ,

$$\int |f_r| d\gamma \geq C(1 - \varphi(r)) \int_{\sqrt{r}}^r \frac{e^{x^2/2}}{x} d\gamma = \frac{C}{\sqrt{2\pi}}(1 - \varphi(r)) \int_{\sqrt{r}}^r x^{-1} dx = \frac{C}{2\sqrt{2\pi}}(1 - \varphi(r)) \log r.$$

On the other hand

$$\|\mathcal{L}f_r\|_1 = \int |\mathcal{L}f_r| d\gamma = 2(1 - \varphi(r)).$$

Hence  $\|f_r\|_1 / \|\mathcal{L}f_r\|_1$  is unbounded as  $r$  goes to infinity, which concludes the proof.  $\square$

## 5.5 The negative dimension case

We now consider a space satisfying the curvature-dimension condition with *negative dimension parameter*  $N$ . We refer to [189, 173, 131] for an introduction to the notion. Since the Bochner inequality in the non-smooth setting seems not to have been investigated yet in the literature (in particular, its possible equivalence with other non-smooth definitions studied in [173]), we shall restrict ourselves to the smooth setting, and assume the Bochner inequality (5.1) holds in a strong sense, as in [131]. As noted in [130], the spectral gap bound of Theorem 62 is sharp for  $N \leq -1$ , but ceases to be sharp when  $N$  is negative and  $|N|$  is small (see also [148] where RCD spaces with negative effective dimension, positive curvature and *infinite* volume are studied). When  $N \leq -1$ , a particular example of a model satisfying a  $CD(1 - N, N)$  condition is given by the generalized Cauchy distribution as presented in Section 5.2.3,

$$d\mu_{\bar{N}} = \frac{(1 + x^2)^{N/2-1}}{Z^-} dx$$

on  $\mathbb{R}$ , with generator

$$\mathcal{L}^- f = (1 + x^2)f'' + Nxf'.$$

Our main result in this section is:

**Theorem 73** (The negative dimensional case). *Let  $(M, g)$  be a smooth Riemannian manifold associated with the probability measure  $\mu$ , satisfying the curvature-dimension condition  $CD(1 - N, N)$  with  $N < -1$ . We assume that the spectral gap  $\lambda_1$  satisfies  $\lambda_1 \leq -N + \varepsilon$  for some  $\varepsilon \in (0, 1)$ , and let  $f$  an eigenfunction of  $\mathcal{L}$  with eigenvalue  $\lambda_1$  such that  $\int \Gamma(f) d\mu = N/(N+1)$ .*

*Assume moreover that  $\Gamma(f) \in L^{1+c}$  for some  $c > 0$ . There is a constant  $C > 0$  depending only on  $N$  and  $c$  such that*

$$W_1(\mu \circ f^{-1}, \mu_{\bar{N}}) \leq C\varepsilon \log(2/\varepsilon).$$

This result shall be a direct consequence of combining Proposition 76 with Corollary 78 below.

### 5.5.1 Estimates on the first eigenfunction

As for Lemma 65 with the same proof, we deduce the following estimate for the Bochner formula:

**Lemma 74.** *Let assume that  $\mathcal{L}$  satisfies the  $CD(1 - N, N)$  condition. We assume that there exists  $f$  an eigenfunction of  $-\mathcal{L}$  with eigenvalue  $-N + \varepsilon$  with  $\varepsilon \in [0, 1]$  such that  $f \in L^2(\mu)$ . Then*

$$(5.20) \quad \|\mathcal{L}(\Gamma(f) + (\varepsilon - 1)f^2)\|_1 \leq 4\varepsilon \frac{(1 - N)^2}{|N|} \|f^2\|_1.$$

*Proof.* Applying the  $CD(1 - N, N)$  condition (5.1) to the eigenfunction  $f$ , we have

$$\frac{1}{2}\mathcal{L}\Gamma(f) + (\varepsilon - 1)\Gamma(f) - \frac{(N - \varepsilon)^2}{N}f^2 \geq 0.$$

Moreover, since  $2\Gamma(f) = \mathcal{L}(f^2) + 2(\varepsilon - N)f^2$ , we have

$$\frac{1}{2}\mathcal{L}(\Gamma(f) + (\varepsilon - 1)f^2) + \varepsilon(\varepsilon - N)\frac{1 - N}{|N|}f^2 \geq 0$$

In particular, for  $h = \Gamma(f) + (\varepsilon - 1)f^2$ ,

$$(\mathcal{L}h)_- \leq 2\varepsilon(\varepsilon - N)\frac{1 - N}{|N|}f^2 \leq -2\frac{\varepsilon}{N}(1 - N)^2f^2,$$

using  $\varepsilon < 1$ . Since  $\mathcal{L}h$  has zero average with respect to  $\mu$ , we get

$$\|\mathcal{L}h\|_1 = 2\|(\mathcal{L}h)_-\|_1 \leq 4\frac{\varepsilon}{|N|}(1 - N)^2\|f^2\|_1,$$

which concludes the proof.  $\square$

Once again, we do not have an ultracontractive estimate for the semigroup when the dimension parameter is negative, so we must rely on Lemma 69 to deduce an  $L^1$ -estimate on  $h = \Gamma(f) + (\varepsilon - 1)f^2$ . This use of Lemma 69 would be justified if both  $\Gamma(f) \in L^p(\mu)$  for some  $p > 1$  and  $f \in L^q(\mu)$  for some  $q > 2$ . The following lemma shows that only the integrability condition on  $\Gamma(f)$  is actually required.

**Lemma 75.** *Assume the couple  $(\mu, \Gamma)$  satisfies a Poincaré inequality with constant  $C_P$ . If  $\Gamma(g) \in L^{1+c}(\mu)$  for some  $c > 0$  then  $g \in L^{2(1+2c)/(1+c)}(\mu)$ , and*

$$\|g\|_{2(1+2c)/(1+c)}^2 \leq \|g\|_2^{2(1+2c)/(1+c)} + 4C_P\|\Gamma(g)\|_{1+c}\|g\|_2^{2c/(1+c)}.$$

*Proof.* The Poincaré inequality insures that for all smooth function  $h$ ,

$$\int h^2 d\mu - \left(\int h d\mu\right)^2 \leq C_P \int \Gamma(h) d\mu.$$

This inequality applied to  $g^p$  gives

$$\|g\|_{2p}^p - \|g\|_p^{2p} \leq p^2 C_P \int \Gamma(g) g^{2(p-1)} d\mu.$$

We then apply the Hölder inequality with exponents  $1 + c$  and  $(c + 1)/c$  to get

$$\|g\|_{2p}^{2p} - \|g\|_p^{2p} \leq p^2 C_P \|\Gamma(g)\|_{1+c} \left(\int g^{2(p-1)(c+1)/c} d\mu\right)^{c/(c+1)}.$$

We then take  $p = 1 + c/(c + 1)$ , since  $p < 2$  we have  $\|g\|_p^{2p} \leq \|g\|_2^{2p}$ , which concludes the proof.  $\square$

**Proposition 76.** *Assume that  $\mathcal{L}$  satisfies the  $CD(1 - N, N)$  condition (with  $N < -1$ ) and assume that  $\mathcal{L}$  admits an eigenfunction  $f$  with eigenvalue  $-N + \varepsilon$  with  $\varepsilon \leq 1$ . We assume moreover that  $f \in L^2(\mu)$ ,  $\Gamma(f) \in L^{1+c}(\mu)$  for some  $c > 0$ , and  $\int \Gamma(f) d\mu = \frac{N}{N+1}$ . Then*

$$\|\Gamma(f) - f^2 - 1\|_1 \leq C\varepsilon \log(2/\varepsilon)$$

for a numerical constant  $C$  that only depends on  $c$  and  $N$ .

*Proof.* The proof is a straightforward combination of Lemmas 69, 74 and 75. Note that with the normalization of  $\int \Gamma(f) d\mu = \frac{N}{N+1}$ , we have

$$\int f^2 d\mu = \frac{N}{(N+1)(\varepsilon - N)}$$

and

$$\int (\Gamma(f) - f^2) d\mu = 1 + \frac{\varepsilon}{(N+1)(N-\varepsilon)}.$$

□

### 5.5.2 Stein's method for generalized Cauchy distributions

In this Section, we shall establish a version of Stein's lemma for generalized Cauchy distributions in dimension one, following the standard approach of explicitly solving a Poisson equation. The result is the following:

**Theorem 77** (Stein's method for Cauchy distributions). *Let  $N < -1$ , the  $W_1$  distance between a probability measure  $\nu$  on  $\mathbb{R}$  to the generalized Cauchy distribution  $\mu_N^-$  satisfies*

$$(5.21) \quad W_1(\nu, \mu_N^-) \leq \sup \left\{ \int [(1+x^2)g'(x) + Nxg(x)] d\nu \right\},$$

where the supremum is running over all absolutely continuous function  $g$  on  $\mathbb{R}$  such that

$$\|g\|_\infty \leq \max \left( \frac{4N+3}{|N|(N+1)}, \frac{9}{2} \left( \frac{N}{N+1} \right)^2 + \frac{N}{N+1} \right),$$

$$\text{and } \|g'\|_\infty \leq 1 + \left( \frac{3}{2} + \frac{N}{N+1} \right) \frac{N}{N+1}.$$

The constants appearing here are not sharp, but have the sharp order of magnitude as  $N$  goes to  $-\infty$ .

As a corollary, we have the analogue of [153, Thm. 1] for generalized Cauchy target distributions:

**Corollary 78.** *Let  $f$  be an eigenfunction of a diffusion operator  $-\mathcal{L}$  with eigenvalue  $\lambda$  and invariant probability measure  $\mu$ , and let  $\nu$  be the pushforward of  $\mu$  by  $f$ . Then for  $N < -1$  we have*

$$W_1(\nu, \mu_N^-) \leq \left( 1 + \left( \frac{3}{2} + \frac{N}{N+1} \right) \frac{N}{N+1} \right) \|\Gamma(f) - f^2 - 1\|_{L^1(\mu)}$$

$$+ |\lambda + N| \max \left( \frac{4|N|+3}{|N||N+1|}, \frac{9}{2} \left( \frac{N}{N+1} \right)^2 + \frac{N}{N+1} \right) \|f^2\|_{L^1(\mu)}.$$

*Proof of Corollary 78.* Once again, let  $\nu = \mu \circ f^{-1}$  and  $g$  be a smooth test function.

$$\begin{aligned} \lambda \int xg(x) d\nu &= \lambda \int fg \circ f d\mu = \int (-\mathcal{L}f)g \circ f d\mu = \int \Gamma(f, g \circ f) d\mu = \int g' \circ f \Gamma(f) d\mu \\ &= \int g' \circ f (\Gamma(f) - f^2 - 1) d\mu + \int (x^2 + 1)g' d\nu. \end{aligned}$$

Hence

$$\left| \int \left[ (1+x^2)g'(x) + Nxg(x) \right] d\nu \right| \leq \|g'\|_\infty \|\Gamma(f) - f^2 - 1\|_1 + |\lambda + N| \|g\|_\infty \|f\|_2.$$

We can then apply Theorem 77 to conclude.  $\square$

The proof of Theorem 77 relies on Stein's method, as briefly presented in Section 5.3.3. To begin with, we notice that the probability measure  $\mu_N^- := \frac{1}{Z^-} (1+x^2)^{\frac{N}{2}-1}$  on  $\mathbb{R}$ , with negative dimension  $N < -1$ , is characterized by the integration-by-parts formula

$$\int (1+x^2)g'(x) d\mu_N(x) = - \int Nxg(x) d\mu_N(x).$$

The associated Stein equation is then

$$(5.22) \quad (1+x^2)g' + Nxg = h - \int h d\mu_N^-$$

for some function  $h$ . The solution of interest is given by

$$(5.23) \quad g(x) = (1+x^2)^{-\frac{N}{2}} \int_{-\infty}^x (1+t^2)^{\frac{N}{2}-1} \left( h(t) - \int h d\mu_N^- \right) dt$$

$$(5.24) \quad = - (1+x^2)^{-\frac{N}{2}} \int_x^{+\infty} (1+t^2)^{\frac{N}{2}-1} \left( h(t) - \int h d\mu_N^- \right) dt$$

*Proof of Theorem 77.* For all absolutely continuous  $h$  such that  $\|h\|_{Lip} \leq 1$ , let  $g_h$  be the solution (5.23) of

$$(1+x^2)g'_h + Nxg_h = h - \int h d\mu_N^-.$$

Then we have

$$W_1(\nu, \mu_N^-) = \sup_{\|h\|_{Lip} \leq 1} \int \left( h - \int h d\mu_N^- \right) d\nu = \sup_{g_h} \int \left( (1+x^2)g'_h + Nxg_h \right) d\nu,$$

and bounds on  $g_h$  and  $g'_h$  given by Lemma 80 below complete the proof.  $\square$

To prove Lemma 80, we need the technical lemma below.

**Lemma 79.** *We have the following estimates on the cumulative distribution function of  $\mu_N^-$ :*

$$\forall x < 0, \quad \int_{-\infty}^x \frac{1}{Z^-} (1+t^2)^{\frac{N}{2}-1} dt \leq \min \left( \frac{1}{2}, \frac{1}{|NZ^-x|} \right) (1+x^2)^{\frac{N}{2}},$$

$$\forall x > 0, \quad \int_x^{+\infty} \frac{1}{Z^-} (1+t^2)^{\frac{N}{2}-1} dt \leq \min \left( \frac{1}{2}, \frac{1}{|NZ^-x|} \right) (1+x^2)^{\frac{N}{2}}.$$

Moreover,

$$\forall x \leq 0, \quad \int_{-\infty}^x \frac{1}{C_N} (1+t^2)^{\frac{N}{2}} dt \leq \frac{1}{2} (1+x^2)^{\frac{N}{2}+1},$$

where  $C_N := \int_{\mathbb{R}} (1+t^2)^{\frac{N}{2}} dt$ , and

$$\forall x < 0, \quad \int_{-\infty}^x \frac{1}{C_N} (1+t^2)^{\frac{N}{2}} dt \leq \frac{1}{(N+1)C_N x} (1+x^2)^{\frac{N}{2}+1}.$$

*Proof.* These bounds can be straightforwardly established by studying the functions and their monotonicity.

Let us show the first estimate. Let  $h(x) := \int_{-\infty}^x \frac{1}{Z^-} (1+t^2)^{\frac{N}{2}-1} dt - \frac{1}{2} (1+x^2)^{\frac{N}{2}}$ .

We have

$$h'(x) = (1+x^2)^{\frac{N}{2}-1} \left[ \frac{1}{Z^-} - \frac{N}{2}x \right].$$

Hence  $h'$  is decreasing on  $(-\infty, \frac{2}{NZ^-})$  and increasing on  $(\frac{2}{NZ^-}, 0)$ . Now,  $h(-\infty) = 0$  and  $h(0) = 0$  because  $\mu_N^-$  is symmetric. Therefore,  $h(x) \leq 0$ .

Let  $h_1(x) := \int_{-\infty}^x \frac{1}{Z^-} (1+t^2)^{\frac{N}{2}-1} dt - \frac{1}{NZ^-x} (1+x^2)^{\frac{N}{2}}$ .

We have

$$h'_1(x) = \frac{1}{NZ^-x^2} (1+x^2)^{\frac{N}{2}} < 0.$$

Hence  $h_1(x) \leq h_1(-\infty) = 0$ .

The second estimate follows from the first one by symmetry.

With regards to the third estimate, let  $h(x) := \int_{-\infty}^x \frac{1}{C_N} (1+t^2)^{\frac{N}{2}} dt - \frac{1}{2} (1+x^2)^{\frac{N}{2}+1}$ . Then

$$h'(x) = (1+x^2)^{\frac{N}{2}} \left( \frac{1}{C_N} - \left( \frac{N}{2} + 1 \right) x \right),$$

and there are two cases to consider. Either  $N \leq -2$  and then  $h$  is decreasing on  $(-\infty, \frac{N/2+1}{C_N})$ , increasing on  $(\frac{N/2+1}{C_N}, 0)$  and  $h(-\infty) = h(0) = 0$ , or  $N \in (-2, -1)$  and then  $h$  is increasing on  $\mathbb{R}_-$  and  $h(0) = 0$ . The result therefore stands in both cases.

Finally, let  $h_1(x) := \int_{-\infty}^x \frac{1}{C_N} (1+t^2)^{\frac{N}{2}} dt - \frac{1}{(N+1)C_N x} (1+x^2)^{\frac{N}{2}+1}$ . Then

$$h'_1(x) = \frac{1}{(N+1)C_N x^2} (1+x^2)^{\frac{N}{2}} < 0,$$

hence  $h_1$  is decreasing, and  $h_1(-\infty) = 0$ . The final estimate immediately follows.  $\square$

We shall now establish a priori bounds on solutions to the Poisson equation (5.22).

**Lemma 80.** *If  $h$  is absolutely continuous and  $\mu_N^-$ -centered, then the solution (5.23) satisfies*

$$\|g\|_\infty \leq L_N \|h'\|_\infty \quad \text{and} \quad \|g'\|_\infty \leq K_N \|h'\|_\infty$$

where

$$L_N = \max \left( \frac{4|N|+3}{N(N+1)}, \frac{9}{2} \left( \frac{N}{N+1} \right)^2 + \frac{N}{N+1} \right),$$

$$\text{and} \quad K_N = 1 + \left( \frac{3}{2} + \frac{N}{N+1} \right) \frac{N}{N+1}.$$

*Proof.* Let  $q(x) := \int_{-\infty}^x \frac{1}{Z^-} (1+t^2)^{\frac{N}{2}-1} dt$ , and let us rewrite the solution  $g$  and its derivative. We have

$$h(x) - \int h d\mu_N^- = \int_{-\infty}^x h'(t)q(t) dt - \int_x^{+\infty} h'(t)(1-q(t)) dt$$

Combined with (5.23), we get

$$(5.25) \quad g(x) = -Z^- \frac{1-q(x)}{(1+x^2)^{\frac{N}{2}}} \int_{-\infty}^x h'(t)q(t) dt - Z^- \frac{q(x)}{(1+x^2)^{\frac{N}{2}}} \int_x^{+\infty} h'(t)(1-q(t)) dt$$

Finally, since  $g$  is solution to (5.22), we obtain

$$\begin{aligned} g'(x) &= \frac{1}{1+x^2} \left( 1 + NZ^-(1-q(x))x(1+x^2)^{-\frac{N}{2}} \right) \int_{-\infty}^x h'(t)q(t) dt \\ &\quad - \frac{1}{1+x^2} \left( 1 - NZ^-q(x)x(1+x^2)^{-\frac{N}{2}} \right) \int_x^{+\infty} h'(t)(1-q(t)) dt. \end{aligned}$$

Hence

$$\|g\|_\infty \leq Z^- \sup_{x \in \mathbb{R}} (b_1(x) + b_2(x)) \|h'\|_\infty \quad \text{and} \quad \|g'\|_\infty \leq \sup_{x \in \mathbb{R}} (a_1(x) + a_2(x)) \|h'\|_\infty$$

where

$$\begin{aligned} a_1(x) &:= \frac{1}{1+x^2} \left| 1 + NZ^-(1-q(x))x(1+x^2)^{-\frac{N}{2}} \right| \int_{-\infty}^x q(t) dt, \\ a_2(x) &:= \frac{1}{1+x^2} \left| 1 - NZ^-q(x)x(1+x^2)^{-\frac{N}{2}} \right| \int_x^{+\infty} (1-q(t)) dt, \\ b_1(x) &:= (1-q(x))(1+x^2)^{-\frac{N}{2}} \int_{-\infty}^x q(t) dt, \end{aligned}$$

and

$$b_2(x) := q(x)(1+x^2)^{-\frac{N}{2}} \int_{-\infty}^x (1-q(t)) dt.$$

It remains to show that  $K_N := \sup_{x \in \mathbb{R}} (a_1(x) + a_2(x))$  and  $L_N := Z^- \sup_{x \in \mathbb{R}} (b_1(x) + b_2(x))$  are finite.

