Contributions to matrix inequalities and some applications

Mohammad Mahdi Ghabries

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Par

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Contributions to Matrix Inequalities and Some Applications

Ph.D Thesis

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Lebanon, 2022

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Acknowledgments

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Introduction to Matrix Theory

Matrix theory has been under study for a long time, it has been a fundamental tool in mathematical disciplines presenting interesting and challenging problems. The most common example of using matrices is solving simultaneous linear equations, wherein solving $n = 2$ simultaneous linear equations is simple while $n \geq 3$ is complicated and requires the matrix method to be solved. We treat the numbers as 1×1 matrices, consequently we always try to generalize definitions, properties and operations for matrices same as numbers. In this book, we will focus on one class of matrices which is the set of all positive semi-definite matrices, in this case these matrices will generalize the non-negative numbers.
The main topics of this thesis are determinantal inequalities, eigenvalue and singular value inequalities and the geometric mean of two positive definite matrices. These concepts arise in many research areas and they play a decisive role in the diffusion tensor imaging field, probability and statistics, information theory, theoretical computer science, quantum mechanics and other mathematical fields.

The thesis is divided to five chapters.
Chapter 1, consists of six sections, highlights basic definitions, notations and properties for matrices that are essential for our work through the thesis.
Section 1.1 presents the most essential and well-known properties of Hermitian, unitary and positive semi-definite matrices, it also present the famous matrix decomposition as spectral decomposition and polar decomposition. Section 1.2 deals with matrix norms, unitarily invariant norms in particular. Section 1.3 shows the importance of Schur complement of a 2×2 complex block matrix. Section 1.4 deals with two types of matrix functions. Section 1.5 focuses on the geometric mean of two positive definite matrices, and Section 1.6 introduces the concepts of majorization relation and the log-majorization relation.

Chapter 2, consists of five sections, deals with determinantal inequalities related to a determinantal inequality arisen in the study of interpolation methods for image processing in diffusion tensor imaging established by Audenaert. These determinantal inequalities are proved for any two positive semi-definite matrices.
Section 2.1 presents determinantal inequalities and two conjectures posed by Lin regarding Au-

denaert's inequality. Section 2.2 shows the preliminaries needed in our investigation. Section 2.3 presents generalizations of the determinantal inequalities established by Audenaert and Lin, various related inequalities are shown. Section 2.4 concerns with the two conjectures, it gives an affirmative answer to the first conjecture in a slightly more general setting namely in the case when the two matrices are Hermitian. In addition, it presents the solution of the second conjecture in two different approaches. To be more precise the first approach gives a partial answer and the second approach gives the answer in the affirmative. Furthermore, a certain log-majoriation was investigated due to its important role in solving the conjectures and implying several inequalities. Lastly, Section 2.5 settles new conjectures concerning determinantal inequalities.

Chapter 3, consists of four sections, presents several applications for some obtained majorization inequalities established in Chapter 2.
Section 3.1 focuses on a new positive definite quantity for two positive definite matrices that has a connection with the geometric mean. Section 3.2 presents an upper bound for a Golden-Thompson type inequality established by Hiai and Petz. Section 3.3 deals with some new results related to the Rényi divergences as an application of the obtained majorization inequalities. Section 3.4 presents some upper bounds for a unitarily invariant norm inequality conjectured by Bhatia, Lim and Yamazaki which was subsequently proved by Dinh, Dumitru and Franco for Schatten p-norms.

Chapter 4, consists of three sections, deals with majorization inequalities concerning the geometric mean based on a previous work done by Zou, Hiai, Lin, Lemos and Soares. These majorization relations concerns eigenvalues and singular values of matrices.
Section 4.1 presents some new inequalities like generalizations of Zou's and Hiai-Lin majorization relations, as well as a reverse Lemos-Soares type inequality. Section 4.2 provides an example in which it shows that one of the conjectures introduced by Lemos and Soares is not valid in its current setting, it also presents a further generalization of Lemos-Soares majorization relation precisely in all the cases where it has been proven valid. Lastly, Section 4.3 deals with several inequalities related to another conjecture posed by Lemos and Soares.

Chapter 5, which consists of two sections, presents singular value inequalities related to a recent result of Lin and a conjecture.
Section 5.1 introduces the concepts of completely positive maps, partial transpose of a block matrix and Liebian functions, it also deals with several properties for 2×2 block matrices. Section 5.2 presents a singular value inequality that gives in particular a solution to Lin's conjecture when one of the diagonal blocks commute with the off diagonal of a 2×2 positives semi-definite block matrix.

As a general rule, the results throughout the thesis are all new. Whenever the result is not new, it is presented with a reference.

1.1. Matrix Theory Fundamentals

The basic definitions, notations and results can be found in [61]. For every natural number n we denote the set of $n \times n$ matrices with entries in the field $\mathbb{C}$ of complex numbers by $\mathbb{M}_n(\mathbb{C})$, but we can simply write $\mathbb{M}_n$, and its identity element is denoted by I_n also known as the identity matrix. The vector space of all n- dimensional complex vectors (all vectors are column vectors) denoted by $\mathbb{C}^n$ is a Hilbert space with the inner product

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \boldsymbol{y}^* \boldsymbol{x} = \sum_{k=1}^{n} x_k \overline{y_k}$$

where $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)^T$ and $\boldsymbol{y} = (y_1, y_2, \ldots, y_n)^T$.

In this thesis we consider bold lower-case letters and upper case letters to denote vectors and matrices, respectively. The transpose of a matrix $A = [a_{ij}]_{i,j=1}^n \in \mathbb{M}_n$ is denoted and defined by $A^T = [a_{ji}]_{i,j=1}^n$. The conjugate transpose of a matrix $A = [a_{ij}]_{i,j=1}^n \in \mathbb{M}_n$ sometimes called the adjoint matrix is symbolized by A^* and defined by $A^* = [\overline{a_{ji}}]_{i,j=1}^n$.

Definition 1.1. *A matrix $A = [a_{ij}]_{i,j=1}^n \in \mathbb{M}_n$ is said to be:*

(i) Diagonal matrix if $a_{ij} = 0$ when $i \neq j$.

(ii) Invertible matrix if there exists an matrix B of order $n \times n$ such that $AB = I_n$. In case A is invertible, we say that A has a unique inverse matrix $A^{-1} \in \mathbb{M}_n$ such that $A^{-1}A = AA^{-1} = I_n$.

(iii) Normal matrix if $AA^* = A^*A$.

(iv) Unitary matrix if $AA^* = A^*A = I_n$.

(v) Hermitian matrix if $A = A^*$.

(vi) Positive semi-definite matrix if $\langle A\boldsymbol{x}, \boldsymbol{x} \rangle \geq 0$ for all $\boldsymbol{x} \in \mathbb{C}^n$.

(vii) Positive definite matrix if $\langle A\boldsymbol{x}, \boldsymbol{x} \rangle > 0$ for all $\boldsymbol{x} \in \mathbb{C}^n \backslash \{0\}$.

Definition 1.2. *Let A and B be two complex matrices of the same order. We say that A and B commute if*

$$AB = BA.$$

Definition 1.3. *Let $A \in \mathbb{M}_n$. A is said to be a singular matrix if A is not invertible.*

The next definition shows a partial order well-known as the Löwner order in which it compares Hermitian matrices of the same order.

Definition 1.4. ***(Löwner Order)*** *Let A and B be two Hermitian matrices of same order n. We say that $A \geq B$ if and only if $A - B$ is a positive semi-definite matrix.*

Remark 1.1. *We can just say that $A \geq 0$ if and only if A is a positive semi-definite matrix. If A is a positive definite matrix, then we can write $A > 0$.*

Definition 1.5. ***(Eigenvalues)*** *A complex number λ is said to be an eigenvalue of a matrix A corresponding to its non-zero eigenvector $\boldsymbol{x}$ if*

$$A\boldsymbol{x} = \lambda\boldsymbol{x}.$$

The multiset of the eigenvalues of A is denoted by $Sp(A)$ and known as the spectrum of A.

Remark 1.2. *Note that any square matrix of order $n \times n$ has exactly n eigenvalues. In general, we denote the eigenvalues of the matrix A as $\lambda_j(A)$ for $j = 1, 2, \ldots, n$.*

Theorem 1.1. *Let $A \in \mathbb{M}_n$ be a positive semi-definite matrix, then A has a unique square root matrix $B \in \mathbb{M}_n$ such that $A = B^2$. We denote the square root by $A^{\frac{1}{2}}$.*

Definition 1.6. ***(Matrix Modulus and Singular values)*** *The modulus of a matrix $A \in \mathbb{M}_n$ is the square root of the matrix A^*A denoted by*

$$|A| = (A^*A)^{\frac{1}{2}}.$$

The eigenvalues of $|A|$ are known as the singular values of A and they are denoted as $\sigma_j(A)$ for $j = 1, 2, \ldots, n$.

Proposition 1.1. *Let $A, B \in \mathbb{M}_n$. We have*

(1) $Sp(AB) = Sp(BA)$.

(2) If A is a Hermitian matrix then $Sp(A) \subset \mathbb{R}$.

(3) A is a positive semi-definite (respectively positive definite) if and only if A is a Hermitian matrix and $Sp(A) \subset \mathbb{R}^+$ (respectively $Sp(A) \subset \mathbb{R}^+ \backslash \{0\}$).

(4) $X^*X \geq 0$ for all $X \in \mathbb{M}_n$.

(5) If $A, B \geq 0$ then $Sp(AB) \subset \mathbb{R}^+$.

Remark 1.3. *Due to the fact of Part (3) and Part (4) of the previous proposition, we can notice that the singular values of any matrix are non-negative.*

Throughout this thesis we will arrange the singular values of a matrix $A \in \mathbb{M}_n$ in decreasing order that is

$$\sigma_1(A) \geq \sigma_2(A) \geq \cdots \geq \sigma_n(A).$$

In addition, if the eigenvalues $\lambda_1(A), \lambda_2(A), \ldots, \lambda_n(A)$ of A are real then we will always assume that they are also arranged in non-increasing order, that is

$$\lambda_1(A) \geq \lambda_2(A) \geq \cdots \geq \lambda_n(A).$$

For a Hermitian matrix $A \in \mathbb{M}_n$, we shall denote

$$\boldsymbol{\lambda(A)} = (\lambda_1(A), \lambda_2(A), \ldots, \lambda_n(A))^T$$

which is clearly a real column vector of order n, and so is the vector

$$\boldsymbol{\sigma(A)} = (\sigma_1(A), \sigma_2(A), \ldots, \sigma_n(A))^T.$$

Theorem 1.2. *Let $A \in \mathbb{M}_n$. Then*

(1) ***(Spectral decomposition)*** *A is a normal matrix if and only if there exists a unitary matrix $U \in \mathbb{M}_n$ such that $U^*AU = D$ where D is a diagonal matrix, and its diagonal entries are the eigenvalues of A.*

(2) ***(Polar decomposition)*** *There exists a unitary matrix $U \in \mathbb{M}_n$ and a positive semi-definite matrix $P \in \mathbb{M}_n$ such that*

$$A = UP.$$

Moreover, if A is invertible then U is unique.

Definition 1.7. ***(Determinant, Trace)*** *For a given matrix $A = [a_{ij}]_{i,j=1}^n \in \mathbb{M}_n$ with $Sp(A) = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$. We have*

- *The determinant of A is denoted and defined by*

$$\det(A) = \sum_{\rho \in \mathbb{S}_n} \left(sgn(\rho) \prod_{i=1}^n a_{i\rho_i} \right) = \prod_{j=1}^n \lambda_j.$$

  where $\mathbb{S}_n$ is the set of all permutations ρ of the set $\mathbb{S} = \{1, 2, \ldots, n\}$. A permutation ρ is a function that rearrange $\mathbb{S}$, the integer in the i^{th} position in ρ is denoted by ρ_i. The signature of a permutation ρ, denoted and defined by $sgn(\rho) = (-1)^\nu$ where ν is the number of interchanges needed to obtain ρ from $\mathbb{S}$.

- *The trace of A is denoted as $Tr(A)$ and is defined as the sum of the of the diagonal entries of A*

$$Tr(A) = \sum_{i=1}^n a_{ii} = \sum_{j=1}^n \lambda_j.$$

Proposition 1.2. *Let $A \in \mathbb{M}_n$. A is a positive semi-definite matrix if and only if for all $P \in \mathbb{M}_n$ invertible matrices P^*AP is a positive semi-definite matrix.*

Proposition 1.3. *Let $A, B \in \mathbb{M}_n$ be two Hermitian matrices with $Sp(A) = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ and $Sp(B) = \{\mu_1, \mu_2, \ldots, \mu_n\}$. Then*

(a) If $A > 0$ and $B > 0$, then $A \geq B$ if and only if $B^{-1} \geq A^{-1}$.

(b) If $A \geq B$, then $X^*AX \geq X^*BX$ for every $X \in \mathbb{M}_n$.

(c) If $A \geq B$, then $\lambda_j \geq \mu_j$ for each $j = 1, 2, \ldots, n$.

(d) If $A \geq B \geq 0$, then $Tr(A) \geq Tr(B) \geq 0$.

(e) If $A \geq B \geq 0$, then $\det(A) \geq \det(B) \geq 0$.

1.2. Matrix Norms

A function $||\cdot|| : \mathbb{M}_n \to \mathbb{R}$ is said to be a matrix norm if for all $A, B \in \mathbb{M}_n$ and $\forall\ \alpha \in \mathbb{C}$ we have:

(1) $||A|| \geq 0$.

(2) $||A|| = 0$ if and only if $A = 0$.

(3) $||\alpha A|| = |\alpha| \cdot ||A||$.

(4) $||A + B|| \leq ||A|| + ||B||$.

In addition, a matrix norm is said to be sub-multiplicative matrix norm if

$$||AB|| \leq ||A|| \cdot ||B||.$$

Definition 1.8. ***(Unitarily invariant norm)*** *A matrix norm is said to be a unitarily invariant norm if for every $A \in \mathbb{M}_n$, we have $||UAV|| = ||A||$ for all $U, V \in \mathbb{M}_n$ unitary matrices. It is denoted as $|||\cdot|||$.*

Let $A = [a_{ij}]_{i,j=1}^n \in \mathbb{M}_n$. The following unitarily invariant norms are frequently used:

(i) The operator norm of A, defined by

$$|||A|||_{op} = \sqrt{\lambda_1(A^*A)} = \sigma_1(A).$$

(ii) The Frobenius norm of A also known as the Hilbert-Schmidt norm is defined as

$$|||A|||_F = \left(\sum_{i,j} |a_{ij}|^2\right)^{\frac{1}{2}} = \left(\sum_{j=1}^n \sigma_j^2(A)\right)^{\frac{1}{2}} = \sqrt{tr(A^*A)}.$$

(iii) The Ky Fan norms, defined as

$$|||A|||_{(k)} = \sum_{j=1}^{k} \sigma_j(A) \quad \text{for } k = 1, 2, \ldots, n.$$

(iv) For every $1 \leq p \leq \infty$, the Schatten p-norm of the matrix A is defined as

$$|||A|||_p = \left(\sum_{j=1}^{n} \sigma_j^p(A) \right)^{\frac{1}{p}}.$$

Of special importance are the cases $p = 1$ named the trace norm, $p = 2$ the Hilbert-Schmidt norm. For $p = \infty$ we can see that

$$\begin{aligned} \lim_{p\to\infty} |||A|||_p &= \lim_{p\to\infty} \left(\sigma_1^p(A) + \sigma_2^p(A) + \cdots + \sigma_n^p(A)\right)^{\frac{1}{p}} \\ &= \sigma_1(A) \lim_{p\to\infty} \left(1 + \left(\frac{\sigma_2(A)}{\sigma_1(A)}\right)^p + \cdots + \left(\frac{\sigma_n(A)}{\sigma_1(A)}\right)^p\right)^{\frac{1}{p}} \\ &= \sigma_1(A) \\ &= |||A|||_{op}. \end{aligned}$$

Now we will introduce one of the most basic theorem for unitarily invariant norms. The theorem is known as the Ky Fan dominance theorem. See more well-known Ky Fan inequalities in [52]

Theorem 1.3. ***(Ky Fan dominance theorem)*** *Let A and B be any two $n \times n$ matrices. The inequalities*

$$|||A|||_{(k)} \leq |||B|||_{(k)} \ (k = 1, 2, \ldots, n) \text{ hold if and only if } |||A||| \leq |||B|||$$

for all unitarily invariant norms $|||\cdot|||$.

1.3. Schur complement

The set of $m \times m$ block matrices whose entries are matrices of order $n \times n$ with complex entries is denoted as $\mathbb{M}_m(\mathbb{M}_n(\mathbb{C}))$. Let $A \in \mathbb{M}_m(\mathbb{M}_n(\mathbb{C}))$. Then A is written as

$$A = \begin{bmatrix} A_{11} & A_{12} & \ldots & A_{1m} \\ A_{21} & A_{22} & \ldots & A_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ A_{m1} & A_{m2} & \ldots & A_{mm} \end{bmatrix}.$$

If $A \in \mathbb{M}_2(\mathbb{M}_n(\mathbb{C}))$, we can say that A is a 2×2 block matrix, and it can be written as

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}.$$

Consider these two 2×2 block matrices

$$M := \begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \text{ and } P := \begin{bmatrix} I_n & -A^{-1}B \\ 0 & I_n \end{bmatrix}$$

where A, B and $C \in \mathbb{M}_n$ such that A is invertible.

Now, clearly we have

$$P^*MP = \begin{bmatrix} A & 0 \\ 0 & C - B^*A^{-1}B \end{bmatrix}. \tag{1.1}$$

Proposition 1.4. *Let X and Y be two square matrices of order n. The 2×2 block matrix*

$$\begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix} \geq 0 \quad \textit{if and only if} \quad X \geq 0 \quad \textit{and} \quad Y \geq 0.$$

Remark 1.4. *Due to the fact of Proposition 1.2, Proposition 1.4 as well as the decomposition in (1.1), it is worthy to note that the matrix M is positive semi-definite if and only if the 2×2 block matrix*

$$\begin{bmatrix} A & 0 \\ 0 & C - B^*A^{-1}B \end{bmatrix}$$

is a positive semi-definite matrix.

Definition 1.9. ***(Schur Complement)*** *The Schur complement of A is denoted and defined by*

$$\widetilde{A} = C - B^*A^{-1}B.$$

Proposition 1.5. *Let M be a 2×2 block matrix written as*

$$M := \begin{bmatrix} A & B \\ B^* & C \end{bmatrix}$$

with $A > 0$. Then $M \geq 0$ if and only if $\widetilde{A} = C - B^*A^{-1}B \geq 0$.

As a consequence of the previous proposition, we have the following results which can be found in [62]. But we will include the proof here for the sake of clarification.

Proposition 1.6. *Let $X, Y \in \mathbb{M}_n$. We have:*

(i) $\begin{bmatrix} |X| & X^* \\ X & |X^*| \end{bmatrix} \geq 0.$

(ii) $\begin{bmatrix} X & X \\ X & X \end{bmatrix} \geq 0$ *if and only if* $X \geq 0$.

(iii) If X *and* Y *Hermitian matrices. Then, the* 2×2 *block matrix*

$$\begin{bmatrix} X & Y \\ Y & X \end{bmatrix} \geq 0 \text{ if and only if } X \geq Y \text{ and } X \geq -Y.$$

Proof. (i) By appealing to Shcur complement, our claim is to show that

$$|X^*| \geq X|X|^{-1}X^*.$$

Using the polar decomposition of the matrix X gives

$$X = U|X| \text{ where } U \text{ is a unitary matrix.}$$

Now, observe that $X^* = |X|U^*$, $U = X|X|^{-1}$ and $U^* = |X|^{-1}X^*$. Hence, we have

$$\begin{aligned} |X^*| &= (XX^*)^{\frac{1}{2}} \\ &= (U|X||X|U^*)^{\frac{1}{2}} \\ &= (U|X|^2U^*)^{\frac{1}{2}} \\ &= U|X|U^* \\ &= X|X|^{-1}|X||X|^{-1}X^* \\ &= X|X|^{-1}X^*. \end{aligned}$$

This completes the proof.

(ii) It is obvious here to check it using Schur complement.

(iii) For the sake of simplicity, let $M = \begin{bmatrix} X & Y \\ Y & X \end{bmatrix}$. The sufficient condition is by using Schur complement. Assume that X is invertible, so our claim is to show that

$$X > 0 \text{ and } X - YX^{-1}Y \geq 0.$$

Due to our given, we can see that

$$2X = X + X \geq Y - Y = 0$$

which gives $X > 0$ since X is invertible and $X \geq 0$.

Now, noticing that $X^{-1} \leq Y^{-1}$ gives

$$Y = YY^{-1}Y \geq YX^{-1}Y.$$

Hence, $M \geq 0$.

For the necessary condition, consider the following decomposition

$$\left(\sqrt{\frac{1}{2}}\begin{bmatrix} I_n & -I_n \\ I_n & I_n \end{bmatrix}\right)\begin{bmatrix} X & Y \\ Y & X \end{bmatrix}\left(\sqrt{\frac{1}{2}}\begin{bmatrix} I_n & I_n \\ -I_n & I_n \end{bmatrix}\right) = \begin{bmatrix} X-Y & 0 \\ 0 & X+Y \end{bmatrix}.$$

So, the eigenvalues of M are the union of the eigenvalues of $X - Y$ and $X + Y$, since the 2×2 block matrix

$$\sqrt{\frac{1}{2}}\begin{bmatrix} I_n & I_n \\ -I_n & I_n \end{bmatrix} \text{ is a Unitary matrix.}$$

Therefore, $X - Y$ and $X + Y$ are positive semi-definite matrices. Thus

$$X \geq Y \text{ and } X \geq -Y.$$

■

Proposition 1.7. *Let A, B and C be any three complex matrices. If*

$$M := \begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \geq 0, \text{ then } \begin{bmatrix} \det(A) & \det(B) \\ \det(B^*) & \det(C) \end{bmatrix} \geq 0.$$

Proof. Without loss of generality we will assume that A is invertible. By appealing to Schur complement we have

$$A > 0 \text{ and } C \geq B^*A^{-1}B$$

which implies that

$$\det(A) > 0 \text{ and } \det(C) \geq \det(B^*A^{-1}B).$$

Hence, $\det(A) > 0$ and $\det(C) \geq (\det(B))^*(\det(A))^{-1}(\det(B))$. Again, by using Schur complement we get for A invertible

$$\begin{bmatrix} \det(A) & \det(B) \\ \det(B^*) & \det(C) \end{bmatrix} \geq 0.$$

In case A is singular. The continuity argument states that there exists a strictly positive real number δ such that for all $0 < \epsilon < \delta$, we have

$$A_\epsilon = A + \epsilon I_n \text{ is invertible.}$$

Now, take

$$M_\epsilon := \begin{bmatrix} A_\epsilon & B \\ B^* & C \end{bmatrix}.$$

Here, M_ϵ is a positive semi-definite 2×2 block matrix since

$$M_\epsilon = \begin{bmatrix} A & B \\ B^* & C \end{bmatrix} + \begin{bmatrix} \epsilon I_n & 0 \\ 0 & 0 \end{bmatrix}.$$

We can now apply the previous case for M_ϵ, so

$$\begin{bmatrix} \det(A_\epsilon) & \det(B) \\ \det(B^*) & \det(C) \end{bmatrix} \geq 0.$$

Therefore, we obtain the result since the determinant function is continuous. ■

1.4. Matrix Functions

The first matrix function we are interested in is $f : \mathbb{M}_n \to \mathbb{M}_n$ which is in fact the extension of the scalar functions. For a given $\mathbb{I} \subset \mathbb{R}$, let $f : \mathbb{I} \to \mathbb{R}$. For every $A \in \mathbb{M}_n$ Hermitian matrix with $Sp(A) = \{\lambda_1, \lambda_2, \ldots, \lambda_n\} \in I$. Using the spectral decomposition we have

$$f(A) = U^* f(D) U$$

where $f(D)$ is a diagonal matrix with diagonal elements $\{f(\lambda_1), f(\lambda_2), \ldots, f(\lambda_n)\}$.

Corollary 1.1. *Let $A \in \mathbb{M}_n$ be a positive semi-definite matrix with $Sp(A) = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$. Then, for every $r \in [0, \infty)$*

$$Sp(A^r) = \{\lambda_1^r, \lambda_2^r, \ldots, \lambda_n^r\}.$$

Definition 1.10. *Let $A, B \in \mathbb{M}_n$ be two Hermitian matrices with spectrum in $\mathbb{I} \subset \mathbb{R}$. A function $f : \mathbb{I} \to \mathbb{R}$ is said to be an operator monotone if*

$$A \leq B \quad \textit{implies} \quad f(A) \leq f(B).$$

Theorem 1.4. ***(Löwner-Heinz Inequality [47])*** *Let A and B be two positive semi-definite matrices such that $A \leq B$. Then*

$$A^r \leq B^r, \qquad 0 \leq r \leq 1.$$

These are some monotone and non-monotone matrix functions:

(i) $f(A) = \ln(A)$ is an operator monotone function.

(ii) For every $0 \leq r \leq 1$, $f(A) = A^r$ is an operator monotone function.

(iii) $f(A) = A^2$ is not a monotone function.

The second matrix function we need is an operation on a matrix producing a scalar such as trace and determinant. Later in Chapter 5, we need the definition of functions of class $\mathbb{L}$ from [8, p.268]; in fact these functions were first defined by E.H. Lieb [39] and they are called Liebian functions.

Definition 1.11. *A continuous complex-valued function f on the space of matrices is called a Liebian function if f satisfies these two conditions:*

1. *If $A \geq B$ then $f(A) \geq f(B)$.*
2. *$|f(A^*B)|^2 \leq f(A^*A)f(B^*B)$ for all A, B.*

These functions are examples of Liebian functions:

(i) $f(A) = \det(A)$.

(ii) $f(A) = Tr(A)$.

(iii) For all $1 \leq k \leq n$, $f_{(k)}(A) = \prod_{j=1}^{k} \sigma_j(A)$.

1.5. Geometric Mean

Average operations are of interests in the context of matrices as well, and various notions of means of two positive definite matrices A and B have been studied, see [9, Chapter 4].

Definition 1.12. *The geometric mean of two non-negative numbers x and y is denoted and defined by*

$$G(x, y) = \sqrt{xy}.$$

The following theorem shows some properties of

Theorem 1.5. *For all $x, y \in \mathbb{R}^+$ and $\forall\ \alpha \in \mathbb{R}^+$, we have:*

1. *$G(x, y) \geq 0$.*
2. *If $0 \leq x \leq y$ then $x \leq G(x, y) \leq y$.*
3. *$G(x, y) = G(y, x)$.*
4. *$G(\alpha x, \alpha y) = \alpha G(x, y)$ and $G(\alpha x\bar{\alpha}, \alpha y\bar{\alpha}) = \alpha G(x, y)\bar{\alpha}$.*
5. *$G(x, y) \leq G(x', y)$ if $x \leq x'$ and $G(x, y) \leq G(x, y')$ if $y \leq y'$.*

In general, the matrix product of any two Hermitian matrices A and B is not Hermitian except when A and B commute. Then the square root of AB is neither Hermitian nor unique, and so it is not possible to define the matrix geometric mean of $A, B \geq 0$ by $(AB)^{\frac{1}{2}}$. However, for $a, b > 0$ the positive quantity $\sqrt{ab}$ can be also written as:

$$\sqrt{ab} = \sqrt{a^{\frac{1}{2}}ba^{\frac{1}{2}}} = \sqrt{aa^{-\frac{1}{2}}ba^{-\frac{1}{2}}a} = a^{\frac{1}{2}}\sqrt{a^{-\frac{1}{2}}ba^{-\frac{1}{2}}}a^{\frac{1}{2}} = a^{\frac{1}{2}}(a^{-\frac{1}{2}}ba^{-\frac{1}{2}})^{\frac{1}{2}}a^{\frac{1}{2}}.$$

Definition 1.13. ***(Geometric Mean [55])*** *Let A and B be two positive definite matrices. The geometric mean of A and B is denoted and defined by*

$$A\#B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\frac{1}{2}}A^{\frac{1}{2}}.$$

As it is well known, this can be extended to the positive semi-definite matrices as follows

$$A\#B = \lim_{\epsilon \to 0^+} (A + \epsilon I_n)\#(B + \epsilon I_n).$$

Theorem 1.6. ***(Riccati Equation [53])*** *Let A and B be two positive definite matrices. The unique positive solution of the quadratic equation defined by*

$$XA^{-1}X = B$$

known as the Riccati equation is $A\#B$.

The next proposition shows another formula of the geometric mean of two positive definite matrices.

Proposition 1.8. *Let A and B be two positive definite matrices. Then*

$$A\#B = \max\left\{X : X^* = X, \begin{bmatrix} A & X \\ X & B \end{bmatrix} \geq 0\right\}.$$

Proposition 1.9. *For all $A, B, X \in \mathbb{M}_n$ such that $A, B > 0$ and $\alpha, \beta \geq 0$.*

(i) $A\#B > 0$.

(ii) $(A\#B)^{-1} = A^{-1}\#B^{-1}$.

(iii) If A, B commutes then $A\#B = A^{\frac{1}{2}}B^{\frac{1}{2}}$.

(iv) If $A \geq B > 0$ *then* $A \geq A\#B \geq B$.

(v) $A\#B = B\#A$.

(vi) If $B_1 \geq B_2 > 0$ *then* $A\#B_1 \geq A\#B_2$.

(vii) $(\alpha A)\#(\beta B) = \sqrt{\alpha\beta}(A\#B)$.

(viii) $(X^*AX)\#(X^*BX) = X^*(A\#B)X$ *for every invertible matrix* X.

The following example shows that last part of Proposition 1.9 is not valid for X singular matrix.

Example 1.1. *Consider* $A = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 20 & 6 \\ 6 & 2 \end{bmatrix}$ *and* $X = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. *Then*

$$X^*(A\#B)X = \begin{bmatrix} 8 & 0 \\ 0 & 0 \end{bmatrix} \neq (X^*AX)\#(X^*BX) = \begin{bmatrix} \sqrt{80} & 0 \\ 0 & 0 \end{bmatrix}.$$

A matrix connection is a binary operation on the set of positive definite matrices denoted by γ and defined by F. Kubo and T. Ando [35].

Definition 1.14. *A binary operation* γ *is said to be a connection if it satisfies the following conditions for all* $A, B, C, D > 0$

(i) If $A \geq C$ *and* $B \geq D$, *then* $A\gamma B \geq C\gamma D$.

(ii) $X^*(A\gamma B)X \leq (X^*AX)\gamma(X^*BX)$ *for invertible matrix* $X \in \mathbb{M}_n$.

(iii) If $A_n \downarrow A$ *and* $B_n \downarrow B$, *then* $(A_n\gamma B_n) \downarrow A\gamma B$ *where* $A_n, B_n \in \mathbb{M}_n$.

For each connection γ, there exists a unique matrix monotone function $f : \mathbb{R}^+ \to \mathbb{R}^+$, such that $f(x)I_n = I_n\gamma(xI)$ for $x \in \mathbb{R}^+$, and for $A, B \geq 0$ such that A is invertible, $A\gamma B$ can be presented as

$$A\gamma B = A^{\frac{1}{2}} f(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}) A^{\frac{1}{2}}. \tag{1.2}$$

It can be extended to any singular matrix A by continuity as follows

$$A\gamma B = \lim_{\epsilon \to 0^+} (A + \epsilon I_n)\gamma B.$$

A special case of (1.2) and a generalization of the geometric mean is the well known weighted geometric mean

$$A\#_t B = A^{\frac{1}{2}}(A^{-\frac{1}{2}} B A^{-\frac{1}{2}})^t A^{\frac{1}{2}} \qquad 0 \leq t \leq 1.$$

In particular, when $t = \frac{1}{2}$ we get the well known geometric mean, that is, $A\#_{\frac{1}{2}}B = A\#B$.

The weighted geometric mean appears in the study of the Riemannian manifold of positive invertible matrices, see [17, 36].

1.6. Majorization Relation

In this section we will define the majorization relation that compares two vectors $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$. It is an important tool in deriving determinantal (see [42, 56]), trace inequalities (see [19, 20])

and norm inequalities (see [4]). A good review on the theory of majorization was given by Marshall, Olkin and Arnold in [48].

Definition 1.15. *Let $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)^T$ be a vector in $\mathbb{R}^n$. By rearranging the components of $\boldsymbol{x}$ in decreasing order we get the vector $\boldsymbol{x}^{\downarrow} = (x_1^{\downarrow}, x_2^{\downarrow}, \ldots, x_n^{\downarrow})^T \in \mathbb{R}^n$ where*

$$x_1^{\downarrow} \geq x_2^{\downarrow} \geq \ldots \geq x_n^{\downarrow}.$$

Example 1.2. *If* $\boldsymbol{x} = \begin{bmatrix} 1 \\ 4 \\ 2 \\ 0 \end{bmatrix}$, *then* $\boldsymbol{x}^{\downarrow} = \begin{bmatrix} 4 \\ 2 \\ 1 \\ 0 \end{bmatrix}$.

Definition 1.16. *Let $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)^T$ and $\boldsymbol{y} = (y_1, y_2, \ldots, y_n)^T$ be two vectors in $\mathbb{R}^n$. We say:*

1. *$\boldsymbol{x}$ is weakly majorized by $\boldsymbol{y}$, denoted by $\boldsymbol{x} \prec_w \boldsymbol{y}$ or $\boldsymbol{y} \succ_w \boldsymbol{x}$ if*

$$\sum_{i=1}^{k} x_i^{\downarrow} \leq \sum_{i=1}^{k} y_i^{\downarrow} \qquad k = 1, 2, \ldots, n \tag{1.3}$$

2. *$\boldsymbol{x}$ is majorized by $\boldsymbol{y}$, denoted by $\boldsymbol{x} \prec \boldsymbol{y}$ if (1.3) is true for $k = 1, 2, \ldots, n-1$ and equality holds for $k = n$.*

Proposition 1.10. *The majorization relation is:*

(i) Reflexive and transitive, however it is not anti-symmetric.

Example: *Let* $\boldsymbol{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ *and* $\boldsymbol{y} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$. *Then $\boldsymbol{x} \succ \boldsymbol{y}$ and $\boldsymbol{x} \prec \boldsymbol{y}$, but $\boldsymbol{x} \neq \boldsymbol{y}$.*

(ii) Not a total relation.

Example: *Let* $\boldsymbol{x} = \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix}$ *and* $\boldsymbol{y} = \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix}$. *We have $\boldsymbol{x} \not\succ \boldsymbol{y}$ and $\boldsymbol{x} \not\prec \boldsymbol{y}$.*

Remark 1.5. *If $\boldsymbol{x}$ is majorized by $\boldsymbol{y}$, then $\boldsymbol{x}'$ is majorized by $\boldsymbol{y}'$, where $\boldsymbol{x}'$ and $\boldsymbol{y}'$ are two vectors in $\mathbb{R}^n$ having the same components as the vectors $\boldsymbol{x}$ and $\boldsymbol{y}$, respectively.*

Definition 1.17. *Let $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)^T$ and $\boldsymbol{y} = (y_1, y_2, \ldots, y_n)^T$ be two vectors in $(\mathbb{R}^+ \backslash \{0\})^n$. If for all $1 \leq k \leq n$, we have*

$$\prod_{i=1}^{k} x_i^{\downarrow} \leq \prod_{i=1}^{k} y_i^{\downarrow}, \tag{1.4}$$

then we say $\boldsymbol{x}$ *is weakly log-majorized by* $\boldsymbol{y}$ *denoted by* $\boldsymbol{x} \prec_{wlog} \boldsymbol{y}$. *The vector* $\boldsymbol{x}$ *is log-majorized by* $\boldsymbol{y}$ *written as* $\boldsymbol{x} \prec_{log} \boldsymbol{y}$ *if* (1.4) *is true for* $1 \leq k \leq n-1$ *and equality is valid for* $k = n$.

Remark 1.6. *For given vectors* $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)^T, \boldsymbol{y} = (y_1, y_2, \ldots, y_n)^T \in (\mathbb{R}^+ \backslash \{0\})^n$, *the log-majorization (respectively weakly log-majorization) relation of* $\boldsymbol{x}$ *and* $\boldsymbol{y}$ *is in fact the majorization (respectively weakly majorization) relation of the the vectors*

$$\ln \boldsymbol{x} = (\ln x_1, \ln x_2, \ldots, \ln x_n)^T \quad \textit{and} \quad \ln \boldsymbol{y} = (\ln y_1, \ln y_2, \ldots, \ln y_n)^T.$$

A very useful relation between the log-majorization inequalities and majorization inequalities is the following theorem.

Theorem 1.7. ***([61, Theorem 10.15])*** *Let* $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)^T, \boldsymbol{y} = (y_1, y_2, \ldots, y_n)^T \in (\mathbb{R}^+ \backslash \{0\})^n$. *Then*

$$\boldsymbol{x} \prec_{wlog} \boldsymbol{y} \quad \Rightarrow \quad \boldsymbol{x} \prec_{w} \boldsymbol{y}.$$

That is,

$$\prod_{i=1}^{k} x_i \leq \prod_{i=1}^{k} y_i, \ k = 1, 2, \ldots, n \quad \Rightarrow \quad \sum_{i=1}^{k} x_i \leq \sum_{i=1}^{k} y_i, \ k = 1, 2, \ldots, n.$$

2

Determinantal Inequalities

The main contributions to this thesis is presented in this chapter. We study a determinantal inequality arisen in the study of interpolation methods for image processing in diffusion tensor imaging showed by Audenaert [3]. Motivated by Audenaert's result, Lin [40] proved a generalization and complement of Audenaert's inequality, and introduced two conjectures concerning determinantal inequalities. We provide a further generalization of Audenaert's inequality as well as generalizations of Lin's inequalities. We then give affirmative answers to the conjectures. Various related inequalities are established. Lastly, we propose some new conjectures based on the work done in this chapter. Throughout this chapter we give step by step the results obtained in this research for the sake of solving these conjectures.

2.1. A Determinantal Inequality Arising from Diffusion Tensor Imaging

In 2015, Audenaert [3] proved the following determinantal inequality.

Theorem 2.1. *Let A and B be two positive semi-definite matrices. Then*

$$\det(A^2 + |BA|) \leq \det(A^2 + AB). \tag{2.1}$$

As a result, he was able to show the following corollary.

Corollary 2.1. *Let A and B be two n-square positive semi-definite matrices and U is a specified unitary matrix such that $BA = U|BA|$. Then*

$$\det(A + U^*B) \leq \det(A + B). \tag{2.2}$$

Proof. Without loss of generality we will assume that A is a positive definite matrix. Dividing both sides of (2.1) with $\det(A) > 0$ gives (2.2).

Now, for $A \geq 0$, for A singular to be precise. The continuity argument states that there exists a strictly positive real number δ such that for all $0 < \epsilon < \delta$, we have

$$A_\epsilon = A + \epsilon I_n$$

is invertible, that is, $A_\epsilon > 0$. Then, $\det(A_\epsilon + U_\epsilon^* B) \leq \det(A_\epsilon + B)$ where U_ϵ is a unitary matrix such that

$$BA_\epsilon = U_\epsilon |BA_\epsilon|.$$

As a matter of fact, the set of all unitary matrices is a compact group, then every sequence has a convergent sub-sequence, hence U_ϵ converges into a unitary matrix U as $\epsilon \to 0^+$. So, we get the outcome as the two functions $f(A) = A^p$ and $g(A, B) = |BA|^p$ are continuous for all $p \geq 0$, as well as the determinant function. ■

In fact, inequality (2.2) was arisen in the study of interpolation methods for image processing in diffusion tensor imaging. Audenaert compared geodesics introduced by different metrics and the matrices studied here are the so called statistical covariance matrices. These covariance matrices are real, positive semi-definite matrices of order $n = 3$, which means inequality (2.2) is actually more general than what is needed as it holds for any complex positive semi-definite matrices of all dimensions $n \geq 1$. See [3] for more details on how inequality (2.2) was formulated.

In 2017, M. Lin [40] proved a generalization and a complement of (2.1).

Theorem 2.2. *Let A and B be two positive semi-definite matrices. Then, for all $0 \leq p \leq 2$*

$$\det(A^2 + |BA|^p) \leq \det(A^2 + A^p B^p). \tag{2.3}$$

Theorem 2.3. *Let A and B be two positive semi-definite matrices. Then*

$$\det(A^2 + |AB|) \geq \det(A^2 + AB). \tag{2.4}$$

Clearly, inequalities (2.1) and (2.4) imply that

$$\det(A^2 + |BA|) \leq \det(A^2 + |AB|). \tag{2.5}$$

In the same paper, it was asked whether it is possible to find a generalization of (2.5) and put forward the following conjecture.

Conjecture 2.1. *Let A and B be two positive semi-definite matrices. Then, for all $p \geq 0$*

$$\det(A^2 + |BA|^p) \leq \det(A^2 + |AB|^p).$$

In addition, he introduced the following conjecture which is a generalization of (2.4) and a complementing result for (2.3).

Conjecture 2.2. *Let A and B be $n \times n$ positive semi-definite matrices. Then, for $0 \leq p \leq 2$ it holds that*

$$\det(A^2 + |AB|^p) \geq \det(A^2 + A^p B^p).$$

Lin [40] was able to prove Conjecture 2.1 for $p = 1$ and for all p positive even integers, and Conjecture 2.2 for $p = 1$ and $p = 2$, the case $p = 0$ is obvious. All other cases for both conjectures remain unsolved.

From the first look at these conjectures, it is obvious that for $A, B \geq 0$ such that A is singular, we have for $k, p \geq 0$

$$\det(A^k + |BA|^p) = \det(A^k + A^p B^p) \leq \det(A^k + |AB|^p).$$

We can see it simply by using the spectral decomposition for the matrix A, that is, there exists U unitary matrix and D diagonal matrix such that $A = UDU^*$, and noticing that for $k, p \geq 0$ we can write

$$\begin{aligned} \det(A^k + A^p B^p) &= \det(UD^k U^* + UD^p U^* B^p) \\ &= \det[U(D^k + D^p U^* B^P U)U^*] \\ &= \det(D^k + D^p X) \end{aligned}$$

where $X = U^* B^p U$ is a positive semi-definite matrix.

We can also write

$$\begin{aligned} \det(A^k + |BA|^p) &= \det(A^k + (AB^2A)^{\frac{p}{2}}) \\ &= \det(UD^k U^* + (UDU^* B^2 UDU^*)^{\frac{p}{2}}) \\ &= \det(UD^k U^* + U(DU^* B^2 UD)^{\frac{p}{2}} U^*) \\ &= \det[U(D^k + (DU^* B^2 UD)^{\frac{p}{2}})U^*] \\ &= \det(D^k + (DYD)^{\frac{p}{2}}) \end{aligned}$$

where $Y = U^* B^2 U$ is a positive semi-definite matrix.

The singularity of the matrix A implies that D has at least one zero eigenvalue and so are D^k and D^p, then $D^p X$ and $(DYD)^{p/2}$ have a zero row on the same position as that of D^k, thus $D^k + D^p X$ and $D^k + (DYD)^{p/2}$ have a zero row. So, for all $k, p \geq 0$, we have

$$\det(A^k + A^p B^p) = \det(A^k + |BA|^p) = 0.$$

The determinantal quantity $\det(A^k + |AB|^p)$ is greater than or equal to zero in general, and it is not necessary equal to zero for same conditions above (see the example below), thus we get

in case A is singular

$$\det(A^k + |AB|^p) \geq \det(A^k + A^p B^p) = \det(A^k + |BA|^p) = 0 \quad \text{for } k, p \geq 0.$$

Example 2.1. *Let* $A = \begin{bmatrix} \sqrt{2} & 0 \\ 0 & 0 \end{bmatrix}$ *and* $B = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$. *We have for* $k = p = 2$

$$\det(A^2 + BA^2B) = 16 \neq 0.$$

The conjectures are not simply observed in other cases.

2.2. Preliminaries

In this section, we will introduce some definitions and theorems we need in our work.

Definition 2.1. ***(Hadamard Product)*** *Let* $A = [a_{ij}]_{i,j=1}^n$ *and* $B = [b_{ij}]_{i,j=1}^n$ *be two complex matrices of order* n. *The Hadamard product of* A *and* B *is defined by*

$$A \circ B = \begin{bmatrix} a_{11}b_{11} & a_{12}b_{12} & \dots & a_{1n}b_{1n} \\ a_{21}b_{21} & a_{22}b_{22} & \dots & a_{2n}b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}b_{n1} & a_{n2}b_{n2} & \dots & a_{nn}b_{nn} \end{bmatrix}.$$

As we can see, the Hadamard product is simply entrywise multiplication. Note that both A and B need to be of same size. The Hadamard product can also be defined for vectors as

$$\boldsymbol{x} \circ \boldsymbol{y} = (x_1y_1, x_2y_2, \dots, x_ny_n)^T$$

where $\boldsymbol{x} = (x_1, x_2, \dots, x_n)^T$ and $\boldsymbol{y} = (y_1, y_2, \dots, y_n)^T$ are vectors in $\mathbb{C}^n$. We will refer to [34, 51] for more information about this product.

Definition 2.2. *1. Let* X *and* Y *be two Hermitian matrices with*

$$\boldsymbol{\lambda(X)} = (\lambda_1(X), \lambda_2(X), \dots, \lambda_n(X))^T \text{ and } \boldsymbol{\lambda(Y)} = (\lambda_1(Y), \lambda_2(Y), \dots, \lambda_n(Y))^T.$$

Then

$$\boldsymbol{\lambda(X)} \circ \boldsymbol{\lambda(Y)} = (\lambda_1(X)\lambda_1(Y), \lambda_2(X)\lambda_2(Y), \dots, \lambda_n(X)\lambda_n(Y))^T.$$

2. For each matrix $X \in \mathbb{M}_n$. *The modulus of the vector* $\boldsymbol{\lambda(X)}$ *is defined as*

$$|\boldsymbol{\lambda(X)}| = (|\lambda_1(X)|, |\lambda_2(X)|, \dots, |\lambda_n(X)|)^T.$$

Notice that the vector $\boldsymbol{\lambda(X)} \circ \boldsymbol{\lambda(Y)}$ is not automatically arranged in decreasing order after the Hadamard product, that is,

$$\lambda_1(X)\lambda_1(Y) \geq \lambda_2(X)\lambda_2(Y) \geq \cdots \geq \lambda_n(X)\lambda_n(Y) \tag{2.6}$$

as the following example shows.

Example 2.2. *Take A and B two Hermitian matrices such that*

$$\boldsymbol{\lambda(A)} = (5, 3, -3)^T \text{ and } \boldsymbol{\lambda(B)} = (4, 2, -10)^T.$$

Then

$$\lambda_3(A)\lambda_3(B) = 30 > \lambda_1(A)\lambda_1(B) = 20.$$

However (2.6) is valid for A and B positive semi-definite matrices.

The next two lemmas are very useful for deriving log-majorization inequalities and their proofs can be found in [61, p. 352, p.353].

Lemma 2.1. *Let* $M = \begin{bmatrix} X & Y \\ Y^* & Z \end{bmatrix} \geq 0$. *Then*

$$|\boldsymbol{\lambda(Y)}| \prec_{log} \boldsymbol{\lambda(X)}^{\frac{1}{2}} \circ \boldsymbol{\lambda(Z)}^{\frac{1}{2}} \quad \text{and} \quad \boldsymbol{\sigma(Y)} \prec_{log} \boldsymbol{\lambda(X)}^{\frac{1}{2}} \circ \boldsymbol{\lambda(Z)}^{\frac{1}{2}}.$$

In particular, we have

$$|\lambda_1(Y)| \leq \lambda_1(X)^{\frac{1}{2}} \cdot \lambda_1(Z)^{\frac{1}{2}} \quad \text{and} \quad \sigma_1(Y) \leq \lambda_1(X)^{\frac{1}{2}} \cdot \lambda_1(Z)^{\frac{1}{2}}.$$

Lemma 2.2. *Let X be any matrix of order n. Then*

$$|\boldsymbol{\lambda(X)}| \prec_{log} \boldsymbol{\sigma(X)}.$$

Now, we will show an interplay between majorization relations and determinantal inequalities, but first we need the following theorem whose proof can be found in [61, p. 343].

Theorem 2.4. *Let $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)^T, \boldsymbol{y} = (y_1, y_2, \ldots, y_n)^T \in \mathbb{R}^n$. Then*

(i) $\boldsymbol{x}$ is majorized by $\boldsymbol{y}$ if and only if $\sum_{i=1}^{n} f(x_i) \leq \sum_{i=1}^{n} f(y_i)$ for every convex function f.

(ii) $\boldsymbol{x}$ is weakly majorized by $\boldsymbol{y}$ if and only if $\sum_{i=1}^{n} f(x_i) \leq \sum_{i=1}^{n} f(y_i)$ for all convex and increasing functions f.

The next lemma is proved by M. Lin in [40, (P1), (P2)]. To cover every aspect, we add the proof here.

Lemma 2.3. *Let A and B be two positive semi-definite matrices. Then*

(i) If $\boldsymbol{\lambda(A)}, \boldsymbol{\lambda(B)} \in (\mathbb{R}_+\backslash\{0\})^n$ *such that* $\boldsymbol{\lambda(A)} \prec \boldsymbol{\lambda(B)}$, *then* $\det(A) \geq \det(B)$.

(ii) If $\boldsymbol{\lambda(A)} \prec_{wlog} \boldsymbol{\lambda(B)}$, *then* $\det(I_n + A) \leq \det(I_n + B)$.

Proof. (i) Let $f(x) = -\ln x$. Clearly f is a convex function on $\mathbb{R}^+\backslash\{0\}$. Applying Theorem 2.4 gives

$$\begin{aligned}
\boldsymbol{\lambda(A)} \prec \boldsymbol{\lambda(B)} &\Rightarrow \sum_{i=1}^{n}(-\ln\lambda_i(A)) \leq \sum_{i=1}^{n}(-\ln\lambda_i(B)) \\
&\Rightarrow \sum_{i=1}^{n}\ln\lambda_i(A) \geq \sum_{i=1}^{n}\ln\lambda_i(B) \\
&\Rightarrow \ln\left(\prod_{i=1}^{n}\lambda_i(A)\right) \geq \ln\left(\prod_{i=1}^{n}\lambda_i(B)\right) \\
&\Rightarrow \prod_{i=1}^{n}\lambda_i(A) \geq \prod_{i=1}^{n}\lambda_i(B) \\
&\qquad \text{(since the function } e^x \text{ is monotone on } \mathbb{R}^+) \\
&\Rightarrow \det(A) \geq \det(B).
\end{aligned}$$

(ii) Let $f(x) = \ln(1 + e^x)$, then $f'(x) = \frac{e^x}{1+e^x} \geq 0$ and $f''(x) = \frac{e^x}{(1+e^x)^2} \geq 0$ so f is a convex and increasing function on $\mathbb{R}$. Hence

$$\begin{aligned}
\boldsymbol{\lambda(A)} \prec_{wlog} \boldsymbol{\lambda(B)} &\Rightarrow \boldsymbol{\ln\lambda(A)} \prec_w \boldsymbol{\ln\lambda(B)} \\
&\Rightarrow \sum_{i=1}^{n} f(\ln\lambda_i(A)) \leq \sum_{i=1}^{n} f(\ln\lambda_i(B)) \\
&\Rightarrow \sum_{i=1}^{n}\ln(1+\lambda_i(A)) \leq \sum_{i=1}^{n}\ln(1+\lambda_i(B)) \\
&\Rightarrow \ln\left(\prod_{i=1}^{n}(1+\lambda_i(A))\right) \leq \ln\left(\prod_{i=1}^{n}(1+\lambda_i(B))\right) \\
&\Rightarrow \prod_{i=1}^{n}(1+\lambda_i(A)) \leq \prod_{i=1}^{n}(1+\lambda_i(B)) \\
&\Rightarrow \det(I_n + A) \leq \det(I_n + B).
\end{aligned}$$

■

Remark 2.1. *The majorization condition in Part (i) of Lemma 2.3 is essential since the weakly majorization does not imply the determinantal inequality as shown in the following example.*

Example 2.3. *Let* $A = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix} \in \mathbb{M}_2$. *Then*

$$\boldsymbol{\lambda(A)} = (4,1)^T \prec_w \boldsymbol{\lambda(B)} = (5,1)^T, \quad \textit{but} \quad \det(A) = 4 < \det(B) = 5.$$

Definition 2.3. *Let $r = 1, 2, \ldots, n$. The r^{th} anti-symmetric tensor product of a matrix $A \in \mathbb{M}_n$ denoted by $\wedge^r A$ is the restriction of the r^{th} tensor power of A denoted by $\otimes^r A$, to the totally anti-symmetric subspace of $(\mathbb{C}^n)^{\otimes r}$.*

The following are some essential properties of this product where the first one is known as the Binet-Cauchy formula.

Proposition 2.1. *Let A and B be any two $n \times n$ matrices. We have*

(i) $\wedge^r(AB) = \wedge^r A \wedge^r B$.

(ii) $(\wedge^r(A))^* = \wedge^r A^*$.

(iii) $(\wedge^r A)^{-1} = \wedge^r A^{-1}$ *in case A invertible.*

(iv) $(\wedge^r A)^p = \wedge^r A^p$ *for every $A \geq 0$ and $p > 0$.*

(v) If A is positive semi-definite or Hermitian matrix then so is $\wedge^r A$.

(vi) If $A \geq 0$, then $\lambda_1(\wedge^r A) = \prod_{i=1}^r \lambda_i(A)$.

The well known Löwner-Heinz inequality mentioned before states that

$$0 \leq A \leq B \quad \text{ensures} \quad 0 \leq A^r \leq B^r \quad \text{for} \quad 0 \leq r \leq 1.$$

However, $0 \leq A \leq B$ does not always ensure that $0 \leq A^r \leq B^r$ for $r \geq 1$. In 1987, T. Furuta [23] proved a genius extension of Löwner-Heinz inequality (see [26] for the elementary proof). This operator inequality can be applied in the theories of other operator inequalities as well as norm inequalities, see [2, 24, 25, 49] for some applications.

Lemma 2.4. ***(Furuta Inequality)*** *Let X and Y be two positive semi-definite matrices such that $X \geq Y$. Then, for all $p \geq 1$, $r \geq 0$,*

$$X^{(p+2r)/p} \geq (X^r Y^p X^r)^{1/p}.$$

The following lemma is a fundamental inequality commonly attributed to Araki, Lieb and Thirring whose proof can be found in [8, p. 258].

Lemma 2.5. *Let X and Y be two positive semi-definite matrices. Then for every unitarily invariant norm, we have*

$$|||X^t Y^t X^t||| \leq |||(XYX)^t||| \qquad 0 \leq t \leq 1, \tag{2.7}$$

and

$$|||X^t Y^t X^t||| \geq |||(XYX)^t||| \qquad t \geq 1 \tag{2.8}$$

Next, we prove the following elementary lemma which in fact is a slight generalization of Lemma A on page 129 of [27].

Lemma 2.6. *Let X and Y be two invertible matrices. Then, for all $t \in \mathbb{R}$,*

$$(X^*Y^*YX)^t = X^*Y^*(YXX^*Y^*)^{t-1}YX.$$

Proof. Let $X^*Y^* = U|X^*Y^*|$ be the polar decomposition of the matrix X^*Y^*, where U is unitary. Then, clearly we obtain $U = X^*Y^*|X^*Y^*|^{-1}$, $U^* = |X^*Y^*|^{-1}YX$, and $YX = |X^*Y^*|U^*$.

$$\begin{aligned}(X^*Y^*YX)^t &= (U|X^*Y^*||X^*Y^*|U^*)^t \\ &= \left(U|X^*Y^*|^2U^*\right)^t \\ &= U|X^*Y^*|^{2t}U^* \\ &= X^*Y^*|X^*Y^*|^{-1}|X^*Y^*|^{2t}|X^*Y^*|^{-1}YX \\ &= X^*Y^*(YXX^*Y^*)^{t-1}YX.\end{aligned}$$

■

2.3. Related Determinantal Inequalities

We start this section with a summary of the determinantal inequalities proved by Audenaert and Lin which are stated in Section 1.

Proposition 2.2. *Let A and B be two positive semi-definite matrices. Then*

(1) ***(Audenaert)*** $\det(A^2 + |BA|) \leq \det(A^2 + AB)$.

(2) ***(Lin)*** $\det(A^2 + |BA|^p) \leq \det(A^2 + A^pB^p)$ *for* $0 \leq p \leq 2$.

(3) ***(Lin)*** $\det(A^2 + |AB|) \geq \det(A^2 + AB)$.

(4) ***(Lin)*** $\det(A^2 + |AB|^2) \geq \det(A^2 + A^2B^2)$.

One would ask if Part (2) of Proposition 2.2 is valid for $p > 2$, which is not as the next example shows.

Example 2.4. *Take* $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ *and* $B = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$. *A simple check shows that*

$$\det(A^2 + |BA|^3) = 139.86 > \det(A^2 + A^3B^3) = 136.$$

However, it turns out that there is a general relation between the two determinantal quantities

$$\det(A^2 + |BA|)^p \text{ and } \det(A^2 + A^pB^p) \text{ for every } p \in \mathbb{R} \text{ and with } A, B \geq 0.$$

In this section, first we will show how the above relation behaves for any $p \in \mathbb{R}$. Next, we will show a further generalization for Part (1) of Proposition 2.2. And motivated by this further generalization, we will show similar generalizations for Part (3) and Part (4) of Proposition 2.2. We will also show a refinement for Part (4) of Proposition 2.2.

P.S. The techniques used in the proof of these related results guided us later to find affirmative answers to Conjecture 2.1 and Conjecture 2.2.

2.3.1. Comparing $\det(A^2 + |BA|^p)$ with $\det(A^2 + A^pB^p)$

The main purpose here is to study the relation between the two determinantal quantities

$$\det(A^2 + |BA|^p) \quad \text{and} \quad \det(A^2 + A^pB^p) \quad \text{for } p \in \mathbb{R} \text{ and with } A, B \geq 0.$$

We shall start with the following lemma which is obtained from a result proved by A. Matsumoto, R. Nakamoto and M. Fujii [50] by using anti-symmetric tensor product argument.

Lemma 2.7. *Let X and Y be two positive semi-definite matrices. Then,*

(i) $\boldsymbol{\lambda(X^{k+t}Y^t) \prec_{log} \lambda(X^{\frac{k}{2}}(X^{\frac{1}{2}}YX^{\frac{1}{2}})^tX^{\frac{k}{2}})}$ *for all* $0 \leq t \leq 1$ *and* $k \geq 0$.

(ii) $\boldsymbol{\lambda(X^{k+t}Y^t) \succ_{log} \lambda(X^{\frac{k}{2}}(X^{\frac{1}{2}}YX^{\frac{1}{2}})^tX^{\frac{k}{2}})}$ *for all* $t \geq 1$ *and* $0 \leq k \leq 1$.

Proof. (i) The authors in [50] proved the following interesting operator norm inequality for all $k \geq 0$

$$|||X^{\frac{k+t}{2}}Y^tX^{\frac{k+t}{2}}|||_{op} \leq |||X^{\frac{k}{2}}(X^{\frac{s}{2}}Y^sX^{\frac{s}{2}})^{\frac{t}{s}}X^{\frac{k}{2}}|||_{op} \qquad \text{for } \ 0 \leq t \leq s \tag{2.9}$$

which is in fact a generalized norm inequality proved by N. Bebiano, R. Lemos and J. Providencia in [6].

Now, using the anti-symmetric tensor product, we have for $1 \leq r \leq n$

$$\wedge^r \left(X^{k/2}(X^{s/2}Y^sX^{s/2})^{\frac{t}{s}}X^{kt/2}\right) = (\wedge^r X)^{k/2}\left((\wedge^r X)^{s/2}(\wedge^r Y)^s(\wedge^r X)^{s/2}\right)^{\frac{t}{s}}(\wedge^r X)^{k/2}$$

and

$$\wedge^r \left(X^{\frac{k+t}{2}}Y^tX^{\frac{k+t}{2}}\right) = (\wedge^r X)^{\frac{k+t}{2}}(\wedge^r Y)^t(\wedge^r X)^{\frac{k+t}{2}}.$$

Replacing X and Y with $\wedge^r X$ and $\wedge^r Y$, respectively, in (2.9) gives

$$\lambda_1\left(\wedge^r\left(X^{\frac{k+t}{2}}Y^tX^{\frac{k+t}{2}}\right)\right) \leq \lambda_1\left(\wedge^r\left(X^{\frac{k}{2}}(X^{\frac{s}{2}}Y^sX^{\frac{s}{2}})^{\frac{t}{s}}X^{\frac{k}{2}}\right)\right).$$

So, for all $1 \leq r \leq n-1$, we have

$$\prod_{i=1}^{r} \lambda_i \left(X^{\frac{k+t}{2}} Y^t X^{\frac{k+t}{2}} \right) \leq \prod_{i=1}^{r} \lambda_i \left(X^{\frac{k}{2}} (X^{\frac{s}{2}} Y^s X^{\frac{s}{2}})^{\frac{t}{s}} X^{\frac{k}{2}} \right).$$

In general, $\det \left(X^{\frac{k}{2}} (X^{\frac{s}{2}} Y^s X^{\frac{s}{2}})^{\frac{t}{s}} X^{\frac{k}{2}} \right) = \det \left(X^{\frac{k+t}{2}} Y^t X^{\frac{k+t}{2}} \right)$, we obtain

$$\prod_{i=1}^{n} \lambda_i \left(X^{\frac{k}{2}} (X^{\frac{s}{2}} Y^s X^{\frac{s}{2}})^{\frac{t}{s}} X^{\frac{k}{2}} \right) = \prod_{i=1}^{n} \lambda_i \left(X^{\frac{k+t}{2}} Y^t X^{\frac{k+t}{2}} \right).$$

Therefore,

$$\boldsymbol{\lambda} \left(\boldsymbol{X^{\frac{k}{2}} (X^{\frac{s}{2}} Y^s X^{\frac{s}{2}})^{\frac{t}{s}} X^{\frac{k}{2}}} \right) \prec_{log} \boldsymbol{\lambda} \left(\boldsymbol{X^{k+t} Y^t} \right).$$

Thus, we get the desired inequality by taking $s = 1$.

(ii) The proof here is done in a similar way as the previous one, but this time by using the operator norm inequality also proved in [50]

$$|||X^{\frac{k+t}{2}} Y^t X^{\frac{k+t}{2}}|||_{op} \geq |||X^{\frac{k}{2}} (X^{\frac{s}{2}} Y^s X^{\frac{s}{2}})^{\frac{t}{s}} X^{\frac{k}{2}}|||_{op} \qquad \text{for} \quad 0 \leq k \leq s \leq t.$$

■

As a consequence, we have the next theorem.

Theorem 2.5. *Let A and B be two positive semi-definite matrices. We have*

(1) In case A and B invertible. Then

- $\det(A^2 + |BA|^p) \leq \det(A^2 + A^p B^p)$ *for* $p \in (-\infty, -2]$, *and*
- $\det(A^2 + |BA|^p) \geq \det(A^2 + A^p B^p)$ *for* $p \in [-2, 0]$.

(2) If $p \in [0, 2]$, then $\det(A^2 + |BA|^p) \leq \det(A^2 + A^p B^p)$.

(3) If $p \in [2, 4]$, then $\det(A^2 + |BA|^p) \geq \det(A^2 + A^p B^p)$.

(4) If $p \in [4, \infty)$, then $\det(A^2 + |BA|^p) \leq \det(A^2 + A^p B^p)$.

Proof. (1) • Taking $k = 1$, $t = -\frac{p}{2}$ and replacing X and Y with A^{-2} and B^{-2}, respectively in Part (ii) of Lemma 2.7 gives

$$\boldsymbol{\lambda(A^{-1}(AB^2A)^{\frac{p}{2}}A^{-1})} \prec_{log} \boldsymbol{\lambda(A^{p-2}B^p)} \qquad p \leq -2.$$

Now, using Part (ii) of Lemma 2.3 on the above majorization relation yields

$$\det(I_n + A^{-1}(AB^2A)^{\frac{p}{2}}A^{-1}) \leq \det(I_n + A^{p-2}B^p) \qquad p \leq -2. \tag{2.10}$$

Hence, the result is obtained after multiplying both sides of (2.10) with $\det(A^2)$.

- For $p \in [-2, 0]$ the proof can be done in a similar fashion as in the first case by making use this time of Part (i) of Lemma 2.7.

(2) The proof for the case where $0 \leq p \leq 2$, is due to [40].

(3) Without loss of generality, we shall assume that A is a positive definite matrix as the singular case of A is true. For $2 \leq p \leq 4$, we can write

$$\begin{aligned}
\boldsymbol{\lambda}\left(\boldsymbol{A^{-1}(AB^2A)^{\frac{p}{2}}A^{-1}}\right) &= \boldsymbol{\lambda}\left(\boldsymbol{A^{-1}(AB(BA^2B)^{\frac{p}{2}-1}BA)A^{-1}}\right) \\
&= \boldsymbol{\lambda}\left(\boldsymbol{B(BA^2B)^{\frac{p}{2}-1}B}\right) \\
&= \boldsymbol{\lambda}\left(\boldsymbol{(B^2)^{\frac{1}{2}}\left((B^2)^{\frac{1}{2}}(A^2)(B^2)^{\frac{1}{2}}\right)^{\frac{p}{2}-1}(B^2)^{\frac{1}{2}}}\right) \\
&\succ_{log} \boldsymbol{\lambda}\left(\boldsymbol{(B^2)^{1+\frac{p}{2}-1}(A^2)^{\frac{p}{2}-1}}\right) \\
&= \boldsymbol{\lambda}\left(\boldsymbol{A^{p-2}B^p}\right).
\end{aligned}$$

Here, the first equality follows from Lemma 2.6 by substituting X and Y with A and B, respectively, and the inequality follows from replacing X with B^2 and Y with A^2, and taking $t = \frac{p}{2} - 1$ in Part (i) of Lemma 2.7.

So,

$$\boldsymbol{\lambda(A^{-1}(AB^2A)^{\frac{p}{2}}A^{-1})} \succ_{log} \boldsymbol{\lambda(A^{p-2}B^p)}. \tag{2.11}$$

Next, applying Part (ii) of Lemma 2.3 on (2.11) gives

$$\det(I_n + A^{-1}(AB^2A)^{\frac{p}{2}}A^{-1}) \geq \det(I_n + A^{p-2}B^p). \tag{2.12}$$

Multiplying both sides of (2.12) by $\det(A^2) > 0$ completes the proof of this case.

(4) The case where $p \geq 4$ can be done with similar methods as the previous case by making use this time of Part (ii) of Lemma 2.7.

■

Remark 2.2. *In the previous theorem, equality holds for* $p \in \{-2, 0, 2, 4\}$.

2.3.2. Generalization of Audeanart's Determinantal Inequality

We will show a generalization for Part (1) and Part (2) of Proposition 2.2, but first we need the following lemma.

Lemma 2.8. *Let A and B be two positive definite matrices. Then, for all $0 \leq t \leq 1$ and $k > 1$, we have*

$$\boldsymbol{\lambda}\left(\boldsymbol{A^{kt/2}(A^{-1/2}BA^{-1/2})^tA^{kt/2}}\right) \prec_{log} \boldsymbol{\lambda}\left(\boldsymbol{A^{(k-1)t}B^t}\right). \tag{2.13}$$

Proof. To achieve (2.13), it is enough to show that

$$A^{\frac{(k-1)t}{2}}B^tA^{\frac{(k-1)t}{2}} \leq I_n \Rightarrow A^{kt/2}(A^{-1/2}BA^{-1/2})^tA^{kt/2} \leq I_n.$$

Assume that $A^{\frac{(k-1)t}{2}}B^tA^{\frac{(k-1)t}{2}} \leq I_n$, so $0 \leq B^t \leq A^{-(k-1)t}$ and by applying Furuta inequality, we get

$$A^{-(k-1)t(p+2r)/p} \geq \left(A^{-(k-1)tr}B^{tp}A^{-(k-1)tr}\right)^{1/p}.$$

Now, by replacing p with $\frac{1}{t} \geq 1$ and r with $\frac{1}{2(k-1)t} > 0$ we obtain

$$A^{-(k-1)t^2\left(\frac{1}{t}+\frac{1}{(k-1)t}\right)} \geq \left(A^{-1/2}BA^{-1/2}\right)^t$$

which implies

$$A^{-kt} \geq \left(A^{-1/2}BA^{-1/2}\right)^t.$$

Therefore $A^{kt/2}(A^{-1/2}BA^{-1/2})^tA^{kt/2} \leq I_n$.

Let $a = \lambda_1(A^{\frac{(k-1)t}{2}}B^tA^{\frac{(k-1)t}{2}})$. If $a = 0$, then it is obvious that (2.13) is true.

If $a > 0$, we observe that

$$A^{\frac{(k-1)t}{2}}B^tA^{\frac{(k-1)t}{2}} \leq a\ I_n \quad \text{and} \quad \left(\frac{1}{a^{1/kt}}A\right)^{\frac{(k-1)t}{2}}\left(\frac{1}{a^{1/kt}}B\right)^t\left(\frac{1}{a^{1/kt}}A\right)^{\frac{(k-1)t}{2}} \leq I_n.$$

This yields

$$\left(\frac{1}{a^{1/kt}}A\right)^{kt/2}\left[\left(\frac{1}{a^{1/kt}}A\right)^{-1/2}\left(\frac{1}{a^{1/kt}}B\right)\left(\frac{1}{a^{1/kt}}A\right)^{-1/2}\right]^t\left(\frac{1}{a^{1/kt}}A\right)^{kt/2} \leq I_n.$$

Thus

$$A^{kt/2}(A^{-1/2}BA^{-1/2})^tA^{kt/2} \leq a\ I_n.$$

And hence

$$\lambda_1\left(A^{kt/2}(A^{-1/2}BA^{-1/2})^tA^{kt/2}\right) \leq \lambda_1\left(A^{\frac{(k-1)t}{2}}B^tA^{\frac{(k-1)t}{2}}\right).$$

Now, using the anti-symmetric tensor product argument, and observing that

$$\det\left(A^{kt/2}(A^{-1/2}BA^{-1/2})^tA^{kt/2}\right) = \det\left(A^{\frac{(k-1)t}{2}}B^tA^{\frac{(k-1)t}{2}}\right),$$

we get the desired inequality. ∎

Theorem 2.6. *Let A and B be two positive semi-definite matrices. Then, for all $0 \leq p \leq 2$ and for all $k > 1$*

$$\det(A^{kp} + |BA|^p) \leq \det(A^{kp} + A^p B^p) \tag{2.14}$$

Proof. By applying Part (ii) of Lemma 2.3 on the log-majorization relation in Lemma 2.8 we get

$$\det(I_n + A^{kt/2}(A^{-1/2}BA^{-1/2})^t A^{kt/2}) \leq \det(I_n + A^{(k-1)t}B^t). \tag{2.15}$$

Taking $A = A^{-2}$, $B = B^2$ and $t = \frac{p}{2}$ in (2.15) gives

$$\det(I_n + A^{-kp/2}(AB^2A)^{p/2}A^{-kp/2}) \leq \det(I_n + A^{-kp+p}B^p) \tag{2.16}$$

Multiplying both sides of (2.16) by $\det(A^{kp})$ leads to the result for $k > 1$ and $0 < p \leq 2$.

Finally, it is easy to see that (2.14) is true for $p = 0$. ∎

Remark 2.3. *Note that inequality (2.13) is equivalent to*

$$\boldsymbol{\lambda}\left(\boldsymbol{A^{k/2}(A^{-1/2}BA^{-1/2})^t A^{k/2}}\right) \prec_{log} \boldsymbol{\lambda}\left(\boldsymbol{A^{k-t}B^t}\right) \quad \text{for } 0 \leq t \leq 1 \text{ and } k > t. \tag{2.17}$$

It is worthy to mention here that the log-majorization relation (2.17) was first proved by T. Furuta [25, Corollary 3.1], and for $k = 1$ it was obtained earlier by J.S. Matharu and J.S. Aujla [49, Theorem 2.10]. Later, for $k = 2$ and $t = \frac{1}{2}$, it was reobtained by Bhatia, Lim and Yamazaki [12, Theorem 2], and also for $k = 2$ it was reobtained by D.T. Hoa [31, Proposition 2.1].

Theorem 2.7. *Let A, B be two positive semi-definite matrices. Then, for $0 \leq p \leq 2$,*

$$\det(A^p + |BA|^p) \leq \det(A^p + A^p B^p) \tag{2.18}$$

Proof. Again, we can assume that A is an invertible matrix, the case of A singular is true. Replacing X with A^{-1}, Y with AB^2A and t with $\frac{p}{2}$ in (2.7) and using spectral norm gives

$$\lambda_1\left(A^{-\frac{p}{2}}(AB^2A)^{\frac{p}{2}}A^{-\frac{p}{2}}\right) \leq \lambda_1\left((A^{-1}AB^2AA^{-1})^{\frac{p}{2}}\right) = \lambda_1(B^p).$$

As $\det(A^{-\frac{p}{2}}(AB^2A)^{\frac{p}{2}}A^{-\frac{p}{2}}) = \det(B^p)$, and by a similar use of the anti-symmetric tensor product argument, we have

$$\boldsymbol{\lambda}\left(\boldsymbol{A^{-\frac{p}{2}}(AB^2A)^{\frac{p}{2}}A^{-\frac{p}{2}}}\right) \prec_{log} \boldsymbol{\lambda(B^p)} \qquad 0 \leq p \leq 2. \tag{2.19}$$

Applying Part (ii) of Lemma 2.3 for (2.19) yields

$$\det(I_n + A^{-p/2}(AB^2A)^{p/2}A^{-p/2}) \leq \det(I_n + B^p).$$

Again, we multiply both sides by $\det(A^p)$ to obtain the result. ∎

The following corollary (we later knew it was first proved in [37]) is a result of Theorem 2.6 and Theorem 2.7.

Corollary 2.2. *Let A and B be two positive semi-definite matrices. Then, for all $0 \leq p \leq 2$ and $k \geq p$*

$$\det(A^k + |BA|^p) \leq \det(A^k + A^p B^p). \tag{2.20}$$

We will show that inequality (2.20) is not true for $k = 0$, however the reverse is true as the next theorem shows.