Let us begin with  $K_N$ . Since  $\mu_N$  is symmetric,  $a_1(-x) = a_2(x)$ , so  $a_1 + a_2$  is symmetric. Moreover, with Lemma 79 and since  $Z^- \leq C_N$ :

$$\begin{aligned} \forall x \leq 0, \quad a_1(x) &= (1+x^2)^{-1} \int_{-\infty}^x q(t) dt + NZ^-(1-q(x))x(1+x^2)^{-\frac{N}{2}-1} \int_{-\infty}^x q(t) dt \\ &\leq (1+x^2)^{-1} \int_{-\infty}^x \frac{1}{2}(1+t^2)^{\frac{N}{2}} dt + NZ^-x(1+x^2)^{-\frac{N}{2}-1} \int_{-\infty}^x \frac{1}{2}(1+t^2)^{\frac{N}{2}} dt \\ &\leq \frac{C_N}{4}(1+x^2)^{-1}(1+x^2)^{\frac{N}{2}+1} + \frac{1}{2} \frac{NZ^-}{N+1} \\ &\leq \frac{C_N}{4} + \frac{1}{2} \frac{NZ^-}{N+1} \leq \frac{1}{2} C_N \left( \frac{1}{2} + \frac{N}{N+1} \right), \end{aligned}$$

and

$$\begin{aligned} \forall x > 0, \quad \int_{-\infty}^x q(t) dt &= \int_{-\infty}^0 q(t) dt + \int_0^x q(t) dt \\ &\leq \frac{1}{2} \int_{-\infty}^0 (1+t^2)^{\frac{N}{2}} dt + \int_0^x dt \\ &= \frac{C_N}{4} + x. \end{aligned}$$

Hence,

$$\begin{aligned} \forall x > 0, \quad a_1(x) &= \left| (1+x^2)^{-1} \int_{-\infty}^x q(t) dt + NZ^-(1-q(x))x(1+x^2)^{-\frac{N}{2}-1} \int_{-\infty}^x q(t) dt \right| \\ &\leq (1+x^2)^{-1} \left( \frac{C_N}{4} + x \right) + (1+x^2)^{-1} \left( \frac{C_N}{4} + x \right) \\ &= 1 + \frac{C_N}{2} \end{aligned}$$

Finally,

$$K_N = \sup_{x \in \mathbb{R}} (a_1(-x) + a_1(x)) \leq 1 + \frac{1}{2} \left( \frac{3}{2} + \frac{N}{N+1} \right) \int_{\mathbb{R}} (1+t^2)^{\frac{N}{2}} dt \leq 1 + \left( \frac{3}{2} + \frac{N}{N+1} \right) \frac{N}{N+1}.$$

To bound  $L_N$  the work is very similar. First, we notice the symmetry  $b_1(-x) = b_2(x)$ , then using many times again Lemma 79, we get

$$\begin{aligned} \forall x \leq -1, \quad b_1(x) &\leq \frac{1}{|N|Z^-}, \\ \forall x \in [-1, 0), \quad b_1(x) &\leq C_N, \\ \forall x \in [0, 1], \quad b_1(x) &\leq \frac{1}{2} \left( \frac{C_N}{4} + 1 \right), \\ \forall x \geq 1, \quad b_1(x) &\leq \frac{-1}{NZ^-} \left( \frac{C_N}{4} + 1 \right). \end{aligned}$$

So finally

$$L_N = Z^- \sup_{x \in \mathbb{R}} (b_1(x) + b_2(x)) \leq \max \left( \frac{1}{|N|} \left( \frac{C_N}{4} + 2 \right), Z^- C_N + \frac{Z^-}{2} \left( \frac{C_N}{4} + 1 \right) \right),$$

which boils down to the result since  $Z^- \leq C_N \leq \frac{2N}{N+1}$ . □





# Chapter 6

## On sharp constants in Bobkov inequalities

In this chapter, we study the sharp constant in Bobkov's inequality for isoperimetric problems associated with the Euclidean and the uniform enlargements. In particular, we study the relations between infinite dimensional isoperimetric inequalities and the existence of these constants. A new bound  $\sqrt{2}$  is derived for the sharp Euclidean Bobkov constant of the logistic distribution. The sharp constant of the one-sided exponential distribution is computed in the weaker formulation of the uniform enlargement. In addition, we present a numerical exploration of stationary sets for product measures  $\mu \otimes \mu_\alpha$  where  $\mu$  is the exponential distribution on  $\mathbb{R}_+$  and  $\mu_\alpha$  denotes its  $\alpha$ -dilatation. Our results are based on Huou's interpretation of Bobkov type inequalities [118, Proposition 5.3.1] and the variational formulation of the isoperimetric problem.

### 6.1 Introduction

The Bobkov inequality for a probability distribution  $\mu$  introduced in [33] is equivalent to a minoration of the isoperimetric profile of  $\mu$  by the Gaussian isoperimetric profile  $I_\gamma$  of the Gaussian distribution  $\gamma$ . This is the content of the following theorem proved by F. Barthe and B. Maurey in 2000.

**Theorem 81.** [23] *A probability distribution  $\mu$  on  $(\mathbb{R}^d, |\cdot|)$  satisfies*

$$(6.1) \quad \exists c > 0, \quad I_\mu \geq cI_\gamma$$

*if, and only if, for all locally Lipschitz functions  $f : \mathbb{R}^d \rightarrow [0, 1]$ ,*

$$(6.2) \quad I_\gamma \left( \int_{\mathbb{R}^d} f d\mu \right) \leq \int_{\mathbb{R}^d} \sqrt{I_\gamma(f)^2 + \frac{1}{c^2} |\nabla f|^2} d\mu.$$

Let us specify that when  $\mu$  is a probability distribution, we denote by

$$I_\mu(p) := \inf \{ \text{Per}_\mu(A) \mid \mu(A) = p \},$$

its isoperimetric profile.

A natural extension of Bobkov's inequality consists in replacing the Gaussian isoperimetric profile  $I_\gamma$  in (6.2) by the isoperimetric profile  $I_\nu$  of another probability distribution  $\nu$ . Theorem 81 leads to the question whether these Bobkov-type inequalities can be characterized in term of a comparison of isoperimetric profiles as Inequality (6.1). In his PhD thesis, B. Huou showed that these inequalities are related to exact isoperimetric problems in the following way.

**Theorem 82.** (Huou's theorem [118, Section 5]) Let  $\mu$  be a probability measure on  $\mathbb{R}^d$ , and  $\nu$  be an even log-concave probability measure on  $\mathbb{R}$  with bounded Lebesgue density. Then the Bobkov type inequality:

$$(6.3) \quad \text{for all locally Lipschitz } f : \mathbb{R}^d \rightarrow [0, 1], \quad I_\nu \left( \int_X f d\mu \right) \leq \int_{\mathbb{R}^d} \sqrt{I_\nu(f)^2 + |\nabla f|^2} d\mu$$

holds if and only if half-spaces of the form  $\mathbb{R}^d \times (-\infty, t)$  are solution to the isoperimetric problem on  $\mathbb{R}^{d+1}$  associated with the Euclidean distance and the product measure  $\mu \otimes \nu$ .

The proof of Theorem 82 relies on the following symmetrization method:

$$\forall f : \mathbb{R}^d \rightarrow [0, 1], \text{ locally Lipschitz,} \quad \text{Per}_{\mu \otimes \nu}(E_\nu(f)) = \int_{\mathbb{R}^d} \sqrt{I_\nu(f)^2 + |\nabla f|^2} d\mu,$$

where

$$E_\nu(f) := \{(x, t) \in \mathbb{R}^d \times \mathbb{R} \mid t < \Phi_\nu^{-1}(f(x))\}.$$

Recall that  $\Phi_\nu$  denotes the cumulative distribution function of  $\nu$ . Let us sketch the proof.

### Proof of Theorem 82

Let  $A \subset \mathbb{R}^d \times \mathbb{R}$ , and set  $A^x := \{t \in \mathbb{R} \mid (x, t) \in A\}$ . Let  $f : \mathbb{R}^d \rightarrow [0, 1]$  be given by  $f(x) = \nu(A^x)$ . Then the symmetrized set  $A^* := E_\nu(f)$  of  $A$  satisfies  $\mu \otimes \nu(A) = \mu \otimes \nu(A^*)$

$$(6.4) \quad \text{Per}_{\mu \otimes \nu}(A^*) \leq \text{Per}_{\mu \otimes \nu}(A).$$

The inequality (6.4) is proven in Proposition 2.7.13 in the thesis [118] and makes use of the symmetry of  $\nu$ . Moreover,

$$q := \mu \otimes \nu(A) = \int_{x \times \mathbb{R}} \mathbf{1}_{A^*} d\nu(t) d\mu(x) = \int_{\mathbb{R}^d} f(x) d\mu(x).$$

So if  $B := \mathbb{R}^d \times (-\infty, \Phi_\nu^{-1}(q))$ , then  $\text{Per}_{\mu \otimes \nu}(B) = I_\nu(q)$ , and hence the Bobkov inequality (6.3) gives

$$\text{Per}_{\mu \otimes \nu}(B) \leq \text{Per}_{\mu \otimes \nu}(A^*) \leq \text{Per}_{\mu \otimes \nu}(A).$$

Conversely, if half-spaces are solution of the product isoperimetric problem, then

$$\text{Per}_{\nu \otimes \mu} \left( \mathbb{R}^d \times \left( -\infty, \Phi_\nu^{-1} \left( \int_{\mathbb{R}^d} f(x) d\mu(x) \right) \right) \right) \leq \text{Per}_{\mu \otimes \nu}(E_\nu(f)),$$

which is exactly (6.3) by the property of symmetrization.  $\square$

Let  $\nu$  be a probability measure in  $\mathbb{R}$  such that for all probability distributions  $\mu$  on  $\mathbb{R}$ , if half-spaces of the form  $\mathbb{R}^d \times (-\infty, t)$  are solution of the isoperimetric problem associated to  $\mu \otimes \nu$ , then  $I_\mu \geq I_\nu$ . Indeed, in this case,  $I_\nu = I_{\mu \otimes \nu} \leq I_\mu$ . However, the converse statement is only valid if  $\nu = \gamma$  is the normal distribution. Indeed, let  $\nu$  be such that for all  $\mu$ , if  $I_\mu \geq I_\nu$ , then half-spaces of the form  $\mathbb{R}^d \times (-\infty, t)$  are solution of the isoperimetric problem associated to  $\mu \otimes \nu$ . Since those horizontal half-spaces are obviously solution of the isoperimetric problem associated to  $\nu \otimes \nu$ , it implies that  $I_\nu \geq I_{\nu \otimes \nu}$ , but the only distribution satisfying this property is the normal distribution (see [38]).

Huou's theorem allows us to link the exact geometric fact for half-spaces to be isoperimetric sets, with the Bobkov-type functional inequality (6.3). Let us apply Huou's theorem in the case where  $\nu = \mu$  is a probability distribution on  $\mathbb{R}$ , equipped with the standard Euclidean distance. Then (6.3) becomes

$$\text{for all } f : \mathbb{R} \rightarrow [0, 1], \text{ locally Lipschitz,} \quad I_\mu \left( \int_{\mathbb{R}} f d\mu \right) \leq \int_{\mathbb{R}} \sqrt{I_\mu(f)^2 + (f')^2} d\mu.$$

Huou's theorem tells us that this functional inequality is satisfied if and only if the horizontal half-spaces are isoperimetric sets for  $\mu \otimes \mu$ . However, we know that this is only the case for the Gaussian distribution. We are therefore led to transform the inequality slightly by adding a constant  $\alpha \geq 1$ :

$$(6.5) \quad \text{for all } f : \mathbb{R} \rightarrow [0, 1], \text{ locally Lipschitz, } I_\mu \left( \int_{\mathbb{R}} f d\mu \right) \leq \int_{\mathbb{R}} \sqrt{I_\mu(f)^2 + (\alpha f')^2} d\mu.$$

For example, the logistic measure satisfies (6.5) with  $\alpha = 2\sqrt{6}$  (see [118, Corollary 5.4.8]). An interesting feature of (6.5) is that it can be translated into (6.3) by dilating the measure  $\nu$ . Indeed, let  $\mu_\alpha$  be the  $\alpha$ -dilatation of  $\mu$ . In other words, if  $X$  is a random variable drawn with probability distribution  $\mu$ , then  $\alpha X$  is drawn with probability distribution  $\mu_\alpha$ . In terms of densities, if  $\phi(t)$  is the density of  $\mu$  with respect to the Lebesgue measure, then the density of  $\mu_\alpha$  is given by  $\frac{1}{\alpha}\phi(\frac{t}{\alpha})$ . Moreover, the cumulative distribution function of  $\alpha X$  is given by  $\Phi_\mu \left( \frac{t}{\alpha} \right)$ , and finally the isoperimetric profile of  $\mu_\alpha$  is given by  $I_{\mu_\alpha}(q) = \frac{I_\mu(q)}{\alpha}$ . Therefore, the inequality (6.5) is equivalent to

$$\text{for all } f : \mathbb{R} \rightarrow [0, 1], \text{ locally Lipschitz, } I_{\mu_\alpha} \left( \int_{\mathbb{R}} f d\mu \right) \leq \int_{\mathbb{R}} \sqrt{I_{\mu_\alpha}(f)^2 + (f')^2} d\mu,$$

which, by Huou's theorem, is itself equivalent in terms of isoperimetry to all horizontal half-spaces being isoperimetric sets for  $\mu \otimes \mu_\alpha$ . Moreover, it is clear that the half-spaces are isoperimetric sets for  $\mu \otimes \mu_\alpha$  if and only if the isoperimetric profile of  $\mu \otimes \mu_\alpha$  is

$$I_{\mu \otimes \mu_\alpha} = I_{\mu_\alpha} = \frac{1}{\alpha} I_\mu.$$

Although the following monotonicity property is not so obvious from the isoperimetric point of view, it is easy to prove with Huou's theorem from the point of view of functional inequalities.

**Lemma 83.** *If  $I_{\mu \otimes \mu_\alpha} = \frac{1}{\alpha} I_\mu$  for some  $\alpha \geq 1$ , then  $\forall \beta \geq \alpha$ ,  $I_{\mu \otimes \mu_\beta} = \frac{1}{\beta} I_\mu$  is also true.*

**Proof** By the theorem 82 we are reduced to proving the same monotonicity property for the Bobkov inequality (6.5), which is obvious.  $\square$

The problem of finding the best constant in the Bobkov inequality (6.5) is then translated into the following exact isoperimetric problem:

**Problem 1.** *Find the best (i.e. the smallest) constant  $\alpha \geq 1$  such that  $I_{\mu \otimes \mu_\alpha} = \frac{1}{\alpha} I_\mu$ , where  $\mu$  is a probability distribution on  $\mathbb{R}$  and  $I_{\mu \otimes \mu_\alpha}$  denotes the isoperimetric profile associated to the Euclidean enlargement.*

In the following we will study Problem 1 for the logistic measure on  $\mathbb{R}$  and the exponential measure on  $\mathbb{R}_+$ . In Section 6.5, we will also study the reformulation of Problem 1 in terms of the uniform enlargement.

## 6.2 Probability distributions between the exponential and the Gaussian measure

A natural extension of the isoperimetric problem consists in deriving infinite-dimensional isoperimetric inequalities, which are inequalities of the form

$$(6.6) \quad I_{\mu^\infty} \geq C I_\mu,$$

where

$$I_{\mu^\infty} := \inf_{n \geq 1} I_{\mu^{\otimes n}}.$$

The infinite dimensional isoperimetric profile  $I_{\mu^\infty}$  is either trivial or greater than a multiple of the exponential isoperimetric profile  $I(p) = \min(p, 1 - p)$  (see [205]). Moreover, a probability measure  $\mu$  satisfying (6.6) has an intermediate behaviour between the exponential and the Gaussian distribution (see [21]).

The existence of a constant  $\alpha$  solving Problem 1 is stronger than the infinite dimensional isoperimetric inequality. Indeed, let  $\mu$  be an even log-concave probability distribution on  $\mathbb{R}$ , and let  $\alpha$  be such that  $I_{\mu \otimes \mu_\alpha} \geq I_{\mu_\alpha}$ . Since  $\mu$  is an even log-concave distribution, we can use the comparison theorem for product measures [18, Theorem 8]. It follows then that

$$I_{\mu \otimes \mu \otimes \mu_\alpha} \geq I_{\mu \otimes \mu_\alpha} \geq I_{\mu_\alpha}.$$

By iterating this procedure, we get that for all  $n \geq 1$ ,

$$I_{\mu^{\otimes n} \otimes \mu_\alpha} \geq I_{\mu \otimes \mu_\alpha} = I_{\mu_\alpha}.$$

Using that  $I_{\mu \otimes \mu} \geq I_{\mu \otimes \mu \otimes \mu_\alpha}$  and that  $I_{\mu_\alpha} = \frac{1}{\alpha} I_\mu$ , we therefore get that  $\mu$  satisfies (6.6), and moreover, the sharp solution  $\alpha$  of Problem 1 and the sharp constant in (6.6) are related by

$$C \geq \frac{1}{\alpha}.$$

The converse statement does not hold. Indeed, for all  $p \in [1, 2]$ , let  $d\mu_p(t) = \frac{1}{Z_p} e^{-|t|^p} dt$ , with  $Z_p$  a normalization constant. In [20], F. Barthe, P. Cattiaux and C. Roberto proved those probability distributions between the exponential and the Gaussian satisfy infinite dimensional isoperimetric inequalities (6.6). Moreover, the constant can be chosen uniformly.

**Theorem 84.** [20, Theorem 46] *It exists a constant  $K > 0$  such that for all  $p \in [1, 2]$ ,*

$$I_{\mu_p^\infty} \geq K I_{\mu_p}.$$

However, for  $p \in (1, 2)$ , there is no  $\alpha$  solving Problem 1. See [118, Proposition 5.5.1]. This is due to the fact that the isoperimetric profile of these distributions  $\mu_p$  is not differentiable at  $\frac{1}{2}$ .

At  $p = 2$ ,  $\mu$  is the Normal distribution and satisfies (6.2) with the isoperimetric profile  $I_\gamma$  and the constant  $c = 1$ , which is the original Bobkov inequality.

At  $p = 1$ ,  $\mu$  is the Laplace distribution, which satisfies (6.5) with the isoperimetric profile  $I_\mu(p) = \min(p, 1 - p)$  and the (non sharp) constant  $\alpha = 2\sqrt{6}$ . The interest in searching for the best constant comes from the fact that the exponential distribution is the limiting case in this family of measures.

The symmetric exponential distribution has a non-smooth density in zero but a smooth isoperimetric profile  $t \mapsto \min(t, 1 - t)$ . Another measure with an exponential tail is the logistic measure, given by the density  $\frac{e^{-x}}{(1+e^{-x})^2}$  with respect to the Lebesgue measure. Thus, the case of the logistic measure can also be considered as a limiting case when  $p = 1$ . The logistic measure satisfies (6.5) with (non sharp) constant  $2\sqrt{6}$ . The advantage of considering the logistic measure instead of the symmetric exponential is that we will be working with a smooth density, but the disadvantage is to have more complicated formulas than with the exponential. Moreover, the isoperimetric profile of the logistic distribution is  $t \mapsto t(1 - t)$  which is equivalent to the one of the exponential.

Another approach to this limiting case  $p = 1$  is to study the exponential distribution  $e^{-x} dx$  on  $\mathbb{R}_+$ . Since this distribution acts on  $\mathbb{R}_+$ , we no longer have the symmetrization method used in the proof of Houou's theorem. However, Problem 1 still makes sense.

## 6.3 The logistic measure

Let  $\mu$  be the logistic measure with density  $f(x) = \frac{e^x}{(e^x+1)^2}$  on  $\mathbb{R}$ . The logistic measure is a log-concave symmetric measure, and its density can be written  $f = e^{-V}$  for some potential  $V$  verifying  $V''(x) = 2f(x)$ .

Let also  $\alpha \geq 1$ . In this section, we study the stationary and stability of the half-spaces  $H_t := \mathbb{R} \times (-\infty, t)$  for the dilated product measure  $\mu \otimes \mu_\alpha$  where  $\mu_\alpha$  has density  $f_\alpha(x) = \frac{1}{\alpha} f(\frac{x}{\alpha})$ . Consequently,  $\mu \otimes \mu_\alpha$  has density

$$e^{\psi(x,y)} = e^{-V(x)-V(\frac{y}{\alpha})-\ln \alpha}.$$

Recall that stationary sets are critical sets for the functional Perimeter restricted to sets of fixed volume, and that stable sets are local minima of the same functional. Since isoperimetric sets are by definition global minima of the Perimeter functional restricted to sets of fixed constant volume, stationarity and stability are necessary (but not sufficient) conditions.

**Fact.** *For all  $\alpha > 0$ , all horizontal half-spaces  $H_t = \mathbb{R} \times (-\infty, t)$  are stationary for  $\mu \otimes \mu_\alpha$ .*

A half-space  $H_t$  is stationary if and only if (Cf. [184]) its generalized mean curvature is constant:

$$H - \nabla \psi \cdot \vec{N} = cte$$

where  $H$  is its mean curvature and  $\vec{N}$  is the Gauss map (a field of normal and normalized vectors on its boundary). Since  $H_t = \mathbb{R} \times (-\infty, t)$ , we have that  $\vec{N}(x, y) = (0, 1)$  is constant, hence  $H = -\operatorname{div} \vec{N} = 0$  and the generalized mean curvature is  $\frac{1}{\alpha} V'(\frac{t}{\alpha})$  so it is constant.