Theorem 2.8. *Let A and B be two positive semi-definite matrices. Then for $0 \leq p \leq 2$,*

$$\det(I_n + |BA|^p) \geq \det(I_n + A^p B^p) \tag{2.21}$$

Proof. Take $X = B^2$, $Y = A$ and the spectral norm in (2.7) gives

$$\lambda_1(B^t A^{2t} B^t) \leq \lambda_1((BA^2B)^t) \qquad 0 \leq t \leq 1.$$

By anti-symmetric tensor product argument and since

$$\det((AB^2A)^t) = \det(A^t B^{2t} A^t),$$

we have for all $0 \leq t \leq 1$

$$\boldsymbol{\lambda(A^t B^{2t} A^t) \prec_{log} \lambda((AB^2A)^t)}.$$

It is enough to assume that A is positive definite matrix. For $t = \frac{p}{2}$ and by Part (ii) of Lemma 2.3 we get

$$\det(I_n + A^p B^p) = \det(I_n + A^{\frac{p}{2}} B^p A^{\frac{p}{2}}) \leq \det(I_n + |BA|^p).$$

■

With a similar proof of Theorem 2.7 and Theorem 2.8 but using this time (2.8), inequalities (2.18) and (2.21) are reversed for $p \geq 2$.

Theorem 2.9. *Let A, B be two positive semi-definite matrices. Then, for all $p \geq 2$*

- $\det(A^p + |BA|^p) \geq \det(A^p + A^p B^p)$.
- $\det(I_n + |BA|^p) \leq \det(I_n + A^p B^p)$.

2.3.3. Generalization of Lin's Determinantal Inequalities

We will start with a generalization for Part (3) of Proposition 2.2, which is also a complement of Corollary 2.2 when $p = 1$.

Theorem 2.10. *Let A and B be two positive semi-definite matrices. Then for every $k \geq 1$*

$$\det(A^k + |AB|) \geq \det(A^k + AB).$$

Proof. Reminding that for $k \geq 0$ and $X, Y \in \mathbb{M}_n(\mathbb{C})$ such that $Y \geq 0$ (see Proposition 1.6), then

$$\begin{bmatrix} |X^*| & X \\ X^* & |X| \end{bmatrix} \geq 0 \text{ and } \begin{bmatrix} Y^k & Y^k \\ Y^k & Y^k \end{bmatrix} \geq 0.$$

Now, replacing X with AB and Y with A, respectively gives

$$\begin{bmatrix} |BA| & AB \\ BA & |AB| \end{bmatrix} + \begin{bmatrix} A^k & A^k \\ A^k & A^k \end{bmatrix} \geq 0.$$

Then,

$$\begin{bmatrix} A^k + |BA| & A^k + AB \\ A^k + BA & A^k + |AB| \end{bmatrix} \geq 0.$$

By using Proposition 1.7 we have

$$\begin{bmatrix} \det(A^k + |BA|) & \det(A^k + AB) \\ \det(A^k + BA) & \det(A^k + |AB|) \end{bmatrix} \geq 0.$$

This yields

$$\det(A^k + |BA|) \cdot \det(A^k + |AB|) \geq \det(A^k + AB)^2.$$

The required determinantal inequality follows by noting that for all $k \geq 1$, we have

$$\det(A^k + |BA|) \leq \det(A^k + AB).$$

■

We can find a more general complement for Corollary 2.2 when $k = 4$ and $p = 2$ as the following theorem shows.

Theorem 2.11. *Let A and B be two Hermitian matrices. Then,*

$$\det(A^4 + |AB|^2) \geq \det(A^4 + A^2B^2).$$

Proof. Assume that A is an invertible matrix, the general case is due to a continuity argument.

Using Schur complement we get $M = \begin{bmatrix} BA^{-2}B & B^2A^{-2} \\ A^{-2}B^2 & A^{-2}BA^2BA^{-2} \end{bmatrix} \geq 0$. Then by Lemma 2.1

$$\lambda_1(B^2A^{-2})^2 \leq \lambda_1(BA^{-2}B) \cdot \lambda_1(A^{-2}BA^2BA^{-2}).$$

If $\lambda_1(B^2A^{-2}) = 0$, then the desired determinantal inequality is true.

Noticing that $\lambda_1(B^2A^{-2}) = \lambda_1(BA^{-2}B) = \lambda_1(A^{-1}B^2A^{-1})$, and dividing both sides by $\lambda_1(B^2A^{-2}) > 0$ gives

$$\lambda_1(A^{-1}B^2A^{-1}) \leq \lambda_1(A^{-2}BA^2BA^{-2}).$$

By the usual anti-symmetric tensor product argument, and as $\det(A^{-1}B^2A^{-1}) = \det(A^{-2}BA^2BA^{-2})$ we obtain

$$\boldsymbol{\lambda(A^{-1}B^2A^{-1})} \prec_{log} \boldsymbol{\lambda(A^{-2}BA^2BA^{-2})}.$$

Using Part (ii) of Lemma 2.3 gives

$$\det(I_n + A^{-1}B^2A^{-1}) \leq \det(I_n + A^{-2}BA^2BA^{-2}).$$

Multiplying both sides by $\det(A^4)$ yields

$$\det(A^4 + A^2B^2) \leq \det(A^4 + BA^2B) = \det(A^4 + |AB|^2).$$

■

The next theorem is a refinement for Part (4) of Proposition 2.2.

Theorem 2.12. *Let A and B be two positive semi-definite matrices. Then*

$$\det(A^2 + |BA|^2) = \det(A^2 + A^2B^2) \leq \det(A^2 + (AB)^2) \leq \det(A^2 + |AB|^2).$$

Proof. Notice that

$$\begin{aligned}
\det(A^2 + (AB)^2) &= \det(A^2 + ABAB) \\
&= \det(A) \cdot \det(A + BAB) \\
&= \det(A) \cdot \det((A^{1/2})^2 + |A^{1/2}B|^2) \\
&\geq \det(A) \cdot \det((A^{1/2})^2 + (A^{1/2})^2B^2) \\
&\qquad \text{(using Part (4) of Proposition 2.3.1)} \\
&= \det(A) \cdot \det(A + AB^2) \\
&= \det(A^2 + A^2B^2).
\end{aligned}$$

Now, for proving the second inequality we realize that using Schur complement we have

$$\begin{bmatrix} |BA|^2 & (AB)^2 \\ (BA)^2 & |AB|^2 \end{bmatrix} \geq 0.$$

Then

$$\begin{bmatrix} A^2 & A^2 \\ A^2 & A^2 \end{bmatrix} + \begin{bmatrix} |BA|^2 & (AB)^2 \\ (BA)^2 & |AB|^2 \end{bmatrix} \geq 0.$$

So,

$$\begin{bmatrix} A^2+|BA|^2 & A^2+(AB)^2 \\ A^2+(BA)^2 & A^2+|AB|^2 \end{bmatrix} \geq 0.$$

Hence

$$\begin{bmatrix} \det(A^2+|BA|^2) & \det(A^2+(AB)^2) \\ \det(A^2+(BA)^2) & \det(A^2+|AB|^2) \end{bmatrix} \geq 0.$$

Therefore,

$$\det(A^2+|BA|^2)\cdot\det(A^2+|AB|^2) \geq \det(A^2+(AB)^2)^2.$$

Since $\det(A^2+(AB)^2) \geq \det(A^2+A^2B^2) = \det(A^2+|BA|^2)$, we obtain the second inequality. ∎

One can ask if it is possible to find a generalization for Theorem 2.11 and Part (4) of Proposition 2.2, as well as a generalization of the new refinement in Theorem 2.12.

Question 1. *Let A be a positive semi-definite matrix and B be a Hermitian matrix. Then, for all $k \geq 0$*

$$\det(A^k+A^2B^2) \leq \det(A^k+|AB|^2).$$

Question 2. *Let A be a positive semi-definite matrix and B be a Hermitian matrix. Then, for all $k \geq 1$*

$$\det(A^k+A^2B^2) \leq \det(A^k+(AB)^2).$$

The following log-majorization inequality given in the next lemma, stands for a larger set of matrices than the one originally proved by L. Plevnik in [54]. The steps in the proof are essentially the same, but we include them here for the sake of completeness and also to assert our claim that it is valid for a wider class of matrices.

Lemma 2.9. *Let X, Y be in $\mathbb{M}_n$. If either one of the following two conditions*

(1) X is Hermitian, $Y \geq 0$ and $p,\ q \in [0,\infty)$, or

(2) X, Y are Hermitian and $p,\ q$ are even positive integers,

is satisfied, then it holds that

$$\boldsymbol{\lambda(XY^pXY^q) \prec_{log} \lambda(X^2Y^{p+q})}.$$

Proof. By appealing to a standard argument (anti-symmetric tensor product), then it suffices to prove that

$$\lambda_1(XY^pXY^q) \leq \lambda_1(X^2Y^{p+q}).$$

Without loss of generality, we shall assume that X is invertible as the general case can be done by continuity argument. In addition, we shall assume that $q \leq p$ $(p > 0)$ and $\lambda_1(X^2Y^{p+q}) = 1$. Now, obviously proving our claim is equivalent to showing that

$$\lambda_1(XY^pXY^q) \leq 1.$$

The fact that the largest eigenvalue of the matrix X^2Y^{p+q} is equal to 1, clearly implies that

$$\lambda_j(X^2Y^{p+q}) \leq 1 \quad \text{for all} \quad 1 \leq j \leq n.$$

But this is equivalent to $XY^{p+q}X \leq I_n$ which in turn gives

$$Y^{p+q} \leq (X^{-1})^2. \tag{2.22}$$

Next, applying Löwner-Heinz inequality on (2.22) for a power $0 \leq \frac{p}{p+q} \leq 1$, yields

$$(Y^{p+q})^{\frac{p}{p+q}} \leq ((X^{-1})^2)^{\frac{p}{p+q}}.$$

Obviously, for both cases in the lemma we see that $(Y^{p+q})^{\frac{p}{p+q}} = Y^p$, and here it is worthy to draw the attention to the fact that $((X^{-1})^2)^{\frac{p}{p+q}}$ is defined since $(X^{-1})^2 > 0$. Thus, we obtain

$$Y^p \leq ((X^{-1})^2)^{\frac{p}{p+q}}. \tag{2.23}$$

Again, taking a power $0 \leq \frac{q}{p} \leq 1$ in both sides of (2.23), we get

$$(Y^p)^{\frac{q}{p}} \leq \left(((X^{-1})^2)^{\frac{p}{p+q}}\right)^{\frac{q}{p}}.$$

As before, in both cases of the lemma we have

$$Y^q \leq \left((X^{-1})^2\right)^{\frac{q}{p+q}} = \left((X^2)^{\frac{q}{p+q}}\right)^{-1}.$$

Hence,

$$(X^2)^{\frac{q}{p+q}} \leq Y^{-q}. \tag{2.24}$$

Therefore,

$$\begin{aligned}
\lambda_1(XY^pXY^q) &= \lambda_1\left(Y^{q/2}XY^pXY^{q/2}\right) \\
&\leq \lambda_1\left(Y^{q/2}X((X^{-1})^2)^{\frac{p}{p+q}}XY^{q/2}\right) && \text{(using (2.23))} \\
&= \lambda_1\left((Y^{q/2}(X^2)^{\frac{q}{p+q}}Y^{q/2}\right) \\
&\leq \lambda_1(I_n) && \text{(using (2.24))} \\
&= 1.
\end{aligned}$$

Now, we prove the following lemma which turns out to be an essential result.

Lemma 2.10. *Let X, Y be in $\mathbb{M}_n$. If either one of the following two conditions*

(1) X is Hermitian, $Y > 0$, $p, q \in [0, \infty)$ and $p \geq 2q$.

(2) X, Y are Hermitian, q and $p \geq 2q$ are even positive integers,

is satisfied, then we have

$$\boldsymbol{\lambda(X^2Y^{p-q}) \prec_{log} \lambda(XY^pXY^{-q})}.$$

Proof. By Schur complement, we know that

$$M = \begin{bmatrix} Y^{\frac{p}{2}}XY^{-q}XY^{\frac{p}{2}} & Y^{\frac{p}{2}}X^2Y^{\frac{p}{2}-q} \\ Y^{\frac{p}{2}-q}X^2Y^{\frac{p}{2}} & Y^{\frac{p}{2}-q}XY^qXY^{\frac{p}{2}-q} \end{bmatrix} \geq 0.$$

Applying Lemma 2.1 on the positive semi-definite matrix M gives

$$\lambda_1\left(Y^{-\frac{q}{2}}X^2Y^{p-\frac{q}{2}}\right)^2 \leq \lambda_1\left(Y^{-\frac{q}{2}}XY^pXY^{-\frac{q}{2}}\right) \cdot \lambda_1\left(Y^{p-\frac{q}{2}}XY^{-p}XY^{p-\frac{p}{2}}\right).$$

That is,

$$\lambda_1(X^2Y^{p-q})^2 \leq \lambda_1(Y^pXY^{-q}X) \cdot \lambda_1(Y^{p-2q}XY^qX). \tag{2.25}$$

In view of Lemma 2.9, and keeping in mind that either Condition (1) or (2) is satisfied, it is worthy to observe that

$$\boldsymbol{\lambda(XY^{p-2q}XY^q) \prec_{log} \lambda(X^2Y^{p-q})}.$$

which is certainly implies that

$$\lambda_1(Y^{p-2q}XY^qX) \leq \lambda_1(X^2Y^{p-q}).$$

Therefore, using (2.25) we conclude that

$$\lambda_1\left(Y^pXY^{-q}X\right) \geq \lambda_1\left(X^2Y^{p-q}\right).$$

By a standard anti-symmetric tensor product argument, and realizing that

$$\det\left(Y^pXY^{-q}X\right) = \det\left(X^2Y^{p-q}\right)$$

the proof is achieved. ∎

As a result of the previous two lemmas, we have the following theorem.

Theorem 2.13. *Let A be a positive semi-definite matrix and B be a Hermitian matrix. Then*

(i) If $0 \leq k \leq 1$, *then* $\det(A^k + |AB|^2) \geq \det(A^k + A^2B^2)$.

(ii) If $k \geq 4$, *then* $\det(A^k + |AB|^2) \geq \det(A^k + A^2B^2)$.

Proof. We shall assume first that A is invertible, and then the general case follows easily by a continuity argument.

1. Let $0 \leq k \leq 1$:
   Replacing X with B, Y with A, q with k, and p with 2 (notice that the chosen real numbers p, q sutisfy $p \geq 2q$) in Lemma 2.10 gives

   $$\boldsymbol{\lambda(B^2A^{2-k}) \prec_{log} \lambda(BA^2BA^{-k})}.$$

   Now using Part (ii) of Lemma 2.3 yields

   $$\det(I_n + B^2A^{2-k}) \leq \det(I_n + BA^2BA^{-k}).$$

   By multiplying both sides by $\det(A^k) > 0$, we obtain the result.

2. Let $k \geq 4$:
   Taking $X = B$, $Y = A^{-1}$, $q = 2$, and $p = k \geq 2 \cdot 2 = 4$ in Lemma 2.10 gives

   $$\boldsymbol{\lambda\left(B^2(A^{-1})^{k-2}\right) \prec_{log} \lambda\left(B(A^{-1})^{-2}B(A^{-1})^k\right)}.$$

   Again using first Part (ii) of Lemma 2.3 which yields

   $$\det(I_n + B^2A^{2-k}) \leq \det(I_n + BA^2BA^{-k}),$$

   and then multiplying both sides by $\det(A^k) > 0$, will prove the result.

■

For our purposes, we need the next two lemmas.

Lemma 2.11. *Let* $A, B \in \mathbb{M}_n$ *such that* A *is positive semi-definite and* B *is Hermitian, and let* α_1 *and* β_1 *be any two real numbers. If, for all* $k \in [\alpha_1, \beta_1]$,

$$\det(A^k + |AB|^2) \geq \det(A^k + A^2B^2), \tag{2.26}$$

then,

$$\det(A^{k'} + (AB)^2) \geq \det(A^{k'} + A^2B^2),$$

for all $k' \in [\alpha_2, \beta_2]$ *with* $\alpha_2 = \frac{\alpha_1}{2} + 1$ *and* $\beta_2 = \frac{\beta_1}{2} + 1$.

Proof. Let $k' \in [\frac{\alpha_1}{2} + 1, \frac{\beta_1}{2} + 1]$, then $2(k' - 1) \in [\alpha_1, \beta_1]$.

Now, replacing A with $A^{1/2}$ and k with $2(k'-1)$ in inequality (2.26) gives

$$\det\left((A^{1/2})^{2(k'-1)}+|A^{1/2}B|^2\right)\geq\det\left((A^{1/2})^{2(k'-1)}+(A^{1/2})^2B^2\right). \tag{2.27}$$

On the other hand, we can write

$$\begin{aligned}
\det(A^{k'}+(AB)^2) &= \det(A^{k'}+ABAB)\\
&= \det(A)\cdot\det(A^{k'-1}+BAB)\\
&= \det(A)\cdot\det\left((A^{1/2})^{2(k'-1)}+|A^{1/2}B|^2\right)\\
&\geq \det(A)\cdot\det\left((A^{1/2})^{2(k'-1)}+(A^{1/2})^2B^2\right) \qquad \text{(using (2.27))}\\
&= \det(A)\cdot\det(A^{k'-1}+AB^2)\\
&= \det(A^{k'}+A^2B^2).
\end{aligned}$$

■

Lemma 2.12. *Let $A,B\in\mathbb{M}_n$ such that A is positive semi-definite and B is Hermitian, and let α and β be any two positive real numbers. If*

$$\det(A^k+(AB)^2)\geq\det(A^k+A^2B^2)$$

is true for $k\in[\alpha,\beta]$, then

$$\det(A^k+|AB|^2)\geq\det(A^k+(AB)^2)\geq\det(A^k+A^2B^2)$$

is also true for $k\in[\alpha,\beta]$.

Proof. Using Schur complement, we can see that

$$\begin{bmatrix}|BA|^2 & (AB)^2\\ (BA)^2 & |AB|^2\end{bmatrix}\geq 0 \text{ and } \begin{bmatrix}A^k & A^k\\ A^k & A^k\end{bmatrix}\geq 0.$$

So,

$$\begin{bmatrix}A^k+|BA|^2 & A^k+(AB)^2\\ A^k+(BA)^2 & A^k+|AB|^2\end{bmatrix}\geq 0,$$

and then

$$\begin{bmatrix}\det(A^k+|BA|^2) & \det(A^k+(AB)^2)\\ \det(A^k+(BA)^2) & \det(A^k+|AB|^2)\end{bmatrix}\geq 0.$$

Using the fact that $\det(A^k+(AB)^2)=\det(A^k+(BA)^2)$, we conclude that

$$\det(A^k+|BA|^2)\cdot\det(A^k+|AB|^2)\geq\det(A^k+(AB)^2)^2.$$

As $\det(A^k + |BA|^2) = \det(A^k + A^2B^2) \leq \det(A^k + (AB)^2)$ for all $k \in [\alpha, \beta]$, then we obtain

$$\det(A^k + |AB|^2) \geq \det(A^k + (AB)^2).$$

Thus, for all real numbers $k \in [\alpha, \beta]$, we have that

$$\det(A^k + |AB|^2) \geq \det(A^k + (AB)^2) \geq \det(A^k + A^2B^2).$$

■

Now, we are ready to prove Question 1.

Theorem 2.14. *Let $A, B \in \mathbb{M}_n$ such that A is positive semi-definite and B is Hermitian. Then, for $k \in [0, \infty)$, we have that*

$$\det(A^k + |AB|^2) \geq \det(A^k + A^2B^2). \tag{2.28}$$

Proof. For the proof, we distinguish between the two cases: $k \in [2, \infty)$ and $k \in [0, 2]$.

Case 1: Let $k \in [2, \infty)$. We construct a recursive sequence $(\alpha_n)_{n\geq 1}$ with $\alpha_1 = 4$, and $\alpha_{i+1} = \frac{\alpha_i}{2} + 1$. Then, in view of Part (ii) in Theorem 2.13, we know that (2.28) is true for all $k \in [\alpha_1, \infty)$. Next, we shall show in two steps the validity of inequality (2.28) for $k \in [\alpha_2, \infty)$, where the new interval is wider, that is, $[\alpha_1, \infty) \supset [\alpha_2, \infty)$.

Step 1: Combining Part (ii) of Theorem 2.13 and Lemma 2.11, implies that for all $k \in [\alpha_2, \infty)$ where $\alpha_2 = \frac{\alpha_1}{2} + 1 = 3$, we have

$$\det(A^k + (AB)^2) \geq \det(A^k + A^2B^2).$$

Step 2: Making use of Lemma 2.12, we first conclude that (2.28) is true for all $k \in [\alpha_2, \infty)$.

Repeating the same process as before, we see that inequality (2.28) is true for all

$$\begin{aligned}
&k \in [\alpha_3, \infty) \ , \quad \left(\alpha_3 = \frac{\alpha_1}{2^2} + \frac{1}{2} + 1 = \frac{5}{2}\right) \\
&k \in [\alpha_4, \infty) \ , \quad \left(\alpha_4 = \frac{\alpha_1}{2^3} + \frac{1}{2^2} + \frac{1}{2} + 1 = \frac{9}{4}\right) \\
&\vdots \\
&k \in [\alpha_{n+2}, \infty) \ , \quad \left(\alpha_{n+2} = \frac{\alpha_1}{2^{n+1}} + \frac{1}{2^n} + \frac{1}{2^{n-1}} + \cdots + \frac{1}{2} + 1 = 2 + \frac{1}{2^n}\right).
\end{aligned}$$

As n approaches ∞, the sequence $(\alpha_n)_n$ tends to 2. Therefore, (2.28) is true for $k \in [2, \infty)$.

Case 2: Let $k \in [0, 2]$. Here we construct another recursive sequence $(\alpha_n)_{n\geq 1}$ with $\alpha_1 = 0$, and $\alpha_{i+1} = \frac{\alpha_i}{2} + 1$. Again, Part (i) of Theorem 2.13 says that (2.28) is true for all $k \in [\alpha_1, \alpha_2]$.

Using a similar argument as earlier, we conclude that inequality (2.28) is valid for all

$$\begin{aligned}
&k \in [\alpha_2, \alpha_3], \quad \left(\alpha_3 = 1 + \frac{1}{2}\right)\\
&k \in [\alpha_3, \alpha_4], \quad \left(\alpha_4 = 1 + \frac{1}{2} + \frac{1}{2^2}\right)\\
&k \in [\alpha_4, \alpha_5], \quad \left(\alpha_5 = 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3}\right)\\
&\vdots\\
&k \in [\alpha_{n-1}, \alpha_n], \quad \left(\alpha_n = 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^n} = 2 - \frac{1}{2^n}\right).
\end{aligned}$$

As n approaches ∞, the sequence α_n also tends to 2. Thus, for all $A \geq 0$, B Hermitian matrix and for $k \in [0, 2]$, we have that

$$\det(A^k + |AB|^2) \geq \det(A^k + A^2B^2).$$

■

As a consequence of the previous theorem and Lemma 2.11, we have the following corollary which gives an affirmative answer to Question 2.

Corollary 2.3. *Let A be a positive semi-definite matrix and B be a Hermitian matrix. Then for all $k \in [1, \infty)$*

$$\det(A^k + (AB)^2) \geq \det(A^k + A^2B^2).$$

We are tempted to study inequality (2.28) for a larger class of matrices, and to see when it might fail.

Theorem 2.15. *Let A and B be two Hermitian matrices. Then for all positive even integers k,*

$$\det(A^k + |AB|^2) \geq \det(A^k + A^2B^2).$$

Proof. The proof for even integers $k \geq 4$ is similar to that of Part (ii) in Theorem 2.13, we just need to apply the second case of Lemma 2.10 when A and B are Hermitian matrices and q, $p \geq 2q$ are even integers. To close the proof, we still have to show the inequality for $k = 0$ and $k = 2$. However, for $k = 0$, it is easy to see that

$$\det(I_n + |AB|^2) = \det(I_n + A^2B^2).$$

Next, for $k = 2$, assume first that A and B are invertible matrices. Observe that in this case $A^2 + BA^2B > 0$ and $A^2 + AB^2A > 0$ for all A and B invertible Hermitian matrices. So that

$$\boldsymbol{\lambda(A^2 + BA^2B)},\ \boldsymbol{\lambda(A^2 + AB^2A)} \in \mathbb{R}^n_+ \backslash \{0\}.$$

From [41], the authors proved the following majorization inequality

$$\boldsymbol{\lambda(XX^* + YY^*) \prec \lambda(X^*X + Y^*Y)}$$

for every X and Y belongs to $\mathbb{M}_n$ such that X^*Y Hermitian.

Now substituting X and Y with A and BA, respectively, gives

$$\boldsymbol{\lambda(A^2 + BA^2B) \prec \lambda(A^2 + AB^2A)}. \tag{2.29}$$

Now, applying Part (i) of Lemma 2.3 on (2.29) yields

$$\det(A^2 + |AB|^2) \geq \det(A^2 + AB^2A) = \det(A^2 + A^2B^2).$$

Finally, by a continuity argument, the proof is complete. ■

Remark 2.4. *It is worthy to note the following.*

(1) For Hermitian matrices A and B and for positive odd integers k, the inequality

$$\det(A^k + |AB|^2) \geq \det(A^k + A^2B^2)$$

as well as its reverse, are not true in general. This can be easily seen by taking

$$A = \begin{bmatrix} -3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & -1 \\ 0 & -1 & -1 \end{bmatrix},$$

and then a simple check shows that

- *for $k = 1$, $\det(A + A^2B^2) = 4872 < \det(A + |AB|^2) = 5766$, and*
- *for $k = 3$, $\det(A^3 + A^2B^2) = -1872 > \det(A^3 + |AB|^2) = -10650$.*

(2) On the other hand, for Hermitian matrices A and B, the following inequality

$$\det(A^k + (AB)^2) \geq \det(A^k + A^2B^2)$$

may also fail for even integers $k \geq 0$. This can be easily seen by taking first

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix} \text{ and } B = \begin{bmatrix} -1 & 2 & 3 \\ 2 & 0 & -i \\ 3 & i & 2 \end{bmatrix}.$$

A simple inspection shows that $k = 2$, we have

$$\det(A^2 + (AB)^2) = 14082 > \det(A^2 + A^2B^2) = 12708.$$

Secondly, if $A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -\frac{1}{2} \end{bmatrix}$ *and* $B = \begin{bmatrix} 0 & 0 & 3 \\ 0 & 0 & -i \\ 3 & i & 2 \end{bmatrix}$, *then again for* $k = 2$, *a simple check shows that*

$$\det(A^2 + (AB)^2) = 82.75 < \det(A^2 + A^2B^2) = 125.$$

2.4. Lin's Conjectures Concerning Determinantal Inequalities

Let us recall the two conjectures formulated by M. Lin [40]. For any positive semi-definite matrices A and B the conjectures are stated as follows

$$\begin{aligned} &\det(A^2 + |AB|^p) \geq \det(A^2 + |BA|^p) \quad \text{for} \quad p \geq 0; \\ &\det(A^2 + |AB|^p) \geq \det(A^2 + A^pB^p) \quad \text{for} \quad 0 \leq p \leq 2. \end{aligned}$$

The main goal of this section is to confirm the first conjecture (Conjecture 2.1) in a slightly more general setting namely in the case when A and B are Hermitian, and also to present two different approaches to prove Conjecture 2.2. The first approach gives a partial answer to the conjecture, when $0 \leq p \leq \frac{4}{3}$ to be precise, and the second approach gives a solution to the conjecture in the affirmative. First, we need to extend a majorization inequality established in the previous section (see Lemma 2.10) due to its important role in the proof of both conjectures. More several inequalities are established in this chapter, Chapter 3 and Chapter 4 as applications of this majorization relation. Furthermore, we study the relation between the two determinantal quantities

$$\det(A^2 + |AB|^p) \text{ and } \det(A^2 + A^pB^p) \text{ for } p \in \mathbb{R} \text{ and with } A, B \geq 0,$$

which is motivated by the relation observed in Section 2.3 between the two determinantal quantities $\det(A^2 + |BA|^p)$ and $\det(A^2 + A^pB^p)$ for every $p \in \mathbb{R}$ and with $A, B \geq 0$.

2.4.1. An Extension of a Log-Majorization Relation

The key for the solution of Conjecture 2.1 and Conjecture 2.2 is in fact Lemma 2.10. Although Lemma 2.10 is incomplete in its current form as it states that for $p, q \geq 0$ such that $p \geq 2q$, we have

$$\boldsymbol{\lambda(XY^pXY^{-q}) \succ_{log} \lambda(X^2Y^{p-q})}.$$

where X is a Hermitian matrix and Y is a positive definite matrix. The log-majorization inequality above should be true for every $p, q \geq 0$ to be considered complete. Revisiting the proof

of Theorem 2.14, we used an argument to extend the interval for which inequality (2.28) is verified. We kept extending the interval by repeating the argument and at the end we obtained the result as the repetition approaches infinity.

The next theorem is the complete version of Lemma 2.10 by using this time of a new argument.

Theorem 2.16. *Let Y be a positive definite matrix and X be a Hermitian matrix. Then for all $p, q \in [0, \infty)$*

$$\boldsymbol{\lambda}(\boldsymbol{XY^pXY^{-q}}) \succ_{log} \boldsymbol{\lambda}(\boldsymbol{X^2Y^{p-q}}). \tag{2.30}$$

Proof. The proof will be divided into three cases.

Case 1: Let $q \geq 2p$. Replacing Y with Y^{-1} in Lemma 2.9 gives the result for the first case.

Case 2: Let $p \leq q \leq 2p$. The idea of the proof here depends on writing the interval $[p, 2p] = \bigcup_{k=2}^{\infty} [\frac{k+1}{k}p, 2p]$ and then proving (2.30) for each subinterval. We start with case $k = 2$ i.e. for $2p \geq q \geq \frac{3p}{2}$. Then,

$$\begin{aligned}
\boldsymbol{\lambda}(\boldsymbol{Y^pXY^{-q}X}) &= \boldsymbol{\lambda}\left(\boldsymbol{Y^{\frac{3p}{2}}}(\boldsymbol{Y^{-\frac{p}{2}}XY^{-\frac{p}{2}}})\boldsymbol{Y^{p-q}}(\boldsymbol{Y^{-\frac{p}{2}}XY^{-\frac{p}{2}}})\boldsymbol{Y^{\frac{p}{2}}}\right) \\
&= \boldsymbol{\lambda}\left(\boldsymbol{Y^{2p}}(\boldsymbol{Y^{-\frac{p}{2}}XY^{-\frac{p}{2}}})\boldsymbol{Y^{-(q-p)}}(\boldsymbol{Y^{-\frac{p}{2}}XY^{-\frac{p}{2}}})\right) \\
&= \boldsymbol{\lambda}\left((\boldsymbol{Y^{-1}})^{\boldsymbol{-2p}}(\boldsymbol{Y^{-\frac{p}{2}}XY^{-\frac{p}{2}}})(\boldsymbol{Y^{-1}})^{\boldsymbol{q-p}}(\boldsymbol{Y^{-\frac{p}{2}}XY^{-\frac{p}{2}}})\right).
\end{aligned}$$

Now considering this last expression and noticing that $2p \geq 2(q-p) \geq 0$, then in view of Case 1; replacing Y with Y^{-1}, X with $Y^{-\frac{p}{2}}XY^{-\frac{p}{2}}$, q with $2p$ and lastly p with $q-p$, we obtain

$$\begin{aligned}
\boldsymbol{\lambda}(\boldsymbol{Y^pXY^{-q}X}) &\succ_{wlog} \boldsymbol{\lambda}\left(\boldsymbol{Y^{3p-q}}(\boldsymbol{Y^{-\frac{p}{2}}XY^{-\frac{p}{2}}})^{\boldsymbol{2}}\right) \\
&= \boldsymbol{\lambda}\left(\boldsymbol{Y^{2p-q}XY^{-p}X}\right) \\
&\succ_{log} \boldsymbol{\lambda}(\boldsymbol{Y^{p-q}X^2}) \quad \text{(again using Case 1 as } p \geq 2(2p-q) \geq 0).
\end{aligned}$$

Next, using a similar argument, we prove (2.30) is true for $k = 3$ i.e. for $2p \geq q \geq \frac{4p}{3}$. As

earlier, we can write

$$\begin{aligned}
\boldsymbol{\lambda}(\boldsymbol{Y}^p\boldsymbol{X}\boldsymbol{Y}^{-q}\boldsymbol{X}) &= \boldsymbol{\lambda}\left(\boldsymbol{Y}^{\frac{3p}{2}}(\boldsymbol{Y}^{-\frac{p}{2}}\boldsymbol{X}\boldsymbol{Y}^{-\frac{p}{2}})\boldsymbol{Y}^{p-q}(\boldsymbol{Y}^{-\frac{p}{2}}\boldsymbol{X}\boldsymbol{Y}^{-\frac{p}{2}})\boldsymbol{Y}^{\frac{p}{2}}\right)\\
&= \boldsymbol{\lambda}\left(\boldsymbol{Y}^{2p}(\boldsymbol{Y}^{-\frac{p}{2}}\boldsymbol{X}\boldsymbol{Y}^{-\frac{p}{2}})\boldsymbol{Y}^{-(q-p)}(\boldsymbol{Y}^{-\frac{p}{2}}\boldsymbol{X}\boldsymbol{Y}^{-\frac{p}{2}})\right)\\
&= \boldsymbol{\lambda}\left((\boldsymbol{Y}^{-1})^{-2p}(\boldsymbol{Y}^{-\frac{p}{2}}\boldsymbol{X}\boldsymbol{Y}^{-\frac{p}{2}})(\boldsymbol{Y}^{-1})^{q-p}(\boldsymbol{Y}^{-\frac{p}{2}}\boldsymbol{X}\boldsymbol{Y}^{-\frac{p}{2}})\right)\\
&\succ_{log} \boldsymbol{\lambda}\left(\boldsymbol{Y}^{3p-q}(\boldsymbol{Y}^{-\frac{p}{2}}\boldsymbol{X}\boldsymbol{Y}^{-\frac{p}{2}})^2\right)\\
&\qquad \text{(in view of of Case 1 as } 2p \geq 2(q-p) \geq 0)\\
&= \boldsymbol{\lambda}\left(\boldsymbol{Y}^{2p-q}\boldsymbol{X}\boldsymbol{Y}^{-p}\boldsymbol{X}\right)\\
&\succ_{log} \boldsymbol{\lambda}(\boldsymbol{Y}^{p-q}\boldsymbol{X}^2)\\
&\qquad \text{(similarly in view of Case } k=2 \text{ as } p \geq \frac{3}{2}(2p-q) \geq 0).
\end{aligned}$$

Now to complete this case, we need to show that for any positive integer k inequality (2.30) is true for all $q, p \geq 0$ with $2p \geq q \geq \frac{k+1}{k}p$. In other words, we will assume that inequality (2.30) is true for all $q, p \geq 0$ with $2p \geq q \geq \frac{k_0}{k_0-1}p$ where k_0 is a positive integer. Our purpose now is to show that (2.30) is still valid for $q, p \geq 0$ with $2p \geq q \geq \frac{k_0+1}{k_0}p$. Again, notice that

$$\begin{aligned}
\boldsymbol{\lambda}(\boldsymbol{Y}^p\boldsymbol{X}\boldsymbol{Y}^{-q}\boldsymbol{X}) &= \boldsymbol{\lambda}\left(\boldsymbol{Y}^{2p}(\boldsymbol{Y}^{-\frac{p}{2}}\boldsymbol{X}\boldsymbol{Y}^{-\frac{p}{2}})\boldsymbol{Y}^{-(q-p)}(\boldsymbol{Y}^{-\frac{p}{2}}\boldsymbol{X}\boldsymbol{Y}^{-\frac{p}{2}})\right)\\
&= \boldsymbol{\lambda}\left((\boldsymbol{Y}^{-1})^{-2p}(\boldsymbol{Y}^{-\frac{p}{2}}\boldsymbol{X}\boldsymbol{Y}^{-\frac{p}{2}})(\boldsymbol{Y}^{-1})^{q-p}(\boldsymbol{Y}^{-\frac{p}{2}}\boldsymbol{X}\boldsymbol{Y}^{-\frac{p}{2}})\right)\\
&\succ_{log} \boldsymbol{\lambda}\left(\boldsymbol{Y}^{3p-q}(\boldsymbol{Y}^{-\frac{p}{2}}\boldsymbol{X}\boldsymbol{Y}^{-\frac{p}{2}})^2\right)\\
&\qquad \text{(in view of of Case 1 as } 2p \geq 2(q-p) \geq 0)\\
&= \boldsymbol{\lambda}\left(\boldsymbol{Y}^{2p-q}\boldsymbol{X}\boldsymbol{Y}^{-p}\boldsymbol{X}\right)\\
&\succ_{log} \boldsymbol{\lambda}(\boldsymbol{Y}^{p-q}\boldsymbol{X}^2)\\
&\qquad \text{(similarly in view of Case } k=k_0-1 \text{ as } p \geq \frac{k_0}{k_0-1}(2p-q) \geq 0).
\end{aligned}$$

Finally, the proof for this case can be achieved by letting k tends to infinity.

Case 3: Let $q \leq p$. To complete the proof of this case, it suffices to apply the preceding two cases on Y^{-1}.

■

As applications of Theorem 2.16 and Lemma 2.9, we will show the relation between the three determinantal quantities

$$\det(A^k + |AB|^2),\ \det(A^k + A^2B^2),\ \text{and}\ \det(A^k + (AB)^2) \text{ for } k \in \mathbb{R}.$$

We can also see that Theorem 2.16 gives a simpler proof for Theorem 2.14.

Theorem 2.17. *Let A be a positive semi-definite matrix and B be a Hermitian matrix. Then*

(1) If $0 \le k \le 1$, then $\det(A^k + |AB|^2) \ge \det(A^k + A^2B^2) \ge \det(A^k + (AB)^2)$.

(2) If $k \ge 1$, then $\det(A^k + |AB|^2) \ge \det(A^k + (AB)^2) \ge \det(A^k + A^2B^2)$.

(3) For $k \ge 0$, and in the event of A being invertible, we have

(i) $\det(A^{-k} + A^2B^2) \ge \det(A^{-k} + (AB)^2) \ge \det(A^{-k} + |AB|^2)$ if $0 \le k \le 1$.

(ii) $\det(A^{-k} + A^2B^2) \ge \det(A^{-k} + |AB|^2) \ge \det(A^{-k} + (AB)^2)$ if $k \ge 1$.

Proof. Assume that A is invertible. Replacing X with B and Y with A respectively, and taking $p = 2$ and $q = -k$ in Theorem 2.16 and Lemma 2.9 give

$$\boldsymbol{\lambda(A^{-k}BA^2B)} \prec_{log} \boldsymbol{\lambda(A^{2-k}B^2)} \qquad k \le 0; \tag{2.31}$$

$$\boldsymbol{\lambda(A^{-k}BA^2B)} \succ_{log} \boldsymbol{\lambda(A^{2-k}B^2)} \qquad k \ge 0. \tag{2.32}$$

Now using Part (ii) of Lemma 2.3 yields for (2.31) and (2.32)

$$\det(I_n + A^{-k}BA^2B) \le \det(I_n + A^{2-k}B^2) \qquad k \le 0; \tag{2.33}$$

$$\det(I_n + A^{-k}BA^2B) \ge \det(I_n + A^{2-k}B^2) \qquad k \ge 0. \tag{2.34}$$

Hence, multiplying both sides with $\det(A^k) > 0$ of (2.33) and (2.34) gives

$$\det(A^k + |AB|^2) \le \det(A^k + A^2B^2) \qquad k \le 0; \tag{2.35}$$

$$\det(A^k + |AB|^2) \ge \det(A^k + A^2B^2) \qquad k \ge 0. \tag{2.36}$$

As a consequence of inequality (2.36) as well as Lemma 2.11 we have

$$\det(A^k + (AB)^2) \ge \det(A^k + A^2B^2) \qquad k \in [1, \infty) \tag{2.37}$$

We can also write for $k \le 1$

$$\begin{aligned} \det(A^k + (AB)^2) &= \det(A) \cdot \det\left((A^{\frac{1}{2}})^{2(k-1)} + |A^{\frac{1}{2}}B|^2\right) \\ &\le \det(A) \cdot \det\left((A^{\frac{1}{2}})^{2(k-1)} + (A^{\frac{1}{2}})^2B^2\right) \qquad \text{(using (2.35))} \\ &= \det(A^k + A^2B^2). \end{aligned}$$

So,

$$\det(A^k + (AB)^2) \le \det(A^k + A^2B^2) \qquad k \le 1 \tag{2.38}$$

(1) Let $0 \le k \le 1$. The first inequality is a case of (2.36), and the second inequality is a case of (2.38).

(2) Let $k \geq 1$. The first inequality here is also a case of (2.36), and the second inequality is actually inequality (2.37).

(3) (i) Let $0 \leq k \leq 1$. The first inequality is a case of inequality (2.38) by replacing k with $-k$. Now, to prove the second inequality, it is suffices to show that

$$\boldsymbol{\lambda}(\boldsymbol{A^{k+1}BAB}) \succ_{log} \boldsymbol{\lambda}(\boldsymbol{A^kBA^2B}) \qquad 0 \leq k \leq 1. \tag{2.39}$$

For $0 \leq k \leq 1$, we have

$$\begin{aligned}\boldsymbol{\lambda}(\boldsymbol{A^kBA^2B}) &= \boldsymbol{\lambda}\left(\boldsymbol{A^{k-1}}(\boldsymbol{A^{\frac{1}{2}}BA^{\frac{1}{2}}})\boldsymbol{A}(\boldsymbol{A^{\frac{1}{2}}BA^{\frac{1}{2}}})\right)\\ &\prec_{log} \boldsymbol{\lambda}\left(\boldsymbol{A^k}(\boldsymbol{A^{\frac{1}{2}}BA^{\frac{1}{2}}})^{\boldsymbol{2}}\right)\\ &\qquad \text{(using Theorem 2.16 for } p = 1 \text{ and } q = 1 - k \geq 0)\\ &= \boldsymbol{\lambda}(\boldsymbol{A^{k+1}BAB}).\end{aligned}$$

Hence, applying Part (ii) of Lemma 2.3 on (2.39), then multiplying both sides with $\det(A^{-k})$ gives the second inequality.

(ii) Let $k \geq 1$. The first inequality here is a case of inequality (2.35) by taking $k = -k$, and the second inequality can be proved with a similar fashion as in the previous case, however using this time of Lemma 2.9 for $p = 1$ and $q = k - 1 \geq 0$.

■

2.4.2. Solution for the First Conjecture

The following result is due to the log-majorization relation in Theorem 2.16 in which it gives an affirmative answer to Conjecture 2.1.

Theorem 2.18. *Let A and B be two Hermitian matrices. Then, for all $p \in [0, \infty)$, we have*

$$\det(A^2 + |BA|^p) \leq \det(A^2 + |AB|^p).$$

Proof. As usual, we shall assume that A and B are invertible, the general case is by continuity argument. As a consequence of Lemma 2.6, substituting X with A, Y with B and t with $\frac{p}{2}$ respectively, we have for all $p \in [0, +\infty)$

$$(AB^2A)^{\frac{p}{2}} = AB(BA^2B)^{\frac{p}{2}-1}BA.$$

Then,

$$\begin{aligned}\boldsymbol{\lambda}\left(\boldsymbol{A^{-1}(AB^2A)^{\frac{p}{2}}A^{-1}}\right) &= \boldsymbol{\lambda}\left[\boldsymbol{A^{-1}}\left(\boldsymbol{AB(BA^2B)^{\frac{p}{2}-1}BA}\right)\boldsymbol{A^{-1}}\right] \\ &= \boldsymbol{\lambda}\left(\boldsymbol{B^2(BA^2B)^{\frac{p}{2}-1}}\right).\end{aligned}$$

Now considering this last expression, if we replace X with B, Y with BA^2B, p with $\frac{p}{2}$ and q with 1 in Theorem 2.16, we obtain

$$\begin{aligned}\boldsymbol{\lambda}\left(\boldsymbol{B^2(BA^2B)^{\frac{p}{2}-1}}\right) &\prec_{log} \boldsymbol{\lambda}\left(\boldsymbol{B(BA^2B)^{\frac{p}{2}}B(BA^2B)^{-1}}\right) \\ &= \boldsymbol{\lambda}\left(\boldsymbol{A^{-1}(BA^2B)^{\frac{p}{2}}A^{-1}}\right).\end{aligned}$$

Hence,

$$\boldsymbol{\lambda}\left(\boldsymbol{A^{-1}(AB^2A)^{\frac{p}{2}}A^{-1}}\right) \prec_{log} \boldsymbol{\lambda}\left(\boldsymbol{A^{-1}(BA^2B)^{\frac{p}{2}}A^{-1}}\right).$$

Next, applying Part (ii) of Lemma 2.3 on the above inequality gives

$$\det(I_n + A^{-1}(AB^2A)^{\frac{p}{2}}A^{-1}) \leq \det(I_n + A^{-1}(BA^2B)^{\frac{p}{2}}A^{-1}).$$

Finally, multiplying both sides with $\det(A^2) > 0$ yields for all $p \in [0, \infty)$

$$\det(A^2 + |BA|^p) \leq \det(A^2 + |AB|^p).$$

■

Remark 2.5. *In [40], M. Lin asked whether the following majorization inequality is true for all $A, B \geq 0$:*

$$\boldsymbol{\lambda(A^2 + |AB|^p)} \prec \boldsymbol{\lambda(A^2 + |BA|^p)} \qquad \textit{for every} \quad p \geq 0 \quad ?$$

Applying Part (i) of Lemma 2.3 on the above majorization relation (if it is valid) implies Conjecture 2.1, which make it another approach for the solution other than the one we presented above.

2.4.3. Comparing $\det(A^2 + |AB|^p)$ with $\det(A^2 + A^pB^p)$

Motivated by Section 2.3, our purpose here is to find for what values $p \in (-\infty, +\infty)$, the two determinantal quantities $\det(A^2 + |AB|^p)$ and $\det(A^2 + A^pB^p)$ have a general relation.

In the next theorem we can see that the determinantal inequality of Theorem 2.18 is reversed when $p \leq 0$ provided that A and B are invertible.

Theorem 2.19. *Let A and B be two invertible Hermitian matrices. Then, for all $p \leq 0$, we have*

$$\det(A^2 + |BA|^p) \geq \det(A^2 + |AB|^p).$$

Proof. For all $p \in (-\infty, 0]$, we have

$$\begin{aligned}
\boldsymbol{\lambda}\left(\boldsymbol{A^{-1}(BA^2B)^{\frac{p}{2}}A^{-1}}\right) &= \boldsymbol{\lambda}\left(\boldsymbol{A^{-1}B^{-1}A^{-1}(AB^2A)^{\frac{p}{2}+1}A^{-1}B^{-1}A^{-1}}\right) \\
&\qquad \text{(using Lemma 2.6)} \\
&= \boldsymbol{\lambda}\left(\boldsymbol{(A^{-1}BA^{-1})^2(AB^2A)^{\frac{p}{2}+1}}\right) \\
&\prec_{log} \boldsymbol{\lambda}\left(\boldsymbol{(A^{-1}B^{-1}A^{-1})(AB^2A)^1(A^{-1}B^{-1}A^{-1})(AB^2A)^{\frac{p}{2}}}\right) \\
&\qquad \text{(using Theorem 2.16)} \\
&= \boldsymbol{\lambda}\left(\boldsymbol{A^{-1}(AB^2A)^{\frac{p}{2}}A^{-1}}\right).
\end{aligned}$$

Similarly as in Theorem 2.18, after using Part (ii) of Lemma 2.3 and multiplying both sides with $\det(A^2)$, we obtain the desired inequality for every $p \leq 0$. ∎

As an analogue of Theorem 2.5, Theorem 2.18, and Theorem 2.19, we have the following result.

Theorem 2.20. *Let A and B be two positive semi-definite matrices. We have*

(1) In case A and B invertible matrices, then for every $p \leq 0$,

$$\det(A^2 + |AB|^p) \leq \det(A^2 + A^pB^p).$$

(2) If $0 \leq p \leq 4$, then $\det(A^2 + |AB|^p) \geq \det(A^2 + A^pB^p)$.

Proof. 1. The proof of the first case is divided into two cases.

Case 1: Let $p \leq -2$. Combining Part (1) of Theorem 2.5 with Theorem 2.19 implies that for all $p \leq -2$

$$\det(A^2 + |AB|^p) \leq \det(A^2 + |BA|^p) \leq \det(A^2 + A^pB^p).$$

Case 2: Let $-2 \leq p \leq 0$. As usual, to prove the desired determinantal inequality we need to prove a log-majorization that implies it, which is

$$\boldsymbol{\lambda}(\boldsymbol{A^{\frac{1}{2}}(B^{\frac{1}{2}}AB^{\frac{1}{2}})^tA^{\frac{1}{2}}}) \prec_{wlog} \boldsymbol{\lambda}(\boldsymbol{A^{1+t}B^t}) \qquad 0 \leq t \leq 1. \tag{2.40}$$

And to prove this log-majorization relation it is enough to show that for $0 \leq t \leq 1$

$$B^{\frac{t}{2}}A^{1+t}B^{\frac{t}{2}} \leq I_n \Rightarrow A^{\frac{1}{2}}(B^{\frac{1}{2}}AB^{\frac{1}{2}})^tA^{\frac{1}{2}} \leq I_n.$$

Assuming that $B^{\frac{t}{2}}A^{1+t}B^{\frac{t}{2}} \leq I_n$ gives $A^{1+t} \leq B^{-t}$. Using Löwner-Heinz inequality for $0 \leq \frac{1}{1+t} \leq 1$ gives

$$A \leq B^{-\frac{t}{1+t}} \quad \text{and} \quad B^{\frac{t}{1+t}} \leq A^{-1}.$$

So,

$$\begin{aligned} A^{\frac{1}{2}}(B^{\frac{1}{2}}AB^{\frac{1}{2}})^tA^{\frac{1}{2}} &\leq A^{\frac{1}{2}}(B^{\frac{1}{2}}B^{-\frac{t}{1+t}}B^{\frac{1}{2}})^tA^{\frac{1}{2}} \\ &= A^{\frac{1}{2}}B^{\frac{t}{1+t}}A^{\frac{1}{2}} \\ &\leq I_n. \end{aligned}$$

Hence, $\lambda_1(A^{\frac{1}{2}}(B^{\frac{1}{2}}AB^{\frac{1}{2}})^tA^{\frac{1}{2}}) \leq \lambda_1(A^{1+t}B^t) \qquad 0 \leq t \leq 1.$

Thus, (2.40) is true by an anti-symmetric tensor product argument. Now, replacing A with A^{-2}, B with B^{-2} and taking $t = -\frac{p}{2}$, we get

$$\boldsymbol{\lambda(A^{-1}(BA^2B)^{\frac{p}{2}}A^{-1}) \prec_{wlog} \lambda(A^{p-2}B^p)} \qquad -2 \leq p \leq 0.$$

Finally, applying Part (ii) of Lemma 2.3 and then multiplying both sides with $\det(A^2)$, we get the desired determinantal inequality.

2. As a result of Theorem 2.18 and Part (3) of Theorem 2.5, we have for all $2 \leq p \leq 4$

$$\det(A^2 + |AB|^p) \geq \det(A^2 + |BA|^p) \geq \det(A^2 + A^pB^p).$$

The proof for the case where $0 \leq p \leq 2$ is in the next subsection. ■

The next example shows that there is no general relation between the two determinantal quantities $\det(A^2 + |AB|^p)$ and $\det(A^2 + A^pB^p)$ for $p \in [4, \infty)$.

Example 2.5. *(1) For* $A = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$ *and* $p = 6$, *we have*

$$\det(A^2 + |AB|^6) = 23.09 < \det(A^2 + A^6B^6) = 26.15.$$

(2) For $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$ *and* $p = 6$, *we have*

$$\det(A^2 + |AB|^6) = 17388 > \det(A^2 + A^6B^6) = 15336.$$

2.4.4. Solution for the Second Conjecture

In this subsection, our purpose is to find for what values of k and t the following majorization inequality

$$\boldsymbol{\lambda(A^{\frac{k}{2}-t}B^t) \prec_{log} \lambda(A^{\frac{k}{4}}(B^{\frac{1}{2}}A^{-1}B^{\frac{1}{2}})^tA^{\frac{k}{4}})} \tag{2.41}$$

is valid, where A and B are positive definite matrices. The first approach gives an indirect proof by showing inequality (2.41) for the case $k \geq 4t$ with $0 \leq t \leq 1$ and the case $k \geq 6t$ with $\frac{1}{2} \leq t \leq 1$. In the second approach we give a direct proof for a slightly more general result stated as

$$\boldsymbol{\lambda}(\boldsymbol{A}^{k-t}\boldsymbol{B}^{t}) \prec_{log} \boldsymbol{\lambda}(\boldsymbol{A}^{\frac{k}{2}}(\boldsymbol{B}^{\frac{s}{2}}\boldsymbol{A}^{-s}\boldsymbol{B}^{\frac{s}{2}})^{\frac{t}{s}}\boldsymbol{A}^{\frac{k}{2}}) \qquad 0 \leq t \leq s \leq k.$$

4.4.1 First Approach

The next theorem shows that (2.41) is valid for all $0 \leq t \leq 1$ and $k \geq 4t$.

Theorem 2.21. *Let A and B be two positive definite matrices. Then, for all $0 \leq t \leq 1$ and $k \geq 4t$*

$$\boldsymbol{\lambda}(\boldsymbol{A}^{\frac{k}{2}-t}\boldsymbol{B}^{t}) \prec_{log} \boldsymbol{\lambda}(\boldsymbol{A}^{\frac{k}{4}}(\boldsymbol{B}^{\frac{1}{2}}\boldsymbol{A}^{-1}\boldsymbol{B}^{\frac{1}{2}})^{t}\boldsymbol{A}^{\frac{k}{4}}).$$

Proof. Let $0 \leq t \leq 1$ and $k \geq 4t$. Using Schur complement, we know that

$$M = \begin{bmatrix} A^{\frac{k}{4}-t}B^{t}(B^{-\frac{1}{2}}AB^{-\frac{1}{2}})^{t}B^{t}A^{\frac{k}{4}-t} & A^{\frac{k}{4}-t}B^{t}A^{\frac{k}{4}} \\ A^{\frac{k}{4}}B^{t}A^{\frac{k}{4}-t} & A^{\frac{k}{4}}(B^{\frac{1}{2}}A^{-1}B^{\frac{1}{2}})^{t}A^{\frac{k}{4}} \end{bmatrix} \geq 0.$$

Applying Lemma 2.1 on the the matrix M gives

$$\left(\lambda_1(A^{\frac{k}{2}-t}B^{t})\right)^2 \leq \lambda_1\left(A^{\frac{k}{4}}(B^{\frac{1}{2}}A^{-1}B^{\frac{1}{2}})^{t}A^{\frac{k}{4}}\right) \cdot \lambda_1\left(A^{\frac{k}{4}-t}B^{t}(B^{-\frac{1}{2}}AB^{-\frac{1}{2}})^{t}B^{t}A^{\frac{k}{4}-t}\right). \tag{2.42}$$

As mentioned earlier, in order to prove our claim, then it is enough to prove that

$$\lambda_1\left(A^{\frac{k}{2}-t}B^{t}\right) \geq \lambda_1\left(A^{\frac{k}{4}-t}B^{t}(B^{-\frac{1}{2}}AB^{-\frac{1}{2}})^{t}B^{t}A^{\frac{k}{4}-t}\right), \tag{2.43}$$

which is in turn equivalent to showing that

$$B^{\frac{t}{2}}A^{\frac{k}{2}-t}B^{\frac{t}{2}} \leq I_n \Rightarrow A^{\frac{k}{4}-t}B^{t}(B^{-\frac{1}{2}}AB^{-\frac{1}{2}})^{t}B^{t}A^{\frac{k}{4}-t} \leq I_n.$$

For this, let $B^{\frac{t}{2}}A^{\frac{k}{2}-t}B^{\frac{t}{2}} \leq I_n$, then $A^{\frac{k}{2}-t} \leq B^{-t}$.

First, making use of Löwner-Heinz inequality for $0 \leq \frac{t}{\frac{k}{2}-t} \leq 1$ gives

$$A^{t} \leq B^{-\frac{t^2}{\frac{k}{2}-t}}.$$

Next, applying Furuta inequality yields

$$\left(B^{-\frac{t^2}{\frac{k}{2}-t}}\right)^{\frac{p+2r}{p}} \geq \left[\left(B^{-\frac{t^2}{\frac{k}{2}-t}}\right)^{r}(A^{t})^{p}\left(B^{-\frac{t^2}{\frac{k}{2}-t}}\right)^{r}\right]^{\frac{1}{p}}.$$

Now, replacing p with $\frac{1}{t} \geq 1$ and r with $\frac{\frac{k}{2}-t}{2t^2} \geq 0$ gives

$$B^{-\frac{\frac{kt}{2}}{\frac{k}{2}-t}} \geq (B^{-\frac{1}{2}}AB^{-\frac{1}{2}})^t.$$

Pre-post multiplying both sides by $B^t > 0$ implies that

$$(B^t)^{\frac{\frac{k}{2}-2t}{\frac{k}{2}-t}} = B^t B^{-\frac{\frac{kt}{2}}{\frac{k}{2}-t}} B^t \geq B^t (B^{-\frac{1}{2}}AB^{-\frac{1}{2}})^t B^t. \tag{2.44}$$

However, $A^{\frac{k}{2}-t} \leq B^{-t}$, so that $A^{-(\frac{k}{2}-t)} \geq B^t$. By again appealing to Löwner-Heinz inequality for $0 \leq \frac{\frac{k}{2}-2t}{\frac{k}{2}-t} \leq 1$ we obtain

$$(A^{-(\frac{k}{2}-t)})^{\frac{\frac{k}{2}-2t}{\frac{k}{2}-t}} \geq (B^t)^{\frac{\frac{k}{2}-2t}{\frac{k}{2}-t}},$$

which gives

$$A^{-(\frac{k}{2}-2t)} \geq (B^t)^{\frac{\frac{k}{2}-2t}{\frac{k}{2}-t}}. \tag{2.45}$$

Now it is worthy to observe that inequalities (2.44) and (2.45) yield

$$A^{-(\frac{k}{2}-2t)} \geq B^t (B^{-\frac{1}{2}}AB^{-\frac{1}{2}})^t B^t.$$

Hence,

$$A^{\frac{k}{4}-t} B^t (B^{-\frac{1}{2}}AB^{-\frac{1}{2}})^t B^t A^{\frac{k}{4}-t} \leq I_n,$$

and therefore, (2.43) is true for all $0 \leq t \leq 1$ and $k \geq 4t$.

On the other hand, using (2.42) and (2.43) gives

$$\lambda_1 \left(A^{\frac{k}{2}-t} B^t \right) \leq \lambda_1 \left(A^{\frac{k}{4}} (B^{\frac{1}{2}} A^{-1} B^{\frac{1}{2}})^t A^{\frac{k}{4}} \right), \qquad 0 \leq t \leq 1, \quad k \geq 4t.$$

Thus, by a standard anti-symmetric tensor product argument, we get

$$\boldsymbol{\lambda} \left(\boldsymbol{A^{\frac{k}{2}-t} B^t} \right) \prec_{wlog} \boldsymbol{\lambda} \left(\boldsymbol{A^{\frac{k}{4}} (B^{\frac{1}{2}} A^{-1} B^{\frac{1}{2}})^t A^{\frac{k}{4}}} \right), \qquad 0 \leq t \leq 1, \quad k \geq 4t.$$

Finally, the proof is complete by making use of the fact that

$$\det \left(A^{\frac{k}{2}-t} B^t \right) = \det \left(A^{\frac{k}{4}} (B^{\frac{1}{2}} A^{-1} B^{\frac{1}{2}})^t A^{\frac{k}{4}} \right).$$

■

Our next goal is to show that (2.41) is also true for all $\frac{1}{2} \leq t \leq 1$ and $k \geq 6t - 2$. First, we need the following lemma.

Lemma 2.13. *Let A and B be two positive definite matrices. Then, for all $\frac{1}{2} \leq t \leq 1$ and $k \geq 6t-2$, we have*

$$\boldsymbol{\lambda}\left(\boldsymbol{A}^{\frac{k+2t}{4}}\left(\boldsymbol{A}^{-\frac{1}{2}}\boldsymbol{B}\boldsymbol{A}^{-\frac{1}{2}}\right)^{\boldsymbol{2t}}\boldsymbol{A}^{\frac{2t+k}{4}}\right) \succ_{log} \boldsymbol{\lambda}(\boldsymbol{A}^{\frac{k}{2}-t}\boldsymbol{B}^{\boldsymbol{2t}}).$$

Proof. As in similar situations, it is enough to prove that for all $\frac{1}{2} \leq t \leq 1$ and $k \geq 6t-2$

$$\lambda_1\left(A^{\frac{k+2t}{4}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{2t}A^{\frac{2t+k}{4}}\right) \geq \lambda_1(A^{\frac{k}{2}-t}B^{2t}).$$

First, in view of Lemma 2.6, we obtain the following equality:

$$A^{\frac{2t+k}{4}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{2t}A^{\frac{2t+k}{4}} = A^{\frac{2t+k-2}{4}}B^{\frac{1}{2}}\left(B^{\frac{1}{2}}A^{-1}B^{\frac{1}{2}}\right)^{2t-1}B^{\frac{1}{2}}A^{\frac{2t+k-2}{4}}.$$

For the sake of clarification, we shall use the following notation. For all $\frac{1}{2} \leq t \leq 1$ and $k \geq 6t-2$, let

$$\begin{aligned} X &= A^{\frac{2t+k-2}{4}}B^{\frac{1}{2}}\left(B^{\frac{1}{2}}A^{-1}B^{\frac{1}{2}}\right)^{2t-1}B^{\frac{1}{2}}A^{\frac{2t+k-2}{4}}, \\ Y &= A^{\frac{2t+k-2}{4}}B^{2t}A^{\frac{k-6t+2}{4}}, \quad \textit{and} \\ Z &= A^{\frac{k-6t+2}{4}}B^{2t-\frac{1}{2}}(B^{-\frac{1}{2}}AB^{-\frac{1}{2}})^{2t-1}B^{2t-\frac{1}{2}}A^{\frac{k-6t+2}{4}}. \end{aligned}$$

Now by making use of Schur complement, the following 2×2 block matrix

$$M = \begin{bmatrix} X & Y \\ Y^* & Z \end{bmatrix}$$

is positive semi-definite and hence $\lambda_1(X) \cdot \lambda_1(Z) \geq \lambda_1(Y)^2$. Our next goal is to show that $\lambda_1(Z) \leq \lambda_1(Y)$ which in turn gives $\lambda_1(X) \geq \lambda_1(Y)$. Noticing that $\lambda_1(Y) = \lambda_1(B^tA^{\frac{k}{2}-t}B^t)$, then in order to prove $\lambda_1(Z) \leq \lambda_1(Y)$, it is suffices to show that

$$B^tA^{\frac{k}{2}-t}B^t \leq I_n \Rightarrow Z \leq I_n.$$

For this purpose, let $B^tA^{\frac{k}{2}-t}B^t \leq I_n$, then clearly $A^{\frac{k}{2}-t} \leq B^{-2t}$. Using Löwner-Heinz inequality for $0 \leq \frac{2t-1}{\frac{k}{2}-t} \leq 1$ gives

$$A^{2t-1} \leq B^{-\frac{2t(2t-1)}{\frac{k}{2}-t}}.$$

Now, applying Furuta inequality yields

$$\left(B^{-\frac{2t(2t-1)}{\frac{k}{2}-t}}\right)^{\frac{p+2r}{p}} \geq \left[\left(B^{-\frac{2t(2t-1)}{\frac{k}{2}-t}}\right)^{r}(A^{2t-1})^{p}\left(B^{-\frac{2t(2t-1)}{\frac{k}{2}-t}}\right)^{r}\right]^{\frac{1}{p}}.$$

Next, taking $p=\frac{1}{2t-1}\geq 1$ and $r=\frac{\frac{k}{2}-t}{4t(2t-1)}\geq 0$ implies that

$$B^{-\frac{(\frac{k}{2}+t)(2t-1)}{\frac{k}{2}-t}}\geq (B^{-\frac{1}{2}}AB^{-\frac{1}{2}})^{2t-1}.$$

Pre-post multiplying both sides with $B^{2t-\frac{1}{2}}>0$ gives

$$B^{2t-\frac{1}{2}}B^{-\frac{(\frac{k}{2}+t)(2t-1)}{\frac{k}{2}-t}}B^{2t-\frac{1}{2}}=\left(B^{2t}\right)^{\frac{\frac{k-6t+2}{2}}{\frac{k}{2}-t}}\geq B^{2t-\frac{1}{2}}(B^{-\frac{1}{2}}AB^{-\frac{1}{2}})^{2t-1}B^{2t-\frac{1}{2}}.$$

Again, noticing that $A^{\frac{k}{2}-t}\leq B^{-2t}$ implies that $A^{-(\frac{k}{2}-t)}\geq B^{2t}$, then by appealing to Löwner-Heinz for $0\leq\frac{\frac{k-6t+2}{2}}{\frac{k}{2}-t}\leq 1$ we obtain

$$(A^{-(\frac{k}{2}-t)})^{\frac{\frac{k-6t+2}{2}}{\frac{k}{2}-t}}\geq (B^{2t})^{\frac{\frac{k-6t+2}{2}}{\frac{k}{2}-t}}$$

which gives

$$A^{-(\frac{k-6t+2}{2})}\geq B^{2t-\frac{1}{2}}(B^{-\frac{1}{2}}AB^{-\frac{1}{2}})^{2t-1}B^{2t-\frac{1}{2}}.$$

Hence,

$$Z=A^{\frac{k-6t+2}{4}}B^{2t-\frac{1}{2}}(B^{-\frac{1}{2}}AB^{-\frac{1}{2}})^{2t-1}B^{2t-\frac{1}{2}}A^{\frac{k-6t+2}{4}}\leq I_n.$$

Therefore, for all $\frac{1}{2}\leq t\leq 1$ and $k\geq 6t-2$,

$$\lambda_1(X)\geq\lambda_1(Y).$$

Thus, by an anti-symmetric tensor product argument and by the fact that $\det(X)=\det(Y)$, we get

$$\boldsymbol{\lambda(X)}\succ_{log}\boldsymbol{\lambda(Y)}.$$

Finally, noting that

$$\boldsymbol{\lambda(X)}=\boldsymbol{\lambda}\left(\boldsymbol{A}^{\frac{k+2t}{4}}\left(\boldsymbol{A}^{-\frac{1}{2}}\boldsymbol{B}\boldsymbol{A}^{-\frac{1}{2}}\right)^{\boldsymbol{2t}}\boldsymbol{A}^{\frac{2t+k}{4}}\right)\text{ and }\boldsymbol{\lambda(Y)}=\boldsymbol{\lambda}(\boldsymbol{A}^{\frac{k}{2}-t}\boldsymbol{B}^{\boldsymbol{2t}}),$$

the proof is complete. ■

Theorem 2.22. *Let A and B be two positive definite matrices. Then, for all $\frac{1}{2}\leq t\leq 1$ and $k\geq 6t-2$*

$$\boldsymbol{\lambda}\left(\boldsymbol{A}^{\frac{k}{4}}\left(\boldsymbol{B}^{\frac{1}{2}}\boldsymbol{A}^{-1}\boldsymbol{B}^{\frac{1}{2}}\right)^{t}\boldsymbol{A}^{\frac{k}{4}}\right)\succ_{log}\boldsymbol{\lambda}(\boldsymbol{A}^{\frac{k}{2}-t}\boldsymbol{B}^{t}).$$

Proof. Let $\frac{1}{2} \leq t \leq 1$ and $k \geq 6t-2$. Then, we can write

$$\begin{aligned}
\boldsymbol{\lambda}\left(\boldsymbol{A}^{\frac{k}{4}}\left(\boldsymbol{B}^{\frac{1}{2}}\boldsymbol{A}^{-1}\boldsymbol{B}^{\frac{1}{2}}\right)^{t}\boldsymbol{A}^{\frac{k}{4}}\right) &= \boldsymbol{\lambda}\left(\boldsymbol{A}^{\frac{k}{4}}\left(\boldsymbol{A}^{\frac{1}{2}}(\boldsymbol{A}^{-\frac{1}{2}}\boldsymbol{B}^{\frac{1}{2}}\boldsymbol{A}^{-1}\boldsymbol{B}^{\frac{1}{2}}\boldsymbol{A}^{-\frac{1}{2}})\boldsymbol{A}^{\frac{1}{2}}\right)^{t}\boldsymbol{A}^{\frac{k}{4}}\right) \\
&\succ_{log} \boldsymbol{\lambda}\left(\boldsymbol{A}^{\frac{k}{2}+t}\left(\boldsymbol{A}^{-\frac{1}{2}}\boldsymbol{B}^{\frac{1}{2}}\boldsymbol{A}^{-1}\boldsymbol{B}^{\frac{1}{2}}\boldsymbol{A}^{-\frac{1}{2}}\right)^{t}\right) \\
&= \boldsymbol{\lambda}\left(\boldsymbol{A}^{\frac{k}{2}+t}\left(\boldsymbol{A}^{-\frac{1}{2}}\boldsymbol{B}^{\frac{1}{2}}\boldsymbol{A}^{-\frac{1}{2}}\right)^{2t}\right) \\
&= \boldsymbol{\lambda}\left(\boldsymbol{A}^{\frac{k+2t}{4}}\left(\boldsymbol{A}^{-\frac{1}{2}}\boldsymbol{B}^{\frac{1}{2}}\boldsymbol{A}^{-\frac{1}{2}}\right)^{2t}\boldsymbol{A}^{\frac{k+2t}{4}}\right) \\
&\succ_{log} \boldsymbol{\lambda}\left(\boldsymbol{A}^{\frac{k}{2}-t}(\boldsymbol{B}^{\frac{1}{2}})^{2t}\right) \qquad \text{(by Lemma 2.13)} \\
&= \boldsymbol{\lambda}\left(\boldsymbol{A}^{\frac{k}{2}-t}\boldsymbol{B}^{t}\right).
\end{aligned}$$

The first inequality is due to replacing X with A and Y with $A^{-\frac{1}{2}}B^{\frac{1}{2}}A^{-1}B^{\frac{1}{2}}A^{-\frac{1}{2}}$, respectively, in Part (i) of Lemma 2.7. ∎

As a result of Theorem 2.21 and Theorem 2.22, we have the following corollary.

Corollary 2.4. *Let A and B be two positive semi-definite matrices. Then, for ($0 \leq p \leq 2$ and $k \geq 2p$) or for ($1 \leq p \leq 2$ and $k \geq 3p-2$) we have*

$$\det(A^k + |AB|^p) \geq \det(A^k + A^pB^p).$$

Proof. Without loss of generality we may assume that A and B are positive definite matrices, the general case is by continuity argument. Suppose that $0 \leq p \leq 2$ and $k \geq 2p$. Then, replacing A with A^{-2}, B with B^2 and t with $0 \leq \frac{p}{2} \leq 1$ in Theorem 2.21 gives

$$\boldsymbol{\lambda}(\boldsymbol{A}^{p-k}\boldsymbol{B}^{p}) \prec_{log} \boldsymbol{\lambda}(\boldsymbol{A}^{-\frac{k}{2}}(\boldsymbol{B}\boldsymbol{A}^{2}\boldsymbol{B})^{\frac{p}{2}}\boldsymbol{A}^{-\frac{k}{2}}). \tag{2.46}$$

Applying Part (ii) of Lemma 2.3 on (2.46) yields

$$\det(I_n + A^{-\frac{k}{2}}(BA^2B)^{\frac{p}{2}}A^{-\frac{k}{2}}) \geq \det(I_n + A^{p-k}B^p).$$

Next, multiplying both sides with $\det(A^k) > 0$, we obtain

$$\det(A^k + |AB|^p) \geq \det(A^k + A^pB^p), \qquad 0 \leq p \leq 2,\ k \geq 2p.$$

For the case when $1 \leq p \leq 2$ and $k \geq 3p-2$, the proof can be done in a similar fashion by making use this time of Theorem 2.22. ∎

As an analogue of Corollary 2.4, we have the following result which gives a partial answer of Conjecture 2.2.

Theorem 2.23. *Let A and B be two positive semi-definite matrices. Then, for all $0 \leq p \leq \frac{4}{3}$, the following holds*

$$\det(A^2 + |AB|^p) \geq \det(A^2 + A^p B^p).$$

4.4.2 Second Approach

We will establish the following log-majorization relation.