**Proposition 85.** *All  $H_t$  are stable for  $\mu \otimes \mu_\alpha$  if and only if  $\alpha \geq \sqrt{2}$ .*

**Proof** A half-space is stable if and only if (Cf. [184]) it is stationary and  $Q(u, u) \geq 0$  for all  $u \in C_0^\infty(\partial H_t)$  such that  $\int_{\partial H_t} f u da = 0$ , with

$$Q(u, u) = \int_{\partial H_t} f \left( |\nabla_{\partial H_t} u|^2 - |\sigma|^2 u^2 + u^2 \nabla^2 \psi(\vec{N}, \vec{N}) \right) da$$

where  $\sigma$  is the vector of the principal curvature (ie: the eigenvalues of the differential of the Gauss map  $\vec{N}$ ). Here,  $\vec{N} = (0, 1)$  is constant, hence  $\sigma = 0$ . Moreover,  $\partial H_t = \mathbb{R} \times \{t\}$ , so  $\nabla_{\partial H_t} = \frac{\partial}{\partial x}$  and  $\nabla^2 \psi(\vec{N}, \vec{N}) = \frac{\partial^2}{\partial y^2} \psi = -\frac{1}{\alpha^2} V''(\frac{t}{\alpha})$ . Thus

$$Q(u, u) = \int_{\mathbb{R}} \left( \left( \frac{\partial}{\partial x} u(x, t) \right)^2 - \frac{1}{\alpha^2} V''(\frac{t}{\alpha}) u(x, t)^2 \right) f(x) dx.$$

The stability condition is therefore nothing more than the following Poincaré inequality for the logistic measure on  $\mathbb{R}$ :

$$\int_{\mathbb{R}} \left( \frac{\partial}{\partial x} u(x, t) \right)^2 d\mu \geq \frac{1}{\alpha^2} V''(\frac{t}{\alpha}) \int_{\mathbb{R}} u(x, t)^2 d\mu$$

so it is true if and only if (Cf. [19])

$$\frac{1}{\alpha^2} V''(\frac{t}{\alpha}) \leq \frac{1}{4}$$

we can conclude because  $V'' = 2f$  and  $\max f = \frac{1}{4}$ .  $\square$

**Corollary 86.** *The smallest constant  $\alpha$  solving Problem 1 for the logistic measure  $\mu$  satisfies  $\alpha \geq \sqrt{2}$ .*

This constant  $\sqrt{2}$  is better than the already known value  $2\sqrt{6}$ . However we do not know if  $\sqrt{2}$  is the sharp constant or not. It is possible to generalize Proposition 85 in the following way.

**Proposition 87.** *Let  $\mu$  be a probability distribution on  $\mathbb{R}$  with smooth Lebesgue density  $f(x) = e^{-V(x)}$ , and satisfying a Poincaré inequality with sharp constant  $C_P(\mu)$ . Then all half-spaces  $H_t$  are stable for  $\mu \otimes \mu_\alpha$  if and only if  $\alpha \geq \sqrt{\frac{\max V''}{C_P(\mu)}}$ .*

**Proof** A half-space is stable if and only if it is stationary and  $Q(u, u) \geq 0$  for all  $u \in C_0^\infty(\partial H_t)$  such that  $\int_{\partial H_t} f u \, da = 0$ , with

$$Q(u, u) = \int_{\partial H_t} f \left( |\nabla_{\partial H_t} u|^2 - |\sigma|^2 u^2 + u^2 \nabla^2 \psi(\vec{N}, \vec{N}) \right) da$$

where  $\sigma$  is the vector of the principal curvature (ie: the eigenvalues of the differential of the Gauss map  $\vec{N}$ ), and  $\psi(x, y) = -V(x) - V\left(\frac{y}{\alpha}\right) - \ln(\alpha)$  is the log-density of  $\mu \otimes \mu_\alpha$ . Here,  $\vec{N} = (0, 1)$  is constant, so  $\sigma = 0$ . Moreover,  $\partial H_t = \mathbb{R} \times \{t\}$ , so  $\nabla_{\partial H_t} = \frac{\partial}{\partial x}$  and  $\nabla^2 \psi(\vec{N}, \vec{N}) = \frac{\partial^2}{\partial y^2} \psi = -\frac{1}{\alpha^2} V''\left(\frac{y}{\alpha}\right)$ . Thus

$$Q(u, u) = \int_{\mathbb{R}} \left( \left( \frac{\partial}{\partial x} u(x, t) \right)^2 - \frac{1}{\alpha^2} V''\left(\frac{t}{\alpha}\right) u(x, t)^2 \right) f(x) dx.$$

Since it is clear that the half-spaces are stationary, the stability condition is none other than the following Poincaré inequality for the measure  $\mu$  on  $\mathbb{R}$ :

$$\int_{\mathbb{R}} \left( \frac{\partial}{\partial x} u(x, t) \right)^2 d\mu \geq \frac{1}{\alpha^2} V''\left(\frac{t}{\alpha}\right) \int_{\mathbb{R}} u(x, t)^2 d\mu$$

so it is true if and only if

$$\frac{1}{\alpha^2} V''\left(\frac{t}{\alpha}\right) \leq C_P(\mu)$$

which gives the result. □

**Remark 4.** *In the case of the normal distribution, we have  $\sqrt{\frac{\max V''}{C_P(\mu)}} = 1$  which is in fact the sharp dilatation parameter  $\alpha$  solving the problem 1.*

**Corollary 88.** *If  $\mu$  is a probability distribution on  $\mathbb{R}$  with smooth Lebesgue density  $f(x) = e^{-V(x)}$ , and satisfying a Poincaré inequality with sharp constant  $C_P(\mu)$ , then the solution  $\alpha$  of Problem 1 satisfies  $\alpha \geq \sqrt{\frac{\max V''}{C_P(\mu)}}$ , provided that such an  $\alpha$  exists.*

## 6.4 Research of stationary sets for the uniform isoperimetric problem associated with the distributions $\mu \otimes \mu_\alpha$ when $\mu$ is the one-sided exponential distribution.

Thanks to the link given by Theorem 82, when  $\mu$  is the logistic measure or the symmetric exponential distribution, Problem 1 reduces to finding the best dilatation  $\alpha$  such that  $\mu \otimes \mu_\alpha$  admits the half-spaces as isoperimetric sets. As this problem is two-dimensional, the differential equations of stationary and stable sets are easier to deal with and seem to allow at least to improve the already known constants. Recall that stationary sets are critical sets for the perimeter functional restricted to sets of fixed volume, and that stable sets are local minima of the same functional. Since isoperimetric sets are by definition global minima of the perimeter

functional restricted to sets of fixed volume, stationarity and stability are necessary (but not sufficient) conditions. Moreover, let us specify that isoperimetric sets have a smooth boundary. Indeed, if  $M$  is a smooth Riemannian manifold of dimension  $d$  without boundary and  $E$  is an isoperimetric set, we know that except for a set of Hausdorff dimension at most  $d - 7$ ,  $E$  is an analytic submanifold (see [165, Corollary 3.8]). Moreover, if the manifold  $M$  has smooth boundary or it is the product of manifold with smooth boundary, there are no differences between the isoperimetric problems for  $M$  or for the interior of  $M$ . We refer the reader to [182]. We study here the isoperimetric problem in the two-dimensional product space  $(\mathbb{R}_+ \times \mathbb{R}_+, |\cdot|_2, \alpha^{-1}e^{-x-\frac{y}{\alpha}}dxdy)$ , hence isoperimetric sets are regular.

**Proposition 89.** *Let  $\gamma = (\gamma_1, \gamma_2) : [0, l] \rightarrow \mathbb{R}_+ \times \mathbb{R}_+$  be a parametric curve with speed 1 (i.e.  $|\gamma'(s)| = 1, \forall s$ ), and of length  $l$ . Then  $\gamma$  is the boundary of a stationnary set for the measure  $\mu \otimes \mu_\alpha$  when  $\mu$  is the one-sided exponential distribution, if, and only if it meets the boundary of  $\mathbb{R}_+ \times \mathbb{R}_+$  orthogonally and satisfies the following differential equation on  $[0, l]$*

$$(6.7) \quad \gamma_2' \gamma_1'' - \gamma_1' \gamma_2'' - \gamma_2' + \frac{1}{\alpha} \gamma_1' = \text{constant}.$$

By definition, we call boundary of  $\mathbb{R}_+ \times \mathbb{R}_+$  "the set  $\{0\} \times \mathbb{R}_+^* \sqcup \mathbb{R}_+^* \times \{0\}$ ".

**Proof** We know from [56, Corollary 3.3] that a set  $E$  of a weighted smooth Riemannian manifold with boundary  $M$  is stationary if and only if it has constant generalized mean curvature and its boundary orthogonally meets the boundary of  $M$ . The generalized mean curvature is defined by

$$H_\psi = (d - 1)H - \langle \nabla \psi, \vec{N} \rangle,$$

where  $d$  is the dimension of  $M$ ,  $\psi$  is the log-density of the weight  $dm = e^\psi dx$ ,  $N$  is the unit normal vector along  $\partial E$ , and  $H = \frac{-1}{d-1} \text{div}_{\partial E} \vec{N}$  is the mean curvature of  $\partial E$ . In our case,  $M = \mathbb{R}_+ \times \mathbb{R}_+$  is the cartesian product of two smooth weighted Riemannian manifold  $(\mathbb{R}_+, |\cdot|, e^{-t} dt)$  and  $(\mathbb{R}_+, |\cdot|, \frac{1}{\alpha} e^{-\frac{t}{\alpha}})$ , so the result can be applied.

Since  $\gamma$  parametrizes the boundary of a set  $E$ , we have that the unit normal is  $\vec{N} = \pm(-\gamma_2', \gamma_1')$  and  $H = \pm \gamma'' \cdot (-\gamma_2', \gamma_1')$  is the curvature of the curve. Moreover,  $\psi(x, y) = -x - \frac{y}{\alpha} - \ln \alpha$ , so  $\nabla \psi = (-1, \frac{-1}{\alpha})$ . Finally, the fact that the generalized mean curvature must be constant gives Equation (6.7).  $\square$

Denoting by  $\vec{N} = (-\gamma_2', \gamma_1')$  the normal vector, we see that  $(\gamma', \vec{N})$  is an orthonormal basis of  $\mathbb{R}^2$ . Moreover, since  $\gamma$  is parametrized at 1, we obtain that  $\gamma''$  and  $\gamma'$  are orthogonal, and thus  $\gamma'' = (\gamma'' \cdot \vec{N}) \vec{N}$ . In addition to this, the differential equation (6.7) can be rewritten

$$\gamma'' \cdot \vec{N} = \text{constant} + \gamma_2' - \frac{1}{\alpha} \gamma_1'.$$

By identification, we finally obtain the following system of differential equations on  $\mathbb{R}_+ \times \mathbb{R}_+$  parametrized by  $c \in \mathbb{R}$ :

$$(6.8) \quad \begin{cases} \gamma_1'' = -\left(c + \gamma_2' - \frac{1}{\alpha} \gamma_1'\right) \gamma_2' \\ \gamma_2'' = \left(c + \gamma_2' - \frac{1}{\alpha} \gamma_1'\right) \gamma_1' \end{cases}$$

We will now proceed to a numerical study of this ODE system. First, we will draw the general shape of the solutions of (6.8). Then, we will keep only the curves meeting orthogonally the boundary of  $\mathbb{R}_+ \times \mathbb{R}_+$ . Finally, we will numerically identify among these curves those which are better competitors for the isoperimetric problem than the half-spaces.



### 6.4.1 General shape of the solutions of (6.8)

In this section, we present the general form of the solutions of (6.8). The graphs have been obtained using the free software GNU Octave. The differential equations (6.8) have been solved numerically with the LSODE package (Livermore solver for ODE's). The LSODE package is the starting point of a collection of ODE initial value solvers called ODEPACK. (see [114])

#### Case $\alpha = 1$

When  $\alpha = 1$ ,  $\mu \otimes \mu_\alpha = \mu \otimes \mu$ , hence by symmetry starting from  $(0, 1)$  we obtain the same curve shapes as starting from  $(1, 0)$ . This can be seen in Figures 6.1 and 6.3. Similarly for the Figures 6.5 and 6.2.

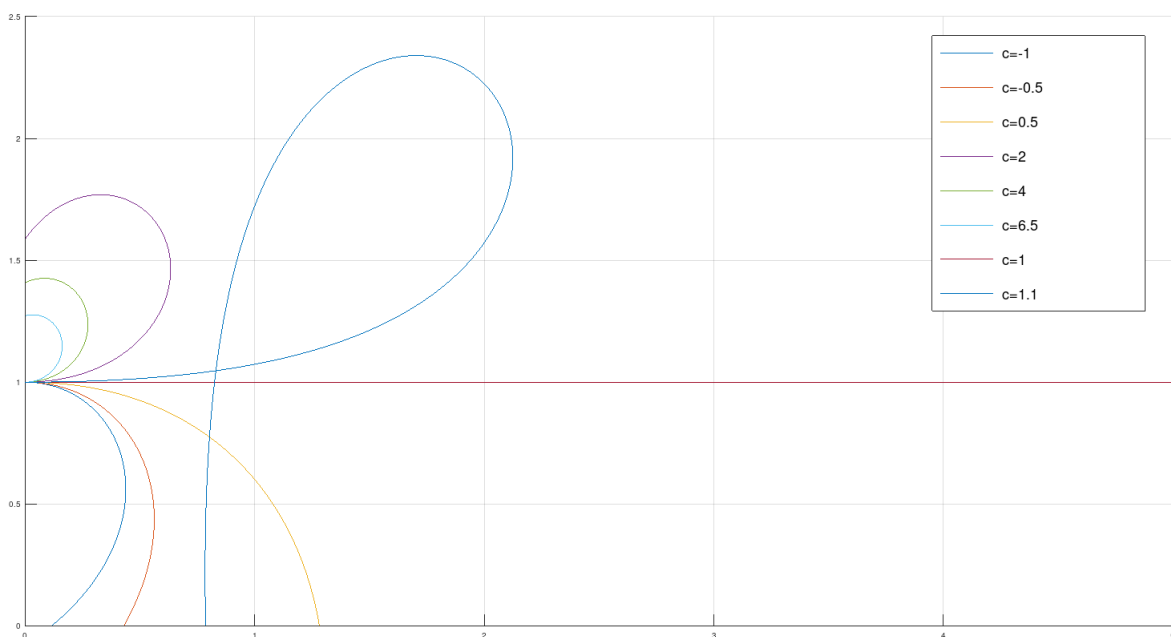


Figure 6.1: The solutions of (6.8) starting at  $(0, 1)$  with speed  $(1, 0)$  for the dilatation parameter  $\alpha = 1$ .

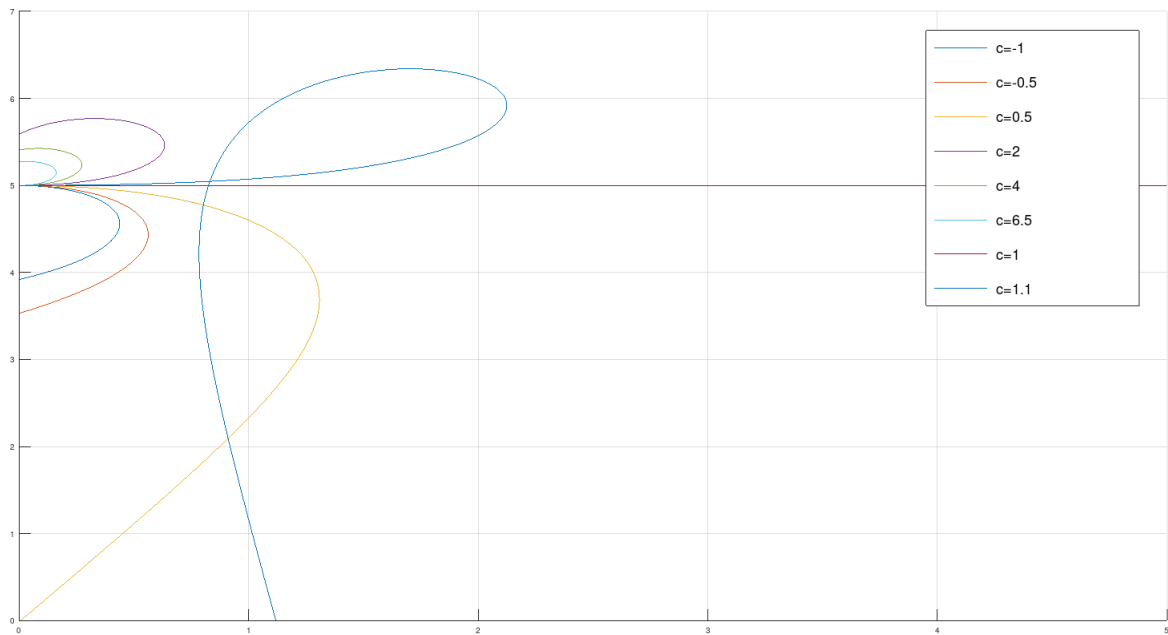


Figure 6.2: The solutions of (6.8) starting at  $(0, 5)$  with speed  $(1, 0)$  for the dilatation parameter  $\alpha = 1$ .

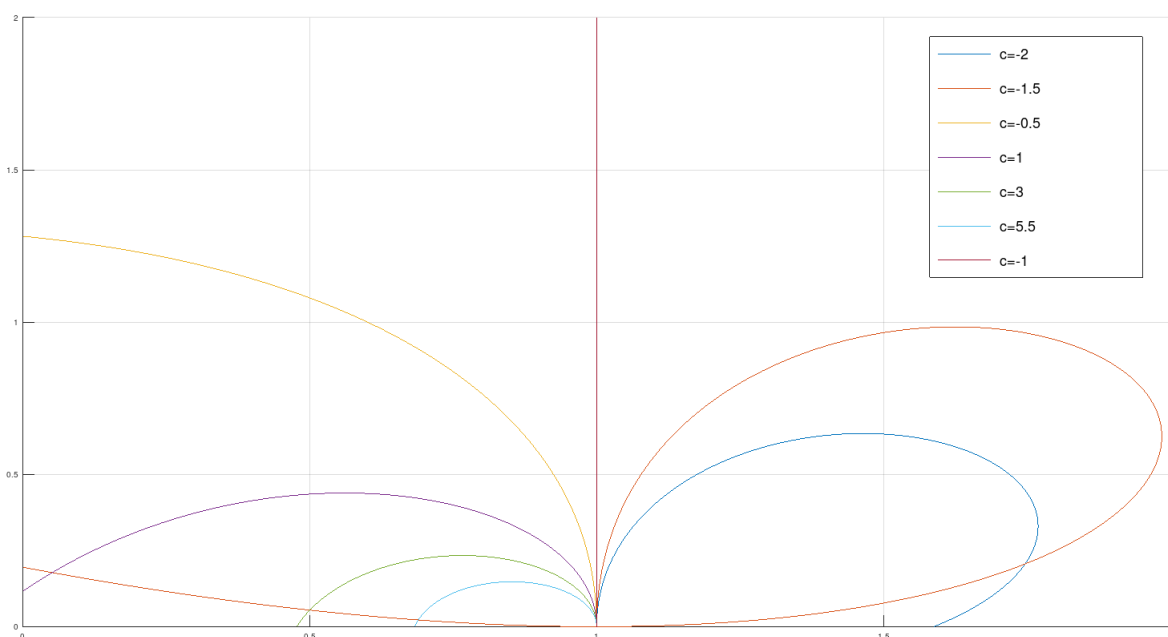


Figure 6.3: The solutions of (6.8) starting at  $(1, 0)$  with speed  $(0, 1)$  for the dilatation parameter  $\alpha = 1$ .

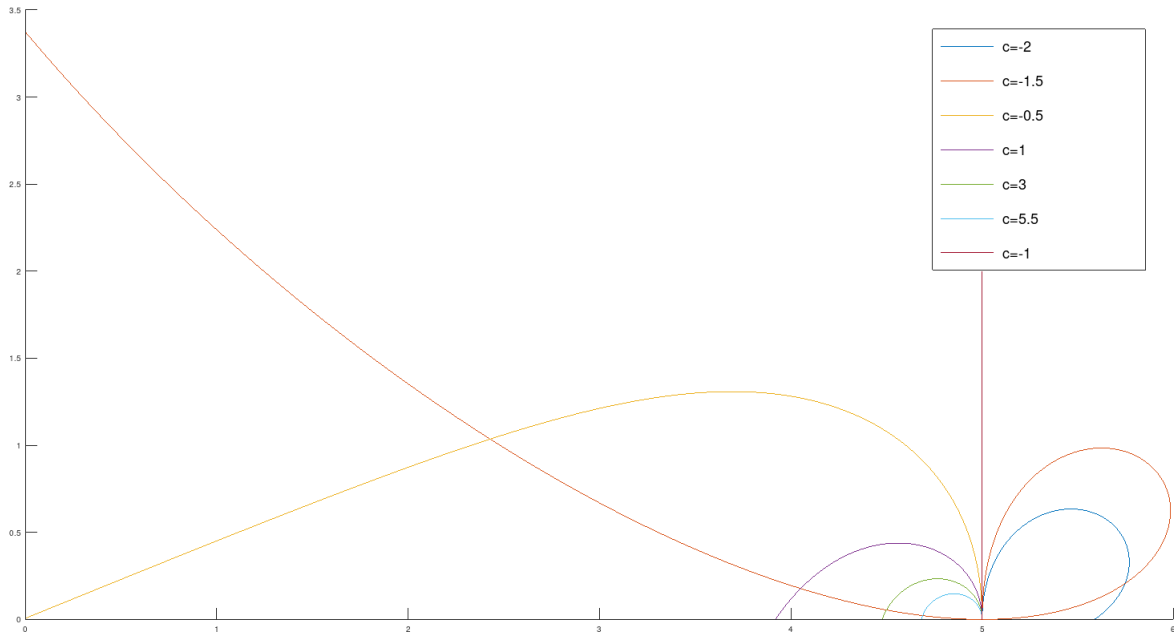


Figure 6.4: The solutions of (6.8) starting at  $(5, 0)$  with speed  $(0, 1)$  for the dilatation parameter  $\alpha = 1$ .