Lemma 2.14. *Let A, B be two positive definite matrices. Then for all $0 \leq t \leq s \leq k$,*

$$\boldsymbol{\lambda}(\boldsymbol{A}^{\frac{k}{2}}(\boldsymbol{B}^{\frac{s}{2}}\boldsymbol{A}^{-s}\boldsymbol{B}^{\frac{s}{2}})^{\frac{t}{s}}\boldsymbol{A}^{\frac{k}{2}}) \succ_{wlog} \boldsymbol{\lambda}(\boldsymbol{A}^{k-t}\boldsymbol{B}^{t}).$$

Proof. As usual in such situation, by a standard anti-symmetric tensor product argument, it is enough to prove that for all $0 \leq t \leq s \leq k$

$$\lambda_1(A^{\frac{k}{2}}(B^{\frac{s}{2}}A^{-s}B^{\frac{s}{2}})^{\frac{t}{s}}A^{\frac{k}{2}}) \geq \lambda_1(A^{k-t}B^t).$$

Without loss of generality, we assume

$$\lambda_1(A^{\frac{k}{2}}(B^{\frac{s}{2}}A^{-s}B^{\frac{s}{2}})^{\frac{t}{s}}A^{\frac{k}{2}}) = 1.$$

This is equivalent to

$$A^{\frac{k}{2}}(B^{\frac{s}{2}}A^{-s}B^{\frac{s}{2}})^{\frac{t}{s}}A^{\frac{k}{2}} \leq I_n$$

which in turn gives

$$(B^{\frac{s}{2}}A^{-s}B^{\frac{s}{2}})^{\frac{t}{s}} \leq A^{-k}. \tag{2.47}$$

Now obviously proving our claim is equivalent to showing that

$$\lambda_1(A^{k-t}B^t) \leq 1.$$

Clearly, inequality (2.47) gives

$$A^k \leq (B^{-\frac{s}{2}}A^sB^{-\frac{s}{2}})^{\frac{t}{s}}.$$

Now using Löwner-Heinz inequality for $0 \leq \frac{k-t}{k} \leq 1$, we obtain

$$A^{k-t} \leq (B^{-\frac{s}{2}}A^sB^{-\frac{s}{2}})^{\frac{t(k-t)}{sk}}. \tag{2.48}$$

Thus, we can write

$$\begin{aligned}
\lambda_1\left(A^{k-t}B^t\right) &= \lambda_1\left(B^{\frac{t}{2}}A^{k-t}B^{\frac{t}{2}}\right)\\
&\leq \lambda_1\left(B^{\frac{t}{2}}(B^{-\frac{s}{2}}A^sB^{-\frac{s}{2}})^{\frac{t(k-t)}{sk}}B^{\frac{t}{2}}\right)\\
&\leq \lambda_1\left(B^{\frac{s}{2}}(B^{-\frac{s}{2}}A^sB^{-\frac{s}{2}})^{\frac{k-t}{k}}B^{\frac{s}{2}}\right)^{\frac{t}{s}}\\
&= \lambda_1\left(B^{\frac{s}{2}}(B^{\frac{s}{2}}A^{-s}B^{\frac{s}{2}})^{\frac{t}{k}-1}B^{\frac{s}{2}}\right)^{\frac{t}{s}}\\
&\leq \lambda_1\left(B^{\frac{s}{2}}(B^{\frac{s}{2}}A^{-s}B^{\frac{s}{2}})^{-1}B^{\frac{s}{2}}(B^{\frac{s}{2}}A^{-s}B^{\frac{s}{2}})^{\frac{t}{k}}\right)^{\frac{t}{s}}\\
&= \lambda_1\left(A^{\frac{s}{2}}(B^{\frac{s}{2}}A^{-s}B^{\frac{s}{2}})^{\frac{t}{k}}A^{\frac{s}{2}}\right)^{\frac{t}{s}}\\
&\leq \lambda_1\left(A^{\frac{k}{2}}(B^{\frac{s}{2}}A^{-s}B^{\frac{s}{2}})^{\frac{t}{s}}A^{\frac{k}{2}}\right)^{\frac{t}{k}}\\
&= (1)^{\frac{t}{k}}\\
&= 1.
\end{aligned}$$

Here, the first inequality follows from (2.48), the second and the last inequality follow from Lemma 2.5 by substituting $0 \leq p = \frac{t}{s} \leq 1$ and $0 \leq p = \frac{s}{k} \leq 1$, respectively, and the third inequality follows from taking $q = 1$, $p = \frac{t}{k}$ and replacing X with $B^{\frac{s}{2}}$ and Y with $B^{\frac{s}{2}}A^{-s}B^{\frac{s}{2}}$ in Theorem 2.16. ■

Finally, we are in a position to present a generalization of Conjecture 2.2.

Theorem 2.24. *Let A and B be two positive semi-definite matrices. Then, for all $0 \leq p \leq s \leq k$,*

$$\det(A^k + |A^{\frac{s}{2}}B^{\frac{s}{2}}|^{\frac{2p}{s}}) \geq \det(A^k + A^pB^p).$$

Proof. We shall assume again that A and B are positive definite matrices as the general case can be obtained by a continuity argument.

With this in mind, replacing now A with A^{-1} and taking $t = p$ in Lemma 2.14 give

$$\boldsymbol{\lambda}(\boldsymbol{A}^{-\frac{\boldsymbol{k}}{\boldsymbol{2}}}(\boldsymbol{B}^{\frac{\boldsymbol{s}}{\boldsymbol{2}}}\boldsymbol{A}^{\boldsymbol{s}}\boldsymbol{B}^{\frac{\boldsymbol{s}}{\boldsymbol{2}}})^{\frac{\boldsymbol{t}}{\boldsymbol{s}}}\boldsymbol{A}^{-\frac{\boldsymbol{k}}{\boldsymbol{2}}}) \succ_{wlog} \boldsymbol{\lambda}(\boldsymbol{A}^{\boldsymbol{t}-\boldsymbol{k}}\boldsymbol{B}^{\boldsymbol{t}}), \quad 0 \leq t \leq s \leq k. \tag{2.49}$$

Applying Part (ii) of Lemma 2.3 to (2.49) yields

$$\det(I_n + A^{-k/2}(B^{\frac{s}{2}}A^sB^{\frac{s}{2}})^{\frac{p}{s}}A^{-k/2}) \geq \det(I_n + A^{p-k}B^p), \quad 0 \leq p \leq s \leq k.$$

Therefore, multiplying both sides with $\det(A^k) > 0$ implies

$$\det(A^k + |A^{\frac{s}{2}}B^{\frac{s}{2}}|^{\frac{2p}{s}}) \geq \det(A^k + A^pB^p), \quad 0 \leq p \leq s \leq k.$$

■

2.5. Open Problems

The following results were presented in this chapter:

$$\det(A^k + |BA|^p) \leq \det(A^k + A^p B^p) \quad \text{for} \quad 0 \leq p \leq 2, \quad \text{and} \quad k \geq p; \tag{2.50}$$
$$\det(A^2 + |AB|^p) \geq \det(A^2 + |BA|^p) \quad \text{for} \quad p \geq 0; \tag{2.51}$$
$$\det(A^2 + |AB|^p) \leq \det(A^2 + |BA|^p) \quad \text{for} \quad p \leq 0; \tag{2.52}$$
$$\det(A^k + |AB|^2) \geq \det(A^k + A^2 B^2) \quad \text{for} \quad k \geq 0; \tag{2.53}$$
$$\det(A^k + |AB|^2) \leq \det(A^k + A^2 B^2) \quad \text{for} \quad k \leq 0; \tag{2.54}$$
$$\det(A^k + |AB|^p) \geq \det(A^k + A^p B^p) \quad \text{for} \quad 0 \leq p \leq 2, \quad \text{and} \quad k \geq 2. \tag{2.55}$$

where A and B are two positive semi-definite matrices (A and B are invertible in (2.52) and (2.54)).

As analogue of (2.53) and (2.55), we can introduce our first problem.

Conjecture 2.3. *Let A and B be two positive semi-definite matrices. Then, for all $k \geq 0$*

$$\det(A^k + |AB|^p) \geq \det(A^2 + A^p B^p) \qquad 0 \leq p \leq 2.$$

As a result of (2.51) and noting that (2.50) and (2.55) implies

$$\det(A^k + |AB|^p) \geq \det(A^k + |BA|^p) \qquad 0 \leq p \leq 2 \text{ and } k \geq 2,$$

we can formulate the following conjecture.

Conjecture 2.4. *Let A and B be two positive semi-definite matrices. Then, for all $k,\ p \geq 0$*

$$\det(A^k + |AB|^p) \geq \det(A^k + |BA|^p).$$

Motivated by some recent work in [37], we will show a determinantal inequality related to inequality (2.55). More explicitly, our main goal here is to show that with the same setting, the inequality in Theorem 2.24 is reversed when replacing A^k by A^{-k} provided that A is invertible.

The starting point here is the following lemma.

Lemma 2.15. *Let A and B be two positive definite matrices. Then for all $0 \leq t \leq s \leq k$,*

$$\boldsymbol{\lambda}(\boldsymbol{A}^{\frac{k}{2}}(\boldsymbol{B}^{\frac{s}{2}}\boldsymbol{A}^{s}\boldsymbol{B}^{\frac{s}{2}})^{\frac{t}{s}}\boldsymbol{A}^{\frac{k}{2}}) \prec_{wlog} \boldsymbol{\lambda}(\boldsymbol{A}^{k+t}\boldsymbol{B}^{t}).$$

Proof. As in similar situations, it is enough to prove that for all $0 \leq t \leq s \leq k$,

$$\lambda_1(A^{\frac{k}{2}}(B^{\frac{s}{2}}A^sB^{\frac{s}{2}})^{\frac{t}{s}}A^{\frac{k}{2}}) \leq \lambda_1(A^{k+t}B^t).$$

Again, without loss of generality, we shall assume that $\lambda_1(A^{k+t}B^t) = 1$. As mentioned earlier, our task now is equivalent to proving

$$B^{\frac{t}{2}}A^{k+t}B^{\frac{t}{2}} \leq I_n,$$

which is in turn equivalent to showing that

$$A^{k+t} \leq B^{-t}.$$

By appealing to Lemma 2.1 for $0 \leq \frac{k}{k+t} \leq 1$, we obtain

$$A^k \leq B^{-\frac{tk}{k+t}}. \tag{2.56}$$

Then, using again a power $\frac{s}{k}$ where $0 \leq \frac{s}{k} \leq 1$ on (2.56), we get

$$A^s \leq B^{-\frac{ts}{k+t}}. \tag{2.57}$$

Now, we can write

$$\begin{aligned}
\lambda_1\left(A^{\frac{k}{2}}(B^{\frac{s}{2}}A^sB^{\frac{s}{2}})^{\frac{t}{s}}A^{\frac{k}{2}}\right) &\leq \lambda_1\left(A^{\frac{k}{2}}(B^{\frac{s}{2}}B^{-\frac{st}{k+t}}B^{\frac{s}{2}})^{\frac{t}{s}}A^{\frac{k}{2}}\right) \qquad \text{(by using (2.57))} \\
&= \lambda_1\left(A^{\frac{k}{2}}B^{\frac{tk}{k+t}}A^{\frac{k}{2}}\right) \\
&= \lambda_1\left(B^{\frac{tk}{2(k+t)}}A^kB^{\frac{tk}{2(k+t)}}\right) \\
&\leq \lambda_1\left(B^{\frac{tk}{2(k+t)}}B^{-\frac{tk}{(k+t)}}B^{\frac{tk}{2(k+t)}}\right) \qquad \text{(by using (2.56))} \\
&= \lambda_1(I_n) \\
&= 1.
\end{aligned}$$

As before, applying anti-symmetric tensor product argument gives for all $0 \leq t \leq s \leq k$,

$$\boldsymbol{\lambda}(\boldsymbol{A}^{\frac{k}{2}}(\boldsymbol{B}^{\frac{s}{2}}\boldsymbol{A}^s\boldsymbol{B}^{\frac{s}{2}})^{\frac{t}{s}}\boldsymbol{A}^{\frac{k}{2}}) \prec_{wlog} \boldsymbol{\lambda}(\boldsymbol{A}^{\boldsymbol{k+t}}\boldsymbol{B}^{\boldsymbol{t}}).$$

■

As a result, we have the following theorem.

Theorem 2.25. *Let A and B be two $n \times n$ complex matrices such that A is positive definite and B is positive semi-definite matrix. Then for all $0 \leq p \leq s \leq k$, it holds that*

$$\det(A^{-k} + |A^{\frac{s}{2}}B^{\frac{s}{2}}|^{\frac{2p}{s}}) \leq \det(A^{-k} + A^pB^p).$$

Proof. As usual we shall assume that B is a positive definite matrix. Applying Lemma 2.2.3 to the majorization inequality of Lemma 2.15 gives

$$\det(I_n + A^{k/2}(B^{\frac{s}{2}}A^sB^{\frac{s}{2}})^{\frac{t}{s}}A^{k/2}) \leq \det(I_n + A^{t+k}B^t), \qquad 0 \leq t \leq s \leq k.$$

Taking $t = p$ and multiplying both sides with $\det(A^{-k}) > 0$ yield

$$\det(A^{-k} + |A^{\frac{s}{2}}B^{\frac{s}{2}}|^{\frac{2p}{s}}) \leq \det(A^{-k} + A^pB^p), \qquad 0 \leq t \leq s \leq k.$$

■

As a consequence of Theorem 2.25, inequality (2.52) and inequality (2.54), we can introduce the following two conjectures.

Conjecture 2.5. *Let A and B be two invertible Hermitian matrices. Then, for all $p \leq 0$ and $k \geq 0$*

$$\det(A^k + |AB|^p) \leq \det(A^k + |BA|^p).$$

Conjecture 2.6. *Let A and B be two $n \times n$ complex matrices such that A is positive definite and B is positive semi-definite matrix. Then, for all $k \leq 0$*

$$\det(A^k + |AB|^p) \leq \det(A^k + A^pB^p) \qquad 0 \leq p \leq 2.$$

3

Applications of Some Obtained Majorization Relations

In the previous chapter, we have established some majorization relations to prove our desired determinantal inequalities. In this chapter, we provide some applications to these obtained majorization inequalities. First, we introduce a new positive definite quantity recently defined by Lin [40]. We study basic properties of this quantity and show its relation with the geometric mean. We also provide majorization inequalities concerning the eigenvalues of this new quantity compared with that of the geometric mean. We then give a complement of a Golden-Thompson type inequality posed by Hiai and Petz in [30]. Furthermore, we show an application to Rényi divergences. Lastly, we present a complement of a unitarily invariant norm inequality which was conjectured by Bhatia, Lim and Yamazaki [12] and recently proved by Dinh, Dumitru and Franco [18] for the Schatten p-norm with $1 \leq p \leq \infty$.

3.1. A Positive Definite Quantity Related to the Matrix Geometric Mean

The following quantities was first introduced by M. Lin [40] for any $n \times n$ positive definite matrices A and B.

Definition 3.1. *Let A and B be two positive semi-definite matrices with A invertible. The matrix $A \natural B$ is defined by*

$$A \natural B = A^{\frac{1}{2}}(B^{\frac{1}{2}}A^{-1}B^{\frac{1}{2}})^{\frac{1}{2}}A^{\frac{1}{2}}.$$

As usual, it can be extended to any singular matrix A by continuity argument as follows:

$$A \natural B = \lim_{\epsilon \to 0^+} (A + \epsilon I_n) \natural B.$$

Definition 3.2. *Let A and B be two positive definite matrices. The weighted version of $A \natural B$ is denoted and defined by*

$$A \natural_t B = A^{\frac{1}{2}}(B^{\frac{1}{2}}A^{-1}B^{\frac{1}{2}})^{t}A^{\frac{1}{2}} \quad \textit{for} \quad t \in \mathbb{R},$$

and it can be extended to $A \geq 0$ *by continuity as follows:*

$$A\natural_t B = \lim_{\epsilon \to 0^+} (A + \epsilon I_n)\natural_t B.$$

In particular, when $t = \frac{1}{2}$, we have $A\natural_{\frac{1}{2}} B = A\natural B$.

Here are some obvious properties of this new quantity.

(i) $A\natural_t B > 0$.

(ii) For $t = 0$ and $t = 1$ we have $A\natural_0 B = A$ and $A\natural_1 B = A^{\frac{1}{2}} B^{\frac{1}{2}} A^{-1} B^{\frac{1}{2}} A^{\frac{1}{2}}$.

(iii) $f(A, B) = A\natural_t B$ is continuous for all $A, B > 0$.

(iv) $(\alpha A)\natural_t(\beta B) = \alpha^{1-t}\beta^t (A\natural_t B)$ for all $\alpha, \beta \in \mathbb{R}\backslash\{0\}$.

(v) $A\natural_t B = A^{1-t} B^t$ in case $AB = BA$.

(vi) $(A\natural_t B)^{-1} = A^{-1}\natural_t B^{-1}$.

The following proposition shows a relation between the two quantities $A\natural B$ and $B\natural A$ as well as the geometric mean.

Proposition 3.1. *Let* A *and* B *be two positive definite matrices. Then*

$$A\natural B = A^{\frac{1}{2}} B^{\frac{1}{2}} (A\#B)^{-1} B^{\frac{1}{2}} A^{\frac{1}{2}}, \text{ and}$$
$$A\natural B = A^{\frac{1}{2}} B^{\frac{1}{2}} A^{-\frac{1}{2}} B^{-\frac{1}{2}} (B\natural A) B^{-\frac{1}{2}} A^{-\frac{1}{2}} B^{\frac{1}{2}} A^{\frac{1}{2}}.$$

Proof. It is easy to see using Lemma 2.6 that

$$A^{\frac{1}{2}} (B^{\frac{1}{2}} A^{-1} B^{\frac{1}{2}})^{\frac{1}{2}} A^{\frac{1}{2}} = A^{\frac{1}{2}} B^{\frac{1}{2}} A^{-\frac{1}{2}} (A^{-\frac{1}{2}} B A^{-\frac{1}{2}})^{-\frac{1}{2}} A^{-\frac{1}{2}} B^{\frac{1}{2}} A^{\frac{1}{2}}.$$

So, clearly we have $A\natural B = A^{\frac{1}{2}} B^{\frac{1}{2}} (A\#B)^{-1} B^{\frac{1}{2}} A^{\frac{1}{2}}$.

Now, for the second equality we have due to the first equality and the fact that the geometric mean is symmetric

$$\begin{aligned} A\natural B &= A^{\frac{1}{2}} B^{\frac{1}{2}} (A\#B)^{-1} B^{\frac{1}{2}} A^{\frac{1}{2}} \\ &= A^{\frac{1}{2}} B^{\frac{1}{2}} (B\#A)^{-1} B^{\frac{1}{2}} A^{\frac{1}{2}} \\ &= A^{\frac{1}{2}} B^{\frac{1}{2}} A^{-\frac{1}{2}} B^{-\frac{1}{2}} B^{\frac{1}{2}} A^{\frac{1}{2}} (B\#A)^{-1} A^{\frac{1}{2}} B^{\frac{1}{2}} B^{-\frac{1}{2}} A^{-\frac{1}{2}} B^{\frac{1}{2}} A^{\frac{1}{2}}. \end{aligned}$$

Hence, we obtain the desired equality since $B\natural A = B^{\frac{1}{2}} A^{\frac{1}{2}} (B\#A)^{-1} A^{\frac{1}{2}} B^{\frac{1}{2}}$. ∎

Remark 3.1. *As a consequence of Proposition 3.1, the positive definite quantity* $A\natural B$ *is not symmetric as* $A\natural B = B\natural A$ *is not true in general. However,* $A\natural B = B\natural A$ *is just valid when* A *and* B *commute.*

Remark 3.2. *It is easy to check that the binary operation $\gamma = \natural$ is not a connection since the function $f(A,B) = A\natural B$ is not monotone for $A, B > 0$, that is, $A_1\natural B_1 \leq A_2\natural B_2$ is not true in general for every $A_1 \leq A_2$ and $B_1 \leq B_2$.*

Example 3.1. *Take three positive definite matrices such that $A \leq C$*

$$A := \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, B := \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \quad \text{and} \quad C := \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}.$$

Then $A\natural B = \begin{bmatrix} 0.86 & 0.53 \\ 0.53 & 1.95 \end{bmatrix}$ *and* $C\natural B = \begin{bmatrix} 0.85 & 0.61 \\ 0.61 & 2.46 \end{bmatrix}$, *but*

$$A\natural B \not\leq C\natural B.$$

Proposition 3.2. *Let A and B be two positive definite matrices. Then*

(1)
- $A \leq A\natural B \leq A^{\frac{1}{2}}B^{\frac{1}{2}}A^{-1}B^{\frac{1}{2}}A^{\frac{1}{2}}$ *if* $A \leq B$ *and*
- $A^{\frac{1}{2}}B^{\frac{1}{2}}A^{-1}B^{\frac{1}{2}}A^{\frac{1}{2}} \leq A\natural B \leq A$ *is valid when* $B \leq A$.

In particular, when $B \leq A$ we have

$$\lambda_1(B) \leq \lambda_1(A\natural B) \leq \lambda_1(A).$$

(2) The matrix $A\natural B$ has a very similar property as the geometric mean

$$A\natural B = \max\left\{X : X = X^*, \begin{bmatrix} A & X \\ X & \left(A^{-\frac{1}{2}}B^{\frac{1}{2}}A^{\frac{1}{2}}\right)^* \left(A^{-\frac{1}{2}}B^{\frac{1}{2}}A^{\frac{1}{2}}\right) \end{bmatrix} \geq 0\right\}.$$

(3) The quantity $A\natural B$ is the unique positive solution of the equation

$$XA^{-1}X = A^{\frac{1}{2}}B^{\frac{1}{2}}A^{-1}B^{\frac{1}{2}}A^{\frac{1}{2}}.$$

Proof. (1)
- The left inequality is easily obtained by the fact that $A \leq B$. Now, to prove the right inequality, we can see that by Lemma 2.6

$$\begin{aligned} A\natural B &= A^{\frac{1}{2}}\left(B^{\frac{1}{2}}A^{-1}B^{\frac{1}{2}}\right)^{\frac{1}{2}}A^{\frac{1}{2}} \\ &= A^{\frac{1}{2}}B^{\frac{1}{2}}A^{-\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{-\frac{1}{2}}A^{-\frac{1}{2}}B^{\frac{1}{2}}A^{\frac{1}{2}} \\ &= A^{\frac{1}{2}}B^{\frac{1}{2}}A^{-\frac{1}{2}}\left(A^{\frac{1}{2}}B^{-1}A^{\frac{1}{2}}\right)^{\frac{1}{2}}A^{-\frac{1}{2}}B^{\frac{1}{2}}A^{\frac{1}{2}} \\ &\leq A^{\frac{1}{2}}B^{\frac{1}{2}}A^{-\frac{1}{2}}\left(A^{\frac{1}{2}}A^{-1}A^{\frac{1}{2}}\right)^{\frac{1}{2}}A^{-\frac{1}{2}}B^{\frac{1}{2}}A^{\frac{1}{2}} \\ &= A^{\frac{1}{2}}B^{\frac{1}{2}}A^{-1}B^{\frac{1}{2}}A^{\frac{1}{2}}. \end{aligned}$$

- The proof of this inequality is done by similar steps as in the case of $A \leq B$. For the particular case, we can easily observe with the help of the fact that $X \leq Y$ implies $\lambda_j(A) \leq \lambda_j(B)$ for every $j = 1, 2, \ldots, n$. So, we have

$$\lambda_1\left(A^{\frac{1}{2}}B^{\frac{1}{2}}A^{-1}B^{\frac{1}{2}}A^{\frac{1}{2}}\right) \leq \lambda_1(A\natural B) \leq \lambda(A).$$

Now, by using Theorem 2.16 for $X = B^{\frac{1}{2}}$ and $Y = A$, we get

$$\lambda_1\left(A^{\frac{1}{2}}B^{\frac{1}{2}}A^{-1}B^{\frac{1}{2}}A^{\frac{1}{2}}\right) = \lambda_1\left(AB^{\frac{1}{2}}A^{-1}B^{\frac{1}{2}}\right) \geq \lambda_1(B).$$

Therefore,

$$\lambda_1(B) \leq \lambda_1(A\natural B) \leq \lambda_1(A).$$

(2) Let $\mathbb{S} := \left\{X : X = X^*, \begin{bmatrix} A & X \\ X & \left(A^{-\frac{1}{2}}B^{\frac{1}{2}}A^{\frac{1}{2}}\right)^* \left(A^{-\frac{1}{2}}B^{\frac{1}{2}}A^{\frac{1}{2}}\right) \end{bmatrix} \geq 0\right\}$. Our proof is divided into two steps. The first step is to show that the quantity $A\natural B$ satisfies

$$M := \begin{bmatrix} A & A\natural B \\ A\natural B & A^{\frac{1}{2}}B^{\frac{1}{2}}A^{-1}B^{\frac{1}{2}}A^{\frac{1}{2}} \end{bmatrix} \geq 0.$$

The second step is to verify that every element X in $\mathbb{S}$ satisfies

$$X \leq A\natural B.$$

step 1: The matrix M can be written as

$$\begin{bmatrix} A^{\frac{1}{2}} \\ A^{\frac{1}{2}}(B^{\frac{1}{2}}A^{-1}B^{\frac{1}{2}})^{\frac{1}{2}} \end{bmatrix} \begin{bmatrix} A^{\frac{1}{2}} \\ A^{\frac{1}{2}}(B^{\frac{1}{2}}A^{-1}B^{\frac{1}{2}})^{\frac{1}{2}} \end{bmatrix}^* = \begin{bmatrix} A & A\natural B \\ A\natural B & A^{\frac{1}{2}}B^{\frac{1}{2}}A^{-1}B^{\frac{1}{2}}A^{\frac{1}{2}} \end{bmatrix}.$$

Hence, $A\natural B \in \mathbb{S}$.

Step 2: Let us assume that X is an element of $\mathbb{S}$. Then

$$\begin{bmatrix} A & X \\ X & A^{\frac{1}{2}}B^{\frac{1}{2}}A^{-1}B^{\frac{1}{2}}A^{\frac{1}{2}} \end{bmatrix} \geq 0.$$

By using Schur's complement we have $A^{\frac{1}{2}}B^{\frac{1}{2}}A^{-1}B^{\frac{1}{2}}A^{\frac{1}{2}} - XA^{-1}X \geq 0$. So

$$A^{\frac{1}{2}}B^{\frac{1}{2}}A^{-1}B^{\frac{1}{2}}A^{\frac{1}{2}} \geq XA^{-1}X.$$

Pre-post multiplying both sides with $A^{-\frac{1}{2}} > 0$ gives

$$B^{\frac{1}{2}}A^{-1}B^{\frac{1}{2}} \geq A^{-\frac{1}{2}}XA^{-1}XA^{-\frac{1}{2}} = (A^{-\frac{1}{2}}XA^{-\frac{1}{2}})^2.$$

Now applying Löwner-Heinz inequality for $r = \frac{1}{2}$ yields

$$(B^{\frac{1}{2}}A^{-1}B^{\frac{1}{2}})^{\frac{1}{2}} \geq A^{-\frac{1}{2}}XA^{-\frac{1}{2}}.$$

Therefore, after pre-post multiplying both sides with $A^{\frac{1}{2}} > 0$, we obtain

$$A\natural B \geq X.$$

(3) First, one can simply see that $A\natural B$ is a solution of the equation

$$XA^{-1}X = A^{\frac{1}{2}}B^{\frac{1}{2}}A^{-1}B^{\frac{1}{2}}A^{\frac{1}{2}}. \tag{3.1}$$

Next, our purpose is to show the uniqueness. We will assume that there exist two solutions X and Y of (3.1), then

$$XA^{-1}X = YA^{-1}Y = A^{\frac{1}{2}}B^{\frac{1}{2}}A^{-1}B^{\frac{1}{2}}A^{\frac{1}{2}}.$$

Then $XA^{-1}X = YA^{-1}Y$. Pre-post multiplying both sides with $A^{-\frac{1}{2}} > 0$ gives

$$A^{-\frac{1}{2}}XA^{-1}XA^{-\frac{1}{2}} = A^{-\frac{1}{2}}YA^{-1}YA^{-\frac{1}{2}},$$

which is equivalent to

$$(A^{-\frac{1}{2}}XA^{-\frac{1}{2}})^2 = (A^{-\frac{1}{2}}YA^{-\frac{1}{2}})^2.$$

So,

$$A^{-\frac{1}{2}}XA^{-\frac{1}{2}} = A^{-\frac{1}{2}}YA^{-\frac{1}{2}}.$$

Finally, pre-post multiplying both sides with $A^{\frac{1}{2}}$ yields $X = Y$, which assures the uniqueness of the solution of the equation (3.1).

■

Theorem 3.1. *For any $n \times n$ positive definite matrices A and B. We have*

(i) If $0 \leq t \leq 1$ then $\boldsymbol{\lambda(A\#_tB) \prec_{log} \lambda(A^{1-t}B^t) \prec_{log} \lambda(A\natural_tB)}$.

(ii) If $-1 \leq t \leq 0$ then $\boldsymbol{\lambda(A\#_tB) \succ_{log} \lambda(A^{1-t}B^t) \succ_{log} \lambda(A\natural_tB)}$.

(iii) If $t \geq 0$ then $\boldsymbol{\lambda(A\#_tB) \prec_{log} \lambda(A\natural_tB)}$.

(iv) If $t \leq 0$ then $\boldsymbol{\lambda(A\#_tB) \succ_{log} \lambda(A\natural_tB)}$.

Proof. Before we start the proof, let us recall some majorization relations from Chapter 2. In Section 2.3, Part (i) of Lemma 2.7 and Lemma 2.8 state that for every $A, B > 0$ we have for all

$0 \leq t \leq 1$,

$$\boldsymbol{\lambda}(\boldsymbol{A}^{\frac{kt}{2}}(\boldsymbol{A}^{-\frac{1}{2}}\boldsymbol{B}\boldsymbol{A}^{-\frac{1}{2}})^{t}\boldsymbol{A}^{\frac{kt}{2}}) \prec_{log} \boldsymbol{\lambda}(\boldsymbol{A}^{(k-1)t}\boldsymbol{B}^{t}) \qquad k > 1; \tag{3.2}$$

$$\boldsymbol{\lambda}(\boldsymbol{A}^{\frac{k}{2}}(\boldsymbol{A}^{\frac{1}{2}}\boldsymbol{B}\boldsymbol{A}^{\frac{1}{2}})^{t}\boldsymbol{A}^{\frac{k}{2}}) \succ_{log} \boldsymbol{\lambda}(\boldsymbol{A}^{k+t}\boldsymbol{B}^{t}) \qquad k \geq 0. \tag{3.3}$$

In Section 2.4 and Section 2.5, we established two majorization relations (see Lemma 2.14 and Lemma 2.15) that show that for any positive definite matrices A and B we have for all $0 \leq t \leq s \leq k$

$$\boldsymbol{\lambda}(\boldsymbol{A}^{k-t}\boldsymbol{B}^{t}) \prec_{wlog} \boldsymbol{\lambda}(\boldsymbol{A}^{\frac{k}{2}}(\boldsymbol{B}^{\frac{s}{2}}\boldsymbol{A}^{-s}\boldsymbol{B}^{\frac{s}{2}})^{\frac{t}{s}}\boldsymbol{A}^{\frac{k}{2}});$$
$$\boldsymbol{\lambda}(\boldsymbol{A}^{k+t}\boldsymbol{B}^{t}) \succ_{wlog} \boldsymbol{\lambda}(\boldsymbol{A}^{\frac{k}{2}}(\boldsymbol{B}^{\frac{s}{2}}\boldsymbol{A}^{s}\boldsymbol{B}^{\frac{s}{2}})^{\frac{t}{s}}\boldsymbol{A}^{\frac{k}{2}}).$$

In particular, when $s = 1$, the following relations hold for all $0 \leq t \leq 1$ and $k \geq 1$

$$\boldsymbol{\lambda}(\boldsymbol{A}^{k-t}\boldsymbol{B}^{t}) \prec_{wlog} \boldsymbol{\lambda}(\boldsymbol{A}^{\frac{k}{2}}(\boldsymbol{B}^{\frac{1}{2}}\boldsymbol{A}^{-1}\boldsymbol{B}^{\frac{1}{2}})^{t}\boldsymbol{A}^{\frac{k}{2}}); \tag{3.4}$$

$$\boldsymbol{\lambda}(\boldsymbol{A}^{k+t}\boldsymbol{B}^{t}) \succ_{wlog} \boldsymbol{\lambda}(\boldsymbol{A}^{\frac{k}{2}}(\boldsymbol{B}^{\frac{1}{2}}\boldsymbol{A}\boldsymbol{B}^{\frac{1}{2}})^{t}\boldsymbol{A}^{\frac{k}{2}}). \tag{3.5}$$

(i) Our purpose is to show that for all $0 \leq t \leq 1$,

$$\boldsymbol{\lambda}(\boldsymbol{A}^{\frac{1}{2}}(\boldsymbol{A}^{-\frac{1}{2}}\boldsymbol{B}\boldsymbol{A}^{-\frac{1}{2}})^{t}\boldsymbol{A}^{\frac{1}{2}}) \prec_{log} \boldsymbol{\lambda}(\boldsymbol{A}^{1-t}\boldsymbol{B}^{t}) \prec_{log} \boldsymbol{\lambda}(\boldsymbol{A}^{\frac{1}{2}}(\boldsymbol{B}^{\frac{1}{2}}\boldsymbol{A}^{-1}\boldsymbol{B}^{\frac{1}{2}})^{t}\boldsymbol{A}^{\frac{1}{2}}). \tag{3.6}$$

It is clear that for $t = 0$ and $t = 1$ the left inequality is true. Now, for $0 < t < 1$, substituting $k = \frac{1}{t} > 1$ in (3.2) we get the left inequality. The inequality on the right is obtained after taking $k = 1$ and $s = 1$ in (3.3) and noting that $\det(A^{1-t}B^{t}) = \det(A\natural_{t}B)$.

(ii) In this case, our object is to prove that (3.6) is reversed when $-1 \leq t \leq 0$. The desired is obtained after replacing B with B^{-1} and t with $-t$ and taking $k = 1$ in (3.3) and (3.5) and noting that $\det(A^{1-t}B^{t}) = \det(A\natural_{t}B)$.

(iii) For $t \geq 0$, our aim is to show that

$$\boldsymbol{\lambda}(\boldsymbol{A}^{\frac{1}{2}}(\boldsymbol{A}^{-\frac{1}{2}}\boldsymbol{B}\boldsymbol{A}^{-\frac{1}{2}})^{t}\boldsymbol{A}^{\frac{1}{2}}) \prec_{log} \boldsymbol{\lambda}(\boldsymbol{A}^{\frac{1}{2}}(\boldsymbol{B}^{\frac{1}{2}}\boldsymbol{A}^{-1}\boldsymbol{B}^{\frac{1}{2}})^{t}\boldsymbol{A}^{\frac{1}{2}}).$$

As a result of Lemma 2.6, we can write

$$A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{t}A^{\frac{1}{2}} = A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}B^{\frac{1}{2}}(B^{\frac{1}{2}}A^{-1}B^{\frac{1}{2}})^{t-1}B^{\frac{1}{2}}A^{-\frac{1}{2}}\right)A^{\frac{1}{2}}.$$

Then

$$\begin{aligned}\boldsymbol{\lambda}\left(\boldsymbol{A}^{\frac{1}{2}}(\boldsymbol{A}^{-\frac{1}{2}}\boldsymbol{B}\boldsymbol{A}^{-\frac{1}{2}})^{t}\boldsymbol{A}^{\frac{1}{2}}\right) &= \boldsymbol{\lambda}\left[\boldsymbol{A}^{\frac{1}{2}}\left(\boldsymbol{A}^{-\frac{1}{2}}\boldsymbol{B}^{\frac{1}{2}}(\boldsymbol{B}^{\frac{1}{2}}\boldsymbol{A}^{-1}\boldsymbol{B}^{\frac{1}{2}})^{t-1}\boldsymbol{B}^{\frac{1}{2}}\boldsymbol{A}^{-\frac{1}{2}}\right)\boldsymbol{A}^{\frac{1}{2}}\right] \\ &= \boldsymbol{\lambda}\left(\boldsymbol{B}(\boldsymbol{B}^{\frac{1}{2}}\boldsymbol{A}^{-1}\boldsymbol{B}^{\frac{1}{2}})^{t-1}\right) \\ &\prec_{log} \boldsymbol{\lambda}\left(\boldsymbol{B}^{\frac{1}{2}}(\boldsymbol{B}^{\frac{1}{2}}\boldsymbol{A}^{-1}\boldsymbol{B}^{\frac{1}{2}})^{t}\boldsymbol{B}^{\frac{1}{2}}(\boldsymbol{B}^{\frac{1}{2}}\boldsymbol{A}^{-1}\boldsymbol{B}^{\frac{1}{2}})^{-1}\right) \\ &= \boldsymbol{\lambda}\left(\boldsymbol{A}^{\frac{1}{2}}(\boldsymbol{B}^{\frac{1}{2}}\boldsymbol{A}^{-1}\boldsymbol{B}^{\frac{1}{2}})^{t}\boldsymbol{A}^{\frac{1}{2}}\right).\end{aligned}$$

Here, the log-majorization inequality follows from Theorem 2.16 by taking $X = B^{\frac{1}{2}}$, $Y = B^{\frac{1}{2}}A^{-1}B^{\frac{1}{2}}$, $p = t$ and $q = 1$.

(iv) Let $t \leq 0$. We can write

$$\begin{aligned}&\boldsymbol{\lambda}\left(\boldsymbol{A}^{\frac{1}{2}}(\boldsymbol{B}^{\frac{1}{2}}\boldsymbol{A}^{-1}\boldsymbol{B}^{\frac{1}{2}})^{t}\boldsymbol{A}^{\frac{1}{2}}\right) \\ &= \boldsymbol{\lambda}\left(\boldsymbol{A}^{\frac{1}{2}}\boldsymbol{B}^{-\frac{1}{2}}\boldsymbol{A}^{\frac{1}{2}}(\boldsymbol{A}^{-\frac{1}{2}}\boldsymbol{B}\boldsymbol{A}^{-\frac{1}{2}})^{t+1}\boldsymbol{A}^{\frac{1}{2}}\boldsymbol{B}^{-\frac{1}{2}}\boldsymbol{A}^{\frac{1}{2}}\right) \\ &= \boldsymbol{\lambda}\left((\boldsymbol{A}^{\frac{1}{2}}\boldsymbol{B}^{-\frac{1}{2}}\boldsymbol{A}^{\frac{1}{2}})^{2}(\boldsymbol{A}^{-\frac{1}{2}}\boldsymbol{B}\boldsymbol{A}^{-\frac{1}{2}})^{1-(-t)}\right) \\ &\prec_{log} \boldsymbol{\lambda}\left((\boldsymbol{A}^{\frac{1}{2}}\boldsymbol{B}^{-\frac{1}{2}}\boldsymbol{A}^{\frac{1}{2}})(\boldsymbol{A}^{-\frac{1}{2}}\boldsymbol{B}\boldsymbol{A}^{-\frac{1}{2}})^{1}(\boldsymbol{A}^{\frac{1}{2}}\boldsymbol{B}^{-\frac{1}{2}}\boldsymbol{A}^{\frac{1}{2}})(\boldsymbol{A}^{-\frac{1}{2}}\boldsymbol{B}\boldsymbol{A}^{-\frac{1}{2}})^{-(-t)}\right) \\ &= \boldsymbol{\lambda}\left(\boldsymbol{A}^{\frac{1}{2}}(\boldsymbol{A}^{-\frac{1}{2}}\boldsymbol{B}\boldsymbol{A}^{-\frac{1}{2}})^{t}\boldsymbol{A}^{\frac{1}{2}}\right).\end{aligned}$$

The first equality follows from Lemma 2.6, and the inequality follows from substituting X with $A^{\frac{1}{2}}B^{\frac{1}{2}}A^{\frac{1}{2}}$ and Y with $A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$ respectively, and taking $p = 1$ and $q = -t > 0$ in Theorem 2.16.

■

Remark 3.3. *It is worthy to mention here that the left inequality in Part (i) of Theorem 3.1 was first proved by J.S. Matharu and J.S. Aujla [49, Theorem 2.10], and the right inequality was first proved by M. Lin [40] in case* $t = \frac{1}{2}$.

Proposition 3.3. *Let* $X, Y \in \mathbb{M}_n$ *such that* $\boldsymbol{\sigma}(\boldsymbol{X}) \prec_{log} \boldsymbol{\sigma}(\boldsymbol{Y})$. *Then*

$$|||X||| \leq |||Y|||,$$

is valid for every unitarily invariant norm $|||\cdot|||$.

Proof. Let us recall the fact that $\boldsymbol{x} \prec_{log} \boldsymbol{y}$ implies $\boldsymbol{x} \prec_{w} \boldsymbol{y}$ for every $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{C}^n$ (see Theorem 1.7), which implies

$$\sum_{j=1}^{k}\sigma_j(X) \leq \sum_{j=1}^{k}\sigma_j(Y) \quad \text{for } k = 1, 2, \ldots, n.$$

This can be written equivalently as

$$|||X|||_{(k)} \leq |||Y|||_{(k)} \quad \text{for } k = 1, 2, \ldots, n.$$

Therefore, using the well-known Ky Fan Dominance Theorem (see Theorem 1.3) we obtain our final result. ■

As analogue of Theorem 3.1 and Proposition 3.3, we have the following unitarily invariant norms.

Corollary 3.1. *For any $n \times n$ positive definite matrices A and B. We have*

(i) If $0 \leq t \leq 1$ then $|||A\#_t B||| \leq |||B^{\frac{t}{2}} A^{1-t} B^{\frac{t}{2}}||| \leq |||A \natural_t B|||$.

(ii) If $-1 \leq t \leq 0$ then $|||A\#_t B||| \geq |||B^{\frac{t}{2}} A^{1-t} B^{\frac{t}{2}}||| \geq |||A \natural_t B|||$.

(iii) If $t \geq 0$ then $|||A\#_t B||| \leq |||A \natural_t B|||$.

(iv) If $t \leq 0$ then $|||A\#_t B||| \geq |||A \natural_t B|||$.

3.2. A Complement of a Golden-Thompson Type Inequality

The well-known Golden-Thompson trace inequality independently proved in 1965 by S. Golden [33] and C.J. Thompson [58] states that

$$Tr(e^{A+B}) \leq Tr(e^A e^B) \tag{3.7}$$

where A and B are two Hermitian matrices (see [7, 22] and references therein for more details on this trace inequality).

In 1993, F. Hiai and D. Petz [30] established a lower bound of (3.7) in terms of the geometric mean of matrices as follows

$$Tr\left(e^{pA}\#e^{pB}\right)^{\frac{2}{p}} \leq Tr\left(e^{A+B}\right) \qquad p \geq 0.$$

In fact, they have proved for all $p \geq 0$ and $0 \leq t \leq 1$

$$Tr\left(e^{pA}\#_t e^{pB}\right)^{\frac{1}{p}} \leq Tr\left(e^{(1-t)A+tB}\right).$$

In this subsection, we will complement this result for $p \geq 1$ as the following theorem shows.

Theorem 3.2. *Let A and B be two Hermitian matrices. Then for all $p \geq 1$ and $0 \leq t \leq 1$*

$$Tr\left(e^{(1-t)A+tB}\right) \leq \left(e^{(1-t)A} e^{tB}\right) \leq Tr\left(e^{pA} \natural_t e^{pB}\right)^{\frac{1}{p}}.$$

Proof. The first inequality is due to Golden-Thompson inequality. Now for the second inequality, replacing A and B with e^{pA} and e^{pB}, respectively in the right inequality of Part (i) of Theorem 3.1 gives

$$\boldsymbol{\lambda}\left((\boldsymbol{e^{pA}})^{\boldsymbol{1-t}}(\boldsymbol{e^{pB}})^{\boldsymbol{t}}\right) \prec_{log} \boldsymbol{\lambda}\left(\boldsymbol{e^{pA}\natural_t e^{pB}}\right).$$

So,

$$\boldsymbol{\lambda}\left(\boldsymbol{e^{(1-t)pA+tpB}}\right) \prec_{w} \boldsymbol{\lambda}\left(\boldsymbol{e^{pA}\natural_t e^{pB}}\right).$$

This implies

$$Tr\left(e^{(1-t)pA+tpB}\right) \leq Tr\left(e^{pA}\natural_t e^{pB}\right). \tag{3.8}$$

Now, if we substitute X and Y with $(e^{tB})^{\frac{1}{2}}$ and $e^{(1-t)A}$, respectively, and we take $t = p$ in Lemma 2.5, we have for $p \geq 1$

$$|||(e^{tB})^{\frac{1}{2}}e^{(1-t)A}(e^{tB})^{\frac{1}{2}}||| \leq ||| \left((e^{tB})^{\frac{p}{2}}(e^{(1-t)A})^{p}(e^{tB})^{\frac{p}{2}}\right)^{1/p} |||.$$

Hence,

$$Tr\left((e^{tB})^{\frac{1}{2}}e^{(1-t)A}(e^{tB})^{\frac{1}{2}}\right) \leq Tr\left((e^{tB})^{\frac{p}{2}}(e^{(1-t)A})^{p}(e^{tB})^{\frac{p}{2}}\right)^{1/p} \tag{3.9}$$

Therefore,

$$\begin{aligned} Tr\left(e^{(1-t)A}e^{tB}\right) &= Tr\left((e^{tB})^{\frac{1}{2}}e^{(1-t)A}(e^{tB})^{\frac{1}{2}}\right) \\ &\leq Tr\left((e^{tB})^{\frac{p}{2}}(e^{(1-t)A})^{p}(e^{tB})^{\frac{p}{2}}\right)^{1/p} \qquad \text{(using (3.9))} \\ &= Tr\left(e^{(1-t)pA}e^{tpB}\right)^{1/p} \\ &\leq Tr\left(e^{pA}\natural_t e^{pB}\right)^{1/p} \qquad \text{(using (3.8)).} \end{aligned}$$

■

We conclude this subsection with the next corollary.

Corollary 3.2. *Let A and B be two Hermitian matrices. Then for all $p \geq 1$*

$$Tr\left(e^{pA}\#e^{pB}\right)^{2/p} \leq Tr\left(e^{A+B}\right) \leq Tr\left(e^{pA}\natural e^{pB}\right)^{2/p}.$$

3.3. Some Applications to Rényi divergences

In this section, we give some new results related to the Rényi divergences. First we present some definitions that came from different types of Rényi divergences. In fact, these definitions are of special interest for density matrices. A matrix is called a density matrix if it is a positive

semi-definite matrix with its trace equal to one. See [29] and the references therein for more information in this topic.

Definition 3.3. *Let A, B be two positive definite matrices with B invertible, and let $\alpha, z > 0$ with $\alpha \neq 1$. We have*

(i) $P_\alpha(A,B) := B\#_\alpha A = B^{\frac{1}{2}}(B^{-\frac{1}{2}}AB^{-\frac{1}{2}})^\alpha B^{\frac{1}{2}}$.

(ii) $Q_{\alpha,z}(A,B) := (B^{\frac{1-\alpha}{2z}}A^{\frac{\alpha}{z}}B^{\frac{1-\alpha}{2z}})^z$.
In particular, the following two variable matrix functions are special versions of $Q_{\alpha,z}(A,B)$:

(a) $Q_\alpha(A,B) := Q_{\alpha,1}(A,B) = B^{\frac{1-\alpha}{2}}A^\alpha B^{\frac{1-\alpha}{2}}$.

(b) $\widetilde{Q}_\alpha(A,B) := Q_{\alpha,\alpha}(A,B) = (B^{\frac{1-\alpha}{2\alpha}}AB^{\frac{1-\alpha}{2\alpha}})^\alpha$.

Definition 3.4. *Let A, B be two positive definite matrices with B invertible and $A \neq 0$, and let $\alpha, z > 0$ with $\alpha \neq 1$. We have*

(i) The standard α-Rényi divergence is defined as

$$D_\alpha(A||B) := \frac{1}{\alpha - 1}\log Tr\ Q_\alpha(A,B).$$

(ii) The sandwiched α-Rényi divergence is defined as

$$\widetilde{D}_\alpha(A||B) := \frac{1}{\alpha - 1}\log Tr\ \widetilde{Q}_\alpha(A,B).$$

(iii) The α-z-Rényi divergence is a generalization of the previous two definitions and it is defined as

$$D_{\alpha,z}(A||B) := \frac{1}{\alpha - 1}\log Tr\ Q_{\alpha,z}(A,B).$$

(iv) The maximal α-Rényi divergence is defined as

$$\widehat{D}_\alpha(A||B) := \frac{1}{\alpha - 1}\log Tr\ P_\alpha(A,B).$$

The next theorem was recently proved by F. Hiai in [29]. It shows a norm relation between the two matrix functions $P_\alpha(A,B)$ and $Q_{\alpha,z}(A,B)$ for all $\alpha, z \geq 0$.

Theorem 3.3. *Let A, B be two positive semi-definite matrices with B invertible, and let $|||\cdot|||$ be any unitarily invariant norm.*

(1) Assume that $0 \leq \alpha \leq 1$. Then for every $z > 0$,

$$|||P_\alpha(A,B)||| \leq |||Q_{\alpha,z}(A,B)|||.$$

(2) Assume that $\alpha > 1$. *Then for every* $0 \leq z \leq \min\{\alpha/2, \alpha - 1\}$,

$$|||P_\alpha(A, B)||| \leq |||Q_{\alpha,z}(A, B)|||.$$

(3) Assume that $\alpha > 1$. *Then for every* $z \geq \max\{\alpha/2, \alpha - 1\}$,

$$|||Q_{\alpha,z}(A, B)) \leq |||P_\alpha(A, B)|||.$$

As consequence, he showed a relation between

$$D_\alpha(A||B),\ \widetilde{D}_\alpha(A||B) \text{ and } \widehat{D}_\alpha(A||B) \text{ for all } A, B > 0.$$

Theorem 3.4. *Let* A, B *be two positive semi-definite matrices with* B *invertible, and* $A \neq 0$. *We have*

(1) If $0 \leq \alpha \leq 2$ *and* $\alpha \neq 1$ *then*

$$\widetilde{D}_\alpha(A||B) \leq D_\alpha(A||B) \leq \widehat{D}_\alpha(A||B).$$

(2) If $\alpha \geq 2$ *then*

$$\widetilde{D}_\alpha(A||B) \leq D_\alpha(A||B) \leq \widehat{D}_\alpha(A||B).$$

Motivated by Hiai's results, we will complement these results for some $\alpha, z \geq 0$ using our new log-majorization relations. But, first we will introduce a new definition that is related to the mentioned types of Rényi divergences.

Definition 3.5. *Let* A, B *be two positive semi-definite matrices with* B *invertible and* $A \neq 0$, *and let* $\alpha, z > 0$ *with* $\alpha \neq 1$. *A matrix function* $\overline{D}_\alpha$ *is defined as*

$$\overline{D}_\alpha(A||B) := \frac{1}{\alpha - 1} \log Tr\ \overline{P}_\alpha(A, B)$$

where $\overline{P}_\alpha(A, B) := B\natural_\alpha A := B^{\frac{1}{2}}(A^{\frac{1}{2}}B^{-1}A^{\frac{1}{2}})^\alpha B^{\frac{1}{2}}$.

Theorem 3.5. *Let* A, B *be two positive definite matrices with* B *invertible. Then for any unitarily invariant norm we have*

(1) If $\alpha \geq 0$, *then* $|||P_\alpha(A, B)||| \leq |||\overline{P}_\alpha(A, B)|||$.

(2) If $0 \leq \alpha \leq 1$ *and* $z \geq \max\{\alpha, 1 - \alpha\}$, *then* $|||Q_{\alpha,z}(A, B)||| \leq |||\overline{P}_\alpha(A, B)|||$.

Proof. (1) As an application of Part (iii) of Corollary 3.1, substituting A with B, B with A and t with α respectively gives

$$|||B\#_\alpha A||| \leq |||B\natural_\alpha A|||.$$

(2) The proof will be divided into two cases here.

Case 1: Let $0 \leq \alpha \leq 1$ and $\max\{\alpha, 1-\alpha\} \leq z \leq 1$. As earlier, using the Ky Fan Dominance Theorem our claim is to show that

$$\boldsymbol{\sigma}\left(\boldsymbol{A}^{\frac{\alpha}{z}}\boldsymbol{B}^{\frac{1-\alpha}{z}}\boldsymbol{A}^{\frac{\alpha}{2z}}\right)^{z} \prec_{log} \boldsymbol{\sigma}\left(\boldsymbol{B}^{\frac{1}{2}}(\boldsymbol{A}^{\frac{1}{2}}\boldsymbol{B}^{-1}\boldsymbol{A}^{\frac{1}{2}})^{\alpha}\boldsymbol{B}^{\frac{1}{2}}\right).$$

As usual, by a standard anti-symmetric tensor product argument, it is enough to prove that for all $0 \leq \alpha \leq 1$ and $\max\{\alpha, 1-\alpha\} \leq z \leq 1$

$$\sigma_1\left(A^{\frac{\alpha}{z}}B^{\frac{1-\alpha}{z}}A^{\frac{\alpha}{2z}}\right)^{z} \leq \sigma_1\left(B^{\frac{1}{2}}(A^{\frac{1}{2}}B^{-1}A^{\frac{1}{2}})^{\alpha}B^{\frac{1}{2}}\right). \tag{3.10}$$

Notice that by the homogeneity on the degree of both expressions of (3.10), our claim is equivalent to showing that

$$\sigma_1\left(B^{\frac{1}{2}}(A^{\frac{1}{2}}B^{-1}A^{\frac{1}{2}})^{\alpha}B^{\frac{1}{2}}\right) = 1 \Rightarrow \sigma_1\left(A^{\frac{\alpha}{z}}B^{\frac{1-\alpha}{z}}A^{\frac{\alpha}{2z}}\right)^{z} \leq 1.$$

Without loss of generality, we assume that

$$\sigma_1\left(B^{\frac{1}{2}}(A^{\frac{1}{2}}B^{-1}A^{\frac{1}{2}})^{\alpha}B^{\frac{1}{2}}\right) = 1.$$

This implies

$$B \leq (A^{\frac{1}{2}}B^{-1}A^{\frac{1}{2}})^{-\alpha}.$$

Now using Löwner-Heinz inequality for $0 \leq \frac{1-\alpha}{z} \leq 1$, we have

$$B^{\frac{1-\alpha}{z}} \leq (A^{\frac{1}{2}}B^{-1}A^{\frac{1}{2}})^{-\alpha\cdot\frac{1-\alpha}{z}}. \tag{3.11}$$

So, we can write

$$\begin{aligned}
\sigma_1\left(A^{\frac{\alpha}{z}}B^{\frac{1-\alpha}{z}}A^{\frac{\alpha}{2z}}\right)^{z} &= \lambda_1\left(A^{\frac{\alpha}{z}}B^{\frac{1-\alpha}{z}}A^{\frac{\alpha}{2z}}\right)^{z} \\
&\leq \lambda_1\left(A^{\frac{\alpha}{z}}(A^{\frac{1}{2}}B^{-1}A^{\frac{1}{2}})^{-\alpha\cdot\frac{1-\alpha}{z}}A^{\frac{\alpha}{z}}\right)^{z} \\
&\qquad \text{(using (3.11))} \\
&\leq \lambda_1\left(A^{\frac{1}{2}}(A^{\frac{1}{2}}B^{-1}A^{\frac{1}{2}})^{\alpha-1}A^{\frac{1}{2}}\right)^{\alpha} \\
&\qquad \text{(using Lemma 2.5 for } 0 \leq p = \frac{\alpha}{z} \leq 1) \\
&\leq \lambda_1\left(A^{\frac{1}{2}}(A^{\frac{1}{2}}B^{-1}A^{\frac{1}{2}})^{-1}A^{\frac{1}{2}}(A^{\frac{1}{2}}B^{-1}A^{\frac{1}{2}})^{\alpha}\right)^{\alpha} \\
&\qquad \text{(using Theorem 2.16 for } p = \alpha \text{ and } q = 1) \\
&= \lambda_1\left(B^{\frac{1}{2}}(A^{\frac{1}{2}}B^{-1}A^{\frac{1}{2}})^{\alpha}B^{\frac{1}{2}}\right)^{\alpha} \\
&= (1)^{\alpha} \\
&= 1.
\end{aligned}$$

Therefore, we obtain the desired norm inequality for the studied case.

Case 2: Let $0 \leq \alpha \leq 1$ and $z \geq 1$. Notice that using Araki-Lieb-Thirring inequality (see Lemma 2.5 inequality (2.7)) for $0 \leq \frac{1}{z} \leq 1$, we have

$$|||Q_{\alpha,z}(A,B)||| = ||| \left(B^{\frac{1-\alpha}{2z}} A^{\frac{\alpha}{z}} B^{\frac{1-\alpha}{2z}} \right)^z ||| \leq |||B^{\frac{1-\alpha}{2}} A^{\alpha} B^{\frac{1-\alpha}{2}}|||. \tag{3.12}$$

On the other hand, observe that

$$|||B^{\frac{1-\alpha}{2}} A^{\alpha} B^{\frac{1-\alpha}{2}}||| \leq |||B \natural_{\alpha} A||| = |||\overline{P}_{\alpha}(A,B)|||. \tag{3.13}$$

Thus, the combination of (3.12) and (3.13) yields our final result.

■

Corollary 3.3. *Let A, B be two positive semi-definite matrices with B invertible. Then for any unitarily invariant norm we have*

(1) If $0 \leq \alpha \leq 1$ and $z \geq \max\{\alpha, 1-\alpha\}$, then

$$|||P_{\alpha}(A,B)||| \leq |||Q_{\alpha,z}(A,B)||| \leq |||\overline{P}_{\alpha}(A,B)|||.$$

(2) If $\alpha \geq 1$ and $z \geq \max\{\alpha/2, \alpha-1\}$, then

$$|||Q_{\alpha,z}(A,B)||| \leq |||P_{\alpha}(A,B)||| \leq |||\overline{P}_{\alpha}(A,B)|||.$$

Corollary 3.4. *Let A, B be two positive semi-definite matrices with B invertible and $A \neq 0$, and let $\alpha, z > 0$ with $\alpha \neq 1$. We have*

(1) If $0 \leq \alpha < 1$ and $z \geq \max\{\alpha, 1-\alpha\}$, then

$$\overline{D}_{\alpha}(A||B) \leq D_{\alpha,z}(A||B) \leq \widehat{D}_{\alpha}(A||B).$$

(2) If $\alpha > 1$ and $z \geq \max\{\alpha/2, \alpha-1\}$, then

$$D_{\alpha,z}(A||B) \leq \widehat{D}_{\alpha}(A||B) \leq \overline{D}_{\alpha}(A||B).$$

Theorem 3.6. *Let A, B be two positive semi-definite matrices with B invertible and $A \neq 0$. We have*

(1) If $0 \leq \alpha \leq \frac{1}{2}$ then

- $\overline{D}_{\alpha}(A||B) \leq D_{\alpha}(A||B) \leq \widehat{D}_{\alpha}(A||B)$, *and*
- $\widetilde{D}_{\alpha}(A||B) \leq D_{\alpha}(A||B) \leq \widehat{D}_{\alpha}(A||B)$.

(2) If $\frac{1}{2} \leq \alpha < 1$ then $\overline{D}_{\alpha}(A||B) \leq \widetilde{D}_{\alpha}(A||B) \leq D_{\alpha}(A||B) \leq \widehat{D}_{\alpha}(A||B)$.

(3) If $1 < \alpha \leq 2$ *then* $\widetilde{D}_\alpha(A||B) \leq D_\alpha(A||B) \leq \widehat{D}_\alpha(A||B) \leq \overline{D}_\alpha(A||B)$.

(4) If $\alpha \geq 2$ *then*

- $\widetilde{D}_\alpha(A||B) \leq \widehat{D}_\alpha(A||B) \leq \overline{D}_\alpha(A||B)$, *and*
- $\widetilde{D}_\alpha(A||B) \leq \widehat{D}_\alpha(A||B) \leq D_\alpha(A||B)$.

3.4. A complement of a norm inequality

In [12], R. Bhatia, Y. Lim and T. Yamazaki showed the following norm inequality for all $A, B \geq 0$,

$$||A + B + 2(A\sharp B)||_p \leq ||A + B + A^{\frac{1}{2}}B^{\frac{1}{2}} + B^{\frac{1}{2}}A^{\frac{1}{2}}||_p \qquad p = 1, 2, \infty. \tag{3.14}$$

Then they conjectured that it is true for every unitarily invariant norm, which was recently proved by T.H. Dinh, R. Dumitru and J.A. Franco [18] for the Schatten p-norms with $1 \leq p \leq \infty$.

First, we shall recall the following basic result (see Part (iii) of Proposition 1.6).

Proposition 3.4. *Let* $M := \begin{bmatrix} A & B \\ B & A \end{bmatrix}$ *where* A *and* B *are in* M_n. *Then, the eigenvalues of* M *are the union of the eigenvalues of* $A + B$ *and* $A - B$.

Theorem 3.7. *Let* A, B *and* C *be positive definite matrices of order* n *such that* $C \geq A + B$. *Then for all* $j = 1, 2, \ldots, n$,

$$s_j(C + A^{\frac{1}{2}}B^{\frac{1}{2}} + B^{\frac{1}{2}}A^{\frac{1}{2}}) \leq s_j(C + A\sharp B + A\natural B).$$

Proof. Let $M = \begin{bmatrix} A\sharp B & B^{\frac{1}{2}}A^{\frac{1}{2}} \\ A^{\frac{1}{2}}B^{\frac{1}{2}} & A\natural B \end{bmatrix} \geq 0$. Then $N = \begin{bmatrix} A\natural B & A^{\frac{1}{2}}B^{\frac{1}{2}} \\ B^{\frac{1}{2}}A^{\frac{1}{2}} & A\sharp B \end{bmatrix} \geq 0$.

Hence,

$$T = M + N + \begin{bmatrix} C & C \\ C & C \end{bmatrix} = \begin{bmatrix} C + A\sharp B + A\natural B & C + A^{\frac{1}{2}}B^{\frac{1}{2}} + B^{\frac{1}{2}}A^{\frac{1}{2}} \\ C + A^{\frac{1}{2}}B^{\frac{1}{2}} + B^{\frac{1}{2}}A^{\frac{1}{2}} & C + A\sharp B + A\natural B \end{bmatrix} \geq 0.$$

From the preceding proposition, we conclude that

$$C + A^{\frac{1}{2}}B^{\frac{1}{2}} + B^{\frac{1}{2}}A^{\frac{1}{2}} \leq C + A\sharp B + A\natural B \quad \text{and} \quad -(C + A^{\frac{1}{2}}B^{\frac{1}{2}} + B^{\frac{1}{2}}A^{\frac{1}{2}}) \leq C + A\sharp B + A\natural B.$$

Thus,

$$\lambda_j(C + A^{\frac{1}{2}}B^{\frac{1}{2}} + B^{\frac{1}{2}}A^{\frac{1}{2}}) \leq \lambda_j(C + A\sharp B + A\natural B).$$

Consequently, from the fact that

$$C + A^{\frac{1}{2}}B^{\frac{1}{2}} + B^{\frac{1}{2}}A^{\frac{1}{2}} \geq A + B + A^{\frac{1}{2}}B^{\frac{1}{2}} + B^{\frac{1}{2}}A^{\frac{1}{2}} = (A^{\frac{1}{2}} + B^{\frac{1}{2}})^2 \geq 0,$$

we get the desired result. ■

The following corollary can be considered as a complement of inequality (3.14) when replacing C by $A + B$ in Theorem 3.7.

Corollary 3.5. *Let A and B be two positive definite matrices of order n. Then for all unitarily invariant norms,*

$$|||A + B + A^{\frac{1}{2}}B^{\frac{1}{2}} + B^{\frac{1}{2}}A^{\frac{1}{2}}||| \leq |||A + B + A\#B + A\natural B|||.$$

Next, we present a further complement of the preceding corollary.

Theorem 3.8. *Let A and B be two positive definite matrices of order n. Then for $p = 1, 2$*

$$||A + B + A\#B + A\natural B||_p \leq ||A + B + 2(A\natural B)||_p.$$

Before presenting the proof, we shall consider the following log-majorization relations. The first lemma is a special case of Theorem 3.1.

Lemma 3.1. *Let A and B be two positive definite matrices. Then,*

$$\boldsymbol{\lambda}(\boldsymbol{A\#B}) \prec_{log} \boldsymbol{\lambda}(\boldsymbol{A\natural B}).$$

Lemma 3.2. *Let A and B be two positive definite matrices. Then,*

$$\boldsymbol{\lambda}(\boldsymbol{A(A\#B)}) \prec_{log} \boldsymbol{\lambda}(\boldsymbol{A(A\natural B)}).$$

Proof. Recalling first the inequality from Lemma 2.8 which can be stated for $A, B > 0$ as follows

$$\boldsymbol{\lambda}\left(\boldsymbol{A^{\frac{kt}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^tA^{\frac{kt}{2}}}\right) \prec_{log} \boldsymbol{\lambda}\left(\boldsymbol{A^{(k-1)t}B^t}\right) \qquad 0 \leq t \leq 1 \text{ and } k > 1$$

which is equivalent to

$$\boldsymbol{\lambda}\left(\boldsymbol{A^{\frac{k}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^tA^{\frac{k}{2}}}\right) \prec_{log} \boldsymbol{\lambda}\left(\boldsymbol{A^{k-t}B^t}\right) \qquad 0 \leq t \leq 1 \text{ and } k > t. \tag{3.15}$$

Recalling another one from Lemma 2.14 which is a complement of (3.15) and can be stated as

$$\boldsymbol{\lambda}\left(\boldsymbol{A^{k-t}B^t}\right) \prec_{log} \boldsymbol{\lambda}\left(\boldsymbol{A^{\frac{k}{2}}(B^{\frac{1}{2}}A^{-1}B^{\frac{1}{2}})^tA^{\frac{k}{2}}}\right) \qquad 0 \leq t \leq 1 \text{ and } k \geq 1. \tag{3.16}$$

Now, taking $k = 2$ and $t = \frac{1}{2}$ in both inequalities (3.15) and (3.16), we obtain

$$\boldsymbol{\lambda}\left(\boldsymbol{A(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\frac{1}{2}}A}\right) \prec_{log} \boldsymbol{\lambda}\left(\boldsymbol{A^{\frac{3}{2}}B^{\frac{1}{2}}}\right) \prec_{log} \boldsymbol{\lambda}\left(\boldsymbol{A(B^{\frac{1}{2}}A^{-1}B^{\frac{1}{2}})^{\frac{1}{2}}A}\right) \tag{3.17}$$

which gives the desired result. ■

Lemma 3.3. *Let A and B be two positive definite matrices. Then*

$$\boldsymbol{\lambda(B(A\#B))} \prec_{log} \boldsymbol{\lambda(B(A\natural B))}.$$

Proof. By Schur complement we know that

$$M = \begin{bmatrix} B^{\frac{1}{2}}A^{\frac{1}{2}}(B^{\frac{1}{2}}A^{-1}B^{\frac{1}{2}})^{\frac{1}{2}}A^{\frac{1}{2}}B^{\frac{1}{2}} & B^{\frac{1}{2}}A^{\frac{1}{2}}B \\ BA^{\frac{1}{2}}B^{\frac{1}{2}} & B(B^{-\frac{1}{2}}AB^{-\frac{1}{2}})^{\frac{1}{2}}B \end{bmatrix} \geq 0.$$

Next, by appealing to Lemma 2.1, we obtain

$$\begin{aligned} \lambda_1\left(B^{\frac{3}{2}}A^{\frac{1}{2}}\right)^2 &\leq \lambda_1\left(B^{\frac{1}{2}}A^{\frac{1}{2}}(B^{\frac{1}{2}}A^{-1}B^{\frac{1}{2}})^{\frac{1}{2}}A^{\frac{1}{2}}B^{\frac{1}{2}}\right)\cdot\lambda_1\left(B(B^{-\frac{1}{2}}AB^{-\frac{1}{2}})^{\frac{1}{2}}B\right) \\ &= \lambda_1\left(B^{\frac{1}{2}}(A\natural B)B^{\frac{1}{2}}\right)\cdot\lambda_1\left(B(B^{-\frac{1}{2}}AB^{-\frac{1}{2}})^{\frac{1}{2}}B\right) \\ &\leq \lambda_1(B(A\natural B))\cdot\lambda_1(B^{\frac{3}{2}}A^{\frac{1}{2}}) \qquad \text{(by using (3.17))} \end{aligned}$$

but this implies that

$$\lambda_1(B(A\#B)) \leq \lambda_1(B^{\frac{3}{2}}A^{\frac{1}{2}}) \leq \lambda_1(B(A\natural B)).$$

Finally, we obtain the result with a standard anti-symmetric tensor product argument. ■

Now, we are in position to prove Theorem 3.8.

proof of Theorem 3.8. Using Lemma 3.1, it is easy to see that

$$Tr(A+B+A\#B+A\natural B) \leq Tr(A+B+2(A\natural B)),$$

so that for $p=1$, the inequality is satisfied. Now, for $p=2$, it is enough to show that

$$\begin{aligned} &Tr(A\#B) \leq Tr(A\natural B);\\ &Tr(A(A\#B)) \leq Tr(A(A\natural B));\\ &Tr(B(A\#B)) \leq Tr(B(A\natural B)), \end{aligned}$$

which, in view of the previous lemmas, these are all true. This completes the proof. ■

Based on our work in this section, we conclude the paper with the following conjecture.

Conjecture 3.1. *Let A and B be two positive definite matrices of order n. Then for $1 \leq p \leq \infty$*

$$||A+B+A\#B+A\natural B||_p \leq ||A+B+2(A\natural B)||_p.$$

Finally, it is worthy to observe that the validation of the previous conjecture would imply that for all $1 \leq p \leq \infty$,

$$||A + B + 2(A\#B)||_p \leq ||A + B + A^{\frac{1}{2}}B^{\frac{1}{2}} + B^{\frac{1}{2}}A^{\frac{1}{2}}||_p \leq ||A + B + 2(A\natural B)||_p.$$

4

New Log-Majorization Inequalities

The main objective of this chapter is to present new log-majorization results related to some obtained inequalities by Zou, Hiai, Lin, Lemos and Soares. These majorization relations concerns eigenvalues and singular values of matrices. First, we introduce these results and the two conjectures posed by Lemos and Soares. We then present several new inequalities like generalizations of Zou's and Hiai-Lin majorization relations, as well as a reverse Lemos-Soares type inequality. In addition, we provide an example in which it shows that one of the conjectures is not valid in its current setting, we also present a further generalization of Lemos-Soares majorization relation precisely in all the cases where it has been proven valid. Lastly, we show several inequalities related to the other conjecture.

4.1. More majorization inequalities concerning the geometric mean

In 2016, L. Zou [63] proved some singular values inequalities. Among these inequalities, he established the following theorem.

Theorem 4.1. *Let A and B be two $n \times n$ positive semi-definite matrices. Then*

$$\prod_{j=1}^{k} \sigma_j \left(A^{\frac{1}{2}} (A \# B) B^{\frac{1}{2}} \right) \leq \prod_{j=1}^{k} \sigma_j (AB), \qquad k = 1, 2, \ldots, n. \tag{4.1}$$

Remark 4.1. *Inequality (4.1) was proved again by R. Lemos and G. Soares [38, Corollary 7.2]. It is worth to mention here that Corollary 7.2 in [38] deals with a more general result.*

Another similar inequality concerning the geometric mean was established by F. Hiai and M. Lin [28].

Theorem 4.2. *Let A and B be two $n \times n$ positive semi-definite matrices. Then for all $0 \leq t \leq 1$*

$$\prod_{j=1}^{k} \lambda_j \left[(A \#_t B)(A \#_{1-t} B) \right] \leq \prod_{j=1}^{k} \lambda_j (AB), \qquad k = 1, 2, \ldots, n. \tag{4.2}$$

In the same paper, the authors asked whether it is possible to replace the eigenvalues with singular values.

Conjecture 4.1. *Let A and B be two $n \times n$ positive semi-definite matrices. Then for all $0 \leq t \leq 1$*

$$\prod_{j=1}^{k} \sigma_j \left[(A\#_t B)(A\#_{1-t} B)\right] \leq \prod_{j=1}^{k} \sigma_j(AB), \qquad k = 1, 2, \ldots, n. \tag{4.3}$$

In 2018, R. Lemos and G. Soares [37] gave another proof of (4.1), and asked whether it is possible to find some generalization of it. More explicitly, their question gives rise to the following conjecture.

Conjecture 4.2. *Let A and B be two positive semi-definite matrices. Then*

$$\boldsymbol{\sigma}\left(\boldsymbol{A^t(A\#_t B)B^{1-t}}\right) \prec_{log} \boldsymbol{\sigma(AB)}, \qquad 0 \leq t \leq 1. \tag{4.4}$$

In the same paper they gave the following notation.

Definition 4.1. *Let A and B be two positive semi-definite matrices with A invertible. Then*

$$A\#_{s,t}B = A^{\frac{s}{2}}(B^{\frac{1}{2}}A^{-1}B^{\frac{1}{2}})^t A^{\frac{s}{2}} \quad \text{for} \quad s, t \in \mathbb{R}.$$

As usual it can be extended to any singular matrix A by continuity argument as follows

$$A\#_{s,t}B = \lim_{\epsilon \to 0^+} (A + \epsilon I_n)\#_{s,t}B.$$

They also generalized inequality (4.2) as the next theorem shows.

Theorem 4.3. *Let A and B be two $n \times n$ positive semi-definite matrices. Then for all $r, s \in \mathbb{R}$*

$$\boldsymbol{\lambda}\left[(\boldsymbol{A\#_{r,t}B})(\boldsymbol{A\#_{s,1-t}B})\right] \prec_{log} \boldsymbol{\lambda(A^{r+s-1}B)}, \qquad 0 \leq t \leq 1. \tag{4.5}$$

More generally, R. Lemos and G. Soares [37] conjectured the following inequality.

Conjecture 4.3. *Let A and B be two $n \times n$ positive semi-definite matrices. Then for all $r, s \in \mathbb{R}$ then*

$$\boldsymbol{\sigma}\left[(\boldsymbol{A\#_{r,t}B})(\boldsymbol{A\#_{s,1-t}B})\right] \prec_{log} \boldsymbol{\sigma}\left(\boldsymbol{A^{r+s-1}B}\right), \qquad 0 \leq t \leq 1. \tag{4.6}$$

Remark 4.2. *The case $r = s = 1$ was proved by F. Hiai and M. Lin precisely for $\frac{1}{4} \leq t \leq \frac{3}{4}$ in [28]. Later, R. Lemos and G. Soares [37] showed that Conjecture 4.3 is valid for $r, s \geq 0$ and $\frac{r}{r+s} \leq 2t \leq \frac{2r+s}{r+s}$. For the latter case, we shall prove further generalization of (4.6).*

The rest of the chapter is organized as follows. In this section, we present new generalizations of Theorem 4.1 and Theorem 4.2. In Section 4.2, we first disprove Conjecture 4.3 in its current setting by providing a counterexample. Then we give a further generalization of (4.3) as well as (4.6) in the cases where its has been shown to be valid. In Section 4.3, new results related to Conjecture 4.2 are established.