**Case  $\alpha = 2$**

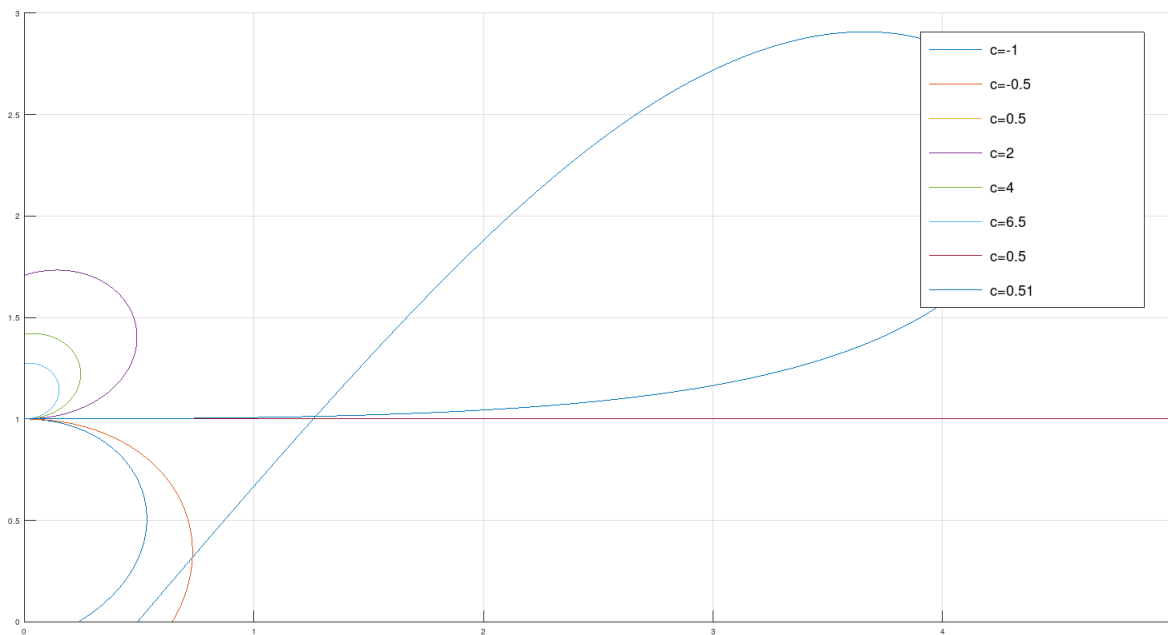


Figure 6.5: The solutions of (6.8) starting at  $(0, 1)$  with speed  $(1, 0)$  for the dilatation parameter  $\alpha = 2$ .

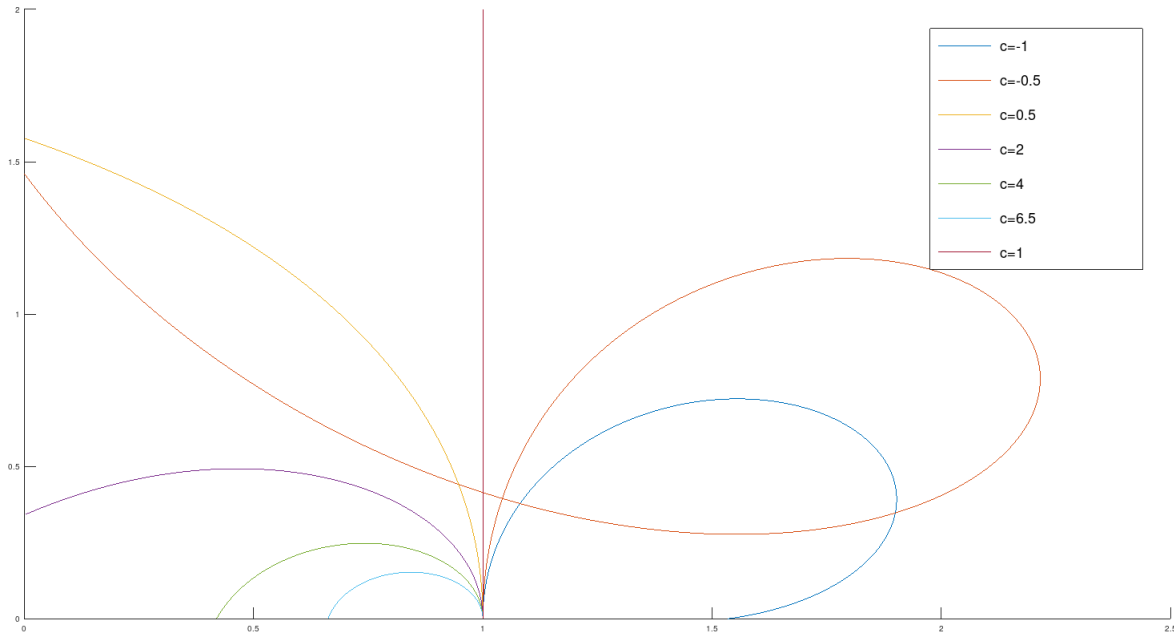


Figure 6.6: The solutions of (6.8) starting at  $(1, 0)$  with speed  $(0, 1)$  for the dilatation parameter  $\alpha = 2$ .

### Qualitative comments on the curves

First of all, we can see that the curves starting from the  $y$ -axis behave in a similar way to those starting from the  $x$ -axis. We will therefore only describe the behaviour of the curves starting from the  $y$ -axis (see figures 6.1, 6.2 and 6.5). The observation is as follows. Given a starting point  $(0, y_0)$ , there are two values  $c_1 < \frac{1}{\alpha} < c_2$  such that

1. for  $c < c_1$ , the curve exits  $\mathbb{R}_+ \times \mathbb{R}_+$  on the  $y$ -axis (see the example  $c = -1$  in Figure 6.2),
2. for  $c = c_1$ , the curve exits through the corner  $(0, 0)$  (see the example  $c = 0.5$  in Figure 6.2),
3. for  $c_1 < c < \frac{1}{\alpha}$ , the curve exits on the  $x$ -axis (see the example  $c = 0.5$  in Figure 6.1),
4. for  $c = \frac{1}{\alpha}$ , the curve is an horizontal straight line,
5. for  $\frac{1}{\alpha} < c < c_2$ , the curve loops and exits through the  $x$ -axis (see the example  $c = 1.1$  in Figure 6.1),
6. for  $c = c_2$ , the curve makes a loop tangent to the  $y$ -axis (it is the same case as the  $c = -1.5$  in Figure 6.3 but transposed to the case of a curve starting from the  $y$ -axis),
7. for  $c > c_2$ , the curve exits on the  $y$ -axis (see the example  $c = 2$  in Figure 6.1).

#### 6.4.2 The solutions of (6.8) that meet the boundary orthogonally

In this section, we are looking for solution curves that meet the boundary of  $\mathbb{R}_+ \times \mathbb{R}_+$  orthogonally. It appears that apart from straight lines (which are the boundary of half-spaces  $\mathbb{R}_+ \times [0, t)$ ), the only good candidates are curves starting from the  $y$ -axis and arriving at the  $x$ -axis and curves starting from the  $x$ -axis and arriving at the  $y$ -axis. Indeed, using the above description, Case 2 and Case 6 do not correspond to orthogonality, Case 5 corresponds to a non-smooth stationary set, and it appears that in Case 1 and Case 7, the exit angle always remains obtuse (respectively acute). The only candidates are therefore the curves of the case 3.

Let us present the method we used. For example, for  $\alpha = 1$ , we see on Figure 6.1 that the orthogonal curve must have its parameter  $c$  between  $-0.5$  and  $0.5$ . We implement a numerical search for the orthogonal parameter  $c$  (see Figure 6.7) between these two bounds, and stop the search when we find an orthogonal curve within  $10^{-1}$ . The orthogonality error is quantified by the first coordinate of the velocity vector of the curve when it exits  $\mathbb{R}_+ \times \mathbb{R}_+$  on the  $x$ -axis.

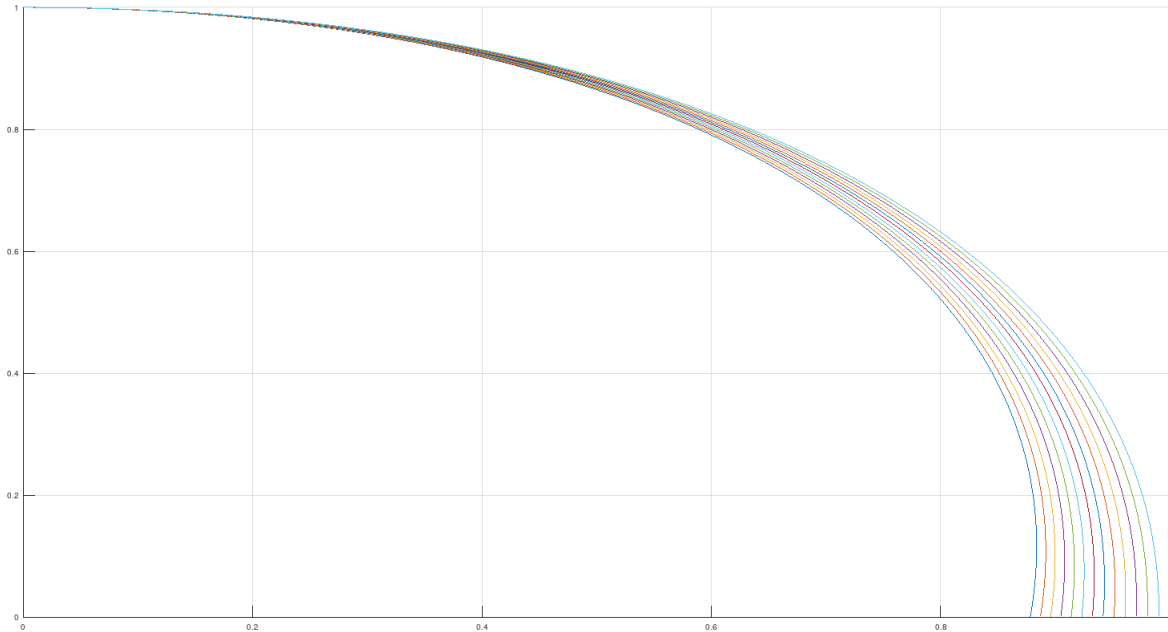


Figure 6.7: Search for orthogonal curves for the dilatation parameter  $\alpha = 1$ .

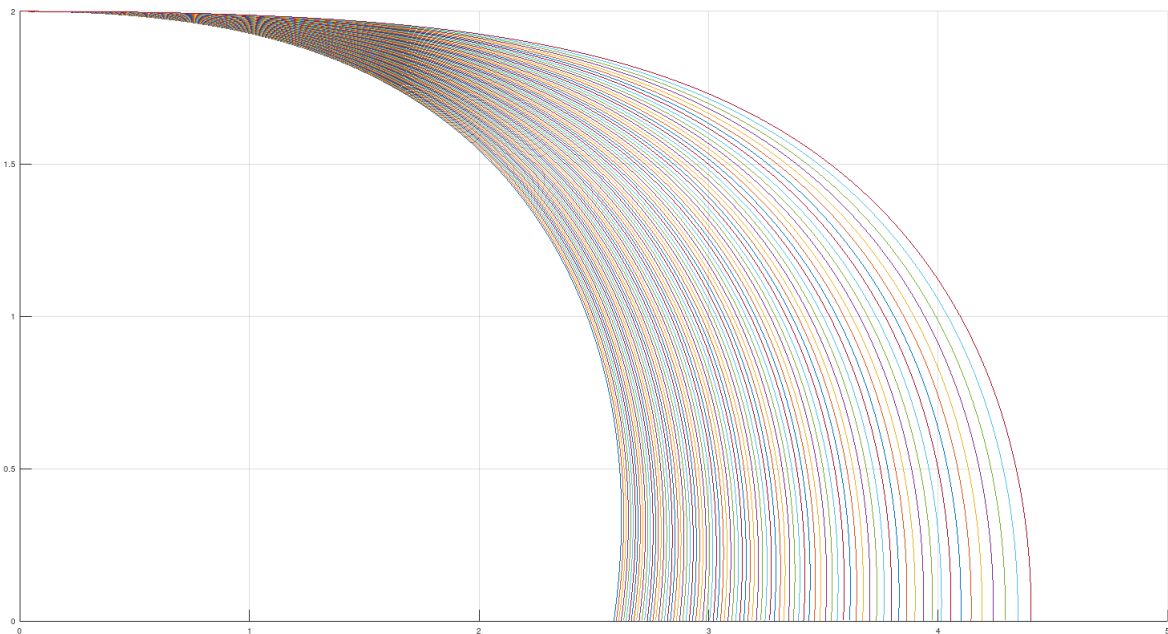


Figure 6.8: Search for orthogonal curves for the dilatation parameter  $\alpha = 2$ .

### Numerical estimation of the perimeter

Once we have an estimate of the orthogonal parameter  $c$ , we therefore have that the curve corresponding to this orthogonal parameter  $c$  is an approximation of a stationary curve for  $\mu \otimes \mu_\alpha$ , i.e. an approximation of the boundary of a stationary set. We can then use the trapezoidal rule to approximate the weighted volume  $(\mu \otimes \mu_\alpha)(S_c)$  of the domain  $S_c$  between

the boundary of  $\mathbb{R}_+ \times \mathbb{R}_+$  and the curve (which contains the corner), and also to approximate the weighted perimeter  $(\mu \otimes \mu_\alpha)^+(S_c)$  of this domain, i.e. the weighted length of the curve. Moreover, we compare it with the weighted perimeter  $(\mu \otimes \mu_\alpha)^+(H_c)$  where  $H_c$  denotes the half-space  $\mathbb{R}_+ \times (0, t_c)$  with  $t_c > 0$  chosen such that  $(\mu \otimes \mu_\alpha)(H_c) = (\mu \otimes \mu_\alpha)(S_c)$ . We summarize the numerical results in Table 6.1. The coloured boxes mean that the competitor  $S_c$  for the isoperimetric problem is better than  $H_c$ . This means that the inequation  $I_{m\mu\otimes\alpha} \geq I_{m\mu\alpha\text{Ipha}}$  does not hold, so such an  $\alpha$  is not a solution to Problem 1.

$\alpha$	starting point	orthogonal $c$	$(\mu \otimes \mu_\alpha)(S_c)$	$(\mu \otimes \mu_\alpha)^+(S_c)$	$(\mu \otimes \mu_\alpha)^+(H_c)$
1	(0, 1)	0.26	0.357	0.442	0.357
1	(0, 2)	0.75	0.717	0.246	0.282
1.1	(0, 2)	0.71	0.713	0.236	0.260
1.5	(0, 2)	0.59	0.678	0.2105	0.2147
2	(0, 1)	0.02	0.245	0.306	0.12250
2	(0, 1.5)	0.34	0.437	0.267	0.218
2	(0, 2)	0.48	0.613	0.19045	0.19062
2	(0, 2.1)	0.499	0.6486	0.1757	0.1757
2.5	(0, 1)	-0.045	0.208	0.264	0.083
2.5	(0, 2)	0.39	0.541	0.18388	0.18376

Table 6.1: Numerical estimates of the perimeter of stationary sets.

It appears that at a starting point lower than (0, 2), the curve is always worse than the competing straight line for the isoperimetric problem, whatever the value of  $\alpha$ . This is explained by the fact that the measure  $\mu \otimes \mu_\alpha$  concentrates most of its mass in the corner of  $\mathbb{R}_+ \times \mathbb{R}_+$ . For  $\alpha = 1$  and (0, 2) as a starting point, the curve is a better competitor than the line in the isoperimetric problem. So this curve may describe an isoperimetric set. The same appears for  $\alpha = 1.1$  and (0, 2) as a starting point. The same seems also to appear for  $\alpha = 1.5$  and (0, 2) as a starting point, but it is slightly less clear because of the precision required. For  $\alpha = 2$  and  $\alpha = 2.5$ , the estimated values of length are too close to the length of the line, so we cannot conclude on the basis of these numerical estimates. We were not able to obtain better estimates due to the precision required for the orthogonal parameter  $c$  in these cases. It shows that the sharp constant  $\alpha$  solving Problem 1 for the one-sided exponential distribution is greater than 1.1. Moreover, with a lower degree of certainty, this suggests that  $\alpha$  lies between 2 and 2.5. We will prove rigorously in Section 6.5.3 that  $\alpha \geq 2$ .

## 6.5 Uniform enlargement

The choice of metric is very important when dealing with the isoperimetric problem for product measures. Until now, the metric used has always been (implicitly or not) the Euclidean distance. In this section, we will use the uniform metric. Let us explain this. If  $m$  is a distribution on a

metric space  $(X, d)$ , its associated weighted perimeter is defined for all measurables  $A \subset X$  by

$$m^+(A) := \liminf_{r \rightarrow 0} \frac{m(A_r \setminus A)}{r},$$

where  $A_r := \{x \in X \mid d(x, A) < r\}$ . There are many possibilities to define the weighted perimeter on  $X \times X$  for the product measure  $m \otimes m$ . The Euclidean enlargement is the most used, and is given by

$$(m \otimes m)_2^+(A) := \liminf_{r \rightarrow 0} \frac{(m \otimes m)(A_r \setminus A)}{r},$$

where  $A_r := \{(x_1, x_2) \in X \times X \mid d_2((x_1, x_2), A) < r\}$ , and

$$d_2((x_1, x_2), (y_1, y_2)) := \sqrt{d(x_1, y_1)^2 + d(x_2, y_2)^2}.$$

Instead, in this section we will use the uniform expansion, which is given by

$$(m \otimes m)_\infty^+(A) := \liminf_{r \rightarrow 0} \frac{(m \otimes m)(A_r \setminus A)}{r},$$

where  $A_r := \{(x_1, x_2) \in X \times X \mid d_\infty((x_1, x_2), A) < r\}$ , and

$$d_\infty((x_1, x_2), (y_1, y_2)) := \max(d(x_1, y_1), d(x_2, y_2)).$$

The fact that  $d_\infty \leq d_2$  means that for any measurable  $A \subset X \times X$ , we have

$$(m \otimes m)_2^+(A) \leq (m \otimes m)_\infty^+(A).$$

Thus the uniform isoperimetric profile controls the Euclidean profile:

$$I_{m \otimes m}^\infty \geq I_{m \otimes m}^{(2)}$$

where  $I_{m \otimes m}^\infty$  denotes the isoperimetric profile for  $m \otimes m$  with respect to the distance  $d_\infty$ , and  $I_{m \otimes m}^{(2)}$  denotes the isoperimetric profile for  $m \otimes m$  with respect to the distance  $d_2$ . In the case of the uniform enlargement, the relevant Bobkov inequality takes the following form (see [34, Section 4]). For any locally Lipschitz function  $f : \mathbb{R}^d \rightarrow [0, 1]$ ,

$$(6.9) \quad I_\nu \left( \int_X f \, d\mu \right) \leq \int_{\mathbb{R}^d} (I_\nu(f) + |\nabla f|_1) \, d\mu,$$

where  $|\cdot|_1$  denotes the  $L^1$  norm of a vector. Using the same symmetrization method as for Theorem 82, we can obtain the following.

**Theorem 90.** *Let  $\mu$  be a probability measure on  $\mathbb{R}^d$ , and  $\nu$  be a log-concave probability measure on  $\mathbb{R}$  with bounded density. Then the Bobkov type inequality:*

$$(6.10) \quad I_\nu \left( \int_X f \, d\mu \right) \leq \int_{\mathbb{R}^d} (I_\nu(f) + |\nabla f|_1) \, d\mu,$$

holds if and only if half-spaces of the type  $\mathbb{R}^d \times (-\infty, t)$  are solution of the isoperimetric problem for the measure  $\mu \otimes \nu$  on  $\mathbb{R}^{d+1}$  associated to the uniform distance.

**Proof** The proof is the same as that of Theorem 82, where the Euclidean symmetrization method is replaced by Lemma 91 below.  $\square$

As with the Euclidean expansion, this theorem leads us to formulate the following problem.

**Problem 2.** *Find the best (i.e. the smallest) constant  $\alpha \geq 1$  such that  $I_{\mu \otimes \mu_\alpha}^\infty = \frac{1}{\alpha} I_\mu$ , where  $\mu$  is a probability distribution on  $\mathbb{R}$  and  $I_{\mu \otimes \mu_\alpha}^\infty$  denotes the isoperimetric profile associated to the uniform enlargement.*

Therefore, if  $\alpha_2$  (respectively  $\alpha_\infty$ ) solves the problem 1 (respectively the problem 2), then,

$$\alpha_2 \geq \alpha_\infty.$$

### 6.5.1 The solutions of Problem 2 satisfy infinite dimensional isoperimetric inequalities

Let us return to infinite dimensional isoperimetric inequalities, which are inequalities of the form

$$(6.11) \quad \exists C \in (0, 1], \quad I_{\mu^\infty} \geq C I_\mu,$$

where

$$I_{\mu^\infty} := \inf_{n \geq 1} I_{\mu^{\otimes n}}$$

is now the infinite dimensional isoperimetric profile associated with the uniform enlargement. The logistic measure on  $\mathbb{R}$  is an example of a distribution  $\mu$  satisfying  $I_{\mu^\infty} = I_\mu$ . Moreover, a probability measure  $\mu$  satisfying (6.11) has an intermediate behavior between the exponential measure and a Gumbel type distribution (see [37]). In this section, we show that if  $\alpha \geq 1$  solves Problem 2, then  $\mu$  satisfies the infinite dimensional isoperimetric inequality (6.11). Let us start with a technical symmetrization Lemma.