4.1.1. A Generalization of Zou's Majorization Relation

Proposition 4.1. *Let A and B be two positive definite matrices. Then*

$$\boldsymbol{\sigma}\left(\boldsymbol{A^t(A\#_tB)B^t}\right) \prec_{log} \boldsymbol{\lambda}\left(\boldsymbol{AB^{2t}}\right), \qquad 0 \leq t \leq \frac{1}{2}.$$

Proof. Consider the following matrix

$$M = \begin{bmatrix} B^tAB^t & B^t(A\#_tB)A^t \\ A^t(A\#_tB)B^t & A^{t+\frac{1}{2}}X^{2t}A^{t+\frac{1}{2}} \end{bmatrix}.$$

Then clearly M can be rewritten as

$$M = \begin{bmatrix} B^tA^{\frac{1}{2}} \\ A^{t+\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^t \end{bmatrix}\begin{bmatrix} B^tA^{\frac{1}{2}} \\ A^{t+\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^t \end{bmatrix}^* \geq 0.$$

Applying Lemma 2.1 on the 2×2 block matrix M gives

$$\begin{aligned}
\boldsymbol{\sigma}\left(\boldsymbol{A^t(A\#_tB)B^t}\right) &\prec_{wlog} \boldsymbol{\lambda}\left(\boldsymbol{B^tAB^t}\right)^{\frac{1}{2}} \circ \boldsymbol{\lambda}\left(\boldsymbol{A^{t+\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{2t}A^{t+\frac{1}{2}}}\right)^{\frac{1}{2}} \\
&= \boldsymbol{\lambda}\left(\boldsymbol{AB^{2t}}\right)^{\frac{1}{2}} \circ \boldsymbol{\lambda}\left(\boldsymbol{A^{\frac{2t+1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{2t}A^{\frac{2t+1}{2}}}\right)^{\frac{1}{2}} \\
&\prec_{log} \boldsymbol{\lambda}\left(\boldsymbol{AB^{2t}}\right)^{\frac{1}{2}} \circ \boldsymbol{\lambda}\left(\boldsymbol{AB^{2t}}\right)^{\frac{1}{2}} \\
&= \boldsymbol{\lambda}\left(\boldsymbol{AB^{2t}}\right).
\end{aligned}$$

Here, the second inequality follows from substituting $k = 2t+1$ and $0 \leq t = 2t \leq 1$ in (2.17).

Now taking into account that the determinants of $A^t(A\#_tB)B^t$ and AB^{2t} are equal we get the desired log-majorization inequality. ∎

As a consequence of Proposition 4.1, we have the following theorem which shows a generalization of Theorem 4.1 different than the inequality (4.4) in Conjecture 4.2.

Theorem 4.4. *Let A and B be two positive definite matrices. Then*

$$\boldsymbol{\sigma}\left(\boldsymbol{A^t(A\#_tB)B^t}\right) \prec_{log} \boldsymbol{\sigma}\left(\boldsymbol{AB^{2t}}\right), \qquad 0 \leq t \leq \frac{1}{2}. \tag{4.7}$$

Proof. Recalling the fact that $|\lambda(X)| \prec_{log} \sigma(X)$ is true for any $X \in \mathbb{M}_n$ gives for all $0 \leq t \leq \frac{1}{2}$

$$\boldsymbol{\sigma}\left(\boldsymbol{A^t(A\#_tB)B^t}\right) \prec_{log} \boldsymbol{\lambda}\left(\boldsymbol{AB^{2t}}\right) \prec_{log} \boldsymbol{\sigma}\left(\boldsymbol{AB^{2t}}\right).$$

∎

The next theorem is also a consequence of Proposition 4.1, which shows that replacing the singular values with eigenvalues in (4.7) is valid.

Theorem 4.5. *Let A and B be two positive definite matrices. Then*

$$|\boldsymbol{\lambda}\left(\boldsymbol{A^t(A\#_tB)B^t}\right)| \prec_{log} \boldsymbol{\lambda}\left(\boldsymbol{AB^{2t}}\right), \qquad 0 \leq t \leq \frac{1}{2}.$$

Proof. Similarly, we have for all $0 \leq t \leq \frac{1}{2}$

$$|\boldsymbol{\lambda}\left(\boldsymbol{A^t(A\#_tB)B^t}\right)| \prec_{log} \boldsymbol{\sigma}\left(\boldsymbol{A^t(A\#_tB)B^t}\right) \prec_{log} \boldsymbol{\lambda}\left(\boldsymbol{AB^{2t}}\right).$$

■

4.1.2. A Generalized Hiai-Lin Majorization Inequality

For the sake of simplicity, throughout this chapter we shall use the following notation

$$X := A^{-\frac{1}{2}}BA^{-\frac{1}{2}},$$

whenever A and B are are both fixed positive semi-definite matrices in $\mathbb{M}_n$.

Theorem 4.6. *Let A and B be two $n \times n$ positive semi-definite matrices. Then*

$$\boldsymbol{\lambda}\left[(\boldsymbol{A\#_{t_1}B})(\boldsymbol{A\#_{t_2}B})\right] \prec_{log} \boldsymbol{\lambda}\left(\boldsymbol{A^{2-(t_1+t_2)}B^{t_1+t_2}}\right), \qquad 0 \leq t_1, t_2 \leq 1.$$

Proof. We shall assume that A is a positive definite matrix, and the general case can then be deduced by a continuity argument. In addition, we shall divide the proof into two cases.

Case 1: If $0 \leq t_1 + t_2 \leq 1$, then observe that

$$\begin{aligned}
\boldsymbol{\lambda}\left[(\boldsymbol{A\#_{t_1}B})(\boldsymbol{A\#_{t_2}B})\right] &= \boldsymbol{\lambda}\left(\boldsymbol{A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{t_1}A(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{t_2}A^{\frac{1}{2}}}\right) \\
&= \boldsymbol{\lambda}\left(\boldsymbol{A(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{t_1}A(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{t_2}}\right) \\
&\prec_{log} \boldsymbol{\lambda}\left(\boldsymbol{A^2(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{t_1+t_2}}\right) \\
&= \boldsymbol{\lambda}\left(\boldsymbol{(A^{\frac{1}{2}})^2(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{t_1+t_2}(A^{\frac{1}{2}})^2}\right) \\
&\prec_{log} \boldsymbol{\lambda}\left(\boldsymbol{A^{2-(t_1+t_2)}B^{t_1+t_2}}\right).
\end{aligned}$$

Here, the first inequality follows from Lemma 2.9 by replacing X and Y with A and $A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$, respectively, and the last inequality follows from taking $0 \leq t = t_1 + t_2 \leq 1$ and $k = 2 \geq t_1 + t_2$ in (2.17).

Case 2: Let $0 \leq t_1, t_2 \leq 1$ such that $t_1 + t_2 \geq 1$. Our aim is to show that for all $0 \leq t_1, t_2 \leq 1$ such

that $t_1 + t_2 \geq 1$,

$$\lambda_1 \left[(A\#_{t_1}B)(A\#_{t_2}B)\right] \leq \lambda_1 \left(A^{2-(t_1+t_2)}B^{t_1+t_2}\right).$$

Notice that $\lambda_1 \left(A^{2-(t_1+t_2)}B^{t_1+t_2}\right) = \lambda_1 \left(A^{1-\frac{t_1+t_2}{2}}B^{t_1+t_2}A^{1-\frac{t_1+t_2}{2}}\right)$ and $\lambda_1 \left[(A\#_{t_1}B)(A\#_{t_2}B)\right] = \lambda_1 \left(X^{\frac{t_2}{2}}AX^{t_1}AX^{\frac{t_2}{2}}\right)$.

As in similar situations, proving our claim is now equivalent to showing that

$$A^{1-\frac{t_1+t_2}{2}}B^{t_1+t_2}A^{1-\frac{t_1+t_2}{2}} \leq I_n \Rightarrow X^{\frac{t_2}{2}}AX^{t_1}AX^{\frac{t_2}{2}} \leq I_n.$$

Without loss of generality, let us assume that $A^{1-\frac{t_1+t_2}{2}}B^{t_1+t_2}A^{1-\frac{t_1+t_2}{2}} \leq I_n$, then this implies that

$$B^{t_1+t_2} \leq A^{t_1+t_2-2}.$$

Now Appealing to Löwner-Heinz inequality for $0 \leq \frac{1}{t_1+t_2} \leq 1$ we obtain

$$B \leq A^{1-\frac{2}{t_1+t_2}}. \tag{4.8}$$

Next, multiplying both sides of (4.8) with $A^{-\frac{1}{2}} > 0$ yields

$$X \leq A^{-\frac{2}{t_1+t_2}},$$

and by appealing again to Löwner-Heinz inequality this time for $0 \leq t_1 \leq 1$ and $0 \leq t_2 \leq 1$, respectively, we obtain

$$X^{t_1} \leq A^{-\frac{2t_1}{t_1+t_2}}, \tag{4.9}$$

and

$$X^{t_2} \leq A^{-\frac{2t_2}{t_1+t_2}}. \tag{4.10}$$

Now, observe that

$$\begin{aligned} X^{\frac{t_2}{2}}AX^{t_1}AX^{\frac{t_2}{2}} &\leq X^{\frac{t_2}{2}}AA^{-\frac{2t_1}{t_1+t_2}}AX^{\frac{t_2}{2}} && \text{(by using (4.9))} \\ &= X^{\frac{t_2}{2}}A^{\frac{2t_2}{t_1+t_2}}X^{\frac{t_2}{2}} \\ &\leq I_n && \text{(by using (4.10)).} \end{aligned}$$

Therefore, for all $0 \leq t_1, t_2 \leq 1$ such that $t_1 + t_2 \geq 1$,

$$\lambda_1 \left[(A\#_{t_1}B)(A\#_{t_2}B)\right] \leq \lambda_1 \left(A^{2-(t_1+t_2)}B^{t_1+t_2}\right). \tag{4.11}$$

Using the anti-symmetric tensor product, we have for $1 \leq r \leq n$

$$\wedge^r \left(A^{2-(t_1+t_2)}B^{t_1+t_2}\right) = (\wedge^r A)^{2-(t_1+t_2)} (\wedge^r B)^{t_1+t_2},$$

and

$$\wedge^r [(A\#_{t_1}B)(A\#_{t_2}B)] = [(\wedge^r A)\#_{t_1}(\wedge^r B)] [(\wedge^r A)\#_{t_2}(\wedge^r B)].$$

Replacing A and B with $\wedge^r A$ and $\wedge^r B$, respectively, in (4.11) gives

$$\lambda_1 [(\wedge^r A\#_{t_1} \wedge^r B) (\wedge^r A\#_{t_2} \wedge^r B)] \leq \lambda_1 \left[(\wedge^r A)^{2-(t_1+t_2)} (\wedge^r B)^{t_1+t_2}\right].$$

This is equivalent to

$$\lambda_1 (\wedge^r [(A\#_{t_1}B)(A\#_{t_2}B)]) \leq \lambda_1 \left[\wedge^r \left(A^{2-(t_1+t_2)}B^{t_1+t_2}\right)\right].$$

So, for all $1 \leq r \leq n$, we have

$$\prod_{i=1}^{r} \lambda_i [(A\#_{t_1}B)(A\#_{t_2}B)] \leq \prod_{i=1}^{r} \lambda_i(A^{2-(t_1+t_2)}B^{t_1+t_2}).$$

In general, $\det [(A\#_{t_1}B)(A\#_{t_2}B)] = \det(A^{2-(t_1+t_2)}B^{t_1+t_2})$.

Thus we get the desired inequality for this case.

■

4.1.3. A Reverse Lemos-Soares Type Inequality

We will show that Theorem 4.3 is reversed when $t \geq 1$ or $t \leq 0$.

Theorem 4.7. *Let A and B be two $n \times n$ positive semi-definite matrices. Then for all $t \geq 1$ or $t \leq 0$*

$$\boldsymbol{\lambda}[(\boldsymbol{A}\#_{r,t}\boldsymbol{B})(\boldsymbol{A}\#_{s,1-t}\boldsymbol{B})] \succ_{log} \boldsymbol{\lambda}(\boldsymbol{A}^{r+s-1}\boldsymbol{B}), \qquad r, s \in \mathbb{R}.$$

Proof. First, observe that

$$\boldsymbol{\lambda}[(\boldsymbol{A}\#_{r,t}\boldsymbol{B})(\boldsymbol{A}\#_{s,1-t}\boldsymbol{B})] = \boldsymbol{\lambda}(\boldsymbol{A}^{\frac{r+s}{2}}\boldsymbol{X}^{t}\boldsymbol{A}^{\frac{r+s}{2}}\boldsymbol{X}^{1-t}).$$

Now, substituting X with $A^{\frac{r+s}{2}}$ and Y with X, respectively, in Theorem 2.16 to get

$$\boldsymbol{\lambda}(\boldsymbol{A}^{\frac{r+s}{2}}\boldsymbol{X}^{t}\boldsymbol{A}^{\frac{r+s}{2}}\boldsymbol{X}^{1-t}) \succ_{log} \boldsymbol{\lambda}(\boldsymbol{A}^{r+s}\boldsymbol{X}) = \boldsymbol{\lambda}(\boldsymbol{A}^{r+s-1}\boldsymbol{B}).$$

Thus we get the desired. ■

Definition 4.2. *Let A and B be two positive semi-definite matrices with A singular. The quantity $A\natural_{s,t}B$ is defined by*

$$A\natural_{s,t}B = A^{\frac{s}{2}}(B^{\frac{1}{2}}A^{-1}B^{\frac{1}{2}})^tA^{\frac{s}{2}} \quad \textit{for} \quad s,t\in\mathbb{R}.$$

As usual, this quantity can be extended to any singular matrix A by continuity argument.

It is interesting to investigate the relation between the positive quantity $A\natural_{s,t}B$ and the generalized geometric mean $A\#_{s,t}B$. We shall present two relations between the vectors

$$\boldsymbol{\lambda}\left[(\boldsymbol{A\natural_{r,t}B})(\boldsymbol{A\natural_{s,1-t}B})\right] \text{ and } \boldsymbol{\lambda}(\boldsymbol{A^{r+s-1}B}) \quad \text{for } r,s,t\in\mathbb{R}.$$

Theorem 4.8. *Let A and B be two $n\times n$ positive semi-definite matrices. Then for all $r,s\leq 0$*

$$\boldsymbol{\lambda}\left[(\boldsymbol{A\natural_{r,t}B})(\boldsymbol{A\natural_{s,1-t}B})\right] \prec_{log} \boldsymbol{\lambda}(\boldsymbol{A^{r+s-1}B}), \qquad 0\leq t\leq 1.$$

Proof. Let $r,s\leq 0$ (i.e $-r-s\geq 0$) and $0\leq t\leq 1$. We can write

$$\begin{aligned}
\boldsymbol{\lambda}\left[(\boldsymbol{A\natural_{r,t}B})(\boldsymbol{A\natural_{s,1-t}B})\right] &= \boldsymbol{\lambda}\left(\boldsymbol{A}^{\frac{r+s}{2}}(\boldsymbol{B}^{\frac{1}{2}}\boldsymbol{A}^{-1}\boldsymbol{B}^{\frac{1}{2}})^t\boldsymbol{A}^{\frac{r+s}{2}}(\boldsymbol{B}^{\frac{1}{2}}\boldsymbol{A}^{-1}\boldsymbol{B}^{\frac{1}{2}})^{1-t}\right)\\
&\prec_{log} \boldsymbol{\lambda}\left(\boldsymbol{A}^{r+s}(\boldsymbol{B}^{\frac{1}{2}}\boldsymbol{A}^{-1}\boldsymbol{B}^{\frac{1}{2}})^{t+1-t}\right)\\
&= \boldsymbol{\lambda}\left((\boldsymbol{A}^{-1})^{-r-s}\boldsymbol{B}^{\frac{1}{2}}(\boldsymbol{A}^{-1})^{1}\boldsymbol{B}^{\frac{1}{2}}\right)\\
&\prec_{log} \boldsymbol{\lambda}\left((\boldsymbol{A}^{-1})^{-r-s+1}\boldsymbol{B}\right)\\
&= \boldsymbol{\lambda}(\boldsymbol{A}^{r+s-1}\boldsymbol{B}).
\end{aligned}$$

Here, both inequalities follow from Lemma 2.9. The first inequality is by replacing X with $A^{\frac{r+s}{2}}$ and Y with $B^{\frac{1}{2}}A^{-1}B^{\frac{1}{2}}$, respectively, and the second inequality is by taking $X=B^{\frac{1}{2}}$ and $Y=A^{-1}$. ∎

Theorem 4.9. *Let A and B be two $n\times n$ positive semi-definite matrices. Then for all $r,s\geq 0$*

$$\boldsymbol{\lambda}\left[(\boldsymbol{A\natural_{r,t}B})(\boldsymbol{A\natural_{s,1-t}B})\right] \succ_{log} \boldsymbol{\lambda}(\boldsymbol{A^{r+s-1}B}), \qquad t\geq 1 \text{ or } t\leq 0.$$

Proof. Let $r,s\geq 0$ and $t\geq 1$ (i.e $t-1\geq 0$). Then

$$\begin{aligned}
\boldsymbol{\lambda}\left[(\boldsymbol{A\natural_{r,t}B})(\boldsymbol{A\natural_{s,1-t}B})\right] &= \boldsymbol{\lambda}\left(\boldsymbol{A}^{\frac{r+s}{2}}(\boldsymbol{B}^{\frac{1}{2}}\boldsymbol{A}^{-1}\boldsymbol{B}^{\frac{1}{2}})^t\boldsymbol{A}^{\frac{r+s}{2}}(\boldsymbol{B}^{\frac{1}{2}}\boldsymbol{A}^{-1}\boldsymbol{B}^{\frac{1}{2}})^{1-t}\right)\\
&= \boldsymbol{\lambda}\left(\boldsymbol{A}^{\frac{r+s}{2}}(\boldsymbol{B}^{\frac{1}{2}}\boldsymbol{A}^{-1}\boldsymbol{B}^{\frac{1}{2}})^t\boldsymbol{A}^{\frac{r+s}{2}}(\boldsymbol{B}^{\frac{1}{2}}\boldsymbol{A}^{-1}\boldsymbol{B}^{\frac{1}{2}})^{-(t-1)}\right)\\
&\succ_{log} \boldsymbol{\lambda}\left(\boldsymbol{A}^{r+s}\boldsymbol{B}^{\frac{1}{2}}\boldsymbol{A}^{-1}\boldsymbol{B}^{\frac{1}{2}}\right)\\
&\succ_{log} \boldsymbol{\lambda}\left(\boldsymbol{A}^{r+s-1}\boldsymbol{B}\right).
\end{aligned}$$

This time both inequalities follow from Theorem 2.16. The first inequality is by replacing X with $A^{\frac{r+s}{2}}$ and Y with $B^{\frac{1}{2}}A^{-1}B^{\frac{1}{2}}$, respectively, and the second inequality is by taking $X=B^{\frac{1}{2}}$

and $Y = A$.

The proof in case $t \leq 0$ is done with similar steps. ■

4.2. Results regarding Conjecture 4.3

This section deals first with presenting a counterexample to Conjecture 4.3. Then some generalization result related to Conjecture 4.3 is given. Finally, based on our results, we conclude with formulating an alternative conjecture.

4.2.1. Case when Conjecture 4.3 fails

The following example shows that Conjecture 4.3 is not valid in its current setting.

Example 4.1. *Consider two real numbers r and s such that $s-1 \leq 0$ (i.e $1-s \geq 0$) and $2r+s-1 \geq 0$. As a consequence of Theorem 2.16, we have for any $A > 0$ and $B \geq 0$*

$$\boldsymbol{\lambda}\left(\boldsymbol{A^{2r+s-1}BA^{-(1-s)}B}\right) \succ_{log} \boldsymbol{\lambda}(\boldsymbol{A^{2r+2s-2}B^2}). \tag{4.12}$$

Now substituting these fixed r and s in the left-hand side of (4.6) for $t = 0$ leads to

$$\begin{aligned}
\boldsymbol{\sigma}\left[(\boldsymbol{A}\#_{r,0}\boldsymbol{B})(\boldsymbol{A}\#_{s,1-0}\boldsymbol{B})\right]^2 &= \boldsymbol{\sigma}\left(\boldsymbol{A}^{\frac{r}{2}}(\boldsymbol{A}^{-\frac{1}{2}}\boldsymbol{B}\boldsymbol{A}^{-\frac{1}{2}})^{\boldsymbol{0}}\boldsymbol{A}^{\frac{r+s}{2}}(\boldsymbol{A}^{-\frac{1}{2}}\boldsymbol{B}\boldsymbol{A}^{-\frac{1}{2}})^{\boldsymbol{1-0}}\boldsymbol{A}^{\frac{s}{2}}\right)^2 \\
&= \boldsymbol{\sigma}\left(\boldsymbol{A}^{\frac{2r+s-1}{2}}\boldsymbol{B}\boldsymbol{A}^{\frac{s-1}{2}}\right)^2 \\
&= \boldsymbol{\lambda}\left(\boldsymbol{A^{2r+s-1}BA^{s-1}B}\right) \\
&= \boldsymbol{\lambda}\left(\boldsymbol{A^{2r+s-1}BA^{-(1-s)}B}\right) \\
&\succ_{log} \boldsymbol{\lambda}(\boldsymbol{A^{2r+2s-2}B^2}) \qquad \textit{(by using (4.12))} \\
&= \boldsymbol{\sigma}\left(\boldsymbol{A^{r+s-1}B}\right)^2.
\end{aligned}$$

Hence, there exist two real numbers r and s such that

$$\boldsymbol{\sigma}\left[(\boldsymbol{A}\#_{r,0}\boldsymbol{B})(\boldsymbol{A}\#_{s,1-0}\boldsymbol{B})\right] \succ_{log} \boldsymbol{\sigma}\left(\boldsymbol{A^{r+s-1}B}\right).$$

4.2.2. A further Generalization of Lemos-Soares Majorization Inequality

In order to prove our main result of this section, the next lemma is needed and it is known as the Furuta inequality with negative powers, whose proof can be found in [57].

Lemma 4.1. *Let X, Y be two invertible matrices satisfying $0 < Y \leq X$. Let $0 < p' \leq 1, 0 < q' \leq 1$ and $-1 \leq r' < 0$. Then, it holds that*

$$X^{\frac{p'+r'}{q'}} \geq \left(X^{\frac{r'}{2}} Y^{p'} X^{\frac{r'}{2}}\right)^{\frac{1}{q'}}$$

as long as the real numbers p', r' and q' satisfy

$$-r'(1-q') \leq p' \leq q' - r'(1-q'), \tag{4.13}$$

and one of the following two conditions:

$$\frac{1}{2} \leq q' \leq 1 \tag{4.14}$$

or

$$0 \leq q' \leq \frac{1}{2} \quad \text{and} \quad \frac{-r'(1-q') - q'}{1-2q'} \leq p' \leq \frac{-r'(1-q')}{1-2q'}. \tag{4.15}$$

Now we are in a position to present our first result which generalizes inequality (4.5) and gives a further generalization of inequality (4.6) precisely in all the cases where it has been proven valid.

Theorem 4.10. *Let A and B be two positive semi-definite matrices. Then, for all $1 \leq p \leq 2$, $r, s \in \mathbb{R}$ with similar signs and $t \in \mathbb{R}$ such that $\frac{rp-r}{(r+s)p} \leq t \leq \frac{rp+s}{(r+s)p}$, we have*

$$\boldsymbol{\lambda}[(\boldsymbol{A}\#_{\boldsymbol{r},\boldsymbol{t}}\boldsymbol{B})^{\boldsymbol{p}}(\boldsymbol{A}\#_{\boldsymbol{s},\boldsymbol{1-t}}\boldsymbol{B})^{\boldsymbol{p}}] \prec_{log} \boldsymbol{\lambda}(\boldsymbol{A}^{\boldsymbol{r+s-1}}\boldsymbol{B})^{\boldsymbol{p}}.$$

Proof. Without loss of generality, we shall assume that A and B are positive definite matrices as the general case can be then obtained by a continuity argument. Let $r, s \in \mathbb{R}$ with similar signs and $t \in \mathbb{R}$ such that $\frac{rp-r}{(r+s)p} \leq t \leq \frac{rp+s}{(r+s)p}$. Note that in order to finish the proof in this case, it is enough to show that for all $A, B > 0$ the following is true

$$A^{\frac{r+s-1}{2}} B A^{\frac{r+s-1}{2}} \leq I_n \Rightarrow (A\#_{s,1-t}B)^{\frac{p}{2}}(A\#_{r,t}B)^{p}(A\#_{s,1-t}B)^{\frac{p}{2}} \leq I_n. \tag{4.16}$$

Now assume that $A^{\frac{r+s-1}{2}} B A^{\frac{r+s-1}{2}} \leq I_n$. Then clearly, this is equivalent to

$$0 < A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \leq A^{-(r+s)}. \tag{4.17}$$

Applying Lemma 4.1 on (4.17) with $X = A^{-(r+s)}$ and $Y = A^{-\frac{1}{2}} B A^{-\frac{1}{2}}$, we then obtain

$$\left(A^{-(r+s)}\right)^{\frac{p'+r'}{q'}} \geq \left[\left(A^{-(r+s)}\right)^{\frac{r'}{2}} \left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}\right)^{p'} \left(A^{-(r+s)}\right)^{\frac{r'}{2}}\right]^{\frac{1}{q'}}. \tag{4.18}$$

Now taking $p' = t$, $r' = -\frac{r}{r+s}$ and $q' = \frac{1}{p}$ in (4.18) for which conditions (4.13) and (4.14) are

satisfied, we get

$$A^{rp-(r+s)pt} \geq \left(A^{\frac{r}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^t A^{\frac{r}{2}}\right)^p$$

which is the same as

$$(A\#_{r,t}B)^p \leq A^{rp-(r+s)pt}. \tag{4.19}$$

Again, replacing this time p' with $1-t$, r' with $-\frac{s}{r+s}$ and q' with $\frac{1}{p}$ in (4.18) where conditions (4.13) and (4.14) are also satisfied, yields

$$A^{(r+s)pt-rp} \geq \left(A^{\frac{s}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{1-t}A^{\frac{s}{2}}\right)^p = (A\#_{s,1-t}B)^p,$$

which is equivalent to

$$A^{rp-(r+s)pt} \leq (A\#_{s,1-t}B)^{-p}. \tag{4.20}$$

Therefore, we can write

$$\begin{aligned}
&(A\sharp_{s,1-t}B)^{\frac{p}{2}}(A\sharp_{r,t}B)^p(A\sharp_{s,1-t}B)^{\frac{p}{2}} \\
&\leq (A\sharp_{s,1-t}B)^{\frac{p}{2}}A^{rp-(r+s)pt}(A\sharp_{s,1-t}B)^{\frac{p}{2}} \qquad \text{(using (4.19))} \\
&\leq (A\sharp_{s,1-t}B)^{\frac{p}{2}}(A\sharp_{s,1-t}B)^{-p}(A\sharp_{s,1-t}B)^{\frac{p}{2}} \qquad \text{(using (4.20))} \\
&= I_n.
\end{aligned}$$

Let $a = \lambda_1(A^{r+s-1}B)^p > 0$. We observe that $\left(A^{\frac{r+s-1}{2}}BA^{\frac{r+s-1}{2}}\right)^p \leq aI_n$, which is equivalent to

$$\left(\frac{1}{a^{\frac{1}{2p(r+s-1)}}}A\right)^{\frac{r+s-1}{2}}\left(\frac{1}{a^{\frac{1}{2p}}}B\right)\left(\frac{1}{a^{\frac{1}{2p(r+s-1)}}}A\right)^{\frac{r+s-1}{2}} \leq I_n.$$

For simplicity, we will replace $\left(\frac{1}{a^{\frac{1}{2p(r+s-1)}}}A\right)$ and $\left(\frac{1}{a^{\frac{1}{2p}}}B\right)$ with A' and B', respectively, then

$$(A')^{\frac{r+s-1}{2}}(B')(A')^{\frac{r+s-1}{2}} \leq I_n$$

which using (4.16) implies that

$$(A'\#_{s,1-t}B')^{\frac{p}{2}}(A'\#_{r,t}B')^p(A'\#_{s,1-t}B')^{\frac{p}{2}} \leq I_n.$$

So,

$$\frac{1}{a}\left[(A\#_{s,1-t}B)^{\frac{p}{2}}(A\#_{r,t}B)^p(A\#_{s,1-t}B)^{\frac{p}{2}}\right] \leq I_n.$$

Hence,

$$(A\#_{s,1-t}B)^{\frac{p}{2}}(A\#_{r,t}B)^p(A\#_{s,1-t}B)^{\frac{p}{2}} \leq \lambda_1(A^{r+s-1}B)^p I_n.$$

Thus, for all $1 \leq p \leq 2$ and for all $r, s \in \mathbb{R}$ with similar signs and such that $\frac{rp-r}{(r+s)p} \leq t \leq \frac{rp+s}{(r+s)p}$, we have that

$$\lambda_1\left[(A\#_{r,t}B)^p(A\#_{s,1-t}B)^p\right] \leq \lambda_1\left(A^{r+s-1}B\right)^p.$$

Finally, as usual by an anti-symmetric tensor product argument the proof is achieved. ■

Remark 4.3. *It is worthy to note here that as a result of Theorem 4.10 and Araki-Lieb-Thirring inequality (see Lemma 2.5), we can conclude that for all* $1 \leq p \leq 2$ *and* $\frac{rp-r}{(r+s)p} \leq t \leq \frac{rp+s}{(r+s)p}$, *the following inequality holds*

$$\boldsymbol{\lambda}\left[(\boldsymbol{A}\#_{r,t}\boldsymbol{B})^p(\boldsymbol{A}\#_{s,1-t}\boldsymbol{B})^p\right] \prec_{log} \boldsymbol{\lambda}\left(\boldsymbol{A}^{r+s-1}\boldsymbol{B}\right)^p \prec_{log} \boldsymbol{\lambda}\left(\boldsymbol{A}^{p(r+s-1)}\boldsymbol{B}^p\right). \tag{4.21}$$

Clearly, the particular cases $p = 1$ *and* $p = 2$ *in (4.21) correspond to R. Lemos and G. Soares results.*

The following corollary is a consequence of Theorem 4.10 for the case $r = s = 1$.

Corollary 4.1. *Let* A *and* B *be two positive semi-definite matrices. Then for all* $1 \leq p \leq 2$

$$\boldsymbol{\lambda}[(\boldsymbol{A}\#_t\boldsymbol{B})^p(\boldsymbol{A}\#_{1-t}\boldsymbol{B})^p] \prec_{log} \boldsymbol{\lambda}(\boldsymbol{AB})^p, \qquad \frac{p-1}{2p} \leq t \leq \frac{p+1}{2p}.$$

The proof of the next theorem can be done in a similar fashion as that of Theorem 4.10 with only minor changes; namely by making use of condition (4.15) instead of (4.14) in Lemma 4.1.

Theorem 4.11. *Let* A *and* B *be two positive semi-definite matrices. Then for all* $p \geq 2$, *it holds that*

$$\boldsymbol{\lambda}[(\boldsymbol{A}\#_t\boldsymbol{B})^p(\boldsymbol{A}\#_{1-t}\boldsymbol{B})^p] \prec_{log} \boldsymbol{\lambda}(\boldsymbol{AB})^p, \qquad \frac{p-1}{2p} \leq t \leq \frac{p+1}{2p}.$$

Taking into account Example 4.1 and in view of Theorem 4.10 as well as Theorem 4.11, we conclude this section with the following more general conjecture.

Conjecture 4.4. *If* $A, B > 0$, $0 \leq t \leq 1$, $p \geq 1$ *and* ($r, s \geq 1$ *or* $r, s \leq 0$), *then*

$$\boldsymbol{\lambda}[(\boldsymbol{A}\#_{r,t}\boldsymbol{B})^p(\boldsymbol{A}\#_{s,1-t}\boldsymbol{B})^p] \prec_{log} \boldsymbol{\lambda}(\boldsymbol{A}^{p(r+s-1)}\boldsymbol{B}^p).$$

4.3. Results related to Conjecture 4.2

This section deals with some results related to Conjecture 4.2. In particular, we shall show that replacing the singular values with eigenvalues in (4.4) is true. However, Conjecture 4.2 is still an open problem.

Theorem 4.12. *Let* A *and* B *be two positive semi-definite matrices. Then*

$$\left|\boldsymbol{\lambda}\left(\boldsymbol{A}^t(\boldsymbol{A}\#_t\boldsymbol{B})\boldsymbol{B}^{1-t}\right)\right| \prec_{log} \boldsymbol{\lambda}(\boldsymbol{AB}), \qquad 0 \leq t \leq 1.$$

Proof. As usual, we shall assume that A is a positive definite matrix, and the general case can then be deduced by a continuity argument. In addition, we shall divide the proof into two cases.

Case 1: If $0 \leq t \leq \frac{1}{2}$, then let

$$M = \begin{bmatrix} B^{\frac{1}{2}}AB^{\frac{1}{2}} & B^{\frac{1}{2}}A^t(A\#_tB)B^{\frac{1}{2}-t} \\ B^{\frac{1}{2}-t}(A\#_tB)A^tB^{\frac{1}{2}} & B^{\frac{1}{2}-t}(A\#_tB)A^{-\frac{1}{2}}A^{2t}A^{-\frac{1}{2}}(A\#_tB)B^{\frac{1}{2}-t} \end{bmatrix},$$

which can be rewritten in terms of $X = A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$ as

$$M = \begin{bmatrix} B^{\frac{1}{2}}AB^{\frac{1}{2}} & B^{\frac{1}{2}}A^{\frac{1}{2}}A^tX^tA^{\frac{1}{2}}B^{\frac{1}{2}-t} \\ B^{\frac{1}{2}-t}A^{\frac{1}{2}}X^tA^tA^{\frac{1}{2}}B^{\frac{1}{2}} & B^{\frac{1}{2}-t}A^{\frac{1}{2}}X^tA^{2t}X^tA^{\frac{1}{2}}B^{\frac{1}{2}-t} \end{bmatrix}.$$

Moreover, M can be also rewritten as $ZZ^* \geq 0$, where

$$Z = \begin{bmatrix} B^{\frac{1}{2}}A^{\frac{1}{2}} \\ B^{\frac{1}{2}-t}A^{\frac{1}{2}}X^tA^t \end{bmatrix},$$

so that $M \geq 0$.

Now, applying Lemma 2.1 on the matrix M gives

$$\left|\boldsymbol{\lambda}\left(\boldsymbol{A^t(A\#_tB)B^{1-t}}\right)\right| \prec_{wlog} \boldsymbol{\lambda}\left(\boldsymbol{B^{\frac{1}{2}}AB^{\frac{1}{2}}}\right)^{\frac{1}{2}} \circ \boldsymbol{\lambda}\left(\boldsymbol{B^{\frac{1}{2}-t}A^{\frac{1}{2}}X^tA^{2t}X^tA^{\frac{1}{2}}B^{\frac{1}{2}-t}}\right)^{\frac{1}{2}}. \tag{4.22}$$

Next, our goal is to prove that for all $0 \leq t \leq \frac{1}{2}$,

$$\boldsymbol{\lambda}\left(\boldsymbol{B^{\frac{1}{2}-t}A^{\frac{1}{2}}X^tA^{2t}X^tA^{\frac{1}{2}}B^{\frac{1}{2}-t}}\right) \prec_{wlog} \boldsymbol{\lambda}\left(\boldsymbol{AB}\right),$$

for which, by using an anti-symmetric tensor product argument, is sufficient to show that for all $0 \leq t \leq \frac{1}{2}$,

$$\lambda_1\left(B^{\frac{1}{2}-t}A^{\frac{1}{2}}X^tA^{2t}X^tA^{\frac{1}{2}}B^{\frac{1}{2}-t}\right) \leq \lambda_1\left(AB\right).$$

Without loss of generality, we shall assume that $\lambda_1(AB) = 1$. Obviously, proving our claim is now equivalent to showing that

$$\lambda_1\left(B^{\frac{1}{2}-t}A^{\frac{1}{2}}X^tA^{2t}X^tA^{\frac{1}{2}}B^{\frac{1}{2}-t}\right) \leq 1.$$

As $\lambda_1(AB) = 1$, then this implies that

$$A^{\frac{1}{2}}BA^{\frac{1}{2}} \leq I_n$$

which in turn gives that

$$B \leq A^{-1}. \tag{4.23}$$

Applying Löwner-Heinz inequality on (4.23) for $0 \leq 1-2t \leq 1$ we obtain

$$B^{1-2t} \leq A^{2t-1}. \tag{4.24}$$

Next, multiplying both sides of (4.23) with $A^{-\frac{1}{2}} > 0$ yields

$$X \leq A^{-2},$$

and by appealing again to Löwner-Heinz inequality this time for $0 \leq t \leq \frac{1}{2}$, we obtain

$$X^t \leq A^{-2t} \tag{4.25}$$

Now, we can write

$$\begin{aligned}
\lambda_1\left(B^{\frac{1}{2}-t}A^{\frac{1}{2}}X^tA^{2t}X^tA^{\frac{1}{2}}B^{\frac{1}{2}-t}\right) &= \lambda_1\left(A^tX^tA^{\frac{1}{2}}B^{1-2t}A^{\frac{1}{2}}X^tA^t\right) \\
&\leq \lambda_1\left(A^tX^tA^{2t}X^tA^t\right) \qquad \text{(using (4.24))} \\
&= \lambda_1\left(A^{2t}X^tA^{2t}X^t\right) \\
&= \lambda_1(A^{2t}X^t)^2 \\
&= \lambda_1(A^tX^tA^t)^2 \\
&\leq \lambda_1(I_n)^2 \qquad \text{(using (4.25))} \\
&= 1.
\end{aligned}$$

Therefore,

$$\boldsymbol{\lambda}\left(\boldsymbol{B^{\frac{1}{2}-t}A^{\frac{1}{2}}X^tA^{2t}X^tA^{\frac{1}{2}}B^{\frac{1}{2}-t}}\right) \prec_{log} \boldsymbol{\lambda}\left(\boldsymbol{AB}\right), \qquad 0 \leq t \leq \frac{1}{2}. \tag{4.26}$$

On the other hand, using (4.22) and (4.26) gives

$$\begin{aligned}
\left|\boldsymbol{\lambda}\left(\boldsymbol{A^t(A\sharp_tB)B^{1-t}}\right)\right| &\prec_{wlog} \boldsymbol{\lambda}\left(\boldsymbol{B^{\frac{1}{2}}AB^{\frac{1}{2}}}\right)^{\frac{1}{2}} \circ \boldsymbol{\lambda}\left(\boldsymbol{B^{\frac{1}{2}-t}A^{\frac{1}{2}}X^tA^{2t}X^tA^{\frac{1}{2}}B^{\frac{1}{2}-t}}\right)^{\frac{1}{2}} \\
&\prec_{wlog} \boldsymbol{\lambda(B^{\frac{1}{2}}AB^{\frac{1}{2}})^{\frac{1}{2}} \circ \lambda(AB)^{\frac{1}{2}}} \\
&= \boldsymbol{\lambda(AB)}.
\end{aligned}$$

Making use of the fact that the determinants of the matrices on the left and on the right are equal, we finally arrive at $\left|\boldsymbol{\lambda}\left(\boldsymbol{A^t(A\#_tB)B^{1-t}}\right)\right| \prec_{log} \boldsymbol{\lambda(AB)}$.

Case 2: Let $\frac{1}{2} \leq t \leq 1$. Replacing A with B, B with A and t with $1-t$ in first case, yields

$$\left|\boldsymbol{\lambda}\left(\boldsymbol{B^{1-t}(B\#_{1-t}A)A^{1-(1-t)}}\right)\right| \prec_{log} \boldsymbol{\lambda(BA)} \qquad 0 \leq 1-t \leq \frac{1}{2}.$$

By noting that $A\#_t B = B\#_{1-t}A$ and that $|\boldsymbol{\lambda}(\boldsymbol{X})| = |\boldsymbol{\lambda}(\boldsymbol{X^*})|$ for all $X \in \mathbb{M}_n$, we thus have that for all $\frac{1}{2} \leq t \leq 1$,

$$\begin{aligned}\left|\boldsymbol{\lambda}\left(\boldsymbol{A^t(A\#_tB)B^{1-t}}\right)\right| &= \left|\boldsymbol{\lambda}\left(\boldsymbol{B^{1-t}(B\#_{1-t}A)A^t}\right)\right| \\ &\prec_{log} \boldsymbol{\lambda(BA)} \\ &= \boldsymbol{\lambda(AB)}.\end{aligned}$$

■

Our next goal is to show a result that strengthens Conjecture 4.2.

Theorem 4.13. *Let A and B be two positive definite matrices. Then,*

1. $\boldsymbol{\sigma}\left(\boldsymbol{A^t(A\#_tB)B^{1-t}}\right) \prec_{log} \boldsymbol{\sigma}\left(\boldsymbol{A^{\frac{3}{2}}BA^{-\frac{1}{2}}}\right)$ *for* $\frac{1}{2} \leq t \leq 1$,
2. $\boldsymbol{\sigma}\left(\boldsymbol{A^t(A\#_tB)B^{1-t}}\right) \prec_{log} \boldsymbol{\sigma}\left(\boldsymbol{B^{\frac{3}{2}}AB^{-\frac{1}{2}}}\right)$ *for* $0 \leq t \leq \frac{1}{2}$.

Proof. 1. Let $\frac{1}{2} \leq t \leq 1$. Without loss of generality, we shall again assume that

$$\lambda_1\left(A^{\frac{3}{2}}BA^{-1}BA^{\frac{3}{2}}\right) = 1.$$

As earlier, our claim now amounts to proving that

$$\lambda_1\left(A^t(A\#_tB)B^{2(1-t)}(A\#_tB)A^t\right) \leq 1.$$

Noting that $\lambda_1\left(A^{\frac{3}{2}}BA^{-1}BA^{\frac{3}{2}}\right) = \lambda_1\left(A^2X^2A^2\right) = 1$, which clearly implies that $X^2 \leq A^{-4}$, then by appealing to Löwner-Heinz inequality we obtain

$$X^{2t} \leq A^{-4t} \tag{4.27}$$

and

$$B^{2(1-t)} \leq A^{-2(1-t)}. \tag{4.28}$$

Now, we can write

$$\begin{aligned}&\lambda_1\left(A^t(A\#_tB)B^{2(1-t)}(A\#_tB)A^t\right) \\ &\leq \lambda_1\left(A^{t+\frac{1}{2}}X^tA^{2t-1}X^tA^{t+\frac{1}{2}}\right) && \text{(using (4.28))} \\ &= \lambda_1\left(A^{2t+1}X^tA^{2t-1}X^t\right) \\ &\leq \lambda_1\left(A^{4t}X^{2t}\right) && \text{(using Lemma 2.9)} \\ &= \lambda_1\left(A^{4t}X^{2t}\right) \\ &\leq \lambda_1(I_n) && \text{(using (4.27))}.\end{aligned}$$

Hence,

$$\lambda_1\left(A^t(A\#_tB)B^{2(1-t)}(A\#_tB)A^t\right) \leq \lambda_1\left(A^3BA^{-1}B\right), \qquad \frac{1}{2}\leq t\leq 1. \tag{4.29}$$

Applying square root on both sides of Inequality (4.29) gives

$$\sigma_1\left(A^t(A\#_tB)B^{1-t}\right) \leq \sigma_1\left(A^{\frac{3}{2}}BA^{-\frac{1}{2}}\right), \qquad \frac{1}{2}\leq t\leq 1.$$

Therefore, with an anti-symmetric tensor product argument the proof can be achieved.

2. If $0\leq t\leq \frac{1}{2}$, then as was done earlier, replacing A with B, B with A and t with $1-t$ in the preceding case gives

$$\boldsymbol{\sigma\left(B^{1-t}(B\#_{1-t}A)A^t\right) \prec_{log} \sigma\left(B^{\frac{3}{2}}AB^{-\frac{1}{2}}\right).}$$

Finally, to complete the proof it is enough to notice that $B\#_{1-t}A = A\#_tB$.

■

Before we close this section, it is worthy to note that in view of Theorem 2.16, we conclude that for all $A, B > 0$,

$$\boldsymbol{\sigma(AB) \prec_{log} \sigma(A^{\frac{3}{2}}BA^{-\frac{1}{2}})} \quad \text{and} \quad \boldsymbol{\sigma(AB) \prec_{log} \sigma(B^{\frac{3}{2}}AB^{-\frac{1}{2}}).}$$

5

Singular Value Inequalities

The study of eigenvalue and singular value inequalities is of central importance in matrix theory. The Cauchy's interlacing theorem [8, page 52] is one of the well known theorems in which it derived some basic eigenvalue and singular value inequalities. Another useful theorem is the Weyl's Monotonicity theorem [8, Page 63]. There is also inequalities shown by R. Bhatia, F. Kittaneh and Drury (see [13, 14, 15, 21]). In 2016, M. Lin [43] conjectured that if

$$\begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \in \mathbb{M}_2(\mathbb{M}_n)$$

is a positive semi-definite matrix then

$$\sigma_j(\Phi(B)) \leq \sigma_j(\Phi(A) \sharp \Phi(C)), \ j = 1, 2, \ldots, n,$$

where $\Phi(X) = X + Tr(X)I_n$. In this chapter, we confirm this conjecture in more general setting in case A and B commute. Some related inequalities are also investigated.

5.1. Preliminaries

In 2014, Lin proved in [44] that the linear map

$$\Phi(X) = X + Tr(X)I_n$$

is completely PPT (see the definition below). This result made us curious about other maps. As we know, the definition of completely positive maps is given for linear maps, so the question is about the use of the definitions (i), (ii) and (iii) (see the definition below) for non-linear maps.

Definition 5.1. *A linear map $\Phi : \mathbb{M}_n \to \mathbb{M}_k$ is said to be*

(i) positive if $A \geq 0$ then $\Phi(A) \geq 0$.

(ii) m-positive if

$$\begin{bmatrix} A_{11} & A_{12} & \dots & A_{1m} \\ A_{21} & A_{22} & \dots & A_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ A_{m1} & A_{m2} & \dots & A_{mm} \end{bmatrix} \geq 0 \text{ then}$$

$$\begin{bmatrix} \Phi(A_{11}) & \Phi(A_{12}) & \dots & \Phi(A_{1m}) \\ \Phi(A_{21}) & \Phi(A_{22}) & \dots & \Phi(A_{2m}) \\ \vdots & \vdots & \ddots & \vdots \\ \Phi(A_{m1}) & \Phi(A_{m2}) & \dots & \Phi(A_{mm}) \end{bmatrix} \geq 0.$$

(iii) completely positive if it is m-positive for any integer $m \geq 1$.

More explanations and results about completely positive maps can be found in [9].

Definition 5.2. *Let $A \in \mathbb{M}_k(\mathbb{M}_n)$ such that*

$$A = \begin{bmatrix} A_{11} & A_{12} & \dots & A_{1k} \\ A_{21} & A_{22} & \dots & A_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ A_{k1} & A_{k2} & \dots & A_{kk} \end{bmatrix}.$$

The partial transpose of A is defined as

$$A^\tau = \begin{bmatrix} A_{11} & A_{21} & \dots & A_{k1} \\ A_{12} & A_{22} & \dots & A_{k2} \\ \vdots & \vdots & \ddots & \vdots \\ A_{1k} & A_{2k} & \dots & A_{kk} \end{bmatrix}.$$

Remark 5.1. *Notice that the partial transpose of a block matrix is different than the usual transpose which is defined as*

$$A^T = \begin{bmatrix} A_{11}^T & A_{21}^T & \dots & A_{k1}^T \\ A_{12}^T & A_{22}^T & \dots & A_{k2}^T \\ \vdots & \vdots & \ddots & \vdots \\ A_{1k}^T & A_{2k}^T & \dots & A_{kk}^T \end{bmatrix}.$$

Definition 5.3. *Let $A \in \mathbb{M}_m(\mathbb{M}_n)$ be a positive semi-definite matrix. The matrix A has a positive partial transpose if $A^\tau \in \mathbb{M}_m(\mathbb{M}_n)$ are positive semi-definite. For simplicity we say that A has a PPT.*

Definition 5.4. *A map $\Phi : \mathbb{M}_n \to \mathbb{M}_k$ is said to be completely PPT if Φ is completely positive and for all $A \in \mathbb{M}_m(\mathbb{M}_n)$ positive semi-definite matrix ($m \geq 1$), A has a PPT.*

Now before presenting our essential maps Ψ_f we need to recall the definition of the Liebian functions defined by E.H. Lieb [39] (or functions of class $\mathbb{L}$ defined in [8, page 268]).

Definition 5.5. *A continuous complex-valued function f on the space of matrices is said to be a Liebian function if f satisfies these two conditions:*

1. *If $A \geq B$ then $f(A) \geq f(B)$.*
2. *$|f(A^*B)|^2 \leq f(A^*A)f(B^*B)$ for all A, B.*

An equivalent result about these Liebian functions is the following.

Proposition 5.1. *A continuous complex-valued function f is a Liebian function if and only if f satisfies the following two conditions*

1. *$f(A) \geq 0$ for all $A \geq 0$,*
2. *$|f(B)|^2 \leq f(A)f(C)$ for all $A, B, C \in \mathbb{M}_n$ such that $\begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \geq 0$.*

Example 5.1. *The following well-known functions are examples of Liebian functions:*

(i) $f(X) = \det(X)$.

(ii) $f(X) = Tr(X)$.

(iii) For all $1 \leq k \leq n$, $f_{(k)}(X) = \prod_{j=1}^{k} \sigma_j(X)$.

In the sequel, we define a generalized map of the linear map $\Phi(X)$.

Definition 5.6. *Let $X \in \mathbb{M}_n$ and let f be any Liebian function. The map $\Psi_f(X)$ is defined as*

$$\Psi_f(X) = X + f(X)I_n.$$

Before we continue to the next section, we need the following lemmas which are essential in our analysis. The first lemma is proved by T. Ando in [1], second one is the necessary condition of Part (iii) of proposition 1.6, third lemma is obtained from the well known Weyl's monotonicity theorem [8, page 63], the fourth is a theorem proved by Y. Tao in [59], and the last lemma is Hiroshima's theorem.

Lemma 5.1. *Let $j = 1, 2$. We have*

$$\text{If } \begin{bmatrix} A_j & B \\ B^* & C_j \end{bmatrix} \geq 0 \text{ then } \begin{bmatrix} A_1 \# A_2 & B \\ B^* & C_1 \# C_2 \end{bmatrix} \geq 0.$$

Lemma 5.2. *Let H and K be two Hermitian matrices of the same size such that*

$$\begin{bmatrix} H & K \\ K & H \end{bmatrix} \geq 0.$$

Then $H \geq K$ and $H \geq -K$.

Lemma 5.3. *If $A \geq B$, then $\lambda_j(A) \geq \lambda_j(B) \qquad j = 1, 2, \ldots, n.$*

Lemma 5.4. *Let $M = \begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \in \mathbb{M}_2(\mathbb{M}_n)$ be a positive semi-definite matrix. Then*

$$2\sigma_j(B) \leq \sigma_j(M) \qquad j = 1, 2, \ldots, n.$$

Lemma 5.5. ***(Hiroshima's Theorem)*** *Let $M = \begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \in \mathbb{M}_2(\mathbb{M}_n)$ be a positive semi-definite matrix. If M has a PPT, that is, $M = \begin{bmatrix} A & B^* \\ B & C \end{bmatrix}$ is also positive semi-definite matrix, then*

$$|||M||| \leq |||A + C|||$$

is valid for any unitarily invariant norm.

Observing that Lemma 5.4 and Lemma 5.5 give the following proposition.

Proposition 5.2. *Suppose that $M = \begin{bmatrix} A & B \\ B^* & C \end{bmatrix}$ has a PPT. Then*

$$2|||B||| \leq |||A + C|||.$$

5.2. Results Related to Geometric Mean and Singular Values

M. Lin introduced in his work [43] the following conjecture.

Conjecture 5.1. *If $M = \begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \in \mathbb{M}_2(\mathbb{M}_n)$ is positive semi-definite matrix then*

$$\sigma_j(\Phi(B)) \leq \sigma_j(\Phi(A)\#\Phi(C)) \qquad j = 1, 2, \ldots, n$$

where $\Phi(X) = X + Tr(X)I_n$.

It has been shown that the weaker case of the conjecture. It is a special case of [1, Theorem 3.3] and [45, Theorem 4.3].

Theorem 5.1. *If $\begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \in \mathbb{M}_2(\mathbb{M}_n)$ is positive semi-definite matrix. Then*

$$|||\Phi(B)||| \leq |||\Phi(A)\#\Phi(C)|||.$$

5.2.1. Solution of Conjecture 5.1 when $AB = BA$

One of the main results in this section is the following theorem which is a generalization of the Conjecture 5.1, when A and B commute. Noted that some interesting results for 2×2 block matrices with the commutative condition are given in [16].

Theorem 5.2. *If* $\begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \in \mathbb{M}_2(\mathbb{M}_n)$ *is positive semi-definite matrix with* A *and* B *commute (or* B *and* C *commute). Then*

$$\sigma_j(\Psi_f(B)) \leq \sigma_j(\Psi_f(A) \# \Psi_f(C)), \qquad j = 1, 2, \ldots, n. \tag{5.1}$$

Proof. Without loss of generality we can assume that $\Psi_f(B)$ is invertible, the general case is due to continuity argument. Using the polar decomposition of $\Psi_f(B)$ gives

$$\Psi_f(B) = U|\Psi_f(B)| \quad \text{for some unitary matrix} \quad U.$$

Notice that A and B commute and since $A = A^*$ then A and $(\Psi_f(B))^*$ commute, so we get A and $(\Psi_f(B))^*(\Psi_f(B))$ commute.

By consequently, A and $|\Psi_f(B)| = [(\Psi_f(B))^*(\Psi_f(B))]^{1/2}$ commute. Finally,

$$AU^*\Psi_f(B) = U^*\Psi_f(B)A = U^*A\Psi_f(B)$$

and thus $AU^* = U^*A$ (that is $\Psi_f(A)$ and U^* commute).

Now as f is a Liebian function then

$$K = \begin{bmatrix} f(A)I_n & f(B)I_n \\ \overline{f(B)}I_n & f(C)I_n \end{bmatrix} \geq 0.$$

Therefore,

$$\begin{bmatrix} \Psi_f(A) & \Psi_f(B) \\ (\Psi_f(B))^* & \Psi_f(C) \end{bmatrix} \geq 0.$$

Consider the following decomposition

$$\begin{bmatrix} U^* & 0 \\ 0 & I_n \end{bmatrix} \begin{bmatrix} \Psi_f(A) & \Psi_f(B) \\ (\Psi_f(B))^* & \Psi_f(C) \end{bmatrix} \begin{bmatrix} U & 0 \\ 0 & I_n \end{bmatrix} = \begin{bmatrix} U^*\Psi_f(A)U & |\Psi_f(B)| \\ |\Psi_f(B)| & \Psi_f(C) \end{bmatrix}$$
$$= \begin{bmatrix} \Psi_f(A) & |\Psi_f(B)| \\ |\Psi_f(B)| & \Psi_f(C) \end{bmatrix}.$$

This decomposition implies that

$$T_f = \begin{bmatrix} \Psi_f(A) & |\Psi_f(B)| \\ |\Psi_f(B)| & \Psi_f(C) \end{bmatrix} \geq 0 \text{ and } R_f = \begin{bmatrix} \Psi_f(C) & |\Psi_f(B)| \\ |\Psi_f(B)| & \Psi_f(A) \end{bmatrix} \geq 0.$$

Therefore, by applying Lemma 5.1 on T_f and R_f, we obtain

$$\begin{bmatrix} \Psi_f(A)\#\Psi_f(C) & |\Psi_f(B)| \\ |\Psi_f(B)| & \Psi_f(A)\#\Psi_f(C) \end{bmatrix} \geq 0.$$

Thus, by Lemma 5.2, $\Psi_f(A)\#\Psi_f(C) \geq |\Psi_f(B)|$, and $\Psi_f(A)\#\Psi_f(C) \geq -|\Psi_f(B)|$.

Consequently, from Lemma 5.3, we get for all $j = 1, 2, \ldots, n$

$$\sigma_j(\Psi_f(A)\#\Psi_f(C)) = \lambda_j(\Psi_f(A)\#\Psi_f(C)) \geq \lambda_j(|\Psi_f(B)|) = \sigma_j(\Psi_f(B)).$$

The inequality (5.1) is still true when B and C commute with a similar proof. ■

M. Lin also proved in [43] the following result.

Theorem 5.3. *Let* $M = \begin{bmatrix} A & B \\ B^* & C \end{bmatrix}$ *be a positive semi-definite matrix. If* M *has a PPT then*

$$\sigma_j(\Phi(B)) \leq \sigma_j(\Phi(A\#C)) \qquad j = 1, 2, \ldots, n. \tag{5.2}$$

The next theorem shows that the inequality (5.2) also holds true when A and B commute.

Theorem 5.4. *If* $\begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \geq 0$ *such that* A *and* B *commute (or* B *and* C *commute), then*

$$\sigma_j(\Psi_f(|B|)) \leq \sigma_j(\Psi_f(A\#C)), \qquad j = 1, 2, \ldots, n.$$

In particular, we have $\sigma_j(\Phi(B)) \leq \sigma_j(\Phi(A\#C))$, *for* $j = 1, 2, \ldots, n$, $(\Psi_{Tr} = \Phi)$.

Proof. As we have mentioned before, $T_f \geq 0$ and $R_f \geq 0$ when A and B commute. Then for $f = 0$ (i.e $\Psi_0(X) = X$ for all $X \in \mathbb{M}_n$) we have

$$T_0 = \begin{bmatrix} A & |B| \\ |B| & C \end{bmatrix} \geq 0 \text{ and } R_0 = \begin{bmatrix} C & |B| \\ |B| & A \end{bmatrix} \geq 0.$$

Lemma 5.1 gives

$$\begin{bmatrix} A\#C & |B| \\ |B| & A\#C \end{bmatrix} \geq 0 \text{ and hence } \begin{bmatrix} \Psi_f(A\#C) & \Psi_f(|B|) \\ \Psi_f(|B|) & \Psi_f(A\#C) \end{bmatrix} \geq 0.$$

By applying Lemma 5.2, we get

$$\Psi_f(A\#C) \geq \Psi_f(|B|)$$

which gives

$$\sigma_j(\Psi_f(|B|)) \leq \sigma_j(\Psi_f(A\#C)) \qquad j = 1, 2, \ldots, n.$$

If $f(X) = Tr(X)$, that is, $\Psi_{Tr} = \Phi$, we have

$$\begin{aligned}
\sigma_j(\Phi(|B|)) &= \sigma_j\left(|B| + Tr(|B|)I_n\right) \\
&= \lambda_j\left(|B| + Tr(|B|)I_n\right) \\
&= \lambda_j(|B|) + Tr(|B|) \\
&= \sigma_j(B) + Tr(|B|) \\
&\geq \sigma_j(B) + |Tr(B)| \\
&\geq \sigma_j\left(B + Tr(B)I_n\right) \\
&= \sigma_j(\Phi(B)).
\end{aligned}$$

Consequently,

$$\sigma_j(\Phi(B)) \leq \sigma_j(\Phi(|B|)) \leq \sigma_j(\Phi(A\#C)), \qquad j = 1, 2, \ldots, n.$$

■

Remark 5.2. *If f is a Liebian function such that $f(|X|) \geq |f(X)|$, then*

$$\sigma_j(\Psi_f(B)) \leq \sigma_j(\Psi_f(|B|)) \leq \sigma_j(\Psi_f(A\#C)).$$

M. Lin also proved in [43] the following interesting theorem.

Theorem 5.5. *If* $\begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \in \mathbb{M}_2(\mathbb{M}_n)$ *is positive semi-definite matrix then*

$$\sigma_j(\Phi(A) + \Phi(C)) \geq 2\sigma_j(\Phi(B)) \qquad j = 1, 2, \ldots, n.$$

As we can see, Conjecture 5.1 implies the above theorem. So, Theorem 5.4 can give us a generalization of the above result when A and B commute (or C and B commute), but first we need the following basic result.

Proposition 5.3. *Let X and Y be two positive semi-definite matrices. Then*

$$X\#Y \leq \frac{X+Y}{2}.$$

Proof. In Chapter 1, Proposition 1.8 implies that for all $X \geq 0$ and $Y \geq 0$ we have

$$\begin{bmatrix} X & X\#Y \\ X\#Y & Y \end{bmatrix} \geq 0,$$

which gives

$$\begin{bmatrix} Y & X\#Y \\ X\#Y & X \end{bmatrix} \geq 0.$$

Hence

$$\begin{bmatrix} X+Y & 2(X\#Y) \\ 2(X\#Y) & X+Y \end{bmatrix} \geq 0.$$

Thus, using Lemma 5.2 gives the desired result. ∎

Theorem 5.6. *Let* $\begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \in \mathbb{M}_2(\mathbb{M}_n)$ *be a positive semi-definite matrix such that* A *and* B *commute. Then for all Liebian functions* f *we have*

$$\sigma_j(\Psi_f(A) + \Psi_f(C)) \geq 2\sigma_j(\Psi_f(B)), \qquad j = 1, 2, \ldots, n. \tag{5.3}$$

Proof. By appealing to the previous proposition and since A and B commute, we obtain

$$|\Psi_f(B)| \leq \Psi_f(A)\#\Psi_f(C) \leq \frac{\Psi_f(A) + \Psi_f(C)}{2}.$$

Hence

$$\sigma_j(\Psi_f(B)) \leq \sigma_j(\Psi_f(A)\#\Psi_f(C)) \leq \sigma_j\left(\frac{\Psi_f(A) + \Psi_f(C)}{2}\right).$$

∎

This means inequality (5.1) is stronger than (5.3).

5.2.2. Theorem 5.6 fails when $AB \neq BA$

The next example shows that (5.3) is not necessary true without the commutative condition of A and B.

Example 5.2. *The determinant is a Liebian function. Take*

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \; C = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \textit{and} \quad B = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Observe that $A \geq 0$, $C \geq 0$ *and*

$$\begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \geq 0.$$

We can easily check that

$$\det(A) = \det(B) = \det(C) = 0$$

which gives

$$\sigma_1(A + C + [\det(A) + \det(C)]I_n) = 1 < 2\sigma_1(B + \det(B)I_n) = 2.$$

The above example also shows that for all $\begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \geq 0$,

$$\sigma_j(A + C + \det(A + C)I_n) \geq 2\sigma_j(B + \det(B)I_n)$$

does not hold true, however it is true in case A and B commute or (B and C commute). In general we have for all $X \geq 0$ and $Y \geq 0$ (see [61, Theorem 7.7])

$$\det(X + Y) \geq \det(X) + \det(Y).$$

So,

$$A + C + \det(A + C)I_n \geq A + C + [\det(A) + \det(C)]I_n.$$

The last inequality gives, for all $1 \leq j \leq n$,

$$\sigma_j(A + C + \det(A + C)I_n) \geq \sigma_j(A + C + [\det(A) + \det(C)]I_n) \geq 2\sigma_j(B + \det(B)I_n).$$

Motivated by M. Lin's results in [44, 46], one can ask if the defined map Ψ_f has a PPT. Let us start the investigation with the map $\Psi_{\det}(X) = X + \det(X)I_n$. Indeed, the considered map is completely positive (see [32, page 445, P25]). Unfortunately, we can see from Example 5.2 that the following norm inequality

$$|||A + C + [\det(A) + \det(C)]I_n||| \geq 2|||B + \det(B)I_n|||$$

is not always true, hence from Proposition 5.2, it is impossible for the 2×2 block matrix

$$\begin{bmatrix} A + \det(A)I_n & B + \det(B)I_n \\ B^* + \overline{\det(B)}I_n & C + \det(C)I_n \end{bmatrix}$$

which is positive semi-definite to have a PPT. In other words, the map

$$\Psi_{\det}(X) = X + \det(X)I_n$$

is not completely PPT in general.

The inequality (5.3) also failed for the following known Liebian functions:

$$f(X) = |||X|||_{op} \text{ and } f_{(2)}(X) = \prod_{i=1}^{2} \sigma_i(X)$$

Example 5.3. *(1) If we choose A, B and C as the previous example and $f(X) = |||X|||_{op}$, we find*

$$|||A|||_{op} = |||C|||_{op} = |||B|||_{op} = 1 \quad \textit{and}$$
$$\sigma_1(A + C + 2I_n) = 3 < 2\sigma_1(B + I_n) = \sqrt{5} + 1.$$

(2) If $f_{(2)}(X) = \prod_{i=1}^{2} \sigma_i(X)$, and $A, B, C \geq 0$ such that

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, C = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \textit{ and } B = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

We have

$$\sigma_1(A) = \sigma_1(B) = \sigma_1(C) = 1 \quad \textit{and} \quad \sigma_2(A) = \sigma_2(B) = \sigma_2(C) = 1$$

which gives

$$f_{(2)}(A) = f_{(2)}(B) = f_{(2)}(C) = 1 \quad \textit{and}$$
$$\sigma_1(A + C + 2I_n) = 3 < 2\sigma_1(B + I_n) = \sqrt{5} + 1.$$

With a similar analysis made for the map $\Psi_{det}(X) = X + det(X)I_n$, we conclude that the two maps

$$\Psi_1(X) = X + |||X|||_{op} I_n \quad \text{and} \quad \Psi_2(X) = X + f_{(2)}(X) I_n$$

are not completely PPT as well.

5.2.3. More singular value inequalities

We end this chapter with the next result.

Theorem 5.7. *Let $M = \begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \in \mathbb{M}_2(\mathbb{M}_n)$ be a positive semi-definite matrix such that A and B commute, and let f_i, for all $1 \leq i \leq t$, be Liebian functions. Then, for all $j = 1, 2, \ldots, n$, we have:*

(i) $\sigma_j \left(A + C + \left[\prod_{i=1}^{t} f_i(A) + \prod_{i=1}^{t} f_i(C) \right] I_n \right) \geq 2\sigma_j \left(B + \prod_{i=1}^{t} f_i(B) I_n \right).$

(ii) $\sigma_j\left(A+C+\left[\sum_{i=1}^{t} f_i(A)+\sum_{i=1}^{t} f_i(C)\right] I_n\right) \geq 2\sigma_j\left(B+\sum_{i=1}^{t} f_i(B) I_n\right).$

(iii) $\sigma_j(A+C+f(A)A+f(C)C) \geq 2\sigma_j(B+f(B)B).$

Proof. Let $h=\prod_{i=1}^{t} f_i$ and $g=\sum_{i=1}^{t} f_i$. It is obvious that h is a Liebian function. To show that g is a Liebian function, we need to prove that $g(X) \geq 0$, for $X \geq 0$, and

$$|g(Y)|^2 \leq g(X)g(Z) \text{ for all } X,Y,Z \in \mathbb{M}_n \text{ such that } \begin{bmatrix} X & Y \\ Y^* & Z \end{bmatrix} \geq 0.$$

The first condition is clear, now observe that

$$H=\begin{bmatrix} g(X) & g(Y) \\ \overline{g(Y)} & g(Z) \end{bmatrix} \geq 0.$$

Therefore $\det(H) \geq 0$ which implies the second condition and then g is a Liebian function. Applying Theorem 5.6 for $f=h$ and $f=g$ gives inequalities (i) and (ii) respectively. The positivity of the matrix M gives

$$L=\begin{bmatrix} f(A)J_n & f(B)J_n \\ \overline{f(B)}J_n & f(C)J_n \end{bmatrix} \geq 0,$$

where $J_n=\begin{bmatrix} 1 & \dots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \dots & 1 \end{bmatrix}$ is the $n \times n$ matrix whose all entries are ones.

Recall that the Hadamard product of two positive semi-definite matrices X and Y, denoted by $X \circ Y$, is a positive semi-definite matrix. Therefore

$$M+M\circ L=\begin{bmatrix} A+f(A)A & B+f(B)B \\ B^*+\overline{f(B)}B^* & C+f(C)C \end{bmatrix} \geq 0.$$

Notice that if A and B commute, so is $A+f(A)A$ and $B+f(B)B$. And with similar steps as before, we get

$$\begin{bmatrix} A+f(A)A & |B+f(B)B| \\ |B+f(B)B| & C+f(C)C \end{bmatrix} \geq 0.$$

Therefore

$$\begin{bmatrix} A+C+f(A)A+f(C)C & 2|B+f(B)B| \\ 2|B+f(B)B| & C+A+f(C)C+f(A)A \end{bmatrix} \geq 0,$$

which gives $A+C+f(A)A+f(C)C \geq 2|B+f(B)B|$ and so the third inequality is proved. ∎

Open Problems

Through out the thesis, the following problems were introduced. We add them all here.

Problem 1. ***(Chapter 2)*** *Let A and B be two positive semi-definite matrices. Then, for all $k \geq 0$*

$$\det(A^k + |AB|^p) \geq \det(A^2 + A^p B^p) \qquad 0 \leq p \leq 2.$$

Problem 2. ***(Chapter 2)*** *Let A and B be two positive semi-definite matrices. Then, for all $k,\ p \geq 0$*

$$\det(A^k + |AB|^p) \geq \det(A^k + |BA|^p).$$

Problem 3. ***(Chapter 2)*** *Let A and B be two invertible Hermitian matrices. Then, for all $p \leq 0$ and $k \geq 0$*

$$\det(A^k + |AB|^p) \leq \det(A^k + |BA|^p).$$

Problem 4. ***(Chapter 2)*** *Let A and B be two $n \times n$ complex matrices such that A is positive definite and B is positive semi-definite matrix. Then, for all $k \leq 0$*

$$\det(A^k + |AB|^p) \leq \det(A^k + A^p B^p) \qquad 0 \leq p \leq 2.$$

Problem 5. ***(Chapter 3)*** *Let A and B be two positive definite matrices of order n. Then for $1 \leq p \leq \infty$*

$$||A + B + A\#B + A\natural B||_p \leq ||A + B + 2(A\natural B)||_p.$$

Problem 6. ***(Chapter 4)*** *If $A, B > 0$, $0 \leq t \leq 1$, $p \geq 1$ and ($r, s \geq 1$ or $r, s \leq 0$), then*

$$\boldsymbol{\lambda}[(\boldsymbol{A}\#_{\boldsymbol{r},\boldsymbol{t}}\boldsymbol{B})^{\boldsymbol{p}}(\boldsymbol{A}\#_{\boldsymbol{s},\boldsymbol{1-t}}\boldsymbol{B})^{\boldsymbol{p}}] \prec_{log} \boldsymbol{\lambda}(\boldsymbol{A}^{\boldsymbol{p(r+s-1)}}\boldsymbol{B}^{\boldsymbol{p}}).$$

- Problem 1 is proved for all $k \geq 2$ and $0 \leq p \leq 2$. It remains unknown for all $0 \leq k \leq 2$ and $0 \leq p \leq 2$.
- Problem 2 is proved for $k = 2$ and $p \geq 0$.
- Problem 3 is proved for all $k = 2$ and $p \leq 0$.
- Problem 4 is proved for all $k \leq -2$ and $0 \leq p \leq 2$. It remains unknown for all $-2 \leq k \leq 0$ and $0 \leq p \leq 2$.

- Problem 5 is only shown for $p = 1$ and $p = 2$.
- Problem 6 is proved for all $1 \leq p \leq 2$ and $r, s \in \mathbb{R}$ with similar signs and $t \in \mathbb{R}$ such that $\frac{rp-r}{(r+s)p} \leq t \leq \frac{rp+s}{(r+s)p}$. It is also valid for $r = s = 1$ and $p \geq 1$.

List of Articles Related to the Thesis

- H. Abbas, M. Ghabries, Some Generalizations and Complements of Determinantal Inequalities, Math. Inequal. Appl. 23, 1 (2020), 169–176.
- H. Abbas, M. Ghabries, B. Mourad, New determinantal inequalities concerning Hermitian and positive semi-definite matrices, Oper. Matrices 15 (1) (2021) 105–116.
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Titre : Contributions aux inégalités matricielles et quelques applications

Mots clés : matrice semi-définie positive, log-Majorisation, inégalité déterminentielle, valeurs propres, moyenne géométrique

Résumé : La théorie matricielle est à l'étude depuis longtemps, elle a été un outil fondamental dans les disciplines mathématiques présentant des problèmes intéressants et stimulants. Dans cette thèse, nous nous concentrons sur une classe de matrices qui est l'ensemble de toutes les matrices semi-définies positives. Les sujets principaux sont les inégalités déterminantales, les inégalités aux valeurs propres et aux valeurs singulières et la moyenne géométrique de deux matrices définies positives. Ces concepts apparaissent dans de nombreux domaines de recherche et jouent un rôle décisif dans la théorie de l'information, la mécanique quantique et d'autres domaines mathématiques. Le point de départ de notre travail est les deux inégalités déterminantes non confirmées suivantes introduites par M. Lin (2017) pour toute matrice semi-définie positive A et B du même ordre :

$\det(A^2 + |AB|^p) \geq \det(A^2 + A^p B^p)$ pour $0 \leq p \leq 2$;

$\det(A^2 + |AB|^p) \geq \det(A^2 + |BA|^p)$ pour $p \geq 0$.

Nous avons montré que la seconde inégalité est vraie pour un plus grand ensemble de matrices ; Matrices hermitiennes. Nous avons également démontré un résultat plus général de la première inégalité énoncé comme suit :

$\det(A^k + |AB|^p) \geq \det(A^k + A^p B^p)$ pour $k \geq 2, 0 \leq p \leq 2$

L'idée principale de la preuve est d'établir des relations de log-majorisation, nous donnons également quelques applications pour ces relations, plus précisément une borne supérieure pour une inégalité de type Golden-Thompson établie par Hiai et Petz, et quelques nouveaux résultats liés aux divergences de Rényi. Plus de conjectures concernant les inégalités de valeurs singulières sont étudiées.

Title : Contributions to Matrix Inequalities and Some Applications

Keywords : positive semi-definite matrix, log-Majorization, determinantal inequalities, eigenvalues, geometric mean

Abstract : Matrix theory has been under study for a long time, it has been a fundamental tool in mathematical disciplines presenting interesting and challenging problems. In this thesis, we focus on one class of matrices which is the set of all positive semi-definite matrices. The main topics are determinantal inequalities, eigenvalue and singular value inequalities and the geometric mean of two positive definite matrices. These concepts arise in many research areas and they play a decisive role in information theory, quantum mechanics and other mathematical fields. The starting point of our work is the following two unconfirmed determinantal inequalities introduced by M. Lin (2017) for all positive semi-definite matrices A and B of the same order :

$\det(A^2 + |AB|^p) \geq \det(A^2 + A^p B^p)$ for $0 \leq p \leq 2$;

$\det(A^2 + |AB|^p) \geq \det(A^2 + |BA|^p)$ for $p \geq 0$.

We have shown that second inequality is true for a larger set of matrices; Hermitian matrices. We have also proved a more general result of the first inequality stated as follows:

$\det(A^k + |AB|^p) \geq \det(A^k + A^p B^p)$ for $k \geq 2, 0 \leq p \leq 2$.

The main idea of the proof is to establish log-majorization relations, we also give some applications for these relations, more precisely, an upper bound for a Golden-Thompson type inequality established by Hiai and Petz, and some new results related to the Rényi divergences. More conjectures regarding singular values inequalities are investigated.