**Lemma 91.** *Let  $m$  be a distribution on  $\mathbb{R}$  with a bounded continuous density  $\phi$  with respect to the Lebesgue measure. Let  $\mu$  be a distribution on a metric space  $(X, d)$ . Let  $f : X \times \mathbb{R} \rightarrow \mathbb{R}$  be a Lipschitz function, and set*

$$S := \{(x, t) \in X \times \mathbb{R} \mid t \leq f(x)\}.$$

Then

$$(\mu \otimes m)_\infty^+(S) \leq \int_X (1 + |\nabla f|(x)) \phi(f(x)) d\mu(x).$$

**Proof** Let  $\varepsilon > 0$  and set

$$S_\varepsilon := \{(x, t) \in X \times \mathbb{R} \mid \exists (a, s) \in S \text{ such that } d(a, x) < \varepsilon, \text{ and } |s - t| < \varepsilon\}.$$

We also define

$$\theta(x, \varepsilon) := \sup \left\{ \frac{|f(x) - f(y)|}{d(x, y)} \mid 0 < d(x, y) < \varepsilon \right\}.$$

Let then  $(x, t) \in S_\varepsilon$ . We get

$$t < s + \varepsilon \leq f(a) + \varepsilon$$

and

$$\begin{aligned} f(a) &\leq f(a) - f(x) + f(x) \\ &\leq |f(a) - f(x)| + f(x) \\ &\leq \theta(x, \varepsilon)d(a, x) + f(x) \\ &\leq \theta(x, \varepsilon)\varepsilon + f(x). \end{aligned}$$

Thus

$$(x, t) \in S_\varepsilon \Rightarrow t < f(x) + \varepsilon(1 + \theta(x, \varepsilon)).$$

So we have

$$\mu \otimes m(S_\varepsilon \setminus S) \leq \int_X \int_{f(x)}^{f(x) + \varepsilon(1 + \theta(x, \varepsilon))} \phi(t) dt d\mu(x)$$

and since  $\phi$  is bounded continuous and  $f$  is Lipschitz, we can apply the dominated convergence theorem and we obtain

$$\begin{aligned} \frac{1}{\varepsilon} \int_{f(x)}^{f(x) + \varepsilon(1 + \theta(x, \varepsilon))} \phi(t) dt &= \frac{F(f(x) + \varepsilon(1 + \theta(x, \varepsilon))) - F(f(x))}{\varepsilon(1 + \theta(x, \varepsilon))} \cdot \frac{\varepsilon(1 + \theta(x, \varepsilon))}{\varepsilon} \\ &\xrightarrow{\varepsilon \rightarrow 0} \phi(f(x)) \cdot (1 + |\nabla f|(x)), \end{aligned}$$



where  $F(u) := \int_{-\infty}^u \phi(t) dt$ . This concludes the proof.  $\square$

We can now state the following isoperimetric product comparison theorem, on which the proof of Theorem 93 is based. This theorem is an adaptation of [18, Theorem 8] which deals with the case of Euclidean expansion. This type of result is deeply related to the symmetrization that we briefly mentioned in the introduction.

**Theorem 92.** *Let  $m$  be an even log-concave distribution on  $\mathbb{R}$  with a bounded continuous density  $\phi$  with respect to the Lebesgue measure. Let  $\mu$  be a distribution on a metric space  $(X, d)$ . If  $I_\mu \geq I_m$  then for all distributions  $\nu$  on  $X$ , we have*

$$I_{\mu \otimes \nu}^\infty \geq I_{m \otimes \nu}^\infty.$$

**Proof** For all Lipschitz functions  $f : X \rightarrow [0, 1]$ , it holds that

$$(6.12) \quad I_{m \otimes \nu}^\infty \left( \int f d\nu \right) \leq \int I_m(f) d\nu + \int |\nabla f|_1 d\nu.$$

Indeed, let  $\Phi : \mathbb{R} \rightarrow [0, 1]$  be the cumulative distribution function of  $m$ . Lemma 91 applied to the function  $\Phi^{-1} \circ f$  and to the product measure  $m \otimes \nu$  gives

$$\begin{aligned} (m \otimes \nu)_\infty^+(S) &\leq \int_X \left( 1 + |\nabla(\Phi^{-1} \circ f)|(x) \right) \phi(\Phi^{-1} \circ f(x)) d\nu(x) \\ &= \int_X I_m(f) d\nu + \int |\nabla f|_1 d\nu, \end{aligned}$$

where the last equality holds because  $I_m = \Phi' \circ \Phi^{-1}$  since  $m$  is an even log-concave distribution on  $\mathbb{R}$ . Let then  $g : X \times X \rightarrow \mathbb{R}$  be a Lipschitz function, and let us apply Inequality (6.12) to  $f : x \mapsto \int g(x, y) d\mu(y)$ . We get

$$\begin{aligned} I_{m \otimes \nu}^\infty \left( \int \int g d\nu \otimes d\mu \right) &\leq \int I_m \left( \int g(x, y) d\mu(y) \right) d\nu(x) + \int \left| \nabla_x \int g(x, y) d\mu(y) \right| d\nu(x) \\ &\leq \int I_m \left( \int g(x, y) d\mu(y) \right) d\nu(x) + \int \int |\nabla_x g(x, y)| d\mu \otimes d\nu \\ &\leq \int \left( 2I_m \left( \int \min(g, 1 - g) d\mu \right) + \int |\nabla_y g| d\mu \right) d\nu + \int \int |\nabla_x g(x, y)| d\mu \otimes d\nu \end{aligned}$$

where we used [18, Theorem 7] in the last inequality. Hence

$$(6.13) \quad I_{m \otimes \nu}^\infty \left( \int g d\nu \otimes \mu \right) \leq 2 \int I_m \left( \int \min(g, 1 - g) d\mu \right) d\nu + \int |\nabla g|_1 d\mu d\nu.$$

By applying Inequality (6.13) to a Lipschitz approximation of an indicator function  $g = \mathbf{1}_A$  we obtain

$$I_{m \otimes \nu}^\infty (\nu \otimes \mu(A)) \leq (\mu \otimes \nu)_\infty^+(A),$$

which gives

$$I_{m \otimes \nu}^\infty \leq I_{\mu \otimes \nu}^\infty.$$

This concludes the proof.  $\square$

**Theorem 93.** *Let  $\mu$  be an even log-concave probability distribution on  $\mathbb{R}$ . Suppose that  $\alpha \geq 1$  solves Problem 2. Then  $\mu$  satisfies an infinite dimensional isoperimetric inequality with constant  $C \geq \frac{1}{\alpha}$ .*

**Proof** The same reasoning as for the Euclidean enlargement can be made. Indeed, let  $\mu$  be an even log-concave probability distribution on  $\mathbb{R}$ , and let  $\alpha$  be such that  $I_{\mu \otimes \mu_\alpha} \geq I_{\mu_\alpha}$ . Since

$\mu$  is an even log-concave distribution, we can use Theorem 92 where  $\mu_\alpha$  plays the role of  $m$ ,  $\mu \otimes \mu_\alpha$  plays the role of  $\mu$ , and  $\mu$  plays the role of  $\nu$ . It follows then that

$$I_{\mu \otimes \mu \otimes \mu_\alpha} \geq I_{\mu \otimes \mu_\alpha} \geq I_{\mu_\alpha}.$$

By iterating this procedure, we get that for all  $n \geq 1$ ,

$$I_{\mu^{\otimes n} \otimes \mu_\alpha} \geq I_{\mu \otimes \mu_\alpha} = I_{\mu_\alpha}.$$

Using that  $I_{\mu^{\otimes n}} \geq I_{\mu^{\otimes n} \otimes \mu_\alpha}$  and that  $I_{\mu_\alpha} = \frac{1}{\alpha} I_\mu$ , we therefore get that  $\mu$  satisfies the infinite dimensional isoperimetric inequality  $I_{\mu^{\otimes n}} \geq \frac{1}{\alpha} I_\mu$ , and moreover, the sharp solution  $\alpha$  of Problem 2 and the sharp constant  $C$  in the infinite dimensional isoperimetric inequality are related by  $C \geq \frac{1}{\alpha}$ .  $\square$

## 6.5.2 Asymptotic of Problem 2

In this section, we study the connection between the infinite dimensional isoperimetric inequality (6.11), and Problem 2 in an asymptotic formulation, for an even log-concave probability distribution  $\mu$  on  $\mathbb{R}$ . In particular, we try to derive sufficient conditions on  $\mu$  to have

$$I_{\mu \otimes \mu_\alpha}^\infty \underset{\alpha \rightarrow \infty}{\sim} I_{\mu_\alpha}^\infty.$$

For this purpose, let

$$D := \{(t, a, b) \in (0, 1) \times [0, 1] \times [0, 1] \mid ab = t \text{ or } 1 - t\},$$

and for all  $\alpha \geq 1$ , set

$$\forall (t, a, b) \in D, \quad f_\alpha(t, a, b) := \alpha a \frac{I_\mu(b)}{I_\mu(t)} + b \frac{I_\mu(a)}{I_\mu(t)}.$$

By [18, Theorem 15] (see also Section 6.5.3), we know that

$$(6.14) \quad \forall t \in (0, 1), \quad \frac{I_{\mu \otimes \mu_\alpha}(t)}{I_{\mu_\alpha}(t)} = \inf_{a, b} f_\alpha(t, a, b)$$

where the supremum runs over all  $(a, b)$  such that  $(t, a, b) \in D$ . We are interested in the quantity

$$A_\alpha := \inf_D f_\alpha.$$

Equation (6.14) and the decay of the isoperimetric profile function with respect to the product of measures give that  $\forall \alpha \geq 1$ ,  $A_\alpha \leq 1$ . Solving Problem 2 is looking for sufficient conditions on the distribution  $\mu$  to ensure the existence of some  $\alpha \geq 1$  reaching the equality case  $A_\alpha = 1$ . Solving the asymptotic of Problem 2 is looking for sufficient conditions on the distribution  $\mu$  to ensure that  $A_\alpha \xrightarrow{\alpha \rightarrow \infty} 1$ .

**Proposition 94.** • *The family  $(f_\alpha)_\alpha$  is non decreasing with respect to the parameter  $\alpha$ . Moreover,  $f_\alpha$  converges pointwise towards  $f_\infty : D \rightarrow \mathbb{R}$  given by*

$$f_\infty(t, a, b) = \begin{cases} 1 & \text{if } b = 1 \text{ and } a = t \text{ or } 1 - t \\ +\infty & \text{else.} \end{cases}$$

- *The family  $\frac{1}{\alpha}(f_\alpha)_\alpha$  is non increasing with respect to the parameter  $\alpha$ . Moreover,  $\frac{1}{\alpha}f_\alpha$  converges pointwise towards  $g_\infty : D \rightarrow \mathbb{R}$  given by*

$$g_\infty(t, a, b) = a \frac{I_\mu(b)}{I_\mu(t)}.$$

*If in addition the function  $h : t \mapsto \frac{I_\mu(t)}{t}$  is essentially non increasing on  $(0, \frac{1}{2}]$  (i.e. there exists a certain  $K > 0$  such that  $\forall 0 < x \leq y \leq \frac{1}{2}$ ,  $K h(x) \geq h(y)$ ), then the above convergence is uniform on  $D$ .*

**Proof** The family  $\frac{1}{\alpha}(f_\alpha)_\alpha$  is clearly non increasing with respect to  $\alpha$  since it is a linear function of  $\frac{1}{\alpha}$  with non negative steering coefficient. The pointwise convergence of  $\frac{1}{\alpha}(f_\alpha)_\alpha$  when  $\alpha$  tends to  $\infty$  is also obvious. Let us show the uniform convergence under the essential monotonicity assumption. One can compute, for all  $x = (t, a, b) \in D$ ,

$$\left| \frac{1}{\alpha} f_\alpha(x) - g_\infty(x) \right| = \frac{b I_\mu(a)}{\alpha I_\mu(t)}.$$

Now, either  $ab = t$  or  $ab = 1 - t$ . Let us treat the first case  $ab = t$ . We then have

$$\frac{b I_\mu(a)}{\alpha I_\mu(t)} = \frac{1 I_\mu(a)/a}{\alpha I_\mu(t)/t}.$$

Since  $b \in [0, 1]$ , it follows that  $a \geq ab = t$ . We will show that the term  $\frac{I_\mu(a)/a}{I_\mu(t)/t}$  is bounded. There are three cases.

1. If  $0 < t \leq a \leq \frac{1}{2}$ , since the function  $h$  is essentially non increasing, there is  $K > 0$  such that  $\frac{I_\mu(a)/a}{I_\mu(t)/t} \leq K$ .
2. If  $\frac{1}{2} \leq t \leq a < 1$ , then  $\frac{I_\mu(a)/a}{I_\mu(t)/t} \leq 2 \frac{I_\mu(a)}{I_\mu(t)} = 2 \frac{I_\mu(1-a)}{I_\mu(1-t)} \leq 2$ , where the last inequality comes from the fact that  $I_\mu$  increases on  $[0, \frac{1}{2}]$ .
3. If  $0 \leq t \leq \frac{1}{2} \leq a \leq 1$ , since  $I_\mu$  is concave and symmetric around  $\frac{1}{2}$ , there is a constant  $C > 0$  such that  $I_\mu(t) \geq \min(t, 1-t)$  on  $[0, 1]$ . Then  $\frac{I_\mu(a)/a}{I_\mu(t)/t} \leq \frac{I_\mu(a)}{I_\mu(t)} \leq C \min(t, 1-t) I_\mu(a) \leq \frac{C}{2} I_\mu(\frac{1}{2})$ , where the last inequality comes from the fact that  $I_\mu$  decreases on  $[\frac{1}{2}, 1]$ .

In the case  $ab = 1 - t$ , we can write

$$b \frac{I_\mu(a)}{I_\mu(t)} = \frac{I_\mu(a)}{a} \cdot \frac{t'}{I_\mu(t')},$$

with  $t' = 1 - t$  and  $a \leq t'$ . Therefore it is reduced to the first case. So the family  $\frac{1}{\alpha}(f_\alpha)_\alpha$  uniformly converges towards  $g_\infty$ . The proof is complete.  $\square$

By [22, Theorem 5], we know that the existence of  $c \in (0, 1)$  such that

$$(6.15) \quad I_{\mu^\infty} \geq c I_\mu$$

is equivalent to the fact that the function  $h : t \mapsto \frac{I_\mu(t)}{t \log(\frac{1}{t})}$  is essentially non decreasing on  $(0, 1)$ .

Moreover, this condition implies that  $t \mapsto \frac{I_\mu(t)}{t}$  is essentially non increasing on  $(0, \frac{1}{2}]$  (see [22, Lemma 5]). Hence by Proposition 94, the infinite dimensional isoperimetric inequality (6.15) implies that  $\frac{1}{\alpha} f_\alpha$  converges uniformly towards  $g_\infty : D \rightarrow \mathbb{R}$  when  $\alpha$  tends to  $\infty$ .

### 6.5.3 The case of the one-sided exponential distribution

In this section, we calculate the exact uniform isoperimetric profile of the product of a one-sided exponential distribution and a dilated one-sided exponential distribution. To do so, we use the following result of F. Barthe (see Theorem 15 in [18] and the remark below).

**Theorem 95.** ([18, Theorem 15]) *Let  $\mu_1$  and  $\mu_2$  be two probability distributions and  $I_1, I_2$  be their respective isoperimetric profiles with respect to the uniform enlargement. If  $I_1$  and  $I_2$  are concave, symmetrical about  $\frac{1}{2}$ , and if  $\forall a \in (0, 1)$ , there exists an extremal set  $A_i, i = 1, 2$  such that  $\mu_i(A_i) = a, \mu_i^+(A_i) = \mu_i^+(A_i^c)$ , and such that the  $\liminf$  in the definition of  $\mu_i^+(A_i)$  is a true limit, then the uniform isoperimetric profile of the product measure  $\mu_1 \otimes \mu_2$  is given by*

$$(6.16) \quad I(t) = \inf\{bI_1(a) + aI_2(b) \mid a, b \in [0, 1], ab = t \text{ or } ab = 1 - t\}.$$

In the case of  $\mu_1(t) = e^{-t}\mathbf{1}_{\mathbb{R}_+}(t)dt$  and  $\mu_2(t) = \frac{1}{\alpha}e^{-\frac{t}{\alpha}}\mathbf{1}_{\mathbb{R}_+}(t)dt$ , we have that  $I_1(t) = \min(t, 1-t)$  and  $I_2(t) = \frac{1}{\alpha}\min(t, 1-t)$ . Using the formula (6.16) and after tedious but elementary calculations, we obtain the following result:

**Proposition 96.** *Let  $\mu(t) = e^{-t}\mathbf{1}_{\mathbb{R}_+}(t)dt$  and  $\mu_\alpha(t) = \frac{1}{\alpha}e^{-\frac{t}{\alpha}}\mathbf{1}_{\mathbb{R}_+}(t)dt$  with  $\alpha \geq 1$ . Then the uniform isoperimetric profile  $I_{\mu \otimes \mu_\alpha}$  of  $\mu \otimes \mu_\alpha$  is*

$$\text{if } \alpha \in [1, 2), \quad I_{\mu \otimes \mu_\alpha}(t) = \begin{cases} \frac{1}{\alpha} \min(t, 1-t) & \text{for } t \in [0, 1 - \frac{1}{\alpha}] \cup [\frac{1}{\alpha}, 1], \\ 2\sqrt{\frac{\max(t, 1-t)}{\alpha}} - \left(1 + \frac{1}{\alpha}\right) \max(t, 1-t), & \text{for } t \in (1 - \frac{1}{\alpha}, \frac{1}{\alpha}). \end{cases}$$

$$\text{and if } \alpha \geq 2, \quad I_{\mu \otimes \mu_\alpha}(t) = \frac{1}{\alpha} \min(t, 1-t).$$

See Figure 6.9 for a plot of the uniform isoperimetric profile  $I_{\mu \otimes \mu_\alpha}$  for some values of  $\alpha$ . Therefore, Problem 2 is solved for the one-sided exponential distribution in the case of the uniform enlargement: the sharp dilation constant  $\alpha_\infty$  is 2.

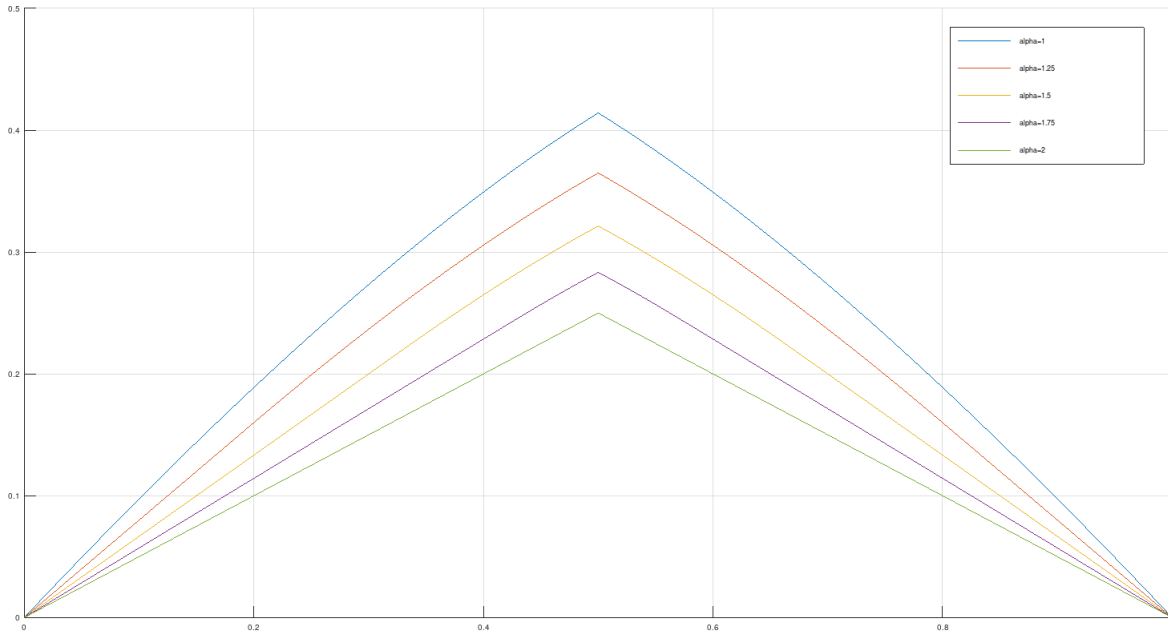


Figure 6.9: The uniform isoperimetric profile of  $\mu \otimes \mu_\alpha$  when  $\mu$  is the one-sided exponential distribution.

## Consequences

For one-dimensional measures, the question of whether to use the Euclidean or uniform enlargement is meaningless. Indeed, there is only one possible isoperimetric profile in this case. Moreover, in dimension at least two, since

$$I_\nu^\infty \geq I_\nu^{(2)},$$

we have that for  $\nu = \mu \otimes \mu_\alpha$ , and for  $\alpha_2$  the solution of Problem 1 associated with the Euclidean enlargement:

$$\frac{1}{\alpha_\infty} I_\mu \geq I_\nu^\infty \geq I_\nu^{(2)} = \frac{1}{\alpha_2} I_\mu,$$

so

$$I_\nu^\infty = I_\nu^{(2)} = \frac{1}{\alpha_2} I_\mu$$

and finally, we obtain

$$\alpha_2 \geq 2$$

since  $2 = \alpha_\infty$  is the solution of Problem 2 associated with the uniform enlargement. This partially proves the observation made in Section 6.4 from numerical estimates.

Moreover, since the isoperimetric profile for the product of one-sided exponential distributions with respect to the uniform enlargement has already been calculated and is

$$\forall t \in [0, 1], \quad I_{\mu_1^{\otimes n}}(t) = n \max(t, 1-t) \left[ \max(t, 1-t)^{-\frac{1}{n}} - 1 \right],$$

we can deduce its isoperimetric profile of infinite dimension with respect to the uniform enlargement :

$$(6.17) \quad \forall t \in [0, 1], \quad I_{\mu_1^\infty}(t) = -(1 - I_{\mu_1}(t)) \log(1 - I_{\mu_1}(t)).$$

Elementary calculations then show that the one-sided exponential distribution satisfies the infinite-dimensional isoperimetric inequality with a sharp constant  $C = \log 2$ :

$$\min_{t \in (0,1)} \frac{I_{\mu_1^\infty}(t)}{I_{\mu_1}(t)} = \log 2.$$

Thus the inequality  $C \geq \frac{1}{\alpha}$  given by Theorem 93 becomes in this case  $\log 2 \geq \frac{1}{2}$  which is a strict inequality.

# Chapter 7

## Perspectives and work in progress

### 7.1 Spectral monotonicity under renormalization

In this section, we generalize a result of B. Klartag and E. Putterman using a renormalization method introduced by R. Bauerschmidt and T. Bodineau. We have discussed this work in more detail in [193]. A related perspective to the stability of the Poincaré constant is the study of its behavior along a flow. One such result along the heat flow is the following.

**Theorem 97.** [128, Theorem 1.1] *Let  $\nu_0$  be a log-concave probability distribution in  $\mathbb{R}^d$ , let  $\gamma_t$  be the Gaussian distribution with zero mean and covariance  $tI_d$ ,  $t \geq 0$ , and set  $\nu_t := \nu_0 * \gamma_t$ , where  $*$  denotes the convolution. Then the Poincaré constant  $C_P(\nu_t)$  is non decreasing in  $t$ .*

From a geometric point of view, the log-concavity assumption is a positive curvature assumption. A natural question is whether the previous result can be generalized to non-log-concave measures. The answer is positive under a dynamic curvature assumption (not necessarily positive), see Assumption 2. To do this, we use the semigroup renormalization method of [24, Section 2.1]. Let  $(C_t)_{t \geq 0}$  be a family of positive semidefinite matrices on  $\mathbb{R}^d$ . We assume that the  $C_t$  matrices increase continuously as quadratic forms from  $C_0 = 0$  to a matrix  $C_\infty$ . We also assume the family to be twice differentiable with respect to the parameter  $t$ . We denote by  $\gamma_{C_t}$  the (possibly degenerate) Gaussian measure with covariance  $C_t$  and mean zero.

Let

$$\nu_0 = \frac{1}{Z} e^{-\langle x, C_\infty x \rangle - V_0(x)} dx$$

be a probability distribution on  $\mathbb{R}^d$ , where  $Z$  is a normalization constant and  $V_0 : \mathbb{R}^d \rightarrow \mathbb{R}$  is a twice continuously differentiable function. The part  $e^{-\langle x, C_\infty x \rangle}$  represents the log-concave part of the measure, while  $e^{-V_0}$  represents the non log-concave part of  $\nu_0$ .

**Definition 98.** *We define the renormalization flow by*

$$\begin{aligned} V_t &:= -\log(\gamma_{C_t} * e^{-V_0}), \\ P_{s,t} f &:= e^{V_t} \gamma_{C_t - C_s} * f e^{-V_s}, \\ \nu_t &:= e^{V_\infty(0)} (\gamma_{C_\infty - C_s} * e^{-V_t})(x) dx \\ &= e^{-\langle x, (C_\infty - C_t)x \rangle - V_t(x) + V_\infty(0)} dx. \end{aligned}$$

The log-concave part of the measure is transformed in the same way as in the Klartag-Putterman result (by convolution with a heat kernel), and the non-convex part  $V_t$  is transformed according to the Polchinski equation (see [24, Proposition 2.1])

$$(7.1) \quad \partial_t V_t = \frac{1}{2} \left( \frac{1}{2} \Delta_{C'_t} + L_t \right) V_t,$$

where

$$(7.2) \quad L_t := \frac{1}{2} \Delta_{C'_t} - \langle \nabla V_t, \nabla \rangle_{C'_t},$$

and the index  $C'_t$  in the Laplacian or the dot product denotes that these operations are computed with respect to the metric tensor  $C'_t$  on  $\mathbb{R}^d$ , i.e.

$$\langle U, V \rangle_{C'_t} := \sum_{i,j} (C'_t)_{i,j} U_i V_j \quad \text{and} \quad \Delta_{C'_t} f := \sum_{i,j} (C'_t)_{i,j} \frac{\partial^2 f}{\partial x_{i,j}^2}.$$

Let us emphasize that  $\nu_t$  is not the equilibrium distribution of the operator  $L_t$ . The assumption of convexity of  $V_0$  is replaced by the following dynamic curvature assumption.

**Assumption 2.** *For all  $t \geq 0$ , there exists  $\lambda'_t \in \mathbb{R}$  (possibly negative) such that*

$$(7.3) \quad \forall x \in \mathbb{R}^d, \quad C'_t \nabla^2 V_t C'_t \geq C''_t + \lambda'_t C'_t,$$

*in the sense of quadratic forms, where  $C'_t$  denotes the derivative of  $t \mapsto C_t$  with respect to the  $t$  parameter. The function  $t \mapsto \lambda'_t$  is assumed to be locally integrable.*

Contrary to the classic convex assumption  $\text{Hess } V_0 \geq 0$ , Assumption 2 is an assumption that bears on the whole flow. Evaluated in  $t=0$ , the hypothesis is empty because we can always find a family of matrices  $(C_t)_t$  satisfying it. However, the hypothesis concerns the possibility of constructing, from  $V_0$  and  $C_\infty$ , a flow with the curvature dynamics (7.3). When  $V_0 = 0$ , then  $V_t = 0$ , and one takes  $C_t = C_\infty^{-1} - C_\infty^{-1} e^{-tC_\infty}$ . Then  $C''_t = -C_\infty C'_t$  is negative semi definite, and then Assumption 2 is satisfied with  $\lambda'_t = 0$ . When  $\nabla^2 V_0 \geq 0$ , then  $\nabla^2 V_t \geq 0$  for all  $t \geq 0$  (see [24, Example 1.3]). Hence by taking  $C_t = C_\infty^{-1} - C_\infty^{-1} e^{-tC_\infty}$  as in the case  $V_0 = 0$ , Assumption 2 is satisfied again with  $\lambda'_t = 0$ . In all the sequel, we will denote  $\lambda_t := \int_0^t \lambda'_s ds$ . We will prove the following monotonicity result along the renormalization flow.

**Theorem 99.** *If Assumption 2 is satisfied by the renormalization flow defined in Definition 98, then we have that for all  $0 \leq s \leq t$ ,*

$$C_P(\nu_s) \leq e^{(\alpha_t - \alpha_s) - 2(\lambda_t - \lambda_s)} C_P(\nu_t),$$

*where  $\lambda_t := \int_0^t \lambda'_s ds$  is defined from the multiscale curvatures  $\lambda'_t$ , and  $\alpha_t := \int_0^t \alpha'_s ds$  is such that*

$$(C'_t)^{\frac{1}{2}} \left( \nabla^2 V_t + (C_\infty - C_t)^{-1} \right) (C'_t)^{\frac{1}{2}} \leq \alpha'_t I_d,$$

*where  $I_d$  is the identity matrix on  $\mathbb{R}^d$ . Since  $C_t$  increases from zero to  $C_\infty$ , the  $\alpha'_t$  will be positive as soon as  $\nabla^2 V_t$  is not too negative.*

Let us underline the following.

1. When  $\nu_0$  is log-concave, then we have seen that  $\lambda_t$  is zero, and hence we find Theorem 97.
2. Contrary to [128, Theorem 1.1], the measure  $\nu_t$  satisfies a weighted Poincaré inequality. Indeed, the metric is adapted to  $\nu_t$ , and the Poincaré constant is defined as the smaller constant  $K_t$  such that for all  $\nu_t$ -centered and compactly supported  $\phi$ ,

$$\int_{\mathbb{R}^d} \phi^2 d\nu_t \leq K_t \int_{\mathbb{R}^d} |\nabla \phi|_{C'_t}^2 d\nu_t.$$

The sequel is devoted to the proof of Theorem 99. For this we will need the following technical lemma.

**Lemma 100.** *Let  $\phi : \mathbb{R}^d \rightarrow \mathbb{R}$  be smooth and compactly supported. We denote by  $\phi_t := P_{0,t}\phi$ . Then the Rayleigh quotient*

$$R_\phi(t) := \frac{\mathbb{E}_{\nu_t} |\nabla \phi_t|_{C'_t}^2}{\mathbb{E}_{\nu_t} \phi_t^2}$$

satisfies for all  $t > s$ ,

$$R_\phi(t) \leq R_\phi(s) \exp((\alpha_t - \alpha_s) - 2(\lambda_t - \lambda_s)),$$

where the  $\lambda_t$  are given by Assumption 2, and  $\alpha'_t$  is the largest eigenvalue of the symmetric matrix  $C'_t (\nabla^2 V_t + (C_\infty - C_t)^{-1})$ .

*Proof.* First, let us underline that  $\mathbb{E}_{\nu_t} \phi_t = 0$ , justifying that  $\mathbb{E}_{\nu_t} \phi_t^2$  is the  $\nu_t$ -variance of  $\phi_t$ . We can compute that (see [24, Proposition 2.1])

$$\frac{\partial}{\partial t} \mathbb{E}_{\nu_t} |\nabla \phi_t|_{C'_t}^2 = \mathbb{E}_{\nu_t} \left[ 2 \langle \nabla L_t \phi_t, \nabla \phi_t \rangle_{C'_t} - L_t |\nabla \phi_t|_{C'_t}^2 + |\nabla \phi_t|_{C'_t}^2 \right]$$

and

$$\frac{\partial}{\partial t} \mathbb{E}_{\nu_t} \phi_t^2 = -\mathbb{E}_{\nu_t} |\nabla \phi_t|_{C'_t}^2.$$

Hence

$$\left( \mathbb{E}_{\nu_t} \phi_t^2 \right)^2 R'(t) = \mathbb{E}_{\nu_t} \phi_t^2 \mathbb{E}_{\nu_t} \left[ 2 \langle \nabla L_t \phi_t, \nabla \phi_t \rangle_{C'_t} - L_t |\nabla \phi_t|_{C'_t}^2 + |\nabla \phi_t|_{C'_t}^2 \right] + \left( \mathbb{E}_{\nu_t} |\nabla \phi_t|_{C'_t}^2 \right)^2.$$

But the Bochner-type formula for the operator  $\Gamma_{2,t}$  associated to the diffusion operator  $L_t$  gives

$$-2 \Gamma_{2,t}(\phi_t) = 2 \langle \nabla L_t \phi_t, \nabla \phi_t \rangle_{C'_t} - L_t |\nabla \phi_t|_{C'_t}^2 = -\|\nabla^2 \phi_t\|_{C'_t}^2 - 2 \langle \nabla^2 V_t C'_t \nabla \phi_t, \nabla \phi_t \rangle_{C'_t}.$$

So using following multiscale Bakry-Emery condition (see Assumption 2),

$$C'_t \nabla^2 V_t C'_t \geq \frac{1}{2} C''_t + \lambda'_t C'_t,$$

one gets

$$-2 \langle \nabla^2 V_t C'_t \nabla \phi_t, \nabla \phi_t \rangle_{C'_t} + \|\nabla \phi_t\|_{C'_t}^2 \leq -2\lambda'_t \|\nabla \phi_t\|_{C'_t}^2.$$

Therefore we obtain

$$\begin{aligned} R'(t) &\leq \frac{\mathbb{E}_{\nu_t} (\phi_t^2) E_{\nu_t} \left[ -\|\nabla^2 \phi_t\|_{C'_t}^2 - 2\lambda'_t \|\nabla \phi_t\|_{C'_t}^2 \right] + \left( \mathbb{E}_{\nu_t} |\nabla \phi_t|_{C'_t}^2 \right)^2}{\left( \mathbb{E}_{\nu_t} \phi_t^2 \right)^2} \\ &= \frac{\left( \mathbb{E}_{\nu_t} |\nabla \phi_t|_{C'_t}^2 \right)^2 - \mathbb{E}_{\nu_t} (\phi_t^2) E_{\nu_t} \left( \|\nabla^2 \phi_t\|_{C'_t}^2 \right)}{\left( \mathbb{E}_{\nu_t} \phi_t^2 \right)^2} - 2\lambda'_t R_\phi(t) \end{aligned}$$

Moreover, by denoting by  $\Gamma_2^{\mathcal{L}_t}$  the  $\Gamma_2$ -operator of  $\mathcal{L}_t$ , we have

$$\begin{aligned} \mathbb{E}_{\nu_t} |\nabla \phi_t|_{C'_t}^2 &= -\mathbb{E}_{\nu_t} (\phi_t \mathcal{L}_t \phi_t) \\ &\leq \left( \mathbb{E}_{\nu_t} \phi_t^2 \right)^{\frac{1}{2}} \left( \mathbb{E}_{\nu_t} ((-\mathcal{L}_t \phi_t)^2) \right)^{\frac{1}{2}} \\ &= \left( \mathbb{E}_{\nu_t} \phi_t^2 \right)^{\frac{1}{2}} \left( \mathbb{E}_{\nu_t} \Gamma_2^{\mathcal{L}_t}(\phi_t) \right)^{\frac{1}{2}} \\ &= \left( \mathbb{E}_{\nu_t} \phi_t^2 \right)^{\frac{1}{2}} \left( E_{\nu_t} \left( \|\nabla^2 \phi_t\|_{C'_t}^2 \right) + \mathbb{E}_{\nu_t} \left( \langle (\nabla^2 V_t + (C_\infty - C_t)^{-1}) C'_t \nabla \phi_t, \nabla \phi_t \rangle_{C'_t} \right) \right)^{\frac{1}{2}}. \end{aligned}$$

Since by definition we have  $C'_t (\nabla^2 V_t + (C_\infty - C_t)^{-1}) C'_t \leq \alpha'_t C'_t$ , we obtain

$$\begin{aligned} &\left( \mathbb{E}_{\nu_t} |\nabla \phi_t|_{C'_t}^2 \right)^2 - \mathbb{E}_{\nu_t} (\phi_t^2) E_{\nu_t} \left( \|\nabla^2 \phi_t\|_{C'_t}^2 \right) \\ &\leq \mathbb{E}_{\nu_t} (\phi_t^2) \mathbb{E}_{\nu_t} \left( \langle (\nabla^2 V_t + (C_\infty - C_t)^{-1}) C'_t \nabla \phi_t, \nabla \phi_t \rangle_{C'_t} \right) \\ &\leq \alpha'_t \mathbb{E}_{\nu_t} (\phi_t^2) \mathbb{E}_{\nu_t} |\nabla \phi_t|_{C'_t}^2, \end{aligned}$$



where  $\alpha_t := \int_0^t \alpha'_s ds$ . Finally, the derivative of  $R_\phi$  satisfies

$$R'_\phi(t) \leq (\alpha'_t - 2\lambda'_t) R_\phi(t),$$

and the Gronwall lemma gives

$$R_\phi(t) \leq R_\phi(s) \exp((\alpha_t - \alpha_s) - 2(\lambda_t - \lambda_s)),$$

where  $\alpha_t := \int_0^t \alpha'_s ds$ . The proof is complete.  $\square$

### Proof of Theorem 99

By definition of the spectral gap, one has

$$\frac{1}{C_P(\nu_0)} = \inf\{R_\phi(0) \mid \phi \in C_c^\infty(\mathbb{R}^d)\},$$

where  $C_c^\infty(\mathbb{R}^d)$  denotes the set of all smooth compactly supported functions on  $\mathbb{R}^d$ . Let then  $t \geq 0$ . For all  $\varepsilon > 0$ , there exists  $\phi \in C_c^\infty(\mathbb{R}^d)$  such that

$$R_\phi(0) < \frac{1}{C_P(\nu_0)} + \varepsilon.$$

By Lemma 100, it follows that

$$e^{2\lambda_t - \alpha_t} R_\phi(t) < \frac{1}{C_P(\nu_0)} + \varepsilon.$$

But by the definition of the Poincaré constant,  $\frac{1}{C_P(\nu_t)} \leq R_\phi(t)$ , so we get

$$\forall \varepsilon > 0, \quad \frac{e^{2\lambda_t - \alpha_t}}{C_P(\nu_t)} < \frac{1}{C_P(\nu_0)} + \varepsilon,$$

from which we get

$$C_P(\nu_t) \geq e^{2\lambda_t - \alpha_t} C_P(\nu_0)$$

by let  $\varepsilon$  go to zero. The theorem is therefore proven thanks to the semigroup property: the idea is to start the flow from the time  $s$  instead of 0, and with the same reasoning, we obtain exactly the result.  $\square$

This result is to be put in relation with the result of Y. Shenfeld [194] which establishes the Lipschitz continuity of the Langevin transport map under a multiscale Bakry-Emery criterion, when the matrices  $C'_t$  are of the form  $C'_t = e^{-t\varepsilon^d(-\Delta^\varepsilon + m)}$ , with  $\Delta^\varepsilon$  the discrete Laplacian and  $m > 0$ . Recall that the Lipschitz continuity of transport maps is a powerful tool to prove functional inequalities [161]. The multiscale Bakry-Emery criterion used in Theorem 99 is more general and corresponds to the Brownian transport map [157]. In another context, it corresponds to Eldan stochastic localization [82]. The study of the Brownian transport map under general multiscale Bakry-Emery criteria seems to be very promising.

## 7.2 Stein's lemma with coupling method

In this section, we present two methods for obtaining a generalized Stein lemma. These methods differ from the direct computation method used to prove Proposition 29 in Chapter 3. Recall that we call a regularity result of the following form a Stein lemma.

**Stein's lemma prototype.** *Let  $L$  be a diffusion generator satisfying some conditions. Then there is some constant  $C > 0$  such that for all 1-Lipschitz functions  $g$ , there exist a solution  $f$  of the Poisson equation  $Lf = g$  such that  $\|\text{Hess } f\|_\infty \leq C$ .*

In Chapter 3, we considered a one-dimensional generator of the form  $Lf(x) = h(x)f''(x) - xf'(x)$ , and the required condition was a growth control on the function  $h$  at the boundary (see Proposition 36). Here, the required condition on the generator  $L$  is a positive curvature dimension condition with infinite dimension. In Section 7.2.2, we need to add to this curvature condition a Lipschitz condition on the tangent process generated by  $L$ .

### 7.2.1 The Bonis approach

This section is based on [39, Section 4] by T. Bonis. Let  $L$  be a diffusion generator on  $\mathbb{R}^d$  satisfying the curvature dimension condition  $CD(\rho, \infty)$ , for  $\rho > 0$ , let  $\Gamma$  be its carré du champ operator, and let  $P_t$  its generated semi group. We assume that  $L$  is of the form

$$(7.4) \quad Lf(x) = \langle a(x), \text{Hess } f(x) \rangle_{HS} - b(x) \cdot \nabla f(x),$$

for some symmetric positive definite matrix valued function  $a$ , and some vector valued function  $b$ . Its carré du champ operator is therefore given by  $\Gamma(f)(x) = \langle \nabla f(x), a(x)\nabla f(x) \rangle$ , and we define  $\Gamma(\nabla f)(x) := \langle \text{Hess } f(x), a(x)\text{Hess } f(x) \rangle_{HS}$ . We will show that such a diffusion generator satisfies a Stein lemma. To do this, we need the following commutation property.

**Proposition 101.** *Then, for all  $\phi \in C_c^2(\mathbb{R}^d)$ , we have*

$$\forall x \in \mathbb{R}^d, \quad \Gamma(\nabla P_t \phi)(x) \leq \frac{\rho e^{-2\rho t}}{e^{2\rho t} - 1} P_t(\Gamma(\phi))(x).$$

**Proof** The following proof is a particular case of the proof of [39, Proposition 5].

- **Step 1** It is easily checked that

$$\Gamma(\nabla P_t \phi)(x) = \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon^2} \Gamma(P_t \phi(\cdot + \varepsilon) - P_t \phi)(x).$$

- **Step 2**

Let  $x \in \mathbb{R}^d$ , and  $\varepsilon > 0$  small enough. Let  $X_t$  be a random walk generated by  $L$  and starting from  $x$ , and let  $X_t^\varepsilon$  starting from  $x + \varepsilon$ . Set  $\psi_\varepsilon(y) := \mathbb{E}(\phi(X_t^\varepsilon) | X_t = y)$ . Then by conditioning, one has:

$$P_t \phi(x + \varepsilon) - P_t \phi(x) = P_t(\psi_\varepsilon - \phi)(x).$$

- **Step 3**

Using [13, Theorem 4.7.2] (this result make use of the curvature condition), one gets:

$$\Gamma(P_t(\psi_\varepsilon - \phi))(x) \leq \frac{\rho}{e^{2\rho t} - 1} P_t(|\psi_\varepsilon - \phi|^2)(x).$$

Using then Jensen's inequality, one concludes that

$$\Gamma(P_t(\psi_\varepsilon - \phi))(x) \leq \frac{\rho}{e^{2\rho t} - 1} \mathbb{E}(|\phi(X_t^\varepsilon) - \phi(X_t)|^2).$$

- **Step 4**

Let us denote  $d$  the Riemannian distance on  $\mathbb{R}^d$  induced by the metric  $g_x(u, v) := \langle u, a(x)^{-1}v \rangle$ . The commutation relation between  $\Gamma$  and  $P_t$  is satisfied because  $L$  satisfies the condition  $CD(\rho, \infty)$ . We can therefore apply the Kuwada theorem [133, Theorem 2.2] with  $p = 1$ ,  $\mu = \delta_x$  and  $\nu = \delta_{x+\varepsilon}$  to get a contractive coupling  $(X_t, X_t^\varepsilon)$ . So  $X_0 = x$ ,  $X_0^\varepsilon = x + \varepsilon$  and  $\forall t \geq 0$ ,  $\mathbb{E}(d(X_t, X_t^\varepsilon)) \leq e^{-\rho t} d(x, x + \varepsilon)$ .

- **Step 5**

We compute:

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon^2} \mathbb{E} \left( |\phi(X_t^\varepsilon) - \phi(X_t)|^2 \right) &= \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon^2} \mathbb{E} \left( \Gamma(\phi)(X_t) |X_t^\varepsilon - X_t|^2 \right) \\ &\stackrel{\text{Step 4}}{\leq} e^{-2\rho t} \mathbb{E} \left( \lim_{\varepsilon \rightarrow 0} \frac{d(x, x + \varepsilon)^2}{\varepsilon^2} \Gamma(\phi)(X_t) \right) \end{aligned}$$

But since  $y \mapsto d(y, x)$  is 1 Lipschitz, we get that  $\lim_{\varepsilon \rightarrow 0} \frac{d(x, x + \varepsilon)^2}{\varepsilon^2} \leq 1$ . So we obtain:

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon^2} \mathbb{E} \left( |\phi(X_t^\varepsilon) - \phi(X_t)|^2 \right) \leq e^{-2\rho t} \mathbb{E} \left( \Gamma(\phi)(X_t) \right).$$

- **Conclusion**

Finally,

$$\begin{aligned} \Gamma((P_t \phi)')(x) &\stackrel{\text{step 1}}{=} \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon^2} \Gamma(P_t \phi(\cdot + \varepsilon) - P_t \phi)(x) \\ &\stackrel{\text{step 2}}{=} \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon^2} \Gamma(P_t(\psi_\varepsilon - \phi))(x) \\ &\stackrel{\text{step 3}}{\leq} \frac{\rho}{e^{2\rho t} - 1} \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon^2} \mathbb{E} \left( |\phi(X_t^\varepsilon) - \phi(X_t)|^2 \right) \\ &\stackrel{\text{step 5}}{\leq} \frac{\rho e^{-2\rho t}}{e^{2\rho t} - 1} \mathbb{E} \left( \Gamma(\phi)(X_t) \right) \\ &= \frac{\rho e^{-2\rho t}}{e^{2\rho t} - 1} P_t(\Gamma(\phi))(x). \end{aligned}$$

The proof is complete. □

We can now state the following regularization result.

**Theorem 102.** (Stein's lemma for  $CD(\rho, \infty)$ ,  $\rho > 0$ )

Let  $L$  be a diffusion generator of the form (7.4) and satisfying the curvature dimension condition  $CD(\rho, \infty)$  for some  $\rho > 0$ . Let  $\mu$  be the reversible measure of the process generated by  $L$ . Let  $g$  be  $\mu$ -centered and such that  $\Gamma(g) \leq 1$ , and let  $\phi$  be the solution of the Poisson equation  $L\phi = g$  given by

$$\phi = - \int_0^\infty P_t g(x) dt.$$

Then the following bound holds

$$\Gamma(\nabla \phi) \leq \frac{2}{\rho}.$$

**Proof** We have

$$\sqrt{\Gamma(\nabla \phi)(x)} \leq \int_0^\infty \sqrt{\Gamma(\nabla P_t(g))(x)} dt \leq \int_0^\infty \frac{\rho e^{-\rho t}}{\sqrt{e^{2\rho t} - 1}} \sqrt{P_t^*(\Gamma^*(g))(x)} dt \leq \int_0^\infty \frac{\rho e^{-\rho t}}{\sqrt{e^{2\rho t} - 1}} dt = \sqrt{\frac{2}{\rho}}$$

where we used Proposition 101 to obtain the second inequality. □

## 7.2.2 The SDE approach

The goal to this section is to derive a Stein lemma for a  $CD(\kappa, \infty)$  diffusion on a Riemannian manifold by stochastic calculation tools. The positive curvature constant will be denoted here as  $\kappa > 0$  because of the use of the notation  $\rho$  for the distance and  $d$  for the Ito differential. This

is based on the work [31] by H. Le, A. Lewis, K. Bharath, and C. Fallaize. Let  $L$  be a diffusion operator on a Riemannian manifold  $M$  of dimension  $d$ . We assume that  $L$  is of the form

$$(7.5) \quad Lf = \Delta f - \nabla V \cdot \nabla f,$$

where  $V : M \rightarrow \mathbb{R}$  is  $\mathbb{C}^1$ , and  $\Delta$  denotes the Laplace-Beltrami operator on  $M$ . Moreover, we assume that  $M$  equipped with the generator  $L$  is a  $CD(\kappa, \infty)$  manifold, for some  $\kappa > 0$ , i.e.

$$(7.6) \quad \text{Hess } V + \text{Ric} \geq \kappa g,$$

where  $g$  denotes the metric tensor and  $\text{Ric}$  is the Ricci curvature. The diffusion operator  $L$  generates a Markov process  $(X_t)_{t \geq 0}$  which is solution of the following Ito SDE:

$$(7.7) \quad dX_t = -\nabla V(X_t)dt + \sqrt{2}d\beta_t$$

where  $\beta_t$  is the Brownian motion intrinsically defined on  $M$ . This SDE can also be written in terms of Stratonovich differential and standard Brownian motion on  $\mathbb{R}$ .

$$(7.8) \quad \begin{aligned} d_s X_t &= \sqrt{2} \Xi_t d_s B_t - \nabla V(X_t)dt, & X_0 &= x_0 \\ d_s \Xi_t &= H_{\Xi_t} d_s X_t, & \Xi_0 &= \xi_0 \end{aligned}$$

where  $d_s$  denotes Stratonovich differential,  $B_t$  is a standard Brownian motion on  $\mathbb{R}$ ,  $\Xi$  is an orthonormal frame above  $X$  (i.e.  $\Xi_t : \mathbb{R}^d \rightarrow T_{X_t}M$  is an isomorphism),  $H_{\Xi}$  denotes the horizontal lift from  $TM$  to the orthonormal frame bundle  $\mathcal{O}(M)$ , i.e.

$$H_{\Xi_t} \oplus \ker(d\pi_{\Xi_t} : T_{\Xi_t}\mathcal{O}(M) \rightarrow T_{X_t}M) = T_{\Xi_t}\mathcal{O}(M),$$

with  $\pi$  the projection of  $\mathcal{O}(M)$  onto  $M$ . In order to derive a regularization result, the following two propositions are necessary.

**Proposition 103.** [31, Theorem 1] *For all  $(x_0, y_0) \in M \times M$ , there exists a coupling  $(X_t, Y_t)$  starting from  $(x_0, y_0)$  such that  $X_t$  and  $Y_t$  both satisfies 7.7 and*

$$(7.9) \quad \forall l \geq 1, \forall t \geq 0, \quad \rho(X_t, Y_t)^l \leq \rho(x_0, y_0)^l e^{-l\kappa t}$$

where  $\rho$  is the Riemannian distance on  $M$ .

**Proof** We sketch a proof in the case where there are no conjugate points. Full proof can be found in the article [31]. The coupling  $(X_t, Y_t)$  is defined as the solution of the following Stratonovich SDE system:

$$\begin{aligned} d_s X_t &= \sqrt{2} \Xi_t d_s B_t - \nabla V(X_t)dt, \\ d_s Y_t &= \sqrt{2} \Upsilon_t d_s B'_t - \nabla V(Y_t)dt, \\ d_s \Xi_t &= H_{\Xi_t} d_s X_t, \\ d_s \Upsilon_t &= H_{\Upsilon_t} d_s Y_t, \\ dB'_t &= (\Upsilon_t^{-1} \Pi_{X_t, Y_t} \Xi_t) dB_t, \end{aligned}$$

with the initial conditions  $X_0 = x_0$ ,  $Y_0 = y_0$ ,  $\Xi_0 = \xi_0$  and  $\Upsilon_0 = v_0$ . We denote  $\Pi_{X_t, Y_t}$  the parallel transport,  $\Xi$  is a lift for  $X$ ,  $\xi_0$  sits above  $x_0$ , i.e.  $\xi_0 : T_{x_0}M \rightarrow \mathbb{R}^d$  is a linear isomorphism, and similarly  $\Upsilon$  is a lift for  $Y$  and  $v_0$  sits above  $y_0$ . The standard Brownian motion on  $\mathbb{R}^d$  is denoted  $B_t$ . Let  $\gamma_t$  be the unit speed geodesic from  $X_t$  to  $Y_t$ . Choose  $u_0 \in \mathbb{R}^d \setminus \{0\}$  such that  $\Xi_t u_0 = \gamma'_t(0)$  and let then  $u_0, u_1, \dots, u_{d-1}$  be an orthonormal basis of  $\mathbb{R}^d$ . Set  $v_i := (\Upsilon_t^{-1} \Pi_{X_t, Y_t} \Xi_t) u_i$ ,  $i = 0, \dots, d-1$ . Applying Itô's lemma to  $\rho(X_t, Y_t)$ , we obtain:

$$\begin{aligned} d\rho(X_t, Y_t) &= \sqrt{2} \Xi_t u_0 \rho(X_t, Y_t) d \langle u_0, B_t \rangle + \sqrt{2} \Upsilon_t v_0 \rho(X_t, Y_t) d \langle v_0, B'_t \rangle \\ &\quad + \sum_{i=0}^{d-1} (\Xi_t u_i + \Upsilon_t v_i)^2 \rho(X_t, Y_t) dt + \langle \nabla V(X_t), \gamma'_t(0) \rangle dt - \langle \nabla V(Y_t), \gamma'_t(\rho(X_t, Y_t)) \rangle dt. \end{aligned}$$

However, by construction we have that

$$\begin{aligned}\Xi_t u_0 \rho &= -\Upsilon_t V_0 \rho, \\ \langle u_0, B_t \rangle &= \langle v_0, B'_t \rangle, \\ (\Xi_t u_0)^2 \rho &= (\Upsilon_t v_0)^2 \rho = (\Xi_t u_0)(\Upsilon_t v_0) \rho = 0.\end{aligned}$$

So  $\rho(X_t, Y_t)$  is deterministic:

$$d\rho(X_t, Y_t) = \left( \sum_{i=1}^{d-1} (\Xi_t u_i + \Upsilon_t v_i)^2 \rho(X_t, Y_t) + \langle \nabla V(X_t), \gamma'_t(0) \rangle - \langle \nabla V(Y_t), \gamma'_t(\rho(X_t, Y_t)) \rangle \right) dt.$$

On the one hand, by using the second variation of the arc length (e.g. [65]), it is possible to show that

$$\sum_{i=1}^{d-1} (\Xi_t u_i + \Upsilon_t v_i)^2 \rho(X_t, Y_t) \leq - \int_0^{\rho(X_t, Y_t)} \text{Ric}_{\gamma_t(s)}(\gamma'_t(s), \gamma'_t(s)) ds.$$

On the other hand, using the fact that  $\nabla_{\gamma'_t} \gamma'_t = 0$  because  $\gamma_t$  is a geodesic, and the definition of the hessian, we can calculate

$$\begin{aligned}\frac{d}{ds} \langle \nabla V(\gamma_t(s)), \gamma'_t(s) \rangle &= \langle \nabla_{\gamma'_t} \nabla V(\gamma_t(s)), \gamma'_t \rangle + \langle \nabla V(\gamma_t(s)), \nabla_{\gamma'_t} \gamma'_t \rangle \\ &= \text{Hess}_{\gamma_t(s)} V(\gamma'_t(s), \gamma'_t(s)).\end{aligned}$$

Finally, by using the curvature dimension condition:

$$\begin{aligned}d\rho(X_t, Y_t) &\leq - \int_0^{\rho(X_t, Y_t)} \left( \text{Hess}_{\gamma_t(s)} V(\gamma'_t(s), \gamma'_t(s)) + \text{Ric}_{\gamma_t(s)}(\gamma'_t(s), \gamma'_t(s)) \right) ds dt \\ &\leq -\kappa \int_0^{\rho(X_t, Y_t)} \|\gamma'_t(s)\| ds dt = -\kappa \rho(X_t, Y_t) dt.\end{aligned}$$

We conclude with the Gronwall lemma, and the case  $l > 1$  follows immediately.  $\square$

The existence of a contractive coupling under a positive curvature condition goes back to the paper [215] where M-K. von Renesse and K-T. Sturm showed that the two are in fact equivalent in the case of an unweighted Riemannian manifold. Let  $\Xi$  be the lift given by (7.8), and recall that for all  $x \in M$ ,  $\Xi(x) : \mathbb{R}^d \rightarrow T_x M$  is a linear isomorphism. We define

$$(7.10) \quad \|D\Xi(x)\|_{op}^2 := \sup_{v \in T_x M, |v|=1} \sum_{i=1}^d \|D_v(\Xi(x)(e_i))\|^2,$$

where  $(e_i)_{1 \leq i \leq d}$  denotes the canonical basis of  $\mathbb{R}^d$ . The map  $x \mapsto \Xi(x)(e_i)$  is from  $M$  to  $T_x M$ , so it is a vector field. So the quantity  $D_v(\Xi(x)(e_i))$  is its derivative along the vector field  $v$ , and  $\|D_v(\Xi(x)(e_i))\|$  denotes its operator norm. Therefore, (7.10) corresponds to the Lipschitz bound of the frame  $\Xi$ . This quantity controls the growth of the tangent process.

**Lemma 104.** [31, Lemma 4 Appendix A2] *Let  $X_t(x)$  be the stochastic flow of the SDE (7.8), i.e.  $X_t(x)$  is the value at time  $t$  of the process starting from  $x \in M$ . Assume that  $\|D\Xi(x)\|_{op} \leq c_1$  for some  $c_1 \geq 0$ . Then the tangent process  $v_t := dF_t(x)v$  satisfies*

$$\mathbb{E} [|v_t|^q] \leq |v|^q \exp\left(q\alpha_q \frac{t}{2}\right)$$

for any  $q > 1$ , where  $\alpha_q := -\kappa + (q-1)c_1^2$ .

**Proof** We only give the idea of the proof. We refer to the article [31] for the full proof. The tangent process satisfies the following Stratonovich SDE

$$Dv_t = D_{v_t} \Xi d_s B_t - \frac{1}{2} \text{Hess} V(v_t, v_t) dt.$$

By using the Ito lemma, we can compute that

$$d|v_t|^2 = 2 \langle v_t, D_{v_t} \Xi dB_t \rangle + \left( -\text{Hess } V(v_t, v_t) - \text{Ric}(v_t, v_t) + \sum_{i=1}^d |D_{v_t} \Xi e_i|^2 \right) dt.$$

So the curvature dimension condition (7.6) and the bound on  $\|D\Xi(x)\|_{op}$  together give

$$d\mathbb{E} [|v_t|^2] \leq (c_1^2 - \kappa) \mathbb{E} [|v_t|^2] dt,$$

and the result is then established in the case  $q = 2$  using Gronwall's lemma. The case of a general  $q > 1$  is proved in a similar way by writing  $|v_t|^q$  in terms of  $|v_t|^2$  and applying Ito's lemma.  $\square$

We deduce from Lemma 104 and Proposition 103 the following generalization of Stein's lemma.

**Theorem 105.** (see [31, Proposition 3])

Let  $L$  be a diffusion generator of the form (7.5) on a Riemannian manifold  $M$  and satisfying the curvature dimension condition  $CD(\kappa, \infty)$  for some  $\kappa > 0$ . Let  $\mu$  be the reversible probability distribution of the process generated by  $L$ . Let  $g : M \rightarrow \mathbb{R}$  be a  $\mu$ -centered and differentiable function such that its differential is Lipschitz, i.e.

$$(7.11) \quad C_1(g) := \sup_{x \neq y} \frac{\|dg(x) - \Pi_{x,y} dg(y)\|_{op}}{\rho(x, y)} < \infty,$$

where  $\Pi$  is the parallel transport, and let  $\phi$  be the solution of the Poisson equation  $L\phi = g$  given by

$$\phi(x) = - \int_0^\infty \mathbb{E}[g(X_t(x))] dt,$$

with  $X_t(x)$  the stochastic flow of the equation (7.7). Assume that  $\|D\Xi\|_{op} < c < \infty$ , where  $\Xi$  is the lift defined in (7.8). Then the differential  $d\phi$  of  $\phi$  is Lipschitz with Lipschitz constant  $C_1(\phi) \leq \frac{C_1(g)}{\kappa}$ .

**Proof** We give the general idea of the proof. More details can be found in the article [31]. Let  $v \in T_x M$ . We have

$$(7.12) \quad d\phi(x)(v) = \int_0^\infty d\mathbb{E}[f(X_t(x))](v) dt = \int_0^\infty \mathbb{E}[df(X_t(x))(v_t)] dt$$

where  $v_t := dF_t(x)v$  is the tangent process. Let then  $(X_t, Y_t)$  be the contractive coupling given by Proposition 103. Let  $q > 1$  and  $p > 1$  so that  $\frac{1}{p} + \frac{1}{q} = 1$ . Using in this order the formulas (7.12), (7.11), the Hölder inequality, Formula (7.9) and the lemma 104, we obtain

$$\begin{aligned} |(d\phi(x) - \Pi_{x,y} d\phi(y))(v)| &\leq \int_0^\infty |\mathbb{E}[(dg(X_t) - \Pi_{x,y} dg(Y_t))(v_t)]| dt \\ &\leq \int_0^\infty C_1(g) \mathbb{E}[|v_t| \rho(X_t, Y_t)] dt \\ &\leq C_1(g) \int_0^\infty \mathbb{E}(\rho(X_t, Y_t)^p)^{\frac{1}{p}} \mathbb{E}[|v_t|^q]^{\frac{1}{q}} dt \\ &\leq C_1(g) \int_0^\infty \rho(x, y) e^{-\kappa t} \mathbb{E}[|v_t|^q]^{\frac{1}{q}} dt \\ &\leq C_1(g) \int_0^\infty \rho(x, y) e^{-\kappa t + \alpha_q \frac{t}{2}} |v| dt \end{aligned}$$

where  $\alpha_q = -\kappa + (q-1)c^2$  is given by the lemma 104. Then we choose  $q = 1 + \frac{\kappa}{c^2}$  so that  $\alpha_q = 0$ . Therefore we have

$$\begin{aligned} |(d\phi(x) - \Pi_{x,y} d\phi(y))(v)| &\leq C_1(g) \int_0^\infty \rho(x, y) e^{-\kappa t} |v| dt \\ &= \frac{C_1(g)}{\kappa} \rho(x, y) |v|, \end{aligned}$$

which concludes the proof.  $\square$

### 7.3 The pushforward by several eigenfunctions

In this section, we revisit the results of Chapter 3 in the case where we consider an orthonormal basis of eigenfunctions instead of a single one. The aim is to obtain more accurate stability results by involving comparisons of multidimensional rather than just one-dimensional probability measures. Such a result already exists for the Gaussian [75, Corollary 2.4]: for all centered probability measures  $\nu$  on  $\mathbb{R}^d$ , normalized such that  $\int |x|^2 d\nu = d$ , it holds

$$(7.13) \quad C_P(\nu) \geq 1 + \frac{W_2(\nu, \gamma)^2}{d}$$

where  $C_P$  denotes the sharp Poincaré constant,  $\gamma$  denotes the Gaussian  $\mathcal{N}(0, I_d)$  and  $W_2$  is the 2-Wasserstein distance. As in Chapter 3, we want to generalise this type of result by replacing the Gaussian  $\gamma$  by the reversible measure  $\mu$  of a diffusion process. Recall that in Chapter 3 we obtain results of the form

$$C_P(\nu) \geq C_P(\mu) + d(f_1^\# \nu, f_1^\# \mu)^2,$$

where  $f_1^\#$  denotes the pushforward by an eigenfunction associated to the first non zero eigenvalue  $\frac{1}{C_P(\mu)}$  of  $-L$ , and  $d$  is a distance between probability distributions. Here we are looking for results of the form

$$(7.14) \quad C_P(\nu) \geq C_P(\mu) + d(f^\# \nu, f^\# \mu)^2,$$

where  $f = (f_1^1, \dots, f_1^m)$  is the vector of eigenfunctions  $f_1^1, \dots, f_1^m$  forming an  $L^2(\mu)$  orthonormal basis of the first eigenspace  $Sp_1(\mu)$ , and  $m = \dim(Sp_1(\mu))$ . In the case of the Gaussian, the basis is formed by all the coordinate functions  $x_i$ , so  $f$  is the identity, so the result (7.13) is of the form (7.14).

#### 7.3.1 Approximate integration by parts formula

Let  $L$  be a diffusion generator on a Riemannian manifold  $M$  of dimension  $d$ . Let  $\mu$  be the reversible measure of  $L$ . Let  $\Gamma$  be the operator carré du champ. Assume that  $\mu$  satisfies the following Poincaré inequality:

$$\forall f \in H^1(\mu), \quad \int f^2 d\mu \leq C_p(\mu) \int \Gamma(f) d\mu$$

where  $H^1(\mu) := \{f \in L^2(\mu) \mid \int f d\mu = 0 \text{ and } \int \Gamma(f) d\mu < \infty\}$  and  $C_p(\mu) < \infty$  is the sharp Poincaré constant. We denote by  $E(\mu)$  the space of vector-valued functions with coordinates in  $H^1(\mu)$ . Let  $f_1 = (f_1^1, \dots, f_1^m) \in E(\mu)$  be the vector of eigenfunctions  $f_1^1, \dots, f_1^m$  forming an  $L^2(\mu)$  orthonormal basis of the first eigenspace  $Sp_1(\mu)$ , i.e.

$$\forall i, j \in \{1, \dots, m\}, \quad \int f_1^i f_1^j d\mu = \delta_{i,j},$$

where  $\delta_{i,j}$  denotes the Kronecker delta. Let  $\nu$  be another probability measure on  $M$  normalized such that

$$\forall i \in \{1, \dots, m\}, \quad \int f_1^i d\nu = 0, \quad \int (f_1^i)^2 d\nu = 1, \quad \text{and} \quad \int \Gamma(f_1^i) d\nu \leq \frac{1}{C_p(\mu)}.$$

Let us also suppose that  $\nu$  satisfies the following Poincaré inequality:

$$\forall f \in H^1(\nu), \quad \int f^2 d\nu \leq C_p(\nu) \int \Gamma(f) d\nu,$$

where  $H^1(\nu)$  is defined in the same way as  $H^1(\mu)$ . Similarly, let  $E(\nu)$  be the space of vector-valued functions with coordinates in  $H^1(\nu)$ . The following approximate integration by parts formula is a generalization of Theorem 18.

**Theorem 106.** *The following inequality holds for all  $g = (g_1, \dots, g_d) \in E(\nu)$ ,*

(7.15)

$$\left| \int f \cdot g \, d\nu - C_P(\nu) \sum_i \int \Gamma(f_i, g_i) \, d\nu \right| \leq \sqrt{2m} (C_p(\nu) - C_p(\mu))^{\frac{1}{2}} \left( \frac{C_P(\nu)}{C_P(\mu)} \right)^{\frac{1}{2}} \left( \sum_i \int \Gamma(g_i) \, d\nu \right)^{\frac{1}{2}}.$$

**Proof** We apply the theorem 18 for each  $g_i$  and sum it up, which gives

$$\left| \sum_i \int f_1^i g_i \, d\nu - C_P(\nu) \sum_i \int \Gamma(f_1^i, g_i) \, d\nu \right| \leq (C_p(\nu) - C_p(\mu))^{\frac{1}{2}} \left( \frac{C_P(\nu)}{C_P(\mu)} \right)^{\frac{1}{2}} \sum_i \left( \int \Gamma(g_i) \, d\nu \right)^{\frac{1}{2}},$$

and we conclude by using that  $\forall x \geq 0, \sum_{i=1}^m \sqrt{x_i} \leq \sqrt{2m} \sqrt{\sum_i x_i}$ .  $\square$

Our goal is now to derive an approximate integration by parts formula for  $\nu^* := f_1^\# \nu$  from (7.15). For this purpose, we will apply Theorem 106 with  $g$  of the form  $g = \phi \circ f_1$  where  $\phi = (\phi_1, \dots, \phi_m) : f_1(M) \subset \mathbb{R}^m \rightarrow \mathbb{R}^m$ . On the one hand,

$$\sum_i \int \Gamma(f_1^i, \phi_i(f_1)) \, d\nu = \sum_i \sum_j \int \frac{\partial \phi_i}{\partial x_j}(f_1) \Gamma(f_1^i, f_1^j) \, d\nu = \int \langle \text{Jac } \phi, \Gamma \rangle_{HS} \, d\nu.$$

where  $\text{Jac}$  denotes the Jacobian matrix,  $\langle A, B \rangle_{HS} = \text{Tr}(A^T B)$  is the Hilbert-Schmidt scalar product, and  $\Gamma = (\Gamma(f_1^i, f_1^j))_{i,j} \in \mathbb{M}_m(\mathbb{R})$  is a  $m \times m$  real matrix. On the other hand,

$$\Gamma(\phi_i(f_1)) = \sum_j \frac{\partial \phi_i}{\partial x_j}(f_1) \Gamma(f_1^j, \phi_i(f_1)) = \sum_{j,k} \frac{\partial \phi_i}{\partial x_j}(f_1) \frac{\partial \phi_i}{\partial x_k}(f_1) \Gamma(f_1^j, f_1^k).$$

Therefore

$$\begin{aligned} \sum_i \int \Gamma(\phi_i \circ f_1) \, d\nu &= \sum_{i,j,k} \int \frac{\partial \phi_i}{\partial x_j}(f_1) \frac{\partial \phi_i}{\partial x_k}(f_1) \Gamma(f_1^j, f_1^k) \, d\nu \\ &= \sum_i \int \nabla \phi_i(f_1)^T \cdot \Gamma \cdot \nabla \phi_i(f_1) \, d\nu \\ &= \langle \Gamma \cdot \text{Jac } \phi, \text{Jac } \phi \rangle_{HS}. \end{aligned}$$

We have thus proved the following result.

**Corollary 107.** *Assume that  $\Gamma = H \circ f_1$ , for some matrix-valued function  $H : f_1(M) \subset \mathbb{R}^m \rightarrow \mathbb{M}_m(\mathbb{R})$ . Then for all  $\phi = (\phi_1, \dots, \phi_m) : f_1(M) \subset \mathbb{R}^m \rightarrow \mathbb{R}^m$  such that  $\phi \circ f_1 \in E(\nu)$  we have:*

$$(7.16) \quad \left| \int_{\mathbb{R}^m} x \cdot \phi(x) \, d\nu^* - C_P(\mu) \int_{\mathbb{R}^m} \langle \text{Jac } \phi(x), H(x) \rangle_{HS} \, d\nu^* \right| \leq \sqrt{2m} \left( \frac{C_p(\nu)^2}{C_p(\mu)} - C_p(\nu) \right)^{\frac{1}{2}} \int_{\mathbb{R}^m} \langle H(x) \cdot \text{Jac } \phi(x), \text{Jac } \phi(x) \rangle_{HS} \, d\nu^*(x).$$

The assumption that  $\Gamma = H \circ f_1$  is the multidimensional generalization of Assumption 2 in Chapter 3 (see Section 3.2) and implies that all Markovian structure on the manifold is mapped onto a Markovian structure in  $\mathbb{R}^m$ . This assumption is less natural in this multidimensional context than in the context of the chapter 3 because we no longer have a general result implying the injectivity of  $f_1$  (see the discussion section 7.3.3). Let us explain the quotient process. For all  $g = (g_1, \dots, g_m) : M \rightarrow \mathbb{R}^m$ , the generator  $L$  can be extended by  $Lg := (Lg_1, \dots, Lg_m)$ . The diffusion property becomes: for all smooth functions  $\phi : \mathbb{R}^m \rightarrow \mathbb{R}$ ,

$$L(\phi \circ f_1) = \sum_{i,j} \partial_{i,j}^2 \phi(f_1) \Gamma(f_1^i, f_1^j) + \nabla \phi(f_1) \cdot Lf_1,$$

which can also be written as

$$L(\phi \circ f_1) = \langle \text{Hess}(\phi)(f_1), H(f_1) \rangle_{HS} + \nabla \phi(f_1) \cdot Lf_1.$$



So the quotient generator is given by

$$\forall u \in \mathbb{R}^m, \quad L^*(\phi)(u) = \langle \text{Hess}(\phi)(u), H(u) \rangle_{HS} - \frac{1}{C_P(\mu)} u \cdot \nabla \phi(u).$$

In addition, the quotient carré du champ is given by

$$\begin{aligned} \Gamma^*(\phi)(f_1(x)) &= \Gamma(\phi \circ f_1)(x) \\ &= \sum_{i,j} \partial_i \phi(f_1(x)) \partial_j \phi(f_1(x)) \Gamma(f_1^i, f_1^j)(x) \\ &= \nabla \phi(f_1(x))^T \cdot H(f_1(x)) \cdot \nabla \phi(f_1(x)) \\ &= \langle H(f_1(x)) \nabla \phi(f_1(x)), \nabla \phi(f_1(x)) \rangle_{\mathbb{R}^m}. \end{aligned}$$

### 7.3.2 An improved stability result for the Brascamp-Lieb inequality

Let  $V : \mathbb{R}^d \rightarrow \mathbb{R}$  be a  $C^2$  strictly convex function (i.e.  $\text{Hess } V > 0$ ) such that  $\mu = e^{-V} dx$  is a probability distribution. According to the Brascamp-Lieb inequality, we have

$$(7.17) \quad \forall f \in H^1(\mu), \quad \text{Var}_\mu(f) \leq \int (\text{Hess } V)^{-1} (\nabla f, \nabla f) d\mu.$$

There exists a diffusion generator  $L$  such that its carré du champ operator is  $\Gamma(f) = (\text{Hess } V)^{-1} (\nabla f, \nabla f)$  and  $\mu$  is its reversible measure. Let us explain it. Its diffusion coefficient must be  $(\text{Hess } V)^{-1}$  because of the value of its carré du champ  $\Gamma$ . Therefore  $L$  is of the form

$$(7.18) \quad Lf = \langle (\text{Hess } V)^{-1}, \text{Hess } f \rangle_{HS} - F \cdot \nabla f,$$

for some vector-valued function  $F : \mathbb{R}^d \rightarrow \mathbb{R}^d$ . Let us denote  $(\text{Hess } V)^{-1} = (a_{i,j})_{i,j}$ . Then  $\mu$  is the equilibrium distribution of  $L$  if, and only if, for all functions  $f$  in its domain it holds

$$\int \langle (\text{Hess } V)^{-1}, \text{Hess } f \rangle_{HS} e^{-V} dx = \int \nabla f \cdot \left[ \sum_j \left( \frac{\partial a_{i,j}}{\partial x_j} - a_{i,j} \frac{\partial V}{\partial x_j} \right) \right]_i e^{-V} dx.$$

Therefore

$$F = \left[ \sum_j \left( \frac{\partial a_{i,j}}{\partial x_j} - a_{i,j} \frac{\partial V}{\partial x_j} \right) \right]_{i=1,\dots,d}.$$

The Brascamp-Lieb inequality (7.17) means that  $\mu$  satisfies a Poincaré inequality with sharp constant 1 with respect to the metric  $\text{Hess } V$ . Moreover, the case of equality is reached for the functions  $\langle \nabla f, c \rangle$  where  $c \in \mathbb{R}$  is a constant, so the functions  $-\frac{\partial V}{\partial x_i}$  form a linear basis of the eigenspace  $S_{p_1}(\mu)$ . Let  $\mu^* = \nabla V \# \mu$  be the moment measure (see [74]). The matrix  $\Gamma(\nabla V)$  can be calculated as

$$\Gamma(\nabla V) := \left[ \Gamma \left( \frac{\partial V}{\partial x_i}, \frac{\partial V}{\partial x_j} \right) \right]_{i,j} = \text{Hess } V.$$

Therefore the quotient process  $L^* := (\nabla f) \# L$  is given by

$$\forall \phi \in \mathcal{C}^2(\nabla V(\mathbb{R}^d), \mathbb{R}), \quad \forall u \in \nabla V(\mathbb{R}^d), \quad L^*(\phi)(u) = \langle \text{Hess}(\phi)(u), H(u) \rangle_{HS} - u \cdot \nabla \phi(u)$$

where  $H = \text{Hess}(V) \circ \nabla V^*$ , with  $V^*$  the Legendre transformation of the convex function  $V$ . In addition, the quotient carré du champ is given by

$$\Gamma^*(\phi)(u) = \langle H(u) \nabla \phi(u), \nabla \phi(u) \rangle_{\mathbb{R}^d}.$$

Let  $\nu$  be another probability measure on  $\mathbb{R}^d$  satisfying for all  $i = 1, \dots, d$ ,

$$\int \frac{\partial V}{\partial x_i} d\nu = 0, \quad \int \left( \frac{\partial V}{\partial x_i} \right)^2 d\nu = \int \left( \frac{\partial V}{\partial x_i} \right)^2 d\mu, \quad \text{and} \quad \int \Gamma \left( \frac{\partial V}{\partial x_i} \right) d\nu \leq \int \Gamma \left( \frac{\partial V}{\partial x_i} \right) d\mu.$$

Let us also assume that  $\nu$  satisfies the following weighted Poincaré inequality:

$$(7.19) \quad \forall f \in H^1(\mu), \quad \text{Var}_\nu(f) \leq C_P(\nu) \int (\text{Hess } V)^{-1}(\nabla f, \nabla f) d\nu.$$

Then by applying (7.19) to  $f = \frac{\partial V}{\partial x_i}$ , we immediately obtain that

$$C_P(\nu) \geq 1 = C_P(\mu).$$

Moreover, we have the following stability result under an additional curvature assertion.

**Theorem 108.** *Let  $V : \mathbb{R}^d \rightarrow \mathbb{R}$  be a  $C^2$  strictly convex function (i.e.  $\text{Hess } V > 0$ ) such that  $\mu = e^{-V} dx$  is a probability distribution. Let  $\nu$  be another probability measure on  $\mathbb{R}^d$  satisfying for all  $i = 1, \dots, d$ ,*

$$\int \frac{\partial V}{\partial x_i} d\nu = 0, \quad \int \left( \frac{\partial V}{\partial x_i} \right)^2 d\nu = \int \left( \frac{\partial V}{\partial x_i} \right)^2 d\mu, \text{ and } \int \Gamma \left( \frac{\partial V}{\partial x_i} \right) d\nu \leq \int \Gamma \left( \frac{\partial V}{\partial x_i} \right) d\mu.$$

Let us also assume that  $\nu$  satisfies the following weighted Poincaré inequality:

$$\forall f \in H^1(\mu), \quad \text{Var}_\nu(f) \leq C_P(\nu) \int (\text{Hess } V)^{-1}(\nabla f, \nabla f) d\nu.$$

If the operator (7.18) satisfies the curvature-dimension condition  $CD(\rho, \infty)$  for some  $\rho > 0$ , then the following stability estimate holds

$$C_P(\nu) (C_P(\nu) - 1) \geq \frac{\rho}{2\sqrt{2d}} W_1(\nu^*, \mu^*)^2,$$

where  $\nu^*$  (respectively  $\mu^*$ ) is the pushforward of  $\nu$  (resp.  $\mu$ ) by  $\nabla V$ , and  $W_1$  is the 1-Wasserstein distance with respect to the metric  $\text{Hess } V$  on  $\mathbb{R}^d$ .

**Proof** The proof is immediate from the corollary 107 and the Stein lemma 102.  $\square$

The theorem 108 is a generalization of the results of Section 3.6.3 of Chapter 3.

### 7.3.3 Discussion

The factorization assumption  $\Gamma = H \circ f_1$  of Corollary 107 is necessary in this approach in order to be able to pushforward the Markov process by the eigenfunctions vector  $f_1$ . This assumption has already been discussed in dimension one in chapter 3 and also in chapter 4 for higher order eigenfunctions. It is automatically satisfied when  $f_1$  is injective. In the case of Section 7.3.2, the function  $f_1$  is the gradient of a convex function, and so the Legendre transformation allows us to overcome this difficulty. However, a solution to this factorization problem seems difficult to find by a general property of injectivity of the eigenfunction vector  $f_1$ . Another way would be to consider conditional expectations. Indeed, one can notice that the quantity  $\Gamma$  only ever appears under integral symbols. Thus, instead of factoring pointwise, we can write, for a random variable  $Y$  drawn with the distribution  $\nu$ ,

$$\begin{aligned} \int \langle \text{Jac } \phi(f_1), \Gamma \rangle_{HS} d\nu &= \mathbb{E} [\langle \text{Jac } \phi(f_1(Y)), \Gamma(Y) \rangle_{HS}] \\ &= \mathbb{E} [\langle \text{Jac } \phi(f_1(Y)), \mathbb{E} [\Gamma(Y) | f_1(Y)] \rangle_{HS}] \\ &= \int \langle \text{Jac } \phi, H \rangle_{HS} d\nu^*, \end{aligned}$$

where  $H$  is a function given by the Doob-Dynkin lemma satisfying  $\mathbb{E} [\Gamma(Y) | f_1(Y)] = H \circ f_1$ . The main problem is now that  $H$  depends on  $\nu$ . We thus need a result of comparison between the conditional expectations  $\mathbb{E} [\Gamma(Y) | f_1(Y)]$  and  $\mathbb{E} [\Gamma(X) | f_1(X)]$  when  $X$  is drawn with the distribution  $\mu$ . We have not been able to establish such results, but we believe that this is an interesting avenue for research.

## 7.4 Stein's method applied to isoperimetric problems

There are stability results in the field of isoperimetry (e.g. [93, Thm 1.1]). Although Stein's method is specifically designed to obtain stability results, it has never been implemented in the context of isoperimetry due to technical difficulties. The main problem is the handling of boundary terms in the Stein equation. In this section, we propose to see isoperimetric problems themselves as applications of Stein's method. The aim is not to establish regularization results that allow us to use Stein's method to derive stability results, but to highlight the parallel between Stein's method and classical isoperimetric inequalities. To do this, we start from the weak formulation of the perimeter that goes back to the work of R. Caccioppoli [50, 51] and E. De Giorgi [78, 79]. The perimeter of a set  $\Omega \subset \mathbb{R}^d$  is considered as the total variation of its characteristic function:

$$(7.20) \quad \text{Per}(\Omega) = \sup \left\{ \int_{\Omega} \text{div } \phi \mid \phi \in \mathcal{C}_0^1(\mathbb{R}^d, \mathbb{R}^d), \|\phi\|_{\infty} \leq 1 \right\}$$

where  $\mathcal{C}_0^1(\mathbb{R}^d, \mathbb{R}^d)$  denotes the set of continuously differentiable functions on  $\mathbb{R}^d$  vanishing at infinity. Similarly, in the case of the Normal distribution  $\gamma$  on  $\mathbb{R}$ , for all subset  $A \subset \mathbb{R}$ , we have

$$(7.21) \quad \gamma^+(A) = \sup \left\{ \int_A (v'(x) - xv(x)) d\gamma(x) \mid v \in \mathcal{C}_0^1(\mathbb{R}, \mathbb{R}), \|v\|_{\infty} \leq 1 \right\}.$$

What we notice is that the previous quantity is nothing but an error term in an integration by parts, as it appears in Stein's method. More precisely, the quantity  $\gamma^+(A)$  quantifies the extent to which the normal measure restricted to the set  $A$  satisfies the classical integration by parts formula of the normal measure on all  $\mathbb{R}$ :

$$\forall v \in C^1(\mathbb{R}), \quad \int v'(x) d\gamma(x) = \int xv(x) d\gamma(x).$$

Let  $V := \{v \in \mathcal{C}_0^1(\mathbb{R}, \mathbb{R}) \mid \|v\|_{\infty} \leq 1\}$ , and  $\gamma_A(B) := \frac{\gamma(A \cap B)}{\gamma(A)}$ , be the normal distribution restricted to  $A \subset \mathbb{R}$ . Then Stein's classical lemma gives that for any  $B \subset \mathbb{R}$ ,

$$v(x) := e^{\frac{x^2}{2}} \int_{-\infty}^x (\mathbf{1}_B - \gamma(B)) d\gamma$$

is a solution of the Stein equation  $v'(x) - xv(x) = \mathbf{1}_B - \gamma(B)$  satisfying  $\|v\|_{\infty} \leq \sqrt{\frac{\pi}{2}}$ . In addition, it is easy to verify that there is a constant  $K > 0$  such that  $|v(x)| \underset{x \rightarrow \pm\infty}{\leq} \frac{K}{|x|}$ . Consequently, the function  $v$  vanishes at infinity, and therefore  $\sqrt{\frac{2}{\pi}}v \in V$ . This makes it possible to obtain

$$\begin{aligned} \gamma^+(A) &= \sup_{v \in V} \int_A (v'(x) - xv(x)) d\gamma(x) \\ &= \gamma(A) \sup_{v \in V} \int (v'(x) - xv(x)) d\gamma_A(x) \\ &\geq \sqrt{\frac{2}{\pi}} \gamma(A) \sup_{B \subset \mathbb{R}} \int (\mathbf{1}_B - \gamma(B)) d\gamma_A(x) \\ &= \sqrt{\frac{2}{\pi}} \gamma(A) d_{TV}(\gamma, \gamma_A) \\ &= 2\sqrt{\frac{2}{\pi}} \gamma(A)(1 - \gamma(A)) \\ &\geq \sqrt{\frac{2}{\pi}} \min(\gamma(A), 1 - \gamma(A)), \end{aligned}$$

which is Cheeger's isoperimetric inequality for the normal distribution with the sharp Cheeger constant  $\sqrt{\frac{2}{\pi}}$ . We were thus able to recover the sharp Cheeger inequality for the Gaussian using

a Stein lemma slightly improved by cancellation conditions at the boundary. It would seem interesting to ask whether Cheeger's inequality can be obtained by this method for other probability measures, and whether this method could be improved in order to obtain isoperimetric inequalities that are more accurate than Cheeger's inequality.



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